Solution Book of Mathematic

Senior 2 Part I

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Chapter 12

Sequence and Series

12.1 Sequence and Series

12.1.1 Practice 1

1. Find the first 5 terms of the sequence $a_n = \frac{2^n}{n+1}$.

   sol. $a_1 = \frac{2}{2} = 1, a_2 = \frac{4}{3}, a_3 = \frac{8}{4}, a_4 = \frac{16}{5}, a_5 = \frac{32}{6}$

2. Write the general term of the sequence $1, 8, 27, 64, \cdots$

   sol. $a_n = n^3$

12.1.2 Practice 2

1. Express the series $\sum_{n=1}^{10} n^2 + 1$ in the form of numbers.

   sol. $$\begin{aligned}&\sum_{n=1}^{10} n^2 + 1\\ &= (1^2+1)+(2^2+1)+(3^2+1)+(4^2+1)\\ &+(5^2+1)+(6^2+1)+(7^2+1)\\ &+(8^2+1)+(9^2+1)+(10^2+1)\\ &= 2+5+10+17+26+37+50+65\\ &+82+101\end{aligned}$$

2. Write the first term, last term and the number of terms of the series $\sum_{n=1}^{10}(3^n - 2^n)$.

   sol. $$\begin{aligned}&First\ term = (3^1 - 2^1) = 1\\ &Last\ term = (3^{10} - 2^{10}) = 59049\\ &Number\ of\ terms = 10\end{aligned}$$

3. Express the series $2\cdot5 + 3\cdot7 + 4\cdot9 + \cdots + 15\cdot31$ in the form of $\sum$.

   sol.
   $$\begin{aligned}a_1 &= 2\cdot5 = 10\\ a_2 &= 3\cdot7 = 21\\ a_3 &= 4\cdot9 = 36\\ a_4 &= 5\cdot11 = 55\\ &\vdots\\ a_{15} &= 15\cdot31 = 465\end{aligned}$$
   $$\begin{aligned}&\therefore 2\cdot5+3\cdot7+4\cdot9+\cdots+15\cdot31\\ &= \sum_{n=1}^{15} a_n\end{aligned}$$

12.1.3 Exercise 12.1

1. Find the general term of the following sequences.

   (a) $5, 8, 11, 14, \ldots$
   sol. $a_n = 3n + 2$

   (b) $2, 4, 8, 16, \ldots$
   sol. $a_n = 2^n$

   (c) $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \cdots$
   sol. $a_n = \frac{n+1}{n}$

   (d) $\frac{2}{5}, \frac{4}{7}, \frac{6}{9}, \frac{8}{11}, \cdots$
   sol. $a_n = \frac{2n}{2n+1}$

2. Find the first 5 terms of the following sequences.

   (a) $a_n = 2n + 3$
   sol. $a_1 = 2\cdot1+3 = 5, a_2 = 2\cdot2+3 = 7, a_3 = 2\cdot3+3 = 9, a_4 = 2\cdot4+3 = 11, a_5 = 2\cdot5+3 = 13$

   (b) $a_n = n(n-2)$
   sol. $a_1 = 1\cdot(-1) = -1, a_2 = 2\cdot0 = 0, a_3 = 3\cdot1 = 3, a_4 = 4\cdot2 = 8, a_5 = 5\cdot3 = 15$

   (c) $a_n = \frac{n}{2n+1}$
   sol. $a_1 = \frac{1}{2\cdot1+1} = \frac{1}{3}, a_2 = \frac{2}{2\cdot2+1} = \frac{2}{5}, a_3 = \frac{3}{2\cdot3+1} = \frac{3}{7}, a_4 = \frac{4}{2\cdot4+1} = \frac{4}{9}, a_5 = \frac{5}{2\cdot5+1} = \frac{5}{11}$

   (d) $a_n = (-3)^n$
   sol. $a_1 = (-3)^1 = -3, a_2 = (-3)^2 = 9, a_3 = (-3)^3 = -27, a_4 = (-3)^4 = 81, a_5 = (-3)^5 = -243$

3. Express the following series in the form of numbers.

   (a) $\sum_{n=1}^{5} n(n+3)$

   sol. $$\begin{aligned}&\sum_{n=1}^{5} n(n+3)\\ &= (1\cdot4)+(2\cdot5)+(3\cdot6)+(4\cdot7)\\ &+(5\cdot8)\\ &= 4+10+18+28+40\end{aligned}$$

(b) $\sum_{n=2}^{6} \frac{1}{3^n}$

sol. $$\sum_{n=2}^{6} \frac{1}{3^n} = \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6} = \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729}$$

(c) $\sum_{n=1}^{6} \frac{1}{n(2n+1)}$

sol. $$\begin{aligned}\sum_{n=1}^{6} \frac{1}{n(2n+1)} &= \frac{1}{1(2\cdot1+1)} + \frac{1}{2(2\cdot2+1)} \\ &+ \frac{1}{3(2\cdot3+1)} + \frac{1}{4(2\cdot4+1)} \\ &+ \frac{1}{5(2\cdot5+1)} + \frac{1}{6(2\cdot6+1)} \\ &= \frac{1}{3} + \frac{1}{10} + \frac{1}{21} + \frac{1}{36} + \frac{1}{55} + \frac{1}{78}\end{aligned}$$

(d) $\sum_{n=2}^{5} \frac{1}{n^2+2}$

sol. $$\sum_{n=2}^{5} \frac{1}{n^2+2} = \frac{1}{4+2} + \frac{1}{9+2} + \frac{1}{16+2} + \frac{1}{25+2} = \frac{1}{6} + \frac{1}{11} + \frac{1}{18} + \frac{1}{27}$$

4. Find the first term, last term and the number of terms of the following series.

(a) $\sum_{n=3}^{10} 2^2$

sol. $a_3 = 2^2 = 4, a_{10} = 2^2 = 4, n = 10-3+1 = 8$

(b) $\sum_{n=1}^{8} \frac{n+2}{n}$

sol. $a_1 = \frac{1+2}{1} = \frac{3}{1} = 3, a_8 = \frac{8+2}{8} = \frac{10}{8} = \frac{5}{4}, n = 8-1+1 = 8$

(c) $\sum_{n=1}^{10} 3n^2 - n$

sol. $a_1 = 3\cdot1^2 - 1 = 2, a_{10} = 3\cdot10^2 - 10 = 290, n = 10-1+1 = 10$

(d) $\sum_{n=9}^{14} n^2(n-7)$

sol. $a_9 = 9^2(9-7) = 9^2\cdot2 = 162, a_{14} = 14^2(14-7) = 14^2\cdot7 = 2744, n = 14-9+1 = 6$

5. Express the following series in the form of $\sum$.

(a) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{30}$

Sol.

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \frac{1}{2} \\ a_3 &= \frac{1}{3} \\ &\vdots \\ a_{30} &= \frac{1}{30} \end{aligned}$$

$$\therefore 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{30} = \sum_{n=1}^{30} \frac{1}{n}$$

(b) $1^3 + 2^3 + 3^3 + \cdots + 50^3$

Sol.

$$\begin{aligned} a_1 &= 1^3 \\ a_2 &= 2^3 \\ a_3 &= 3^3 \\ &\vdots \\ a_{50} &= 50^3 \end{aligned}$$

$$\therefore 1^3 + 2^3 + 3^3 + \cdots + 50^3 = \sum_{n=1}^{50} n^3$$

(c) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16}$

Sol.

$$\begin{aligned} a_1 &= (-\frac{1}{2})^{1-1} \\ a_2 &= (-\frac{1}{2})^{2-1} \\ a_3 &= (-\frac{1}{2})^{3-1} \\ a_4 &= (-\frac{1}{2})^{4-1} \\ a_5 &= (-\frac{1}{2})^{5-1} \end{aligned}$$

$$\therefore 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} = \sum_{n=1}^{5} (-\frac{1}{2})^{n-1}$$

(d) $2\cdot4 + 4\cdot7 + 6\cdot10 + 8\cdot13 + 10\cdot16$

Sol.

$$\begin{aligned} a_1 &= 2\cdot1\cdot(3\cdot1+1) \\ a_2 &= 2\cdot2\cdot(3\cdot2+1) \\ a_3 &= 2\cdot3\cdot(3\cdot3+1) \\ a_4 &= 2\cdot4\cdot(3\cdot4+1) \\ a_5 &= 2\cdot5\cdot(3\cdot5+1) \end{aligned}$$

$$\therefore 2\cdot4 + 4\cdot7 + 6\cdot10 + 8\cdot13 + 10\cdot16 = \sum_{n=1}^{5} 2n(3n+1)$$

12.2 Arithmetic Progression

General term of an Arithmetic Progression (AP) is given by

$$a_n = a_1 + (n-1)d$$

where a_1 is the first term, d is the common difference and n is the number of terms.

12.2.1 Practice 3

1. Find the number of terms of the AP $-4 - 2\frac{3}{4} - 1\frac{1}{2} - \frac{1}{4} + \cdots + 16$.

$$\begin{aligned} a_1 &= -4 \\ a_n &= 16 \\ d &= -2\frac{3}{4} - (-4) \\ &= -2\frac{3}{4} + 4 \\ &= \frac{5}{4} \\ 16 &= -4 + (n-1)\frac{5}{4} \\ 20 &= \frac{5}{4}(n-1) \\ 80 &= 5(n-1) \\ n-1 &= 16 \\ n &= 17 \end{aligned}$$

2. Given that $a_2 = 4$ and $a_6 = -8$, find the 10th term of the AP.

Sol.

$$\begin{aligned} a_2 &= 4 \\ a + (2-1)d &= 4 \\ a_6 &= -8 \\ a + (6-1)d &= -8 \end{aligned}$$

$$\begin{cases} a + d = 4 & (1) \\ a + 5d = \text{-}8 & (2) \end{cases}$$

$$\begin{aligned} (2) - (1):\ 4d &= -12 \\ d &= -3 \\ a + (-3) &= 4 \\ a &= 7 \\ \therefore a_{10} &= 7 + (10-1)(-3) \\ &= 7 - 27 \\ &= -20 \end{aligned}$$

3. How many multiples of 7 are there between 50 and 500?

Sol.

$$\begin{aligned} a_1 &= 56 \\ a_n &= 497 \\ d &= 7 \\ 497 &= 56 + (n-1)7 \\ 441 &= 7(n-1) \\ n-1 &= 63 \\ n &= 64 \end{aligned}$$

4. Find 5 numbers between 30 and 54 such that these numbers form an AP.

Sol.

$$\begin{aligned} a_1 &= 30 \\ a_7 &= 54 \\ 54 &= 30 + (7-1)d \\ 24 &= 6d \\ d &= 4 \end{aligned}$$

$\therefore$ *These* 5 *numbers are* 34, 38, 42, 46, *and* 50.

Arithmetic mean

If A is in between x and y, and x, A, y are in AP, then

$$A = \frac{x+y}{2}$$

12.2.2 Practice 4

1. If 9, x, 17 are in AP, find x.

Sol.

$$\begin{aligned} x &= \frac{9+17}{2} \\ &= \frac{26}{2} \\ &= 13 \end{aligned}$$

2. Find the arithmetic mean of 26 and -11.

Sol.

$$\begin{aligned} A &= \frac{26-11}{2} \\ &= \frac{15}{2} \end{aligned}$$

3. Find x and y when 3, x, 12, y, 21 are in AP.

Sol.

$$\begin{aligned} x &= \frac{3+12}{2} \\ &= \frac{15}{2} \\ y &= \frac{12+21}{2} \\ &= \frac{33}{2} \end{aligned}$$

Summation of Arithmetic Progression

The summation formula for AP is given by

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

or

$$S_n = \frac{n}{2}(a_1 + a_n)$$

12.2.3 Practice 5

1. Find the sum of the first 16 terms of the AP $22 + 18 + 14 + 10 + \cdots$

Sol.

$$\begin{aligned} a_1 &= 22 \\ n &= 16 \\ d &= -4 \\ S_n &= \frac{16}{2}(2 \cdot 22 + (-4)(16-1)) \\ &= \frac{16}{2}(44 + (-4)(15)) \\ &= \frac{16}{2}(44 - 60) \\ &= \frac{16}{2}(-16) \\ &= -128 \end{aligned}$$

2. If the sum of AP $23+19+15+\cdots$ is 72, find the number of terms.

Sol.

$$\begin{aligned} a_1 &= 23 \\ S_n &= 72 \\ d &= -4 \\ 72 &= \frac{n}{2}(2 \cdot 23 + (-4)(n-1)) \\ 72 &= \frac{n}{2}(46 + (-4)(n-1)) \\ 144 &= n(46 + (-4)(n-1)) \\ 144 &= n(46 - 4n + 4) \\ 144 &= n(50 - 4n) \\ 144 &= 50n - 4n^2 \\ 72 &= 25n - 2n^2 \\ 2n^2 - 25n + 72 &= 0 \\ (n-8)(2n-9) &= 0 \\ n &= 8 \end{aligned}$$

3. Given that $S_n = 2n + 3n^2$, find the first term and the common difference of the AP.

Sol.

$$\begin{aligned} S_n &= 2n + 3n^2 \\ 2n + 3n^2 &= \frac{n}{2}(2a + (n-1)d) \\ 4n + 6n^2 &= n(2a + (n-1)d) \\ 4n + 6n^2 &= 2na + (n-1)nd \\ 4n + 6n^2 &= 2na + n^2 d - nd \\ 4n + 6n^2 &= (2a-d)n + dn^2 \end{aligned}$$

$$\begin{aligned} &Comparing\ both\ sides, \\ 2a - d &= 4 \\ d &= 6 \\ a &= 5 \end{aligned}$$

12.2.4 Exercise 12.2

1. Find the 10th terms of the AP $5, 13, 21, \cdots$

Sol.

$$\begin{aligned} a_1 &= 5 \\ n &= 10 \\ d &= 8 \\ a_{10} &= 5 + (10-1) \cdot 8 \\ &= 5 + 72 \\ &= 77 \end{aligned}$$

2. Find the 8th term of the AP $5, 4\frac{1}{4}, 3\frac{1}{2}, 2\frac{3}{4}, \cdots$

Sol.

$$
\begin{aligned}
a_1 &= 5 \\
n &= 8 \\
d &= -\frac{3}{4} \\
a_8 &= 5 + (8-1) \cdot -\frac{3}{4} \\
&= 5 - \frac{3}{4} \cdot 7 \\
&= 5 - \frac{21}{4} \\
&= -\frac{1}{4}
\end{aligned}
$$

3. Find the number of terms of the following AP.

   (a) $4, 9, \ldots, 64$

   Sol.

$$
\begin{aligned}
a_1 &= 4 \\
a_n &= 64 \\
d &= 5 \\
64 &= 4 + (n-1) \cdot 5 \\
60 &= 5(n-1) \\
12 &= n-1 \\
n &= 13
\end{aligned}
$$

   (b) $4\frac{1}{3}, 3\frac{2}{3}, 3, \ldots, -10\frac{1}{3}$

   Sol.

$$
\begin{aligned}
a_1 &= 4\frac{1}{3} \\
a_n &= -10\frac{1}{3} \\
d &= -\frac{2}{3} \\
-10\frac{1}{3} &= 4\frac{1}{3} + (n-1) \cdot -\frac{2}{3} \\
-\frac{31}{3} &= \frac{13}{3} - \frac{1}{3}(n-1) \\
-31 &= 13 - 2n + 2 \\
-46 &= 2n \\
n &= 23
\end{aligned}
$$

4. The 6th term of an AP is 43, and its 10th term is 75. Find the first term and common difference of this AP.

   Sol.

$$
\begin{aligned}
a_6 &= 43 \\
a_{10} &= 75 \\
43 &= a + (6-1)d \\
75 &= a + (10-1)d \\
32 &= 4d \\
d &= 8 \\
43 &= a + 5 \cdot 8 \\
43 &= a + 40 \\
3 &= a \\
a &= 3 \\
\therefore\ a_1 &= 3,\ d = 8
\end{aligned}
$$

5. The 7th term of an AP is -10, and the 12th term -25, find the 15th term of this AP.

   Sol.

$$
\begin{aligned}
a_7 &= -10 \\
a_{12} &= -25 \\
-10 &= a + (7-1)d \\
-25 &= a + (12-1)d \\
-15 &= 5d \\
d &= -3 \\
-10 &= a + 6 \cdot -3 \\
-10 &= a - 18 \\
a &= 8 \\
a_{15} &= 8 + (15-1) \cdot -3 \\
&= 8 - 42 \\
&= -34
\end{aligned}
$$

6. How many multiples of 7 are there between 100 and 200?

   Sol.

$$
\begin{aligned}
a &= 105 \\
d &= 7 \\
a_n &= 196 \\
196 &= 105 + (n-1) \cdot 7 \\
91 &= 7(n-1) \\
13 &= n-1 \\
n &= 14
\end{aligned}
$$

7. Find the arithmetic mean o fthe following number pairs.

   (a) $(8, 20)$

   Sol.

$$\frac{8+20}{2} = 14$$

   (b) $(-9, 17)$

Sol.

$$\frac{-9+17}{2} = 4$$

8. Find 5 numbers between 22 and 58 such that these 7 numbers are in AP.

Sol.

$$\begin{aligned} a_1 &= 22 \\ a_7 &= 58 \\ 58 &= 22 + (7-1)d \\ 36 &= 6d \\ d &= 6 \\ \therefore\ &These\ 5\ numbers\ are\ 22, 28, 34, 40, 46 \end{aligned}$$

9. Find the sum of first 20 terms of AP $12 + 15 + 18 + \cdots$

Sol.

$$\begin{aligned} a_1 &= 12 \\ n &= 20 \\ d &= 3 \\ S_{20} &= \frac{20}{2}(2 \cdot 12 + (20-1) \cdot 3) \\ &= 10(24 + 57) \\ &= 10(81) \\ &= 810 \end{aligned}$$

10. Find the sum of first 12 terms of the AP $18 + 10 + 2 - 6 - \ldots$

Sol.

$$\begin{aligned} a_1 &= 18 \\ n &= 12 \\ d &= -8 \\ S_{12} &= \frac{12}{2}(2 \cdot 18 + (12-1) \cdot -8) \\ &= 6(36 - 88) \\ &= 6(-52) \\ &= -312 \end{aligned}$$

11. Find the sum of first 14 terms of the AP $\frac{1}{6} + \frac{4}{3} + \frac{5}{2} + \cdots$

Sol.

$$\begin{aligned} a_1 &= \frac{1}{6} \\ n &= 14 \\ d &= \frac{7}{6} \\ S_{14} &= \frac{14}{2}(2 \cdot \frac{1}{6} + (14-1) \cdot \frac{7}{6}) \\ &= 7(\frac{1}{3} + \frac{91}{6}) \\ &= 7 \cdot \frac{93}{6} \\ &= 7 \cdot \frac{31}{2} \\ &= \frac{217}{2} \end{aligned}$$

12. Find the sum of all the multiples of 13 in between 200 and 800.

Sol.

$$\begin{aligned} a_1 &= 208 \\ a_n &= 793 \\ d &= 13 \\ 793 &= 208 + (n-1) \cdot 13 \\ 585 &= 13(n-1) \\ 45 &= n - 1 \\ n &= 46 \end{aligned}$$

$$\begin{aligned} S_{46} &= \frac{46}{2}(2 \cdot 208 + (46-1) \cdot 13) \\ &= 23(416 + 585) \\ &= 23(1001) \\ &= 23023 \end{aligned}$$

13. If the sum of first n terms of the AP $-3, -7, -11, \cdots$ is -903, find the value of n.

Sol.

$$\begin{aligned} a_1 &= -3 \\ d &= -4 \\ -903 &= \frac{n}{2}(2 \cdot (-3) - 4(n-1)) \\ -1806 &= -2n - 4n^2 \\ 4n^2 + 2n - 1806 &= 0 \\ 2n^2 + n - 903 &= 0 \\ (n-21)(2n+43) &= 0 \\ n &= 21, -43(invalid) \\ \therefore n &= 21 \end{aligned}$$

14. Given that the first 3 terms of an AP are $x,\ 3x-4,\ 2x+7$, find:

    (a) The value of x

Sol.

$$\begin{aligned} 3x-4 &= \frac{x+2x+7}{2} \\ 6x-8 &= 3x+7 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

(b) The common difference

Sol.

$$\begin{aligned} a_1 &= x = 5 \\ a_2 &= 3x-4 = 3\cdot 5 - 4 = 11 \\ d &= 11-5 \\ &= 6 \end{aligned}$$

(c) The sum of first 10 terms.

Sol.

$$\begin{aligned} a_1 &= x = 5 \\ n &= 10 \\ d &= 6 \\ S_{10} &= \frac{10}{2}(2\cdot 5 + (10-1)\cdot 6) \\ &= 5(10+54) \\ &= 5(64) \\ &= 320 \end{aligned}$$

15. Let the sum of the first n terms of an AP to be $S_n = \frac{n(n+1)}{4}$, find:

(a) The first term

Sol.

$$\begin{aligned} \frac{n(n+1)}{4} &= \frac{n}{2}(2a+(n-1)d) \\ n(n+1) &= 2n(2a+dn-d) \\ n^2+n &= 4na+2dn^2-2nd \\ n^2+n &= 2dn^2+(4a-2d)n \end{aligned}$$

$$\begin{aligned} &Comparing\ both\ sides, \\ 2d &= 1 \\ d &= \frac{1}{2} \\ 4a-2d &= 1 \\ 4a-1 &= 1 \\ 4a &= 2 \\ a &= \frac{1}{2} \end{aligned}$$

(b) The common difference

Sol.

$$d = \frac{1}{2}$$

gg

(c) The 6th terms

Sol.

$$\begin{aligned} a_1 &= \frac{1}{2} \\ n &= 6 \\ d &= \frac{1}{2} \\ a_6 &= \frac{1}{2} + (6-1)\cdot\frac{1}{2} \\ &= \frac{1}{2}+\frac{5}{2} \\ &= 3 \end{aligned}$$

(d) The sum from 6th term to 10th term

Sol.

$$\begin{aligned} a &= \frac{1}{2} \\ d &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} S_{10} &= \frac{10}{2}(2\cdot\frac{1}{2} + (10-1)\cdot\frac{1}{2}) \\ &= \frac{10}{2}(1+\frac{9}{2}) \\ &= 5\cdot\frac{11}{2} \\ &= \frac{55}{2} \end{aligned}$$

$$\begin{aligned} S_5 &= \frac{5}{2}(2\cdot\frac{1}{2} + (5-1)\cdot\frac{1}{2}) \\ &= \frac{5}{2}(1+2) \\ &= \frac{15}{2} \end{aligned}$$

$$\begin{aligned} S_{10} - S_6 &= \frac{55}{2} - \frac{15}{2} \\ &= \frac{40}{2} \\ &= 20 \end{aligned}$$

16. Given three numbers in an AP, the sum of these three numbers is 30, and the sum of square of these numbers is 318, find these three numbers.

Sol.

$$
\begin{aligned}
a_1 + a_2 + a_3 &= 30 \\
a_1^2 + a_2^2 + a_3^2 &= 318 \\
a_2 - a_1 &= a_3 - a_2 \\
a_1 - 2a_2 + a_3 &= 0 \\
3a_2 &= 30 \\
a_2 &= 10 \\
a_1 - 20 + a_3 &= 0 \\
a_1 + a_3 &= 20 \\
a_3 &= 20 - a_1 \\
a_1^2 + 100 + (20 - a_1)^2 &= 318 \\
a_1^2 + 100 + 400 + a_1^2 - 40a_1 &= 318 \\
2a_1^2 - 40a_1 + 182 &= 0 \\
a_1^2 - 20a_1 + 91 &= 0 \\
(a_1 - 7)(a_1 - 13) &= 0 \\
a_1 &= 7 or a_1 = 13
\end{aligned}
$$

$$\therefore These\ three\ numbers\ are\ 7,\ 10,\ and\ 13$$

17. Find the sum of all the numbers between 100 and 200 that are both the multiples of 2 and 3.

Sol.

$$
\begin{aligned}
a_1 &= 102 \\
d &= 6 \\
a_n &= 198 \\
198 &= 102 + (n-1) \cdot 6 \\
96 &= 6(n-1) \\
6n - 6 &= 96 \\
6n &= 102 \\
n &= 17
\end{aligned}
$$

$$
\begin{aligned}
S_{17} &= \frac{17}{2}(2 \cdot 102 + (17-1) \cdot 6) \\
&= \frac{17}{2}(204 + 96) \\
&= \frac{17}{2}(300) \\
&= 150 \cdot 17 \\
&= 2550
\end{aligned}
$$

18. Given an AP $-100 - 96 - 92 - \cdots$:

(a) Find the term where the number become positive.

Sol.

$$
\begin{aligned}
a_1 &= -100 \\
d &= 4 \\
a_n = -100 + (n-1) \cdot 4 &> 0 \\
-100 + 4n - 4 &> 0 \\
4n &> 104 \\
n &> 26
\end{aligned}
$$

$$\therefore n = 27$$

(b) Find the term where the sum of this AP becomes positive.

Sol.

$$
\begin{aligned}
S_n = \frac{n}{2}(2(-100) + (n-1) \cdot (4)) &> 0 \\
\frac{n}{2}(-200 + 4n - 4) &> 0 \\
\frac{n}{2}(-204 + 4n) &> 0 \\
n(2n - 102) &> 0 \\
n(n - 51) &> 0 \\
n &> 51
\end{aligned}
$$

$$\therefore n = 52$$

19. Find the first negative term of the AP $20, 19\frac{1}{5}, 18\frac{2}{5}, \cdots$

Sol.

$$
\begin{aligned}
a_1 &= 20 \\
d &= -\frac{4}{5} \\
a_n = 20 + (n-1) \cdot (-\frac{4}{5}) &< 0 \\
100 - 4n + 4 &< 0 \\
4n &> 104 \\
n &> 26
\end{aligned}
$$

$$\therefore n = 27$$

20. Given an AP $10 + 9\frac{1}{5} + 8\frac{2}{5} + \cdots$, what is the first negative term? When will the sum of the terms become negative, and what's the value of it?

Sol.

$$\begin{aligned} a_n = 10 + (n-1)\cdot(-\frac{4}{5}) &< 0 \\ 10 - \frac{4}{5}(n-1) &< 0 \\ 50 - 4n + 4 &< 0 \\ -4n &< -54 \\ n &> 13\frac{1}{2} \\ \\ \therefore n &= 14 \end{aligned}$$

$$\begin{aligned} S_n = \frac{n}{2}(2\cdot 10 + (n-1)\cdot(-\frac{4}{5})) &< 0 \\ \frac{n}{2}(20 - \frac{4}{5}(n-1)) &< 0 \\ 20n - \frac{4}{5}(n^2 - n) &< 0 \\ 100n - 4n^2 + 4n &< 0 \\ 25n - n^2 + n &< 0 \\ 26n - n^2 &< 0 \\ n(n-26) &> 0 \\ n &> 26 \\ \\ \therefore n &= 27 \end{aligned}$$

$$\begin{aligned} S_{27} &= \frac{27}{2}(2\cdot 10 + (27-1)\cdot(-\frac{4}{5})) \\ &= \frac{27}{2}(20 - \frac{4}{5}(27-1)) \\ &= \frac{27}{2}(20 - \frac{4}{5}(26)) \\ &= \frac{27}{2}\cdot(-\frac{4}{5}) \\ &= -\frac{54}{5} \end{aligned}$$

$\therefore The\ first\ negative\ term\ is\ the\ 14th\ term$

$\therefore The\ first\ term\ where\ the\ sum\ of\ the\ terms\ becomes\ negative\ is\ the\ 27th\ term$

$\therefore The\ value\ of\ the\ sum\ of\ the\ terms\ when\ it\ becomes\ negative\ is\ -\frac{54}{5}$

21. Given a polygon which all their internal angles are in AP. The common difference of this AP is 6°, the largest angle is 135°. How many sides does this polygon have?

Sol.

$$\begin{aligned} a_1 &= 135 \\ d &= -6 \\ \frac{n}{2}(2\cdot 135 + (n-1)\cdot(-6)) &= 180(n-2) \\ n(270 - 6(n-1)) &= 360(n-2) \\ n(276 - 6n) &= 360n - 720 \\ 276n - 6n^2 &= 360n - 720 \\ 46n - n^2 &= 60n - 120 \\ n^2 + 14n - 120 &= 0 \\ (n+20)(n-6) &= 0 \\ n &= -20\ (invalid) \\ n &= 6 \\ \therefore The\ number\ of\ sides\ is\ 6 & \end{aligned}$$

22. Given an AP which its 5th term is 3 and the sum of its first 10 terms is $26\frac{1}{4}$. Whcih term in this AP is 0?

Sol.

$$\begin{aligned} a_5 = a + (5-1)d &= 3 \\ a + 4d &= 3 \\ S_{10} = \frac{10}{2}(2a + (10-1)d) &= 26\frac{1}{4} \\ 5(2a + 9d) &= 26\frac{1}{4} \\ 20(2a + 9d) &= 105 \\ 4(2a + 9d) &= 21 \\ 8a + 36d &= 21 \\ 8a + 32d &= 24 \\ 4d &= -3 \\ d &= -\frac{3}{4} \\ a &= 3 + \frac{3}{4}\cdot 4 \\ &= 6 \\ a_n = 6 + (n-1)\cdot(-\frac{3}{4}) &= 0 \\ 6 - \frac{3}{4}(n-1) &= 0 \\ 24 - 3n + 3 &= 0 \\ 3n &= 27 \\ n &= 9 \end{aligned}$$

23. Given that the sum of the first 6 terms of an AP is 96, and the sum of the first 20 terms is 3 times the sum of the first 10 terms of this AP. Find the first term and the 10th term of it.

Sol.

$$
\begin{aligned}
S_6 = \frac{6}{2}(2a + (6-1)d) &= 96 \\
3(2a + 5d) &= 96 \\
2a + 5d &= 32 \\
S_{20} &= 3S_{10} \\
\frac{20}{2}(2a + (20-1)d) &= 3 \cdot \frac{10}{2}(2a \\
&+ (10-1)d) \\
10(2a + 19d) &= 15(2a + 9d) \\
2(2a + 19d) &= 3(2a + 9d) \\
4a + 38d &= 6a + 27d \\
2a - 11d &= 0 \\
16d &= 32 \\
d &= 2 \\
a &= \frac{11 \cdot 2}{2} \\
&= 11 \\
a_{10} &= 11 + (10-1) \cdot 2 \\
&= 29
\end{aligned}
$$

24. Given that $5^2 \cdot 5^4 \cdot 5^6 \cdot \cdots \cdot 5^{2n} = (0.04)^{-28}$, find the value of n.

Sol.

$$
\begin{aligned}
(0.04)^{-28} &= \frac{1}{25}^{-28} \\
&= (5^(- 2))^{-28} \\
&= 5^{56} \\
\because n^a \cdot n^b &= n^{a+b} \\
2 + 4 + 6 + \cdots + 2n &= 56 \\
S_n = \frac{n}{2}(2 \cdot 2 + (n-1) \cdot 2) &= 56 \\
n(4 + 2(n-1)) &= 112 \\
n(2 + 2n) &= 112 \\
2n^2 + 2n &= 112 \\
n^2 + n - 56 &= 0 \\
(n+8)(n-7) &= 0 \\
n &= -8 \ (invalid) \\
n &= 7
\end{aligned}
$$

25. Given that the 9th term of an AP is double the 5th term of it. Find the ratio of the sum of first 9 terms and the sum of first 5 terms of the AP.

Sol.

$$
\begin{aligned}
a_9 &= 2a_5 \\
a + (9-1)d &= 2(a + (5-1)d) \\
a + 8d &= 2a + 8d \\
a &= 0 \\
S_9 : S_5 &= \frac{9}{2}(2a + a_9) : \frac{5}{2}(2a + a_5) \\
&= \frac{9}{2}(2a + 2a_5) : \frac{5}{2}(2a + a_5) \\
&= 9(a + a_5) : \frac{5}{2}(2a + a_5) \\
\frac{S_9}{S_5} &= \frac{9(a + a_5)}{\frac{5}{2}(2a + a_5)} \\
&= \frac{18(a + a_5)}{5(2a + a_5)} \\
&= \frac{18 \cdot a_5}{5 \cdot a_5} \\
&= \frac{18}{5} \\
\therefore S_9 : S_5 &= 18 : 5
\end{aligned}
$$

12.3 Geometric Progression

The general formula of a geometric progression (GP) is given by

$$a_n = a_1 \cdot r^{n-1}$$

where a_1 is the first term, r is the common ratio, and n is the number of terms.

12.3.1 Practice 6

1. Find the 6th term of the GP $12, -18, 27, \cdots$

Sol.

$$
\begin{aligned}
a_1 &= 12 \\
r &= \frac{-18}{12} \\
&= -\frac{3}{2} \\
a_6 &= 12 \cdot (-\frac{3}{2})^{6-1} \\
&= 12 \cdot (-\frac{3}{2})^{5} \\
&= 12 \cdot (-\frac{243}{32}) \\
&= -\frac{729}{8}
\end{aligned}
$$

2. Find the number of terms of GP $\frac{1}{64} - \frac{1}{32} + \frac{1}{16} - \frac{1}{8} + \cdots - 512$

Sol.

$$\begin{aligned}
a_1 &= \frac{1}{64} \\
r &= \frac{-\frac{1}{32}}{\frac{1}{64}} \\
&= -2 \\
-512 &= \frac{1}{64}(-2)^{n-1} \\
(-2)^9 &= \frac{1}{2^6}(-2)^{n-1} \\
(-2)^{15} &= (-2)^{n-1} \\
n-1 &= 15 \\
n &= 16
\end{aligned}$$

3. The 5th term of a GP is 3, and its 9th term is $\frac{1}{27}$, find the first term and the common ratio of this GP.

   Sol.

$$\begin{aligned}
a_5 &= ar^4 = 3 \\
a_9 &= ar^8 = \frac{1}{27} \\
r^4 &= \frac{1}{27} \cdot \frac{1}{3} \\
&= \frac{1}{81} \\
r &= \frac{1}{3} \\
a_1 &= 3 \cdot 81 \\
&= 243
\end{aligned}$$

4. Find 5 numbers between $\frac{1}{2}$ and $frac1128$ such that these 7 numbers are in GP.

   Sol.

$$\begin{aligned}
a_1 &= \frac{1}{2} \\
n &= 7 \\
\frac{1}{128} &= \frac{1}{2}r^{7-1} \\
r^6 &= \frac{1}{64} \\
r &= \frac{1}{2}
\end{aligned}$$

$$\therefore These\ 5\ numbers\ are\ \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$$

Geometric Mean

The geometric mean G of two numbers x and y is given by

$$\begin{aligned}
\frac{G}{x} &= \frac{G}{y} \\
G^2 &= xy \\
G &= \mp\sqrt[2]{xy}
\end{aligned}$$

12.3.2 Practice 7

Find the geometric mean of $\frac{27}{8}$ and $\frac{2}{3}$.

Sol.

$$\begin{aligned}
G &= \pm\sqrt[2]{\frac{27}{8} \cdot \frac{2}{3}} \\
&= \pm\sqrt[2]{\frac{9}{4}} \\
&= \pm\frac{3}{2}
\end{aligned}$$

Summation of Geometric Progression

The sum of n terms of a GP is given by

$$S_n = \frac{a_1(1-r^n)}{1-r} \ (r \neq 1)$$

12.3.3 Practice 8

1. Find the sum of the first 8 terms of GP $3+6+12+\cdots$

   Sol.

$$\begin{aligned}
a_1 &= 3 \\
r &= \frac{6}{3} \\
&= 2 \\
n &= 8 \\
S_n &= \frac{3(1-2^8)}{1-2} \\
&= \frac{3(1-256)}{1-2} \\
&= 3 \cdot 255 \\
&= 765
\end{aligned}$$

2. Find the sum of the GP $1+\sqrt{3}+3+\cdots+81$

Sol.

$$
\begin{aligned}
a_1 &= 1 \\
r &= \sqrt{3} \\
81 &= 1 \cdot (\sqrt{3})^{n-1} \\
3^4 &= (\sqrt{3})^{n-1} \\
(\sqrt{3})^8 &= (\sqrt{3})^{n-1} \\
n-1 &= 8 \\
n &= 9 \\
S_n &= \frac{1(1-(\sqrt{3})^9)}{1-\sqrt{3}} \\
&= \frac{1-81\sqrt{3}}{1-\sqrt{3}} \\
&= \frac{(1-81\sqrt{3})(1+\sqrt{3})}{-2} \\
&= \frac{1-81\sqrt{3}+\sqrt{3}-243}{-2} \\
&= \frac{-242-80\sqrt{(3)}}{-2} \\
&= 121+40\sqrt{3}
\end{aligned}
$$

3. Given that the sum of the first n terms of GP $4\frac{4}{5}, 1\frac{3}{5}, \frac{8}{15}, \cdots$ is $7\frac{145}{729}$, find n.

Sol.

$$
\begin{aligned}
a_1 &= \frac{24}{5} \\
r &= \frac{8}{5} \cdot \frac{5}{24} \\
&= \frac{1}{3} \\
S_n &= \frac{24}{5} \cdot \frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}} \\
\frac{5248}{729} &= \frac{24}{5} \cdot \frac{1-(\frac{1}{3})^n}{\frac{2}{3}} \\
\frac{5248}{729} \cdot \frac{5}{24} \cdot \frac{2}{3} &= 1-(\frac{1}{3})^n \\
\frac{6560}{6561} &= 1-(\frac{1}{3})^n \\
-\frac{1}{6561} &= -(\frac{1}{3})^n \\
(\frac{1}{3})^8 &= (\frac{1}{3})^n \\
n &= 8
\end{aligned}
$$

Summation of Infinite Geometric Progression

The sum of infinite GP is given by

$$S_\infty = \frac{a_1}{1-r} \; (-1 < r < 1)$$

12.3.4 Practice 9

1. Find the sum of the following infinite GP.

(a) $16 + 8 + 4 + \cdots$

Sol.

$$
\begin{aligned}
a_1 &= 16 \\
r &= \frac{8}{16} \\
&= \frac{1}{2} \\
S_\infty &= \frac{16}{1-\frac{1}{2}} \\
&= \frac{16}{\frac{1}{2}} \\
&= 32
\end{aligned}
$$

(b) $18 - 12 + 8 + \cdots$

Sol.

$$
\begin{aligned}
a_1 &= 18 \\
r &= \frac{8}{-12} \\
&= -\frac{2}{3} \\
S_\infty &= \frac{18}{1+\frac{2}{3}} \\
&= \frac{18}{\frac{5}{3}} \\
&= \frac{54}{5}
\end{aligned}
$$

(c) $1 + \frac{3}{4} + \frac{9}{16} + \cdots$

Sol.

$$
\begin{aligned}
a_1 &= 1 \\
r &= \frac{9}{16} \cdot \frac{16}{9} \\
&= \frac{3}{4} \\
S_\infty &= \frac{1}{1-\frac{3}{4}} \\
&= \frac{1}{\frac{1}{4}} \\
&= 4
\end{aligned}
$$

(d) $\sqrt{2} + 1 + \frac{1}{\sqrt{2}} + \cdots$

Sol.

$$
\begin{aligned}
a_1 &= \sqrt{2} \\
r &= \frac{1}{\sqrt{2}} \\
S_\infty &= \frac{\sqrt{2}}{1-\frac{1}{\sqrt{2}}} \\
&= \frac{\sqrt{2}}{\frac{\sqrt{2}-1}{\sqrt{2}}} \\
&= \frac{2}{\sqrt{2}-1} \\
&= 2(\sqrt{2}+1)
\end{aligned}
$$

2. Convert the following recurring decimals to fraction using the summation of inifinite geometric series.

(a) $0.\overline{3}$

Sol.

$$
\begin{aligned}
a_1 &= 0.3 \\
r &= 0.1 \\
S_\infty &= \frac{0.3}{1-0.1} \\
&= \frac{0.3}{0.9} \\
&= \frac{1}{3}
\end{aligned}
$$

$$\therefore 0.\overline{3} = \frac{1}{3}$$

(b) $0.5\overline{3}$

Sol.

$$
\begin{aligned}
a_1 &= 0.03 \\
r &= 0.01 \\
S_\infty &= \frac{0.03}{1-0.01} \\
&= \frac{0.03}{0.99} \\
&= \frac{3}{99}
\end{aligned}
$$

$$
\begin{aligned}
\therefore 0.5\overline{3} &= \frac{5}{10} + \frac{3}{99} \\
&= \frac{53}{99}
\end{aligned}
$$

12.3.5 Exercise 12.3

1. Find the 10th term of the GP $2, 4, 8, \cdots$

Sol.

$$
\begin{aligned}
a_1 &= 2 \\
r &= \frac{4}{2} \\
&= 2 \\
a_{10} &= 2 \cdot 2^{10-1} \\
&= 2 \cdot 512 \\
&= 1024
\end{aligned}
$$

2. Find the 8th term of the GP $243, -162, 108, \cdots$

Sol.

$$
\begin{aligned}
a_1 &= 243 \\
r &= \frac{-162}{243} \\
&= -\frac{2}{3} \\
a_8 &= 243 \cdot (-\frac{2}{3})^{8-1} \\
&= 243 \cdot (-\frac{128}{2187}) \\
&= -\frac{128}{9}
\end{aligned}
$$

3. Find the number of terms of the following GP.

(a) $8, 4, 2, 1, \ldots, \frac{1}{64}$

Sol.

$$
\begin{aligned}
a_1 &= 8 \\
r &= \frac{4}{8} \\
&= \frac{1}{2} \\
\frac{1}{64} &= 8 \cdot (\frac{1}{2})^{n-1} \\
\frac{1}{512} &= (\frac{1}{2})^{n-1} \\
\frac{1}{2^9} &= (\frac{1}{2})^{n-1} \\
n-1 &= 9 \\
n &= 10
\end{aligned}
$$

(b) $6, -18, 54, \ldots, -13122$

Sol.

$$
\begin{aligned}
a_1 &= 6 \\
r &= \frac{-18}{6} \\
&= -3 \\
-13122 &= 6 \cdot (-3)^{n-1} \\
-2187 &= (-3)^{n-1} \\
(-3)^7 &= (-3)^{n-1} \\
n-1 &= 7 \\
n &= 8
\end{aligned}
$$

(c) $54, 36, 24, \dots, 3\frac{13}{81}$

Sol.

$$\begin{aligned} a_1 &= 54 \\ r &= \frac{36}{54} \\ &= \frac{2}{3} \\ \frac{256}{81} &= 54 \cdot (\frac{2}{3})^{n-1} \\ \frac{256}{81} \cdot \frac{1}{54} &= (\frac{2}{3})^{n-1} \\ \frac{128}{2187} &= (\frac{2}{3})^{n-1} \\ (\frac{2}{3})^7 &= (\frac{2}{3})^{n-1} \\ n-1 &= 7 \\ n &= 8 \end{aligned}$$

4. Given that the 2nd term of a GP is 12, and its 4th term is 108, find the first term and the common ratio of it.

Sol.

$$\begin{aligned} a_2 &= ar = 12 \\ a_4 &= ar^3 = 109 \\ r^2 &= 9 \\ r &= \pm 3 \\ a_1 &= \pm 4 \\ \therefore a_1 &= 4, r = 3 \text{ or } a_1 = -4, r = -3 \end{aligned}$$

5. Given that the 3rd term of an GP is $1\frac{1}{3}$, and its 8th term is $-10\frac{1}{8}$. Find the 5th term of this AP.

Sol.

$$\begin{aligned} a_3 &= ar^2 = \frac{4}{3} \\ a_8 &= ar^7 = -\frac{81}{8} \\ r^5 &= -\frac{81}{8} \cdot \frac{3}{4} \\ &= -\frac{243}{32} \\ &= (-\frac{3}{2})^5 \\ r &= -\frac{3}{2} \\ a &= \frac{4}{3} \cdot \frac{4}{9} \\ &= \frac{16}{27} \\ a_5 &= \frac{16}{27} \cdot (\frac{3}{2})^4 \\ &= \frac{16}{27} \cdot \frac{81}{16} \\ &= 3 \end{aligned}$$

6. Find the geometric mean of 2 and 18.

Sol.

$$\begin{aligned} G &= \pm \sqrt[2]{2 \cdot 18} \\ &= \pm \sqrt[2]{36} \\ &= \pm 6 \end{aligned}$$

7. Given that x+12, x+4 and x-2 are in GP, find the value of x and the common ratio of this GP.

Sol.

$$\begin{aligned} x+4 &= \pm\sqrt{(x+12)(x-2)} \\ x^2 + 8x + 16 &= x^2 + 10x - 24 \\ 2x &= 40 \\ x &= 20 \\ a_1 &= 20 + 12 = 32 \\ a_2 &= 20 + 4 = 24 \\ r &= \frac{24}{32} \\ &= \frac{3}{4} \end{aligned}$$

8. Find 3 numbers between 14 and 224 such that these 5 numbersare in GP.

Sol.

$$\begin{aligned} a_1 &= 14 \\ a_5 &= 224 \\ 244 &= 14 \cdot r^4 \\ 16 &= r^4 \\ (\pm 2)^4 &= r^4 \\ r &= \pm 2 \end{aligned}$$

$$\begin{aligned} &\therefore These\ 3\ numbers\ are\ 28,\ 56,\ 112 \\ &\quad or\ -28, 56, -112 \end{aligned}$$

9. Calculate the sum of the first 6 terms of the GP $2 + 6 + 18 + \cdots$

Sol.

$$\begin{aligned} a_1 &= 2 \\ r &= \frac{6}{2} \\ &= 3 \\ S_6 &= \frac{2(1-3^6)}{1-3} \\ &= \frac{2(1-729)}{-2} \\ &= 728 \end{aligned}$$

10. Calculate the sum of the first 8 terms of the GP $32 - 16 + 8 - \cdots$

Sol.

$$\begin{aligned} a_1 &= 32 \\ r &= \frac{-16}{32} \\ &= -\frac{1}{2} \\ S_8 &= \frac{32(1-(\frac{1}{2})^8)}{1+\frac{1}{2}} \\ &= \frac{32(1-\frac{1}{256})}{\frac{3}{2}} \\ &= 32 \cdot \frac{255}{256} \cdot \frac{2}{3} \\ &= \frac{85}{4} \end{aligned}$$

11. Find the sum of the GP $14 - 28 + 56 - \cdots + 3584$

Sol.

$$\begin{aligned} a_1 &= 14 \\ r &= \frac{-28}{14} = -2 \\ 3584 &= 14 \cdot (-2)^{n-1} \\ 256 &= (-2)^{n-1} \\ (-2)^8 &= (-2)^{n-1} \\ n-1 &= 8 \\ n &= 9 \\ S_9 &= \frac{14(1-(-2)^9)}{1-(-2)} \\ &= \frac{14(1+512)}{3} \\ &= \frac{14 \cdot 513}{3} \\ &= 2394 \end{aligned}$$

12. If the first term of a GP is 7, its common ratio is 3, and the sum of its terms is 847, find the number of terms and the last term of this GP.

Sol.

$$\begin{aligned} a_1 &= 7 \\ r &= 3 \\ S_n &= \frac{7(1-3^n)}{1-3} = 847 \\ 7(1-3^n) &= -1694 \\ 1-3^n &= -242 \\ 3^n &= 243 \\ 3^n &= 3^5 \\ n &= 5 \\ a_5 &= 7 \cdot 3^4 = 567 \end{aligned}$$

13. Find the sum of the following infinite GP.

(a) $24 + 18 + 13\frac{1}{2} + \cdots$

Sol.

$$\begin{aligned} a_1 &= 24 \\ r &= \frac{18}{24} = \frac{3}{4} \\ S_\infty &= \frac{24}{1-\frac{3}{4}} \\ &= \frac{24}{\frac{1}{4}} \\ &= 96 \end{aligned}$$

(b) $27 - 9 + 3 - 1 + \cdots$

Sol.

$$\begin{aligned} a_1 &= 27 \\ r &= \frac{-9}{27} = -\frac{1}{3} \\ S_\infty &= \frac{27}{1+\frac{1}{3}} \\ &= \frac{27}{\frac{4}{3}} \\ &= \frac{81}{4} \end{aligned}$$

(c) $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \cdots$

Sol.

$$\begin{aligned} a_1 &= 2 \\ r &= \frac{-\frac{1}{2}}{2} = -\frac{1}{4} \\ S_\infty &= \frac{2}{1+\frac{1}{4}} \\ &= \frac{2}{\frac{5}{4}} \\ &= \frac{8}{5} \end{aligned}$$

14. Given an infinite GP which has a sum of 24 and first term of 30, find the common difference.

Sol.

$$\begin{aligned} a_1 &= 30 \\ S_\infty &= 24 \\ 24 &= \frac{30}{1-r} \\ 24(1-r) &= 30 \\ 24 - 24r &= 30 \\ -24r &= 6 \\ r &= -\frac{1}{4} \end{aligned}$$

15. Convert the following recurrring decimals into fractions.

(a) $0.\overline{45}$

Sol.

$$\begin{aligned}a_1 &= 0.45\\ r &= 0.01\\ S_\infty &= \frac{0.45}{1-0.01}\\ &= \frac{0.45}{0.99}\\ &= \frac{45}{99}\\ &= \frac{5}{11}\end{aligned}$$

$$\therefore 0.\overline{45} = \frac{5}{11}$$

(b) $0.\overline{037}$

Sol.

$$\begin{aligned}a_1 &= 0.037\\ r &= 0.001\\ S_\infty &= \frac{0.037}{1-0.001}\\ &= \frac{0.037}{0.999}\\ &= \frac{37}{999}\\ &= \frac{1}{27}\end{aligned}$$

$$\therefore 0.\overline{037} = \frac{1}{27}$$

(c) $0.2\overline{18}$

Sol.

$$\begin{aligned}a_1 &= 0.018\\ r &= 0.01\\ S_\infty &= \frac{0.018}{1-0.01}\\ &= \frac{0.018}{0.99}\\ &= \frac{18}{990}\\ &= \frac{1}{55}\end{aligned}$$

$$\begin{aligned}\therefore 0.2\overline{18} &= \frac{1}{5} + \frac{1}{55}\\ &= \frac{12}{55}\end{aligned}$$

(d) $1.\overline{3}$

Sol.

$$\begin{aligned}a_1 &= 0.3\\ r &= 0.1\\ S_\infty &= \frac{0.3}{1-0.1}\\ &= \frac{0.3}{0.9}\\ &= \frac{1}{3}\end{aligned}$$

$$\begin{aligned}\therefore 1.\overline{3} &= 1 + \frac{1}{3}\\ &= \frac{4}{3}\end{aligned}$$

16. Three integers are in GP, summing up to 42 while accumulating up to 512, find these three integers.

Sol.

$$\begin{aligned}a_1 + a_2 + a_3 &= 42\\ a_1a_2a_3 &= 512\\ a_2 &= \pm\sqrt{a_1a_3}\\ a_1a_3 &= a_2^2\\ a_2^3 &= 512\\ a_2 &= \sqrt[3]{512}\\ &= 8\\ a_1a_3 &= 64\\ a_3 &= \frac{64}{a_1}\\ a_1 + 8 + \frac{64}{a_1} &= 42\\ a_1 + \frac{64}{a_1} &= 34\\ a_1^2 + 64 &= 34a_1\\ a_1^2 - 34a_1 + 64 &= 0\\ (a_1 - 32)(a_1 - 2) &= 0\\ a_1 &= 32 \ or \ a_1 = 2\end{aligned}$$

$$\therefore These\ three\ integers\ are\ 2,\ 8,\ 32$$

17. The sum of first 6 term of a GP is 9 times the sum of first 3 terms. Find the common ratio.

Sol.

$$\begin{aligned}
S_6 &= 9S_3\\
\frac{a(1-r^6)}{1-r} &= 9\cdot\frac{a(1-r^3)}{1-r}\\
a(1-r^6) &= 9a(1-r^3)\\
1-r^6 &= 9(1-r^3)\\
&= 9-9r^3\\
r^6-9r^3+8 &= 0\\
(r^3-8)(r^3-1) &= 0\\
r^3 &= 8\ or\ r^3=1\\
r &= 1\ (invalid)\\
r &= 2
\end{aligned}$$

18. Given a GP, its first term is 16, last term is $\frac{1}{2}$ and its sum is $31\frac{1}{2}$, find its common ratio and number of terms.

Sol.

$$\begin{aligned}
a_1 &= 16\\
\frac{1}{2} &= 16r^{n-1}\\
\frac{1}{32} &= r^{n-1}\\
&= r^n\cdot\frac{1}{r}\\
r^n &= \frac{r}{32}\\
\frac{63}{2} &= \frac{16(1-r^n)}{1-r}\\
63(1-r) &= 32(1-r^n)\\
63-63r &= 32-32r^n\\
-31 &= 32r^n-63r\\
-31 &= r-63r\\
-31 &= -62r\\
r &= \frac{1}{2}\\
(\frac{1}{2})^{n-1} &= \frac{1}{32}\\
&= (\frac{1}{2})^5\\
n-1 &= 5\\
n &= 6
\end{aligned}$$

19. Given a GP, its 3rd term is 6 less than its 2nd term, ant its 2nd term is 9 less than its 1st term. Find the 4th term and the sum of the first 4 terms.

Sol.

$$\begin{aligned}
Let\ x &= a_2\\
a_3 &= x-6\\
a_1 &= x+9\\
x &= \pm\sqrt{(x-6)(x+9)}\\
x^2 &= x^2+3x-54\\
3x-54 &= 0\\
x &= 18\\
a_2 &= 18\\
a_1 &= 27\\
r &= \frac{12}{18}\\
&= \frac{2}{3}\\
a_4 &= 27\cdot(\frac{2}{3})^3\\
&= 8\\
S_4 &= \frac{27(1-(\frac{16}{3})^4)}{1-\frac{2}{3}}\\
&= \frac{27(1-\frac{8}{81})}{\frac{1}{3}}\\
&= 81\cdot\frac{65}{81}\\
&= 65
\end{aligned}$$

20. GIven an infinite GP, its common ratio is positive and the sum of it is 9. The sum of the first two terms is 5, find the 4th term.

Sol.

$$
\begin{aligned}
S_\infty &= \frac{a}{1-r} = 9 \\
a &= 9(1-r) \\
&= 9-9r \\
S_2 &= \frac{a(1-r^2)}{1-r} = 5 \\
a - ar^2 &= 5-5r \\
9-9r-(9-9r)r^2 &= 5-5r \\
9-9r-9r^2+9r^3 &= 5-5r \\
4-4r-9r^2+9r^3 &= 0 \\
4(1-r)-9r^2(1-r) &= 0 \\
(4-9r^2)(1-r) &= 0 \\
(9r^2-4)(r-1) &= 0 \\
(3r^2+2)(3r^2-2)(r-1) &= 0 \\
r &= 1\ (invalid) \\
r &= -\frac{2}{3}(invalid) \\
r &= \frac{2}{3} \\
a &= 9(1-\frac{2}{3}) \\
&= 3 \\
a_4 &= 3(\frac{2}{3})^3 \\
&= 3 \cdot \frac{8}{27} \\
&= \frac{8}{9}
\end{aligned}
$$

21. If $x+1, x-2, \frac{1}{2}x$ are the first three terms of an infinite GP, find:

(a) The value of x

Sol.

$$
\begin{aligned}
x-2 &= \pm\sqrt{(x+1)(\frac{1}{2}x)} \\
x^2-4x+4 &= \frac{1}{2}x(x+1) \\
2x^2-8x+8 &= x^2+x \\
x^2-9x+8 &= 0 \\
(x-8)(x-1) &= 0 \\
x &= 8 \ or \ x = 1
\end{aligned}
$$

(b) The common ratio

Sol.

$$
\begin{aligned}
&When\ x = 8, \\
r &= \frac{8-2}{8+1} \\
&= \frac{6}{9} \\
&= \frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
&When\ x = 1, \\
r &= \frac{1-2}{1+1} \\
&= -\frac{1}{2}
\end{aligned}
$$

(c) The sum of the GP

Sol.

$$
\begin{aligned}
&When\ x = 8, \\
S_\infty &= \frac{a}{1-r} \\
&= \frac{9}{1-\frac{2}{3}} \\
&= 9 \cdot 3 \\
&= 27
\end{aligned}
$$

$$
\begin{aligned}
&When\ x = 1, \\
S_\infty &= \frac{a}{1-r} \\
&= \frac{2}{1+\frac{1}{2}} \\
&= 2 \cdot \frac{2}{3} \\
&= \frac{4}{3}
\end{aligned}
$$

12.4 Simple Summation of Special Series

Sum formula of natural number:

$$\sum_{i=1}^{n} k = \frac{n(n+1)}{2}$$

Sum formula of square of natural number:

$$\sum_{i=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum formula of cube of natural number:

$$\sum_{i=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

12.4.1 Practice 10

1. Find the sum of the following series.

(a) $\sum_{k=1}^{8} 3k$

Sol.

$$\begin{aligned}\sum_{k=1}^{8} 3k &= 3\sum_{k=1}^{8} k\\ &= 3\cdot\frac{8(8+1)}{2}\\ &= 3\cdot\frac{8\cdot 9}{2}\\ &= 3\cdot\frac{72}{2}\\ &= 3\cdot 36\\ &= 108\end{aligned}$$

(b) $\sum_{k=1}^{12} k^2$

Sol.

$$\begin{aligned}\sum_{k=1}^{12} k^2 &= \frac{12(12+1)(2\cdot 12+1)}{6}\\ &= \frac{12\cdot 13\cdot 25}{6}\\ &= 650\end{aligned}$$

(c) $\sum_{k=3}^{10}(2k-3)$

Sol.

$$\begin{aligned}&\sum_{k=3}^{10}(2k-3)\\ &= 2\sum_{k=3}^{10} k - \sum_{k=3}^{10} 3\\ &= 2\left[\sum_{k=1}^{10} k - \sum_{k=1}^{2} k\right] - (30-6)\\ &= 2\left[\frac{10(10+1)}{2} - \frac{2(2+1)}{2}\right] - 8\\ &= 2(55-3) - 24\\ &= 2\cdot 52 - 24\\ &= 104 - 24\\ &= 80\end{aligned}$$

(d) $\sum_{k=7}^{13} 3k^2$

Sol.

$$\begin{aligned}&\sum_{k=7}^{13} 3k^2\\ &= 3\left[\sum_{k=1}^{13} k^2 - \sum_{k=1}^{6} k^2\right]\\ &= 3\cdot\left[\frac{13(13+1)(2\cdot 13+1)}{6}\right.\\ &\quad\left. - \frac{6(6+1)(2\cdot 6+1)}{6}\right]\\ &= 3\cdot\left[\frac{13\cdot 14\cdot 27}{6} - \frac{6\cdot 7\cdot 13}{6}\right]\\ &= 3\cdot\left[\frac{4914}{6} - \frac{546}{6}\right]\\ &= 3\cdot\frac{4368}{6}\\ &= 3\cdot 728\\ &= 2184\end{aligned}$$

2. Given that the nth term of a series is n (n+3), find the sum of the first 20 terms of the series.

Sol.

$$\begin{aligned}&\sum_{k=1}^{20} k(k+3)\\ &= \sum_{k=1}^{20} k^2 + 3k\\ &= \sum_{k=1}^{20} k^2 + 3\sum_{k=1}^{20} k\\ &= \frac{20(20+1)(2\cdot 20+1)}{6} + 3\cdot\frac{20(20+1)}{2}\\ &= \frac{20\cdot 21\cdot 41}{6} + 3\cdot\frac{20\cdot 21}{2}\\ &= 2870 + 630\\ &= 3500\end{aligned}$$

3. Find the sum of series $1\cdot 3+2\cdot 4+3\cdot 5+\cdots+n(n+2)$.

Sol.

$$\begin{aligned}
&\sum_{k=1}^{n} k(k+2) \\
&= \sum_{k=1}^{n} k^2 + 2k \\
&= \sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k \\
&= \frac{n(n+1)(2n+1)}{6} + 2 \cdot \frac{n(n+1)}{2} \\
&= \frac{n(n+1)(2n+1)}{6} + n(n+1) \\
&= \frac{n(n+1)(2n+1) + 6n(n+1)}{6} \\
&= \frac{n(n+1)(2n+7)}{6}
\end{aligned}$$

12.4.2 Exercise 12.4

1. Find the sum of the following series.

(a) $\sum k = 1^8 5k^2$

Sol.

$$\begin{aligned}
\sum_{k=1}^{8} 5k^2 = 5\sum_{k=1}^{8} k^2 &= 5 \cdot \frac{8(8+1)(2\cdot 8+1)}{6} \\
&= 5 \cdot \frac{8\cdot 9\cdot 17}{6} \\
&= 5 \cdot \frac{1368}{6} \\
&= 5 \cdot 204 \\
&= 1020
\end{aligned}$$

(b) $\sum_{k=1}^{9} k^3$

Sol.

$$\begin{aligned}
\sum_{k=1}^{9} k^3 &= \left[\frac{9(9+1)}{2}\right]^2 \\
&= 45^2 \\
&= 2025
\end{aligned}$$

(c) $\sum_{n=1}^{10}(3n-5)$

Sol.

$$\begin{aligned}
\sum_{n=1}^{10}(3n-5) = 3\sum_{n=1}^{10} n - 5\sum_{n=1}^{10} 1 &= 3 \cdot \frac{10(10+1)}{2} - 5 \cdot 10 \\
&= 3 \cdot \frac{10 \cdot 11}{2} - 5 \cdot 10 \\
&= 3 \cdot 55 - 50 \\
&= 3 \cdot 5 - 50 \\
&= 165 - 50 \\
&= 115
\end{aligned}$$

(d) $\sum_{k=3}^{6} 2k^3$

Sol.

$$\begin{aligned}
\sum_{k=3}^{6} 2k^3 = 2\sum_{k=3}^{6} k^3 &= 2\left(\sum_{k=1}^{6} k^3 - \sum_{k=1}^{2} k^3\right) \\
&= 2\left\{\left[\frac{6(6+1)}{2}\right]^2 \right. \\
&\qquad \left. - \left[\frac{2(2+1)}{2}\right]^2\right\} \\
&= 2(21^2 - 3^2) \\
&= 2(441 - 9) \\
&= 2 \cdot 432 \\
&= 864
\end{aligned}$$

(e) $\sum_{k=6}^{10}(2k^2+3)$

Sol.

$$\begin{aligned}
&\sum_{k=6}^{10}(2k^2+3) \\
&= 2\sum_{k=6}^{10} k^2 + 3\sum_{k=6}^{10} 1 \\
&= 2\left(\sum_{k=1}^{10} k^2 - \sum_{k=1}^{5} k^2\right) \\
&\quad + 3 \cdot (10-5) \\
&= 2 \cdot \left[\frac{10\cdot 11\cdot 21}{6} - \frac{5\cdot 6\cdot 11}{6}\right] \\
&\quad + 3 \cdot 5 \\
&= 2 \cdot \left[\frac{2310}{6} - \frac{330}{6}\right] + 3 \cdot 5 \\
&= 2 \cdot \frac{1980}{6} + 3 \cdot 5 \\
&= 2 \cdot 330 + 3 \cdot 5 \\
&= 660 + 15 \\
&= 675
\end{aligned}$$

(f) $\sum_{n=11}^{15}(n^2+2n)$

Sol.

$$\begin{aligned}
&\sum_{n=11}^{15}(n^2+2n)\\
&=\sum_{n=11}^{15}n^2+2\sum_{n=11}^{15}n\\
&=\left[\sum_{n=1}^{15}n^2-\sum_{n=1}^{10}n^2\right]\\
&\quad+2\left[\sum_{n=1}^{15}n-\sum_{n=1}^{10}n\right]\\
&=\left[\frac{15\cdot16\cdot31}{6}-\frac{10\cdot11\cdot21}{6}\right]\\
&\quad+2\left[\frac{15\cdot16}{2}-\frac{10\cdot11}{2}\right]\\
&=985
\end{aligned}$$

(g) $\sum_{n=2}^{6}n(n^2-n+1)$

Sol.

$$\begin{aligned}
&\sum_{n=2}^{6}n(n^2-n+1)\\
&=\sum_{n=2}^{6}n^3-\sum_{n=2}^{6}n^2+\sum_{n=2}^{6}n\\
&=\left[\sum_{n=1}^{6}n^3-\sum_{n=1}^{1}n^3\right]-\left[\sum_{n=1}^{6}n^2-\sum_{n=1}^{1}n^2\right]\\
&\quad+\left[\sum_{n=1}^{6}n-\sum_{n=1}^{1}n\right]\\
&=\left[\left(\frac{6\cdot7}{2}\right)^2-\left(\frac{1\cdot2}{2}\right)^2\right]\\
&\quad-\left(\frac{6\cdot7\cdot13}{6}-\frac{1\cdot2\cdot3}{6}\right)\\
&\quad+\left(\frac{6\cdot7}{2}-\frac{1\cdot2}{2}\right)\\
&=21^2-1^2-(7\cdot13-1)+(3\cdot7-1)\\
&=440-90+20\\
&=370
\end{aligned}$$

2. Fiven that the nth term of a series is $3n^2+n$, find the sum of the first 10 terms of the series.

Sol.

$$\begin{aligned}
\sum_{n=1}^{10}3n^2+n&=3\sum_{n=1}^{10}n^2+\sum_{n=1}^{10}n\\
&=3\left(\frac{10\cdot11\cdot21}{6}\right)+\left(\frac{10\cdot11}{2}\right)\\
&=3\cdot\frac{2310}{6}+\frac{110}{2}\\
&=3\cdot385+55\\
&=1210
\end{aligned}$$

3. Find the sum of first nth term of series $1\cdot3+2\cdot7+3\cdot11+\cdots$

Sol.

$$\begin{aligned}
&\sum_{n=1}^{n}n\cdot(4n-1)\\
&=4\sum_{n=1}^{n}n^2-\sum_{n=1}^{n}n\\
&=4\left(\frac{n(n+1)(2n+1)}{6}\right)-\left(\frac{n(n+1)}{2}\right)\\
&=\frac{4n(n+1)(2n+1)-3n(n+1)}{6}\\
&=\frac{n(n+1)(8n+1)}{6}
\end{aligned}$$

4. Find the sum fo the series $1^2+3^2+5^2+\cdots+15^2$

Sol.

$$\begin{aligned}
\sum_{n=1}^{8}(2n-1)^2&=\sum_{n=1}^{8}(4n^2-4n+1)\\
&=4\sum_{n=1}^{8}n^2-4\sum_{n=1}^{8}n+\sum_{n=1}^{8}1\\
&=4\left(\frac{8\cdot9\cdot17}{6}\right)-4\left(\frac{8\cdot9}{2}\right)+8\\
&=4\cdot204-4\cdot36+8\\
&=816-144+8\\
&=680
\end{aligned}$$

12.5 Revision Exercise 12

1. Express the following series in form of $\sum$.

(a) $\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\cdots+\frac{49}{50}$

Sol.

$$\begin{aligned}
a_1&=\frac{2\cdot1-1}{2\cdot1}\\
a_2&=\frac{2\cdot2-1}{2\cdot2}\\
a_3&=\frac{2\cdot3-1}{2\cdot3}\\
&\vdots\\
a_{25}&=\frac{2\cdot25-1}{2\cdot25}\\
\therefore\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\cdots+\frac{49}{50}&=\sum_{n=1}^{25}\frac{2n-1}{2n}
\end{aligned}$$

(b) $6-7+8-9+\cdots$

Sol.

$$a_1 = (-1)^6 \cdot 6$$
$$a_2 = (-1)^7 \cdot 7$$
$$a_3 = (-1)^8 \cdot 8$$
$$\vdots$$
$$a_n = (-1)^n n \therefore \quad 6 - 7 + 8 - 9 + \cdots = \sum_{n=1}^{\infty} (-1)^n n$$

(c) $2 \cdot 5 + 3 \cdot 7 + 4 \cdot 9 + \cdots + 15 \cdot 31$

Sol.

$$\begin{aligned}
a_1 &= (1+1)(2 \cdot 1 + 3) \\
a_2 &= (2+1)(2 \cdot 2 + 3) \\
a_3 &= (3+1)(2 \cdot 3 + 3) \\
&\vdots \\
a_{14} &= (14+1)(2 \cdot 14 + 3) \\
\therefore\ & 2 \cdot 5 + 3 \cdot 7 + 4 \cdot 9 + \cdots + 15 \cdot 31 \\
&= \sum_{n=1}^{14} (n+1)(2n+3)
\end{aligned}$$

2. Given a general formula $a_n = \frac{3^n}{2n-3}$, state the first 5 terms of the sequence.

Sol.

$$\begin{aligned}
a_1 &= \frac{3^1}{2 \cdot 1 - 3} = -3 \\
a_2 &= \frac{3^2}{2 \cdot 2 - 3} = 9 \\
a_3 &= \frac{3^3}{2 \cdot 3 - 3} = 9 \\
a_4 &= \frac{3^4}{2 \cdot 4 - 3} = \frac{81}{5} \\
a_5 &= \frac{3^5}{2 \cdot 5 - 3} = \frac{243}{7}
\end{aligned}$$

3. Express the series $\sum_{k=1}^{10} (2k^2 - 3)$

Sol.

$$\begin{aligned}
&\sum_{k=1}^{10} (2k^2 - 3) \\
&= (2 \cdot 1^2 - 3) + (2 \cdot 2^2 - 3) + (2 \cdot 3^2 - 3) \\
&\quad + (2 \cdot 4^2 - 3) + (2 \cdot 5^2 - 3) + (2 \cdot 6^2 - 3) \\
&\quad + (2 \cdot 7^2 - 3) + (2 \cdot 8^2 - 3) + (2 \cdot 9^2 - 3) \\
&\quad + (2 \cdot 10^2 - 3) \\
&= -1 + 5 + 15 + 29 + 47 + 69 + 95 + 125 \\
&\quad + 159 + 197
\end{aligned}$$

4. State the first term, last term and the number of terms of theh series $\sum_{k=3}^{7} (3^k - 2^k - k)$

Sol.

$$\begin{aligned}
a_3 &= 3^3 - 2^3 - 3 = 27 - 8 - 3 = 16 \\
a_7 &= 3^7 - 2^7 - 7 = 2187 - 128 - 7 = 2052 \\
n &= 5
\end{aligned}$$

5. Find the number of terms of the AP $-4 - 2\frac{3}{4} - 112 - \frac{1}{4} + \cdots + 16$

Sol.

$$\begin{aligned}
a &= -4 \\
d &= \frac{5}{4} \\
16 &= -4 + (n-1)\frac{5}{4} \\
20 &= \frac{5}{4}(n-1) \\
5n - 5 &= 80 \\
5n &= 85 \\
n &= 17
\end{aligned}$$

6. If x+1, 2x+1, x-3 are the first 3 terms of AP, find:

(a) The value of x

Sol.

$$\begin{aligned}
2x + 1 &= \frac{x + 1 + x - 3}{2} \\
4x + 2 &= 2x - 2 \\
2x &= -4 \\
x &= -2
\end{aligned}$$

(b) Sum from the 10th term to the 20th term

Sol.

$$\begin{aligned}
a_1 &= -1 \\
a_2 &= -3 \\
r &= -2 \\
S &= S_{20} - S_9 \\
&= \frac{20}{2}(-2 + (20-1)(-2)) \\
&\quad - \frac{9}{2}(-2 + (9-1)(-2)) \\
&= 10 \cdot (-40) - 9 \cdot (-9) \\
&= -400 + 81 \\
&= -319
\end{aligned}$$

7. Find 4 numbers between 28 and -12 such that these 6 numbers form an AP.

Sol.

$$\begin{aligned}a_1 &= 28\\ a_n &= -12\\ n &= 6\\ -12 &= 28 + 5d\\ 5d &= 40\\ d &= 8\end{aligned}$$

$$\therefore These\ 4\ numbers\ are\ -4, 4, 12, 20$$

8. Find the sum of the following AP.

   (a) $7 + 11 + 15 + \cdots$ up to the 10th term
   Sol.

$$\begin{aligned}a_1 &= 7\\ d &= 4\\ n &= 10\\ S_{10} &= \frac{10}{2}(2 \cdot 7 + (10-1)4)\\ &= 5(14 + 36)\\ &= 250\end{aligned}$$

   (b) $20 + 18\frac{1}{2} + 17 + \cdots$ up to the 16tm term
   Sol.

$$\begin{aligned}a_1 &= 20\\ d &= -\frac{3}{2}\\ n &= 16\\ S_{16} &= \frac{16}{2}(2 \cdot 20 + (16-1)(-\frac{3}{2}))\\ &= 8(40 - \frac{45}{2})\\ &= 8 \cdot \frac{35}{2}\\ &= 140\end{aligned}$$

   (c) $2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} + \cdots + 13\sqrt{2}$
   Sol.

$$\begin{aligned}a_1 &= 2\sqrt{2}\\ d &= \sqrt{2}\\ n &= 12\\ S_{12} &= \frac{12}{2}(2 \cdot 2\sqrt{2} + (12-1)\sqrt{2})\\ &= 6(4\sqrt{2} + 11\sqrt{2})\\ &= 6 \cdot 15\sqrt{2}\\ &= 90\sqrt{2}\end{aligned}$$

9. Given an AP which the sum of the first n terms $S_n = n(1 + 2n)$, find:

   (a) First term

   Sol.

$$\begin{aligned}\frac{n}{2}(2a + (n-1)d) &= n(1 + 2n)\\ n(2a + (n - 1d)) &= 2n(1 + 2n)\\ 2an + dn^2 - dn &= 2n - 4n^2\\ (2a - d)n + dn^2 &= 2n - 4n^2\end{aligned}$$

$$\begin{aligned}&Comparing\ both\ sides,\\ &a = 3\\ &d = 4\end{aligned}$$

   (b) Common Difference

   Sol.

$$\begin{aligned}&According\ to\ the\ sol.\ of\ (a),\\ &d = 4\end{aligned}$$

   (c) Sum of the first 20 terms.

