Senior 2 Math Part I

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Chapter 1

Sequence and Series

1.1 Sequence and Series

1.1.1 Practice 1

1. Find the first 5 terms of the sequence $a_n = \frac{2^n}{n+1}$.

   Ans. $a_1 = \frac{2}{2} = 1, a_2 = \frac{4}{3}, a_3 = \frac{8}{4}, a_4 = \frac{16}{5}, a_5 = \frac{32}{6}$

2. Write the general term of the sequence 1, 8, 27, 64, ...

   Ans. $a_n = n^3$

1.1.2 Practice 2

1. Express the series $\sum_{n=1}^{10} n^2 + 1$ in the form of numbers.

   Ans. $$\sum_{n=1}^{10} n^2 + 1$$
   $$= (1^2+1) + (2^2+1) + (3^2+1) + (4^2+1) + (5^2+1)$$
   $$+ (6^2+1) + (7^2+1) + (8^2+1) + (9^2+1) + (10^2+1)$$
   $$= 2 + 5 + 10 + 17 + 26 + 37 + 50 + 65 + 82 + 101$$

2. Write the first term, last term and the number of terms of the series $\sum_{n=1}^{10} (3^n - 2^n)$.

   Ans. $$First\ term = (3^1 - 2^1) = 1$$
   $$Last\ term = (3^{10} - 2^{10}) = 59049$$
   $$Number\ of\ terms = 10$$

3. Express the series $2 \times 5 + 3 \times 7 + 4 \times 9 + \ldots + 15 \times 31$ in the form of $\sum$.

Ans.

$$a_1 = 2 \times 5 = 10$$
$$a_2 = 3 \times 7 = 21$$
$$a_3 = 4 \times 9 = 36$$
$$a_4 = 5 \times 11 = 55$$
$$\vdots$$
$$a_{15} = 15 \times 31 = 465$$
$$\therefore 2 \times 5 + 3 \times 7 + 4 \times 9 + \ldots + 15 \times 31 = \sum_{n=1}^{15} a_n$$

1.1.3 Exercise 12.1

1. Find the general term of the following sequences.

   (a) 5, 8, 11, 14, ...
   Ans. $a_n = 3n + 2$

   (b) 2, 4, 8, 16, ...
   Ans. $a_n = 2^n$

   (c) $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$
   Ans. $a_n = \frac{n+1}{n}$

   (d) $\frac{2}{5}, \frac{4}{7}, \frac{6}{9}, \frac{8}{11}, \ldots$
   Ans. $a_n = \frac{2n}{2n+1}$

2. Find the first 5 terms of the following sequences.

   (a) $a_n = 2n + 3$
   Ans. $a_1 = 2 \times 1 + 3 = 5, a_2 = 2 \times 2 + 3 = 7, a_3 = 2 \times 3 + 3 = 9, a_4 = 2 \times 4 + 3 = 11, a_5 = 2 \times 5 + 3 = 13$

   (b) $a_n = n(n-2)$
   Ans. $a_1 = 1 \times (-1) = -1, a_2 = 2 \times 0 = 0, a_3 = 3 \times 1 = 3, a_4 = 4 \times 2 = 8, a_5 = 5 \times 3 = 15$

   (c) $a_n = \frac{n}{2n+1}$
   Ans. $a_1 = \frac{1}{2\times1+1} = \frac{1}{3}, a_2 = \frac{2}{2\times2+1} = \frac{2}{5}, a_3 = \frac{3}{2\times3+1} = \frac{3}{7}, a_4 = \frac{4}{2\times4+1} = \frac{4}{9}, a_5 = \frac{5}{2\times5+1} = \frac{5}{11}$

   (d) $a_n = (-3)^n$
   Ans. $a_1 = (-3)^1 = -3, a_2 = (-3)^2 = 9, a_3 = (-3)^3 = -27, a_4 = (-3)^4 = 81, a_5 = (-3)^5 = -243$

3. Express the following series in the form of numbers.

   (a) $\sum_{n=1}^{5} n(n+3)$

Ans. $\sum_{n=1}^{5} n(n+3)$

$= (1 \times 4) + (2 \times 5) + (3 \times 6) + (4 \times 7) + (5 \times 8)$

$= 4 + 10 + 18 + 28 + 40$

(b) $\sum_{n=2}^{6} \frac{1}{3^n}$

Ans. $\sum_{n=2}^{6} \frac{1}{3^n}$

$= \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6}$

$= \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729}$

(c) $\sum_{n=1}^{6} \frac{1}{n(2n+1)}$

Ans. $\sum_{n=1}^{6} \frac{1}{n(2n+1)}$

$= \frac{1}{1(2 \times 1 + 1)} + \frac{1}{2(2 \times 2 + 1)} + \frac{1}{3(2 \times 3 + 1)}$

