Senior 2 Math Part I

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Chapter 12

Sequence and Series

12.1 Sequence and Series

12.1.1 Practice 1

1. Find the first 5 terms of the sequence $a_n = \frac{2^n}{n+1}$.

   Sol. $a_1 = \frac{2}{2} = 1, a_2 = \frac{4}{3}, a_3 = \frac{8}{4}, a_4 = \frac{16}{5}, a_5 = \frac{32}{6}$

2. Write the general term of the sequence 1, 8, 27, 64, ...

   Sol. $a_n = n^3$

12.1.2 Practice 2

1. Express the series $\sum_{n=1}^{10} n^2 + 1$ in the form of numbers.

   Sol. $$\sum_{n=1}^{10} n^2 + 1$$
   $$= (1^2+1) + (2^2+1) + (3^2+1) + (4^2+1)$$
   $$+ (5^2+1) + (6^2+1) + (7^2+1)$$
   $$+ (8^2+1) + (9^2+1) + (10^2+1)$$
   $$= 2 + 5 + 10 + 17 + 26 + 37 + 50 + 65$$
   $$+ 82 + 101$$

2. Write the first term, last term and the number of terms of the series $\sum_{n=1}^{10} (3^n - 2^n)$.

   Sol. $First\ term = (3^1 - 2^1) = 1$

   $Last\ term = (3^{10} - 2^{10}) = 59049$

   $Number\ of\ terms = 10$

3. Express the series $2\times5+3\times7+4\times9+\ldots+15\times31$ in the form of $\sum$.

   Sol.
   $$a_1 = 2\times5 = 10$$
   $$a_2 = 3\times7 = 21$$
   $$a_3 = 4\times9 = 36$$
   $$a_4 = 5\times11 = 55$$
   $$\vdots$$
   $$a_{15} = 15\times31 = 465$$
   $$\therefore 2\times5+3\times7+4\times9+\ldots+15\times31$$
   $$= \sum_{n=1}^{15} a_n$$

12.1.3 Exercise 12.1

1. Find the general term of the following sequences.

   (a) 5, 8, 11, 14, ...
   Sol. $a_n = 3n + 2$

   (b) 2, 4, 8, 16, ...
   Sol. $a_n = 2^n$

   (c) $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$
   Sol. $a_n = \frac{n+1}{n}$

   (d) $\frac{2}{5}, \frac{4}{7}, \frac{6}{9}, \frac{8}{11}, \ldots$
   Sol. $a_n = \frac{2n}{2n+1}$

2. Find the first 5 terms of the following sequences.

   (a) $a_n = 2n + 3$
   Sol. $a_1 = 2\times1+3 = 5, a_2 = 2\times2+3 = 7, a_3 = 2\times3+3 = 9, a_4 = 2\times4+3 = 11, a_5 = 2\times5+3 = 13$

   (b) $a_n = n(n-2)$
   Sol. $a_1 = 1\times(-1) = -1, a_2 = 2\times0 = 0, a_3 = 3\times1 = 3, a_4 = 4\times2 = 8, a_5 = 5\times3 = 15$

   (c) $a_n = \frac{n}{2n+1}$
   Sol. $a_1 = \frac{1}{2\times1+1} = \frac{1}{3}, a_2 = \frac{2}{2\times2+1} = \frac{2}{5}, a_3 = \frac{3}{2\times3+1} = \frac{3}{7}, a_4 = \frac{4}{2\times4+1} = \frac{4}{9}, a_5 = \frac{5}{2\times5+1} = \frac{5}{11}$

   (d) $a_n = (-3)^n$
   Sol. $a_1 = (-3)^1 = -3, a_2 = (-3)^2 = 9, a_3 = (-3)^3 = -27, a_4 = (-3)^4 = 81, a_5 = (-3)^5 = -243$

3. Express the following series in the form of numbers.

(a) $\sum_{n=1}^{5} n(n+3)$

Sol. $\sum_{n=1}^{5} n(n+3)$
$= (1 \times 4) + (2 \times 5) + (3 \times 6) + (4 \times 7)$
$+ (5 \times 8)$
$= 4 + 10 + 18 + 28 + 40$

(b) $\sum_{n=2}^{6} \frac{1}{3^n}$

Sol. $\sum_{n=2}^{6} \frac{1}{3^n}$
$= \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6}$
$= \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729}$

(c) $\sum_{n=1}^{6} \frac{1}{n(2n+1)}$

Sol. $\sum_{n=1}^{6} \frac{1}{n(2n+1)}$
$= \frac{1}{1(2 \times 1 + 1)} + \frac{1}{2(2 \times 2 + 1)}$
$+ \frac{1}{3(2 \times 3 + 1)} + \frac{1}{4(2 \times 4 + 1)}$
$+ \frac{1}{5(2 \times 5 + 1)} + \frac{1}{6(2 \times 6 + 1)}$
$= \frac{1}{3} + \frac{1}{10} + \frac{1}{21} + \frac{1}{36} + \frac{1}{55} + \frac{1}{78}$

(d) $\sum_{n=2}^{5} \frac{1}{n^2+2}$

Sol. $\sum_{n=2}^{5} \frac{1}{n^2+2}$
$= \frac{1}{4+2} + \frac{1}{9+2} + \frac{1}{16+2} + \frac{1}{25+2}$
$= \frac{1}{6} + \frac{1}{11} + \frac{1}{18} + \frac{1}{27}$

4. Find the first term, last term and the number of terms of the following series.

(a) $\sum_{n=3}^{10} 2^2$
Sol. $a_3 = 2^2 = 4, a_{10} = 2^2 = 4, n = 10 - 3 + 1 = 8$

(b) $\sum_{n=1}^{8} \frac{n+2}{n}$
Sol. $a_1 = \frac{1+2}{1} = \frac{3}{1} = 3, a_8 = \frac{8+2}{8} = \frac{10}{8} = \frac{5}{4}, n = 8 - 1 + 1 = 8$

(c) $\sum_{n=1}^{10} 3n^2 - n$
Sol. $a_1 = 3 \times 1^2 - 1 = 2, a_{10} = 3 \times 10^2 - 10 = 290, n = 10 - 1 + 1 = 10$

(d) $\sum_{n=9}^{14} n^2(n-7)$
Sol. $a_9 = 9^2(9-7) = 9^2 \times 2 = 162, a_{14} = 14^2(14-7) = 14^2 \times 7 = 2744, n = 14 - 9 + 1 = 6$

5. Express the following series in the form of $\sum$.

(a) $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{30}$

Sol.

$$
\begin{aligned}
a_1 &= 1 \\
a_2 &= \frac{1}{2} \\
a_3 &= \frac{1}{3} \\
&\vdots \\
a_{30} &= \frac{1}{30} \\
\therefore 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{30} &= \sum_{n=1}^{30} \frac{1}{n}
\end{aligned}
$$

