Report 2

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0 Information and formulas

0.1 Discrete alpha-stable vector

In case when Γ is a discrete spectral measure with a finite number of point masses, i.e.,

$$\Gamma(\cdot) = \sum_{j=1}^{n} \gamma_j \delta_{\mathbf{s}_j}(\cdot), \tag{1}$$

where γ_j are the weights, and $\delta_{\mathbf{s}_j}$ are point masses at the points $\mathbf{s}_j \in S_d, j = 1, 2 \dots, n$. $\left(S_d = \left\{\mathbf{x} \in \mathbb{R}^d : \|x\| = 1\right\}\right)$. For such discrete spectral measure (1), the characteristic function of $\mathbf{X} \sim S_{\alpha,d}\left(\Gamma, \mu^0 = \mathbf{0}\right)$

$$\mathbb{E} \exp\{i\langle \mathbf{X}, \mathbf{t}\rangle\},$$

takes the form

$$\phi^*(\mathbf{t}) = \exp\left(-\sum_{j=1}^{n} \psi_\alpha\left(\langle \mathbf{t}, \mathbf{s}_j\rangle\right)\gamma_j\right), \tag{2}$$

where ψ_α is given by

$$\psi_\alpha(u) = \begin{cases} |u|^\alpha (1 - i \operatorname{sign}(u)) \tan \frac{\pi\alpha}{2}, & \alpha \neq 1, \\ |u| \left(1 + i\frac{2}{\pi} \operatorname{sign}(u)\right) \log |u|, & \alpha = 1. \end{cases}$$

Following result from Modarres and Nolan, if $\mathbf{X}$ has a characteristic function (2), then

$$\mathbf{X} \overset{D}{=} \begin{cases} \sum_{j=1}^{n} \gamma_j^{1/\alpha} Z_j \mathbf{s}_j, & \alpha \neq 1, \\ \sum_{j=1}^{n} \gamma_j^{1/\alpha} \left(Z_j + \frac{2}{\pi} \log \gamma_j\right) \mathbf{s}_j, & \alpha = 1, \end{cases}$$

where $Z_1, Z_2, \dots, Z_n$ are i.i.d. totally skewed, standardized one dimensional α-stable random variables, i.e. $Z_i \sim S_\alpha\left(\beta = 1, \gamma = 1, \delta = 0\ \right)$. (When $\mu^{\mathbf{0}} \neq \mathbf{0}$, both cases above have a $+\mu^{\mathbf{0}}$ in them.)

0.2 Stable vector

The characteristic function of $\mathbf{X} \sim S_{\alpha,d}\left(\Gamma, \mu^0\right)$ is

$$\phi_{\mathbf{X}}(\mathbf{t}) = \mathbf{E} \exp\{i < \mathbf{X}, \mathbf{t} >\} = \exp\left(-I_{\mathbf{X}}(\mathbf{t}) + i < \mu^0, t >\right),$$

where the function in the exponent is

$$I_{\mathbf{X}}(\mathbf{t}) = \int_{S_d} \psi_\alpha(<\mathbf{t}, \mathbf{s}>)\Gamma(ds).$$

Here $<\mathbf{t}, \mathbf{s}>= t_1 s_1 + \cdots + t_d s_d$ is the inner product, and

$$\psi_\alpha(u) = \begin{cases} |u|^\alpha \left(1 - i \operatorname{sign}(u) \tan \frac{\pi\alpha}{2}\right) & \alpha \neq 1, \\ |u| \left(1 + i \frac{2}{\pi} \operatorname{sign}(u) \ln |u|\right) & \alpha = 1. \end{cases}$$

0.3 Sub-Gaussian vectors

Choose a random variable $A \sim S_{\alpha/2}\left(\gamma = \left(\cos \frac{\pi\alpha}{4}\right)^{2/\alpha}, \beta = 1, \delta = 0\right)$ with $\alpha < 2$. Let $\mathbf{G} = [G_1, G_2, \ldots, G_d]$ be a zero mean Gaussian vector in $\mathbb{R}^d$ independent of A. Then the random vector

$$\mathbf{X} = \left[A^{1/2}G_1, A^{1/2}G_2, \ldots, A^{1/2}G_d\right], \tag{3}$$

has symmetric α-stable distribution in $\mathbb{R}^d$ and is called a sub-Gaussian SaS random vector in $\mathbb{R}^d$ with underlying Gaussian vector $\mathbf{G}$. Your goal is to check if the following statements, assuming the underlying Gaussian vector $\mathbf{G}$ has i.i.d. components with variance σ^2.

