Mathematics

Senior 3 Part I

MELVIN CHIA

Started on 10 April 2023

Finished on XX XX 2023

Actual time spent: XX days

Introduction

Why this book?

Disclaimer

Acknowledgements

Contents

Introduction **1**

21.1 Compound Interest and Annuity 3

21.1 Compound Interest and Annuity

Simple interest and compound interest are two different methods of calculating interest. Simple interest is calculated on the principal amount of a loan only. Compound interest is calculated on the principal amount and also on the accumulated interest of previous periods, and can thus be regarded as interest on interest.

For example, a fund amounted to RMp is deposited into a bank account with a yearly interest rate of $r\%$.

$$\text{Principal amount} = \text{RM}p$$

When $t = 1$,

$$\begin{aligned}
\text{Interest earned} &= p \times r\% = \frac{pr}{100} \\
\text{Accumulated amount} &= p + \frac{pr}{100} = p\left(1 + \frac{r}{100}\right)
\end{aligned}$$

When $t = 2$,

$$\begin{aligned}
\text{Interest earned} &= \left(p + \frac{pr}{100}\right) \times r\% = \frac{pr}{100}\left(1 + \frac{r}{100}\right) \\
\text{Accumulated amount} &= p\left(1 + \frac{r}{100}\right) + \frac{pr}{100}\left(1 + \frac{r}{100}\right) \\
&= p\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right) \\
&= p\left(1 + \frac{r}{100}\right)^2
\end{aligned}$$

When $t = 3$,

$$\begin{aligned}
\text{Interest earned} &= \left(p\left(1 + \frac{r}{100}\right)^2\right) \times r\% = \frac{pr}{100}\left(1 + \frac{r}{100}\right)^2 \\
\text{Accumulated amount} &= p\left(1 + \frac{r}{100}\right)^2 + \frac{pr}{100}\left(1 + \frac{r}{100}\right)^2 \\
&= p\left(1 + \frac{r}{100}\right)^2\left(1 + \frac{r}{100}\right) \\
&= p\left(1 + \frac{r}{100}\right)^3
\end{aligned}$$

In general, the accumulated amount after t years is given by

$$\boxed{A = p\left(1 + \frac{r}{100}\right)^t}$$

where p is called the *present value* of A.

If the interest is compounded m times per year, then the accumulated amount is given by

$$A = p\left(1 + \frac{r}{100m}\right)^{mt}$$

Annuity and Present Value of Annuity

An annuity is a series of equal payments made at equal intervals of time according to some kind of contract, standing order or the amount received. For example, all sorts of instalment, insurance premiums, house rent, car loan, etc. are annuities. In this book, we will only consider annuities with equal payments made or received at equal intervals of time.

Note that the annuity is not limited to once a year.

We can compare which payment plan is better by comparing the present values of the annuities. From the formula $A = p(1 + r\%)^t$, we can know that the present value $p = \dfrac{A}{(1 + r\%)^t}$. If the yearly interest rate is $r\%$, the annuity is RMA, the payment is made once per year, then the present value of the amount paid after a year is $A(1 + r\%)^{-1}$, the present value of the amount paid after two years is $A(1 + r\%)^{-2}$, and so on. The present value of the amount paid after n years is $A(1 + r\%)^{-n}$. Hence, the sum of the present values of the amount paid after n years is

$$\begin{aligned}
&\frac{A}{1 + r\%} + \frac{A}{(1 + r\%)^2} + \cdots + \frac{A}{(1 + r\%)^n} \\
&= A\left[\frac{1}{1 + r\%} + \frac{1}{(1 + r\%)^2} + \cdots + \frac{1}{(1 + r\%)^n}\right] \\
&= A\left[\frac{1 - \dfrac{1}{(1 + r\%)^n}}{1 - \dfrac{1}{1 + r\%}}\right] \\
&= \frac{A}{r\%}\left(1 - \frac{1}{(1 + r\%)^n}\right)
\end{aligned}$$

Annuity that is paid indefinitely is called *perpetuity*, $n \to \infty$, $\dfrac{1}{(1 + r\%)^n} \to 0$. From that, we can know that the present value of perpetuity is $\dfrac{A}{r\%}$.

Practice 8

1. Given that the principal amount is RM75,000, interest rate is 4.5%. Using composite interest method, find the accumulated amount after 10 years.

   Sol.

$$\begin{aligned} A &= p\left(1+\frac{r}{100}\right)^{t} \\ &= 75000\left(1+\frac{4.5}{100}\right)^{10} \\ &= 75000\left(\frac{104.5}{100}\right)^{10} \\ &= 75000(1.045)^{10} \\ &= \text{RM}116,472.71 \end{aligned}$$

2. A person has deposited RM40,000 into a bank account. The bank pays 8% interest per annum compounded half yearly. Using the compound interest method, find the amount in the account after 3 years.

   Sol.

$$\begin{aligned} A &= p\left(1+\frac{r}{100m}\right)^{mt} \\ &= 40000\left(1+\frac{8}{100\times 2}\right)^{2\times 3} \\ &= 40000\left(\frac{208}{200}\right)^{6} \\ &= 40000(1.04)^{6} \\ &= \text{RM}50,612.76 \end{aligned}$$

3. Given that the interest rate is 6%, the interest is compounded half yearly. Using the compound interest method, the accumulated amount after 5 years is RM4031.75, find the principal amount.

