Calculus

Melvin Chia

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Contents

Chapter 1

Limits

1.1 Arithmetic Properties of Limits

If $\lim_{x \to x_0} f(x) = A$, $\lim_{x \to x_0} g(x) = B$, then:

(a) $\lim_{x \to x_0} (f(x) \pm g(x)) = \lim_{x \to x_0} f(x) \pm \lim_{x \to x_0} g(x) = A \pm B$

(b) $\lim_{x \to x_0} (f(x) \cdot g(x)) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x) = A \cdot B$

(c) $\lim_{x \to x_0} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)} = \frac{A}{B}$, $(B \neq 0)$

(d) If k is a constant, then $\lim_{x \to x_0} k = k$

(e) If k is a constant, then $\lim_{x \to x_0} k \cdot f(x) = k \lim_{x \to x_0} f(x) = kA$

(f) If $n \in \mathbb{R}$, and $\lim_{x \to x_0} f(x) > 0$, then $\lim_{x \to x_0} [f(x)]^n = \left[\lim_{x \to x_0} f(x)\right]^n = A^n$

(g) If $\lim_{x \to x_0} f(x) = 0$, then $\lim_{x \to x_0} \frac{1}{f(x)} = \infty$

(h) L'Hopital's Rule: If $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ is indeterminate, then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$

Squeeze Theorem or Sandwich Rule

$$\begin{aligned} &\text{Near point } x_0, \\ &\text{If } f(x) \leq g(x) \leq h(x) \\ \text{and } &\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = A, \\ \text{then } &\lim_{x \to x_0} g(x) = A. \end{aligned}$$

1. $\lim_{x \to 3} 3x$

Sol.

$$\begin{aligned} \lim_{x \to 3} 3x &= 3 \cdot 3 \\ &= 9 \quad \square \end{aligned}$$

2. $\lim_{x \to -1} (x^2 + 4x)$

Sol.

$$\begin{aligned}\lim_{x\to-1}(x^2+4x) &= \lim_{x\to-1}x^2+\lim_{x\to-1}4x\\ &= (-1)^2+4(-1)\\ &= 1-4\\ &= -3 \quad \square\end{aligned}$$

3. $\lim_{x\to3}(9-x^2)$

Sol.

$$\begin{aligned}\lim_{x\to3}(9-x^2) &= \lim_{x\to3}9-\lim_{x\to3}x^2\\ &= 9-3^2\\ &= 9-9\\ &= 0 \quad \square\end{aligned}$$

4. $\lim_{n\to-2}(x^2-2x+1)$

Sol.

$$\begin{aligned}\lim_{n\to-2}(x^2-2x+1) &= \lim_{n\to-2}(x-1)^2\\ &= (-3)^2\\ &= 9 \quad \square\end{aligned}$$

5. $\lim_{x\to-4}x^2(x+2)$

Sol.

$$\begin{aligned}\lim_{x\to-4}x^2(x+2) &= \lim_{x\to-4}x^2\lim_{x\to-4}(x+2)\\ &= (-4)^2\cdot(-4+2)\\ &= 16\cdot(-2)\\ &= -32 \quad \square\end{aligned}$$

6. $\lim_{h\to2}(h^2\ -4h+4)$

Sol.

$$\begin{aligned}\lim_{h\to2}(h^2\ -4h+4) &= \lim_{h\to2}(h-2)^2\\ &= (2-2)^2\\ &= 0 \quad \square\end{aligned}$$

7. $\lim_{a\to-1}(a+3)(a-4)$

Sol.

$$\begin{aligned}\lim_{a\to-1}(a+3)(a-4) &= \lim_{a\to-1}(a+3)\lim_{a\to-1}(a-4)\\ &= (-1+3)\cdot(-1-4)\\ &= 2\cdot-5\\ &= -10 \quad \square\end{aligned}$$

8. $\lim_{x\to3}\frac{x^2-5}{x+2}$

Sol.

$$\begin{aligned}\lim_{x\to3}\frac{x^2-5}{x+2} &= \lim_{x\to3}\frac{x^2-5}{x+2}\\ &= \frac{3^2-5}{3+2}\\ &= \frac{4}{5} \quad \square\end{aligned}$$

9. $\lim_{x\to-3}\frac{(x+5)(x+3)}{x+3}$

Sol.

$$\begin{aligned}\lim_{x\to-3}\frac{(x+5)(x+3)}{x+3} &= \lim_{x\to-3}\frac{(x+5)(x+3)}{x+3}\\ &= \lim_{x\to-3}(x+5)\\ &= -3+5\\ &= 2 \quad \square\end{aligned}$$

10. $\lim_{x\to0}\frac{x^2+5x}{x}$

Sol.

$$\begin{aligned}\lim_{x\to0}\frac{x^2+5x}{x} &= \lim_{x\to0}\frac{x^2+5x}{x}\\ &= \lim_{x\to0}\frac{x(x+5)}{x}\\ &= \lim_{x\to0}(x+5)\\ &= 0+5\\ &= 5 \quad \square\end{aligned}$$

11. $\lim_{x\to-2}\frac{x^3+8}{x+2}$

Sol.

