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Stabilization of a nonlinear underactuated hovercraft

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SUMMARY

We consider the control of a hovercraft having only two control inputs with three degrees of freedom. The model is obtained from equations of a simplified ship which is nonlinear and underactuated. Using a co-ordinate transformation the model is given by polynomial equations which describe its kinematics and dynamics. Two control laws are proposed. The first one controls the velocity of the hovercraft. The other one stabilizes both its position and the (underactuated) side velocity and provides global convergence to the origin. The convergence analysis is based on a Lyapunov approach. Copyright © 2000 John Wiley & Sons, Ltd.

KEY WORDS: nonlinear systems; underactuated ships; Lyapunov functions

1. INTRODUCTION

Nowadays control problems of underactuated vehicles motivate the development of new nonlinear control design methodologies. Such systems are vehicles with fewer independent control inputs than degrees of freedom to be controlled.

In order to capture the essential nonlinear behaviour of an underactuated ship, we have simplified its model as found in Reference [1]. We have neglected the damping, we have considered that the shape of the ship is symmetric with respect to three axes, mainly a circle, and we have considered that the two propellers are situated at the centre of mass. Therefore, after these simplifications, we obtain the model of a hovercraft which has two propellers to move the vehicle forwards (and backwards) and to make it turn. The main difference with respect to a two-wheel mobile robot is that a hovercraft can move freely sideways even though this degree of freedom is not actuated. The hovercraft model presented here will be used to design a control strategy and the purpose is to promote the development of new control design methods, such as the studies of other highly nonlinear mechanical systems like the ball and beam and the inverted pendulum have done.

We will first consider the problem of regulating the surge, the sway and the angular velocities to zero. We will also propose strategies for positioning the hovercraft at the origin.

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Various control algorithms for controlling underactuated vessels have appeared in the literature. Leonard [2] was the first to control a dynamic autonomous underwater vessel model (AUV model) with force and torque control inputs. It was shown how open-loop periodic time-varying control can be used to control underwater vehicles.

Pettersen and Egeland [3] developed a stability result involving continuous time-varying feedback laws that exponentially stabilize both the position and orientation of a surface vessel having only two control inputs. This result was extended to include integral action in Reference [4].

Other approaches also exist in the literature like the one in Reference [5] which considers a nonlinear ship model including the hydrodynamic effects due to time-varying speed and wave frequency. This involves a non-symmetrical inertia matrix and non-positive damping at high speed. They use a backstepping technique for tracking control design.

Bullo and Leonard [6] develop high-level motion procedures which solve point-to-point reconfiguration, local exponential stabilization and static interpolation problems for underactuated vehicles.

Strand *et al.* [7] propose a stabilizing controller for moored and free-floating ships (but not underactuated) constructed by backstepping. They propose a locally asymptotically convergent algorithm based on H_∞-optimal control. They also present a global result using inverse optimality for the nonlinear system.

Pettersen and Nijmeijer [8] proposed a time-varying feedback control law that provides global practical stabilization and tracking, using a combined integrator backstepping and averaging approach. In Reference [9] they proposed a tracking control law which steers both position and course angle of the surface vessel, providing semi-global exponential stabilization of the desired trajectory.

Berge *et al.* [10] develop a tracking controller for the underactuated ship using partial feedback linearization. The control law makes the position and velocities converge exponentially to the reference trajectory, while the course angle of the ship is not controlled.

One of the difficulties encountered in the stabilization of underactuated vehicles is that classical nonlinear techniques in nonlinear control theory like feedback linearization are not applicable. Therefore, new design methodologies should be explored.

In the present paper we propose two different control strategies. The first controller globally asymptotically stabilizes the surge, sway and angular velocities with a differentiable controller. In this case we consider the surge force and the angular torque as inputs. In the second controller we globally asymptotically stabilizes the position and the sway velocities at the origin using the surge and the angular velocities as inputs. The proposed controller is discontinuous. In both cases the analysis is based on a Lyapunov approach.

