Report 2

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0 Information and formulas

0.1 Discreate alpha-stable vectore

In case when Γ is a discrete spectral measure with a finite number of point masses, i.e.,

$$\Gamma(\cdot)=\sum_{j=1}^{n}\gamma_j\delta_{\mathbf{s}_j}(\cdot)$$

where γ_j are the weights, and $\delta_{\mathbf{s}_j}$'s are point masses at the points $\mathbf{s}_j\in S_d, j=1,2\ldots,n$. $\left(S_d=\left\{\mathbf{x}\in\mathbb{R}^d:\|x\|=1\right\}\right)$ For such discrete spectral measure (1), the characteristic function of $\mathbf{X}\sim S_{\alpha,d}\left(\Gamma,\mu^0=\mathbf{0}\right)$

$$\mathbb{E}\exp\{i\langle\mathbf{X},\mathbf{t}\rangle\}$$

takes the form

$$\phi^*(\mathbf{t})=\exp\left(-\sum_{j=1}^{n}\psi_\alpha\left(\langle\mathbf{t},\mathbf{s}_j\rangle\right)\gamma_j\right)$$

where ψ_α is given by

$$\psi_\alpha(u)=\begin{cases}|u|^\alpha(1-i\operatorname{sign}(u))\tan\frac{\pi\alpha}{2}, & \alpha\neq 1,\\ |u|\left(1+i\frac{2}{\pi}\operatorname{sign}(u)\right)\log|u|, & \alpha=1.\end{cases}$$

Following result from Modarres and Nolan, if $\mathbf{X}$ has a characteristic function (3), then:

$$\mathbf{X}\overset{D}{=}\begin{cases}\sum_{j=1}^{n}\gamma_j^{1/\alpha}Z_j\mathbf{s}_j, & \alpha\neq 1,\\ \sum_{j=1}^{n}\gamma_j^{1/\alpha}\left(Z_j+\frac{2}{\pi}\log\gamma_j\right)\mathbf{s}_j, & \alpha=1,\end{cases}$$

where $Z_1,Z_2,\ldots,Z_n$ are iid totally skewed, standardized one dimensional α-stable random variables, i.e. $Z_i\sim S_\alpha\left(\beta=1,\gamma=1,\delta=0\ \right)$. (When $\mu^{\mathbf{0}}\neq\mathbf{0}$, both cases above have a $+\mu^{\mathbf{0}}$ in them.)

0.2 Stable vector

The characteristic function of $\mathbf{X}\sim S_{\alpha,d}\left(\Gamma,\mu^0\right)$ is

$$\phi_{\mathbf{X}}(\mathbf{t})=\mathbf{E}\exp\{i<\mathbf{X},\mathbf{t}>\}=\exp\left(-I_{\mathbf{X}}(\mathbf{t})+i<\mu^0,t>\right),$$

where the function in the exponent is

$$I_{\mathbf{X}}(\mathbf{t}) = \int_{S_d} \psi_\alpha(<\mathbf{t}, \mathbf{s}>)\Gamma(ds).$$

Here $<\mathbf{t}, \mathbf{s}>= t_1 s_1 + \cdots + t_d s_d$ is the inner product, and

$$\psi_\alpha(u) = \begin{cases} |u|^\alpha \left(1 - i \operatorname{sign}(u) \tan \frac{\pi\alpha}{2}\right) & \alpha \neq 1 \\ |u| \left(1 + i \frac{2}{\pi} \operatorname{sign}(u) \ln |u|\right) & \alpha = 1 \end{cases}$$

0.3 Sub-Gaussian vectors

Choose a random variable $A \sim S_{\alpha/2}\left(\gamma = \left(\cos \frac{\pi\alpha}{4}\right)^{2/\alpha}, \beta = 1, \delta = 0\right)$ with $\alpha < 2$. Let $\mathbf{G} = [G_1, G_2, \ldots, G_d]$ be a zero mean Gaussian vector in $\mathbb{R}^d$ independent of A. Then the random vector

$$\mathbf{X} = \left[A^{1/2}G_1, A^{1/2}G_2, \ldots, A^{1/2}G_d\right]$$

has symmetric α-stable distribution in $\mathbb{R}^d$ and is called a sub-Gaussian SaS random vector in $\mathbb{R}^d$ with underlying Gaussian vector $\mathbf{G}$. Your goal is to check if following statements, assuming the underlying Gaussian vector $\mathbf{G}$ has i.i.d. components with variance σ^2

0.4 Estimatation of spectral measure

We use the Rachev-Xin-Cheng method (RXC). A value is picked for r and it is used to estimate the measure of as set $A \subset S_d$ by:

$$\widehat{\Gamma}(A) = \text{ const. } \frac{\#\left\{\mathbf{X}_i : |\mathbf{X}_i| > r, \mathbf{X}_i \in \operatorname{Cone}(A)\right\}}{\#\left\{\mathbf{X}_i : |\mathbf{X}_i| > r\right\}}.$$

