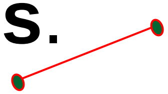
DISMATH
Discrete Mathematics and Its Applications

GRAPHS

Graph Theory

- **Graph** – discrete structures consisting of vertices and edges that connect these vertices.
- A graph G = (V, E) consists of V, a nonempty set of **vertices** (or nodes) and E, a set of **edges**. Each edge has either one or two vertices associated with it, called its **endpoints**.

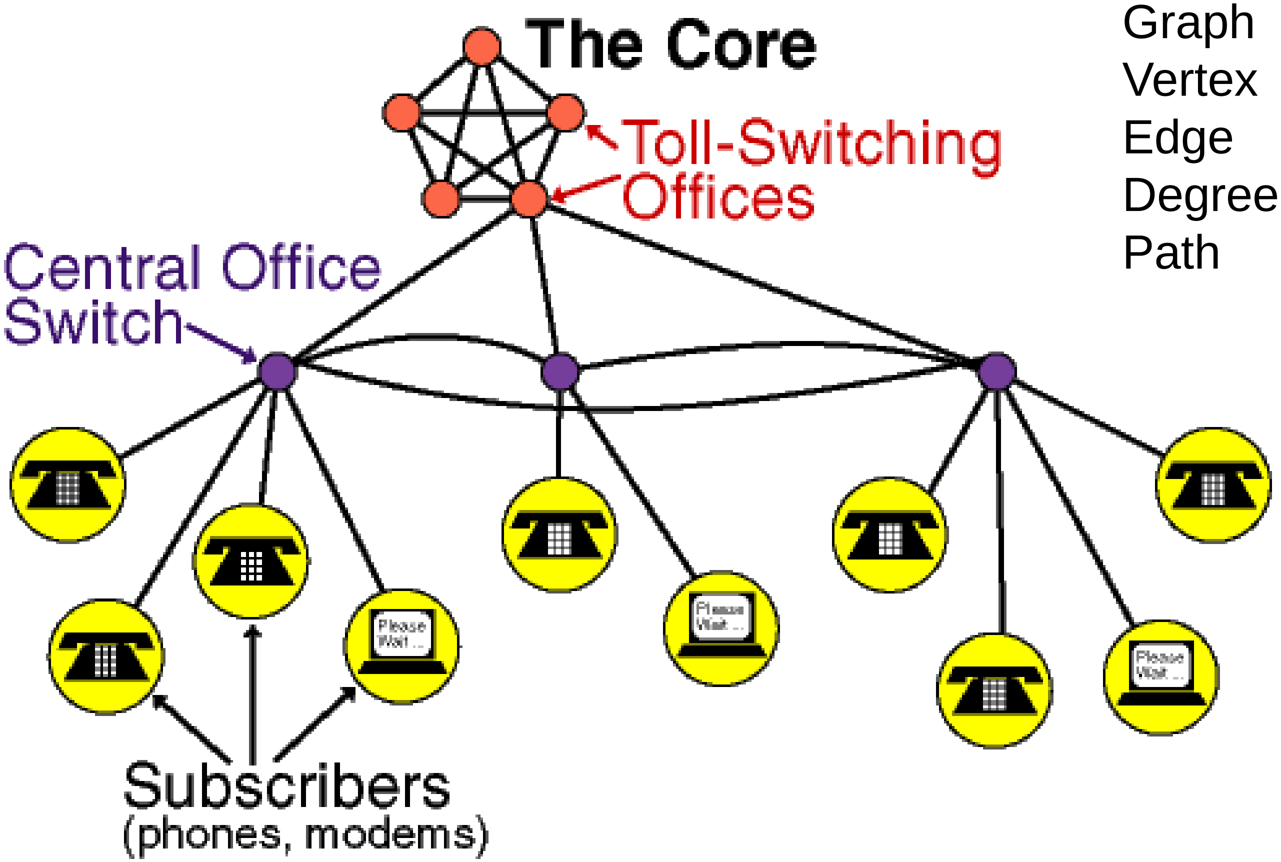


Applications:

- Networks (LAN, MAN, Social Networks, etc.)
- Job assignments
- Representing computational models
- Developing a bot to retrieve info off www

Application

- Telephone network



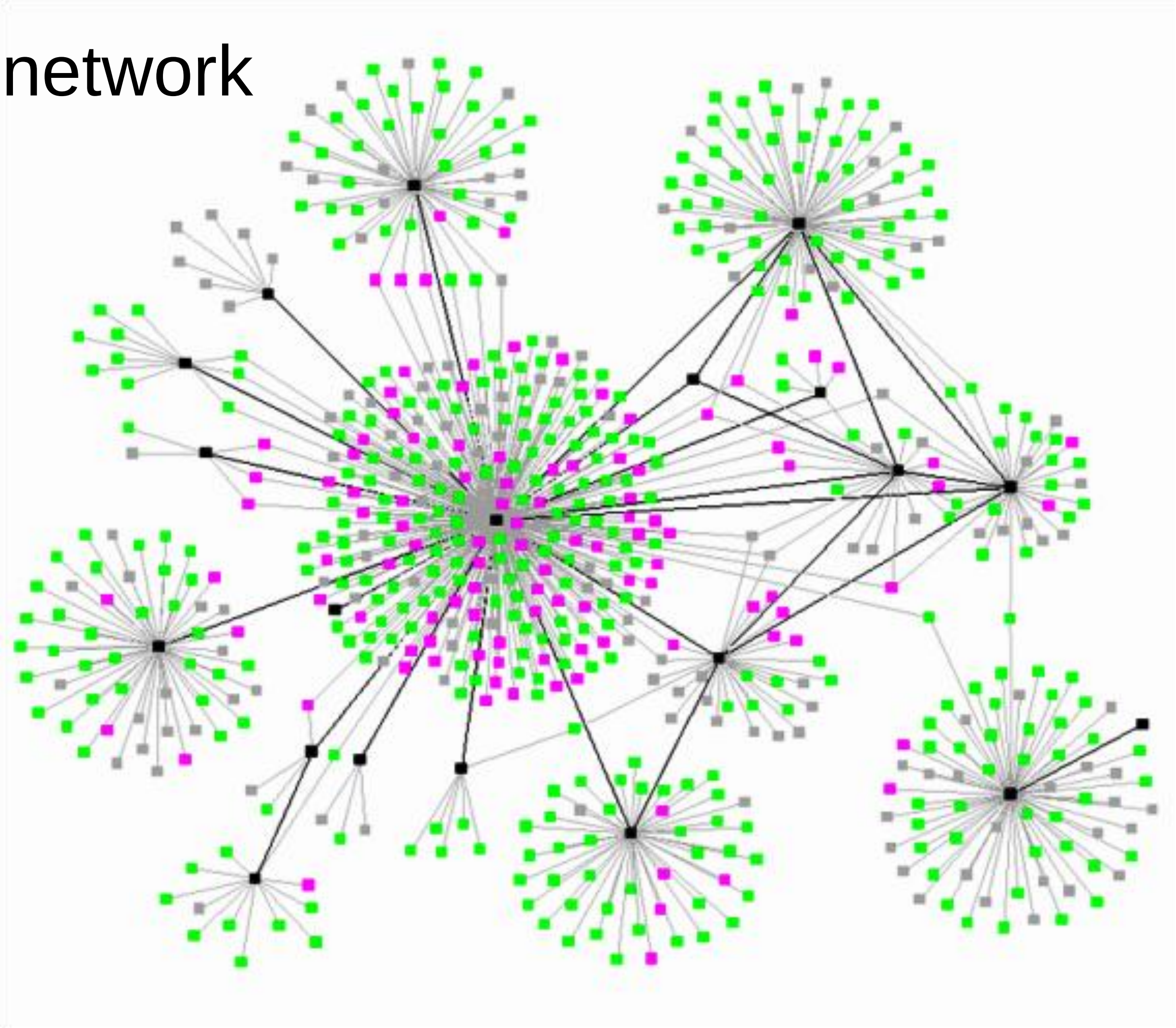
Graph Theory	Network Applications
Graph	Network
Vertex	Node
Edge	Arc
Degree	Order
Path	Route

Acquaintanceship Graphs

- We can use graph models to represent various relationships between people. For example, we can use a simple graph to represent whether two people know each other, that is, whether they are acquainted.
- Each person in a particular group of people is represented by a **vertex**. An undirected edge is used to connect two people when these people know each other.