   Sol.

$$\begin{aligned}&According\ to\ the\ sol.\ of\ (a),\\ &a = 3\\ &d = 4\\ &n = 20\\ S_{20} &= \frac{20}{2}(2 \cdot 3 + (20-1)4)\\ &= 10(6 + 76)\\ &= 10 \cdot 82\\ &= 820\end{aligned}$$

10. Given an AP $33 + 27 + 21 + \cdots$

   (a) If the first sum of the first n terms is 105, find the value of n.

   Sol.

$$\begin{aligned}a_1 &= 33\\ d &= -6\\ 105 &= \frac{n}{2}(2 \cdot 33 + (n-1) \cdot (-6))\\ 210 &= n(66 - (n-1)6)\\ 35 &= 11n - n^2 + n\\ n^2 - 12n + 35 &= 0\\ (n-7)(n-5) &= 0\\ n &= 7\ or\ n = 5\end{aligned}$$

   (b) If the sum of the first n terms is negative value, find the minimum value of n.

Sol.

$$\begin{aligned} a_1 &= 33 \\ d &= -6 \\ \frac{n}{2}(2 \cdot 33 + (n-1) \cdot (-6)) &< 0 \\ n(66 - 6n + 6) &< 0 \\ 12n - n^2 &< 0 \\ n(12 - n) &< 0 \\ n &> 12 \end{aligned}$$

$$\therefore The\ minimum\ value\ of\ n\ is\ 13$$

11. Find the sum of the numbers between 150 and 300 that are multiple of both 5 and 3.

Sol.

$$\begin{aligned} a_1 &= 165 \\ a_n &= 285 \\ d &= 15 \\ 285 &= 165 + (n-1) \cdot 15 \\ 8 &= n - 1 \\ n &= 9 \end{aligned}$$

$$\begin{aligned} S_9 &= \frac{9}{2}(2 \cdot 165 + (9-1) \cdot 15) \\ &= \frac{9}{2} \cdot 450 \\ &= 2025 \end{aligned}$$

12. Find the sum of all the numbers between 100 and 200 that can be divided by 2 or 3.

Sol.

$$\begin{aligned} a_1 &= 102 \\ a_n &= 198 \end{aligned}$$

When $d = 2$,

$$\begin{aligned} 198 &= 102 + (n-1) \cdot 2 \\ 48 &= n - 1 \\ n &= 49 \\ S_{49} &= \frac{49}{2}(2 \cdot 102 + (49-1) \cdot 2) \\ &= \frac{49}{2} \cdot (204 + 96) \\ &= 7350 \end{aligned}$$

When $d = 3$,

$$\begin{aligned} 198 &= 102 + (n-1) \cdot 3 \\ 32 &= n - 1 \\ n &= 33 \\ S_{33} &= \frac{33}{2}(2 \cdot 102 + (33-1) \cdot 3) \\ &= \frac{33}{2} \cdot (204 + 96) \\ &= 4950 \end{aligned}$$

When $d = 6$,

$$\begin{aligned} 198 &= 102 + (n-1) \cdot 6 \\ 16 &= n - 1 \\ n &= 17 \\ S_{17} &= \frac{17}{2}(2 \cdot 102 + (17-1) \cdot 6) \\ &= \frac{17}{2} \cdot (204 + 96) \\ &= 2550 \end{aligned}$$

$$\begin{aligned} \therefore S &= 7350 + 4950 - 2550 \\ &= 9750 \end{aligned}$$

13. Find the sum of the numbers between 50 and 100 that cannot be divided by 5.

Sol.

$$\begin{aligned}
&\text{When } d = 1,\\
a_1 &= 51\\
a_n &= 99\\
99 &= 51 + (n-1)\cdot 1\\
48 &= n - 1\\
n &= 49\\
S_{49} &= \frac{49}{2}(2\cdot 51 + (49-1)\cdot 1)\\
&= \frac{49}{2}\cdot(102+48)\\
&= 3675
\end{aligned}$$

$$\begin{aligned}
&\text{When } d = 5,\\
a_1 &= 55\\
a_n &= 95\\
95 &= 55 + (n-1)\cdot 5\\
8 &= n - 1\\
n &= 9\\
S_9 &= \frac{9}{2}(2\cdot 55 + (9-1)\cdot 5)\\
&= \frac{9}{2}\cdot(110+40)\\
&= 675
\end{aligned}$$

$$\begin{aligned}
\therefore S &= 3675 - 675\\
&= 3000
\end{aligned}$$

14. Which term is the first negative term of the AP $20 + 16\frac{1}{4} + 12\frac{1}{2} + \cdots$?

Sol.

$$\begin{aligned}
a_1 &= 20\\
d &= -\frac{15}{4}\\
a_n = 20 - (n-1)\cdot\frac{15}{4} &< 0\\
80 - 15(n-1) &< 0\\
16 - 3n + 3 &< 0\\
3n &> 19\\
n &> 6\frac{1}{3}
\end{aligned}$$

$$\therefore The\ first\ negative\ term\ is\ 7$$

15. Three numbers are in AP, thier sum is 15 while the sum of the square of these numbers is 83. Find this three numbers.

Sol.

$$\begin{aligned}
a_1 + a_2 + a_3 &= 15\\
a_1^2 + a_2^2 + a_3^2 &= 83\\
a_2 - a_1 &= a_3 - a_2\\
a_1 + a_3 &= 2a_2\\
3a_2 &= 15\\
a_2 &= 5\\
a_3 &= 10 - a_1\\
a_1^2 + a_3^2 &= 83 - 25\\
&= 58\\
a_1^2 + (10 - a_1)^2 &= 58\\
a_1^2 + 100 - 20a_1 + a_1^2 &= 58\\
2a_1^2 - 20a_1 + 100 &= 58\\
2a_1^2 - 20a_1 + 42 &= 0\\
a_1^2 - 10a_1 + 21 &= 0\\
(a_1 - 7)(a_1 - 3) &= 0\\
a_1 &= 7\ or\ a_1 = 3
\end{aligned}$$

$$\therefore The\ three\ numbers\ are\ 7, 5, 3$$

16. Find the sum of the series $18^2 - 17^2 + 16^2 - 15^2 + 14^2 - 13^2 + \cdots + 2^2 - 1^2$

Sol.

$$\begin{aligned}
&18^2 - 17^2 + 16^2 - 15^2 + \cdots + 2^2 - 1^2\\
&= (18^2 - 17^2) + (16^2 - 15^2) + \cdots + (2^2 - 1^2)\\
&= ((2\cdot 9)^2 - (2\cdot 9 - 1)^2) + ((2\cdot 8)^2 - (2\cdot 8 - 1)^2)\\
&\quad + \cdots + ((2\cdot 1)^2 - (2\cdot 1 - 1)^2)\\
&= \sum_{n=1}^{9}\left[(2n)^2 - (2n-1)^2\right]\\
&= \sum_{n=1}^{9}(4n - 1)\\
&= 4\sum_{n=1}^{9} n - \sum_{n=1}^{9} 1\\
&= 4\cdot\frac{9\cdot 10}{2} - 9\\
&= 180 - 9\\
&= 171
\end{aligned}$$

17. State the general formula of the series $20, -10, 5, -2\frac{1}{2}, \cdots$

Sol.

$$a_1 = 20$$
$$r = -\frac{1}{2}$$
$$a_n = 20(-\frac{1}{2})^{n-1}$$

18. Given three integers x-3, x+1, 4x-2 that are in GP. If the sum of this GP is S, common ratio is r, find the value of S+r.

Sol.

$$x + 1 = \pm\sqrt{(x-3)(4x-2)}$$
$$x^2 + 2x + 1 = 4x^2 - 14x + 6$$
$$3x^2 - 16^x + 5 = 0$$
$$(3x-1)(x-5) = 0$$
$$x = 5 \text{ or } x = \frac{1}{3}$$

$$a_1 = x - 3 = 5 - 3 = 2$$
$$a_2 = x + 1 = 5 + 1 = 6$$
$$a_3 = 4x - 2 = 4(5) - 2 = 18$$

$$\begin{aligned} S &= a_1 + a_2 + a_3 \\ &= 2 + 6 + 18 \\ &= 26 \end{aligned}$$

$$r = \frac{a_3}{a_2} = \frac{18}{6} = 3$$

$$\begin{aligned} \therefore S + r &= 26 + 3 \\ &= 29 \end{aligned}$$

19. Find the geometric mean of $\frac{1}{3}$ and $\frac{1}{5}$

Sol.

$$\begin{aligned} G &= \pm\sqrt{\frac{1}{3} \cdot \frac{1}{5}} \\ &= \pm\sqrt{\frac{1}{15}} \\ &= \pm\frac{1}{\sqrt{15}} \\ &= \pm\frac{\sqrt{15}}{15} \end{aligned}$$

20. Find 5 numbers between $-\frac{1}{4}$ and $-\frac{1}{256}$ such that these 7 numbers form a GP.

Sol.

$$a_1 = -\frac{1}{4}$$
$$n = 7$$
$$-\frac{1}{256} = -\frac{1}{4}r^6$$
$$\frac{1}{64} = r^6$$
$$\left(\pm\frac{1}{2}\right)^6 = r^6$$
$$r = \pm\frac{1}{2}$$

When $r = \frac{1}{2}$,
These 5 numbers are
$\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$

When $r = -\frac{1}{2}$,
These 5 numbers are
$\frac{1}{8}, -\frac{1}{16}, \frac{1}{32}, -\frac{1}{64}, \frac{1}{128}$

21. Find the sum of the series $\sum_{n=5}^{15} n^2(3n+1)$

Sol.

$$\begin{aligned} \sum_{n=5}^{15} n^2(3n+1) &= \sum_{n=5}^{15} n^3 + \sum_{n=5}^{15} 3n^2 \\ &= 3\sum_{n=5}^{15} n^3 + \sum_{n=5}^{15} n^2 \\ &= 3\left[\sum_{n=1}^{15} n^3 - \sum_{n=1}^{4} n^3\right] \\ &\quad + \left[\sum_{n=1}^{15} n^2 - \sum_{n=1}^{4} n^2\right] \\ &= 3\left[\left(\frac{15 \cdot 16}{2}\right)^2 - \left(\frac{4 \cdot 5}{2}\right)^2\right] \\ &\quad + \left[\frac{15 \cdot 16 \cdot 31}{6} - \frac{4 \cdot 5 \cdot 9}{6}\right] \\ &= 3\left[(15 \cdot 8)^2 - (2 \cdot 5)^2\right] \\ &\quad + 1240 - 30 \\ &= 3(14400 - 100) + 1210 \\ &= 42900 + 1210 \\ &= 44110 \end{aligned}$$

22. Find the sum of the series $5^2 + 7^2 + 9^2 + \cdots + 25^2$

Sol.

$$\sum_{n=1}^{11} (2n+3)^2$$
$$= \sum_{n=1}^{11} 4n^2 + 12n + 9$$
$$= 4\sum_{n=1}^{11} n^2 + 12\sum_{n=1}^{11} n + 11$$
$$= 4\left[\frac{11 \cdot 12 \cdot 23}{6}\right] + 12\left[\frac{11 \cdot 12}{2}\right] + 99$$
$$= 2024 + 792 + 99$$
$$= 2915$$

23. Find the sum of the series $2 \cdot 3 + 3 \cdot 12 + 4 \cdot 27 + \cdots + (n+1) \cdot 3n^2$

Sol.

$$\sum_{n=1}^{n} (n+1)3n^2$$
$$= \sum_{n=1}^{n} 3n^3 + \sum_{n=1}^{n} 3n^2$$
$$= 3\left[\sum_{n=1}^{n} n^3 + \sum_{n=1}^{n} n^2\right]$$
$$= 3\left[\left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6}\right]$$
$$= 3\left[\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}\right]$$
$$= 3\left[\frac{3n^2(n+1)^2 + 2n(n+1)(2n+1)}{12}\right]$$
$$= \frac{n(n+1)\left[3n^2 + 3n + 4n + 2\right]}{4}$$
$$= \frac{n(n+1)\left[3n^2 + 7n + 2\right]}{4}$$
$$= \frac{n(n+1)(n+2)(3n+1)}{4}$$

Chapter 13

System of Equations

13.1 System of Equations with Two Variables

13.1.1 Practice 1

Solve the following system of equations.

1.

$$\begin{cases} 2x - 3y &= 11 \\ xy &= -5 \end{cases}$$

Sol.

$$\begin{cases} 2x - 3y = 11 & (1) \\ xy = -5 & (2) \end{cases}$$

$$\begin{aligned} (2) \Rightarrow y &= -\frac{5}{x} && (3) \\ \text{Sub (3) into (1)} \Rightarrow 2x - \frac{15}{x} &= 11 \\ 2x^2 - 15 &= 11x \\ 2x^2 - 11x - 15 &= 0 \\ (2x-5)(x-3) &= 0 \\ x = 3 \text{ or } x &= \frac{5}{2} \\ \text{Sub x = 3 into (2)} \Rightarrow y &= -\frac{5}{3} \\ \text{Sub x} = \frac{5}{2} \text{ into (2)} \Rightarrow y &= -\frac{5}{\frac{5}{2}} \\ \Rightarrow y &= -\frac{5}{5} \\ \Rightarrow y &= -1 \end{aligned}$$

$$\therefore \begin{cases} x = 3 \\ y = -\frac{5}{3} \end{cases} or \begin{cases} x = \frac{5}{2} \\ y = -1 \end{cases}$$

2.

$$\begin{cases} 3x + y &= 5 \\ x^2 - 2xy &= 8 \end{cases}$$

Sol.

$$\begin{cases} 3x + y = 5 & (1) \\ x^2 - 2xy = 8 & (2) \end{cases}$$

$$\begin{aligned} 3(1) \Rightarrow y &= 5 - 3x && (3) \\ \text{Sub (3) into (2)} \Rightarrow x^2 - 2x(5-3x) &= 8 \\ x^2 - 10x + 6x^2 &= 8 \\ 7x^2 - 10x + 8 &= 0 \\ (7x+4)(x-2) &= 0 \\ x = -\frac{4}{7} \text{ or } x &= 2 \\ \text{Sub } x = -\frac{4}{7} \text{ into (1)} \Rightarrow y &= 5 - 3\left(-\frac{4}{7}\right) \\ \Rightarrow y &= \frac{47}{7} \\ \text{Sub } x = 2 \text{ into (1)} \Rightarrow y &= -1 \end{aligned}$$

$$\therefore \begin{cases} x = -\frac{4}{7} \\ y = \frac{47}{7} \end{cases} or \begin{cases} x = 2 \\ y = -1 \end{cases}$$

13.1.2 Exercise 13.1

Solve the following system of equations.

1.

$$\begin{cases} x - y &= 1 \\ xy &= 6 \end{cases}$$

Sol.

$$\begin{cases} x - y = 1 & (1) \\ xy = 6 & (2) \end{cases}$$

$$\begin{aligned} (1) \Rightarrow y &= x - 1 && (3) \\ \text{Sub (3) into (2)} \Rightarrow x(x-1) &= 6 \\ x^2 - x &= 6 \\ x^2 - x - 6 &= 0 \\ (x+2)(x-3) &= 0 \\ x = -2 \text{ or } x &= 3 \\ \text{Sub } x = -2 \text{ into (1)} \Rightarrow y &= -2 - 1 \\ \Rightarrow y &= -3 \\ \text{Sub } x = 3 \text{ into (1)} \Rightarrow y &= 3 - 1 \\ \Rightarrow y &= 2 \end{aligned}$$

$$\therefore \begin{cases} x = -2 \\ y = -3 \end{cases} or \begin{cases} x = 3 \\ y = 2 \end{cases}$$

2.

$$\begin{cases} 3x - y = 4 \\ xy = 4 \end{cases}$$

Sol.

$$\begin{cases} 3x - y = 4 & (1) \\ xy = 4 & (2) \end{cases}$$

$$\begin{aligned} (1) &\Rightarrow y = 3x - 4 \quad (3) \\ \text{Sub (3) into (2)} &\Rightarrow x(3x-4) = 4 \\ & 3x^2 - 4x = 4 \\ & 3x^2 - 4x - 4 = 0 \\ & (3x+2)(x-2) = 0 \\ & x = -\frac{2}{3} \text{ or } x = 2 \\ \text{Sub } x = -\frac{2}{3} \text{ into (1)} &\Rightarrow y = 3\left(-\frac{2}{3}\right) - 4 \\ &\Rightarrow y = -6 \\ \text{Sub } x = 2 \text{ into (1)} &\Rightarrow y = 3(2) - 4 \\ &\Rightarrow y = 2 \end{aligned}$$

$$\therefore \begin{cases} x = -\frac{2}{3} \\ y = -6 \end{cases} \text{ or } \begin{cases} x = 2 \\ y = 2 \end{cases}$$

3.

$$\begin{cases} 3x + 4y = -39 \\ xy = 30 \end{cases}$$

Sol.

$$\begin{cases} 3x + 4y = -39 & (1) \\ xy = 30 & (2) \end{cases}$$

$$\begin{aligned} (2) &\Rightarrow y = \frac{30}{x} \quad (3) \\ \text{Sub (3) into (1)} &\Rightarrow 3x + 4\frac{30}{x} = -39 \\ & 3x^2 + 120 = -39x \\ & 3x^2 + 39x + 120 = 0 \\ & x^2 + 13x + 40 = 0 \\ & (x+5)(x+8) = 0 \\ & x = -5 \text{ or } x = -8 \\ \text{Sub } x = -5 \text{ into (1)} &\Rightarrow y = \frac{30}{-5} - 39 \\ &\Rightarrow y = -6 \\ \text{Sub } x = -8 \text{ into (1)} &\Rightarrow y = \frac{30}{-8} - 39 \\ &\Rightarrow y = -\frac{15}{4} \end{aligned}$$

$$\therefore \begin{cases} x = -5 \\ y = -6 \end{cases} \text{ or } \begin{cases} x = -8 \\ y = -\frac{15}{4} \end{cases}$$

4.

$$\begin{cases} y = 2x + 3 \\ y = x^2 - 2x + 1 \end{cases}$$

Sol.

$$\begin{cases} y = 2x + 3 & (1) \\ y = x^2 & (2) \end{cases}$$

$$\begin{aligned} (1) = (2) &\Rightarrow 2x + 3 = x^2 \\ &\Rightarrow x^2 - 2x - 3 = 0 \\ &\Rightarrow (x+1)(x-3) = 0 \\ &\Rightarrow x = -1 \text{ or } x = 3 \\ \text{Sub } x = -1 \text{ into (1)} &\Rightarrow y = 2(-1) + 3 \\ &\Rightarrow y = 1 \\ \text{Sub } x = 3 \text{ into (1)} &\Rightarrow y = 2(3) + 3 \\ &\Rightarrow y = 9 \end{aligned}$$

$$\therefore \begin{cases} x = -1 \\ y = 1 \end{cases} \text{ or } \begin{cases} x = 3 \\ y = 9 \end{cases}$$

5.

$$\begin{cases} x - y = 1 \\ x^2 + y^2 = 25 \end{cases}$$

Sol.

$$\begin{cases} x - y = 1 & (1) \\ x^2 + y^2 = 25 & (2) \end{cases}$$

$$\begin{aligned} (1) &\Rightarrow x = y + 1 \quad (3) \\ \text{Sub (3) into (2)} &\Rightarrow (y+1)^2 + y^2 = 25 \\ &\Rightarrow y^2 + 2y + 1 + y^2 = 25 \\ &\Rightarrow 2y^2 + 2y = 24 \\ &\Rightarrow y^2 + y = 12 \\ &\Rightarrow y^2 + y - 12 = 0 \\ &\Rightarrow (y+4)(y-3) = 0 \\ &\Rightarrow y = -4 \text{ or } y = 3 \\ \text{Sub } y = -4 \text{ into (1)} &\Rightarrow x = -4 + 1 \\ &\Rightarrow x = -3 \\ \text{Sub } y = 3 \text{ into (1)} &\Rightarrow x = 3 + 1 \\ &\Rightarrow x = 4 \end{aligned}$$

$$\therefore \begin{cases} x = -3 \\ y = -4 \end{cases} \text{ or } \begin{cases} x = 4 \\ y = 3 \end{cases}$$

6.

$$\begin{cases} 5x - y = 3 \\ y^2 - 6x^2 = 25 \end{cases}$$

Sol.

$$\begin{cases} 5x - y = 3 & (1) \\ y^2 - 6x^2 = 25 & (2) \end{cases}$$

$$\begin{aligned} (1) &\Rightarrow y = 5x - 3 \quad (3) \\ \text{Sub (3) into (2)} &\Rightarrow (5x-3)^2 - 6x^2 = 25 \\ &\Rightarrow 25x^2 - 30x + 9 \\ &\quad - 6x^2 = 25 \\ &\Rightarrow 19x^2 - 30x + 16 = 0 \\ &\Rightarrow (19x+8)(x-2) = 0 \\ &\Rightarrow x = -\frac{8}{19} \textit{ or } x = 2 \\ \text{Sub } x = -\frac{8}{19} \text{ into (1)} &\Rightarrow y = 5(-\frac{8}{19}) - 3 \\ &\Rightarrow y = -\frac{97}{19} \\ \text{Sub } x = 2 \text{ into (1)} &\Rightarrow y = 7 \end{aligned}$$

$$\therefore \begin{cases} x = -\frac{8}{19} \\ y = -\frac{97}{19} \end{cases} \textit{or} \begin{cases} x = 2 \\ y = 7 \end{cases}$$

7.

$$\begin{cases} x + y = 3 \\ (x+2)(y+3) = 12 \end{cases}$$

Sol.

$$\begin{cases} x + y = 3 & (1) \\ (x+2)(y+3) = 12 & (2) \end{cases}$$

$$\begin{aligned} (1) &\Rightarrow x = 3 - y \quad (3) \\ \text{Sub (3) into (2)} &\Rightarrow (3 - y + 2)(y+3) = 12 \\ &\Rightarrow (5-y)(y+3) = 12 \\ &\Rightarrow 5y + 15 - y^2 - 3y = 12 \\ &\Rightarrow 2y - y^2 = -3 \\ &\Rightarrow y^2 - 2y - 3 = 0 \\ &\Rightarrow (y+1)(y-3) = 0 \\ &\Rightarrow y = -1 \textit{ or } y = 3 \\ \text{Sub } y = -1 \text{ into (1)} &\Rightarrow x = 4 \\ \text{Sub } y = 3 \text{ into (1)} &\Rightarrow x = 0 \end{aligned}$$

$$\therefore \begin{cases} x = 4 \\ y = -1 \end{cases} \textit{or} \begin{cases} x = 0 \\ y = 3 \end{cases}$$

8.

$$\begin{cases} 5x - 6y = -1 \\ 25x^2 + 36y^2 = 61 \end{cases}$$

Sol.

$$\begin{cases} 5x - 6y = -1 & (1) \\ 25x^2 + 36y^2 = 61 & (2) \end{cases}$$

$$\begin{aligned} (1) &\Rightarrow y = \frac{5x+1}{6} \quad (3) \\ \text{Sub (3) into (2)} &\Rightarrow 25x^2 + 36\left(\frac{5x+1}{6}\right)^2 \\ &\quad = 61 \\ &\Rightarrow 25x^2 + 36\left(\frac{5x+1}{6}\right)^2 \\ &\quad + 36 = 61 \\ &\Rightarrow 25x^2 + 25x^2 + 10x \\ &\quad + 1 = 61 \\ &\Rightarrow 50x^2 + 10x = 60 \\ &\Rightarrow 5x^2 + x - 6 = 0 \\ &\Rightarrow (5x+6)(x-1) = 0 \\ &\Rightarrow x = -\frac{6}{5} \textit{ or } x = 1 \\ \text{Sub } x = -\frac{6}{5} \text{ into (1)} &\Rightarrow y = \frac{5(-\frac{6}{5}) + 1}{6} \\ &\Rightarrow y = -\frac{5}{6} \\ \text{Sub } x = 1 \text{ into (1)} &\Rightarrow y = \frac{5(1)+1}{6} \\ &\Rightarrow y = \frac{6}{6} \\ &\Rightarrow y = 1 \end{aligned}$$

$$\therefore \begin{cases} x = -\frac{6}{5} \\ y = -\frac{5}{6} \end{cases} \textit{or} \begin{cases} x = 1 \\ y = 1 \end{cases}$$

9.

$$\begin{cases} x + 4y = 5 \\ 2x^2 + 21xy + 27y^2 = 0 \end{cases}$$

Sol.

$$\begin{cases} x + 4y = 5 & (1) \\ 2x^2 + 21xy + 27y^2 = 0 & (2) \end{cases}$$

$$
\begin{aligned}
(1) &\Rightarrow x = 5 - 4y \qquad (3)\\
\text{Sub (3) into (2)} &\Rightarrow 2(5-4y)^2 + 21(5-4y)y\\
&\quad + 27y^2 = 0\\
&\Rightarrow 2\left(25 - 40y + 16y^2\right)\\
&\quad + 105y - 84y^2 + 27y^2 = 0\\
&\Rightarrow 50 - 80y + 32y^2 + 105y\\
&\quad - 57y^2 = 0\\
&\Rightarrow 25y^2 - 25y - 50 = 0\\
&\Rightarrow y^2 - y - 2\\
&\Rightarrow (y+1)(y-2) = 0\\
&\Rightarrow y = -1 \text{ or } y = 2\\
\text{Sub } y = -1 \text{ into (1)} &\Rightarrow x = 5 - 4(-1) = 9\\
\text{Sub } y = 2 \text{ into (1)} &\Rightarrow x = 5 - 4(2) = -3
\end{aligned}
$$

$$
\therefore \begin{cases} x = 9 \\ y = -1 \end{cases} \text{or} \begin{cases} x = -3 \\ y = 2 \end{cases}
$$

10.

$$
\begin{cases} \frac{x}{3} - \frac{y}{10} = \frac{5}{6} \\ x(y-2) = 2y+3 \end{cases}
$$

Sol.

$$
\begin{cases} \dfrac{x}{3} - \dfrac{y}{10} = \dfrac{5}{6} & (1)\\ x(y-2) = 2y+3 & (2) \end{cases}
$$

$$
\begin{aligned}
(1) &\Rightarrow 10x - 3y = 25 \qquad (3)\\
(2) &\Rightarrow x = \frac{2y+3}{y-2} \qquad (4)\\
\text{Sub (4) int 3} &\Rightarrow 10\left(\frac{2y+3}{y-2}\right) - 3y = 25\\
&\Rightarrow 10(2y+3) - 3y(y-2)\\
&\quad = 25(y-2)\\
&\Rightarrow 20y + 30 - 3y^2 + 6y\\
&\quad = 25y - 50\\
&\Rightarrow 3y^2 - y - 80 = 0\\
&\Rightarrow (y+5)(3y-16) = 0\\
&\Rightarrow y = -5 \text{ or } y = \frac{16}{3}\\
\text{Sub } y = -5 \text{ into (1)} &\Rightarrow 10x - 3(-5) = 25\\
&\Rightarrow 10x + 15 = 25\\
&\Rightarrow 10x = 10\\
&\Rightarrow x = 1\\
\text{Sub } y = \frac{16}{3} \text{ into (1)} &\Rightarrow 10x - 3\left(\frac{16}{3}\right) = 25\\
&\Rightarrow 10x = 41\\
&\Rightarrow x = \frac{41}{10}
\end{aligned}
$$

$$
\therefore \begin{cases} x = 1 \\ y = -5 \end{cases} \text{or} \begin{cases} x = \frac{41}{10} \\ y = \frac{16}{3} \end{cases}
$$

13.2 System of Equations with Three Variables

13.2.1 Practice 2

Solve the system of equation

$$
\begin{cases} x + 2y - z = -5 \\ 2x - y + z = 6 \\ x - y - 3z = -3 \end{cases}
$$

Sol.

$$
\begin{cases} x + 2y - z = -5 & (1)\\ 2x - y + z = 6 & (2)\\ x - y - 3z = -3 & (3) \end{cases}
$$

$$
\begin{aligned}
(1)\cdot 3 &\Rightarrow 3x+6y-3z=-15 && (4)\\
(2)\cdot 3 &\Rightarrow 6x-3y+3z=18 && (5)\\
(3)+(5) &\Rightarrow 7x-4y=15 && (6)\\
(4)+(5) &\Rightarrow 9x+3y=3 && (7)\\
(6)\cdot 3 &\Rightarrow 21x-12y=45 && (8)\\
(7)\cdot 4 &\Rightarrow 36x+12y=12 && (9)\\
(8)+(9) &\Rightarrow 57x=57 && (10)\\
&\Rightarrow x=1\\
\text{Sub } x=1 \text{ into } (7) &\Rightarrow -4y=8\\
&\Rightarrow y=-2\\
\text{Sub } y=-2 \text{ and } x=1 \text{ into } (1) &\Rightarrow -z=-2\\
&\Rightarrow z=2
\end{aligned}
$$

$\therefore x=1, y=-2, z=2$

13.2.2 Exercise 13.2

Solve the following system of equations.

1.

$$
\begin{cases}
x+y-z=1\\
2x-3y+z=0\\
2x+y+2z=5
\end{cases}
$$

Sol.

$$
\begin{cases}
x+y-z=1 & (1)\\
2x-3y+z=0 & (2)\\
2x+y+2z=5 & (3)
\end{cases}
$$

$$
\begin{aligned}
(1)\cdot 2 &\Rightarrow 2x+2y-2z=2 && (4)\\
(4)-(3) &\Rightarrow y-4z=-3 && (5)\\
(3)-(2) &\Rightarrow 4y+z=5 && (6)\\
(5)\cdot 4 &\Rightarrow 4y-16z=-12 && (7)\\
(6)-(7) &\Rightarrow 17z=17\\
&\Rightarrow z=1\\
\text{Sub } z=1 \text{ into } (5) &\Rightarrow y=1\\
\text{Sub } y=1 \text{ and } z=1 \text{ into } (1) &\Rightarrow z=1
\end{aligned}
$$

$\therefore x=1, y=1, z=1$

2.

$$
\begin{cases}
x-2y=5\\
2x+y-3z=8\\
x+4y-z=0
\end{cases}
$$

Sol.

$$
\begin{cases}
x-2y=5 & (1)\\
2x+y-3z=8 & (2)\\
x+4y-z=0 & (3)
\end{cases}
$$

$$
\begin{aligned}
(3)\cdot 3 &\Rightarrow 3x+12y-3z=0 && (4)\\
(4)-(2) &\Rightarrow x+11y=-8 && (5)\\
(5)-(1) &\Rightarrow 13y=-13\\
&\Rightarrow y=-1\\
\text{Sub } y=-1 \text{ into } (1) &\Rightarrow x+2=5\\
&\Rightarrow x=3\\
\text{Sub } x=3 &\\
\text{and } y=-1 \text{ into } (2) &\Rightarrow -3z=3\\
&\Rightarrow z=-1
\end{aligned}
$$

$\therefore x=3, y=-1, z=-1$

3.

$$
\begin{cases}
x+y=z-5\\
y+z=x-3\\
z+x=y+1
\end{cases}
$$

Sol.

$$
\begin{cases}
x+y=z-5 & (1)\\
y+z=x-3 & (2)\\
z+x=y+1 & (3)
\end{cases}
$$

$$
\begin{aligned}
(1) &\Rightarrow x+y-z=-5 && (4)\\
(2) &\Rightarrow -x+y+z=-3 && (5)\\
(3) &\Rightarrow x-y+z=1 && (6)\\
(4)+(5) &\Rightarrow 2y=-8\\
&\Rightarrow y=-4\\
(5)+(6) &\Rightarrow 2z=-2\\
&\Rightarrow z=-1\\
\text{Sub } y=-4 &\\
\text{and } z=-1 \text{ into } (2) &\Rightarrow x-3=-5\\
&\Rightarrow x=-2
\end{aligned}
$$

$\therefore x=-2, y=-4, z=-1$

4.

$$
\begin{cases}
x+4y+2z=4\\
2x-2y+z=4\\
x-2y+3z=3
\end{cases}
$$

Sol.

$$
\begin{cases}
x+4y+2z=4 & (1)\\
2x-2y+z=4 & (2)\\
x-2y+3z=3 & (3)
\end{cases}
$$

$$
\begin{aligned}
(1)\cdot 2 &\Rightarrow 2x+8y+4z=8 && (4)\\
(3)\cdot 2 &\Rightarrow 2x-4y+6z=6 && (5)\\
(4)-(2) &\Rightarrow 10y+3z=4 && (6)\\
(5)-(4) &\Rightarrow -12y+2z=-2 && (7)\\
(6)\cdot 2 &\Rightarrow 20y+6z=8 && (8)\\
(7)\cdot 3 &\Rightarrow -36y+6z=-6 && (9)\\
(8)-(9) &\Rightarrow 56y=14\\
&\Rightarrow y=\frac{1}{4}\\
\text{Sub } y=\frac{1}{4} \text{ into (6)} &\Rightarrow 6z=3\\
&\Rightarrow z=\frac{1}{2}\\
\text{Sub } y=\frac{1}{4} \text{ and } z=\frac{1}{2} \text{ into (1)} &\Rightarrow x+1+1=4\\
&\Rightarrow x=2
\end{aligned}
$$

$\therefore x=2, y=\frac{1}{4}, z=\frac{1}{2}$

5.

$$
\begin{cases}
x-y-z=0\\
3x+2y=13\\
y-3z=-1
\end{cases}
$$

Sol.

$$
\begin{cases}
x-y-z=0 & (1)\\
3x+2y=13 & (2)\\
y-3z=-1 & (3)
\end{cases}
$$

$$
\begin{aligned}
(3) &\Rightarrow y=3z-1 && (4)\\
\text{Sub (4) into (1)} &\Rightarrow x-(3z-1)-z=0\\
&\Rightarrow x-4z=-1 && (5)\\
\text{Sub (4) into (2)} &\Rightarrow 3x+2(3z-1)=13\\
&\Rightarrow 3x+6z=15 && (6)\\
(5)\cdot 3 &\Rightarrow 3x-12z=-3 && (7)\\
(6)-(7) &\Rightarrow 18z=18\\
&\Rightarrow z=1\\
\text{Sub } z=1 \text{ into (4)} &\Rightarrow y=2\\
\text{Sub } z=1 \text{ into (5)} &\Rightarrow x-4=-1\\
&\Rightarrow x=3
\end{aligned}
$$

$\therefore x=3, y=2, z=1$

6.

$$
\begin{cases}
2x+2y-z=-1\\
x+3y+z=-8\\
3x-2y+3z=9
\end{cases}
$$

Sol.

$$
\begin{cases}
2x+2y-z=-1 & (1)\\
x+3y+z=-8 & (2)\\
3x-2y+3z=9 & (3)
\end{cases}
$$

$$
\begin{aligned}
(1)\cdot 3 &\Rightarrow 6x+6y-3z=-3 && (4)\\
(2)\cdot 3 &\Rightarrow 3x+9y+3z=-24 && (5)\\
(3)+(4) &\Rightarrow 9x+4y=6 && (6)\\
(4)+(5) &\Rightarrow 9x+15y=-27 && (7)\\
(7)-(6) &\Rightarrow 11y=-33\\
&\Rightarrow y=-3\\
\text{Sub } y=-3 \text{ into (6)} &\Rightarrow 9x=18\\
&\Rightarrow x=2\\
\text{Sub } x=2\\
\text{and } y=-3 \text{ into (2)} &\Rightarrow -7+z=-8\\
&\Rightarrow z=-1
\end{aligned}
$$

$\therefore x=2, y=-3, z=-1$

7.

$$
\begin{cases}
\frac{3}{x}+\frac{1}{y}+\frac{4}{z}=0\\
\frac{1}{x}+\frac{4}{y}-\frac{2}{z}=4\\
\frac{2}{x}-\frac{3}{y}-\frac{1}{z}=-11
\end{cases}
$$

Sol.

$$
\begin{cases}
\dfrac{3}{x}+\dfrac{1}{y}+\dfrac{4}{z}=0 & (1)\\
\dfrac{1}{x}+\dfrac{4}{y}-\dfrac{2}{z}=4 & (2)\\
\dfrac{2}{x}-\dfrac{3}{y}-\dfrac{1}{z}=-11 & (3)
\end{cases}
$$

$$\text{Let } u = \frac{1}{x}, v = \frac{1}{y}, w = \frac{1}{z}$$

$$\begin{aligned}
(1) &\Rightarrow 3u + v + 4w = 0 && (4)\\
(2) &\Rightarrow u + 4v - 2w = 4 && (5)\\
(3) &\Rightarrow 2u - 3v - w = -11 && (6)\\
(5)\cdot 2 &\Rightarrow 2u + 8v - 4w = 8 && (7)\\
(6)\cdot 4 &\Rightarrow 8u - 12v - 4w = -44 && (8)\\
(4)+(7) &\Rightarrow 5u + 9v = 8 && (9)\\
(4)+(8) &\Rightarrow 11u - 11v = -44 &&\\
&\Rightarrow u - v = -4 && (10)\\
(10)\cdot 5 &\Rightarrow 5u - 5v = -20 && (11)\\
(9)-(11) &\Rightarrow 14v = 28 && (12)\\
&\Rightarrow v = 2 &&\\
\text{Sub } v = 2 \text{ into } (10) &\Rightarrow u = -2 &&\\
\text{Sub } u = -2 &&&\\
\text{and } v = 2 \text{ into } (4) &\Rightarrow -4 + 4w = 0 &&\\
&\Rightarrow w = 1 &&
\end{aligned}$$

$$\therefore u = -2, v = 2, w = 1$$

$$\therefore x = -\frac{1}{2}, y = \frac{1}{2}, z = 1$$

13.3 Revision Exercise 13

Solve the following system of equations.

1.

$$\begin{cases} 3x + 4y = 24 \\ xy = 12 \end{cases}$$

Sol.

$$\begin{cases} 3x + 4y = 24 & (1) \\ xy = 12 & (2) \end{cases}$$

$$\begin{aligned}
(2) &\Rightarrow y = \frac{12}{x} && (3)\\
\text{Sub (3) into (1)} &\Rightarrow 3x + 4(\frac{12}{x}) = 24 &&\\
&\Rightarrow 3x^2 + 48 = 24x &&\\
&\Rightarrow x^2 - 8x + 16 = 0 &&\\
&\Rightarrow (x-4)^2 = 0 &&\\
&\Rightarrow x = 4, x = -4 &&\\
\text{Sub } x = 4 \text{ into (3)} &\Rightarrow y = \frac{12}{4} = 3 &&\\
\text{Sub } x = -4 \text{ into (3)} &\Rightarrow y = \frac{12}{-4} = -3 &&
\end{aligned}$$

$$\therefore \begin{cases} x = 4 \\ y = 3 \end{cases} \text{ or } \begin{cases} x = -4 \\ y = -3 \end{cases}$$

2.

$$\begin{cases} x + 2y = 5 \\ 5x^2 + 4y^2 + 12x = 29 \end{cases}$$

Sol.

$$\begin{cases} x + 2y = 5 & (1) \\ 5x^2 + 4y^2 + 12x = 29 & (2) \end{cases}$$

$$\begin{aligned}
(1) &\Rightarrow x = 5 - 2y && (3)\\
\text{Sub (3) into (2)} &\Rightarrow 5(5-2y)^2 + 4y^2 &&\\
&\quad + 12(5-2y) = 29 &&\\
&\Rightarrow 5(25 - 20y + 4y^2) &&\\
&\quad + 4y^2 + 60 - 24y = 29 &&\\
&\Rightarrow 125 - 100y + 20y^2 &&\\
&\quad + 4y^2 + 60 - 24y = 29 &&\\
&\Rightarrow 24y^2 + 124y + 156 = 0 &&\\
&\Rightarrow 6y^2 + 31y + 39 = 0 &&\\
&\Rightarrow (y-3)(6y-13) = 0 &&\\
&\Rightarrow y = 3, y = \frac{13}{6} &&\\
\text{Sub } y = 3 \text{ into (1)} &\Rightarrow x = 5 - 2(3) = -1 &&\\
\text{Sub } y = \frac{13}{6} \text{ into (1)} &\Rightarrow x = 5 - 2(\frac{13}{6}) = \frac{2}{3} &&
\end{aligned}$$

$$\therefore \begin{cases} x = -1 \\ y = 3 \end{cases} \text{ or } \begin{cases} x = \frac{2}{3} \\ y = \frac{13}{6} \end{cases}$$

3.

$$\begin{cases} 2x + y = 7 \\ x^2 - xy + y^2 = 7 \end{cases}$$

Sol.

$$\begin{cases} 2x + y = 7 & (1) \\ x^2 - xy + y^2 = 7 & (2) \end{cases}$$

$$
\begin{aligned}
(1) &\Rightarrow y = 7 - 2x && (3)\\
\text{Sub (3) into (2)} &\Rightarrow x^2 - x(7-2x)\\
&\quad + (7-2x)^2 = 7\\
&\Rightarrow x^2 - 7x + 2x^2 - 28x\\
&\quad + 49 + 4x^2 = 7\\
&\Rightarrow 7x^2 - 35x + 42 = 0\\
&\Rightarrow x^2 - 5x + 6 = 0\\
&\Rightarrow (x-2)(x-3) = 0\\
&\Rightarrow x = 2, x = 3\\
\text{Sub } x = 2 \text{ into (3)} &\Rightarrow y = 7 - 2(2) = 3\\
\text{Sub } x = 3 \text{ into (3)} &\Rightarrow y = 7 - 2(3) = 1
\end{aligned}
$$

$$
\therefore \begin{cases} x = 2 \\ y = 3 \end{cases} \text{ or } \begin{cases} x = 3 \\ y = 1 \end{cases}
$$

4.

$$
\begin{cases} 2x + 3y = 7 \\ x^2 + xy + y^2 = 7 \end{cases}
$$

Sol.

$$
\begin{cases} 2x + 3y = 7 & (1) \\ x^2 + xy + y^2 = 7 & (2) \end{cases}
$$

$$
\begin{aligned}
(1) &\Rightarrow y = \frac{7-2x}{3} && (3)\\
\text{Sub (3) into (2)} &\Rightarrow x^2 + x(\frac{7-2x}{3})\\
&\quad + (\frac{7-2x}{3})^2 = 7\\
&\Rightarrow x^2 + \frac{7x - 2x^2}{3}\\
&\quad + \frac{49 - 28x + 4x^2}{9} = 7\\
&\Rightarrow 9x^2 + 21x - 6x^2 + 49\\
&\quad - 28x + 4x^2 = 63\\
&\Rightarrow 7x^2 - 7x - 14 = 0\\
&\Rightarrow x^2 - x - 2 = 0\\
&\Rightarrow (x+1)(x-2) = 0\\
&\Rightarrow x = -1, x = 2\\
\text{Sub } x = -1 \text{ into (3)} &\Rightarrow y = \frac{7 - 2(-1)}{3} = 3\\
\text{Sub } x = 2 \text{ into (3)} &\Rightarrow y = \frac{7 - 2(2)}{3} = 1
\end{aligned}
$$

$$
\therefore \begin{cases} x = -1 \\ y = 3 \end{cases} \text{ or } \begin{cases} x = 2 \\ y = 1 \end{cases}
$$

5.

$$
\begin{cases} 4x - 3y + 2 = 0 \\ 2y + 5z - 19 = 0 \\ 5x - 7z + 16 = 0 \end{cases}
$$

Sol.

$$
\begin{cases} 4x - 3y + 2 = 0 & (1) \\ 2y + 5z - 19 = 0 & (2) \\ 5x - 7z + 16 = 0 & (3) \end{cases}
$$

$$
\begin{aligned}
(1)\cdot 2 &\Rightarrow 8x - 6y + 4 = 0 && (4)\\
(2)\cdot 3 &\Rightarrow 6y + 15z - 57 = 0 && (5)\\
(4)+(5) &\Rightarrow 8x + 15z - 53 = 0 && (6)\\
(3)\cdot 8 &\Rightarrow 40x - 56z + 128 = 0 && (7)\\
(6)\cdot 5 &\Rightarrow 40x + 75z - 265 = 0 && (8)\\
(7)-(8) &\Rightarrow -131z + 393 = 0 && (9)\\
&\Rightarrow 131z = 393\\
&\Rightarrow z = 3\\
\text{Sub } z = 3 \text{ into (8)} &\Rightarrow 40x + 75(3) - 265 = 0\\
&\Rightarrow 40x + 225 - 265 = 0\\
&\Rightarrow 40x - 40 = 0\\
&\Rightarrow x = 1\\
\text{Sub } z = 3 \text{ into (2)} &\Rightarrow 6y - 12 = 0\\
&\Rightarrow y = 2
\end{aligned}
$$

$$
\therefore x = 1, y = 2, z = 3
$$

6.

$$
\begin{cases} x + y + z = 9 \\ 3x + y - 2z = 1 \\ x - 2y + z = 0 \end{cases}
$$

Sol.

$$
\begin{cases} x + y + z = 9 & (1) \\ 3x + y - 2z = 1 & (2) \\ x - 2y + z = 0 & (3) \end{cases}
$$

$$
\begin{aligned}
(1) &\Rightarrow x+z=9-y && (4)\\
\text{Sub (4) into (3)} &\Rightarrow 9-y-2y=0\\
&\Rightarrow 3y=9\\
&\Rightarrow y=3\\
\text{Sub } y=3 \text{ into (2)} &\Rightarrow 3x-2z=-2 && (5)\\
\text{Sub } y=3 \text{ into (3)} &\Rightarrow x+z=6 && (6)\\
(6)\cdot 2 &\Rightarrow 2x+2z=12 && (7)\\
(5)+(7) &\Rightarrow 5x=10\\
&\Rightarrow x=2\\
\text{Sub } x=2 \text{ into (6)} &\Rightarrow z=4
\end{aligned}
$$

$$\therefore x=2,\ y=3,\ z=4$$

7.

$$
\begin{cases}
2x-3y-z=4\\
4x+y+2z=3\\
x-4y-3z=2
\end{cases}
$$

Sol.

$$
\begin{cases}
2x-3y-z=4 & (1)\\
4x+y+2z=3 & (2)\\
x-4y-3z=2 & (3)
\end{cases}
$$

$$
\begin{aligned}
(1)\cdot 2 &\Rightarrow 4x-6y-2z=8 && (4)\\
(3)\cdot 4 &\Rightarrow 4x-16y-12z=8 && (5)\\
(2)-(4) &\Rightarrow 7y+4z=-5 && (6)\\
(4)-(5) &\Rightarrow 10y+10z=0\\
&\Rightarrow y+z=0\\
&\Rightarrow y=-z\\
\text{Sub } y=-z \text{ into (6)} &\Rightarrow 7(-z)+4z=-5\\
&\Rightarrow 3z=5\\
&\Rightarrow z=\frac{5}{3}\\
y=-z &\Rightarrow y=-\frac{5}{3}\\
\text{Sub } y=-\frac{5}{3}\\
\text{and } z=\frac{5}{3} \text{ into (1)} &\Rightarrow 2x-3(-\frac{5}{3})-\frac{5}{3}=4\\
&\Rightarrow 2x-\frac{5}{3}=-1\\
&\Rightarrow 2x=\frac{2}{3}\\
&\Rightarrow x=\frac{1}{3}
\end{aligned}
$$

$$\therefore x=\frac{1}{3},\ y=-\frac{5}{3},\ z=\frac{5}{3}$$

8.

$$
\begin{cases}
\frac{3}{x+1}-\frac{1}{y+2}+\frac{1}{z-1}=2\\
\frac{2}{x+1}-\frac{3}{y+2}-\frac{1}{z-1}=7\\
\frac{1}{x+1}+\frac{1}{y+2}-\frac{4}{z-1}=8
\end{cases}
$$

Sol.

$$
\begin{cases}
\dfrac{3}{x+1}-\dfrac{1}{y+2}+\dfrac{1}{z-1}=2 & (1)\\
\dfrac{2}{x+1}-\dfrac{3}{y+2}-\dfrac{1}{z-1}=7 & (2)\\
\dfrac{1}{x+1}+\dfrac{1}{y+2}-\dfrac{4}{z-1}=8 & (3)
\end{cases}
$$

Let $u=\dfrac{1}{x+1}, v=\dfrac{1}{y+2}, w=\dfrac{1}{z-1}$

$$
\begin{aligned}
(1) &\Rightarrow 3u-v+w=2 && (4)\\
(2) &\Rightarrow 2u-3v-w=7 && (5)\\
(3) &\Rightarrow u+v-4w=8 && (6)\\
(4)\cdot 3 &\Rightarrow 9u-3v+3w=6 && (7)\\
(6)\cdot 3 &\Rightarrow 3u+3v-12w=24 && (8)\\
(5)+(8) &\Rightarrow 5u-13w=31 && (9)\\
(7)+(8) &\Rightarrow 12u-9w=30\\
&\Rightarrow 4u-3w=10 && (10)\\
(9)\cdot 4 &\Rightarrow 20u-52w=124 && (11)\\
(10)\cdot 5 &\Rightarrow 20u-15w=50 && (12)\\
(12)-(11) &\Rightarrow 37w=-74 && (13)\\
&\Rightarrow w=-2\\
\text{Sub } w=-2 \text{ into (10)} &\Rightarrow 4u=4\\
&\Rightarrow u=1\\
\text{Sub } u=1\\
\text{and } w=-2 \text{ into (6)} &\Rightarrow 9+v=8\\
&\Rightarrow v=-1
\end{aligned}
$$

$$\therefore u=1,\ v=-1,\ w=-2$$

$$\therefore x=0,\ y=-3,\ z=\frac{1}{2}$$

Chapter 14

Marix and Determinant

14.1 Matrix

Definition of Matrix

A matrix is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. It is generally denoted as:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where m is the number of rows and n is the number of columns.

Each number in the matrix is called *an entry of the matrix*, the number in the i^{th} row and j^{th} column is denoted as a_{ij}. Thus, a matrix can also be denoted as $A = (a_{ij})$, or $A = (a_{ij})_{mn}$ where m is the number of rows and n is the number of columns.

A matrix with m rows and n columns is called an $m \cdot n$ matrix, where $m \cdot n$ is called the *order of the matrix*. For example, the following matrix is a **3 · 4 matrix**:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

When $m = n$, the matrix is called a *square matrix*. For example, the following matrix is a **third-order square matrix**:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

When $m = 1$, the matrix is called a *row matrix*. For example, the following matrix is a **row matrix**:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

When $n = 1$, the matrix is called a *column matrix*. For example, the following matrix is a **column matrix**:

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

Equal Matrices

Two matrices A and B are equal if they have the same order and the same entries. That is, $A = B$ if and only if $A_{ij} = B_{ij}$ for all i and j.

Zero Matrix

The matrix with all entries equal to zero is called the *zero matrix* and is denoted as O. Zero matrix can be in any order. For exmaple, the matrix below is a **2 · 2 zero matrix** or a **second-order square zero matrix**:

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Identity Matrix

The matrix with all entries equal to zero except the entries on the main diagonal, which are equal to one, is called the *identity matrix* and is denoted as I. Identity matrix can be in any order. The form of an identity matrix is:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Transpose Matrix

The transpose of a matrix A is denoted as A', A^t or A^T and is obtained by interchanging the rows and columns of A. For example, given the matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

The transpose of A is:

$$A' = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

Thus, we know that the transpose matrix of $m \cdot n$ matrix is a $n \cdot m$ matrix.

14.1.1 Exercise 14.1

1. State the order of the following matrices.

   (a) $A = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$
   Sol. A is a matrix with order $\mathbf{3 \cdot 1}$.

   (b) $B = \begin{pmatrix} 2 & 1 & 3 & 0 \\ 4 & 2 & 1 & 5 \end{pmatrix}$
   Sol. B is a matrix with order $\mathbf{2 \cdot 4}$.

   (c) $C = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 3 & 0 \\ 2 & 1 & -1 \end{pmatrix}$
   Sol. C is a matrix with order $\mathbf{3 \cdot 3}$.

2. Given $A = \begin{pmatrix} 1 & 5 & -2 & 4 \\ 2 & -4 & 3 & 1 \\ 0 & 6 & 4 & 7 \end{pmatrix}$, what is a_{23} and a_{34}?
   Sol. $a_{23} = 3$ and $a_{34} = 7$.

3. If $\begin{pmatrix} 2 & 0 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & x \end{pmatrix}$, what is x?
   Sol. $x = -4$.

14.2 Matrix Addition and Substraction

Given two matrices A and B of the same order, the sum of A and B is defined as the matrix $A + B$ whose (i, j)-th entry is the sum of the (i, j)-th entries of A and B. That is:

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

The difference of A and B is defined as the matrix $A - B$ whose (i, j)-th entry is the difference of the (i, j)-th entries of A and B. That is:

$$A - B = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdots & a_{2n} - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \cdots & a_{mn} - b_{mn} \end{pmatrix}$$

Note that the order of A and B must be the same. For example, the following metrices cannot be added or subtracted:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

The addition of matrices has the following properties:

- Commutative: $A + B = B + A$.
- Associative: $(A + B) + C = A + (B + C)$.
- Identity: $A \pm O = A$.
- Inverse: $A + (-A) = O$.
- Transpose: $(A \pm B)' = A' \pm B'$.

where A, B, C are matrices of the same order and O is the zero matrix of the same order as A.

Given a matrix A, if $A = A'$, then A is called a *symmetric matrix*. If $A = -A'$, then A is called an *antisymmetric matrix*.

For any given matrix A, $A + A'$ is symmetric, and $A - A'$ is antisymmetric.

14.2.1 Practice 1

Let $A = \begin{pmatrix} -4 & 2 & -7 \\ 5 & 4 & 0 \\ 3 & -2 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & -5 \\ 4 & -1 & 1 \end{pmatrix}$. Find the following:

1. $A + B'$.

   Sol.

$$B' = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 1 & -1 \\ 2 & -5 & 1 \end{pmatrix}$$

$$A + B' = \begin{pmatrix} -3 & 5 & -3 \\ 8 & 5 & -1 \\ 5 & -7 & -2 \end{pmatrix}$$

2. (A-B)’

   Sol.

$$A - B = \begin{pmatrix} -5 & -1 & -9 \\ 2 & 3 & 5 \\ -1 & -1 & -4 \end{pmatrix}$$

$$(A - B)' = \begin{pmatrix} -5 & 2 & -1 \\ -1 & 3 & -1 \\ -9 & 5 & -4 \end{pmatrix}$$

14.2.2 Exercise 14.2

Let $P = \begin{pmatrix} -5 & 4 & 2 \\ 6 & -4 & 3 \\ -2 & 1 & 6 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & 1 \\ 0 & 0 & 4 \end{pmatrix}$. Evaluate the following:

1. $(P + Q)'$

   Sol.

$$P + Q = \begin{pmatrix} -4 & 2 & 2 \\ 9 & -2 & 4 \\ -2 & 1 & 10 \end{pmatrix}$$

$$\therefore (P + Q)' = \begin{pmatrix} -4 & 9 & -2 \\ 2 & -2 & 1 \\ 2 & 4 & 10 \end{pmatrix}$$

2. $Q' - P'$

   Sol.

$$Q - P = \begin{pmatrix} 6 & -6 & 2 \\ -3 & 6 & -2 \\ 2 & -1 & -2 \end{pmatrix}$$

$$\therefore Q' - P' = (Q - P)' = \begin{pmatrix} 6 & -3 & 2 \\ -6 & 6 & -1 \\ 2 & -2 & -2 \end{pmatrix}$$

3. $(P' - Q)'$

   Sol.

$$P' = \begin{pmatrix} -5 & 6 & -2 \\ 4 & -4 & 1 \\ 2 & 3 & 6 \end{pmatrix}$$

$$P' - Q = \begin{pmatrix} -6 & 8 & -2 \\ 1 & -6 & 0 \\ 2 & 3 & 2 \end{pmatrix}$$

$$\therefore (P' - Q)' = \begin{pmatrix} -6 & 1 & 2 \\ 8 & -6 & 3 \\ -2 & 0 & 2 \end{pmatrix}$$

4. $P' - (I - Q)'$

   Sol.

$$\begin{aligned} P' - (I - Q)' &= P' - I' + Q' \\ &= (P + Q)' - I' \\ &= (P + Q - I)' \end{aligned}$$

$$P + Q - I = \begin{pmatrix} -4 & 2 & 2 \\ 9 & -2 & 4 \\ -2 & 1 & 10 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 2 & 2 \\ 9 & -3 & 4 \\ -2 & 1 & 9 \end{pmatrix}$$

$$\begin{aligned} \therefore P' - (I - Q)' &= (P + Q - I)' \\ &= \begin{pmatrix} -5 & 9 & -2 \\ 2 & -3 & 1 \\ 2 & 4 & 9 \end{pmatrix} \end{aligned}$$

14.3 Scalar Product of Matrices

Let $A = (a_{ij})_{m \cdot n}$ be an $m \cdot n$ matrix, k be any real number, then $kA = (ka_{ij})_{m \cdot n}$. This is called scalar product of a matrix

A and scalar k. For example:

$$k\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}=\begin{pmatrix}k & 2k & 3k\\ 4k & 5k & 6k\end{pmatrix}$$

The scalar product of a matrix has the following properties:

- $r(A+B)=rA+sB$
- $(r+s)A=rA+sA$
- $(rs)A=r(sA)$

14.3.1 Practice 2

Let $A=\begin{pmatrix}2 & 0\\ -3 & 5\end{pmatrix}$ and $B=\begin{pmatrix}1 & -1\\ 2 & -4\end{pmatrix}$. Evaluate the following:

1. $3A+B$

   Sol.

$$\begin{aligned}3A+B&=\begin{pmatrix}6 & 0\\ -9 & 15\end{pmatrix}+\begin{pmatrix}1 & -1\\ 2 & -4\end{pmatrix}\\ &=\begin{pmatrix}7 & -1\\ -7 & 11\end{pmatrix}\end{aligned}$$

2. $2A-3B$

   Sol.

$$\begin{aligned}2A-3B&=\begin{pmatrix}4 & 0\\ -6 & 10\end{pmatrix}-\begin{pmatrix}3 & -3\\ 6 & -12\end{pmatrix}\\ &=\begin{pmatrix}1 & 3\\ -12 & 22\end{pmatrix}\end{aligned}$$

3. $4B-2A$

   Sol.

$$\begin{aligned}4B-2A&=2(2B-A)\\ &=2\left(\begin{pmatrix}2 & -2\\ 4 & -8\end{pmatrix}-\begin{pmatrix}2 & 0\\ -3 & 5\end{pmatrix}\right)\\ &=2\begin{pmatrix}0 & -2\\ 7 & -13\end{pmatrix}\\ &=\begin{pmatrix}0 & -4\\ 14 & -26\end{pmatrix}\end{aligned}$$

4. $A'-2B'$

   Sol.

$$\begin{aligned}A'-2B'&=(A-2B)'\\ &=\left(\begin{pmatrix}2 & 0\\ -3 & 5\end{pmatrix}-\begin{pmatrix}2 & -2\\ 4 & -8\end{pmatrix}\right)'\\ &=\left(\begin{pmatrix}0 & 2\\ -7 & 13\end{pmatrix}\right)'\\ &=\begin{pmatrix}0 & -7\\ 2 & 13\end{pmatrix}\end{aligned}$$

14.3.2 Exercise 14.3

1. Let $A=\begin{pmatrix}2 & 3\\ 1 & 0\end{pmatrix}$, $B=\begin{pmatrix}6 & 1\\ 3 & 2\end{pmatrix}$, $C=\begin{pmatrix}3 & 1\\ 1 & 0\end{pmatrix}$, Calculate the following:

   (a) $2A-3B+4C$
   Sol.

$$\begin{aligned}&2A-3B+4C\\ &=\begin{pmatrix}4 & 6\\ 2 & 0\end{pmatrix}-\begin{pmatrix}18 & 3\\ 9 & 6\end{pmatrix}+\begin{pmatrix}12 & 4\\ 4 & 0\end{pmatrix}\\ &=\begin{pmatrix}16 & 10\\ 6 & 0\end{pmatrix}-\begin{pmatrix}18 & 3\\ 9 & 6\end{pmatrix}\\ &=\begin{pmatrix}-2 & 7\\ -3 & -6\end{pmatrix}\end{aligned}$$

   (b) $4A'-(C+B)'$
   Sol.

$$\begin{aligned}&4A'-(C+B')\\ &=4\begin{pmatrix}2 & 1\\ 3 & 0\end{pmatrix}-\left(\begin{pmatrix}3 & 1\\ 1 & 0\end{pmatrix}+\begin{pmatrix}6 & 3\\ 1 & 2\end{pmatrix}\right)\\ &=\begin{pmatrix}8 & 4\\ 12 & 0\end{pmatrix}-\begin{pmatrix}9 & 4\\ 2 & 2\end{pmatrix}\\ &=\begin{pmatrix}-1 & 0\\ 10 & -2\end{pmatrix}\end{aligned}$$

2. Let $A=\begin{pmatrix}1 & 2\\ 0 & 1\\ 3 & 1\end{pmatrix}$, $B=\begin{pmatrix}2 & 1\\ 4 & 3\\ 1 & 0\end{pmatrix}$, $C=\begin{pmatrix}4 & -1\\ 3 & 1\\ 2 & -3\end{pmatrix}$, evaluate the following:

   (a) $3A+B-2C$
   Sol.

$$\begin{aligned}3A+B-2C&=\begin{pmatrix}3 & 6\\ 0 & 3\\ 9 & 3\end{pmatrix}+\begin{pmatrix}2 & 1\\ 4 & 3\\ 1 & 0\end{pmatrix}-\begin{pmatrix}8 & -2\\ 6 & 2\\ 4 & -6\end{pmatrix}\\ &=\begin{pmatrix}5 & 7\\ 4 & 6\\ 10 & 3\end{pmatrix}-\begin{pmatrix}8 & -2\\ 6 & 2\\ 4 & -6\end{pmatrix}\\ &=\begin{pmatrix}-3 & 9\\ -2 & 4\\ 6 & -3\end{pmatrix}\end{aligned}$$

   (b) $3(A+C)'-B'$

Sol.

$$\begin{aligned}
&3(A+C)'-B'\\
&=3\left(\begin{pmatrix}1&2\\0&1\\3&1\end{pmatrix}+\begin{pmatrix}4&-1\\3&1\\2&-3\end{pmatrix}\right)'-\begin{pmatrix}2&1\\4&3\\1&0\end{pmatrix}'\\
&=3\left(\begin{pmatrix}5&1\\3&2\\5&-2\end{pmatrix}\right)'-\begin{pmatrix}2&4&1\\1&3&0\end{pmatrix}\\
&=\begin{pmatrix}15&9&15\\3&6&-6\end{pmatrix}-\begin{pmatrix}2&4&1\\1&3&0\end{pmatrix}\\
&=\begin{pmatrix}13&5&14\\2&3&-6\end{pmatrix}
\end{aligned}$$

3. Given $A=\begin{pmatrix}1&2&3\\0&1&1\end{pmatrix}$, $B=\begin{pmatrix}2&1&3\\1&2&0\end{pmatrix}$, Find the matrix X in the following expression:

(a) $X+4A=3(X+B)-A$

Sol.

$$\begin{aligned}
X+4A&=3(X+B)-A\\
&=3X+3B-A\\
2X&=5A-3B\\
2X&=5\begin{pmatrix}1&2&3\\0&1&1\end{pmatrix}-3\begin{pmatrix}2&1&3\\1&2&0\end{pmatrix}\\
&=\begin{pmatrix}5&10&15\\0&5&5\end{pmatrix}-\begin{pmatrix}6&3&9\\3&6&0\end{pmatrix}\\
&=\begin{pmatrix}-1&7&6\\-3&-1&5\end{pmatrix}\\
x&=\begin{pmatrix}-\frac{1}{2}&\frac{7}{2}&3\\-\frac{3}{2}&-\frac{1}{2}&\frac{5}{2}\end{pmatrix}
\end{aligned}$$

(b) $2A-B+X'=B$

Sol.

$$\begin{aligned}
2A-B+X'&=B\\
X'&=-2A+2B\\
&=-2(A-B)\\
&=-2\left(\begin{pmatrix}1&2&3\\0&1&1\end{pmatrix}\right.\\
&\left.-\begin{pmatrix}2&1&3\\1&2&0\end{pmatrix}\right)\\
&=-2\begin{pmatrix}-1&1&0\\-1&-1&1\end{pmatrix}\\
&=\begin{pmatrix}2&-2&0\\2&2&-2\end{pmatrix}\\
X&=\begin{pmatrix}2&2\\-2&2\\0&-2\end{pmatrix}
\end{aligned}$$

14.4 Multiplication of Matrices

Let A and B be matrices of order $m\cdot n$ and $n\cdot p$ respectively. Then the product of A and B is defined as the matrix AB of order $m\cdot p$ such that the $(i,j)^{th}$ element of AB is given by

$$(AB)_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}$$

The multiplication of matrices has the following properties:

- Associative: $A(BC)=(AB)C$
- Distributive: $A(B+C)=AB+AC$ and $(B+C)A=BA+CA$
- $k(AB)\neq(kA)B$ for $k\neq 0$

14.4.1 Practice 3

Evaluate the following:

1. $\begin{pmatrix}-1&-1\\2&3\end{pmatrix}\begin{pmatrix}2&1\\-1&2\end{pmatrix}$

Sol.

$$\begin{aligned}
&\begin{pmatrix}-1&-1\\2&3\end{pmatrix}\begin{pmatrix}2&1\\-1&2\end{pmatrix}\\
&=\begin{pmatrix}-1(-1)+(-1)(2)&-1(2)+(-1)(1)\\2(2)+3(-1)&2(1)+3(2)\end{pmatrix}\\
&=\begin{pmatrix}1&-3\\1&8\end{pmatrix}
\end{aligned}$$

2. $\begin{pmatrix}3&4\\-1&1\end{pmatrix}\begin{pmatrix}0&1&2\\3&-3&2\end{pmatrix}$

Sol.

$$\begin{aligned}
&\begin{pmatrix}3&4\\-1&1\end{pmatrix}\begin{pmatrix}0&1&2\\3&-3&2\end{pmatrix}\\
&=\begin{pmatrix}3(0)+4(3)&3(1)+4(-3)&3(2)+4(2)\\-1(0)+1(3)&-1(1)+1(-3)&-1(2)+1(2)\end{pmatrix}\\
&=\begin{pmatrix}12&-9&14\\3&-4&0\end{pmatrix}
\end{aligned}$$

3. $\begin{pmatrix}1&0\\2&4\\3&-5\end{pmatrix}\begin{pmatrix}6&1&5\\-2&3&2\end{pmatrix}$

Sol.

$$\begin{aligned}
&\begin{pmatrix}1&0\\2&4\\3&-5\end{pmatrix}\begin{pmatrix}6&1&5\\-2&3&2\end{pmatrix}\\
&=\begin{pmatrix}1(6)+0(-2)&1(1)+0(3)&1(5)+0(2)\\2(6)+4(-2)&2(1)+4(3)&2(5)+4(2)\\3(6)+(-5)(-2)&3(1)+(-5)(3)&3(5)+(-5)(2)\end{pmatrix}\\
&=\begin{pmatrix}6&1&5\\4&14&18\\28&-12&5\end{pmatrix}
\end{aligned}$$

4. $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 1 \\ -1 & 3 \end{pmatrix}$

Sol.

$$\begin{aligned} &\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 1 \\ -1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1(1)+3(2)+2(-1) & 1(3)+3(1)+2(3) \\ 0(1)+1(2)+5(-1) & 0(3)+1(1)+5(3) \end{pmatrix} \\ &= \begin{pmatrix} 5 & 12 \\ -3 & 16 \end{pmatrix} \end{aligned}$$

14.4.2 Exercise 14.4

Calculate the following products (Question 1 to 8):

1. $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Sol.

$$\begin{aligned} &\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 1(1)+2(2)+3(3) \end{pmatrix} \\ &= \begin{pmatrix} 14 \end{pmatrix} \end{aligned}$$

2. $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

Sol.

$$\begin{aligned} &\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \end{aligned}$$

3. $\begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Sol.

$$\begin{aligned} &\begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2(1)+(-3)(0) & 2(0)+(-3)(1) \\ 1(1)+5(0) & 1(0)+5(1) \end{pmatrix} \\ &= \begin{pmatrix} 2 & -3 \\ 1 & 5 \end{pmatrix} \end{aligned}$$

4. $\begin{pmatrix} -6 & -4 & 2 \\ 7 & 8 & -5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

Sol.

$$\begin{aligned} &\begin{pmatrix} -6 & -4 & 2 \\ 7 & 8 & -5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -6(1)+(-4)(2)+2(3) \\ 7(1)+8(2)+(-5)(3) \end{pmatrix} \\ &= \begin{pmatrix} -8 \\ 8 \end{pmatrix} \end{aligned}$$

5. $\begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 3 & 1 \\ 4 & 2 \end{pmatrix}$

Sol.

$$\begin{aligned} &\begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 3 & 1 \\ 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2(2)+3(3)+4(4) & 2(0)+3(1)+4(2) \\ 0(2)+1(3)+2(4) & 0(0)+1(1)+2(2) \end{pmatrix} \\ &= \begin{pmatrix} 27 & 11 \\ 11 & 5 \end{pmatrix} \end{aligned}$$

6. $\begin{pmatrix} 6 & 4 & 2 \\ 5 & -2 & 0 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$

Sol.

$$\begin{aligned} &\begin{pmatrix} 6 & 4 & 2 \\ 5 & -2 & 0 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 6(5)+4(2)+2(3) \\ 5(5)+(-2)(2)+0(3) \\ 0(5)+3(2)+1(3) \end{pmatrix} \\ &= \begin{pmatrix} 44 \\ 21 \\ 9 \end{pmatrix} \end{aligned}$$

7. $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Sol.

$$\begin{aligned} &\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{bmatrix} 0(0)+1(1)+0(0) & 0(1)+1(0)+0(0) & 0(0)+1(0)+0(1) \\ 1(0)+0(1)+0(0) & 1(1)+0(0)+0(0) & 1(0)+0(0)+0(1) \\ 0(0)+0(1)+1(0) & 0(1)+0(0)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

8. $\begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix}$

Sol.

$$\begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$
$$= \begin{bmatrix} 1+(-2)+1 & 2+(-4)+2 & 3+(-6)+3 \\ (-3)+4+(-1) & (-6)+8+(-2) & (-9)+12+(-3) \\ (-2)+2+0 & (-4)+4+0 & (-6)+6+0 \end{bmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

14.5 Determinants

The determinant of an n-order matrix $A = (a_{ij})_{n \cdot n}$ is denoted as $\det(A)$. When $n \leq 2$, the deternimant can also be denoted as $|A|$. The determinant is a value.

