

## 高一第十章

# 三角方程式

(作答题)

1. 解方程式  $\sin 4x + \cos 2x = 0$ 。

解:

$$\begin{aligned}\sin 4x + \cos 2x &= 0 \\ 2 \sin 2x \cos 2x + \cos 2x &= 0 \\ \cos 2x(2 \sin 2x + 1) &= 0 \\ \cos 2x = 0 \quad \text{or} \quad \sin 2x &= -\frac{1}{2} \\ 2x = k\pi + \frac{\pi}{2} \quad \text{or} \quad 2x = k\pi + (-1)^{k+1} \frac{\pi}{6} \\ x = k\pi + \frac{\pi}{4} \quad \text{or} \quad x = k\pi + (-1)^{k+1} \frac{\pi}{12} \quad \text{where } k \in \mathbb{Z} \quad \blacksquare\end{aligned}$$

2. 解  $\frac{\cos x - \sin x}{\cos x + \sin x} = \cos^2 x - \sin^2 x$ 。

解:

$$\begin{aligned}\frac{\cos x - \sin x}{\cos x + \sin x} &= \cos^2 x - \sin^2 x \\ \frac{(\cos x - \sin x)^2}{(\cos^2 x - \sin^2 x)} &= \cos^2 x - \sin^2 x \\ \frac{\cos^2 x - 2 \sin x \cos x + \sin^2 x}{\cos 2x} &= \cos 2x \\ 1 - \sin 2x &= \cos^2 2x \\ 1 - \sin 2x &= 1 - \sin^2 2x \\ \sin^2 2x - \sin 2x &= 0 \\ \sin 2x(\sin 2x - 1) &= 0 \\ \sin 2x = 0 \quad \text{or} \quad \sin 2x &= 1 \\ 2x = k\pi \quad \text{or} \quad 2x = 2k\pi + \frac{\pi}{2} \\ x = k\pi \quad \text{or} \quad x = k\pi + \frac{\pi}{4} \quad \text{where } k \in \mathbb{Z} \quad \blacksquare\end{aligned}$$

3. 解方程式  $\tan\left(\frac{\pi}{4} - x\right) + \cot\left(\frac{\pi}{4} - x\right) = 4$ 。

解:

$$\begin{aligned}
 \tan\left(\frac{\pi}{4} - x\right) + \cot\left(\frac{\pi}{4} - x\right) &= 4 \\
 \tan\left(\frac{\pi}{4} - x\right) + \frac{1}{\tan\left(\frac{\pi}{4} - x\right)} &= 4 \\
 \frac{\tan^2\left(\frac{\pi}{4} - x\right) + 1}{\tan\left(\frac{\pi}{4} - x\right)} &= 4 \\
 \frac{\sec^2\left(\frac{\pi}{4} - x\right)}{\tan\left(\frac{\pi}{4} - x\right)} &= 4 \\
 \frac{1}{\sin\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - x\right)} &= 4 \\
 \frac{1}{2\sin\left(\frac{\pi}{2} - 2x\right)} &= 2 \\
 \sin\left(\frac{\pi}{2} - 2x\right) &= \frac{1}{2} \\
 \cos 2x &= \frac{1}{2} \\
 2x &= 2k\pi \pm \frac{\pi}{3} \\
 x &= k\pi \pm \frac{\pi}{6} \quad \text{where } k \in \mathbb{Z}
 \end{aligned}$$

4. 试证  $4\cos x - 3\sin x \leq 5$ 。若  $4\cos x - 3\sin x = 5$ , 求  $x$  之一般值。

解:

$$\begin{aligned}
 y &= 4\cos x - 3\sin x \\
 \frac{dy}{dx} &= -4\sin x - 3\cos x = 0 \\
 -4\sin x &= 3\cos x \\
 \tan x &= -\frac{3}{4} \\
 x &\approx -36.87^\circ \\
 \frac{d^2y}{dx^2} &= -4\cos x + 3\sin x
 \end{aligned}$$

When  $x = -36.87^\circ$ ,

$$\begin{aligned}
 \frac{d^2y}{dx^2} &= -4\cos x + 3\sin x \\
 &= -4\cos(-36.87^\circ) + 3\sin(-36.87^\circ) \\
 &\approx -4 \times 0.8 + 3 \times -0.6 \\
 &\approx -5
 \end{aligned}$$

$\therefore y$  is maximum when  $x = -36.87^\circ$  and  $y = 5 \implies 4\cos x - 3\sin x \leq 5$ .

