Mathematics

Senior 3 Part II

MELVIN CHIA

Started on 12 June 2023

Finished on XX XX 2023

Actual time spent: XX days

Preface

Why this book?

Disclaimer

Acknowledgements

Contents

Chapter 26

Applications of Differentiation

26.1 Revision Exercise 26

1. Find the equation of the tangent of the curve $y = x^3 - 3x$ at the point where $x = 3$.

Sol.

$$y = x^3 - 3x$$
$$\frac{dy}{dx} = 3x^2 - 3$$

At $x = 3$, $y = (3)^3 - 3(3) = 18$.

$$\begin{aligned}\text{Gradient of tangent } \frac{dy}{dx} &= 3(3)^2 - 3\\ &= 27 - 3\\ &= 24\end{aligned}$$

$$\begin{aligned}\therefore \text{Equation of tangent is } y - 18 &= 24(x - 3)\\ y - 18 &= 24x - 72\\ y &= 24x - 54\end{aligned}$$

2. Find the equation of the normal of the curve $y = x(x-4)(x+1)$ at the points of intersection of the curve and the x-axis.

Sol.

$$\begin{aligned}y &= x(x-4)(x+1)\\ &= x(x^2 - 3x - 4)\\ &= x^3 - 3x^2 - 4x\\ \frac{dy}{dx} &= 3x^2 - 6x - 4\end{aligned}$$

$$\begin{aligned}\text{When } x = 0, y &= 0\\ x(x-4)(x+1) &= 0\\ x = 0 \text{ or } x &= 4 \text{ or } x = -1\end{aligned}$$

When $x = 0$,

$$\begin{aligned}\because \text{Gradient of tangent } \frac{dy}{dx} &= 3(0)^2 - 6(0) - 4 = -4\\ \therefore \text{Gradient of normal } &= \frac{1}{4}\\ \therefore \text{Equation of normal is } y - 0 &= \frac{1}{4}(x - 0)\\ y &= \frac{1}{4}x\\ x - 4y &= 0\end{aligned}$$

When $x = 4$,

$$\begin{aligned}\because \text{Gradient of tangent } \frac{dy}{dx} &= 3(4)^2 - 6(4) - 4 = 20\\ \therefore \text{Gradient of normal } &= -\frac{1}{20}\\ \therefore \text{Equation of normal is } y - 0 &= -\frac{1}{20}(x - 4)\\ x + 20y - 4 &= 0\end{aligned}$$

When $x = -1$,

$$\begin{aligned}\because \text{Gradient of tangent } \frac{dy}{dx} &= 3(-1)^2 - 6(-1) - 4 = 5\\ \therefore \text{Gradient of normal } &= -\frac{1}{5}\\ \therefore \text{Equation of normal is } y - 0 &= -\frac{1}{5}(x + 1)\\ x + 5y + 1 &= 0\end{aligned}$$

Hence, the equations of the normals are $x - 4y = 0$, $x + 20y - 4 = 0$ and $x + 5y + 1 = 0$.

3. Given that the curve $y = ax^2 + bx - 10$ passes through the point $(2, 0)$, and that the gradient of the curve at the point is 3. Find the values of a and b.

Sol.

$$\begin{aligned} y &= ax^2 + bx - 10 \\ \frac{dy}{dx} &= 2ax + b \end{aligned}$$

Since the curve passes through $(2, 0)$,

$$\begin{aligned} 0 &= a(2)^2 + b(2) - 10 \\ 0 &= 4a + 2b - 10 \\ 4a &= 10 - 2b \\ a &= \frac{10 - 2b}{4} \\ &= \frac{5 - b}{2} \quad \cdots \ (1) \end{aligned}$$

Since the gradient of the curve at the point is 3,

$$\begin{aligned} 3 &= 2a(2) + b \\ 3 &= 4a + b \quad \cdots \ (2) \end{aligned}$$

Substituting (1) into (2),

$$\begin{aligned} 3 &= 4\left(\frac{5 - b}{2}\right) + b \\ 3 &= 2(5 - b) + b \\ 3 &= 10 - 2b + b \\ b &= 7 \end{aligned}$$

Substituting $b = 7$ into (1),

$$\begin{aligned} a &= \frac{5 - 7}{2} \\ &= -1 \end{aligned}$$

Hence, $a = -1$ and $b = 7$.

4. Find the equation of the normal of the curve $y = x + \frac{2}{x}$ at the point $(2, 3)$. If the normal line intersects with the x-axis and y-axis at A and B respectively, find the length of AB.

Sol.

$$\begin{aligned} y &= x + \frac{2}{x} \\ \frac{dy}{dx} &= 1 - \frac{2}{x^2} \end{aligned}$$

At $x = 2$,

$$\begin{aligned} \frac{dy}{dx} &= 1 - \frac{2}{2^2} \\ &= \frac{1}{2} \end{aligned}$$

Hence, the gradient of the normal at the point $(2, 3)$ is -2.

Therefore, the equation of the normal is

$$\begin{aligned} y - 3 &= -2(x - 2) \\ y &= -2x + 7 \end{aligned}$$

When $y = 0$,

$$\begin{aligned} 0 &= -2x + 7 \\ x &= \frac{7}{2} \\ \therefore A &= \left(\frac{7}{2}, 0\right) \end{aligned}$$

When $x = 0$,

$$\begin{aligned} y &= -2(0) + 7 \\ y &= 7 \\ \therefore B &= (0, 7) \end{aligned}$$

$$\begin{aligned} AB &= \sqrt{\left(\frac{7}{2} - 0\right)^2 + (0 - 7)^2} \\ &= \sqrt{\frac{49}{4} + 49} \\ &= \sqrt{\frac{245}{4}} \\ &= \frac{\sqrt{245}}{2} \\ &= \frac{7\sqrt{5}}{2} \end{aligned}$$

Of the following functions, which intervals are the function increasing or decreasing? (Question 5 to 6)

5. $f(x) = 2x^2(6 - x)$

   Sol.

$$\begin{aligned} f(x) &= 2x^2(6 - x) \\ &= 12x^2 - 2x^3 \\ f'(x) &= 24x - 6x^2 \\ f'(x) &= 0 \\ 24x - 6x^2 &= 0 \\ x(x-4) &= 0 \\ x &= 0 \text{ or } x = 4 \end{aligned}$$

   At the interval $(-\infty, 0)$, $f'(x) < 0$, hence $f(x)$ is decreasing at the interval $(-\infty, 0]$.

   At the interval $(0, 4)$, $f'(x) > 0$, hence $f(x)$ is increasing at the interval $[0, 4]$.

   At the interval $(4, \infty)$, $f'(x) < 0$, hence $f(x)$ is decreasing at the interval $[4, \infty)$.