(b) $1^3 + 2^3 + 3^3 + \ldots + 50^3$

Sol.

$$
\begin{aligned}
a_1 &= 1^3 \\
a_2 &= 2^3 \\
a_3 &= 3^3 \\
&\vdots \\
a_{50} &= 50^3 \\
\therefore 1^3 + 2^3 + 3^3 + \ldots + 50^3 &= \sum_{n=1}^{50} n^3
\end{aligned}
$$

(c) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16}$

Sol.

$$\begin{aligned}
a_1 &= (-\frac{1}{2})^{1-1}\\
a_2 &= (-\frac{1}{2})^{2-1}\\
a_3 &= (-\frac{1}{2})^{3-1}\\
a_4 &= (-\frac{1}{2})^{4-1}\\
a_5 &= (-\frac{1}{2})^{5-1}\\
\therefore 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16}\\
&= \sum_{n=1}^{5}(-\frac{1}{2})^{n-1}
\end{aligned}$$

(d) $2 \times 4 + 4 \times 7 + 6 \times 10 + 8 \times 13 + 10 \times 16$

Sol.

$$\begin{aligned}
a_1 &= 2 \times 1 \times (3 \times 1 + 1)\\
a_2 &= 2 \times 2 \times (3 \times 2 + 1)\\
a_3 &= 2 \times 3 \times (3 \times 3 + 1)\\
a_4 &= 2 \times 4 \times (3 \times 4 + 1)\\
a_5 &= 2 \times 5 \times (3 \times 5 + 1)\\
\therefore 2 \times 4 + 4 \times 7 + 6 \times 10 + 8 \times 13\\
+ 10 \times 16 &= \sum_{n=1}^{5} 2n(3n+1)
\end{aligned}$$

12.2 Arithmetic Progression

General term of an Arithmetic Progression (AP) is given by

$$a_n = a_1 + (n-1)d$$

where a_1 is the first term, d is the common difference and n is the number of terms.

12.2.1 Practice 3

1. Find the number of terms of the AP $-4 - 2\frac{3}{4} - 1\frac{1}{2} - \frac{1}{4} + \ldots + 16$.

$$\begin{aligned}
a_1 &= -4\\
a_n &= 16\\
d &= -2\frac{3}{4} - (-4)\\
&= -2\frac{3}{4} + 4\\
&= \frac{5}{4}\\
16 &= -4 + (n-1)\frac{5}{4}\\
20 &= \frac{5}{4}(n-1)\\
80 &= 5(n-1)\\
n - 1 &= 16\\
n &= 17
\end{aligned}$$

2. Given that $a_2 = 4$ and $a_6 = -8$, find the 10th term of the AP.

Sol.

$$\begin{aligned}
a_2 &= 4\\
a + (2-1)d &= 4\\
a_6 &= -8\\
a + (6-1)d &= -8
\end{aligned}$$

$$\begin{cases} a + d &= 4 \quad (12.1)\\ a + 5d &= \text{-8} \quad (12.2) \end{cases}$$

$$\begin{aligned}
(2) - (1) : 4d &= -12\\
d &= -3\\
a + (-3) &= 4\\
a &= 7\\
\therefore a_{10} &= 7 + (10-1)(-3)\\
&= 7 - 27\\
&= -20
\end{aligned}$$

3. How many multiples of 7 are there between 50 and 500?

Sol.

$$\begin{aligned}
a_1 &= 56\\
a_n &= 497\\
d &= 7\\
497 &= 56 + (n-1)7\\
441 &= 7(n-1)\\
n - 1 &= 63\\
n &= 64
\end{aligned}$$

4. Find 5 numbers between 30 and 54 such that these numbers form an AP.

   Sol.

   $$\begin{aligned} a_1 &= 30 \\ a_7 &= 54 \\ 54 &= 30 + (7-1)d \\ 24 &= 6d \\ d &= 4 \end{aligned}$$

   $$\therefore \textit{These } 5 \textit{ numbers are } 34,\ 38,\ 42,\ 46, \textit{ and } 50.$$

Arithmetic mean

If A is in between x and y, and x, A, y are in AP, then

$$A = \frac{x+y}{2}$$

12.2.2 Practice 4

1. If 9, x, 17 are in AP, find x.

   Sol.

   $$\begin{aligned} x &= \frac{9+17}{2} \\ &= \frac{26}{2} \\ &= 13 \end{aligned}$$

2. Find the arithmetic mean of 26 and -11.

   Sol.

   $$\begin{aligned} A &= \frac{26-11}{2} \\ &= \frac{15}{2} \end{aligned}$$

3. Find x and y when 3, x, 12, y, 21 are in AP.

   Sol.

   $$\begin{aligned} x &= \frac{3+12}{2} \\ &= \frac{15}{2} \\ y &= \frac{12+21}{2} \\ &= \frac{33}{2} \end{aligned}$$

Summation of Arithmetic Progression

The summation formula for AP is given by

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

or

$$S_n = \frac{n}{2}(a_1 + a_n)$$

12.2.3 Practice 5

1. Find the sum of the first 16 terms of the AP $22 + 18 + 14 + 10 + \ldots$

   Sol.

   $$\begin{aligned} a_1 &= 22 \\ n &= 16 \\ d &= -4 \\ S_n &= \frac{16}{2}(2 \times 22 + (-4)(16-1)) \\ &= \frac{16}{2}(44 + (-4)(15)) \\ &= \frac{16}{2}(44 - 60) \\ &= \frac{16}{2}(-16) \\ &= -128 \end{aligned}$$

2. If the sum of AP $23 + 19 + 15 + \ldots$ is 72, find the number of terms.

   Sol.

   $$\begin{aligned} a_1 &= 23 \\ S_n &= 72 \\ d &= -4 \\ 72 &= \frac{n}{2}(2 \times 23 + (-4)(n-1)) \\ 72 &= \frac{n}{2}(46 + (-4)(n-1)) \\ 144 &= n(46 + (-4)(n-1)) \\ 144 &= n(46 - 4n + 4) \\ 144 &= n(50 - 4n) \\ 144 &= 50n - 4n^2 \\ 72 &= 25n - 2n^2 \\ 2n^2 - 25n + 72 &= 0 \\ (n-8)(2n-9) &= 0 \\ n &= 8 \end{aligned}$$

3. Given that $S_n = 2n + 3n^2$, find the first term and the common difference of the AP.

   Sol.