Let  $t = \tan \frac{x}{2}$ , then  $\cos x = \frac{1-t^2}{1+t^2}$  and  $\sin x = \frac{2t}{1+t^2}$ .

$$\begin{aligned}
 4 \cos x - 3 \sin x &= 5 \\
 4 \left( \frac{1-t^2}{1+t^2} \right) - 3 \left( \frac{2t}{1+t^2} \right) &= 5 \\
 4 - 4t^2 - 6t &= 5 + 5t^2 \\
 9t^2 + 6t + 1 &= 0 \\
 (3t+1)^2 &= 0 \\
 t &= -\frac{1}{3} \\
 \tan \frac{x}{2} &= -\frac{1}{3} \\
 \frac{x}{2} &= 180^\circ k - 18.43^\circ \\
 x &= 360^\circ k - 36.87^\circ \quad \text{where } k \in \mathbb{Z}
 \end{aligned}$$

■

5. 求满足方程式  $4 \sin x - 2 \cos x = 3$  的所有自  $0^\circ$  至  $360^\circ$  的角。

解:

Let  $t = \tan \frac{x}{2}$ , then  $\cos x = \frac{1-t^2}{1+t^2}$  and  $\sin x = \frac{2t}{1+t^2}$ .

$$\begin{aligned}
 4 \sin x - 2 \cos x &= 3 \\
 4 \left( \frac{2t}{1+t^2} \right) - 2 \left( \frac{1-t^2}{1+t^2} \right) &= 3 \\
 8t - 2 + 2t^2 &= 3 + 3t^2 \\
 t^2 - 8t + 5 &= 0 \\
 t &= \frac{8 \pm \sqrt{64 - 4 \times 5}}{2} \\
 &= 4 \pm \sqrt{11} \\
 \tan \frac{x}{2} &= 4 \pm \sqrt{11} \\
 \frac{x}{2} &= 180^\circ k + 82.22^\circ \quad \text{or} \quad 180^\circ k - 34.35^\circ \\
 x &= 360^\circ k + 164.44^\circ \quad \text{or} \quad 360^\circ k - 68.7^\circ \quad \text{where } k \in \mathbb{Z}
 \end{aligned}$$

■

6. 解方程式  $3 \sin 2x = 2 \tan x$ , 其中  $0 \leq x \leq 2\pi$ 。

解:

$$\begin{aligned}
 3 \sin 2x &= 2 \tan x \\
 3 \sin x \cos x &= \frac{\sin x}{\cos x} \\
 3 \sin x \cos^2 x - \sin x &= 0 \\
 \sin x (3 \cos^2 x - 1) &= 0 \\
 \sin x = 0 \quad \text{or} \quad 3 \cos^2 x - 1 &= 0
 \end{aligned}$$

$$x = k\pi \quad \text{or} \quad \cos x = \pm \frac{1}{\sqrt{3}}$$

$$x = k\pi \quad \text{or} \quad x = 2k\pi \pm 0.955$$

When  $k = 0$ ,  $x = 0$  or  $x = \pm 0.955$ .

When  $k = 1$ ,  $x = \pi$  or  $x = \pi \pm 0.955$ .

When  $k = 2$ ,  $x = 2\pi$  or  $x = 2\pi \pm 0.955$ .

Since  $x \leq 2\pi$ , the solutions are  $0, \pi, 2\pi, 0.955, \pi \pm 0.955, 2\pi - 0.955$ . ■

7. 求满足方程式  $2 \sin 3x + \cos 2x = 1$  在  $0^\circ \leq x \leq 360^\circ$  范围内  $x$  的角度。

解:

$$2 \sin 3x + \cos 2x = 1$$

$$2(\sin x \cos 2x + \cos x \sin 2x) + 1 - 2 \sin^2 x = 1$$

$$2(\sin x(2 \cos^2 x - 1) + \cos x(2 \sin x \cos x)) - 2 \sin^2 x = 0$$

$$2(2 \cos^2 x \sin x - \sin x + 2 \sin x \cos^2 x) - 2 \sin^2 x = 0$$

$$2(4 \sin x \cos^2 x - \sin x) - 2 \sin^2 x = 0$$

$$2(4 \sin x(1 - \sin^2 x) - \sin x) - 2 \sin^2 x = 0$$

$$2(4 \sin x - 4 \sin^3 x - \sin x) - 2 \sin^2 x = 0$$

$$2(3 \sin x - 4 \sin^3 x) - 2 \sin^2 x = 0$$

$$6 \sin x - 8 \sin^3 x - 2 \sin^2 x = 0$$

$$\sin x(4 \sin^2 x + \sin x - 3) = 0$$

$$\sin x(\sin x + 1)(4 \sin x - 3) = 0$$

$$\sin x = 0 \quad \text{or} \quad \sin x = -1 \quad \text{or} \quad \sin x = \frac{3}{4}$$

$$x = 180^\circ k \quad \text{or} \quad x = 360^\circ k - 90^\circ \quad \text{or} \quad x = 180^\circ k + (-1)^k \cdot 48.59^\circ$$

When  $k = 0$ ,  $x = 0$  or  $x = -90$  or  $x = 48.59$ .