The paper is organized as follows. In Section 2, the model of the simplified ship is recalled. Section 3 presents the control algorithm to stop the hovercraft. Section 4 is devoted to the control strategy for positioning of the hovercraft. Section 5 gives simulation results.

2. THE HOVERCRAFT MODEL

The nonlinear model for the underactuated hovercraft presented next is obtained from the ship model in References [1, 8, 9]. We have neglected damping, considered that the shape of the ship is a disc and that the propellers are located at the centre of mass as shown in Figure 1. The

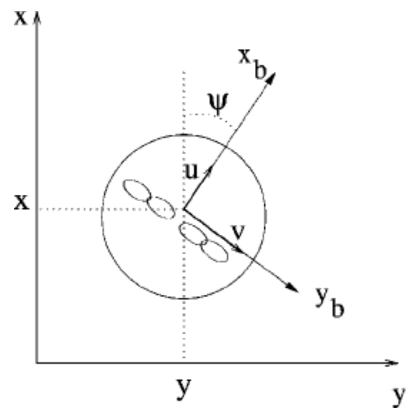

Figure 1. The hovercraft.

kinematics are described by

$$\begin{aligned}\dot{x} &= \cos(\psi)u - \sin(\psi)v \\ \dot{y} &= \sin(\psi)u + \cos(\psi)v \\ \dot{\psi} &= r\end{aligned} \tag{1}$$

where x, y and ψ denote the position and the orientation of the hovercraft in the earth-fixed frame and u, v, r denote the linear velocities in surge, sway and the angular velocity in yaw, respectively.

We will in the following consider the problem of controlling the position, not the yaw angle ψ and thus disregard the latter equation in (1). In order to achieve simpler polynomial kinematic equations and to eliminate ψ, we use the following co-ordinate transformation as in Reference [3], which is a global diffeomorphism:

$$\begin{aligned}z_1 &= \cos(\psi)x + \sin(\psi)y \\ z_2 &= -\sin(\psi)x + \cos(\psi)y \\ z_3 &= \psi\end{aligned} \tag{2}$$

The resulting model is then (see Reference [3])

$$\begin{aligned}\dot{u} &= vr + \tau_u \\ \dot{v} &= -ur \\ \dot{r} &= \tau_r \\ \dot{z}_1 &= u + z_2 r \\ \dot{z}_2 &= v - z_1 r\end{aligned} \tag{3}$$

where τ_u is the control force in surge and τ_r is the control torque in yaw.

Note that in order to obtain a simple model capturing the essential nonlinearities of the ship, we assumed the inertia matrix to be diagonal and equal to the identity matrix. This explains the idea of a hovercraft. Moreover, we cancelled the hydrodynamic damping, which is not essential in controlling the system.

In the second equation of system (3), the right term represents Coriolis and centripetal forces.

2.1. Controllability of the linearized system

Since the third equation ($\dot{r} = \tau_r$) in (3) is directly controllable, let us consider the linearization of the four other equations.

The system can be rewritten as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & r & 0 & 0 \\ -r & 0 & 0 & 0 \\ 1 & 0 & 0 & r \\ 0 & 1 & -r & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\tau_u = AX + B\tau_u$$

We then have

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad AB = \begin{bmatrix} 0 \\ -r \\ 1 \\ 0 \end{bmatrix}, \quad A^2B = \begin{bmatrix} -r^2 \\ 0 \\ 0 \\ -2r \end{bmatrix}, \quad A^3B = \begin{bmatrix} 0 \\ r^3 \\ -3r^2 \\ 0 \end{bmatrix}$$

and $\det(B|AB|A^2B|A^3B) = 4r^4$.

Therefore the linearized system is controllable if $r \neq 0$. A very simple control strategy can be obtained by fixing r to a constant different from zero and computing a linear controller for the input τ_u. This controller will exponentially stabilize (u, v, z_1, z_2) to the origin but r will not converge to zero.