0.4 Estimation of spectral measure

We use the Rachev-Xin-Cheng method (RXC). A value is picked for r and it is used to estimate the measure of as set $A \subset S_d$ by

$$\widehat{\Gamma}(A) = \text{ const. } \frac{\#\left\{\mathbf{X}_i : |\mathbf{X}_i| > r, \mathbf{X}_i \in \operatorname{Cone}(A)\right\}}{\#\left\{\mathbf{X}_i : |\mathbf{X}_i| > r\right\}}.$$

0.5 Estimation of characteristic function

The empirical characteristic function (ECF) method is straightforward. Given an i.i.d. sample $\mathbf{X}_1, \ldots, \mathbf{X}_k$ of α-stable random vectors with spectral measure Γ, let $\widehat{\varphi}_k(\mathbf{t})$ and $\widehat{I}_k$ be the empirical counterparts of ϕ and I, i.e. $\widehat{\varphi}_k(\mathbf{t}) = (1/k)\sum_{j=1}^k \exp\left(i < \mathbf{t}, \mathbf{X}_j >\right)$ is the sample characteristic function, and $\widehat{I}_k(\mathbf{t}) = -\ln \widehat{\varphi}_k(\mathbf{t})$. Given a grid $\mathbf{t}_1, \ldots, \mathbf{t}_n \in S_d, \vec{I}_{ECF,k} = \left[\hat{I}_k\left(\mathbf{t}_1\right), \ldots, \hat{I}_k\left(\mathbf{t}_n\right)\right]'$ is the ECF estimate of $I_{\mathbf{X}}(\cdot)$.

0.6 Codifference estimator

Let us define the codiffence between random variables X and Y in the following way:

$$\tau_{X,Y} = \log(\mathbb{E}\exp(i(X - Y))) - \log(\mathbb{E}\exp(iX)) - \log(\mathbb{E}\exp(-iY)). \tag{4}$$

1 Multivariate stable distribution generator

We checked if our generator was implemented correctly, by generating 20000 of two-dim vectors

1.1 Symmetric stable vector

For following parameters, we generated vectors and we created plot 1 - dependence of x and y obtained in vectors. Symmetric case $\alpha = 0.9$ and $n = 6$ point masses

- $\gamma_1 = 0.25$ at $\mathbf{s}_1 = (1, 0)$,
- $\gamma_2 = 0.125$ at $\mathbf{s}_2 = (1/2, \sqrt{3}/2)$,
- $\gamma_3 = 0.25$ at $\mathbf{s}_3 = (-1/2, \sqrt{3}/2)$,
- $\gamma_4 = 0.25$ at $\mathbf{s}_4 = (-1, 0)$,
- $\gamma_5 = 0.125$ at $\mathbf{s}_5 = (-1/2, -\sqrt{3}/2)$,
- $\gamma_6 = 0.25$ at $\mathbf{s}_6 = (1/2, -\sqrt{3}/2)$.

