Praktis 3
Integration

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Praktis Formatif

3.1 Integration as the Inverse of Differentiation

1. (a) Given $\frac{d}{dx}(2x^3+5x^2-7x)=6x^2+10x-7$, find $\int 6x^2+10x-7\,dx$.

   Sol.

$$\int 6x^2+10x-7\,dx = 2x^3+5x^2-7x \quad \square$$

   (b) Given $\frac{d}{dx}(5x^4+3x^2+x)=20x^3+6x+1$, find $\int 20x^3+6x+1\,dx$.

   Sol.

$$\int 20x^3+6x+1\,dx = 5x^4+3x^2+x \quad \square$$

2. (a) Given $\frac{d}{dx}(4x-5x^2+2x^3)=4-10x+6x^2$, find $\int 2-5x+3x^2\,dx$.

   Sol.

$$\begin{aligned}\int 2-5x+3x^2\,dx &= \frac{2}{2}\int 2-5x+3x^2\,dx\\ &= \frac{1}{2}\int 4-10x+6x^2\,dx\\ &= \frac{1}{2}(4x-5x^2+2x^3)\\ &= 2x-\frac{5}{2}x^2+x^3 \quad \square\end{aligned}$$

   (b) Given $\frac{d}{dx}\left(2x-\frac{3}{x^4}\right)=2+\frac{12}{x^5}$, find $\int 6+\frac{36}{x^5}\,dx$

   Sol.

$$\begin{aligned}\int 6+\frac{36}{x^5}\,dx &= 6\int 1+\frac{6}{x^5}\,dx\\ &= 3\int 2+\frac{12}{x^5}\,dx\\ 3&\left(2x-\frac{3}{x^4}\right)\\ &= 6x-\frac{9}{x^4} \quad \square\end{aligned}$$

   (c) Given $f(x)=\frac{d}{dx}[g(x)]$, find $\int 2f(x)\,dx$.

   Sol.

$$\begin{aligned}\int 2f(x)\,dx &= 2\int f(x)\,dx\\ &= 2g(x) \quad \square\end{aligned}$$

   (d) Differentiate $\frac{2x^2}{3x-1}$ with respect to x and hence, find $\int \frac{6x(3x-2)}{(3x-1)^2}\,dx$.

   Sol.

$$\begin{aligned}\frac{d}{dx}\left(\frac{2x^2}{3x-1}\right) &= \frac{4x(3x-1)-3(2x^2)}{(3x-1)^2}\\ &= \frac{12x^2-4x-6x^2}{(3x-1)^2}\\ &= \frac{6x^2-4x}{(3x-1)^2}\\ &= \frac{2x(3x-2)}{(3x-1)^2}\\ \int \frac{6x(3x-2)}{(3x-1)^2}\,dx &= 3\int \frac{2x(3x-2)}{(3x-1)^2}\,dx\\ &= 3\left(\frac{2x^2}{3x-1}\right)\\ &= \frac{6x^2}{3x-1} \quad \square\end{aligned}$$

3. The daily production of bread of a bakery shop is given b y the function $R(x)=-50(x^2-12x)$, where x represents the number of bakers who work in the shop with condition x is not more than 6.

   (a) Find the rate of daily production of bread in terms of x.

   Sol.

$$R'(x) = -100x+600 \quad \square$$

   (b) If the rate of daily production of bread becomes $300-50x$ on a particular day, calculate the revenue of the bakery shop if all the loaves of bread baked by three bakers on that day are sold out at a price of RM5.50 for each loaf.

   Sol.

$$\begin{aligned}\int 300-50x\,dx &= \frac{1}{2}\int(600-100x)\,dx\\ &= \frac{1}{2}(-50x^2+600x)\\ &= -25x^2+300x\\ R(3) &= -25(3)^2+300(3)\\ &= -225+900\\ &= 675\end{aligned}$$

$$\begin{aligned}\text{Revenue} &= 675\times 5.50\\ &= \text{RM}3712.50 \quad \square\end{aligned}$$

4. Given $f(x)=x^4-2x^3$ and $f'(x)=4x^3-6x^2$. Express $f'(x)\int f'(x)\,dx$ in factored form.

   Sol.

$$\begin{aligned}f'(x)\int f'(x)\,dx &= (4x^3-6x^2)(x^4-2x^3)\\ &= 2x^5(2x-3)(x-2) \quad \square\end{aligned}$$

5. Given $y = \dfrac{2x-6}{x}$.

   (a) Find $\dfrac{dy}{dx}$.

   Sol.

$$\begin{aligned} \frac{dy}{dx} &= \frac{2x-2x-6}{x^2} \\ &= -\frac{6}{x^2} \quad \square \end{aligned}$$

   (b) Solve $4 + \displaystyle\int \left(\frac{dy}{dx}\right) dx = 0$.

   Sol.

$$\begin{aligned} 4 + \int \left(\frac{dy}{dx}\right) dx &= 0 \\ 4 + \int \left(-\frac{6}{x^2}\right) dx &= 0 \\ 4 + \frac{2x-6}{x} &= 0 \\ 4x + 2x - 6 &= 0 \\ 6x &= 6 \\ x &= 1 \quad \square \end{aligned}$$

6. Given $f'(x) = g(x)$. Find $\dfrac{3f(x)}{\displaystyle\int g(x)dx}$.

   Sol.

$$\begin{aligned} f'(x) &= g(x) \\ f(x) &= \int g(x)dx \\ \frac{3f(x)}{\displaystyle\int g(x)dx} &= \frac{3f(x)}{f(x)} \\ &= 3 \quad \square \end{aligned}$$

7. The population of town A is given by a function $P(t) = \frac{5}{6}(2.72^{1.2t}) - t^2 + 1495$ and the population continues to increase at the rate of $2.72^{1.2t} - 2t$ people per year where t is the number of years. Given that the population of town b increases at twice the rate of the population of town A based on the same model, find, to the nearest integer,

   (a) the rate of increase of the population of town B at $t = 5$ years.