When  $k = 1$ ,  $x = 180$  or  $x = 270$  or  $x = 131.41$ .

When  $k = 2$ ,  $x = 360$  or  $x = 630$  or  $x = 408.59$ .

Since  $0 \leq x \leq 360$ , the solutions are  $0^\circ, 180^\circ, 270^\circ, 360^\circ, 48.59^\circ, 131.41^\circ$ . ■

8. 求满足方程式  $5 \cos 2x + 8 \sin x = 3$  的所有自  $0^\circ$  至  $360^\circ$  的  $x$  值。

解:

$$5 \cos 2x + 8 \sin x = 3$$

$$5(1 - 2 \sin^2 x) + 8 \sin x = 3$$

$$5 - 10 \sin^2 x + 8 \sin x = 3$$

$$10 \sin^2 x - 8 \sin x - 2 = 0$$

$$5 \sin^2 x - 4 \sin x - 1 = 0$$

$$\begin{aligned}
(5 \sin x + 1)(\sin x - 1) &= 0 \\
\sin x &= -\frac{1}{5} \quad \text{or} \quad \sin x = 1 \\
x &= 180^\circ k + (-1)^{k+1} \cdot 11.54^\circ \quad \text{or} \quad x = 180^\circ k + 90^\circ
\end{aligned}$$

When  $k = 0$ ,  $x = -11.54$  or  $x = 90$ .

When  $k = 1$ ,  $x = 191.54$  or  $x = 270$ .

When  $k = 2$ ,  $x = 348.46$  or  $x = 450$ .

Since  $0 \leq x \leq 360$ , the solutions are  $90^\circ, 270^\circ, 191.54^\circ, 348.46^\circ$ . ■

9. 解方程式  $20 \cos x - 15 \sin x = 9$ , 式中  $0^\circ \leq x \leq 360^\circ$ 。

解:

Let  $t = \tan \frac{x}{2}$ , then  $\cos x = \frac{1-t^2}{1+t^2}$  and  $\sin x = \frac{2t}{1+t^2}$ .

$$\begin{aligned}
20 \cos x - 15 \sin x &= 9 \\
20 \left( \frac{1-t^2}{1+t^2} \right) - 15 \left( \frac{2t}{1+t^2} \right) &= 9 \\
20 - 20t^2 - 30t &= 9 + 9t^2 \\
29t^2 + 30t - 11 &= 0
\end{aligned}$$

$$\begin{aligned}
t &= \frac{-30 \pm \sqrt{30^2 - 4 \times 29 \times -11}}{2 \times 29} \\
&= \frac{-15 \pm 4\sqrt{34}}{29} \\
\tan \frac{x}{2} &= \frac{-15 \pm 4\sqrt{34}}{29} \\
\frac{x}{2} &= 180^\circ k + \arctan \left( \frac{-15 \pm 4\sqrt{34}}{29} \right) \\
x &= 360^\circ k + 32.03^\circ \quad \text{or} \quad x = 360^\circ k - 105.8^\circ \quad \text{where } k \in \mathbb{Z}
\end{aligned}$$

When  $k = 0$ ,  $x = 32.03$  or  $x = -105.8$ .

When  $k = 1$ ,  $x = 392.03$  or  $x = 254.2$ .

Since  $0 \leq x \leq 360$ , the solutions are  $32.03^\circ, 254.2^\circ$ . ■

10. 试不用计算机或对数表, 求三角方程式  $\cos \theta + \sqrt{3} \sin \theta = \sqrt{2}$  之一般解。

解:

$$\cos \theta + \sqrt{3} \sin \theta = R \cos(\theta - \alpha) = R \cos \theta \cos \alpha + R \sin \theta \sin \alpha$$

$$\begin{cases} R \cos \alpha = 1 & \cdots (1) \\ R \sin \alpha = \sqrt{3} & \cdots (2) \end{cases}$$

$$(1)^2 + (2)^2 \Rightarrow R^2(\cos^2 \alpha + \sin^2 \alpha) = 1 + 3$$

$$R^2 = 4$$

$$R = \pm 2$$

$$\frac{(2)}{(1)} \Rightarrow \tan \alpha = \sqrt{3}$$

$$\alpha = \frac{\pi}{3}$$

$$\cos \theta + \sqrt{3} \sin \theta = \sqrt{2}$$

$$\pm 2 \cos \left( \theta - \frac{\pi}{3} \right) = \sqrt{2}$$

$$\cos \left( \theta - \frac{\pi}{3} \right) = \pm \frac{\sqrt{2}}{2}$$

$$\theta - \frac{\pi}{3} = 2k\pi \pm \frac{\pi}{4}$$

$$\theta = 2k\pi + \frac{\pi}{3} \pm \frac{\pi}{4}$$

$$\theta = 2k\pi + \frac{7\pi}{12} \quad \text{or} \quad \theta = 2k\pi + \frac{\pi}{12} \quad \text{where } k \in \mathbb{Z} \quad \blacksquare$$