$+ \frac{1}{4(2 \times 4 + 1)} + \frac{1}{5(2 \times 5 + 1)} + \frac{1}{6(2 \times 6 + 1)}$

$= \frac{1}{3} + \frac{1}{10} + \frac{1}{21} + \frac{1}{36} + \frac{1}{55} + \frac{1}{78}$

(d) $\sum_{n=2}^{5} \frac{1}{n^2+2}$

Ans. $\sum_{n=2}^{5} \frac{1}{n^2+2}$

$= \frac{1}{4+2} + \frac{1}{9+2} + \frac{1}{16+2} + \frac{1}{25+2}$

$= \frac{1}{6} + \frac{1}{11} + \frac{1}{18} + \frac{1}{27}$

4. Find the first term, last term and the number of terms of the following series.

   (a) $\sum_{n=3}^{10} 2^2$
   Ans. $a_3 = 2^2 = 4, a_{10} = 2^2 = 4, n = 10 - 3 + 1 = 8$

   (b) $\sum_{n=1}^{8} \frac{n+2}{n}$
   Ans. $a_1 = \frac{1+2}{1} = \frac{3}{1} = 3, a_8 = \frac{8+2}{8} = \frac{10}{8} = \frac{5}{4}, n = 8 - 1 + 1 = 8$

(c) $\sum_{n=1}^{10} 3n^2 - n$
Ans. $a_1 = 3\times1^2-1 = 2, a_{10} = 3\times10^2-10 = 290, n = 10-1+1 = 10$

(d) $\sum_{n=9}^{14} n^2(n-7)$
Ans. $a_9 = 9^2(9-7) = 9^2 \times 2 = 162, a_{14} = 14^2(14-7) = 14^2 \times 7 = 2744, n = 14-9+1 = 6$

5. Express the following series in the form of $\sum$.

(a) $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{30}$

$$\begin{aligned}
\textbf{Ans.} a_1 &= 1 \\
a_2 &= \frac{1}{2} \\
a_3 &= \frac{1}{3} \\
&\vdots \\
a_{30} &= \frac{1}{30} \\
\therefore 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{30} &= \sum_{n=1}^{30} \frac{1}{n}
\end{aligned}$$

(b) $1^3 + 2^3 + 3^3 + \ldots + 50^3$

$$\begin{aligned}
\textbf{Ans.} a_1 &= 1^3 \\
a_2 &= 2^3 \\
a_3 &= 3^3 \\
&\vdots \\
a_{50} &= 50^3 \\
\therefore 1^3 + 2^3 + 3^3 + \ldots + 50^3 &= \sum_{n=1}^{50} n^3
\end{aligned}$$

(c) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16}$

$$\begin{aligned}
\textbf{Ans.} a_1 &= (-\frac{1}{2})^{1-1} \\
a_2 &= (-\frac{1}{2})^{2-1} \\
a_3 &= (-\frac{1}{2})^{3-1} \\
a_4 &= (-\frac{1}{2})^{4-1} \\
a_5 &= (-\frac{1}{2})^{5-1} \\
\therefore 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} &= \sum_{n=1}^{5} (-\frac{1}{2})^{n-1}
\end{aligned}$$

(d) $2 \times 4 + 4 \times 7 + 6 \times 10 + 8 \times 13 + 10 \times 16$

$$\begin{aligned}
\textbf{Ans.} a_1 &= 2 \times 1 \times (3 \times 1 + 1) \\
a_2 &= 2 \times 2 \times (3 \times 2 + 1) \\
a_3 &= 2 \times 3 \times (3 \times 3 + 1) \\
a_4 &= 2 \times 4 \times (3 \times 4 + 1) \\
a_5 &= 2 \times 5 \times (3 \times 5 + 1) \\
\therefore 2 \times 4 + 4 \times 7 + 6 \times 10 + 8 \times 13 + 10 \times 16 &= \sum_{n=1}^{5} 2n(3n + 1)
\end{aligned}$$

1.2 Arithmetic Progression

General term of an Arithmetic Progression (AP) is given by

$$a_n = a_1 + (n - 1)d$$

where a_1 is the first term, d is the common difference and n is the number of terms.