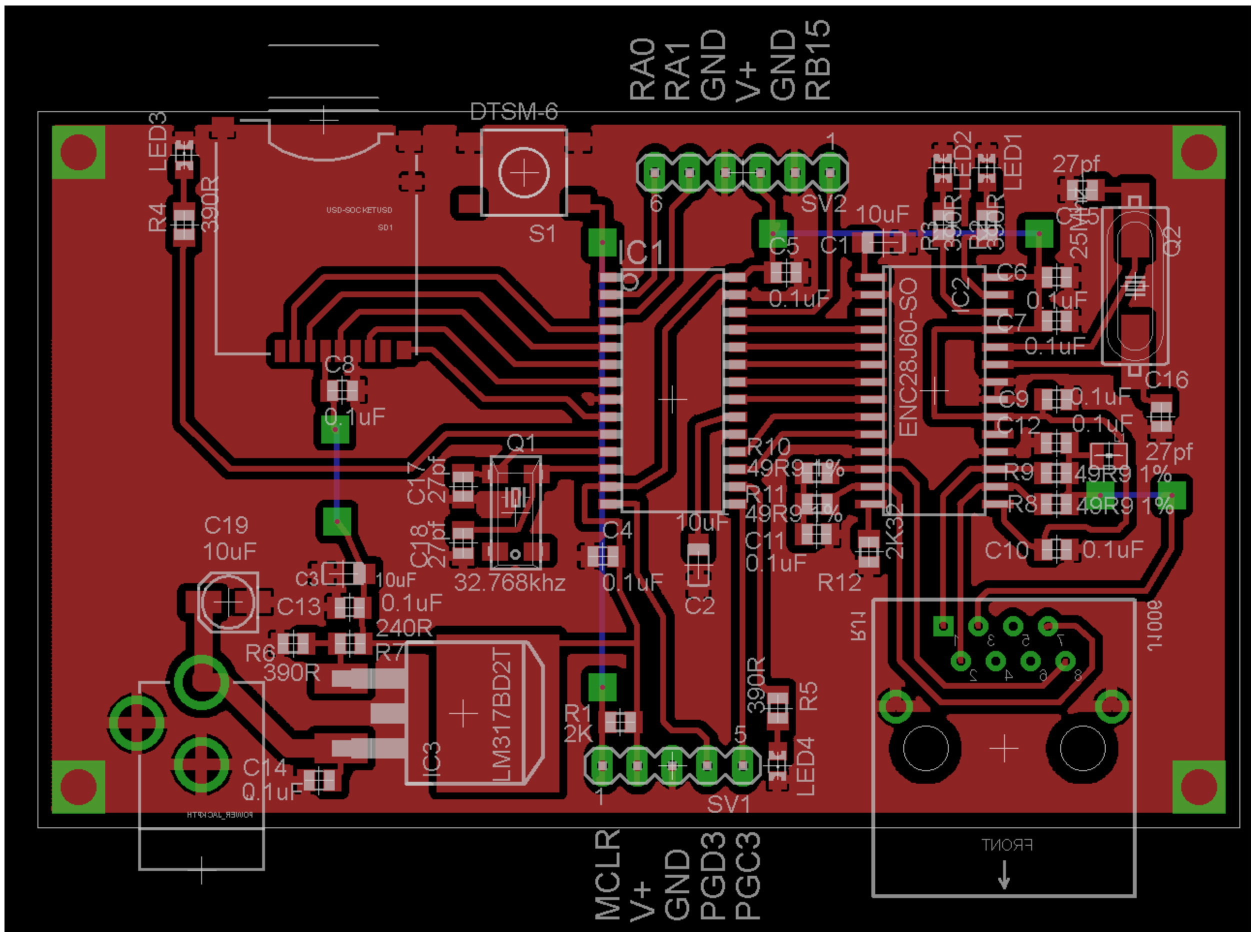
Application

- Social network



Application

- PCB Design

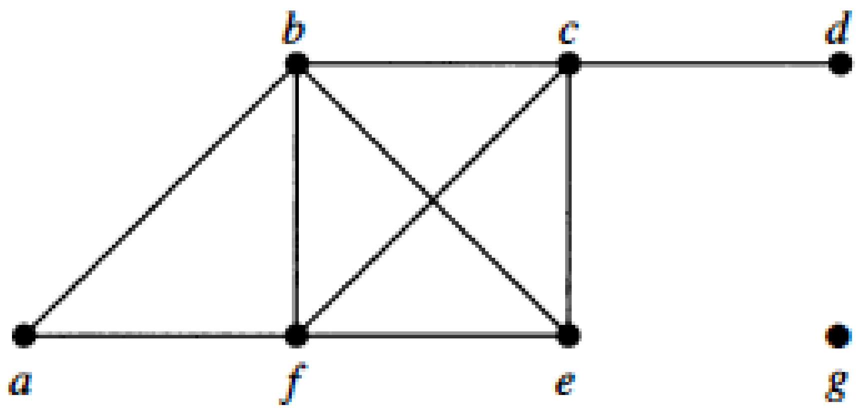


Basic Terminology

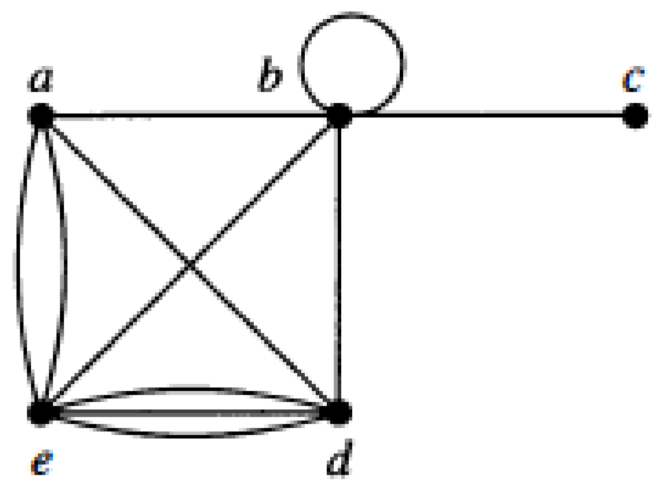
- Two vertices u and v in an undirected graph G are called **adjacent** (or neighbors) in G if u and v are endpoints of an edge of G . If e is associated with { u , v}, the edge e is called **incident** with the vertices u and v. The edge e is also said to connect u and v. The vertices u and v are called **endpoints** of an edge associated with { u , v}.

Basic Terminology

- The **degree** of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.



deg(a) = 2 , deg(b) = deg(c) = deg(f) = 4
deg(d) = 1 , deg(e) = 3 , and deg(g) = 0.



deg(a) = 4, deg(b) = deg(e) = 6,
deg(c) = 1 , and deg(d) = 5

Trivia

- A vertex of degree zero is called

Ans. **isolated**.

- It follows that an isolated vertex is not adjacent to any vertex.

- A vertex is ________ if and only if it has degree one.

Ans. **pendant**

Handshaking Theorem

- Let G = (V, E) be an undirected graph with e edges. Then

$$2e = \sum_{v \in V} \deg(v)$$

- Ex. How many edges are there in a graph with 10 vertices each of degree six?

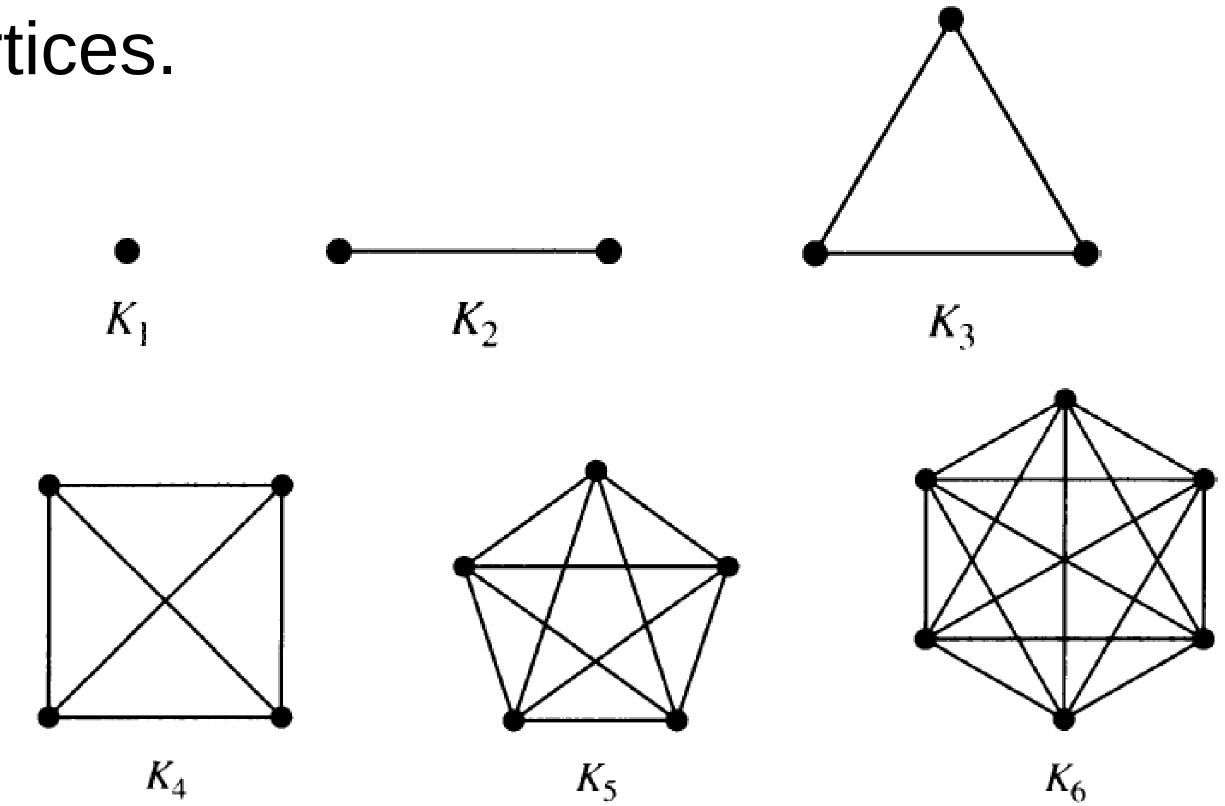
$$\sum_{v \in V} \deg(v) = 6 \times 10 = 60$$