   Sol.

$$\begin{aligned} A &= p\left(1+\frac{r}{100m}\right)^{mt} \\ p &= \frac{A}{\left(1+\frac{r}{100m}\right)^{mt}} \\ &= \frac{4031.75}{\left(1+\frac{6}{100\times 2}\right)^{2\times 5}} \\ &= \frac{4031.75}{1.03^{10}} \\ &= \text{RM}3000 \end{aligned}$$

4. Given that the interest rate is 4%, the annuity is RM3,500, the payment is made once per year. The payment has since been made for 15 years continuously. Find the present value. Hence, find the present value of the perpetuity.

Sol.

$$\begin{aligned} p &= \frac{A}{r\%}\left(1 - \frac{1}{(1+r\%)^n}\right) \\ &= \frac{3500}{4\%}\left(1 - \frac{1}{(1+4\%)^{15}}\right) \\ &= \text{RM}38,914.36 \end{aligned}$$

$$\begin{aligned} \text{Present value of perpetuity} &= \frac{3500}{4\%} \\ &= \text{RM}87,500 \end{aligned}$$

Exercise 23.6

1. Given that the principal amount is RM90,000, the interest rate is 5%. Compounding the interest once per year, find the accumulated amount after 10 years.

Sol.

$$\begin{aligned} A &= p\left(1 + \frac{r}{100}\right)^t \\ &= 90000\left(\frac{105}{100}\right)^{10} \\ &= 90000(1.05)^{10} \\ &= \text{RM}146,600.52 \end{aligned}$$

2. A person has deposited a fund into a bank account. The bank pays 8% interest per annum compounded yearly. The amount in the account after 3 years has increased by RM779.14. Find the amount of the fund deposited.

Sol.

$$\begin{aligned} A - p &= 779.14 \\ p\left(1 + \frac{8}{100}\right)^3 - p &= 779.14 \\ p(1.08)^3 - p &= 779.14 \\ p(1.08^3 - 1) &= 779.14 \\ p &= \frac{779.14}{1.08^3 - 1} \\ &= \text{RM}3,000.02 \end{aligned}$$

3. RM80,000 was deposited into a financial institution. The interest rate is 8% per annum compounded once per three months. Find the amount in the account after 5 years.

Sol.

$$\begin{aligned}
A &= p\left(1+\frac{r}{100m}\right)^{mt} \\
&= 80000\left(1+\frac{8}{100\times 4}\right)^{4\times 5} \\
&= 80000\left(\frac{408}{400}\right)^{20} \\
&= 80000(1.02)^{20} \\
&= \text{RM}118,875.79
\end{aligned}$$

4. Prove that the accumulated amount after being compounded with an interest of 5 for 15 years will exceed twice the principal amount.

Proof.

$$\begin{aligned}
A &= p\left(1+\frac{r}{100}\right)^{t} \\
&= p\left(1+\frac{5}{100}\right)^{15} \\
&= p(1.05)^{15} \\
&\approx 2.078p > 2p \quad \square
\end{aligned}$$

5. Given that the principal amount is RM15,000, the interest rate is 6% being compounded once per year. How long does it take for the accumulated amount to be more than RM300,000?

Sol.

$$\begin{aligned}
A &= p\left(1+\frac{r}{100}\right)^{t} \\
&= 15000\left(1+\frac{6}{100}\right)^{t} \\
&= 15000(1.06)^{t} > 300000
\end{aligned}$$

$$\begin{aligned}
1.06^{t} &> 2 \\
\log 1.06^{t} &> \log 20 \\
t \log 1.06 &> \log 20 \\
t &> \frac{\log 20}{\log 1.06} \\
t &> 51.41 \\
t &= 52 \text{ years}
\end{aligned}$$

6. Given that the principal amount is RM120,000, the interest rate is 5.5% being compounded half yearly. How long does it take for the accumulated amount to be more than RM200,000?

Sol.

$$
\begin{aligned}
A &= p\left(1 + \frac{r}{100m}\right)^{mt} \\
&= 120000\left(1 + \frac{5.5}{100 \times 2}\right)^{2t} \\
&= 120000\left(\frac{205.5}{200}\right)^{2t} \\
&= 120000(1.0275)^{2t} > 200000 \\
1.0275^{2t} &> \frac{5}{3} \\
2t \log 1.0275 &> \log \frac{5}{3} \\
t &> \frac{\log \frac{5}{3}}{2 \log 1.0275} \\
t &> 9.41 \\
t &= 9.5 \text{ years}
\end{aligned}
$$

7. A person deposited RM2,500 into his bank account at the beginning of every year, the interest rate is 4.5% compounded once per year. Find the amount in the account after 15 years.