11. (a) 解方程式  $3 \cos x - \sin x = 1$ , 式中  $0^\circ \leq x \leq 360^\circ$ 。

解:

Let  $t = \tan \frac{x}{2}$ , then  $\cos x = \frac{1-t^2}{1+t^2}$  and  $\sin x = \frac{2t}{1+t^2}$ .

$$3 \cos x - \sin x = 1$$

$$3 \left( \frac{1-t^2}{1+t^2} \right) - \left( \frac{2t}{1+t^2} \right) = 1$$

$$3 - 3t^2 - 2t = 1 + t^2$$

$$4t^2 + 2t - 2 = 0$$

$$2t^2 + t - 1 = 0$$

$$(2t-1)(t+1) = 0$$

$$t = \frac{1}{2} \quad \text{or} \quad t = -1$$

$$\tan \frac{x}{2} = \frac{1}{2} \quad \text{or} \quad \tan \frac{x}{2} = -1$$

$$\frac{x}{2} = 180^\circ k + 26.57^\circ \quad \text{or} \quad \frac{x}{2} = 180^\circ k - 45^\circ$$

$$x = 360^\circ k + 53.14^\circ \quad \text{or} \quad x = 360^\circ k - 90^\circ$$

When  $k = 0$ ,  $x = 53.14$  or  $x = -90$ .

When  $k = 1$ ,  $x = 413.14$  or  $x = 270$ .

Since  $0 \leq x \leq 360$ , the solutions are  $53.14^\circ, 270^\circ$ . \blacksquare

(b) 求下列方程式的一般解:

i.  $\sin 2\theta + \cos^2 \theta = 1$ ;

解:

$$\sin 2\theta + \cos^2 \theta = 1$$

$$2 \sin \theta \cos \theta + 1 - \sin^2 \theta = 1$$

$$2 \sin \theta \cos \theta - \sin^2 \theta = 0$$

$$\sin \theta (2 \cos \theta - \sin \theta) = 0$$

$$\sin \theta = 0 \text{ or } 2 \cos \theta - \sin \theta = 0$$

$$\sin \theta = 0 \text{ or } \tan \theta = 2$$

$$\theta = k\pi \text{ or } \theta = k\pi + 1.107 \quad \text{where } k \in \mathbb{Z} \quad \blacksquare$$

ii.  $\cos 3\theta + 2 \cos \theta = 0$ 。

解:

$$\cos 3\theta + 2 \cos \theta = 0$$

$$\cos 2\theta \cos \theta - \sin 2\theta \sin \theta + 2 \cos \theta = 0$$

$$(1 - 2 \sin^2 \theta) \cos \theta - 2 \sin^2 \theta \cos \theta + 2 \cos \theta = 0$$

$$3 \cos \theta - 4 \sin^2 \theta \cos \theta = 0$$

$$\cos \theta (3 - 4 \sin^2 \theta) = 0$$

$$\cos \theta = 0 \text{ or } 3 - 4(1 - \cos^2 \theta) = 0$$

$$\cos \theta = 0 \text{ or } \cos^2 \theta = \frac{1}{4}$$

$$\cos \theta = 0 \text{ or } 2 \cos^2 \theta = \frac{1}{2}$$

$$\cos \theta = 0 \text{ or } 1 - 2 \cos^2 \theta = -\frac{1}{2}$$

$$\cos \theta = 0 \text{ or } \cos 2\theta = -\frac{1}{2}$$

$$\theta = k\pi + \frac{\pi}{2} \text{ or } 2\theta = 2k\pi \pm \frac{2\pi}{3}$$

$$\theta = k\pi + \frac{\pi}{2} \text{ or } \theta = k\pi \pm \frac{\pi}{3} \quad \text{where } k \in \mathbb{Z} \quad \blacksquare$$

12. 求满足方程式  $\sin 5x + \sin 3x = \sin 8x$  在  $0 \leq x \leq \pi$  的所有  $x$  之值。

解:

$$\sin 5x + \sin 3x = \sin 8x$$

$$2 \sin 4x \cos x = 2 \sin 4x \cos 4x$$

$$\sin 4x (\cos x - \cos 4x) = 0$$

$$\sin 4x \left( 2 \sin \frac{5x}{2} \sin \frac{3x}{2} \right) = 0$$

$$\sin 4x \sin \frac{5x}{2} \sin \frac{3x}{2} = 0$$

$$\begin{aligned}\sin 4x = 0 \text{ or } \sin \frac{5x}{2} = 0 \text{ or } \sin \frac{3x}{2} = 0 \\ 4x = k\pi \text{ or } \frac{5x}{2} = k\pi \text{ or } \frac{3x}{2} = k\pi \\ x = \frac{k\pi}{4} \text{ or } x = \frac{2k\pi}{5} \text{ or } x = \frac{2k\pi}{3}\end{aligned}$$

When  $k = 0$ ,  $x = 0$  or  $x = 0$  or  $x = 0$ .