$$2e = 60$$

$$e = 30$$

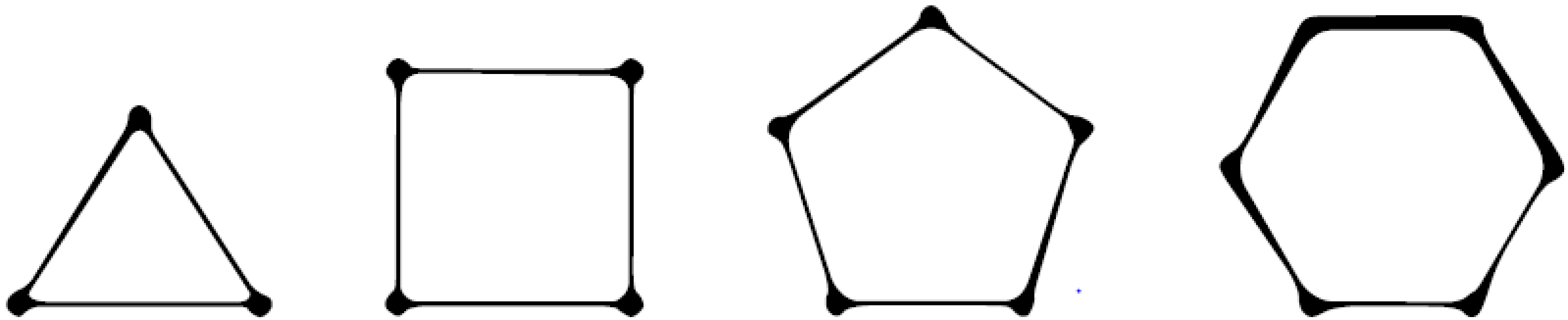
Some Special Simple Graphs

- The complete graph on *n* vertices, denoted by K_n , is the simple graph that contains exactly one edge between each pair of distinct vertices.



Cycles

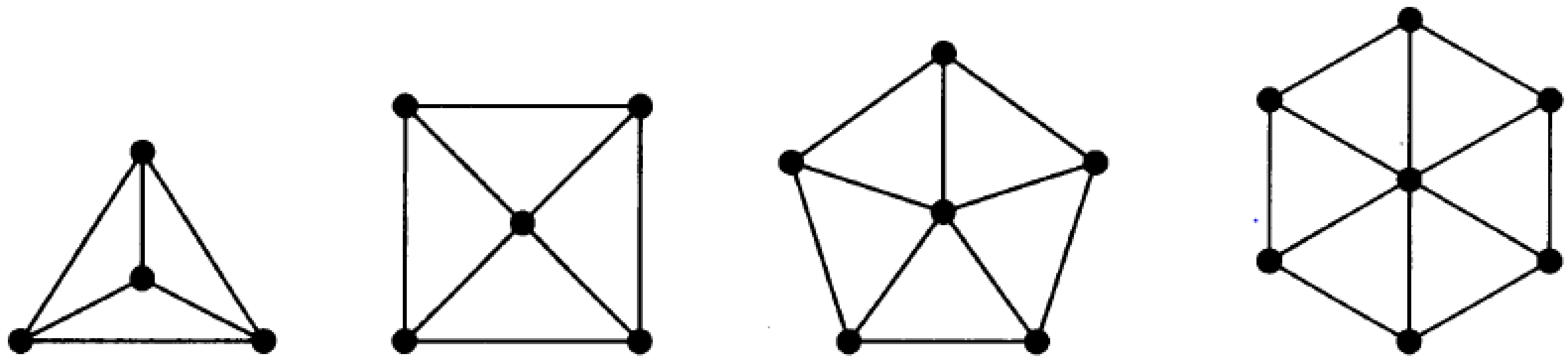
- The cycle C_n, $n>=3$, consists of n vertices V_1, V_2 , . . . , V_n and edges { V_1 , V_2 } , { V_2 , V_3 } , . . . , { v_{n-1} , v_n } , and { v_n , v_1 }.



C_3 C_4 C_5 C_6

Wheels

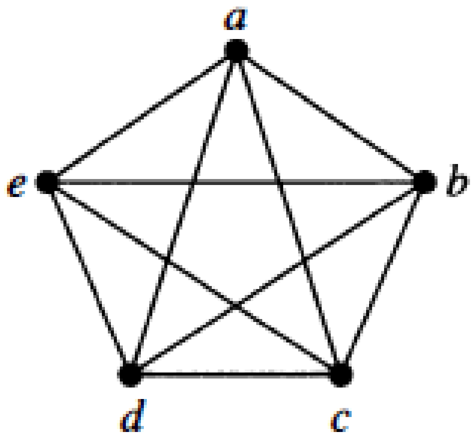
- We obtain the wheel W_n when we add an additional vertex to the cycle en , for n >=3 , and connect this new vertex to each o f the n vertices in C_n, by new edges.



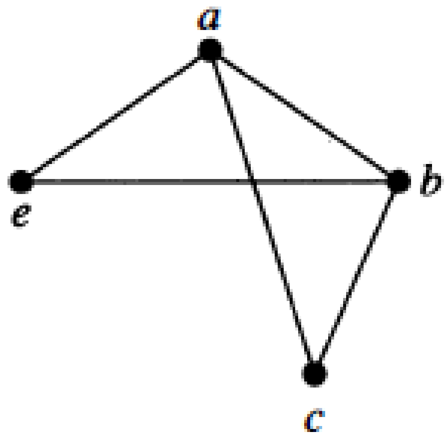
W_3 W_4 W_5 W_6

Basic Terminology

- A **subgraph** of a graph G = (V, E) is a graph H = (W, F), where W $\subseteq$ V and F$\subseteq$ E . A subgraph H of G is a proper subgraph of G if H $\neq$ G.