Sol.

$$
\begin{aligned}
A &= 2500\left(1 + \frac{4.5}{100}\right) + 2500\left(1 + \frac{4.5}{100}\right)^2 + \cdots + 15 \cdot 2500\left(1 + \frac{4.5}{100}\right)^{15} \\
&= 2500(1.045 + 1.045^2 + \cdot + 1.045^{15}) \\
&= 1000 \times \frac{1.045(1.045^{15} - 1)}{1.045 - 1} \\
&= \text{RM}54,298.34
\end{aligned}
$$

8. If the present value is RM15,443.46, the interest rate is 5%, find the annuity if the payment is made for 10 years.

Sol.

$$
\begin{aligned}
\text{Present value} &= \frac{A}{r\%}\left(1 - \frac{1}{(1 + r\%)^t}\right) \\
15443.46 &= \frac{A}{5\%}\left(1 - \frac{1}{1.05^{10}}\right) \\
A &= \frac{15443.46 \times 5\%}{1 - \frac{1}{1.05^{10}}} \\
&= \text{RM}2,000
\end{aligned}
$$

9. Given that the annuity is RM5,000, the interest rate is 5%, the payment is made once per year for 25 years. Find the present value. Hence, find the present value of the perpetuity.

Sol.

$$\begin{aligned}\text{Present value} &= \frac{A}{r\%}\left(1-\frac{1}{(1+r\%)^t}\right)\\ &= \frac{5000}{5\%}\left(1-\frac{1}{1.05^{25}}\right)\\ &= \text{RM}70,469.72\end{aligned}$$

$$\text{Present value of perpetuity} = \frac{A}{r\%} = \frac{5000}{5\%} = \text{RM}100,000$$

10. Given that the annuity is RM2,500, the interest rate is 4.5%, the payment is made once per year. How many years does it take for the present value to exceed RM30,000?

Sol.

$$\begin{aligned}\text{Present value} &= \frac{A}{r\%}\left(1-\frac{1}{(1+r\%)^t}\right)\\ &= \frac{2500}{4.5\%}\left(1-\frac{1}{1.045^t}\right) > 30000\\ 2500\left(1-\frac{1}{1.045^t}\right) &> 1350\\ 1-\frac{1}{1.045^t} &> \frac{27}{50}\\ 1.045^t &> \frac{50}{23}\\ t\log 1.045 &> \log\frac{50}{23}\\ t &> \frac{\log\frac{50}{23}}{\log 1.045} \approx 17.64\\ t &= 18 \text{ years}\end{aligned}$$

11. If a bank has introduced an annuity scheme, the investors can receive RM1,000 per year for life after paying RM20,000. If the annuity plan is considered approximately to be a perpetuity, find the interest rate.

Sol.

$$\begin{aligned}\text{Present value of perpetuity} &= \frac{A}{r\%}\\ &= \frac{1000}{r\%}\\ &= 20000\\ r\% &= \frac{1000}{20000} = 5\%\end{aligned}$$

Revision Exercise 23

1. Without using a calculator, find the value of the following:

   (a) $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^6 + \left(-\frac{1}{2}\right)^{-2}$

   (b) $5^{\frac{1}{2}} + 5^{-\frac{1}{2}} - \left(\dfrac{1}{5}\right)^{\frac{1}{2}} + \left(\dfrac{1}{5}\right)^{-\frac{1}{2}}$

   (c) $\left(\frac{1}{3}\right)^2 \times \left(-\frac{1}{3}\right)^2 \times \left(\frac{1}{2}\right)^{-2}$

   (d) $\sqrt{2\sqrt[3]{3}} \div \sqrt[3]{\dfrac{\sqrt{8}}{3}}$

2. Simplify the following expressions:

   (a) $\left(\dfrac{b}{2a^2}\right)^3 \div \left(\dfrac{2b^2}{3a}\right)^0 \times \left(-\dfrac{b}{a}\right)^{-3}$

   (b) $\dfrac{3^{n+2} - 2 \times 3^n}{5(3^{n+1})}$

   (c) $\dfrac{\left(x^{-1} + y^{-1}\right)\left(x^{-1} - y^{-1}\right)}{x^{-2}y^{-2}}$

   (d) $\left(x^{\frac{1}{4}} - y^{-\frac{1}{4}}\right)\left(x^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)\left(x^{\frac{1}{4}} + y^{-\frac{1}{4}}\right)$

   (e) $\dfrac{\left(\sqrt[4]{p^3}\right)^{\frac{1}{6}} \sqrt[9]{p^{-3}}}{\left(\sqrt{p^{-7}}\right)^{\frac{1}{6}}}$