0.5 Estimatation of characteristic function

The empirical characteristic function (ECF) method is straightforward. Given an i.i.d. sample $\mathbf{X}_1, \ldots, \mathbf{X}_k$ of α-stable random vectors with spectral measure Γ, let $\widehat{\varphi}_k(\mathbf{t})$ and $\widehat{I}_k$ be the empirical counterparts of ϕ and I, i.e. $\widehat{\varphi}_k(\mathbf{t}) = (1/k)\sum_{j=1}^k \exp\left(i < \mathbf{t}, \mathbf{X}_j >\right)$ is the sample characteristic function, and $\widehat{I}_k(\mathbf{t}) = -\ln \widehat{\varphi}_k(\mathbf{t})$. Given a grid $\mathbf{t}_1, \ldots, \mathbf{t}_n \in S_d, \vec{I}_{ECF,k} = \left[\hat{I}_k\left(\mathbf{t}_1\right), \ldots, \hat{I}_k\left(\mathbf{t}_n\right)\right]'$ is the ECF estimate of $I_{\mathbf{X}}(\cdot)$.

0.6 Codifference estimator

Let us define the codiffence between random variables X and Y in the following way:

$$\tau_{X,Y} = \log(\mathbb{E}\exp(i(X - Y))) - \log(\mathbb{E}\exp(iX)) - \log(\mathbb{E}\exp(-iY))$$

1 Multivariate stable distribution generator

We checked if our generator was impolemented correctly, by generating 20000 of two-dim vectors Aby sprawdzić poprawność naszego generatora, wygenerowaliśmy po 20000 supplies dwu wymiarowych wektorów alpha stabilnych dla każdego z poniżej podanych typów.

1.1 Symmetric stable vector

For following parameters we generated vectors and we created plot 1 - dependence of x and y obtained in vectors. Symmetric case $\alpha = 0.9$ and $n = 6$ point masses

$$\begin{aligned}
\gamma_1 &= 0.25 \text{ at } \mathbf{s}_1 = (1, 0) \\
\gamma_2 &= 0.125 \text{ at } \mathbf{s}_2 = (1/2, \sqrt{3}/2) \\
\gamma_3 &= 0.25 \text{ at } \mathbf{s}_3 = (-1/2, \sqrt{3}/2) \\
\gamma_4 &= 0.25 \text{ at } \mathbf{s}_4 = (-1, 0) \\
\gamma_5 &= 0.125 \text{ at } \mathbf{s}_5 = (-1/2, -\sqrt{3}/2) \\
\gamma_6 &= 0.25 \text{ at } \mathbf{s}_6 = (1/2, -\sqrt{3}/2)
\end{aligned}$$