When  $k = 1$ ,  $x = \frac{\pi}{4}$  or  $x = \frac{2\pi}{5}$  or  $x = \frac{2\pi}{3}$ .

When  $k = 2$ ,  $x = \frac{\pi}{2}$  or  $x = \frac{4\pi}{5}$  or  $x = \frac{4\pi}{3}$ .

When  $k = 3$ ,  $x = \frac{3\pi}{4}$  or  $x = \frac{6\pi}{5}$  or  $x = \frac{6\pi}{3}$ .

When  $k = 4$ ,  $x = \pi$  or  $x = \frac{8\pi}{5}$  or  $x = 2\pi$ .

Since  $0 \leq x \leq \pi$ , the solutions are  $0, \frac{\pi}{4}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{4\pi}{5}, \pi$ . ■

13. 解方程式  $3 \sin 2\theta - 4 \cos 2\theta = 2$ , 式中  $0^\circ \leq \theta \leq 180^\circ$ 。

解:

$$\begin{aligned}3 \sin 2\theta - 4 \cos 2\theta &= 2 \\ 6 \sin \theta \cos \theta - 4(\cos^2 \theta - \sin^2 \theta) &= 2 \\ 6 \sin \theta \cos \theta - 4 \cos^2 \theta + 4 \sin^2 \theta &= 2 \\ 3 \sin \theta \cos \theta - 2 \cos^2 \theta + 2 \sin^2 \theta &= 1 \\ 3 \sin \theta \cos \theta - 2 \cos^2 \theta + 2 \sin^2 \theta &= \sin^2 \theta + \cos^2 \theta \\ 3 \sin \theta \cos \theta - 3 \cos^2 \theta + \sin^2 \theta &= 0 \\ \tan^2 \theta + 3 \tan \theta - 3 &= 0 \\ \tan \theta &= \frac{-3 \pm \sqrt{21}}{2} \\ \theta &= 180^\circ k + \arctan \left( \frac{-3 \pm \sqrt{21}}{2} \right) \\ \theta &= 180^\circ k + 38.35^\circ \quad \text{or} \quad \theta = 180^\circ k - 75.22^\circ\end{aligned}$$

When  $k = 0$ ,  $\theta = 38.35$  or  $\theta = -75.22$ .

When  $k = 1$ ,  $\theta = 218.35$  or  $\theta = 104.78$ .

Since  $0 \leq \theta \leq 180$ , the solutions are  $38.35^\circ, 104.78^\circ$ . ■



14. 已知  $\alpha$  是锐角且  $\cos \alpha = x - 1$ , 证明  $\cos 2\alpha - 3 \cos \alpha \sin^2 \alpha = 3x^3 - 7x^2 + 2x + 1$ 。然后解方程式  $\cos 2\alpha - 3 \cos \alpha \sin^2 \alpha + 1 = 0$ 。

解:

$$\begin{aligned}
 \cos 2\alpha - 3 \cos \alpha \sin^2 \alpha &= 2 \cos^2 \alpha - 1 - 3 \cos \alpha (1 - \cos^2 \alpha) \\
 &= 2 \cos^2 \alpha - 1 - 3 \cos \alpha + 3 \cos^3 \alpha \\
 &= 2(x-1)^2 - 1 - 3(x-1) + 3(x-1)^3 \\
 &= 2(x^2 - 2x + 1) - 1 - 3x + 3 + 3(x^3 - 3x^2 + 3x - 1) \\
 &= 2x^2 - 4x + 2 - 1 - 3x + 3 + 3x^3 - 9x^2 + 9x - 3 \\
 &= 3x^3 - 7x^2 + 2x + 1
 \end{aligned}$$

■

$$\begin{aligned}
 \cos 2\alpha - 3 \cos \alpha \sin^2 \alpha + 1 &= 0 \\
 3x^3 - 7x^2 + 2x + 1 + 1 &= 0 \\
 3x^3 - 7x^2 + 2x + 2 &= 0 \\
 (x-1)(3x^2 - 4x - 2) &= 0 \\
 x-1 = 0 \text{ or } 3x^2 - 4x - 2 &= 0 \\
 \cos \alpha = 1 \text{ or } \cos \alpha + 1 &= \frac{2 \pm \sqrt{10}}{3} \\
 \alpha = k\pi + \frac{\pi}{2} \text{ or } \alpha &= 180^\circ k \pm 43.88^\circ
 \end{aligned}$$

When  $n = 0$ ,  $\alpha = 90^\circ$  or  $\alpha = 43.88^\circ$  or  $\alpha = -43.88^\circ$ .