1.2.1 Practice 3

1. Find the number of terms of the AP $-4 - 2\frac{3}{4} - 1\frac{1}{2} - \frac{1}{4} + \ldots + 16$.

Ans.

$$\begin{aligned}
a_1 &= -4\\
a_n &= 16\\
d &= -2\frac{3}{4} - (-4)\\
&= -2\frac{3}{4} + 4\\
&= \frac{5}{4}\\
16 &= -4 + (n-1)\frac{5}{4}\\
20 &= \frac{5}{4}(n-1)\\
80 &= 5(n-1)\\
n-1 &= 16\\
n &= 17
\end{aligned}$$

2. Given that $a_2 = 4$ and $a_6 = -8$, find the 10th term of the AP.

Ans.

$$\begin{aligned}
a_2 &= 4\\
a + (2-1)d &= 4\\
a_6 &= -8\\
a + (6-1)d &= -8
\end{aligned}$$

$$\begin{cases} a + d & = 4 \qquad (1.1)\\ a + 5d & = \text{-8} \qquad (1.2) \end{cases}$$

$$\begin{aligned}
(2)-(1): 4d &= -12\\
d &= -3\\
a + (-3) &= 4\\
a &= 7\\
\therefore a_{10} &= 7 + (10-1)(-3)\\
&= 7 - 27\\
&= -20
\end{aligned}$$

3. How many multiples of 7 are there between 50 and 500?

Ans.

$$\begin{aligned}
a_1 &= 56 \\
a_n &= 497 \\
d &= 7 \\
497 = 56 + (n-1)7 \\
441 = 7(n-1) \\
n - 1 &= 63 \\
n &= 64
\end{aligned}$$

4. Find 5 numbers between 30 and 54 such that these numbers form an AP.

Ans.

$$\begin{aligned}
&a_1 = 30 \\
&a_7 = 54 \\
&54 = 30 + (7-1)d \\
&24 = 6d \\
&\ d = 4
\end{aligned}$$

$$\therefore These\ 5\ numbers\ are\ 34,\ 38,\ 42,\ 46,\ and\ 50.$$

Arithmetic mean

If A is in between x and y, and x, A, y are in AP, then

$$A = \frac{x+y}{2}$$

1.2.2 Practice 4

1. If 9, x, 17 are in AP, find x.

Ans.

$$\begin{aligned}
x &= \frac{9+17}{2} \\
&= \frac{26}{2} \\
&= 13
\end{aligned}$$

2. Find the arithmetic mean of 26 and -11.

Ans.

$$\begin{aligned} A &= \frac{26 - 11}{2} \\ &= \frac{15}{2} \end{aligned}$$

3. Find x and y when 3, x, 12, y, 21 are in AP.

Ans.

$$\begin{aligned} x &= \frac{3 + 12}{2} \\ &= \frac{15}{2} \\ y &= \frac{12 + 21}{2} \\ &= \frac{33}{2} \end{aligned}$$

Summation of Arithmetic Progression

The summation formula for AP is given by

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

or

$$S_n = \frac{n}{2}(a_1 + a_n)$$

1.2.3 Practice 5

1. Find the sum of the first 16 terms of the AP $22 + 18 + 14 + 10 + \ldots$

Ans.

$$
\begin{aligned}
a_1 &= 22\\
n &= 16\\
d &= -4\\
S_n &= \frac{16}{2}(2 \times 22 + (-4)(16-1))\\
&= \frac{16}{2}(44 + (-4)(15))\\
&= \frac{16}{2}(44 - 60)\\
&= \frac{16}{2}(-16)\\
&= -128
\end{aligned}
$$

2. If the sum of AP $23 + 19 + 15 + \ldots$ is 72, find the number of terms.

Ans.

$$
\begin{aligned}
a_1 &= 23\\
S_n &= 72\\
d &= -4\\
72 &= \frac{n}{2}(2 \times 23 + (-4)(n-1))\\
72 &= \frac{n}{2}(46 + (-4)(n-1))\\
144 &= n(46 + (-4)(n-1))\\
144 &= n(46 - 4n + 4)\\
144 &= n(50 - 4n)\\
144 &= 50n - 4n^2\\
72 &= 25n - 2n^2\\
2n^2 - 25n + 72 &= 0\\
(n-8)(2n-9) &= 0\\
n &= 8
\end{aligned}
$$

3. Given that $S_n = 2n + 3n^2$, find the first term and the common difference of the AP.

Ans.

$$S_n = 2n + 3n^2$$
$$2n + 3n^2 = \frac{n}{2}(2a + (n-1)d)$$
$$4n + 6n^2 = n(2a + (n-1)d)$$
$$4n + 6n^2 = 2na + (n-1)nd$$
$$4n + 6n^2 = 2na + n^2d - nd$$
$$4n + 6n^2 = (2a-d)n + dn^2$$

Comparing both sides,

$$2a - d = 4$$
$$a = 6$$
$$d = 2$$