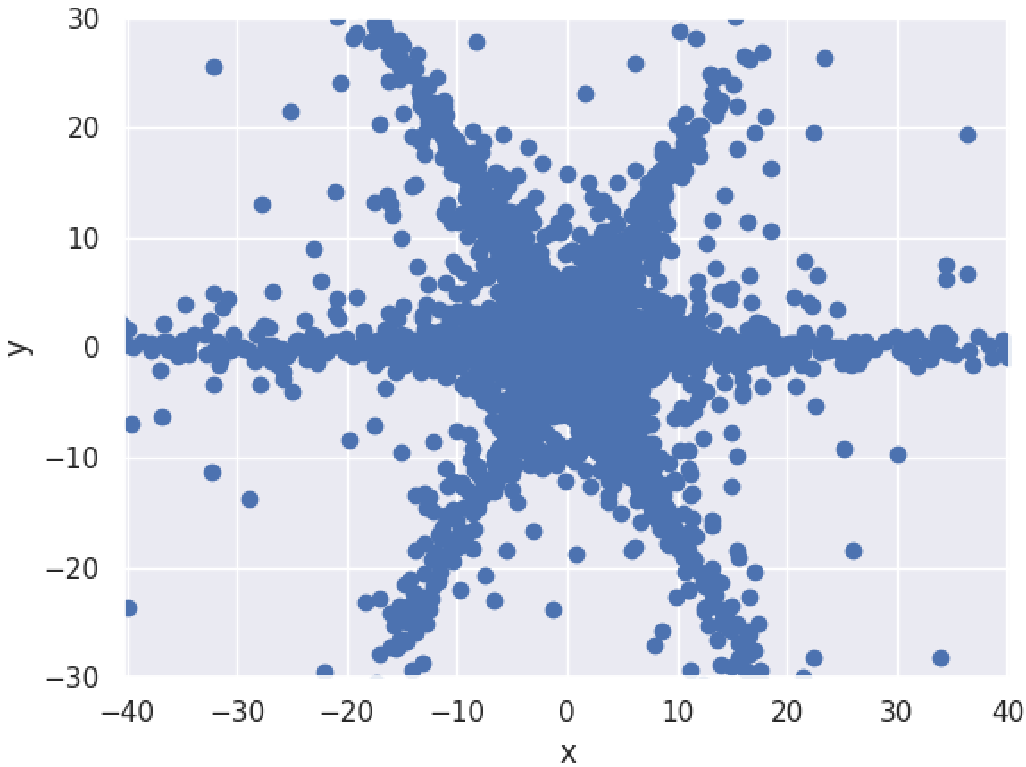


Figure 1: Result of simulation of symmetric stable vector for $\alpha = 0.9$ and $n = 6$.

We can observe, that points are located on six diffeent direction lines, so we can conclude, that the implementation of our generator in this case should be correct.

1.2 Stable vector with independent components

In this case we generated vectors using formula from 0.3.

We used following covariance matrix

$$\Sigma(G) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

.

By using covariance matrix with zeros on anti-diagonal, we obtain i.i.d. vectors A and G and elements of generated gaussian vector are independent.

We are checking the correctness of the simulation in section 2.

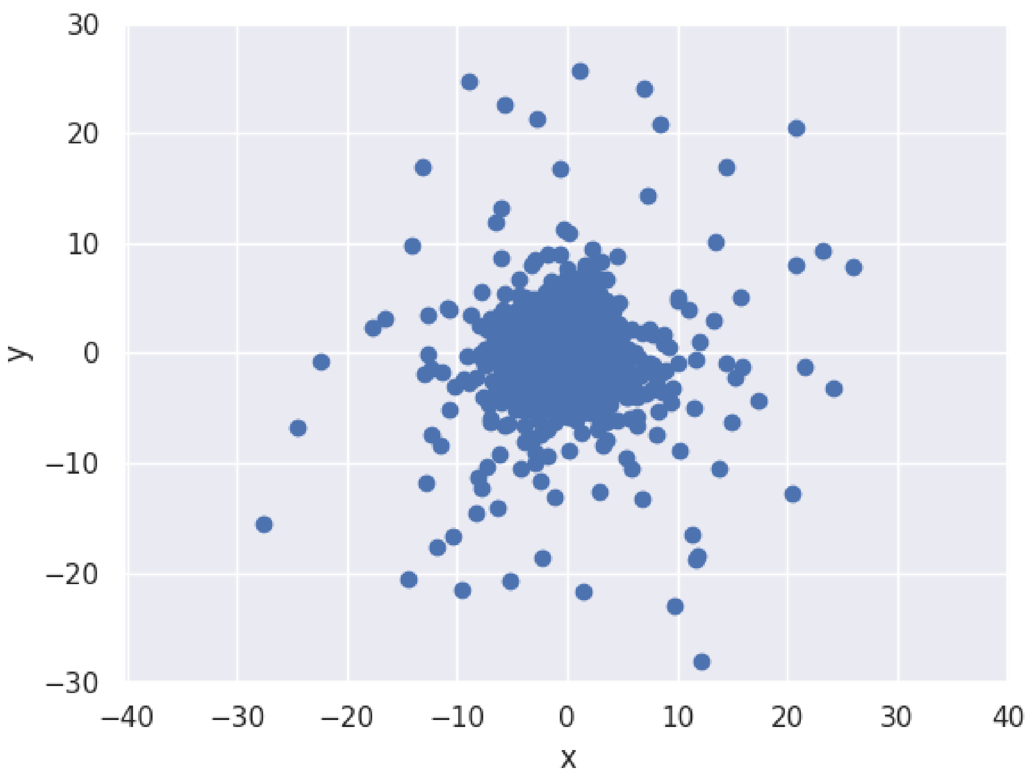


Figure 2: Result of stable vector with independent components for $\alpha = 1.6$.

Thanks to results we obtained and are depicted on graph 2, we can conclude, that in this case, vectors are also simulated correctly.

1.3 Stable vector which is not symmetric and has not independent components.

In this case matrix of cavariance is equal:

$$\Sigma(G) = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.7 \end{bmatrix},$$

so covariance of vector G is not zero and elements of this vector are not independent. It means, that vector components of vector X are not independent. What's more, in vector G, variance of first element is different from the second one, so generated vectors are not symmetric.