   Sol.

$$\begin{aligned} P'_B(5) &= 2[2.72^{1.2(5)} - 2(5)] \\ &= 2[404.96 - 10] \\ &= 2(394.96) \\ &= 789.92 \\ &= 790 \text{ people per year} \quad \square \end{aligned}$$

   (b) the population of town B after 5 years.

   Sol.

$$\begin{aligned} P_B(5) &= 2\left[\frac{5}{6}(2.72^{1.2\cdot 5}) - (5)^2 + 1495\right] \\ &= \frac{5}{3}(2.72^6) - 50 + 2990 \\ &= 3614.93 \\ &= 3615 \text{ people} \quad \square \end{aligned}$$

3.2 Indefinite Integral

8. By using the indefinite integral formula, find the integral of each of the following constants or algebraic functions.

   (a) $\displaystyle\int 3\,dx$

   Sol.

$$\int 3\,dx = 3x + C \quad \square$$

   (b) $\displaystyle\int 24x\,dx$

   Sol.

$$\int 24x\,dx = 12x^2 + C \quad \square$$

   (c) $\displaystyle\int 6x^2\,dx$

   Sol.

$$\int 6x^2\,dx = 2x^3 + C \quad \square$$

   (d) $\displaystyle\int 3x^2 + 4x\,dx$

   Sol.

$$\int 3x^2 + 4x\,dx = x^3 + 2x^2 + C \quad \square$$

   (e) $\displaystyle\int \frac{2}{x^4}\,dx$

   Sol.

$$\int \frac{2}{x^4}\,dx = -\frac{2}{x^3} + C \quad \square$$

   (f) $\displaystyle\int x^2(x-3)\,dx$

   Sol.

$$\begin{aligned} \int x^2(x-3)\,dx &= \int x^3 - 3x^2\,dx \\ &= \frac{1}{4}x^4 - x^3 + C \quad \square \end{aligned}$$

   (g) $\displaystyle\int (x+2)(2x^4-1)\,dx$

   Sol.

$$\begin{aligned} &\int (x+2)(2x^4-1)\,dx \\ &= \int 2x^5 - x + 4x^4 - 2 \\ &= \frac{1}{3}x^6 + \frac{4}{5}x^5 - \frac{1}{2}x^2 - 2x + C \quad \square \end{aligned}$$

(h) $\displaystyle\int \frac{x^2+3x+2}{x+2}\,dx$

Sol.

$$\begin{aligned}\int \frac{x^2+3x+2}{x+2}\,dx &= \int \frac{(x+2)(x+1)}{x+2}\,dx\\ &= \int x+1\,dx\\ &= \frac{1}{2}x^2+x+C \quad \square\end{aligned}$$

9. Find the indefinite integral for each of the following by using

(a) the substitution method.

(b) the indefinite integral formula.

i. $\displaystyle\int \frac{2}{(x+2)^5}\,dx$

Sol.

(a) Let $v=(x+2)$.

$$\begin{aligned}\int \frac{2}{(x+2)^5}\,dx &= \int \frac{2}{v^5}\,dv\\ &= \int 2v^{-5}\,dv\\ &= -\frac{1}{2}v^{-4}+C\\ &= -\frac{1}{2v^4}+C\\ &= -\frac{1}{2(x+2)^4}+C \quad \square\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{2}{(x+2)^5}\,dx &= \int 2(x+2)^{-5}\,dx\\ &= 2\int (x+2)^{-5}\,dx\\ &= 2\left[\frac{(x+2)^{-4}}{-4}\right]+C\\ &= -\frac{1}{2(x+2)^4}+C \quad \square\end{aligned}$$

ii. $\displaystyle\int \frac{3}{5}(3x+2)^8\,dx$

Sol.

(a) Let $v=3x+2$, $\dfrac{dv}{dx}=3$.

$$\begin{aligned}\int \frac{3}{5}(3x+2)^8\,dx &= \int \frac{3}{5}v^8\,dv\\ &= \int \frac{3}{5}v^8\frac{dv}{3}\\ &= \int \frac{1}{5}v^8\,dv\\ &= \frac{1}{45}v^9+C\\ &= \frac{(3x+2)^9}{45}+C \quad \square\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{3}{5}(3x+2)^8\,dx &= \frac{3}{5}\int (3x+2)^8\,dx\\ &= \frac{3}{5}\left[\frac{(3x+2)^9}{27}\right]+C\\ &= \frac{(3x+2)^9}{45}+C \quad \square\end{aligned}$$

10. Determine the equation of a curve based on the following information.

(a) The gradient function of the curve is $\dfrac{dy}{dx}=3x^2+x-2$ and it passes through the point $p(2,15)$.