   (f) $\dfrac{\left(a^2 + a^{-2} + 2\right)^2}{\left(a^2 + 1\right)^4}$

3. Without using a calculator, compare the value of the following:

   (a) 2.3^{-2} and 2.3^{-1}

   (b) $0.15^{-\frac{1}{2}}$ and $0.15^{-\frac{1}{3}}$

   (c) $\left(\dfrac{1}{3}\right)^{\frac{2}{5}}$ and $3^{-\frac{5}{3}}$

   (d) $\left(\dfrac{3}{5}\right)^2$ and $\left(\dfrac{5}{3}\right)^{3.1}$

4. Without using a calculator, compare the value of the following:

   (a) $\log_{3.2} 3$ and $\log_{3.2} 2$

   (b) $\log_{0.5} 5.3$ and $\log_{0.5} 3.5$

   (c) $\log_3 2$ and $\log_2 3$

   (d) $\log_2 2.3$ and $\log_4 4.8$

5. Find the domain of the following functions:

(a) $y=\log_{0.5}\left(16-x^2\right)$

(b) $y=\log_2\left(2x^2-5x-12\right)$

(c) $y=\sqrt{3-3^x}$

(d) $y=\log_2{(x-3)}^2$

(e) $y=\log_5\left(x^2-2x\right)$

(f) $y=\log_3\dfrac{2}{3-x}$

(g) $y=\dfrac{1}{\log(x+1)-1}$

(h) $y=\dfrac{\log_3{(2-x)}}{\log_3{(2+x)}}$

(i) $y=\sqrt{\log_3(x-2)}$

(j) $y=\dfrac{2}{\sqrt{1-\log x}}$

6. Simplify the following expressions:

(a) $\log_3 27^x$

(b) $\log_x b^{a\log_b x}$

(c) $\log_5{(25^x\cdot 5^y)}$

(d) $3^{2\log_3 x-\log_3 y}$

(e) $5^{-2\log_{25} x}$

7. If $\log_2 5=p$, express $\log_2 100$ in terms of p.

8. If $\log_3 12=a$, express the following in terms of a:

(a) $\log_3 24$

(b) $\log_9 36$

9. Without using a calculator, find the value of the following:

(a) $\log_c\dfrac{1}{5}+\log_c 5$

Sol.

$$\begin{aligned}\log_c\frac{1}{5}+\log_c 5&=\log_c\frac{1}{5}\times 5\\&=\log_c 1\\&=0\end{aligned}$$

(b) $\log_2\left(2\sqrt{2}\right)-2\log_2\sqrt{2}$

Sol.

$$\begin{aligned}\log_2\left(2\sqrt{2}\right) - 2\log_2\sqrt{2} &= \log_2\left(2\sqrt{2}\right) - \log_2 2\\ &= \log_2\left(\frac{2\sqrt{2}}{2}\right)\\ &= \log_2\sqrt{2}\\ &= \frac{1}{2}\end{aligned}$$

(c) $\log_8\frac{2}{7} - \log_8(-2)^2 - \log_8\frac{1}{7}$

Sol.

$$\begin{aligned}\log_8\frac{2}{7} - \log_8(-2)^2 - \log_8\frac{1}{7} &= \log_8\frac{2}{7} - \log_8 4 - \log_8\frac{1}{7}\\ &= \log_8\left(\frac{2}{7}\times\frac{1}{4}\times 7\right)\\ &= \log_8\frac{1}{2}\\ &= -\log_8 2\\ &= -\frac{1}{3}\end{aligned}$$

(d) $\log\frac{5}{32} - 2\log\frac{5}{6} + \log\frac{40}{9}$

Sol.

$$\begin{aligned}\log\frac{5}{32} - 2\log\frac{5}{6} + \log\frac{40}{9} &= \log\frac{5}{32} - \log\left(\frac{5}{6}\right)^2 + \log\frac{40}{9}\\ &= \log\frac{5}{32} - \log\frac{25}{36} + \log\frac{40}{9}\\ &= \log\left(\frac{5}{32}\times\frac{36}{25}\times\frac{40}{9}\right)\\ &= \log 1\\ &= 0\end{aligned}$$

(e) $\left(\log_2 3\right)\left(\log_3 4\right)$

Sol.

$$\begin{aligned}\left(\log_2 3\right)\left(\log_3 4\right) &= \log_2 3\cdot\frac{\log_2 4}{\log_2 3}\\ &= \frac{1}{\log_3 2}\cdot\frac{2}{\log_2 3}\\ &= 2\end{aligned}$$

(f) $\dfrac{\log_{16} 5}{\log_{32} 5}$

Sol.

$$\begin{aligned}\frac{\log_5 32}{\log_5 16} &= \frac{\log_5 2^5}{\log_5 2^4} \\ &= \frac{5\log_5 2}{4\log_5 2} \\ &= \frac{5}{4}\end{aligned}$$

(g) $\log_3 5 \cdot \log_5 7 \cdot \log_7 27$

Sol.

$$\begin{aligned}\log_3 5 \cdot \frac{\log_3 7}{\log_3 5} \cdot \frac{\log_3 27}{\log_3 7} &= \log_3 5 \cdot \frac{\log_3 7}{\log_3 5} \cdot \frac{3\log_3 3}{\log_3 7} \\ &= 3\log_3 3 \\ &= 3\end{aligned}$$

(h) $\log_2 \dfrac{1}{9} \cdot \log_3 \dfrac{1}{25} \cdot \log_5 \sqrt{8}$

Sol.

$$\begin{aligned}\log_2 \frac{1}{9} \cdot \log_3 \frac{1}{25} \cdot \log_5 \sqrt{8} &= \frac{\log_3 \frac{1}{9}}{\log_3 2} \cdot \frac{\log_3 \frac{1}{25}}{\log_3 3} \cdot \frac{\log_3 \sqrt{8}}{\log_3 5} \\ &= \frac{-2\log_3 3}{\log_3 2} \cdot \frac{-2\log_3 5}{1} \cdot \frac{\frac{3}{2}\log_3 2}{\log_3 5} \\ &= -2 \cdot (-2) \cdot \frac{3}{2} \\ &= 6\end{aligned}$$