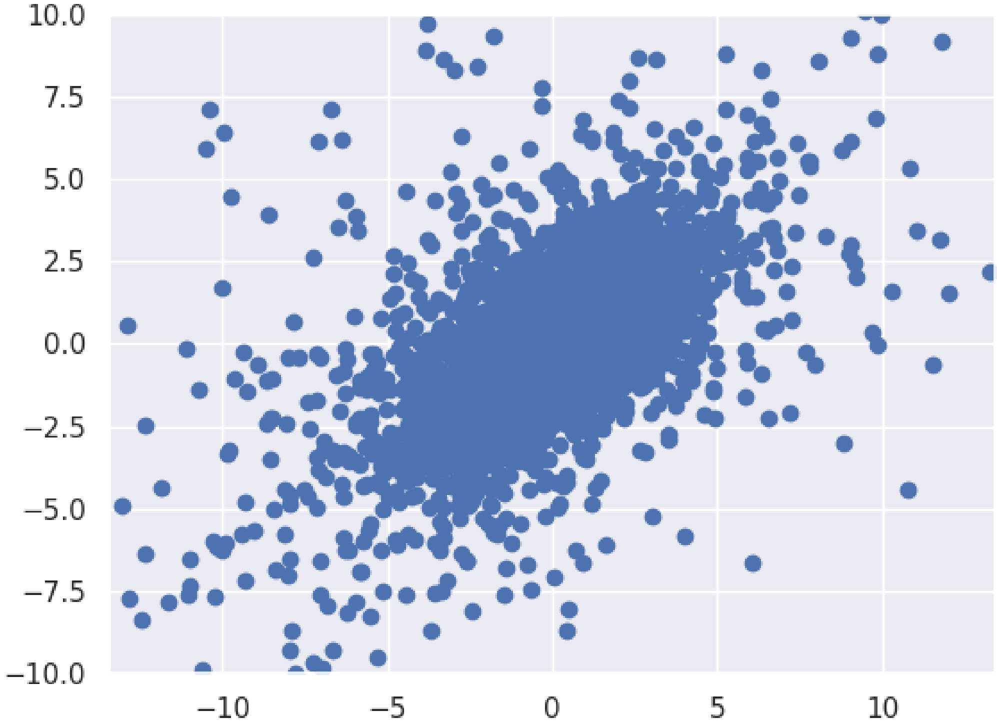
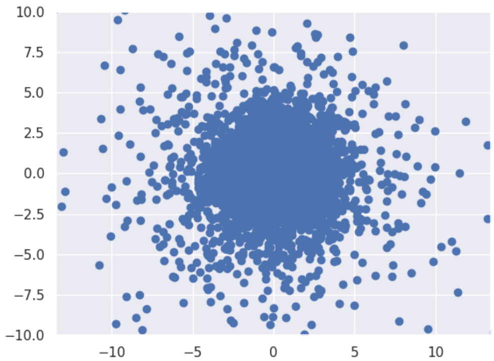
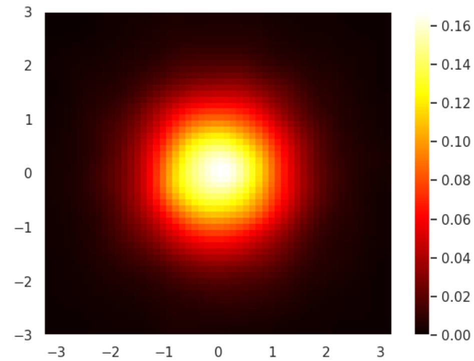


Figure 3: Result of stable vector which is not symmetric and has not independent components for $\alpha = 1.6$.

The results, shown on the graph 3 confirm the theoretical assumptions, so we can assume that our implementation works correctly in this case as well.



(a) Generated sub-gaussian random vectors



(b) Two-dimentional density of generated vectors.

2 Sub-Gaussian random vector generator.

To check if our implementation of sub-gaussian random vector generator, we generated 20000 of random vectors using method described in subsection 0.3, where $\alpha = 1.6$ and following covariance matrix:

$$\Sigma(G) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

.

At the graph 4a we presented the dependences of first and second values of generated vectors. In addition, on graph 4b, we presented empirical two-dimentional density of the scatterplot. We can see, that results are symmetic.

