影像處理
Chapter 04 頻率域影像處理

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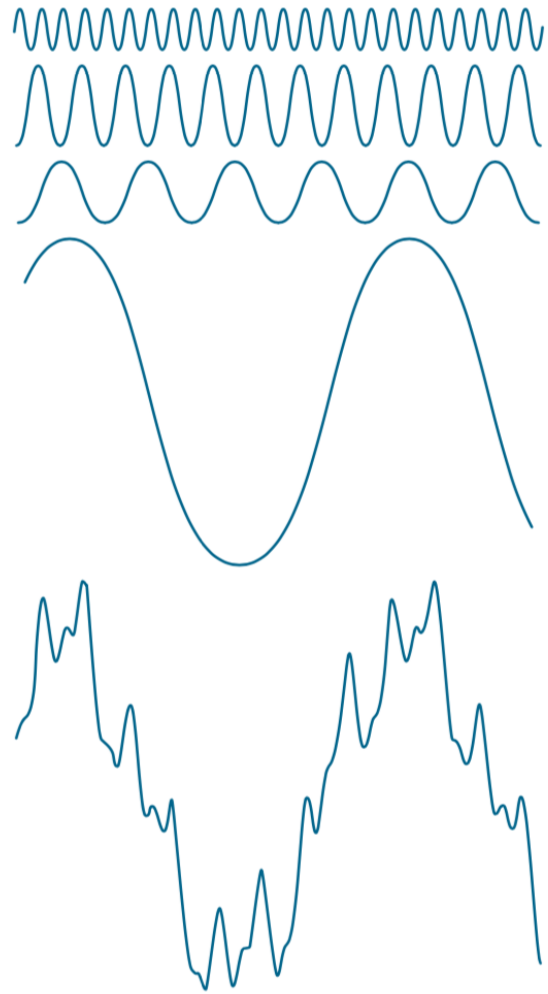
基本概念

圖 6-1　影像局部區域與頻率域的關係

Fourier Series

FIGURE 4.1
The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.



4.2 PRELIMINARY CONCEPTS

We pause briefly to introduce several of the basic concepts that underlie the material in later sections.

COMPLEX NUMBERS

A complex number, C, is defined as

$$C = R + jI \tag{4-3}$$

where R and I are real numbers and $j = \sqrt{-1}$. Here, R denotes the *real part* of the complex number and I its *imaginary part*. Real numbers are a subset of complex numbers in which $I = 0$. The *conjugate* of a complex number C, denoted C^*, is defined as

$$C^* = R - jI \tag{4-4}$$

Complex numbers can be viewed geometrically as points on a plane (called the *complex plane*) whose abscissa is the *real axis* (values of R) and whose ordinate is the *imaginary axis* (values of I). That is, the complex number $R + jI$ is point (R, I) in the coordinate system of the complex plane.

Sometimes it is useful to represent complex numbers in polar coordinates,

$$C = |C|(\cos\theta + j\sin\theta) \tag{4-5}$$

where $|C| = \sqrt{R^2 + I^2}$ is the length of the vector extending from the origin of the complex plane to point (R, I), and θ is the angle between the vector and the real axis. Drawing a diagram of the real and complex axes with the vector in the first quadrant will show that $\tan\theta = (I/R)$ or $\theta = \arctan(I/R)$. The arctan function returns angles in the range $[-\pi/2, \pi/2]$. But, because I and R can be positive and negative independently, we need to be able to obtain angles in the full range $[-\pi, \pi]$. We do this

FOURIER SERIES

As indicated in the previous section, a function $f(t)$ of a continuous variable, t, that is periodic with a period, T, can be expressed as the sum of sines and cosines multiplied by appropriate coefficients. This sum, known as a *Fourier series*, has the form

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T}t} \tag{4-8}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T}t} dt \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots \tag{4-9}$$

are the coefficients. The fact that Eq. (4-8) is an expansion of sines and cosines follows from Euler's formula, Eq. (4-6).

練習計算

EXAMPLE 4.1: Obtaining the Fourier transform of a simple continuous function.

The Fourier transform of the function in Fig. 4.4(a) follows from Eq. (4-20):

$$
\begin{aligned}
F(\mu) &= \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} dt = \int_{-W/2}^{W/2} A e^{-j2\pi\mu t} dt \\
&= \frac{-A}{j2\pi\mu}\Big[e^{-j2\pi\mu t}\Big]_{-W/2}^{W/2} = \frac{-A}{j2\pi\mu}\Big[e^{-j\pi\mu W} - e^{j\pi\mu W}\Big] \\
&= \frac{A}{j2\pi\mu}\Big[e^{j\pi\mu W} - e^{-j\pi\mu W}\Big] \\
&= AW \frac{\sin(\pi\mu W)}{(\pi\mu W)}
\end{aligned}
$$

Euler's formula,

$$e^{j\theta} = \cos\theta + j\sin\theta$$

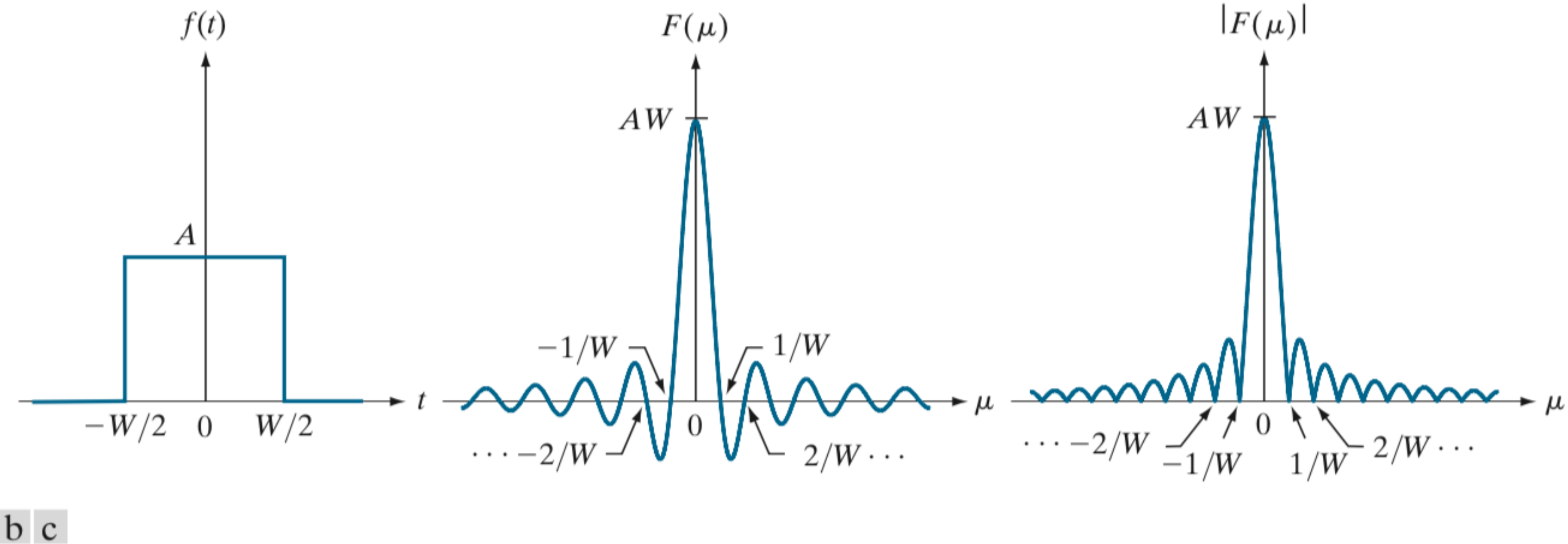
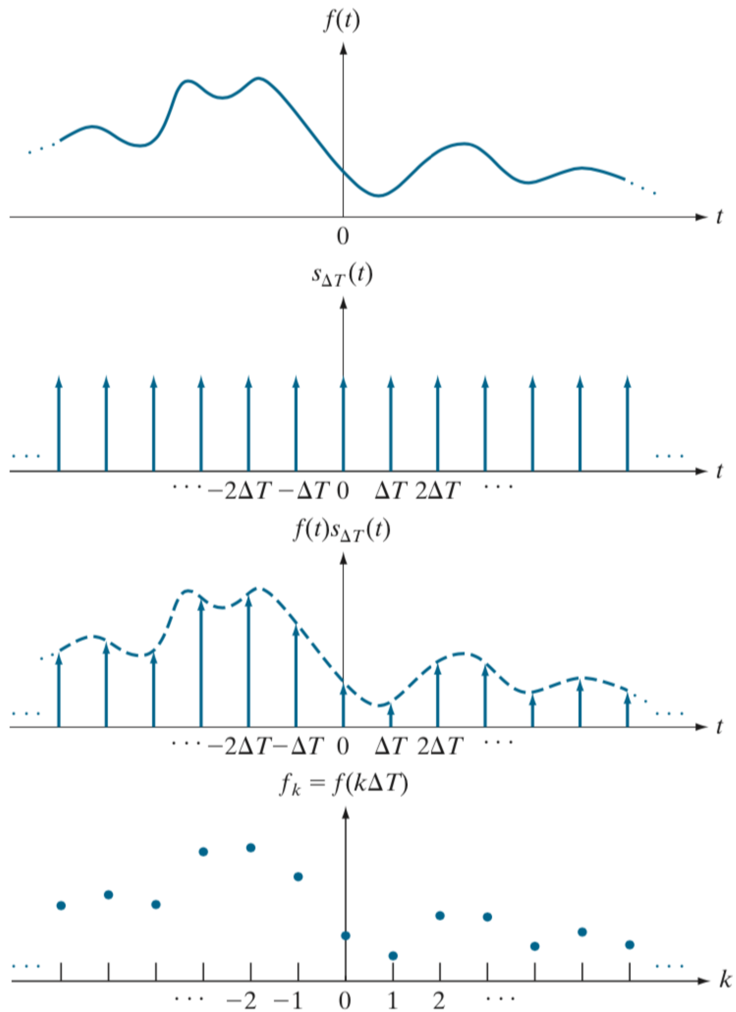