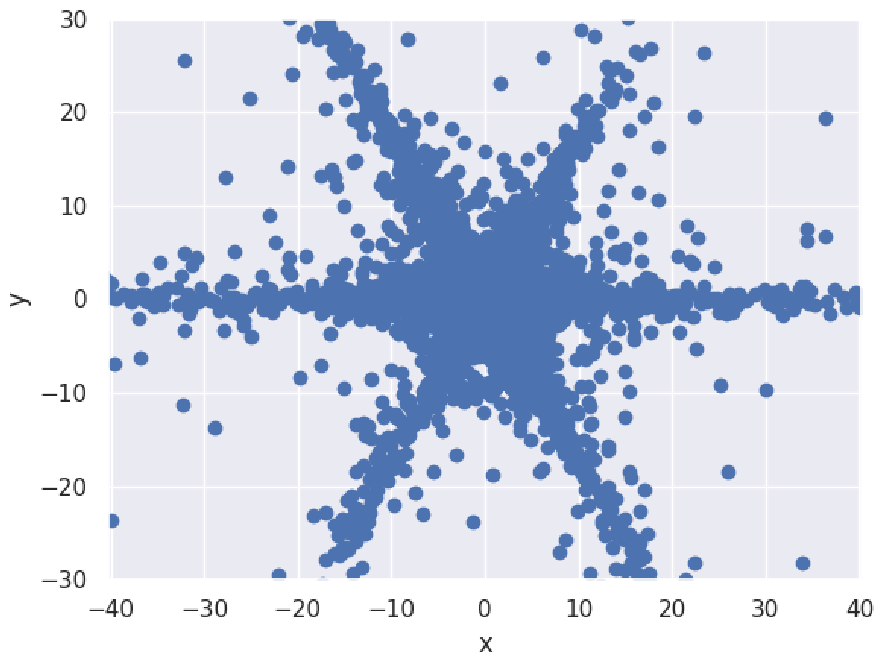


Figure 1: Result of simulation of symmetric stable vector for $\alpha = 0.9$ and $n = 6$.

We can observe that points are located on six different direction lines, so we can conclude, that the implementation of our generator in this case should be correct.

1.2 Stable vector with independent components

- $\alpha = 1.6$,
- $\gamma_1 = 0.25$ at $\mathbf{s}_1 = (1, 0)$,
- $\gamma_2 = 0.25$ at $\mathbf{s}_2 = (0, 1)$,
- $\gamma_3 = 0.25$ at $\mathbf{s}_3 = (-1, 0)$,
- $\gamma_4 = 0.25$ at $\mathbf{s}_4 = (0, -1)$.

For discrite case with parameters mentioned above, we simulated independent α-stable vector.

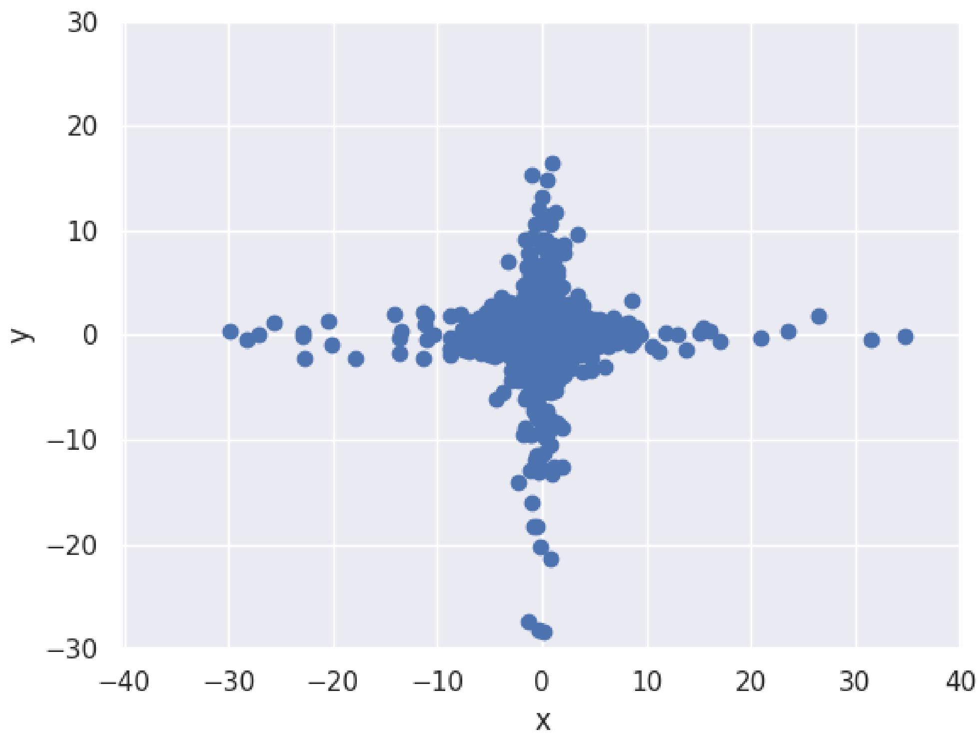


Figure 2: Result of stable vector with independent components for $\alpha = 1.6$.

Thanks to the results we obtained and are depicted on graph 2, we can conclude, that in this case, vectors are also simulated correctly.

