Calculus

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March 9, 2023

Contents

Chapter 1

Limits

1.1 Arithmetic Properties of Limits

If $\lim_{x\to x_0} f(x) = A$, $\lim_{x\to x_0} g(x) = B$, then:

(a) $\lim_{x\to x_0} (f(x) \pm g(x)) = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x) = A \pm B$

(b) $\lim_{x\to x_0} (f(x) \cdot g(x)) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x) = A \cdot B$

(c) $\lim_{x\to x_0} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} = \frac{A}{B}$, $(B \neq 0)$

(d) If k is a constant, then $\lim_{x\to x_0} k = k$

(e) If k is a constant, then $\lim_{x\to x_0} k \cdot f(x) = k \lim_{x\to x_0} f(x) = kA$

(f) If $n \in \mathbb{R}$, and $\lim_{x\to x_0} f(x) > 0$, then $\lim_{x\to x_0} [f(x)]^n = \left[\lim_{x\to x_0} f(x)\right]^n = A^n$

(g) If $\lim_{x\to x_0} f(x) = 0$, then $\lim_{x\to x_0} \frac{1}{f(x)} = \infty$

Squeeze Theorem or Sandwich Rule

$$\begin{aligned} &\text{Near point } x_0, \\ &\text{If } f(x) \le g(x) \le h(x) \\ &\text{and } \lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = A, \\ &\text{then } \lim_{x\to x_0} g(x) = A. \end{aligned}$$

1. $\lim_{x\to 3} 3x$

Sol.

$$\begin{aligned} \lim_{x\to 3} 3x &= 3 \cdot 3 \\ &= 9 \quad \square \end{aligned}$$

2. $\lim_{x\to -1} (x^2 + 4x)$

Sol.

$$\begin{aligned} \lim_{x\to -1} (x^2 + 4x) &= \lim_{x\to -1} x^2 + \lim_{x\to -1} 4x \\ &= (-1)^2 + 4(-1) \\ &= 1 - 4 \\ &= -3 \quad \square \end{aligned}$$

3. $\lim_{x\to 3} (9 - x^2)$

Sol.

$$\begin{aligned}\lim_{x\to3}(9-x^2) &= \lim_{x\to3}9-\lim_{x\to3}x^2\\ &= 9-3^2\\ &= 9-9\\ &= 0 \quad \square\end{aligned}$$

4. $\lim_{n\to-2}(x^2-2x+1)$

Sol.

$$\begin{aligned}\lim_{n\to-2}(x^2-2x+1) &= \lim_{n\to-2}(x-1)^2\\ &= (-3)^2\\ &= 9 \quad \square\end{aligned}$$

5. $\lim_{x\to-4}x^2(x+2)$

Sol.

$$\begin{aligned}\lim_{x\to-4}x^2(x+2) &= \lim_{x\to-4}x^2\lim_{x\to-4}(x+2)\\ &= (-4)^2\cdot(-4+2)\\ &= 16\cdot(-2)\\ &= -32 \quad \square\end{aligned}$$

6. $\lim_{h\to2}(h^2-4h+4)$

Sol.

$$\begin{aligned}\lim_{h\to2}(h^2-4h+4) &= \lim_{h\to2}(h-2)^2\\ &= (2-2)^2\\ &= 0 \quad \square\end{aligned}$$

7. $\lim_{a\to-1}(a+3)(a-4)$

Sol.

$$\begin{aligned}\lim_{a\to-1}(a+3)(a-4) &= \lim_{a\to-1}(a+3)\lim_{a\to-1}(a-4)\\ &= (-1+3)\cdot(-1-4)\\ &= 2\cdot-5\\ &= -10 \quad \square\end{aligned}$$

8. $\lim_{x\to3}\frac{x^2-5}{x+2}$

Sol.

$$\begin{aligned}\lim_{x\to3}\frac{x^2-5}{x+2} &= \lim_{x\to3}\frac{x^2-5}{x+2}\\ &= \frac{3^2-5}{3+2}\\ &= \frac{4}{5} \quad \square\end{aligned}$$