$\therefore \alpha$  is acute,  $\therefore \alpha = 43.88^\circ$ .

■

15. (a) 若  $5 \cos \theta - 12 \sin \theta = R \cos(\theta + \alpha)$ , 式中  $R$  为常数,  $\alpha$  为锐角, 试求  $R$  和  $\alpha$  之值如果  $5 \cos \theta - 12 \sin \theta = k$ , 试证  $-13 \leq k \leq 13$ 。

由此, 试解方程式  $5 \cos \theta - 12 \sin \theta = 4$ , 式中  $0^\circ \leq \theta \leq 360^\circ$ 。

解:

$$5 \cos \theta - 12 \sin \theta = R \cos(\theta + \alpha) = R \cos \theta \cos \alpha - R \sin \theta \sin \alpha$$

$$\begin{cases} R \cos \alpha = 5 & \cdots (1) \\ R \sin \alpha = 12 & \cdots (2) \end{cases}$$

$$(1)^2 + (2)^2 \Rightarrow R^2(\cos^2 \alpha + \sin^2 \alpha) = 5^2 + 12^2$$

$$R = 13$$

■

$$\frac{(2)}{(1)} \Rightarrow \tan \alpha = \frac{12}{5}$$

$$\alpha = 67.38^\circ$$

■

$$\begin{aligned}
5 \cos \theta - 12 \sin \theta &= k \\
13 \cos (\theta + 67.38^\circ) &= k \\
-1 &\leq \cos (\theta + 67.38^\circ) \leq 1 \\
-13 &\leq 13 \cos (\theta + 67.38^\circ) \leq 13 \\
-13 &\leq k \leq 13
\end{aligned}$$

$$\begin{aligned}
5 \cos \theta - 12 \sin \theta &= 4 \\
13 \cos (\theta + 67.38^\circ) &= 4 \\
\cos (\theta + 67.38^\circ) &= \frac{4}{13} \\
\theta + 67.38^\circ &= 360^\circ k \pm 72.08^\circ \\
\theta &= 360^\circ k - 67.38^\circ \pm 72.08^\circ \\
\theta &= 360^\circ k + 4.7^\circ \quad \text{or} \quad \theta = 360^\circ k - 139.46^\circ
\end{aligned}$$

When  $k = 0$ ,  $\theta = 4.7$  or  $\theta = -139.46$ .

When  $k = 1$ ,  $\theta = 364.7$  or  $\theta = 220.54$ .

Since  $0 \leq \theta \leq 360$ , the solutions are  $4.7^\circ, 220.54^\circ$ . ■

(b) 因式分解  $8 \cos^3 x + 6 \cos^2 x - 3 \cos x - 1$ 。

由此试求满足方程式  $8 \cos^3 x + 6 \cos^2 x - 3 \cos x - 1 = 0$  的所有  $x$  之值, 且  $0^\circ \leq \theta \leq 360^\circ$ 。

解:

Let  $u = \cos x$ .

$$\begin{aligned}
8 \cos^3 x + 6 \cos^2 x - 3 \cos x - 1 &= 8u^3 + 6u^2 - 3u - 1 \\
&= (2u - 1)(4u^2 + 5u + 1) \\
&= (2u - 1)(u + 1)(4u + 1) \\
&= (2 \cos x - 1)(\cos x + 1)(4 \cos x + 1)
\end{aligned}$$
■

$$\begin{aligned}
8 \cos^3 x + 6 \cos^2 x - 3 \cos x - 1 &= 0 \\
(2 \cos x - 1)(\cos x + 1)(4 \cos x + 1) &= 0 \\
2 \cos x - 1 = 0 \quad \text{or} \quad \cos x + 1 = 0 \quad \text{or} \quad 4 \cos x + 1 &= 0 \\
\cos x = \frac{1}{2} \quad \text{or} \quad \cos x = -1 \quad \text{or} \quad \cos x = -\frac{1}{4} \\
x = 360^\circ k \pm 60^\circ \quad \text{or} \quad x = 360^\circ k + 180^\circ \quad \text{or} \quad x = 360^\circ k \pm 104.48^\circ
\end{aligned}$$

When  $k = 0$ ,  $x = 60$  or  $x = -60$  or  $x = 180$  or  $x = 104.48$  or  $x = -104.48$ .