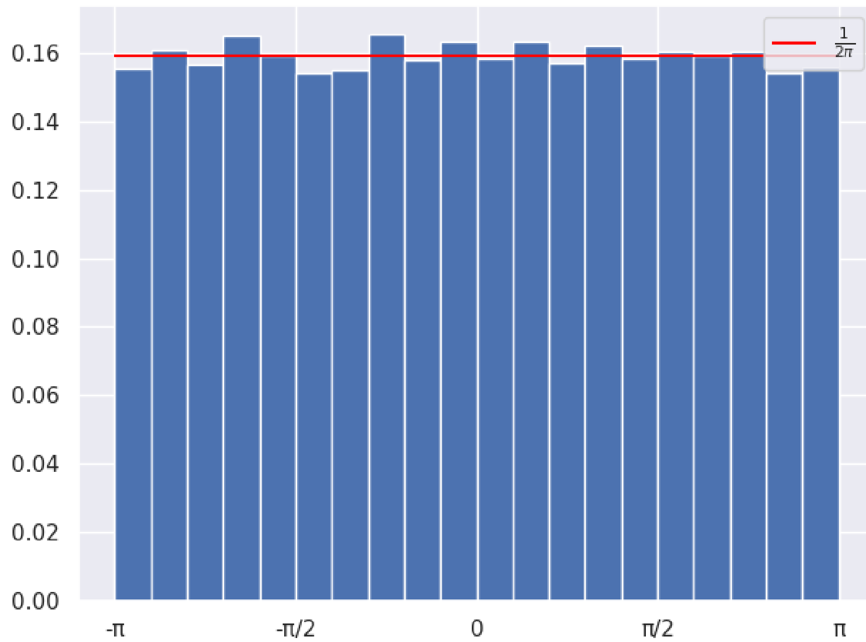


Figure 5: Estimated spectral measure.

We also checked spectral measure, the results are depicted at the graph 4b, where a horizontal red line is equal to $\frac{1}{2\pi}$ what is the value of density of uniform distribution from $-\pi$ to π. We also checked using KS-test, if results are uniformly distributed. We got p-value $= 0.96$, so based on the this graph and the result of KS-test, we can conclude, that spectral measure is uniformly distributed.

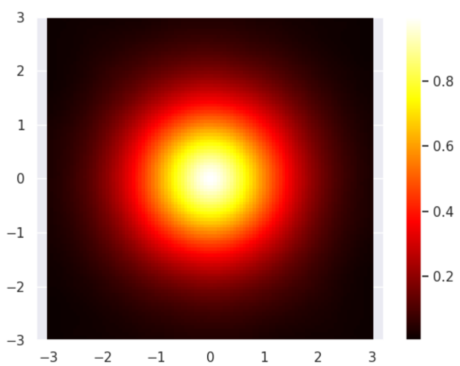
We estimated CF of SaS random vector, which is presented at graph 6a and at graph 6b we have included errors between the estimated and the theoretical function. Additionally, in table 1, we included basic statistics of this errors. We can observe, that errors are really small.

count	mean	std	min	25%	50%	75%	max
10000.0	-0.000252	0.00322	-0.00809	-0.002605	-0.000049	0.001813	0.007082

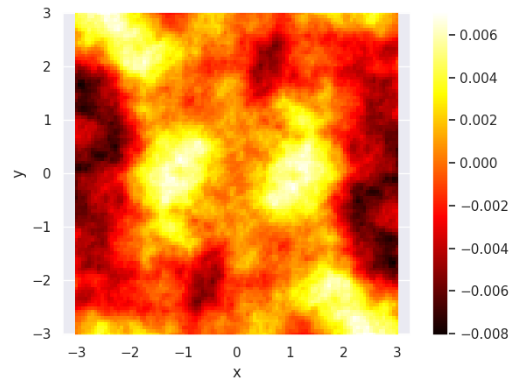
Table 1: Basic statistics of errors between the estimated and the theoretical characteristic function.

3 Estimation of α and spectral measure Γ.

Using the method described in 0.5, we were able to fit correct function to double-logarithm charactersitic function and based on knowledge of CF from section 4 we estimated parameter α.



(a) ECF of generated vectors.



(b) Errors of comparsion of theoretical and empirical CF of SaS random vector

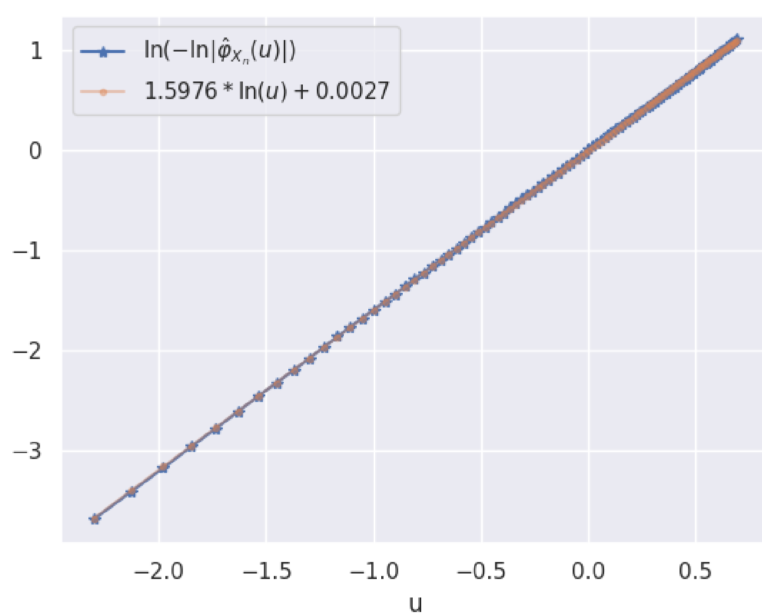
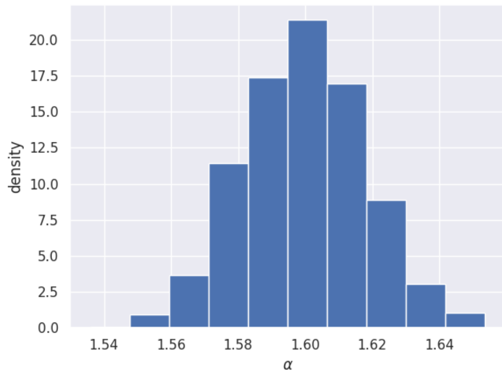