$$\begin{aligned}\lim_{x\to-2}\frac{x^3+8}{x+2} &= \lim_{x\to-2}\frac{(x+2)(x^2-x+4)}{x+2}\\ &= \lim_{x\to-2}(x^2-x+4)\\ &= (-2)^2-(-2)+4\\ &= 10 \quad \square\end{aligned}$$

12. $\lim_{x\to4}\frac{x^2-5x+6}{x-3}$

Sol.

$$\begin{aligned}\lim_{x\to4}\frac{x^2-5x+6}{x-3} &= \lim_{x\to4}\frac{(x-3)(x-2)}{x-3}\\ &= \lim_{x\to4}(x-2)\\ &= 4-2\\ &= 2 \quad \square\end{aligned}$$

13. $\lim_{x\to3}\frac{3x}{x+2}$

Sol.

$$\begin{aligned}\lim_{x\to3}\frac{3x}{x+2} &= \lim_{x\to3}\frac{3x}{x+2}\\ &= \frac{3(3)}{3+2}\\ &= \frac{9}{5} \quad \square\end{aligned}$$

14. $\lim_{x\to 5} \frac{x-5}{2x^2-9x-5}$

Sol.

$$\begin{aligned}\lim_{x\to 5} \frac{x-5}{2x^2-9x-5} &= \lim_{x\to 5} \frac{x-5}{(2x+1)(x-5)}\\ &= \lim_{x\to 5} \frac{1}{2x+1}\\ &= \frac{1}{2(5)+1}\\ &= \frac{1}{11} \quad \square\end{aligned}$$

15. $\lim_{x\to 1} \frac{x-1}{x^2+x-2}$

Sol.

$$\begin{aligned}\lim_{x\to 1} \frac{x-1}{x^2+x-2} &= \lim_{x\to 1} \frac{x-1}{(x-1)(x+2)}\\ &= \lim_{x\to 1} \frac{1}{x+2}\\ &= \frac{1}{1+2}\\ &= \frac{1}{3} \quad \square\end{aligned}$$

16. $\lim_{x\to 4} \frac{x-1}{x^2+x-2}$

Sol.

$$\begin{aligned}\lim_{x\to 4} \frac{x-1}{x^2+x-2} &= \lim_{x\to 4} \frac{x-1}{(x-1)(x+2)}\\ &= \lim_{x\to 4} \frac{1}{x+2}\\ &= \frac{1}{4+2}\\ &= \frac{1}{6} \quad \square\end{aligned}$$

17. $\lim_{x\to -2} \frac{x-2}{x^2-4}$

Sol.

$$\begin{aligned}\lim_{x\to -2} \frac{x-2}{x^2-4} &= \lim_{x\to -2} \frac{x-2}{(x+2)(x-2)}\\ &= \lim_{x\to -2} \frac{1}{x+2}\\ &= \infty \quad \square\end{aligned}$$

18. $\lim_{h\to 0} \frac{2x^2h \ +3h}{h}$

Sol.

$$\begin{aligned}\lim_{h\to 0} \frac{2x^2h \ +3h}{h} &= \lim_{h\to 0} \frac{h(2x^2+3)}{h}\\ &= \lim_{h\to 0} (2x^2+3)\\ &= 2x^2+3 \quad \square\end{aligned}$$

19. $\lim_{h\to 0} \frac{(2+h)^2-4}{h}$

Sol.

$$\begin{aligned}\lim_{h\to 0} \frac{(2+h)^2-4}{h} &= \lim_{h\to 0} \frac{[(2+h)+2][(2+h)-2]}{h}\\ &= \lim_{h\to 0} \frac{(4+h)h}{h}\\ &= \lim_{h\to 0} (4+h)\\ &= 4+0\\ &= 4 \quad \square\end{aligned}$$

20. $\lim_{h\to 0} \frac{(1+h)^3-1}{h}$

Sol.

$$\begin{aligned}&\lim_{h\to 0} \frac{(1+h)^3-1}{h}\\ &= \lim_{h\to 0} \frac{[(1+h)-1][(1+h)^2+(1+h)+1]}{h}\\ &= \lim_{h\to 0} \frac{h(1+2h+h^2+1+h+1)}{h}\\ &= \lim_{h\to 0} (h^2+3h+3)\\ &= (0)^2+3(0)+3\\ &= 3 \quad \square\end{aligned}$$

21. $\lim_{x\to -1} 2x(x^2-4)$

Sol.

$$\begin{aligned}\lim_{x\to -1} 2x(x^2-4) &= \lim_{x\to -1} 2x(x^2-4)\\ &= -2(-1)^2[(-1)^2-4]\\ &= -2(-3)\\ &= 6 \quad \square\end{aligned}$$

22. $\lim_{x\to 3} \frac{x^2+2}{x+1}$

Sol.

$$\begin{aligned}\lim_{x\to 3} \frac{x^2+2}{x+1} &= \frac{3^2+2}{3+1}\\ &= \frac{11}{4} \quad \square\end{aligned}$$

23. $\lim_{x\to 2} \left(x^2-3x+5\right)$

Sol.

$$\begin{aligned}\lim_{x\to 2} \left(x^2-3x+5\right) &= 2^2-3(2)+5\\ &= 3 \quad \square\end{aligned}$$

24. $\lim_{x\to 1} \frac{2x^2+1}{3x^2+4x-1}$

Sol.