FIGURE 4.4 (a) A box function, (b) its Fourier transform, and (c) its spectrum. All functions extend to infinity in both directions. Note the inverse relationship between the width, W, of the function and the zeros of the transform.

a
b
c
d

FIGURE 4.5
(a) A continuous function. (b) Train of impulses used to model sampling. (c) Sampled function formed as the product of (a) and (b). (d) Sample values obtained by integration and using the sifting property of impulses. (The dashed line in (c) is shown for reference. It is not part of the data.)



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- 4.1 FFT Sample

離散傅立葉轉換

定義 離散傅立葉轉換 (1D)

給定離散序列 $x[n], n = 0, 1, \cdots, N-1$ ，則**離散傅立葉轉換** (Discrete Fourier Transform, DFT) 可以定義爲：

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N}, k = 0, 1, \ldots, N-1$$

其**反轉換** (Inverse DFT) 爲：

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j2\pi kn/N}, n = 0, 1, 2, \ldots, N-1$$

定義 離散傅立葉轉換 (2D)

給定數位影像，則**離散傅立葉轉換** (DFT) 可以定義為：

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi(ux/M+vy/N)}$$

其**反轉換** (Inverse DFT) 為：

$$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\, e^{j2\pi(ux/M+vy/N)}$$

EXAMPLE 4.4: The mechanics of computing the DFT.

Figure 4.12(a) shows four samples of a continuous function, $f(t)$, taken ΔT units apart. Figure 4.12(b) shows the samples in the x-domain. The values of x are 0, 1, 2, and 3, which refer to the number of the samples in sequence, counting up from 0. For example, $f(2) = f(t_0 + 2\Delta T)$, the third sample of $f(t)$.

From Eq. (4-44), the first value of $F(u)$ [i.e., $F(0)$] is

$$F(0) = \sum_{x=0}^{3} f(x) = \left[f(0) + f(1) + f(2) + f(3) \right] = 1 + 2 + 4 + 4 = 11$$

The next value of $F(u)$ is

$$F(1) = \sum_{x=0}^{3} f(x) e^{-j2\pi(1)x/4} = 1e^{0} + 2e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2} = -3 + 2j$$

Similarly, $F(2) = -(1 + 0j)$ and $F(3) = -(3 + 2j)$. Observe that *all* values of $f(x)$ are used in computing *each* value of $F(u)$.

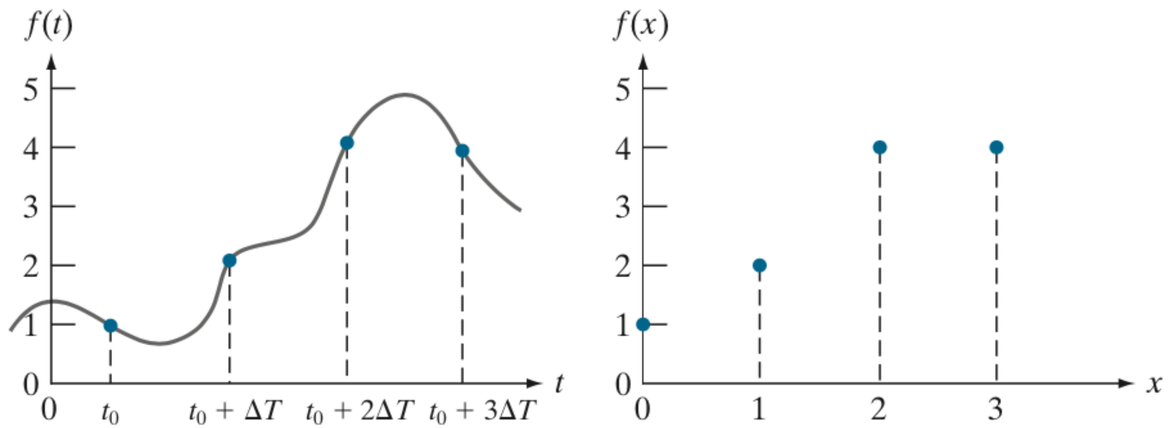
If we were given $F(u)$ instead, and were asked to compute its inverse, we would proceed in the same manner, but using the inverse Fourier transform. For instance,

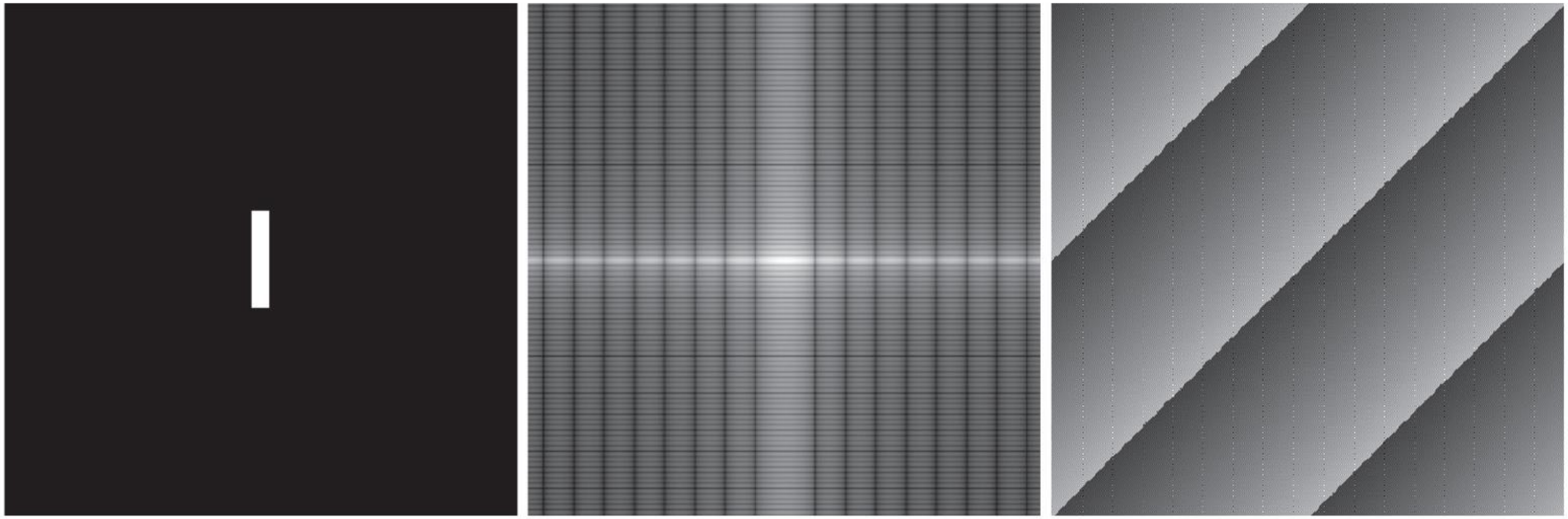
$$f(0) = \frac{1}{4} \sum_{u=0}^{3} F(u) e^{j2\pi u(0)} = \frac{1}{4} \sum_{u=0}^{3} F(u) = \frac{1}{4}\left[11 - 3 + 2j - 1 - 3 - 2j\right] = \frac{1}{4}[4] = 1$$

which agrees with Fig. 4.12(b). The other values of $f(x)$ are obtained in a similar manner.