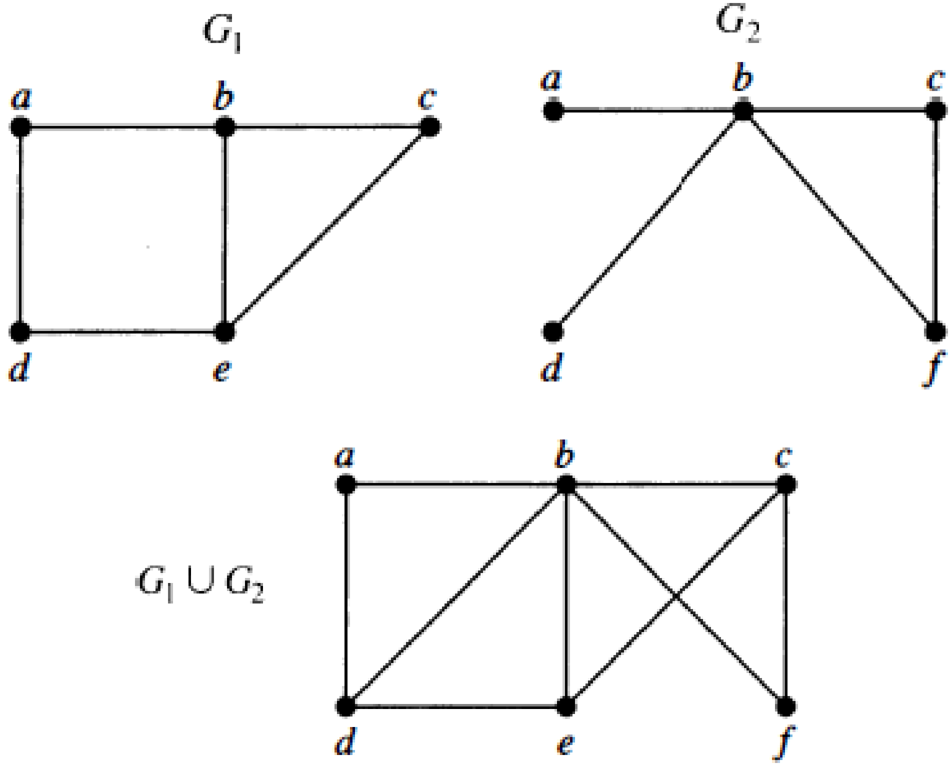
G



subgraph of G

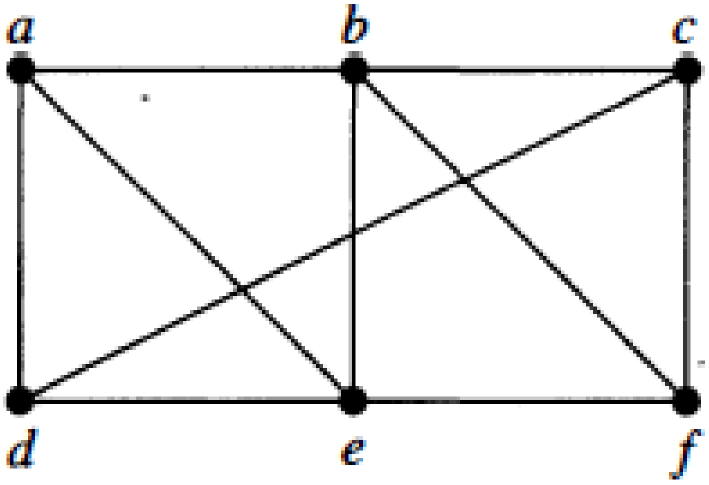
Basic Terminology

- The **union** of two simple graphs G1 = (V1, E1) and G2 = (V2, E2) is the simple graph with vertex set V1 U V2 and edge set E1 U E2 .



Paths

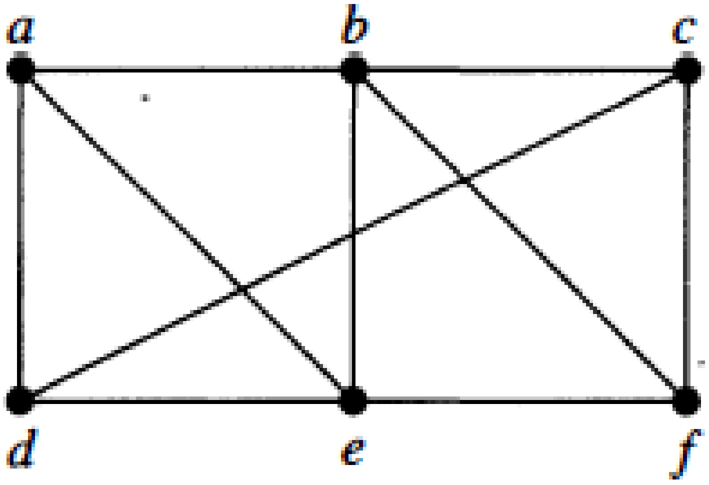
- A **path** is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.



- Is a, d, c, f, e a path?
- **YES**. {a , d}, {d, c} , {c, f}, and {f, e} are all edges

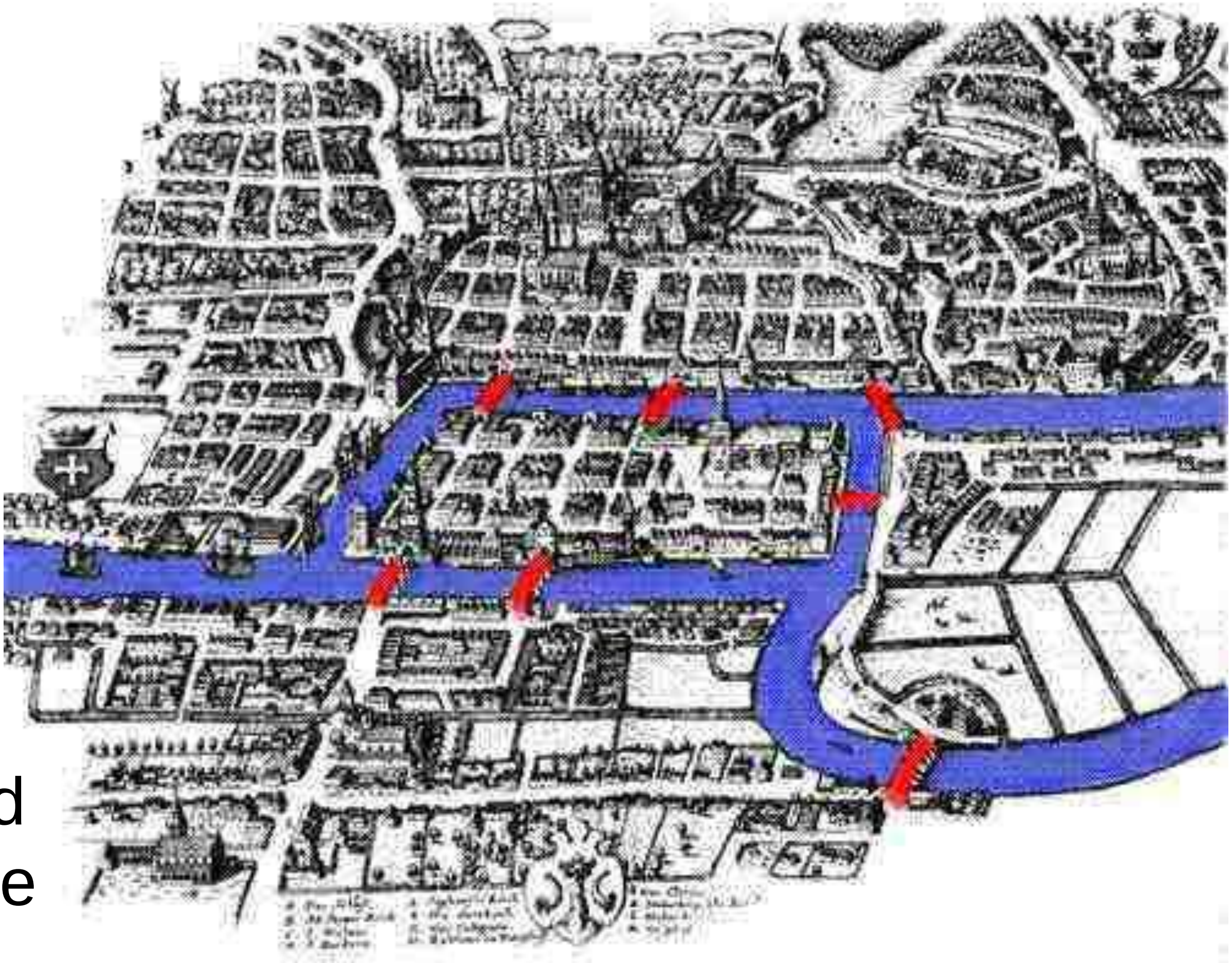
Paths

- A **path** is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.