Figure 7: Fitted function to empirical double-logarithm CF.

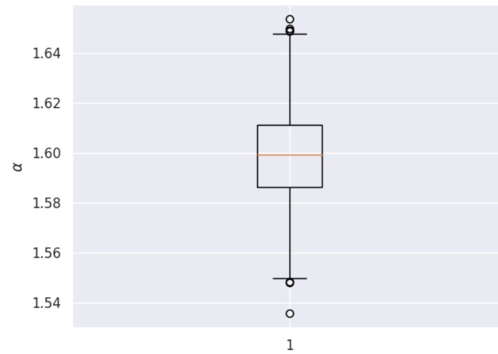
In this case we used the same parameters as in section 2. We estimated α 10000 times using samples with size of 10000.

We created two graphs (8a, 8b), histogram and boxplot of estimated parameter α.

Basic statistics of our estimations we have located in table 2



(a) Histogram of estimated α



(b) Boxplot of estimated α

Figure 8: Estimaton's results of α

count	mean	std	min	25%	50%	75%	max
1000.0	1.599539	0.01839	1.535825	1.586632	1.599462	1.611326	1.65366

Table 2: Basic statistics of esitmated α.

We can see, that mean is close to true α and variance is low. We obtained also small range.

4 Estimation of the characteristic function for multivariate data

Using method described in 0.5 by function $I(x)$ We are able to estimate CF for stable vector. As an example we estimated CF for vectors with the same parameters as in section 2. (Graph 9)

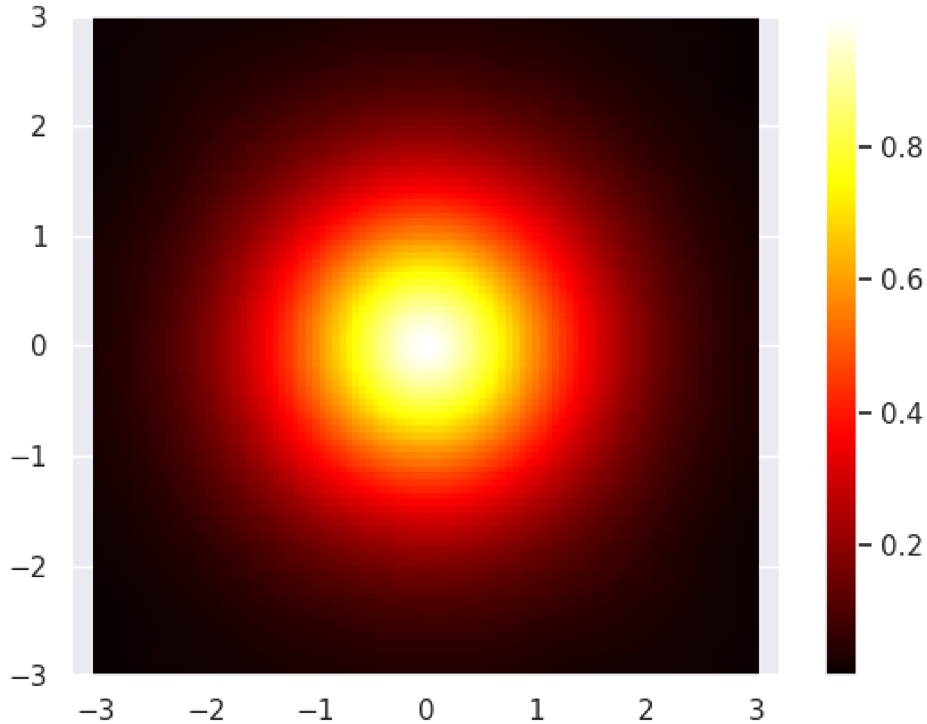


Figure 9: CF of sub-gaussian

5 Estimation of codifference measure

To check correctness of our estimator implementation, we have generated 100000 samples thanks which we estimated codifference for particular cases.