$$\begin{aligned} S_n &= 2n + 3n^2 \\ 2n + 3n^2 &= \frac{n}{2}(2a + (n-1)d) \\ 4n + 6n^2 &= n(2a + (n-1)d) \\ 4n + 6n^2 &= 2na + (n-1)nd \\ 4n + 6n^2 &= 2na + n^2d - nd \\ 4n + 6n^2 &= (2a-d)n + dn^2 \end{aligned}$$

$$\begin{aligned} &\textit{Comparing both sides,} \\ &2a - d = 4 \\ &a = 6 \\ &d = 2 \end{aligned}$$

12.2.4 Exercise 12.2

1. Find the 10th terms of the AP $5, 13, 21, \ldots$

   Sol.

$$\begin{aligned} a_1 &= 5 \\ n &= 10 \\ d &= 8 \\ a_{10} &= 5 + (10-1) \times 8 \\ &= 5 + 72 \\ &= 77 \end{aligned}$$

2. Find the 8th term of the AP $5, 4\frac{1}{4}, 3\frac{1}{2}, 2\frac{3}{4}, \ldots$

   Sol.

$$\begin{aligned} a_1 &= 5 \\ n &= 8 \\ d &= -\frac{3}{4} \\ a_8 &= 5 + (8-1) \times -\frac{3}{4} \\ &= 5 - \frac{3}{4} \times 7 \\ &= 5 - \frac{21}{4} \\ &= -\frac{1}{4} \end{aligned}$$

3. Find the number of terms of the following AP.

   (a) $4, 9, \ldots, 64$

   Sol.

$$\begin{aligned} a_1 &= 4 \\ a_n &= 64 \\ d &= 5 \\ 64 &= 4 + (n-1) \times 5 \\ 60 &= 5(n-1) \\ 12 &= n - 1 \\ n &= 13 \end{aligned}$$

   (b) $4\frac{1}{3}, 3\frac{2}{3}, 3, \ldots, -10\frac{1}{3}$

   Sol.

$$\begin{aligned} a_1 &= 4\frac{1}{3} \\ a_n &= -10\frac{1}{3} \\ d &= -\frac{2}{3} \\ -10\frac{1}{3} &= 4\frac{1}{3} + (n-1) \times -\frac{2}{3} \\ -\frac{31}{3} &= \frac{13}{3} - \frac{1}{3}(n-1) \\ -31 &= 13 - 2n + 2 \\ -46 &= 2n \\ n &= 23 \end{aligned}$$

4. The 6th term of an AP is 43, and its 10th term is 75. Find the first term and common difference of this AP.

   Sol.

$$\begin{aligned} a_6 &= 43 \\ a_{10} &= 75 \\ 43 &= a + (6-1)d \\ 75 &= a + (10-1)d \\ 32 &= 4d \\ d &= 8 \\ 43 &= a + 5 \times 8 \\ 43 &= a + 40 \\ 3 &= a \\ a &= 3 \\ &\therefore a_1 = 3,\ d = 8 \end{aligned}$$

5. The 7th term of an AP is -10, and the 12th term -25, find the 15th term of this AP.

Sol.

$$\begin{aligned} a_7 &= -10 \\ a_{12} &= -25 \\ -10 &= a + (7-1)d \\ -25 &= a + (12-1)d \\ -15 &= 5d \\ d &= -3 \\ -10 &= a + 6 \times -3 \\ -10 &= a - 18 \\ a &= 8 \\ a_{15} &= 8 + (15-1) \times -3 \\ &= 8 - 42 \\ &= -34 \end{aligned}$$

6. How many multiples of 7 are there between 100 and 200?

   Sol.

$$\begin{aligned} a &= 105 \\ d &= 7 \\ a_n &= 196 \\ 196 &= 105 + (n-1) \times 7 \\ 91 &= 7(n-1) \\ 13 &= n - 1 \\ n &= 14 \end{aligned}$$

7. Find the arithmetic mean o fthe following number pairs.

   (a) $(8, 20)$

   Sol.

$$\frac{8+20}{2} = 14$$

   (b) $(-9, 17)$

   Sol.

$$\frac{-9+17}{2} = 4$$

8. Find 5 numbers between 22 and 58 such that these 7 numbers are in AP.

   Sol.

$$\begin{aligned} a_1 &= 22 \\ a_7 &= 58 \\ 58 &= 22 + (7-1)d \\ 36 &= 6d \\ d &= 6 \\ &\therefore \mathit{These}\ 5\ \mathit{numbers\ are}\ 22, 28, 34, 40, 46 \end{aligned}$$

9. Find the sum of first 20 terms of AP $12 + 15 + 18 + \dots$

   Sol.

$$\begin{aligned} a_1 &= 12 \\ n &= 20 \\ d &= 3 \\ S_{20} &= \frac{20}{2}(2 \times 12 + (20-1) \times 3) \\ &= 10(24 + 57) \\ &= 10(81) \\ &= 810 \end{aligned}$$

10. Find the sum of first 12 terms of the AP $18 + 10 + 2 - 6 - \dots$

    Sol.

$$\begin{aligned} a_1 &= 18 \\ n &= 12 \\ d &= -8 \\ S_{12} &= \frac{12}{2}(2 \times 18 + (12-1) \times -8) \\ &= 6(36 - 88) \\ &= 6(-52) \\ &= -312 \end{aligned}$$

11. Find the sum of first 14 terms of the AP $\frac{1}{6} + \frac{4}{3} + \frac{5}{2} + \dots$

Sol.

$$\begin{aligned} a_1 &= \frac{1}{6} \\ n &= 14 \\ d &= \frac{7}{6} \\ S_{14} &= \frac{14}{2}(2 \times \frac{1}{6} + (14-1) \times \frac{7}{6}) \\ &= 7(\frac{1}{3} + \frac{91}{6}) \\ &= 7 \times \frac{93}{6} \\ &= 7 \times \frac{31}{2} \\ &= \frac{217}{2} \end{aligned}$$

12. Find the sum of all the multiples of 13 in between 200 and 800.

Sol.

$$\begin{aligned} a_1 &= 208 \\ a_n &= 793 \\ d &= 13 \\ 793 &= 208 + (n-1) \times 13 \\ 585 &= 13(n-1) \\ 45 &= n-1 \\ n &= 46 \end{aligned}$$

$$\begin{aligned} S_{46} &= \frac{46}{2}(2 \times 208 + (46-1) \times 13) \\ &= 23(416 + 585) \\ &= 23(1001) \\ &= 23023 \end{aligned}$$

13. If the sum of first n terms of the AP $-3, -7, -11, \ldots$ is -903, find the value of n.

Sol.

$$\begin{aligned} a_1 &= -3 \\ d &= -4 \\ -903 &= \frac{n}{2}(2 \times (-3) - 4(n-1)) \\ -1806 &= -2n - 4n^2 \\ 4n^2 + 2n - 1806 &= 0 \\ 2n^2 + n - 903 &= 0 \\ (n-21)(2n+43) &= 0 \\ n &= 21, -43(invalid) \\ \therefore\ n &= 21 \end{aligned}$$

14. Given that the first 3 terms of an AP are $x,\ 3x-4,\ 2x+7$, find:

(a) The value of x

Sol.

$$\begin{aligned} 3x - 4 &= \frac{x + 2x + 7}{2} \\ 6x - 8 &= 3x + 7 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

(b) The common difference

Sol.

$$\begin{aligned} a_1 &= x = 5 \\ a_2 &= 3x - 4 = 3 \times 5 - 4 = 11 \\ d &= 11 - 5 \\ &= 6 \end{aligned}$$

(c) The sum of first 10 terms.