9. $\lim_{x\to-3}\frac{(x+5)(x+3)}{x+3}$

Sol.

$$\begin{aligned}\lim_{x\to-3}\frac{(x+5)(x+3)}{x+3} &= \lim_{x\to-3}\frac{(x+5)(x+3)}{x+3}\\ &= \lim_{x\to-3}(x+5)\\ &= -3+5\\ &= 2 \quad \square\end{aligned}$$

10. $\lim_{x\to0}\frac{x^2+5x}{x}$

Sol.

$$\begin{aligned}\lim_{x\to0}\frac{x^2+5x}{x} &= \lim_{x\to0}\frac{x^2+5x}{x}\\ &= \lim_{x\to0}\frac{x(x+5)}{x}\\ &= \lim_{x\to0}(x+5)\\ &= 0+5\\ &= 5 \quad \square\end{aligned}$$

11. $\lim_{x\to-2}\frac{x^3+8}{x+2}$

Sol.

$$\begin{aligned}\lim_{x\to-2}\frac{x^3+8}{x+2} &= \lim_{x\to-2}\frac{(x+2)(x^2-x+4)}{x+2}\\ &= \lim_{x\to-2}(x^2-x+4)\\ &= (-2)^2-(-2)+4\\ &= 10 \quad \square\end{aligned}$$

12. $\lim_{x\to4}\frac{x^2-5x+6}{x-3}$

Sol.

$$\begin{aligned}\lim_{x\to4}\frac{x^2-5x+6}{x-3} &= \lim_{x\to4}\frac{(x-3)(x-2)}{x-3}\\ &= \lim_{x\to4}(x-2)\\ &= 4-2\\ &= 2 \quad \square\end{aligned}$$

13. $\lim_{x\to3}\frac{3x}{x+2}$

Sol.

$$\begin{aligned}\lim_{x\to3}\frac{3x}{x+2} &= \lim_{x\to3}\frac{3x}{x+2}\\ &= \frac{3(3)}{3+2}\\ &= \frac{9}{5} \quad \square\end{aligned}$$

14. $\lim_{x\to5}\frac{x-5}{2x^2-9x-5}$

Sol.

$$\begin{aligned}\lim_{x\to5}\frac{x-5}{2x^2-9x-5} &= \lim_{x\to5}\frac{x-5}{(2x+1)(x-5)}\\ &= \lim_{x\to5}\frac{1}{2x+1}\\ &= \frac{1}{2(5)+1}\\ &= \frac{1}{11} \quad \square\end{aligned}$$

15. $\lim_{x\to 1} \frac{x-1}{x^2+x-2}$

Sol.

$$\begin{aligned}\lim_{x\to 1} \frac{x-1}{x^2+x-2} &= \lim_{x\to 1} \frac{x-1}{(x-1)(x+2)} \\ &= \lim_{x\to 1} \frac{1}{x+2} \\ &= \frac{1}{1+2} \\ &= \frac{1}{3} \quad \square\end{aligned}$$

16. $\lim_{x\to 4} \frac{x-1}{x^2+x-2}$

Sol.

$$\begin{aligned}\lim_{x\to 4} \frac{x-1}{x^2+x-2} &= \lim_{x\to 4} \frac{x-1}{(x-1)(x+2)} \\ &= \lim_{x\to 4} \frac{1}{x+2} \\ &= \frac{1}{4+2} \\ &= \frac{1}{6} \quad \square\end{aligned}$$

17. $\lim_{x\to -2} \frac{x-2}{x^2-4}$

Sol.

$$\begin{aligned}\lim_{x\to -2} \frac{x-2}{x^2-4} &= \lim_{x\to -2} \frac{x-2}{(x+2)(x-2)} \\ &= \lim_{x\to -2} \frac{1}{x+2} \\ &= \infty \quad \square\end{aligned}$$

18. $\lim_{h\to 0} \frac{2x^2h\ +3h}{h}$

Sol.

$$\begin{aligned}\lim_{h\to 0} \frac{2x^2h\ +3h}{h} &= \lim_{h\to 0} \frac{h(2x^2+3)}{h} \\ &= \lim_{h\to 0} (2x^2+3) \\ &= 2x^2+3 \quad \square\end{aligned}$$

