Mathematics

Senior 3 Part I

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Introduction

Why this book?

Disclaimer

Acknowledgements

Contents

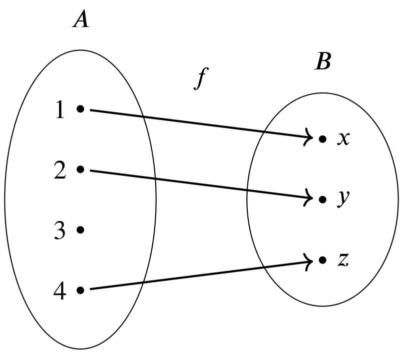
Chapter 22

Function

22.1 Definition of a Function

Mapping, Preimage and Image

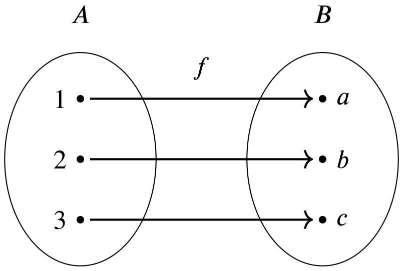
For two non-empty sets A and B, If an element a inside set A has a corresponding element b inside set B, denoted as $a \to b$, then we say that a is mapped to b or a and b are paired. The mapping between two sets is normally denoted as f, g, h, etc. The mapping shown in the diagram below can be denoted as $f : 1 \to x,\ 2 \to y,\ 4 \to z$.



Let $f : A \to B$ is a mapping, a is an element in A. If a is mapped to b under the mapping f, then b is said to be the image of a under the mapping f, denoted as $b = f(a)$; a is said to be the preimage of b under the mapping f. In the diagram above, under the mapping f, the image of 1, 2, and 4 are x, y, and z respectively, while the preimage of x, y, and z are 1, 2, and 4 respectively.

> Let A and B be two non-empty sets, f is a mapping from A to B such that for all elements in A, there is a unique corresponding element in B, then f is a function or a mapping from A to B, denoted as $f : A \to B$.

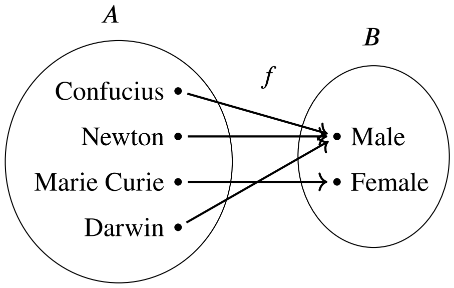
The mapping shown in the diagram below is a function.



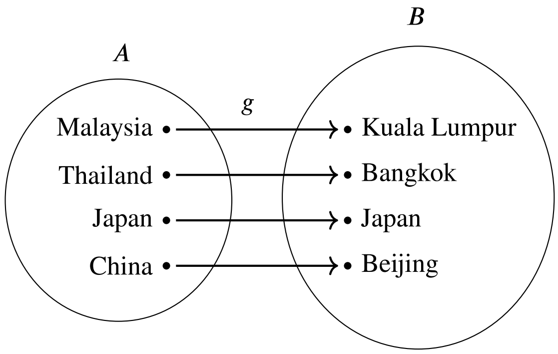
22.1.1 Practice 1

1. For the following mappings, list the image of each element in A and the preimage of each element in B, and determine whether the mapping is a function or not:

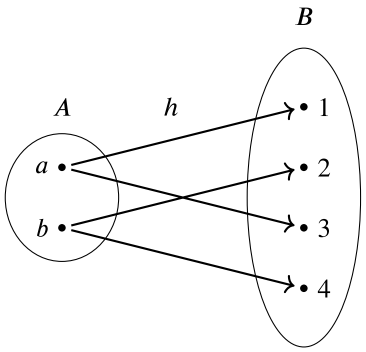
(a)



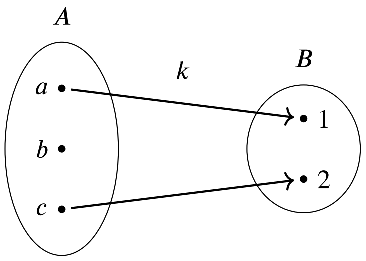
(b)



