# DISMATH
# Discrete Mathematics and Its Applications

Planar Graphs

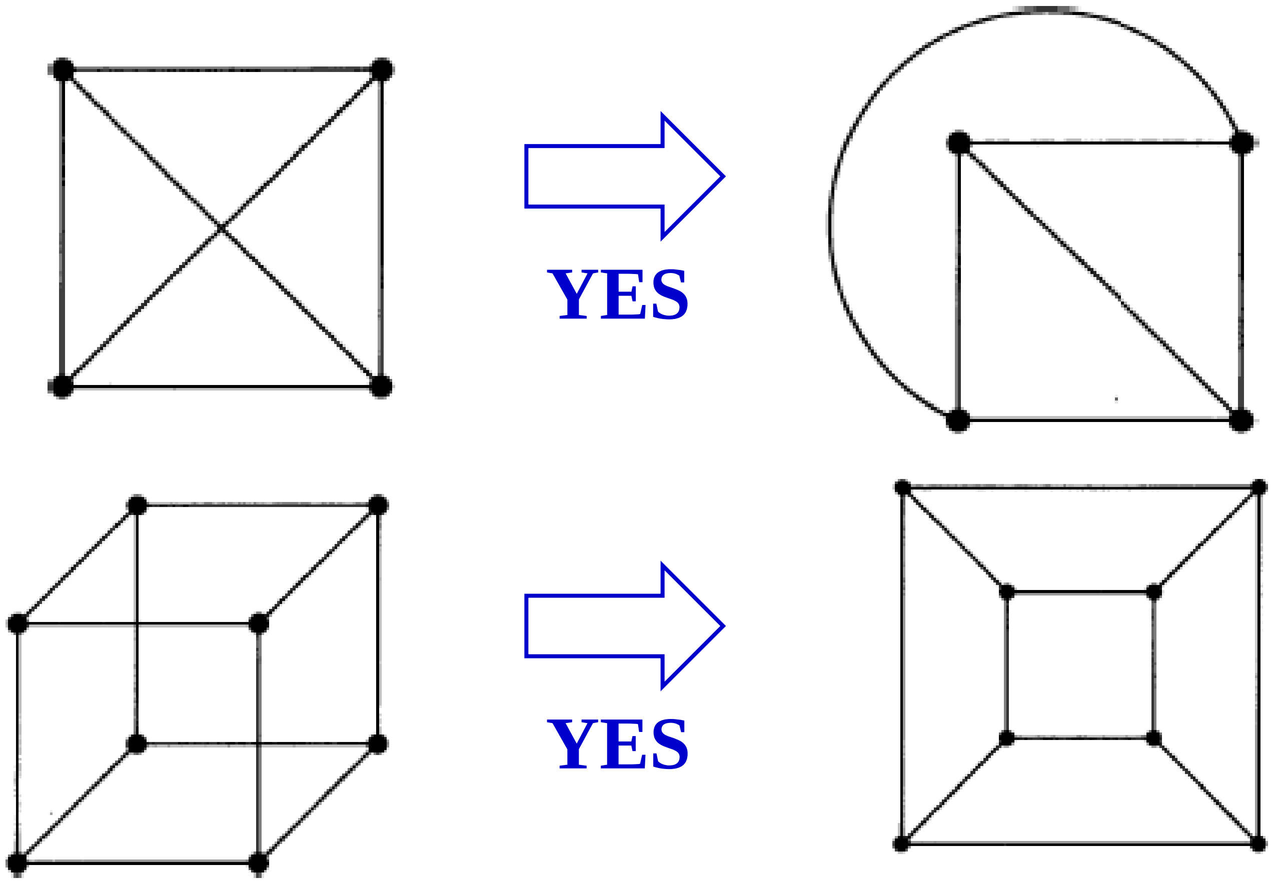
# Planar Graphs

- **Planar graph** – are graphs that can be drawn in the plane without edges having to cross.

- Any graph representation of maps/ topographical information is planar.
  - graph algorithms often specialized to planar graphs (e.g. traveling salesman)
- Circuits usually represented by planar graphs
  - PCB circuit design

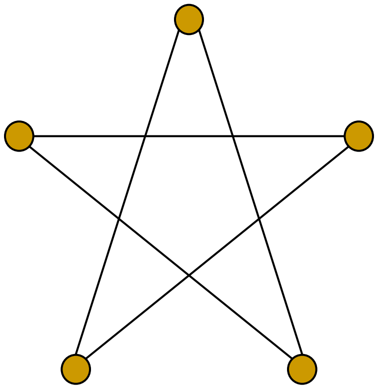
# Example

- Are the following graphs planar?



# Drill

- Is the following graph planar?



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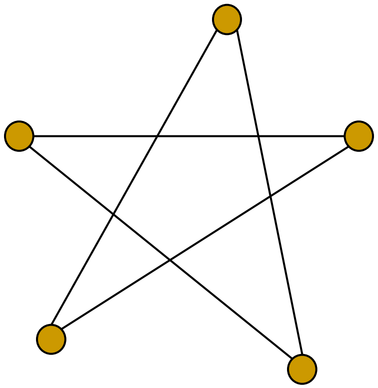

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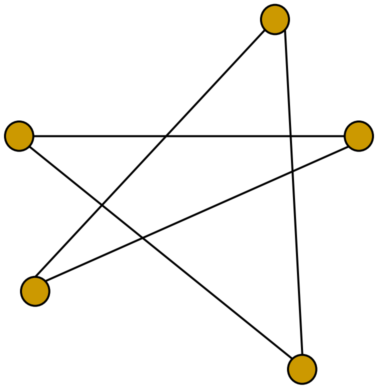

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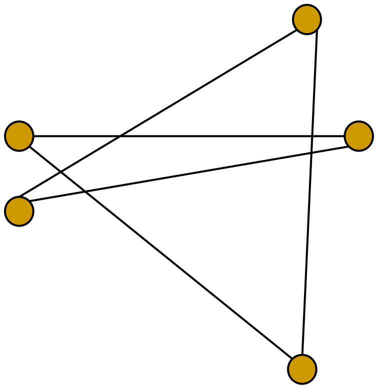

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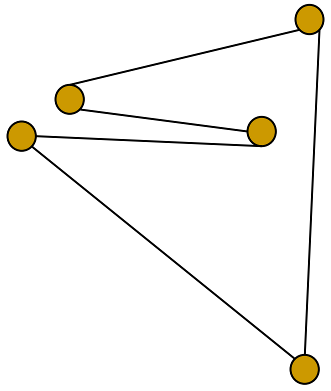

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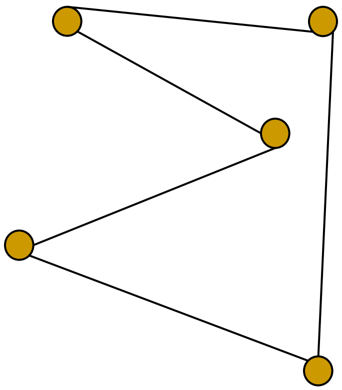

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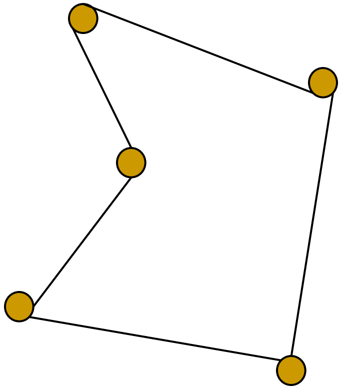

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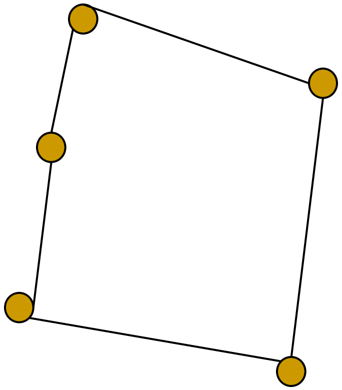

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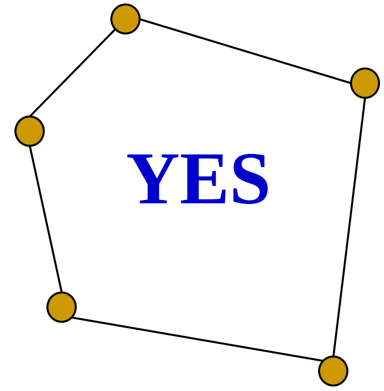

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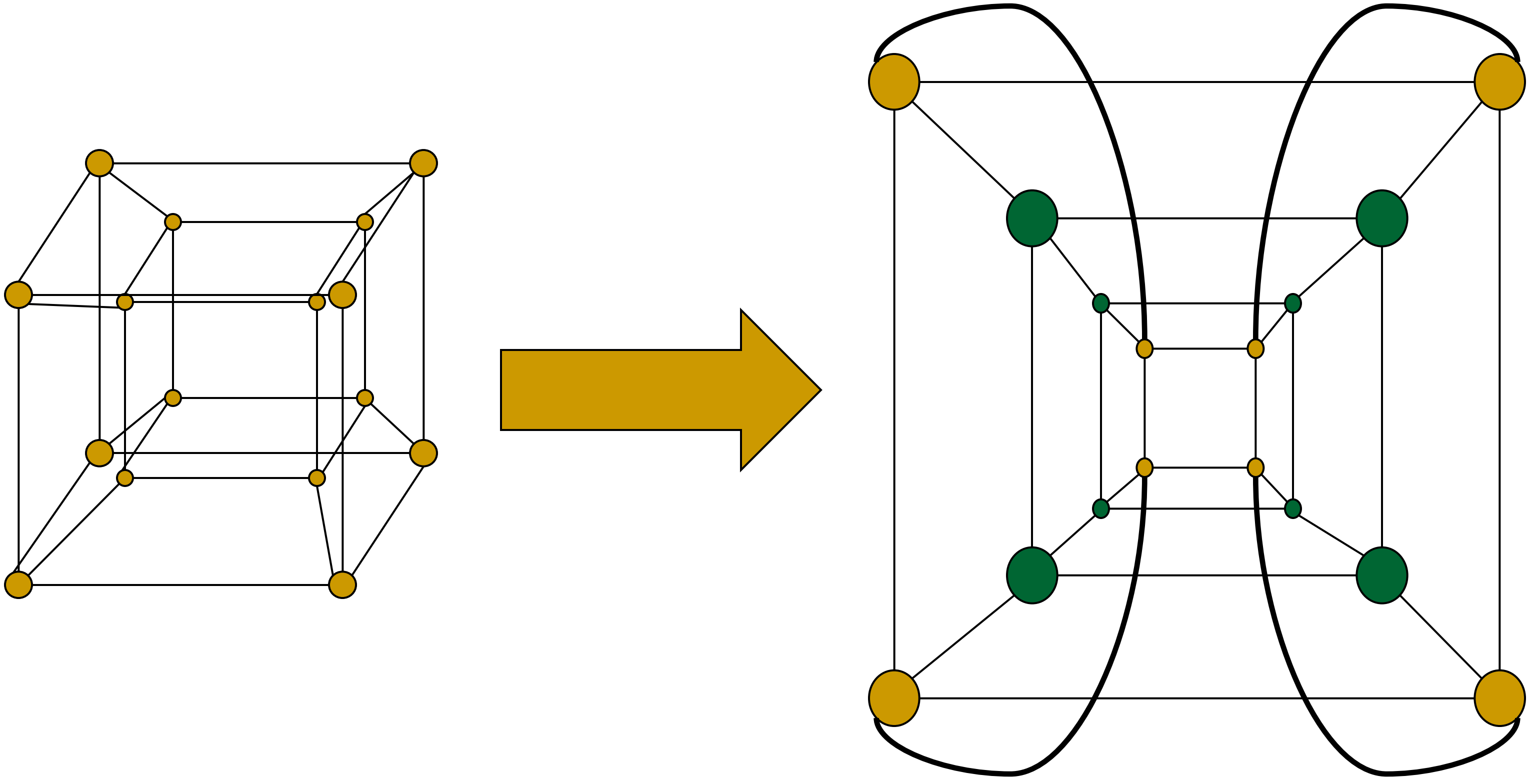