6. $f(x) = 4x^3 - 3x^2 - 6x + 1$

   Sol.

$$\begin{aligned} f(x) &= 4x^3 - 3x^2 - 6x + 1 \\ f'(x) &= 12x^2 - 6x - 6 \\ f'(x) &= 0 \\ 12x^2 - 6x - 6 &= 0 \\ 2x^2 - x - 1 &= 0 \\ (2x+1)(x-1) &= 0 \\ x &= -\frac{1}{2} \text{ or } x = 1 \end{aligned}$$

   At the interval $\left(-\infty, -\frac{1}{2}\right)$, $f'(x) > 0$, hence $f(x)$ is increasing at the interval $\left(-\infty, -\frac{1}{2}\right]$.
   At the interval $\left(-\frac{1}{2}, 1\right)$, $f'(x) < 0$, hence $f(x)$ is decreasing at the interval $\left[-\frac{1}{2}, 1\right]$.
   At the interval $(1, \infty)$, $f'(x) > 0$, hence $f(x)$ is increasing at the interval $[1, \infty)$.

7. If $x - y = 3$, find the relative minimum value of x^2y.

Sol.

$$x - y = 3$$
$$y = x - 3$$

Let $f(x) = x^2y$,

$$\begin{aligned} f(x) &= x^2y \\ &= x^2(x-3) \\ &= x^3 - 3x^2 \\ f'(x) &= 3x^2 - 6x \\ f'(x) &= 0 \\ 3x^2 - 6x &= 0 \\ x(x-2) &= 0 \\ x &= 0 \text{ or } x = 2 \end{aligned}$$

$f''(x) = 6x - 6$

$\because f''(0) = -6 < 0,\ f''(2) = 6 > 0$

$\therefore f(2) = -4$ is a relative minimum value.

8. If $2x^2 + y^2 = 6x$, find the relative maximum value of $x^2 + y^2 + 2x$.

Sol.

$$2x^2 + y^2 = 6x$$
$$y^2 = 6x - 2x^2$$

Let $f(x) = x^2 + y^2 + 2x$,

$$\begin{aligned} f(x) &= x^2 + y^2 + 2x \\ &= x^2 + 6x - 2x^2 + 2x \\ &= -x^2 + 8x \\ f'(x) &= -2x + 8 \\ f'(x) &= 0 \\ -2x + 8 &= 0 \\ x - 4 &= 0 \\ x &= 4 \end{aligned}$$

$f''(x) = -2$

$\because f''(4) = -2 < 0$

$\therefore f(4) = 16$ is a relative maximum value.

9. Given that $y = 18x^2 + 12x + 7$ has a relative minimum value q and the point where $x = p$. Find the value of p and q.

Sol.

$$\begin{aligned} y &= 18x^2 + 12x + 7 \\ y' &= 36x + 12 \\ y' &= 0 \\ 36x + 12 &= 0 \\ 3x + 1 &= 0 \\ p = x &= -\frac{1}{3} \end{aligned}$$

When $x = -\frac{1}{3},\ y = 5$

$y'' = 36 > 0$

$\therefore$ The relative minimum value is $q = 5$.

10. There's a rectangular field where one side of it is a wall and the other three sides are fenced. If the total length of the fence is $40m$, find the width and height of the field such that the area of the field is the maximum.

Sol. Let x be the length of the field and y be the width of the field.

$$\begin{aligned}
2x + y &= 40 \\
y &= 40 - 2x \\
A &= xy \\
&= x(40 - 2x) \\
&= 40x - 2x^2 \\
\frac{dA}{dx} &= 40 - 4x \\
\frac{dA}{dx} &= 0 \\
40 - 4x &= 0 \\
x &= 10
\end{aligned}$$

$\because \dfrac{d^2A}{dx^2} = -4 < 0$

$\therefore$ The area of the field is the maximum when $x = 10$.When $x = 10,\ y = 20$.

$\therefore$ The field has a width of $20m$ and a height of $10m$ when the area is the maximum.

11. One side of a rectangle with a perimeter of $18cm$ is revolved about one side to form a cylinder. If the volume of the cylinder is the maximum, find the dimensions of the rectangle and the maximum volume of the cylinder.

Sol.

Let the length of the rectangle be x and the width of the rectangle be y.

$$\begin{aligned}
2x + 2y &= 18 \\
x + y &= 9 \\
y &= 9 - x
\end{aligned}$$

$$\begin{aligned}
V &= \pi r^2 h \\
&= \pi x^2 y \\
&= \pi(9x^2 - x^3) \\
\frac{dV}{dx} &= \pi(18x - 3x^2) \\
\frac{dV}{dx} &= 0 \\
\pi(18x - 3x^2) &= 0 \\
x^2 - 6x &= 0 \\
x(x - 6) &= 0 \\
x &= 6,\ x = 0 \text{ (rejected, } x > 0)
\end{aligned}$$

$\because \dfrac{d^2V}{dx^2} = \pi(18 - 6x) = -18\pi < 0$

$\therefore$ The volume of the cylinder is the maximum when $x = 6$.When $x = 6,\ y = 3$.

$\therefore$ The rectangle has a length of $6cm$ and a width of $3cm$ when the volume is the maximum.

Also, the maximum volume of the cylinder is $V = \pi(6)^2(3) = 108\pi\ \text{cm}^3$ when the volume is the maximum.

12. The cross section of a tunnel is a rectangle with a semicircle on top of it. If the area of the cross section is fixed, find the ratio of the radius of the semicircle to the height of the rectangle such that the perimeter of the cross section is the minimum.

Sol.

Let the radius of the semicircle be r and the height of the rectangle be h.

$$\begin{aligned}
A &= \frac{1}{2}\pi r^2 + 2rh \\
2rh &= A - \frac{1}{2}\pi r^2 \\
h &= \frac{A - \frac{1}{2}\pi r^2}{2r} \\
&= \frac{A}{2r} - \frac{1}{4}\pi r \\
P &= \pi r + 2h + 2r \\
&= (\pi + 2)r + \frac{A}{r} - \frac{1}{2}\pi r \\
\frac{dP}{dr} &= \pi + 2 - \frac{A}{r^2} - \frac{1}{2}\pi \\
&= \frac{1}{2}\pi + 2 - \frac{A}{r^2} \\
\frac{dP}{dr} &= 0 \\
\frac{1}{2}\pi + 2 - \frac{A}{r^2} &= 0 \\
\frac{1}{2}\pi + 2 - \left(\frac{1}{2}\pi r^2 + 2rh\right) \cdot \frac{1}{r^2} &= 0 \\
\frac{1}{2}\pi + 2 - \frac{1}{2}\pi - \frac{2}{r}h &= 0 \\
2 - \frac{2}{r}h &= 0 \\
2 &= \frac{2}{r}h \\
2r &= 2h \\
r &= h
\end{aligned}$$

Hence, the ratio of the radius of the semicircle to the height of the rectangle is 1 : 1.