$$\begin{aligned}\lim_{x\to 1} \frac{2x^2+1}{3x^2+4x-1} &= \frac{2(1)^2+1}{3(1)^2+4(1)-1}\\ &= \frac{3}{6}\\ &= \frac{1}{2} \quad \square\end{aligned}$$

25. $\lim_{x\to 1} \frac{x^2-5x+6}{x^2-9}$

Sol.

$$\begin{aligned}\lim_{x\to 1} \frac{x^2-5x+6}{x^2-9} &= \frac{(x-3)(x-2)}{(x+3)(x-3)} \\ &= \frac{x-2}{x+3} \\ &= \frac{1-2}{1+3} \\ &= -\frac{1}{4} \quad \square\end{aligned}$$

26. $\lim_{x\to -1} \frac{x^3+1}{x+1}$

Sol.

$$\begin{aligned}\lim_{x\to -1} \frac{x^3+1}{x+1} &= \lim_{x\to -1} \frac{(x+1)(x^2-x+1)}{x+1} \\ &= \lim_{x\to -1}(x^2-x+1) \\ &= (-1)^2-(-1)+1 \\ &= 3 \quad \square\end{aligned}$$

27. $\lim_{x\to 1} \frac{x^3-1}{x-1}$

Sol.

$$\begin{aligned}\lim_{x\to 1} \frac{x^3-1}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1} \\ &= \lim_{x\to 1}(x^2+x+1) \\ &= 1^2+1+1 \\ &= 3 \quad \square\end{aligned}$$

28. $\lim_{x\to 0} \frac{2x^3+3x^2}{x^3}$

Sol.

$$\begin{aligned}\lim_{x\to 0} \frac{2x^3+3x^2}{x^3} &= \lim_{x\to 0} \left(\frac{2x^3}{x^3}+\frac{3x^2}{x^3}\right) \\ &= \lim_{x\to 0} \left(2+\frac{3}{x}\right) \\ &= 2 \quad \square\end{aligned}$$

29. $\lim_{k\to 0} \frac{(x-k)^2-2kx^3}{x(x+k)}$

Sol.

$$\begin{aligned}\lim_{k\to 0} \frac{(x-k)^2-2kx^3}{x(x+k)} &= \frac{(x-0)^2-2(0)(x^3)}{x(x+0)} \\ &= \frac{x^2}{x^2} \\ &= 1 \quad \square\end{aligned}$$

30. $\lim_{x\to 1} \frac{x^2-2x+5}{x^2+7}$

Sol.

$$\begin{aligned}\lim_{x\to 1} \frac{x^2-2x+5}{x^2+7} &= \frac{(1)^2-2(1)+5}{(1)^2+7} \\ &= \frac{4}{8} \\ &= \frac{1}{2} \quad \square\end{aligned}$$

31. $\lim_{x\to -2} \frac{x^4-16}{x^3-2}$

Sol.

$$\begin{aligned}\lim_{x\to -2} \frac{x^4-16}{x^3-2} &= \frac{(-2)^4-16}{(-2)^3-2} \\ &= \frac{16-16}{-8-2} \\ &= \frac{0}{-10} \\ &= 0 \quad \square\end{aligned}$$

32. $\lim_{x\to 1} \frac{x-1}{x^2-1}$

Sol.

$$\begin{aligned}\lim_{x\to 1} \frac{x-1}{x^2-1} &= \lim_{x\to 1} \frac{x-1}{(x-1)(x+1)} \\ &= \lim_{x\to 1} \frac{1}{x+1} \\ &= \frac{1}{1+1} \\ &= \frac{1}{2} \quad \square\end{aligned}$$

33. $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$

Sol.

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} &= \lim_{x\to 0} \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)} \\ &= \lim_{x\to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)} \\ &= \lim_{x\to 0} \frac{x}{x(\sqrt{1+x}+1)} \\ &= \lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} \\ &= \frac{1}{\sqrt{1+0}+1} \\ &= \frac{1}{2}\end{aligned}$$

34. $\lim_{x\to 2} \frac{x^2+4}{x^2+1}$

Sol.

$$\begin{aligned}\lim_{x\to 2} \frac{x^2+4}{x^2+1} &= \frac{(2)^2+4}{(2)^2+1} \\ &= \frac{8}{5}\end{aligned}$$

35. $\lim_{x \to 0} \frac{x^2+3x+2}{x^2+2}$

Sol.

$$\begin{aligned}\lim_{x \to 0} \frac{x^2+3x+2}{x^2+2} &= \frac{(0)^2+3(0)+2}{(0)^2+2} \\ &= \frac{2}{2} \\ &= 1 \quad \square\end{aligned}$$

36. $\lim_{x \to 1} \frac{x^2-2x+1}{x^2 \; -1}$

Sol.

$$\begin{aligned}\lim_{x \to 1} \frac{x^2-2x+1}{x^2 \; -1} &= \frac{(x-1)^2}{(x-1)(x+1)} \\ &= \frac{x-1}{x+1} \\ &= \frac{1-1}{1+1} \\ &= 0 \quad \square\end{aligned}$$

$\mathbf{a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})}$

Proof.