# Example

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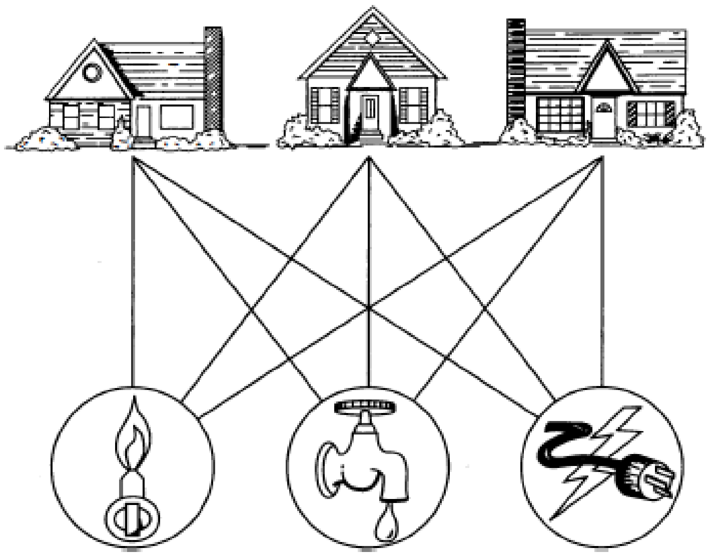
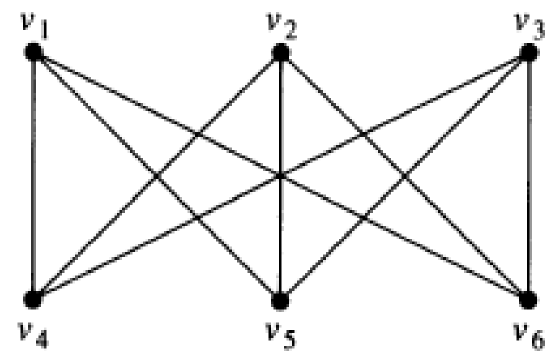
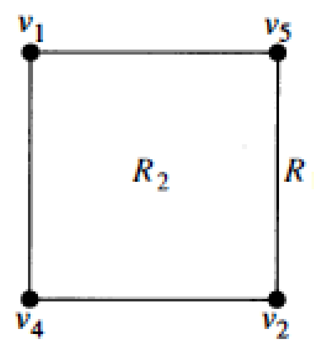
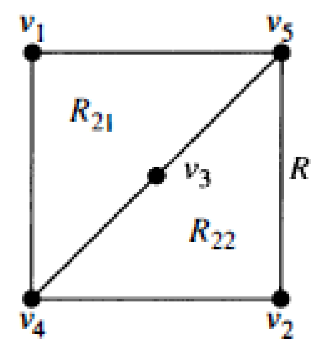




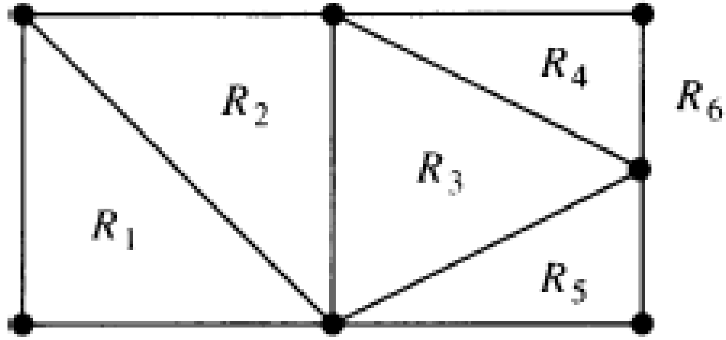



# Euler's Formula

- Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G . Then

$$\boxed{r = e - v + 2}$$

- A planar representation of a graph splits the plane into regions, including an unbounded region.

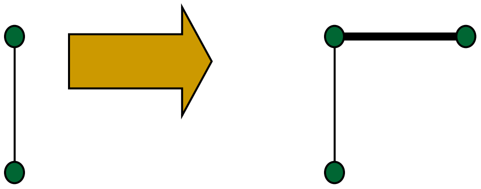


# Euler's Formula

- The formula is proved by showing that the quantity (chi) $\chi = r - |E| + |V|$ must equal 2 for planar graphs. $\chi$ is called the ***Euler characteristic***.
- The idea is that any connected planar graph can be built up from a vertex through a sequence of vertex and edge additions.

# Euler Characteristic

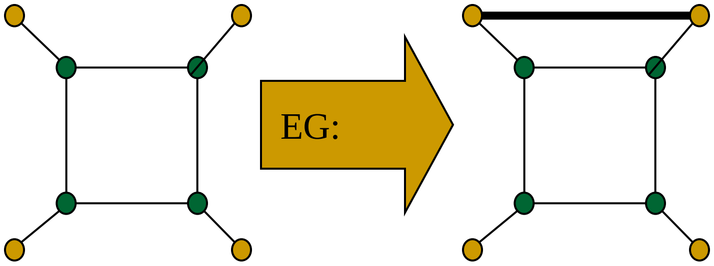
Check that moves don't change $\chi$ :



1.) Adding a degree 1 vertex:

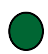
*r* is unchanged. |*E* | increases by 1. |*V* | increases by 1. $\chi$ += (0-1+1)

2) Adding an edge between pre-existing vertices:



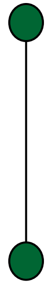

*r* increases by 1. |*E* | increases by 1. |*V* | unchanged. $\chi$ += (1-1+0)

# Euler Characteristic



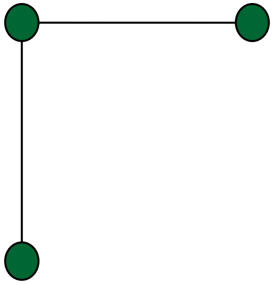
| $\|V\|$ | $\|E\|$ | $r$ | $\chi = r - \|E\| + \|V\|$ |
|---|---|---|---|
| 1 | 0 | 1 | 2 |

# Euler Characteristic



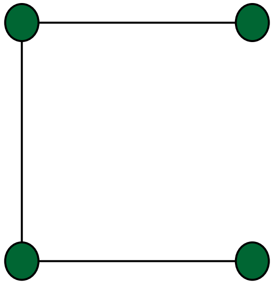
| $\|V\|$ | $\|E\|$ | $r$ | $\chi =$ $r - \|E\| + \|V\|$ |
|---|---|---|---|
| 2 | 1 | 1 | 2 |

# Euler Characteristic



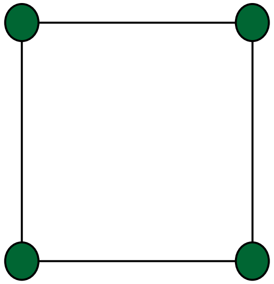
| $\|V\|$ | $\|E\|$ | $r$ | $\chi = r - \|E\| + \|V\|$ |
|---|---|---|---|
| 3 | 2 | 1 | 2 |

# Euler Characteristic



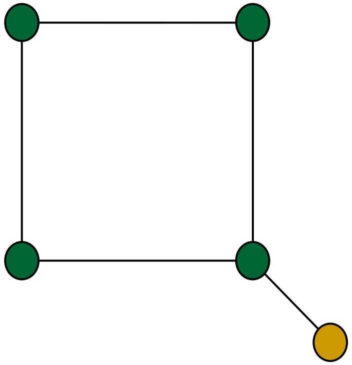
| $\|V\|$ | $\|E\|$ | $r$ | $\chi =$ $r - \|E\| + \|V\|$ |
|---|---|---|---|
| 4 | 3 | 1 | 2 |

# Euler Characteristic



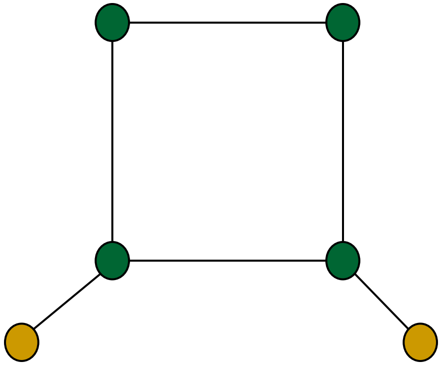
| $\|V\|$ | $\|E\|$ | $r$ | $\chi = r - \|E\| + \|V\|$ |
|---|---|---|---|
| 4 | 4 | 2 | 2 |

# Euler Characteristic



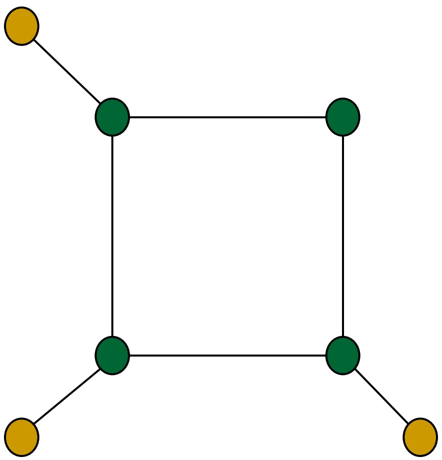
| $\|V\|$ | $\|E\|$ | $r$ | $\chi = r - \|E\| + \|V\|$ |
|---|---|---|---|
| 5 | 5 | 2 | 2 |

# Euler Characteristic



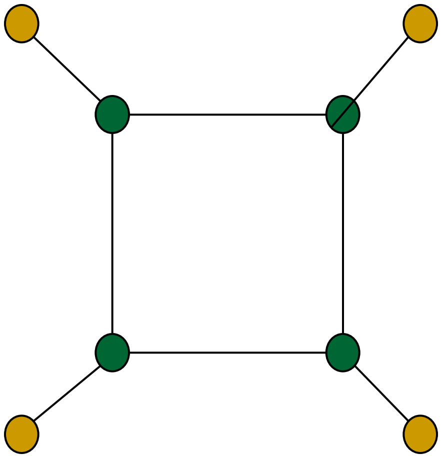
| $\|V\|$ | $\|E\|$ | $r$ | $\chi = r - \|E\| + \|V\|$ |
|---|---|---|---|
| 6 | 6 | 2 | 2 |

# Euler Characteristic



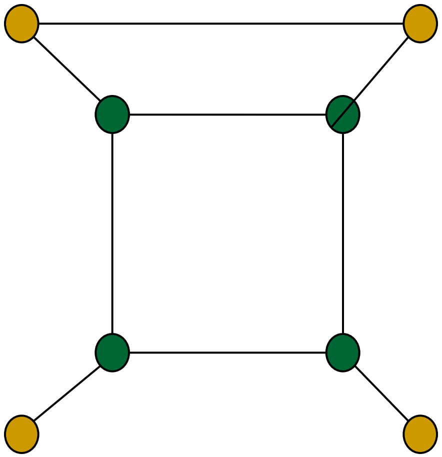
| $\|V\|$ | $\|E\|$ | $r$ | $\chi = r - \|E\| + \|V\|$ |
|---|---|---|---|
| 7 | 7 | 2 | 2 |