- Is a, d, e, c, f a path?
- **NO**. because {e, c} is not an edge

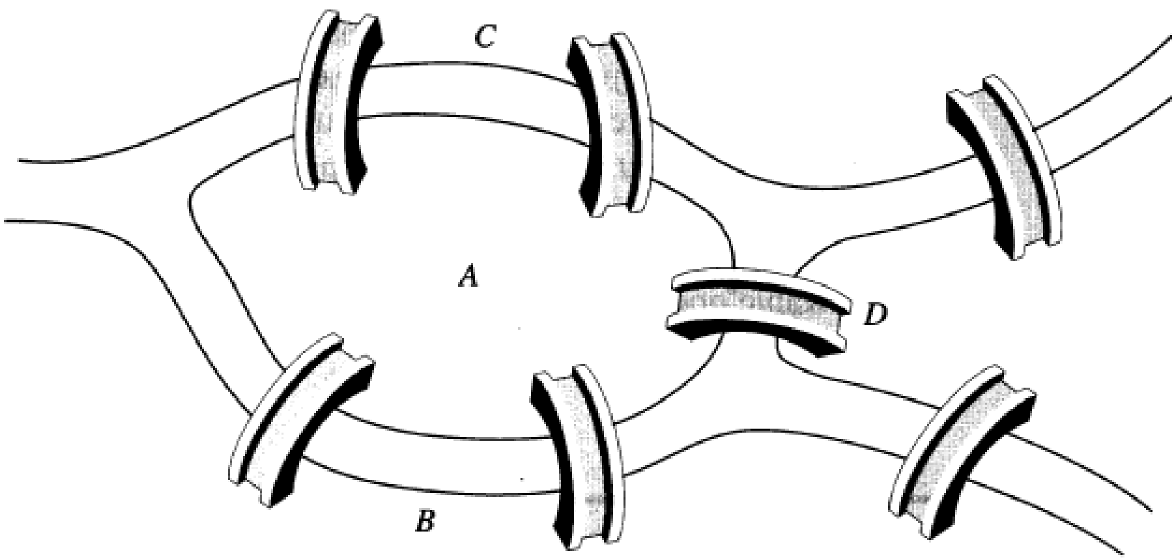
Königsberg Problem

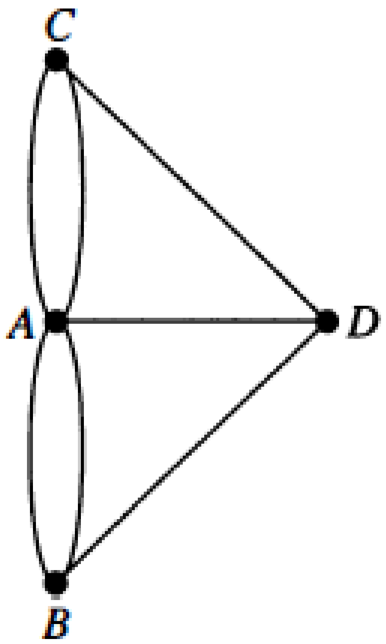


Find a walk through the city that would cross each bridge once and only once

Euler Circuit and Paths

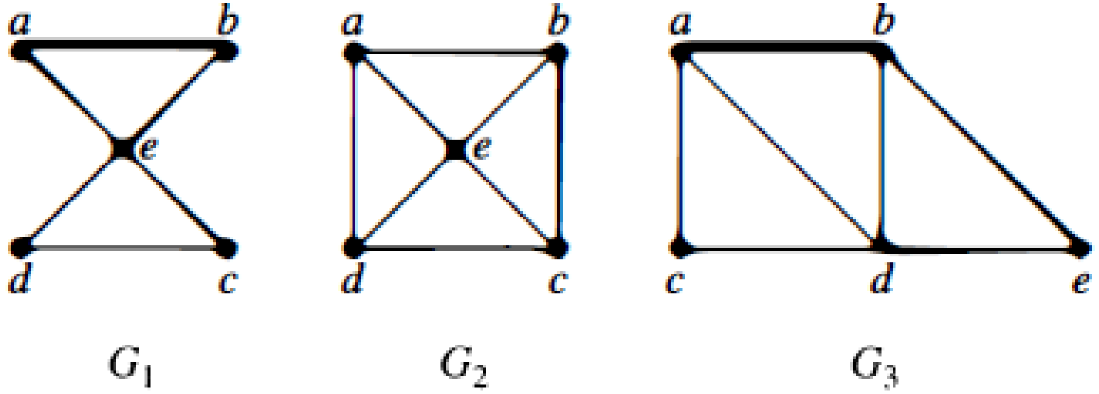
- **Euler circuit** in a graph G is a simple circuit containing every edge of G. An **Euler path** in G is a simple path containing every edge of G .





Drill

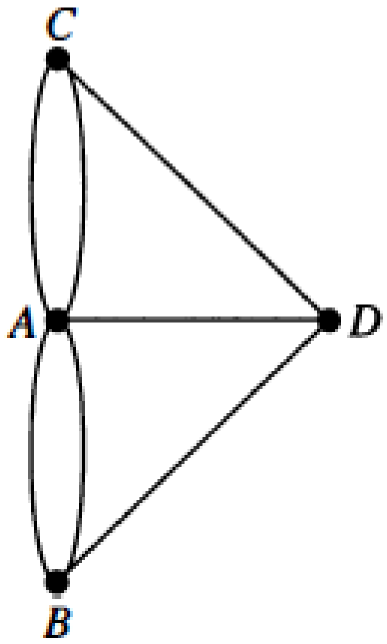
- Which of the undirected graphs have an Euler circuit? Of those that do not, which have an Euler path?



- G1 has an Euler circuit, for example, a , e, c, d, e, b, a. Neither of the graphs G2 or G3 has an Euler circuit. G3 has an Euler path, namely, a , c, d, e, b, d, a , b.
- G2 does not have an Euler path

Euler Circuit and Paths

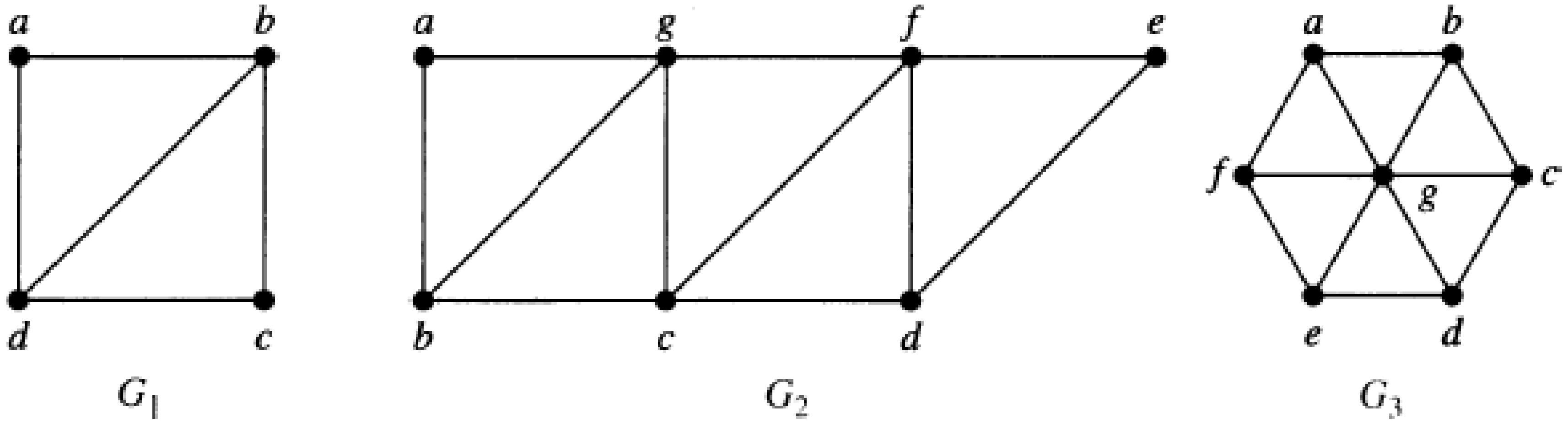
- A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.



Since it has four vertices of **odd** degree, it does not have an Euler circuit. There is no way to start at a given point, cross each bridge exactly once, and return to the starting point.

Euler Circuit and Paths

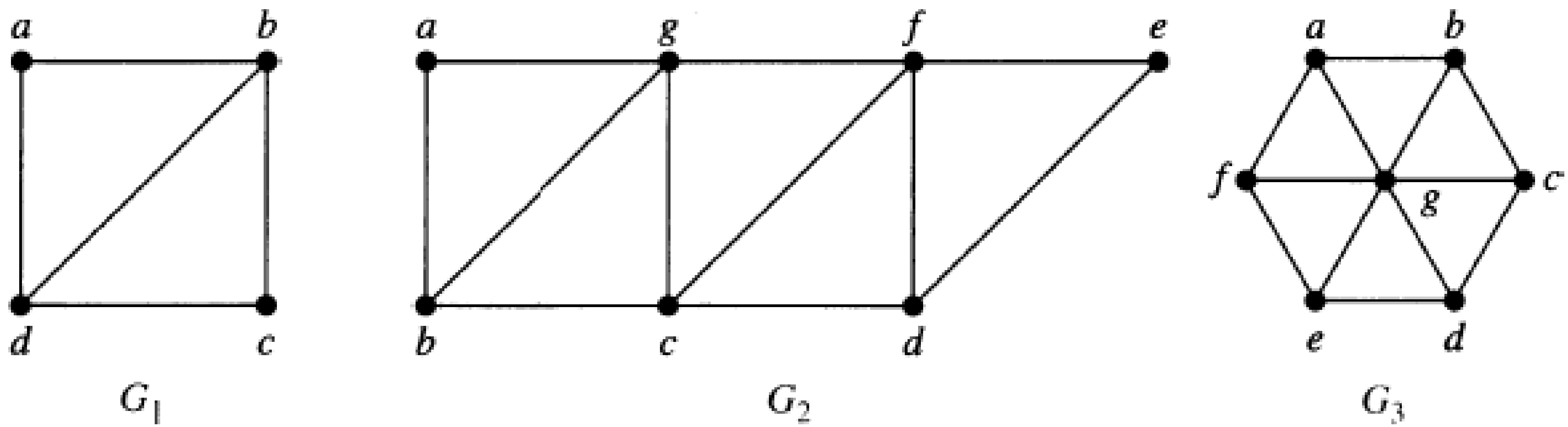
- A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.