$$\begin{aligned}
(a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) &= (a-b)\left(\sum_{k=0}^{n-1} a^{n-k-1}b^k\right) \\
&= a\sum_{k=0}^{n-1} a^{n-k-1}b^k - b\sum_{k=0}^{n-1} a^{n-k-1}b^k \\
&= \sum_{k=0}^{n-1} a^{n-k}b^k - \sum_{k=0}^{n-1} a^{n-k-1}b^{k+1} \\
&= a^n + \sum_{k=1}^{n-1} a^{n-k}b^k - \sum_{l=0}^{n-2} a^{n-l-1}b^{l+1} - b^n \\
&= a^n + \sum_{k=1}^{n-1} a^{n-k}b^k - \sum_{k=1}^{n-1} a^{n-k}b^k - b^n \quad (l = k-1) \\
&= a^n - b^n \quad \square
\end{aligned}$$

37. $\displaystyle\lim_{x\to a} \frac{x^n - a^n}{x-a} \quad (n \in \mathbb{N})$

Sol.

$$\begin{aligned}
\lim_{x\to a} \frac{x^n - a^n}{x-a} &= \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{(x-a)} \\
&= \lim_{x\to a}(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\
&= a^{n-1} + a^{n-2}a + \cdots + a^{n-2}a + a^{n-1} \\
&= a^{n-1} + a^{n-1} + \cdots + a^{n-1} \\
&= na^{n-1} \quad \square
\end{aligned}$$

38. $\displaystyle\lim_{x\to 1}(3x^2 - 6x + 5)$

Sol.

$$\begin{aligned}
\lim_{x\to 1}(3x^2 - 6x + 5) &= 3(1)^2 - 6(1) + 5 \\
&= 3 - 6 + 5 \\
&= 2 \quad \square
\end{aligned}$$

39. $\displaystyle\lim_{x\to 1} \frac{2x^2 - 1}{3x^3 - 6x^2 + 5}$

Sol.

$$\begin{aligned}
\lim_{x\to 1} \frac{2x^2 - 1}{3x^3 - 6x^2 + 5} &= \frac{2(1)^2 - 1}{3(1)^3 - 6(1)^2 + 5} \\
&= \frac{1}{2} \quad \square
\end{aligned}$$

40. $\displaystyle\lim_{x\to 1} \frac{x^2 - 1}{x^2 - 4x + 3}$

Sol.

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2 - 1}{x^2 - 4x + 3} &= \lim_{x\to 3} \frac{(x+1)(x-1)}{(x-1)(x-3)} \\
&= \lim_{x\to 3} \frac{x+1}{x-3} \\
&= -1 \quad \square
\end{aligned}$$

41. $\displaystyle\lim_{x\to 3} \frac{x^2 - 5x + 6}{7x^2 - 22x + 3}$

Sol.

$$\begin{aligned}
\lim_{x\to 3} \frac{x^2 - 5x + 6}{7x^2 - 22x + 3} &= \lim_{x\to 3} \frac{(x-3)(x-2)}{(7x-1)(x-3)} \\
&= \lim_{x\to 3} \frac{x-2}{7x-1} \\
&= \frac{3-2}{7(3)-1} \\
&= \frac{1}{20} \quad \square
\end{aligned}$$

42. $\displaystyle\lim_{x\to 3} \frac{x^3 - 2x^2 - 2x + 4}{x^3 + x^2 - 10x + 8}$

Sol.

$$\begin{aligned}\lim_{x\to 3}\frac{x^3-2x^2-2x+4}{x^3+x^2-10x+8} &= \lim_{x\to 3}\frac{x^2(x-2)-2(x-2)}{(x-2)(x+4)(x-1)}\\ &= \lim_{x\to 3}\frac{(x-2)(x^2-2)}{(x-2)(x+4)(x-1)}\\ &= \lim_{x\to 3}\frac{x^2-2}{(x+4)(x-1)}\\ &= \frac{3^2-2}{(3+4)(3-1)}\\ &= \frac{1}{2} \quad \square\end{aligned}$$

43. $\displaystyle\lim_{x\to 1}\frac{x^4+2x^2-3}{x^2-3x+2}$

Sol.

$$\begin{aligned}\lim_{x\to 1}\frac{x^4+2x^2-3}{x^2-3x+2} &= \lim_{x\to 3}\frac{(x-1)(x+1)(x^2+3)}{(x-2)(x-1)}\\ &= \lim_{x\to 3}\frac{(x+1)(x^2+3)}{x-2}\\ &= \frac{(1+1)(1^2+3)}{1-2}\\ &= \frac{8}{-1}\\ &= -8 \quad \square\end{aligned}$$

44. $\displaystyle\lim_{x\to 1}\frac{1-\sqrt[4]{x}}{1-\sqrt[3]{x}}$

Sol.

$$\begin{aligned}\lim_{x\to 1}\frac{1-\sqrt[4]{x}}{1-\sqrt[3]{x}} &= \lim_{x\to 1}\frac{\left(x^{\frac{1}{4}}-1\right)'}{\left(x^{\frac{1}{3}}-1\right)'}\\ &= \lim_{x\to 1}\frac{\frac{1}{4}x^{-\frac{3}{4}}}{\frac{1}{3}x^{-\frac{2}{3}}}\\ &= \lim_{x\to 1}\frac{\frac{1}{4x^{\frac{3}{4}}}}{\frac{1}{3x^{\frac{2}{3}}}}\\ &= \lim_{x\to 1}\frac{3x^{\frac{2}{3}}}{4x^{\frac{3}{4}}}\\ &= \frac{3(1)^{\frac{2}{3}}}{4(1)^{\frac{3}{4}}}\\ &= \frac{3}{4} \quad \square\end{aligned}$$

45. $\displaystyle\lim_{x\to 0}\frac{\sqrt[n]{1+x}-1}{x} \quad (n\in\mathbb{W})$

Sol.