# Euler Characteristic



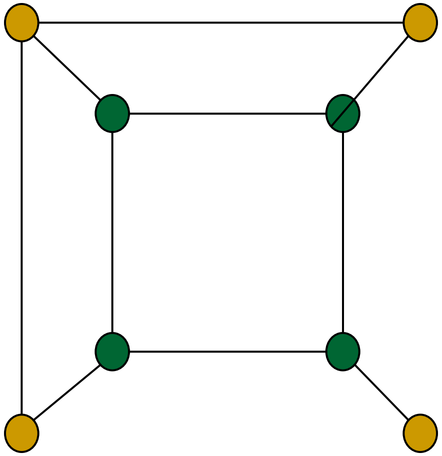
| $\|V\|$ | $\|E\|$ | $r$ | $\chi = r - \|E\| + \|V\|$ |
|---|---|---|---|
| 8 | 8 | 2 | 2 |

# Euler Characteristic



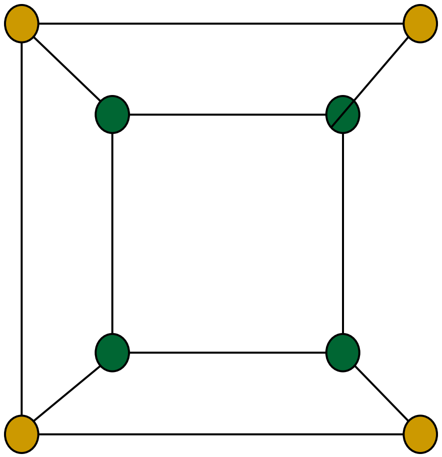
| $\lvert V \rvert$ | $\lvert E \rvert$ | $r$ | $\chi = r - \lvert E \rvert + \lvert V \rvert$ |
|---|---|---|---|
| 8 | 9 | 3 | 2 |

# Euler Characteristic



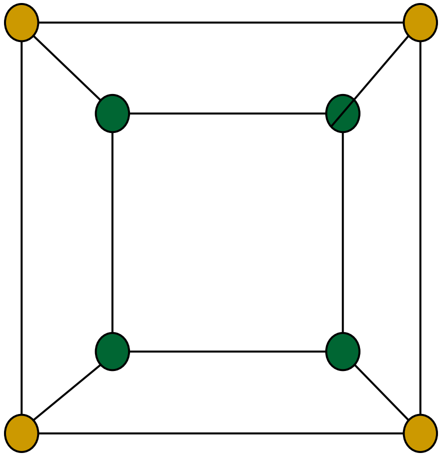
| $\|V\|$ | $\|E\|$ | $r$ | $\chi = r - \|E\| + \|V\|$ |
|---|---|---|---|
| 8 | 10 | 4 | 2 |

# Euler Characteristic



| $\|V\|$ | $\|E\|$ | $r$ | $\chi = r - \|E\| + \|V\|$ |
|---|---|---|---|
| 8 | 11 | 5 | 2 |

# Euler Characteristic



| $\lvert V \rvert$ | $\lvert E \rvert$ | $r$ | $\chi = r - \lvert E \rvert + \lvert V \rvert$ |
|---|---|---|---|
| 8 | 12 | 6 | 2 |

# Example

- Suppose that a connected planar simple graph has 20 vertices, each of degree 3 . Into how many regions does a representation of this planar graph split the plane?

$$\boxed{r = e - v + 2}$$

$$2e = \sum_{v \in V} \deg(v)$$

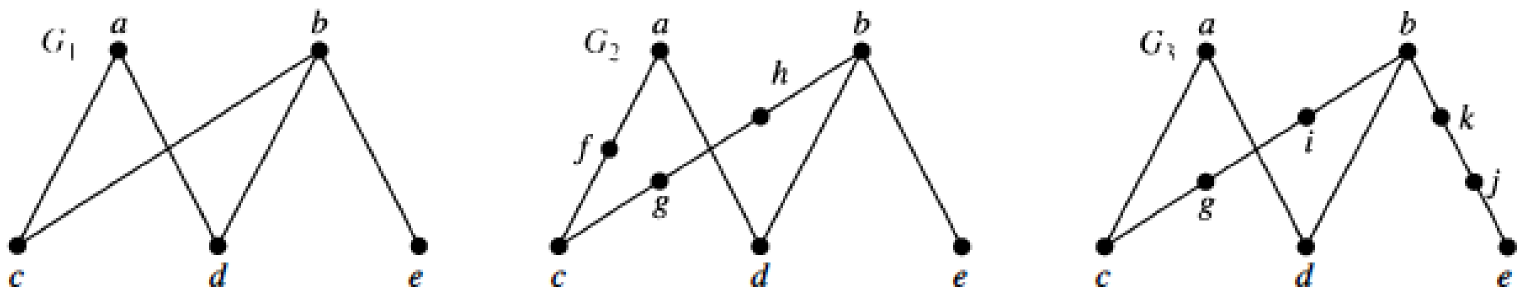
$$2e = 3(20)$$

$$e = 30$$

$$\begin{aligned} r &= e - v + 2 \\ &= 30 - 20 + 2 \\ &= 12 \end{aligned}$$

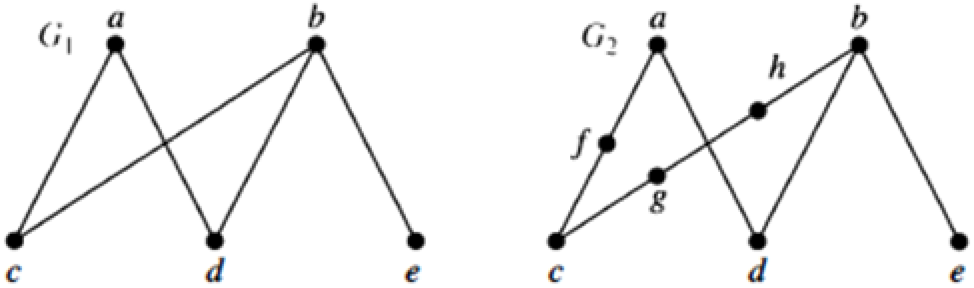
# Homeomorphic Graphs

- Elementary subdivision - If a graph is planar, so will be any graph obtained by removing an edge { u, v} and adding a new vertex w together with edges { u, w } and {w, v} .
- The graphs G1= ( $V_1$, $E_1$ ) and $G_2$ = ( $V_2$ , $E_2$) are called **homeomorphic** if they can be obtained from the same graph by a sequence of elementary subdivisions.



# Example

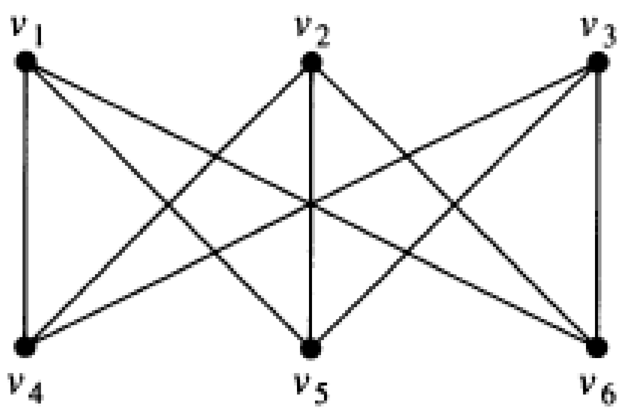
- Show that the graphs G1, and G2 are homeomorphic.

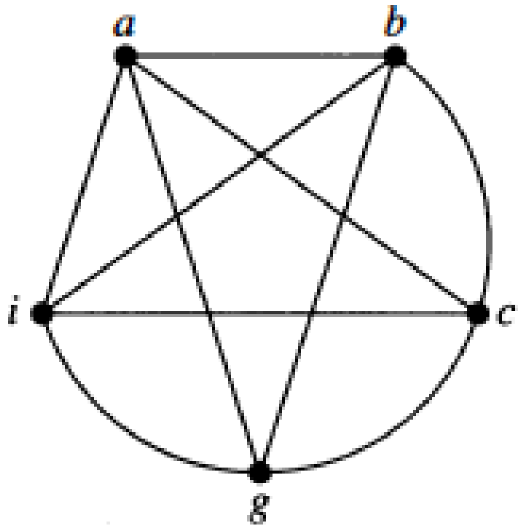



- G1 can be obtained from itself by an empty sequence of elementary subdivisions.
- G2 from G1: (i) remove the edge {a, c}, add the vertex f, and add the edges {a, f} and { j, c}; (ii) remove the edge {b, c}, add the vertex g, and add the edges {b, g} and {g, c } ; and (iii) remove the edge {b, g}, add the vertex h , and add the edges {g, h } and {b, h }

# Kuratowski's Theorem

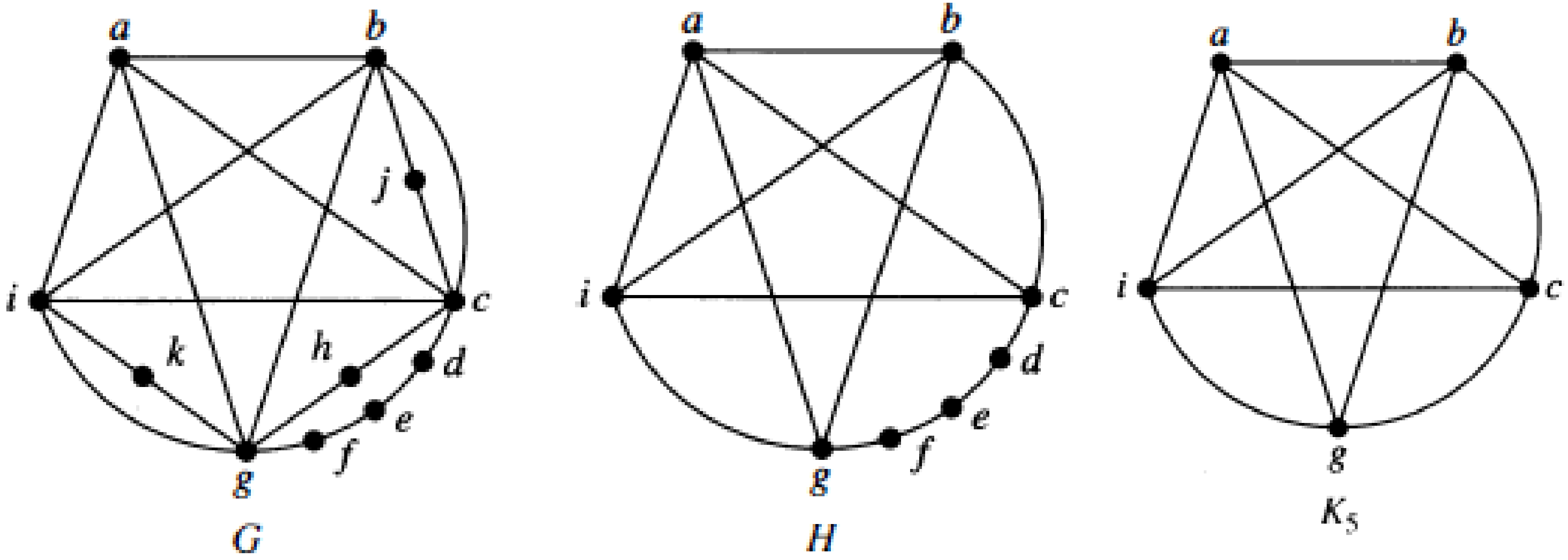
- A graph is nonplanar if and only if it contains a subgraph homeomorphic to the following graphs



# Example

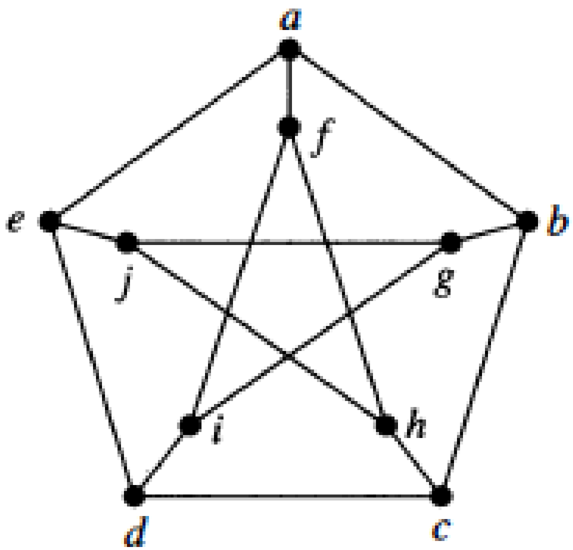
- Determine whether graph G is planar.