Sol.

$$\begin{aligned}\frac{dy}{dx} &= 3x^2+x-2\\ y &= \int 3x^2+x-2\,dx\\ &= x^3+\frac{x^2}{2}-2x+C\end{aligned}$$

When $x=2$, $y=15$,

$$\begin{aligned}15 &= 2^3+\frac{2^2}{2}-2(2)+C\\ 15 &= 8+2-4+C\\ 15 &= 6+C\\ C &= 9\end{aligned}$$

Hence, the equation of the curve is $y=x^3+\dfrac{x^2}{2}-2x+9$. $\square$

(b) The gradient function of the curve is $f'(x)=2x+9$ and $f(3)=21$.

Sol.

$$\begin{aligned}f'(x) &= 2x+9\\ f(x) &= \int 2x+9\,dx\\ &= x^2+9x+C\\ f(3) &= 3^2+9(3)+C\\ 21 &= 9+27+C\\ C &= -15\end{aligned}$$

Hence, the equation of the curve is $f(x)=x^2+9x-15$. $\square$

(c) The gradient function of the curve is given by $g(t)=\dfrac{5t^2-6t+1}{t^3(t-1)}$ and it passes through the point $(1,3)$.

Sol.

$$\begin{aligned} g(t) &= \frac{5t^2-6t+1}{t^3(t-1)} \\ &= \frac{(5t-1)(t-1)}{t^3(t-1)} \\ &= \frac{5t-1}{t^3} \\ &= \frac{5}{t^2}-\frac{1}{t^3} \\ &= 5t^{-2}-t^{-3} \\ f(t) &= \int 5t^{-2}-t^{-3}\,dt \\ &= -\frac{5}{t}+\frac{1}{2t^2}+C \end{aligned}$$

When $t=1$, $f(1)=3$,

$$\begin{aligned} 3 &= -5+\frac{1}{2}+C \\ 3 &= -\frac{9}{2}+C \\ C &= \frac{15}{2} \end{aligned}$$

Hence, the equation of the curve is $f(t) = -\frac{5}{t}+\frac{1}{2t^2}+\frac{15}{2}$. ▯

11. Tommy moves in his roller skates at the rate of change in displacement, $\frac{ds}{dt}=t^2+9$ metres per second, where t is the time in seconds. At $t=3$ seconds, Tommy is 4 metres away from his starting place. Find the displacement, s metres, when $t=10$ seconds.

Sol.

$$\begin{aligned} \frac{ds}{dt} &= t^2+9 \\ s &= \int t^2+9\,dt \\ &= \frac{t^3}{3}+9t+C \end{aligned}$$

When $t=3$, $s=4$,

$$\begin{aligned} 4 &= \frac{3^3}{3}+9(3)+C \\ 4 &= 9+27+C \\ 4 &= 54+C \\ C &= -32 \\ s &= \frac{t^3}{3}+9t-32 \end{aligned}$$

When $t=10$,

$$\begin{aligned} s &= \frac{10^3}{3}+9(10)-32 \\ &= 333+90-32 \\ &= 391\frac{1}{3}m \quad ▯ \end{aligned}$$

12. Given the gradient function of a curve is $\frac{dy}{dx}=kx^2+2x$ where k is a constant. The curve passes through point $A(1,6)$ and point $B(-2,0)$. Determine the equation of the curve.

Sol.

$$\begin{aligned} \frac{dy}{dx} &= kx^2+2x \\ y &= \int kx^2+2x\,dx \\ &= \frac{kx^3}{3}+x^2+C \end{aligned}$$

When $x=1$, $y=6$,

$$\begin{aligned} 6 &= \frac{k}{3}+1+C \\ k+3C &= 15 \quad (1) \end{aligned}$$

When $x=-2$, $y=0$,

$$\begin{aligned} 0 &= -\frac{8k}{3}+4+C \\ 8k-3C &= 12 \quad (2) \end{aligned}$$

$$\begin{aligned} (2)+(1)\cdot 8:\ 9k &= 27 \\ k &= 3 \\ C &= 4 \end{aligned}$$

Hence, the equation of the curve is $y=x^3+x^2+4$ ▯

3.3 Definite Integral

13. Calculate each of the following.

(a) $\int_2^1 \left(\sqrt{x}+\frac{1}{x}\right)$

Sol.

$$\begin{aligned} \int_2^1 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right) &= \int_1^2 \left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right) \\ &= \left[\frac{2\sqrt{x^3}}{3}+2\sqrt{x}\right]_1^2 \\ &= \left[\frac{4\sqrt{2}}{3}+2\sqrt{2}\right]-\left[\frac{2}{3}+2\right] \\ &= \frac{10\sqrt{2}}{3}-\frac{8}{3} \\ &= \frac{10\sqrt{2}-8}{3} \\ &\approx 2.0474 \quad ▯ \end{aligned}$$

(b) $\int_0^3 \left(\frac{x^4+3x}{x}\right)\,dx$

Sol.

$$\begin{aligned}\int_0^3 \left(\frac{x^4+3x}{x}\right) dx &= \int_0^3 \left(x^3+3\right) dx \\ &= \left[\frac{1}{4}x^4+3x\right]_0^3 \\ &= \left[\frac{1}{4}(3^4)+3\cdot 3\right]-0 \\ &= \frac{81}{4}+9 \\ &= \frac{117}{4} \\ &= 29.25 \quad \square\end{aligned}$$

(c) $\displaystyle\int_{-2}^{-1} \left(\frac{(4-x)(3-x)}{x^5}\right) dx$

Sol.