13. Split 28 into two parts such that the sum of the squares of the one part and the cube of the other part is the minimum.

Sol.

Let the two parts be x and y.

$$x + y = 28$$
$$y = 28 - x$$
$$S = x^2 + y^3$$
$$= x^2 + (28 - x)^3$$
$$\frac{dS}{dx} = 2x - 3(28 - x)^2$$
$$\frac{dS}{dx} = 0$$
$$2x - 3(28 - x)^2 = 0$$
$$2x - 3(784 - 56x + x^2) = 0$$
$$2x - 2352 + 168x - 3x^2 = 0$$
$$3x^2 - 170x + 2352 = 0$$
$$(3x - 98)(x - 24) = 0$$
$$x = 24 \text{ or } x = \frac{98}{3}$$
$$\frac{d^2S}{dx^2} = 2 + 6(28 - x)$$
$$= 2 + 168 - 6x$$
$$= -6x + 170$$

When $x = 24$, $\frac{d^2S}{dx^2} = -6(24) + 170$
$$= 26 > 0$$

When $x = \frac{98}{3}$, $\frac{d^2S}{dx^2} = -6\left(\frac{98}{3}\right) + 170$
$$= -26 < 0$$

$\because$ When $x = 24$, $\frac{d^2S}{dx^2} > 0$,

$\therefore$ The sum of the squares of the one part and the cube of the other part is the minimum when $x = 24$.

$\because$ When $x = 24$, $y = 4$.

$\therefore$ The two parts are 24 and 4.

14. The capacity of a cylindrical can is fixed. If the material used to make the can is the minimum, what should be the ratio of the radius of the base to the height of the can?

    Sol.

    Let the radius of the base be r and the height of the can be h.

$$\begin{aligned}
V &= \pi r^2 h \\
h &= \frac{V}{\pi r^2} \\
A &= 2\pi r^2 + 2\pi r h \\
&= 2\pi r^2 + \frac{2V}{r} \\
\frac{dA}{dr} &= 4\pi r - \frac{2V}{r^2} \\
\frac{dA}{dr} &= 0 \\
4\pi r - \frac{2V}{r^2} &= 0 \\
2\pi r^3 - \pi r^2 h &= 0 \\
2r^3 - r^2 h &= 0 \\
2r - h &= 0 \\
2r &= h \\
\frac{r}{h} &= \frac{1}{2}
\end{aligned}$$

    Hence, the ratio of the radius of the base to the height of the can is $1 : 2$.

Find the coordinate of the point of inflection of the following functions. (Question 15 to 16)

15. $y = x^3 - 2$

16. $3x + (2 - x)^3$

17. Given the function $y = \dfrac{x}{1 - x^2}$. Find the extreme values of the function, and determine the coordinates of the convex intervals and the point of inflection.

18. Given the function $y = \dfrac{x}{x^2 + 1}$.

    (a) Find the coordinates of the stationary points.

    (b) Determine which intervals the function is increasing or decreasing.

    (c) Find the coordinates of the convex intervals and the point of inflection.

Construct the graph of the following functions. (Question 19 to 20)

19. $y = x^3 - 5x^2 + 3x - 2$

20. $y = x^3 - 3x^2 + 4$

21. In a container, the relationship between the volume of water V (cm^3) and the depth of water x (cm) is given by the equation $V = 4x^2 + \frac{1}{6}x^3$. If the water is poured into the container at a rate of 6 cm^3 per second, find the rate of change of the depth of water when $x = 2$ cm.

22. The water is poured into a conical pool with a width and a base radius of 20m and 10m respectively at a rate of 5m^3/min. When the height of the water is 10cm, find

    (a) the rate of increasing of the height of the water.

(b) the rate of change of the radius of the water surface.

23. The radius of a spherical container decreases from 4cm to 3.95cm. Find the approximate amount of decrease in the volume and the surface area of the container.

24. The capacity of water of a spherical container is given by $V = \left[\frac{\pi h^2}{3}(15-h)\right]\text{cm}^3$, where h is the depth of the water. Find the approximate amount of increase in the capacity of the container when the depth of the water increases from 4cm to 4.01cm.

25. In a bowl, when the height of the water is $h$cm, the volume of the water is given by $V = \left(h^2 + 3h^2 + 11h\right)\text{cm}^3$. When the height of the water is $7cm$, pour an additional $\Delta V\text{cm}^3$ of water into the bowl. Find the approximate amount of increase in the height of the water.

26. If $y = \frac{1}{\sqrt[3]{[x]}}$, find $\frac{dy}{dx}$. Hence, find the approximate value of $\frac{1}{\sqrt[3]{[130}}$. (Correct to 3 decimal places)

Chapter 27

Indefinite Integrals

27.1 Indefinite Integrals as the Inverse of Differentiation

Let function $F(x)$ and $f(x)$ be defined at the interval (a, b). If any point x in this interval satisfies $F'(x) = f(x)$, then $F(x)$ is is the preimage of $f(x)$ at the interval (a, b).

According to the definition above, to find the preimage of a function $f(x)$, we need to find the function $F(x)$ that satisfies $F'(x) = f(x)$. For example,

$$\begin{aligned}(x^2)' &= 2x \\ (x^2+1)' &= 2x \\ (x^2-2)' &= 2x \\ &\vdots\end{aligned}$$

For any constant C, the derivative of $x^2 + C$ is $2x$. Hence, the preimage of $2x$ is $x^2 + C$, where C is an arbitary constant.

Since $[F(x) + C]' = F'(x) + 0 = F'(x)$, if the function $F(x)$ is a preimage of $f(x)$, then $F(x) + C$ (C is a constant) is also a preimage of $f(x)$. That is to say, there are infinite number of preimages of a function $f(x)$.

In the other hand, if $F(x)$ and $G(x)$ are both preimages of $f(x)$, then $F'(x) = f(x)$, $G'(x) = f(x)$.

$$\begin{aligned}\because\ [G(x) - F(x)]' &= G'(x) - F'(x) \\ &= f(x) - f(x) \\ &= 0\end{aligned}$$

$$\begin{aligned}&\therefore G(x) - F(x) = C \\ &\quad \text{that is, } G(x) = F(x) + C\end{aligned}$$

This shows that for any two preimages of $f(x)$, the difference between them is a constant. If $F(x)$ is a preimage of $f(x)$, then all the preimages of $f(x)$ can be expressed as $F(x) + C$, where C is a constant.