When  $k = 1$ ,  $x = 420$  or  $x = 300$  or  $x = 540$  or  $x = 464.48$  or  $x = 255.52$ .

Since  $0 \leq x \leq 360$ , the solutions are  $60^\circ, 180^\circ, 300^\circ, 104.48^\circ, 255.52^\circ$ . ■

16. 试在  $0 \leq x \leq 2\pi$  范围内解方程式:  $\sin x + \sin 3x + \sin 5x = 0$ 。

解:

$$\sin x + \sin 3x + \sin 5x = 0$$

$$\sin x + 2 \sin 4x \cos x = 0$$

$$\sin x + 4 \sin 2x \cos 2x \cos x = 0$$

$$\sin x + 4 \sin 2x(1 - 2 \sin^2 x) \cos x = 0$$

$$\sin x + 4 \sin 2x \cos x - 8 \sin^2 x \sin 2x \cos x = 0$$

$$\sin x + 4(2 \sin x \cos x) \cos x - 8 \sin^2 x(2 \sin x \cos x) \cos x = 0$$

$$\sin x + 8 \sin x \cos^2 x - 16 \sin^3 x \cos^2 x = 0$$

$$\sin x + 8 \sin x(1 - \sin^2 x) - 16 \sin^3 x(1 - \sin^2 x) = 0$$

$$\sin x + 8 \sin x - 8 \sin^3 x - 16 \sin^3 x + 16 \sin^5 x = 0$$

$$16 \sin^5 x - 24 \sin^3 x + 9 \sin x = 0$$

$$\sin x(16 \sin^4 x - 24 \sin^2 x + 9) = 0$$

$$\sin x(4 \sin^2 x - 3)^2 = 0$$

$$\sin x = 0 \text{ or } 4 \sin^2 x - 3 = 0$$

$$x = k\pi \text{ or } \sin x = \pm \frac{\sqrt{3}}{2}$$

$$x = k\pi \text{ or } x = k\pi \pm (-1)^k \left(\frac{\pi}{3}\right)$$

$$x = k\pi \text{ or } x = k\pi \pm \frac{\pi}{3}$$

When  $k = 0$ ,  $x = 0$  or  $x = \frac{\pi}{3}$  or  $x = -\frac{\pi}{3}$ .

When  $k = 1$ ,  $x = \pi$  or  $x = \frac{4\pi}{3}$  or  $x = \frac{2\pi}{3}$ .

When  $k = 2$ ,  $x = 2\pi$  or  $x = \frac{5\pi}{3}$  or  $x = \frac{7\pi}{3}$ .

Since  $0 \leq x \leq 2\pi$ , the solutions are  $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$ . ■

17. 如果  $3 \sin x + 2 \cos x \equiv R \sin(x + \alpha)$ , 式中  $R$  是常数,  $\alpha$  是锐角, 求  $R$  及  $\alpha$  的值。据此或其他方法, 求:

解:

$$3 \sin x + 2 \cos x = R \sin(x + \alpha) = R \sin x \cos \alpha + R \cos x \sin \alpha$$

$$3 \sin x + 2 \cos x = R \sin x \cos \alpha + R \cos x \sin \alpha$$

$$\begin{cases} R \cos \alpha = 3 \cdots (1) \\ R \sin \alpha = 2 \cdots (2) \end{cases}$$

$$(1)^2 + (2)^2 \Rightarrow R^2(\cos^2 \alpha + \sin^2 \alpha) = 3^2 + 2^2$$

$$R = \sqrt{13}$$
■

$$\frac{(2)}{(1)} \Rightarrow \tan \alpha = \frac{2}{3}$$

$$\alpha = 33.69^\circ$$

■

(a)  $\frac{1}{(3 \sin x + 2 \cos x)^2}$  的最小值;

解:

$$\frac{1}{(3 \sin x + 2 \cos x)^2} = \frac{1}{13 \sin^2(x + 33.69^\circ)}$$

Since  $\sin^2(x + 33.69^\circ) \leq 1$ , the minimum value is  $\frac{1}{13}$ .

■

(b) 方程式  $3 \sin x + 2 \cos x = 3$  在  $0^\circ \leq x \leq 180^\circ$  的解。

解:

$$3 \sin x + 2 \cos x = 3$$

$$\sqrt{13} \sin(x + 33.69^\circ) = 3$$

$$\sin(x + 33.69^\circ) = \frac{3}{\sqrt{13}}$$

$$x + 33.69^\circ = 180^\circ k + (-1)^k \cdot 56.31^\circ$$

$$x = 180^\circ k + (-1)^k \cdot 56.31^\circ - 33.69^\circ$$

When  $k = 0$ ,  $x = 22.62$ .

When  $k = 1$ ,  $x = 90$ .