(c)



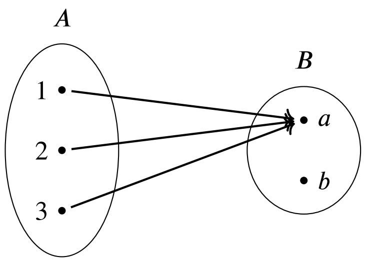
(d)



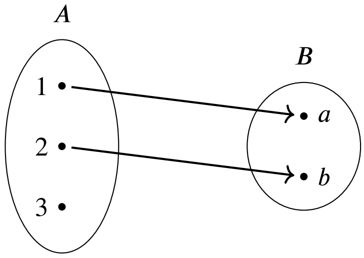
2. Given a mapping $g : x \to x + 3$, $x \in \{-2, -1, 0, 1, 2, 3\}$, find the image of each x.

3. Determine whether the following mappings are functions.

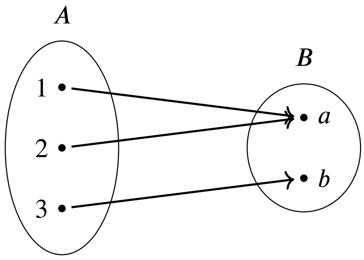
(a)



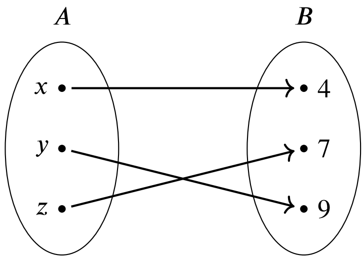
(b)



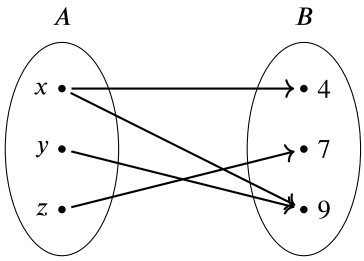
(c)



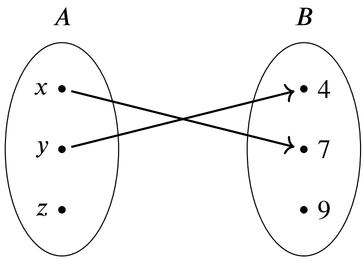
(d)



(e)



(f)



The function $f : A \to B$ can be written as $y = f(x)$, x is the element of A and y is the element of B. When x changes, y changes as well. x is called independent variable, while y is called dependent variable.

Keep in mind that $f(x)$ is NOT the product of f and x.

Representation of Functions

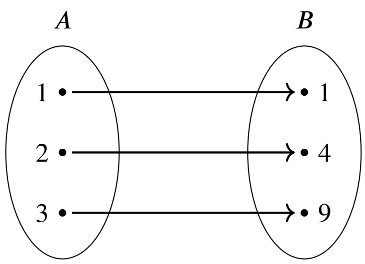
Generally speaking, there are a few ways to represent a function:

1. **Narrative Form**: express the function of two sets in words. For example, Let $A = \{1, 2, 3\}$ and $B = \{1, 4, 9\}$, f is a function from A to B, its definition is that for any element x in A, its corresponding element is x^2 in B.

2. **Arrow Method**: draw an arrow to connect the preimage and image of a function such that the preimage is corresponding to the image. To express the example above, we express it as $f : 1 \to 1,\ 2 \to 4,\ 3 \to 9$.

3. **Analytical Method**: express the function in the form of mathematical expression to represent the relationship between the independent variable and the dependent variable. For example, $f(x) = x^2, x \in A$.