The Concept of Indefinite Integral

Let function $F(x)$ be a preimage of $f(x)$. All the preimages $F(x) + C$ (C is a constant) of a function $f(x)$ is called the indefinite integral of $f(x)$, denoted by $\int f(x)dx$. That is, $\int f(x)dx = F(x) + C$. $\int$ is called the integral sign, $f(x)$ is called the integrand, C is called the constant of integration.

Finding the indefinite integral of a function $f(x)$ is equivalent to finding all the preimages of $f(x)$. From the explanation above, we just need to find one preimage of $f(x)$, then add a constant C to it to get the indefinite integral of $f(x)$.

27.2 Arithmetic Properties of Indefinite Integrals

Basic Formulas of Indefinite Integrals

In order to learn the methods and skills of finding indefinite integrals, we must first learn some basic formulas of indefinite integrals. We know that finding the indefinite integral is equivalent to finding the anti-derivative. Therefore, we can get the formulas of indefinite integrals from the corresponding formulas of derivatives.

For example, when $n \neq -1$, $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$.

Hence, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

Similarly, we can also get the other basic formulas of indefinite integrals. The basic formulas of indefinite integrals are listed below:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1 \qquad \int \frac{1}{x} dx = \ln|x| + C$$

$$\int e^x dx = e^x + C \qquad \int a^x dx = \frac{a^x}{\ln a} + C,$$

$$\int \sin x dx = -\cos x + C \qquad \int \cos x dx = \sin x + C$$

$$\int \sec^2 x dx = \tan x + C \qquad \int \csc^2 x dx = -\cot x + C$$

$$\int \csc x \cot x dx = -\csc x + C \qquad \int \sec x \tan x dx = \sec x + C$$

27.2.1 Practice 1

Find the following indefinite integrals:

1. $\int x^3 dx$
2. $\int \sqrt[3]{x} dx$
3. $\int \frac{1}{x^3} dx$
4. $\int \frac{1}{\sqrt[3]{x}} dx$

27.2.2 Exercise 27.2a

Find the following indefinite integrals:

1. $\int 3x dx$
2. $\int 5x^4 dx$
3. $\int 5 dx$

4. $\int x^{-9} dx$

5. $\int x^{\frac{1}{2}} dx$

6. $\int 2x^{-\frac{1}{2}} dx$

7. $\int \frac{1}{x^5} dx$

8. $\int \left(\frac{1}{x}\right)^4 dx$

9. $\int \sqrt{3x} dx$

10. $\int x^3 \sqrt[3]{x^2} dx$

11. $\int \cos(-x) dx$

12. $\int \frac{2}{\csc x} dx$

13. $\int \frac{1}{\sin^2 x} dx$

14. $\int \frac{1}{1 - \sin^2 x} dx$

15. $\int \frac{\sin x}{\cos^2 x} dx$

16. $\int \frac{\cos x}{\sin^2 x} dx$

Using the arithmetic properties of derivatives, we can also get the following arithmetic properties of indefinite integrals:

1. Constant Multiple Rule: $\int kf(x)dx = k\int f(x)dx$, where k is a constant.
2. Sum and Difference Rule: $\int [f(x) \pm g(x)]\, dx = \int f(x)dx \pm \int g(x)dx$

Note that when integrating by parts, the result of each part of the integral contains a constant of integration. However, the sum (or difference) of multiple constants of integration is still a constant of integration. Therefore, when integrating by parts, we only need to add one constant of integration to the final result.

27.2.3 Practice 2

Find the following indefinite integrals:

1. $\int \frac{1}{4}x^3 dx$
2. $\int \left(\frac{2}{\sqrt{x}} - \sqrt[3]{x^2}\right) dx$
3. $\int \frac{x^2+3x-1}{x^3} dx$
4. $\int (\sin x - 3\cos x + 2^x)\, dx$

27.2.4 Practice 3

1. Prove that $\frac{d}{dx}\left(\frac{x^2-2}{x+1}\right) = \frac{x^2+2x+2}{(x+1)^2}$. Hence, find $\int \frac{x^2+2x+2}{2(x+1)^2} dx$.
2. Given the function $y = \frac{3}{(5x+7)^3}$, and $\frac{dy}{dx} = 5g(x)$, find $\int [3 - g(x)]\, dx$.

27.2.5 Exercise 27.2b

Find the following indefinite integrals (Question 1 to 20):

1. $\int (x^3 - 3x + 1)dx$
2. $\int \left(5x^4 + 2\sqrt{x}\right) dx$
3. $\int \left(\frac{x^2}{2} - \frac{2}{x^2}\right) dx$
4. $\int (\sin x - 3\cos x)dx$
5. $\int (x-5)^2 dx$
6. $\int (x-1)(x-2)dx$

7. $\int (x^2+2)\sqrt{x}dx$

8. $\int \frac{x^4-5}{x^2}dx$

9. $\int \frac{x+5}{\sqrt{x}}dx$

10. $\int \frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}}dx$

11. $\int \frac{x^2-9}{x+3}dx$

12. $\int \frac{x^3-8}{x-2}dx$

13. $\int \sqrt[3]{2}\left(\sqrt{x}-\frac{1}{x}\right)dx$

14. $\int \frac{(2x+1)^2}{x}dx$

15. $\int \frac{(x+1)(3x^2-4)}{2x^3}dx$

16. $\int \left(\frac{x-1}{x^2}\right)^2 dx$

17. $\int \left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2 dx$

18. $\int \tan^2 x dx$

19. $\int \left(e^2+\frac{1}{4x}\right)dx$

20. $\int (2e)^x dx$

21. Given the function $y=\frac{5x}{3-x}$, find $\frac{dy}{dx}$. Hence, find $\int \frac{1}{(3-x)^2}dx$.

22. If the function $y=\frac{2x^2}{3x-1}$, find $\frac{dy}{dx}$. Hence, find $\int \frac{2x-3x^2}{(3x-1)^2}dx$.

23. Prove that $\frac{d}{dx}\left(\frac{3x^2}{x^2+2}\right)=\frac{12x}{(x^2+2)^2}$. Hence, find $\int \frac{4x}{(x^2+2)^2}dx$.

24. Given that $\frac{d}{dx}\left(\frac{3x^2-1}{5x^2+7}\right)=f(x)$, find $\int \left[3x^2-1-2f(x)\right]dx$.

25. Given that $\frac{d}{dx}\left(\frac{2+x^3}{2-x^3}\right)=3g(x)$, find $\int [g(x)-3x+2]\,dx$.

27.3 Integration by Substitution

In the last section, we learned to find the indefinite integral of some functions using some basic formulas of indefinite integrals and two arithmetic properties of indefinite integrals. However, for the indefinite integral of some more complicated functions like $\int 3\sqrt{3x+1}dx$, $\int 2\sin 2x dx$, etc., we cannot find their indefinite integrals straight away using the basic formulas of indefinite integrals and the arithmetic properties of indefinite integrals. Hence, we need to learn some other methods to find the indefinite integral of these functions. Here we will introduce a method called integration by substitution.