For generated vectors, we calculated following covariance of marginals of vector, which is equal to -0.02. It is really close to zero, so we can assume, that we sucessfully generated independent case.

$$\mathrm{Cov}(X, Y) = -0.02$$

1.3 Stable vector which is not symmetric and has not independent components.

In this case, matrix of covariance is equal

$$\Sigma(G) = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.7 \end{bmatrix},$$

Every element of vector X is multiplied by proper weight.

$$\mathbf{X} = \left[w_1 A^{1/2} G_1,\ w_2 A^{1/2} G_2 \right], \tag{5}$$

where $w = (w_1, 1 - w_1) = (\frac{1}{3}, \frac{2}{3})$.

In our case the covariance of marginals of vector G is not zero and elements of this vector are not independent. It means that vector components of vector X are not independent. In addition, using different weights on symetric case, provide us an non symetric output.

Figure 3: Result of α stable vector which is not symmetric and has not independent components for $\alpha = 1.6$.

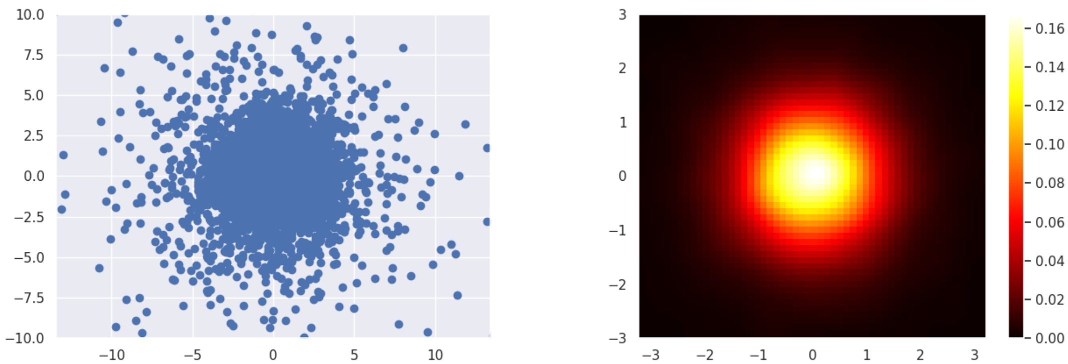
The results, shown on the graph 3 confirm the theoretical assumptions, so we can assume that our implementation works correctly in this case as well.

We are checking the correctness of the simulation sub-Gaussian vector in section 2.

2 Sub-Gaussian random vector generator.

To check if our implementation of sub-Gaussian random vector generator, we generated 20000 of random vectors using the method described in subsection 0.3, where $\alpha = 1.6$ and the following covariance matrix:

$$\Sigma(G) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$



(a) Generated sub-Gaussian random vectors.

(b) Two-dimensional density of generated vectors.

Figure 4: Result of simulation of Sub-Gaussian random vector for $\alpha = 1.6$.

At the graph 4a we presented the dependence of first and second values of generated vectors. In addition, on graph 4b, we presented empirical two-dimensional density of the scatter plot. We can see that the results are symmetric.

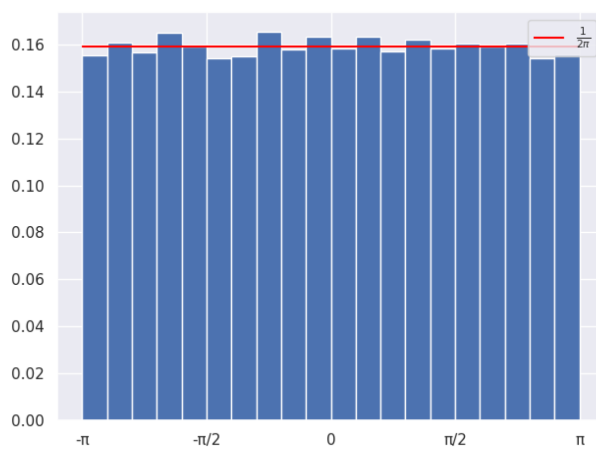
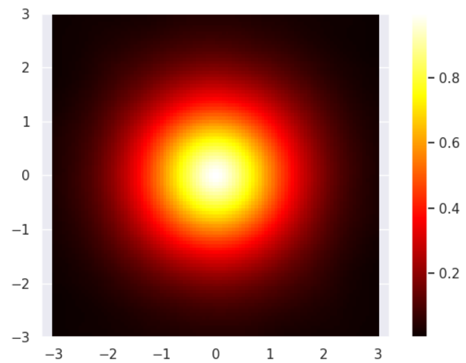