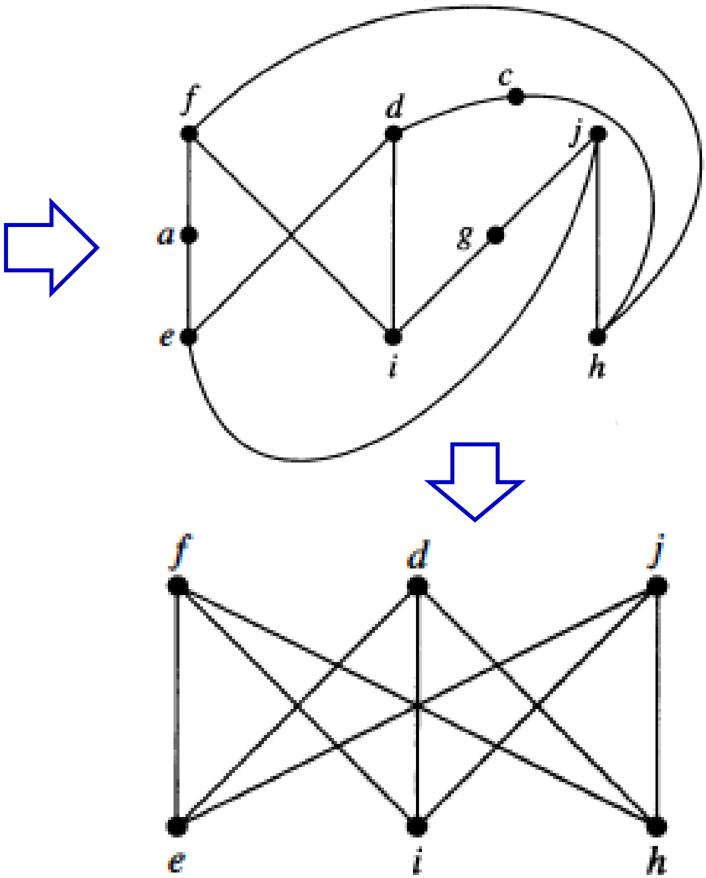



- H is obtained by deleting h, j, and k and all edges incident with these vertices. H is homeomorphic to K5 because it can be obtained from K5 by a sequence of elementary subdivisions.

# Example

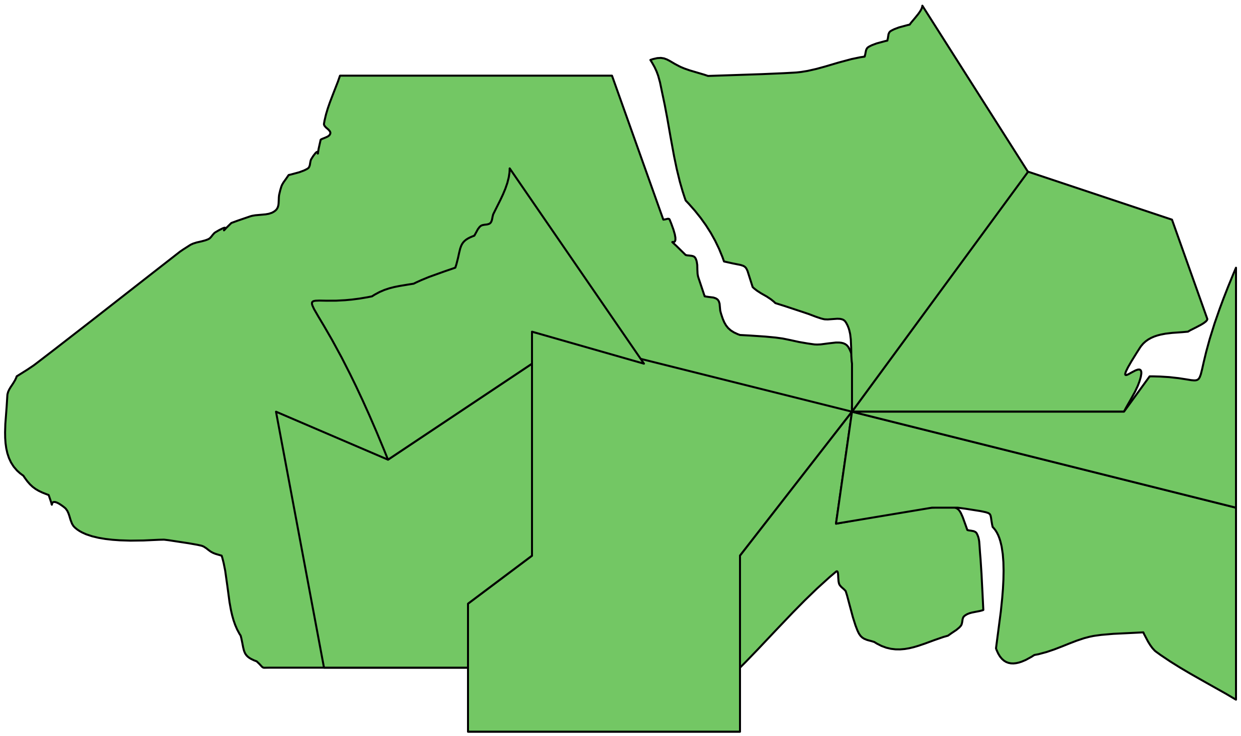
- Is the Petersen graph shown below planar?



# Graph Coloring

- Consider a map.



# Graph Coloring

- Consider a map.

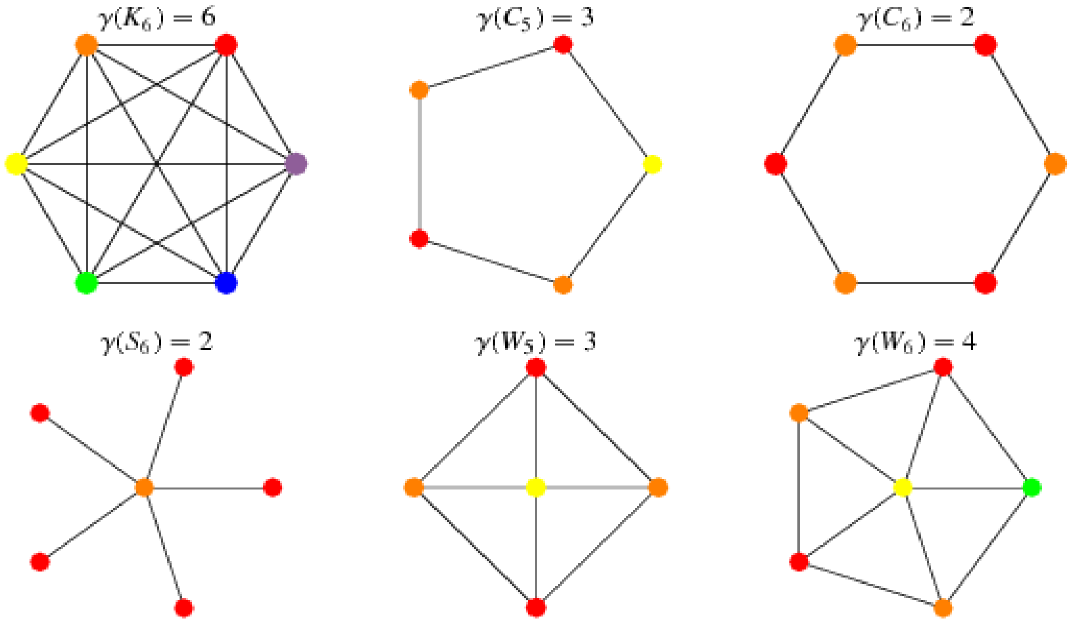
# Graph Coloring

# Colored Graphs

- A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.
- The chromatic number of a graph $\chi(G)$ is the least number of colors needed for a coloring of this graph.
- The best algorithms known for finding the chromatic number of a graph have exponential worst-case time complexity

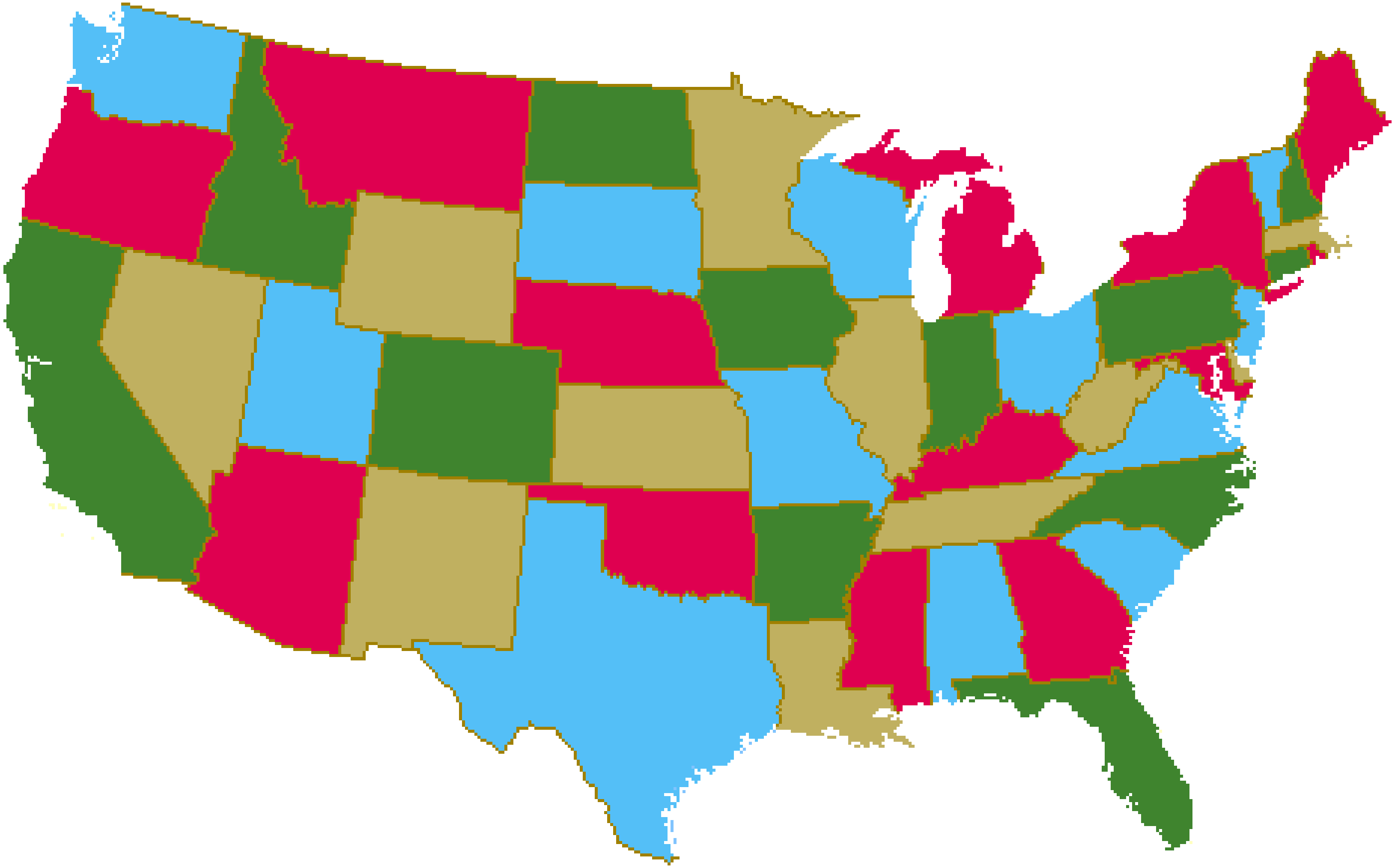
# FOUR COLOR THEOREM

- The chromatic number of a planar graph is no greater than four.