Consider a function $F(u)$, where u is a function of x, that is, $u = g(x)$.

Using the chain rule, we have $\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x)$. Hence,

$$\boxed{\int F'(g(x)) \cdot g'(x)dx = F(g(x)) + C}$$

Generally speaking, during the calculation process, we let $u = g(x)$.

$$\begin{aligned}
\therefore \int F'(g(x)) \cdot g'(x)dx &= \int F'(u)\frac{du}{dx}dx \\
&= \int F'(u)du \\
&= F(u) + C \\
&= F(g(x)) + C
\end{aligned}$$

For the functions that we cannot find their indefinite integrals straight away using the basic formulas of indefinite integrals, if it can be expressed in the form of $\int F'(g(x)) \cdot g'(x)dx$, we can perform substitution using $u = g(x)$ and express the indefinite integral as $\int F'(u)du$ to find its indefinite integral.

27.3.1 Practice 4

Find the following indefinite integral:

1. $\int \sqrt[3]{3x+1}dx$
2. $\int \frac{1}{1-4x}dx$
3. $\int 2^{4x+3}dx$
4. $\int \sin x \cos x dx$

27.3.2 Exercise 27.3a

Find the following indefinite integral:

1. $\int (2x+1)^3 dx$
2. $\int (3x+2)^5 dx$

3. $\int (3-x)^6 dx$

4. $\int (2x-1)^{-3} dx$

5. $\int 4\sqrt{2x-1} dx$

6. $\int 2(3x+1)^2 dx$

7. $\int \frac{dx}{(2x+5)^8}$

8. $\int \frac{2}{(3-2x)^2} dx$

9. $\int x\sqrt{x^2+1} dx$

10. $\int 3x^2 \left(x^3+4\right)^3 dx$

11. $\int 15x^2 \left(x^3-1\right)^4 dx$

12. $\int (2x+1)\left(x^2+x+2\right)^5 dx$

13. $\int \left(x^2-2x\right)\left(x^3-3x^2+1\right)^4 dx$

14. $\int \frac{x+1}{x^2+2x+3} dx$

15. $\int x^2 \cos\left(x^3+2\right) dx$

16. $\int \sin\frac{x}{2} dx$

17. $\int \frac{\ln^2 x}{x} dx$

18. $\int e^{1-2x} dx$

19. $\int \left(e^x+e^{-x}\right) dx$

20. $\int xe^{x^2} dx$

27.3.3 Practice 5

Find the following indefinite integral:

1. $\int \sin 2x \cos 2x dx$

2. $\int \cos^2 2x dx$

3. $\int \sin^3 x dx$

4. $\displaystyle\int \cos^3 x dx$

5. $\displaystyle\int \tan^4 x \sec^2 x dx$

6. $\displaystyle\int \tan^4 \frac{x}{2} dx$

27.3.4 Exercise 27.3b

Find the following indefinite integral:

1. $\displaystyle\int \sin^2 \frac{x}{2} dx$

2. $\displaystyle\int \tan^2 5x dx$

3. $\displaystyle\int \frac{1}{\sec^2 4x} dx$

4. $\displaystyle\int \cos^2(3x-1) dx$

5. $\displaystyle\int \sec 5x \tan 5x dx$

6. $\displaystyle\int -\csc 3x \cot 3x dx$

7. $\displaystyle\int \left(\sin \frac{x}{8} - \sec^2 2x\right) dx$

8. $\displaystyle\int \left(\sin \frac{x}{2} + \cos \frac{x}{2}\right)^2 dx$

9. $\displaystyle\int (\sec x + \tan x)^2 dx$

10. $\displaystyle\int (2 - \sin x)^2 dx$

11. $\displaystyle\int \cos^4 x dx$

12. $\displaystyle\int \left(1 + \tan^2 x\right)\left(1 - \tan^2 x\right) dx$

13. $\displaystyle\int \sin^2 4x \cos 4x dx$

14. $\displaystyle\int 3 \cot^3 3x \csc^2 3x dx$

15. $\displaystyle\int \tan^2 x \sec^4 x dx$

16. $\displaystyle\int \sec x \cdot \tan^3 x dx$

27.4 Integration by Partial Fractions

Generally speaking, in order to find the indefinite integral of a proper fraction, we first have to decompose the proper fraction into partial fractions, then perform integration on each term. This method of integration is called integration by partial fractions.

27.4.1 Practice 6

Find the following indefinite integral:

1. $\displaystyle\int \frac{x}{(x-1)(x-2)}dx$
2. $\displaystyle\int \frac{x}{(x-1)^2}dx$

27.4.2 Exercise 27.4

Find the following indefinite integral:

1. $\displaystyle\int \frac{1}{x(x+1)}dx$
2. $\displaystyle\int \frac{x}{(x+1)(x-3)}dx$
3. $\displaystyle\int \frac{4x-13}{2x^2+x-6}dx$
4. $\displaystyle\int \frac{5x-1}{1-x^2}dx$
5. $\displaystyle\int \frac{x^2+5}{(x+1)(x-1)}dx$
6. $\displaystyle\int \frac{x^3+2}{x^2-1}dx$
7. $\displaystyle\int \frac{2x-1}{x^2+2x+1}dx$
8. $\displaystyle\int \frac{4x-3}{(2x+1)^2}dx$

27.5 Applications of Indefinite Integrals

We can use derivative to find the slope of a tangent line to a curve at any point on the curve. Conversely, if the slope of a tangent line to a curve at a point on the curve is known, we can use indefinite integration to find the equation of the curve.

27.5.1 Practice 7

A curve $y = f(x)$ passes through the point $\left(\frac{1}{2}, \frac{3}{2}\right)$. If the gradient of the tangent line of a point on the curve is $\frac{dy}{dx} = 3 - \frac{1}{x^2}$, find the equation of the curve.

27.5.2 Exercise 27.5

1. Given that $\frac{dy}{dx} = 4x^3 - 6x^2 + 3$, and when $x = 2$, $y = 7$. Express y in terms of x.
2. The gradient of the tangent line of a point on a curve is $\frac{dy}{dx} = x^2 + 2x - 4$, and the curve passes through point $(3, 3)$. Find the equation of the curve.
3. The gradient of a curve at the point $(1, -1)$ is -4, and $\frac{dy}{dx} = \sqrt{x} + k$. Find
   (a) The value of k.
   (b) The equation of the curve.