When $n = 1$, the determinant is the value of the only element in the matrix.

Determinant of a 2x2 matrix

For a 2x2 matrix, the determinant is defined as:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

14.5.1 Practice 4

Find the value of the following determinants.

1. $\begin{vmatrix} 2 & -3 \\ 5 & 7 \end{vmatrix}$

   Sol.

$$\begin{vmatrix} 2 & -3 \\ 5 & 7 \end{vmatrix}$$
$$= 2(7) - (-3)(5)$$
$$= 14 + 15$$
$$= 29$$

2. $\begin{vmatrix} -6 & -7 \\ -8 & -9 \end{vmatrix}$

   Sol.

$$\begin{vmatrix} -6 & -7 \\ -8 & -9 \end{vmatrix}$$
$$= (-6)(-9) - (-7)(-8)$$
$$= 54 - 56$$
$$= -2$$

3. $\begin{vmatrix} 12 & -20 \\ -21 & 35 \end{vmatrix}$

   Sol.

$$\begin{vmatrix} 12 & -20 \\ -21 & 35 \end{vmatrix}$$
$$= 12(35) - (-20)(-21)$$
$$= 420 - 420$$
$$= 0$$

Determinant of a 3x3 matrix

For a 3x3 matrix, the determinant is defined as:

$$\det(A) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$
$$= a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - a_3 b_2 c_1 - b_3 c_2 a_1$$
$$- c_3 a_2 b_1$$

A 3x3 matrix can be expanded using the Sarrus method. The Sarrus method is defined as:

$$\begin{array}{ccc:cc} + & + & + & & \\ a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \\ - & - & - & & \end{array}$$

Note that the Sarrus method is only applicable to 3x3 matrices.

14.5.2 Practice 5

Calculate the value of the following determinants.

1. $\begin{vmatrix} 1 & 5 & 1 \\ 1 & 6 & 3 \\ 9 & 8 & 9 \end{vmatrix}$

   Sol.

$$\begin{array}{ccc:cc} + & + & + & & \\ 1 & 5 & 1 & 1 & 5 \\ 1 & 6 & 3 & 1 & 6 \\ 9 & 8 & 9 & 9 & 8 \\ - & - & - & & \end{array}$$

$$\begin{vmatrix} 1 & 5 & 1 \\ 1 & 6 & 3 \\ 9 & 8 & 9 \end{vmatrix} = 54 + 135 + 8 - 54 - 24 - 45$$
$$= 74$$

2. $\begin{vmatrix} 3 & 1 & -2 \\ 0 & -1 & 1 \\ 4 & 2 & 5 \end{vmatrix}$

Sol.

$$\begin{array}{ccccc} + & + & + & & \\ 3 & 1 & -2 & 3 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ 4 & 2 & 5 & 4 & 2 \\ - & - & - & & \end{array}$$

$$\begin{vmatrix} 3 & 1 & -2 \\ 0 & -1 & 1 \\ 4 & 2 & 5 \end{vmatrix} = -15 + 4 - 0 - 8 - 6 - 0$$
$$= -25$$

Minor and Cofactor

The minor of an element in a matrix is the determinant of the matrix obtained by deleting the row and column containing the element. Take $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ as an example. The minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, the minor of c_2 is $\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$, and so on.

The cofactor of an element in a matrix is the minor of the element multiplied by $(-1)^{i+j}$, where i and j are the row and column indices of the element. The cofactor of a_1 is $(-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, the cofactor of c_2 is $(-1)^{3+2}\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$, and so on.

Let A_1, B_1, C_1 are the cofactors of a_1, b_1, c_1 respectively. Then

$$A_1 = (-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$
$$B_1 = (-1)^{1+2}\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$
$$C_1 = (-1)^{1+3}\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Thus,
$$|A| = a_1A_1 + a_2B_1 + a_3C_1$$

That is, the value of the determinant is the elements of the first row multiplied by the cofactors of the elements of the first row.

The sign of the cofactor is determined by the sum of the row and column indices of the element. If the sum is even, the cofactor is positive; if the sum is odd, the cofactor is negative.

Generally, a 3x3 determinant has the following theorem:

Theorem 1. *The determinant of a 3x3 matrix is the sum of the elements of any row or column multiplied by the cofactors of the elements of that row or column.*

That is, we can use the cofactor expansion to calculate the determinant of a 3x3 matrix.

$$\begin{aligned} |A| &= a_1A_1 + b_1B_1 + c_1C_1 \\ &= a_2B_2 + b_2B_2 + c_2C_2 \\ &= a_3C_3 + b_3C_3 + c_3C_3 \\ &= a_1A_1 + a_2A_2 + a_3A_3 \\ &= b_1B_1 + b_2B_2 + b_3B_3 \\ &= c_1C_1 + c_2C_2 + c_3C_3 \end{aligned}$$

The determinant of any order matrix can also be calculated by the cofactor expansion.

Theorem 2. *The product of the elements of any row or column and the cofactor of corresponding elements of another row or column of a determinant is 0.*

For example, the product of the elements of the second row and the corresponding element of the cofactor of first row of the determinant is 0. That is,

$$\begin{aligned} & a_2B_1 + b_2B_1 + c_2C_1 \\ &= a_2\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_2\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_2\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_2b_2c_3 + a_2b_3c_2 - a_2b_2c_3 + a_3b_2c_2 + a_2b_3c_2 - a_3b_2c_2 \\ &= 0 \end{aligned}$$

14.5.3 Practice 6

Find the value of the following 3x3 determinants.

1. $\begin{vmatrix} 4 & -2 & 1 \\ 1 & -3 & 0 \\ 2 & 7 & -1 \end{vmatrix}$

Sol.

$$\begin{aligned} & \begin{vmatrix} 4 & -2 & 1 \\ 1 & -3 & 0 \\ 2 & 7 & -1 \end{vmatrix} \\ &= 4\begin{vmatrix} -3 & 0 \\ 7 & -1 \end{vmatrix} - \begin{vmatrix} -2 & 1 \\ 7 & -1 \end{vmatrix} + 2\begin{vmatrix} -2 & 1 \\ -3 & 0 \end{vmatrix} \\ &= 4(3-0) - (2-7) + 2(0+3) \\ &= 12 + 5 + 6 \\ &= 23 \end{aligned}$$

2. $\begin{vmatrix} 5 & -4 & 2 \\ 1 & 0 & -3 \\ 1 & -1 & 2 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 5 & -4 & 2 \\ 1 & 0 & -3 \\ 1 & -1 & 2 \end{vmatrix}$$
$$= 5\begin{vmatrix} 0 & -3 \\ -1 & 2 \end{vmatrix} - \begin{vmatrix} -4 & 2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} -4 & 2 \\ 0 & -3 \end{vmatrix}$$
$$= 5(0-3) - (-8+2) + (12+0)$$
$$= -15 + 6 + 12$$
$$= 3$$

3. $\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{vmatrix}$$
$$= 2\begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix}$$
$$= 2(-2-0) + 2(-2)$$
$$= -4 - 4$$
$$= -8$$

14.5.4 Exercise 14.5a

Find the value of the following determinants.

1. $\begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix}$$
$$= 3(-4) - 2(1)$$
$$= -12 - 2$$
$$= -14$$

2. $\begin{vmatrix} 35 & -2 \\ -11 & 5 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 35 & -2 \\ -11 & 5 \end{vmatrix}$$
$$= 35(5) - (-2)(-11)$$
$$= 175 - 22$$
$$= 153$$

3. $\begin{vmatrix} 1 & a \\ -a & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 1 & a \\ -a & 1 \end{vmatrix}$$
$$= 1(1) - a(-a)$$
$$= 1 + a^2$$

4. $\begin{vmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{vmatrix}$

Sol.

$$\begin{vmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{vmatrix}$$
$$= \sin x \sin x - (-\cos x)(\cos x)$$
$$= \sin^2 x + \cos^2 x$$
$$= 1$$

5. $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -4 \\ 3 & -2 & 5 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -4 \\ 3 & -2 & 5 \end{vmatrix}$$
$$= 1\begin{vmatrix} 3 & -4 \\ -2 & 5 \end{vmatrix} - 2\begin{vmatrix} -2 & 3 \\ -2 & 5 \end{vmatrix} + 3\begin{vmatrix} -2 & 3 \\ 3 & -4 \end{vmatrix}$$
$$= 1(15-8) - 2(-10+6) + 3(8-9)$$
$$= 7 + 8 - 3$$
$$= 12$$

6. $\begin{vmatrix} 1 & -3 & 4 \\ 2 & 0 & -5 \\ 3 & -1 & 7 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 1 & -3 & 4 \\ 2 & 0 & -5 \\ 3 & -1 & 7 \end{vmatrix}$$
$$= \begin{vmatrix} 0 & -5 \\ -1 & 7 \end{vmatrix} - 2\begin{vmatrix} -3 & 4 \\ -1 & 7 \end{vmatrix} + 3\begin{vmatrix} -3 & 4 \\ 0 & -5 \end{vmatrix}$$
$$= (0-5) - 2(-21+4) + 3(15-0)$$
$$= -5 + 34 + 45$$
$$= 74$$

7. $\begin{vmatrix} -1 & 3 & -2 \\ -3 & 2 & 0 \\ 4 & 0 & 5 \end{vmatrix}$

Sol.

$$\begin{vmatrix} -1 & 3 & -2 \\ -3 & 2 & 0 \\ 4 & 0 & 5 \end{vmatrix}$$
$$= -1\begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} + 3\begin{vmatrix} 3 & -2 \\ 0 & 5 \end{vmatrix} + 4\begin{vmatrix} 3 & -2 \\ 2 & 0 \end{vmatrix}$$
$$= -1(10) + 3(-15+0) + 4(0+4)$$
$$= -10 + 45 + 16$$
$$= 51$$

8. $\begin{vmatrix} 0 & -q & -r \\ q & 0 & -s \\ r & s & 0 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 0 & -q & -r \\ q & 0 & -s \\ r & s & 0 \end{vmatrix}$$
$$= 0\begin{vmatrix} 0 & -s \\ s & 0 \end{vmatrix} - q\begin{vmatrix} -q & -r \\ s & 0 \end{vmatrix} + r\begin{vmatrix} -q & -r \\ 0 & -s \end{vmatrix}$$
$$= 0 - q(0+sr) + r(0+qs)$$
$$= 0$$

9. $\begin{vmatrix} p & -q & r \\ q & r & -s \\ -r & s & p \end{vmatrix}$

Sol.

$$\begin{vmatrix} p & -q & r \\ q & r & -s \\ -r & s & p \end{vmatrix}$$
$$= p\begin{vmatrix} r & -s \\ s & p \end{vmatrix} - q\begin{vmatrix} -q & r \\ s & p \end{vmatrix} + r\begin{vmatrix} -q & r \\ r & -s \end{vmatrix}$$
$$= p(rp+s^2) - q(-qp-sr) - r(qs-r^2)$$
$$= rp^2 + ps^2 + q^2p + qsr - qsr + r^3$$
$$= rp^2 + s^2p + q^2p - r^3$$

10. $\begin{vmatrix} 1 & x & a \\ 1 & y & b \\ 1 & z & c \end{vmatrix}$

Sol.

$$\begin{vmatrix} 1 & x & a \\ 1 & y & b \\ 1 & z & c \end{vmatrix}$$
$$= \begin{vmatrix} y & b \\ z & c \end{vmatrix} - \begin{vmatrix} x & a \\ z & c \end{vmatrix} + \begin{vmatrix} x & a \\ y & b \end{vmatrix}$$
$$= (yc - bz) - (xc - az) + (xb - ay)$$
$$= bx + cy + az - cx - ay - bz$$

Identities of Determinants

Theorem 1. *The value of a determinant is the same as the value of its transpose, aka* $|A| = |A'|$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

Theorem 2. *Switching any two rows or columns of a determinants results in the opposite value.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

14.5.5 Practice 7

Given $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 10$, find $\begin{vmatrix} b & c & a \\ e & f & d \\ h & i & g \end{vmatrix}$.

Sol.

$$\begin{vmatrix} b & c & a \\ e & f & d \\ h & i & g \end{vmatrix} = -\begin{vmatrix} a & c & b \\ d & f & e \\ g & i & h \end{vmatrix}$$
$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$
$$= 10$$

Theorem 3. *If two rows or cols of a determinant are identical, the value of the determinant is zero.*

$$\begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix} = 0$$

Theorem 4. *If all elements of a row (or column) of a determinant are multiplied by some scalar number k, the value of the new determinant is k times of the given determinant.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

14.5.6 Practice 8

Using the identities of determinants, prove that $\begin{vmatrix} 10 & -12 & 2 \\ -15 & 18 & 3 \\ 5 & 6 & -1 \end{vmatrix} = 180\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$.

Sol.

$$\begin{vmatrix} 10 & -12 & 2 \\ -15 & 18 & 3 \\ 5 & 6 & -1 \end{vmatrix}$$

$$= 5 \cdot 6 \begin{vmatrix} 2 & -2 & 2 \\ -3 & 3 & 3 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 5 \cdot 6 \cdot 2 \cdot 3 \cdot \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= 180 \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Theorem 5. *In a determinant each element in any row (or column) consists of the sum of two terms, then the determinant can be expressed as sum of two determinants of same order.*

$$\begin{vmatrix} a_1 + d_1 & b_1 & c_1 \\ a_2 + d_2 & b_2 & c_2 \\ a_3 + d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Theorem 6. *If a determinant is obtained by adding a row or column multiplied by a some scalar number k to a different row or column, then the value of the new determinant is the same as the original determinant.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

14.5.7 Practice 9

Prove that $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$.

Sol.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} \qquad \text{(Adding row 1 multiplied by -1 to row 2 and 3)}$$

$$= 2 \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} \qquad \text{(Theorem 4)}$$

$$= 0 \qquad \text{(Theorem 3)}$$

Theorem 7. *The determinant of product of two matrices of equal size is equal to the product of determinants of each matrix, aka $|AB| = |A||B|$.*

14.5.8 Practice 10

Let $A = \begin{vmatrix} -2 & 1 \\ 4 & 3 \end{vmatrix}$ and $B = \begin{vmatrix} 1 & x \\ 2 & 3 \end{vmatrix}$. Given that $|AB| = -18$, find x.

Sol.

$$\because |AB| = |A||B| = -18$$

$$\therefore \begin{vmatrix} -2 & 1 \\ 4 & 3 \end{vmatrix} \begin{vmatrix} 1 & x \\ 2 & 3 \end{vmatrix} = -18$$

$$-2(3 - 2x) = -18$$

$$3 - 2x = 9$$

$$-2x = 6$$

$$x = -3$$

14.5.9 Exercise 14.5b

1. Given $\begin{vmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix} = -1$, Find the value of the following determinants.

   (a) $\begin{vmatrix} 2 & 0 & -1 \\ -2 & -1 & 2 \\ 3 & -2 & 1 \end{vmatrix}$

   Sol.

   $$\begin{vmatrix} 2 & 0 & -1 \\ -2 & -1 & 2 \\ 3 & -2 & 1 \end{vmatrix}$$

   $$= \left| \begin{pmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{pmatrix}' \right|$$

   $$= -1 \qquad \text{(Theorem 1)}$$

   (b) $\begin{vmatrix} 2 & 0 & -1 \\ 3 & -2 & 1 \\ -2 & -1 & 2 \end{vmatrix}$

   Sol.

   $$\begin{vmatrix} 2 & 0 & -1 \\ 3 & -2 & 1 \\ -2 & -1 & 2 \end{vmatrix}$$

   Sol.

   $$\begin{vmatrix} 2 & 0 & -1 \\ 3 & -2 & 1 \\ -2 & -1 & 2 \end{vmatrix}$$

   $$= - \begin{vmatrix} 2 & 0 & -1 \\ -2 & -1 & 2 \\ 3 & -2 & 1 \end{vmatrix}$$

   $$= - \left| \begin{pmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{pmatrix}' \right|$$

   $$= 1 \qquad \text{(Theorem 1)}$$

   (c) $\begin{vmatrix} 4 & -2 & 3 \\ 0 & -2 & -4 \\ -2 & 2 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 4 & -2 & 3 \\ 0 & -2 & -4 \\ -2 & 2 & 1 \end{vmatrix}$$

$$= 2 \cdot 2 \begin{vmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= -4$$

(d) $\begin{vmatrix} -3 & -2 & -2 \\ 2 & -1 & 0 \\ -1 & 2 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} -3 & -2 & -2 \\ 2 & -1 & 0 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -2 & 2 \\ -2 & -1 & 0 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= 1$$

(e) $\begin{vmatrix} 0 & 2 & 5 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 0 & 2 & 5 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 + (2 \cdot (-1)) & -2 + (-2 \cdot 2) & 3 + (2 \cdot 1) \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= -1$$

(f) $\begin{vmatrix} 2 & 4 & 3 \\ 0 & -5 & -2 \\ -1 & 4 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 2 & 4 & 3 \\ 0 & -5 & -2 \\ -1 & 4 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 + (2 \cdot 3) & 3 \\ 0 & -1 + (2 \cdot (-2)) & -2 \\ -1 & 2 + (2 \cdot 1) & 1 \end{vmatrix}$$

$$= -1$$

2. Prove the following equations using identities of determinants without expanding them.

(a) $\begin{vmatrix} 1 & 0 & 3 \\ 4 & 6 & 12 \\ 3 & 3 & 9 \end{vmatrix} = 0$

Proof.

$$L.H.S. = \begin{vmatrix} 1 & 0 & 3 \\ 4 & 6 & 12 \\ 3 & 3 & 9 \end{vmatrix}$$

$$= 2 \cdot 3 \begin{vmatrix} 1 & 0 & 3 \\ 2 & 3 & 6 \\ 1 & 1 & 3 \end{vmatrix}$$

$$= 2 \cdot 3 \cdot 3 \begin{vmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 0 = R.H.S. \qquad \text{(Theorem 3)}$$

(b) $\begin{vmatrix} 2 & -1 & 3 \\ -1 & 1 & 2 \\ 4 & -2 & 6 \end{vmatrix} = 0$

Proof.

$$L.H.S. = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 1 & 2 \\ 4 & -2 & 6 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 2 & -1 & 3 \\ -1 & 1 & 2 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 0 = R.H.S. \qquad \text{(Theorem 3)}$$

(c) $\begin{vmatrix} 3 & 2 & 2 \\ 9 & 6 & 5 \\ 12 & 8 & 8 \end{vmatrix} = 0$

Proof.

$$L.H.S. = \begin{vmatrix} 3 & 2 & 2 \\ 9 & 6 & 5 \\ 12 & 8 & 8 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 3 & 2 & 2 \\ 9 & 6 & 5 \\ 3 & 2 & 2 \end{vmatrix}$$

$$= 0 = R.H.S. \qquad \text{(Theorem 3)}$$

(d) $\begin{vmatrix} 10 & 8 & 2 \\ 15 & 12 & 3 \\ 20 & 32 & 12 \end{vmatrix} = 0$

Proof.

$$L.H.S. = \begin{vmatrix} 10 & 8 & 2 \\ 15 & 12 & 3 \\ 20 & 32 & 12 \end{vmatrix}$$

$$= 2 \cdot 3 \cdot 4 \begin{vmatrix} 5 & 4 & 1 \\ 5 & 4 & 1 \\ 5 & 8 & 3 \end{vmatrix}$$

$$= 2 \cdot 3 \cdot 4 \cdot 5 \cdot 4 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 0 = R.H.S. \qquad \text{(Theorem 3)}$$

(e) $\begin{vmatrix} 1 & -4 & 5 \\ 2 & 3 & -2 \\ 7 & 3 & 0 \end{vmatrix} = \begin{vmatrix} -4 & 3 & 3 \\ 5 & -2 & 0 \\ 1 & 2 & 7 \end{vmatrix}$

Proof.

$$L.H.S = \begin{vmatrix} 1 & -4 & 5 \\ 2 & 3 & -2 \\ 7 & 3 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} 5 & -4 & 1 \\ -2 & 3 & 2 \\ 0 & 3 & 7 \end{vmatrix}$$

$$= \begin{vmatrix} -4 & 5 & 1 \\ 3 & -2 & 2 \\ 3 & 0 & 7 \end{vmatrix} \quad \text{(Theorem 2)}$$

$$= \left|\begin{pmatrix} -4 & 5 & 1 \\ 3 & -2 & 2 \\ 3 & 0 & 7 \end{pmatrix}'\right| \quad \text{(Theorem 1)}$$

$$= \begin{vmatrix} -4 & 3 & 3 \\ 5 & -2 & 0 \\ 1 & 2 & 7 \end{vmatrix} = R.H.S.$$

(f) $\begin{vmatrix} -6 & 6 & 3 \\ 0 & -9 & -3 \\ 3 & -3 & -6 \end{vmatrix} = -27\begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ -2 & 2 & -1 \end{vmatrix}$

Proof.

$L.H.S.$

$$= \begin{vmatrix} -6 & 6 & 3 \\ 0 & -9 & -3 \\ 3 & -3 & -6 \end{vmatrix}$$

$$= 3 \cdot 3 \cdot 3 \begin{vmatrix} -2 & 2 & 1 \\ 0 & -3 & -1 \\ 1 & -1 & -2 \end{vmatrix}$$

$$= -27 \begin{vmatrix} 1 & -1 & -2 \\ 0 & -3 & -1 \\ -2 & 2 & 1 \end{vmatrix} \quad \text{(Theorem 2)}$$

$$= 27 \begin{vmatrix} -1 & 1 & -2 \\ -3 & 0 & -1 \\ 2 & -2 & 1 \end{vmatrix} \quad \text{(Theorem 2)}$$

$$= -27 \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ -2 & 2 & -1 \end{vmatrix} = R.H.S. \quad \text{(Theorem 4)}$$

(g) $\begin{vmatrix} 1 & 0 & -3 \\ 3 & -2 & 4 \\ 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ -1 & 2 & 4 \\ 7 & 3 & -2 \end{vmatrix}$

Proof.

$L.H.S.$

$$= \begin{vmatrix} 1 & 0 & -3 \\ 3 & -2 & 4 \\ 1 & 3 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 + (2 \cdot 0) & 0 & -3 \\ 3 + (2 \cdot (-2)) & -2 & 4 \\ 1 + (2 \cdot 3) & 3 & -2 \end{vmatrix} \quad \text{(Theorem 6)}$$

$$= \begin{vmatrix} 1 & 0 & -3 \\ -1 & 2 & 4 \\ 7 & 3 & -2 \end{vmatrix} = R.H.S.$$

(h) $\begin{vmatrix} 5 & 1 & -1 \\ 2 & -1 & -1 \\ 1 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 0 \\ 4 & -1 & -2 \\ 5 & -2 & 2 \end{vmatrix}$

Proof.

$L.H.S.$

$$= \begin{vmatrix} 5 & 1 & -1 \\ 2 & -1 & -1 \\ 1 & -2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 5 + (-2 \cdot 1) & 1 & -1 + 1 \\ 2 + (-2 \cdot (-1)) & -1 & -2 - 1 \\ 1 + (-2 \cdot (-2)) & -2 & 4 - 2 \end{vmatrix} \quad \text{(Theorem 6)}$$

$$= \begin{vmatrix} 3 & 1 & 0 \\ 4 & -1 & -2 \\ 5 & -2 & 2 \end{vmatrix} = R.H.S.$$

3. Let $A = \begin{pmatrix} 7 & -4 \\ -3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2x+1 & -2 \\ x & 1 \end{pmatrix}$. Given that $|AB| = -22$, find the value of x.

Sol.

$$\because |AB| = |A||B| = -22$$

$$\therefore \begin{vmatrix} 7 & -4 \\ -3 & 2 \end{vmatrix} \begin{vmatrix} 2x+1 & -2 \\ x & 1 \end{vmatrix} = -22$$

$$2(2x + 1 + 2x) = -22$$

$$4x + 1 = -11$$

$$4x = -12$$

$$x = -3$$

4. Let $P = \begin{pmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $Q = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$. Given that $PQ = \begin{pmatrix} 30 & -18 & -33 \\ -6 & 4 & 6 \\ -11 & 6 & 14 \end{pmatrix}$, find the value of $|Q|$.

Sol.

$$\because |P||Q| = |PQ|$$

$$\therefore \begin{vmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = \begin{vmatrix} 30 & -18 & -33 \\ -6 & 4 & 6 \\ -11 & 6 & 14 \end{vmatrix}$$

$$3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} 30 & -18 & -33 \\ -6 & 4 & 6 \\ -11 & 6 & 14 \end{vmatrix}$$

$$3\left(\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\right)^2 = 3\begin{vmatrix} 10 & -6 & -11 \\ -6 & 4 & 6 \\ -11 & 6 & 14 \end{vmatrix}$$

$$\left(\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\right)^2 = \begin{vmatrix} 10 & -6 & -11 \\ -6 & 4 & 6 \\ -11 & 6 & 14 \end{vmatrix}$$

$$= 4$$

$$\therefore |Q| = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

$$= \left|\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}'\right|$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= \pm 2$$

Find the value of x in the following equations.

5. $\begin{vmatrix} x & x \\ -2x & -1 \end{vmatrix} = 6$

Sol.

$$\begin{vmatrix} x & x \\ -2x & -1 \end{vmatrix} = 6$$

$$x\begin{vmatrix} 1 & 1 \\ -2x & -1 \end{vmatrix} = 6$$

$$x(-1 + 2x) = 6$$

$$-x + 2x^2 = 6$$

$$2x^2 - x - 6 = 0$$

$$(x - 2)(2x + 3) = 0$$

$$x = 2 \text{ or } x = -\frac{3}{2}$$

6. $\begin{vmatrix} 2 & 4 & 0 \\ 2 & 5 & 6 \\ 3 & x & 9 \end{vmatrix} = 0$

Sol.

$$\begin{vmatrix} 2 & 4 & 0 \\ 2 & 5 & 6 \\ 3 & x & 9 \end{vmatrix} = 0$$

$$2 \cdot 3\begin{vmatrix} 1 & 2 & 0 \\ 2 & 5 & 2 \\ 3 & x & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 0 \\ 2 & 5 & 2 \\ 3 & x & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 5 & 2 \\ x & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ x & 3 \end{vmatrix} + 3\begin{vmatrix} 2 & 0 \\ 5 & 2 \end{vmatrix} = 0$$

$$15 - 2x - 12 + 12 = 0$$

$$-2x = -15$$

$$x = \frac{15}{2}$$

7. $\begin{vmatrix} 1 & 1 & x \\ 1 & x & 1 \\ x & 1 & 1 \end{vmatrix} = 0$

Sol.

$$\begin{vmatrix} 1 & 1 & x \\ 1 & x & 1 \\ x & 1 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & x \\ 1 & 1 \end{vmatrix} + x\begin{vmatrix} 1 & x \\ x & 1 \end{vmatrix} = 0$$

$$x - 1 - 1 + x + x - x^3 = 0$$

$$-x^3 + 3x - 2 = 0$$

$$x^3 - 3x + 2 = 0$$

$$(x + 2)(x^2 - 2x + 1) = 0$$

$$x = -2 \text{ or } x = 1$$

8. $\begin{vmatrix} 2x - 7 & 6 & 9 \\ 3x - 5 & 5 & 4 \\ x - 3 & 0 & 1 \end{vmatrix} = 0$

Sol.

$$
\begin{aligned}
\begin{vmatrix} 2x-7 & 6 & 9 \\ 3x-5 & 5 & 4 \\ x-3 & 0 & 1 \end{vmatrix} &= 0 \\
\begin{vmatrix} 2x & 6 & 9 \\ 3x & 5 & 4 \\ x & 0 & 1 \end{vmatrix} + \begin{vmatrix} -7 & 6 & 9 \\ -5 & 5 & 4 \\ -3 & 0 & 1 \end{vmatrix} &= 0 \\
\begin{vmatrix} 2x & 6 & 9 \\ 3x & 5 & 4 \\ x & 0 & 1 \end{vmatrix} &= -58 \\
x\begin{vmatrix} 6 & 9 \\ 5 & 4 \end{vmatrix} + \begin{vmatrix} 2x & 6 \\ 3x & 5 \end{vmatrix} &= -58 \\
-21x + 10x - 18x &= -58 \\
-29x &= -58 \\
x &= 2
\end{aligned}
$$

9. $\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0$

Sol.

$$
\begin{aligned}
\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} &= 0 \\
\begin{vmatrix} 15 & 11 & 10 \\ 11 & 17 & 16 \\ 7 & 14 & 13 \end{vmatrix} + \begin{vmatrix} -2x & 11 & 10 \\ -3x & 17 & 16 \\ -x & 14 & 13 \end{vmatrix} &= 0 \\
\begin{vmatrix} -2x & 11 & 10 \\ -3x & 17 & 16 \\ -x & 14 & 13 \end{vmatrix} &= 36 \\
-2x\begin{vmatrix} 17 & 16 \\ 14 & 13 \end{vmatrix} + 3x\begin{vmatrix} 11 & 10 \\ 14 & 13 \end{vmatrix} - x\begin{vmatrix} 11 & 10 \\ 17 & 16 \end{vmatrix} &= 36 \\
x\left(2\begin{vmatrix} 17 & 16 \\ 14 & 13 \end{vmatrix} - 3\begin{vmatrix} 11 & 10 \\ 14 & 13 \end{vmatrix} + \begin{vmatrix} 11 & 10 \\ 17 & 16 \end{vmatrix}\right) &= -36 \\
(-6-9+6)x &= -36 \\
-9x &= -36 \\
x &= 4
\end{aligned}
$$

10. $\begin{vmatrix} x-1 & 0 & x-3 \\ 1 & x-2 & 1 \\ 2 & x-2 & 2 \end{vmatrix} = 0$

Sol.

$$
\begin{aligned}
\begin{vmatrix} x-1 & 0 & x-3 \\ 1 & x-2 & 1 \\ 2 & x-2 & 2 \end{vmatrix} &= 0 \\
(x-2)\begin{vmatrix} x-1 & 0 & x-3 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{vmatrix} &= 0 \\
(x-2)\left(-\begin{vmatrix} x-1 & x-3 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} x-1 & x-3 \\ 1 & 1 \end{vmatrix}\right) &= 0 \\
(x-2)\left[-(2x-2-2x+6) + (x-1-x-3)\right] &= 0 \\
(x-2) &= 0 \\
x &= 2
\end{aligned}
$$

14.6 Inverse Matrix

If two square matrices A and B are of the same order such that $AB = BA = I$, while I is an identity matrix that has the same order as A and B, then A and B are said to be inverse matrices of each other, and can be denoted as $B = A^{-1}$ and $A = B^{-1}$.

Note that only square matrix have inverse matrix. If a matrix has an inverse matrix, then it is said to be invertible, and the inverse matrix is unique.

Inverse Matrix of a 2x2 Matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2x2 matrix. Then

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad (ad-bc \neq 0)$$

If $|A| = ad - bc = 0$, then A is said to be non-invertible.

14.6.1 Practice 11

Determine if the following matrices are invertible. If they are, find their inverse matrices.

1. $\begin{pmatrix} 6 & 3 \\ 7 & 5 \end{pmatrix}$

Sol.

$|A| = 6 \cdot 5 - 3 \cdot 7 = 9 \neq 0$

$\therefore A$ is invertible.

$$
\begin{aligned}
A^{-1} &= \frac{1}{9}\begin{pmatrix} 5 & -3 \\ -7 & 6 \end{pmatrix} \\
&= \begin{pmatrix} \frac{5}{9} & -\frac{1}{1} \\ -\frac{7}{9} & \frac{2}{3} \end{pmatrix}
\end{aligned}
$$

2. $\begin{pmatrix} -3 & -2 \\ 6 & 4 \end{pmatrix}$

Sol.

$$|A| = -3 \cdot 4 - (-2) \cdot 6 = 0$$
$$\therefore A \text{ is non-invertible.}$$

3. $\begin{pmatrix} 2 & -6 \\ 3 & -5 \end{pmatrix}$

Sol.

$$|A| = 2 \cdot -5 - (-6) \cdot 3 = 8 \neq 0$$
$$\therefore A \text{ is invertible.}$$
$$\begin{aligned} A^{-1} &= \frac{1}{8}\begin{pmatrix} -5 & 6 \\ -3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{5}{8} & \frac{3}{4} \\ -\frac{3}{8} & \frac{1}{4} \end{pmatrix} \end{aligned}$$

4. If $\begin{pmatrix} 2b+1 & 2 \\ -3b-3 & -4 \end{pmatrix}$ is non-invertible, find the value of b.

Sol.

$$\because \text{The matrix is non-invertible}$$
$$\begin{aligned} \therefore \begin{vmatrix} 2b+1 & 2 \\ -3b-3 & -4 \end{vmatrix} &= 0 \\ -8b - 4 + 6b + 6 &= 0 \\ -2b + 2 &= 0 \\ b &= 1 \end{aligned}$$

Inverse Matrix of a 3x3 Matrix

Let a 3x3 matrix A be of the form $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$. Arrange all the cofactors of elements in A into a matrix:

$$\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$$

Then the transpose of the matrix is the adjoint matrix of A, and can be denoted as adj A. That is:

$$\text{adj } A = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

The inverse matrix of A is:

$$A^{-1} = \frac{1}{|A|} \text{adj } A \quad (|A| \neq 0)$$

14.6.2 Practice 12

Find the inverse matrix of the following matrices.

1. $\begin{pmatrix} -1 & 2 & 3 \\ 3 & 0 & 1 \\ 2 & -1 & -2 \end{pmatrix}$

Sol.

$$|A| = \begin{vmatrix} -1 & 2 & 3 \\ 3 & 0 & 1 \\ 2 & -1 & -2 \end{vmatrix} = 6$$

$$\begin{pmatrix} \begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 2 & -2 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 3 & 0 \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 8 & -3 \\ 1 & -4 & 3 \\ 2 & 10 & -6 \end{pmatrix}$$
$$\text{adj } A = \begin{pmatrix} 1 & 1 & 2 \\ 8 & -4 & 10 \\ -3 & 3 & -6 \end{pmatrix}$$
$$\therefore A^{-1} = \frac{1}{6}\begin{pmatrix} 1 & 1 & 2 \\ 8 & -4 & 10 \\ -3 & 3 & -6 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{4}{3} & -\frac{2}{3} & \frac{5}{3} \\ -\frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}$$

2. $\begin{pmatrix} 1 & -2 & -1 \\ -1 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix}$

Sol.

$$|A| = \begin{vmatrix} 1 & -2 & -1 \\ -1 & 2 & -3 \\ 1 & 0 & 1 \end{vmatrix} = 8$$

$$\begin{pmatrix} \begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & -3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} -2 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} -2 & -1 \\ 2 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ -1 & -3 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 2 & -2 & -2 \\ 2 & 2 & -2 \\ 8 & 4 & 0 \end{pmatrix}$$
$$\text{adj } A = \begin{pmatrix} 2 & 2 & 8 \\ -2 & 2 & 4 \\ -2 & -2 & 0 \end{pmatrix}$$
$$\therefore A^{-1} = \frac{1}{8}\begin{pmatrix} 2 & 2 & 8 \\ -2 & 2 & 4 \\ -2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 1 \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & 0 \end{pmatrix}$$

Solving Systems of Linear Equations

Binary and ternary systems of linear equations can be solved by using the inverse matrix of the coefficient matrix. Note that the coefficient matrix must be invertible for this method to work.

14.6.3 Practice 13

Solve the following systems of linear equations using the inverse matrix method.

1. $\begin{cases} 3x - 2y = 12 \\ 7x + 5y = -1 \end{cases}$

   Sol.

$$\begin{aligned}
\text{Let } A &= \begin{pmatrix} 3 & -2 \\ 7 & 5 \end{pmatrix} \\
A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 12 \\ -1 \end{pmatrix} \\
A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} &= A^{-1}\begin{pmatrix} 12 \\ -1 \end{pmatrix} \\
\begin{pmatrix} x \\ y \end{pmatrix} &= A^{-1}\begin{pmatrix} 12 \\ -1 \end{pmatrix} \\
&= \frac{1}{29}\begin{pmatrix} 5 & 2 \\ -7 & 3 \end{pmatrix}\begin{pmatrix} 12 \\ -1 \end{pmatrix} \\
&= \frac{1}{29}\begin{pmatrix} 58 \\ -87 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ -3 \end{pmatrix} \\
\therefore x = 2,&\ y = -3
\end{aligned}$$

2. $\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ x - y - 2z = -7 \end{cases}$

   Sol.

$$\begin{aligned}
\text{Let } A &= \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & -2 \end{pmatrix} \\
A\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 6 \\ 3 \\ -7 \end{pmatrix} \\
A^{-1}A\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= A^{-1}\begin{pmatrix} 6 \\ 3 \\ -7 \end{pmatrix} \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= A^{-1}\begin{pmatrix} 6 \\ 3 \\ -7 \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{7} & \frac{1}{7} & \frac{2}{7} \\ \frac{5}{7} & -\frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix}\begin{pmatrix} 6 \\ 3 \\ -7 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\
\therefore x = 1,&\ y = 2,\ z = 3
\end{aligned}$$

14.6.4 Exercise 14.6

Determine if the following second-order matrices are invertible. If they are, find their inverse matrix.

1. $\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$

   Sol.

   $|A| = 5 \cdot 3 - 2 \cdot 7 = 1 \neq 0$

   $\therefore A$ is invertible

   $A^{-1} = \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$

2. $\begin{pmatrix} 4 & -8 \\ -1 & 2 \end{pmatrix}$

   Sol.

   $|A| = 4 \cdot 2 - (-8) \cdot (-1) = 0$

   $\therefore A$ is not invertible

3. $\begin{pmatrix} 10 & 5 \\ -6 & -3 \end{pmatrix}$

   Sol.

   $|A| = 10 \cdot (-3) - 5 \cdot (-6) = 0$

   $\therefore A$ is not invertible

4. $\begin{pmatrix} 4 & -5 \\ -7 & 9 \end{pmatrix}$

   Sol.

   $|A| = 4 \cdot 9 - (-5) \cdot (-7) = 1 \neq 0$

   $\therefore A$ is invertible

   $A^{-1} = \begin{pmatrix} 9 & 5 \\ 7 & 4 \end{pmatrix}$

5. $\begin{pmatrix} -2 & -1 \\ 6 & 3 \end{pmatrix}$

   Sol.

   $|A| = (-2) \cdot 3 - (-1) \cdot 6 = 0$

   $\therefore A$ is not invertible

6. $\begin{pmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix}$

   Sol.

   $|A| = \sin\alpha \cdot \sin\alpha - (-\cos\alpha) \cdot \cos\alpha = 1 \neq 0$

   $\therefore A$ is invertible

   $A^{-1} = \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix}$

7. Given that the inverse matrix of matrix $\begin{pmatrix} -2 & 5 \\ 1 & x \end{pmatrix}$ is $\begin{pmatrix} x & y \\ -1 & -2 \end{pmatrix}$, find the value of x and y.

Sol.

$$\begin{vmatrix} -2 & 5 \\ 1 & x \end{vmatrix} = -2x - 5$$

$$(-2x-5)\begin{pmatrix} x & -5 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} x & y \\ -1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} -2x^2 - 10x & 10x + 25 \\ 2x + 5 & 4x + 10 \end{pmatrix} = \begin{pmatrix} x & y \\ -1 & -2 \end{pmatrix}$$

Comparing coefficients,

$$\begin{cases} -2x^2 - 10x = x \\ 10x + 25 = y \\ 2x + 5 = -1 \\ 4x + 10 = -2 \end{cases}$$

$$\begin{aligned} 2x &= -6 \\ x &= -3 \\ -30 + 25 &= y \\ y &= -5 \\ \therefore x &= -3,\ y = -5 \end{aligned}$$

8. If the matrix $\begin{pmatrix} 3 & x \\ -2 & 4 \end{pmatrix}$ is not invertible, find the value of x.

Sol.

$$\begin{vmatrix} 3 & x \\ -2 & 4 \end{vmatrix} = 3 \cdot 4 - x \cdot (-2) = 0$$

$$12 + 2x = 0$$

$$x = -6$$

9. Given the matrix $\begin{pmatrix} y^2 - 7 & -2 \\ 6 & 2y \end{pmatrix}$, find the range of y such that the matrix is invertible.

Sol.

$$\begin{vmatrix} y^2 - 7 & -2 \\ 6 & 2y \end{vmatrix} = (y^2 - 7) \cdot 2y + 12 \neq 0$$

$$y^3 - 7y + 6 \neq 0$$

$$(y-1)(y+3)(y-2) \neq 0$$

$$y \in \mathbb{R},\ y \neq -3, 1, 2$$

10. Given the matrix $\begin{pmatrix} x & 2 & 1 \\ -1 & x-1 & -2 \\ 1-x & 1 & 1 \end{pmatrix}$, find the range of x such that the matrix is not invertible.

Sol.

$$\begin{vmatrix} x & 2 & 1 \\ -1 & x-1 & -2 \\ 1-x & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & x-1 \\ 1-x & 1 \end{vmatrix} + 2\begin{vmatrix} x & 2 \\ 1-x & 1 \end{vmatrix} + \begin{vmatrix} x & 2 \\ -1 & x-1 \end{vmatrix}$$

$$= 1 + x^2 - 2x + 1 + 2x - 4 + 4x + x^2 - x + 2$$

$$= 2x^2 + 3x - 4 = 0$$

$$(x+2)(2x-1) = 0$$

$$x = -2 \text{ or } x = \frac{1}{2}$$

11. Given an identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $A = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$. If $AJA = J$, and $A + A^{-1} = 3I$, find A.

Sol.

$$AJA = J$$

$$A^{-1}AJA = A^{-1}J$$

$$JA = A^{-1}J$$

$$A^{-1} = 3I - A$$

$$= \begin{pmatrix} 3-a & -1 \\ -1 & 3-b \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} = \begin{pmatrix} 3-a & -1 \\ -1 & 3-b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ -a & -1 \end{pmatrix} = \begin{pmatrix} 1 & 3-a \\ -3+b & -1 \end{pmatrix}$$

$$b = 3 - a$$

$$\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} + \frac{1}{ab-1}\begin{pmatrix} b & -1 \\ -1 & a \end{pmatrix} = 3I$$

$$\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} + \begin{pmatrix} \frac{b}{ab-1} & \frac{-1}{ab-1} \\ \frac{-1}{ab-1} & \frac{a}{ab-1} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{a^2b-a+b}{ab-1} & \frac{ab-2}{ab-1} \\ \frac{ab-2}{ab-1} & \frac{ab^2-b+a}{ab-1} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\frac{ab-2}{ab-1} = 0$$

$$a(3-a) - 2 = 0$$

$$a^2 - 3a + 2 = 0$$

$$(a-2)(a-1) = 0$$

$$a = 2 \text{ or } a = 1$$

When $a = 2,\ b = 1$, and when $a = 1,\ b = 2$

$$\therefore A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ or } A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

12. Given that $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$, and

$C = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{pmatrix}$, if $AB = C$, find A.

Sol.

$$\begin{aligned} AB &= C \\ ABB^{-1} &= CB^{-1} \\ A &= CB^{-1} \\ A &= \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -9 & -2 \\ 0 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & -4 & 0 \\ -7 & 20 & 5 \end{pmatrix} \end{aligned}$$

Find the inverse matrix of the following matrices.

13. $\begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & 3 \\ 2 & 1 & 0 \end{pmatrix}$

Sol.

$$\begin{vmatrix} 1 & 0 & 2 \\ 4 & 1 & 3 \\ 2 & 1 & 0 \end{vmatrix} = 1$$

$$\begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -6 & 2 \\ -2 & -4 & -1 \\ -2 & 5 & 1 \end{pmatrix}$$

$$\operatorname{adj} A = \begin{pmatrix} 3 & -2 & -2 \\ -6 & -4 & 5 \\ 2 & -1 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 3 & -2 & -2 \\ -6 & -4 & 5 \\ 2 & -1 & 1 \end{pmatrix}$$

14. $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -1 & 0 & -2 \end{pmatrix}$

Sol.

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -1 & 0 & -2 \end{vmatrix} = 9$$

$$\begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -1 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 5 & 1 \\ 4 & -3 & -2 \\ 1 & -3 & -5 \end{pmatrix}$$

$$\operatorname{adj} A = \begin{pmatrix} -2 & 4 & 1 \\ 6 & -3 & -3 \\ 1 & -2 & -5 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{9}\begin{pmatrix} -2 & 4 & 1 \\ 6 & -3 & -3 \\ 1 & -2 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{2}{5} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{9} & -\frac{2}{9} & -\frac{5}{9} \end{pmatrix}$$

15. $\begin{pmatrix} 1 & -1 & 3 \\ 0 & -4 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

Sol.

$$\begin{vmatrix} 1 & -1 & 3 \\ 0 & -4 & 3 \\ 2 & 3 & 1 \end{vmatrix} = 5$$

$$\begin{pmatrix} \begin{vmatrix} -4 & 3 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & -4 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} -1 & 3 \\ -4 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & -4 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -13 & 6 & 8 \\ 10 & -5 & -5 \\ 9 & -3 & -4 \end{pmatrix}$$

$$\operatorname{adj} A = \begin{pmatrix} -13 & 10 & 9 \\ 6 & -5 & -3 \\ 8 & -5 & -4 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{5}\begin{pmatrix} -13 & 10 & 9 \\ 6 & -5 & -3 \\ 8 & -5 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{13}{5} & \frac{10}{5} & \frac{9}{5} \\ \frac{6}{5} & -\frac{1}{5} & -\frac{3}{5} \\ \frac{8}{5} & -\frac{1}{5} & -\frac{4}{5} \end{pmatrix}$$

Solve the following systems of linear equations using the inverse matrix method.

16. $\begin{cases} 3x + 2y = 1 \\ 4x - y = 5 \end{cases}$

Sol.

$$
\begin{aligned}
\text{Let } A &= \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix} \\
A \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\
A^{-1} A \begin{pmatrix} x \\ y \end{pmatrix} &= A^{-1} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\
&= A^{-1} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\
&= -\frac{1}{11} \begin{pmatrix} -1 & -2 \\ -4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \\
&= -\frac{1}{11} \begin{pmatrix} -11 \\ 11 \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\
\therefore x &= 1, \ y = 1
\end{aligned}
$$

17. $\begin{cases} 2x - 7y = 8 \\ 9x - 4y = -19 \end{cases}$

Sol.

$$
\begin{aligned}
\text{Let } A &= \begin{pmatrix} 2 & -7 \\ 9 & -4 \end{pmatrix} \\
A \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 8 \\ -19 \end{pmatrix} \\
A^{-1} A \begin{pmatrix} x \\ y \end{pmatrix} &= A^{-1} \begin{pmatrix} 8 \\ -19 \end{pmatrix} \\
\begin{pmatrix} x \\ y \end{pmatrix} &= A^{-1} \begin{pmatrix} 8 \\ -19 \end{pmatrix} \\
&= \frac{1}{55} \begin{pmatrix} -4 & 7 \\ -9 & 2 \end{pmatrix} \begin{pmatrix} 8 \\ -19 \end{pmatrix} \\
&= \frac{1}{55} \begin{pmatrix} -165 \\ -110 \end{pmatrix} \\
&= \begin{pmatrix} -3 \\ -2 \end{pmatrix} \\
\therefore x &= -3, \ y = -2
\end{aligned}
$$

18. $\begin{cases} 2x + 4y - 3z = 3 \\ 3x - 8y + 6z = 1 \\ 8x - 2y - 9z = 4 \end{cases}$

Sol.

$$
\begin{aligned}
\text{Let } A &= \begin{pmatrix} 2 & 4 & -3 \\ 3 & -8 & 6 \\ 8 & -2 & -9 \end{pmatrix} \\
A \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= A^{-1} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} \frac{2}{7} & \frac{1}{7} & 0 \\ \frac{25}{98} & \frac{1}{49} & -\frac{1}{14} \\ \frac{29}{147} & \frac{6}{49} & -\frac{2}{21} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \\
\therefore x &= 1, \ y = \frac{1}{2}, \ z = \frac{1}{3}
\end{aligned}
$$

19. $\begin{cases} 3x - y + 4z = 0 \\ 5x + 4y - 3z = 0 \\ 2x - 3y - z = 0 \end{cases}$

Sol.

$$
\begin{aligned}
\text{Let } A &= \begin{pmatrix} 3 & -1 & 4 \\ 5 & 4 & -3 \\ 2 & -3 & -1 \end{pmatrix} \\
A \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= A^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\therefore x &= 0, \ y = 0, \ z = 0
\end{aligned}
$$

20. $\begin{cases} 3x - y = 14 \\ 2y + z = 5 \\ 5z - x = 10 \end{cases}$

Sol.

$$\text{Let } A = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ -1 & 0 & 5 \end{pmatrix}$$

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ 5 \\ 10 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 14 \\ 5 \\ 10 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{10}{31} & \frac{5}{31} & -\frac{1}{31} \\ -\frac{1}{31} & \frac{15}{31} & -\frac{3}{31} \\ \frac{2}{31} & \frac{1}{31} & \frac{6}{31} \end{pmatrix}\begin{pmatrix} 14 \\ 5 \\ 10 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$$

$$\therefore x = 5,\ y = 1,\ z = 3$$

14.7 Gauss Elimination

The concept of Gauss elimination is to eliminate the variables in the equations one by one, through the use of elementary row operations. The elementary row operations are as follows:

1. Interchange two rows:

   $R_i \leftrightarrow R_j$: interchange row i and row j.

2. Multiply a row by a nonzero constant:

   $R_i \to kR_i$: multiply row i by k, where k is a nonzero constant.

3. Add a multiple of one row to another row:

   $R_i \to R_i + kR_j$: add k times row j to row i.

14.7.1 Practice 14

Solve the following system of equations by Gauss elimination:

1. $\begin{cases} 3x - 2y - z = 4 \\ 2x + y - 4z = 4 \\ x + 2y - 3z = 4 \end{cases}$

Sol.

$$\left(\begin{array}{ccc|c} 3 & -2 & -1 & 4 \\ 2 & 1 & -4 & 4 \\ 1 & 2 & -3 & 4 \end{array}\right) \xrightarrow{R_1 \to R_1 + R_3} \left(\begin{array}{ccc|c} 4 & 0 & -4 & 8 \\ 2 & 1 & -4 & 4 \\ 1 & 2 & -3 & 4 \end{array}\right)$$

$$\xrightarrow{R_1 \to \frac{1}{4}R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & 1 & -4 & 4 \\ 1 & 2 & -3 & 4 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & 1 & -4 & 4 \\ 0 & 2 & -2 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 \to \frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & 1 & -4 & 4 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \to R_2 + 2R_3]{R_1 \to R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\therefore x = 3,\ y = 2,\ z = 1$$

2. $\begin{cases} 3x + y + 2z = 5 \\ 2x - 2y + 5z = 3 \\ x - 3y + 4z = 0 \end{cases}$

Sol.

$$\left(\begin{array}{ccc|c} 3 & 1 & 2 & 5 \\ 2 & -2 & 5 & 3 \\ 1 & -3 & 4 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 + 2R_1} \left(\begin{array}{ccc|c} 3 & 1 & 2 & 5 \\ 8 & 0 & 9 & 13 \\ 10 & 0 & 10 & 15 \end{array}\right)$$

$$\xrightarrow[R_1 \leftrightarrow R_2]{R_3 \to \frac{1}{10}R_3} \left(\begin{array}{ccc|c} 8 & 0 & 9 & 13 \\ 3 & 1 & 2 & 5 \\ 1 & 0 & 1 & \frac{3}{2} \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 - 8R_3]{R_2 \to R_2 - 2R_3} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & \frac{3}{2} \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - R_3} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & \frac{3}{2} \\ 1 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

$$\therefore x = \frac{1}{2},\ y = \frac{3}{2},\ z = 1$$

Gauss elimination can also be used to find the inverse of a matrix. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a invertible matrix, that is, $|A| \neq 0$. Now we arrange the matrix A and the identity matrix I into a 3 by 6 augmented matrix $A|I$ as follows:

$$\left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$

We then apply Gauss elimination to the augmented matrix $A|I$ to obtain the following matrix such that the left hand side of this matrix become an identity matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right)$$

where b_{ij} are constants, the right hand side of the augmented matrix is the inverse of A:

$$A^{-1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

14.7.2 Practice 15

Using the method of Gauss elimination, find the inverse of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & -4 \end{pmatrix}$.

Sol.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & -6 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -8 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to -\frac{1}{8} R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 + R_3]{R_2 \to R_2 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{17}{8} & -\frac{7}{8} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{5}{4} & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{17}{8} & -\frac{7}{8} & -\frac{1}{8} \\ -\frac{5}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{pmatrix}$$

14.7.3 Exercise 14.7

Solve the following system of linear equations using the method of Gauss elimination:

1. $\begin{cases} 3x - y - 14 = 0 \\ 2y + z - 5 = 0 \\ x - 5z + 10 = 0 \end{cases}$

Sol.

$$\begin{cases} 3x - y = 14 \\ 2y + z = 5 \\ x - 5z = -10 \end{cases}$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 0 & 14 \\ 0 & 2 & 1 & 5 \\ 1 & 0 & -5 & -10 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - 3R_3} \left(\begin{array}{ccc|c} 0 & -1 & 15 & 44 \\ 0 & 2 & 1 & 5 \\ 1 & 0 & -5 & -10 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 + 2R_1} \left(\begin{array}{ccc|c} 0 & -1 & 15 & 44 \\ 0 & 0 & 31 & 93 \\ 1 & 0 & -5 & -10 \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{1}{31} R_2} \left(\begin{array}{ccc|c} 0 & -1 & 15 & 44 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & -5 & -10 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 - 15R_2]{R_3 \to R_3 + 5R_2} \left(\begin{array}{ccc|c} 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 5 \end{array}\right)$$

$$\xrightarrow[R_1 \to -R_1]{R_3 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\therefore x = 5,\ y = 1,\ z = 3$$

2. $\begin{cases} x + y + z = 6 \\ x + 2y + 3z = 10 \\ 2x + 3y - 4z = 8 \end{cases}$

Sol.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 2 & 3 & -4 & 8 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & -6 & -4 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -8 & -8 \end{array}\right)$$

$$\xrightarrow{R_3 \to -\frac{1}{8} R_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 + R_3]{R_2 \to R_2 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\therefore x = 3,\ y = 2,\ z = 1$$

3. $\begin{cases} -x+y+z=5 \\ 2x-7y+4z=1 \\ 2x-5y+3z=-2 \end{cases}$

Sol.

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 5 \\ 2 & -7 & 4 & 1 \\ 2 & -5 & 3 & -2 \end{array}\right)$$

$$\xrightarrow[R_2\to R_2+2R_1]{R_3\to R_3+2R_1} \left(\begin{array}{ccc|c} -1 & 1 & 1 & 5 \\ 0 & -5 & 6 & 11 \\ 0 & -3 & 5 & 8 \end{array}\right)$$

$$\xrightarrow{R_2\to R_2-R3} \left(\begin{array}{ccc|c} -1 & 1 & 1 & 5 \\ 0 & -2 & 1 & 3 \\ 0 & -3 & 5 & 8 \end{array}\right)$$

$$\xrightarrow[R_1\to R_1-R_2]{R_3\to R_3-5R_2} \left(\begin{array}{ccc|c} -1 & 3 & 0 & 2 \\ 0 & -2 & 1 & 3 \\ 0 & 7 & 0 & -7 \end{array}\right)$$

$$\xrightarrow[R_1\to -R_1]{R_3\to \frac{1}{7}R_3} \left(\begin{array}{ccc|c} 1 & -3 & 0 & -2 \\ 0 & -2 & 1 & 3 \\ 0 & 1 & 0 & -1 \end{array}\right)$$

$$\xrightarrow[R_1\to R_1+3R_3]{R_2\to R_2+2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{array}\right)$$

$$\xrightarrow{R_2\leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\therefore x=-5,\ y=-1,\ z=1$$

4. $\begin{cases} 4x-y-7z=0 \\ 5x-2y-z=1 \\ 3x+3y+5z=2 \end{cases}$

Sol.

$$\left(\begin{array}{ccc|c} 4 & -1 & -7 & 0 \\ 5 & -2 & -1 & 1 \\ 3 & 3 & 5 & 2 \end{array}\right)$$

$$\xrightarrow{R_2\to R_2-2R_1} \left(\begin{array}{ccc|c} 4 & -1 & -7 & 0 \\ -3 & 0 & 13 & 1 \\ 3 & 3 & 5 & 2 \end{array}\right)$$

$$\xrightarrow{R_3\to R_3+R_2} \left(\begin{array}{ccc|c} 4 & -1 & -7 & 0 \\ -3 & 0 & 13 & 1 \\ 0 & 3 & 18 & 3 \end{array}\right)$$

$$\xrightarrow{R_3\to \frac{1}{3}R_3} \left(\begin{array}{ccc|c} 4 & -1 & -7 & 0 \\ -3 & 0 & 13 & 1 \\ 0 & 1 & 6 & 1 \end{array}\right)$$

$$\xrightarrow{R_1\to R_1+R_3} \left(\begin{array}{ccc|c} 4 & 0 & -1 & 1 \\ -3 & 0 & 13 & 1 \\ 0 & 1 & 6 & 1 \end{array}\right)$$

$$\xrightarrow{R_3\to 4R_3} \left(\begin{array}{ccc|c} 4 & 0 & -1 & 1 \\ -12 & 0 & 52 & 4 \\ 0 & 1 & 6 & 1 \end{array}\right)$$

$$\xrightarrow{R_2\to R_2+3R_1} \left(\begin{array}{ccc|c} 4 & 0 & -1 & 1 \\ 0 & 0 & 49 & 7 \\ 0 & 1 & 6 & 1 \end{array}\right)$$

$$\xrightarrow{R_2\to \frac{1}{49}R_2} \left(\begin{array}{ccc|c} 4 & 0 & -1 & 1 \\ 0 & 0 & 1 & \frac{1}{7} \\ 0 & 1 & 6 & 1 \end{array}\right)$$

$$\xrightarrow[R_3\to R_3-6R_2]{R_1\to R_1+R_2} \left(\begin{array}{ccc|c} 4 & 0 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} \end{array}\right)$$

$$\xrightarrow[R_1\to \frac{1}{4}R_1]{R_2\leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{7} \\ 0 & 1 & 0 & \frac{1}{7} \\ 0 & 0 & 1 & \frac{1}{7} \end{array}\right)$$

$$\therefore x=\frac{2}{7},\ y=\frac{1}{7},\ z=\frac{1}{7}$$

Find the inverse of the following matrices using the method of Gauss Jordan elimination.

5. $\begin{pmatrix} 1 & -1 & 0 \\ 5 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}$

Sol.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 5 & 2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2\to R_2+2R_1]{R_3\to R_3-2R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 7 & 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2\to R_2-7R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 7 & 1 & -5 & 1 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2\to R_2-R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & -3 & 1 & -1 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2\to \frac{1}{5}R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1\to R_1+R_2]{R_3\to R_3-2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -\frac{3}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{4}{5} & -\frac{2}{5} & \frac{7}{5} \end{array}\right)$$

$$\therefore A^{-1}=\begin{pmatrix} \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{4}{5} & -\frac{2}{5} & \frac{7}{5} \end{pmatrix}$$

6. $\begin{pmatrix} 3 & 14 & 0 \\ 2 & 5 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

Sol.

$$\left(\begin{array}{ccc|ccc} 3 & 14 & 0 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1\to R_1-3R_3]{R_2\to R_2-2R_3} \left(\begin{array}{ccc|ccc} 0 & 8 & -3 & 1 & 0 & -3 \\ 0 & 1 & -1 & 0 & 1 & -2 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1\to R_1-3R_2]{R_3\to R_3+R_2} \left(\begin{array}{ccc|ccc} 0 & 5 & 0 & 1 & -3 & 3 \\ 0 & 1 & -1 & 0 & 1 & -2 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1\to \frac{1}{5}R_1]{R_3\to R_3+R_2} \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{5} & -\frac{3}{5} & \frac{3}{5} \\ 0 & 1 & -1 & 0 & 1 & -2 \\ 1 & 3 & 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow[R_2\to R_2-R_1]{R_3\to R_3-3R_1} \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{5} & -\frac{3}{5} & \frac{3}{5} \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{8}{5} & -\frac{13}{5} \\ 1 & 0 & 0 & -\frac{3}{5} & \frac{14}{5} & -\frac{14}{5} \end{array}\right)$$

$$\xrightarrow[R_1\leftrightarrow R_3]{R_2\to -R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & \frac{14}{5} & -\frac{14}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{8}{5} & \frac{13}{5} \\ 0 & 1 & 0 & \frac{1}{5} & -\frac{3}{5} & \frac{3}{5} \end{array}\right)$$

$$\xrightarrow{R_2\leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & \frac{14}{5} & -\frac{14}{5} \\ 0 & 1 & 0 & \frac{1}{5} & -\frac{3}{5} & \frac{3}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{8}{5} & \frac{13}{5} \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} -\frac{3}{5} & \frac{14}{5} & -\frac{14}{5} \\ \frac{1}{5} & -\frac{3}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{8}{5} & \frac{13}{5} \end{pmatrix}$$

14.8 Cramer's Rule

When using this method, the determinant of the coefficient matrix is not zero.