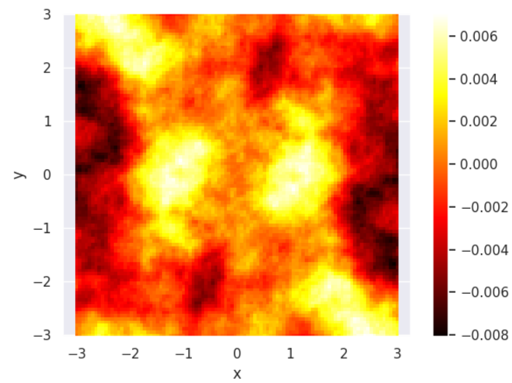
Figure 5: Estimated spectral measure.

We also checked spectral measure, the results are depicted at the graph 5, where a horizontal red line is equal to $\frac{1}{2\pi}$ what is the value of density of uniform distribution from $-\pi$ to π. We also checked, using KS-test, if results are uniformly distributed. We got p-value = 0.96 , so based on the graph and the result of the KS-test, we can conclude, that the spectral measure is uniformly distributed.

We estimated CF of SαS random vector, which is presented at graph 6a and at graph 6b we have included errors between the estimated and the theoretical function. Additionally, in table 1, we included basic statistics of these errors. We can observe that errors are really small.



(a) ECF of generated vectors.



(b) Errors of comparison of theoretical and empirical CF of SaS random vector.

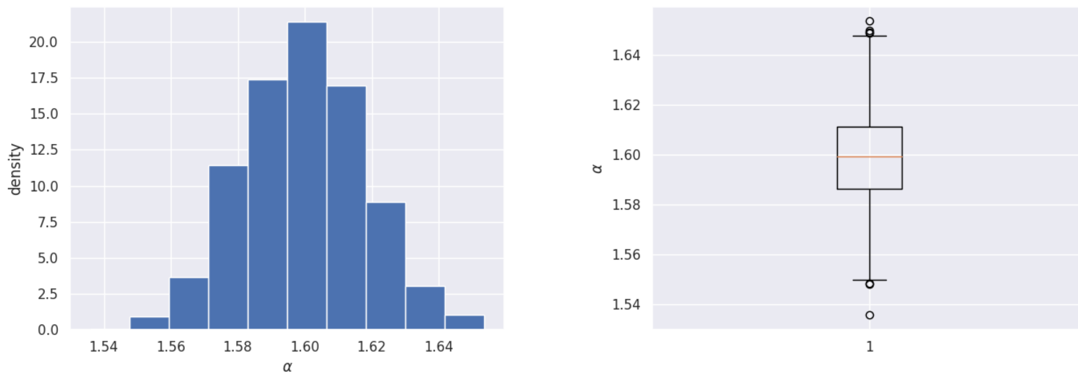
Figure 6: Result of estimate characteristic function for sub-Gaussian stable vector for $\alpha = 1.6$.

count	mean	std	min	25%	50%	75%	max
10000.0	-0.000252	0.00322	-0.00809	-0.002605	-0.000049	0.001813	0.007082

Table 1: Basic statistics of errors between the estimated and the theoretical characteristic function.

3 Estimation of α nd spectral measure Γ

Using the method described in 0.5, we were able to fit the correct function to double-logarithm characteristic function and based on knowledge of CF from section 4 we estimated parameter α.



(a) Histogram of estimated α.

(b) Boxplot of estimated α.

Figure 8: Estimation's results of α.