Revision Exercise 27

Find the following indefinite integral (Question 1 to 34):

1. $\int 2x^{\frac{1}{5}}dx$

Sol.

$$\begin{aligned}\int 2x^{\frac{1}{5}}dx &= 2\cdot\frac{5}{6}x^{\frac{6}{5}}+C\\ &= \frac{5}{3}x^{\frac{6}{5}}+C\end{aligned}$$

2. $\int (2x-1)^3dx$

Sol.

$$\begin{aligned}\int (2x-1)^3dx &= \frac{1}{2}\int (2x-1)^3d(2x-1)\\ &= \frac{1}{2}\cdot\frac{1}{4}(2x-1)^4+C\\ &= \frac{1}{8}(2x-1)^4+C\end{aligned}$$

3. $\int (x+4)^{100}dx$

Sol.

$$\begin{aligned}\int (x+4)^{100}dx &= \int (x+4)^{100}d(x+4)\\ &= \frac{1}{101}(x+4)^{101}+C\end{aligned}$$

4. $\int \left(\frac{5}{x^2}+2x^{\frac{1}{2}}+3\right)dx$

Sol.

$$\int \left(\frac{5}{x^2}+2x^{\frac{1}{2}}+3\right)dx$$
$$=5\int x^{-2}+2\int x^{\frac{1}{2}}+3\int dx$$
$$=-\frac{5}{x}+\frac{4}{3}x^{\frac{3}{2}}+3x+C$$

5. $\int \left(3x^2+\frac{1}{x^2}-\sin x\right)dx$

Sol.

$$\int \left(3x^2+\frac{1}{x^2}-\sin x\right)dx$$
$$=3\int x^2+\int x^{-2}-\int \sin x dx$$
$$=x^3-x^{-1}+\cos x+C$$
$$=x^3+\frac{1}{x}+\cos x+C$$

6. $\int \left(4\cos x+\frac{1}{x}+x^3\right)dx$

Sol.

$$\int \left(4\cos x+\frac{1}{x}+x^3\right)dx$$
$$=4\int \cos x+\int x^{-1}+\int x^3dx$$
$$=4\sin x+\ln|x|+\frac{1}{4}x^4+C$$

7. $\int \frac{3x^3-2x^2+x^{-1}}{x^2}dx$

Sol.

$$\int \frac{3x^3-2x^2+x^{-1}}{x^2}dx$$
$$=\int (3x-2+x^{-3})dx$$
$$=\frac{3}{2}x^2-2x-\frac{1}{2}x^{-2}+C$$

8. $\int (2x-1)(x+2)dx$

Sol.

$$\int (2x-1)(x+2)dx$$
$$=\int (2x^2+3x-2)dx$$
$$=\frac{2}{3}x^3+\frac{3}{2}x^2-2x+C$$

9. $\int \left(x - \frac{1}{x^2}\right)^2 dx$

Sol.

$$\int \left(x - \frac{1}{x^2}\right)^2 dx$$
$$= \int (x^2 - 2x^{-1} + x^{-4}) dx$$
$$= \frac{1}{3}x^3 - 2\ln|x| - \frac{1}{3}x^{-3} + C$$

10. $\int \left(x + \frac{1}{x}\right)^3 dx$

Sol.

$$\int \left(x + \frac{1}{x}\right)^3 dx$$
$$= \int (x^3 + 3x + 3x^{-1} + x^{-3}) dx$$
$$= \frac{1}{4}x^4 + \frac{3}{2}x^2 + 3\ln|x| - \frac{1}{2}x^{-2} + C$$

11. $\int 10^{-x} dx$

Sol.

$$\int 10^{-x} dx$$
$$= -\int 10^{-x} d(-x)$$
$$= -\frac{10^{-x}}{\ln 10} + C$$
$$= -\frac{1}{10^x \ln 10} + C$$

12. $\int (e^x - e^{-x})^2 dx$

Sol.

$$\int (e^x - e^{-x})^2 dx$$
$$= \int (e^{2x} - 2 + e^{-2x}) dx$$
$$= \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + C$$

13. $\int 2x(x^2 - 1)^4 dx$

Sol.

$$\int 2x(x^2 - 1)^4 dx$$
$$= \int (x^2 - 1)^4 d(x^2 - 1)$$
$$= \frac{1}{5}(x^2 - 1)^5 + C$$

14. $\int 3x^2(x^3 + 1)^4 dx$

Sol.

$$\int 3x^2(x^3 + 1)^4 dx$$
$$= \int (x^3 + 1)^4 d(x^3 + 1)$$
$$= \frac{1}{5}(x^3 + 1)^5 + C$$

15. $\int \frac{x+1}{(x^2 + 2x + 5)^3} dx$

Sol.

$$\int \frac{x+1}{(x^2 + 2x + 5)^3} dx$$
$$= \frac{1}{2}\int \frac{1}{(x^2 + 2x + 5)^3} d(x^2 + 2x + 5)$$
$$= -\frac{1}{4(x^2 + 2x + 5)^2} + C$$

16. $\int \frac{2x}{\sqrt{x^2 - 4}} dx$

Sol.

$$\int \frac{2x}{\sqrt{x^2 - 4}} dx$$
$$= \int \frac{1}{\sqrt{x^2 - 4}} d(x^2 - 4)$$
$$= 2\sqrt{x^2 - 4} + C$$

17. $\int \frac{x-2}{\sqrt{(x-1)(x-3)}}dx$

Sol.

$$
\begin{aligned}
&\int \frac{x-2}{\sqrt{(x-1)(x-3)}}dx \\
&= \int \frac{x-2}{\sqrt{x^2-4x+3}}dx \\
&= \frac{1}{2}\int \frac{1}{\sqrt{x^2-4x+3}}d(x^2-4x+3) \\
&= \frac{1}{2}\cdot 2\sqrt{x^2-4x+3}+C \\
&= \sqrt{x^2-4x+3}+C
\end{aligned}
$$

18. $\int \frac{7}{2x^2+5x-3}dx$

Sol.

$$\int \frac{7}{2x^2+5x-3}dx = \int \frac{7}{(2x-1)(x+3)}dx$$

$$\text{Let } \frac{7}{(2x-1)(x+3)} = \frac{A}{2x-1}+\frac{B}{x+3}$$

$$A(x+3)+B(2x-1)=7$$

$$(A+2B)x+(3A-B)=7$$

Comparing coefficients,

$$
\begin{aligned}
A+2B&=0\\
3A-B&=7\\
A=2,\ B&=-1
\end{aligned}
$$

$$
\begin{aligned}
&\therefore \int \frac{7}{2x^2+5x-3}dx \\
&= \int \left(\frac{2}{2x-1}-\frac{1}{x+3}\right)dx \\
&= \int \frac{2}{2x-1}dx - \int \frac{1}{x+3}dx \\
&= \ln|2x-1| - \ln|x+3| + C \\
&= \ln\left|\frac{2x-1}{x+3}\right| + C
\end{aligned}
$$

