Raport 1

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title

1 Information and formulas

Stable random variable

There are two parameterizations of a random variable from an alpha stable distribution $S(\alpha, \beta, \gamma, \delta; 0)$ and $S(\alpha, \beta, \gamma, \delta; 1)$. They are uniquely determined by the characteristic function.

Definition 1. *A random variable X is stable if and only if $X =^d aZ + b$, with $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $a \neq 1$, $b \in \mathbb{R}$ and Z is a random variable with characteristic function*

$$\varphi_Z(u) = \exp(iuZ) = \begin{cases} \exp\left(-|u|^\alpha (1 - i\beta \tan(\frac{\pi\alpha}{2})(\operatorname{sign} u)\right) & \alpha \neq 1, \\ \exp\left(-|u|^\alpha (1 + i\beta \frac{2}{\pi}(\operatorname{sign} u) \ln |u|\right) & \alpha = 1. \end{cases} \tag{1}$$

Definition 2. *Let $X \sim S(\alpha, \beta, \gamma, \delta; 0)$ with $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\gamma \geqslant 0$, $\delta \in \mathbb{R}$ then*

$$X =^d \begin{cases} \gamma(Z - \beta \tan(\frac{\pi\alpha}{2}) + \delta) & \alpha \neq 1, \\ \gamma Z + \delta & \alpha = 1, \end{cases}$$

where $Z = Z(\alpha, \beta)$ is given by 1.

Definition 3. *Let $X \sim S(\alpha, \beta, \gamma, \delta; 1)$ with $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\gamma \geqslant 0$, $\delta \in \mathbb{R}$ then*

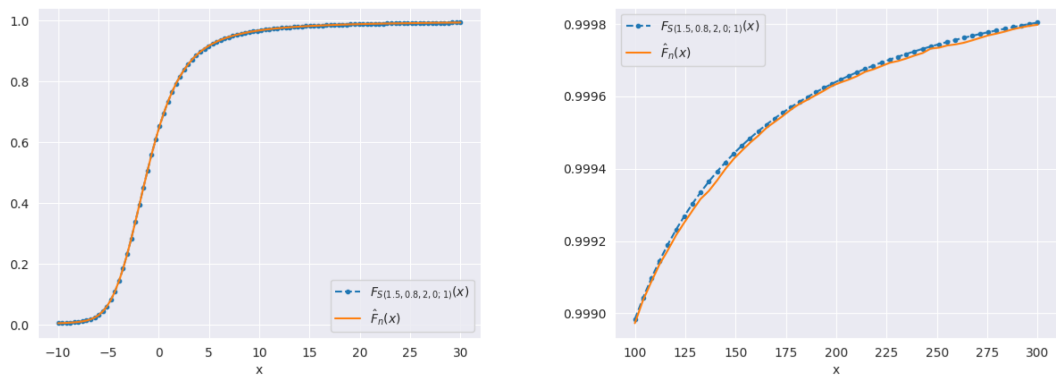
$$X =^d \begin{cases} \gamma Z + \delta & \alpha \neq 1, \\ \gamma Z + (\delta + \beta \frac{2}{\pi} \ln \gamma) & \alpha = 1, \end{cases}$$

where $Z = Z(\alpha, \beta)$ is given by 1.

Above we defined the general stable law in the 0-parameterization and 1-parameterization. Alternatively, we can swap between the parameterizations using the following theorem;

Theorem 1. *Let $Z \sim S(\alpha, \beta, 1, 0; 0)$ with $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\gamma \geqslant 0$, $\delta \in \mathbb{R}$ then*

$$\begin{cases} \gamma Z + \delta + \beta\gamma \tan\left(\frac{\pi\alpha}{2}\right) & \alpha \neq 1, \\ \gamma Z + \delta + \beta \frac{2}{\pi} \ln \gamma & \alpha = 1 \end{cases} \sim S(\alpha, \beta, \gamma, \delta; 1),$$



(a) C

(b) B

Rysunek 1: A

The tail exponent estimation method gives us the information about the index of stability. The tails of stable random variable are asymptotically power laws.