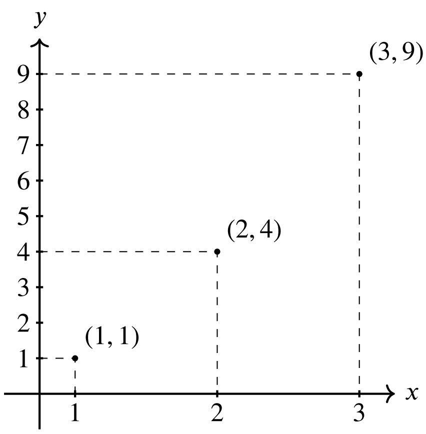
4. **Venn Diagram**: draw arrows between the venn diagram of two sets to represent the function, as shown below:



5. **Table Method**: express the function in the form of table, showing the relationship of the chosen value between independent variable x and the value of its corresponding dependent variable y, as shown below:

x	1	2	3
y	1	4	9

6. **Graphical Method**: draw a graph to represent the function of the two variables, as shown below:



22.1.2 Practice 2

Express the following functions using analytical method, venn diagram, table method and graphical method.

(a) f mapping each integers from -3 to 3 to its squares plus 4.

(b) g mapping each natural numbers from 1 to 4 to its cubes.

22.1.3 Exercise 22.1

1. Express the mapping from set A to set B, and determine which of the following mappings are functions.

	Set A	Set B	Mapping
(a)	{0, 3, 9, 12}	{0, 1, 2, 3}	Divide by 3
(b)	{-2, -1, 0, 1, 2}	{0, 1, 4, 9, 16}	Power of 4
(c)	{-2, -1, 0, 1, 2}	{0, 1, 4}	Square
(d)	{30°, 45°, 60°}	$\left\{\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right\}$	Sine
(e)	{-1, 0, 1, 2}	{-1, 0, 1}	Cube

2. Let function $f(x) = 3x^2 + 1$.

   (a) Find the image of the following elements:

      i. -3

ii. -2

iii. 0

iv. 2

v. 5

(b) Find the preimage of the following elements:

i. 13

ii. 28

iii. 1

iv. 0

v. 4

3. Let function $g(x) = 5x - 2$. Find:

   (a) $g(-2)$

   (b) $g(-1)$

   (c) $g(0)$

4. Let function $f(x) = \begin{cases} 2x, & x \leq -1 \\ x - 1, & -1 \leq x < 3 \\ 4x + 2, & x \geq 3 \end{cases}$, find

   (a) $f(-5)$

   (b) $f(-2)$

   (c) $f(0)$

   (d) $f(2)$

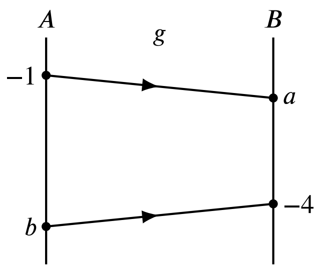
   (e) $f(10)$

5. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$. Find the image of -1, 0, 1, and 2 under f.

6. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$. Find the preimage of 0, 1, and 4 under f.

   In $\mathbb{R}$, which element does not have a preimage?

7. In the diagram below, given that function $g : A \to B$ is defined as $g : x \to 2x - 8$. Find the value of a and b.

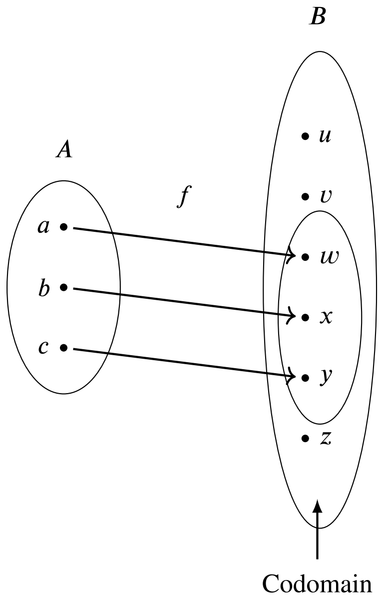


8. Using narrative form, arrow method, venn diagram, table method and graphical method, express the function $f(x) = 2x$, $x \in \{-2, -1, 0, 1, 2\}$.

22.2 Domain and Range

Let f is a function from set A to set B, then set A is called the domain of f, denoted by D_f; set B is called the codomain of f; the set of the images of all elements of A under f is called the range of f, denoted by R_f.

If the domain A and range B of function $f : A \to B$ are both subsets of real number set $\mathbb{R}$, then this function is called real valued function / real function. This book primarily discusses about real valued functions. When the domain of a real function is not mentioned and only the mapping rule is given, its domain is assumed to be the set of all real numbers that yield defined values $f(x)$. After the domain and the mapping rule are determined, the range of a function will then be determined.



