Solution Book of Mathematic

Ssnior 2 Part I

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Contents

Gauss elimination can also be used to find the inverse of a matrix. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a invertible matrix, that is, $|A| \neq 0$. Now we arrange the matrix A and the identity matrix I into a 3 by 6 augmented matrix $A|I$ as follows:

$$\left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$

We then apply Gauss elimination to the augmented matrix $A|I$ to obtain the following matrix such that the left hand side of this matrix become an identity matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right)$$

where b_{ij} are constants, the right hand side of the augmented matrix is the inverse of A:

$$A^{-1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

11.0.1 Practice 15

Using the method of Gauss elimination, find the inverse of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & -4 \end{pmatrix}$.

Sol.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & -6 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -8 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to -\frac{1}{8} R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 + R_3]{R_2 \to R_2 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{17}{8} & -\frac{7}{8} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{5}{4} & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{17}{8} & -\frac{7}{8} & -\frac{1}{8} \\ -\frac{5}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{pmatrix}$$

11.0.2 Exercise 14.7

Solve the following system of linear equations using the method of Gauss elimination:

1. $\begin{cases} 3x - y - 14 = 0 \\ 2y + z - 5 = 0 \\ x - 5z + 10 = 0 \end{cases}$

Sol.

$$\begin{cases} 3x - y = 14 \\ 2y + z = 5 \\ x - 5z = -10 \end{cases}$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 0 & 14 \\ 0 & 2 & 1 & 5 \\ 1 & 0 & -5 & -10 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - 3R_3} \left(\begin{array}{ccc|c} 0 & -1 & 15 & 44 \\ 0 & 2 & 1 & 5 \\ 1 & 0 & -5 & -10 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 + 2R_1} \left(\begin{array}{ccc|c} 0 & -1 & 15 & 44 \\ 0 & 0 & 31 & 93 \\ 1 & 0 & -5 & -10 \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{1}{31} R_2} \left(\begin{array}{ccc|c} 0 & -1 & 15 & 44 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & -5 & -10 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 - 15R_2]{R_3 \to R_3 + 5R_2} \left(\begin{array}{ccc|c} 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 5 \end{array}\right)$$

$$\xrightarrow[R_1 \to -R_1]{R_3 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\therefore x = 5,\ y = 1,\ z = 3$$

2. $\begin{cases} x + y + z = 6 \\ x + 2y + 3z = 10 \\ 2x + 3y - 4z = 8 \end{cases}$

Sol.

$$
\begin{aligned}
&\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 2 & 3 & -4 & 8 \end{array}\right) \\
\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} &\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & -6 & -4 \end{array}\right) \\
\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - R_2} &\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -8 & -8 \end{array}\right) \\
\xrightarrow{R_3 \to -\frac{1}{8} R_3} &\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right) \\
\xrightarrow[R_1 \to R_1 + R_3]{R_2 \to R_2 - 2R_3} &\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \\
&\therefore x = 3,\ y = 2,\ z = 1
\end{aligned}
$$

3. $\begin{cases} -x + y + z = 5 \\ 2x - 7y + 4z = 1 \\ 2x - 5y + 3z = -2 \end{cases}$

Sol.

$$
\begin{aligned}
&\left(\begin{array}{ccc|c} -1 & 1 & 1 & 5 \\ 2 & -7 & 4 & 1 \\ 2 & -5 & 3 & -2 \end{array}\right) \\
\xrightarrow[R_2 \to R_2 + 2R_1]{R_3 \to R_3 + 2R_1} &\left(\begin{array}{ccc|c} -1 & 1 & 1 & 5 \\ 0 & -5 & 6 & 11 \\ 0 & -3 & 5 & 8 \end{array}\right) \\
\xrightarrow{R_2 \to R_2 - R3} &\left(\begin{array}{ccc|c} -1 & 1 & 1 & 5 \\ 0 & -2 & 1 & 3 \\ 0 & -3 & 5 & 8 \end{array}\right) \\
\xrightarrow[R_1 \to R_1 - R_2]{R_3 \to R_3 - 5R_2} &\left(\begin{array}{ccc|c} -1 & 3 & 0 & 2 \\ 0 & -2 & 1 & 3 \\ 0 & 7 & 0 & -7 \end{array}\right) \\
\xrightarrow[R_1 \to -R_1]{R_3 \to \frac{1}{7} R_3} &\left(\begin{array}{ccc|c} 1 & -3 & 0 & -2 \\ 0 & -2 & 1 & 3 \\ 0 & 1 & 0 & -1 \end{array}\right) \\
\xrightarrow[R_1 \to R_1 + 3R_3]{R_2 \to R_2 + 2R_3} &\left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{array}\right) \\
\xrightarrow{R_2 \leftrightarrow R_3} &\left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \\
&\therefore x = -5,\ y = -1,\ z = 1
\end{aligned}
$$

4. $\begin{cases} 4x - y - 7z = 0 \\ 5x - 2y - z = 1 \\ 3x + 3y + 5z = 2 \end{cases}$

Sol.

$$
\begin{aligned}
&\left(\begin{array}{ccc|c} 4 & -1 & -7 & 0 \\ 5 & -2 & -1 & 1 \\ 3 & 3 & 5 & 2 \end{array}\right) \\
\xrightarrow{R_2 \to R_2 - 2R_1} &\left(\begin{array}{ccc|c} 4 & -1 & -7 & 0 \\ -3 & 0 & 13 & 1 \\ 3 & 3 & 5 & 2 \end{array}\right) \\
\xrightarrow{R_3 \to R_3 + R_2} &\left(\begin{array}{ccc|c} 4 & -1 & -7 & 0 \\ -3 & 0 & 13 & 1 \\ 0 & 3 & 18 & 3 \end{array}\right) \\
\xrightarrow{R_3 \to \frac{1}{3} R_3} &\left(\begin{array}{ccc|c} 4 & -1 & -7 & 0 \\ -3 & 0 & 13 & 1 \\ 0 & 1 & 6 & 1 \end{array}\right) \\
\xrightarrow{R_1 \to R_1 + R_3} &\left(\begin{array}{ccc|c} 4 & 0 & -1 & 1 \\ -3 & 0 & 13 & 1 \\ 0 & 1 & 6 & 1 \end{array}\right) \\
\xrightarrow{R_3 \to 4R_3} &\left(\begin{array}{ccc|c} 4 & 0 & -1 & 1 \\ -12 & 0 & 52 & 4 \\ 0 & 1 & 6 & 1 \end{array}\right) \\
\xrightarrow{R_2 \to R_2 + 3R_1} &\left(\begin{array}{ccc|c} 4 & 0 & -1 & 1 \\ 0 & 0 & 49 & 7 \\ 0 & 1 & 6 & 1 \end{array}\right) \\
\xrightarrow{R_2 \to \frac{1}{49} R_2} &\left(\begin{array}{ccc|c} 4 & 0 & -1 & 1 \\ 0 & 0 & 1 & \frac{1}{7} \\ 0 & 1 & 6 & 1 \end{array}\right) \\
\xrightarrow[R_3 \to R_3 - 6R_2]{R_1 \to R_1 + R_2} &\left(\begin{array}{ccc|c} 4 & 0 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} \end{array}\right) \\
\xrightarrow[R_1 \to \frac{1}{4} R_1]{R_2 \leftrightarrow R_3} &\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{7} \\ 0 & 1 & 0 & \frac{1}{7} \\ 0 & 0 & 1 & \frac{1}{7} \end{array}\right) \\
&\therefore x = \frac{2}{7},\ y = \frac{1}{7},\ z = \frac{1}{7}
\end{aligned}
$$