19. $\lim_{h\to 0} \frac{(2+h)^2-4}{h}$

Sol.

$$\begin{aligned}\lim_{h\to 0} \frac{(2+h)^2-4}{h} &= \lim_{h\to 0} \frac{[(2+h)+2][(2+h)-2]}{h} \\ &= \lim_{h\to 0} \frac{(4+h)h}{h} \\ &= \lim_{h\to 0} (4+h) \\ &= 4+0 \\ &= 4 \quad \square\end{aligned}$$

20. $\lim_{h\to 0} \frac{(1+h)^3-1}{h}$

Sol.

$$\begin{aligned}&\lim_{h\to 0} \frac{(1+h)^3-1}{h} \\ &= \lim_{h\to 0} \frac{[(1+h)-1][(1+h)^2+(1+h)+1]}{h} \\ &= \lim_{h\to 0} \frac{h(1+2h+h^2+1+h+1)}{h} \\ &= \lim_{h\to 0} (h^2+3h+3) \\ &= (0)^2+3(0)+3 \\ &= 3 \quad \square\end{aligned}$$

21. $\lim_{x\to -1} 2x(x^2-4)$

Sol.

$$\begin{aligned}\lim_{x\to -1} 2x(x^2-4) &= \lim_{x\to -1} 2x(x^2-4) \\ &= -2(-1)^2[(-1)^2-4] \\ &= -2(-3) \\ &= 6 \quad \square\end{aligned}$$

22. $\lim_{x\to 3} \frac{x^2+2}{x+1}$

Sol.

$$\begin{aligned}\lim_{x\to 3} \frac{x^2+2}{x+1} &= \frac{3^2+2}{3+1} \\ &= \frac{11}{4} \quad \square\end{aligned}$$

23. $\lim_{x\to 2} \left(x^2-3x+5\right)$

Sol.

$$\begin{aligned}\lim_{x\to 2} \left(x^2-3x+5\right) &= 2^2-3(2)+5 \\ &= 3 \quad \square\end{aligned}$$

24. $\lim_{x\to 1} \frac{2x^2+1}{3x^2+4x-1}$

Sol.

$$\begin{aligned}\lim_{x\to 1} \frac{2x^2+1}{3x^2+4x-1} &= \frac{2(1)^2+1}{3(1)^2+4(1)-1} \\ &= \frac{3}{6} \\ &= \frac{1}{2} \quad \square\end{aligned}$$

25. $\lim_{x\to 1} \frac{x^2-5x+6}{x^2-9}$

Sol.

$$\begin{aligned}\lim_{x\to 1} \frac{x^2-5x+6}{x^2-9} &= \frac{(x-3)(x-2)}{(x+3)(x-3)} \\ &= \frac{x-2}{x+3} \\ &= \frac{1-2}{1+3} \\ &= -\frac{1}{4} \quad \square\end{aligned}$$

26. $\lim_{x\to -1} \frac{x^3+1}{x+1}$

Sol.

$$\begin{aligned}\lim_{x\to -1}\frac{x^3+1}{x+1} &= \lim_{x\to -1}\frac{(x+1)(x^2-x+1)}{x+1}\\ &= \lim_{x\to -1}(x^2-x+1)\\ &= (-1)^2-(-1)+1\\ &= 3 \quad \square\end{aligned}$$

27. $\lim_{x\to 1} \frac{x^3-1}{x-1}$

Sol.

$$\begin{aligned}\lim_{x\to 1}\frac{x^3-1}{x-1} &= \lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{x-1}\\ &= \lim_{x\to 1}(x^2+x+1)\\ &= 1^2+1+1\\ &= 3 \quad \square\end{aligned}$$

28. $\lim_{x\to 0} \frac{2x^3+3x^2}{x^3}$

Sol.

$$\begin{aligned}\lim_{x\to 0}\frac{2x^3+3x^2}{x^3} &= \lim_{x\to 0}\left(\frac{2x^3}{x^3}+\frac{3x^2}{x^3}\right)\\ &= \lim_{x\to 0}\left(2+\frac{3}{x}\right)\\ &= 2 \quad \square\end{aligned}$$