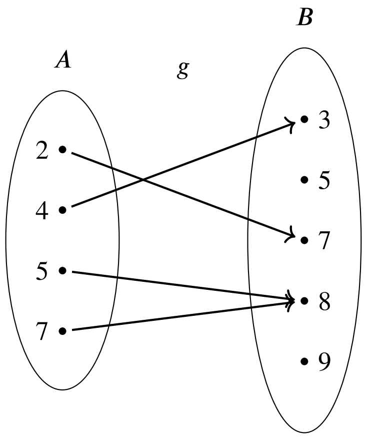
Interval Notation

Let a and b be two real number, $a < b$.

Intervals	Set Notations
(a, b)	$x \mid x \in \mathbb{R}, a < x < b$
$[a, b)$	$x \mid x \in \mathbb{R}, a \leq x < b$
$(a, b]$	$x \mid x \in \mathbb{R}, a < x \leq b$
$[a, b]$	$x \mid x \in \mathbb{R}, a \leq x \leq b$
(a, ∞)	$x \mid x \in \mathbb{R}, x > a$
$[a, \infty)$	$x \mid x \in \mathbb{R}, x \leq a$
$(-\infty, a)$	$x \mid x \in \mathbb{R}, x < a$
$(-\infty, a]$	$x \mid x \in \mathbb{R}, x \leq a$

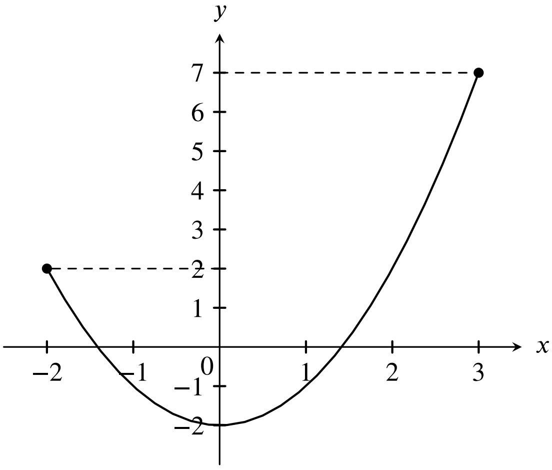
22.2.1 Practice 3

1. Let $A = \{2, 4, 5, 7\}$ and $B = \{3, 5, 7, 8, 9\}$, the definition of function g is given by the diagram below. Find the domain, codomain and range of function g.



2. Let $A = \{-2, -1, 0, 1, 2\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = x^2 - 1$. Find the domain and range of f.

3. The curve in the diagram below represents the function $y = f(x)$, $-2 \le x \le 3$. Find the domain and range of f.



4. Find the domain and range of the following functions:

   (a) $f(x) = -4x + 5$

   (b) $g(x) = x^2 - 1$

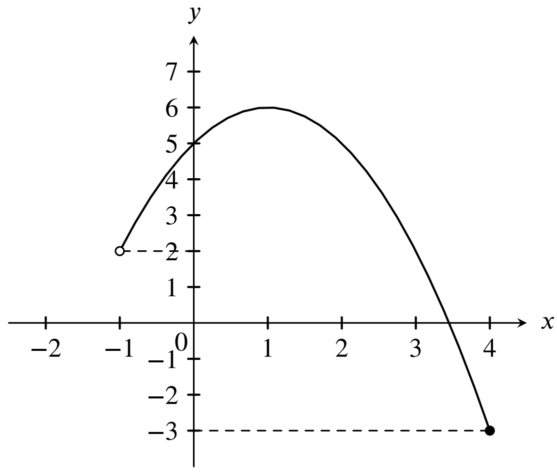
   (c) $h(x) = \dfrac{1}{4x + 7}$

   (d) $k(x) = \sqrt{6 - x}$

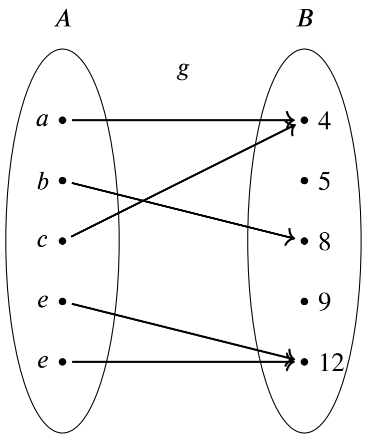
22.2.2 Exercise 22.2

1. Let $X = \{a, b, c, d\}$ and $Y = \{-1, 2, 9, 11\}$, function $f : X \to Y$ is defined by $f(a) = 2$, $f(b) = -1$, $f(c) = 2$, $f(d) = 9$. Find the domain and range of the f.