Sol.

$$\begin{aligned} a_1 &= x = 5 \\ n &= 10 \\ d &= 6 \\ S_{10} &= \frac{10}{2}(2 \times 5 + (10-1) \times 6) \\ &= 5(10 + 54) \\ &= 5(64) \\ &= 320 \end{aligned}$$

15. Let the sum of the first n terms of an AP to be $S_n = \frac{n(n+1)}{4}$, find:

(a) The first term

Sol.

$$\begin{aligned}\frac{n(n+1)}{4} &= \frac{n}{2}(2a+(n-1)d)\\ n(n+1) &= 2n(2a+dn-d)\\ n^2+n &= 4na+2dn^2-2nd\\ n^2+n &= 2dn^2+(4a-2d)n\end{aligned}$$

$$\begin{aligned}&Comparing\ both\ sides,\\ 2d &= 1\\ d &= \frac{1}{2}\\ 4a-2d &= 1\\ 4a-1 &= 1\\ 4a &= 2\\ a &= \frac{1}{2}\end{aligned}$$

(b) The common difference

Sol.

$$d = \frac{1}{2}$$

gg

(c) The 6th terms

Sol.

$$\begin{aligned}a_1 &= \frac{1}{2}\\ n &= 6\\ d &= \frac{1}{2}\\ a_6 &= \frac{1}{2}+(6-1)\times\frac{1}{2}\\ &= \frac{1}{2}+\frac{5}{2}\\ &= 3\end{aligned}$$

(d) The sum from 6th term to 10th term

Sol.

$$\begin{aligned}a &= \frac{1}{2}\\ d &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}S_{10} &= \frac{10}{2}(2\times\frac{1}{2}+(10-1)\times\frac{1}{2})\\ &= \frac{10}{2}(1+\frac{9}{2})\\ &= 5\times\frac{11}{2}\\ &= \frac{55}{2}\end{aligned}$$

$$\begin{aligned}S_5 &= \frac{5}{2}(2\times\frac{1}{2}+(5-1)\times\frac{1}{2})\\ &= \frac{5}{2}(1+2)\\ &= \frac{15}{2}\end{aligned}$$

$$\begin{aligned}S_{10}-S_6 &= \frac{55}{2}-\frac{15}{2}\\ &= \frac{40}{2}\\ &= 20\end{aligned}$$

16. Given three numbers in an AP, the sum of these three numbers is 30, and the sum of square of these numbers is 318, find these three numbers.

Sol.

$$\begin{aligned}
a_1 + a_2 + a_3 &= 30\\
a_1^2 + a_2^2 + a_3^2 &= 318\\
a_2 - a_1 &= a_3 - a_2\\
a_1 - 2a_2 + a_3 &= 0\\
3a_2 &= 30\\
a_2 &= 10\\
a_1 - 20 + a_3 &= 0\\
a_1 + a_3 &= 20\\
a_3 &= 20 - a_1\\
a_1^2 + 100 + (20 - a_1)^2 &= 318\\
a_1^2 + 100 + 400 + a_1^2 - 40a_1 &= 318\\
2a_1^2 - 40a_1 + 182 &= 0\\
a_1^2 - 20a_1 + 91 &= 0\\
(a_1 - 7)(a_1 - 13) &= 0\\
a_1 &= 7 or a_1 = 13
\end{aligned}$$

$$\therefore\ These\ three\ numbers\ are\ 7,\ 10,\ and\ 13$$

17. Find the sum of all the numbers between 100 and 200 that are both the multiples of 2 and 3.

Sol.

$$\begin{aligned}
a_1 &= 102\\
d &= 6\\
a_n &= 198\\
198 &= 102 + (n-1) \times 6\\
96 &= 6(n-1)\\
6n - 6 &= 96\\
6n &= 102\\
n &= 17
\end{aligned}$$

$$\begin{aligned}
S_{17} &= \frac{17}{2}(2 \times 102 + (17-1) \times 6)\\
&= \frac{17}{2}(204 + 96)\\
&= \frac{17}{2}(300)\\
&= 150 \times 17\\
&= 2550
\end{aligned}$$

18. Given an AP $-100 - 96 - 92 - \ldots$:

(a) Find the term where the number become positive.

Sol.

$$\begin{aligned}
a_1 &= -100\\
d &= 4\\
a_n &= -100 + (n-1) \times 4 > 0\\
-100 + 4n - 4 &> 0\\
4n &> 104\\
n &> 26
\end{aligned}$$

$$\therefore\ n = 27$$

(b) Find the term where the sum of this AP becomes positive.

Sol.

$$\begin{aligned}
S_n = \frac{n}{2}(2(-100) + (n-1) \times (4)) &> 0\\
\frac{n}{2}(-200 + 4n - 4) &> 0\\
\frac{n}{2}(-204 + 4n) &> 0\\
n(2n - 102) &> 0\\
n(n - 51) &> 0\\
n &> 51
\end{aligned}$$

$$\therefore\ n = 52$$

19. Find the first negative term of the AP $20, 19\frac{1}{5}, 18\frac{2}{5}, \ldots$

Sol.

$$\begin{aligned}
a_1 &= 20\\
d &= -\frac{4}{5}\\
a_n = 20 + (n-1) \times (-\frac{4}{5}) &< 0\\
100 - 4n + 4 &< 0\\
4n &> 104\\
n &> 26
\end{aligned}$$

$$\therefore\ n = 27$$

20. Given an AP $10 + 9\frac{1}{5} + 8\frac{2}{5} + \ldots$, what is the first negative term? When will the sum of the terms become negative, and what's the value of it?