19. $\int \frac{8-7x}{2+x-3x^2}dx$

Sol.

$$
\begin{aligned}
\int \frac{8+7x}{2+x-3x^2}dx &= \int \frac{-7x-8}{3x^2-x-2}dx \\
&= -\int \frac{7x+8}{(3x+2)(x-1)}dx
\end{aligned}
$$

$$\text{Let } \frac{7x+8}{(3x+2)(x-1)} = \frac{A}{3x+2}+\frac{B}{x-1}$$

$$A(x-1)+B(3x+2)=7x+8$$

$$(A+3B)x+(-A+2B)=7x+8$$

Comparing coefficients,

$$
\begin{aligned}
A+3B&=7\\
-A+2B&=8\\
A=-2,\ B&=3
\end{aligned}
$$

$$
\begin{aligned}
&\therefore \int \frac{8-7x}{2+x-3x^2}dx \\
&= -\int \left(-\frac{2}{3x+2}+\frac{3}{x-1}\right)dx \\
&= -\int -\frac{2}{3x+2}dx - \int \frac{3}{x-1}dx \\
&= \frac{2}{3}\ln|3x+2| - 3\ln|x-1| + C
\end{aligned}
$$

20. $\int \frac{x+1}{(3x+2)(5x+3)}dx$

Sol.

$$\text{Let } \frac{x+1}{(3x+2)(5x+3)} = \frac{A}{3x+2} + \frac{B}{5x+3}$$

$$A(5x+3) + B(3x+2) = x+1$$

$$(5A+3B)x + (3A+2B) = x+1$$

Comparing coefficients,

$$5A + 3B = 1$$

$$3A + 2B = 1$$

$$A = -1,\ B = 2$$

$$\therefore \int \frac{x+1}{(3x+2)(5x+3)}dx$$
$$= \int \left(\frac{-1}{3x+2} + \frac{2}{5x+3}\right) dx$$
$$= -\int \frac{1}{3x+2}dx + \int \frac{2}{5x+3}dx$$
$$= -\frac{1}{3}\ln|3x+2| + \frac{2}{5}\ln|5x+3| + C$$

21. $\int \frac{2x^2+5x-2}{2x^2+x-3}dx$

Sol.

$$\int \frac{2x^2+5x-2}{2x^2+x-3}dx$$
$$= \int \left(1 + \frac{4x+1}{2x^2+x-3}\right) dx$$
$$= \int \left[1 + \frac{(2x^2+x-3)'}{2x^2+x-3}\right] dx$$
$$= x + \ln|2x^2+x-3| + C$$

22. $\int \left(\frac{x+1}{x-1}\right)^2 dx$

Sol.

$$\int \left(\frac{x+1}{x-1}\right)^2 dx$$
$$= \int \left(1 + \frac{2}{x-1}\right)^2 dx$$
$$= \int \left[1 + \frac{4}{x-1} + \frac{4}{(x-1)^2}\right] dx$$
$$= x + 4\ln|x-1| - \frac{4}{x-1} + C$$

23. $\int \frac{(x-1)^3}{(x-2)^2}dx$

Sol.

Let $u = x-2$

$\therefore x = u+2$

$dx = du$

$$\int \frac{(x-1)^3}{(x-2)^2}dx$$
$$= \int \frac{(u+1)^3}{u^2}du$$
$$= \int \left(\frac{u^3+3u^2+3u+1}{u^2}\right) du$$
$$= \int \left(u + 3 + \frac{3}{u} + \frac{1}{u^2}\right) du$$
$$= \frac{u^2}{2} + 3u + 3\ln|u| - \frac{1}{u} + C$$

$$= \frac{(x-2)^2}{2} + 3(x-2) + 3\ln|x-2| - \frac{1}{x-2} + C$$
$$= \frac{x^2-4x+4+6x-12}{2} + 3\ln|x-2| - \frac{1}{x-2} + C$$
$$= \frac{x^2+2x-8}{2} + 3\ln|x-2| - \frac{1}{x-2} + C$$
$$= \frac{1}{2}x^2 + x + 3\ln|x-2| - \frac{1}{x-2} + C$$

24. $\displaystyle\int \frac{x^2}{(x+2)^3}dx$

Sol.

Let $u = x+2$

$\therefore x = u-2$

$dx = du$

$$\int \frac{x^2}{(x+2)^3}dx$$
$$= \int \frac{(u-2)^2}{u^3}du$$
$$= \int \left(\frac{u^2-4u+4}{u^3}\right)du$$
$$= \int \left(\frac{1}{u} - \frac{4}{u^2} + \frac{4}{u^3}\right)du$$
$$= \ln|u| + \frac{4}{u} - \frac{2}{u^2} + C$$
$$= \ln|x+2| + \frac{4}{x+2} - \frac{2}{(x+2)^2} + C$$

25. $\displaystyle\int \left(3\sin 2x - 4e^{3x}\right)dx$

Sol.

$$\int \left(3\sin 2x - 4e^{3x}\right)dx$$
$$= \frac{3}{2}\int \sin 2x d(2x) - \int 4e^{3x}dx$$
$$= -\frac{3}{2}\cos 2x - \frac{4}{3}e^{3x} + C$$

26. $\displaystyle\int \sin(5x-6)dx$

Sol.

$$\int \sin(5x-6)dx$$
$$= \frac{1}{5}\int \sin(5x-6)d(5x-6)$$
$$= -\frac{1}{5}\cos(5x-6) + C$$

27. $\displaystyle\int \left(\cos 6x + \sec^2 4x\right)dx$

Sol.

$$\int \left(\cos 6x + \sec^2 4x\right)dx$$
$$= \frac{1}{6}\int \cos 6x d(6x) + \frac{1}{4}\int \sec^2 4x d(4x)$$
$$= \frac{1}{6}\sin 6x + \frac{1}{4}\tan 4x + C$$

28. $\displaystyle\int \left(\sin\frac{x}{2} + \cos 2x - \cos\frac{x}{7}\right)dx$

Sol.

$$\int \left(\sin\frac{x}{2} + \cos 2x - \cos\frac{x}{7}\right)dx$$
$$= 2\int \sin\frac{x}{2}d\left(\frac{x}{2}\right) + \int \cos 2x d(2x)$$
$$-7\int \cos\frac{x}{7}d\left(\frac{x}{7}\right)$$
$$= -2\cos\frac{x}{2} + \frac{1}{2}\sin 2x - 7\sin\frac{x}{7} + C$$

29. $\displaystyle\int \tan^2 3x dx$

Sol.

$$\int \tan^2 3x dx$$
$$= \int \left(\sec^2 3x - 1\right)dx$$
$$= \frac{1}{3}\int \sec^2 3x d(3x) - \int dx$$
$$= \frac{1}{3}\tan 3x - x + C$$