2. The curve in the diagram below represents the function $y = f(x)$, $-1 < x \le 4$. Find the domain and range of f.



3. Let $A = \{a, b, c, d, e\}$ and $B = \{4, 5, 8, 9, 12\}$, the definition of function $g : A \to B$ is given by the digram below. Find the domain, codomain and range of function g.



4. Let $A = \{-1, 0, 1, 2\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = 3x^2 - 2$, find the domain and range of f.

5. Let $A = \{-1, 0, 2, 5, 11\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = x^2 - x - 2$, find the domain and range of f.

6. Find the domain and range of the following functions:

   (a) $f(x) = x^3$

   (b) $g(x) = \sqrt{1 - x^2}$

   (c) $h(x) = \dfrac{1}{2x + 3}$

   (d) $k(x) = x^2 - 2x + 4$

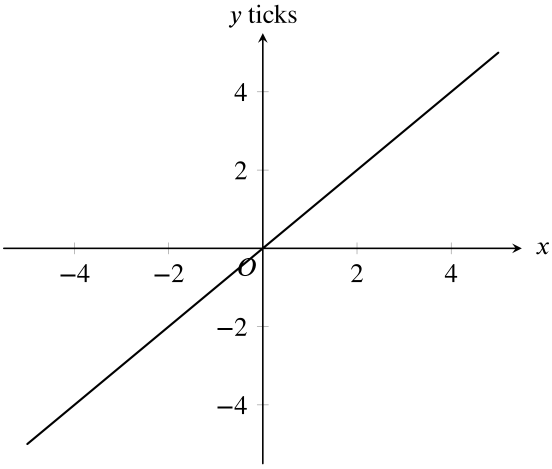
22.3 Graphs of Functions and Their Transformations

Graphs of Simple Functions

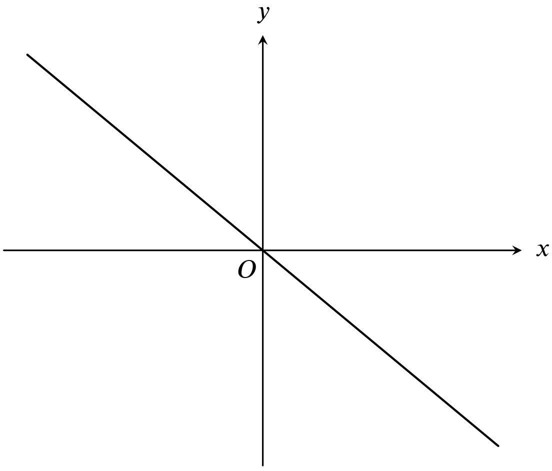
On a Cartesian plane, the graphs formed by all the point (x, y) that satisfied the equation $y = f(x)$ are called graphs of function f. Below are some examples of graphs of simple functions.

Note that any line that is parallel to the y-axis intersects the graph of a function at most once.