$$\begin{aligned}&\int_{-2}^{-1} \left(\frac{(4-x)(3-x)}{x^5}\right) dx \\ &= \int_{-2}^{-1} \left(\frac{x^2-7x+12}{x^5}\right) dx \\ &= \int_{-2}^{-1} \left(\frac{1}{x^3}-\frac{7}{x^4}+\frac{12}{x^5}\right) dx \\ &= \left[-\frac{1}{2x^2}+\frac{7}{3x^3}-\frac{3}{x^4}\right]_{-2}^{-1} \\ &= \left[-\frac{1}{2}-\frac{7}{3}-3\right]-\left[-\frac{1}{8}-\frac{7}{24}-\frac{3}{16}\right] \\ &= -\frac{35}{6}+\frac{29}{48} \\ &= -5\frac{11}{48} \quad \square\end{aligned}$$

14. Given $\displaystyle\int_a^b f(x)\,dx = 5$, $\displaystyle\int_b^c f(x)\,dx = 8$ and $\displaystyle\int_a^b g(x)\,dx = 2$. Find each of the following. [answer can be in terms of a and/or b.]

(a) $\displaystyle\int_a^b 3f(x)\,dx$

Sol.

$$\begin{aligned}\int_a^b 3f(x)\,dx &= 3\int_a^b f(x)\,dx \\ &= 3\cdot 5 \\ &= 15 \quad \square\end{aligned}$$

(b) $\displaystyle\int_a^c f(x)\,dx$

Sol.

$$\begin{aligned}\int_a^c f(x)\,dx &= \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \\ &= 5+8 \\ &= 13 \quad \square\end{aligned}$$

(c) $\displaystyle\int_a^b [f(x)+g(x)]\,dx$

Sol.

$$\begin{aligned}\int_a^b [f(x)+g(x)]\,dx &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\ &= 5-2 \\ &= 3 \quad \square\end{aligned}$$

(d) $\displaystyle\int_c^a f(x)\,dx$

Sol.

$$\begin{aligned}\int_c^a f(x)\,dx &= -\int_a^c f(x)\,dx \\ &= -13 \quad \square\end{aligned}$$

(e) $\displaystyle\int_a^b [g(x)+3]\,dx$

Sol.

$$\begin{aligned}\int_a^b [g(x)+3]\,dx &= \int_a^b g(x)\,dx + \int_a^b 3\,dx \\ &= 2+3(b-a) \\ &= 3b-3a+2 \quad \square\end{aligned}$$

(f) $\displaystyle\int_a^a f(x)\,dx$

Sol.

$$\int_a^a f(x)\,dx = 0 \quad \square$$

(g) The value of k such that $\displaystyle\int_b^a [f(x)+kx]\,dx = 25$ if $a=1$ and $b=4$.

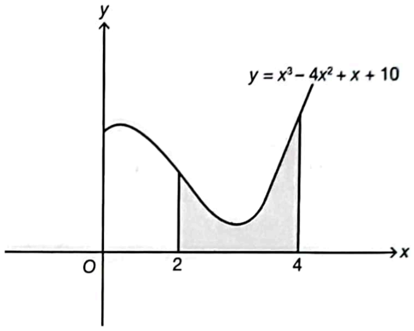
Sol.

$$\begin{aligned}\int_b^a [f(x)+kx]\,dx &= \int_b^a f(x)\,dx + \int_b^a kx\,dx \\ &= -5 + \int_b^a kx\,dx\end{aligned}$$

$$\begin{aligned}-5+\int_1^4 kx\,dx &= 25 \\ \int_4^1 kx\,dx &= 30 \\ k\left[\frac{x^2}{2}\right]_4^1 &= 30 \\ k\left(\frac{1}{2}-\frac{16}{2}\right) &= 30 \\ -\frac{15k}{2} &= 30 \\ -15k &= 60 \\ k &= -4 \quad \square\end{aligned}$$

15. Find the area of the shaded region for each of the following diagrams.

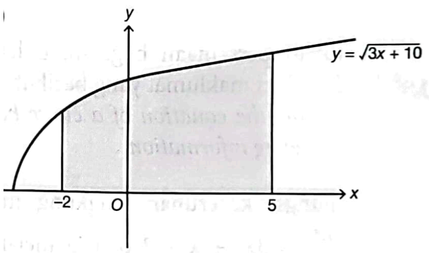
(a)



Sol.

$$A = \int_2^4 (x^3 - 4x^2 + x + 10)\,dx$$
$$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{1}{2}x^2 + 10x\right]_2^4$$
$$= \left(64 - \frac{256}{3} + 8 + 40\right) - \left(4 - \frac{32}{3} + 2 + 20\right)$$
$$= \frac{80}{3} - \frac{46}{3}$$
$$= 11\frac{1}{3} \text{ units}^2 \quad ▯$$

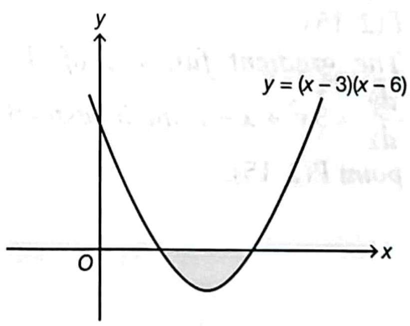
(b)



Sol.

$$A = \int_{-2}^{5} \sqrt{3x+10}\,dx$$
$$= \int_{-2}^{5} (3x+10)^{\frac{1}{2}}\,dx$$
$$= \left[\frac{2(3x+10)^{\frac{3}{2}}}{9}\right]_{-2}^{5}$$
$$= \frac{2(25)^{\frac{3}{2}}}{9} - \frac{2(4)^{\frac{3}{2}}}{9}$$
$$= \frac{250}{9} - \frac{16}{9}$$
$$= \frac{234}{9}$$
$$= 26 \text{ units}^2 \quad ▯$$