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt[n]{1+x}-1}{x} &= \lim_{x\to 1}\frac{\left(\sqrt[n]{1+x}-1\right)'}{x'}\\ &= \lim_{x\to 1}\frac{\left(\sqrt[n]{1+x}\right)'-1'}{1}\\ &= \lim_{x\to 1}\left[(1+x)^{\frac{1}{n}}\right]'\\ &= \lim_{x\to 1}\frac{1}{n}\left[(1+x)^{\frac{1-n}{n}}\right]\\ &= \frac{1}{n}(1+0)^{\frac{1-n}{n}}\\ &= \frac{1}{n}(1)^{\frac{1-n}{n}}\\ &= \frac{1}{n} \quad \square\end{aligned}$$

46. $\displaystyle\lim_{x\to 1}\frac{2-\sqrt{x+3}}{x^2-1}$

Sol.

$$\begin{aligned}
\lim_{x\to 1}\frac{2-\sqrt{x+3}}{x^2-1} &= \lim_{x\to 1}\frac{\left(2-\sqrt{x+3}\right)'}{\left(x^2-1\right)'}\\
&= \lim_{x\to 1}\frac{2'-\left(\sqrt{x+3}\right)'}{2x}\\
&= \lim_{x\to 1}\frac{-\left[(x+3)^{\frac{1}{2}}\right]'}{2x}\\
&= \lim_{x\to 1}\frac{-\frac{1}{2}(x+3)^{-\frac{1}{2}}}{2x}\\
&= \lim_{x\to 1}\frac{-\frac{1}{2\sqrt{x+3}}}{2x}\\
&= \frac{-\frac{1}{2\sqrt{1+3}}}{2(1)}\\
&= \frac{-\frac{1}{4}}{2}\\
&= -\frac{1}{8}
\end{aligned}$$

47. $\displaystyle\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4}$

Sol.

$$\begin{aligned}
\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4} &= \lim_{x\to 16}\frac{\left(\sqrt[4]{x}-2\right)'}{\left(\sqrt{x}-4\right)'}\\
&= \lim_{x\to 16}\frac{\left(\sqrt[4]{x}\right)'-2'}{\left(\sqrt{x}\right)'-4'}\\
&= \lim_{x\to 16}\frac{\frac{1}{4}x^{-\frac{3}{4}}}{\frac{1}{2}x^{-\frac{1}{2}}}\\
&= \lim_{x\to 16}\frac{\frac{1}{4x^{\frac{3}{4}}}}{\frac{1}{2x^{\frac{1}{2}}}}\\
&= \lim_{x\to 16}\frac{x^{\frac{1}{2}}}{2x^{\frac{3}{4}}}\\
&= \frac{16^{\frac{1}{2}}}{2(16)^{\frac{3}{4}}}\\
&= \frac{4}{2(8)}\\
&= \frac{1}{4} \quad \square
\end{aligned}$$

48. $\displaystyle\lim_{x\to 2}(x^2+3x-1)$

Sol.

$$\begin{aligned}
\lim_{x\to 2}(x^2+3x-1) &= (2)^2+3(2)-1 && (1)\\
&= 4+6-1 && (2)\\
&= 9 \quad \square && (3)
\end{aligned}$$

49. $\displaystyle\lim_{x\to -1}\frac{x^2+2}{x^2+x+3}$

Sol.

$$\begin{aligned}
\lim_{x\to -1}\frac{x^2+2}{x^2+x+3} &= \frac{(-1)^2+2}{(-1)^2+(-1)+3}\\
&= \frac{1+2}{1-1+3}\\
&= 1 \quad \square
\end{aligned}$$

50. $\displaystyle\lim_{x\to -1}\frac{x^3+1}{x^2-1}$

Sol.

$$\begin{aligned}
\lim_{x\to -1}\frac{x^3+1}{x^2-1} &= \frac{(x+1)(x^2-x+1)}{(x+1)(x-1)}\\
&= \frac{x^2-x+1}{x-1}\\
&= \frac{(-1)^2-(-1)+1}{-1-1}\\
&= -\frac{3}{2} \quad \square
\end{aligned}$$

51. $\displaystyle\lim_{x\to 1}\frac{x^5-x^4}{x^3-x}$

Sol.

$$\begin{aligned}
\lim_{x\to 1}\frac{x^5-x^4}{x^3-x} &= \lim_{x\to 1}\frac{\left(x^5-x^4\right)'}{\left(x^3-x\right)'}\\
&= \lim_{x\to 1}\frac{5x^4-4x^3}{3x^2-1}\\
&= \frac{5(1)^4-4(1)^3}{3(1)^2-1}\\
&= \frac{1}{2} \quad \square
\end{aligned}$$

52. $\displaystyle\lim_{x\to a}\frac{x^2+ax-2a^2}{x^2-a^2}, a\neq 0$

Sol.