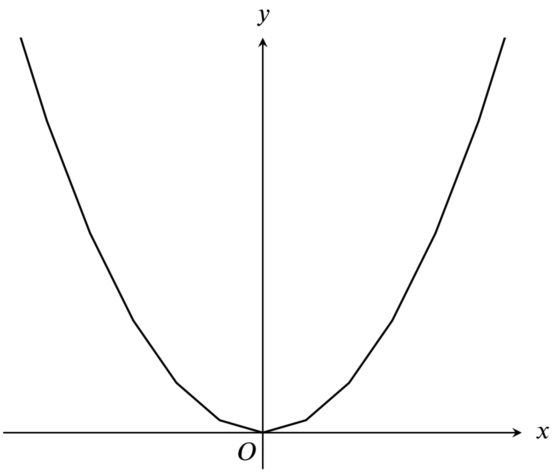
(a) $y = x$



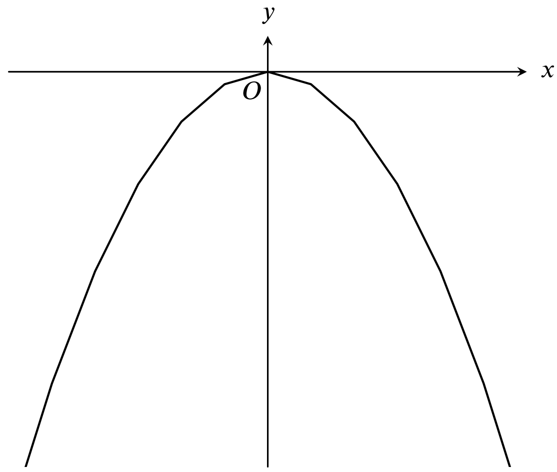
(b) $y = -x$



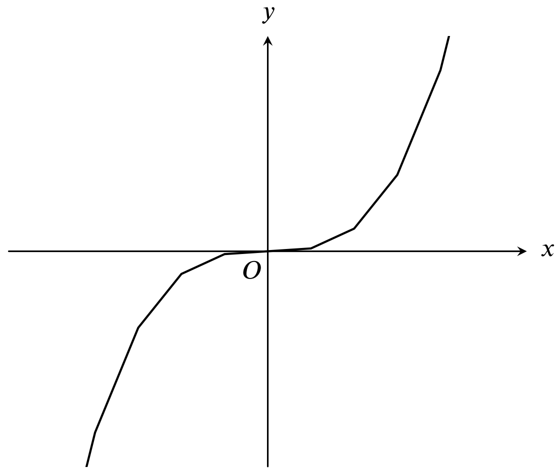
(c) $y = x^2$



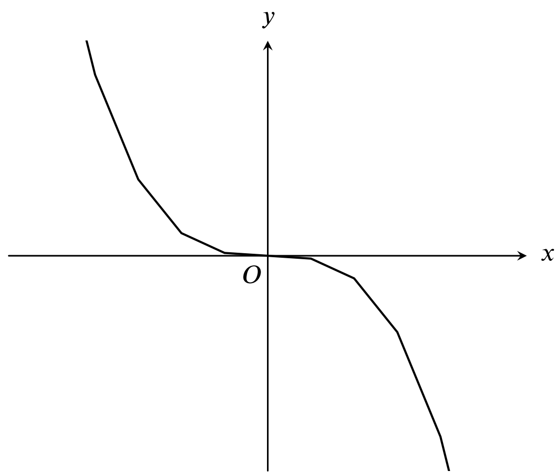
(d) $y = x^2$



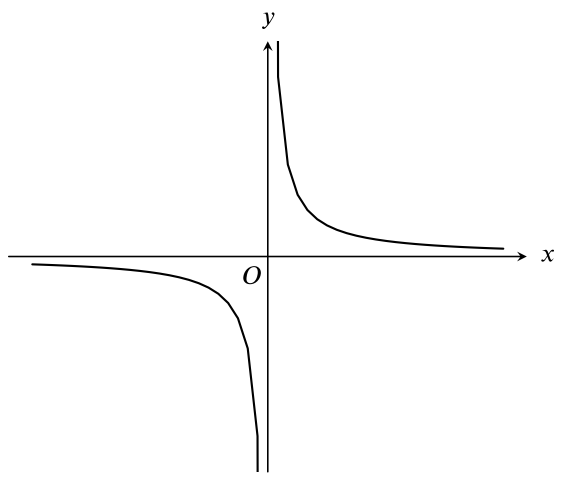
(e) $y = x^3$



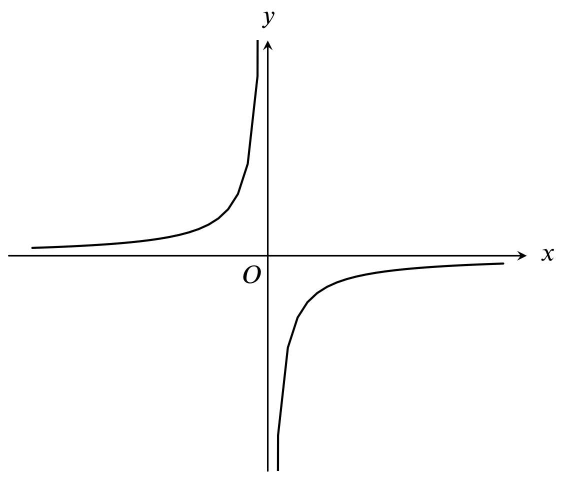
(f) $y = -x^3$



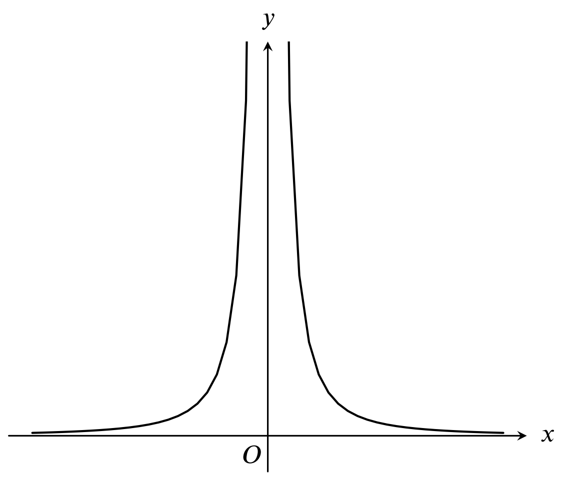
(g) $y = \dfrac{1}{x}$



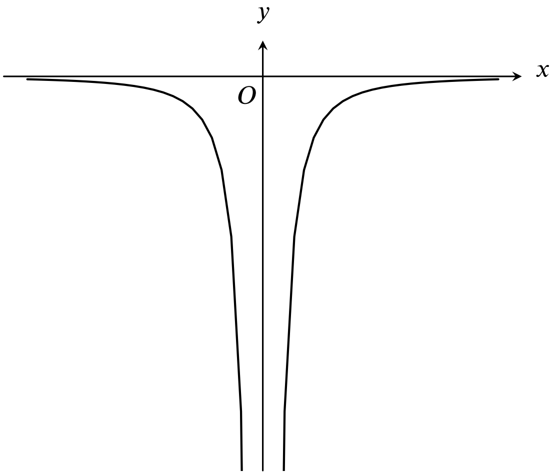
(h) $y = -\dfrac{1}{x}$



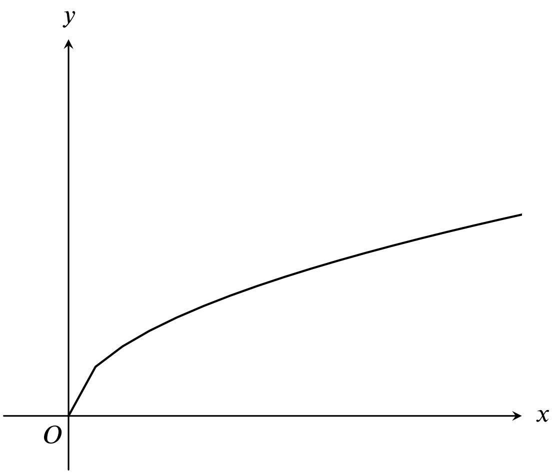
(i) $y = \dfrac{1}{x^2}$



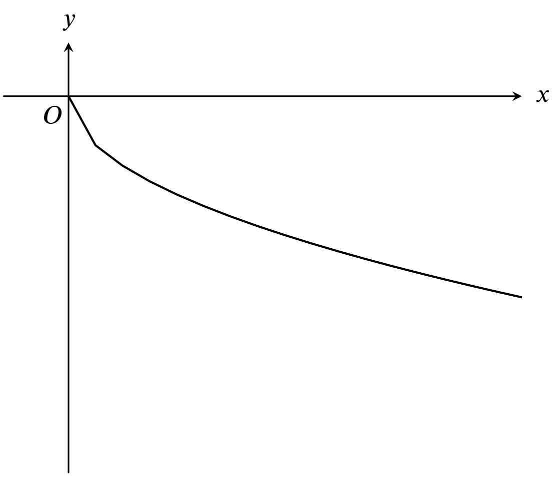
(j) $y = -\dfrac{1}{x^2}$



(k) $y = \sqrt{x}$

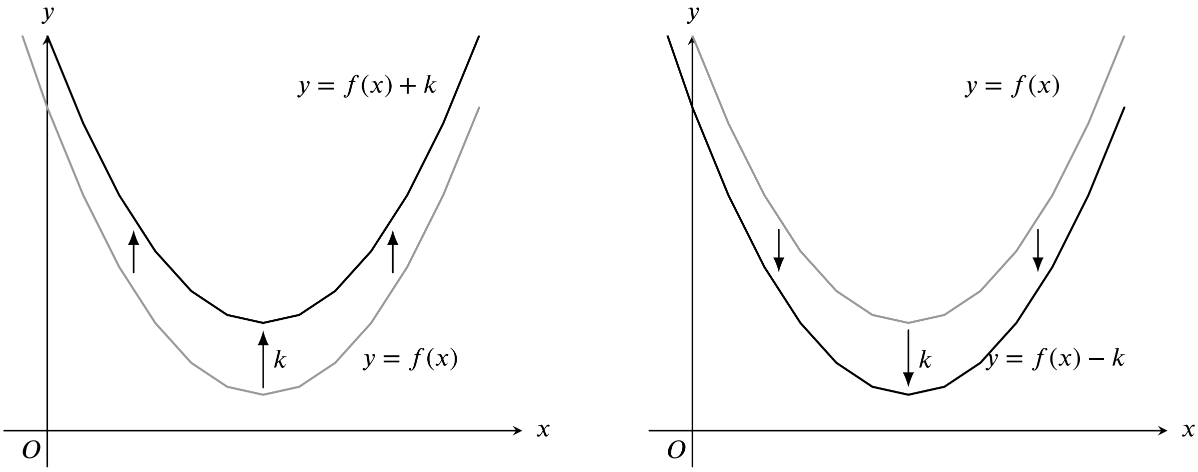


(l) $y = -\sqrt{x}$



Transformations of Graphs