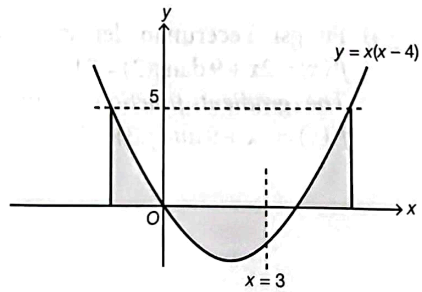
(c)



Sol.

$$A = \left|\int_3^6 (x-3)(x-6)\,dx\right|$$
$$= \left|\int_3^6 (x^2 - 9x + 18)\,dx\right|$$
$$= \left|\left[\frac{1}{3}x^3 - \frac{9}{2}x^2 + 18x\right]_3^6\right|$$
$$= \left|(72 - 162 + 108) - \left(9 - \frac{81}{2} + 54\right)\right|$$
$$= \left|18 - \frac{45}{2}\right|$$
$$= 4.5 \text{ units}^2 \quad ▯$$

(d)



Sol.

When $y = 5$,

$$\begin{aligned} x(x-4) &= 5 \\ x^2 - 4x - 5 &= 0 \\ (x-5)(x+1) &= 0 \\ x &= -1 \text{ or } x = 5 \end{aligned}$$

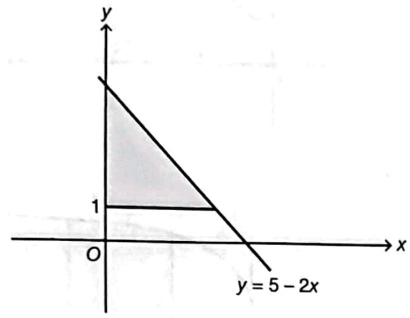
When $y = 0$,

$$\begin{aligned} x(x-4) &= 0 \\ x &= 0 \text{ or } x = 4 \end{aligned}$$

$$A = \int_{-1}^{0} x(x-4)\,dx + \left|\int_0^3 x(x-4)\,dx\right| + \int_4^5 x(x-4)\,dx$$
$$= \int_{-1}^{0} (x^2-4x)\,dx + \left|\int_0^3 (x^2-4x)\,dx\right| + \int_4^5 (x^2-4x)\,dx$$
$$= \left[\frac{1}{3}x^3 - 2x^2\right]_{-1}^{0} + \left|\left[\frac{1}{3}x^3 - 2x^2\right]_0^3\right| + \left[\frac{1}{3}x^3 - 2x^2\right]_4^5$$
$$= 0 - \left(-\frac{1}{3} - 2\right) + |(9 - 18) - 0| + \left(\frac{125}{3} - 50\right) - \left(\frac{64}{3} - 32\right)$$
$$= 13\frac{2}{3} \text{ units}^2 \quad ▯$$

16. Determine the area bounded by the curve, the horizontal line(s) and the y-axis.

(a)



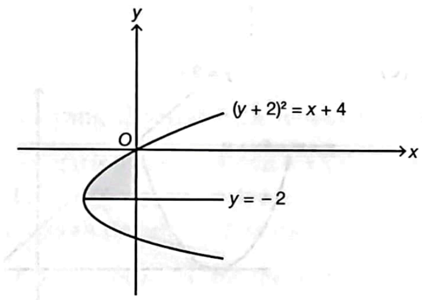
Sol. When $x = 0$,

$$\begin{aligned} y &= 5 - 2x \\ y &= 5 - 2(0) \\ &= 5 \end{aligned}$$

$$\begin{aligned} y &= 5 - 2x \\ 2x &= 5 - y \\ x &= \frac{5 - y}{2} \end{aligned}$$

$$\begin{aligned} A &= \int_1^5 \frac{5 - y}{2}\, dy \\ &= \int_1^5 \left(\frac{5}{2} - \frac{1}{2}y\right) dy \\ &= \left[\frac{5}{2}y - \frac{1}{4}y^2\right]_1^5 \\ &= \left(\frac{25}{2} - \frac{25}{4}\right) - \left(\frac{5}{2} - \frac{1}{4}\right) \\ &= \frac{25}{4} - \frac{9}{4} \\ &= 4 \text{ units}^2 \quad \square \end{aligned}$$

(b)

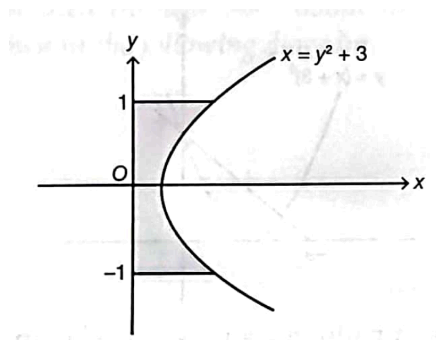


Sol.

$$\begin{aligned} (y+2)^2 &= x + 4 \\ x &= (y+2)^2 - 4 \\ &= y^2 + 4y + 4 - 4 \\ &= y^2 + 4y \end{aligned}$$

$$\begin{aligned} A &= \int_{-2}^0 (y^2 + 4y)\, dy \\ &= \left[\frac{1}{3}y^3 + 2y^2\right]_{-2}^0 \\ &= 0 - \left(-\frac{8}{3} + 8\right) \\ &= 5\frac{1}{3} \text{ units}^2 \quad \square \end{aligned}$$