$$\begin{aligned}
\lim_{x\to a}\frac{x^2+ax-2a^2}{x^2-a^2} &= \lim_{x\to a}\frac{(x+2a)(x-a)}{(x+a)(x-a)}\\
&= \lim_{x\to a}\frac{x+2a}{x+a}\\
&= \frac{a+2a}{a+a}\\
&= \frac{3}{2} \quad \square
\end{aligned}$$

53. $\displaystyle\lim_{x\to a}\frac{\sqrt{3x-a}-\sqrt{x+a}}{x-a}$

Sol.

$$\begin{aligned}
&\lim_{x\to a}\frac{\sqrt{3x-a}-\sqrt{x+a}}{x-a}\\
&=\lim_{x\to a}\frac{(\sqrt{3x-a}-\sqrt{x+a})'}{(x-a)'}\\
&=\lim_{x\to a}\left[(\sqrt{3x-a})'-(\sqrt{x+a})'\right]\\
&=\lim_{x\to a}\left[(\sqrt{3x-a})'-(\sqrt{x+a})'\right]\\
&=\lim_{x\to a}\left[\frac{1}{2\sqrt{3x-a}}(3x-a)'-\frac{1}{2\sqrt{x+a}}\right]\\
&=\lim_{x\to a}\left[\frac{3}{2\sqrt{3x-a}}-\frac{1}{2\sqrt{x+a}}\right]\\
&=\frac{3}{2\sqrt{3a-a}}-\frac{1}{2\sqrt{a+a}}\\
&=\frac{2}{2\sqrt{2a}}\\
&=\frac{1}{\sqrt{2a}}
\end{aligned}$$

54. Given that $f(x) = x^2 - 3x$, find $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Sol.

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&=\lim_{h\to 0}\frac{(x+h)^2-3(x+h)-x^2+3x}{h}\\
&=\lim_{h\to 0}\frac{(x^2+2xh+h^2)-3x-3h-x^2+3x}{h}\\
&=\lim_{h\to 0}\frac{2xh+h^2-3h}{h}\\
&=\lim_{h\to 0}(2x+2h-3)\\
&=2x-3 \quad \square
\end{aligned}$$

55. $\displaystyle\lim_{x\to 2}\sqrt{2x^2+1}$

Sol.

$$\begin{aligned}
\lim_{x\to 2}\sqrt{2x^2+1}&=\sqrt{2(2)^2+1}\\
&=\sqrt{9}\\
&=3 \quad \square
\end{aligned}$$

56. $\displaystyle\lim_{x\to 7}\frac{x^2\sqrt{x+2}}{x^2+14}$

Sol.

$$\begin{aligned}
\lim_{x\to 7}\frac{x^2\sqrt{x+2}}{x^2+14}&=\frac{(7)^2\sqrt{7+2}}{(7)^2+14}\\
&=\frac{49(3)}{49+14}\\
&=\frac{7}{3}
\end{aligned}$$

57. $\displaystyle\lim_{x\to 0}\frac{\sqrt{3x+4}-2}{x}$

Sol.

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{3x+4}-2}{x}&=\lim_{x\to 0}\frac{\left(\sqrt{3x+4}\right)'-2'}{x'}\\
&=\lim_{x\to 0}\left[\frac{1}{2\sqrt{3x+4}}(3x+4)'\right]\\
&=\lim_{x\to 0}\left(\frac{3}{2\sqrt{3x+4}}\right)\\
&=\frac{3}{2\sqrt{3(0)+4}}\\
&=\frac{3}{4} \quad \square
\end{aligned}$$

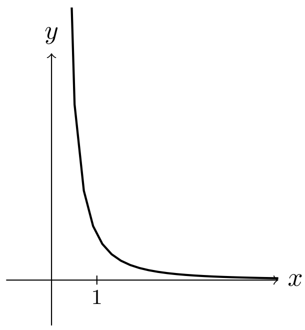
58. $\displaystyle\lim_{x\to 0}\frac{1}{x^2}$

Sol.

$$\lim_{x\to 0}\frac{1}{x^2}=\infty \quad \square$$

59. $\displaystyle\lim_{x\to 1}\frac{1}{x-1}$

Sol.



$$\lim_{x\to 1}\frac{1}{x-1}=\infty$$

60. $\displaystyle\lim_{x\to 1}\frac{4x-3}{x^2-5x+4}$

Sol.

$$\begin{aligned}
\lim_{x\to 1}\frac{4x-3}{x^2-5x+4}&=\lim_{x\to 1}\frac{(4x-3)'}{(x^2-5x+4)'}\\
&=\lim_{x\to 1}\frac{4}{2x-5}\\
&=\frac{4}{2(1)-5}\\
&=-\frac{4}{3}
\end{aligned}$$

61. $\displaystyle\lim_{x\to\infty}\frac{3x^3-4x^2+2}{7x^3+5x^2-3}$

Sol.

$$\lim_{x\to\infty}\frac{3x^3-4x^2+2}{7x^3+5x^2-3}=\lim_{x\to\infty}\frac{\frac{3x^3}{x^3}-\frac{4x^2}{x^3}+\frac{2}{x^3}}{\frac{7x^3}{x^3}+\frac{5x^2}{x^3}-\frac{3}{x^3}}$$
$$=\lim_{x\to\infty}\frac{3-\frac{4}{x}+\frac{2}{x^3}}{7+\frac{5}{x}-\frac{3}{x^3}}$$
$$=\frac{3-0+0}{7+0-0}$$
$$=\frac{3}{7}\quad\square$$