a b

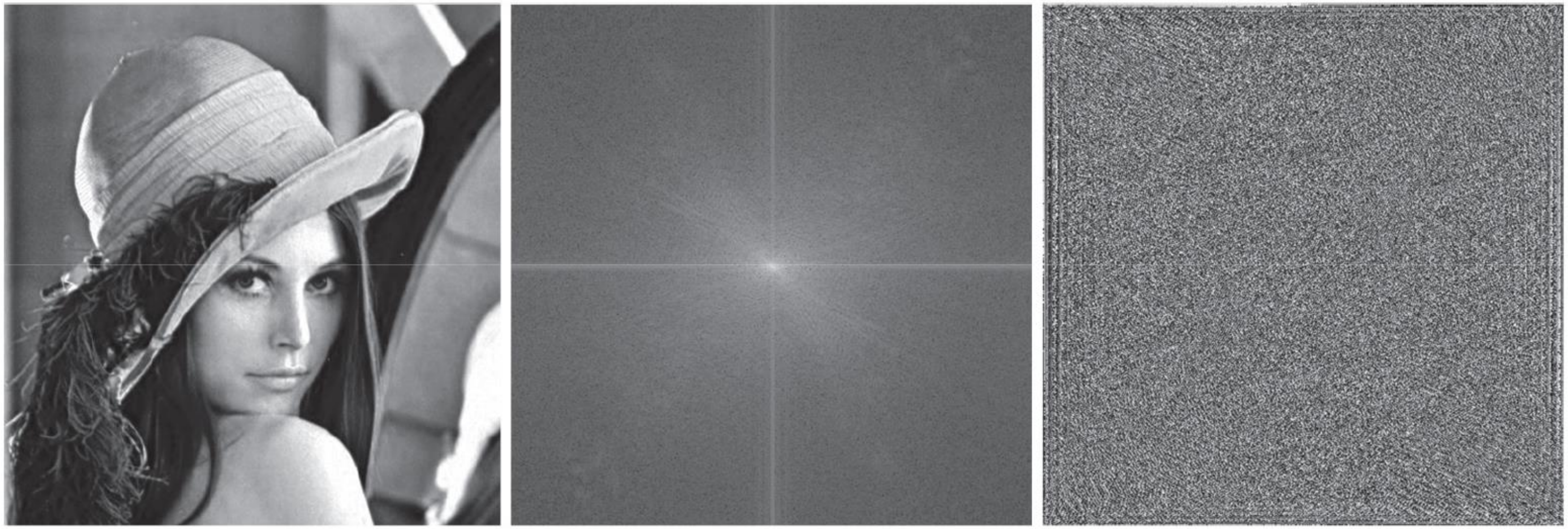
FIGURE 4.12
(a) A continuous function sampled ΔT units apart.
(b) Samples in the x-domain.
Variable t is continuous, while x is discrete.





原始影像　　頻率頻譜　　相位頻譜

圖 6-2　數位影像的頻率頻譜與相位頻譜



原始影像　　頻率頻譜　　相位頻譜

圖 6-3　數位影像的頻率頻譜與相位頻譜

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- 4.2 Fourier Spectrum

FOURIER SPECTRUM AND PHASE ANGLE

Because the 2-D DFT is complex in general, it can be expressed in polar form:

$$\begin{aligned} F(u,v) &= R(u,v) + jI(u,v) \\ &= |F(u,v)| e^{j\phi(u,v)} \end{aligned} \qquad (4\text{-}86)$$

where the magnitude

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v) \right]^{1/2} \qquad (4\text{-}87)$$

is called the *Fourier* (or *frequency*) *spectrum*, and

$$\phi(u,v) = \arctan\left[\frac{I(u,v)}{R(u,v)} \right] \qquad (4\text{-}88)$$

is the *phase angle* or *phase spectrum*. Recall from the discussion in Section 4.2 that the arctan must be computed using a four-quadrant arctangent function, such as MATLAB's **atan2(Imag, Real)** function.

Finally, the *power spectrum* is defined as

$$\begin{aligned} P(u,v) &= |F(u,v)|^2 \\ &= R^2(u,v) + I^2(u,v) \end{aligned} \tag{4-89}$$

As before, R and I are the real and imaginary parts of $F(u,v)$, and all computations are carried out for the discrete variables $u = 0, 1, 2, \ldots, M-1$ and $v = 0, 1, 2, \ldots, N-1$. Therefore, $|F(u,v)|$, $\phi(u,v)$, and $P(u,v)$ are arrays of size $M \times N$.

The Fourier transform of a real function is conjugate symmetric [see Eq. (4-85)], which implies that the spectrum has *even* symmetry about the origin:

$$|F(u,v)| = |F(-u,-v)| \tag{4-90}$$

The phase angle exhibits *odd* symmetry about the origin:

$$\phi(u,v) = -\phi(-u,-v) \tag{4-91}$$

It follows from Eq. (4-67) that

$$F(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

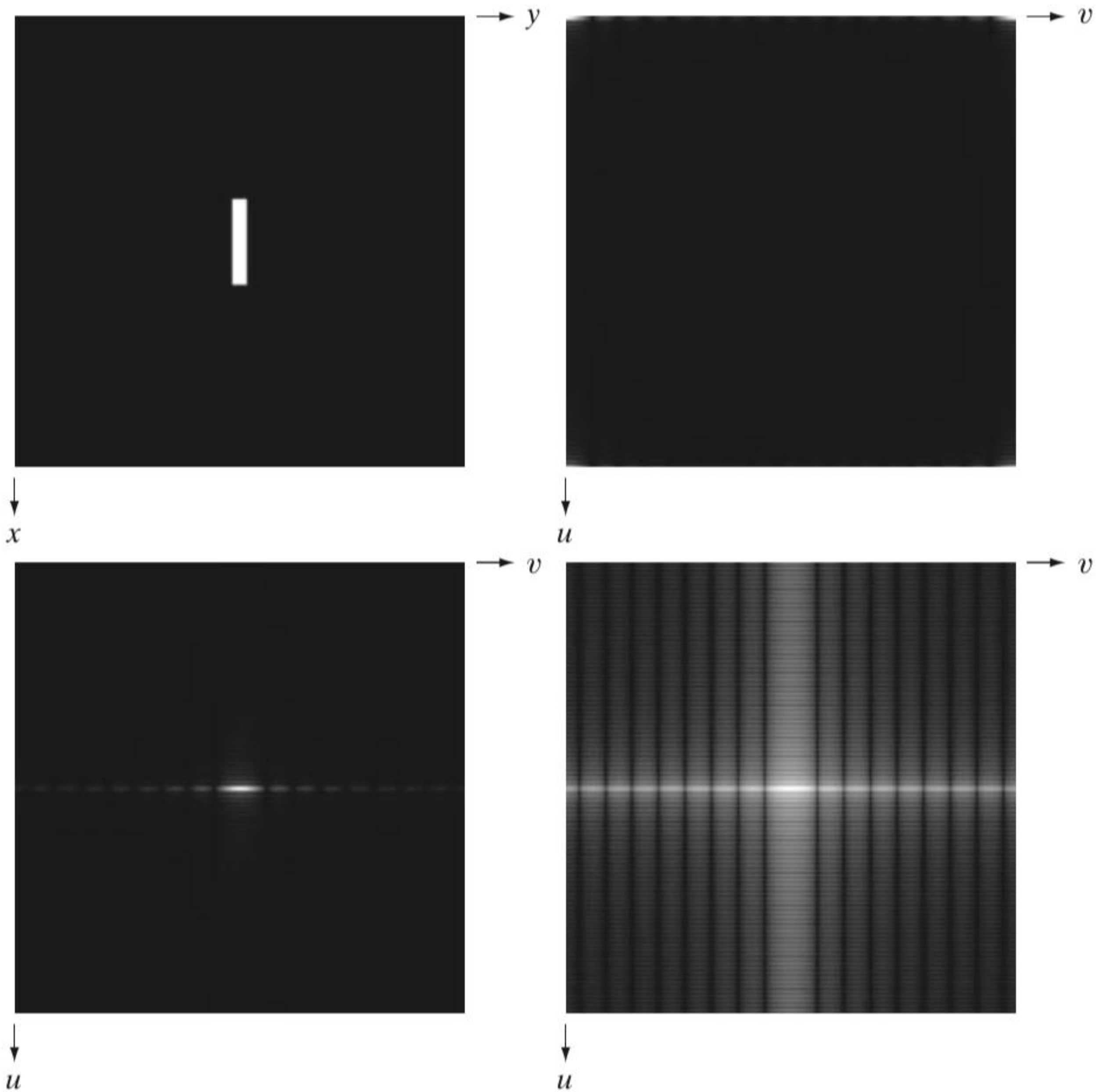
which indicates that the zero-frequency term of the DFT is proportional to the average of $f(x,y)$. That is,