Sol.

$$\begin{aligned} a_n = 10 + (n-1) \times (-\frac{4}{5}) &< 0 \\ 10 - \frac{4}{5}(n-1) &< 0 \\ 50 - 4n + 4 &< 0 \\ -4n &< -54 \\ n &> 13\frac{1}{2} \\ \\ \therefore\ n &= 14 \end{aligned}$$

$$\begin{aligned} S_n = \frac{n}{2}(2 \times 10 + (n-1) \times (-\frac{4}{5})) &< 0 \\ \frac{n}{2}(20 - \frac{4}{5}(n-1)) &< 0 \\ 20n - \frac{4}{5}(n^2 - n) &< 0 \\ 100n - 4n^2 + 4n &< 0 \\ 25n - n^2 + n &< 0 \\ 26n - n^2 &< 0 \\ n(n-26) &> 0 \\ n &> 26 \\ \\ \therefore\ n &= 27 \end{aligned}$$

$$\begin{aligned} S_{27} &= \frac{27}{2}(2 \times 10 + (27-1) \times (-\frac{4}{5})) \\ &= \frac{27}{2}(20 - \frac{4}{5}(27-1)) \\ &= \frac{27}{2}(20 - \frac{4}{5}(26)) \\ &= \frac{27}{2} \times (-\frac{4}{5}) \\ &= -\frac{54}{5} \end{aligned}$$

$\therefore$ *The first negative term is the 14th term*

$\therefore$ *The first term where the sum of the terms becomes negative is the 27th term*

$\therefore$ *The value of the sum of the terms when it becomes negative is* $-\dfrac{54}{5}$

21. Given a polygon which all their internal angles are in AP. The common difference of this AP is $6°$, the largest angle is $135°$. How many sides does this polygon have?

Sol.

$$\begin{aligned} a_1 &= 135 \\ d &= -6 \\ \frac{n}{2}(2 \times 135 + (n-1) \times (-6)) &= 180(n-2) \\ n(270 - 6(n-1)) &= 360(n-2) \\ n(276 - 6n) &= 360n - 720 \\ 276n - 6n^2 &= 360n - 720 \\ 46n - n^2 &= 60n - 120 \\ n^2 + 14n - 120 &= 0 \\ (n+20)(n-6) &= 0 \\ n &= -20\ (invalid) \\ n &= 6 \\ \therefore\ The\ number\ of\ sides\ is\ 6 \end{aligned}$$

22. Given an AP which its 5th term is 3 and the sum of its first 10 terms is $26\frac{1}{4}$. Whcih term in this AP is 0?

Sol.

$$\begin{aligned} a_5 = a + (5-1)d &= 3 \\ a + 4d &= 3 \\ S_{10} = \frac{10}{2}(2a + (10-1)d) &= 26\frac{1}{4} \\ 5(2a + 9d) &= 26\frac{1}{4} \\ 20(2a + 9d) &= 105 \\ 4(2a + 9d) &= 21 \\ 8a + 36d &= 21 \\ 8a + 32d &= 24 \\ 4d &= -3 \\ d &= -\frac{3}{4} \\ a &= 3 + \frac{3}{4} \times 4 \\ &= 6 \\ a_n = 6 + (n-1) \times (-\frac{3}{4}) &= 0 \\ 6 - \frac{3}{4}(n-1) &= 0 \\ 24 - 3n + 3 &= 0 \\ 3n &= 27 \\ n &= 9 \end{aligned}$$

23. Given that the sum of the first 6 terms of an AP is 96, and the sum of the first 20 terms is 3 times the sum of the first 10 terms of this AP. Find the first term and the 10th term of it.

Sol.

$$\begin{aligned}
S_6 = \frac{6}{2}(2a + (6-1)d) &= 96 \\
3(2a + 5d) &= 96 \\
2a + 5d &= 32 \\
S_{20} &= 3S_{10} \\
\frac{20}{2}(2a + (20-1)d) &= 3 \times \frac{10}{2}(2a \\
&\quad + (10-1)d) \\
10(2a + 19d) &= 15(2a + 9d) \\
2(2a + 19d) &= 3(2a + 9d) \\
4a + 38d &= 6a + 27d \\
2a - 11d &= 0 \\
16d &= 32 \\
d &= 2 \\
a &= \frac{11 \times 2}{2} \\
&= 11 \\
a_{10} &= 11 + (10-1) \times 2 \\
&= 29
\end{aligned}$$

24. Given that $5^2 \times 5^4 \times 5^6 \times \ldots \times 5^{2n} = (0.04)^{-28}$, find the value of n.

Sol.

$$\begin{aligned}
(0.04)^{-28} &= \frac{1}{25}^{-28} \\
&= (5^{-2})^{-28} \\
&= 5^{56} \\
\because n^a \times n^b &= n^{a+b} \\
2 + 4 + 6 + \ldots + 2n &= 56 \\
S_n = \frac{n}{2}(2 \times 2 + (n-1) \times 2) &= 56 \\
n(4 + 2(n-1)) &= 112 \\
n(2 + 2n) &= 112 \\
2n^2 + 2n &= 112 \\
n^2 + n - 56 &= 0 \\
(n+8)(n-7) &= 0 \\
n &= -8 \ (invalid) \\
n &= 7
\end{aligned}$$

25. Given that the 9th term of an AP is double the 5th term of it. Find the ratio of the sum of first 9 terms and the sum of first 5 terms of the AP.

Sol.

$$\begin{aligned}
a_9 &= 2a_5 \\
a + (9-1)d &= 2(a + (5-1)d) \\
a + 8d &= 2a + 8d \\
a &= 0 \\
S_9 : S_5 &= \frac{9}{2}(2a + a_9) : \frac{5}{2}(2a + a_5) \\
&= \frac{9}{2}(2a + 2a_5) : \frac{5}{2}(2a + a_5) \\
&= 9(a + a_5) : \frac{5}{2}(2a + a_5) \\
\frac{S_9}{S_5} &= \frac{9(a + a_5)}{\frac{5}{2}(2a + a_5)} \\
&= \frac{18(a + a_5)}{5(2a + a_5)} \\
&= \frac{18 \times a_5}{5 \times a_5} \\
&= \frac{18}{5} \\
\therefore S_9 : S_5 &= 18 : 5
\end{aligned}$$

12.3 Geometric Progression

The general formula of a geometric progression (GP) is given by

$$a_n = a_1 \times r^{n-1}$$

where a_1 is the first term, r is the common ratio, and n is the number of terms.