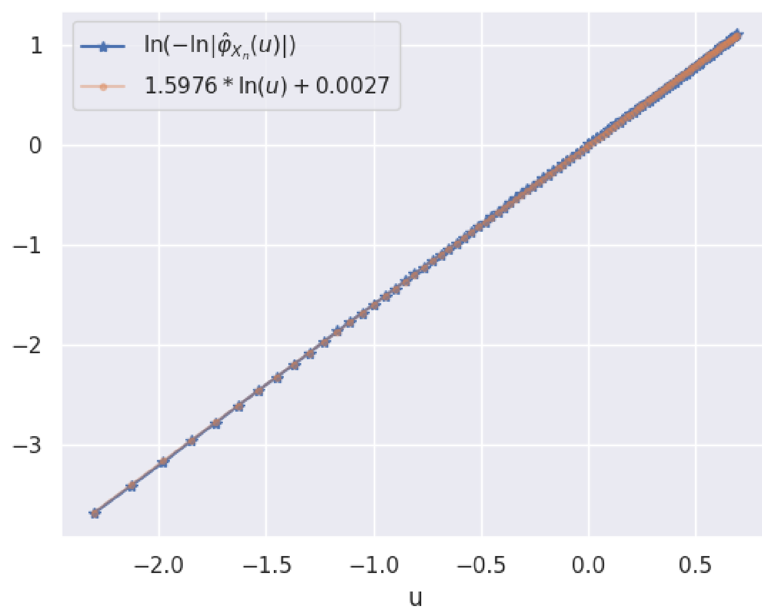


Figure 7: Fitted function to empirical double-logarithm CF.

In this case we used the same parameters as in section 2. We estimated α 1000 times using samples with size of 10000.

We created two graphs (8a, 8b), histogram and boxplot of estimated parameter α.

Basic statistics of our estimations we have located in table 2

count	mean	std	min	25%	50%	75%	max
1000.0	1.599539	0.01839	1.535825	1.586632	1.599462	1.611326	1.65366

Table 2: Basic statistics of estimated α.

We can see that the mean is close to true α and variance is low. We also obtained a small range.

4 Estimation of the characteristic function for multivariate data

Using the method described in 0.5 by function $I(x)$ We can estimate CF for a stable vector. As an example, we estimated CF for vectors with the same parameters as in section 2. (Graph 9)

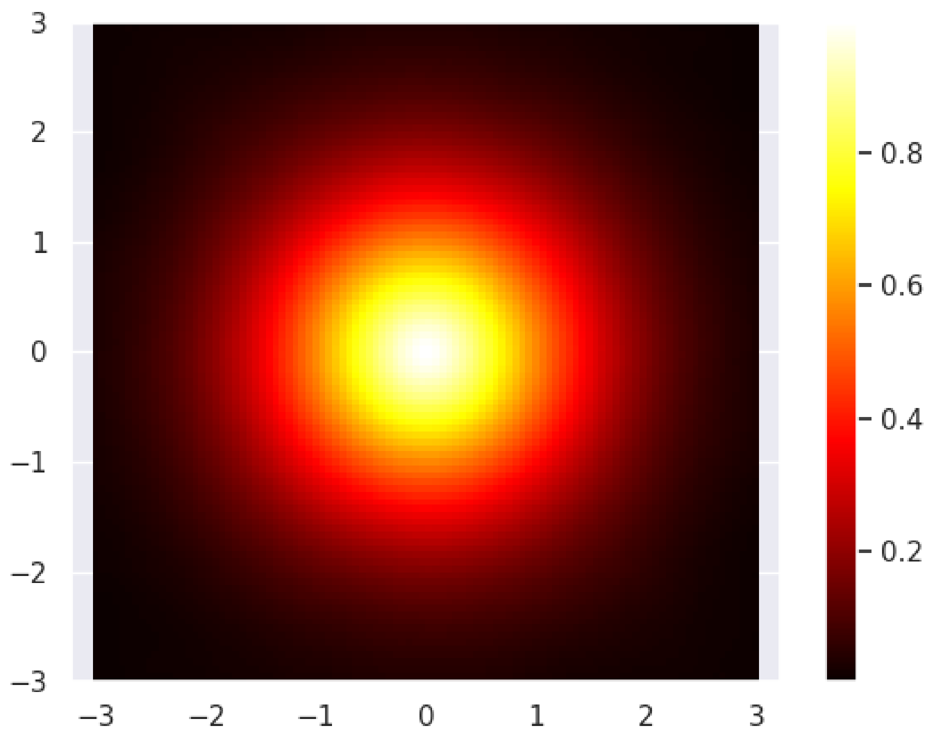


Figure 9: Characteristic function of sub-gaussian.

5 Estimation of codifference measure

To check the correctness of our estimator implementation, we have generated 100000 samples thanks to which we estimated codifference for particular cases.

5.1 First case

Parameters:

- $\alpha = 1.6$,
- $\gamma_1 = 0.25$ at $\mathbf{s}_1 = (1, 0)$,
- $\gamma_2 = 0.25$ at $\mathbf{s}_2 = (0, 1)$,
- $\gamma_3 = 0.25$ at $\mathbf{s}_3 = (-1, 0)$,
- $\gamma_4 = 0.25$ at $\mathbf{s}_4 = (0, -1)$.