Furthermore, if r is time varying, we could use Silverman's criterion to check the controllability of the system, i.e.

$$\operatorname{rank} C(t) = \left[b(t), \left(A(t) - \frac{\mathrm{d}}{\mathrm{d}t} \right) b(t), \ldots, \left(A(t) - \frac{\mathrm{d}}{\mathrm{d}t} \right)^{n-1} b(t) \right] = 4 \tag{4}$$

In our case, $\det(C(t)) = 4r^4$. Therefore, if r is time varying, the system is controllable at all time if $r(t) \neq 0$, $\forall t$.

3. STABILIZING CONTROL LAW FOR THE VELOCITY OF THE HOVERCRAFT

The dynamics of the system are given as follows:

$$\begin{aligned} \dot{u} &= vr + \tau_u \\ \dot{v} &= -ur \\ \dot{r} &= \tau_r \end{aligned} \tag{5}$$

The objective is to stop the hovercraft, i.e. to control 'the state vector $[u\ v\ r]^{\mathrm{T}}$' with the two inputs 'τ_u and τ_r'. τ_u and τ_r are the surge control force and the yaw control torque provided by the main propellers.

We propose the control law

$$\tau_u = -k_u u \tag{6}$$

$$\tau_r = -ur - k_r(r - v) \tag{7}$$

where k_u and k_r are strictly positive constants. Consider the candidate Lyapunov function

$$V_1(u,v,r) = \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2 + \tfrac{1}{2}(r - r_d)^2 \tag{8}$$

with $r_d = v$. The time derivative of V_1 is then

$$\begin{aligned}\dot{V}_1 &= u(vr + \tau_u) - uvr + (r - v)(\tau_r + ur)\\ &= -k_u u^2 - k_r(r - v)^2\end{aligned} \tag{9}$$

Using LaSalle's invariance principle, we consider the set $\Omega = \{(u,v,r)\colon \dot{V}_1(u,v,r) = 0\} = \{(u,v,r)\colon u = 0, r = v\}$. From (6) we see that $\tau_u = 0$ in Ω and from (5) this implies $r = v$ has to be zero to stay in Ω. Thus, Ω contains no trajectory of (5) other than the trivial trajectory, and the continuous control law in (6) and (7) globally asymptotically stabilizes the origin of the state $[u \;\; v \;\; r]^{\mathrm{T}}$.

4. STABILIZATION OF THE POSITION

4.1. First approach for stabilization

In this section we will develop a control law for positioning the hovercraft using the surge and angular velocities u and r as virtual control inputs. The model in (3) reduces to

$$\begin{aligned}\dot{z}_1 &= u + z_2 r\\ \dot{z}_2 &= v - z_1 r\\ \dot{v} &= -ur\end{aligned} \tag{10}$$

Note that the above system satisfies Brockett's condition while it would not if we have added the equation for the course angle: $\dot{\psi} = r$.

Consider the following candidate Lyapunov function:

$$V_2 = \tfrac{1}{2}z_1^2 + \tfrac{1}{2}z_2^2 + \tfrac{1}{2}v^2 \tag{11}$$

Then

$$\dot{V}_2 = z_1 u + z_2 v - uvr = z_1 u + v(z_2 - ur) \tag{12}$$

We propose

$$\begin{aligned}ur &= z_2 + v\\ u &= -\operatorname{sign}(z_1)\phi\end{aligned} \tag{13}$$