(i) $\dfrac{1}{3}\log_2 8 + \log_3 27 - \dfrac{1}{4}\log_4 16$

Sol.

$$\begin{aligned}\frac{1}{3}\log_2 8 + \log_3 27 - \frac{1}{4}\log_4 16 &= \frac{1}{3} \cdot 3 + 3 - \frac{1}{4} \cdot 2 \\ &= 1 + 3 - \frac{1}{2} \\ &= \frac{7}{2}\end{aligned}$$

(j) $\log^2 2 + \log 2 \cdot \log 5 + \log 5$

Sol.

$$\begin{aligned}
\log^2 2 + \log 2 \cdot \log 5 + \log 5 &= \log 2 \log 2 + \log 2 \cdot \log 5 + \log 5 \\
&= \log 2(\log 2 + \log 5) + \log 5 \\
&= \log 2 \log 10 + \log 5 \\
&= \log 2 + \log 5 \\
&= \log 10 \\
&= 1
\end{aligned}$$

(k) $2\log_3 15 + 3\log_3 12 - \log_3 25 - 6\log_3 2$

Sol.

$$\begin{aligned}
2\log_3 15 + 3\log_3 12 - \log_3 25 - 6\log_3 2 &= \log_3 15^2 + \log_3 12^3 - \log_3 25 - \log_3 2^6 \\
&= \log_3 \frac{15^2 \cdot 12^3}{5^2 \cdot 2^6} \\
&= \log_3 \frac{3^2 \cdot 5^2 \cdot 2^3 \cdot 3^3 \cdot 2^3}{5^2 \cdot 2^6} \\
&= \log_3 3^5 \qquad = 5
\end{aligned}$$

(l) $\dfrac{\log\sqrt{27} + \log\sqrt{8} - \log\sqrt{125}}{\log 6 - \log 5}$

(m) $\log_8\left(\log_2\sqrt{8+4\sqrt{3}} + \log_2\sqrt{8-4\sqrt{3}}\right)$

10. If $\log_3 2 = a$, $\log_2 5 = b$, prove that $\log_5 3 = \dfrac{1}{a(b+1)}$.

11. If $\log_2 3 = p$, $\log_3 7 = q$, prove that $\log_2 114 = \dfrac{pq+1}{p(q+1)}$.

12. Given that $2\log_5(x+y) = 1 + \log_5 x + \log_5 y$. Prove that $x^2 + y^2 = 3xy$.

13. Given that $x = 5^k$ and $y = 5^n$. Express the following in terms of k and n:

    (a) $\log_5 \dfrac{xy^3}{125}$

    (b) $\log_{25}\left(5\sqrt{xy}\right)$

14. Given that $2 + \log_4 y = 2\log_1 6x$. Express x in terms of y.

15. Solve the following exponential equations:

    (a) $3^{3x-2} = 243$

Sol.

$$\begin{aligned}
3^{3x-2} &= 243 \\
\log 3^{3x-2} &= \log 3^5 \\
(3x-2)\log 3 &= 5\log 3 \\
3x-2 &= 5 \\
3x &= 7 \\
x &= \frac{7}{3}
\end{aligned}$$

(b) $4^{1-x} = \left(\frac{1}{8}\right)^{2x}$

Sol.

$$\begin{aligned}
4^{1-x} &= \left(\frac{1}{8}\right)^{2x} \\
\log 4^{1-x} &= \log \left(\frac{1}{8}\right)^{2x} \\
(1-x)\log 2^2 &= 2x\log 2^{-3} \\
2-2x &= -6x \\
-4x &= 2 \\
x &= -\frac{1}{2}
\end{aligned}$$

(c) $2^{x^2} = (2^x)^2$

Sol.

$$\begin{aligned}
2^{x^2} &= (2^x)^2 \\
x^2 &= 2x \\
x^2 - 2x &= 0 \\
x(x-2) &= 0 \\
x &= 0 \text{ or } x = 2
\end{aligned}$$

(d) $3^{5^x} = 3$

Sol.

$$\begin{aligned}
3^{5^x} &= 3 \\
\log 3^{5^x} &= \log 3 \\
5^x \log 3 &= \log 3 \\
5^x &= 1 \\
\log 5^x &= \log 1 \\
x\log 5 &= 0 \\
x &= 0
\end{aligned}$$

(e) $5^{8^x} = 625$

Sol.

$$\begin{aligned}
5^{8^x} &= 625 \\
\log 5^{8^x} &= \log 625 \\
8^x \log 5 &= \log 5^4 \\
8^x &= 4 \\
\log 8^x &= \log 4 \\
x\log 8 &= \log 4 \\
x &= \frac{\log 4}{\log 8} \\
&= \frac{2\log 2}{3\log 2} \\
&= \frac{2}{3}
\end{aligned}$$

(f) $3^{x+1} = 6^x$

Sol.