12.3.1 Practice 6

1. Find the 6th term of the GP $12, -18, 27, \ldots$

Sol.

$$\begin{aligned} a_1 &= 12 \\ r &= \frac{-18}{12} \\ &= -\frac{3}{2} \\ a_6 &= 12 \times (-\frac{3}{2})^{6-1} \\ &= 12 \times (-\frac{3}{2})^5 \\ &= 12 \times (-\frac{243}{32}) \\ &= -\frac{729}{8} \end{aligned}$$

2. Find the number of terms of GP $\frac{1}{64} - \frac{1}{32} + \frac{1}{16} - \frac{1}{8} + \ldots - 512$

   Sol.

$$\begin{aligned} a_1 &= \frac{1}{64} \\ r &= \frac{-\frac{1}{32}}{\frac{1}{64}} \\ &= -2 \\ -512 &= \frac{1}{64}(-2)^{n-1} \\ (-2)^9 &= \frac{1}{2^6}(-2)^{n-1} \\ (-2)^{15} &= (-2)^{n-1} \\ n-1 &= 15 \\ n &= 16 \end{aligned}$$

3. The 5th term of a GP is 3, and its 9th term is $\frac{1}{27}$, find the first term and the common ratio of this GP.

   Sol.

$$\begin{aligned} a_5 &= ar^4 = 3 \\ a_9 &= ar^8 = \frac{1}{27} \\ r^4 &= \frac{1}{27} \times \frac{1}{3} \\ &= \frac{1}{81} \\ r &= \frac{1}{3} \\ a_1 &= 3 \times 81 \\ &= 243 \end{aligned}$$

4. Find 5 numbers between $\frac{1}{2}$ and $frac1128$ such that these 7 numbers are in GP.

   Sol.

$$\begin{aligned} a_1 &= \frac{1}{2} \\ n &= 7 \\ \frac{1}{128} &= \frac{1}{2}r^{7-1} \\ r^6 &= \frac{1}{64} \\ r &= \frac{1}{2} \end{aligned}$$

$$\therefore \textit{ These } 5 \textit{ numbers are } \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$$

Geometric Mean

The geometric mean G of two numbers x and y is given by

$$\begin{aligned} \frac{G}{x} &= \frac{G}{y} \\ G^2 &= xy \\ G &= \mp \sqrt[2]{xy} \end{aligned}$$

12.3.2 Practice 7

Find the geometric mean of $\frac{27}{8}$ and $\frac{2}{3}$.

Sol.

$$\begin{aligned} G &= \pm \sqrt[2]{\frac{27}{8} \times \frac{2}{3}} \\ &= \pm \sqrt[2]{\frac{9}{4}} \\ &= \pm \frac{3}{2} \end{aligned}$$

Summation of Geometric Progression

The sum of n terms of a GP is given by

$$S_n = \frac{a_1(1-r^n)}{1-r} \; (r \neq 1)$$

12.3.3 Practice 8

1. Find the sum of the first 8 terms of GP $3 + 6 + 12 + \ldots$

Sol.

$$\begin{aligned} a_1 &= 3 \\ r &= \frac{6}{3} \\ &= 2 \\ n &= 8 \\ S_n &= \frac{3(1-2^8)}{1-2} \\ &= \frac{3(1-256)}{1-2} \\ &= 3 \times 255 \\ &= 765 \end{aligned}$$

2. Find the sum of the GP $1+\sqrt{3}+3+\ldots+81$

   Sol.

$$\begin{aligned} a_1 &= 1 \\ r &= \sqrt{3} \\ 81 &= 1 \times (\sqrt{3})^{n-1} \\ 3^4 &= (\sqrt{3})^{n-1} \\ (\sqrt{3})^8 &= (\sqrt{3})^{n-1} \\ n-1 &= 8 \\ n &= 9 \\ S_n &= \frac{1(1-(\sqrt{3})^9)}{1-\sqrt{3}} \\ &= \frac{1-81\sqrt{3}}{1-\sqrt{3}} \\ &= \frac{(1-81\sqrt{3})(1+\sqrt{3})}{-2} \\ &= \frac{1-81\sqrt{3}+\sqrt{3}-243}{-2} \\ &= \frac{-242-80\sqrt{(3)}}{-2} \\ &= 121+40\sqrt{3} \end{aligned}$$

3. Given that the sum of the first n terms of GP $4\frac{4}{5}, 1\frac{3}{5}, \frac{8}{15}, \ldots$ is $7\frac{145}{729}$, find n.

   Sol.

$$\begin{aligned} a_1 &= \frac{24}{5} \\ r &= \frac{8}{5} \times \frac{5}{24} \\ &= \frac{1}{3} \\ S_n &= \frac{24}{5} \times \frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}} \\ \frac{5248}{729} &= \frac{24}{5} \times \frac{1-(\frac{1}{3})^n}{\frac{2}{3}} \\ \frac{5248}{729} \times \frac{5}{24} \times \frac{2}{3} &= 1-(\frac{1}{3})^n \\ \frac{6560}{6561} &= 1-(\frac{1}{3})^n \\ -\frac{1}{6561} &= -(\frac{1}{3})^n \\ (\frac{1}{3})^8 &= (\frac{1}{3})^n \\ n &= 8 \end{aligned}$$

Summation of Infinite Geometric Progression

The sum of infinite GP is given by

$$S_\infty = \frac{a_1}{1-r} \ (-1 < r < 1)$$

12.3.4 Practice 9

1. Find the sum of the following infinite GP.

   (a) $16+8+4+\ldots$

   Sol.

$$\begin{aligned} a_1 &= 16 \\ r &= \frac{8}{16} \\ &= \frac{1}{2} \\ S_\infty &= \frac{16}{1-\frac{1}{2}} \\ &= \frac{16}{\frac{1}{2}} \\ &= 32 \end{aligned}$$

   (b) $18-12+8+\ldots$

Sol.

$$\begin{aligned} a_1 &= 18 \\ r &= \frac{8}{-12} \\ &= -\frac{2}{3} \\ S_\infty &= \frac{18}{1+\frac{2}{3}} \\ &= \frac{18}{\frac{5}{3}} \\ &= \frac{54}{5} \end{aligned}$$

(c) $1 + \frac{3}{4} + \frac{9}{16} + \ldots$

Sol.

$$\begin{aligned} a_1 &= 1 \\ r &= \frac{9}{16} \times \frac{16}{9} \\ &= \frac{3}{4} \\ S_\infty &= \frac{1}{1-\frac{3}{4}} \\ &= \frac{1}{\frac{1}{4}} \\ &= 4 \end{aligned}$$

(d) $\sqrt{2} + 1 + \frac{1}{\sqrt{2}} + \ldots$

Sol.

$$\begin{aligned} a_1 &= \sqrt{2} \\ r &= \frac{1}{\sqrt{2}} \\ S_\infty &= \frac{\sqrt{2}}{1-\frac{1}{\sqrt{2}}} \\ &= \frac{\sqrt{2}}{\frac{\sqrt{2}-1}{\sqrt{2}}} \\ &= \frac{2}{\sqrt{2}-1} \\ &= 2(\sqrt{2}+1) \end{aligned}$$

2. Convert the following recurring decimals to fraction using the summation of inifinite geometric series.

(a) $0.\overline{3}$

Sol.

$$\begin{aligned} a_1 &= 0.3 \\ r &= 0.1 \\ S_\infty &= \frac{0.3}{1-0.1} \\ &= \frac{0.3}{0.9} \\ &= \frac{1}{3} \end{aligned}$$

$$\therefore 0.\overline{3} = \frac{1}{3}$$

(b) $0.5\overline{3}$

Sol.