where sign(0) = 1 and ϕ is a positive-definite function defined by

$$\phi = [\tfrac{1}{2}(z_1^2 + z_2^2 + v^2)]^{1/4} \tag{14}$$

The resulting control input r is

$$r = \frac{z_2 + v}{-\operatorname{sign}(z_1)\phi} \tag{15}$$

Obviously r is a discontinuous function. The time derivative of V_2 is given by

$$\dot{V}_2 = -|z_1|\phi - v^2 \tag{16}$$

It follows that $\dot{V}_2$ is negative and so V_2 converges. Therefore z_1, z_2 and v remain bounded. Note that although r in (15) is a discontinuous function, r is bounded on any compact set. Integrating (16) it follows that $\int_0^t v^2\,\mathrm{d}t$ and $\int_0^t |z_1|\phi\,\mathrm{d}t$ are finite. From (10) and (13) it follows that $\dot{v}$ is bounded which implies that v is uniformly continuous. Using Barbalat's Lemma it follows that $v \to 0$. Then, since $\dot{z}_2 = v - z_1 r$ is bounded, z_2 is uniformly continuous. From (10) and (13) we have $\dot{v} = -ur = -z_2 + v$, then $\dot{v}$ is uniformly continuous. It follows that $\dot{v} \to 0$, using Barbalat's Lemma. Using again $\dot{v} = -ur = -z_2 + v$ and $v \to 0$, it also follows that $z_2 \to 0$. Since V_2 converges, it follows that z_1 converges to a constant $z_1(\infty)$. We will study two different cases:

- *Case 1*: If $z_1(\infty) = 0$, the state (z_1, z_2, v) converges asymptotically to the origin and the inputs u and r converge to zero.
- *Case 2*: If $z_1(\infty) \neq 0$ then there exists a finite time T such that

$$|z_1| > \tfrac{1}{2}|z_1(\infty)| \quad \forall t \geqslant T$$

Therefore,

$$\int_T^t |z_1|\phi\,\mathrm{d}t \geqslant \frac{|z_1(\infty)|}{2}\int_T^t \phi(t)\,\mathrm{d}t$$

Since the left-hand side of the above is finite and ϕ is uniformly continuous, it follows from Barbalat's lemma that $\phi \to 0$.

Finally, we conclude that the state (z_1, z_2, v) and the inputs u and r converge asymptotically to zero.

4.2. *Second approach for stabilization of the position*

In this section we present an alternative control scheme for achieving positioning of the hovercraft. The advantage of the control strategy proposed here is that the control inputs are smoother than those of the control proposed in the previous Section 4.1. We will prove also that the state (z_1, z_2, v) and the control inputs u converge to zero. However we will only be able to prove that r remains bounded.

The main idea is to choose u and r such that (see (10))

$$\dot{v} + \dot{z_2} = v - (u + z_1)r \triangleq -(v + z_2) \tag{17}$$

We propose the candidate Lyapunov function

$$V_3 = \tfrac{1}{2}(z_1^2 + z_2^2) + \tfrac{1}{4}(v + z_2)^2 \tag{18}$$

Differentiating (18) and using (17), it follows that (see (10))

$$\begin{aligned}\dot{V}_3 &= z_1 u + z_2 v - \tfrac{1}{2}(v + z_2)^2 \\ &= z_1 u - \frac{v^2}{2} - \frac{z_2^2}{2}\end{aligned} \tag{19}$$

Considering the following control inputs u and r

$$u = -z_1 + \sqrt{\frac{v^2}{4} + \frac{z_2^2}{4}} \tag{20}$$

and

$$r = \frac{4v + 2z_2}{\sqrt{v^2 + z_2^2}} \tag{21}$$

The time derivative of V_3 becomes

$$\dot{V}_3 = -z_1^2 - \tfrac{1}{2}v^2 - \frac{z_2^2}{2} + z_1\sqrt{\frac{v^2}{4} + \frac{z_2^2}{4}}$$

and by completion of squares we get that

$$\dot{V}_3 \leqslant -\tfrac{3}{4}z_1^2 - \tfrac{1}{4}v^2 - \tfrac{1}{4}z_2^2 \tag{22}$$

Since V_3 and $-\dot{V}_3$ are both positive definite and since

$$\tfrac{1}{2}(z_1^2 + v^2 + z_2^2) \leqslant \tfrac{1}{2}z_1^2 + \tfrac{1}{8}v^2 + \tfrac{1}{4}z_2^2 \leqslant V_3$$

and

$$V_3 \leqslant \tfrac{1}{2}z_1^2 + z_2^2 + \tfrac{1}{2}v^2 \leqslant z_1^2 + z_2^2 + v^2$$

We have thus proved that the origin of system (10) is globally exponentially stable. Moreover, u converges to zero and r is bounded ($|r| \leqslant 6$).