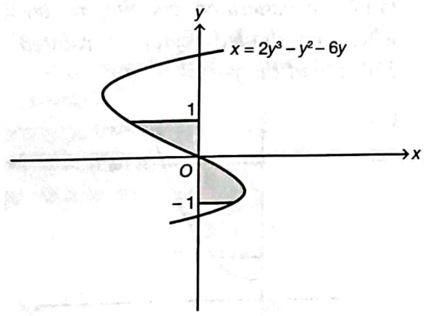
(c)



Sol.

$$\begin{aligned} A &= \int_{-1}^1 (y^2 + 3)\, dy \\ &= \left[\frac{1}{3}y^3 + 3y\right]_{-1}^1 \\ &= \left(\frac{1}{3} + 3\right) - \left(-\frac{1}{3} - 3\right) \\ &= 6\frac{2}{3} \text{ units}^2 \quad \square \end{aligned}$$

(d)

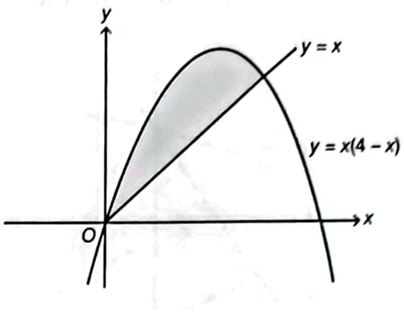


Sol.

$$\begin{aligned} A &= \int_{-1}^0 (2y^3 - y^2 - 6y)\, dy + \left|\int_0^1 (2y^3 - y^2 - 6y)\, dy\right| \\ &= \left[\frac{1}{2}y^4 - \frac{1}{3}y^3 - 3y^2\right]_{-1}^0 + \left|\left[\frac{1}{2}y^4 - \frac{1}{3}y^3 - 3y^2\right]_0^1\right| \\ &= 0 - \left(\frac{1}{2} + \frac{1}{3} - 3\right) + \left|\left(\frac{1}{2} - \frac{1}{3} - 3\right) - 0\right| \\ &= \frac{13}{6} + \frac{17}{6} \\ &= 5 \text{ units}^2 \quad \square \end{aligned}$$

17. Find the area of the shaded region for each of the following.

(a)

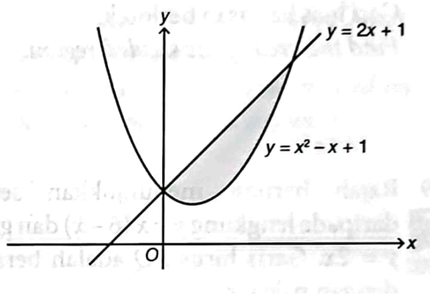


Sol.

$$\begin{aligned} x &= x(4-x) \\ x &= 4x - x^2 \\ x^2 - 3x &= 0 \\ x(x-3) &= 0 \\ x &= 0 \text{ or } x = 3 \end{aligned}$$

$$\begin{aligned} A &= \int_0^3 [x(4-x) - x]\ dx \\ &= \int_0^3 \left[4x - x^2 - x\right]\ dx \\ &= \int_0^3 \left[3x - x^2\right]\ dx \\ &= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3 \\ &= \left(\frac{27}{2} - 9\right) - 0 \\ &= 4.5 \text{ units}^2 \quad \square \end{aligned}$$

(b)

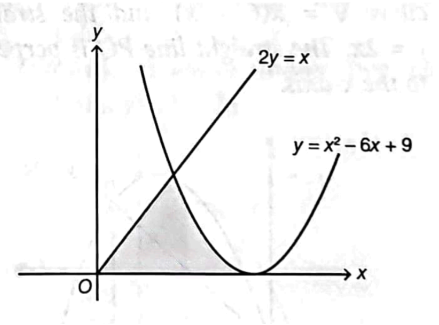


Sol.

$$\begin{aligned} 2x + 1 &= x^2 - x + 1 \\ x^2 - 3x &= 0 \\ x(x-3) &= 0 \\ x &= 0 \text{ or } x = 3 \end{aligned}$$

$$\begin{aligned} A &= \int_0^3 \left[2x + 1 - x^2 + x - 1\right]\ dx \\ &= \int_0^3 \left[-x^2 + 3x\right]\ dx \\ &= \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2\right]_0^3 \\ &= \left(-9 + \frac{27}{2}\right) - 0 \\ &= 4.5 \text{ units}^2 \quad \square \end{aligned}$$

(c)

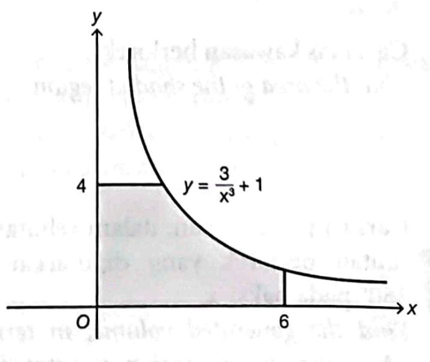


Sol.