- G1 contains exactly two vertices of odd degree, namely, b and d. Hence, it has an Euler path that must have b and d as its endpoints. One such Euler path is d, a , b, c, d, b.

Euler Circuit and Paths

- G2 has exactly two vertices of odd degree, namely, b and d. So it has an Euler path that must have b and d as endpoints. One such Euler path is b, a , g, j, e, d, c, g, b, c, j, d.
- G3 has no Euler path because it has six vertices of odd degree.



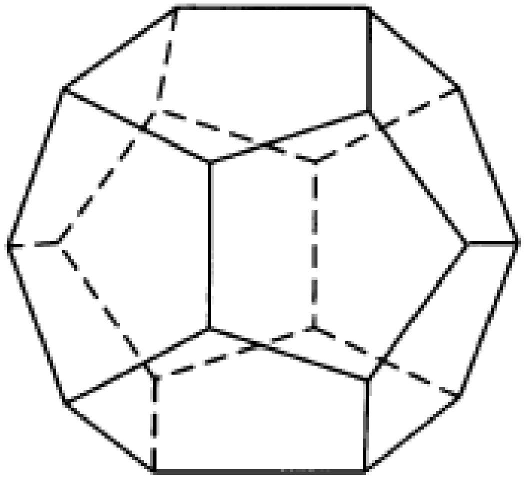
In a Nutshell

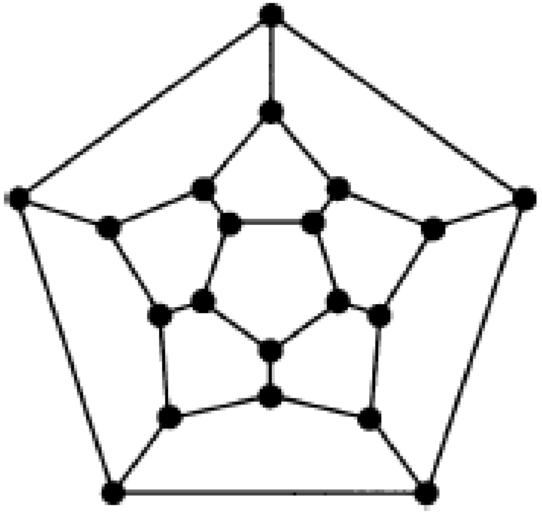
- An Eulerian path in a graph is a path that travels along every edge of the graph exactly once. An Eulerian path might pass through individual vertices of the graph more than once.

- An Eulerian path which begins and ends in the same place is called an Eulerian circuit or an eulerian cycle

Hamilton Paths and Circuits

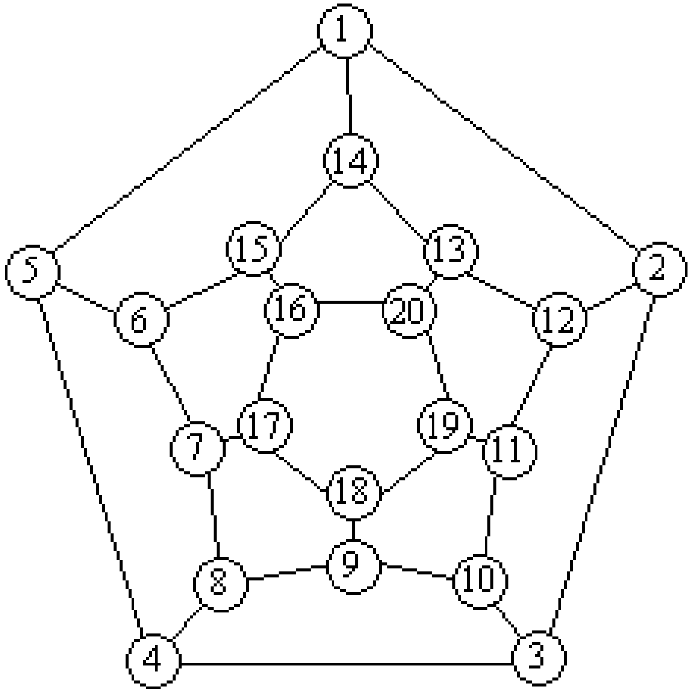
- A simple path in a graph G that passes through every vertex exactly once is called a **Hamilton path**, and a simple circuit in a graph G that passes through every vertex exactly once is called a **Hamilton circuit**.

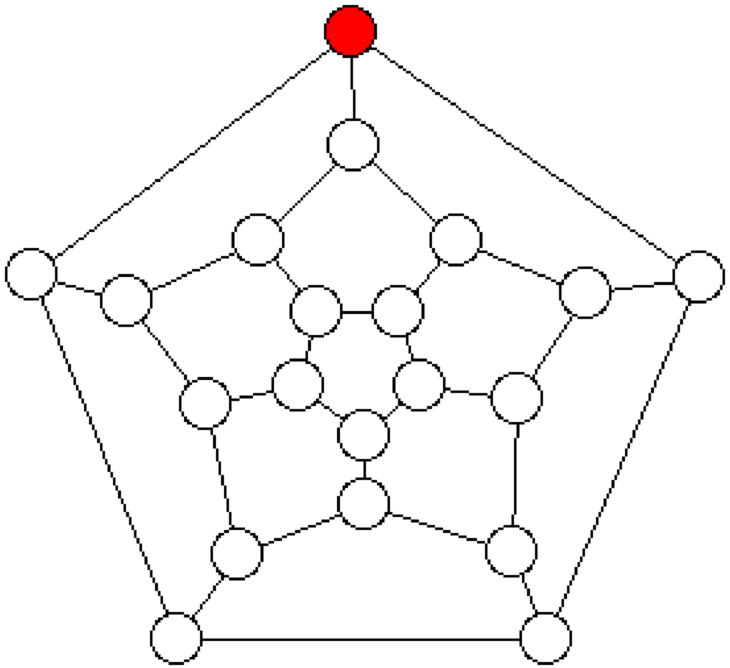




Example

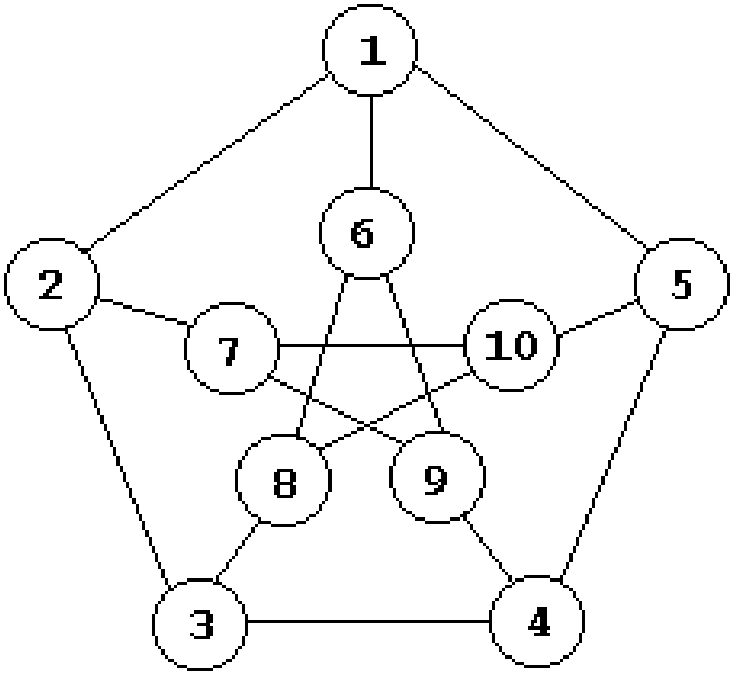
- Hamiltonian cycle/circuit in a dodecahedron

