Notes for Calculus III

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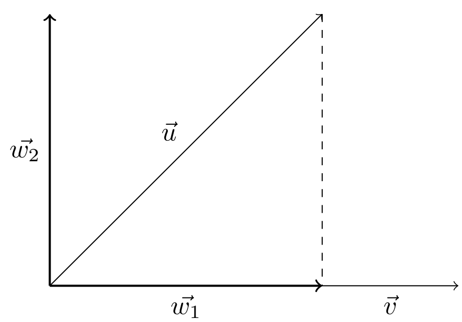
Contents

Chapter 4

Projections

Given two vectors $\vec{v}$ and $\vec{u}$. Construct a vector $\vec{w_1}$ from the terminal point of $\vec{u}$ perpendicular to $\vec{v}$. The vector that starts from the initial point of $\vec{u}$ and ends at the intersection of the line and $\vec{v}$ is called the **projection of $\vec{u}$ onto $\vec{v}$**, which is also known as the **vector component of $\vec{u}$ along $\vec{v}$**.



From the diagram, it is not hard to see that $\vec{u} = \vec{w_1} + \vec{w_2}$
Hence, the vector component of $\vec{u}$ orthogonal to $\vec{v}$ is given by $\vec{w_2} = \vec{u} - \vec{w_1}$
Let $\vec{w_1} = t\vec{v}$, for some scalar t.
Then $\vec{w_2} = \vec{u} - t\vec{v}$ is orthogonal to $\vec{v}$, which implies that $\vec{w_2} \cdot \vec{v} = 0$.

$$\begin{aligned} \vec{w_2} \cdot \vec{v} = (\vec{u} - t\vec{v}) \cdot \vec{v} &= 0 \\ \vec{u} \cdot \vec{v} - t\vec{v} \cdot \vec{v} &= 0 \\ t &= \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \end{aligned}$$

Therefore, The projection of $\vec{u}$ onto $\vec{v}$ is given by

$$proj_{\vec{v}}\vec{u} = \vec{w_1} = t\vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right)\vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right)\vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}\right)\frac{\vec{v}}{\|\vec{v}\|}$$

where $\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$ is the **scalar projection** of $\vec{u}$ onto $\vec{v}$, denoted by $comp_{\vec{v}}\vec{u}$.

Example 1. Find the projection of $\vec{u} = \langle 6, 7\rangle$ onto $\vec{v} = \langle 1, 4\rangle$. Hence, find the vector component of $\vec{u}$ orthogonal to $\vec{v}$.

$$\begin{aligned} proj_{\vec{v}}\vec{u} &= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v} \\ &= \left(\frac{6(1)+7(4)}{1^2+4^2}\right)\langle 1, 4\rangle \\ &= \left(\frac{34}{17}\right)\langle 1, 4\rangle \\ &= \langle 2, 8\rangle \end{aligned}$$

$$\begin{aligned} \vec{w_2} &= \vec{u} - \vec{w_1} \\ &= \langle 6, 7\rangle - \langle 2, 8\rangle \\ &= \langle 4, -1\rangle \end{aligned}$$

Example 2. Find the projection of $\vec{u} = 2i + 3j$ onto $\vec{v} = 5i + j$. Hence, find the vector component of $\vec{u}$ orthogonal to $\vec{v}$.

$$\begin{aligned} proj_{\vec{v}}\vec{u} &= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v} \\ &= \left(\frac{2(5)+3(1)}{5^2+1^2}\right)(5i+j) \\ &= \left(\frac{13}{26}\right)(5i+j) \\ &= \left(\frac{5}{2}\right)i + \left(\frac{1}{2}\right)j \end{aligned}$$

$$\begin{aligned} \vec{w_2} &= \vec{u} - \vec{w_1} \\ &= (2i+3j) - \left(\left(\frac{5}{2}\right)i + \left(\frac{1}{2}\right)j\right) \\ &= \left(-\frac{1}{2}\right)i + \left(\frac{5}{2}\right)j \end{aligned}$$

Example 3. Find the scalar projection of the force $\vec{F} = 4i - 2j + 3k$ in the direction of the vector $v = i - j + 2k$.

Solution.

$$\begin{aligned}
comp_{\vec{v}}\vec{F} &= \frac{\vec{F}\cdot\vec{v}}{\|\vec{v}\|} \\
&= \frac{4(1) + (-2)(-1) + 3(2)}{\sqrt{1^2 + (-1)^2 + 2^2}} \\
&= \frac{4+2+6}{\sqrt{6}} \\
&= \frac{12}{\sqrt{6}} \\
&= 2\sqrt{6}
\end{aligned}$$

Notes: Selected exercises are mixed in the exercises of the previous chapters.