- Note that the Four Color Theorem applies only to planar graphs. Nonplanar graphs can have arbitrarily large chromatic numbers.

# Graph Coloring



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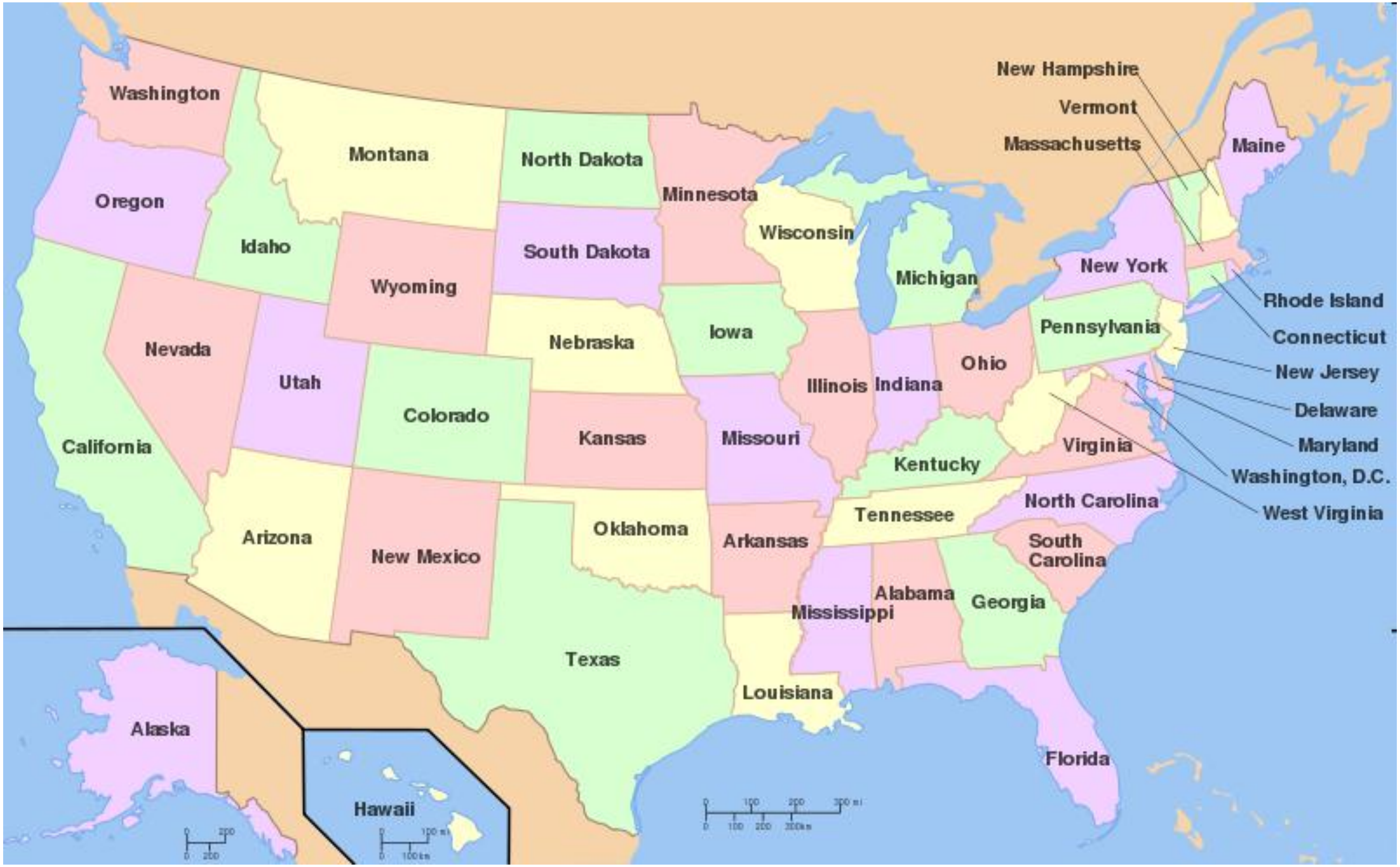
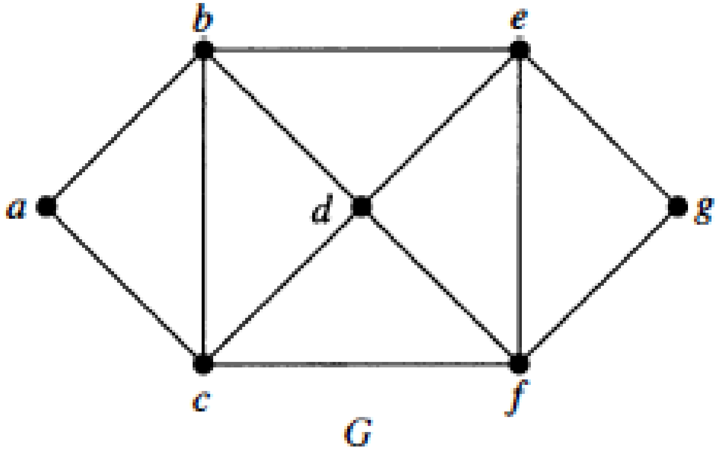
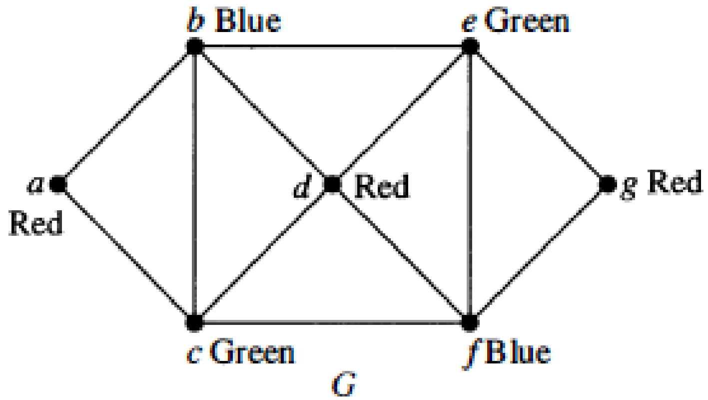


# Example

- What is the chromatic numbers of the graphs G?

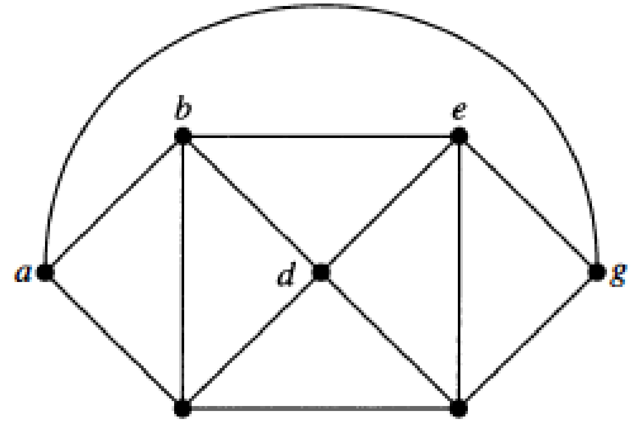


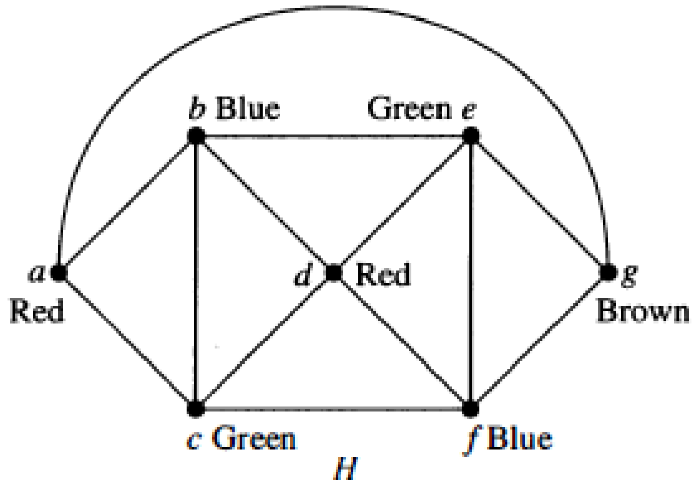




$\chi(G) = 3$

# Example

- What is the chromatic numbers of the graphs H?

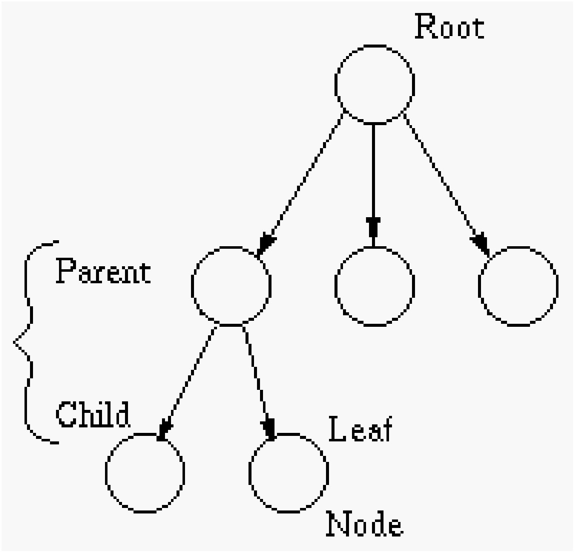



$\chi(G) = 4$

# Trees

- A **tree** is a connected undirected graph with no simple circuits.
- It is a data structure that emulates a hierarchical tree structure with a set of linked nodes.

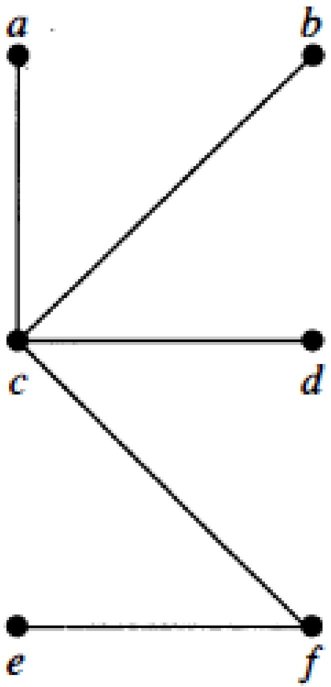


# Why study Trees?

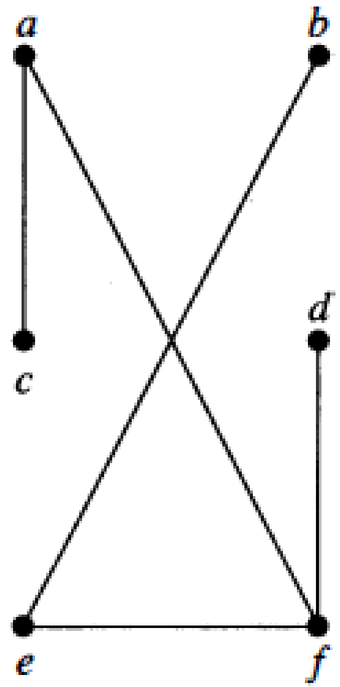
- used to construct efficient algorithms for locating items in a list (binary search trees…)
- used in algorithms, such as Huffman coding, that construct efficient codes saving costs in **data transmission** and **storage**
- used to study games such as checkers and chess and can help determine winning strategies for playing these games
- used to model procedures carried out using a sequence of decisions
- etc.