29. $\lim_{k\to 0} \frac{(x-k)^2-2kx^3}{x(x+k)}$

Sol.

$$\begin{aligned}\lim_{k\to 0}\frac{(x-k)^2-2kx^3}{x(x+k)} &= \frac{(x-0)^2-2(0)(x^3)}{x(x+0)}\\ &= \frac{x^2}{x^2}\\ &= 1 \quad \square\end{aligned}$$

30. $\lim_{x\to 1} \frac{x^2-2x+5}{x^2+7}$

Sol.

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-2x+5}{x^2+7} &= \frac{(1)^2-2(1)+5}{(1)^2+7}\\ &= \frac{4}{8}\\ &= \frac{1}{2} \quad \square\end{aligned}$$

31. $\lim_{x\to -2} \frac{x^4-16}{x^3-2}$

Sol.

$$\begin{aligned}\lim_{x\to -2}\frac{x^4-16}{x^3-2} &= \frac{(-2)^4-16}{(-2)^3-2}\\ &= \frac{16-16}{-8-2}\\ &= \frac{0}{-10}\\ &= 0 \quad \square\end{aligned}$$

32. $\lim_{x\to 1} \frac{x-1}{x^2-1}$

Sol.

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{x^2-1} &= \lim_{x\to 1}\frac{x-1}{(x-1)(x+1)}\\ &= \lim_{x\to 1}\frac{1}{x+1}\\ &= \frac{1}{1+1}\\ &= \frac{1}{2} \quad \square\end{aligned}$$

33. $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$

Sol.

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} &= \lim_{x\to 0}\frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)}\\ &= \lim_{x\to 0}\frac{1+x-1}{x(\sqrt{1+x}+1)}\\ &= \lim_{x\to 0}\frac{x}{x(\sqrt{1+x}+1)}\\ &= \lim_{x\to 0}\frac{1}{\sqrt{1+x}+1}\\ &= \frac{1}{\sqrt{1+0}+1}\\ &= \frac{1}{2}\end{aligned}$$

34. $\lim_{x\to 2} \frac{x^2+4}{x^2+1}$

Sol.

$$\begin{aligned}\lim_{x\to 2}\frac{x^2+4}{x^2+1} &= \frac{(2)^2+4}{(2)^2+1}\\ &= \frac{8}{5}\end{aligned}$$

35. $\lim_{x\to 0} \frac{x^2+3x+2}{x^2+2}$

Sol.

$$\begin{aligned}\lim_{x\to 0}\frac{x^2+3x+2}{x^2+2} &= \frac{(0)^2+3(0)+2}{(0)^2+2}\\ &= \frac{2}{2}\\ &= 1 \quad \square\end{aligned}$$

36. $\lim_{x\to 1} \frac{x^2-2x+1}{x^2\ -1}$

Sol.

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-2x+1}{x^2\ -1} &= \frac{(x-1)^2}{(x-1)(x+1)}\\ &= \frac{x-1}{x+1}\\ &= \frac{1-1}{1+1}\\ &= 0 \quad \square\end{aligned}$$

$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$

Proof.

$$\begin{aligned}
(a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) &= (a-b)\left(\sum_{k=0}^{n-1} a^{n-k-1}b^k\right) \\
&= a\sum_{k=0}^{n-1} a^{n-k-1}b^k - b\sum_{k=0}^{n-1} a^{n-k-1}b^k \\
&= \sum_{k=0}^{n-1} a^{n-k}b^k - \sum_{k=0}^{n-1} a^{n-k-1}b^{k+1} \\
&= a^n + \sum_{k=1}^{n-1} a^{n-k}b^k - \sum_{l=0}^{n-2} a^{n-l-1}b^{l+1} - b^n \\
&= a^n + \sum_{k=1}^{n-1} a^{n-k}b^k - \sum_{k=1}^{n-1} a^{n-k}b^k - b^n \quad (l = k-1) \\
&= a^n - b^n \quad \square
\end{aligned}$$