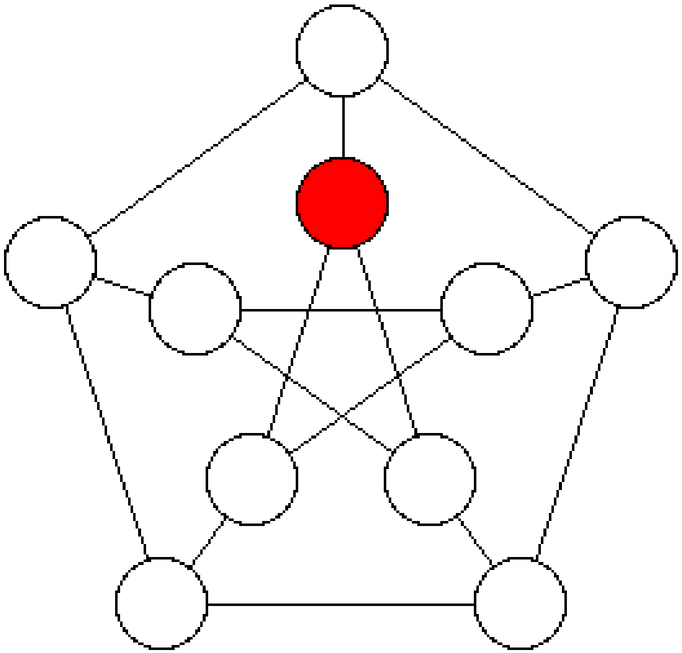




Example

- Hamiltonian cycle/circuit for a Petersen graph





Petersen graph has a Hamiltonian path but no Hamiltonian cycle.

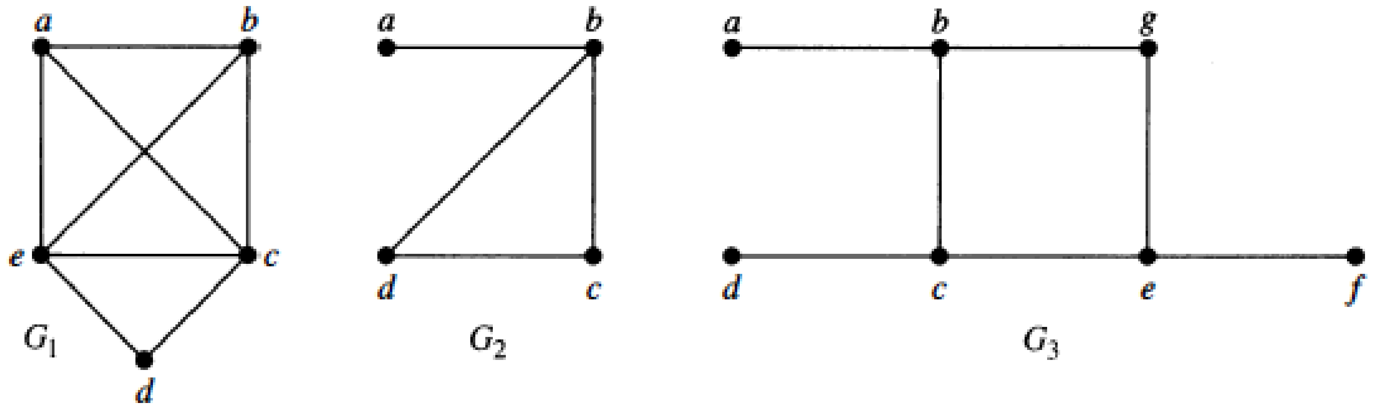
In a Nutshell

- A **hamiltonian path** in a graph is a path that passes through every vertex in the graph exactly once. It does not necessarily pass through all the edges of the graph.

- A hamiltonian path which ends in the same place in which it began is called a **hamiltonian circuit** or a **hamiltonian cycle**.

Drill

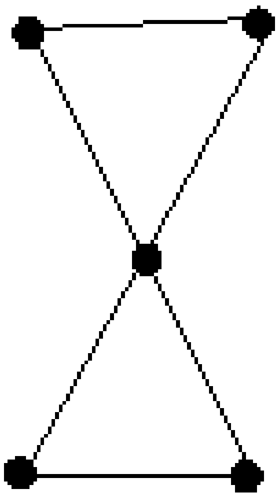
- Which of the simple graphs have a Hamilton circuit or, if not, a Hamilton path?

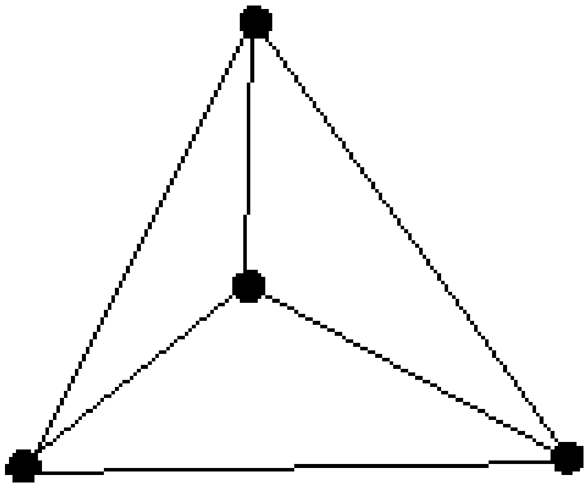


- G1 has a Hamilton circuit: a, b, c, d, e, a. There is no Hamilton circuit in G2 but it does have a Hamilton path, namely, a , b, c, d. G3 has neither a Hamilton circuit nor a Hamilton path, because any path containing all vertices must contain one of the edges {a, b}, {e, f}, and {c, d} more than once.

In a Nutshell

- Which is Eulerian and Hamiltonian?





Mnemonic: **E** for edge & **E** for Euler

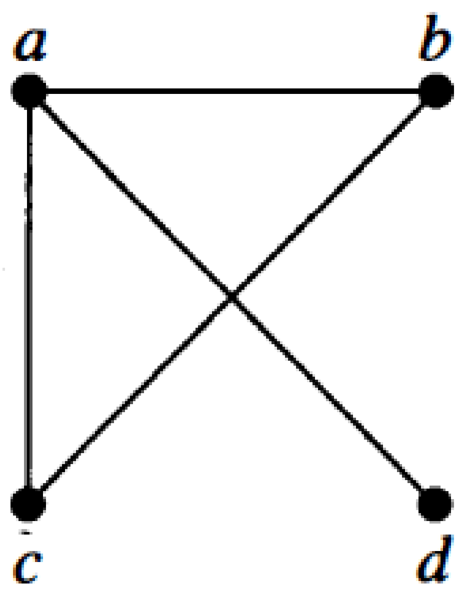
Matrices of Graphs

- The adjacency matrix A (or A_G) of G, with respect to this listing of the vertices, is the nxn zero-one matrix with 1 as its (i, j)th entry when v_i and v_j are adj acent, and 0 as its (i, j)th entry when they are not adjacent

$$a_{i,j} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge} \\ 0 & \end{cases}$$

Example

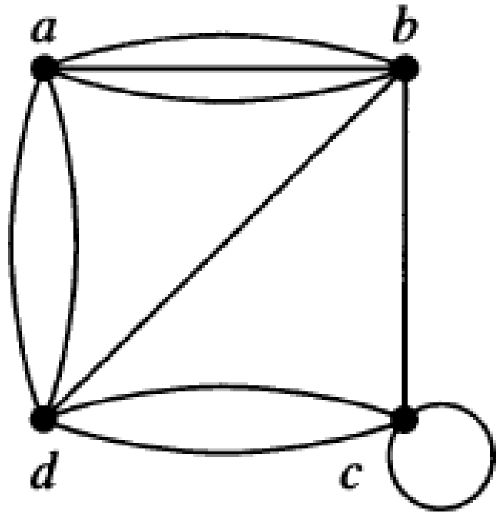
- Use an adjacency matrix to represent the following graph.



$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Drill

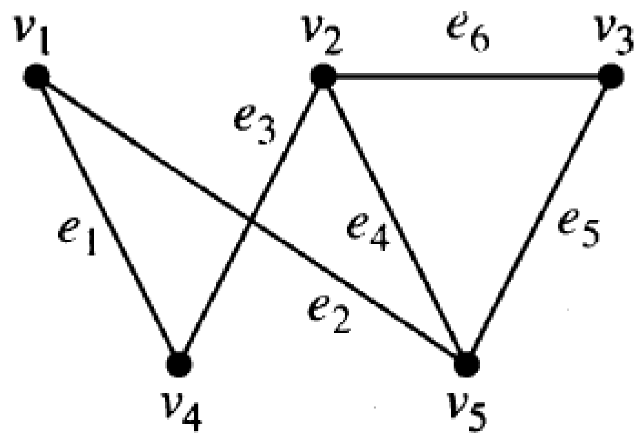
- Use an adjacency matrix to represent the shown.