$$\begin{aligned}
3^{x+1} &= 6^x \\
\log 3^{x+1} &= \log 6^x \\
(x+1)\log 3 &= x\log 6 \\
x\log 3 + \log 3 &= x\log 6 \\
x\log 3 - x\log 6 &= -\log 3 \\
x(\log 3 - \log 6) &= -\log 3 \\
x\log\frac{1}{2} &= -\log 3 \\
x &= \frac{-\log 3}{\log\frac{1}{2}} \\
&\approx 1.5850
\end{aligned}$$

(g) $7^x - 7^{x-1} = 6$

Sol.

$$
\begin{aligned}
7^x - 7^{x-1} &= 6 \\
7^x - \frac{7^x}{7} &= 6 \\
\text{Let } y &= 7^x \\
y - \frac{y}{7} &= 6 \\
7y - y &= 42 \\
6y &= 42 \\
y &= 7 \\
7^x &= 7 \\
\log 7^x &= \log 7 \\
x \log 7 &= \log 7 \\
x &= 1
\end{aligned}
$$

(h) $3^{x+1} = 10\,(3^x) - 3$

Sol.

$$
\begin{aligned}
3^{2x+1} &= 10\,(3^x) - 3 \\
3^{2x} \cdot 3 &= 10\,(3^x) - 3 \\
\text{Let } y &= 3^x \\
3y^2 &= 10y - 3 \\
3y^2 - 10y + 3 &= 0 \\
(3y-1)(y-3) &= 0 \\
y &= \frac{1}{3} \text{ or } y = 3
\end{aligned}
$$

$$
\begin{aligned}
\text{When } y &= \frac{1}{3}, \\
3^x &= \frac{1}{3} \\
\log 3^x &= \log 3^{-1} \\
x \log 3 &= -\log 3 \\
x &= -1
\end{aligned}
$$

$$
\begin{aligned}
\text{When } y &= 3, \\
3^x &= 3 \\
\log 3^x &= \log 3 \\
x \log 3 &= \log 3 \\
x &= 1
\end{aligned}
$$

$$\therefore x = -1 \text{ or } x = 1$$

(i) $2^{2x+1} = 3(2^x) - 1$

Sol.

$$
\begin{aligned}
2^{2x+1} &= 3(2^x) - 1 \\
2^{2x} \cdot 2 &= 3(2^x) - 1 \\
\text{Let } y &= 2^x \\
2y^2 &= 3y - 1 \\
2y^2 - 3y + 1 &= 0 \\
(2y-1)(y-1) &= 0 \\
y &= \frac{1}{2} \text{ or } y = 1
\end{aligned}
$$

$$
\begin{aligned}
\text{When } y &= \frac{1}{2}, \\
2^x &= \frac{1}{2} \\
\log 2^x &= \log 2^{-1} \\
x \log 2 &= -\log 2 \\
x &= -1
\end{aligned}
$$

$$
\begin{aligned}
\text{When } y &= 1, \\
2^x &= 1 \\
\log 2^x &= \log 1 \\
x \log 2 &= 0 \\
x &= 0
\end{aligned}
$$

$$\therefore x = -1 \text{ or } x = 0$$

(j) $5^{2x+1} = 26(5^x) - 5$

Sol.

$$
\begin{aligned}
5^{2x+1} &= 26(5^x) - 5 \\
5^{2x} \cdot 5 &= 26(5^x) - 5 \\
\text{Let } y &= 5^x \\
5y^2 &= 26y - 5 \\
5y^2 - 26y + 5 &= 0 \\
(5y-1)(y-5) &= 0 \\
y &= \frac{1}{5} \text{ or } y = 5
\end{aligned}
$$

$$
\begin{aligned}
\text{When } y &= \frac{1}{5}, \\
5^x &= \frac{1}{5} \\
\log 5^x &= \log 5^{-1} \\
x \log 5 &= -\log 5 \\
x &= -1
\end{aligned}
$$

$$
\begin{aligned}
\text{When } y &= 5, \\
5^x &= 5 \\
\log 5^x &= \log 5 \\
x \log 5 &= \log 5 \\
x &= 1
\end{aligned}
$$

$$\therefore x = -1 \text{ or } x = 1$$

(k) $2^{2x+3} - 2^x = 1 - 2^{x+3}$

Sol.

$$
\begin{aligned}
2^{2x+3} - 2^x &= 1 - 2^{x+3} \\
2^{2x} \cdot 8 - 2^x &= 1 - 2^x \cdot 8 \\
\text{Let } y &= 2^x \\
8y^2 - y &= 1 - 8y \\
8y^2 + 7y - 1 &= 0 \\
(8y-1)(y+1) &= 0 \\
y &= \frac{1}{8} \text{ or } y = -1
\end{aligned}
$$

$$
\begin{aligned}
\text{When } y &= \frac{1}{8}, \\
2^x &= \frac{1}{8} \\
\log 2^x &= \log 2^{-3} \\
x &= -3
\end{aligned}
$$

$$
\begin{aligned}
\text{When } y &= -1, \\
2^x &= -1 \\
\because 2^x &> 0,\ \text{no solution for } x
\end{aligned}
$$

$$\therefore x = -3$$

(l) $2^{2x+8} - 32\,(2^x) + 1 = 0$

Sol.