$$\begin{aligned} 2y &= x \\ y &= \frac{1}{2}x \\ \frac{1}{2}x &= x^2 - 6x + 9 \\ x &= 2x^2 - 12x + 18 \\ 2x^2 - 13x + 18 &= 0 \\ (2x-9)(x-2) &= 0 \\ x &= 2 \text{ or } x = 4.5 \\ x^2 - 6x + 9 &= 0 \\ (x-3)^2 &= 0 \\ x &= 3 \end{aligned}$$

$$\begin{aligned} A &= \int_0^2 \frac{x}{2}\, dx + \int_2^3 (x^2 - 6x + 9)\, dx \\ &= \left[\frac{1}{4}x^2\right]_0^2 + \left[\frac{1}{3}x^3 - 3x^2 + 9x\right]_2^3 \\ &= 1 - 0 + (9 - 27 + 27) - \left(\frac{8}{3} - 12 + 18\right) \\ &= 10 - \frac{26}{3} \\ &= 1\frac{1}{3} \text{ units}^2 \quad \square \end{aligned}$$

(d)

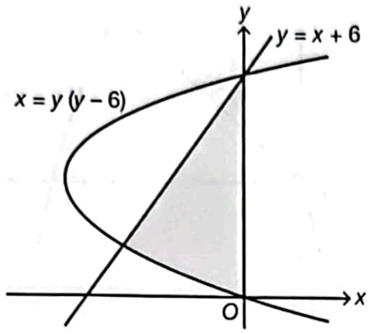


Sol.

$$\begin{aligned} \frac{3}{x^3} + 1 &= 4 \\ \frac{3}{x^3} &= 3 \\ x^3 &= 1 \\ x &= 1 \end{aligned}$$

$$
\begin{aligned}
A &= 1\cdot 4 + \int_1^6 \left(\frac{3}{x^3}+1\right) dx \\
&= 4 + \left[-\frac{3}{2x^2}+x\right]_1^6 \\
&= 4 + \left(-\frac{1}{24}+6\right) - \left(-\frac{3}{2}+6\right) \\
&= 4 + \frac{143}{24} + \frac{9}{2} \\
&= 14\frac{11}{24}\ \text{units}^2 \quad \square
\end{aligned}
$$

18. The following diagram shows a part of the curve $x = y(y-6)$ and the straight line $y = x+6$.



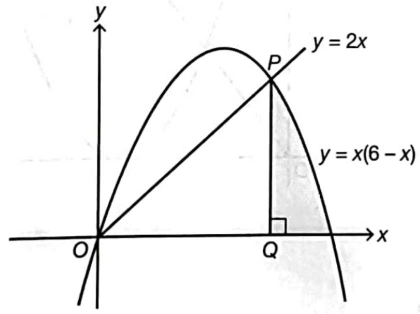
Find the area of the shaded region.

Sol.

$$
\begin{aligned}
y &= x+6 \\
x &= y-6 \\
y-6 &= y(y-6) \\
&= y^2-6y \\
y^2-7y+6 &= 0 \\
(y-6)(y-1) &= 0 \\
y = 6 &\text{ or } y = 1 \\
1 &= x+6 \\
x &= -5
\end{aligned}
$$

$$
\begin{aligned}
A &= \left|\int_0^1 y(y-6)\,dy\right| + \frac{1}{2}(5)(5) \\
&= \left|\int_0^1 (y^2-6y)\,dy\right| + \frac{25}{2} \\
&= \left|\left[\frac{1}{3}y^3-3y^2\right]_0^1\right| + \frac{25}{2} \\
&= \left|\frac{1}{3}-3-0\right| + \frac{25}{2} \\
&= \frac{8}{3}+\frac{25}{2} \\
&= 15\frac{1}{6}\ \mathit{units}^2 \quad \square
\end{aligned}
$$

19. The following diagram shows a part of the curve $y = x(6-x)$ and a straight line $y = 2x$. The straight line PQ is perpendicular to the x-axis.



Find the area of the shaded region.

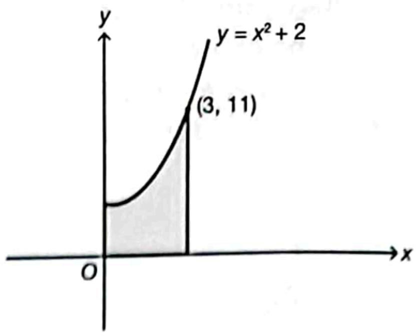
Sol.

$$
\begin{aligned}
x(6-x) &= 0 \\
x = 0 &\text{ or } x = 6 \\
2x &= x(6-x) \\
2x &= 6x-x^2 \\
x^2-4x &= 0 \\
x(x-4) &= 0 \\
x = 0 &\text{ or } x = 4
\end{aligned}
$$

$$
\begin{aligned}
A &= \int_4^6 x(6-x)\,dx \\
&= \int_4^6 (6x-x^2)\,dx \\
&= \left[3x^2-\frac{1}{3}x^3\right]_4^6 \\
&= (108-72) - \left(48-\frac{64}{3}\right) \\
&= 36-\frac{80}{3} \\
&= 9\frac{1}{3}\ \mathit{units}^2 \quad \square
\end{aligned}
$$

20. Find the generated volume, in terms of π, when the shaded region is rotated through 360° about the x-axis.

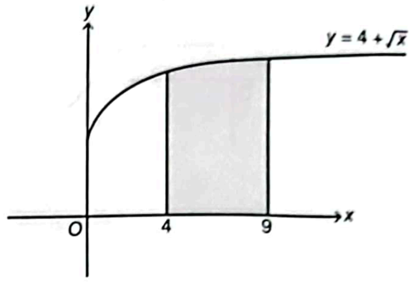
(a)