# Example

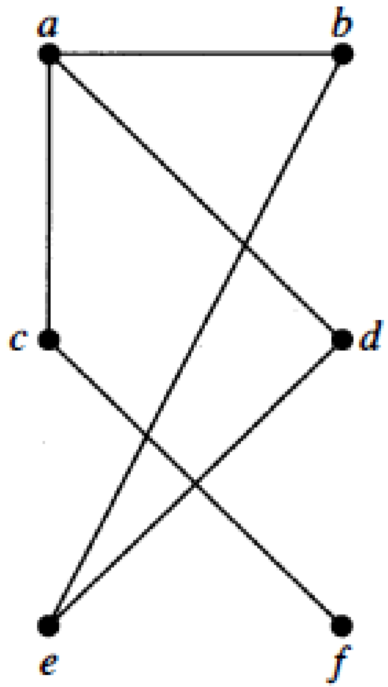
- Which of the graphs shown below are trees?



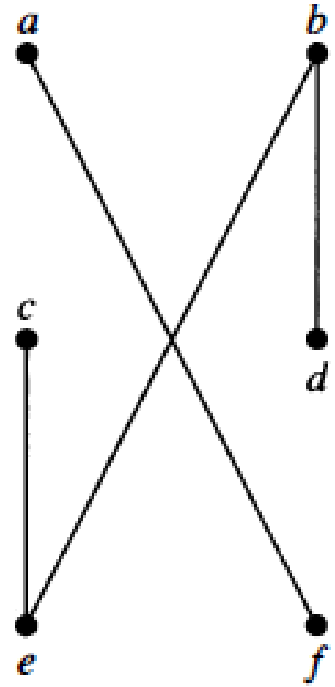

G1




G2



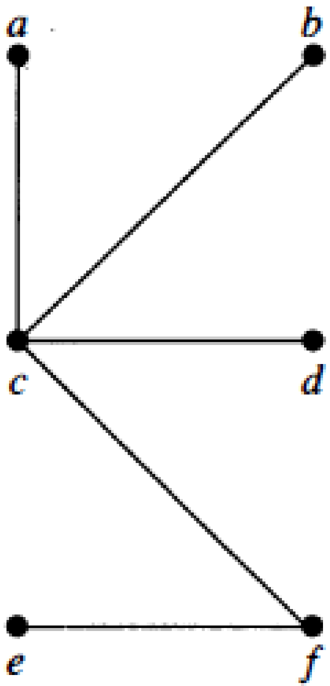

G3



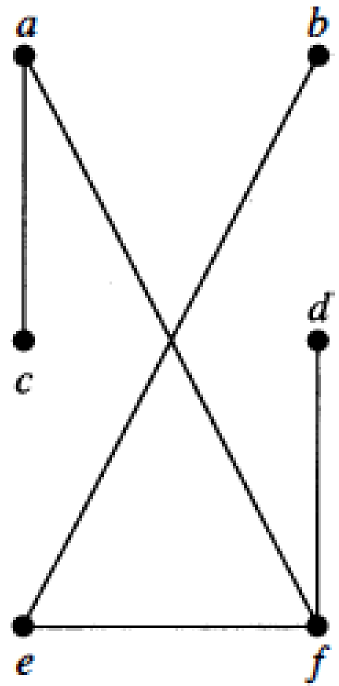

G4

# Example

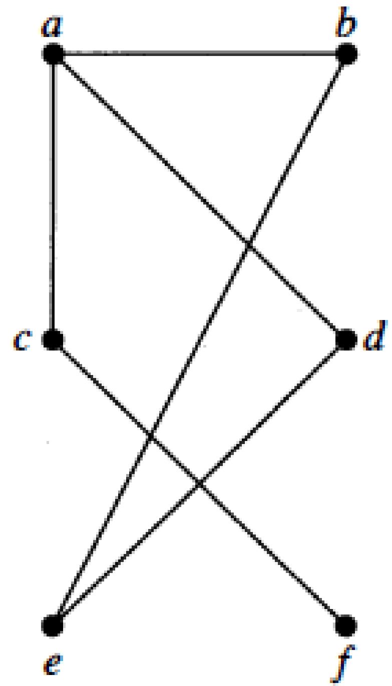
- Which of the graphs shown below are trees?



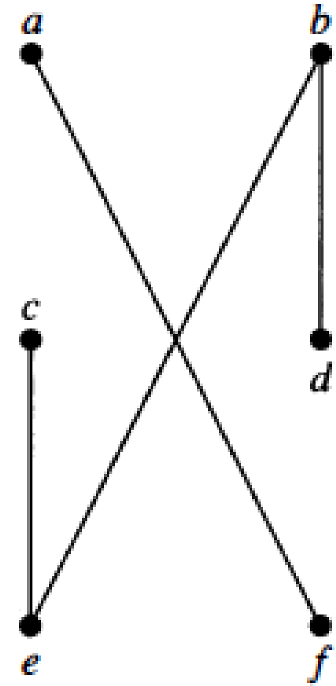

Tree




Tree



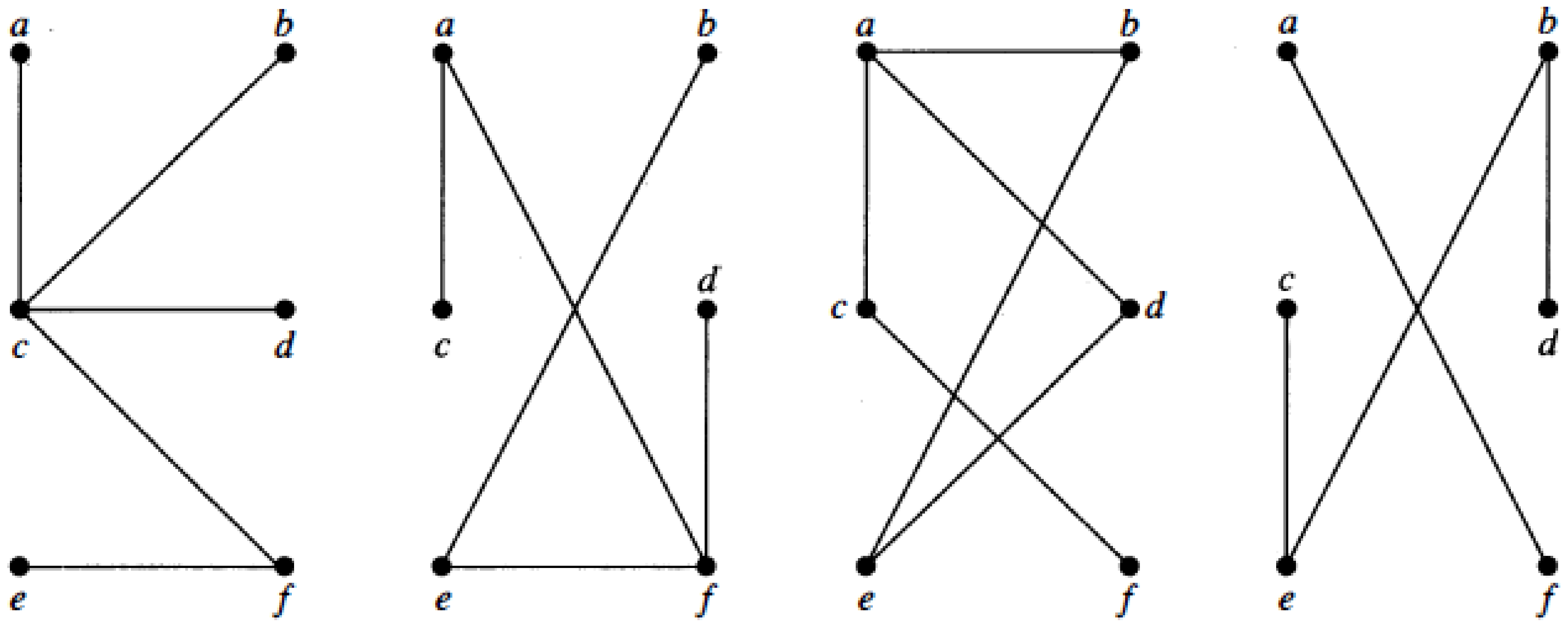

NOT a Tree




NOT a Tree

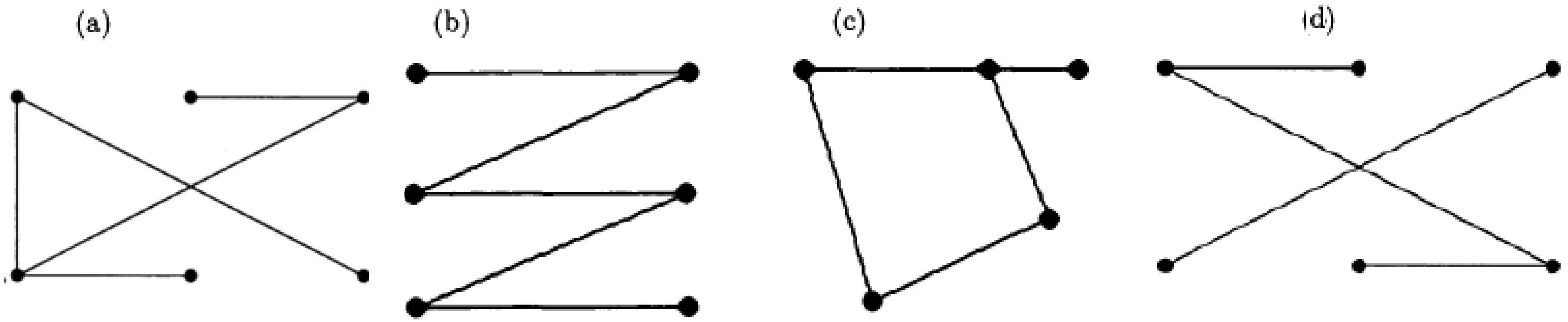
# Theorem

- An undirected graph is a tree if and only if there is a unique simple path between any two of its vertices.



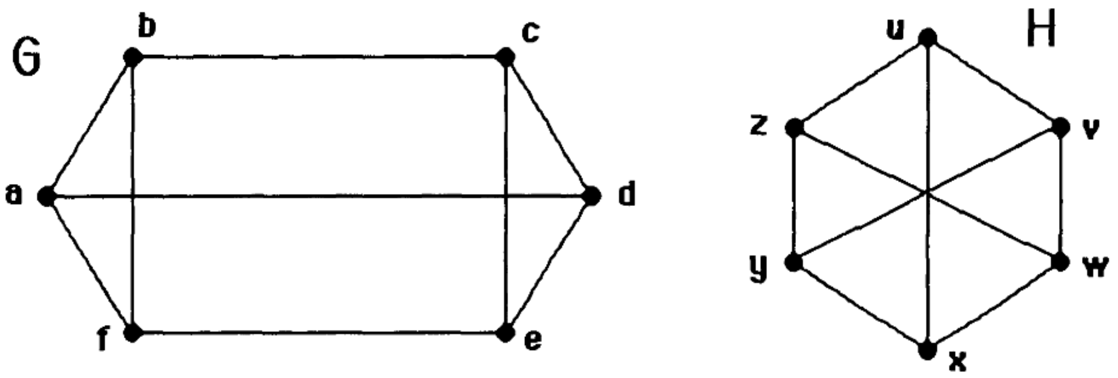
# Q2

- Which of the following is a tree



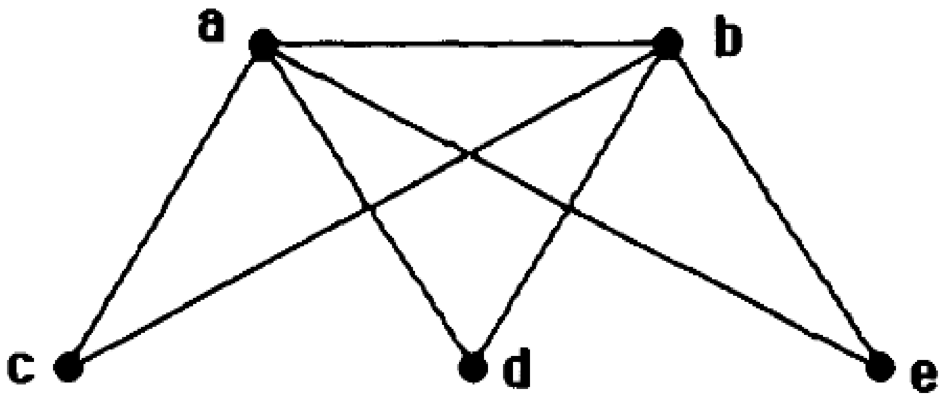
# Q2

- Are the following graphs isomorphic?