$$\begin{aligned} F(0,0) &= MN \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \\ &= MN\bar{f} \end{aligned} \tag{4-92}$$

where $\bar{f}$ (a scalar) denotes the average value of $f(x, y)$. Then,

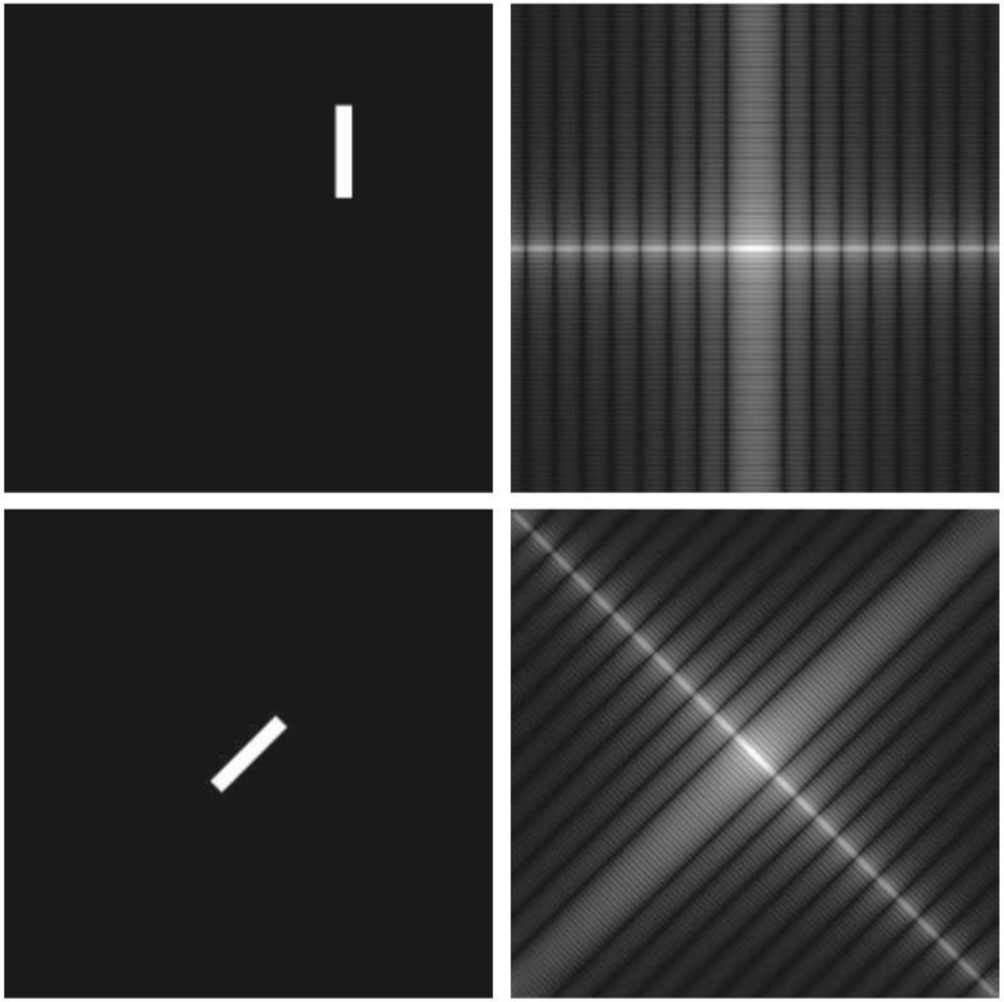
$$|F(0,0)| = MN\left|\bar{f}\right| \tag{4-93}$$

Because the proportionality constant MN usually is large, $|F(0,0)|$ typically is the largest component of the spectrum by a factor that can be several orders of magnitude larger than other terms. Because frequency components u and v are zero at the origin, $F(0,0)$ sometimes is called the *dc component* of the transform. This terminology is from electrical engineering, where "dc" signifies direct current (i.e., current of zero frequency).



a b
c d

FIGURE 4.23
(a) Image.
(b) Spectrum, showing small, bright areas in the four corners (you have to look carefully to see them).
(c) Centered spectrum.
(d) Result after a log transformation. The zero crossings of the spectrum are closer in the vertical direction because the rectangle in (a) is longer in that direction. The right-handed coordinate convention used in the book places the origin of the spatial and frequency domains at the top left (see Fig. 2.19).



a b
c d

FIGURE 4.24
(a) The rectangle in Fig. 4.23(a) translated.
(b) Corresponding spectrum.
(c) Rotated rectangle.
(d) Corresponding spectrum.
The spectrum of the translated rectangle is identical to the spectrum of the original image in Fig. 4.23(a).