30. $\int \tan x \sec^2 x dx$

Sol.

$$\begin{aligned}
&\int \tan x \sec^2 x dx \\
&= \int \tan x d(\tan x) \\
&= \frac{1}{2}\tan^2 x + C
\end{aligned}$$

31. $\int \frac{3\sin x}{\cos 2x + 1} dx$

Sol.

$$\begin{aligned}
&\int \frac{3\sin x}{\cos 2x + 1} dx \\
&= \int \frac{3\sin x}{2\cos^2 x} dx \\
&= \frac{3}{2}\int \frac{\sin x}{\cos^2 x} dx \\
&= \frac{3}{2}\int \sec x \tan x dx \\
&= \frac{3}{2}\sec x + C
\end{aligned}$$

32. $\int \frac{\sec^2 x}{\tan x + 2} dx$

Sol.

$$\begin{aligned}
&\int \frac{\sec^2 x}{\tan x + 2} dx \\
&= \int \frac{1}{\tan x + 2} d(\tan x) \\
&= \ln|\tan x + 2| + C
\end{aligned}$$

33. $\int \cot 2x \csc^3 2x dx$

Sol.

$$\begin{aligned}
&\int \cot 2x \csc^3 2x dx \\
&= \int \csc 2x \cot 2x \csc^2 2x dx \\
&= \frac{1}{2}\int \csc^2 2x d(\csc 2x) \\
&= -\frac{1}{6}\csc^3 2x + C
\end{aligned}$$

34. $\int \tan^3 x \sec^3 x dx$

Sol.

$$\begin{aligned}
&\int \tan^3 x \sec^3 x dx \\
&= \int \tan^2 x \sec^2 x \sec x \tan x dx \\
&= \int \left(\sec^2 x - 1\right) \sec^2 x \sec x \tan x dx \\
&= \int \left(\sec^4 x - \sec^2 x\right) d(\sec x) \\
&= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C
\end{aligned}$$

35. If the function $y = \ln x - \frac{3}{x}$, find $\frac{dy}{dx}$. Hence, find $\int \frac{3+x}{3x^2} dx$.

Sol.

$$\begin{aligned}
y &= \ln x - \frac{3}{x} \\
\frac{dy}{dx} &= \frac{1}{x} + \frac{3}{x^2} \\
&= \frac{x+3}{x^2}
\end{aligned}$$

$$\begin{aligned}
\int \frac{3+x}{3x^2} dx &= \frac{1}{3}\int \frac{x+3}{x^2} d(x^3) \\
&= \frac{1}{3}\ln x - \frac{1}{x} + C
\end{aligned}$$

36. If the function $y = \dfrac{1}{\sqrt{4x^2 - 1}}$, find $\dfrac{dy}{dx}$. Hence, find $\displaystyle\int \frac{x}{\sqrt{\left(4x^2-1\right)^3}}$.

Sol.

$$\begin{aligned} y &= \left[\left(4x^2-1\right)^{-\frac{1}{2}}\right]' \\ &= -\frac{1}{2}\left(4x^2-1\right)^{-\frac{3}{2}} \cdot 8x \\ &= -\frac{4x}{\sqrt{\left(4x^2-1\right)^3}} \end{aligned}$$

$$\begin{aligned} \int \frac{x}{\sqrt{\left(4x^2-1\right)^3}} dx &= -\frac{1}{4} \int -\frac{4x}{\sqrt{\left(4x^2-1\right)^3}} d(4x^2) \\ &= -\frac{1}{4\sqrt{4x^2-1}} + C \end{aligned}$$

37. Given the function $y = \dfrac{x^2+3}{1-x}$, and $\dfrac{dy}{dx} = \dfrac{1}{2} f(x)$, find $\displaystyle\int \left[3 - x^2 - f(x)\right] dx$.

Sol.

$$\begin{aligned} \frac{d}{dx}\left(\frac{x^2+3}{1-x}\right) &= \frac{1}{2} f(x) \\ f(x) &= 2\left[\frac{d}{dx}\left(\frac{x^2+3}{1-x}\right)\right] \end{aligned}$$

$$\begin{aligned} \int \left[3 - x^2 - f(x)\right] dx &= \int 3dx - \int x^2 dx - f(x)dx \\ &= \int 3dx - \int x^2 dx - \int 2\left[\frac{d}{dx}\left(\frac{x^2+3}{1-x}\right)\right] dx \\ &= 3x - \frac{x^3}{3} - \frac{2(x^2+3)}{1-x} + C \end{aligned}$$

38. Given the function $\dfrac{d}{dx}(x \ln x) = g(x)$, find $\displaystyle\int [g(x) - 2x]\, dx$.

Sol.

$$\begin{aligned} \int [g(x) - 2x]\, dx &= \int g(x)dx - \int 2xdx \\ &= \int \frac{d}{dx}(x \ln x)dx - \int 2xdx \\ &= x \ln x - x^2 + C \end{aligned}$$

39. The gradient of the tangent at any point on a curve is 3 times the x-coordinate of the point, and the curve passes through $(-2, 5)$. Find the equation of the curve.

Sol.

$$\begin{aligned}
\frac{dy}{dx} &= 3x \\
dy &= 3x dx \\
y &= \int 3x dx \\
&= \frac{3}{2}x^2 + C
\end{aligned}$$

Given that the curve passes through $(-2, 5)$,

$$\begin{aligned}
5 &= \frac{3(-2)^2}{2} + C \\
C &= 1
\end{aligned}$$

Therefore, the equation of the curve is $y = \frac{3}{2}x^2 + 1$.

40. The gradient of the tangent at any point on a curve is $\frac{dy}{dx} = 3x^2 - 8x + 1$, and the curve intersect with x-axis at point $(2, 0)$, find the other point of intersection of the curve and the x-axis.

Sol.

$$\begin{aligned}
\frac{dy}{dx} &= 3x^2 - 8x + 1 \\
y &= \int (3x^2 - 8x + 1)dx \\
&= x^3 - 4x^2 + x + C
\end{aligned}$$

Given that the curve passes through $(2, 0)$,

$$\begin{aligned}
0 &= 2^3 - 4(2)^2 + 2 + C \\
C &= -6
\end{aligned}$$

Therefore, the equation of the curve is $y = x^3 - 4x^2 + x - 6$.

When the curve intersects with the x-axis, $y = 0$,

$$\begin{aligned}
x^3 - 4x^2 + x + 6 &= 0 \\
(x - 2)(x^2 - 2x - 3) &= 0 \\
(x - 2)(x - 3)(x + 1) &= 0
\end{aligned}$$

Therefore, the curve also intersects with the x-axis at $(-1, 0)$ and $(3, 0)$.

Chapter 28

Definite Integrals