$$\begin{aligned} a_1 &= 0.03 \\ r &= 0.01 \\ S_\infty &= \frac{0.03}{1-0.01} \\ &= \frac{0.03}{0.99} \\ &= \frac{3}{99} \end{aligned}$$

$$\begin{aligned} \therefore 0.5\overline{3} &= \frac{5}{10} + \frac{3}{99} \\ &= \frac{53}{99} \end{aligned}$$

12.3.5 Exercise 12.3

1. Find the 10th term of the GP $2, 4, 8, \ldots$

Sol.

$$\begin{aligned} a_1 &= 2 \\ r &= \frac{4}{2} \\ &= 2 \\ a_{10} &= 2 \times 2^{10-1} \\ &= 2 \times 512 \\ &= 1024 \end{aligned}$$

2. Find the 8th term of the GP $243, -162, 108, \ldots$

Sol.

$$
\begin{aligned}
a_1 &= 243 \\
r &= \frac{-162}{243} \\
&= -\frac{2}{3} \\
a_8 &= 243 \times (-\frac{2}{3})^{8-1} \\
&= 243 \times (-\frac{128}{2187}) \\
&= -\frac{128}{9}
\end{aligned}
$$

3. Find the number of terms of the following GP.

(a) $8, 4, 2, 1, \ldots, \frac{1}{64}$

Sol.

$$
\begin{aligned}
a_1 &= 8 \\
r &= \frac{4}{8} \\
&= \frac{1}{2} \\
\frac{1}{64} &= 8 \times (\frac{1}{2})^{n-1} \\
\frac{1}{512} &= (\frac{1}{2})^{n-1} \\
\frac{1}{2^9} &= (\frac{1}{2})^{n-1} \\
n-1 &= 9 \\
n &= 10
\end{aligned}
$$

(b) $6, -18, 54, \ldots, -13122$

Sol.

$$
\begin{aligned}
a_1 &= 6 \\
r &= \frac{-18}{6} \\
&= -3 \\
-13122 &= 6 \times (-3)^{n-1} \\
-2187 &= (-3)^{n-1} \\
(-3)^7 &= (-3)^{n-1} \\
n-1 &= 7 \\
n &= 8
\end{aligned}
$$

(c) $54, 36, 24, \ldots, 3\frac{13}{81}$

Sol.

$$
\begin{aligned}
a_1 &= 54 \\
r &= \frac{36}{54} \\
&= \frac{2}{3} \\
\frac{256}{81} &= 54 \times (\frac{2}{3})^{n-1} \\
\frac{256}{81} \times \frac{1}{54} &= (\frac{2}{3})^{n-1} \\
\frac{128}{2187} &= (\frac{2}{3})^{n-1} \\
(\frac{2}{3})^7 &= (\frac{2}{3})^{n-1} \\
n-1 &= 7 \\
n &= 8
\end{aligned}
$$

4. Given that the 2nd term of a GP is 12, and its 4th term is 108, find the first term and the common ratio of it.

Sol.

$$
\begin{aligned}
a_2 &= ar = 12 \\
a_4 &= ar^3 = 109 \\
r^2 &= 9 \\
r &= \pm 3 \\
a_1 &= \pm 4 \\
\therefore a_1 &= 4, r = 3 \text{ or } a_1 = -4, r = -3
\end{aligned}
$$

5. Given that the 3rd term of an GP is $1\frac{1}{3}$, and its 8th term is $-10\frac{1}{8}$. Find the 5th term of this AP.

Sol.

$$
\begin{aligned}
a_3 &= ar^2 = \frac{4}{3} \\
a_8 &= ar^7 = -\frac{81}{8} \\
r^5 &= -\frac{81}{8} \times \frac{3}{4} \\
&= -\frac{243}{32} \\
&= (-\frac{3}{2})^5 \\
r &= -\frac{3}{2} \\
a &= \frac{4}{3} \times \frac{4}{9} \\
&= \frac{16}{27} \\
a_5 &= \frac{16}{27} \times (\frac{3}{2})^4 \\
&= \frac{16}{27} \times \frac{81}{16} \\
&= 3
\end{aligned}
$$

6. Find the geometric mean of 2 and 18.

Sol.

$$
\begin{aligned}
G &= \pm\sqrt[2]{2 \times 18} \\
&= \pm\sqrt[2]{36} \\
&= \pm 6
\end{aligned}
$$

7. Given that x+12, x+4 and x-2 are in GP, find the value of x and the common ratio of this GP.

Sol.

$$
\begin{aligned}
x + 4 &= \pm\sqrt{(x+12)(x-2)} \\
x^2 + 8x + 16 &= x^2 + 10x - 24 \\
2x &= 40 \\
x &= 20 \\
a_1 &= 20 + 12 = 32 \\
a_2 &= 20 + 4 = 24 \\
r &= \frac{24}{32} \\
&= \frac{3}{4}
\end{aligned}
$$

8. Find 3 numbers between 14 and 224 such that these 5 numbersare in GP.

Sol.

$$
\begin{aligned}
a_1 &= 14 \\
a_5 &= 224 \\
244 &= 14 \times r^4 \\
16 &= r^4 \\
(\pm 2)^4 &= r^4 \\
r &= \pm 2
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ & \textit{These } 3 \textit{ numbers are } 28,\ 56,\ 112 \\
& \textit{or } -28,\ 56,\ -112
\end{aligned}
$$

9. Calculate the sum of the first 6 terms of the GP $2 + 6 + 18 + \ldots$

Sol.

$$
\begin{aligned}
a_1 &= 2 \\
r &= \frac{6}{2} \\
&= 3 \\
S_6 &= \frac{2(1-3^6)}{1-3} \\
&= \frac{2(1-729)}{-2} \\
&= 728
\end{aligned}
$$

10. Calculate the sum of the first 8 terms of the GP $32 - 16 + 8 - \ldots$

Sol.

$$
\begin{aligned}
a_1 &= 32 \\
r &= \frac{-16}{32} \\
&= -\frac{1}{2} \\
S_8 &= \frac{32(1-(\frac{1}{2})^8)}{1+\frac{1}{2}} \\
&= \frac{32(1-\frac{1}{256})}{\frac{3}{2}} \\
&= 32 \times \frac{255}{256} \times \frac{2}{3} \\
&= \frac{85}{4}
\end{aligned}
$$

11. Find the sum of the GP $14 - 28 + 56 - \ldots + 3584$

Sol.

$$\begin{aligned}
a_1 &= 14\\
r &= \frac{-28}{14} = -2\\
3584 &= 14 \times (-2)^{n-1}\\
256 &= (-2)^{n-1}\\
(-2)^8 &= (-2)^{n-1}\\
n-1 &= 8\\
n &= 9\\
S_9 &= \frac{14(1-(-2)^9)}{1-(-2)}\\
&= \frac{14(1+512)}{3}\\
&= \frac{14 \times 513}{3}\\
&= 2394
\end{aligned}$$

12. If the first term of a GP is 7, its common ratio is 3, and the sum of its terms is 847, find the number of terms and the last term of this GP.