4.3. *Third approach for stabilization of the position*

We will finally propose a last alternative control scheme for controlling the position of the hovercraft. This latter is based on the main idea (17) and on the candidate Lyapunov function V_3(18), which we call V_4 in this approach. The advantage of the control strategy presented here is that the state (z_1, z_2, v) converges exponentially to zero whereas the convergence is only asymptotic in Section 4.1. Moreover, both control inputs u and r converge to zero.

Since $\dot{v} + \dot{z}_2 \triangleq -(v + z_2)$ persists, the time derivative of V_4 remains (see (19))

$$\begin{aligned}\dot{V}_4 &= z_1 u + z_2 v - \tfrac{1}{2}(v + z_2)^2 \\ &= z_1 u - \frac{v^2}{2} - \frac{z_2^2}{2}\end{aligned} \tag{23}$$

We propose

$$u = -z_1 - \operatorname{sign}(z_1)\sqrt{|2v + z_2|} \tag{24}$$

and

$$r = -\operatorname{sign}(z_1(2v + z_2))\sqrt{|2v + z_2|} \tag{25}$$

The time derivative of V_4 becomes

$$\dot{V}_4 = -z_1^2 - |z_1|\sqrt{|2v + z_2|} - \tfrac{1}{2}v^2 - \frac{z_2^2}{2} \tag{26}$$

By completion of squares (as in Section 4.2) it is easy to show that this implies that the origin of system (10) is globally exponentially stable. Furthermore, u and r converge to zero.

5. SIMULATION RESULTS

In order to observe the results of the different proposed control laws we have performed simulations.

Figure 2 shows the results for the stabilization of system (5) using the control law in (6) and (7), with $k_u = 1$ and $k_r = 1$. The initial velocities are $u(0) = 10$, $v(0) = 10$ and $r(0) = 1$.

Figure 3 shows the simulations for the stabilization of the position of system (10) with the control law in (13)–(15). The initial positions are $z_1(0) = 0.1$, $z_2(0) = 0.1$ and $v(0) = 0$.

Figure 4 shows the results of the control law in (20)–(21) for system (10), with initial positions $z_1(0) = 10$, $z_2(0) = 10$ and $v(0) = 1$. We can choose larger initial positions, because the control does not saturate since the control law is smoother than those using the *sign*-function.

Finally, Figure 5 shows the results of the control law in (24)–(25) for system (10), with initial positions $z_1(0) = 0.1$, $z_2(0) = 0.1$ and $v(0) = 0$.

6. CONCLUSIONS

We have presented a model of an underactuated hovercraft with three degrees of freedom and two control inputs. We have proposed a control scheme based on a Lyapunov approach to stabilize

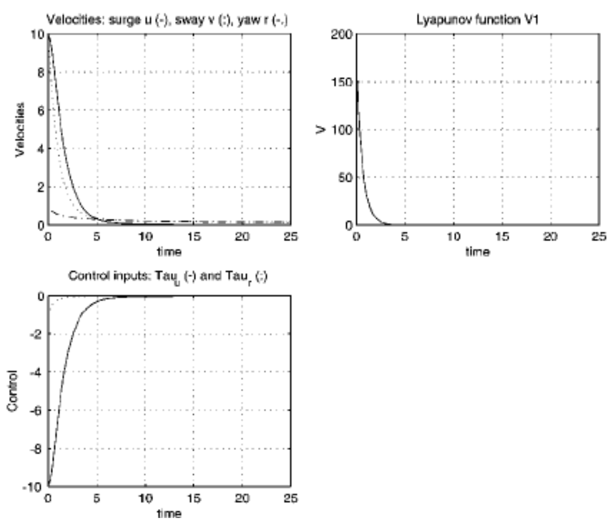

Figure 2. Control of the velocity using the algorithm in Section 3.