Considering a ternary system of equations, we have the following:

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

The coefficient matrix of this system is

$$\Delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

Now we replace the coefficient of x, y and z in Δ with the constants d_1, d_2 and d_3 respectively, and we get the following:

$$\Delta_x = \begin{pmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{pmatrix} \Delta_y = \begin{pmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{pmatrix} \Delta_z = \begin{pmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{pmatrix}$$

The solution of the system of equations is

$$\begin{cases} x = \frac{\Delta_x}{\Delta} \\ y = \frac{\Delta_y}{\Delta} \\ z = \frac{\Delta_z}{\Delta} \end{cases} \qquad \Delta \neq 0$$

14.8.1 Practice 16

Solve the following system of equations using Cramer's Rule:

1. $\begin{cases} 2x + 3y + 4z = 5 \\ 3x + 4y + 5z = 2 \\ 4x + 5y + 2z = 3 \end{cases}$

   Sol.

$$\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 2 \end{vmatrix} = 4$$

$$\Delta_x = \begin{vmatrix} 5 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 2 \end{vmatrix} = -60$$

$$\Delta_y = \begin{vmatrix} 2 & 5 & 4 \\ 3 & 2 & 5 \\ 4 & 3 & 2 \end{vmatrix} = 52$$

$$\Delta_z = \begin{vmatrix} 2 & 3 & 5 \\ 3 & 4 & 2 \\ 4 & 5 & 3 \end{vmatrix} = -4$$

$$\therefore x = \frac{-60}{4} = -15,\ y = \frac{52}{4} = 13,\ z = \frac{-4}{4} = -1$$

2. $\begin{cases} 3x - y + 2z = 4 \\ 2x + 3y - z = 0 \\ 3x - 2y + z = -1 \end{cases}$

   Sol.

$$\Delta = \begin{vmatrix} 3 & -1 & 2 \\ 2 & 3 & -1 \\ 3 & -2 & 1 \end{vmatrix} = -18$$

$$\Delta_x = \begin{vmatrix} 4 & -1 & 2 \\ 0 & 3 & -1 \\ -1 & -2 & 1 \end{vmatrix} = 9$$

$$\Delta_y = \begin{vmatrix} 3 & 4 & 2 \\ 2 & 0 & -1 \\ 3 & -1 & 1 \end{vmatrix} = -27$$

$$\Delta_z = \begin{vmatrix} 3 & -1 & 4 \\ 2 & 3 & 0 \\ 3 & -2 & -1 \end{vmatrix} = -63$$

$$\therefore x = \frac{9}{-18} = -\frac{1}{2},\ y = \frac{-27}{-18} = \frac{3}{2},\ z = \frac{-63}{-18} = \frac{7}{2}$$

14.8.2 Exercise 14.8

Solve the following system of equations using Cramer's Rule:

1. $\begin{cases} x + 3y + 2z = -4 \\ 2x + y + 4z = -3 \\ 3x + 4y + z = -2 \end{cases}$

Sol.

$$\Delta = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{vmatrix} = 25$$

$$\Delta_x = \begin{vmatrix} -4 & 3 & 2 \\ -3 & 1 & 4 \\ -2 & 4 & 1 \end{vmatrix} = 25$$

$$\Delta_y = \begin{vmatrix} 1 & -4 & 2 \\ 2 & -3 & 4 \\ 3 & -2 & 1 \end{vmatrix} = -25$$

$$\Delta_z = \begin{vmatrix} 1 & 3 & -4 \\ 2 & 1 & -3 \\ 3 & 4 & -2 \end{vmatrix} = -25$$

$$\therefore x = \frac{25}{25} = 1,\ y = \frac{-25}{25} = -1,\ z = \frac{-25}{25} = -1$$

2. $\begin{cases} 2x + 3y - 5z = -4 \\ 4x - y + 3z = 2 \\ 3x + 2y + 4z = 1 \end{cases}$

Sol.

$$\Delta = \begin{vmatrix} 2 & 3 & -5 \\ 4 & -1 & 3 \\ 3 & 2 & 4 \end{vmatrix} = -96$$

$$\Delta_x = \begin{vmatrix} -4 & 3 & -5 \\ 2 & -1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 0$$

$$\Delta_y = \begin{vmatrix} 2 & -4 & -5 \\ 4 & 2 & 3 \\ 3 & 1 & 4 \end{vmatrix} = 48$$

$$\Delta_z = \begin{vmatrix} 2 & 3 & -4 \\ 4 & -1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = -48$$

$$\therefore x = \frac{0}{-96} = 0,\ y = \frac{48}{-96} = -\frac{1}{2},\ z = \frac{-48}{-96} = \frac{1}{2}$$

3. $\begin{cases} x + 2y - 3z = 4 \\ 2x + 3y - z = 5 \\ 3x - y + z = 6 \end{cases}$

Sol.

$$\Delta = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & -1 \\ 3 & -1 & 1 \end{vmatrix} = 25$$

$$\Delta_x = \begin{vmatrix} 4 & 2 & -3 \\ 5 & 3 & -1 \\ 6 & -1 & 1 \end{vmatrix} = 55$$

$$\Delta_y = \begin{vmatrix} 1 & 4 & -3 \\ 2 & 5 & -1 \\ 3 & 6 & 1 \end{vmatrix} = 0$$

$$\Delta_z = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & -1 & 6 \end{vmatrix} = -15$$

$$\therefore x = \frac{55}{25} = \frac{11}{5},\ y = \frac{0}{25} = 0,\ z = \frac{-15}{25} = -\frac{3}{5}$$

4. $\begin{cases} \frac{3}{x} + \frac{1}{y} - \frac{1}{z} = 3 \\ \frac{1}{x} - \frac{1}{y} + \frac{2}{z} = 13 \\ \frac{1}{x} + \frac{4}{y} - \frac{1}{z} = -9 \end{cases}$

Sol.

Let $a = \frac{1}{x},\ b = \frac{1}{y},\ c = \frac{1}{z}$

$$\begin{cases} 3a + b - c = 3 \\ a - b + 2c = 13 \\ a + 4b - c = -9 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = -23$$

$$\Delta_a = \begin{vmatrix} 3 & 1 & -1 \\ 13 & -1 & 2 \\ -9 & 4 & -1 \end{vmatrix} = -69$$

$$\Delta_b = \begin{vmatrix} 3 & 3 & -1 \\ 1 & 13 & 2 \\ 1 & -9 & -1 \end{vmatrix} = 46$$

$$\Delta_c = \begin{vmatrix} 3 & 1 & 3 \\ 1 & -1 & 13 \\ 1 & 4 & -9 \end{vmatrix} = -92$$

$$a = \frac{-69}{-23} = 3,\ b = \frac{46}{-23} = -2,\ c = \frac{-92}{-23} = 4$$

$$\therefore x = \frac{1}{3},\ y = -\frac{1}{2},\ z = \frac{1}{4}$$

14.9 Revision Exercise 14

Calculate the following (Question 1 to 4):

1. $5\begin{pmatrix} -3 & -1 \\ 3 & 4 \end{pmatrix} + 4\begin{pmatrix} 6 & 2 \\ 1 & -1 \end{pmatrix}$

Sol.

$$5\begin{pmatrix}-3 & -1\\ 3 & 4\end{pmatrix}+4\begin{pmatrix}6 & 2\\ 1 & -1\end{pmatrix}$$
$$=\begin{pmatrix}-15 & -5\\ 15 & 20\end{pmatrix}+\begin{pmatrix}24 & 8\\ 4 & -4\end{pmatrix}$$
$$=\begin{pmatrix}9 & 3\\ 19 & 16\end{pmatrix}$$

2. $-4\begin{pmatrix}3 & 0\\ -1 & 5\end{pmatrix}-3\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}$

Sol.

$$-4\begin{pmatrix}3 & 0\\ -1 & 5\end{pmatrix}-3\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}$$
$$=\begin{pmatrix}-12 & 0\\ 4 & -20\end{pmatrix}-\begin{pmatrix}3 & 0\\ -3 & -3\end{pmatrix}$$
$$=\begin{pmatrix}-15 & 0\\ 7 & 17\end{pmatrix}$$

3. $\begin{pmatrix}2 & 6 & -1\\ 8 & -4 & 3\\ 5 & 7 & -2\end{pmatrix}+\begin{pmatrix}4 & -2 & 1\\ -2 & 1 & 0\\ -3 & 5 & -4\end{pmatrix}$

Sol.

$$\begin{pmatrix}2 & 6 & -1\\ 8 & -4 & 3\\ 5 & 7 & -2\end{pmatrix}+\begin{pmatrix}4 & -2 & 1\\ -2 & 1 & 0\\ -3 & 5 & -4\end{pmatrix}$$
$$=\begin{pmatrix}6 & 4 & 0\\ 6 & -3 & 3\\ 2 & 12 & -6\end{pmatrix}$$

4. $2\begin{pmatrix}1 & -3 & 5\\ 7 & 2 & 0\\ 2 & 4 & -4\end{pmatrix}-\begin{pmatrix}-2 & -6 & 2\\ -5 & 1 & 1\\ 0 & 0 & -2\end{pmatrix}$

Sol.

$$2\begin{pmatrix}1 & -3 & 5\\ 7 & 2 & 0\\ 2 & 4 & -4\end{pmatrix}-\begin{pmatrix}-2 & -6 & 2\\ -5 & 1 & 1\\ 0 & 0 & -2\end{pmatrix}$$
$$=\begin{pmatrix}2 & -6 & 10\\ 14 & 4 & 0\\ 4 & 8 & -8\end{pmatrix}-\begin{pmatrix}-2 & -6 & 2\\ -5 & 1 & 1\\ 0 & 0 & -2\end{pmatrix}$$
$$=\begin{pmatrix}4 & 0 & 8\\ 19 & 3 & -1\\ 4 & 8 & -6\end{pmatrix}$$

5. Given that $\begin{pmatrix}2\\ -3\end{pmatrix}+3\begin{pmatrix}5\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$, find the value of x and y.

Sol.

$$\begin{pmatrix}2\\ -3\end{pmatrix}+3\begin{pmatrix}5\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$
$$\begin{pmatrix}2\\ -3\end{pmatrix}+\begin{pmatrix}15\\ 3y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$
$$\begin{pmatrix}17\\ -3+3y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$
$$x=17$$
$$y=-3+3y$$
$$2y=3$$
$$y=\frac{3}{2}$$
$$\therefore x=17,\ y=\frac{3}{2}$$

6. Let $P=\begin{pmatrix}3 & -2 & 1\\ -1 & 2 & -3\\ 4 & 0 & -2\end{pmatrix}$, $Q=\begin{pmatrix}1 & -5 & -4\\ -2 & 0 & 6\\ 3 & 2 & 3\end{pmatrix}$ and $R=\begin{pmatrix}4 & 0 & 5\\ 1 & -2 & -7\\ 0 & -2 & 1\end{pmatrix}$. Find the following:

(a) $2Q+R'$

Sol.

$$2Q+R'=2\begin{pmatrix}1 & -5 & -4\\ -2 & 0 & 6\\ 3 & 2 & 3\end{pmatrix}+\begin{pmatrix}4 & 0 & 5\\ 1 & -2 & -7\\ 0 & -2 & 1\end{pmatrix}'$$
$$=\begin{pmatrix}2 & -10 & -8\\ -4 & 0 & 12\\ 6 & 4 & 6\end{pmatrix}+\begin{pmatrix}4 & 1 & 0\\ 0 & -2 & -2\\ 5 & -7 & 1\end{pmatrix}$$
$$=\begin{pmatrix}6 & -9 & -8\\ -4 & -2 & 10\\ 11 & -3 & 7\end{pmatrix}$$

(b) $(P-R)+2Q'$

Sol.

$$(P-R)+2Q'$$
$$=\begin{pmatrix}3 & -2 & 1\\ -1 & 2 & -3\\ 4 & 0 & -2\end{pmatrix}-\begin{pmatrix}4 & 0 & 5\\ 1 & -2 & -7\\ 0 & -2 & 1\end{pmatrix}$$
$$+2\begin{pmatrix}1 & -5 & -4\\ -2 & 0 & 6\\ 3 & 2 & 3\end{pmatrix}'$$
$$=\begin{pmatrix}-1 & -2 & -4\\ -2 & 4 & 4\\ 4 & 2 & -3\end{pmatrix}+\begin{pmatrix}2 & -4 & 6\\ -10 & 0 & 4\\ -8 & 12 & 6\end{pmatrix}$$
$$=\begin{pmatrix}1 & 6 & 2\\ -12 & 4 & 8\\ -4 & 14 & 3\end{pmatrix}$$

(c) $[2(Q-P)]'$

Sol.

$$[2(Q-P)]'$$
$$=\left\{2\left[\begin{pmatrix}1&-5&-4\\-2&0&6\\3&2&3\end{pmatrix}-\begin{pmatrix}3&-2&1\\-1&2&-3\\4&0&-2\end{pmatrix}\right]\right\}'$$
$$=\left[2\begin{pmatrix}-2&-3&-5\\-1&-2&9\\-1&2&5\end{pmatrix}\right]'$$
$$=\begin{pmatrix}-4&-6&-10\\-2&-4&18\\-2&4&10\end{pmatrix}'$$
$$=\begin{pmatrix}-4&-2&-2\\-6&-4&4\\-10&18&10\end{pmatrix}$$

(d) $(R'-Q)'$

Sol.

$$(R'-Q)'=\left[\begin{pmatrix}4&0&5\\1&-2&-7\\0&-2&1\end{pmatrix}'-\begin{pmatrix}1&-5&-4\\-2&0&6\\3&2&3\end{pmatrix}\right]'$$
$$=\begin{pmatrix}4&0&5\\1&-2&-7\\0&-2&1\end{pmatrix}-\begin{pmatrix}1&-5&-4\\-2&0&6\\3&2&3\end{pmatrix}'$$
$$=\begin{pmatrix}4&0&5\\1&-2&-7\\0&-2&1\end{pmatrix}-\begin{pmatrix}1&-2&3\\-5&0&2\\-4&6&3\end{pmatrix}$$
$$=\begin{pmatrix}3&2&2\\6&-2&-9\\4&-8&-2\end{pmatrix}$$

7. Let $M=\begin{pmatrix}-1&0\\4&-3\\2&4\end{pmatrix}$ and $N=\begin{pmatrix}3&6\\7&-1\\-4&2\end{pmatrix}$. Find the matrix X in the following equations:

(a) $2N-3M=2M-X$

Sol.

$$\begin{aligned}2N-3M&=2M-X\\X&=5M-2N\\&=5\begin{pmatrix}-1&0\\4&-3\\2&4\end{pmatrix}-2\begin{pmatrix}3&6\\7&-1\\-4&2\end{pmatrix}\\&=\begin{pmatrix}-5&0\\20&-15\\10&20\end{pmatrix}-\begin{pmatrix}6&12\\14&-2\\-8&4\end{pmatrix}\\&=\begin{pmatrix}-11&-12\\6&-13\\18&16\end{pmatrix}\end{aligned}$$

(b) $2(M-2N)+X=M+N$

Sol.

$$\begin{aligned}2(M-2N)+X&=M+N\\X&=M+N-2(M-2N)\\&=M+N-2M+4N\\&=-M+5N\\&=-\begin{pmatrix}-1&0\\4&-3\\2&4\end{pmatrix}+5\begin{pmatrix}3&6\\7&-1\\-4&2\end{pmatrix}\\&=\begin{pmatrix}1&0\\-4&3\\-2&-4\end{pmatrix}+\begin{pmatrix}15&30\\35&-5\\-20&10\end{pmatrix}\\&=\begin{pmatrix}16&30\\31&-2\\-22&6\end{pmatrix}\end{aligned}$$

(c) $(M+2N)'=X$

Sol.

$$\begin{aligned}(M+2N)'&=X\\X&=(M+2N)'\\&=\left[\begin{pmatrix}-1&0\\4&-3\\2&4\end{pmatrix}+2\begin{pmatrix}3&6\\7&-1\\-4&2\end{pmatrix}\right]'\\&=\left[\begin{pmatrix}-1&0\\4&-3\\2&4\end{pmatrix}+\begin{pmatrix}6&12\\14&-2\\-8&4\end{pmatrix}\right]\\&=\begin{pmatrix}5&12\\18&-5\\-6&8\end{pmatrix}'\\&=\begin{pmatrix}5&18&-6\\12&-5&8\end{pmatrix}\end{aligned}$$

(d) $3N'-M'=2X$

Sol.

$$\begin{aligned}3N'-M'&=2X\\2X&=(3N-M)'\\&=\left[3\begin{pmatrix}3&6\\7&-1\\-4&2\end{pmatrix}-\begin{pmatrix}-1&0\\4&-3\\2&4\end{pmatrix}\right]'\\&=\left[\begin{pmatrix}9&18\\21&-3\\-12&6\end{pmatrix}-\begin{pmatrix}-1&0\\4&-3\\2&4\end{pmatrix}\right]'\\&=\begin{pmatrix}10&18\\17&0\\-14&2\end{pmatrix}'\\&=\begin{pmatrix}10&17&-14\\18&0&2\end{pmatrix}\\X&=\begin{pmatrix}5&\frac{17}{2}&-7\\9&0&1\end{pmatrix}\end{aligned}$$

Of the following matrices, determine if AB and BA are defined. If any of them is defined, find the value of them (Question 8 to 11):

8. $A = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

Sol.

$\because$ The number of columns of A is not equal to the number of rows of B
$\therefore AB$ is not defined

$\because$ The number of columns of B is equal to the number of rows of A
$\therefore BA$ is defined

$$\therefore BA = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ -2 \\ 0 \end{pmatrix}$$

9. $A = \begin{pmatrix} -5 & 4 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 \\ -3 & 0 \\ 1 & -4 \end{pmatrix}$

Sol.

$\because$ The number of columns of A is equal to the number of rows of B
$\therefore AB$ is defined

$$\therefore AB = \begin{pmatrix} -5 & 4 & 3 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ -3 & 0 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -17 \\ -5 & -11 \end{pmatrix}$$

$\because$ The number of columns of B is equal to the number of rows of A
$\therefore BA$ is defined

$$\therefore BA = \begin{pmatrix} -2 & 1 \\ -3 & 0 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} -5 & 4 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 11 & -6 & -3 \\ 13 & -12 & -9 \\ -9 & -4 & -9 \end{pmatrix}$$

10. $A = \begin{pmatrix} 3 & 2 & -1 \\ 5 & 4 & 0 \\ -2 & 6 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 9 & 0 \\ -7 & 1 \\ 2 & 3 \end{pmatrix}$

Sol.

$\because$ The number of columns of A is equal to the number of rows of B
$\therefore AB$ is defined

$$\therefore AB = \begin{pmatrix} 3 & 2 & -1 \\ 5 & 4 & 0 \\ -2 & 6 & 1 \end{pmatrix}\begin{pmatrix} 9 & 0 \\ -7 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 11 & -1 \\ 17 & 4 \\ -58 & 9 \end{pmatrix}$$

$\because$ The number of columns of B is not equal to the number of rows of A
$\therefore BA$ is not defined

11. $A = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 5 & 2 \end{pmatrix}$

Sol.

$\because$ The number of columns of A is not equal to the number of rows of B
$\therefore AB$ is not defined

$\because$ The number of columns of B is equal to the number of rows of A
$\therefore BA$ is defined

$$\therefore BA = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 10 & 4 \\ 18 & 2 \end{pmatrix}$$

12. Given that $A = \begin{pmatrix} a & 3a \\ 2b & b \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $AB = \begin{pmatrix} 45 \\ 48 \end{pmatrix}$, find the value of a and b.

Sol.

$$AB = \begin{pmatrix} a & 3a \\ 2b & b \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 45 \\ 48 \end{pmatrix}$$
$$\begin{pmatrix} 3a + 6a \\ 6b + 2b \end{pmatrix} = \begin{pmatrix} 45 \\ 48 \end{pmatrix}$$
$$9a = 45$$
$$8b = 48$$
$$a = 5$$
$$b = 6$$

13. Given that $A = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $A + B = AB$, find the value of a, b and c.

Sol.

$$A + B = AB$$
$$\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$
$$\begin{pmatrix} 3 + a & b \\ 0 & 4 + c \end{pmatrix} = \begin{pmatrix} 3a & 3b \\ 0 & 4c \end{pmatrix}$$
$$3 + a = 3a$$
$$2a = 3$$
$$b = 3b$$
$$2b = 0$$
$$4 + c = 4c$$
$$3c = 4$$
$$a = \frac{3}{2},\ b = 0,\ c = \frac{4}{3}$$

Find the value of the following determinants (Question 14 to 22):

14. $\begin{vmatrix} 20 & 15 \\ 8 & 6 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 20 & 15 \\ 8 & 6 \end{vmatrix} = 20 \cdot 6 - 15 \cdot 8$$
$$= 120 - 120$$
$$= 0$$

15. $\begin{vmatrix} 6 & -7 \\ 15 & -2 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 6 & -7 \\ 15 & -2 \end{vmatrix} = 6 \cdot -2 - (-7) \cdot 15$$
$$= -12 + 105$$
$$= 93$$

16. $\begin{vmatrix} -4 & -10 \\ 12 & 7 \end{vmatrix}$

Sol.

$$\begin{vmatrix} -4 & -10 \\ 12 & 7 \end{vmatrix} = -4 \cdot 7 - (-10) \cdot 12$$
$$= -28 + 120$$
$$= 92$$

17. $\begin{vmatrix} 3 & 3 & -6 \\ 2 & -2 & 0 \\ 3 & 0 & -3 \end{vmatrix}$

$$\begin{array}{ccc:cc} + & + & + & & \\ 3 & 3 & -6 & 3 & 3 \\ 2 & -2 & 0 & 2 & -2 \\ 3 & 0 & -3 & 3 & 0 \\ - & - & - & & \end{array}$$

Sol.

$$\begin{vmatrix} 3 & 3 & -6 \\ 2 & -2 & 0 \\ 3 & 0 & -3 \end{vmatrix} = 18 + 0 + 0 - 36 - 0 + 18$$
$$= 0$$

18. $\begin{vmatrix} 5 & 7 & 1 \\ -3 & 6 & 9 \\ 4 & 7 & 3 \end{vmatrix}$

$$\begin{array}{ccc:cc} + & + & + & & \\ 5 & 7 & 1 & 5 & 7 \\ -3 & 6 & 9 & -3 & 6 \\ 4 & 7 & 3 & 4 & 7 \\ - & - & - & & \end{array}$$

Sol.

$$\begin{vmatrix} 5 & 7 & 1 \\ -3 & 6 & 9 \\ 4 & 7 & 3 \end{vmatrix} = 90 + 252 - 21 - 24 - 315 + 63$$
$$= 45$$

19. $\begin{vmatrix} -2 & 7 & -4 \\ 3 & -5 & 2 \\ -1 & 0 & -3 \end{vmatrix}$

$$\begin{array}{ccc:cc} + & + & + & & \\ -2 & 7 & -4 & -2 & 7 \\ 3 & -5 & 2 & 3 & -5 \\ -1 & 0 & -3 & -1 & 0 \\ - & - & - & & \end{array}$$

Sol.

$$\begin{vmatrix} -2 & 7 & -4 \\ 3 & -5 & 2 \\ -1 & 0 & -3 \end{vmatrix} = -30 - 14 - 0 + 20 + 0 + 63$$
$$= -39$$

20. $\begin{vmatrix} 1 & 0 & -1 \\ 3 & -2 & 5 \\ -1 & 1 & 3 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 1 & 0 & -1 \\ 3 & -2 & 5 \\ -1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -2 & 5 \\ 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ -2 & 5 \end{vmatrix}$$
$$= -11 - 3 + 2 = -12$$

21. $\begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 1 \\ -2 & -6 & 5 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 1 \\ -2 & -6 & 5 \end{vmatrix} = 3\begin{vmatrix} 2 & 2 & 4 \\ 1 & 1 & 1 \\ -2 & -2 & 5 \end{vmatrix}$$
$$= 0 \quad \text{(col 1 and 2 are the same)}$$

22. $\begin{vmatrix} 10 & 8 & -2 \\ 15 & 16 & -3 \\ -5 & -4 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 10 & 8 & -2 \\ 15 & 16 & -3 \\ -5 & -4 & 1 \end{vmatrix} = -5\begin{vmatrix} 2 & 8 & 2 \\ 3 & 16 & 3 \\ -1 & -4 & -1 \end{vmatrix}$$
$$= 0 \quad \text{(col 1 and 3 are the same)}$$

Using the identities of determinant, prove the following equations (Question 23 to 24):

23. $\begin{vmatrix} bc & 1 & bc(b+c) \\ ca & 1 & ca(c+a) \\ ab & 1 & ab(a+b) \end{vmatrix} = 0$

Proof.

$$\begin{vmatrix} bc & 1 & bc(b+c) \\ ca & 1 & ca(c+a) \\ ab & 1 & ab(a+b) \end{vmatrix}$$

$$= a^2b^2c^2 \begin{vmatrix} 1 & \frac{1}{bc} & b+c \\ 1 & \frac{1}{ca} & c+a \\ 1 & \frac{1}{ab} & a+b \end{vmatrix}$$

$$= a^2b^2c^2 \begin{vmatrix} 1 & \frac{1}{bc} & -a \\ 1 & \frac{1}{ca} & -b \\ 1 & \frac{1}{ab} & -c \end{vmatrix} \qquad C_3 \to C_3 + (a+b+c)C_1$$

$$= a^2b^2c^2 \begin{vmatrix} 1 & a & -a \\ 1 & b & -b \\ 1 & c & -c \end{vmatrix} \qquad C_2 \to C_2 + abcC_1$$

$$= -a^2b^2c^2 \begin{vmatrix} 1 & a & a \\ 1 & b & b \\ 1 & c & c \end{vmatrix}$$

$$= 0 \qquad C_2 = C_3$$

24. $\begin{vmatrix} a & 1 & a^2(b+c) \\ b & 1 & b^2(c+a) \\ c & 1 & c^2(a+b) \end{vmatrix} = 0$

Proof.

$$\begin{vmatrix} a & 1 & a^2(b+c) \\ b & 1 & b^2(c+a) \\ c & 1 & c^2(a+b) \end{vmatrix}$$

$$= \begin{vmatrix} a & 1 & a^2(b+c) \\ b-a & 0 & b^2(c+a)-a^2(b+c) \\ c-a & 0 & c^2(a+b)-a^2(b+c) \end{vmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$= \begin{vmatrix} b-a & b^2(c+a)-a^2(b+c) \\ c-a & c^2(a+b)-a^2(b+c) \end{vmatrix}$$

$$= (b-a)[c^2(a+b)-a^2(b+c)]$$
$$\quad -(c-a)[b^2(c+a)-a^2(b+c)]$$
$$= c^2(b-a)(b+a)-a^2(b+c)(c-a)$$
$$\quad -b^2(c-a)(c+a)+a^2(b+c)(c-a)$$
$$= c^2(b^2-a^2)-b^2(c^2-a^2)$$
$$= b^2c^2-a^2c^2-b^2c^2+a^2c^2$$
$$= 0$$

Find the value of x in the following expressions (Question 25 to 26):

25. $\begin{vmatrix} 2x & 3 & x+5 \\ -3 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 5x-1$

Sol.

$$\begin{vmatrix} 2x & 3 & x+5 \\ -3 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 5x-1$$

$$x+5\begin{vmatrix} -3 & 2 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 2x & 3 \\ 2 & 1 \end{vmatrix} = 5x-1$$
$$-7(x+5)-(2x-6) = 5x-1$$
$$-7x-35-2x+6 = 5x-1$$
$$-14x = 28$$
$$x = -2$$

26. $\begin{vmatrix} x+3 & 1 & 0 \\ x & 3 & 0 \\ 1 & 0 & -x-2 \end{vmatrix} = x+6$

Sol.

$$\begin{vmatrix} x+3 & 1 & 0 \\ x & 3 & 0 \\ 1 & 0 & -x-2 \end{vmatrix} = x+6$$

$$-x-2\begin{vmatrix} x+3 & 1 \\ x & 3 \end{vmatrix} = x+6$$
$$-(x+2)(3x+9-x) = x+6$$
$$(x+2)(2x+9) = -x-6$$
$$2x^2+13x+18 = -x-6$$
$$2x^2+14x+24 = 0$$
$$x^2+7x+12 = 0$$
$$(x+4)(x+3) = 0$$
$$x = -4 \text{ or } x = -3$$

27. Given an identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $(2I+J)^{-1} = rI + sJ$, find the value of r and s.

Sol.

$$
\begin{aligned}
(2I+J)^{-1} &= \left[2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]^{-1}\\
&= \left[\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}+\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]^{-1}\\
&= \begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}^{-1}\\
&= \frac{1}{5}\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}\\
&= \begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}\\
rI+sJ &= r\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+s\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\\
&= \begin{pmatrix}r & 0\\ 0 & r\end{pmatrix}+\begin{pmatrix}0 & s\\ -s & 0\end{pmatrix}\\
&= \begin{pmatrix}r & s\\ -s & r\end{pmatrix}\\
(2I+J)^{-1} &= rI+sJ\\
\begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix} &= \begin{pmatrix}r & s\\ -s & r\end{pmatrix}\\
\therefore r = \frac{2}{5},\ & s = -\frac{1}{5}
\end{aligned}
$$

Find the value of a in the following matrices if they are non-inversible (Question 28 to 31):

28. $\begin{pmatrix}3 & a\\ -2 & 6\end{pmatrix}$

Sol.

$$
\begin{aligned}
\begin{vmatrix}3 & a\\ -2 & 6\end{vmatrix} &= 0\\
18+2a &= 0\\
2a &= -18\\
a &= -9
\end{aligned}
$$

29. $\begin{pmatrix}5a+2 & 4\\ 6 & a\end{pmatrix}$

Sol.

$$
\begin{aligned}
\begin{vmatrix}5a+2 & 4\\ 6 & a\end{vmatrix} &= 0\\
5a^2+2a-24 &= 0\\
(x-2)(5x+12) &= 0\\
x = 2 \text{ or } x &= \frac{12}{5}
\end{aligned}
$$

30. $\begin{pmatrix}-7 & a & 3\\ 2 & -3 & 1\\ 0 & -a & 4\end{pmatrix}$

Sol.

$$
\begin{aligned}
\begin{vmatrix}-7 & a & 3\\ 2 & -3 & 1\\ 0 & -a & 4\end{vmatrix} &= 0\\
-7\begin{vmatrix}-3 & 1\\ -a & 4\end{vmatrix}-2\begin{vmatrix}a & 3\\ -a & 4\end{vmatrix} &= 0\\
-7(-12+a)-2(4a+3a) &= 0\\
84-7a-14a &= 0\\
21a &= 84\\
a &= 4
\end{aligned}
$$

31. $\begin{pmatrix}a & -1 & 0\\ 4 & 0 & -2\\ a+4 & a & -8\end{pmatrix}$

Sol.

$$
\begin{aligned}
\begin{vmatrix}a & -1 & 0\\ 4 & 0 & -2\\ a+4 & a & -8\end{vmatrix} &= 0\\
a\begin{vmatrix}0 & -2\\ a & -8\end{vmatrix}+\begin{vmatrix}4 & -2\\ a+4 & -8\end{vmatrix}\\
2a^2-32+2a+8 &= 0\\
2a^2+2a-24 &= 0\\
a^2+a-12 &= 0\\
(a+4)(a-3) &= 0\\
a = -4 \text{ or } a &= 3
\end{aligned}
$$

Find the inverse of the following matrices (Question 32 to 37):

32. $\begin{pmatrix}2 & 5\\ 2 & 3\end{pmatrix}$

Sol.

$$
\begin{aligned}
\begin{pmatrix}2 & 5\\ 2 & 3\end{pmatrix}^{-1} &= -\frac{1}{4}\begin{pmatrix}3 & -5\\ -2 & 2\end{pmatrix}\\
&= \begin{pmatrix}-\frac{3}{4} & \frac{5}{4}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}
\end{aligned}
$$

33. $\begin{pmatrix}-2 & -1\\ 4 & 6\end{pmatrix}$

Sol.

$$
\begin{aligned}
\begin{pmatrix}-2 & -1\\ 4 & 6\end{pmatrix}^{-1} &= -\frac{1}{8}\begin{pmatrix}6 & 1\\ -4 & -2\end{pmatrix}\\
&= \begin{pmatrix}-\frac{3}{8} & -\frac{1}{8}\\ \frac{1}{2} & \frac{1}{4}\end{pmatrix}
\end{aligned}
$$

34. $\begin{pmatrix}1 & 0 & 3\\ 3 & 1 & 9\\ -2 & 2 & -4\end{pmatrix}$

Sol.

$$\begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & 9 \\ -2 & 2 & -4 \end{vmatrix} = 2$$

$$\operatorname{adj}\begin{pmatrix} 1 & 0 & 3 \\ 3 & 1 & 9 \\ -2 & 2 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{vmatrix} 1 & 9 \\ 2 & -4 \end{vmatrix} & -\begin{vmatrix} 3 & 9 \\ -2 & -4 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 1 & 9 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \end{pmatrix}'$$

$$= \begin{pmatrix} -22 & -6 & 8 \\ 6 & 2 & -2 \\ -3 & 0 & 1 \end{pmatrix}'$$

$$= \begin{pmatrix} -22 & 6 & -3 \\ -6 & 2 & 0 \\ 8 & -2 & 1 \end{pmatrix}'$$

$$\therefore \begin{pmatrix} 1 & 0 & 3 \\ 3 & 1 & 9 \\ -2 & 2 & -4 \end{pmatrix}^{-1}$$

$$= \frac{1}{2}\begin{pmatrix} -22 & 6 & -3 \\ -6 & 2 & 0 \\ 8 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -11 & 3 & -\frac{3}{2} \\ -3 & 1 & 0 \\ 4 & -1 & \frac{1}{2} \end{pmatrix}$$

35. $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & -4 \\ -2 & 5 & 1 \end{pmatrix}$

Sol.

$$\begin{vmatrix} 1 & 2 & -3 \\ 2 & 1 & -4 \\ -2 & 5 & 1 \end{vmatrix} = -3$$

$$\operatorname{adj}\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & -4 \\ -2 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{vmatrix} 1 & -4 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -2 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & -3 \\ 1 & -4 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \end{pmatrix}'$$

$$= \begin{pmatrix} 21 & 6 & 12 \\ -17 & -5 & -9 \\ -5 & -2 & -3 \end{pmatrix}'$$

$$= \begin{pmatrix} 21 & -17 & -5 \\ 6 & -5 & -2 \\ 12 & -9 & -3 \end{pmatrix}'$$

$$\therefore \begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & -4 \\ -2 & 5 & 1 \end{pmatrix}^{-1}$$

$$= -\frac{1}{3}\begin{pmatrix} 21 & -17 & -5 \\ 6 & -5 & -2 \\ 12 & -9 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -7 & -\frac{17}{3} & -\frac{5}{3} \\ -2 & -\frac{5}{3} & -\frac{2}{3} \\ -4 & 3 & 1 \end{pmatrix}$$

36. $\begin{pmatrix} 4 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & -4 & 4 \end{pmatrix}$

Sol.

$$\begin{vmatrix} 4 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & -4 & 4 \end{vmatrix} = 16$$

$$\operatorname{adj}\begin{pmatrix} 4 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & -4 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{vmatrix} -2 & 3 \\ -4 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ 0 & -4 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ -4 & 4 \end{vmatrix} & \begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 0 & -4 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ -2 & 3 \end{vmatrix} & -\begin{vmatrix} 4 & 0 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 0 & -2 \end{vmatrix} \end{pmatrix}'$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ -4 & 16 & 16 \\ 3 & -12 & -8 \end{pmatrix}'$$

$$= \begin{pmatrix} 4 & -4 & 0 \\ 0 & 16 & -12 \\ 9 & 16 & -8 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 4 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & -4 & 4 \end{pmatrix}^{-1}$$

$$= \frac{1}{16}\begin{pmatrix} 4 & -4 & 3 \\ 0 & 16 & -12 \\ 0 & 16 & -8 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{3}{16} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}$$

37. $\begin{pmatrix} 3 & 2 & 4 \\ -1 & -3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$

Sol.

$$\begin{vmatrix} 3 & 2 & 4 \\ -1 & -3 & 0 \\ 1 & 0 & 3 \end{vmatrix} = -9$$

$$\operatorname{adj}\begin{pmatrix} 3 & 2 & 4 \\ -1 & -3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{vmatrix} -3 & 0 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} -1 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} -1 & -3 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & 4 \\ -3 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -1 & -3 \end{vmatrix} \end{pmatrix}'$$

$$= \begin{pmatrix} -9 & 3 & 3 \\ -6 & 5 & 2 \\ 12 & -4 & -7 \end{pmatrix}'$$

$$= \begin{pmatrix} -9 & -6 & 12 \\ 3 & 5 & -4 \\ 3 & 2 & -7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 3 & 2 & 4 \\ -1 & -3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

$$= \frac{1}{-9}\begin{pmatrix} -9 & -6 & 12 \\ 3 & 5 & -4 \\ 3 & 2 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & \frac{2}{3} & -\frac{4}{3} \\ -\frac{1}{3} & -\frac{5}{9} & \frac{4}{9} \\ -\frac{1}{3} & -\frac{2}{9} & \frac{7}{9} \end{pmatrix}$$

Solve the following system of equations using the method of Gauss elimination (Question 38 to 41):

38. $\begin{cases} 2x - y + 4z = 5 \\ 2x + 3y - 4z = -7 \\ x + y + z = 2 \end{cases}$

Sol.

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 5 \\ 2 & 3 & -4 & -7 \\ 1 & 1 & 1 & 2 \end{array}\right)$$

$$\xrightarrow[R_1\to R_1+R_3]{R_2\to R_2+R_1}\left(\begin{array}{ccc|c} 3 & 0 & 5 & 7 \\ 4 & 2 & 0 & -2 \\ 1 & 1 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_2\to \frac{1}{2}R_2}\left(\begin{array}{ccc|c} 3 & 0 & 5 & 7 \\ 2 & 1 & 0 & -1 \\ 1 & 1 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_3\to R_3-R_2}\left(\begin{array}{ccc|c} 3 & 0 & 5 & 7 \\ 2 & 1 & 0 & -1 \\ -1 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_1\to R_1+3R_3}\left(\begin{array}{ccc|c} 0 & 0 & 8 & 16 \\ 2 & 1 & 0 & -1 \\ -1 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_1\to \frac{1}{8}R_1}\left(\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & -1 \\ -1 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_3\to R_3-R_1}\left(\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3\to -R3]{R_2\to R_2+2R_3}\left(\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{array}\right)$$

$$\xrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\therefore x=-1,\ y=1,\ z=2$$

39. $\begin{cases} x-2y-3z=-4 \\ 3x+y-4z=-5 \\ 2x+4y-z=-5 \end{cases}$

Sol.

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & -4 \\ 3 & 1 & -4 & -5 \\ 2 & 4 & -1 & -5 \end{array}\right)$$

$$\xrightarrow[R_2\to R_2-3R_1]{R_3\to R_3-2R_1}\left(\begin{array}{ccc|c} 1 & -2 & -3 & -4 \\ 0 & 7 & 5 & 7 \\ 0 & 8 & 5 & 3 \end{array}\right)$$

$$\xrightarrow{R_3\to R_3-R_1}=\left(\begin{array}{ccc|c} 1 & -2 & -3 & -4 \\ 0 & 7 & 5 & 7 \\ 0 & 1 & 0 & -4 \end{array}\right)$$

$$\xrightarrow[R_2\to R_2-7R_3]{R_1\to R_1+2R_3}=\left(\begin{array}{ccc|c} 1 & 0 & -3 & -12 \\ 0 & 0 & 5 & 35 \\ 0 & 1 & 0 & -4 \end{array}\right)$$

$$\xrightarrow[R_2\leftrightarrow R_3]{R_2\to \frac{1}{5}R_2}=\left(\begin{array}{ccc|c} 1 & 0 & -3 & -12 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 7 \end{array}\right)$$

$$\xrightarrow{R_1\to R_1+3R_2}=\left(\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 7 \end{array}\right)$$

$$\therefore x=9,\ y=-4,\ z=7$$

40. $\begin{cases} x-2y-z=3 \\ 4x-y+2z=1 \\ x+3y=5 \end{cases}$

Sol.

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ 4 & -1 & 2 & 1 \\ 1 & 3 & 0 & 5 \end{array}\right)$$

$$\xrightarrow[R_2\to R_2-4R_3]{R_1\to R_1-R_3}\left(\begin{array}{ccc|c} 0 & -5 & -1 & -2 \\ 0 & -13 & 2 & -19 \\ 1 & 3 & 0 & 5 \end{array}\right)$$

$$\xrightarrow{R_2\to R_2+2R_1}\left(\begin{array}{ccc|c} 0 & -5 & -1 & -2 \\ 0 & -23 & 0 & -23 \\ 1 & 3 & 0 & 5 \end{array}\right)$$

$$\xrightarrow[R_1\leftrightarrow R_3]{R_2\to -\frac{1}{23}R_2}\left(\begin{array}{ccc|c} 1 & 3 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & -5 & -1 & -2 \end{array}\right)$$

$$\xrightarrow[R_3\to R_3+5R_2]{R_1\to R_1-3R_2}\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

$$\therefore x=2,\ y=1,\ z=-3$$

41. $\begin{cases} 2x-y-z=0 \\ 4x-3y+2z=1 \\ 3x-2y-4z=-1 \end{cases}$

Sol.

$$
\begin{aligned}
&\left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 4 & -3 & 2 & 1 \\ 3 & -2 & -4 & -1 \end{array}\right) \\
\xrightarrow{R_2 \to R_2 - 2R_1} &\left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & -1 & 4 & 1 \\ 3 & -2 & -4 & -1 \end{array}\right) \\
\xrightarrow[R_1 \to R_1 - R_2]{R_3 \to R_3 + R_2} &\left(\begin{array}{ccc|c} 2 & 0 & -5 & -1 \\ 0 & -1 & 4 & 1 \\ 3 & -3 & 0 & 0 \end{array}\right) \\
\xrightarrow{R_3 \to \frac{1}{3} R_3} &\left(\begin{array}{ccc|c} 2 & 0 & -5 & -1 \\ 0 & -1 & 4 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right) \\
\xrightarrow{R_2 \to R_2 - R_3} &\left(\begin{array}{ccc|c} 2 & 0 & -5 & -1 \\ -1 & 0 & 4 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right) \\
\xrightarrow{R_1 \to R_1 + 2R_2} &\left(\begin{array}{ccc|c} 0 & 0 & 3 & 1 \\ -1 & 0 & 4 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right) \\
\xrightarrow{R_1 \to \frac{1}{3} R_1} &\left(\begin{array}{ccc|c} 0 & 0 & 1 & \frac{1}{3} \\ -1 & 0 & 4 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right) \\
\xrightarrow[R_3 \leftrightarrow R_2]{R_2 \to R_2 - 4R_1} &\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right) \\
\xrightarrow{R_2 \to -R_2} &\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right) \\
\xrightarrow{R_1 \to R_1 - R_2} &\left(\begin{array}{ccc|c} 0 & -1 & 0 & -\frac{1}{3} \\ 1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right) \\
\xrightarrow{R_1 \to -R_1} &\left(\begin{array}{ccc|c} 0 & 1 & 0 & \frac{1}{3} \\ 1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right) \\
\xrightarrow{R_1 \leftrightarrow R_2} &\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{array}\right)
\end{aligned}
$$

$$\therefore x = \frac{1}{3},\ y = \frac{1}{3},\ z = \frac{1}{3}$$

Solve the following system of equations using the Cramer's rule (Question 42 to 45):

42. $\begin{cases} x - 3y - 2z = 1 \\ 7x + 4y - 5z = 0 \\ 3x + 9y + z = -1 \end{cases}$

Sol.

$$\Delta = \begin{vmatrix} 1 & -3 & -2 \\ 7 & 4 & -5 \\ 3 & 9 & 1 \end{vmatrix} = 13$$

$$\Delta_x = \begin{vmatrix} 1 & -3 & -2 \\ 0 & 4 & -5 \\ -1 & 9 & 1 \end{vmatrix} = 26$$

$$\Delta_y = \begin{vmatrix} 1 & 1 & -2 \\ 7 & 0 & -5 \\ 3 & -1 & 1 \end{vmatrix} = -13$$

$$\Delta_z = \begin{vmatrix} 1 & -3 & 1 \\ 7 & 4 & 0 \\ 3 & 9 & -1 \end{vmatrix} = 26$$

$$\therefore x = \frac{26}{13} = 2,\ y = \frac{-13}{13} = -1,\ z = \frac{26}{13} = 2$$

43. $\begin{cases} x - 2y + 3z = 6 \\ 2x + 3y - 4z = 20 \\ 3x - 2y - 5z = 6 \end{cases}$

Sol.

$$\Delta = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -4 \\ 3 & -2 & -5 \end{vmatrix} = -58$$

$$\Delta_x = \begin{vmatrix} 6 & -2 & 3 \\ 20 & 3 & -4 \\ 6 & -2 & -5 \end{vmatrix} = -464$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 20 & -4 \\ 3 & 6 & -5 \end{vmatrix} = -232$$

$$\Delta_z = \begin{vmatrix} 1 & -2 & 6 \\ 2 & 3 & 20 \\ 3 & -2 & 6 \end{vmatrix} = -116$$

$$\therefore x = \frac{-464}{-58} = 8,\ y = \frac{-232}{-58} = 4,\ z = \frac{-116}{-58} = 2$$

44. $\begin{cases} 2x - 2y - 4z + 3 = 0 \\ 2x + 3y + 4z - 2 = 0 \\ 7x + 3y - 2z - 2 = 0 \end{cases}$

Sol.

$$\begin{cases} 2x - 2y - 4z = -3 \\ 2x + 3y + 4z = 2 \\ 7x + 3y - 2z = 2 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & -2 & -4 \\ 2 & 3 & 4 \\ 7 & 3 & -2 \end{vmatrix} = -40$$

$$\Delta_x = \begin{vmatrix} -3 & -2 & -4 \\ 2 & 3 & 4 \\ 2 & 3 & -2 \end{vmatrix} = 30$$

$$\Delta_y = \begin{vmatrix} 2 & -3 & -4 \\ 2 & 2 & 4 \\ 7 & 2 & -2 \end{vmatrix} = -80$$

$$\Delta_z = \begin{vmatrix} 2 & -2 & -3 \\ 2 & 3 & 2 \\ 7 & 3 & 2 \end{vmatrix} = 25$$

$$\therefore x = \frac{30}{-40} = -\frac{3}{4},\ y = \frac{-80}{-40} = 2,\ z = \frac{25}{-40} = -\frac{5}{8}$$

45. $\begin{cases} \frac{2}{x} - \frac{5}{y} + \frac{4}{z} = -3 \\ \frac{4}{x} + \frac{1}{y} - \frac{2}{z} = 7 \\ \frac{7}{x} - \frac{3}{z} = 4 \end{cases}$

Sol.

Let $a = \dfrac{1}{x},\ b = \dfrac{1}{y},\ c = \dfrac{1}{z}$

$$\begin{cases} 2a - 5b + 4c = -3 \\ 4a + b - 2c = 7 \\ 7a - 3c = 4 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & -5 & 4 \\ 4 & 1 & -2 \\ 7 & 0 & -3 \end{vmatrix} = -24$$

$$\Delta_a = \begin{vmatrix} -3 & -5 & 4 \\ 7 & 1 & -2 \\ 4 & 0 & -3 \end{vmatrix} = -72$$

$$\Delta_b = \begin{vmatrix} 2 & -3 & 4 \\ 4 & 7 & -2 \\ 7 & 4 & -3 \end{vmatrix} = -152$$

$$\Delta_c = \begin{vmatrix} 2 & -5 & -3 \\ 4 & 1 & 7 \\ 7 & 0 & 4 \end{vmatrix} = -136$$

$$\therefore a = \frac{-72}{-24} = 3,\ b = \frac{-152}{-24} = \frac{19}{3},\ c = \frac{-136}{-24} = \frac{17}{3}$$

$$\therefore x = \frac{1}{3},\ y = \frac{3}{19},\ z = \frac{3}{17}$$

Chapter 15

Inequalities and Linear Programming

15.1 Inequalities and its Identities

Inequalities

An inequality is a relation which makes a non-equal comparison between two numbers or other mathematical expressions. For example:

$$11 > 10$$
$$x^2 + 5 < 6x$$

- $a < b$ means a is lesser than b
- $a > b$ means a is greater than b
- $a \leq b$ means a is lesser than or equal to b
- $a \geq b$ means a is greater than or equal to b

For any real number a and b, the following are true:

1. If $a - b > 0$, then $a > b$
2. If $a - b < 0$, then $a < b$

That means, if we want to compare between two numbers, we just have to calculate their difference.

15.1.1 Practice 1

Compare the following algebraic expressions:

1. $(x + 3)(x - 1)$ and $(x + 4)(x - 2)$

   Sol.

$$\begin{aligned} & (x+3)(x-1) - (x+4)(x-2) \\ & = x^2 + 2x - 3 - (x^2 + 2x - 8) \\ & = x^2 + 2x - 3 - x^2 - 2x + 8 \\ & = 5 > 0 \\ \therefore\ & (x+3)(x-1) > (x+4)(x-2) \end{aligned}$$

2. $(x + 8)(x + 10)$ and $(x + 9)^2$

   Sol.

$$\begin{aligned} & (x+8)(x+10) - (x+9)^2 \\ & = x^2 + 18x + 80 - (x^2 + 18x + 81) \\ & = x^2 + 18x + 80 - x^2 - 18x - 81 \\ & = -1 < 0 \\ \therefore\ & (x+8)(x+10) < (x+9)^2 \end{aligned}$$

3. $x^2 + 6x$ and $4x - 2$

   Sol.

$$\begin{aligned} & x^2 + 6x - 4x + 2 \\ & = x^2 + 2x + 2 \\ & = (x+1)^2 - 1 + 2 \\ & = (x+1)^2 + 1 \\ \because\ & (x+1)^2 > 0 \\ \therefore\ & (x+1)^2 + 1 > 0 \\ \therefore\ & x^2 + 6x > 4x - 2 \end{aligned}$$

Identities of Inequalities

Theorem 1. *If $a > b$, $b > c$, then $a > c$*

Theorem 2. *If $a > b$ then $a + c > b + c$*

Theorem 3. *If $a > b$, $c > d$, then $a + c > b + d$*

Theorem 4. *If $a > b$, then:*

1. *When $c > 0$, $ac > bc$*
2. *When $c = 0$, $ac = bc$*
3. *When $c < 0$, $ac < bc$*

15.1.2 Practice 2

Given that $y < x < 0$, use inequality signs to complete the following statements:

1. $x + 1$ and $y + 1$

   Sol.

$$\begin{aligned} \because\ & y < x \\ \therefore\ & y + 1 < x + 1 \end{aligned}$$

2. $2y$ and $2x$

   Sol.

$$\begin{aligned} \because\ & y < x,\ 2 > 0 \\ \therefore\ & 2y < 2x \end{aligned}$$

3. $-x + 1$ and $-y + 2$

Sol.

$$\begin{aligned}
&\because y < x \\
&\therefore -x < -y \\
&\because 1 < 2,\ -x < -y \\
&\therefore -x + 1 < -y + 2
\end{aligned}$$

4. $3x$ and $4y$

Sol.

$$\begin{aligned}
&\because y < x \\
&\therefore 3y < 3x && \cdots (1) \\
&and,\ y < 0 && \cdots (2) \\
&(1) + (2) : 3y + y < 3x \\
&\qquad 4y < 3x
\end{aligned}$$

15.1.3 Exercise 15.1

Compare the following algebraic expressions (Question 1 to 5):

1. $(x-4)^2$ and $(x-6)(x-2)$

Sol.

$$\begin{aligned}
&(x-4)^2 - (x-6)(x-2) \\
&= x^2 - 8x + 16 - (x^2 - 8x + 12) \\
&= x^2 - 8x + 16 - x^2 + 8x - 12 \\
&= 4 > 0 \\
&\therefore (x-4)^2 > (x-6)(x-2)
\end{aligned}$$

2. $x^2 + 13$ and $4x$

Sol.

$$\begin{aligned}
&x^2 + 13 - 4x \\
&= x^2 - 4x + 13 \\
&= (x-2)^2 - 4 + 13 \\
&= (x-2)^2 + 9 \\
&\because (x-2)^2 > 0 \\
&\therefore (x-2)^2 + 9 > 0 \\
&\therefore x^2 + 13 > 4x
\end{aligned}$$

3. $(x-1)(x^2+x+1)$ and $(x+1)(x^2-x+1)$

Sol.

$$\begin{aligned}
&(x-1)(x^2+x+1) - (x+1)(x^2-x+1) \\
&= x^3 - 1 - x^3 - 1 \\
&= -2 < 0 \\
&\therefore (x-1)(x^2+x+1) < (x+1)(x^2-x+1)
\end{aligned}$$

4. $(x^2-x+1)(x^2+x+1)$ and $x^4 + x^2 - 1$

Sol.

$$\begin{aligned}
&(x^2-x+1)(x^2+x+1) - x^4 - x^2 + 1 \\
&= x^4 + x^3 + x^2 - x^3 - x^2 - x + x^2 + x + 1 - x^4 - x^2 + 1 \\
&= 2 > 0 \\
&\therefore (x^2-x+1)(x^2+x+1) > x^4 + x^2 - 1
\end{aligned}$$

5. $(1-2x)(1+2x)$ and $(x^2-6)^2$

Sol.

$$\begin{aligned}
&(x^2-6)^2 - (1-2x)(1+2x) \\
&= x^4 - 12x^2 + 36 - 1 + 4x^2 \\
&= x^4 - 8x^2 + 35 \\
&= (x^2-4)^2 - 16 + 35 \\
&= (x^2-4)^2 + 19 \\
&\because (x^2-4)^2 > 0 \\
&\therefore (x^2-4)^2 + 19 > 0 \\
&\therefore (x^2-6)^2 > (1-2x)(1+2x)
\end{aligned}$$

6. Given that $y < x < 0$, use inequality signs to complete the following:

(a) $2x - 3$ and $2y - 5$

Sol.

$$\begin{aligned}
&\because y < x,\ 2 > 0 \\
&\therefore 2y < 2x \\
&\because -3 > -5,\ 2x > 2y \\
&\therefore 2x - 3 > 2y - 5
\end{aligned}$$

(b) x^2 and y^2

Sol.

$$\begin{aligned}
&\because y < x,\ x^2 > 0,\ y^2 > 0 \\
&\therefore y^2 < x^2
\end{aligned}$$

15.2 Linear Inequalities

Solving Linear Inequalities

The general form of a linear inequality is $ax + b \le c$, where $a \neq 0$.

15.2.1 Practice 3

Solve the following linear inequalities:

1. $2x > x + 9$

Sol.

$$\begin{aligned}
&2x > x + 9 \\
&\ x > 9
\end{aligned}$$

2. $11 - 2x \leq -7$

Sol.

$$\begin{aligned} 11 - 2x &\leq -7 \\ -2x &\leq -18 \\ 2x &\geq 18 \\ x &\geq 9 \end{aligned}$$

3. $2(x+2) \geq \frac{2}{3} + \frac{2x+3}{4}$

Sol.

$$\begin{aligned} 2(x+2) &\geq \frac{2}{3} + \frac{2x+3}{4} \\ 2x + 4 &\geq \frac{2}{3} + \frac{2x+3}{4} \\ 24x + 48 &\geq 8 + 6x + 9 \\ 24x + 48 &\geq 17 + 6x \\ 18x &\geq -31 \\ x &\geq -\frac{31}{18} \end{aligned}$$

4. $2x - \frac{x}{3} + \frac{1}{3} < 3x - \frac{1}{2} + \frac{x}{6}$

Sol.

$$\begin{aligned} 2x - \frac{x}{3} + \frac{1}{3} &< 3x - \frac{1}{2} + \frac{x}{6} \\ 12x - 2x + 2 &< 18x - 3 + x \\ 10x + 2 &< 19x - 3 \\ -9x &< -5 \\ x &> \frac{5}{9} \end{aligned}$$

5. $10 \leq x + 3 \leq 12$

Sol.

$$\begin{aligned} 10 \leq x + 3 \leq 12 \\ 7 \leq x \leq 9 \end{aligned}$$

6. $-3 < 7 - 2x < 9$

Sol.

$$\begin{aligned} -3 < 7 - 2x < 9 \\ -10 < -2x < 2 \\ -2 < 2x < 10 \\ -1 < x < 5 \end{aligned}$$

15.2.2 Exercise 15.2a

Solve the following linear inequalities:

1. $4x - 3 > x + 9$

Sol.

$$\begin{aligned} 4x - 3 &> x + 9 \\ 3x &> 12 \\ x &> 4 \end{aligned}$$

2. $-4x > 1 - x$

Sol.

$$\begin{aligned} -4x &> 1 - x \\ -3x &> 1 \\ 3x &< -1 \\ x &< -\frac{1}{3} \end{aligned}$$

3. $3x + 20 \geq 34 - 4x$

Sol.

$$\begin{aligned} 7x &\geq 14 \\ 7x &\geq 14 \end{aligned}$$

4. $5x + 8 \leq 6x - 7$

Sol.

$$\begin{aligned} 5x + 8 &\leq 6x - 7 \\ -x &\leq -15 \\ x &\geq 15 \end{aligned}$$

5. $1 \leq 6(x - 7)$

Sol.

$$\begin{aligned} 1 &\leq 6(x - 7) \\ 1 &\leq 6x - 42 \\ 6x &\geq 43 \\ x &\geq \frac{43}{6} \end{aligned}$$

6. $2(x + 7) \leq 5x + 14$

Sol.

$$\begin{aligned} 2(x + 7) &\leq 5x + 14 \\ 2x + 14 &\leq 5x + 14 \\ -3x &\leq 0 \\ x &\geq 0 \end{aligned}$$

7. $\frac{x}{2} + \frac{2-3x}{5} > -\frac{7}{2} + \frac{x+1}{5}$

Sol.

$$\begin{aligned} \frac{x}{2} + \frac{2-3x}{5} &> -\frac{7}{2} + \frac{x+1}{5} \\ 5x + 4 - 6x &> -35 + 2x + 2 \\ -x + 4 &> -33 + 2x \\ -3x &> -37 \\ x &< \frac{37}{3} \end{aligned}$$

8. $-5 < 12 - x < -1$

Sol.

$$
\begin{aligned}
-5 &< 12 - x < -1 \\
-17 &< -x < -13 \\
13 &< x < 17
\end{aligned}
$$

9. $-\frac{3}{5} < \frac{x}{2} - \frac{1}{2} < \frac{2}{5}$

Sol.

$$
\begin{aligned}
-6 &< 5x - 5 < 4 \\
-1 &< 5x < 9 \\
-\frac{1}{5} &< x < \frac{9}{5}
\end{aligned}
$$

10. $-2 < \frac{2x}{3} + \frac{1}{2} \leq 4$

Sol.

$$
\begin{aligned}
-12 &< 4x + 3 \leq 24 \\
-15 &< 4x \leq 21 \\
-\frac{15}{4} &< x \leq \frac{21}{4}
\end{aligned}
$$

Solution of the System of Linear Inequalities

The system of iniqualities formed by more than one linear inequality is called a system of linear inequalities. The solution of a system of linear inequalities is the set of all points that satisfy all the inequalities in the system, and can be represented by a numberline.

15.2.3 Practice 4

Solve the following system of linear inequalities.

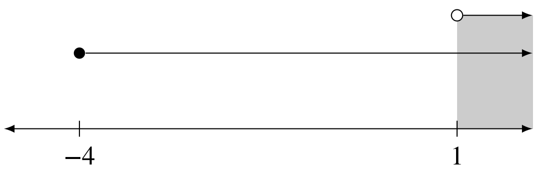
1.

$$
\begin{cases}
3x + 2 \geq 2x - 2 & (1) \\
4x - 3 > 3x - 2 & (2)
\end{cases}
$$

Sol.

$$
\begin{aligned}
(1) &: x \geq 4 \\
(2) &: x > 1
\end{aligned}
$$

$$\therefore x > 1$$



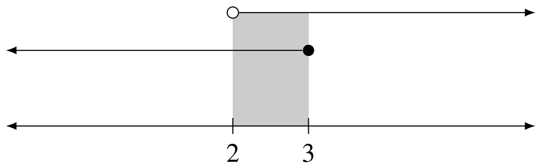
2.

$$
\begin{cases}
5x - 4 \leq 2x + 5 & (1) \\
7 - x < 3 + x & (2)
\end{cases}
$$

Sol.

$$
\begin{aligned}
(1) &: 3x \leq 9 \\
&\quad x \leq 3 \\
(2) &: -2x < -4 \\
&\quad x > 2
\end{aligned}
$$

$$\therefore 2 < x \leq 3$$



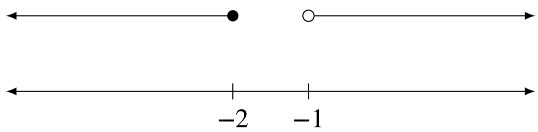
3.

$$
\begin{cases}
2 - x < 4 + x & (1) \\
1 - 2x \geq 3x + 11 & (2)
\end{cases}
$$

Sol.

$$
\begin{aligned}
(1) &: -2x < 2 \\
&\quad x > -1 \\
(2) &: -5x \geq 10 \\
&\quad x \leq -2
\end{aligned}
$$

$\therefore$ No solution



4. $2 - x < 2x - 7 \leq x - 9$

Sol.

$$
\begin{cases}
2 - x < 2x - 7 & (1) \\
2x - 7 \leq x - 9 & (2)
\end{cases}
$$

$$
\begin{aligned}
(1) &: -3x < -9 \\
&\quad x \geq 3 \\
(2) &: x \leq -2
\end{aligned}
$$

$\therefore$ No solution

15.2.4 Exercise 15.2b

Solve the following system of linear inequalities.

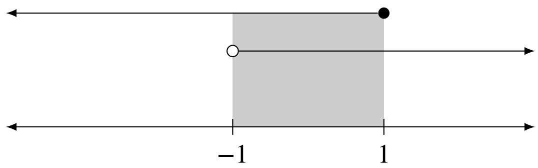
1.

$$\begin{cases} 5 - x < 6 & (1) \\ 7 - 3x \geq 4 & (2) \end{cases}$$

Sol.

$$\begin{aligned} (1):\ -x &< 1 \\ x &> -1 \\ (2):\ -3x &\geq -3 \\ x &\leq 1 \end{aligned}$$

$$\therefore -1 < x \leq 1$$



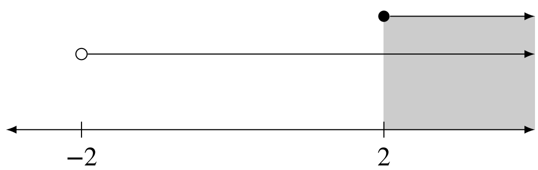
2.

$$\begin{cases} x + 2 > 0 & (1) \\ 2x + 1 \leq 4x - 3 & (2) \end{cases}$$

Sol.

$$\begin{aligned} (1):\ x &> -2 \\ (2):\ -2x &\leq -4 \\ x &\geq 2 \end{aligned}$$

$$\therefore x \geq 2$$



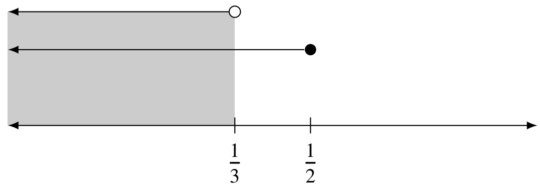
3.

$$\begin{cases} 3x - 1 < 0 & (1) \\ 1 - 2x \geq 0 & (2) \end{cases}$$

Sol.

$$\begin{aligned} (1):\ 3x &< 1 \\ x &< \frac{1}{3} \\ (2):\ -2x &\geq -1 \\ x &\leq \frac{1}{2} \end{aligned}$$

$$\therefore x < \frac{1}{3}$$



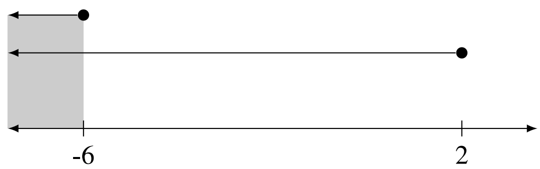
4.

$$\begin{cases} 4x - 6 \geq 5x & (1) \\ 3x + 5 \leq x + 9 & (2) \end{cases}$$

Sol.

$$\begin{aligned} (1):\ -x &\geq 6 \\ x &\leq -6 \\ (2):\ 2x &\leq 4 \\ x &\geq 2 \end{aligned}$$

$$\therefore x \leq -6$$



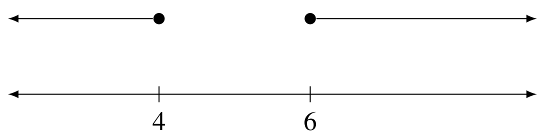
5.

$$\begin{cases} 2(x + 2) > 3x & (1) \\ 6x - 8 > 4(x + 1) & (2) \end{cases}$$

Sol.

$$\begin{aligned} (1):\ 2x + 4 &> 3x \\ -x &> -4 \\ x &< 4 \\ (2):\ 6x - 8 &> 4x + 4 \\ 2x &> 12 \\ x &> 6 \end{aligned}$$

$\therefore$ No solution



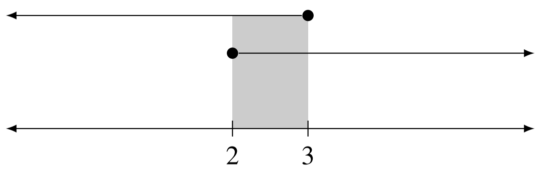
6.

$$\begin{cases} 4x + 4 \leq 3x + 7 & (1) \\ \dfrac{5x}{2} - 1 \leq 3x - 2 & (2) \end{cases}$$

Sol.

$$
\begin{aligned}
(1) &: x \le 3 \\
(2) &: 5x - 2 \le 6x - 4 \\
&\quad -x \le -2 \\
&\quad x \ge 2
\end{aligned}
$$

$$\therefore 2 \le x \le 3$$



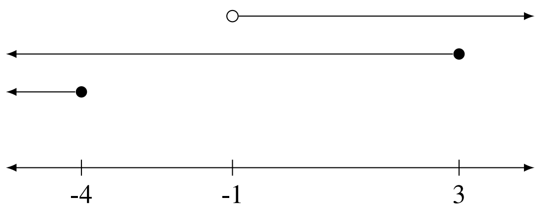
7.

$$
\begin{cases}
3x + 4 > 1 & (1) \\
3x - 1 \le 2x + 2 & (2) \\
1 - 2x > 5 - x & (3)
\end{cases}
$$

Sol.

$$
\begin{aligned}
(1) &: 3x > -3 \\
&\quad x > -1 \\
(2) &: x \le 3 \\
(3) &: -x > 4 \\
&\quad x < -4
\end{aligned}
$$

$\therefore$ No solution



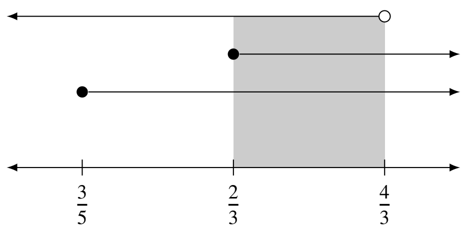
8.

$$
\begin{cases}
2x - \dfrac{1}{3} < 3 - \dfrac{x}{2} & (1) \\
2(1 - x) \le \dfrac{4x}{3} & (2) \\
4(3x - 1) > 1 + \dfrac{9x}{2} & (3)
\end{cases}
$$

Sol.

$$
\begin{aligned}
(1) &: 12 - 2 < 18 - 3x \\
&\quad 15x < 20 \\
&\quad x < \frac{4}{3} \\
(2) &: 2 - 2x \le \frac{4x}{3} \\
&\quad 6 - 6x \le 4x \\
&\quad -10x \le -6 \\
&\quad x \ge \frac{3}{5} \\
(3) &: 12x - 4 > 1 + \frac{9x}{2} \\
&\quad 24x - 8 > 2 + 9x \\
&\quad 15x > 10 \\
&\quad x > \frac{2}{3}
\end{aligned}
$$

$$\therefore \frac{2}{3} < x < \frac{4}{3}$$



9. $-4 + x \le 6 - x \le 10$

Sol.

$$
\begin{cases}
-4 + x \le 6 - x & (1) \\
6 - x \le 10 & (2)
\end{cases}
$$

$$
\begin{aligned}
(1) &: 2x \le 10 \\
&\quad x \le 5 \\
(2) &: -x \le 4 \\
&\quad x \ge 4
\end{aligned}
$$

$$\therefore 4 \le x \le 5$$

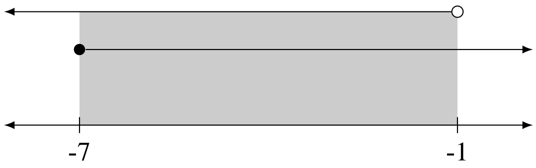
10. $x - 2 \le 2x + 5 < 3$

Sol.

$$
\begin{cases}
x - 2 \le 2x + 5 & (1) \\
2x + 5 < 3 & (2)
\end{cases}
$$

$$
\begin{aligned}
(1) &: -x \le 7 \\
&\quad x \ge -7 \\
(2) &: 2x < -2 \\
&\quad x < -1
\end{aligned}
$$

$$\therefore -7 \le x < -1$$



15.3 Quadratic Inequalities

Solution of Quadratic Inequalities

The inequalities containing only one variable, and the highest exponent of the variable is 2, are called quadratic inequalities.