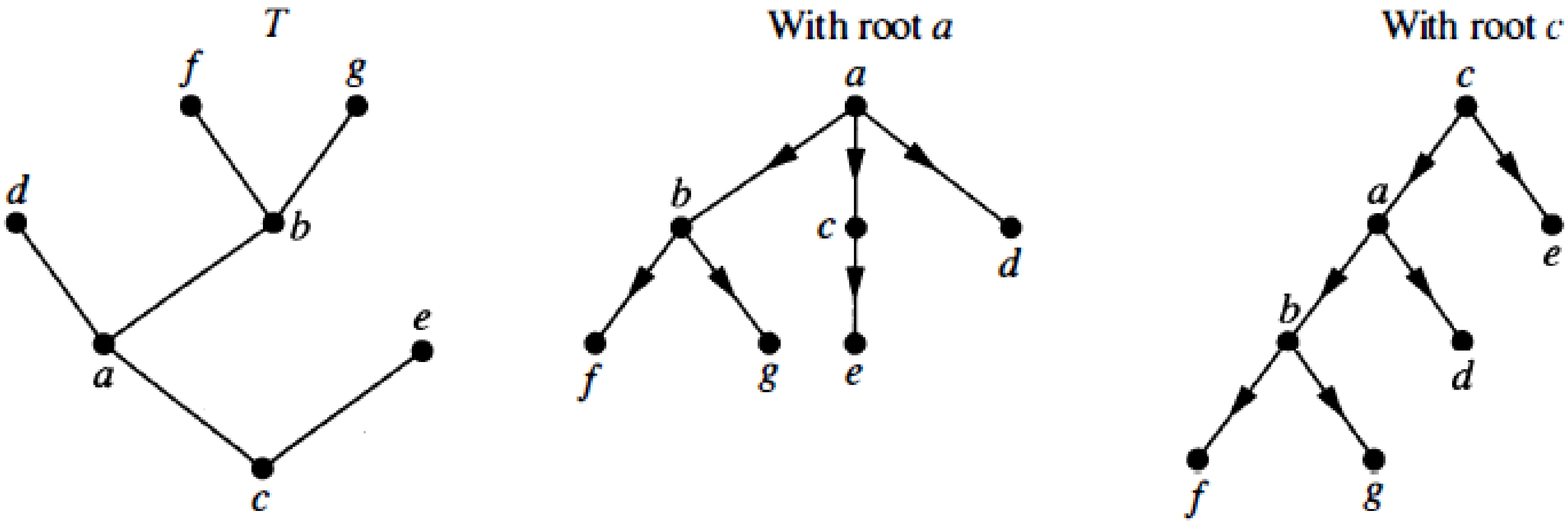
# Q2

- Is the following graph planar? If so draw it without any edges crossing. If it is not, prove that it is not planar.



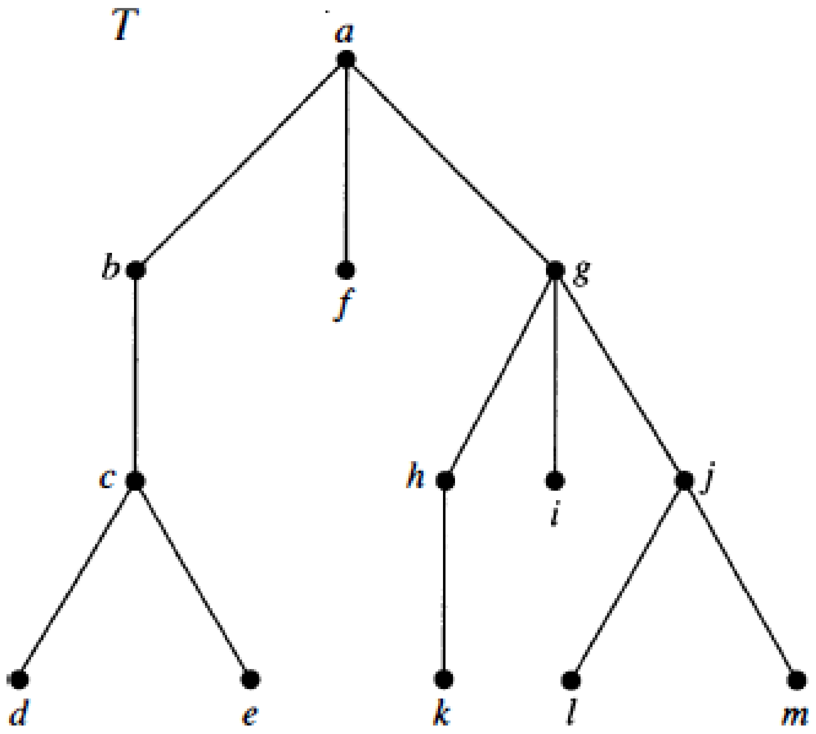
# Rooted Tree

- Rooted Tree - a tree in which one vertex has been designated as the root and every edge is directed away from the root.



# Example

- In the rooted tree T (with root a) shown below, find the parent of c, the children of g, the siblings of h , all ancestors of e, all descendants of b, all internal vertices, and all leaves.
- What is the subtree rooted at g?

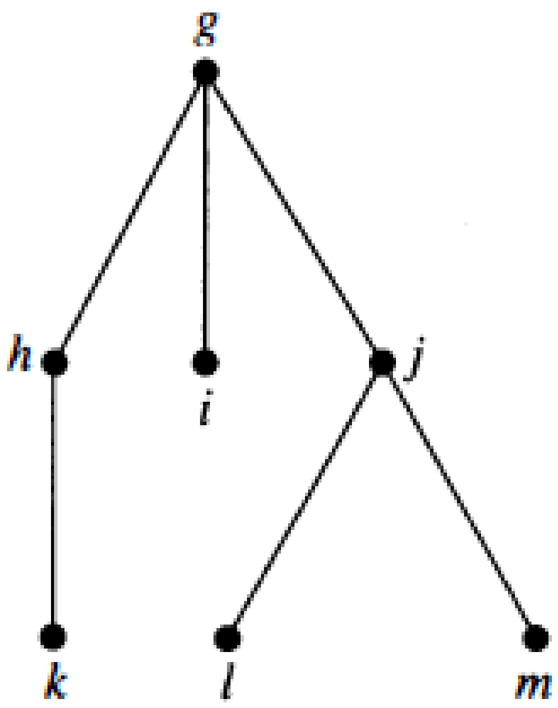


# Example

- In the rooted tree T (with root a) shown below, find the parent of c, the children of g, the siblings of h , all ancestors of e, all descendants of b, all internal vertices, and all leaves.
- The parent of c is b.
- The children of g are h, i, and j.
- The siblings of h are i and j.
- The ancestors of e are c, b, and a.
- The descendants of b are c, d, and e.
- The internal vertices are a, b, c, g, h, and j.
- The leaves are d, e, f, i , k, l, and m.

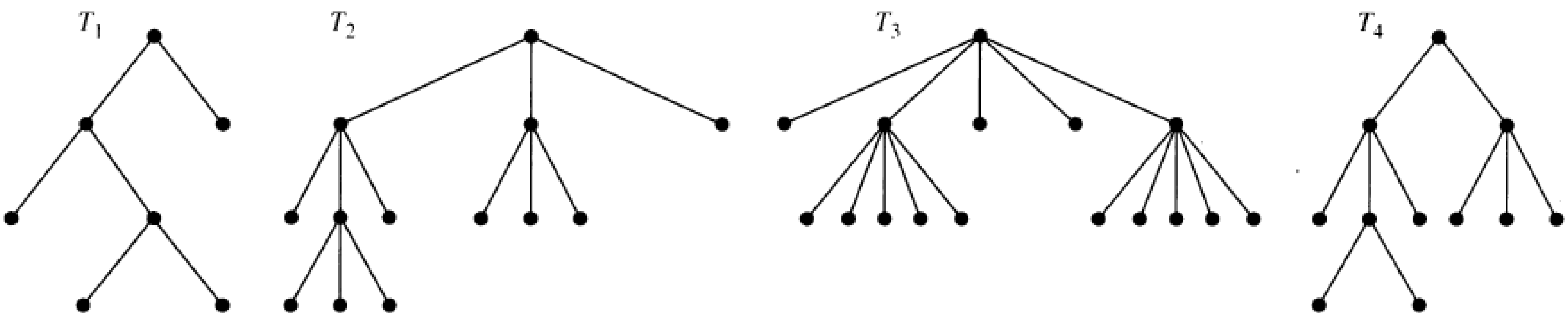
# Example

- What is the subtree rooted at g?



# m−ary tree

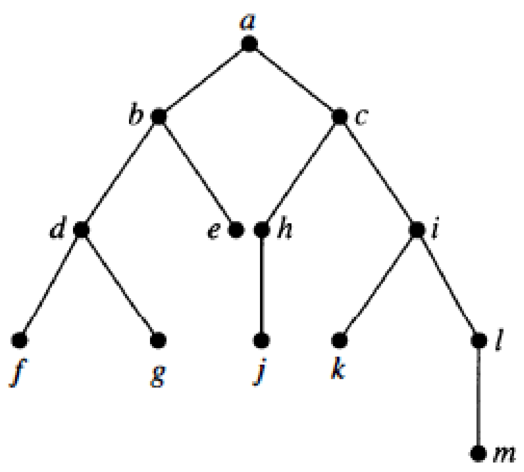
- A rooted tree is called an m-ary tree if every internal vertex has no more than m children. The tree is called a full m-ary tree if every internal vertex has exactly m children.
- An m –ary tree with m = 2 is called a binary tree.

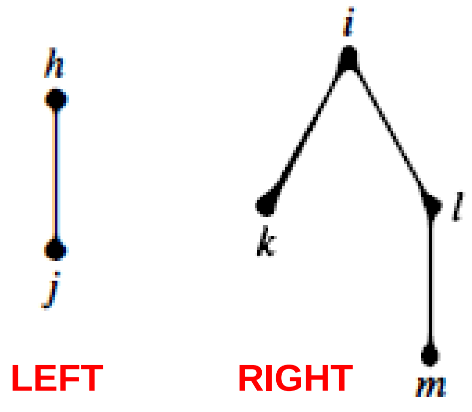



**Full binary tree** **Full 3-ary tree** **Full 5-ary tree** **n/a**

# Ordered rooted tree

- Ordered rooted tree - a rooted tree where the children of each internal vertex are ordered.
- Ex. What are the left and right children of d in the binary tree T shown below (where the order is that implied by the drawing)? What are the left and right subtrees of c?