- If $k > 0$, translate the graph of $y = f(x)$ vertically upwards by k units, the graph of $y = f(x) + k$ is obtained.
- If $k > 0$, translate the graph of $y = f(x)$ vertically downwards by k units, the graph of $y = f(x) - k$ is obtained.



- If $h > 0$, translate the graph of $y = f(x)$ horizontally to the right by h units, the graph of $y = f(x + h)$ is obtained.
- If $h > 0$, translate the graph of $y = f(x)$ horizontally to the left by h units, the graph of $y = f(x - h)$ is obtained.

- If $k > 0$, reflect the graph of $y = f(x)$ about the x-axis, the graph of $y = -f(x)$ is obtained.
- If $k > 0$, reflect the graph of $y = f(x)$ about the y-axis, the graph of $y = f(-x)$ is obtained.

If $a > 0$, zooming (when $a > 1$) or shrinking (when $0 < a < 1$) the graph of $y = f(x)$ by a factor of a in the y-direction, the graph of $y = af(x)$ is obtained.

If $a > 0$, shrinking (when $a > 1$) or zooming (when $0 < a < 1$) the graph of $y = f(x)$ by a factor of $\frac{1}{a}$ in the x-direction, the graph of $y = f(ax)$ is obtained.

22.4 Composite Functions

22.5 One to One Function, Onto Function and One to One Onto Function

22.6 Inverse Functions

Chapter 23

Exponents and Logarithms

23.1 Exponents

23.2 Logarithms

23.3 Arithmetic Properties of Logarithms and Base Changing Formula

23.4 Exponential Equations

23.5 Logarithmic Equations

23.6 Compound Interest and Annuity

Chapter 24

Limits

24.1 Concept of Limits

24.2 Limits of Functions

24.3 Arithmetic Properties of Limits of Functions

Chapter 25

Differentiation