5.1 First case

Parameters:

$$\begin{aligned}\alpha &= 1.6\\ \gamma_1 &= 0.25 \text{ at } \mathbf{s}_1 = (1,0),\\ \gamma_2 &= 0.25 \text{ at } \mathbf{s}_2 = (0,1),\\ \gamma_3 &= 0.25 \text{ at } \mathbf{s}_3 = (-1,0),\\ \gamma_4 &= 0.25 \text{ at } \mathbf{s}_4 = (0,-1).\end{aligned}$$

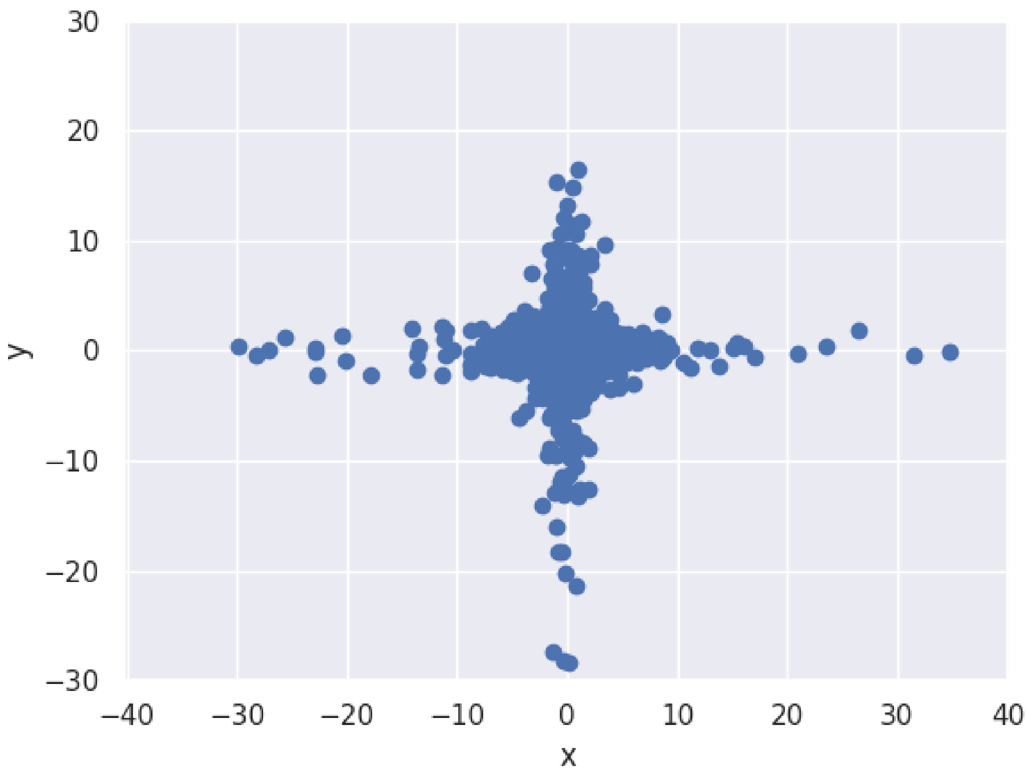


Figure 10: CF of sub-gaussian

Codifference = 0.0158

5.2 Second case

Parameters:

$$\begin{aligned}
\alpha &= 0.9 \\
\gamma_1 &= 0.25\text{at } \mathbf{s}_1 = (\sqrt{2}/2, \sqrt{2}/2), \\
\gamma_2 &= 0.25 \text{ at } \mathbf{s}_2 = (-\sqrt{2}/2, \sqrt{2}/2), \\
\gamma_3 &= 0.25 \text{ at } \mathbf{s}_3 = (-\sqrt{2}/2, -\sqrt{2}/2), \\
\gamma_4 &= 0.25 \text{ at } \mathbf{s}_4 = (\sqrt{2}/2, -\sqrt{2}/2).
\end{aligned}$$