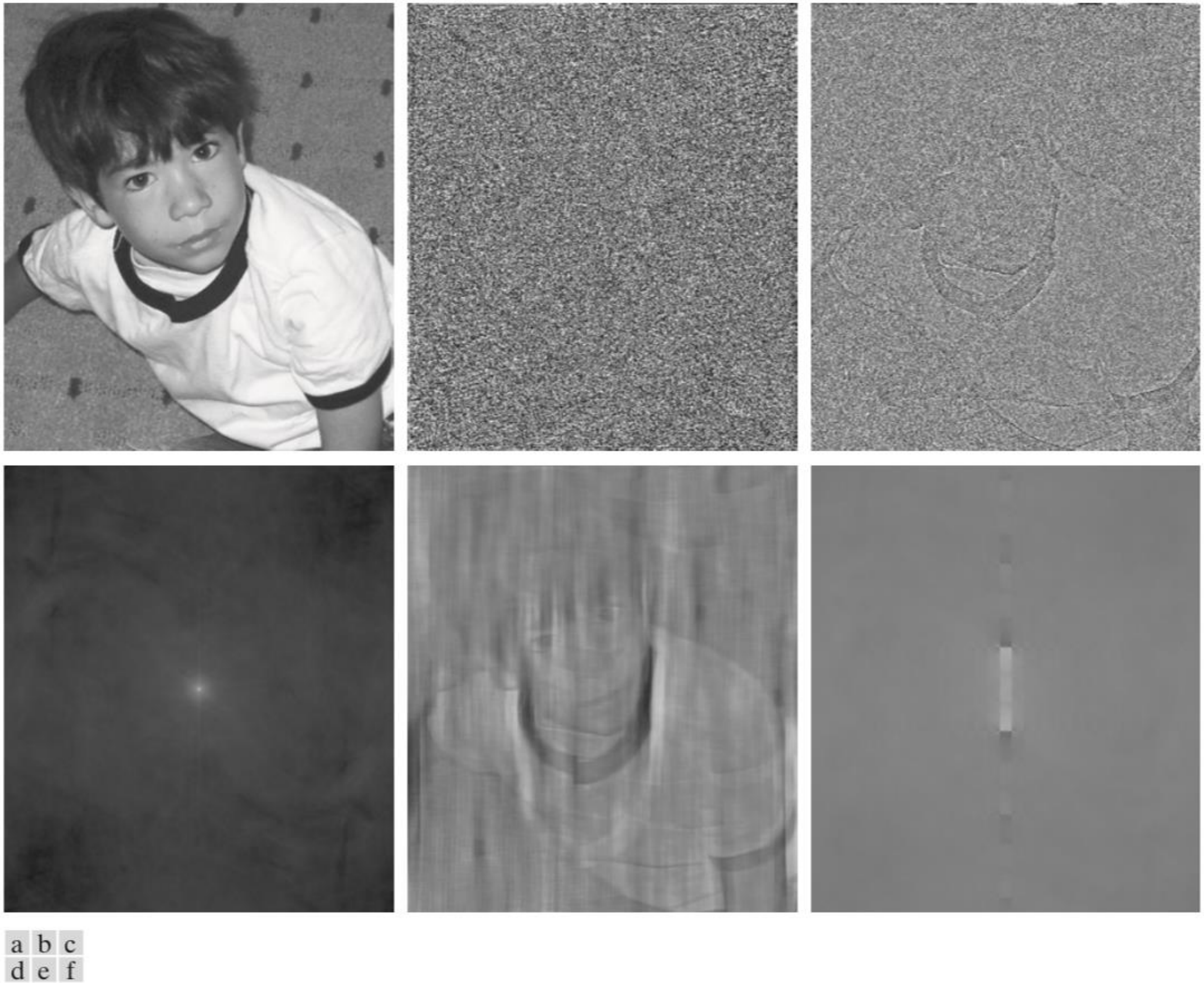
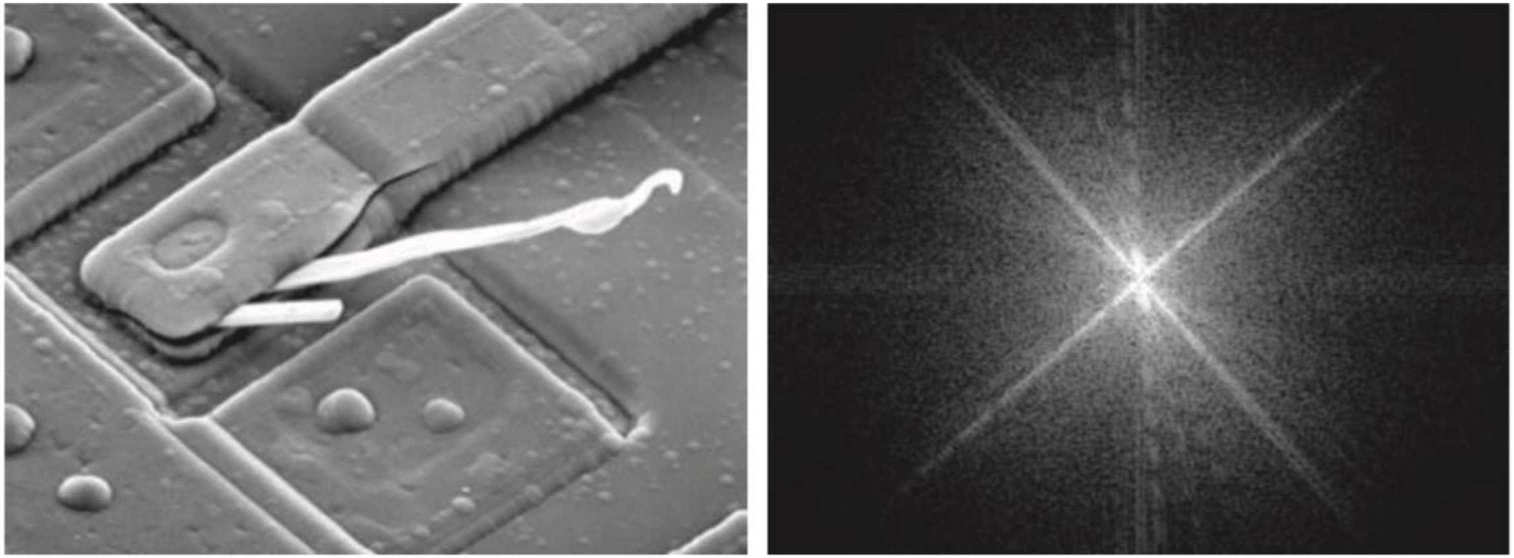


FIGURE 4.26 (a) Boy image. (b) Phase angle. (c) Boy image reconstructed using only its phase angle (all shape features are there, but the intensity information is missing because the spectrum was not used in the reconstruction). (d) Boy image reconstructed using only its spectrum. (e) Boy image reconstructed using its phase angle and the spectrum of the rectangle in Fig. 4.23(a). (f) Rectangle image reconstructed using its phase and the spectrum of the boy's image.

頻率域濾波

$(-1)^{x+y}\cdot f(x,y)$ $G(u,v) = H(u,v)\cdot F(u,v)$

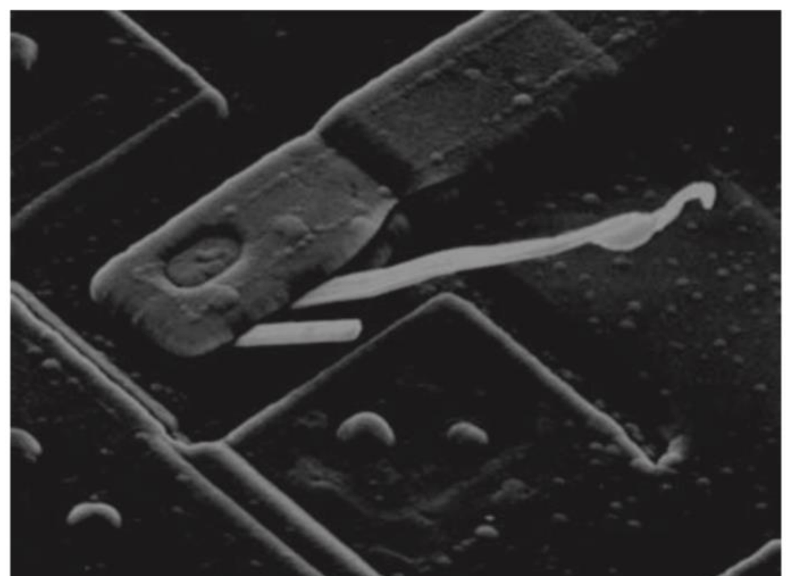
圖 6-4 數位影像的頻率域濾波



a b

FIGURE 4.28 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

FIGURE 4.29 Result of filtering the image in Fig. 4.28(a) with a filter transfer function that sets to 0 the dc term, $F(P/2, Q/2)$, in the centered Fourier transform, while leaving all other transform terms unchanged.