62. $\lim\limits_{n\to\infty} x^2$

Sol.

$$\lim_{n\to\infty} x^2=\infty\quad\square$$

63. $\lim\limits_{x\to\infty}\frac{3x^2-2x-1}{2x^3-x^2+5}$

Sol.

$$\lim_{x\to\infty}\frac{3x^2-2x-1}{2x^3-x^2+5}=\lim_{x\to\infty}\frac{\frac{3x^2}{x^3}-\frac{2x}{x^3}-\frac{1}{x^3}}{\frac{2x^3}{x^3}-\frac{x^2}{x^3}+\frac{5}{x^3}}$$
$$=\lim_{x\to\infty}\frac{\frac{3}{x}-\frac{2}{x^2}-\frac{1}{x^3}}{2-\frac{1}{x}+\frac{5}{x^3}}$$
$$=\frac{0-0-0}{2-0+0}$$
$$=0\quad\square$$

64. $\lim\limits_{n\to\infty}\frac{1}{n^2}(1+2+3+\cdots+n)$

Sol.

$$\lim_{n\to\infty}\frac{1}{n^2}(1+2+3+\cdots+n)=\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}k$$
$$=\lim_{n\to\infty}\frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$$
$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{n+1}{2}\right)$$
$$=\lim_{n\to\infty}\frac{n+1}{2n}$$
$$=\lim_{n\to\infty}\frac{\frac{n}{n}+\frac{1}{n}}{\frac{2n}{n}}$$
$$=\frac{1+0}{2}$$
$$=\frac{1}{2}\quad\square$$

65. $\lim\limits_{n\to\infty}\left[\frac{1+2+3+\cdots+n}{n+2}-\frac{n}{2}\right]$

Sol.

$$\lim_{n\to\infty}\left[\frac{1+2+3+\cdots+n}{n+2}-\frac{n}{2}\right]$$
$$=\lim_{n\to\infty}\left[\frac{\sum\limits_{k=1}^{n}k}{n+2}-\frac{n}{2}\right]$$
$$=\lim_{n\to\infty}\left[\frac{n(n+1)}{2(n+2)}-\frac{n}{2}\right]$$
$$=\lim_{n\to\infty}\left[\frac{n(n+1)-n(n+2)}{2(n+2)}\right]$$
$$=\lim_{n\to\infty}\left[\frac{n(n+1-n-2)}{2(n+2)}\right]$$
$$=\lim_{n\to\infty}\left[-\frac{\frac{n}{n}}{\frac{2n}{n}+\frac{4}{n}}\right]$$
$$=-\frac{1}{2+0}$$
$$=-\frac{1}{2}\quad\square$$

66. $\lim\limits_{n\to\infty}\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}\right]$

Sol.

$$\lim_{n\to\infty}\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}\right]$$
$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k(k+1)}$$
$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$
$$=\lim_{n\to\infty}\left[\sum_{k=1}^{n}\left(\frac{1}{k}\right)-\sum_{k=1}^{n}\left(\frac{1}{k+1}\right)\right]$$
$$=\lim_{n\to\infty}\left[\sum_{k=1}^{n}\left(\frac{1}{k}\right)-\sum_{k=2}^{n+1}\left(\frac{1}{k}\right)\right]$$
$$=\lim_{n\to\infty}\left[1+\sum_{k=2}^{n}\left(\frac{1}{k}\right)-\sum_{k=2}^{n}\left(\frac{1}{k}\right)-\frac{1}{n+1}\right]$$
$$=\lim_{n\to\infty}\left[1-\frac{1}{n+1}\right]$$
$$=\lim_{n\to\infty}\left[\frac{n+1-1}{n+1}\right]$$
$$=\lim_{n\to\infty}\left[\frac{n}{n+1}\right]$$
$$=\lim_{n\to\infty}\left[\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right]$$
$$=\frac{1}{1+0}$$
$$=1\quad\square$$

67. $\lim\limits_{x\to\infty}\frac{5x^3+4x^2-6x+2}{8x^3-7x^2+4x-1}$

Sol.

$$\begin{aligned}
&\lim_{x\to\infty}\frac{5x^3+4x^2-6x+2}{8x^3-7x^2+4x-1}\\
&=\lim_{x\to\infty}\frac{\dfrac{5x^3}{x^3}+\dfrac{4x^2}{x^3}-\dfrac{6x}{x^3}+\dfrac{2}{x^3}}{\dfrac{8x^3}{x^3}-\dfrac{7x^2}{x^3}+\dfrac{4x}{x^3}-\dfrac{1}{x^3}}\\
&=\lim_{x\to\infty}\frac{5+\dfrac{4}{x}-\dfrac{6}{x^2}+\dfrac{2}{x^3}}{8-\dfrac{7}{x}+\dfrac{4}{x^2}-\dfrac{1}{x^3}}\\
&=\frac{5+0-0+0}{8-0+0-0}\\
&=\frac{5}{8}\quad\square
\end{aligned}$$