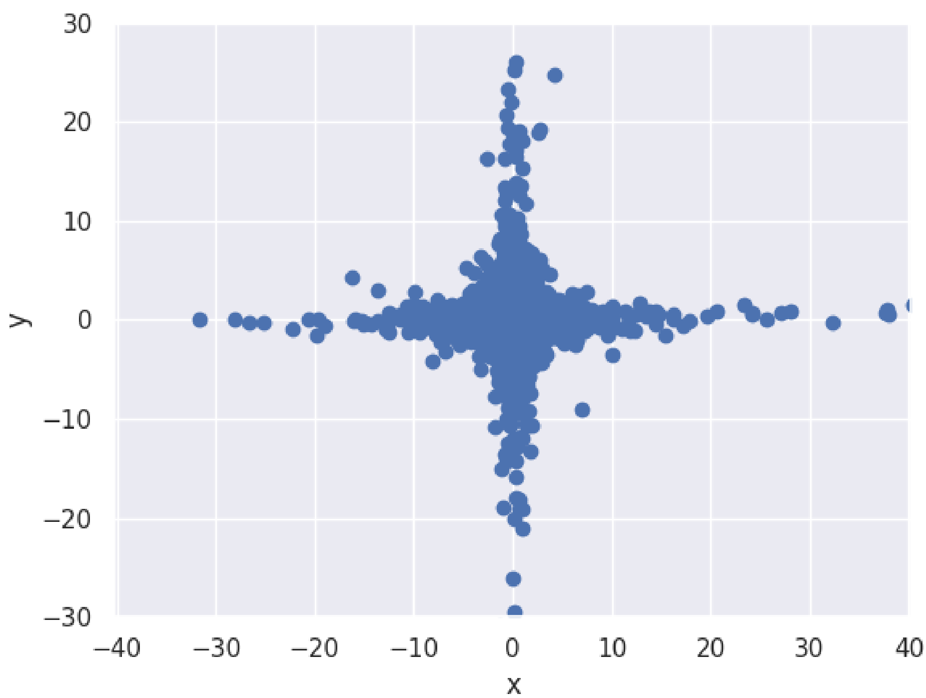


Figure 10: Result of simulation.

Codifference $= 0.0158$.

5.2 Second case

Parameters:

- $\alpha = 0.9$,
- $\gamma_1 = 0.25$at $\mathbf{s}_1 = (\sqrt{2}/2, \sqrt{2}/2)$,
- $\gamma_2 = 0.25$ at $\mathbf{s}_2 = (-\sqrt{2}/2, \sqrt{2}/2)$,
- $\gamma_3 = 0.25$ at $\mathbf{s}_3 = (-\sqrt{2}/2, -\sqrt{2}/2)$,
- $\gamma_4 = 0.25$ at $\mathbf{s}_4 = (\sqrt{2}/2, -\sqrt{2}/2)$.