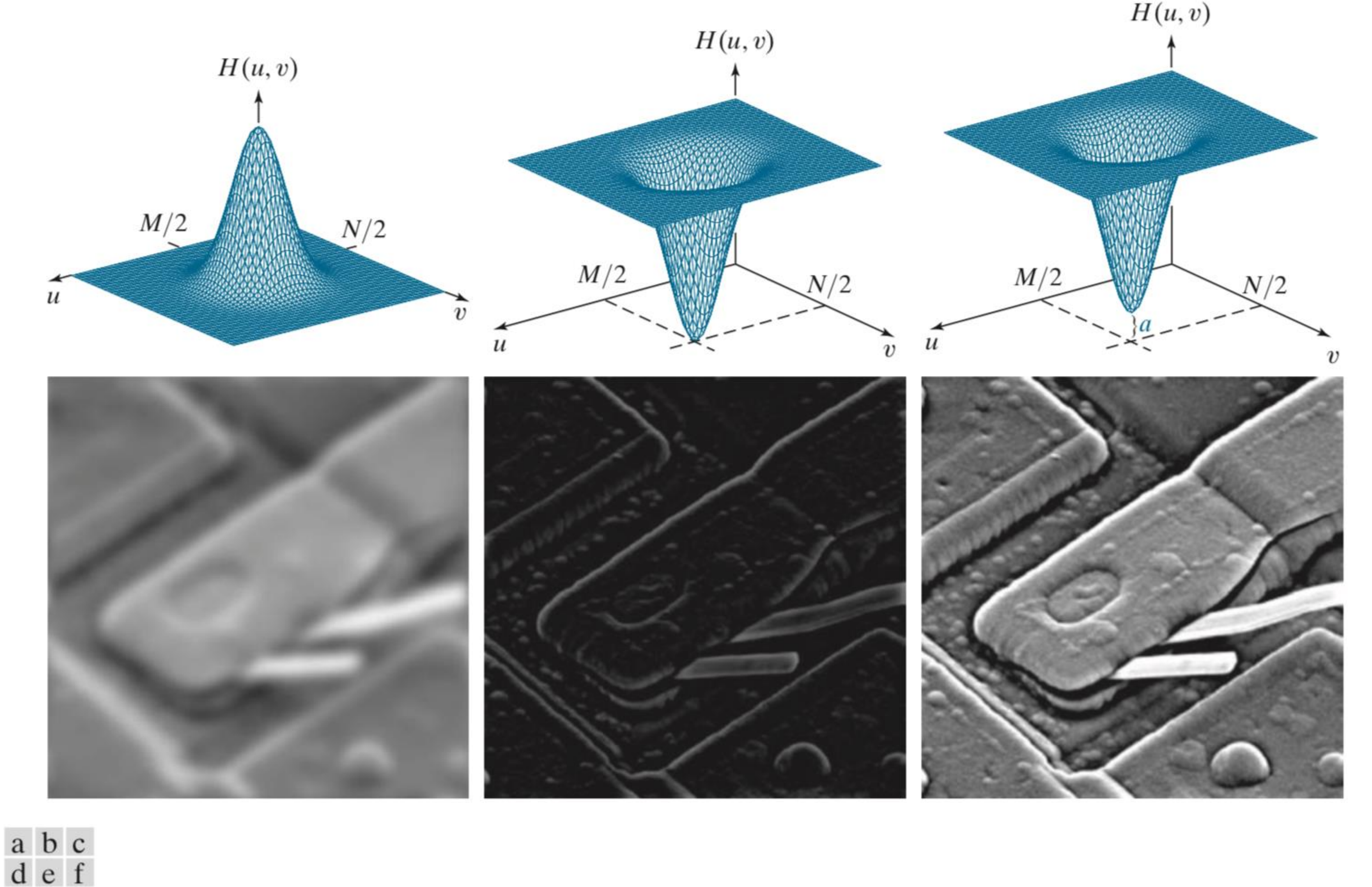
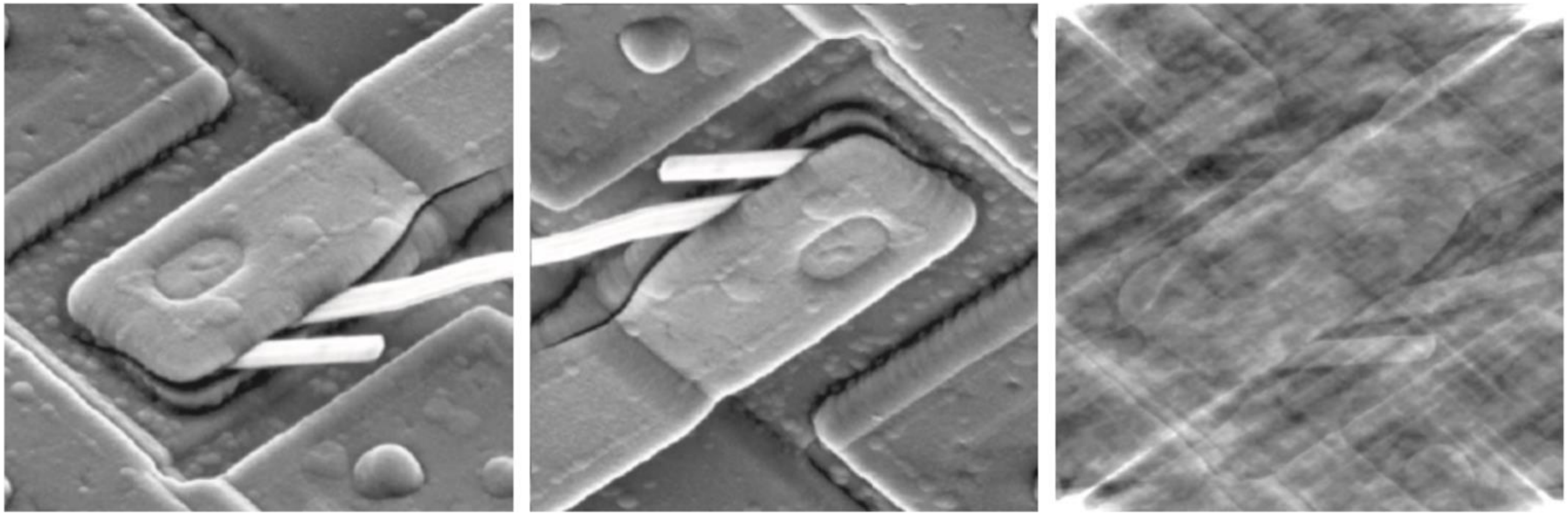


FIGURE 4.30 Top row: Frequency domain filter transfer functions of (a) a lowpass filter, (b) a highpass filter, and (c) an offset highpass filter. Bottom row: Corresponding filtered images obtained using Eq. (4-104). The offset in (c) is $a = 0.85$, and the height of $H(u,v)$ is 1. Compare (f) with Fig. 4.28(a).

a b c

FIGURE 4.31 (a) A simple image. (b) Result of blurring with a Gaussian lowpass filter without padding. (c) Result of lowpass filtering with zero padding. Compare the vertical edges in (b) and (c).



a b c

FIGURE 4.34 (a) Original image. (b) Image obtained by multiplying the phase angle array by –1 in Eq. (4-86) and computing the IDFT. (c) Result of multiplying the phase angle by 0.25 and computing the IDFT. The magnitude of the transform, $|F(u,v)|$, used in (b) and (c) was the same.

頻率域濾波器

定義 **理想低通濾波器**

理想低通濾波器 (Ideal Lowpass Filter, ILPF) 可以定義爲：

$$H(u,v)=\begin{cases}1 & \text{if } D(u,v)\le D_0\\ 0 & \text{if } D(u,v)>D_0\end{cases}$$

其中，$D(u, v)$ 爲距離頻率域中心點的距離，D_0 稱爲**截止頻率** (Cutoff Frequency)。

定義 理想高通濾波器

理想高通濾波器 (Ideal Highpass Filter, IHPF) 可以定義爲：

$$H(u,v)=\begin{cases}0 & if\ D(u,v)\le D_0\\ 1 & if\ D(u,v)>D_0\end{cases}$$

其中，$D(u, v)$ 爲距離頻率域中心點的距離，D_0 稱爲**截止頻率** (Cutoff Frequency)。

原始影像　低通濾波　高通濾波

圖 6-5　理想低通與高通濾波

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- 4.3 Fourier Ideal ighpass filter
- 4.4 Please finish ideal lowpass filter by referencing the code given in 4.3