68. $\displaystyle\lim_{x\to\infty}\frac{x^4-2x^3+x^2+3}{x^5-x^4+1}$

Sol.

$$\begin{aligned}
&\lim_{x\to\infty}\frac{x^4-2x^3+x^2+3}{x^5-x^4+1}\\
&=\lim_{x\to\infty}\frac{\dfrac{x^4}{x^5}-\dfrac{2x^3}{x^5}+\dfrac{x^2}{x^5}+\dfrac{3}{x^5}}{\dfrac{x^5}{x^5}-\dfrac{x^4}{x^5}+\dfrac{1}{x^5}}\\
&=\lim_{x\to\infty}\frac{\dfrac{1}{x}-\dfrac{2}{x^2}+\dfrac{1}{x^3}+\dfrac{3}{x^5}}{1-\dfrac{1}{x}+\dfrac{1}{x^5}}\\
&=\frac{0-0+0+0}{1-0+0}\\
&=0\quad\square
\end{aligned}$$

69. $\displaystyle\lim_{x\to\infty}\frac{x^3-8x^2+4x-1}{x^2-6x+3}$

Sol.

$$\begin{aligned}
&\lim_{x\to\infty}\frac{x^3-8x^2+4x-1}{x^2-6x+3}\\
&=\lim_{x\to\infty}\frac{\dfrac{x^3}{x^3}-\dfrac{8x^2}{x^3}+\dfrac{4x}{x^3}-\dfrac{1}{x^3}}{\dfrac{x^2}{x^3}-\dfrac{6x}{x^3}+\dfrac{3}{x^3}}\\
&=\lim_{x\to\infty}\frac{1-\dfrac{8x}{x}+\dfrac{4}{x^2}-\dfrac{1}{x^3}}{\dfrac{1}{x}-\dfrac{6}{x^2}+\dfrac{3}{x^3}}\\
&=\lim_{x\to\infty}\frac{1-\dfrac{8x}{x}+\dfrac{4}{x^2}-\dfrac{1}{x^3}}{\dfrac{1}{x}-\dfrac{6}{x^2}+\dfrac{3}{x^3}}\\
&=\frac{1-0+0-0}{0-0+0}\\
&=\infty\quad\square
\end{aligned}$$

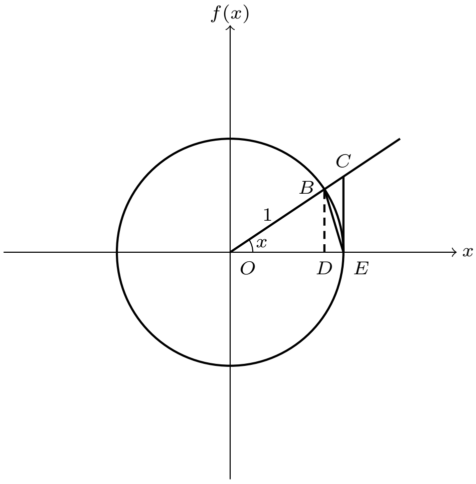
70. $\displaystyle\lim_{x\to\infty}(\sqrt{x^4+1}-x^2)$

Sol.

$$\begin{aligned}
\lim_{x\to\infty}(\sqrt{x^4+1}-x^2) &= \lim_{x\to\infty}\sqrt{x^4+1}-x^2\\
&=\lim_{x\to\infty}\frac{1}{\sqrt{x^4+1}+x^2}\\
&=\frac{\lim_{x\to\infty}1}{\lim_{x\to\infty}\left(\sqrt{x^4+1}+x^2\right)}\\
&=\frac{1}{\infty}\\
&=0\quad\square
\end{aligned}$$

$\lim_{\mathbf{x}\to\mathbf{0}} \frac{\sin\mathbf{x}}{\mathbf{x}} = \mathbf{1}$

Proof.



From the diagram above, we can see that:

$$BD = \sin x$$
$$\widehat{\mathrm{BD}} = x$$
$$EC = \tan x$$

Also, the area of sector EOB is greater than that of $\triangle EOB$, but less than that of $\triangle EOC$. That is,

$$(\text{area of sector } EOB) < (\text{area of } \triangle EOB) < (\text{area of } \triangle EOC)$$

Hence $\quad \frac{1}{2}\times 1\times \sin x < \frac{1}{2}\times 1^2 < \frac{1}{2}\times 1 \times \tan x$

$\therefore \quad \sin x < x < \tan x$

Dividing by $\sin x$ ($\sin x \neq 0$), we get

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

But $\quad \lim_{x\to 0^+} \frac{1}{\cos x} = \frac{x}{\lim_{x\to 0^+} \sin x} = 1$

Hence, $\lim_{x\to 0^+} \frac{x}{\sin x}$ is in between 1 and $\lim_{x\to 0^+} \frac{1}{\cos x}$, which is in between 1 and 1. Therefore, $\lim_{x\to 0^+} \frac{x}{\sin x}$ must be equal to 1 (Squeeze Theorem).

That is $\quad \lim_{x\to 0^+} \frac{x}{\sin x} = 1,$

$\therefore \quad \lim_{x\to 0^+} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{1}{\frac{x}{\sin x}} = \lim_{x\to 0^+} \frac{1}{\lim_{x\to 0^+} \frac{x}{\sin x}} = 1 \quad \square$