# Properties of Trees

- A tree with $n$ vertices has $n - 1$ edges.
- A full $m$-ary tree with $i$ internal vertices contains $n = mi + 1$ vertices.
- A full m-ary tree with

(i) $n$ vertices has $i = (n - 1)/ m$ internal vertices and $l = [(m - 1)n + 1 ]/ m$ leaves,

(ii) $i$ internal vertices has $n = mi + 1$ vertices and $l = (m - 1)i + 1$ leaves,

(iii ) $l$ leaves has $n = (ml - 1 )/(m - 1)$ vertices and $i = (l - 1 )/(m - 1)$ internal vertices.

# Example

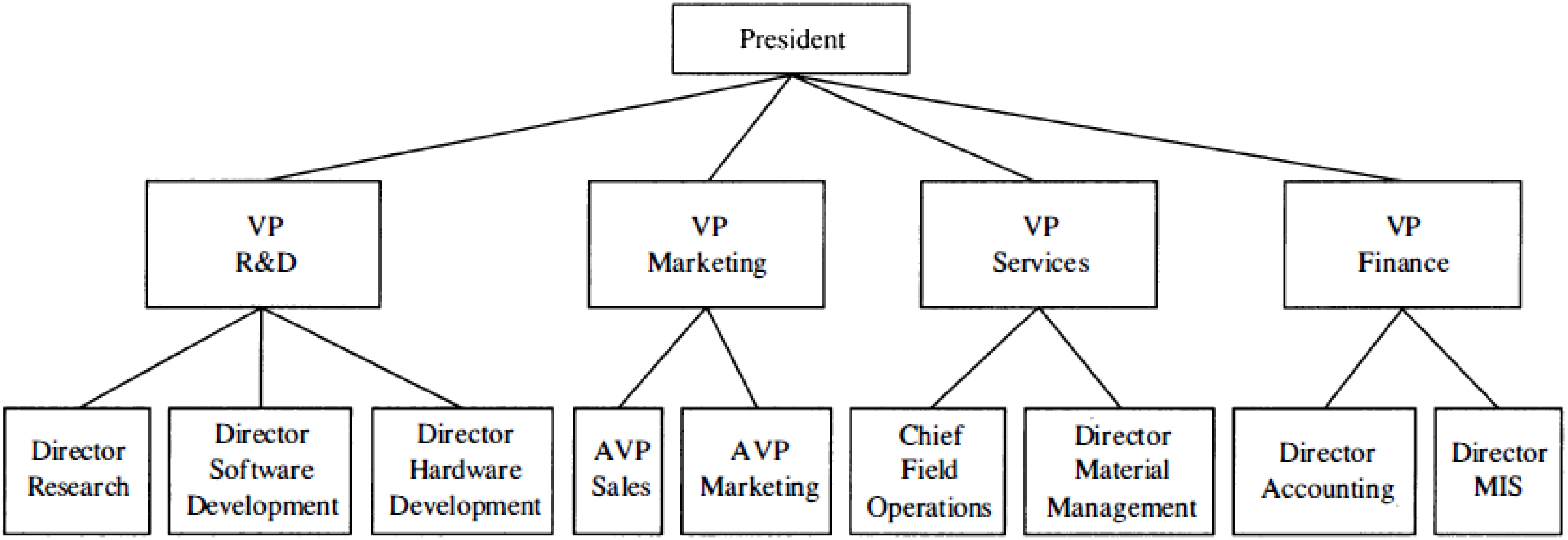
- Suppose that someone starts a chain letter. Each person who receives the letter is asked to send it on to four other people. Some people do this, but others do not send any letters.
- How many people have seen the letter, including the first person, if no one receives more than one letter and if the chain letter ends after there have been 100 people who read it but did not send it out?
- How many people sent out the letter?

# Solution

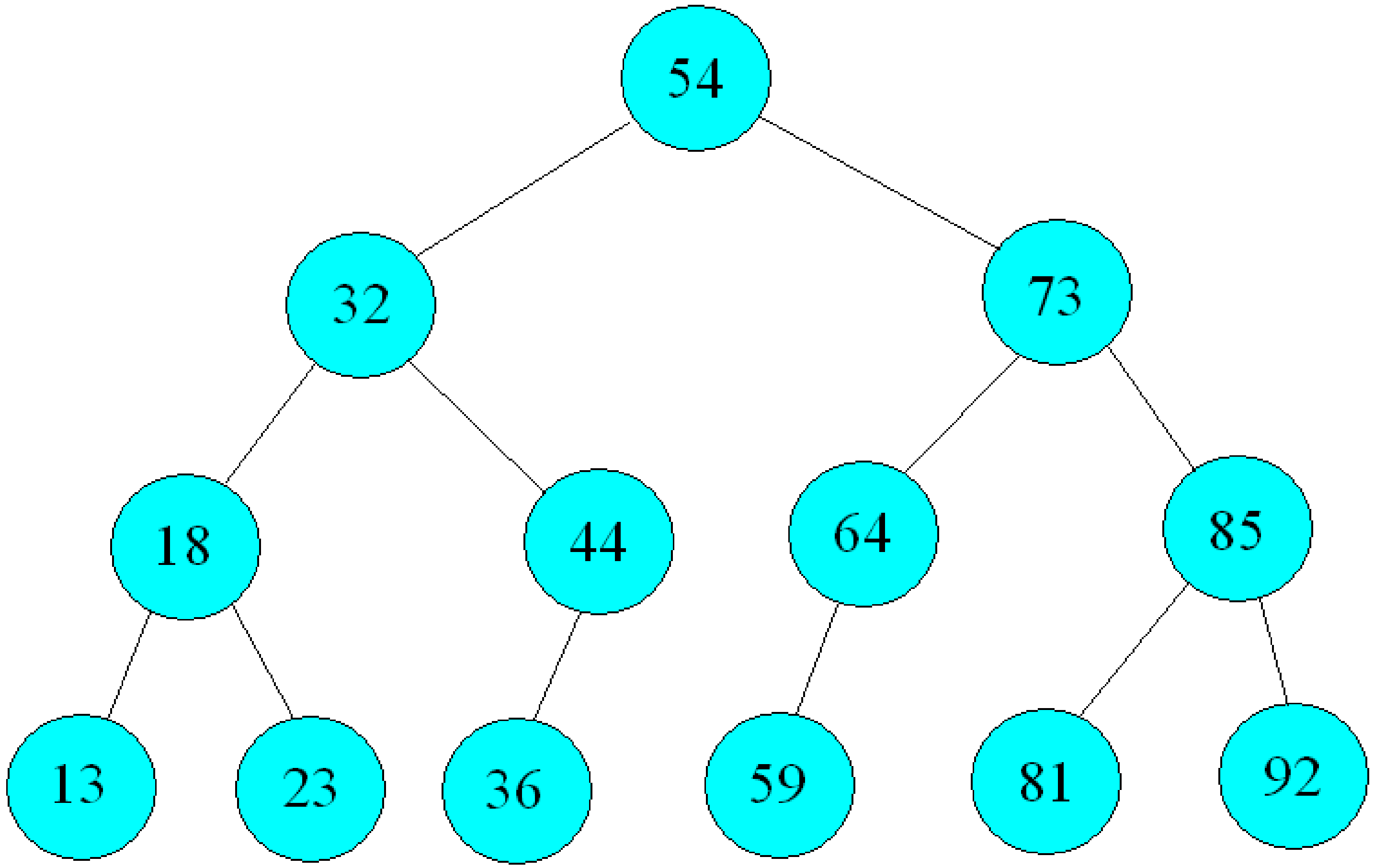
- The chain letter can be represented using a 4-ary tree. The internal vertices correspond to people who sent out the letter, and the leaves correspond to people who did not send it out.
- Because 100 people did not send out the letter, the number of leaves in this rooted tree is $l = 100$.
- The number of people who have seen the letter is $n = (4 \cdot 100 - 1)/(4 - 1) = 133$ .
- The number of internal vertices is $133 - 100 = 33$, so 33 people sent out the letter.

# Applications

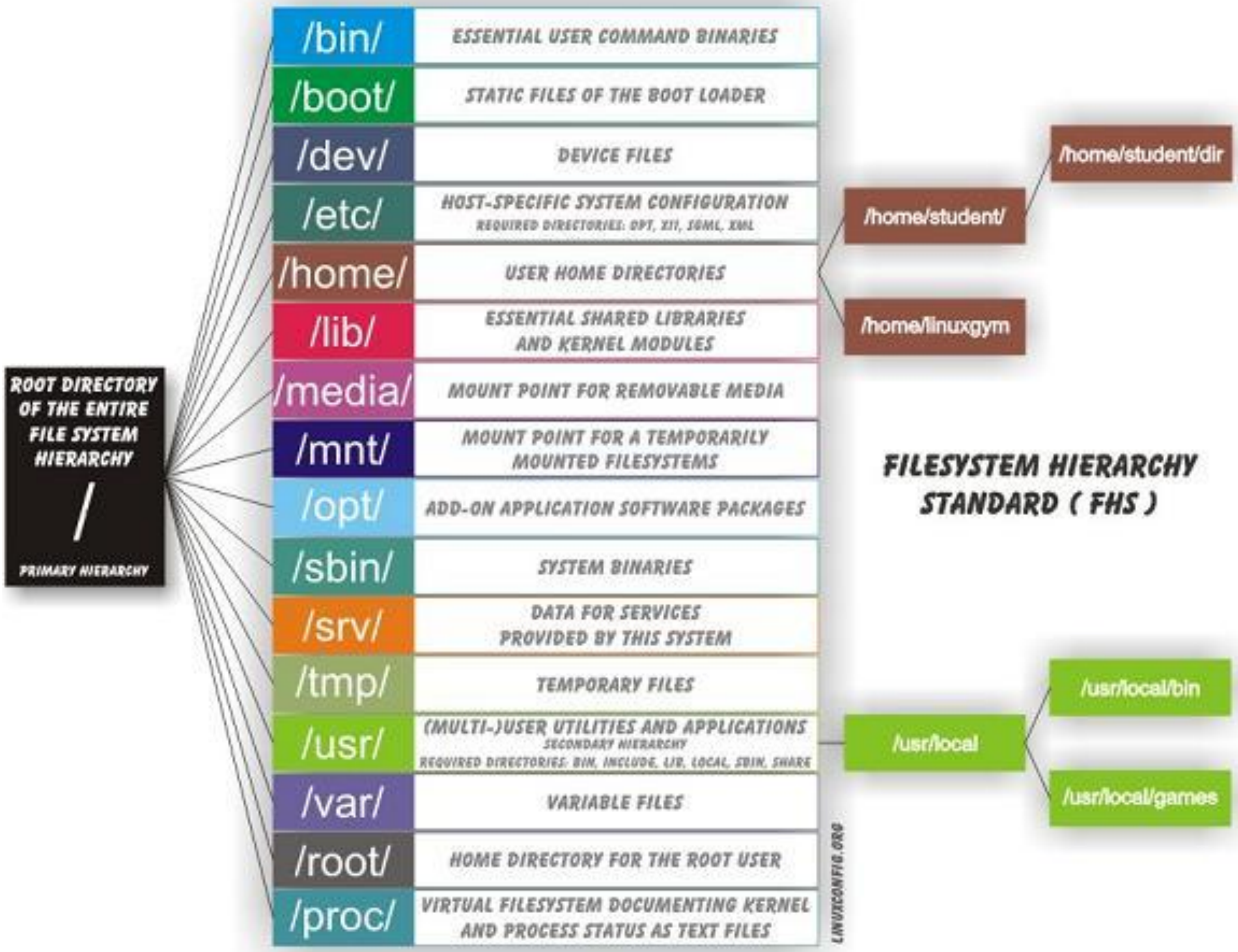
- Trees as models (Ex. Representing Organizations)



# Binary Search Tree



# Tree Example–Linux Directory



# Thank you for listening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