$$
\begin{aligned}
2^{2x+8} - 32\,(2^x) + 1 &= 0 \\
2^{2x} \cdot 256 - 32 \cdot 2^x + 1 &= 0 \\
\text{Let } y &= 2^x \\
256y^2 - 32y + 1 &= 0 \\
(16y-1)^2 &= 0 \\
y &= \frac{1}{16}
\end{aligned}
$$

$$
\begin{aligned}
\text{When } y &= \frac{1}{16}, \\
2^x &= \frac{1}{16} \\
\log 2^x &= \log 2^{-4} \\
x &= -4
\end{aligned}
$$

16. If $\log_2 x + \log_4 x = \frac{9}{2}$, find the value of x.

Sol.

$$
\begin{aligned}
\log_2 x + \log_4 x &= \frac{9}{2} \\
\log_2 x + \frac{1}{2}\log_2 x &= \frac{9}{2} \\
\frac{3}{2}\log_2 x &= \frac{9}{2} \\
\log_2 x &= 3 \\
x &= 2^3 \\
&= 8
\end{aligned}
$$

17. Solve the following logarithmic equations:

(a) $2\log x - 3\log 4 = 2$

Sol.

$$\begin{aligned} 2\log x - 3\log 4 &= 2 \\ \log x^2 &= \log 4^3 + \log 100 \\ \log x^2 &= \log(4^3 \cdot 100) \\ x^2 &= 6400 \\ x &= \pm 80 \end{aligned}$$

After checking, $x = 80$ is the only solution, while $x = -80$ is an extraneous solution.

(b) $2\log x = \log 32 + \log 2$

Sol.

$$\begin{aligned} 2\log x &= \log 32 + \log 2 \\ \log x^2 &= \log(32 \cdot 2) \\ x^2 &= 64 \\ x &= \pm 8 \end{aligned}$$

After checking, $x = 8$ is the only solution, while $x = -8$ is an extraneous solution.

(c) $\log x + \log(x+3) = \log(x+8)$

Sol.

$$\begin{aligned} \log x + \log(x+3) &= \log(x+8) \\ \log \frac{x(x+3)}{x+8} &= 0 \\ \frac{x(x+3)}{x+8} &= 1 \\ x^2 + 3x &= x + 8 \\ x^2 + 2x - 8 &= 0 \\ (x+4)(x-2) &= 0 \\ x &= -4 \text{ or } x = 2 \end{aligned}$$

After checking, $x = 2$ is the only solution, while $x = -4$ is an extraneous solution.

(d) $\left(\log_2 x\right)^2 = \log_2 x + 6$

Sol.

$$\begin{aligned} \text{Let } y &= \log_2 x \\ y^2 &= y + 6 \\ y^2 - y - 6 &= 0 \\ (y-3)(y+2) &= 0 \\ y &= 3 \text{ or } y = -2 \end{aligned}$$

$$\begin{aligned} \text{When } y &= 3, \\ \log_2 x &= 3 \\ x &= 2^3 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \text{When } y &= -2, \\ \log_2 x &= -2 \\ x &= 2^{-2} \\ &= \frac{1}{4} \end{aligned}$$

After checking, $x = 8$ and $x = \frac{1}{4}$ are both solutions.

(e) $\log_3 x + 6\log_x 3 = 5$

Sol.

$$\log_3 x + 6\frac{1}{\log_3 x} = 5$$

$$\text{Let } y = \log_3 x$$

$$y + \frac{6}{y} = 5$$

$$y^2 - 5y + 6 = 0$$

$$(y-2)(y-3) = 0$$

$$y = 2 \text{ or } y = 3$$

When $y = 2$,

$$\log_3 x = 2$$

$$x = 3^2$$

$$= 9$$

When $y = 3$,

$$\log_3 x = 3$$

$$x = 3^3$$

$$= 27$$

After checking, $x = 9$ and $x = 27$ are both solutions.

(f) $4^{\log x} = 2^{\log x + 1}$

Sol.

$$4^{\log x} = 2^{\log x+1}$$

$$\log 4^{\log x} = \log 2^{\log x+1}$$

$$\log x \log 4 = (\log x + 1)\log 2$$

$$\log x \log 4 = \log x \log 2 + \log 2$$

$$\log x \log 4 - \log x \log 2 = \log 2$$

$$\log x(\log 4 - \log 2) = \log 2$$

$$\log x \log 2 = \log 2$$

$$\log x = 1$$

$$x = 10$$

(g) $\log_{x+1}\left(x^2 - 5x - 13\right) = 2$

Sol.

$$\log_{x+1}(x^2 - 5x - 13) = 2$$

$$x^2 - 5x - 13 = (x+1)^2$$

$$x^2 - 5x - 13 = x^2 + 2x + 1$$

$$-7x = 14$$

$$x = -2$$

After checking, $x = -2$ not a solution. Hence, no solution.