Sol.

$$
\begin{aligned}
V_x &= \int_0^3 \pi y^2 \, dx \\
&= \pi \int_0^3 (x^2+2)^2 \, dx \\
&= \pi \int_0^3 (x^4+4x^2+4) \, dx \\
&= \pi \left[\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x \right]_0^3 \\
&= \left[\left(\frac{243}{5} + 36 + 12 \right) - 0 \right] \pi \\
&= 96.6\pi \textit{ units}^3 \quad \square
\end{aligned}
$$

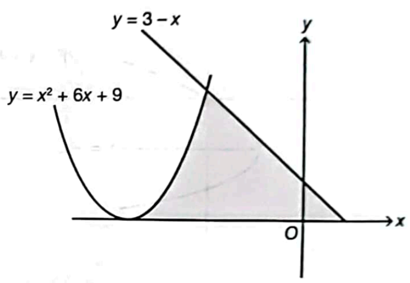
(b)



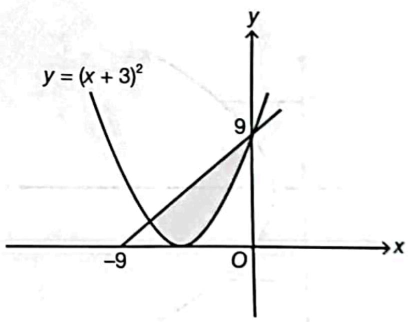
Sol.

$$
\begin{aligned}
V_x &= \int_4^9 \pi (4+\sqrt{x})^2 \, dx \\
&= \pi \int_4^9 (16 + 8\sqrt{x} + x) \, dx \\
&= \pi \left[16x + \frac{16x^{\frac{3}{2}}}{3} + \frac{1}{2}x^2 \right]_4^9 \\
&= \pi \left[\left(144 + 144 + \frac{81}{2} \right) - \left(64 + \frac{128}{3} + 8 \right) \right] \\
&= 213.83\pi \textit{ units}^3 \quad \square
\end{aligned}
$$

(c)

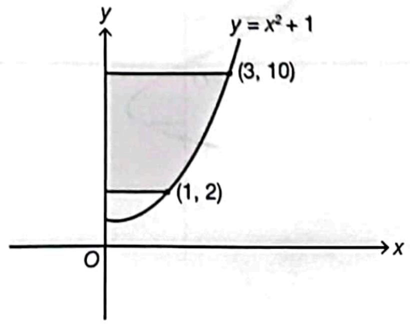


(d)

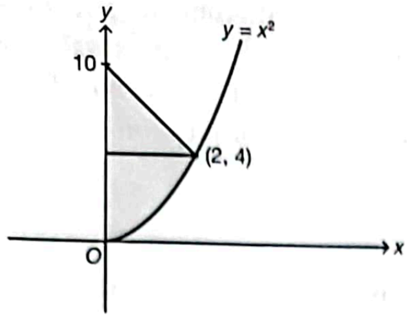


21. Find the generated volume, in terms of π, when the shaded region is rotated through $360°$ about the y-axis.

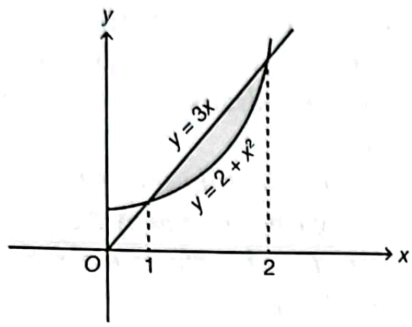
(a)



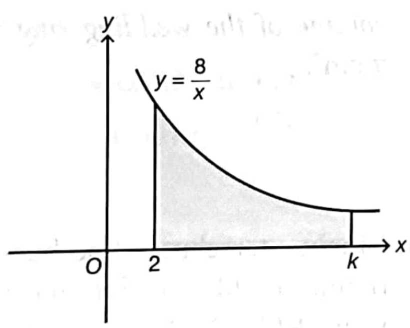
(b)



(c)

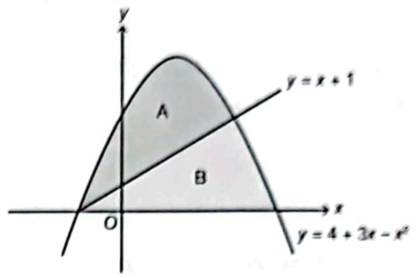


22. The region bounded by the curve $y = \frac{8}{x}$, the x-axis, and the straight line $x = 2$ and $x = k$ is rotated through $360°$ about the x-axis as shown in the following diagram.



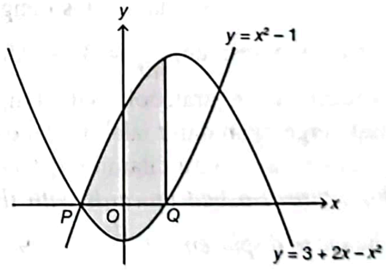
Express the volume generated by the region in terms of k. If the value of k becomes extremely large, deduce the nearest value of volume.

23. The following diagram shows a part of the curve $y = 4 + 3x - x^2$ and the straight line $y = x + 1$.



Find the ration of the area of the shaded region A to the area of the shaded region B.

24. The following diagram shows two curves $y = x^2 - 1$ and $y = 3 + 2x - x^2$.



Find the coordinate of the points P and Q. Hence, calculate the area of the shaded region.