We can solve the quadratic inequalities by first arraging the terms in the form of $ax^2+bx+c > 0$ or $ax^2+bx+c < 0$, where $a > 0$, then solve the quadratic equation $ax^2+bx+c = 0$, and finally, compare the solutions with the inequality sign.

Note that for all real numbers, thier square is always positive.

15.3.1 Practice 5

Solve the following inequalities:

1. $x^2 + 3x \leq 54$

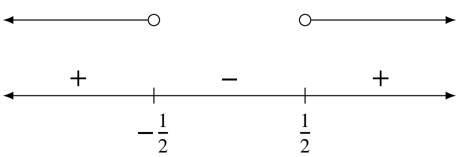
   Sol.

$$\begin{aligned} x^2 + 3x &\leq 54 \\ x^2 + 3x - 54 &\leq 0 \\ (x-6)(x+9) &\leq 0 \\ x \leq -9 \text{ or } x &\geq 6 \end{aligned}$$

2. $4x^2 > 1$

   Sol.

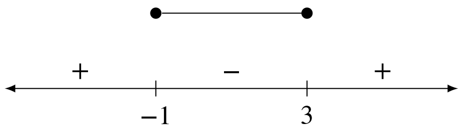
$$\begin{aligned} 4x^2 &> 1 \\ 4x^2 - 1 &> 0 \\ (2x-1)(2x+1) &> 0 \\ x < -\frac{1}{2} \text{ or } x &> \frac{1}{2} \end{aligned}$$



3. $3 + 2x - x^2 \geq 0$

   Sol.

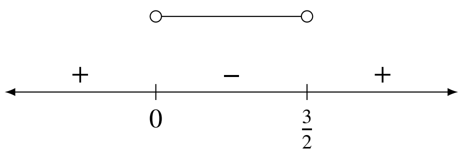
$$\begin{aligned} 3 + 2x - x^2 &\geq 0 \\ -x^2 + 2x + 3 &\geq 0 \\ x^2 - 2x - 3 &\leq 0 \\ (x-3)(x+1) &\leq 0 \\ -1 \leq x &\leq 3 \end{aligned}$$



4. $2x^2 < 3x$

   Sol.

$$\begin{aligned} 2x^2 - 3x &< 0 \\ x(2x-3) &< 0 \\ 0 < x &< \frac{3}{2} \end{aligned}$$



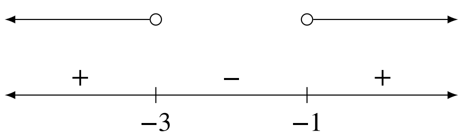
15.3.2 Exercise 15.3a

Solve the following inequalities:

1. $x^4 + 4x + 3 > 0$

   Sol.

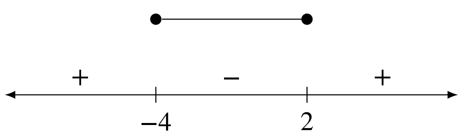
$$\begin{aligned} x^2 + 4x + 3 &> 0 \\ (x+3)(x+1) &> 0 \\ x < -3 \text{ or } x &> -1 \end{aligned}$$



2. $x^2 + 2x - 8 \leq 0$

   Sol.

$$\begin{aligned} x^2 + 2x - 8 &\leq 0 \\ (x+4)(x-2) &\leq 0 \\ -4 \leq x &\leq 2 \end{aligned}$$



3. $4x + 12 > x^2$

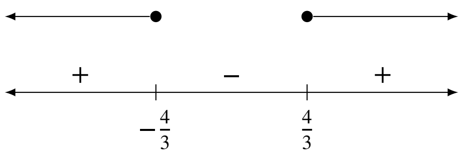
Sol.

$$\begin{aligned} 4x + 12 &> x^2 \\ x^2 - 4x - 12 &< 0 \\ (x-6)(x+2) &< 0 \\ -2 < x &< 6 \end{aligned}$$

4. $9x^2 \geq 16$

Sol.

$$\begin{aligned} 9x^2 - 16 &\geq 0 \\ (3x+4)(3x-4) &\geq 0 \\ x \leq -\frac{4}{3} &\text{ or } x \geq \frac{4}{3} \end{aligned}$$



5. $(x+2)(x-3) \leq 6$

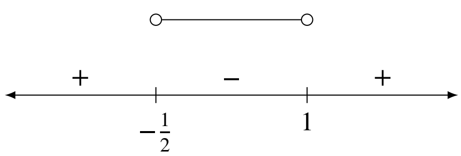
Sol.

$$\begin{aligned} (x+2)(x-3) &\leq 6 \\ x^2 - x - 6 &\leq 6 \\ x^2 - x - 12 &\leq 0 \\ (x-4)(x+3) &\leq 0 \\ -3 \leq x &\leq 4 \end{aligned}$$

6. $x(x+2) < x(3-x) + 1$

Sol.

$$\begin{aligned} x(x+2) &< x(3-x) + 1 \\ x^2 + 2x &< 3x - x^2 + 1 \\ 2x^2 - x - 1 &< 0 \\ (x-1)(2x+1) &< 0 \\ -\frac{1}{2} < x &< 1 \end{aligned}$$



7. $16x^2 - 3x + 1 \geq 5x$

Sol.

$$\begin{aligned} 16x^2 - 3x + 1 &\geq 5x \\ 16x^2 - 8x + 1 &\geq 0 \\ (4x-1)^2 &\geq 0 \\ x &\in \mathbb{R} \end{aligned}$$

8. $(x-4)^2 + (x-6)^2 \leq 2$

Sol.

$$\begin{aligned} (x-4)^2 + (x-6)^2 &\leq 2 \\ x^2 - 8x + 16 + x^2 - 12x + 36 &\leq 2 \\ 2x^2 - 20x + 52 &\leq 2 \\ x^2 - 10x + 26 &\leq 1 \\ x^2 - 10x + 25 &\leq 0 \\ (x-5)^2 &\leq 0 \\ x &= 5 \end{aligned}$$

9. $1 < 4x(1-x)$

Sol.

$$\begin{aligned} 1 &< 4x(1-x) \\ 4x - 4x^2 - 1 &> 0 \\ 4x^2 - 4x + 1 &< 0 \\ (2x-1)^2 &< 0 \\ &\text{No solution} \end{aligned}$$

10. $x^2 - 3x + 9 > 3x(3-x)$

Sol.

$$\begin{aligned} x^2 - 3x + 9 &> 3x(3-x) \\ x^2 - 3x + 9 &> 9x - 3x^2 \\ 4x^2 - 12x + 9 &> 0 \\ (2x-3)^2 &> 0 \\ x \in \mathbb{R}, x &\neq \frac{3}{2} \end{aligned}$$

Solution of System of Quadratic Inequalities

To solve a system of quadratic inequalities, we need to solve each inequality separately and then find the intersection of the solutions.

15.3.3 Practice 6

Solve the following system of inequalities:

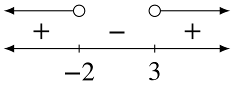
1.

$$\begin{cases} x + 1 < 0 & (1) \\ x^2 - x - 6 > 0 & (2) \end{cases}$$

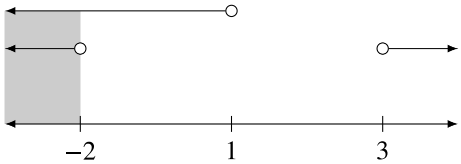
Sol.

(1) : $x < 1$
(2) : $(x-3)(x+2) > 0$

$x < -2$ or $x > 3$



$\therefore x < -2$



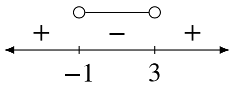
2.

$$\begin{cases} x^2 - x - 3 < 0 & (1) \\ x^2 + 3x - 4 \leq 0 & (2) \end{cases}$$

Sol.

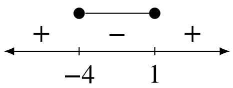
(1) : $(x+1)(x-3) < 0$

$-1 < x < 3$

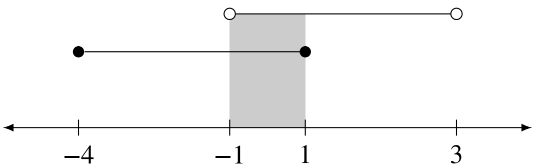


(2) : $(x+4)(x-1) \leq 0$

$-4 \leq x \leq 1$



$\therefore -1 < x \leq 1$



15.3.4 Exercise 15.3b

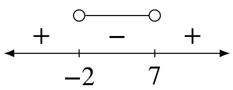
3.

$$\begin{cases} 3x - 4 \geq x - 6 & (1) \\ x^2 - 3x < 2x + 14 & (2) \end{cases}$$

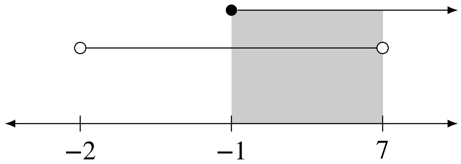
Sol.

(1) : $2x \geq -2$
$x \geq -1$
(2) : $(x-7)(x+2) < 0$

$-2 < x < 7$



$\therefore -1 \leq x < 7$



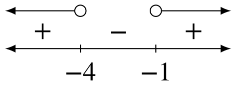
4.

$$\begin{cases} x^2 + 5x + 4 > 0 & (1) \\ x^2 + 10x + 21 \geq 0 & (2) \end{cases}$$

Sol.

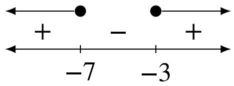
(1) : $(x+4)(x+1) > 0$

$x < -4$ or $x > -1$

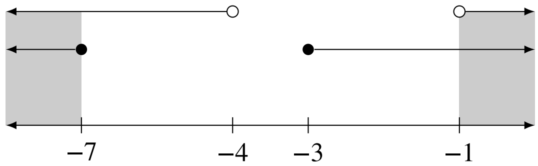


(2) : $(x+7)(x+3) \geq 0$

$x \leq -7$ or $x \geq -3$



$\therefore x \leq -7$ or $x > -1$



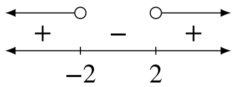
5.

$$\begin{cases} x^2 > 4 & (1) \\ 4x(x-1) \leq 15 & (2) \end{cases}$$

Sol.

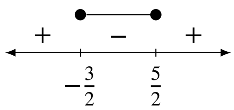
(1) : $x^2 - 4 > 0$
$(x+2)(x-2) > 0$

$x < -2$ or $x > 2$

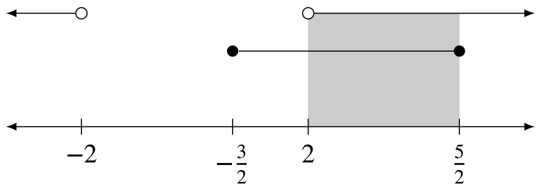


(2) : $4x^2 - 4x - 15 \leq 0$
$(2x+3)(2x-5) \leq 0$

$-\frac{3}{2} \leq x \leq \frac{5}{2}$



$\therefore 2 < x \leq \frac{5}{2}$



6.

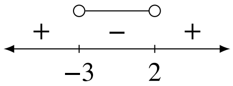
$$\begin{cases} x^2 + x < 6 & (1) \\ 4(2x+3) < (2-x)(1+x) & (2) \end{cases}$$

Sol.

(1) : $x^2 + x - 6 < 0$

$(x+3)(x-2) < 0$

$-3 < x < 2$

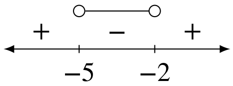


(2) : $8x + 12 < 2 + x - x^2$

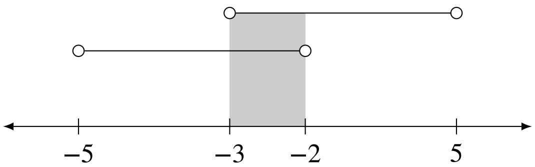
$x^2 + 7x + 10 < 0$

$(x+5)(x+2) < 0$

$-5 < x < -2$



$\therefore\ -3 < x < -2$



7.

$$\begin{cases} (x-1)(x+1) > 11 + 4x & (1) \\ x^2 + 4 \le 7x - 2 & (2) \end{cases}$$

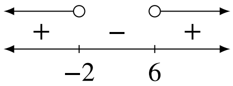
Sol.

(1) : $x^2 - 1 > 11 + 4x$

$x^2 - 4x - 12 > 0$

$(x-6)(x+2) > 0$

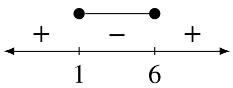
$x < -2$ or $x > 6$



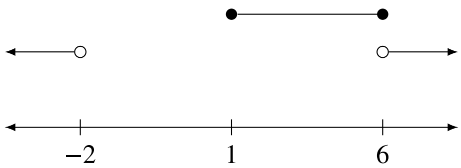
(2) : $x^2 - 7x + 6 \le 0$

$(x-6)(x-1) \le 0$

$1 \le x \le 6$



$\therefore$ No solution



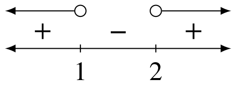
8.

$$\begin{cases} x^2 - 3x + 2 > 0 & (1) \\ x^2 + 3x \ge 0 & (2) \end{cases}$$

Sol.

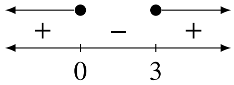
(1) : $(x-2)(x-1) > 0$

$x < 1$ or $x > 2$

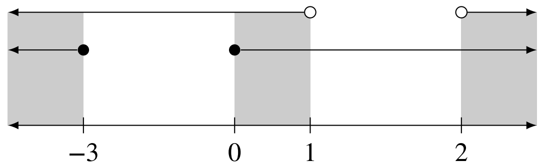


(2) : $x(x+3) \ge 0$

$x \le -3$ or $x \ge 0$



$\therefore x < -3,\ 0 \le x < 1,\ x > 2$



9.

$$\begin{cases} (x-3)(x+3) \ge 16 & (1) \\ x^2 + 3 \ge 13(x-3) & (2) \end{cases}$$

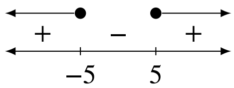
Sol.

$x^2 - 9 \ge 16$

$x^2 - 25 \ge 0$

$(x+5)(x-5) \ge 0$

$x \le -5$ or $x \ge 5$

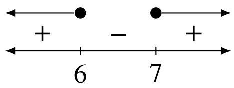


(2) : $x^2 + 3 \ge 13x - 39$

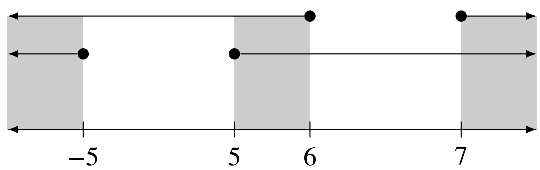
$x^2 - 13x + 42 \ge 0$

$(x-7)(x-6) \ge 0$

$x \le 6$ or $x \ge 7$



$\therefore x \le 5,\ 5 \le x \le 6,\ x \ge 7$



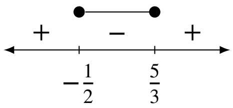
10.

$$\begin{cases} x^2 - x - 1 \le \dfrac{x-1}{6} & (1) \\ (2x-1)(x-6) \ge 13 & (2) \end{cases}$$

Sol.

(1) : $6x^2 - 6x - 6 \le x - 1$

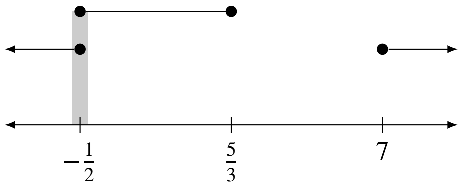
$$\begin{aligned} 6x^2 - 7x - 5 &\le 0 \\ (3x-5)(2x-1) &\le 0 \\ -\frac{1}{2} \le x &\le \frac{5}{3} \end{aligned}$$



(2) : $2x^2 - 13x + 6 \ge 13$

$$\begin{aligned} 2x^2 - 13x - 7 &\ge 0 \\ (2x+1)(x-7) &\ge 0 \\ x \le -\frac{1}{2},\ x &\ge 7 \end{aligned}$$

$\therefore x = -\frac{1}{2}$



15.4 Solution of Linear Inequalities of Higher Degree

A linear inequality thay contains a variable raised to a power greater than 2 is called a linear inequality of higher degree. To solve this type of inequality, we move all the terms with the variable to one side of the inequality, and make the coefficient of the polynomial to be positive.

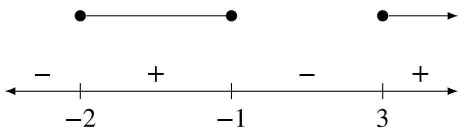
15.4.1 Practice 7

Solve the following inequalities:

1. $x^3 - 7x - 6 \ge 0$

   Sol.

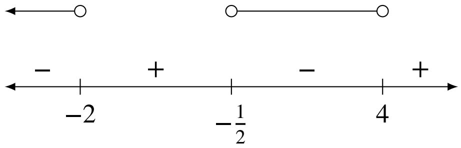
$$\begin{aligned} x^3 - 7x - 6 &\ge 0 \\ (x+2)(x^2 - 2x - 3) &\ge 0 \\ (x+2)(x-3)(x+1) &\ge 0 \\ -2 \le x \le -1 \text{ or } x &\ge 3 \end{aligned}$$



2. $3x^2 + 18x + 8 > 2x^3$

   Sol.

$$\begin{aligned} 3x^2 + 18x + 8 - 2x^3 &> 0 \\ 2x^3 - 3x^2 - 18x - 8 &< 0 \\ (x+2)(2x^2 - 7x - 4) &< 0 \\ (x+2)(2x+1)(x-4) &< 0 \\ x < -2 \text{ or } -\frac{1}{2} < x &< 4 \end{aligned}$$



3. $x^4 + x^3 \le 3x^2 + x - 2$

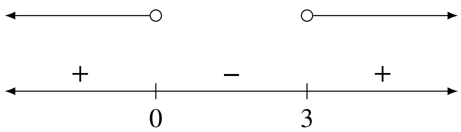
   Sol.

$$\begin{aligned} x^4 + x^3 - 3x^2 - x + 2 &\le 0 \\ (x+2)[x^2(x-1) - (x-1)] &\le 0 \\ (x+2)(x+1)(x-1)^2 &\le 0 \\ \text{When } x \ne 1,\ (x+1)^2 &> 0 \\ (x+2)(x+1) &\le 0 \\ -2 \le x &\le -1 \\ \text{When } x = 1,\ (x+1)^2 &= 0 \\ \therefore x \text{ is the solution} & \\ \therefore\ -2 \le x \le -1 & \end{aligned}$$

4. $x^4 - x^3 - 5x^2 - 3x > 0$

   Sol.

$$\begin{aligned} x^4 - x^3 - 5x^2 - 3x &> 0 \\ x(x+1)(x^2 - 2x - 3) &> 0 \\ x(x-3)(x+1)^2 &> 0 \\ \text{When } x \ne -1,\ (x+1)^2 &> 0 \\ x(x-3) &> 0 \\ x < 0 \text{ or } x &> 3 \\ \text{When } x = -1,\ (x+1)^2 &= 0 \\ \therefore x \text{ is not the solution} & \\ \therefore x < 0 \text{ or } x > 3, x \ne -1 & \end{aligned}$$



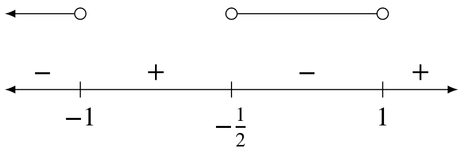
15.4.2 Exercise 15.4

Solve the following inequalities:

1. $(x-1)(x+1)(2x+1) < 0$

Sol.

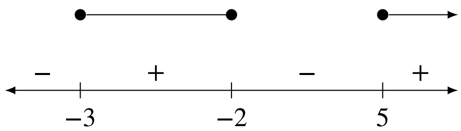
$$(x-1)(x+1)(2x+1) < 0$$
$$\therefore x < -1 \text{ or } -\frac{1}{2} < x < 1$$



2. $(3x+6)(x+3)(5-x) \le 0$

Sol.

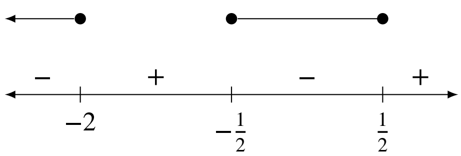
$$(3x+6)(x+3)(5-x) \le 0$$
$$-3(x+2)(x+3)(x-5) \le 0$$
$$(x+2)(x+3)(x-5) \ge 0$$
$$\therefore\ -3 \le x \le -2 \text{ or } x \ge 5$$



3. $4x^3+8x^2-x-2 \le 0$

Sol.

$$4x^2(x+2)-(x+2) \le 0$$
$$(4x^2-1)(x+2) \le 0$$
$$(2x+1)(2x-1)(x+2) \le 0$$
$$\therefore x \le -2 \text{ or } -\frac{1}{2} \le x \le \frac{1}{2}$$



4. $x^3-3x^2+3x-1 \ge 0$

Sol.

$$x^3-3x^2+3x-1 \ge 0$$
$$(x-1)^3 \ge 0$$
$$(x-1)(x-1)^2 \ge 0$$
$$\because (x-1)^2 \ge 0 \text{ for all real numbers } x$$
$$\therefore (x-1) \ge 0$$
$$x \ge 1$$

5. $x^4 > 81$

Sol.

$$x^4 > 81$$
$$x^4-81 > 0$$
$$(x^2-9)(x^2+9) > 0$$
$$(x+3)(x-3)(x^2+9) > 0$$
$$\because x^2+9 > 0 \text{ for all real numbers } x$$
$$\therefore (x+3)(x-3) > 0$$
$$x < -3 \text{ or } x > 3$$

6. $x^3(x+2)^2(x+3) > 0$

Sol.

$$x^3(x+2)^2(x+3) > 0$$
$$x^2(x+2)^2[x(x+3)] > 0$$
For all real numbers x,
$$x^2 > 0 \text{ when } x \ne 0$$
$$(x+2)^2 > 0 \text{ when } x \ne -2$$
$$x(x+3) > 0$$
$$\therefore x < -3 \text{ or } x > 0$$
When $x=-2$ or $x=0$,
$$x^3(x+2)^2(x+3) = 0$$
$\therefore x=0$ and $x=-2$ are not solution.

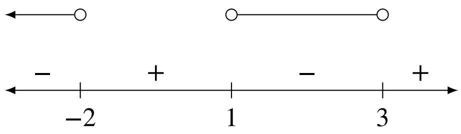
$$\therefore x < -3 \text{ or } x > 0,\ x \ne -2$$

7. $(x-3)^5(x-1)^3(x+2) < 0$

Sol.

$$(x-3)^4(x-1)^2[(x-3)(x-1)(x+2)] < 0$$
For all real numbers x,
$$(x-3)^4 > 0 \text{ when } x \ne 3$$
$$(x-1)^2 > 0 \text{ when } x \ne 1$$
$$(x-3)(x-1)(x+2) < 0$$
$$\therefore x < -2 \text{ or } 1 < x < 3$$
When $x=3$ or $x=1$,
$$(x-3)^4(x-1)^2(x+2) = 0$$
$\therefore x=1$ and $x=3$ are not solution.

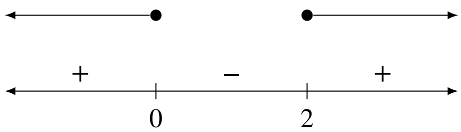
$$\therefore x < -2 \text{ or } 1 < x < 3$$



8. $x^3(x-2) \geq x(2x-1)(x-2)$

Sol.

$$\begin{aligned} x^3(x-2) - x(2x-1)(x-2) &\geq 0 \\ x(x-2)(x^2-2x+1) &\geq 0 \\ x(x-2)(x-1)^2 &\geq 0 \\ \text{When } x \neq 1,\ (x-1)^2 &> 0 \\ x(x-2) &\geq 0 \\ \therefore x \leq 0 \text{ or } x &\geq 2 \\ \text{When } x = 1,\ x(x-2)(x-1)^2 &= 0 \\ \therefore x = 1 \text{ is the solution} & \\ \therefore x \leq 0 \text{ or } x \geq 2 \text{ or } x &= 1 \end{aligned}$$



15.5 Fractional Inequalities

Inequalities that involve fractional expressions are called fractional inequalities. To solve a fractional inequality, we manipulate the inequality until the right side is zero.

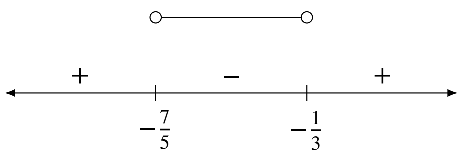
15.5.1 Practice 8

Solve the following inequalities:

1. $\frac{x-5}{3x+1} > 2$

Sol.

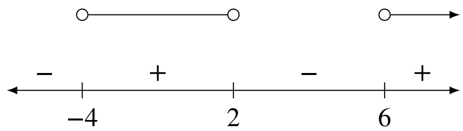
$$\begin{aligned} \frac{x-5}{3x+1} &> 2 \\ \frac{x-5}{3x+1} - 2 &> 0 \\ \frac{x-5-2(3x+1)}{3x+1} &> 0 \\ \frac{x-5-6x-2}{3x+1} &> 0 \\ \frac{-5x-7}{3x+1} &> 0 \\ -\frac{5x+7}{3x+1} &> 0 \\ \frac{5x+7}{3x+1} &< 0 \\ -\frac{7}{5} < x &< -\frac{1}{3} \end{aligned}$$



2. $\frac{x+22}{x-2} < x+1$

Sol.

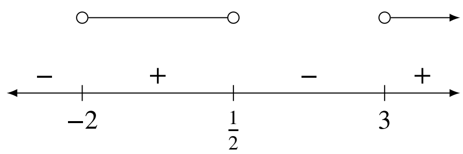
$$\begin{aligned} \frac{x+22}{x-2} &< x+1 \\ \frac{x+22-(x-2)(x+1)}{x-2} &< 0 \\ \frac{x+22-x^2+x+2}{x-2} &< 0 \\ \frac{-x^2+2x+24}{x-2} &< 0 \\ \frac{x^2-2x-24}{x-2} &> 0 \\ \frac{(x-6)(x+4)}{x-2} &> 0 \\ -4 < x < 2 \text{ or } x &> 6 \end{aligned}$$



3. $\frac{1}{x-3} \geq \frac{1}{2x-1}$

Sol.

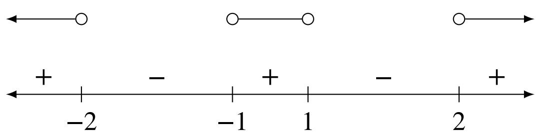
$$\begin{aligned} \frac{1}{x-3} &\geq \frac{1}{2x-1} \\ \frac{2x-1-x+3}{(x-3)(2x-1)} &\geq 0 \\ \frac{x+2}{(x-3)(2x-1)} &\geq 0 \\ \text{When } \frac{x+2}{(x-3)(2x-1)} &= 0,\ x = -2 \\ \text{When } \frac{x+2}{(x-3)(2x-1)} &> 0,\ -2 < x < \frac{1}{2} \text{ or } x > 3 \\ \therefore\ -2 \leq x < \frac{1}{2} & \text{ or } x > 3 \end{aligned}$$



4. $\frac{x^2-7}{1-x^2} \leq 1$

Sol.

$$\begin{aligned}
\frac{x^2-7}{1-x^2} &\le 1\\
\frac{x^2-7-1+x^2}{1-x^2} &\le 0\\
\frac{2x^2-8}{(1+x)(1-x)} &\le 0\\
\frac{2(x+2)(x-2)}{-(x+1)(x-1)} &\le 0\\
\frac{(x+2)(x-2)}{(x+1)(x-1)} &\ge 0\\
\text{When } \frac{(x+2)(x-2)}{(x+1)(x-1)} &= 0,\ x=-2 \text{ or } x=2\\
\text{When } \frac{(x+2)(x-2)}{(x+1)(x-1)} &> 0,\\
x<-2 \text{ or } -1<x<1 &\text{ or } x>2\\
\therefore x\le -2 \text{ or } -1\le x<1 &\text{ or } x\ge 2
\end{aligned}$$



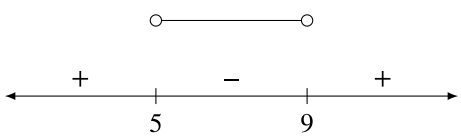
15.5.2 Exercise 15.5

Solve the following inequalities:

1. $\frac{7-x}{9-x} > \frac{1}{2}$

   Sol.

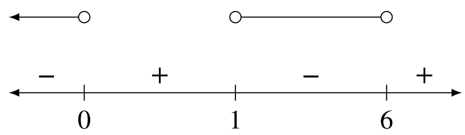
$$\begin{aligned}
\frac{7-x}{9-x} &> \frac{1}{2}\\
\frac{2(7-x)-9+x}{2(9-x)} &> 0\\
\frac{14-2x-9+x}{9-x} &> 0\\
\frac{5-x}{9-x} &> 0\\
\frac{x-5}{x-9} &> 0\\
\therefore x<5 \text{ or } x &> 9
\end{aligned}$$



2. $\frac{5-x}{2} \ge \frac{3-x}{x}$

   Sol.

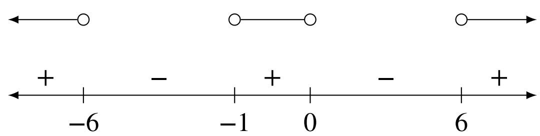
$$\begin{aligned}
\frac{5-x}{2} &\ge \frac{3-x}{x}\\
\frac{x(5-x)-2(3-x)}{2x} &\ge 0\\
\frac{5x-x^2-6+2x}{x} &\ge 0\\
\frac{-x^2+7x-6}{x} &\ge 0\\
\frac{(x-6)(x-1)}{x} &\le 0\\
\text{When } \frac{(x-6)(x-1)}{x} &= 0,\ x=6 \text{ or } x=1\\
\text{When } \frac{(x-6)(x-1)}{x} &< 0, x<0 \text{ or } 1<x<6\\
\therefore x<0 \text{ or } 1\le x\le 6
\end{aligned}$$



3. $\frac{x-4}{x+6} > \frac{1}{x}$

   Sol.

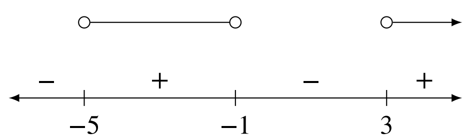
$$\begin{aligned}
\frac{x(x-4)-x-6}{x(x+6)} &> 0\\
\frac{x^2-4x-x-6}{x(x+6)} &> 0\\
\frac{x^2-5x-6}{x(x+6)} &> 0\\
\frac{(x-6)(x+1)}{x(x+6)} &> 0\\
\therefore x<-6,\ -1<x<0 \text{ or } x>6
\end{aligned}$$



4. $\frac{1}{x-3} \ge \frac{1}{2x+2}$

   Sol.

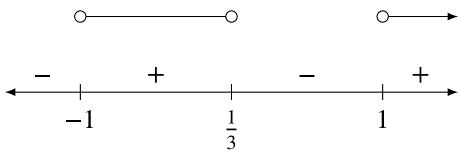
$$\begin{aligned}
\frac{1}{x-3} &\ge \frac{1}{2x+2}\\
\frac{2x+2-x+3}{2(x+1)(x-3)} &\ge 0\\
\frac{x+5}{(x+1)(x-3)} &\ge 0\\
\text{When } \frac{x+5}{(x+1)(x-3)} &= 0,\ x=-5\\
\therefore -5\le x<-1 \text{ or } x>3
\end{aligned}$$



5. $\frac{x-1}{x+1} - \frac{1}{x-1} \le 1$

Sol.

$$\frac{x^2-2x-1-x-1}{(x+1)(x-1)} \le 1$$
$$\frac{x^2-3x-2-x^2+1}{(x+1)(x-1)} \le 0$$
$$\frac{-3x+1}{(x+1)(x-1)} \le 0$$
$$\frac{3x-1}{(x+1)(x-1)} \ge 0$$
$$\text{When } \frac{3x-1}{(x+1)(x-1)} = 0,\ x = \frac{1}{3}$$
$$\text{When } \frac{3x-1}{(x+1)(x-1)} > 0,\ -1 < x < \frac{1}{3} \text{ or } x > 1$$
$$\therefore\ -1 < x \le \frac{1}{3} \text{ or } x > 1$$

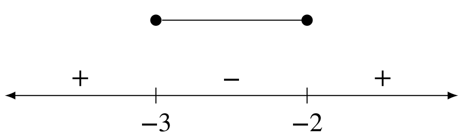


6. $1 + \frac{1}{x-2} \le \frac{x-2}{x-1}$
Sol.

$$\frac{x-2+1}{x-2} \le \frac{x-2}{x-1}$$
$$\frac{x-1}{x-2} - \frac{x-2}{x-1} \le 0$$
$$\frac{x^2-2x+1-x^2+4x-4}{(x-2)(x-1)} \le 0$$
$$\frac{2x-3}{(x-2)(x-1)} \le 0$$
$$\text{When } \frac{2x-3}{(x-2)(x-1)} \le 0,\ x = \frac{3}{2}$$
$$\text{When } \frac{2x-3}{(x-2)(x-1)} < 0,\ x < 1 \text{ or } \frac{3}{2} < x < 2$$
$$\therefore x < 1 \text{ or } \frac{3}{2} \le x < 2$$

7. $\frac{x^2+x-6}{x^2+4x+4} \le 0$
Sol.

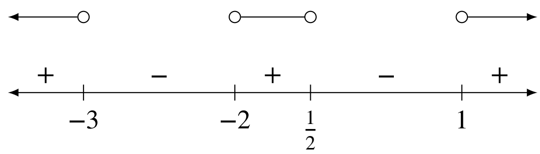
$$\frac{x^2+x-6}{x^2+4x+4} \le 0$$
$$\frac{(x+3)(x-2)}{(x+2)^2} \le 0$$
$$\because (x+2)^2 \ge 0 \text{ for all numbers } x,$$
$$(x+3)(x-2) \le 0\ (x \ge -2)$$
$$\therefore -3 \le x \le 2,\ x \ne -2$$



8. $\frac{2x^2-3x+1}{x^2+5x+6} \ge 0$

Sol.

$$\frac{2x^2-3x+1}{x^2+5x+6} \ge 0$$
$$\frac{(x-1)(2x-1)}{(x+2)(x+3)} \ge 0$$
$$\text{When } \frac{(x-1)(2x-1)}{(x+2)(x+3)} = 0,\ x = 1 \text{ or } x = \frac{1}{2}$$
$$\text{When } \frac{(x-1)(2x-1)}{(x+2)(x+3)} > 0,$$
$$x < -3 \text{ or } -2 < x < \frac{1}{2} \text{ or } x > 1$$
$$\therefore x < -3 \text{ or } -2 < x \le \frac{1}{2} \text{ or } x \ge 1$$



15.6 Intequalities containing absolute values

Given a positive real number x, its absolute value is denoted by $|x|$.

$$|x| = \begin{cases} x, & \text{for } x \ge 0 \\ -x, & \text{for } x < 0 \end{cases}$$

Given a real number a,

- When $a > 0$,

$$|x| < a \iff -a < x < a$$
$$|x| \le a \iff -a \le x \le a$$
$$|x| > a \iff x < -a \text{ or } x > a$$
$$|x| \ge a \iff x \le -a \text{ or } x \ge a$$

- When $a < 0$,

$$|x| < a \iff \text{no solution}$$
$$|x| \le a \iff \text{no solution}$$
$$|x| > a \iff \text{all real numbers}$$
$$|x| \ge a \iff \text{all real numbers}$$

- When $a = 0$, n is an integer,

$$|x-n| < 0 \iff \text{no solution}$$
$$|x-n| \le 0 \iff x = n$$
$$|x-n| > 0 \iff \text{all real numbers except } n$$
$$|x-n| \ge 0 \iff \text{all real numbers}$$

15.6.1 Practice 9

Solve the following inequalities:

1. $|x| > 5$

Sol.

$$\begin{aligned}|x| &> 5\\ x < -5 &\text{ or } x > 5\end{aligned}$$

2. $|x| < 9$

   Sol.

$$\begin{aligned}|x| &< 9\\ -9 < x &< 9\end{aligned}$$

3. $|x + 4| \geq 7$

   Sol.

$$\begin{aligned}|x + 4| &\geq 7\\ x + 4 \geq 7 &\text{ or } x + 4 \leq -7\\ x \leq -11 &\text{ or } x \geq 3\end{aligned}$$

4. $-1 \leq |2x - 3| < 3$

   Sol.

$$\begin{cases} -1 \leq |2x - 3| & (1)\\ |2x - 3| < 3 & (2)\end{cases}$$

$$\begin{aligned}(1):\ &|2x - 3| \geq -1\\ &x \text{ is any real number}\end{aligned}$$

$$\begin{aligned}(2):\ |2x - 3| &< 3\\ -3 < 2x - 3 &< 3\\ 0 < 2x &< 6\\ 0 < x &< 3\end{aligned}$$

$$\therefore 0 < x < 3$$

15.6.2 Exercise 15.6

Solve the following inequalities:

1. $|x - 5| > 3$

   Sol.

$$\begin{aligned}|x - 5| &> 3\\ x - 5 < -3 &\text{ or } x - 5 > 3\\ x < 2 &\text{ or } x > 8\end{aligned}$$

2. $2|x + 1| - 3 > 7$

   Sol.

$$\begin{aligned}2|x + 1| &> 10\\ |x + 1| &> 5\\ x + 1 < -5 &\text{ or } x + 1 > 5\\ x < -6 &\text{ or } x > 4\end{aligned}$$

3. $|2x - 5| < 7$

   Sol.

$$\begin{aligned}|2x - 5| &< 7\\ -7 < 2x - 5 &< 7\\ -2 < 2x &< 12\\ -1 < x &< 6\end{aligned}$$

4. $|5x - 3| \leq 1$

   Sol.

$$\begin{aligned}|5x - 3| &\leq 1\\ -1 \leq 5x - 3 &\leq 1\\ 2 \leq 5x &\leq 4\\ \frac{2}{5} \leq x &\leq \frac{4}{5}\end{aligned}$$

5. $|2 - 3x| \geq 8$

   Sol.

$$\begin{aligned}|2 - 3x| &\geq 8\\ 2 - 3x \leq -8 &\text{ or } 2 - 3x \geq 8\\ -3x \leq -10 &\text{ or } -3x \geq 6\\ x \leq -2 &\text{ or } x \geq \frac{10}{3}\end{aligned}$$

6. $1 < |3 - 2x| \leq 9$

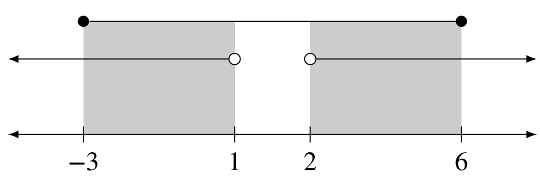
   Sol.

$$\begin{cases} |3 - 2x| > 1 & (1)\\ |3 - 2x| \leq 9 & (2)\end{cases}$$

$$\begin{aligned}(1):\ 3 - 2x < -1 &\text{ or } 3 - 2x > 1\\ -2x < -4 &\text{ or } -2x > -2\\ x > 2 &\text{ or } x < 1\end{aligned}$$

$$\begin{aligned}(2):\ -9 \leq 3 - 2x &\leq 9\\ -12 \leq -2x &\leq 6\\ -3 \leq x &\leq 6\end{aligned}$$

$$\therefore -3 \leq x < 1 \text{ or } 2 < x \leq 6$$



7. $9 \leq 2|x + 2| \leq 19$

   Sol.

$$\begin{cases} 2|x + 2| \geq 9 & (3)\\ 2|x + 2| \leq 19 & (4)\end{cases}$$

$$(1): |x+2| \geq \frac{9}{2}$$

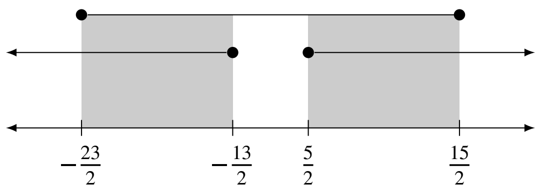
$$x+2 \leq -\frac{9}{2} \text{ or } x+2 \geq \frac{9}{2}$$

$$x \leq -\frac{13}{2} \text{ or } x \geq \frac{5}{2}$$

$$(2): |x+2| \leq \frac{19}{2}$$

$$-\frac{19}{2} \leq x+2 \leq \frac{19}{2}$$

$$-\frac{23}{2} \leq x \leq \frac{15}{2}$$



8. $\frac{2}{|x+1|} - 3 \geq 4$

Sol.

$$\frac{2}{|x+1|} \geq 7$$

$$2 \geq 7|x+1|$$

$$|x+1| \leq \frac{2}{7}$$

$$-\frac{2}{7} \leq x+1 \leq \frac{2}{7}$$

$$-\frac{9}{7} \leq x \leq -\frac{5}{7}$$

When $x = -1$, the fraction is undefined.

$$\therefore -\frac{9}{7} \leq x \leq -\frac{5}{7},\ x \neq -1$$

15.7 Linear Inequalities of Two Variables

Solution of Linear Inequalities of Two Variables

A linear inequality of two variables is inequality with two variables involved.

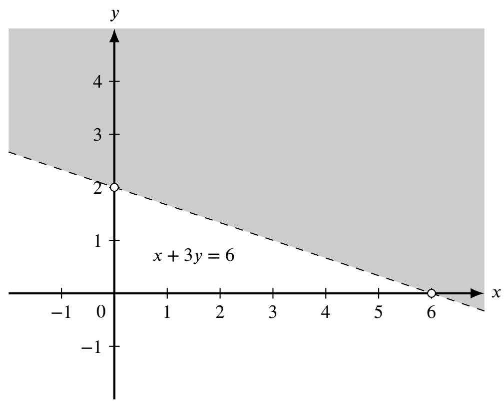
For any linear equation of two variables, there are infinitely many solutions. These solutions can be graphed in the appropriate half of a rectangular coordinate plane.

15.7.1 Practice 10

Express the solution of the following linear inequalities in graph form:

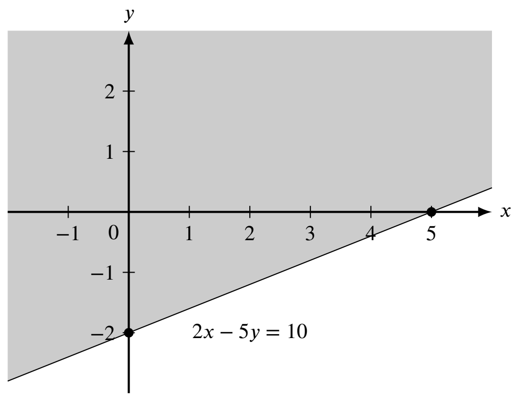
1. $x + 3y < 6$

   Sol.



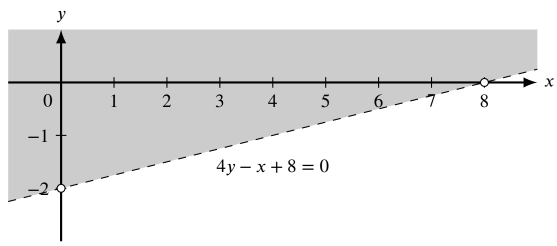
2. $2x - 5y \leq 10$

   Sol.

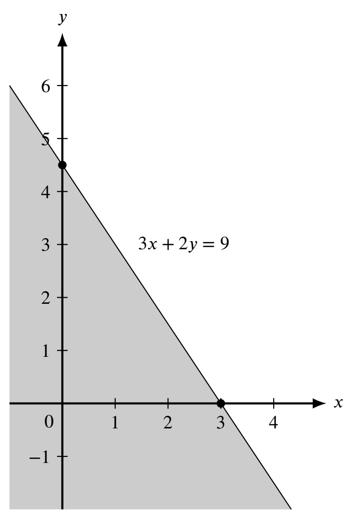


3. $4y - x + 8 > 0$

   Sol.



4. $3x + 2y \leq 9$



Solution of System of Linear Inequalities of Two Variables

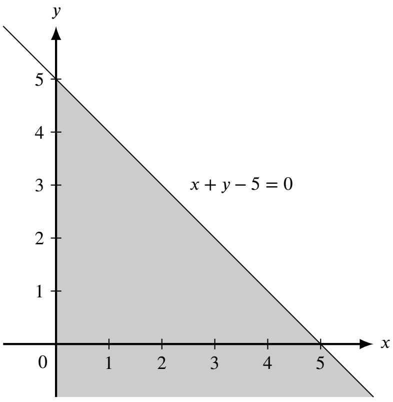
The solution of the system of linear inequalities of two variables is the intersection of the solution of individual inequalities. That is, the region bounded by the lines representing each inequalitiy.

15.7.2 Practice 11

Solve the following system of inequalities:

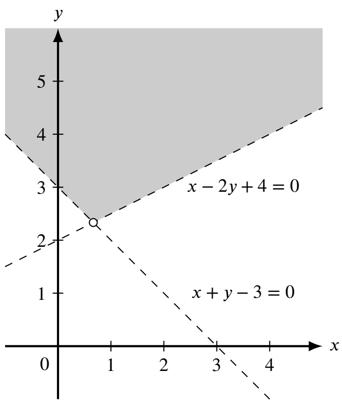
1. $\begin{cases} x \geq 0 \\ x + y - 5 \leq 0 \end{cases}$

   Sol.



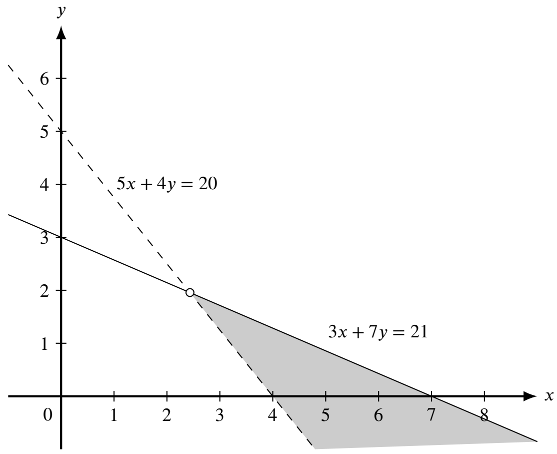
2. $\begin{cases} x + y - 3 > 0 \\ x - 2y + 4 < 0 \end{cases}$

   Sol.



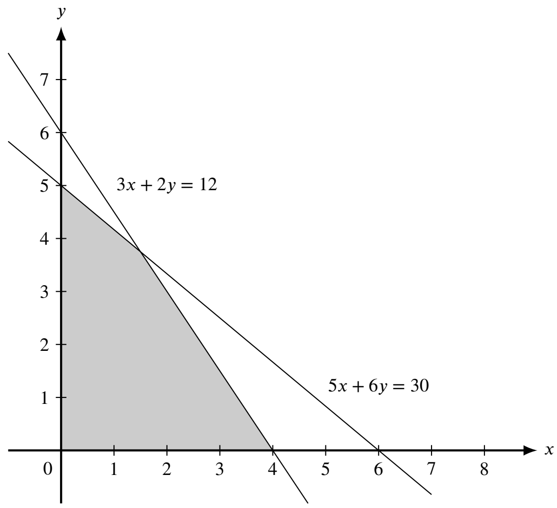
3. $\begin{cases} 3x + 7y \leq 21 \\ 5x + 4y > 20 \end{cases}$

   Sol.



4. $\begin{cases} 5x + 6y \leq 30 \\ 3x + 2y \leq 12 \\ x \geq 0 \\ y \geq 0 \end{cases}$

   Sol.

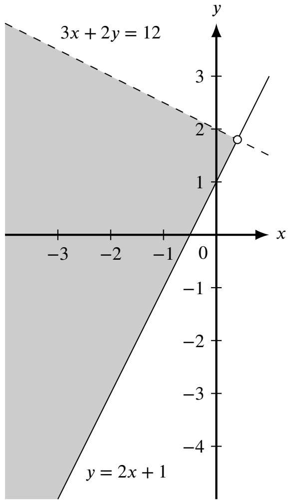


15.7.3 Exercise 15.7

Solve the following system of inequalities:

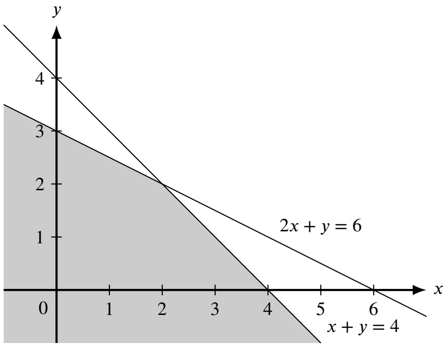
1. $\begin{cases} y \geq 2x + 1 \\ x + 2y < 4 \end{cases}$

   Sol.



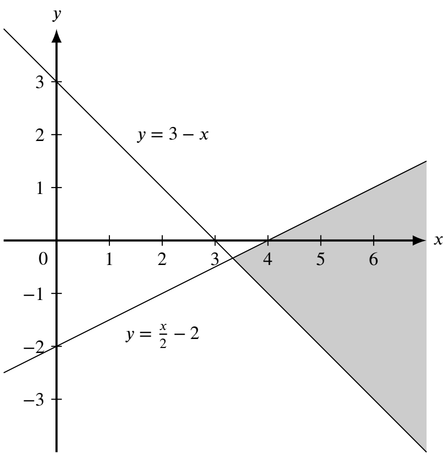
2. $\begin{cases} x + y \leq 4 \\ x + 2y \leq 6 \end{cases}$

Sol.



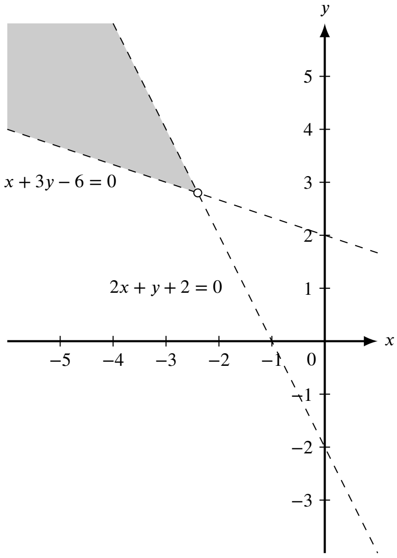
3. $\begin{cases} y \geq 3 - x \\ y \leq \frac{x}{2} - 2 \end{cases}$

Sol.



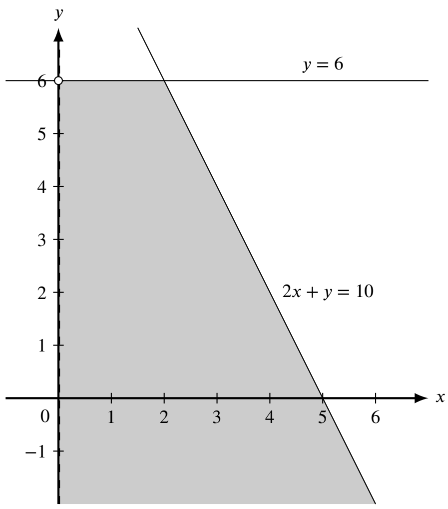
4. $\begin{cases} x + 3y - 6 > 0 \\ 2x + y + 2 < 0 \end{cases}$

Sol.



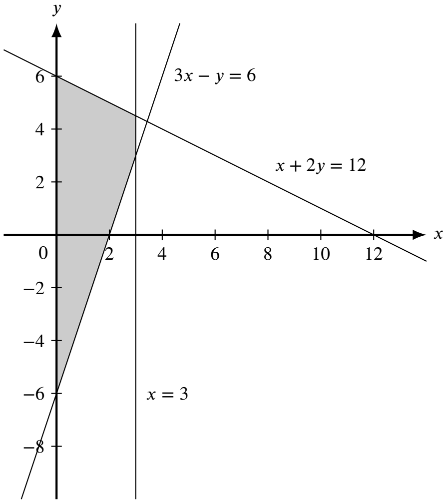
5. $\begin{cases} x > 0 \\ 2x + y \leq 10 \\ y \leq 6 \end{cases}$

Sol.



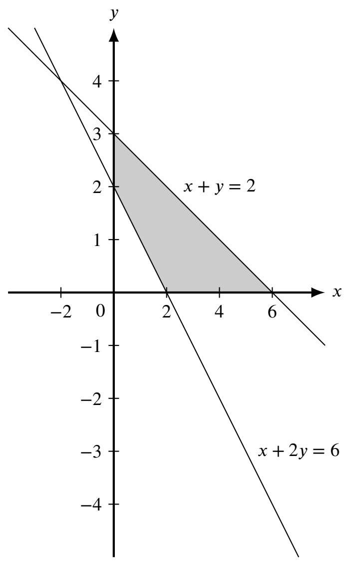
6. $\begin{cases} x + 2y \leq 12 \\ 3x - y \leq 6 \\ 0 \leq x \leq 3 \end{cases}$

Sol.

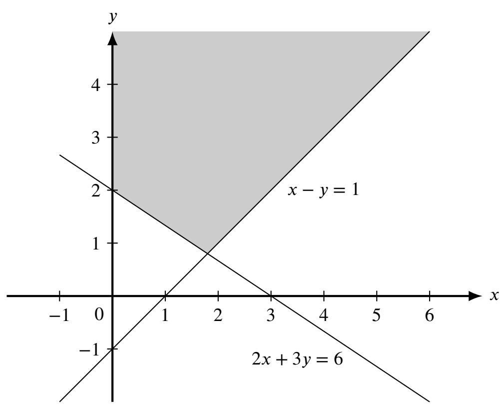


7. $\begin{cases} x + y \geq 2 \\ x + 2y \leq 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$

Sol.

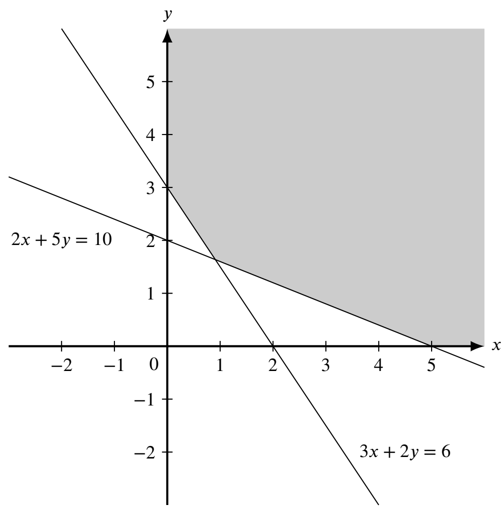


8. $\begin{cases} x - y \leq 1 \\ 2x + 3y \geq 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$



Write a system of inequalities that represents the region bounded by the following graphs:

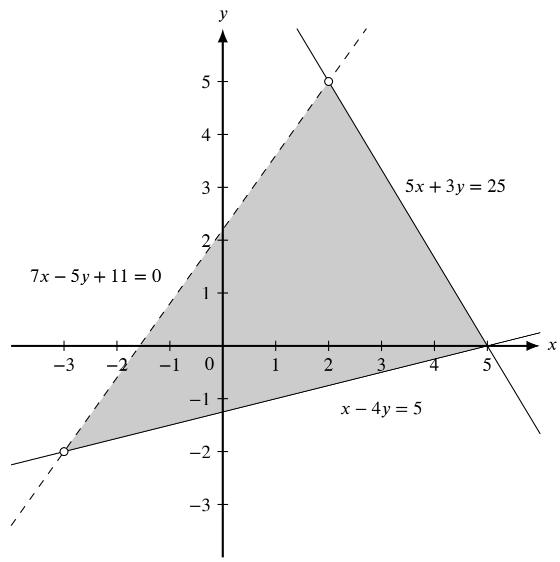
9.



Sol.

$$\begin{cases} 2x + 5y \geq 10 \\ 3x - 2y \geq 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

10.



Sol.

$$\begin{cases} 7x - 5y + 11 > 0 \\ 5x + 3y \leq 25 \\ x - 4y \leq 5 \end{cases}$$

15.8 Linear Programming

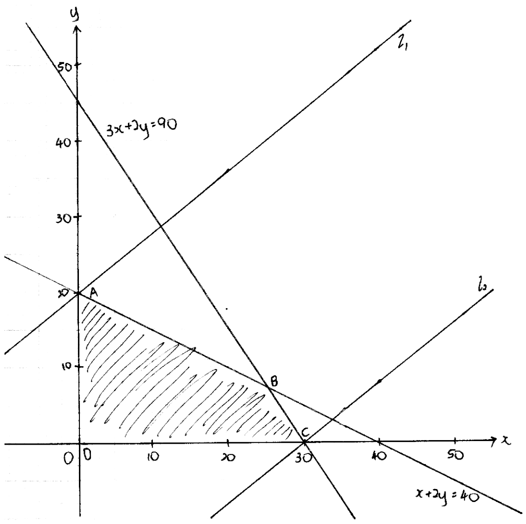
15.8.1 Practice 12

Find the maximum and minimum value of $z = 8x - 10y$ subject to the following constraints:

$$\begin{cases} x + 2y \leq 40 \\ 3x + 2y \leq 90 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Sol.

$$\text{Objective function: } z = 8x - 10y$$
$$10y = 8x - z$$
$$y = \frac{4}{5}x - \frac{z}{10}$$



When $y = \frac{4}{5}x - \frac{z}{10}$ translates towards bottom right of the feasible region, the value of z increases. Therefore, the maximum value of the objective function is the value of z in l_2. The point of intersection C of l_2 and the feasible region makes the objective function to have its maximum value. Since C is also the point of intersection of $3x + 2y = 90$ and $y = 0$,

$$\begin{cases} 3x + 2y = 90 \\ y = 0 \end{cases}$$

$D = (30, 0)$

$z_{\text{max}} = 8(30) - 0 = 240$

When $y = x - z$ translates towards top left of the feasible region, the value of z decreases. Therefore, the minimum value of the objective function is the value of z in l_1. The point of intersection A of l_1 and the feasible region makes the objective function to have its minimum value. Since A is also the point of intersection of $x + 2y = 40$ and $x = 0$,

$$\begin{cases} x + 2y = 40 \\ x = 0 \end{cases}$$

$A = (0, 20)$

$z_{\text{min}} = 0 - 10(20) = -200$

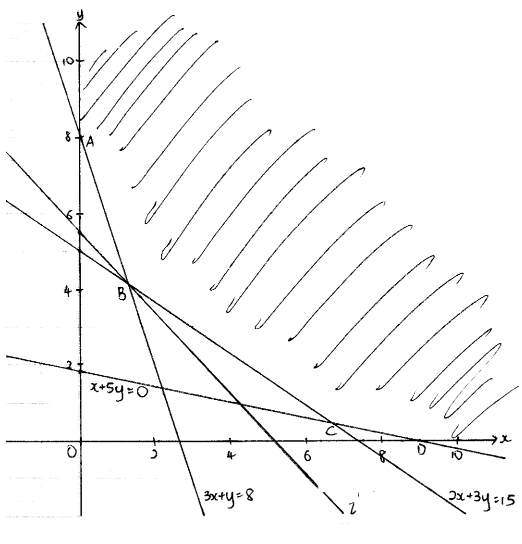
15.8.2 Exercise 15.8

1. Find the minimum value of $z = 10x + 12y$, subject to the following constraints:

$$\begin{cases} 3x + y \geq 8 \\ 2x + 3y \geq 15 \\ x + 5y \geq 9 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Sol.

$$\text{Objective function: } z = 10x + 12y$$
$$12y = -10x + z$$
$$y = -\frac{5}{6}x + \frac{z}{12}$$



The minimum value of the objective function is the value of z in l. The point of intersection B of l and the feasible region makes the objective function to have its minimum value. Since B is also the point of intersection of $3x + y = 8$ and $2x + 3y = 15$,

$$\begin{cases} 3x + y = 8 \\ 2x + 3y = 15 \end{cases}$$

$$
\begin{aligned}
2x + 3(8 - 3x) &= 15 \\
2x + 24 - 9x &= 15 \\
-7x &= -9 \\
x &= \frac{9}{7} \\
\frac{27}{7} + y &= 8 \\
y = 8 - \frac{27}{7} &= \frac{29}{7} \\
B &= (\frac{9}{7}, \frac{29}{7}) \\
z_{\min} &= 10(\frac{9}{7}) + 12(\frac{29}{7}) \\
&= 62\frac{4}{7}
\end{aligned}
$$

2. A housing developer owns a tract of land that is $2,400m^2$ in area and a construction capital of $\$4,600,000$. The developer wishes to build two types of houses: type A and type B. Given that each type A house requires $150m^2$ of land and $\$250,000$ of construction fees, can earn $\$55,000$ in profit; and each type B house requires $200m^2$ of land and $\$400,000$ of construction fees, can earn $\$80,000$ in profit. Assume that all houses built can be sold, how many of each type of house should be built to maximize the profit? Find the maximum profit.

Sol.

Let x be the number of type A houses and y be the number of type B houses.

	A (x units)	**B (y units)**	**Limit**
Area (m^2)	$150x$	$200x$	$2,400$
Cost($)	$250,000x$	$400,000y$	$4,600,000$
Profit($)	$55,000x$	$80,000y$	

The total profit is $z = 55,000x + 80,000y$, this is the objective function. According to the descriptions above, we find the maximum value of it.

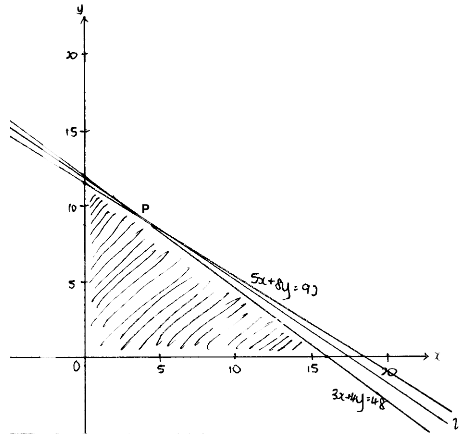
The constraints are:

$$
\begin{cases}
150x + 200y \leq 2,400 \\
250,000x + 400,000y \leq 4,600,000 \\
x \geq 0 \\
y \geq 0
\end{cases}
$$

After simplifying the constraints, we get:

$$
\begin{cases}
3x + 4y \leq 48 \\
5x + 8y \leq 92 \\
x \geq 0 \\
y \geq 0
\end{cases}
$$

The feasible region is as follows:



Let l : $55,000x + 80,000y = z$.

When the line is at l, the value of z is at its maximum. The point of intersection P of l and the feasible region makes the objective function to have its maximum value. Since P is also the point of intersection of $3x + 4y = 48$ and $5x + 8y = 92$,

$$
\begin{cases}
3x + 4y = 48 & (1) \\
5x + 8y = 92 & (2)
\end{cases}
$$

$$
\begin{aligned}
(1) \times 2 &: 6x + 8y = 96 \\
(1) - (2) &: x = 4 \\
\text{Sub } x = 4 \text{ into } (1) &: 12 + 4y = 48 \\
&\quad y = 9 \\
P &= (4, 9) \\
z_{\max} &= 55,000(4) + 80,000(9) \\
&= 940,000
\end{aligned}
$$

Thus, the maximum profit of $\$940,000$ can be obtained by building 4 type A houses and 9 type B houses.

3. One has a building lot that is $180m^2$ in area. He plans to pay $\$7,000$ to split the lot into two type of rooms and rent them out to students: each bigger room is $20m^2$ in area and can accommodate 5 students with a monthly rent of $\$225$ per student; each smaller room is $15m^2$ in area and can accommodate 3 students with a monthly rent of $\$250$ per student. The renovation cost for each bigger room is $\$700$ and for each smaller room is $\$600$. Assume that the source of tenants is stable, how many of each type of room should be divided into to maximize the profit? Find the maximum profit.

Sol.

Let x be the number of bigger rooms and y be the number of smaller rooms.

	Big (x unit)	**Small (y unit)**	**Limit**
Area (m^2)	$20x$	$15y$	180
Cost($)	$700x$	$600y$	$7,000$
Profit($)	$1125x$	$750y$	

The total profit is $z = 1125x + 750y$, this is the objective function. According to the descriptions above, we find the maximum value of it.

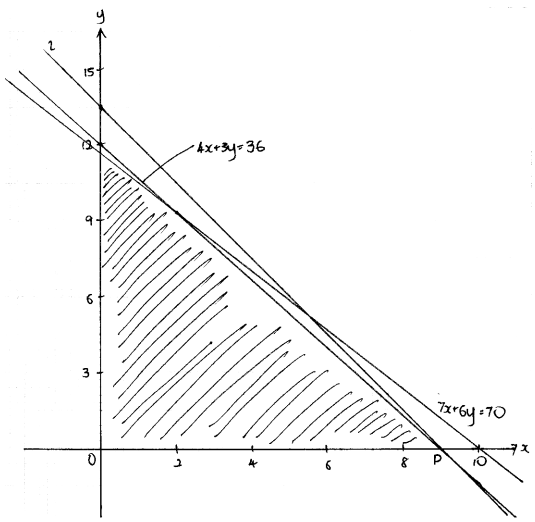
The constraints are:

$$\begin{cases} 20x + 15y \leq 180 \\ 700x + 600y \leq 7,000 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

After simplifying the constraints, we get:

$$\begin{cases} 4x + 3y \leq 36 \\ 7x + 6y \leq 70 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The feasible region is as follows:



Let l : $1125x + 750y = z$.

When the line is at l, the value of z is at its maximum. The point of intersection P of l and the feasible region makes the objective function to have its maximum value. Since P is also the point of intersection of $4x + 3y = 36$ and $y = 0$,

$$\begin{cases} 4x + 3y = 36 \\ y = 0 \end{cases}$$

$P = (9, 0)$

$z_{\max} = 1125(9) + 750(0) = 10,125$

Thus, the maximum profit of \$10,125 can be obtained by spliting the building lot into 9 bigger rooms.

4. Ms. Tan is a tuition teacher who teaches Mathematic subject to junior 3 and senior 3 students. There are a total of 5 students in each junior 3 class, each student pays tuition fees of \$50 per month, and each class is held for 4 hours per week. There are a total of 3 students in each senior 3 class, each student pays tuition fees of \$120 per month, and each class is held for 6 hours per week. Assume that here is a stable source of students, but the number of junior 3 students cannot exceed 2 times the number of senior 3 students. If Ms. Tan is wiling to earn at least \$6,600 per month, how many junior 3 and senior 3 classes should she held per week to minimize the number hours she has to teach? What's the mimimum number of hours she has to teach?

Sol.

Let x be the number of junior 3 classes and y be the number of senior 3 classes.

The number of hours she has to teach is $z = 4x + 6y$, this is the objective function. According to the descriptions above, we find the minimum value of it.

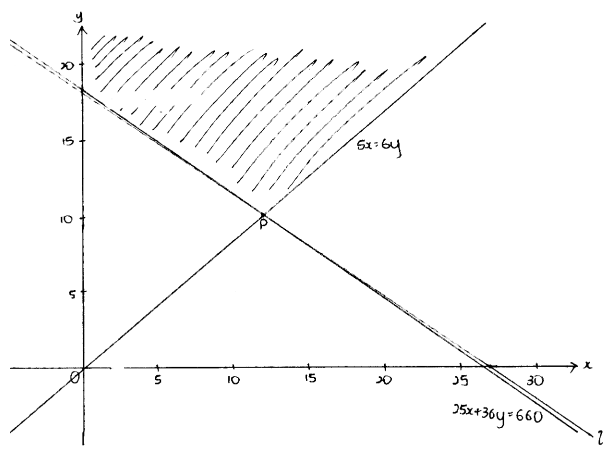
The constraints are:

$$\begin{cases} 5 \times 50 \times x + 3 \times 120 \times y \geq 6,600 \\ 5x \leq 2 \times 3y \\ x \geq 0 \\ y \geq 0 \end{cases}$$

After simplifying the constraints, we get:

$$\begin{cases} 25x + 36y \geq 660 \\ 5x \leq 6y \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The feasible region is as follows:



Let l : $4x + 6y = z$.