37. $\lim\limits_{x\to a} \frac{x^n - a^n}{x-a} \quad (n \in \mathbb{N})$

Sol.

$$\begin{aligned}
\lim_{x\to a} \frac{x^n - a^n}{x-a} &= \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{(x-a)} \\
&= \lim_{x\to a}(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\
&= a^{n-1} + a^{n-2}a + \cdots + a^{n-2}a + a^{n-1} \\
&= a^{n-1} + a^{n-1} + \cdots + a^{n-1} \\
&= na^{n-1} \quad \square
\end{aligned}$$

38. $\lim\limits_{x\to 1}(3x^2 - 6x + 5)$

39. $\lim\limits_{x\to 1} \frac{2x^2-1}{3x^3-6x^2+5}$

40. $\lim\limits_{x\to 1} \frac{x^2-1}{x^2-4x+3}$

41. $\lim\limits_{x\to 3} \frac{x^2-5x+6}{7x^2-22x+3}$

42. $\lim\limits_{x\to 3} \frac{x^3-2x^2-2x+4}{x^3+x^2-10x+8}$

43. $\lim\limits_{x\to 1} \frac{x^4+2x^2-3}{x^2-3x+2}$

44. $\lim\limits_{x\to 1} \frac{1-\sqrt[4]{x}}{1-\sqrt[3]{x}}$

45. $\lim\limits_{x\to 0} \frac{\sqrt[n]{1+x}-1}{x} \quad (n \in \mathbb{W})$

46. $\lim\limits_{x\to 1} \frac{2-\sqrt{x+3}}{x^2-1}$

47. $\lim\limits_{x\to 16} \frac{\sqrt[4]{x}-2}{\sqrt{x}-4}$

48. $\lim\limits_{x\to 2}(x^2 + 3x - 1)$

49. $\lim\limits_{x\to -1} \frac{x^2+2}{x^2+x+3}$

50. $\lim\limits_{x\to 2} \frac{x^3+1}{x^2-1}$

51. $\lim\limits_{x\to 1} \frac{x^5-x^4}{x^3-x}$

52. $\lim\limits_{x\to a} \frac{x^2+ax-2a^2}{x^2-a^2}, a \neq 0$

53. $\lim\limits_{x\to a} \frac{\sqrt{3x-a}-\sqrt{x+a}}{x-a}$

54. Given that $f(x) = x^2 - 3x$, find $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$

55. $\lim\limits_{x\to 2} \sqrt{2x^2+1}$

56. $\lim\limits_{x\to 7} \frac{x^2\sqrt{x+2}}{x^2+14}$

57. $\lim\limits_{x\to 0} \frac{\sqrt{3x+4-2}}{x}$

58. $\lim\limits_{x\to 0} \frac{1}{x^2}$

59. $\lim\limits_{x\to 1} \frac{1}{x-1}$

60. $\lim\limits_{x\to 1} \frac{4x-3}{x^2-5x+4}$

61. $\lim\limits_{x\to \infty} \frac{3x^3-4x^2+2}{7x^3+5x^2-3}$

62. $\lim\limits_{n\to \infty} x^2$

63. $\lim\limits_{x\to \infty} \frac{3x^2-2x-1}{2x^3-x^2+5}$

64. $\lim\limits_{n\to \infty} \frac{1}{n^2}(1+2+3+\cdots+n)$

65. $\lim\limits_{n\to\infty}\left[\frac{1+2+3+\cdots+n}{n+2}-\frac{n}{2}\right]$

66. $\lim\limits_{n\to\infty}\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}\right]$

67. $\lim\limits_{x\to\infty}\frac{5x^3+4x^2-6x+2}{8x^3-7x^2+4x-1}$

68. $\lim\limits_{x\to\infty}\frac{x^4-2x^3+x^2+3}{x^5-x^4+1}$

69. $\lim\limits_{x\to\infty}\frac{x^3-8x^2+4x-1}{x^2-6x+3}$

70. $\lim\limits_{n\to\infty}(\sqrt{x^4+1}-x^2)$