$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Incidence Matrices

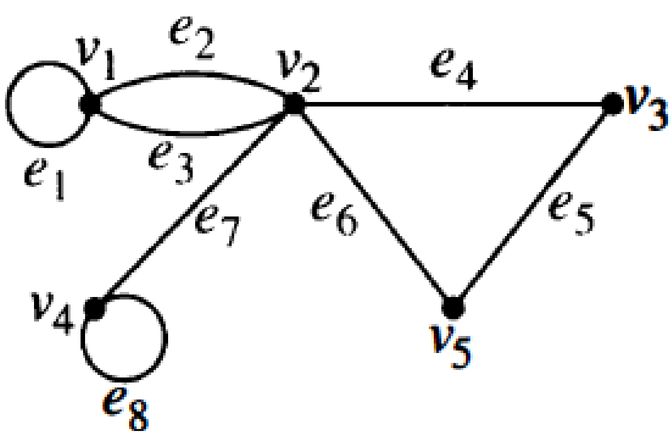
- Represent the graph shown with an incidence matrix.



$$
\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}
\begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.
$$

Drill

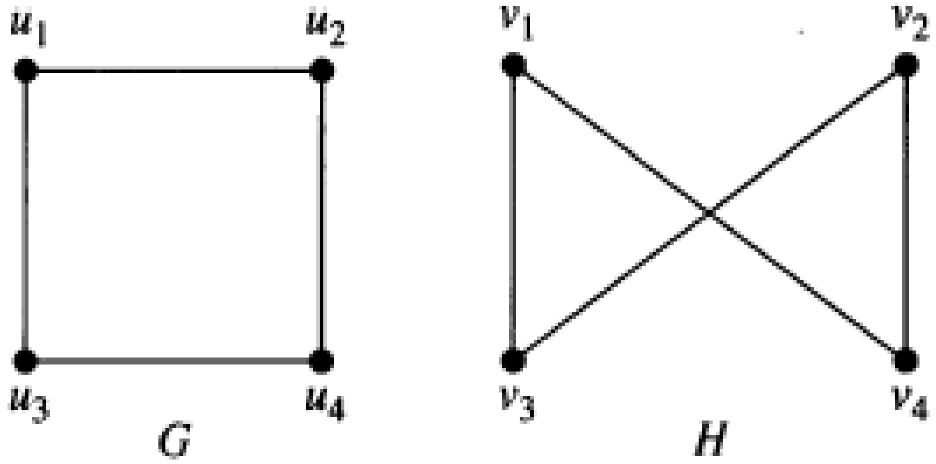
- Represent the graph using an incidence matrix.



$$
\begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
\end{array}
$$

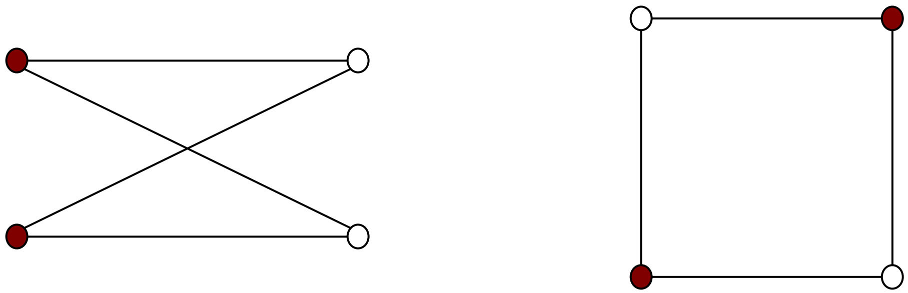
Isomorphism of Graphs

- Simple graphs G1 = (V1, E1) & G2 = (V2, E2) are isomorphic if there is a one-to-one and onto function f from V1 to V2 with the property that a and b are adjacent in G1 if and only if f(a) and f(b) are adjacent in G2 , for all a and b in V1 .



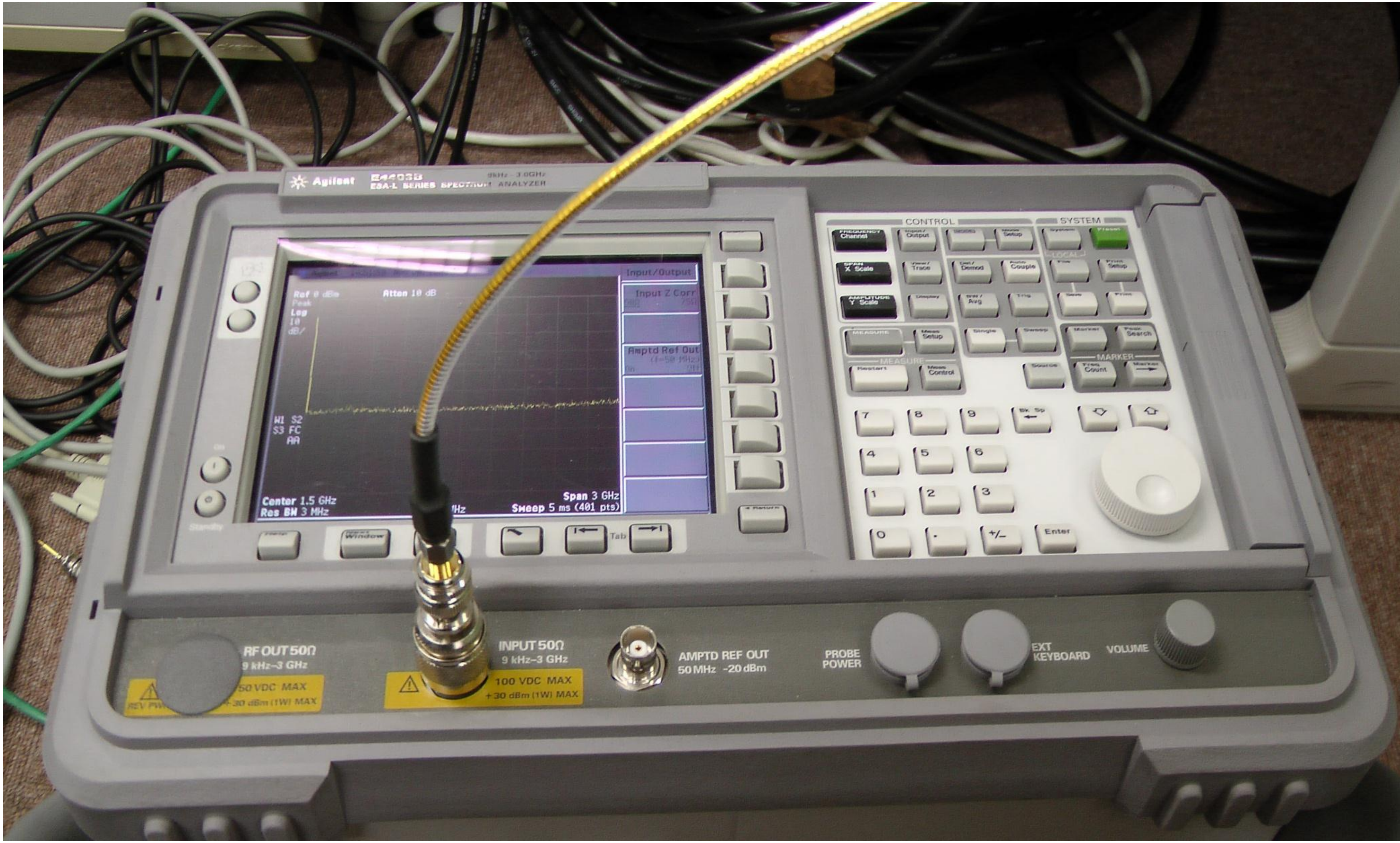
Example

- The function f with f(u1) = V1, f(u2) = V4 , f(u3) = V3 , and f(u4) = V2 is a one-to-one correspondence between V and W.



- Adjacent vertices in G are u1 and u2 , u1 and u3, u2 and u4, and u3 and u4, and each of the pairs f(u1) = V1 and f(u2) = V4, f(u1) = V1 and f(u3) = V3 , f(u2) = V4 and f(u4) = V2 , and f(u3) = V3 and f(u4) = V2 are adjacent in H.

Thank you for listening



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