When the line is at l, the value of z is at its minimum. The point of intersection P of l and the feasible region makes the objective function to have its minimum value. Since P is also the point of intersection of $25x + 36y = 660$ and $5x = 6y$,

$$\begin{cases} 25x + 36y = 660 & (1) \\ 5x = 6y & (2) \end{cases}$$

$$\begin{aligned} \text{Sub (2) into (1)}: 25x + 30x &= 660 \\ 55x &= 660 \\ x &= 12 \\ \text{Sub } x = 12 \text{ into (2)}: 60 &= 6y \\ y &= 10 \end{aligned}$$

$$\begin{aligned} P &= (12, 10) \\ z_{\min} &= 4(12) + 6(10) = 108 \end{aligned}$$

Thus, Ms. Tan should hold 12 junior 3 classes and 10 senior 3 classes per week, and she has to teach for at lest 108 hours per week.

5. A company can produce a product with two types of raw materials. Each ton of the first type of raw material cost \$300, freight cost \$50, and can produce $90kg$ of the product; each ton of the second type of raw material cost \$700, freight cost \$40, and can produce $100kg$ of the product. If the company has a total of \$2, 100 to spend on raw materials and \$200 to spend on freight every day, what's the maximum amount of product that can be produced every day? How many tons of each type of raw material should be used?

Sol.

Let x be the number of tons of the first type of raw material and y be the number of tons of the second type of raw material.

	M1 (x t)	**M2 (y t)**	**Limit**
Cost (\$)	$300x$	$700y$	$2,100$
Freight(\$)	$50x$	$40y$	200
Product(kg)	$90x$	$100y$	

The objective function is $z = 90x + 100y$, which is the amount of product that can be produced every day. According to the descriptions above, we find the maximum value of it.

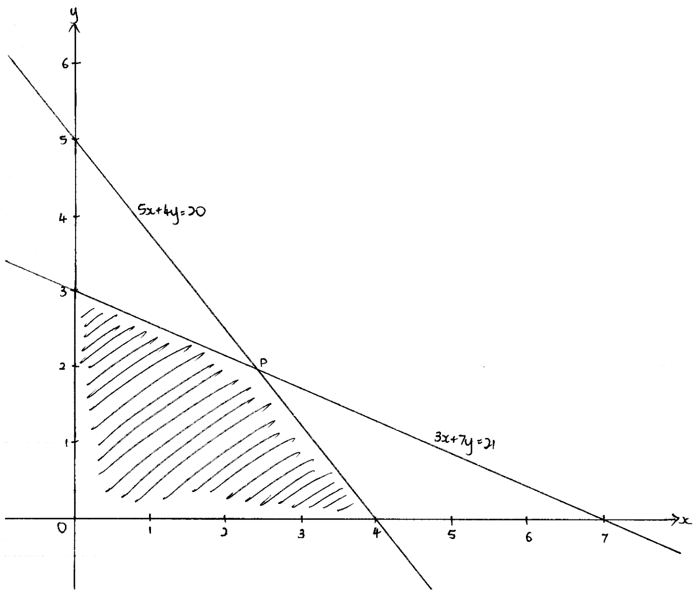
The constraints are:

$$\begin{cases} 300x + 700y \leq 2,100 \\ 50x + 40y \leq 200 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

After simplifying the constraints, we get:

$$\begin{cases} 3x + 7y \leq 21 \\ 5x + 4y \leq 20 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The feasible region is as follows:



Let $l : 90x + 100y = z$.

When the line is at l, the value of z is at its maximum. The point of intersection P of l and the feasible region makes the objective function to have its maximum value. Since P is also the point of intersection of $3x + 7y = 21$ and $5x + 4y = 20$,

$$\begin{cases} 3x + 7y = 21 & (1) \\ 5x + 4y = 20 & (2) \end{cases}$$

$$
\begin{aligned}
(1)\times 5 &: 15x + 35y = 105 \\
(2)\times 3 &: 15x + 12y = 60 \\
(1)-(2) &: 23y = 45 \\
y &= \frac{45}{23} \\
&= 1.96
\end{aligned}
$$

$$
\begin{aligned}
\text{Sub } y = \frac{45}{23} \text{ into (2)} &: 5x + \frac{180}{23} = 20 \\
5x &= \frac{280}{23} \\
x &= \frac{56}{23} \\
&= 2.43
\end{aligned}
$$

$$
\begin{aligned}
P &= (2.43, 1.96) \\
z_{\max} &= 90\left(\frac{56}{23}\right) + 100\left(\frac{45}{23}\right) \\
&= 414.78
\end{aligned}
$$

Thus, the company should use 2.43 tons of the first type of raw material and 1.96 tons of the second type of raw material, and the maximum amount of product that can be produced every day is $414.78kg$.

6. A factory uses four types of raw materials: a, b, c, and d to produce two types of products: A and B, the stock of raw materials a, b, c, and d are 22, 14, 15, and 18 units respectively. Given that the required amount of raw materials a, b, c, and d for producing one unit of product A is 3, 2, 0, 3 units respectively, and the required amount of raw materials a, b, c, and d for producing one unit of product B is 2, 1, 3, 0 units respectively. If each product A can make a profit of \$7,000 and each product B can make a profit of \$5,000, how many units of each product should be produced to maximize the profit with the current stock of raw materials?

Sol.

Let x be the number of units of product A and y be the number of units of product B.

	A (x unit)	**B (y unit)**	**Limit**
a (unit)	$3x$	$2y$	22
b (unit)	$2x$	y	14
c (unit)	$0x$	$3y$	15
d (unit)	$3x$	$0y$	18
Profit (\$)	$7,000x$	$5,000y$	

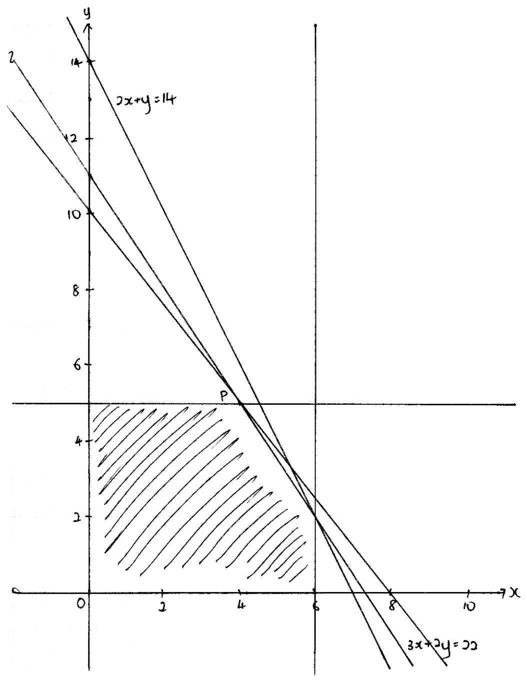
The objective function is $z = 7,000x + 5,000y$, which is the profit. According to the descriptions above, we find the maximum value of it. The constraints are:

$$
\begin{cases}
3x + 2y \le 22 \\
2x + y \le 14 \\
3y \le 15 \\
3x \le 18 \\
x \ge 0 \\
y \ge 0
\end{cases}
$$

After simplifying the constraints, we get:

$$
\begin{cases}
3x + 2y \le 22 \\
2x + y \le 14 \\
0 \le y \le 5 \\
0 \le x \le 6
\end{cases}
$$

The feasible region is as follows:



Let l : $7,000x + 5,000y = z$.

When the line is at l, the value of z is at its maximum. The point of intersection P of l and the feasible region makes the objective function to have its maximum value. Since P is also the point of intersection of $3x + 2y = 22$ and $y = 5$,

$$
\begin{cases}
3x + 2y = 22 \\
y = 5
\end{cases}
$$

$$P = (4, 5)$$

Thus, the company should produce 4 units of product A and 5 units of product B to maximize the profit.

7. Mr. Wong is willing to mix two types of drinks: A and B to produce a new drink. Drink A cost \$2 per litre, contains $20mg$ of vitamin C, $3mg$ of coloring agent, and $150g$ of sugar; drink B cost \$4 per litre, contains $35mg$ of vitamin C, $2mg$ of coloring agent, and $100g$ of sugar. Mr. Tan is willing to mix at least 50 litres of the new drink, but each litre of the new drink has to contain at least $30mg$ of vitamin C, the total amount of sugar cannot exceed $6kg$, and the total cost cannot exceed \$180. How many litres of each type of drink should be mixed to minimize the amount of coloring agent?

   Sol.

   Let x be the number of litres of drink A and y be the number of litres of drink B.

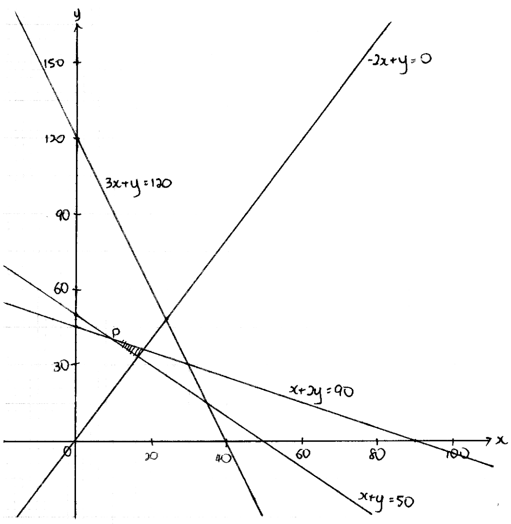
   The objective function is $z = 3x + 2y$, which is the amount of coloring agent. According to the descriptions above, we find the minimum value of it. The constraints are:

$$\begin{cases} x + y \geq 50 \\ 20x + 35y \geq 30(x + y) \\ 150x + 100y \leq 6,000 \\ 2x + 4y \leq 180 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

   After simplifying the constraints, we get:

$$\begin{cases} x + y \geq 50 \\ -2x + y \geq 0 \\ 3x + 2y \leq 120 \\ x + 2y \leq 90 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

   The feasible region is as follows:



   Let l : $3x + 2y = z$.

   When the line is at l, the value of z is at its minimum. The point of intersection P of l and the feasible region makes the objective function to have its minimum value. Since P is also the point of intersection of $x + y = 50$ and $x + 2y = 90$,

$$\begin{cases} x + y = 50 \\ x + 2y = 90 \end{cases}$$

$$P = (10, 40)$$

   Thus, the company should produce 10 litres of drink A and 40 litres of drink B to minimize the amount of coloring agent.

8. A bakery bakes two types of cake: A and B. The ingredients required for baking one cake of type A is $1kg$ of flour, 5 eggs, and $300g$ of sugar; the ingredients required for baking one cake of type B is $800g$ of flour, 8 eggs, and $200g$ of sugar. The bakery has 3 bakers, each of them works for at least 8 hours per day, and the total time required for each baker to bake one cake of type A and B is 40 minutes and 50 minutes respectively. If the bakery has to bake at least 32 cakes every day, and the everyday supply of ingredients is limited to 220 eggs and $9kg$ of sugar. Due to the shortage of flour, the bakery needs to lower the usage of it. How many cakes of each type should be baked to minimize the usage of flour? What's the minimum amount of flour used?

   Sol.

   Let x be the number of cakes of type A and y be the number of cakes of type B.

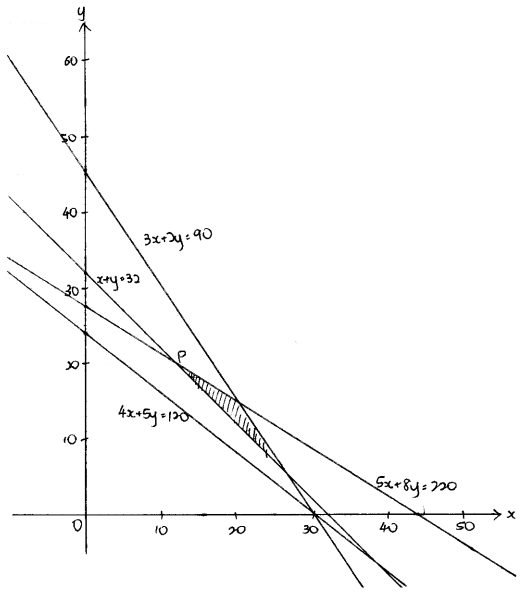
   The objective function is $z = x + 0.8y$, which is the amount of flour used. According to the descriptions above, we find the minimum value of it. The constraints are:

$$\begin{cases} 5x + 8y \leq 220 \\ 300x + 200y \leq 9,000 \\ 40x + 50y \geq 480 \\ x + y \geq 32 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

   After simplifying the constraints, we get:

$$\begin{cases} 5x + 8y \leq 220 \\ 3x + 2y \leq 90 \\ 4x + 5y \geq 120 \\ x + y \geq 32 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The feasible region is as follows:



Let l : $4x + 5y = z$.

When the line is at l, the value of z is at its minimum. The point of intersection P of l and the feasible region makes the objective function to have its minimum value. Since P is also the point of intersection of $x + y = 32$ and $5x + 8y = 220$,

$$\begin{cases} x + y = 32 & (1) \\ 5x + 8y = 220 & (2) \end{cases}$$

$$\begin{aligned} (1) : y &= 32 - x (3) \\ \text{Sub (3) into (2)} : 5x + 8(32 - x) &= 220 \\ 5x + 256 - 8x &= 220 \\ -3x &= -36 \\ x &= 12 \\ y &= 20 \end{aligned}$$

$$\begin{aligned} P &= (12, 20) \\ z &= 12 + 0.8(20) = 28 \end{aligned}$$

Thus, the bakery should bake 12 cakes of type A and 20 cakes of type B to minimize the amount of flour used. The minimum amount of flour used is $28kg$

15.9 Revision Exercise 15

Compare the algebraic expressions in the following questions (Question 1 to 2):

1. $(x - 3)(4 - x)$ and $(6 - x)(x - 1)$

   Sol.

$$\begin{aligned} &(x - 3)(4 - x) - (6 - x)(x - 1) \\ &= -x^2 + 7x - 12 - (-x^2 + 7x - 6) \\ &= -x^2 + 7x - 12 + x^2 - 7x + 6 \\ &= -6 < 0 \\ &\therefore (x - 3)(4 - x) < (6 - x)(x - 1) \end{aligned}$$

2. $6 - x^2$ and $4x - 2x^2$

   Sol.

$$\begin{aligned} &6 - x^2 - (4x - 2x^2) \\ &= 6x - x^2 - 4x + 2x^2 \\ &= x^2 - 2x \\ &= (x - 1)^2 + 1 \\ &\because (x - 1)^2 + 1 > 0 \\ &\therefore (x - 1)^2 + 1 > 0 \\ &\therefore 6 - x^2 > 4x - 2x^2 \end{aligned}$$

Solve the following inequalities (Question 3 to 16):

3. $4(x - 1) > x + 6$

   Sol.

$$\begin{aligned} 4(x - 1) &> x + 6 \\ 4x - 4 &> x + 6 \\ 3x &> 10 \\ x &> \frac{10}{3} \end{aligned}$$

4. $3(3 - x) \geq 2(x + 3)$

   Sol.

$$\begin{aligned} 3(3 - x) &\geq 2(x + 3) \\ 9 - 3x &\geq 2x + 6 \\ -5x &\geq -3 \\ 5x &\leq 3 \\ x &\leq \frac{3}{5} \end{aligned}$$

5. $3 - \frac{x-1}{4} \geq 2 + \frac{3(x+1)}{8}$

   Sol.

$$\begin{aligned} 3 - \frac{x - 1}{4} &\geq 2 + \frac{3(x + 1)}{8} \\ 24 - 2(x - 1) &\geq 16 + 3(x + 1) \\ 24 - 2x + 2 &\geq 16 + 3x + 3 \\ 26 - 2x &\geq 19 + 3x \\ -5x &\geq -7 \\ 5x &\leq 7 \\ x &\leq \frac{7}{5} \end{aligned}$$

6. $x - \frac{x-1}{2} \leq \frac{2x-1}{3} + \frac{x+1}{2}$

Sol.

$$\begin{aligned} x-\frac{x-1}{2} &\le \frac{2x-1}{3}+\frac{x+1}{2} \\ 6x-3(x-1) &\le 2(2x-1)+3(x+1) \\ 6x-3x+3 &\le 4x-2+3x+3 \\ 3x+3 &\le 7x+1 \\ -4x &\le -2 \\ 4x &\ge 2 \\ x &\ge \frac{1}{2} \end{aligned}$$

7. $-1 < \frac{1}{2}x+3 < 7$

Sol.

$$\begin{aligned} -1 < \frac{1}{2}&x+3 < 7 \\ -2 < x&+6 < 14 \\ -8 < &x < 8 \end{aligned}$$

8. $-\frac{3}{2} < 1-3x \le 8$

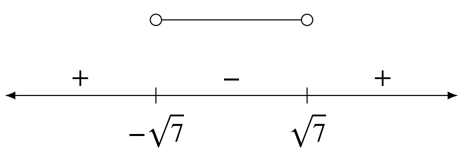
Sol.

$$\begin{aligned} -\frac{3}{2} &< 1-3x \le 8 \\ -3 &< 2-6x \le 16 \\ -5 &< -6x \le 14 \\ -14 &\le 6x < 5 \\ -\frac{7}{3} &\le x < \frac{5}{6} \end{aligned}$$

9. $x^2 < 7$

Sol.

$$\begin{aligned} x^2 &< 7 \\ x^2-7 &< 0 \\ (x+\sqrt{7})(x-\sqrt{7}) &< 0 \\ -\sqrt{7} < x &< \sqrt{7} \end{aligned}$$



10. $x^2+10x-200 \ge 0$

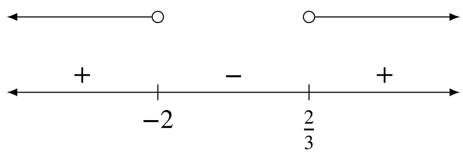
Sol.

$$\begin{aligned} x^2+10x-200 &\ge 0 \\ (x+20)(x-10) &\ge 0 \\ x \le -20 \text{ or } x &\ge 10 \end{aligned}$$

11. $4 < 3x^2+4x$

Sol.

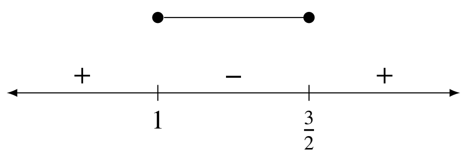
$$\begin{aligned} 4 &< 3x^2+4x \\ 3x^2+4x-4 &> 0 \\ (3x-2)(x+2) &> 0 \\ x < -2 \text{ or } x &> \frac{2}{3} \end{aligned}$$



12. $5x-3 \ge 2x^2$

Sol.

$$\begin{aligned} 5x-3 &\ge 2x^2 \\ 2x^2-5x+3 &\le 0 \\ (2x-3)(x-1) &\le 0 \\ 1 \le x &\le \frac{3}{2} \end{aligned}$$



13. $x^2-x(x-6) > 5(x-1)$

Sol.

$$\begin{aligned} x^2-x(x-6) &> 5(x-1) \\ x^2-x^2+6x &> 5x-5 \\ 6x &> 5x-5 \\ x &> -5 \end{aligned}$$

14. $(2x+1)^2+5 \le 4(x+2)^2$

Sol.

$$\begin{aligned} (2x+1)^2+5 &\le 4(x+2)^2 \\ 4x^2+4x+1+5 &\le 4(x^2+4x+4) \\ 4x^2+4x+6 &\le 4x^2+16x+16 \\ -12x &\le 10 \\ 12x &\ge -10 \\ x &\ge -\frac{5}{6} \end{aligned}$$

15. $9x^2+2 \le 12x-2$

Sol.

$$\begin{aligned} 9x^2+2 &\le 12x-2 \\ 9x^2-12x+4 &\le 0 \\ (3x-2)^2 &\le 0 \\ x &= \frac{2}{3} \end{aligned}$$

16. $4(x^2+7) > 3-20x$

Sol.

$$\begin{aligned} 4(x^2+7) &> 3-20x \\ 4x^2+28 &> 3-20x \\ 4x^2+20x-25 &> 0 \\ (2x+5)^2 &> 0 \\ x \in \mathbb{R}, x &\ne -\frac{5}{2} \end{aligned}$$

Solve the following system of inequalities (Question 17 to 28):

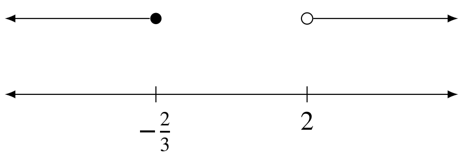
17.

$$\begin{cases} 3x+2 \le 0 & (1) \\ 4-x < x & (2) \end{cases}$$

Sol.

$$\begin{aligned} (1):\ & 3x \le -2 \\ & x \le -\frac{2}{3} \\ (2):\ & -2x < -4 \\ & x > 2 \end{aligned}$$

$\therefore$ No solution.



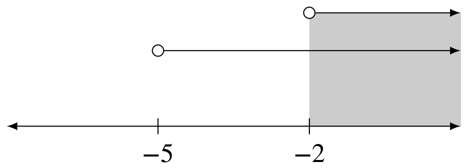
18.

$$\begin{cases} x+4 > -x & (1) \\ \dfrac{3x-1}{2} < 2(x+1) & (2) \end{cases}$$

Sol.

$$\begin{aligned} (1):\ & 2x > -4 \\ & x > -2 \\ (2):\ & 3x-1 < 4(x+1) \\ & 3x-1 < 4x+4 \\ & -x < 5 \\ & x > -5 \end{aligned}$$

$\therefore x > -2$



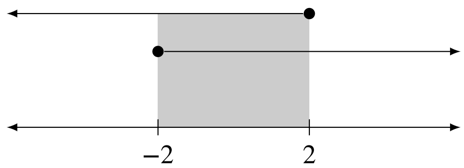
19.

$$\begin{cases} x-3 \le 5-3x & (1) \\ 4+(2x-1) \le 4x+7 & (2) \end{cases}$$

Sol.

$$\begin{aligned} (1):\ & 4x \le 8 \\ & x \le 2 \\ (2):\ & 3+2x \le 4x+7 \\ & -2x \le 4 \\ & x \ge -2 \end{aligned}$$

$\therefore -2 \le x \le 2$



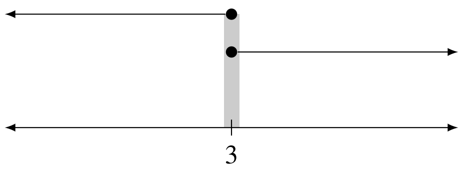
20.

$$\begin{cases} 4x-5 \ge 2x+1 & (1) \\ x+\dfrac{2}{3} \le \dfrac{2x+5}{3} & (2) \end{cases}$$

Sol.

$$\begin{aligned} (1):\ & 2x \ge 6 \\ & x \ge 3 \\ (2):\ & 3x+2 \le 2x+5 \\ & x \le 3 \end{aligned}$$

$\therefore x = 3$



21. $5 < 2x - 7 < x + 1$

Sol.

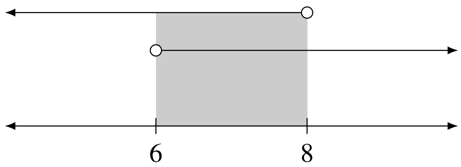
$$\begin{cases} 5 < 2x - 7 & (1) \\ 2x - 7 < x + 1 & (2) \end{cases}$$

$(1): 12 < 2x$

$x > 6$

$(2): x < 8$

$\therefore 6 < x < 8$



22. $4 < 6 + 2x \le 4x$

Sol.

$$\begin{cases} 4 < 6 + 2x & (1) \\ 6 + 2x \le 4x & (2) \end{cases}$$

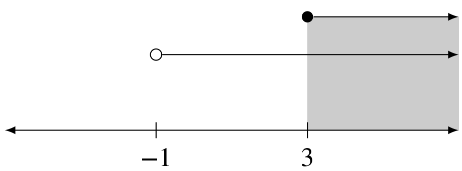
$(1): -2 < 2x$

$x > -1$

$(2): 6 \le 2x$

$x \ge 3$

$\therefore x \ge 3$



23.

$$\begin{cases} x - \frac{1}{2} \ge 1 - \frac{x}{2} & (1) \\ 2 - \frac{x}{3} < \frac{2x}{3} - 3 & (2) \\ \frac{x}{3} + \frac{1}{4} \ge \frac{x}{2} - \frac{3}{4} & (3) \end{cases}$$

Sol.

$(1): 2x - 1 \ge 2 - x$

$3x \ge 3$

$x \ge 1$

$(2): 6 - x < 2x - 9$

$-3x < -15$

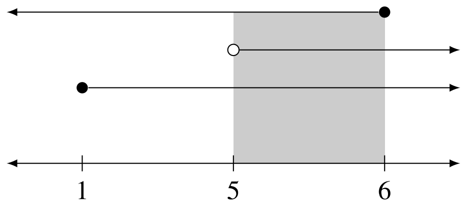
$x > 5$

$(3): 4x + 3 \ge 6x - 9$

$-2x \ge -12$

$x \le 6$

$\therefore 5 < x \le 6$



24.

$$\begin{cases} x + \frac{13}{2} > \frac{7 - x}{2} & (1) \\ 2\left(x + \frac{1}{3}\right) < 2 - x & (2) \\ x^2 \ge \frac{5x}{2} & (3) \end{cases}$$

Sol.

$(1): 2x + 13 > 7 - x$

$3x > -6$

$x > -2$

$(2): 2x + \frac{2}{3} < 2 - x$

$6x + 2 < 6 - 3x$

$9x < 4$

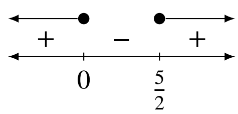
$x < \frac{4}{9}$

$(3): 2x^2 \ge 5x$

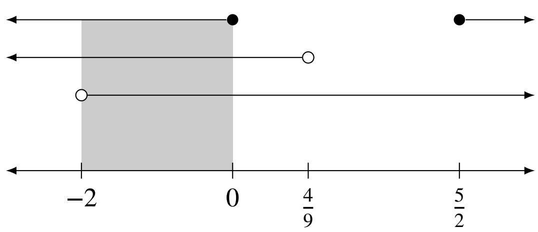
$2x^2 - 5x \ge 0$

$x(2x - 5) \ge 0$

$x \le 0$ or $x \ge \frac{5}{2}$



$\therefore -2 < x \le 0$



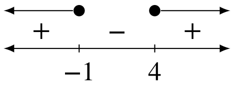
25.

$$\begin{cases} x^2 - 3x - 4 \geq 0 & (1) \\ 2x^2 - x - 6 > 0 & (2) \end{cases}$$

Sol.

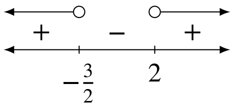
(1) : $(x-4)(x+1) \geq 0$

$x \leq -1$ or $x \geq 4$

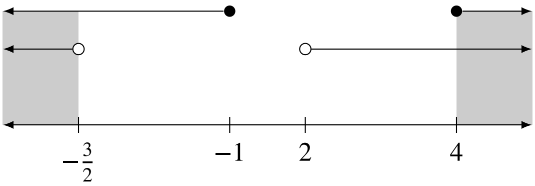


(2) : $(2x+3)(x-2) > 0$

$x < -\frac{3}{2}$ and $x > 2$



$\therefore x < -\frac{3}{2}$ or $x \geq 4$



26.

$$\begin{cases} (2x-1)(x-2) \leq 8x - 9 & (1) \\ 3(x^2 - 2) < 7x & (2) \end{cases}$$

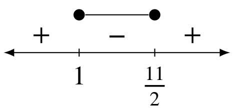
Sol.

(1) : $2x^2 - 5x + 2 \leq 8x - 9$

$2x^2 - 13x + 11 \leq 0$

$(2x-11)(x-1) \leq 0$

$1 \leq x \leq \frac{11}{2}$

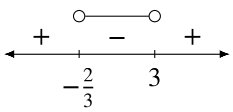


(2) : $3x^2 - 6 < 7x$

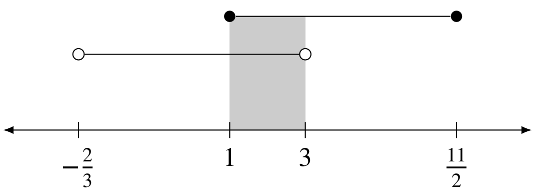
$3x^2 - 7x - 6 < 0$

$(3x+2)(x-3) < 0$

$-\frac{2}{3} < x < 3$



$\therefore 1 \leq x < 3$



27.

$$\begin{cases} 2(x^2+3) \geq 7 - x & (1) \\ (x+3)^2 > 1 & (2) \end{cases}$$

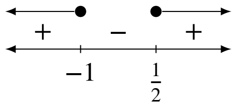
Sol.

(1) : $2x^2 + 6 \geq 7 - x$

$2x^2 + x - 1 \geq 0$

$(2x-1)(x+1) \geq 0$

$x \leq -1$ or $x \geq \frac{1}{2}$

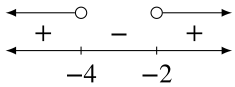


(2) : $x^2 + 6x + 9 > 1$

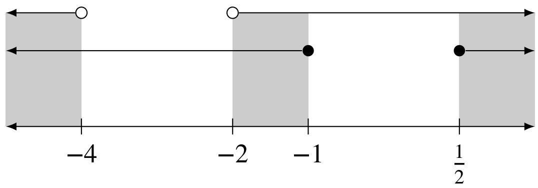
$x^2 + 6x + 8 > 0$

$(x+4)(x+2) > 0$

$x < -4$ or $x > -2$



$\therefore x < -4$ or $-2 < x \leq -1$ or $x \geq \frac{1}{2}$



28.

$$\begin{cases} x(x-1) \leq 2 & (1) \\ x(x+1) \geq 6 & (2) \end{cases}$$

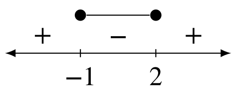
Sol.

(1) : $x^2 - x \leq 2$

$x^2 - x - 2 \leq 0$

$(x-2)(x+1) \leq 0$

$-1 \leq x \leq 2$

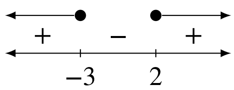


(2) : $x^2 + x \geq 6$

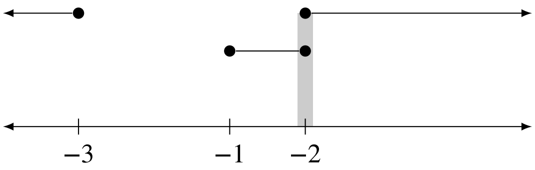
$x^2 + x - 6 \geq 0$

$(x+3)(x-2) \geq 0$

$x \leq -3$ or $x \geq 2$



$\therefore x = 2$

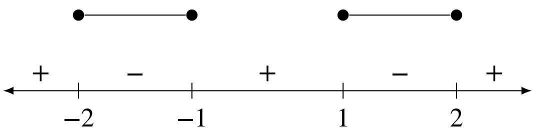


Solve the following inequalities (Question 29 to 40):

29. $x^4 - 5x^2 + 4 \le 0$

Sol.

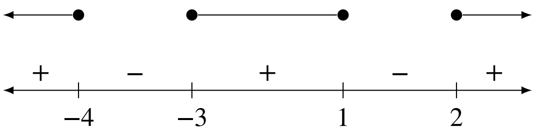
$$\begin{aligned} x^4 - 5x^2 + 4 &\le 0 \\ (x-1)(x^3 + x^2 - 4x - 4) &\le 0 \\ (x-1)(x+1)(x^2 - 4) &\le 0 \\ (x-1)(x+1)(x-2)(x+2) &\le 0 \\ -2 \le x \le -1 \text{ or } 1 \le x &\le 2 \end{aligned}$$



30. $(x^2 + 2x - 8)(x^2 + 2x - 3) > 0$

Sol.

$$\begin{aligned} (x+4)(x-2)(x+3)(x-1) &> 0 \\ x < -4 \text{ or } -3 < x < -1 &\text{ or } x > 2 \end{aligned}$$



31. $(2x+1)^2(x^2 + 3x - 10) < 0$

Sol.

For all real numbers x,

$$\begin{aligned} (2x+1)^2 > 0 \text{ when } x &\ne -\frac{1}{2} \\ x^2 + 3x - 10 &< 0 \\ (x+5)(x-2) &< 0 \\ \therefore\ -5 < x &< -2 \end{aligned}$$

When $x = -\frac{1}{2}$,

$$(2x+1)^2(x^2 + 3x - 10) = 0$$

$\therefore x = -\frac{1}{2}$ is not a solution.

$$\therefore\ -5 < x < -2,\ x \ne -\frac{1}{2}$$

32. $(x-1)^2(6x^2 + 13x + 6) \le 0$

Sol.

For all real numbers x,

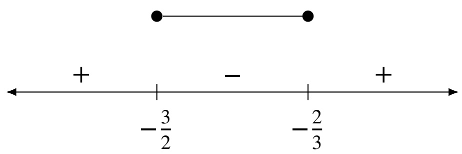
$$\begin{aligned} (x-1)^2 > 0 \text{ when } x &\ne 1 \\ 6x^2 + 13x + 6 &\le 0 \\ (3x+2)(2x+3) &\le 0 \\ \therefore\ -\frac{3}{2} \le x &\le -\frac{2}{3} \end{aligned}$$

When $x = 1$,

$$(x-1)^2(6x^2 + 13x + 6) = 0$$

$\therefore x = 1$ is a solution.

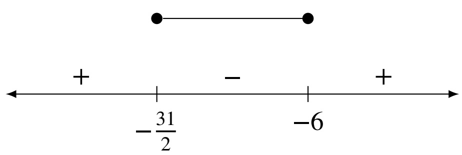
$$\therefore\ -\frac{3}{2} \le x \le -\frac{2}{3} \text{ or } x = 1$$



33. $\frac{2x-7}{x+6} \ge 4$

Sol.

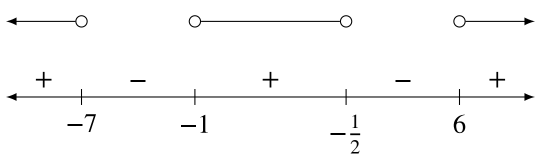
$$\begin{aligned} \frac{2x - 7 - 4(x+6)}{x+6} &\ge 0 \\ \frac{2x - 7 - 4x - 24}{x+6} &\ge 0 \\ \frac{-2x - 31}{x+6} &\ge 0 \\ -\frac{2x + 31}{x+6} &\ge 0 \\ \frac{2x + 31}{x+6} &\le 0 \\ -\frac{31}{2} \le x &\le -6 \end{aligned}$$



34. $\frac{x}{2x+1} > \frac{6}{x+7}$

Sol.

$$
\begin{aligned}
\frac{x}{2x+1} &> \frac{6}{x+7} \\
\frac{x}{2x+1} - \frac{6}{x+7} &> 0 \\
\frac{x(x+7) - 6(2x+1)}{(2x+1)(x+7)} &> 0 \\
\frac{x^2 + 7x - 12x - 6}{(2x+1)(x+7)} &> 0 \\
\frac{x^2 - 5x - 6}{(2x+1)(x+7)} &> 0 \\
\frac{(x-6)(x+1)}{(2x+1)(x+7)} &> 0 \\
x < -7 \text{ or } -1 < x < -\frac{1}{2} &\text{ or } x > 6
\end{aligned}
$$



35. $\frac{(x+3)(x-2)^2}{x^2-1} \leq 0$

Sol.

For all real number x,

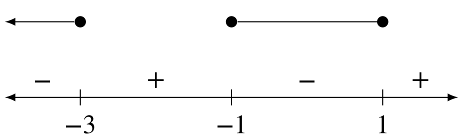
$(x-2)^2 > 0$ when $x \neq 2$

$$
\begin{aligned}
\frac{x+3}{x^2-1} &\leq 0 \\
\frac{x+3}{(x+1)(x-1)} &\leq 0 \\
\therefore x \leq -3 \text{ or } -1 \leq x &\leq 1
\end{aligned}
$$

When $x = 2$, $\frac{(x+3)(x-2)^2}{x^2-1} = 0$

$\therefore x = 2$ is a solution.

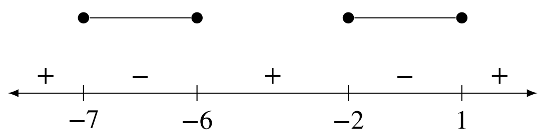
$\therefore x \leq -3$ or $-1 \leq x \leq 1$ or $x = 2$



36. $4 + \frac{7}{x+6} \leq \frac{15}{x+2}$

Sol.

$$
\begin{aligned}
4 + \frac{7}{x+6} &\leq \frac{15}{x+2} \\
4 + \frac{7}{x+6} - \frac{15}{x+2} &\leq 0 \\
\frac{4(x+6)(x+2) + 7(x+2) - 15(x+6)}{(x+6)(x+2)} &\leq 0 \\
\frac{4(x^2+8x+12) + 7x + 14 - 15x - 90}{(x+6)(x+2)} &\leq 0 \\
\frac{4x^2 + 32x + 48 + 7x + 14 - 15x - 90}{(x+6)(x+2)} &\leq 0 \\
\frac{4x^2 + 24x - 28}{(x+6)(x+2)} &\leq 0 \\
\frac{x^2 + 6x - 7}{(x+6)(x+2)} &\leq 0 \\
\frac{(x+7)(x-1)}{(x+6)(x+2)} &\leq 0 \\
-7 \leq x \leq -6 \text{ or } -2 \leq x &\leq 1 \\
\because x \neq -6 \text{ and } x &\neq 2, \\
\therefore -7 \leq x < -6 \text{ or } -2 < x &\leq 1
\end{aligned}
$$



37. $|3 - 5x| \geq 7$

Sol.

$$
\begin{aligned}
&|3 - 5x| \geq 7 \\
&3 - 5x \geq 7 \text{ or } 3 - 5x \leq -7 \\
&-5x \geq 4 \text{ or } -5x \leq -10 \\
&5x \leq -4 \text{ or } 5x \geq 10 \\
&x \leq -\frac{4}{5} \text{ or } x \geq 2
\end{aligned}
$$

38. $2 < |x - 5| < 9$

Sol.

$$
\begin{cases}
|x-5| > 2 & (1) \\
|x-5| < 9 & (2)
\end{cases}
$$

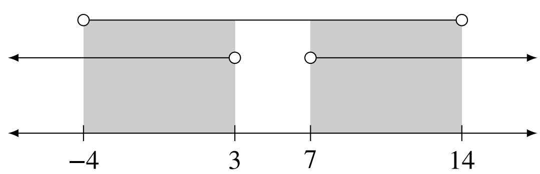
$(1): x - 5 < -2$ or $x - 5 > 2$

$x < 3$ or $x > 7$

$(2): -9 < x - 5 < 9$

$-4 < x < 14$

$\therefore -4 < x < 3$ or $7 < x < 14$



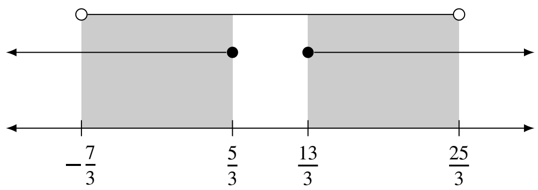
39. $1 \le \left|\frac{3x-1}{4} - 2\right| < 4$

$$\begin{cases} \left|\dfrac{3x-1}{4} - 2\right| \ge 1 & (1) \\ \left|\dfrac{3x-1}{4} - 2\right| < 4 & (2) \end{cases}$$

$$\begin{aligned} (1):\ \frac{3x-1}{4} - 2 \le -1 &\text{ or } \frac{3x-1}{4} - 2 \ge 1 \\ \frac{3x-1}{4} \le 1 &\text{ or } \frac{3x-1}{4} \ge 3 \\ 3x - 1 \le 4 &\text{ or } 3x - 1 \ge 12 \\ 3x \le 5 &\text{ or } 3x \ge 13 \\ x \le \frac{5}{3} &\text{ or } x \ge \frac{13}{3} \end{aligned}$$

$$\begin{aligned} (2):\ -4 < \frac{3x-1}{4} - 2 &< 4 \\ -2 < \frac{3x-1}{4} &< 6 \\ -8 < 3x - 1 &< 24 \\ -7 < 3x &< 25 \\ -\frac{7}{3} < x &< \frac{25}{3} \end{aligned}$$

$$\therefore\ -\frac{7}{3} < x \le \frac{5}{3} \text{ or } \frac{13}{3} \le x < \frac{25}{3}$$



40. $\frac{4}{|x+3|} - 5 \le 3$

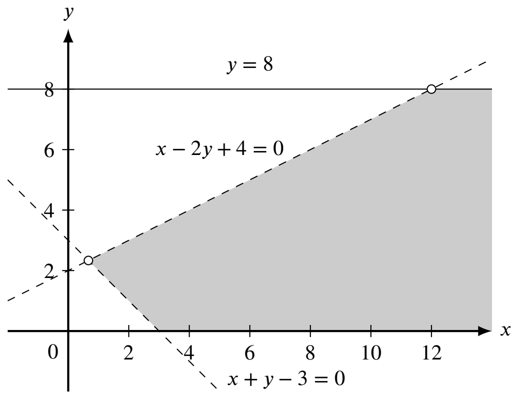
Sol.

$$\begin{aligned} \frac{4}{|x+3|} - 5 &\le 3 \\ \frac{4}{|x+3|} &\le 8 \\ 4 &\le 8|x+3| \\ |x+3| &\ge \frac{1}{2} \\ x + 3 \le -\frac{1}{2} &\text{ or } x + 3 \ge \frac{1}{2} \\ x \le -\frac{7}{2} &\text{ or } x \ge -\frac{5}{2} \end{aligned}$$

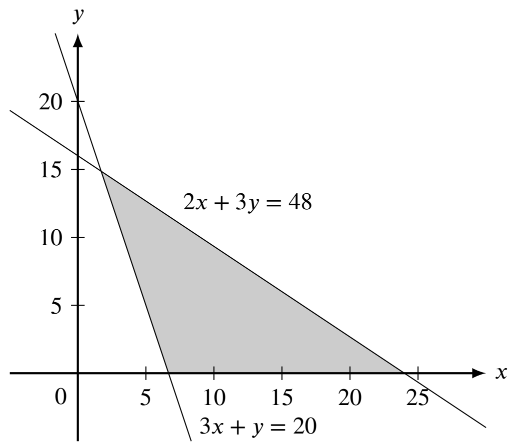
Solve the following system of inequalities with graphs (Question 41 to 42):

41. $\begin{cases} x + y - 3 > 0 \\ x - 2y + 4 > 0 \\ 0 \le y \le 8 \end{cases}$

Sol.



42. $\begin{cases} 3x + y \ge 20 \\ 2x + 3y \le 48 \\ x \ge 0 \\ y \ge 0 \end{cases}$

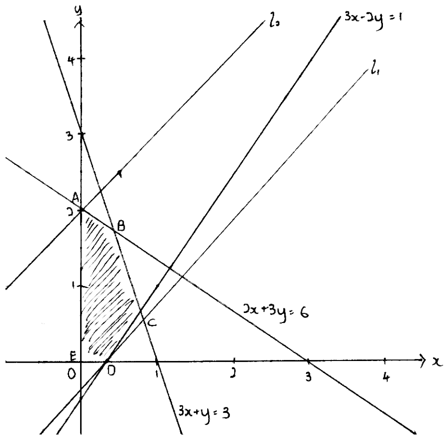


43. Find the maximum and minimum value of $z = x - y$, subject to the following constraints:

$$\begin{cases} 3x + y \le 3 \\ 2x + 3y \le 6 \\ 3x - 2y \le 1 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

Sol.

$$\text{Objective function: } z = x - y$$
$$y = x - z$$



When $y = x - z$ translates towards bottom right of the feasible region, the value of z increases. Therefore, the maximum value of the objective function is the value of z in l_1. The point of intersection D of l_1 and the feasible region makes the objective function to have its maximum value. Since D is also the point of intersection of $3x - 2y = 1$ and $y = 0$,

$$\begin{cases} 3x - 2y = 1 \\ y = 0 \end{cases}$$

$$D = \left(\frac{1}{3}, 0\right)$$

$$z_{\max} = \frac{1}{3} - 0 = \frac{1}{3}$$

When $y = x - z$ translates towards top left of the feasible region, the value of z decreases. Therefore, the minimum value of the objective function is the value of z in l_2. The point of intersection A of l_2 and the feasible region makes the objective function to have its minimum the point of intersection of $2x + 3y = 6$ and $x = 0$,

$$\begin{cases} 2x + 3y = 6 \\ x = 0 \end{cases}$$

$$A = (0, 2)$$

$$z_{\min} = 0 - 2 = -2$$

44. A factory produces two types of products: A and B. The ingredients used in each kilogram of these two products are as follows:

Product (per kg)	Ingr. X (kg)	Ingr. Y (kg)
A	0.6	0.5
B	0.3	0.7

The profit of each kilogram of product A and B is \$3 and \$5 respectively. The factory has $24kg$ of ingredient X and $28kg$ of ingredient Y. How many kilograms of each product should be produced to maximize the profit?

Sol.

Let x be the number of kilograms of product A and y be the number of kilograms of product B.

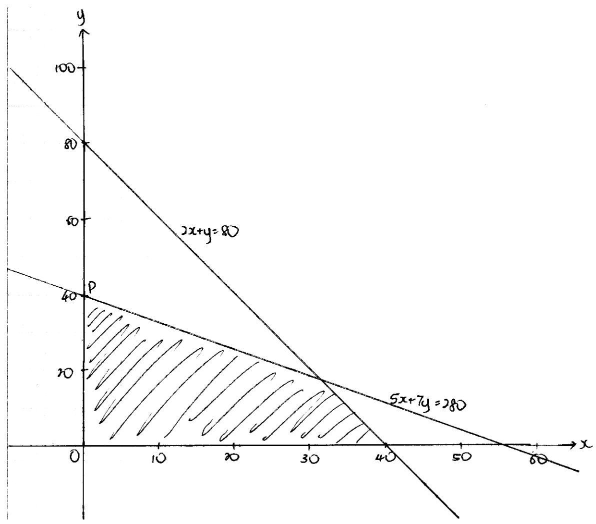
The objective function is $z = 3x + 5y$, which is the profit of the factory. According to the given information, we find the maximum of it.

$$\begin{cases} 0.6x + 0.3y \leq 24 \\ 0.5x + 0.7y \leq 28 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

After simplifying, we get:

$$\begin{cases} 2x + y \leq 80 \\ 5x + 7y \leq 280 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The feasible region is as follows:



When the line is at l, the value of z is at its maximum. The point of intersection P of l and the feasible region makes the objective function to have its maximum value. Since P is also the point of intersection of $5x + 7y = 280$ and $x = 0$,

$$\begin{cases} 5x + 7y = 280 \\ x = 0 \end{cases}$$

$$P = (0, 40)$$

Thus, only $40kg$ of product B should be produced to maximize the profit.

45. An animal must consume three different kind of nutrients: X, Y and Z at least $11units$, $13units$ and $15units$ respectively every day. There are two types of animal food: A and B that contain the following nutrients:

Food	X (unit)	Y (unit)	Z (unit)
A	1	3	2
B	2	1	2

The animal food A costs \$300 per kilogram and the animal food B costs \$400 per kilogram. How many kilograms of each food should be consumed to meet the daily nutrient requirement at the minimum cost? Find the minimum cost.

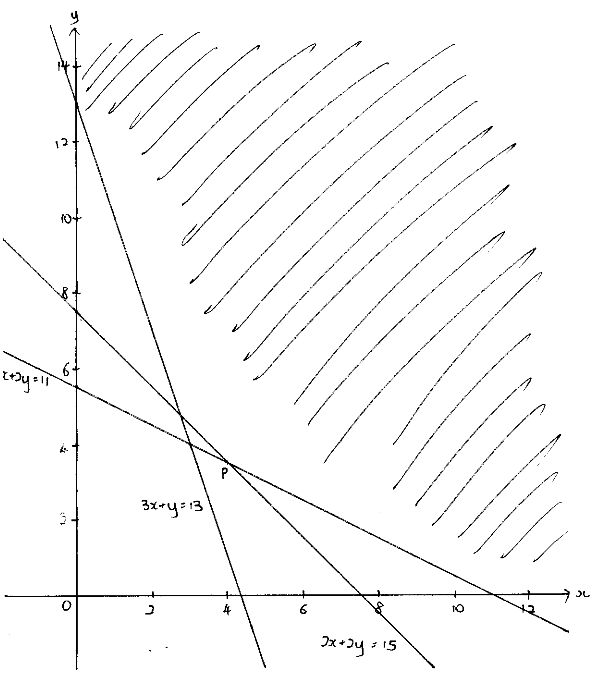
Sol.

Let x be the number of kilograms of food A and y be the number of kilograms of animal food B.

The objective function is $z = 300x+400y$, which is the cost of the food. According to the given information, we find the minimum of it. The constraints are:

$$\begin{cases} x + 2y \geq 11 \\ 3x + y \geq 13 \\ 2x + 2y \geq 15 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The feasible region is as follows:



When the line is at l, the value of z is at its minimum. The point of intersection P of l and the feasible region makes the objective function to have its minimum value. Since P is also the point of intersection of $x + 2y = 11$ and $2x + 2y = 15$,

$$\begin{cases} x + 2y = 11 & (1) \\ 2x + 2y = 15 & (2) \end{cases}$$

$$\begin{aligned} (1) &: 2y = 11 - x \\ \text{Sub (1) into (2)} &: 2x + 11 - x = 15 \\ & \qquad x = 4 \\ & \qquad y = 3.5 \end{aligned}$$

$$\begin{aligned} P &= (4, 3.5) \\ z_{\text{min}} &= 300(4) + 400(3.5) = 2,600 \end{aligned}$$

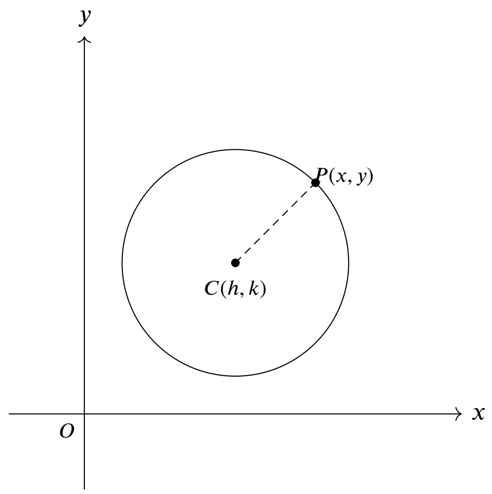
Thus, $4kg$ of food A and $3.5kg$ of food B should be consumed to meet the daily nutrient requirement at a minimum cost of \$2,600.

Chapter 16

Circle

16.1 Standard Equation of a Circle

The circle is a locus of points in a plane that are equidistant from a fixed point called the centre of the circle. The length from the centre to the points on the circle is called the radius of the circle.



The standard equation of a circle is given by

$$(x-h)^2+(y-k)^2=r^2$$

where (h,k) is the centre of the circle and r is the radius of the circle.

If the centre of the circle is at the origin, then the equation of the circle is

$$x^2+y^2=r^2 \quad (r>0)$$

16.1.1 Practice 1

1. Find the equation of the circle with centre $(3,-1)$ and radius 2.

   Sol.

$$\begin{aligned}\text{Equation}: (x-3)^2+[y+(-1)]^2&=2^2\\ (x-3)^2+(y+1)^2&=4\end{aligned}$$

2. Find the equation of the circle with centre $(-2,9)$ and passing through the point $(1,5)$.

   Sol.

$$\begin{aligned}r&=\sqrt{(1-(-2))^2+(5-9)^2}\\ &=\sqrt{9+16}\\ &=\sqrt{25}\\ &=5\end{aligned}$$

$$\begin{aligned}\therefore \text{Equation}: [x-(-2)]^2+(y-9)^2&=5^2\\ (x+2)^2+(y-9)^2&=25\end{aligned}$$

16.1.2 Exercise 16.1

1. Find the equation of the circle with centre at the origin and radius 7.

   Sol.

$$\begin{aligned}\text{Equation}: x^2+y^2&=7^2\\ x^2+y^2&=49\end{aligned}$$

2. Find the equation of circle of each of the following description:

   (a) Passing through the points $(5,-3)$ and centre at $(2,1)$.

   Sol.

$$\begin{aligned}r&=\sqrt{(5-2)^2+(-3-1)^2}\\ &=\sqrt{9+16}\\ &=\sqrt{25}\\ &=5\end{aligned}$$

$$\begin{aligned}\therefore \text{Equation}: (x-2)^2+(y-1)^2&=5^2\\ (x-2)^2+(y-1)^2&=25\end{aligned}$$

   (b) Centre at $(3,2)$ and radius 4.

   Sol.

$$\begin{aligned}\text{Equation}: (x-3)^2+(y-2)^2&=4^2\\ (x-3)^2+(y-2)^2&=16\end{aligned}$$

   (c) Centre at (a,b) and radius $a+b$.

   Sol.

$$\text{Equation}: (x-a)^2+(y-b)^2=(a+b)^2$$

3. Given that the coordinates of two points on the end of the diameter of a circle are $(5,-3)$ and $(3,1)$, find the equation of the circle.

Sol.

$$\begin{aligned}C &= \left(\frac{5+3}{2}, \frac{-3+1}{2}\right)\\ &= \left(\frac{8}{2}, \frac{-2}{2}\right)\\ &= (4, -1)\end{aligned}$$

$$\begin{aligned}r &= \sqrt{(5-4)^2 + (-3-(-1))^2}\\ &= \sqrt{1+4}\\ &= \sqrt{5}\end{aligned}$$

$$\begin{aligned}\therefore \text{Equation}: (x-4)^2 + [y-(-1)]^2 &= (\sqrt{5})^2\\ (x-4)^2 + (y+1)^2 &= 5\end{aligned}$$

4. Find the equation of the circle with a diameter connected by the points $(-3, 4)$ and $(9, 2)$.

   Sol.

$$\begin{aligned}C &= \left(\frac{-3+9}{2}, \frac{4+2}{2}\right)\\ &= \left(\frac{6}{2}, \frac{6}{2}\right)\\ &= (3, 3)\end{aligned}$$

$$\begin{aligned}r &= \sqrt{(-3-3)^2 + (4-3)^2}\\ &= \sqrt{36+1}\\ &= \sqrt{37}\end{aligned}$$

$$\begin{aligned}\therefore \text{Equation}: (x-3)^2 + (y-3)^2 &= (\sqrt{37})^2\\ (x-3)^2 + (y-3)^2 &= 37\end{aligned}$$

5. Given two points $P(-2, 2)$ and $Q(4, 6)$, find the equation of the circle wuth line PQ as its diameter.

   Sol.

$$\begin{aligned}C &= \left(\frac{-2+4}{2}, \frac{2+6}{2}\right)\\ &= \left(\frac{2}{2}, \frac{8}{2}\right)\\ &= (1, 4)\end{aligned}$$

$$\begin{aligned}r &= \sqrt{(-2-1)^2 + (2-4)^2}\\ &= \sqrt{9+4}\\ &= \sqrt{13}\end{aligned}$$

$$\begin{aligned}\therefore \text{Equation}: (x-1)^2 + (y-4)^2 &= (\sqrt{13})^2\\ (x-1)^2 + (y-4)^2 &= 13\end{aligned}$$

6. Turn the equation $x^2 + y^2 - 6x + 12y + 41 = 0$ into the standard form, and find the centre and radius of the circle.

   Sol.

$$\begin{aligned}x^2 + y^2 - 6x + 12y + 41 &= 0\\ x^2 + y^2 - 6x + 12y &= -41\\ (x^2 - 6x + 9) - 9 + (y^2 + 12y + 36) - 36 &= -41\\ (x-3)^2 + (y-6)^2 &= 4\end{aligned}$$

$$\therefore \text{Centre}: (3, 6),\ \text{Radius}: 2$$

16.2 General Equation of a Circle

Expand the standard equation of a circle, we get

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$$

Let $g = -h$, $f = -k$, $c = h^2 + k^2 - r^2$, we get the general equation of a circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

From $c = h^2 + k^2 - r^2$, we have $r^2 = h^2 + k^2 - c$

$$\begin{aligned}r &= \sqrt{h^2 + k^2 - c}\\ &= \sqrt{(-g)^2 + (-f)^2 - c}\\ &= \sqrt{g^2 + f^2 - c}\end{aligned}$$

Thus,

1. When $g^2 + f^2 - c > 0$, the image is a real circle with centre (g, f) and radius $\sqrt{g^2 + f^2 - c}$.
2. When $g^2 + f^2 - c = 0$, the image is point (g, f).
3. When $g^2 + f^2 - c < 0$, the image does not exist.

16.2.1 Practice 2

1. Find the centre and radius of the circle with equation $x^2 + y^2 - 6x - 8y + 21 = 0$.

   Sol.

$$\begin{aligned}x^2 + y^2 - 6x - 8y + 21 &= 0\\ \therefore 2g = -6,\ 2f = -8,\ c &= 21\\ g = -3,\ f = -4,\ c &= 21\end{aligned}$$

$$\therefore C = (3, 4)$$

$$\begin{aligned}r &= \sqrt{(-3)^2 + (-4)^2 - 21}\\ &= \sqrt{9 + 16 - 21}\\ &= \sqrt{4}\\ &= 2\end{aligned}$$

2. Find the equaton of the circle that passes through the following points:

(a) $A(0,0)$, $B(2,0)$, $C(0,-3)$.

Sol.

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases} 0+0+0g+0f+c=0 \\ 4+0+4g+0f+c=0 \\ 0+9+0g-6f+c=0 \end{cases}$$

$$\begin{cases} c=0 \\ 4+4g+c=0 \\ 9-6f+c=0 \end{cases}$$

$$\begin{cases} c=0 \\ 4+4g=0 \\ 9-6f=0 \end{cases}$$

$$\begin{cases} c=0 \\ 4g=-4 \\ -6f=-9 \end{cases}$$

$$\begin{cases} c=0 \\ g=-1 \\ f=\frac{3}{2} \end{cases}$$

$$\begin{aligned} \therefore \text{Equation}: x^2+y^2+2(-1)+2(\frac{3}{2})+0&=0 \\ x^2+y^2-2+3&=0 \\ x^2+y^2+1&=0 \end{aligned}$$

(b) $K(0,3)$, $L(1,2)$, $M(2,-1)$.

Sol.

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases} 0+9+0g+6f+c=0 \\ 1+4+2g+4f+c=0 \\ 4+1+4g-2f+c=0 \end{cases}$$

$$\begin{cases} 6f+c=-9 \\ 2g+4f+c=-5 \\ 4g-2f+c=-5 \end{cases}$$

$$g=3,\ f=1,\ c=-15$$

$$\begin{aligned} \therefore \text{Eq.}: x^2+y^2+2(3)x+2(1)y+(-15)&=0 \\ x^2+y^2+6x+2y-15&=0 \end{aligned}$$

3. Given that the vertices of ΔABC are $(1,2)$, $(2,5)$ and $(-1,2)$, find the equation of the circumcircle of ΔABC.

Sol.

Let the equation of the circumcircle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases} 1+4+2g+4f+c=0 \\ 4+25+8g+10f+c=0 \\ 1+4-2g+4f+c=0 \end{cases}$$

$$\begin{cases} 2g+4f+c=-5 \\ 8g+10f+c=-29 \\ -2g+4f+c=-5 \end{cases}$$

$$g=0,\ f=-4,\ c=11$$

$$\begin{aligned} \therefore \text{Eq.}: x^2+y^2+2(0)x+2(-4)y+11&=0 \\ x^2+y^2-8y+11&=0 \end{aligned}$$

16.2.2 Exercise 16.2

1. Find the centre and radius of the circle with the following equation:

(a) $x^2+y^2-64=0$

Sol.

$$\begin{aligned} x^2+y^2-64&=0 \\ 2g=0,\ 2f=0,\ c&=-64 \\ g=0,\ f=0,\ c&=-64 \end{aligned}$$

$$\begin{aligned} \therefore C&=(0,0) \\ r&=\sqrt{0^2+0^2-(-64)} \\ &=8 \end{aligned}$$

(b) $x^2+y^2-4x-8y=44$

Sol.

$$\begin{aligned} x^2+y^2-4x-8y&=44 \\ x^2+y^2-4x-8y-44&=0 \\ 2g=-4,\ 2f=-8,\ c&=-44 \\ g=-2,\ f=-4,\ c&=-44 \end{aligned}$$

$$\begin{aligned} \therefore C&=(2,4) \\ r&=\sqrt{(-2)^2+(-4)^2-(-44)} \\ &=\sqrt{4+16+44} \\ &=\sqrt{64} \\ &=8 \end{aligned}$$

(c) $x^2+y^2-8x=0$

Sol.

$$\begin{aligned} x^2+y^2-8x&=0\\ 2g=-8,\ 2f=0,\ c&=0\\ g=-4,\ f=0,\ c&=0 \end{aligned}$$

$$\begin{aligned} \therefore C&=(4,0)\\ r&=\sqrt{(-4)^2+0^2-0}\\ &=4 \end{aligned}$$

(d) $9x^2+9y^2+2x-6y-6=0$
Sol.

$$\begin{aligned} 9x^2+9y^2+2x-6y-6&=0\\ x^2+y^2+\frac{2}{9}x-\frac{2}{3}y-\frac{2}{3}&=0\\ 2g=\frac{2}{9},\ 2f=-\frac{2}{3},\ c&=-\frac{2}{3}\\ g=\frac{1}{9},\ f=-\frac{1}{3},\ c&=-\frac{2}{3} \end{aligned}$$

$$\begin{aligned} \therefore C&=\left(-\frac{1}{9},\frac{1}{3}\right)\\ r&=\sqrt{\left(\frac{1}{9}\right)^2+\left(-\frac{1}{3}\right)^2-\left(-\frac{2}{3}\right)}\\ &=\sqrt{\frac{64}{81}}\\ &=\frac{8}{9} \end{aligned}$$

2. Find the equation of the circle that passes through the following points:

(a) $A(1,1)$, $B(1,-1)$, $C(-2,1)$
Sol. Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases} 1+1+2g+2f+c=0\\ 1+1+2g-2f+c=0\\ 4+1-4g+2f+c=0 \end{cases}$$

$$\begin{cases} 2g+2f+c=-2\\ 2g-2f+c=-2\\ -4g+2f+c=-5 \end{cases}$$

$$g=\frac{1}{2},\ f=0,\ c=-3$$

$$\begin{aligned} \therefore \text{Eq}:\ x^2+y^2+2\left(\frac{1}{2}\right)x+2(0)y+(-3)&=0\\ x^2+y^2+x-3&=0 \end{aligned}$$

(b) $F(0,0)$, $G(3,-3)$, $H(-1,0)$

Sol. Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases} 0+0+0g+0f+c=0\\ 9+9+6g-6f+c=0\\ 1+0-2g+0f+c=0 \end{cases}$$

$$\begin{cases} c=0\\ 6g-6f=-18\\ -2g=-1 \end{cases}$$

$$\begin{cases} c=0\\ g-f=-3\\ g=\frac{1}{2} \end{cases}$$

$$\begin{cases} c=0\\ f=\frac{7}{2}\\ g=\frac{1}{2} \end{cases}$$

$$\begin{aligned} \therefore \text{Eq}:\ x^2+y^2+2\left(\frac{1}{2}\right)x+2\left(\frac{7}{2}\right)y+0&=0\\ x^2+y^2+x+7y&=0 \end{aligned}$$

(c) $P(1,0)$, $Q(0,-3)$, $R(3,4)$

Sol. Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases} 1+0+2g+0f+c=0\\ 0+9+0g-6f+c=0\\ 9+16+6g+8f+c=0 \end{cases}$$

$$\begin{cases} 2g+c=-1\\ -6f+c=-9\\ 6g+8f+c=-25 \end{cases}$$

$$g=-26,\ f=10,\ c=51$$

$$\begin{aligned} \therefore \text{Eq}:\ x^2+y^2+2(-26)x+2(10)y+51&=0\\ x^2+y^2-52x+20y+51&=0 \end{aligned}$$

3. A circle passes through point $A(2,2)$ and $B(5,3)$ while intersecting the line $x+y=4$ at y-axis. Find the equation of the circle.

Sol.

$$\begin{aligned} x+y&=4\\ \text{When } x=0,\ y&=4\\ \therefore \text{Another point: } &C(0,4) \end{aligned}$$

Let the equation of the circle be $x^2+y^2+2gx+2fy+$

$c = 0$,

$$\begin{cases} 4+4+4g+4f+c=0 \\ 25+9+10g+6f+c=0 \\ 0+16+0g+8f+c=0 \end{cases}$$

$$\begin{cases} 4g+4f+c=-8 \\ 10g+6f+c=-34 \\ 8f+c=-16 \end{cases}$$

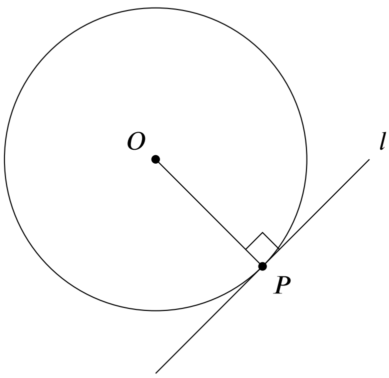
$$g=-\frac{11}{4},\ f=-\frac{19}{4}, c=22$$

$$\therefore \text{Eq}: x^2+y^2+2\left(-\frac{11}{4}\right)x+2\left(-\frac{19}{4}\right)y+22=0$$
$$x^2+y^2-\frac{11}{2}x-\frac{19}{2}y+22=0$$

16.3 Problems Related to Circles

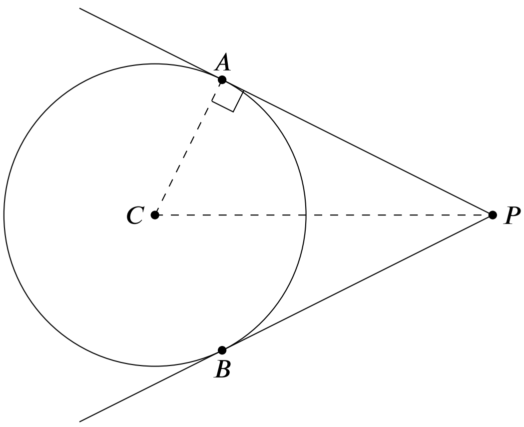
Tangent to a Circle

When a straight line l and a circle intersect at a point P, the line l is called a tangent to the circle, and the point P is called the point of contact. The tangent line is perpendicular to the radius at the point of contact. That is to say, when the length from the point of tangency to the centre of the circle is equal to the radius of the circle, the line is a tangent to the circle.



Length of a Tangent

According to the theorem of length of tangent, the lengths of tangents drawn from an external point to a circle are equal.



Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$, the external point P be (x_1, y_1). Connect PC and CA, $\angle CPA = 90°$, the coordinate of centre of the circle C be $(-g, -f)$.

$$\therefore CA = \sqrt{g^2+f^2-c},\ PC = \sqrt{(x_1+g)^2+(y_1+f)^2}$$

From the Pythagorean theorem,

$$\begin{aligned} PA^2 &= PC^2 - CA^2 \\ &= (x_1+g)^2+(y_1+f)^2-(g^2+f^2-c) \\ &= x_1^2+y_1^2+2gx_1+2fy_1+c \end{aligned}$$

Thus, the length of the tangent is given by

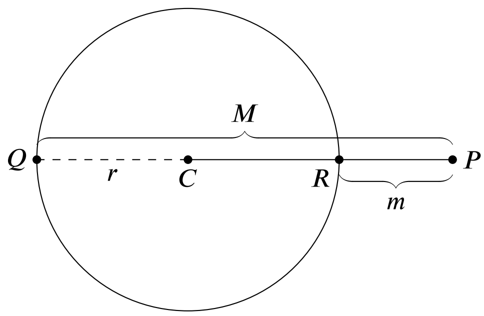
$$PA = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

Note that the coefficient of x_1 and y_1 in the above equation must be 1.