定義 高斯低通濾波器

高斯低通濾波器 (Gaussian Lowpass Filter, GLPF) 可以定義爲：

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

其中，$D(u, v)$ 爲距離頻率域中心點的距離，σ 爲**標準差**。

定義 高斯高通濾波器

高斯高通濾波器 (Gaussian Highpass Filter, GHPF) 可以定義爲：

$$H(u,v) = 1 - e^{-D^2(u,v)/2\sigma^2}$$

其中，$D(u, v)$ 爲距離頻率域中心點的距離，σ 爲**標準差**。

原始影像　　低通濾波　　高通濾波

圖 6-6　高斯低通與高通濾波

定義 巴特沃斯低通濾波器

巴特沃斯低通濾波器 (Butterworth Lowpass Filter, BLPF) 可以定義爲：

$$H(u,v)=\frac{1}{1+\left[D(u,v)/D_0\right]^{2n}}$$

其中，$D(u, v)$ 爲距離頻率域中心點的距離，D_0 稱爲**截止頻率** (Cutoff Frequency)，n 稱爲**階數** (Order)。

定義 巴特沃斯高通濾波器

巴特沃斯高通濾波器 (Butterworth Highpass Filter, BHPF) 可以定義為：

$$H(u,v)=\frac{1}{1+\left[D_0/D(u,v)\right]^{2n}}$$

其中，$D(u, v)$ 為距離頻率域中心點的距離，D_0 稱為**截止頻率** (Cutoff Frequency)，n 稱為**階數** (Order)。

原始影像　低通濾波　高通濾波

圖 6-7　巴特沃斯低通與高通濾波

SUMMARY OF STEPS FOR FILTERING IN THE FREQUENCY DOMAIN

The process of filtering in the frequency domain can be summarized as follows:

1. Given an input image $f(x, y)$ of size $M \times N$, obtain the padding sizes P and Q using Eqs. (4-102) and (4-103); that is, $P = 2M$ and $Q = 2N$.

2. Form a padded† image $f_p(x, y)$ of size $P \times Q$ using zero-, mirror-, or replicate padding (see Fig. 3.39 for a comparison of padding methods).
3. Multiply $f_p(x, y)$ by $(-1)^{x+y}$ to center the Fourier transform on the $P \times Q$ frequency rectangle.
4. Compute the DFT, $F(u, v)$, of the image from Step 3.
5. Construct a real, symmetric filter transfer function, $H(u, v)$, of size $P \times Q$ with center at $(P/2, Q/2)$.
6. Form the product $G(u, v) = H(u, v)F(u, v)$ using elementwise multiplication; that is, $G(i, k) = H(i, k)F(i, k)$ for $i = 0, 1, 2, \ldots, M-1$ and $k = 0, 1, 2, \ldots, N-1$.
7. Obtain the filtered image (of size $P \times Q$) by computing the IDFT of $G(u, v)$:

$$g_p(x, y) = \left(\text{real}\left[\Im^{-1}\left\{G(u, v)\right\}\right]\right)(-1)^{x+y}$$

8. Obtain the final filtered result, $g(x, y)$, of the same size as the input image, by extracting the $M \times N$ region from the top, left quadrant of $g_p(x, y)$.

a b c

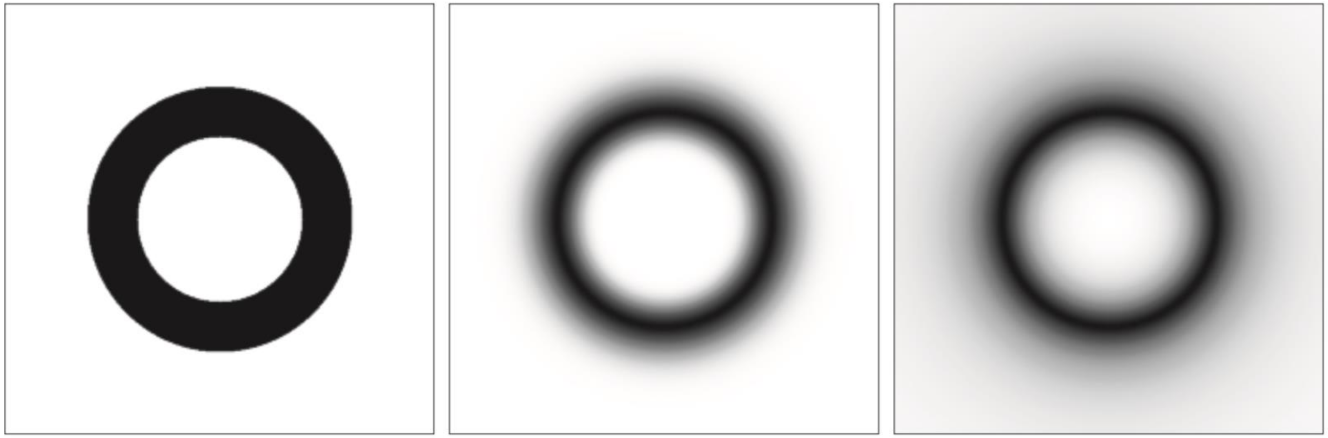

FIGURE 4.63
(a) The ideal, (b) Gaussian, and (c) Butterworth bandpass transfer functions from Fig. 4.62, shown as images. (The thin border lines are not part of the image data.)

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- 4.5 Please finish the Gaussian lowpass filter and Butterworth lowpass filter

Notch Filter

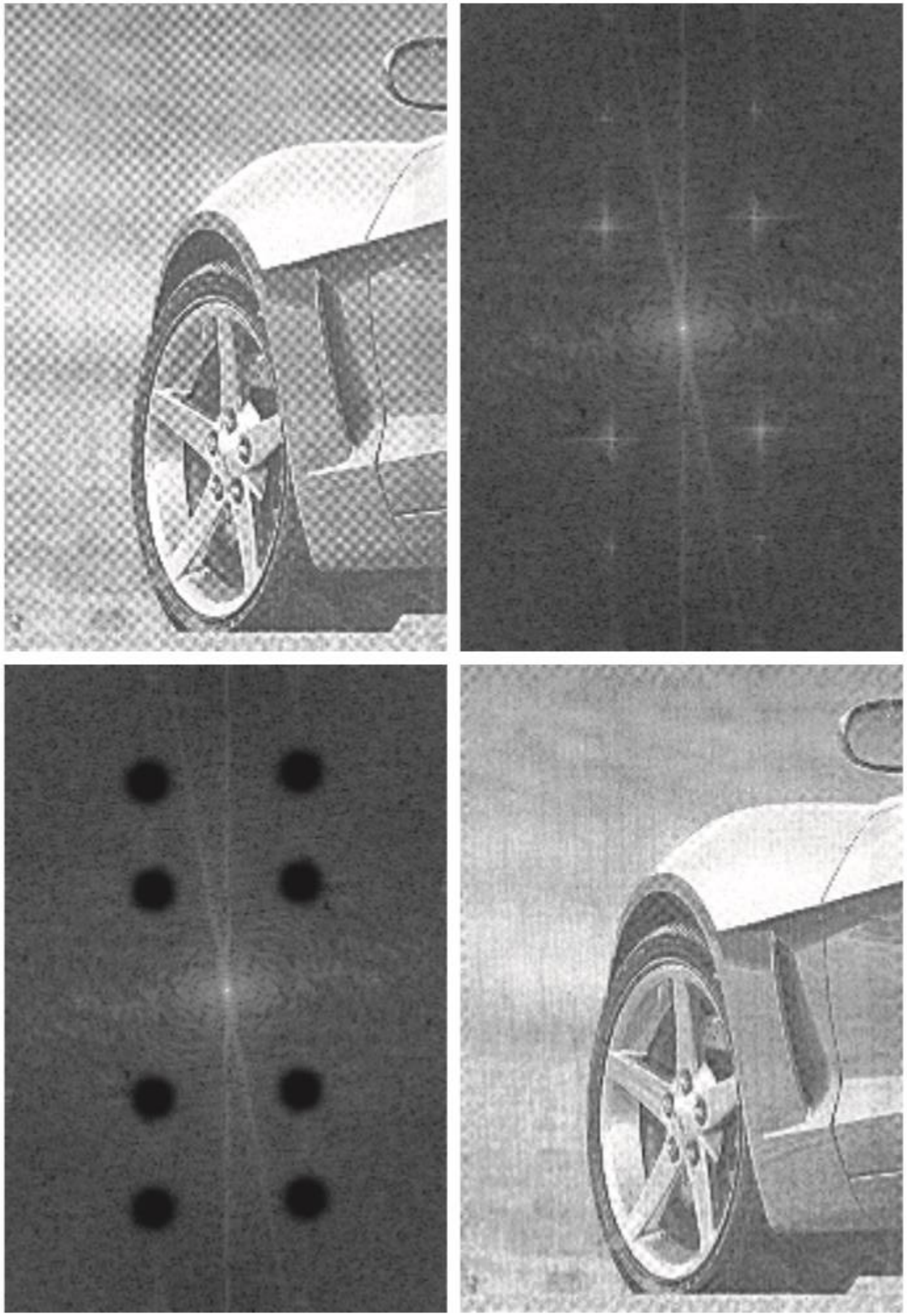
- the most useful of the selective filters
- rejects (or passes) frequencies in a predefined neighborhood of the frequency rectangle
- be symmetric about the origin
- *Notch reject* filter transfer functions are constructed as products of highpass filter transfer functions whose centers have been translated to the centers of the notches. The general form is:

$$H_{\mathrm{NR}}(u,v) = \prod_{k=1}^{Q} H_k(u,v) H_{-k}(u,v)$$

- where $H_k(u,v)$ and $H_{-k}(u,v)$ are highpass filter transfer functions whose centers are at (u_k, v_k) and $(-u_k, -v_k)$, respectively

For example, the following is a Butterworth notch reject filter transfer function of order n, containing three notch pairs:

$$H_{\mathrm{NR}}(u,v) = \prod_{k=1}^{3}\left[\frac{1}{1+\left[D_{0k}/D_{k}(u,v)\right]^{n}}\right]\left[\frac{1}{1+\left[D_{0k}/D_{-k}(u,v)\right]^{n}}\right] \qquad (4\text{-}154)$$

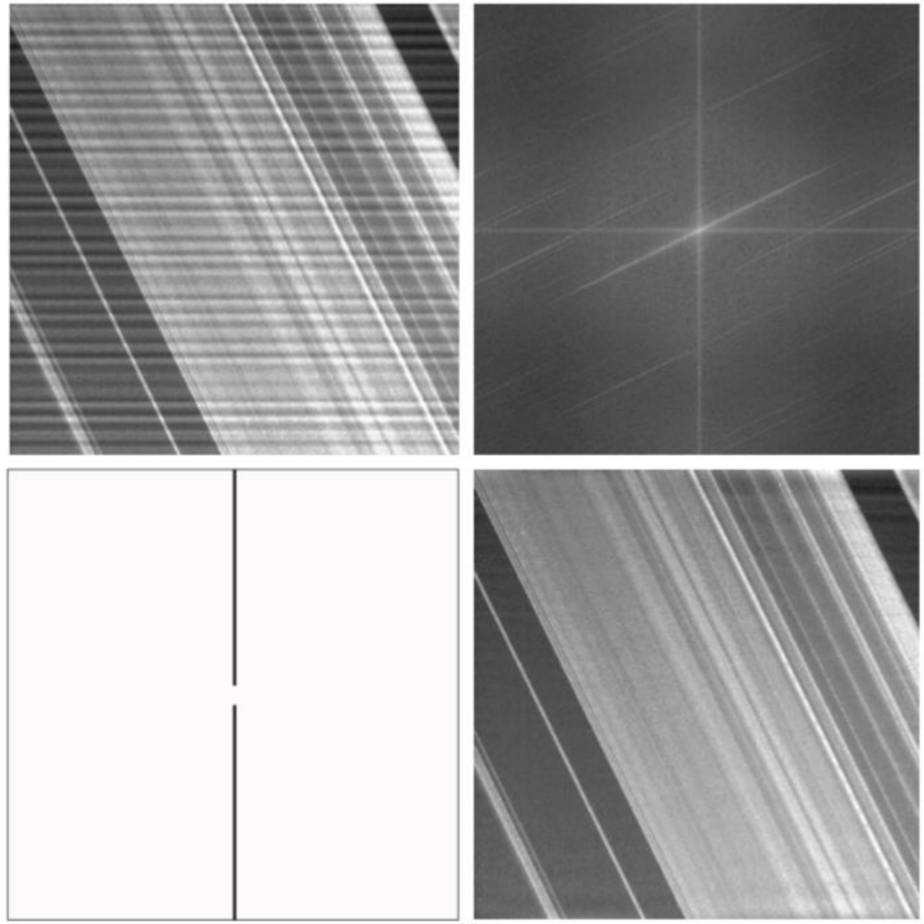


a b
c d

FIGURE 4.64 (a) Sampled newspaper image showing a moiré pattern. (b) Spectrum. (c) Fourier transform multiplied by a Butterworth notch reject filter transfer function. (d) Filtered image.

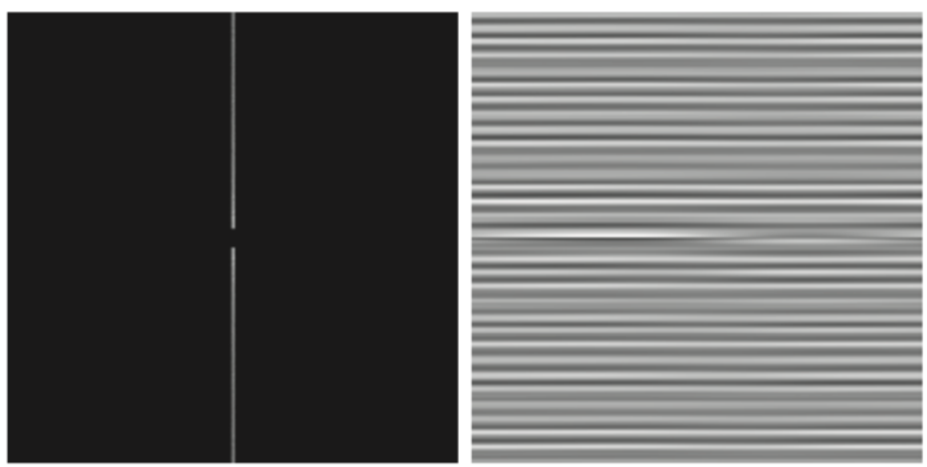
a b
c d

FIGURE 4.65 (a) Image of Saturn rings showing nearly periodic interference. (b) Spectrum. (The bursts of energy in the vertical axis near the origin correspond to the interference pattern). (c) A vertical notch reject filter transfer function. (d) Result of filtering. (The thin black border in (c) is not part of the data.) (Original image courtesy of Dr. Robert A. West, NASA/JPL.)



a b

FIGURE 4.66 (a) Notch pass filter function used to isolate the vertical axis of the DFT of Fig. 4.65(a). (b) Spatial pattern obtained by computing the IDFT of (a).



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- 4.6 Notch Filter
- Please find appropriate notch filter to separate the given image to two images with horizontal and vertical information individually.

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- 4.7 1-D signal FFT analysis
- 4.8 Extract texture features from FFT coefficients

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- 4.9 Texture features distribution plotting
  - 1. Please find texture features(FFT mean and standard deviation) for every non-overlapped kxl from input textures(k and l are factors of image size m and n, respectively), brodatz14.bmp, brodatz16.bmp, brodatz20.bmp, brodat21.bmp.
  - 2. plot features extracted from 4 texture images together by different color and label

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- 4.10 Texture features distribution plotting
  - 1. Please find texture features(mean and standard deviation) for every non-overlapped kxl from input textures(k and l are factors of image size m and n, respectively), brodatz14.bmp, brodatz16.bmp, brodatz20.bmp, brodat21.bmp.
  - 2. plot features extracted from 4 texture images together by different color and label

Check IP04.ipynb

- **4.11 use Python to sharpening an image through FFT**
  - 高頻增強： 透過高通濾波，增強影像的高頻成分，使影像更銳利。
  - 函數名稱:def fft_sharpen(img, s_range, alpha)
  - img: 輸入影像
  - s_range: 控制高頻的範圍(>s_range為高頻部分)
  - alpha: 高頻的強化倍數