(h) $\log_x \sqrt{2x^2 - 5x + 6} = 1$

Sol.

$$\log_x \sqrt{2x^2 - 5x + 6} = 1$$

$$\sqrt{2x^2 - 5x + 6} = x$$

$$2x^2 - 5x + 6 = x^2$$

$$x^2 - 5x + 6 = 0$$

$$(x-2)(x-3) = 0$$

$$x = 2 \text{ or } x = 3$$

After checking, $x = 2$ and $x = 3$ are both solutions.

(i) $\log_2(x+1) + \log_2(x+3) = 3 + \log_2 x$

Sol.

$$\begin{aligned}
\log_2(x+1) + \log_2(x+3) &= 3 + \log_2 x \\
\log_2(x+1)(x+3) &= \log_2 8 + \log_2 x \\
\log_2(x^2+4x+3) &= \log_2 8x \\
x^2+4x+3 &= 8x \\
x^2-4x+3 &= 0 \\
(x-1)(x-3) &= 0 \\
x = 1 \text{ or } x &= 3
\end{aligned}$$

After checking, $x = 1$ and $x = 3$ are both solutions.

(j) $\log_2 \left[\log_3 \left(\log_5 x\right)\right] = 0$

Sol.

$$\begin{aligned}
\log_2 \left[\log_3 \left(\log_5 x\right)\right] &= 0 \\
\log_3 \left(\log_5 x\right) &= 1 \\
\log_5 x &= 3 \\
x &= 5^3 \\
&= 125
\end{aligned}$$

(k) $3\log_8 x - 2\log_2 x + 2 = 0$

Sol.

$$\begin{aligned}
3\log_8 x - 2\log_2 x + 2 &= 0 \\
\frac{3\log_2 x}{\log_2 8} - 2\log_2 x + 2 &= 0 \\
\text{Let } u &= \log_2 x \\
\frac{3u}{3} - 2u + 2 &= 0 \\
u - 2u + 2 &= 0 \\
u &= 2
\end{aligned}$$

$$\begin{aligned}
\log_2 x &= 2 \\
x &= 2^2 \\
&= 4
\end{aligned}$$

(l) $\log_4(x+4) + 1 = \log_2(x+1)$

Sol.

$$\begin{aligned}
\log_4(x+4) + 1 &= \log_2(x+1) \\
\frac{\log_2(x+4)}{\log_2 4} &= \log_2(x+1) + 1 \\
\log_2(x+4) &= \log_2 (x+1)^2 - \log_2 4 \\
&= \log_2 \frac{(x+1)^2}{4} \\
x+4 &= \frac{(x+1)^2}{4} \\
4x+16 &= (x+1)^2 \\
4x+16 &= x^2+2x+1 \\
x^2-2x-15 &= 0 \\
(x-5)(x+3) &= 0 \\
x = 5 \text{ or } x &= -3
\end{aligned}$$

After checking, $x = 5$ is the only solution.

(m) $2\log_2 x \cdot \log_8 x = \log_2 x + \log_8 x$

Sol.

$$
\begin{aligned}
2\log_2 x \cdot \log_8 x &= \log_2 x + \log_8 x \\
2\log_2 x \cdot \frac{\log_2 x}{\log_2 8} &= \log_2 x + \frac{\log_2 x}{\log_2 8} \\
2\log_2 x \cdot \frac{\log_2 x}{3} &= \log_2 x + \frac{\log_2 x}{3}
\end{aligned}
$$

Let $u = \log_2 x$

$$
\begin{aligned}
2u \cdot \frac{u}{3} &= u + \frac{u}{3} \\
\frac{2u^2}{3} &= \frac{4u}{3} \\
2u^2 &= 4u \\
u^2 - 2u &= 0 \\
u(u-2) &= 0 \\
u = 0 &\text{ or } u = 2
\end{aligned}
$$

When $u = 0$,

$$
\begin{aligned}
\log_2 x &= 0 \\
x &= 2^0 \\
&= 1
\end{aligned}
$$

When $u = 2$,

$$
\begin{aligned}
\log_2 x &= 2 \\
x &= 2^2 \\
&= 4
\end{aligned}
$$

After checking, $x = 1$ and $x = 4$ are both solutions.

18. A person has deposited a fund into a bank account that pays 5.5% interest compounded annually. If the balance in the account has increased by RM1,432.95 after 4 years, how much was deposited initially?

19. Given that there is a principal of RM75,000 at an interest rate of 3.5% per annum compounded once per three months. Find the accumulated value of the principal after 8 years.

20. Given that there is a principal of RM150,000 at an interest rate of 5.25% per annum compounded once per annum. How many years does it take to accumulate at least RM300,000?

21. If the present value is RM24,924.44, the interest rate is 5% per annum compounded once per annum, find the annuity payment if the payment is made for 20 years.

22. Given that the annuity payment is RM8,000, the interest rate is 4.5% per annum compounded once per annum, and the payment is made for 15 years. Find the present value. Hence, find the present value of the perpetuity.

23. Given that the annuity amount is RM4,500, the interest rate is 4.5% per annum compounded once per annum. How many years does it take for the present value to exceed RM50,000?

24. The price of a branded laptop is RM2,500, the payment can be paid in full or by instalment. If the payment is made by instalment, the monthly payment is RM110 for 2 years. If the interest rate is 4% per annum compounded once per month, which payment method is more economical considering the present value of the payment?