Sol.

$$\begin{aligned}
a_1 &= 7\\
r &= 3\\
S_n &= \frac{7(1-3^n)}{1-3} = 847\\
7(1-3^n) &= -1694\\
1-3^n &= -242\\
3^n &= 243\\
3^n &= 3^5\\
n &= 5\\
a_5 &= 7 \times 3^4 = 567
\end{aligned}$$

13. Find the sum of the following infinite GP.

(a) $24 + 18 + 13\frac{1}{2} + \dots$

Sol.

$$\begin{aligned}
a_1 &= 24\\
r &= \frac{18}{24} = \frac{3}{4}\\
S_\infty &= \frac{24}{1-\frac{3}{4}}\\
&= \frac{24}{\frac{1}{4}}\\
&= 96
\end{aligned}$$

(b) $27 - 9 + 3 - 1 + \dots$

Sol.

$$\begin{aligned}
a_1 &= 27\\
r &= \frac{-9}{27} = -\frac{1}{3}\\
S_\infty &= \frac{27}{1+\frac{1}{3}}\\
&= \frac{27}{\frac{4}{3}}\\
&= \frac{81}{4}
\end{aligned}$$

(c) $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \dots$

Sol.

$$\begin{aligned}
a_1 &= 2\\
r &= \frac{-\frac{1}{2}}{2} = -\frac{1}{4}\\
S_\infty &= \frac{2}{1+\frac{1}{4}}\\
&= \frac{2}{\frac{5}{4}}\\
&= \frac{8}{5}
\end{aligned}$$

14. Given an infinite GP which has a sum of 24 and first term of 30, find the common difference.

Sol.

$$\begin{aligned}
a_1 &= 30\\
S_\infty &= 24\\
24 &= \frac{30}{1-r}\\
24(1-r) &= 30\\
24 - 24r &= 30\\
-24r &= 6\\
r &= -\frac{1}{4}
\end{aligned}$$

15. Convert the following recurrring decimals into fractions.

(a) $0.\overline{45}$

Sol.

$$\begin{aligned} a_1 &= 0.45 \\ r &= 0.01 \\ S_\infty &= \frac{0.45}{1-0.01} \\ &= \frac{0.45}{0.99} \\ &= \frac{45}{99} \\ &= \frac{5}{11} \end{aligned}$$

$$\therefore 0.\overline{45} = \frac{5}{11}$$

(b) $0.\overline{037}$

Sol.

$$\begin{aligned} a_1 &= 0.037 \\ r &= 0.001 \\ S_\infty &= \frac{0.037}{1-0.001} \\ &= \frac{0.037}{0.999} \\ &= \frac{37}{999} \\ &= \frac{1}{27} \end{aligned}$$

$$\therefore 0.\overline{037} = \frac{1}{27}$$

(c) $0.2\overline{18}$

Sol.

$$\begin{aligned} a_1 &= 0.018 \\ r &= 0.01 \\ S_\infty &= \frac{0.018}{1-0.01} \\ &= \frac{0.018}{0.99} \\ &= \frac{18}{990} \\ &= \frac{1}{55} \end{aligned}$$

$$\begin{aligned} \therefore\ 0.2\overline{18} &= \frac{1}{5} + \frac{1}{55} \\ &= \frac{12}{55} \end{aligned}$$

(d) $1.\overline{3}$

Sol.

$$\begin{aligned} a_1 &= 0.3 \\ r &= 0.1 \\ S_\infty &= \frac{0.3}{1-0.1} \\ &= \frac{0.3}{0.9} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \therefore\ 1.\overline{3} &= 1 + \frac{1}{3} \\ &= \frac{4}{3} \end{aligned}$$

16. Three integers are in GP, summing up to 42 while accumulating up to 512, find these three integers.

Sol.

$$\begin{aligned} a_1 + a_2 + a_3 &= 42 \\ a_1 a_2 a_3 &= 512 \\ a_2 &= \pm\sqrt{a_1 a_3} \\ a_1 a_3 &= a_2^2 \\ a_2^3 &= 512 \\ a_2 &= \sqrt[3]{512} \\ &= 8 \\ a_1 a_3 &= 64 \\ a_3 &= \frac{64}{a_1} \\ a_1 + 8 + \frac{64}{a_1} &= 42 \\ a_1 + \frac{64}{a_1} &= 34 \\ a_1^2 + 64 &= 34a_1 \\ a_1^2 - 34a_1 + 64 &= 0 \\ (a_1 - 32)(a_1 - 2) &= 0 \\ a_1 &= 32 \textit{ or } a_1 = 2 \end{aligned}$$

$$\therefore \textit{ These three integers are } 2,\ 8,\ 32$$

17. The sum of first 6 term of a GP is 9 times the sum of first 3 terms. Find the common ratio.

Sol.

$$
\begin{aligned}
S_6 &= 9S_3 \\
\frac{a(1-r^6)}{1-r} &= 9 \times \frac{a(1-r^3)}{1-r} \\
a(1-r^6) &= 9a(1-r^3) \\
1-r^6 &= 9(1-r^3) \\
&= 9-9r^3 \\
r^6-9r^3+8 &= 0 \\
(r^3-8)(r^3-1) &= 0 \\
r^3 &= 8 \ or \ r^3 = 1 \\
r &= 1 \ (invalid) \\
r &= 2
\end{aligned}
$$

18. Given a GP, its first term is 16, last term is $\frac{1}{2}$ and its sum is $31\frac{1}{2}$, find its common ratio and number of terms.

Sol.

$$
\begin{aligned}
a_1 &= 16 \\
\frac{1}{2} &= 16r^{n-1} \\
\frac{1}{32} &= r^{n-1} \\
&= r^n \times \frac{1}{r} \\
r^n &= \frac{r}{32} \\
\frac{63}{2} &= \frac{16(1-r^n)}{1-r} \\
63(1-r) &= 32(1-r^n) \\
63-63r &= 32-32r^n \\
-31 &= 32r^n - 63r \\
-31 &= r - 63r \\
-31 &= -62r \\
r &= \frac{1}{2} \\
(\frac{1}{2})^{n-1} &= \frac{1}{32} \\
&= (\frac{1}{2})^5 \\
n-1 &= 5 \\
n &= 6
\end{aligned}
$$