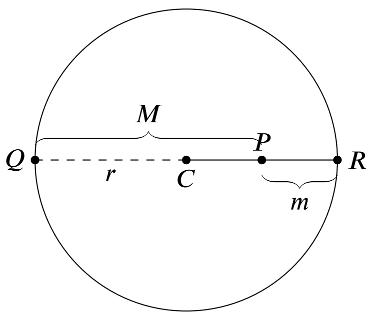
Maximum and Minimum Distance of a Point from a Circle

Given a circle with centre C and radius r and a point P anywhere on the plane,

When $PC > r$, point P is said to be outside the circle, the maximum distance of P from the circle is $M = PC + r$, and the minimum distance of P from the circle is $m = PC - r$.



When $PC < r$, point P is said to be inside the circle, the maximum distance of P from the circle is $M = PC + r$, and the minimum distance of P from the circle is $m = r - PC$.



16.3.1 Practice 3

1. Find the euqaiton of the circle with centre $(3, 4)$ and is tangent to the line $x + 2y - 6 = 0$.

Sol.

$$\begin{aligned} r &= \left| \frac{1(3)+2(4)-6}{\sqrt{1^2+2^2}} \right| \\ &= \left| \frac{3+8-6}{\sqrt{5}} \right| \\ &= \left| \frac{5}{\sqrt{5}} \right| \\ &= \frac{5\sqrt{5}}{5} \\ &= \sqrt{5} \end{aligned}$$

$$\begin{aligned} g = -3,\ f = -4,\ c &= (-3)^2 + (-4)^2 - (\sqrt{5})^2 \\ &= 9 + 16 - 5 \\ &= 20 \end{aligned}$$

$$\begin{aligned} \therefore \text{Eq} : x^2 + y^2 + 2(-3)x + 2(-4)y - 20 &= 0 \\ x^2 + y^2 - 6x - 8y - 20 &= 0 \end{aligned}$$

2. A circle passes through the points $(2, -3)$ and $(-2, -5)$, and its centre is on the line $x - 2y = 3$. Find the equation of the circle.

   Sol.

   Let the centre of the circle be $C(h, k)$, point (2, -3) be A and point (-2, -5) be B.

   $\because C$ is on the line $x - 2y = 3$
   $$h - 2k = 3 \qquad (1)$$

   $$\begin{aligned} CA &= CB \\ \sqrt{(2-h)^2 + (-3-k)^2} &= \sqrt{(-2-h)^2 + (-5-k)^2} \\ h^2 - 4h + 4 + k^2 + 6k + 9 &= h^2 + 4h + 4 + k^2 + 10k + 25 \\ -4h + 6k + 13 &= 4h + 10k + 29 \\ -8h - 4k &= 16 \\ 2h + k &= -4 \qquad (2) \end{aligned}$$

   Solving (1) and (2), $h = -1,\ k = -2$

   $$\begin{aligned} \therefore C = (-1, -2),\ r &= \sqrt{(2-(-1))^2 + (-3-(-2))^2} \\ &= \sqrt{3^2 + 1^2} \\ &= \sqrt{10} \end{aligned}$$

   $$\begin{aligned} g = 1,\ f = -2,\ c &= (-1)^2 + (-2)^2 - (\sqrt{10})^2 \\ &= 1 + 4 - 10 \\ &= -5 \end{aligned}$$

   $$\begin{aligned} \therefore \text{Eq} : x^2 + y^2 + 2(-1)x + 2(-2)y - 5 &= 0 \\ x^2 + y^2 - 2x - 4y - 5 &= 0 \end{aligned}$$

3. A circle with radius $\sqrt{5}$ are tangent with the line $x - 2y - 1 = 0$ at the point (3, 1). Find the equation of the circle.

   Sol.

   Let the centre of the circle be $C(h, k)$, point (3, 1) be P.

   $$\begin{aligned} x - 2y - 1 &= 0 \\ 2y &= x - 1 \\ y &= \frac{1}{2}x - \frac{1}{2} \\ m &= \frac{1}{2} \end{aligned}$$

   Let the line that passes through P and is perpendicular to $x - 2y - 1 = 0$ be l.

   $$\begin{aligned} m_l \times m &= -1 \\ m_l &= -2 \\ l : y - 1 &= -2(x - 3) \\ y - 1 &= -2x + 6 \\ y &= -2x + 7 \end{aligned}$$

   $\because C(h, k)$ is on the line l
   $$k = -2h + 7 \qquad (1)$$

   $$\begin{aligned} \sqrt{(3-h)^2 + (1-k)^2} &= \sqrt{5} \\ h^2 - 6h + 9 + k^2 - 2k + 1 &= 5 \qquad (2) \end{aligned}$$

   Sub (1) in (2),
   $$\begin{aligned} h^2 - 6h + 9 + (-2h+7)^2 - 2(-2h+7) + 1 &= 5 \\ h^2 - 6h + 9 + 4h^2 - 28h + 49 + 4h - 14 + 1 &= 5 \\ 5h^2 - 30h + 40 &= 0 \\ h^2 - 6h + 8 &= 0 \\ (h-4)(h-2) &= 0 \\ h = 4 \text{ or } h &= 2 \end{aligned}$$

   Sub $h = 4$ in (1), $k = -2(4) + 7 = -1$
   Sub $h = 2$ in (1), $k = -2(2) + 7 = 3$

   $\therefore C = (4, -1)$ or $C = (2, 3)$

   When $C = (4, -1)$,
   $$\begin{aligned} g = -4,\ f = 1,\ c &= 4^2 + (-1)^2 - (\sqrt{5})^2 \\ &= 16 + 1 - 5 \\ &= 12 \end{aligned}$$

$$\therefore \text{Eq}: x^2+y^2+2(-4)x+2(1)y+12=0$$
$$x^2+y^2-8x+2y+12=0$$

When $C=(2,3)$,

$$g=-2,\ f=-3,\ c=2^2+3^2-(\sqrt{5})^2$$
$$=4+9-5$$
$$=8$$

$$\therefore \text{Eq}: x^2+y^2+2(-2)x+2(-3)y+8=0$$
$$x^2+y^2-4x-6y+8=0$$

4. Prove the following lines are tangent to the following circles:

(a) $3x-y-5=0$, $x^2+y^2-16x+2y+25=0$

Proof.

$$C=(8,-1)$$
$$r=\sqrt{(-8)^2+1^2-25}=2\sqrt{10}$$
$$d=\left|\frac{3(8)-1(-1)-5}{\sqrt{3^2+(-1)^2}}\right|$$
$$=\left|\frac{20}{\sqrt{10}}\right|$$
$$=\frac{20\sqrt{10}}{10}$$
$$=2\sqrt{10}$$

$\because d=r$

$\therefore$ The line $3x-y-5=0$ is tangent to the circle $x^2+y^2-16x+2y+25=0$

(b) $2x-y-1=0$, $x^2+y^2+2x-4y=0$

Proof.

$$C=(-1,2)$$
$$r=\sqrt{1^2+(-2)^2}=\sqrt{5}$$
$$d=\left|\frac{2(-1)-1(2)-1}{\sqrt{2^2+(-1)^2}}\right|$$
$$=\left|\frac{-5}{\sqrt{5}}\right|$$
$$=\frac{5\sqrt{5}}{5}$$
$$=\sqrt{5}$$

$\because d=r$

$\therefore$ The line $2x-y-1=0$ is tangent to the circle $x^2+y^2+2x-4y=0$

5. Find the length of the tangent from the point $P(8,3)$ to the circle $x^2+y^2-8=0$.

Sol.

$$d=\sqrt{8^2+3^2+2(0)(8)+2(0)(3)-8}=\sqrt{65}$$

16.3.2 Exercise 16.3

1. Find the equation of the circle that passes through the points $(1,4)$ and $(0,-3)$, and its centre is on the line $x-2y=4$.

Sol.

Let the centre of the circle be $C(h,k)$, point (1, 4) be A and point (0, -3) be B.

$\because C$ is on the line $x-2y=4$

$$h-2k=4 \qquad (1)$$

$$CA=CB$$
$$\sqrt{(1-h)^2+(4-k)^2}=\sqrt{(0-h)^2+(-3-k)^2}$$
$$h^2-2h+1+k^2-8k+16=h^2+k^2+6k+9$$
$$-2h+-14k=-8$$
$$h+7k=4 \qquad (2)$$

Solving (1) and (2), $h=4,\ k=0$

$$\therefore C=(4,0),\ r=\sqrt{(1-4)^2+(4-0)^2}$$
$$=\sqrt{(-3)^2+4^2}$$
$$=\sqrt{25}$$
$$=5$$
$$g=-4,\ f=0,\ c=(-4)^2+0^2-5^2$$
$$=16-25$$
$$=-9$$

$$\therefore \text{Eq}: x^2+y^2+2(-4)x+2(0)y-9=0$$
$$x^2+y^2-8x-9=0$$

2. Find the equation of the circle that passes through the points $(3,2)$ and $(-4,-5)$, and its centre is on the line $3x+y+6=0$.

Sol.

Let the centre of the circle be $C(h,k)$, point (3, 2) be A and point (-4, -5) be B.

$\because C$ is on the line $3x+y+6=0$

$$3h+k+6=0 \qquad (1)$$

$$CA = CB$$

$$\sqrt{(3-h)^2 + (2-k)^2} = \sqrt{(-4-h)^2 + (-5-k)^2}$$

$$h^2 - 6h + 9 + k^2 = h^2 + 8h + 16 + k^2$$

$$-4k + 4 \quad + 10k + 25$$

$$-14h - 14k = 28$$

$$h + k = -2 \qquad (2)$$

Solving (1) and (2), $h = -2,\ k = 0$

$$\therefore C = (-2, 0),\ r = \sqrt{[3 - (-2)]^2 + (2-0)^2}$$

$$= \sqrt{5^2 + 2^2}$$

$$= \sqrt{29}$$

$$g = 2,\ f = 0,\ c = (-2)^2 + 0^2 - \sqrt{29}^2$$

$$= 4 - 29$$

$$= -25$$

$$\therefore \text{Eq} : x^2 + y^2 + 2(2)x + 2(0)y - 25 = 0$$

$$x^2 + y^2 + 4x - 25 = 0$$

3. Find the equation of the circle that passes through the points $A(5, 2)$ and $B(-3, 0)$, and its centre is on the y-axis.

   Sol.

   Let the centre of the circle be $C(0, k)$, point (5, 2) be A and point (-3, 0) be B.

$$CA = CB$$

$$\sqrt{(5-0)^2 + (2-k)^2} = \sqrt{(-3-0)^2 + (0-k)^2}$$

$$25 + k^2 - 4k + 4 = 9 + k^2$$

$$-4k = -20$$

$$k = 5$$

$$\therefore C = (0, 5),\ r = \sqrt{(5-0)^2 + (2-5)^2}$$

$$= \sqrt{36}$$

$$= 6$$

$$g = 0,\ f = -5,\ c = 0^2 + (-5)^2 - 6^2$$

$$= 25 - 36$$

$$= -9$$

$$\therefore \text{Eq} : x^2 + y^2 + 2(0)x + 2(-5)y - 9 = 0$$

$$x^2 + y^2 - 10y - 9 = 0$$

4. Find the equation of the circle with centre at the origin and is tangent to the line $3x - 4y + 20 = 0$.

   Sol.

$$r = \left| \frac{3(-0) - 4(-0) + 20}{\sqrt{3^2 + (-4)^2}} \right|$$

$$= \frac{20}{5}$$

$$= 4$$

$$g = 0,\ f = 0,\ c = 0^2 + 0^2 - 4^2$$

$$= -16$$

$$\therefore \text{Eq} : x^2 + y^2 + 2(0)x + 2(0)y - 16 = 0$$

$$x^2 + y^2 - 16 = 0$$

5. Find the equation of the circle with centre $A(-5, 4)$, and is tangent to the x-axis.

   Sol.

$$r = \left| \frac{0(-5) + 1(4) + 0}{\sqrt{0^2 + 1^2}} \right|$$

$$= \left| \frac{4}{1} \right|$$

$$= 4$$

$$g = 5,\ f = -4,\ c = 5^2 + (-4)^2 - 4^2$$

$$= 25$$

$$\therefore \text{Eq} : x^2 + y^2 + 2(5)x + 2(-4)y + 25 = 0$$

$$x^2 + y^2 + 10x - 8y + 25 = 0$$

6. Find the equation of the circle with centre $(-4, 2)$, and is tangent to the line $3x + 2y = 5$.

   Sol.

$$r = \left| \frac{3(-4) + 2(2) - 5}{\sqrt{3^2 + 2^2}} \right|$$

$$= \left| \frac{-13}{\sqrt{13}} \right|$$

$$= \frac{13\sqrt{13}}{13}$$

$$= \sqrt{13}$$

$$g = 4,\ f = -2,\ c = 4^2 + (-2)^2 - \sqrt{13}^2$$

$$= 16 + 4 - 13$$

$$= 7$$

$$\therefore \text{Eq} : x^2 + y^2 + 2(4)x + 2(-2)y + 7 = 0$$

$$x^2 + y^2 + 8x - 4y + 7 = 0$$

7. Find the equation of the circle that passes through the

point $(3, 0)$, and is tangent to the line $2x - 3y - 24 = 0$ at point $(3, -6)$.

Sol.

Let the centre of the circle be $C(h, k)$, point (3, -6) be P.

$$\begin{aligned} 2x - 3y - 24 &= 0 \\ 3y &= 2x - 24 \\ y &= \frac{2}{3}x - 8 \\ m &= \frac{2}{3} \end{aligned}$$

Let the line that passes through P and is perpendicular to $2x - 3y - 24 = 0$ be l.

$$\begin{aligned} m_l \times m &= -1 \\ m_l &= -\frac{3}{2} \\ l : y + 6 &= -\frac{3}{2}(x - 3) \\ 2y + 12 &= -3x + 9 \\ 3x + 2y &= -3 \end{aligned}$$

$\because C(h, k)$ is on the line l

$$3h + 2k = -3 \qquad (1)$$

$$\begin{aligned} \sqrt{(3-h)^2 + (0-k)^2} &= \sqrt{(3-h)^2 + (-6-k)^2} \\ h^2 - 6h + 9 + k^2 &= h^2 - 6h + 9 + k^2 + 12k + 36 \\ 12k + 36 &= 0 \\ k &= -3 \\ \text{Sub } k &= -3 \text{ into (1),} \\ 3h &= 3 \text{ into (1),} \\ h &= 1 \end{aligned}$$

$$\begin{aligned} \therefore C = (1, -3),\ r &= \sqrt{(3-1)^2 + [0 - (-3)]^2} \\ &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} g = -1,\ f = 3,\ c &= (-1)^2 + 3^2 - \sqrt{13}^2 \\ &= 1 + 9 - 13 \\ &= -3 \end{aligned}$$

$$\begin{aligned} \therefore \text{Eq} : x^2 + y^2 + 2(-1)x + 2(3)y - 3 &= 0 \\ x^2 + y^2 - 2x + 6y - 3 &= 0 \end{aligned}$$

8. Given a circle C_1 and another circle $C_2 : x^2 + y^2 - 4x - 6y + 8 = 0$ shares the same centre, and C_1 is tangent to the line $3x + 4y - 13 = 0$. Find the equation of the circle C_1.

Sol.

$$\begin{aligned} C_{C1} &= C_{C2} = (2, 3) \\ r_{C1} &= \left| \frac{3(2) + 4(3) - 13}{\sqrt{3^2 + 4^2}} \right| \\ &= \left| \frac{5}{\sqrt{25}} \right| \\ &= 1 \end{aligned}$$

$$\begin{aligned} g = -2,\ f = -3,\ c &= (-2)^2 + (-3)^2 - 1^2 \\ &= 4 + 9 - 1 \\ &= 12 \end{aligned}$$

$$\begin{aligned} \therefore \text{Eq} : x^2 + y^2 + 2(-2)x + 2(-3)y + 12 &= 0 \\ x^2 + y^2 - 4x - 6y + 12 &= 0 \end{aligned}$$

9. Prove the following lines are tangent to the following circles:

   (a) $6x + 5y - 31 = 0$, $x^2 + y^2 + 4x - 5y - 5 = 0$
   Sol.

$$\begin{aligned} C &= \left(-2, \frac{5}{2}\right) \\ r &= \sqrt{2^2 + \left(-\frac{5}{2}\right)^2 + 5} \\ &= \frac{\sqrt{61}}{2} \\ d &= \left| \frac{6(-2) + 5\left(\frac{5}{2}\right) - 31}{\sqrt{6^2 + 5^2}} \right| \\ &= \left| \frac{-12 + \frac{25}{2} - 31}{\sqrt{61}} \right| \\ &= \frac{61}{2\sqrt{61}} \\ &= \frac{61\sqrt{61}}{2 \times 61} \\ &= \frac{\sqrt{61}}{2} \end{aligned}$$

$\because d = r$
$\therefore$ The line $6x + 5y - 31 = 0$ is tangent to the circle $x^2 + y^2 + 4x - 5y - 5 = 0$

   (b) $3x + 1 = 0$, $9x^2 + 9y^2 + 3x + 6y + 1 = 0$
   Sol.

$$\begin{aligned} 9x^2 + 9y^2 + 3x + 6y + 1 &= 0 \\ x^2 + y^2 + \frac{1}{3}x + \frac{2}{3}y + \frac{1}{9} &= 0 \end{aligned}$$

$$C = \left(-\frac{1}{6}, -\frac{1}{3}\right)$$
$$\begin{aligned} r &= \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2 - \frac{1}{9}} \\ &= \frac{1}{6} \end{aligned}$$
$$\begin{aligned} d &= \left| \frac{3(-\frac{1}{6}) + 0\left(-\frac{1}{3}\right) + 1}{\sqrt{3^2 + 0^2}} \right| \\ &= \left| \frac{-\frac{1}{2} + 1}{3} \right| \\ &= \frac{1}{6} \end{aligned}$$

$\because d = r$

$\therefore$ The line $3x + 1 = 0$ is tangent to the circle $x^2 + y^2 + \frac{1}{3}x + \frac{2}{3}y + \frac{1}{9} = 0$

10. Find the length of the tangent from the following circles to the following circles:

(a) $(-2, 3)$, $x^2 + y^2 - 6x - 2y = 0$

Sol.

$$\begin{aligned} d &= \sqrt{(-2)^2 + (3)^2 + 2(-3)(-2) + 2(-1)(3) + 0} \\ &= \sqrt{4 + 9 + 12 - 6} \\ &= \sqrt{19} \end{aligned}$$

(b) $(-6, 0)$, $x^2 + y^2 - 6x + 2y + 8 = 0$

Sol.

$$\begin{aligned} d &= \sqrt{(-6)^2 + (0)^2 + 2(-6)(-3) + 2(0)(1) + 8} \\ &= \sqrt{36 + 0 + 36 + 0 + 8} \\ &= \sqrt{80} \\ &= 4\sqrt{5} \end{aligned}$$

(c) $(2, 2)$, $2x^2 + 2y^2 + 2x + 4y - 3 = 0$

Sol.

$$\begin{aligned} 2x^2 + 2y^2 + 2x + 4y - 3 &= 0 \\ x^2 + y^2 + x + 2y - \frac{3}{2} &= 0 \end{aligned}$$

$$\begin{aligned} d &= \sqrt{(2)^2 + (2)^2 + 2(2)\left(\frac{1}{2}\right) + 2(2)(1) - \frac{3}{2}} \\ &= \sqrt{4 + 4 + 2 + 4 - \frac{3}{2}} \\ &= \sqrt{\frac{25}{2}} \\ &= \frac{5\sqrt{2}}{2} \end{aligned}$$

11. If the following lines and circles are tengant to each other, find the value of k:

(a) $4x + 3y - k = 0$, $x^2 + y^2 - 6x + 4y - 12 = 0$

Sol.

$$C = (3, -2)$$
$$\begin{aligned} r &= \sqrt{(-3)^2 + 2^2 - (-12)} \\ &= \sqrt{9 + 4 + 12} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$$\begin{aligned} \left| \frac{4(3) + 3(-2) - k}{\sqrt{4^2 + 3^2}} \right| &= 5 \\ \frac{6 - k}{5} &= \pm 5 \\ 6 - k &= \pm 25 \\ k = 6 \pm 25 &= \pm 19 \\ k = 6 + 25 &\text{ or } k = 6 - 25 \\ k = 31 &\text{ or } k = -19 \end{aligned}$$

(b) $x + 3y + k = 0$, $2x^2 + 2y^2 + 12y + 13 = 0$

Sol.

$$\begin{aligned} 2x^2 + 2y^2 + 12y + 13 &= 0 \\ x^2 + y^2 + 6y + \frac{13}{2} &= 0 \end{aligned}$$

$$C = (0, -3)$$
$$\begin{aligned} r &= \sqrt{0^2 + 3^2 - \frac{13}{2}} \\ &= \sqrt{9 - \frac{13}{2}} \\ &= \sqrt{\frac{5}{2}} \end{aligned}$$

$$\left|\frac{1(0)+3(-3)+k}{\sqrt{1^2+3^2}}\right| = \sqrt{\frac{5}{2}}$$

$$\frac{k-9}{\sqrt{10}} = \pm\sqrt{\frac{5}{2}}$$

$$\frac{(k-9)^2}{10} = \frac{5}{2}$$

$$(k-9)^2 = 25$$

$$k-9 = \pm 5$$

$$k = 9 \pm 5$$

$$k = 9+5 \text{ or } k = 9-5$$

$$k = 14 \text{ or } k = 4$$

12. Find the maximum and minimum distance of the point $P(-2,5)$ from the circle $x^2+y^2-2x-2y+1=0$.

Sol.

$$\begin{aligned} C &= (1,1) \\ r &= \sqrt{(-1)^2+(-1)^2-1} \\ &= 1 \\ PC &= \sqrt{(-2-1)^2+(5-1)^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$\because PC > r$

$\therefore P$ is outside the circle

$d_{\max} = PC + r = 5+1 = 6$

$d_{\min} = PC - r = 5-1 = 4$

13. Find the maximum and minimum distance of the point $Q(0,1)$ from the circle $x^2+y^2-6x-10y-2=0$.

Sol.

$$\begin{aligned} C &= (3,5) \\ r &= \sqrt{(-3)^2+(-5)^2-(-2)} \\ &= \sqrt{9+25+2} \\ &= \sqrt{36} \\ &= 6 \\ QC &= \sqrt{(0-3)^2+(1-5)^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$\because PC < r$

$\therefore P$ is inside the circle

$d_{\max} = QC + r = 5+6 = 11$

$d_{\min} = r - QC = 6-5 = 1$

14. Assume that the maximum and minimum distance of the point $R(5,2)$ from the circle $x^2+y^2-4x+4y-1=0$ are M and N respectively, find the product of M and N.

Sol.

$$\begin{aligned} C &= (2,-2) \\ r &= \sqrt{(-2)^2+2^2-(-1)} \\ &= \sqrt{4+4+1} \\ &= \sqrt{9} \\ &= 3 \\ RC &= \sqrt{(5-2)^2+[2-(-2)]^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

$\because RC > r$

$\therefore R$ is outside the circle

$d_{\max} = RC + r = 5+3 = 8$

$d_{\min} = RC - r = 5-3 = 2$

$\therefore MN = 8 \times 2 = 16$

16.4 Revision Exercise 16

1. Find the euqation of the following circles:

(a) A circle with centre $(1,-1)$ and radius 3.

Sol.

$$g=-1,\ f=1,\ c=1^2+(-1)^2-3^2=-7$$

$$\begin{aligned}\therefore \text{Eq}:\ x^2+y^2+2(-1)x+2(-1)y-7&=0\\ x^2+y^2-2x+2y-7&=0\end{aligned}$$

(b) A circle with centre $(2,-3)$ and radius 7.
Sol.

$$g=-2,\ f=3,\ c=(-2)^2+3^2-7^2=-36$$

$$\begin{aligned}\therefore \text{Eq}:\ x^2+y^2+2(-2)x+2(3)y-36&=0\\ x^2+y^2-4x+6y-36&=0\end{aligned}$$

2. Find the equation of the circle with centre at the origin and passes through the point $(2,-1)$.
Sol.

$$r=\sqrt{(2-0)^2+(-1-0)^2}=\sqrt{5}$$
$$g=0,\ f=0,\ c=0^2+0^2-(\sqrt{5})^2=-5$$

$$\begin{aligned}\therefore \text{Eq}:\ x^2+y^22(0)x+2(0)y-5&=0\\ x^2+y^2-5&=0\end{aligned}$$

3. Find the equation of the circle with centre at $(2,3)$ and passes through the point $(-5,6)$.
Sol.

$$r=\sqrt{(2-(-5))^2+(3-6)^2}=\sqrt{58}$$
$$g=-2,\ f=-3,\ c=(-2)^2+(-3)^2-(\sqrt{58})^2=-45$$

$$\begin{aligned}\therefore \text{Eq}:\ x^2+y^2+2(-2)x+2(-3)y-45&=0\\ x^2+y^2-4x-6y-45&=0\end{aligned}$$

4. Find the equation of the circle with diameter connecting the points $(2,-5)$ and $(8,-1)$.
Sol.

$$\begin{aligned}M&=\left(\frac{2+8}{2},\frac{-5-1}{2}\right)\\ &=(5,-3)\end{aligned}$$
$$r=\sqrt{(5-2)^2+(-3-(-5))^2}=\sqrt{13}$$

$$g=-5,\ f=3,\ c=(-5)^2+3^2-(\sqrt{13})^2=21$$

$$\begin{aligned}\therefore \text{Eq}:\ x^2+y^2+2(-5)x+2(3)y+21&=0\\ x^2+y^2-10x+6y+21&=0\end{aligned}$$

5. Find the centre and radius of the following circle:

(a) $x^2+y^2-6x+14y+50=0$
Sol.

$$\begin{aligned}2g=-6,\ 2f=14,\ c=50\\ g=-3,\ f=7,\ c=50\end{aligned}$$

$$\begin{aligned}C&=(3,-7)\\ r&=\sqrt{(-3)^2+7^2-50}\\ &=\sqrt{8}\\ &=2\sqrt{2}\end{aligned}$$

(b) $x^2+y^2+5x-2y+1=0$
Sol.

$$\begin{aligned}2g=5,\ 2f=-2,\ c=1\\ g=\frac{5}{2},\ f=-1,\ c=1\end{aligned}$$

$$\begin{aligned}C&=\left(-\frac{5}{2},1\right)\\ r&=\sqrt{\frac{5}{2}^2+(-1)^2-1}\\ &=\frac{\sqrt{25}}{2}\end{aligned}$$

(c) $3x^2+3y^2+6x-12y+1=0$
Sol.

$$\begin{aligned}3x^2+3y^2+6x-12y+1&=0\\ x^2+y^2+2x-4y+\frac{1}{3}&=0\end{aligned}$$

$$\begin{aligned}2g=2,\ 2f=-4,\ c=\frac{1}{3}\\ g=1,\ f=-2,\ c=\frac{1}{3}\end{aligned}$$

$$\begin{aligned}C&=(-1,2)\\ r&=\sqrt{1^2+(-2)^2-\frac{1}{3}}\\ &=\sqrt{\frac{14}{3}}\\ &=\frac{\sqrt{14}\sqrt{3}}{3}\\ &=\frac{\sqrt{42}}{3}\end{aligned}$$

(d) $4x^2+4y^2-12x+16y-7=0$
Sol.

$$\begin{aligned}4x^2+4y^2-12x+16y-7&=0\\ x^2+y^2-3x+4y-\frac{7}{4}&=0\end{aligned}$$

$$2g=-3,\ 2f=4,\ c=-\frac{7}{4}$$
$$g=-\frac{3}{2},\ f=2,\ c=-\frac{7}{4}$$

$$C=\left(\frac{3}{2},-2\right)$$
$$r=\sqrt{(-\frac{3}{2})^2+2^2-(-\frac{7}{4})}$$
$$=\sqrt{8}$$
$$=2\sqrt{2}$$

6. Find the equation of the circle that passes through the following three points:

(a) $(-1,-1), (-3,5), (1,3)$
Sol. Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases}1+1-2g-2f+c=0\\9+25-6g+10f+c=0\\1+9+2g+6f+c=0\end{cases}$$
$$\begin{cases}2g+2f-c=2\\6g-10f-c=34\\2g+6f+c=-10\end{cases}$$

$$g=2,\ f=-2,\ c=-2$$

$$\begin{aligned}\text{Eq}:\ x^2+y^2+2(2)x+2(-2)y-2&=0\\x^2+y^2+4x-4y-2&=0\end{aligned}$$

(b) $(2,1), (2,-4), (3,-5)$
Sol. Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases}4+1+4g+2f+c=0\\4+16+4g-8f+c=0\\9+25+6g-10f+c=0\end{cases}$$
$$\begin{cases}4g+2f+c=-5\\4g-8f+c=-20\\6g-10f+c=-34\end{cases}$$

$$g=-\frac{11}{2}, f=\frac{3}{2}, c=14$$

$$\begin{aligned}\text{Eq}:\ x^2+y^2-2\frac{11}{2}x+\frac{3}{2}y+14&=0\\x^2+y^2-11x+\frac{3}{2}y+14&=0\end{aligned}$$

(c) $(0,0), (0,a), (b,0)$
Sol. Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases}0+0+0g+0f+c=0\\0+a^2+0g+2af+c=0\\b^2+0+2bg+0f+c=0\end{cases}$$
$$\begin{cases}c=0\\2af=-a^2\\2bg=-b^2\end{cases}$$
$$\begin{cases}c=0\\f=-\frac{a}{2}\\g=-\frac{b}{2}\end{cases}$$

$$\begin{aligned}\text{Eq}:\ x^2+y^2-2\left(\frac{b}{2}\right)x-2\left(\frac{a}{2}\right)y+0&=0\\x^2+y^2-bx-ay&=0\end{aligned}$$

7. Given the radius of of the circle $x^2+y^2-8x+10y+c$ is 9, find the value of c.

Sol.

$$g=-4,\ f=5,\ c=(-4)^2+5^2-9^2=-40$$

8. Given two circles $x^2+y^2-2x-4y-95=0$ and $x^2+y^2-8x-12y+48=0$, find the distance between their centres.

Sol.

$$C_1=(1,2)$$
$$C_2=(4,6)$$

$$d=\sqrt{(4-1)^2+(6-2)^2}$$
$$=\sqrt{25}$$
$$=5$$

9. Find the equation of the circle with centre at $(1,-1)$ and is tangent to the line $5x-12y+9=0$.

Sol.

$$r=\left|\frac{5(1)-12(-1)+9}{\sqrt{5^2+12^2}}\right|$$
$$=\left|\frac{5+12+9}{\sqrt{169}}\right|$$
$$=\left|\frac{26}{13}\right|$$
$$=2$$
$$g=-1,\ f=1,\ c=(-1)^2+1^2-2^2=-2$$

$$\begin{aligned}\therefore \text{Eq}:\ x^2+y^2+2(-1)x+2(1)y-2&=0\\x^2+y^2-2x+2y-2&=0\end{aligned}$$

10. Find the equation of the circle that passes through the points $(1,-1)$ and $(1,1)$, and is tangent to the line $x-2=0$.

Sol.

Let the centre of the circle be $C(h,k)$,

$$\sqrt{(1-h)^2+(-1-k)^2} = \sqrt{(1-h)^2+(1-k)^2}$$
$$h^2-2h+1+k^2+2k+1 = h^2-2h+1+k^2-2k+1$$
$$k = 0$$

$$\left|\frac{1(h)+0(0)-2}{\sqrt{1^2+0^2}}\right| = \sqrt{(1-h)^2+(1-k)^2}$$
$$(h-2)^2 = h^2-2h+1+k^2-2k+1$$
$$h^2-4h+4 = h^2-2h+2$$
$$-2h = -2$$
$$h = 1$$

$C = (1,0)$

$r = \sqrt{(1-1)^2+(1-0)^2} = 1$

$g=-1,\ f=0,\ c=(-1)^2+0^2-1^2=0$

$$\therefore \text{Eq}:\ x^2+y^2+2(-1)x+2(0)y+0=0$$
$$x^2+y^2-2x+0=0$$

11. Find the equation of the circle that passes through the points $(6,-4)$ and $(1,7)$, and its centre is on the line $2x-3y=6$.

Sol.

Let the centre of the circle be $C(h,k)$, point (6, -4) be A and point (1, 7) be B.

$\because C$ is on the line $2x-3y=6$

$$2h-3k=6 \qquad (1)$$

$$\sqrt{(6-h)^2+(-4-k)^2} = \sqrt{(1-h)^2+(7-k)^2}$$
$$h^2-12h+36+k^2+8k+16 = h^2-2h+1+k^2-14k+49$$
$$8k-12h+52 = -2h-14k+50$$
$$-10h+22k=-2$$
$$5h-11k=1 \qquad (2)$$

Solving (1) and (2), $h=9,\ k=4$

$C=(9,4)$

$r=\sqrt{(6-9)^2+(-4-4)^2}=\sqrt{73}$

$g=-9,\ f=-4,\ c=(-9)^2+(-4)^2-(\sqrt{73})^2=24$

$$\therefore \text{Eq}:\ x^2+y^2+2(-9)x+2(-4)y+24=0$$
$$x^2+y^2-18x-8y+24=0$$

12. Find the equation of the circle that passes through the points $(-1,1)$ and $(1,3)$, and its centre is on x-axis.

Sol.

Let the centre of the circle be $C(h,0)$, point (-1, 1) be A and point (1, 3) be B.

$$\sqrt{(-1-h)^2+(1-0)^2} = \sqrt{(1-h)^2+(3-0)^2}$$
$$h^2+2h+1+1 = h^2-2h+1+9$$
$$4h=8$$
$$h=2$$

$C=(2,0)$

$r=\sqrt{(-1-2)^2+(1-0)^2}=\sqrt{10}$

$g=-2,\ f=0,\ c=(-2)^2+0^2-(\sqrt{10})^2=-6$

$$\therefore \text{Eq}:\ x^2+y^2+2(-2)x+2(0)y-6=0$$
$$x^2+y^2-4x-6=0$$

13. Find the equation of the circle that is tangent to the line $3x+4y+18=0$ at point $(-2,-3)$, and its centre is on the line $x-y=0$.

Sol.

Let the centre of the circle be $C(h,k)$,

$\because C$ is on the line $x-y=0$

$$h-k=0$$
$$h=k \qquad (1)$$

$$\left|\frac{3h+4k+18}{\sqrt{3^2+4^2}}\right| = \sqrt{(-2-h)^2+(-3-k)^2}$$

Sub (1),

$$\left|\frac{3h+4h+18}{5}\right| = \sqrt{(-2-h)^2+(-3-h)^2}$$
$$(7h+18)^2 = 25(h^2+4h+4+h^2+6h+9)$$
$$49h^2+252h+324 = 25(2h^2+10h+13)$$
$$49h^2+252h+324 = 50h^2+250h+325$$
$$h^2-2h+1=0$$
$$(h-1)^2=0$$
$$h=1$$
$$k=h=1$$

$C=(1,1)$

$r=\sqrt{(-2-1)^2+(-3-1)^2}=5$

$g=-1,\ f=-1,\ c=(-1)^2+(-1)^2-5^2=-23$

$$\therefore \text{Eq}: x^2+y^2+2(-1)x+2(-1)y-23=0$$
$$x^2+y^2-2x-2y-23=0$$

14. If the following lines and circles are tangent to each other, find the value of k:

    (a) $2x-y+k=0$, $x^2+y^2-1=0$

    Sol.

$$C=(0,0)$$
$$r=1$$

$$\left|\frac{2(0)-1(0)+k}{\sqrt{2^2+(-1)^2}}\right|=1$$
$$\frac{k^2}{5}=1$$
$$k=\pm\sqrt{5}$$

    (b) $2x+3y+3\sqrt{13}=0$, $x^2+y^2=k$

    Sol.

$$C=(0,0)$$
$$r=\sqrt{k}$$

$$\frac{2(0)+3(0)+3\sqrt{13}}{\sqrt{2^2+3^2}}=\sqrt{k}$$
$$\frac{117}{13}=k$$
$$k=9$$

    (c) $y=x+k$, $x^2+y^2=9$

    Sol.

$$C=(0,0)$$
$$r=3$$

$$\frac{1(0)-1(0)+k}{\sqrt{1^2+(-1)^2}}=3$$
$$\frac{k^2}{2}=9$$
$$k^2=18$$
$$k=\pm3\sqrt{2}$$

15. If the circle $x^2+y^2-6y-4y+k=0$ is tangent to the x-axis, find the value of k and the coordinates of the point of tangency.

    Sol.

Let the point of tangency be $P(h,0)$,

$$C=(3,2)$$
$$\left|\frac{0(3)+1(2)+0}{\sqrt{0^2+1^2}}\right|=\sqrt{13-k}$$
$$4=13-k$$
$$k=9$$

$$r=\sqrt{(-3)^2+(-2)^2-k}$$
$$=\sqrt{13-9}$$
$$=2$$

$$\sqrt{(3-h)^2+(2-0)^2}=2$$
$$h^2-6h+9+4=4$$
$$(h-3)^2=0$$
$$h=3$$
$$\therefore P=(3,0)$$

16. Given the coordinates and equations of the following points and circles respectively, find the length of the tangent from the point to the circle:

    (a) $(1,6)$, $x^2+y^2+2x-19=0$

    Sol.

$$d=\sqrt{1^2+6^2+2(1)(1)+2(0)(6)+(-19)}$$
$$=\sqrt{1+36+2+0-19}$$
$$=\sqrt{20}$$
$$=2\sqrt{5}$$

    (b) $(2,4)$, $x^2+y^2-2x+6y+9=0$

    Sol.

$$d=\sqrt{2^2+4^2+2(-1)(2)+2(3)(4)+9}$$
$$=\sqrt{4+16-4+24+9}$$
$$=\sqrt{49}$$
$$=7$$

    (c) $(3,2)$, $2x^2+2y^2+10x+11y-52=0$

    Sol.

$$2x^2+2y^2+10x+11y-52=0$$
$$x^2+y^2+5x+\frac{11}{2}y-26=0$$

$$d = \sqrt{3^2 + 2^2 + 2\left(\frac{5}{2}\right)(3) + 2\left(\frac{11}{4}\right)(2) - 26}$$
$$= \sqrt{9 + 4 + 15 + 11 - 26}$$
$$= \sqrt{13}$$

(d) $(0, 0)$, $x^2 + y^2 - 2ax + 4ay + 4a^2 = 0$

Sol.

$$d = \sqrt{0^2 + 0^2 + 2(-a)(0) + 2(2a)(0) + 4a^2}$$
$$= \sqrt{4a^2}$$
$$= 2|a| \text{ since } a \text{ must } >= 0$$

17. Prove that the distance of the tangent from the point $A(3, -4)$ to the circle $C_1 : x^2 + y^2 - 10x - 7y + 13 = 0$ is equal to the distance of the tangent to the circle $C_2 : 2x^2 + 2y^2 - 3x - 12y - 17 = 0$.

Sol.

$$d_1 = \sqrt{3^2 + (-4)^2 + 2(-5)(3) + 2(-\frac{7}{2})(-4) + 13}$$
$$= \sqrt{9 + 16 - 30 + 28 + 13}$$
$$= \sqrt{36}$$
$$= 6$$
$$C_2 : 2x^2 + 2y^2 - 3x - 12y - 17 = 0$$
$$x^2 + y^2 - \frac{3}{2}x - 6y - \frac{17}{2} = 0$$

$$d_2 = \sqrt{3^2 + (-4)^2 + 2(-\frac{3}{4})(3) + 2(-3)(-4) - \frac{17}{2}}$$
$$= \sqrt{9 + 16 - \frac{9}{2} + 24 - \frac{17}{2}}$$
$$= \sqrt{36}$$
$$= 6$$

$$\therefore d_1 = d_2$$

18. Prove that line $Y = 2x$ is tangent to the circle $x^2 + y^2 + 16x + 12y + 80 = 0$, and find the coordinate of its point of tangency.

19. Find the longest and the shortest distance of the point $P(-5, -12)$ to the circle $x^2 + y^2 - 25 = 0$.

Sol.

$$C = (0, 0)$$
$$r = 5$$
$$PC = \sqrt{(-5 - 0)^2 + (-12 - 0)^2}$$
$$= \sqrt{169}$$
$$= 13$$

$\because PC > r$
$\therefore P$ is outside the circle
$$d_{\max} = PC + r = 13 + 5 = 18$$
$$d_{\min} = PC - r = 13 - 5 = 8$$

20. Given the equation of circle $x^2 + y^2 + 8y - 6y = 0$.

(a) Find the centre and radius of the circle.

Sol.

$$2g = 8,\ 2f = -6,\ c = 0$$
$$g = 4,\ f = -3,\ c = 0$$

$$\therefore C = (-4, 3)$$
$$r = \sqrt{4^2 + (-3)^2 - 0}$$
$$= 5$$

(b) Prove that $P(-2, 7)$ is inside the circle.

Sol.

$$PC = \sqrt{(-2 - (-4))^2 + (7 - 3)^2}$$
$$= \sqrt{4 + 16}$$
$$= \sqrt{20}$$
$$= 2\sqrt{5}$$

$\because PC < r$
$\therefore P$ is inside the circle

(c) Find the equation of chord of the circle that is split into two equal parts by the point $P(-2, 7)$.

Sol.

Let the equation of the chord be l.

$\because l$ is split into two equal parts by P
$\therefore l$ is perpendicular to PC

$$
\begin{aligned}
m_{PC} &= \frac{7-3}{-2-(-4)} \\
&= 2 \\
M_l M_{PC} &= -1 \\
M_l &= -\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\therefore l \,:\, y-7 &= -\frac{1}{2}(x+2) \\
2y-14 &= -x-2 \\
x+2y-12 &= 0
\end{aligned}
$$

Chapter 17

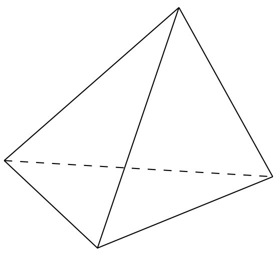
Solid Geometry, Longitude and Latitude

17.1 Solid Geometry

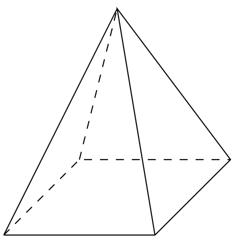
Polyhedron

A polyhedron is a solid bounded by a finite amount of flat polygon, and each side of the polygons must be the common edge of two polygons. Polyhedron can be classified into tetrahedron, pentahedron, hexahedron, etc. based on the number of flat surfaces, aka the *faces* of the polyhedron. The common side of two faces of a polyhedron is called an edge, and the common vertex of three edges is called an *apex*.

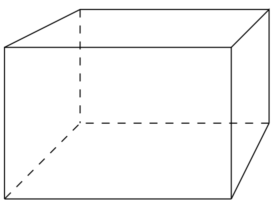
Besides, the angles formed by the faces intersecting at the same apex are called *polyhedral angles* or *solid angles*. The line segment connecting two apexes at different faces is called a *diagonal*.



Tetrahedron



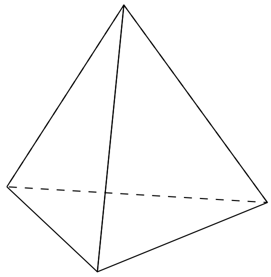
Pentahedron



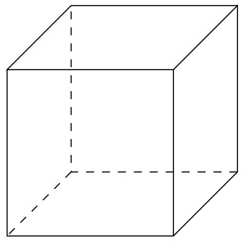
Hexahedron

Regular Polyhedron

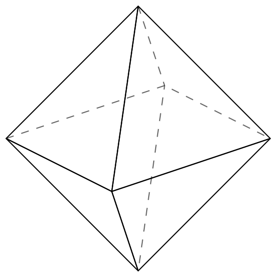
A *regular polyhedron* is a polyhedron with all faces being regular polygons, and all polyhedral angles being equal. The regular polyhedron can be classified into 5 types: *regular tetrahedron*, *regular octahedron*, *regular hexahedron*, *regular dodecahedron* and *regular icosahedron*.



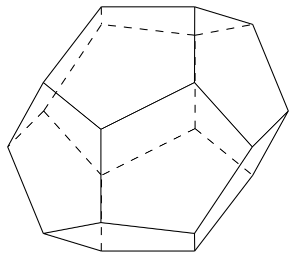
Regular Tetrahedron



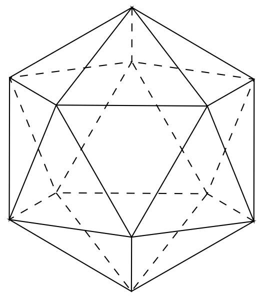
Regular Hexahedron



Regular Octahedron



Regular Dodecahedron

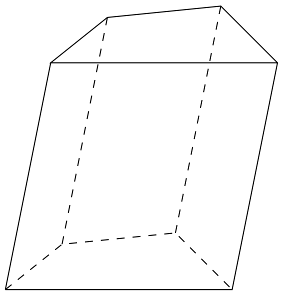


Regular Icosahedron

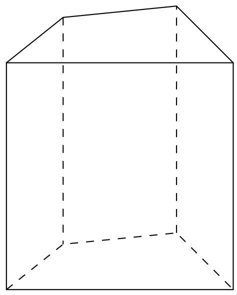
Prism

If two faces of a polyhedron are parallel, while the other faces intersect in sequence to form parallel lines, then the polyhedron is called a *prism*. The two faces which are parallel to each other are called the *bases of the prism*, and the other faces are called the *lateral faces of the prism*. The common sides that two adjacent lateral faces share is called the *lateral edges of the prism*. The distance between two bases is called the *height of the prism*.

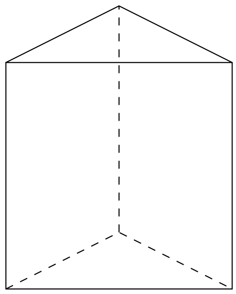
Prism with lateral edges that aren't parallel to each other are called *oblique prism*; prism with lateral edges that are parallel to each other are called *right prism*; regular prism with regular bases are called *regular prism*.



Oblique Prism

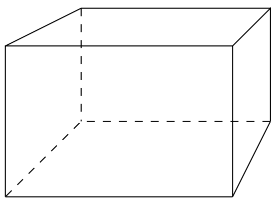


Right Prism

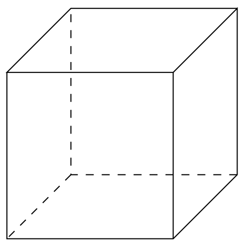


Regular Prism

Prism with bases of parallelogram are called *parallelepiped*. Parallelepiped with lateral edges that are parallel to each other are called *right parallelepiped*. Right parallelepiped with regular bases are called *cuboid*, and a cuboid with equal width, height, and depth is called a *cube*.



Cuboid

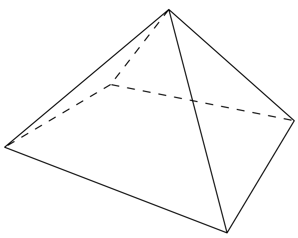


Cube

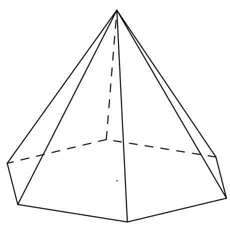
Pyramid

If a polyhedron has a polygonal base and all its lateral faces are triangles that shares a common apex, then the polyhedron is called a *pyramid*.

If the foot point of a pyramid is the centre of its base, then the pyramid is called a *right pyramid*. If the base of a right pyramid is a regular polygon, then the pyramid is called a *regular pyramid*.



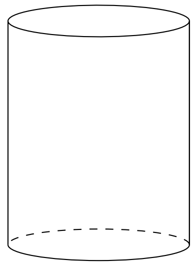
Right Pyramid



Regular Pyramid

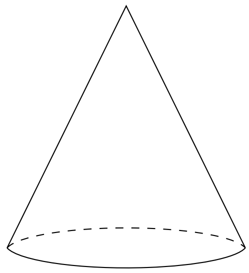
Right Circular Cylinder

A *right circular cylinder* is the solid of revolution generated by rotating a rectangle about one of its sides.



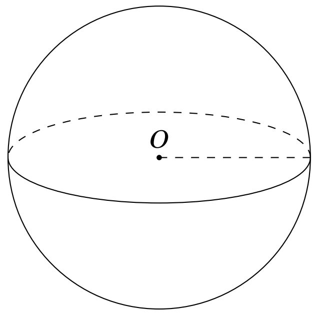
Right Circular Cone

A *right circular cone* is the solid of revolution generated by rotating a right-angled triangle about one of its sides.



Sphere

The surface of revolution generated by rotating a semicircle about its diameter is called a *spherical surface*, and the solid covered by it is called a *sphere*.

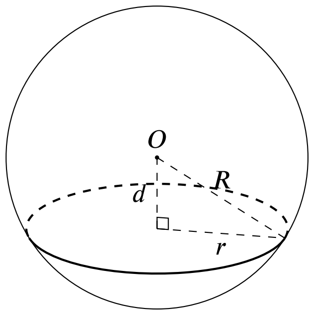


If the circle is cut with a plane, the plane has the following properties:

1. The line joining the centre of the sphere to the centre of the plane are perpendicular to the plane.

2. The distance of the plane from the centre of the sphere d, the radius of the sphere R and the radius of the plane r has the following relation:

$$r = \sqrt{R^2 - d^2}$$

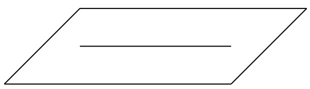


The circle cut by a plane passing through the centre of the sphere is called a *great circle*; the circle cut by a plane that does not pass through the centre of the sphere is called a *small circle*.

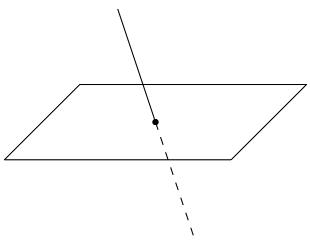
17.2 Angle Formed by Planes and Straight Lines

There are three types of positional relationship between a plane and a straight line:

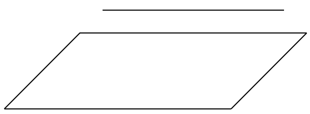
1. The line is on the plane



2. The line only intersects the plane at one point



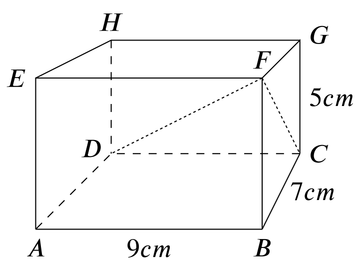
3. The line does not intersect the plane



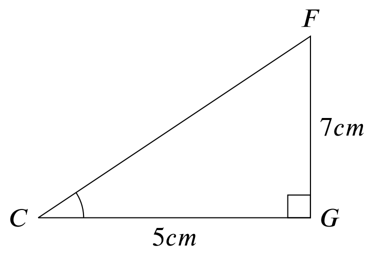
The angle formed by a line and the orthoprojection of the line on the plane is called *the angle formed by the line and the plane*. This angle represents the inclination of the line with respect to the plane, thus it is called *the tilt angle of the line with respect to the plane*.

17.2.1 Practice 1

1. In the diagram below, $AB = 9cm$, $BC = 7cm$, $CG = 5cm$. Find:



(a) The angle formed by line CF and plane $GHDC$.
Sol.
The angle formed by line CF and plane $GHDC$ is $\angle FCG$.

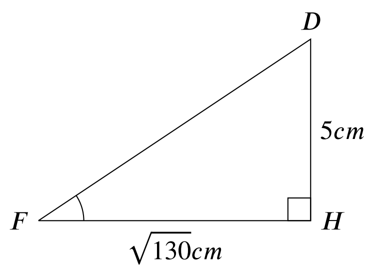


$$\begin{aligned} \tan\angle FCG &= \frac{FG}{CG} \\ &= \frac{7}{5} \\ \angle FCG &\approx 54.46^\circ \end{aligned}$$

(b) The angle formed by line DF and plane $EFGH$.
Sol.

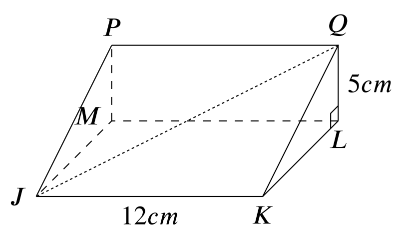
$$\begin{aligned} \text{In } EFGH,\ HF &= \sqrt{EF^2 + EH^2} \\ &= \sqrt{9^2 + 7^2} \\ &= \sqrt{130}cm \end{aligned}$$

The angle formed by line DF and plane $EFGH$ is $\angle DFH$.



$$\begin{aligned} \tan\angle DFH &= \frac{DH}{FH} \\ &= \frac{5}{\sqrt{130}} \\ \angle DFH &\approx 23.68^\circ \end{aligned}$$

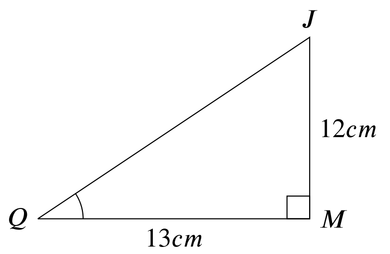
2. The diagram below shows a right prism, its base KQL is a right-angled triangle, $JKLM$ is a square. Given that $JK = 12cm$, $LQ = 5cm$, find the angle formed by line JQ and plane $PQLM$.



Sol.

$$\text{In } PQLM,\ QM = \sqrt{JK^2 + KL^2} = \sqrt{12^2 + 5^2} = 13cm$$

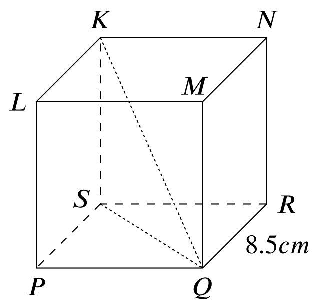
The angle formed by line JQ and plane $PQLM$ is $\angle JQM$.



$$\tan\angle JQM = \frac{JM}{QM} = \frac{12}{13}$$
$$\angle JQM \approx 42.71°$$

17.2.2 Exercise 17.2

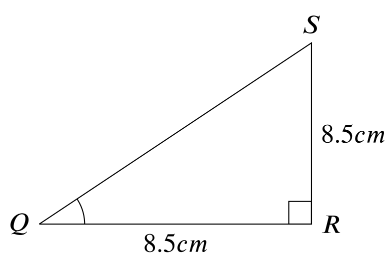
1. The diagram below shows a cube with side length of $8.5cm$. Find:



(a) The angle formed by line QS and plane $MNRQ$.

Sol.

The angle formed by line QS and plane $MNRQ$ is $\angle SQR$.



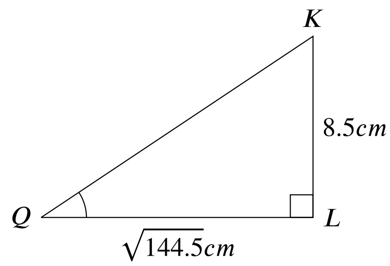
$$\tan\angle SQR = \frac{SR}{QR} = \frac{8.5}{8.5} = 1$$
$$\angle SQR = 45°$$

(b) The angle formed by line KQ and plane $PQML$.

Sol.

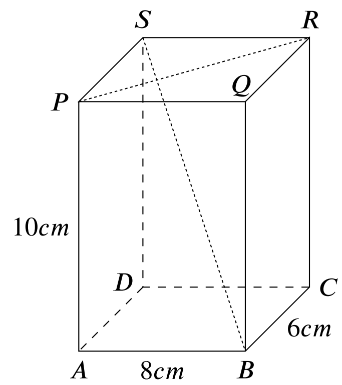
$$\text{In } KLMN,\ KM = \sqrt{8.5^2 + 8.5^2} = \sqrt{144.5}\text{cm}$$

The angle formed by line KQ and plane $PQML$ is $\angle KQL$.



$$\tan\angle KQL = \frac{KL}{QL} = \frac{8.5}{\sqrt{144.5}}$$
$$\angle KQL \approx 35.26°$$

2. THe diagram below shows a cuboid, $AB = 8cm$, $BC = 6cm$, $AP = 10cm$. Find:



(a) The length of PR.

Sol.

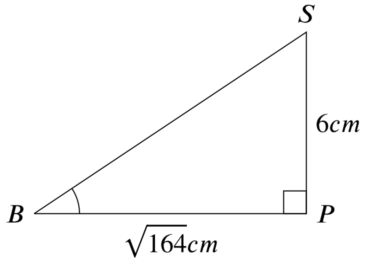
$$PR = \sqrt{PQ^2 + QR^2} = \sqrt{8^2 + 6^2} = 10cm$$

(b) The angle formed by line SB and plane $APQB$.

Sol.

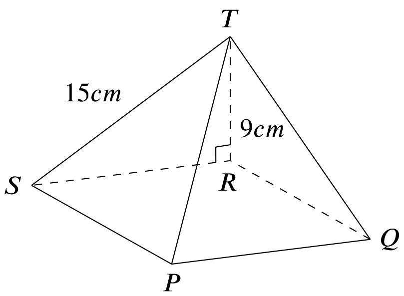
$$\text{In } APQB,\ PB = \sqrt{PA^2 + AB^2}$$
$$= \sqrt{10^2 + 8^2}$$
$$= \sqrt{164}cm$$

The angle formed by line SB and plane $APQB$ is $\angle SBP$.



$$\tan \angle SBP = \frac{SP}{BP}$$
$$= \frac{6}{\sqrt{164}}$$
$$\angle SBP \approx 25.10°$$

3. The diagram below shows a pyramid. Given that its base $PQRS$ is a square, TR is perpendicular to the base, $TS = 15cm$, $TR = 9cm$. Find:



(a) The length of RS.

Sol.

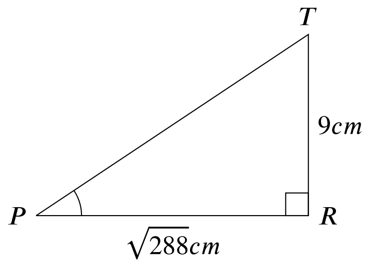
$$RS = \sqrt{ST^2 - TR^2}$$
$$= \sqrt{15^2 - 9^2}$$
$$= 12cm$$

(b) The angle formed by line PT and plane $PQRS$.

Sol.

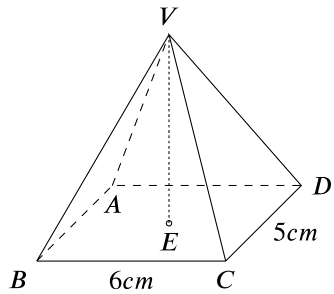
$$\text{In } PQRS,\ PR = \sqrt{PQ^2 + RQ^2}$$
$$= \sqrt{12^2 + 12^2}$$
$$= \sqrt{288}cm$$

The angle formed by line PT and plane $PQRS$ is $\angle TPR$.



$$\tan \angle TPR = \frac{TR}{PR}$$
$$= \frac{9}{\sqrt{288}}$$
$$\angle TPR \approx 27.94°$$

4. The diagram below shows a right pyramid with height of $8cm$, its base is a rectangle, E is the foot point from V to the base. Given that $CD = 5cm$, $BC = 6cm$. Find:

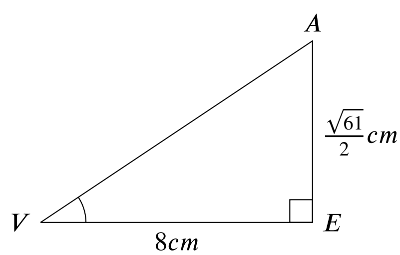


(a) The angle formed by line VA and line VE.

Sol.

$$\text{In } ABCD,\ AC = \sqrt{AB^2 + BC^2}$$
$$= \sqrt{5^2 + 6^2}$$
$$= \sqrt{61}cm$$
$$AE = \frac{AC}{2}$$
$$= \frac{\sqrt{61}}{2}cm$$

The angle formed by line VA and line VE is $\angle AVE$.



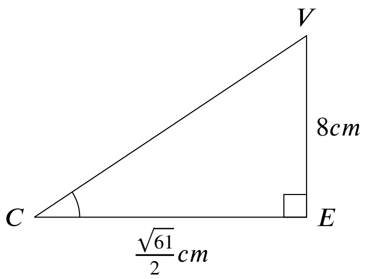
$$\tan \angle AVE = \frac{AE}{VE}$$
$$= \frac{\frac{\sqrt{61}}{2}}{8}$$
$$\angle AVE \approx 26.02°$$

(b) The angle formed by line VC and plane $ABCD$.

Sol.

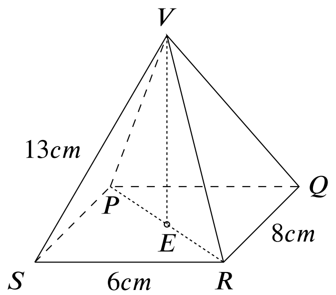
$$\text{In } ABCD,\ EC = \frac{AC}{2} = \frac{\sqrt{61}}{2}cm$$

The angle formed by line VC and plane $ABCD$ is $\angle VCE$.



$$\tan \angle VCE = \frac{VE}{CE} = \frac{8}{\frac{\sqrt{61}}{2}}$$
$$\angle VCE \approx 63.98°$$

5. The diagram below shows a right pyramid, its base $PQRS$ is a regtangle. Given that $SR = 6cm$, $QR = 8cm$, $VS = 13cm$. Find:



(a) The length of PR.

Sol.

$$PR = \sqrt{SR^2 + SP^2} \quad (3)$$
$$= \sqrt{6^2 + 8^2} \quad (4)$$
$$= 10cm \quad (5)$$

(b) The height of the pyramid.

Sol.

Let the foot point of the pyramid be E.

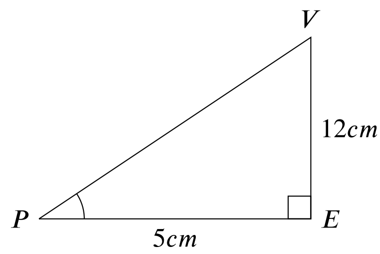
Sol.

$$\text{In } PQRS,\ PE = \frac{PR}{2} = \frac{10}{2} = 5cm$$

$$VE = \sqrt{VP^2 - PE^2} = \sqrt{13^2 - 5^2} = 12cm$$

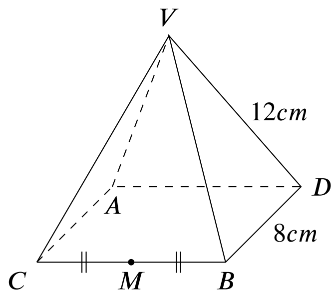
(c) The angle of the line VP and plane $PQRS$.

Sol. The angle of the line VP and plane $PQRS$ is $\angle VPE$.



$$\tan \angle VPE = \frac{VE}{PE} = \frac{12}{5}$$
$$\angle VPE \approx 67.38°$$

6. The diagram below shows a regular pyramid, the length of its lateral edge is $12cm$, its base $ABCD$ is a square with side length of $8cm$, M is the midpoint of BC. Find:



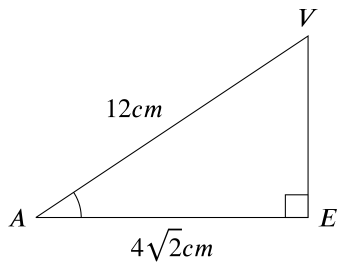
(a) The angle formed by the lateral edge and the base of the pyramid.

Sol.

Let the foot point of the pyramid be E.

$$
\begin{aligned}
\text{In } ABCD,\ AB &= \sqrt{AD^2 + BD^2} \\
&= \sqrt{8^2 + 8^2} \\
&= \sqrt{128}cm \\
&= 8\sqrt{2} \\
AE &= \frac{AB}{2} \\
&= 4\sqrt{2}cm
\end{aligned}
$$

The angle formed by the lateral edge and the base of the pyramid is $\angle VAE$.



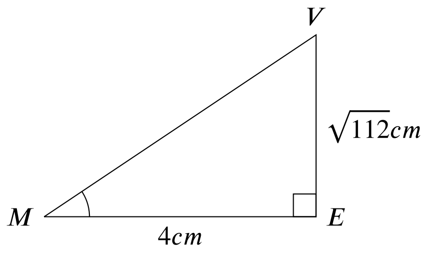
$$
\begin{aligned}
\cos\angle VAE &= \frac{AE}{AV} \\
&= \frac{4\sqrt{2}}{12} \\
&= \frac{\sqrt{2}}{3} \\
\angle VAE &\approx 61.87^\circ
\end{aligned}
$$

(b) The angle formed by line VM and the base of the pyramid.

Sol.

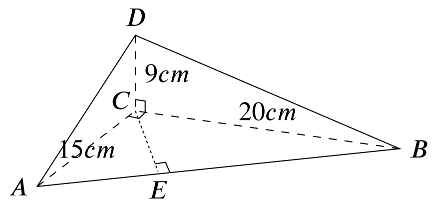
$$
\begin{aligned}
EM &= \frac{BD}{2} \\
&= \frac{8}{2} \\
&= 4cm \\
VE &= \sqrt{AV^2 + AE^2} \\
&= \sqrt{12^2 - (4\sqrt{2})^2} \\
&= \sqrt{144 - 32} \\
&= \sqrt{112}
\end{aligned}
$$

The angle formed by line VM and the base of the pyramid is $\angle VME$.



$$
\begin{aligned}
\cos\angle VME &= \frac{VE}{VM} \\
&= \frac{\sqrt{112}}{4} \\
\angle VME &\approx 69.30^\circ
\end{aligned}
$$

7. In the pyramid shown below, ΔABC is a right-angled triangle, CD is perpendicular to plane ABC, CE is perpendicular to AB. Given that $AC = 15cm$, $BC = 20cm$ and $CD = 9cm$. Find:

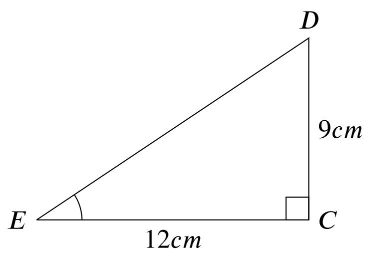


(a) The length of CE.

Sol.

$$
\begin{aligned}
\text{In } \Delta ABC,\ \tan\angle CBA &= \frac{AC}{AB} \\
&= \frac{15}{20} \\
&= \frac{3}{4} \\
\angle CBA &\approx 36.87^\circ \\
\text{In } \Delta CBE,\ \sin\angle CBE &= \frac{CE}{CB} \\
\sin 36.87^\circ &= \frac{CE}{20} \\
CE &\approx 20\sin 36.87^\circ \\
&= 12cm
\end{aligned}
$$

(b) $\angle DEC$.

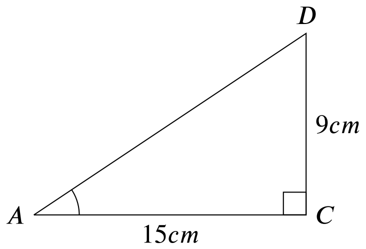


$$\begin{aligned}\cos\angle DEC &= \frac{DC}{EC}\\ &= \frac{9}{12}\\ &= \frac{3}{4}\\ \angle DEC &\approx 36.87^\circ\end{aligned}$$

(c) The angle formed by line AD and plane ABC.

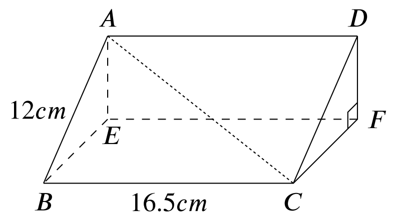
Sol.

The angle formed by line AD and plane ABC is $\angle DAC$.



$$\begin{aligned}\cos\angle DAC &= \frac{DC}{AC}\\ &= \frac{9}{15}\\ &= \frac{3}{5}\\ \angle DAC &\approx 30.96^\circ\end{aligned}$$

8. The diagram below shows a right prism, its base CDF is a right-angled triangle. Given that $BC = 16.5cm$ and $AB = 12cm$. Assume that $CF = 2DF$, find:

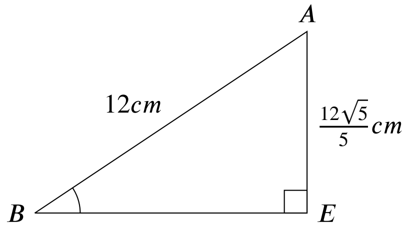


(a) The angle formed by line AB and plane $BCFE$.

Sol.

$$\begin{aligned}CF &= 2DF\\ DF^2 + (2DF)^2 &= 12^2\\ DF^2 + 4DF^2 &= 144\\ 5DF^2 &= 144\\ DF^2 &= \frac{144}{5}\\ DF &= \frac{12}{\sqrt{5}}\\ &= \frac{12\sqrt{5}}{5}cm\end{aligned}$$

The angle formed by line AB and plane $BCFE$ is $\angle ABE$.