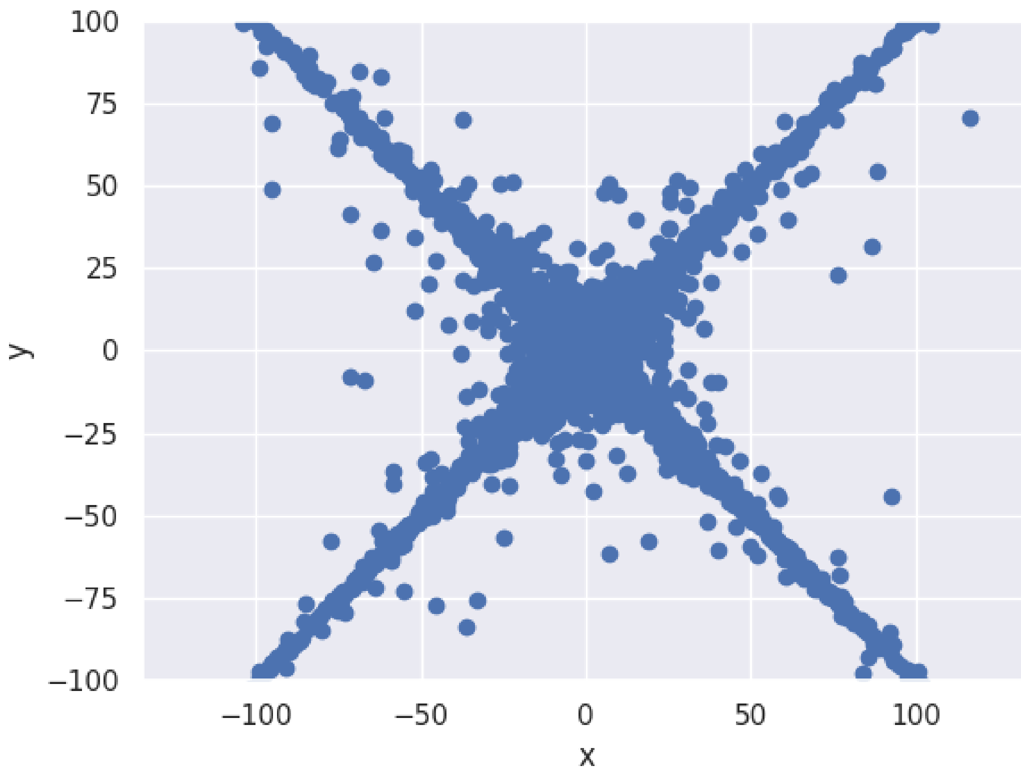


Figure 11: CF of sub-gaussian

Codifference = 0.79

5.3 Third case

Sub-Gaussian vactor with $\alpha = 1.6$ covariance matrix:

$$\Sigma(G) = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

.

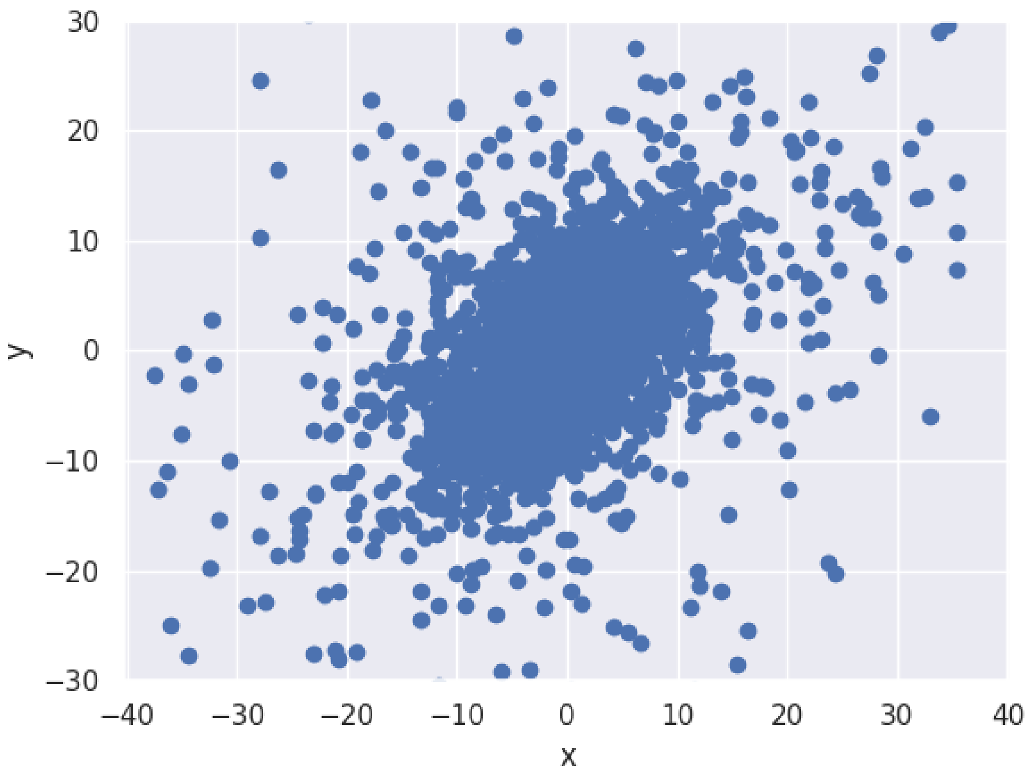


Figure 12: CF of sub-gaussian

Codifference = 0.5778

6 Summary

We checked correctness of implementation of all methods and estimators. In every case we obtained good results.