Theorem 2 (Tail approximation)**.** *Let $X \sim S(\alpha, \beta, \gamma, \delta; k)$ with $\alpha \in (0, 2)$, $\beta \in (-1, 1]$, $k = 0, 1$ then as $x \to \infty$:*

$$\begin{aligned} 1 - F_X(x) &\sim \gamma^\alpha c_a (1 + \beta) x^{-\alpha}, \\ f_X(x) &\sim \alpha \gamma^\alpha c_a (1 + \beta) x^{-(\alpha+1)} \end{aligned}$$

where $c_a = \sin(\frac{\pi\alpha}{2})\Gamma(\alpha)/\pi$ and $f(x) \sim g(x)$ as $x \to a$ means $\lim_{x\to a} h(x)/f(x) = 1$. Using the reflection property, the lower tail properties are similar: for $\beta \in [-1, 1)]$ as $x \to \infty$:

$$\begin{aligned} F_X(-x) &\sim \gamma^\alpha c_a (1 - \beta) x^{-\alpha}, \\ f_X(-x) &\sim \alpha \gamma^\alpha (1 - \beta) c_a x^{-(\alpha+1)} \end{aligned}$$

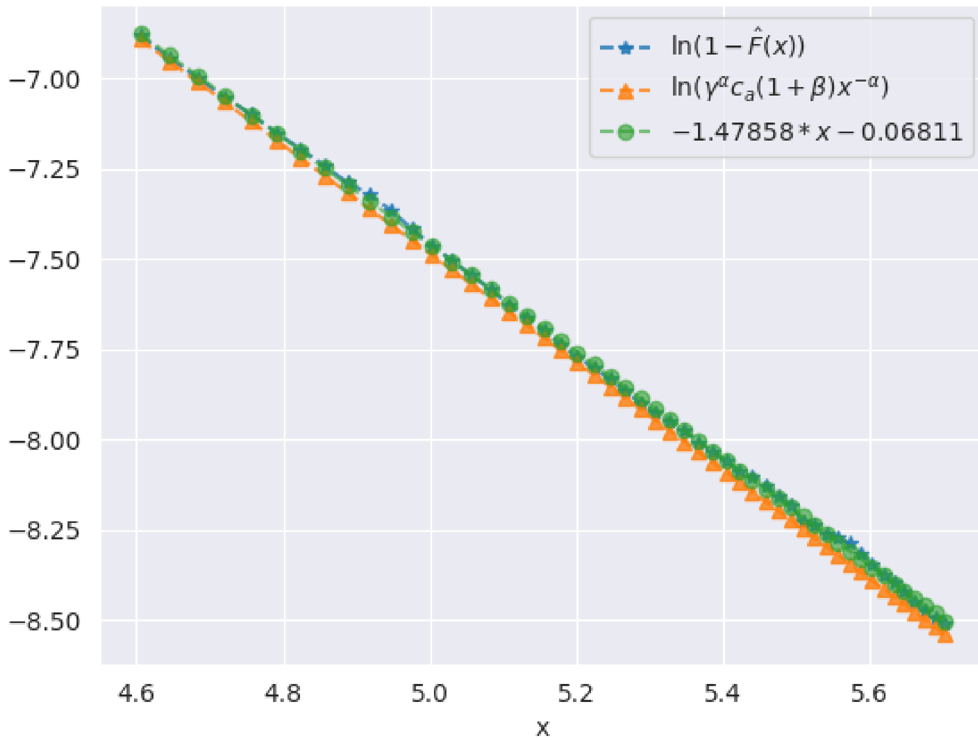
Definition 4. *Let $X_1, \ldots, X_n$ be i.i.d. random variables with the common cumulative distribution function $F(t)$. Then the empirical distribution function is defined as*

$$\hat{F_n}(x) = \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{\{X_k \leqslant x\}}$$

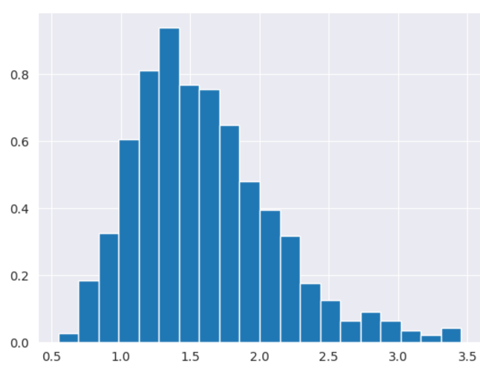
2 Compare two estimators of α parameter we introduced in the laboratories

2.1 Based on the ECDF

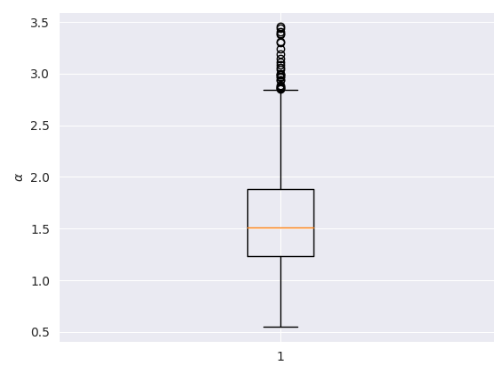
We will be conducting our simulations on consider one set of parameters $(\alpha, \beta, \gamma, \delta) = (1.5, 0.8, 2, 0)$.



Rysunek 2: C



(a) C



(b) B

Rysunek 3: A