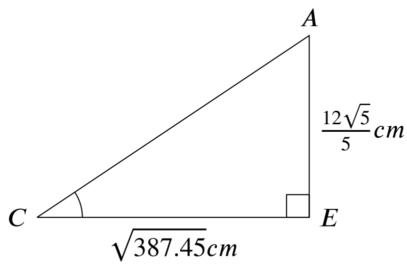
$$\begin{aligned}\sin\angle ABE &= \frac{AE}{AB}\\ &= \frac{\frac{12\sqrt{5}}{5}}{12}\\ &= \frac{\sqrt{5}}{5}\\ \angle ABE &\approx 26.57^\circ\end{aligned}$$

(b) The angle formed by line AC and plan $BCFE$.

Sol.

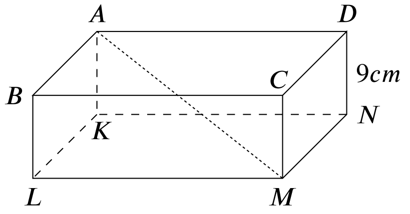
$$\begin{aligned}CF &= 2DF\\ &= \frac{24}{\sqrt{5}}\\ \text{In } BCEF,\ EC &= \sqrt{BC^2 + BE^2}\\ &= \sqrt{16.5^2 + \frac{24^2}{5}}\\ &= \sqrt{387.45}\end{aligned}$$

The angle formed by line AC and plane $BCFE$ is $\angle ACE$.



$$\begin{aligned}\sin\angle ACE &= \frac{AE}{EC}\\ &= \frac{\frac{12\sqrt{5}}{5}}{\sqrt{387.45}}\\ \angle ACE &\approx 15.25^\circ\end{aligned}$$

9. The diagram below shows a cuboid with volume of $300cm^3$. Given that $AD = 2DC$ and $DN = 9cm$. Find the angle formed by line AM and plane $KLMN$.



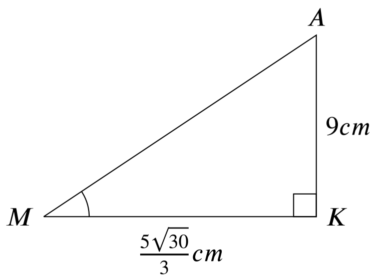
Sol.

$$AD = 2DC$$
$$AD \times DC \times DN = 300$$
$$2DC \times DC \times 9 = 300$$
$$2DC^2 \times 9 = 300$$
$$2DC^2 = \frac{100}{3}$$
$$DC^2 = \frac{50}{3}$$
$$DC = \frac{5\sqrt{2}}{\sqrt{3}}$$
$$= \frac{5\sqrt{6}}{3}$$
$$AD = 2DC$$
$$= \frac{10\sqrt{6}}{3}$$

In $KLMN$, $KM = \sqrt{MN^2 + KN^2}$

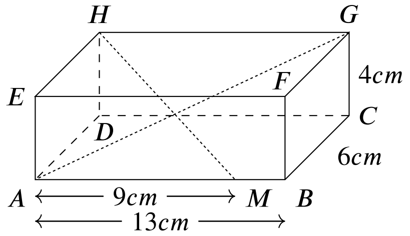
$$= \sqrt{\left(\frac{5\sqrt{6}}{3}\right)^2 + \left(\frac{10\sqrt{6}}{3}\right)^2}$$
$$= \sqrt{\frac{50}{3} + \frac{200}{3}}$$
$$= \frac{5\sqrt{30}}{3}$$

The angle formed by line AM and plane $KLMN$ is $\angle AMK$.



$$\tan\angle AMK = \frac{AK}{MK}$$
$$= \frac{9}{\frac{5\sqrt{30}}{3}}$$
$$\angle AMK \approx 44.59°$$

10. The diagram below shows a cuboid. Given that $AB = 13cm$, $BC = 6cm$, $CG = 4cm$. M is a point on AB, $AM = 9cm$. Find:



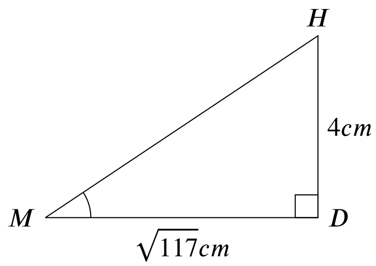
(a) The angle formed by line HM and plane $ABCD$.

Sol.

In $ABCD$, $DM = \sqrt{AM^2 + AD^2}$

$$= \sqrt{9^2 + 6^2}$$
$$= \sqrt{117}cm$$

The angle formed by line HM and plane $ABCD$ is $\angle HMD$.



$$\tan\angle HMD = \frac{HD}{MD}$$
$$= \frac{4}{\sqrt{117}}$$
$$\angle HMD \approx 20.29°$$

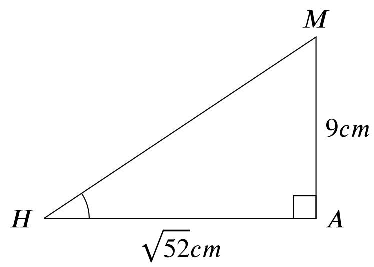
(b) The angle formed by line HM and plane $HDAE$.

Sol.

In $HDAE$, $HA = \sqrt{AD^2 + HD^2}$

$$= \sqrt{6^2 + 4^2}$$
$$= \sqrt{52}cm$$

The angle formed by line HM and plane $HDAE$ is $\angle MHA$.

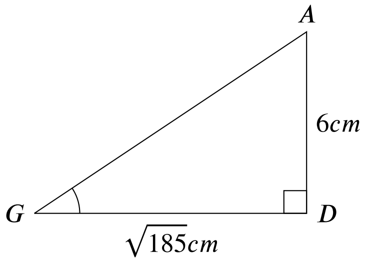


$$\tan \angle MHA = \frac{MA}{HA}$$
$$= \frac{9}{\sqrt{52}}$$
$$\angle MHA \approx 51.30°$$

(c) The angle formed by line AG and plane $CDHG$.
Sol.

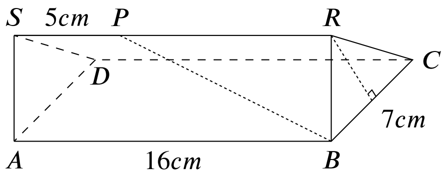
$$\text{In } CDHG,\ DG = \sqrt{DC^2 + GC^2}$$
$$= \sqrt{13^2 + 4^2}$$
$$= \sqrt{185}cm$$

The angle formed by line AG and plane $CDHG$ is $\angle AGD$.



$$\tan \angle AGD = \frac{AD}{GD}$$
$$= \frac{6}{\sqrt{185}}$$
$$\angle AGD \approx 23.80°$$

11. The diagram below shows a regular prism, its bases ADS and BCR are equiliteral triangles. Given that $AB = 16cm$, $BC = 7cm$, $SP = 5cm$. Find:



(a) The length of BP.
Sol.

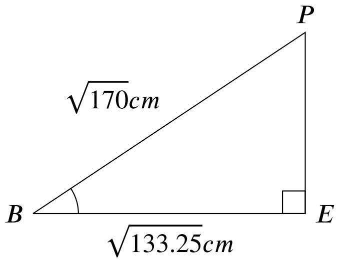
$$PR = SR - SP$$
$$= 16 - 5$$
$$= 11cm$$
$$BP = \sqrt{BR^2 = PR^2}$$
$$= \sqrt{7^2 + 11^2}$$
$$= \sqrt{170}$$
$$\approx 13.04cm$$

(b) The angle formed by line BP and plane $ABCD$.
Sol.

Let the foot point of P be E.

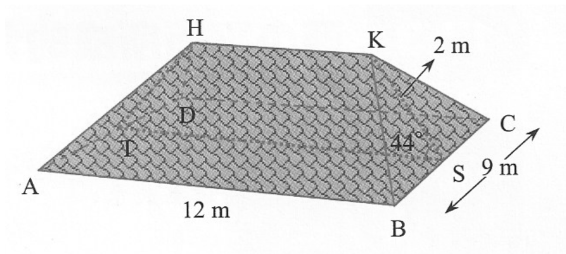
$$EB = \sqrt{3.5^2 + 11^2}$$
$$= \sqrt{133.25}cm$$

The angle formed by line BP and plane $ABCD$ is $\angle PBE$.



$$\cos \angle PBE = \frac{BE}{BP}$$
$$= \frac{\sqrt{133.25}}{\sqrt{170}}$$
$$\angle PBE \approx 27.71°$$

12. The diagram below shows a roof, HK is the ridge of the roof, its edges HA, HD, KB, KC are euqal in length. Both of the planes HAD and KBC form a 44^o angle with plane $ABCD$. Given that S and T are the midpoints of BC and AD respectively. Find:



(a) The distance from line HK to plane $ABCD$.

Sol.

Let the foot point of K on plane $ABCD$ be P.

$$\text{In } \Delta KPS, \sin \angle KSP = \frac{KP}{KS}$$
$$\sin 44° = \frac{KP}{2}$$
$$KP = 2\sin 44°$$
$$\approx 1.39m$$

(b) The length of HK.

Sol.

$$\cos\angle KSP = \frac{PS}{KS}$$
$$\cos 44^\circ = \frac{PS}{2}$$
$$PS = 2\cos 44^\circ$$
$$\approx 1.44m$$
$$HK \approx 12 - 2PS$$
$$\approx 12 - 2.88$$
$$\approx 9.12m$$

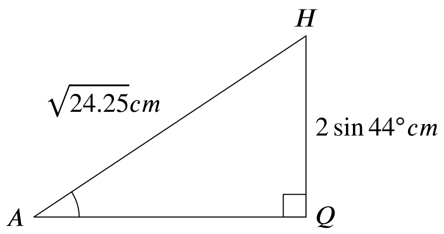
(c) The angle formed by line HA and plane $ABCD$.

Sol.

Let the foot point of H on plane $ABCD$ be Q.

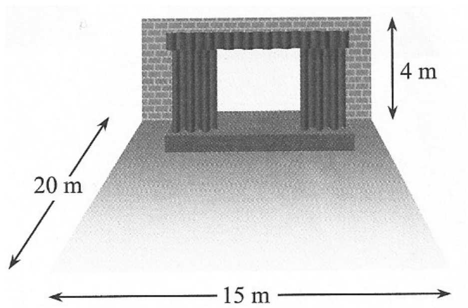
$$HA = \sqrt{HT^2 + AT^2}$$
$$= \sqrt{2^2 + 4.5^2}$$
$$= \sqrt{24.25}cm$$

The angle formed by line HA and plane $ABCD$ is $\angle HAQ$.



$$\sin\angle HAQ = \frac{HQ}{HA}$$
$$\sin\angle HAQ = \frac{2\sin 44^\circ}{\sqrt{24.25}}$$
$$\angle HAQ \approx 16.38^\circ$$

13. The length, width and height of a hall are $20m$, $15m$, and $4m$ respectively. Find:



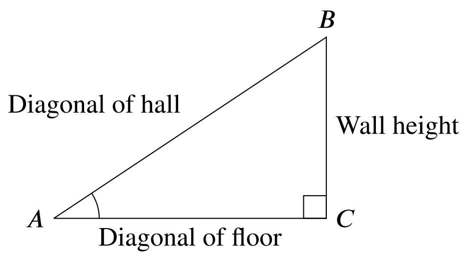
(a) The length of the diagonal of the hall.

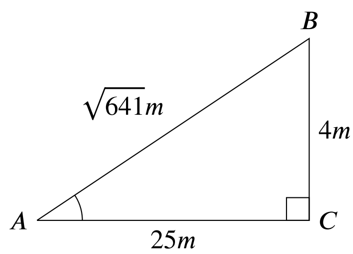
Sol.

$$\text{Diagonal of floor} = \sqrt{20^2 + 15^2}$$
$$= \sqrt{625}m$$
$$= 25m$$
$$\text{Diagonal of hall} = \sqrt{4^2 + 25^2}$$
$$= \sqrt{641}m$$
$$= 25.32m$$

(b) The angle formed by the diagonal and the floor of the hall.

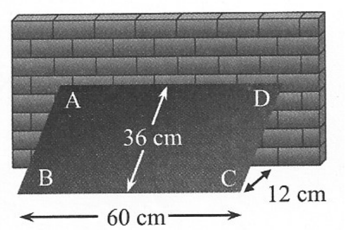
Sol.





$$\tan\angle BAC = \frac{4}{25}$$
$$\angle BAC \approx 9.09^\circ$$

14. In the diagram below, $ABCD$ represents a rectangular plank with length and width of $60cm$ and $36cm$ respectively, its base BC is on the ground and the top of it lies on the wall. Assume that the distance between BC and the corner of the wall is $12cm$, find the angle formed by the diagonal BD of the plank and the ground.

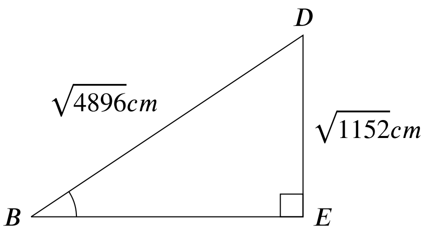


Sol.

Let the footpoint of D on the ground be E.

$$
\begin{aligned}
BD &= \sqrt{BC^2 + CD^2} \\
&= \sqrt{60^2 + 36^2} \\
&= \sqrt{4896}cm \\
DE &= \sqrt{DC^2 - CE^2} \\
&= \sqrt{36^2 - 12^2} \\
&= \sqrt{1152}cm
\end{aligned}
$$

The angle formed by the diagonal BD and the ground is $\angle DBE$.

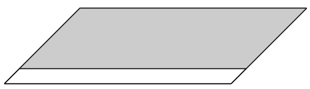


$$
\begin{aligned}
\sin \angle DBE &= \frac{\sqrt{1152}}{\sqrt{4896}} \\
\angle DBE &\approx 29.02^\circ
\end{aligned}
$$

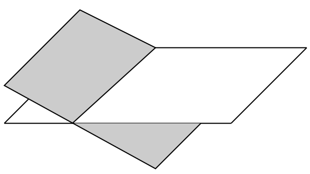
17.3 Angle Formed by Two Planes

There are three types positional relationship between two planes:

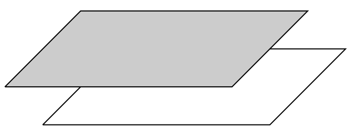
1. Two planes coincide with each other.



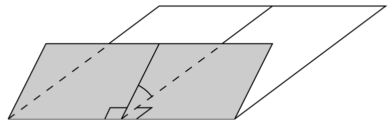
2. Two planes intersect with each other at a line.



3. Two planes are parallel to each other and do not intersect with each other.

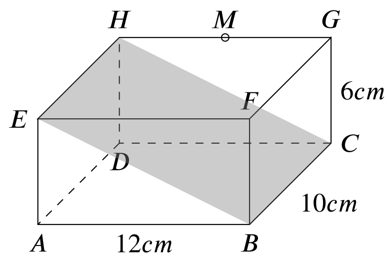


Two non-parallel planes intersect with each other at a line, the line is called the *common edge*. At any point on the common edge, draw a line perpendicular to the common edge on each plane, the acute angles formed by these two perpendicular lines are called *the angle formed by the two planes*.



17.3.1 Practice 2

1. The diagram below shows a cuboid with length of $12cm$, width of $10cm$ and height of $6cm$.

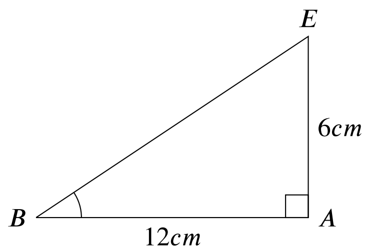


(a) Find the angle formed by plane $EBCH$ and plane $ABCD$.

Sol.

$\because$ BC is the common edge of plane $EBCH$ and plane $ABCD$, $AB \perp BC$ and $EB \perp BC$.

$\therefore$ The angle formed by plane $EBCH$ and plane $ABCD$ is $\angle EBA$.



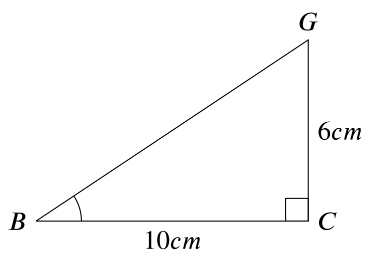
$$
\begin{aligned}
\tan \angle EAB &= \frac{6}{12} \\
&= \frac{1}{2} \\
\angle EAB &\approx 26.57^\circ
\end{aligned}
$$

(b) Assume that M is a point on HG, find the angle formed by plane MAB and plane $ABCD$.

Sol.

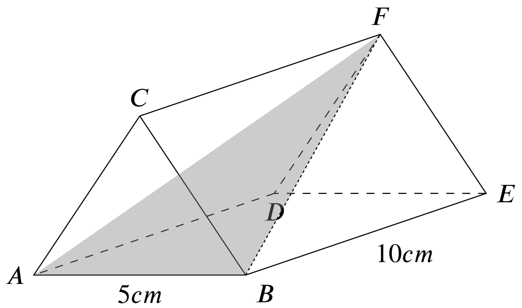
$\because$ AB is the common edge of plane MAB and plane $ABCD$, M is on HG, $HG \perp AB$, $BC \perp AB$.

$\therefore$ The angle formed by plane MAB and plane $ABCD$ is $\angle GBC$.



$$\tan \angle GBC = \frac{6}{10}$$
$$= \frac{3}{5}$$
$$\angle GBC \approx 30.96°$$

2. The diagram below shows a regular prism, its bases ABC and DEF are equilateral triangles with side length of $5cm$. Given that the height of the prism is $10cm$, find:



(a) The length of BF.

Sol.

$$BF = \sqrt{EF^2 + BE^2}$$
$$= \sqrt{10^2 + 5^2}$$
$$= \sqrt{125}$$
$$\approx 11.18cm$$

(b) The angle formed by plane ABF and plane ABC.

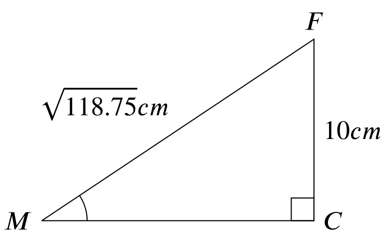
Sol.

Let the midpoint of AB be M.

$$MF = \sqrt{FB^2 - BM^2}$$
$$= \sqrt{125 - 2.5^2}$$
$$= \sqrt{118.75}cm$$

$\because$ AB is the common edge of plane ABF and plane ABC, $MF \perp AB$, $CF \perp AB$.

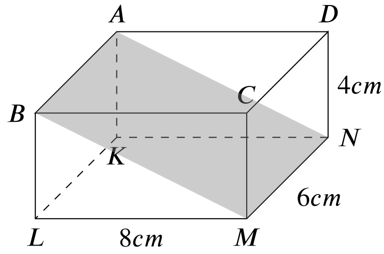
$\therefore$ The angle formed by plane ABF and plane ABC is $\angle FMC$.



$$\sin \angle FMC = \frac{FC}{MF}$$
$$= \frac{10}{\sqrt{118.75}}$$
$$\angle FMC \approx 66.59°$$

17.3.2 Exercise 17.3

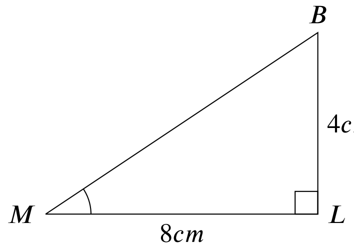
1. The diagram below shows a cuboid with length of $8cm$, width of $6cm$ and height of $4cm$. Find the angle formed by plane $ABMN$ and $KLMN$.



Sol.

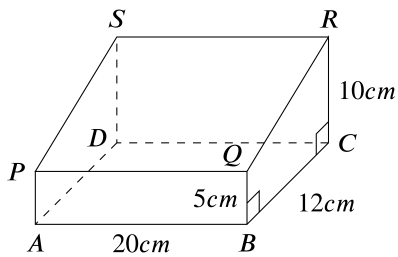
$\because$ MN is the common edge of $ABMN$ and $KLMN$, $LM \perp MN$ and $BM \perp MN$.

$\therefore$ The angle formed by plane $ABMN$ and $KLMN$ is $\angle BML$.



$$\tan \angle BML = \frac{BL}{LM}$$
$$= \frac{4}{8}$$
$$= \frac{1}{2}$$
$$\angle BML \approx 26.57°$$

2. In the right prism shown below, $ABCD$ is a rectangle with length of $20cm$ and width of $12cm$, $BCRQ$ is a trapezoid, $\angle QBC$ and $\angle RCB$ are both right angles, $BQ = 5cm$, $CR = 10cm$. Find the angle formed by plane $PQRS$ and plane $ABCD$.

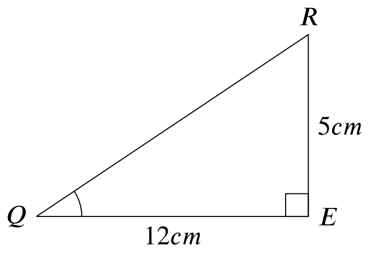


Sol.

Let the midpoint of RC and SD be E and F respectively.

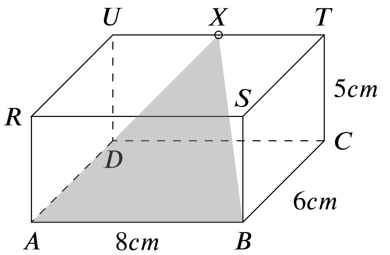
$\because$ $PQEF \parallel ABCD$, PQ is the common edge of $PQRS$ and $PQER$, $PQ \perp QE$, and $PQ \perp QR$.

$\therefore$ The angle formed by plane $PQRS$ and $ABCD$ is $\angle RQE$.



$$\tan\angle RQE = \frac{RE}{QE} = \frac{5}{12}$$
$$\angle RQE \approx 22.62^\circ$$

3. The diagram below shows a cuboid, $AB = 8cm$, $BC = 6cm$, $CT = 5cm$, X is the midpoint of TU. Find:



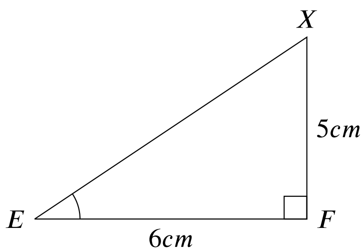
(a) The angle formed by plane XAB and plane $ABCD$.

Sol.

Let the midpoint of AB and CD be E and F respectively.

$\because$ AB is the common edge of $ABCD$ and XAB, $AB \perp XE$, and $AB \perp EF$.

$\therefore$ The angle formed by plane $ABCD$ and XAB is $\angle XEF$.



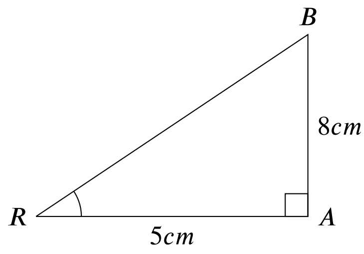
$$\tan\angle XEF = \frac{XF}{EF} = \frac{5}{6}$$
$$\angle XEF \approx 39.81^\circ$$

(b) The angle formed by plane $BCUR$ and plane $ADUR$.

Sol.

$\because$ UR is the common edge of $BCUR$ and $ADUR$, $UR \perp RB$, and $UR \perp AR$.

$\therefore$ The angle formed by plane $BCUR$ and $ADUR$ is $\angle BRA$.



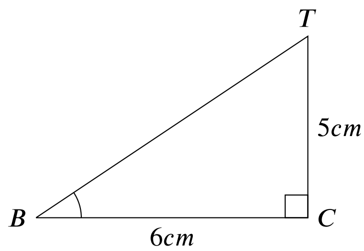
$$\tan\angle BRA = \frac{BA}{RA} = \frac{8}{5}$$
$$\angle BRA \approx 57.99^\circ$$

(c) The angle formed by plane $ABTU$ and plane $ABCD$.

Sol.

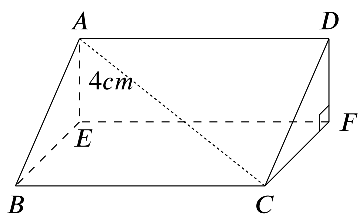
$\because$ AB is the common edge of $ABTU$ and $ABCD$, $AB \perp TB$, and $AB \perp BC$.

$\therefore$ The angle formed by plane $ABTU$ and $ABCD$ is $\angle TBC$.



$$\tan\angle TBC = \frac{TC}{BC} = \frac{5}{6}$$
$$\angle TBC \approx 39.81^\circ$$

4. The diagram below shows a right pyramid, its bases ABE and DCF are right-angled triangles. Given that $AE = 4cm$, $BE = \frac{2}{3}EF$, $EF = 4DF$, find the angle formed by plane $ABCD$ and plane $BCFE$.

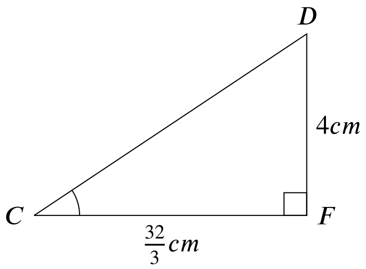


Sol.

$$\begin{aligned} EF &= 4DF \\ &= 4 \times 4 \\ &= 16cm \\ BE &= \frac{2}{3}EF \\ &= \frac{32}{3}cm \end{aligned}$$

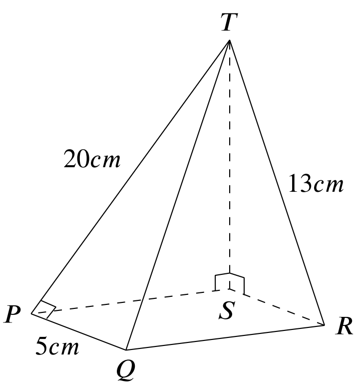
$\because$ BC is the common edge of $ABCD$ and $BCFE$, $BC \perp CD$, and $BC \perp CF$.

$\therefore$ The angle formed by plane $ABCD$ and $BCFE$ is $\angle DCF$.



$$\begin{aligned} \tan\angle DCF &= \frac{DF}{CF} \\ &= \frac{4}{\frac{32}{3}} \\ \angle DCF &\approx 20.56^\circ \end{aligned}$$

5. In the pyramid shown below, PQT, SPT, and SRT are all right-angled triangles, $PQRS$ is a triangle. Given that $PQ = 5cm$, $RT = 13cm$, $PT = 20cm$. Find:



(a) The height of the prism.

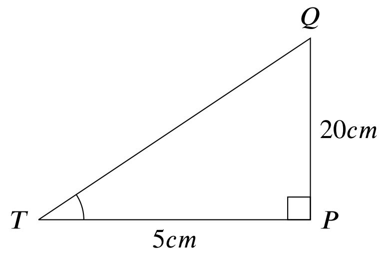
Sol.

$$\begin{aligned} \text{Height of the prism} &= TS \\ &= \sqrt{TR^2 - RS^2} \\ &= \sqrt{13^2 - 5^2} \\ &= 12cm \end{aligned}$$

(b) The angle formed by line TQ and plane PST.

Sol.

The angle formed by line TQ and plane PST is $\angle QTP$.



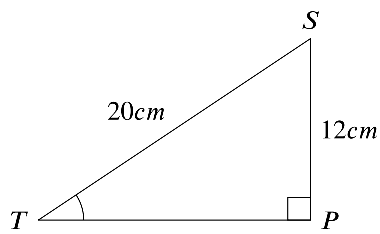
$$\begin{aligned} \tan\angle QTP &= \frac{PQ}{PT} \\ &= \frac{5}{20} \\ &= \frac{1}{4} \\ \angle QTP &\approx 14.04^\circ \end{aligned}$$

(c) The angle formed by plane RST and PQT.

Sol.

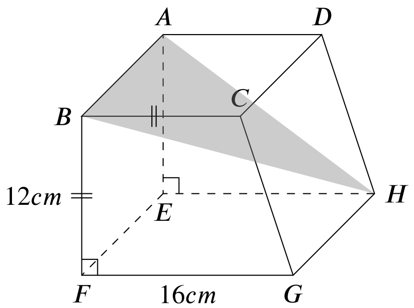
The angle formed by plane RST and PQT is $\angle STP$.

$$\begin{aligned} \text{In } \Delta TRS,\ TS &= \sqrt{TR^2 - SR^2} \\ &= \sqrt{13^2 - 5^2} \\ &= 12cm \end{aligned}$$



$$\begin{aligned} \sin\angle STP &= \frac{SP}{TS} \\ &= \frac{12}{20} \\ &= \frac{3}{5} \\ \angle STP &\approx 53.13^\circ \end{aligned}$$

6. The diagram below shows a right prism, its base $BCGF$ is a trapezoid, $BC = BF = 12cm$, $FG = 16cm$. The lateral face $EFGH$ is a square, and is perependicular to another lateral face $ABFE$. Find:



(a) The angle formed by plane $CDHG$ and plane $EFGH$.

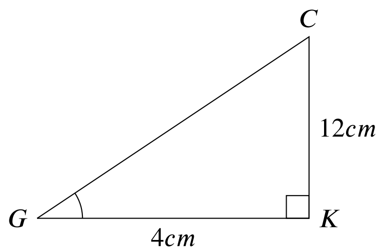
Sol.

Let the foot point of C be K.

$\because$ GH is the common edge of the plane $CDHG$ and plane $EFGH$, $CG \perp GH$, and $KG \perp GH$.

$\therefore$ The angle formed by plane $CDHG$ and plane $EFGH$ is $\angle CGK$.

$$\begin{aligned} KG &= FG - FK \\ &= 16 - 12 \\ &= 4cm \end{aligned}$$



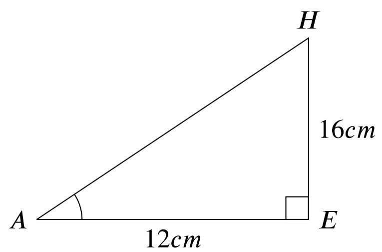
$$\begin{aligned} \tan \angle CGK &= \frac{CK}{KG} \\ &= \frac{12}{4} \\ &= 3 \\ \angle CGK &\approx 71.57^\circ \end{aligned}$$

(b) The angle formed by plane ABH and plane $ABFE$.

Sol.

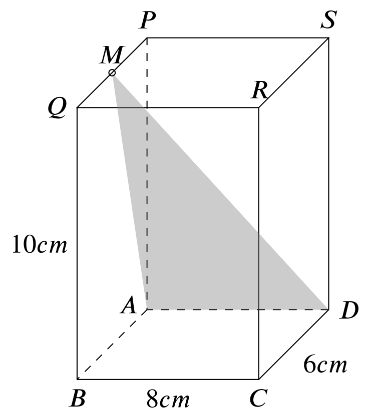
$\because$ AB is the common edge of the plane ABH and plane $ABFE$, $AB \perp AH$ and $AB \perp AE$.

$\therefore$ The angle formed by plane ABH and plane $ABFE$ is $\angle HAE$.



$$\begin{aligned} \tan \angle HAE &= \frac{HE}{AE} \\ &= \frac{16}{12} \\ &= \frac{4}{3} \\ \angle HAE &\approx 53.13^\circ \end{aligned}$$

7. In the cuboid shown below, $BC = 8cm$, $CD = 6cm$, $BQ = 10cm$. Given that M is the midpoint of PQ. Find:

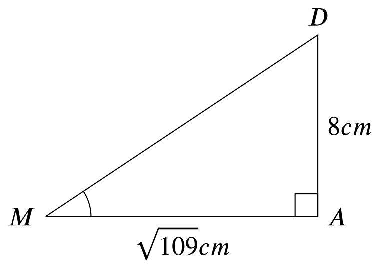


(a) The angle formed by line MD and plane $PQBA$.

Sol.

The angle formed by line MD and plane $PQBA$ is $\angle DMA$.

$$\begin{aligned} \text{In } \Delta MPA,\ MA &= \sqrt{PA^2 + MP^2} \\ &= \sqrt{10^2 + 3^2} \\ &= \sqrt{109}cm \end{aligned}$$



$$\begin{aligned} \tan \angle DMA &= \frac{DA}{MA} \\ &= \frac{8}{\sqrt{109}} \\ \angle DMA &\approx 37.46^\circ \end{aligned}$$

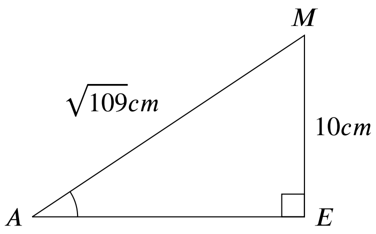
(b) The angle formed by plane AMD and plane $ABCD$.

Sol.

Let the midpoint of AB be E.

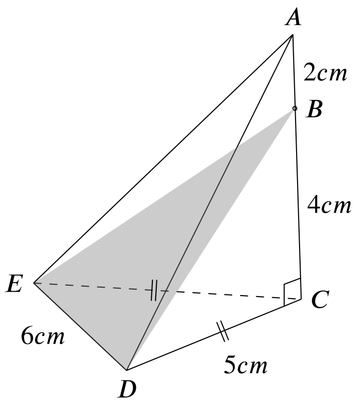
$\because$ AD is the common edge of plane AMD and plane $ABCD$, $AM \perp AD$, and $EA \perp AD$.

$\therefore$ The angle formed by plane AMD and plane $ABCD$ is $\angle MAE$.



$$\sin\angle MAE = \frac{ME}{MA} = \frac{10}{\sqrt{109}}$$
$$\angle MAE \approx 73.30°$$

8. The diagram below shows a pyramid with an isoceles triangle base. Given that $CD = CE = 5cm$, $ED = 6cm$, ACD is a right-angled triangle, B is a point on AC, $AD = 2cm$, $BC = 4cm$. Find:



(a) The angle formed by plane BDE and plane CDE.

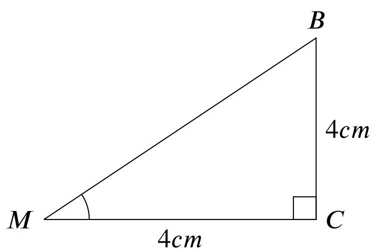
Sol.

Let the midpoint of ED be M.

$\because$ DE is the common edge of plane BDE and plane CDE, $BM \perp DE$, and $CM \perp DE$.

$\therefore$ The angle formed by plane BDE and plane CDE is $\angle BMC$.

$$MC = \sqrt{DC^2 - DM^2} = \sqrt{5^2 - 3^2} = 4cm$$



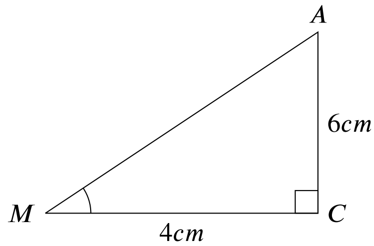
$$\tan\angle BMC = \frac{BC}{CM} = \frac{4}{4} = 1$$
$$\angle BMC = 45°$$

(b) The angle formed by the plane ADE and CDE.

Sol.

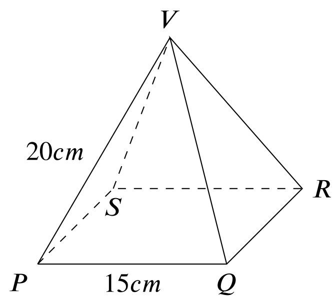
$\because$ DE is the common edge of plane ADE and plane CDE, $CM \perp DE$, and $AM \perp DE$.

$\therefore$ The angle formed by plane ADE and plane CDE is $\angle AMC$.



$$\tan\angle AMC = \frac{AC}{CM} = \frac{6}{4} = \frac{3}{2}$$
$$\angle AMC \approx 56.31°$$

9. The diagram below shows a regular pyramid with a square base. Given that $PQ = 15cm$, $PV = 20cm$. Find:



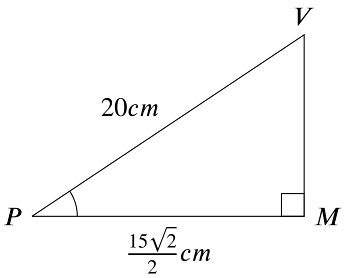
(a) The angle formed by line PV and plane $PQRS$.

Sol.

Let the footpoint of V be M.

The angle formed by line PV and plane $PQRS$ is $\angle VPM$.

$$PR = \sqrt{PQ^2 + QR^2} = \sqrt{15^2 + 15^2} = 15\sqrt{2}cm$$
$$PM = \frac{PR}{2} = \frac{15\sqrt{2}}{2}$$



$$\cos\angle VPM = \frac{PM}{PV}$$
$$= \frac{\frac{15\sqrt{2}}{2}}{20}$$
$$= \frac{3\sqrt{2}}{8}$$
$$\angle VPM \approx 57.97°$$

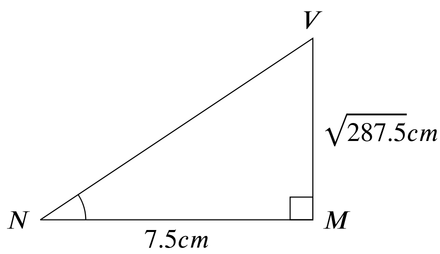
(b) The angle formed by the lateral faces and the base of the pyramid.

Sol.

$$VM = \sqrt{VP^2 - PM^2}$$
$$= \sqrt{20^2 - \left(\frac{15\sqrt{2}}{2}\right)^2}$$
$$= \sqrt{287.5}cm$$

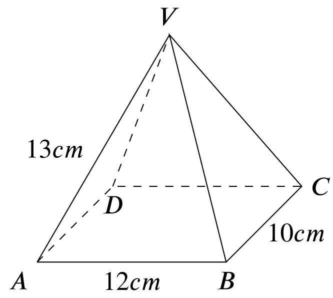
Let the midpoint of PQ be N.

The angle formed by the lateral faces and the base of the pyramid is $\angle VNM$.



$$\tan\angle VNM = \frac{VM}{NM}$$
$$= \frac{\sqrt{287.5}}{7.5}$$
$$\angle VNM \approx 66.14°$$

10. The diagram below shows a right pyramid with lateral edges of $13cm$. Its base $ABCD$ is a rectangle with length of $12cm$ and width of $10cm$. Find:



(a) The angle formed by plane VBC and plane $ABCD$.

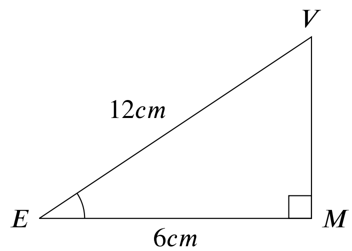
Sol.

Let the midpoint of BC be E, and the footpoint of V be M.

$\because$ BC is the common edge of plane VBC and plane $ABCD$, $VE \perp BC$, and $ME \perp BC$.

$\therefore$ The angle formed by plane VBC and plane $ABCD$ is $\angle VEM$.

$$VE = \sqrt{VB^2 - BE^2}$$
$$= \sqrt{13^2 - 5^2}$$
$$= 12cm$$



$$\cos\angle VEM = \frac{ME}{EV}$$
$$= \frac{1}{2}$$
$$\angle VEM = 60°$$

(b) The angle formed by plane VCD and plane $ABCD$.

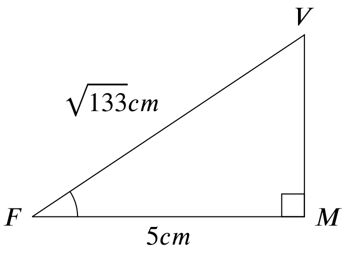
Sol.

Let the midpoint of CD be F

$\because$ CD is the common edge of plane VCD and plane $ABCD$, $VF \perp CD$, and $MF \perp CD$.

$\therefore$ The angle formed by plane VCD and plane $ABCD$ is $\angle VFM$.

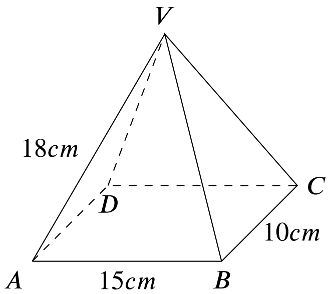
$$VF = \sqrt{VD^2 - DF^2}$$
$$= \sqrt{13^2 - 6^2}$$
$$= \sqrt{133}cm$$



$$
\begin{aligned}
\cos\angle VEM &= \frac{MF}{VF}\\
&= \frac{5}{\sqrt{133}}\\
\angle VEM &\approx 64.31^\circ
\end{aligned}
$$

RIGHT HERE LMAO

11. The diagram below shows a right pyramid with lateral edges of $18cm$, its base $ABCD$ is a rectangle with length of $15cm$ and width of $10cm$. Find:



(a) The height of the pyramid.

Sol.

Let the footpoint of V on $ABCD$ be M.

$$
\begin{aligned}
AC &= \sqrt{AB^2 + BC^2}\\
&= \sqrt{15^2 + 10^2}\\
&= 5\sqrt{13}cm\\
AM &= \frac{AC}{2}\\
&= \frac{5\sqrt{13}}{2}\\
\text{Height of the pyramid} &= VM\\
&= \sqrt{AV^2 - AM^2}\\
&= \sqrt{18^2 - \left(\frac{5\sqrt{13}}{2}\right)^2}\\
&= \sqrt{242.75}\\
&\approx 15.58cm
\end{aligned}
$$

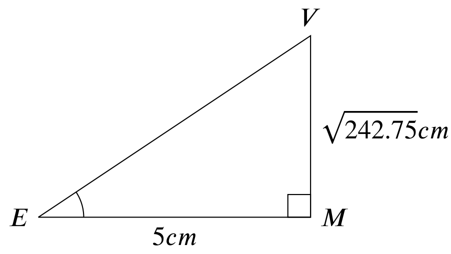
(b) The angle formed by plane VAB and plane $ABCD$.

Sol.

Let the midpoint of AB be E.

$\because$ AB is the common edge of plane VAB and plane $ABCD$, $ME \perp AB$, and $VE \perp AB$.

$\therefore$ The angle formed by plane VAB and plane $ABCD$ is $\angle VEM$.



$$
\begin{aligned}
\tan\angle VEM &= \frac{VM}{ME}\\
&= \frac{\sqrt{242.75}}{5}\\
\angle VEM &\approx 72.21^\circ
\end{aligned}
$$

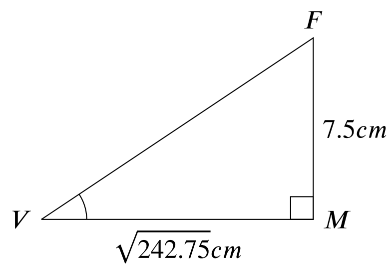
(c) The angle formed by plane VBC and plane VAD.

Sol.

Let the midpoint of AD and BC be F and G respectively.

The angle formed by plane VBC and plane VAD is $\angle FVG$.

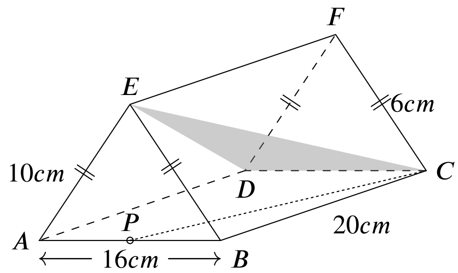
$$
\begin{aligned}
\angle FVG &= \angle FVM + \angle MVG\\
&= 2\angle FVM
\end{aligned}
$$



$$
\begin{aligned}
\tan\angle FVM &= \frac{FM}{VM}\\
&= \frac{7.5}{\sqrt{242.75}}\\
\angle FVM &\approx 25.705^\circ
\end{aligned}
$$

$$
\begin{aligned}
FVG &= 2\angle FVM\\
&\approx 2 \times 25.705^\circ\\
&\approx 51.41^\circ
\end{aligned}
$$

12. The diagram below shows a right prism with isoceles triangle bases. The side length and base length of the triangle base are $10cm$ and $16cm$ respectively, the height of the prism is $20cm$. Given that P is the midpoint of AB. Find:



(a) The length of PC.

Sol.

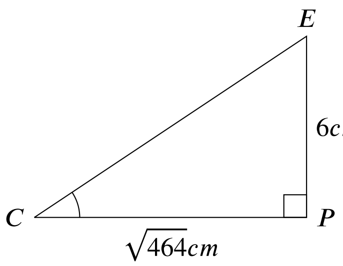
$$
\begin{aligned}
PC &= \sqrt{BC^2 - PB^2} \\
&= \sqrt{20^2 + 8^2} \\
&= \sqrt{464} \\
&\approx 21.54cm
\end{aligned}
$$

(b) The angle formed by line EC and plane $ABCD$.

Sol.

The angle formed by line EC and plane $ABCD$ is $\angle ECP$.

$$
\begin{aligned}
EP &= \sqrt{AE^2 - AP^2} \\
&= \sqrt{10^2 - 8^2} \\
&= 6cm
\end{aligned}
$$



$$
\begin{aligned}
\tan \angle ECP &= \frac{EP}{CP} \\
&= \frac{6}{\sqrt{464}} \\
\angle ECP &\approx 15.56^\circ
\end{aligned}
$$

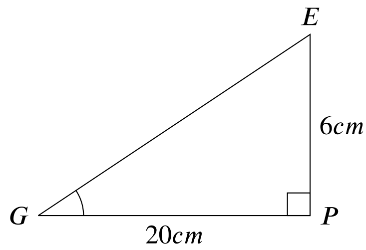
(c) The angle formed by plane DCE and plane $ABCD$.

Sol.

Let the midpoint of CD be G.

$\because$ CD is the common edge of plane DCE and plane $ABCD$, $PG \perp CD$, and $EG \perp CD$.

$\therefore$ The angle formed by plane DCE and plane $ABCD$ is $\angle EGP$.



$$
\begin{aligned}
\tan \angle EGP &= \frac{EP}{GP} \\
&= \frac{6}{20} \\
\angle EGP &= 16.70^\circ
\end{aligned}
$$

17.4 Longitude and Latitude

The earth is approximately spherical in shape, its radius is about $6,370km$, and its axis is a line that passes through the north (N) and south (S) poles. The earth rotating around its axis once is called a day, and the earth rotating around the sun once is called a year.

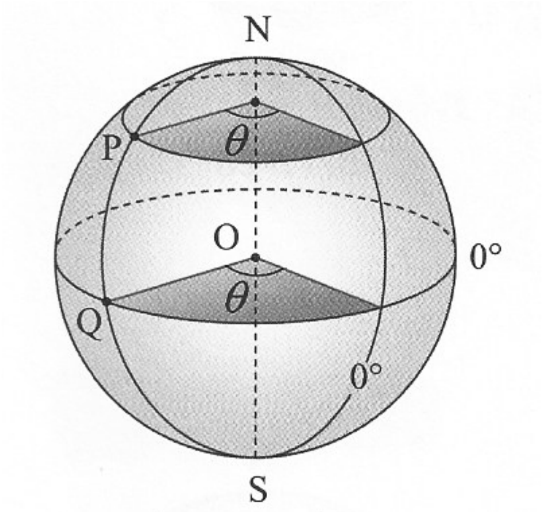
Any point on the earth's surface can be identified by two angles, the first is the angle between the point and the equator, called the *latitude* of the point, and the second is the angle between the point and the prime meridian, called the *longitude* of the point.

Longitude and Lines of Longitude

The two semicircles that are formed by the intersection of the earth's surface with the plane that passes through the north and south poles are called the *lines of longitude*, also called *meridians*. The lines of longitude that passes through the *Greenwich Observatory* in England are considered as 0° longitude, called the *Greenwich Meridian* or *prime meridian*.

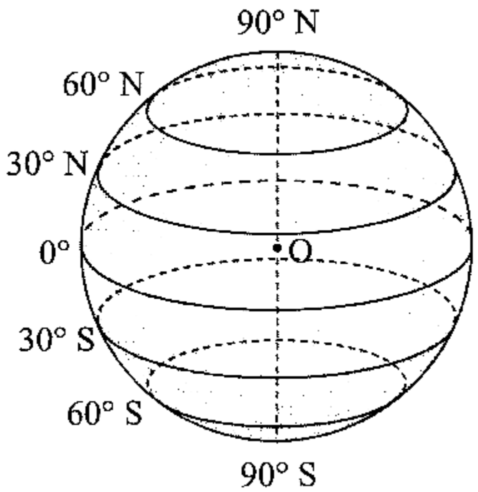
Prime meridian

The angle between the Greenwich Meridian and the line of longitude that passes through the point P is called the *longitude of P*. There are 360 degrees of longitude (+180° eastward and −180° westward.). The prime meridian divides the world into the Eastern Hemisphere and the Western Hemisphere. $180° E$ and $180° W$ coincide with each other at the same line of longitude, called the 180^{th} *Meridian* or *Antimeridian*.

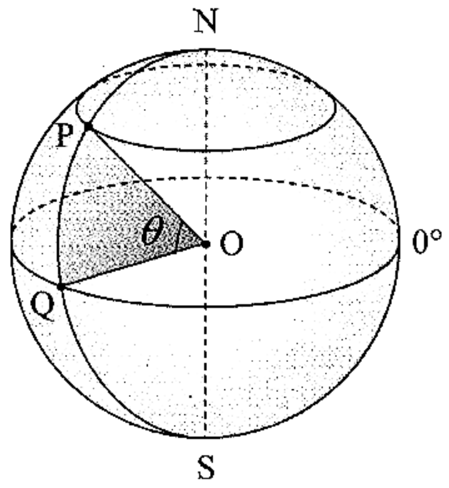


Latitude and Parallels of Latitude

The lines of latitude are the circles that are perpendicular to the plane that passes through the north and south poles. The *equator* is the one and only great circle among the parallels of latitude.

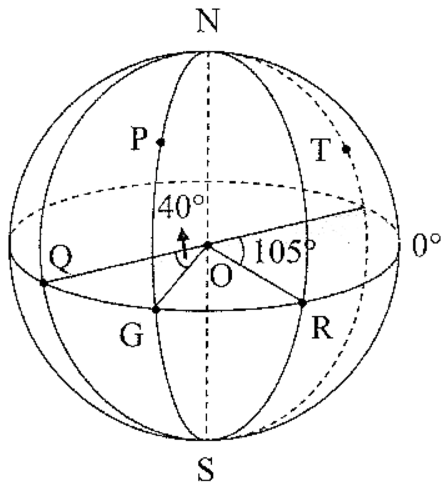


The angle between the equator and the line of latitude that passes through the point P is called the *latitude of* P. There are 180 degrees of latitude (+90° northward and −90° southward). The equator divides the world into the Northern Hemisphere and the Southern Hemisphere.



17.4.1 Practice 3

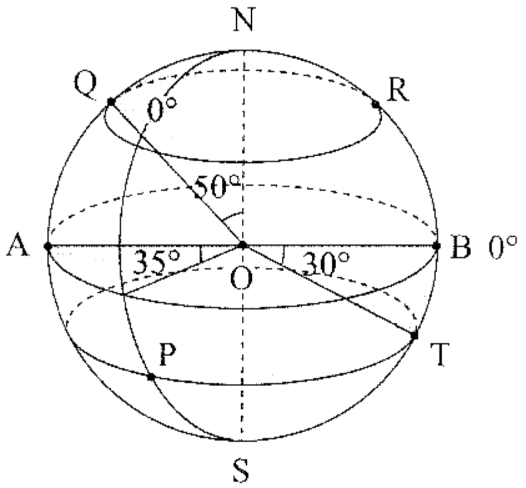
1. In the diagram below, NGS is the prime meridian, O is the centre of the earth. Find the longitude of locations P, Q, R and T.



Sol.

Lon. $P = 0°$
Lon. $Q = 40° W$
Lon. $R = 35° E$
Lon. $T = 140° E$

2. In the diagram below, O is the centre of the earth, location A and B are on the equator. Find the location of P, Q, R and T.



Sol.

$$\begin{aligned} \text{Lon. } P &= 0° \\ \text{Lat. } P &= 30°S \\ \therefore P&(30°S, 0°) \end{aligned}$$

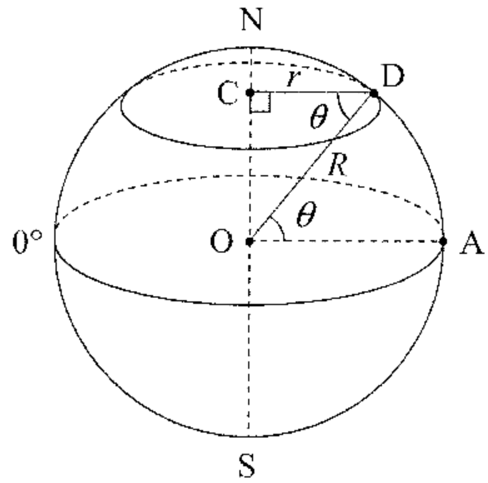
$$\begin{aligned} \text{Lon. } Q &= 35°W \\ \text{Lat. } Q &= 40°N \\ \therefore Q&(40°N, 35°W) \end{aligned}$$

$$\begin{aligned} \text{Lon. } R &= 145°E \\ \text{Lat. } R &= 40°N \\ \therefore R&(40°N, 145°E) \end{aligned}$$

$$\begin{aligned} \text{Lon. } T &= 145°E \\ \text{Lat. } T &= 30°S \\ \therefore T&(30°S, 145°E) \end{aligned}$$

Radius of the Parallel of Latitude

Let R be the radius of the earth, r be the radius of latitude θ, then $r = R\cos\theta$.



Nautical Miles

The arc length corresponding to $1'$ $(= \frac{1}{60}^{\circ})$ of the great circle on earth is called a *nautical mile* ($1NM$), that is, $1NM = \frac{1}{60\times360} \times 2\pi \times 6370km = 1.853km$.

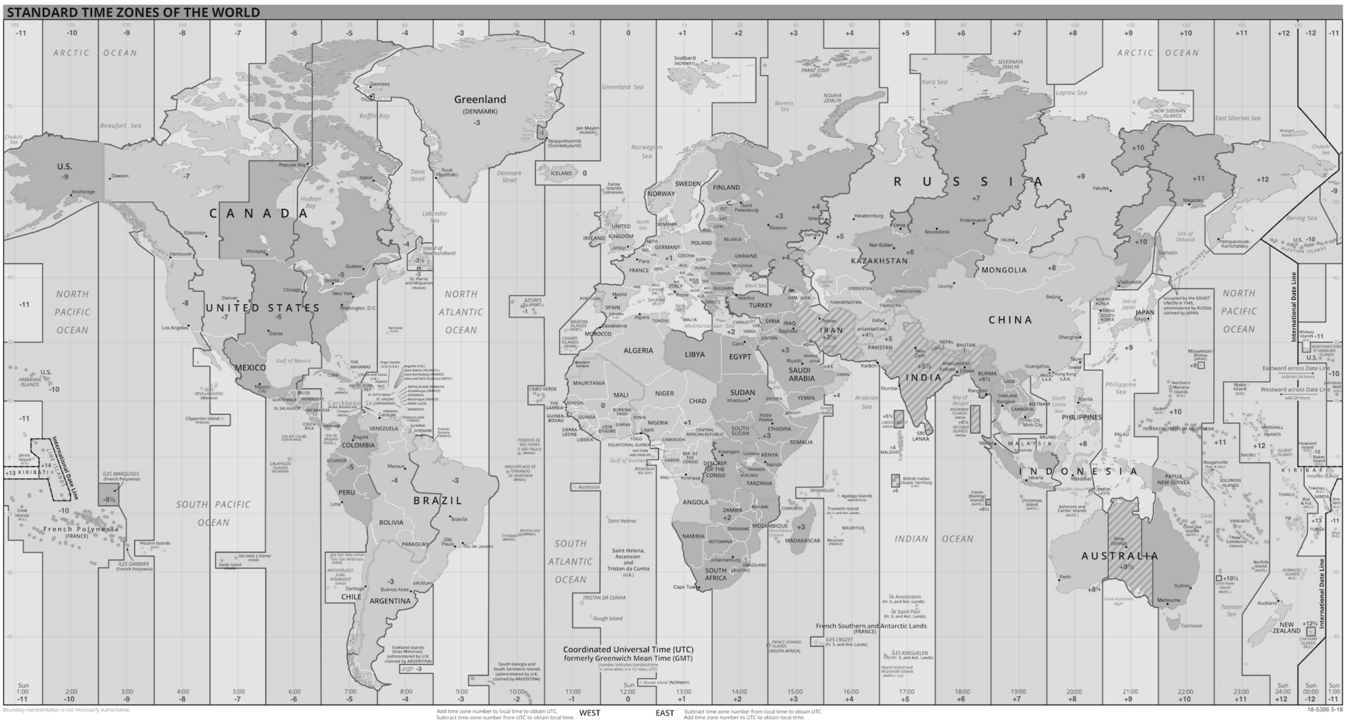
Time Difference and Longitude

The time is calculated by the rotation of the earth around its axis. The earth rotates around its axis from west to east once in $24h$. That is, the earth rotates 15° in $1h$. Thus, the time difference between two locations on the earth is equal to the difference of their longitudes. Thus, the time difference is $1hr$ per 15° of longitude difference.

1. **Local Time**
   The local time is the time at a location on the earth. The local time for any location on the same line of longitude is the same.

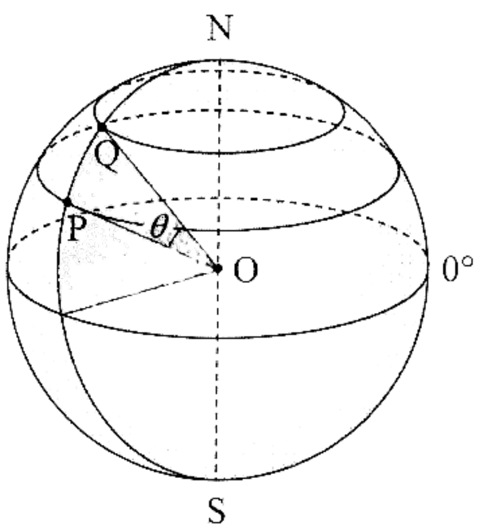
2. **Standard Time**
   Back in the year 1844, International Meridian Conference was held in Washington DC. The conference decided to divide the world into 24 time zones based on the Greenwich Meridian, called the *Greenwich Meridian Time (GMT)*. There is zero time offset 7.5° eastward and 7.5° westward of the Greenwich Meridian. The time offset is $1hr$ per 15° of longitude difference. All places in the same time zone share the same local time with the location located on the line of longitude that passes through the centre of the time zone, called the *standard time* or *zone time*.

   When entering a new time zone from the east, the local time is advanced by $1hr$ per 15° of longitude difference. When entering a new time zone from the west, the local time is delayed by $1hr$ per 15° of longitude difference.



17.5 Distance of Two Locations on the Same Line of Longitude

The distance of two location on the same line of longitude is the arc length corresponding to the difference of their latitudes. Given two location P and Q on the same line of longitude, according to the definition of nautical mile, the distance between P and Q can be acquired by the arc length of PQ. That is, $\overset{\frown}{PQ} = \theta \times 60NM$, where θ is the difference of their latitudes.



17.5.1 Practice 4

1. Given that location A and B are on the same line of longitude. Based on the following longitude, find the distance between A and B (Express your answer in nautical miles):

   (a) $A(58°N)$, $B(75°N)$

   Sol.

$$\begin{aligned}\overset{\frown}{AB} &= (75° - 58°) \times 60NM \\ &= 17 \times 60NM \\ &= 1020NM\end{aligned}$$

   (b) $A(0°)$, $B(42°S)$

   Sol.

$$\begin{aligned}\overset{\frown}{AB} &= (42° - 0°) \times 60NM \\ &= 42 \times 60NM \\ &= 2520NM\end{aligned}$$

   (c) $A(43°N)$, $B(38°S)$

   Sol.

$$\begin{aligned}\overset{\frown}{AB} &= (38° + 43°) \times 60NM \\ &= 81 \times 60NM \\ &= 4860NM\end{aligned}$$

2. Given that location P and Q are on the same line of longitude. The distane between two locations is $1000NM$, P is located at $7°30'$ north of the equator. Based on the following criteria, find the latitude of Q:

$$\begin{aligned}\overset{\frown}{PQ} &= \theta \times 60 \\ 1000 &= \theta \times 60 \\ \theta &= \frac{1000}{60} \\ &= 16°40'\end{aligned}$$

   (a) Q is located at the north of P

Sol.

$$\begin{aligned}\text{Lat. } Q &= (7°30' + 16°40')N \\ &= 24°10'N\end{aligned}$$

(b) Q is located at the south of P

Sol.

$$\begin{aligned}\text{Lat. } Q &= |7°30' - 16°40'|S \\ &= 9°10'S\end{aligned}$$

17.5.2 Exercise 17.5

1. Given that A and B are on the same line of longitude. Based on the following difference of latitude of two locations, find the distance between A and B (Express your answer in nautical miles):

(a) $\theta = 39°$

Sol.

$$\begin{aligned}\overset{\frown}{AB} &= 39 \times 60NM \\ &= 2340NM\end{aligned}$$

(b) $\theta = 80°30'$

Sol.

$$\begin{aligned}\overset{\frown}{AB} &= (80°30') \times 60NM \\ &= 4830NM\end{aligned}$$

(c) $\theta = 64°20'$

Sol.

$$\begin{aligned}\overset{\frown}{AB} &= (64°20') \times 60NM \\ &= 3860NM\end{aligned}$$

2. Given that A and B are on the same line of longitude. Based on the following distance between two locations, find the difference of latitude of A and B (Round your answer to the nearest minute):

(a) $700NM$

Sol.

$$\begin{aligned}\overset{\frown}{AB} &= \theta \times 60 \\ 700 &= \theta \times 60 \\ \theta &= \frac{700}{60} \\ &= 11°40'\end{aligned}$$

(b) $318NM$

Sol.

$$\begin{aligned}\overset{\frown}{AB} &= \theta \times 60 \\ 318 &= \theta \times 60 \\ \theta &= \frac{318}{60} \\ &= 5°18'\end{aligned}$$

(c) $3450NM$

Sol.

$$\begin{aligned}\overset{\frown}{AB} &= \theta \times 60 \\ 3450 &= \theta \times 60 \\ \theta &= \frac{3450}{60} \\ &= 57°30'\end{aligned}$$

3. Find the distance between two locations along the same line of longitude:

(a) $A(21°S, 110°E)$, $B(33°S, 110°E)$

Sol.

$$\begin{aligned}\overset{\frown}{AB} &= (33° - 21°) \times 60NM \\ &= 12 \times 60NM \\ &= 720NM\end{aligned}$$

(b) $X(38°N, 40°W)$, $Y(19°N, 40°W)$

Sol.

$$\begin{aligned}\overset{\frown}{XY} &= (38° - 19°) \times 60NM \\ &= 19 \times 60NM \\ &= 1140NM\end{aligned}$$

(c) $E(34°45'S, 80°E)$, $F(0°, 80°E)$

Sol.

$$\begin{aligned}\overset{\frown}{EF} &= (34°45' - 0°) \times 60NM \\ &= 34°45' \times 60NM \\ &= 2085NM\end{aligned}$$

(d) $P(18°15'N, 90°W)$, $Q(43°30'N, 90°W)$

Sol.

$$\begin{aligned}\overset{\frown}{PQ} &= (43°30' - 18°15') \times 60NM \\ &= 25°15' \times 60NM \\ &= 1515NM\end{aligned}$$

(e) $T(15°30'N, 120°E)$, $M(24°30'S, 120°E)$

Sol.

$$\begin{aligned}\overset{\frown}{TM} &= (24°30' + 15°30') \times 60NM \\ &= 40° \times 60NM \\ &= 2400NM\end{aligned}$$

4. Location X and Y are on the same line of longitude, the distane between them is $400NM$. Find the difference of latitude of X and Y.

Sol.

$$\begin{aligned} \overset{\frown}{XY} &= \theta \times 60 \\ 400 &= \theta \times 60 \\ \theta &= \frac{400}{60} \\ &= 6^\circ 40' \end{aligned}$$

5. Location P and Q are on the same line of longitude, and their distance along the line of longitude is $600NM$, find the difference between their latitude.

   Sol.

$$\begin{aligned} \overset{\frown}{PQ} &= \theta \times 60 \\ \frac{600}{1.853} &= \theta \times 60 \\ \theta &= \frac{600}{1.853 \times 60} \\ &\approx 5.24^\circ \end{aligned}$$

6. X city and Y city are on the same line of longitude, the latitude of X city is $2^\circ 15'$ north of the equator, the latitude of Y city is 6° north of the equator. Find the distance between X city and Y city (Express your answer in kilometers).

   Sol.

$$\begin{aligned} \overset{\frown}{XY} &= (6^\circ - 2^\circ 15') \times 60NM \\ &= 3^\circ 45' \times 60NM \\ &= 225NM \\ &= 225 \times 1.853km \\ &= 416.93km \end{aligned}$$

7. A plane is flying $1000km$ due north from airport $A(15^\circ N, 115^\circ E)$ to airport B. Find the longitude and latitude of airport B.

   Sol.

$$\begin{aligned} \overset{\frown}{AB} &= \theta \times 60NM \\ \frac{1000}{1.853} &= \theta \times 60 \\ \theta &= \frac{1000}{1.853 \times 60} \\ &= 9^\circ \\ \text{Lat.}B &= (15^\circ + 9^\circ)N \\ &= 24^\circ N \end{aligned}$$

$$\therefore B(24^\circ N, 115^\circ E)$$

8. A plane is flying $1500km$ due south from airport $A(5^\circ N, 100^\circ E)$ to airport B. Find the longitude and latitude of airport B.

   Sol.

$$\begin{aligned} \overset{\frown}{AB} &= \theta \times 60NM \\ \frac{1500}{1.853} &= \theta \times 60 \\ \theta &= \frac{1500}{1.853 \times 60} \\ &= 13^\circ 30' \\ \text{Lat.}B &= |5^\circ - 13^\circ 30'|S \\ &= 8^\circ 30' S \end{aligned}$$

$$\therefore B(8^\circ 30' S, 100^\circ E)$$

9. Find the distance from $A(18^\circ 30' S)$ to the north pole along the same line of longitude.

   Sol.

$$\begin{aligned} \overset{\frown}{AN} &= (90^\circ + 18^\circ 30') \times 60NM \\ &= 108^\circ 30' \times 60NM \\ &= 6510NM \end{aligned}$$

10. The distance between location C and D is $700NM$, C is located at the south of D. Assume that C is located at $5^\circ 30'$ north of the equator. Find the latitude of D.

    Sol.

$$\begin{aligned} \overset{\frown}{CD} &= \theta \times 60NM \\ 700 &= \theta \times 60 \\ \theta &= \frac{700}{60} \\ &= 11^\circ 40' \end{aligned}$$

$$\begin{aligned} \therefore \text{Lat.}D &= (35^\circ 30' + 11^\circ 40')N \\ &= 47^\circ 10' N \end{aligned}$$

11. A plane takes off from $P(60^\circ N, 60^\circ E)$ and flies pass north pole along the great circle route to $Q(50^\circ N, 120^\circ W)$. Find the flying distance.

    Sol.

$$\begin{aligned} \overset{\frown}{PN} &= (90^\circ - 60^\circ) \times 60NM \\ &= 30^\circ \times 60NM \\ &= 1800NM \\ \overset{\frown}{NQ} &= (90^\circ - 50^\circ) \times 60NM \\ &= 40^\circ \times 60NM \\ &= 2400NM \\ \overset{\frown}{PQ} &= \overset{\frown}{PN} + \overset{\frown}{NQ} \\ &= 1800 + 2400 \\ &= 4200NM \end{aligned}$$

12. A ship sails from $P(50^\circ S, 160^\circ E)$ due north to another port $Q(30^\circ N, 160^\circ E)$. The sailing time is 10 days.

Find the average speed of the ship. (Express your answer in NM/hr)

Sol.

$$\begin{aligned}\overset{\frown}{PQ} &= (30^\circ + 50^\circ) \times 60NM \\ &= 80^\circ \times 60NM \\ &= 4800NM \\ \text{Average speed} &= \frac{\overset{\frown}{PQ}}{10 \times 24} \\ &= 20NM/hr\end{aligned}$$

13. Given that PQ is the diameter of the parallel of latitude $35^\circ S$. A plane takes off from location P, flies pass the south pole along the line of longitude, and lands at location Q after $13hrs40mins$. Find the average speed of the plane for the whole flight duration. (Express your answer in NM/hr)

Sol.

$$\begin{aligned}\overset{\frown}{PQ} &= 2(90^\circ - 35^\circ) \times 60NM \\ &= 110^\circ \times 60NM \\ &= 6600NM \\ \text{Average speed} &= \frac{\overset{\frown}{PQ}}{13\frac{40}{60}hr} \\ &= \frac{6600NM}{\frac{41}{3}hr} \\ &= 482.93NM/hr\end{aligned}$$