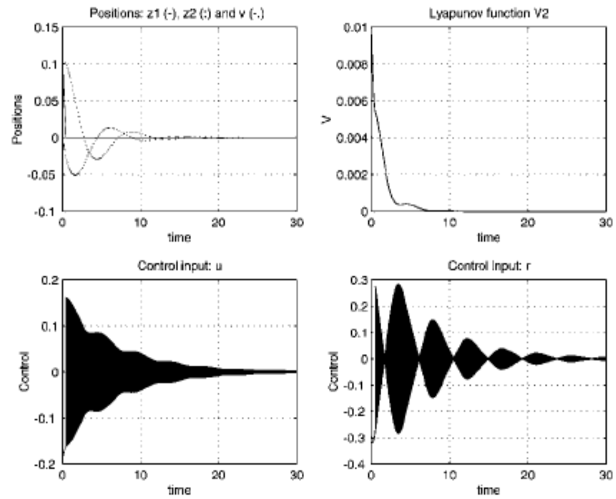

Figure 3. Stabilization of the position using the algorithm in Section 4: controller (13)–(15).

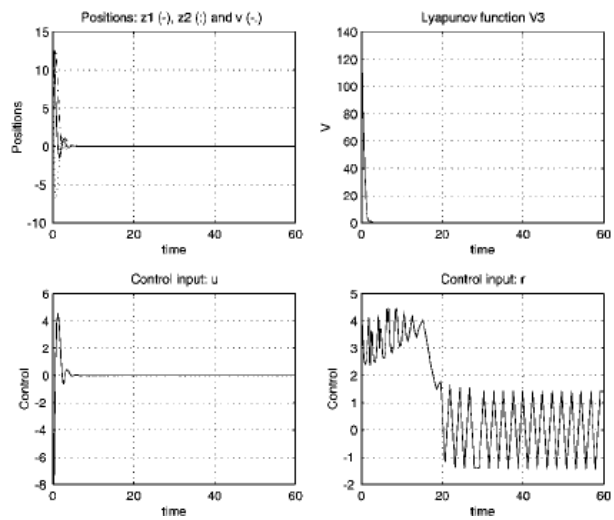

Figure 4. Stabilization of the position using the algorithm in Section 4: controller (20)–(21).

the surge, sway and angular velocities. We have also proposed three control strategies for positioning the vehicle using the surge and the angular velocity as virtual inputs. The three positioning controllers are discontinuous. One of the controllers is such that the origin is globally asymptotically stable and the two inputs converge to zero. The second controller is such that the origin is globally exponentially stable and one of the inputs (u) converges to zero while the other (r) is only proved to be bounded. The third controller is such that the origin is globally exponentially stable and both inputs (u) and (r) converge to zero. The proposed control presents an undesired chattering behaviour. Further studies are underway to better understand the control of the underactuated hovercraft model presented in this paper. Modifications are still

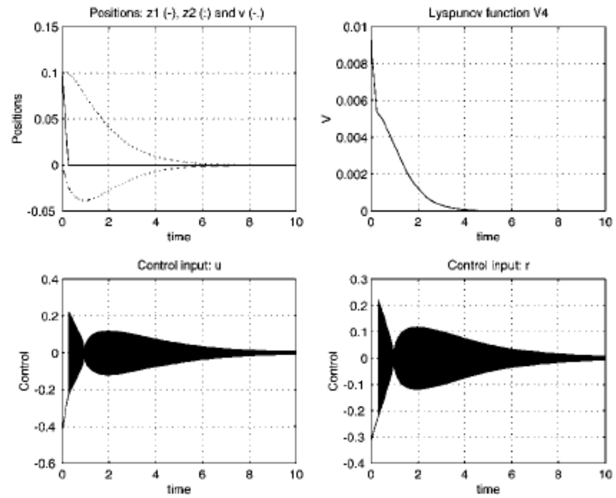

Figure 5. Control of the velocity using the algorithm in Section 4: (24)–(25)

required to reduce the high-frequency oscillations observed in simulations in order to render the controller applicable to a real system.

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