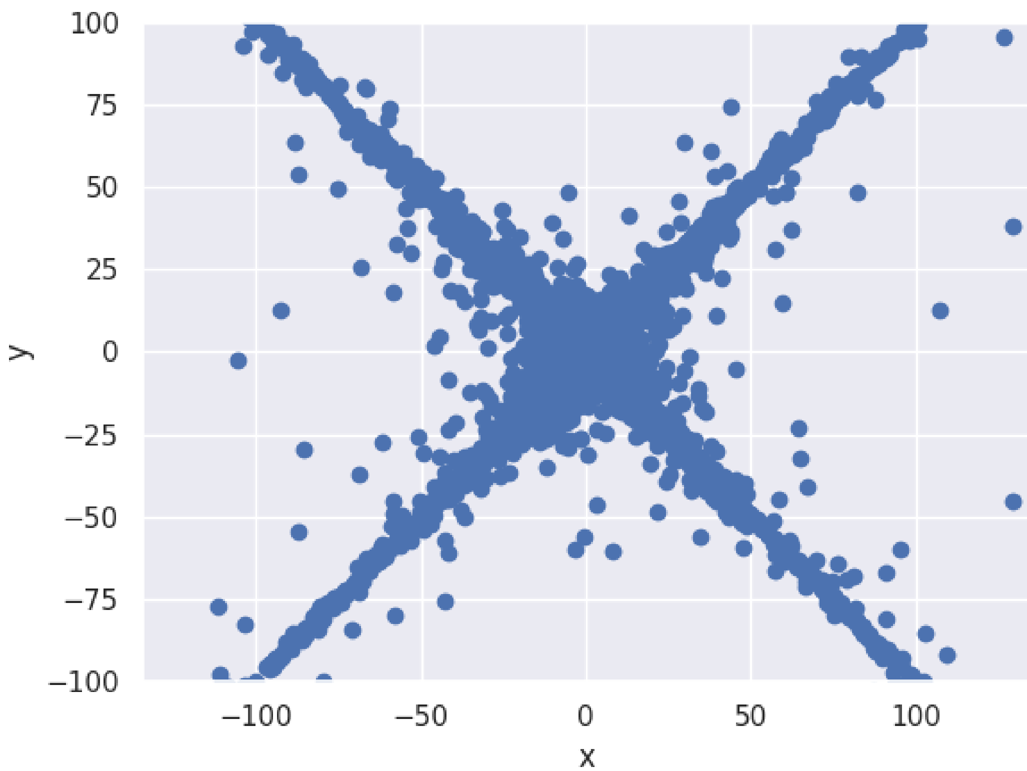


Figure 11: Result of simulation.

Codifference = 0.7933.

5.3 Third case

Sub-Gaussian vector with $\alpha = 1.6$ covariance matrix

$$\Sigma(G) = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}.$$

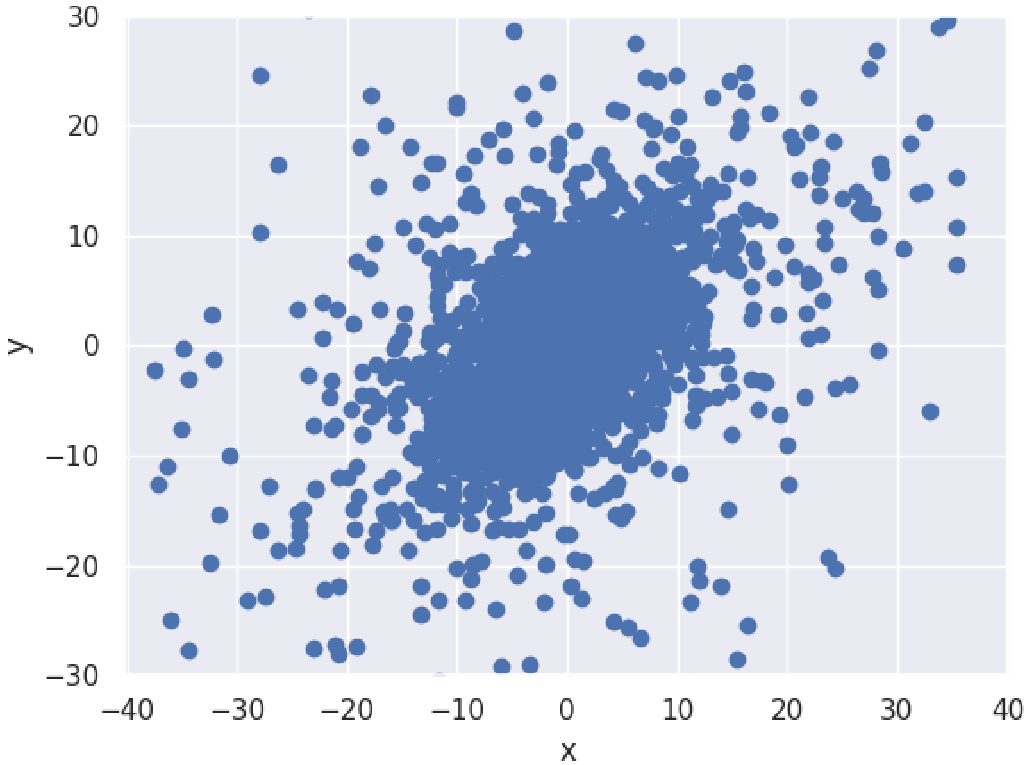


Figure 12: Result of simulation.

Codifference = 0.5778.

6 Summary

We checked correctness of implementation of all methods and estimators. In every case, we obtained good results.