Since  $0 \leq x \leq 180$ , the solutions are  $22.62^\circ, 90^\circ$ .

■

18. 解方程式  $3 \cos x + 4 \sin x = 2$ , 式中  $0^\circ \leq x \leq 360^\circ$ 。

解:

$$3 \cos x + 4 \sin x = R \sin(x + \alpha) = R \sin x \cos \alpha + R \cos x \sin \alpha$$

$$\begin{cases} R \sin \alpha = 3 \cdots (1) \\ R \cos \alpha = 4 \cdots (2) \end{cases}$$

$$(1)^2 + (2)^2 \Rightarrow R^2(\sin^2 \alpha + \cos^2 \alpha) = 3^2 + 4^2$$

$$R = 5$$

$$\frac{(1)}{(2)} \Rightarrow \tan \alpha = \frac{3}{4}$$

$$\alpha = 36.87^\circ$$

$$3 \cos x + 4 \sin x = 2$$

$$5 \sin(x + 36.87^\circ) = 2$$

$$\sin(x + 36.87^\circ) = \frac{2}{5}$$

$$x + 36.87^\circ = 180^\circ k + (-1)^k \cdot 23.58^\circ$$

$$x = 180^\circ k + (-1)^k \cdot 23.58^\circ - 36.87^\circ$$

When  $k = 0$ ,  $x = -13.29$

When  $k = 1$ ,  $x = 119.55$

When  $k = 2$ ,  $x = 346.71$

Since  $0 \leq x \leq 360$ , the solutions are  $119.55^\circ, 346.71^\circ$ . ■

19. 求方程式  $\cos x + \cos 7x = \cos 4x$  的一般解, 答案以弧度表示。

解:

$$\cos x + \cos 7x = \cos 4x$$

$$2 \cos 4x \cos 3x = \cos 4x$$

$$\cos 4x(2 \cos 3x - 1) = 0$$

$$\cos 4x = 0 \text{ or } 2 \cos 3x - 1 = 0$$

$$4x = 2k\pi \pm \frac{\pi}{2} \text{ or } 3x = 2k\pi \pm \frac{\pi}{3}$$

$$x = \frac{k\pi}{2} \pm \frac{\pi}{8} \text{ or } x = \frac{2k\pi}{3} \pm \frac{\pi}{9} \text{ where } k \in \mathbb{Z} \quad \blacksquare$$

20. 求方程式  $2 \cos^2 \theta + \sqrt{3} \sin \theta + 1 = 0$  的一般解。

解:

$$2 \cos^2 \theta + \sqrt{3} \sin \theta + 1 = 0$$

$$2(1 - \sin^2 \theta) + \sqrt{3} \sin \theta + 1 = 0$$

$$2 - 2 \sin^2 \theta + \sqrt{3} \sin \theta + 1 = 0$$

$$2 \sin^2 \theta - \sqrt{3} \sin \theta - 3 = 0$$

$$(\sin \theta - \sqrt{3})(2 \sin \theta + \sqrt{3}) = 0$$

$$\sin \theta = \sqrt{3} \text{ or } \sin \theta = -\frac{\sqrt{3}}{2}$$

$$\theta = k\pi + (-1)^k \frac{\pi}{3} \text{ or } \theta = k\pi + (-1)^{k+1} \frac{\pi}{3}$$

$\therefore \theta = k\pi + (-1)^k \frac{\pi}{3}$  is included in  $\theta = k\pi + (-1)^{k+1} \frac{\pi}{3}$ ,

$\therefore$  the general solution is  $\theta = k\pi + (-1)^{k+1} \frac{\pi}{3}$ . ■

21. 试证  $\cos \theta + 2 \cos 2\theta + \cos 3\theta = 4 \cos 2\theta \cos^2 \frac{1}{2}\theta$ 。

据此, 求满足方程式  $\cos \theta + 2 \cos 2\theta + \cos 3\theta = 0$  的所有  $\theta$  的值, 且  $0 \leq \theta \leq 2\pi$ 。

解:

$$\begin{aligned}
 L.H.S. &= \cos \theta + 2 \cos 2\theta + \cos 3\theta = \cos \theta + 2(2 \cos^2 \theta - 1) + \cos 2\theta \cos \theta - \sin 2\theta \sin \theta \\
 &= \cos \theta + 4 \cos^2 \theta - 2 + (2 \cos^2 \theta - 1) \cos \theta - 2 \sin^2 \theta \cos \theta \\
 &= 4 \cos^2 \theta - 2 + 2 \cos^3 \theta - 2(1 - \cos^2 \theta) \cos \theta \\
 &= 4 \cos^2 \theta - 2 + 2 \cos^3 \theta - 2 \cos \theta + 2 \cos^3 \theta \\
 &= 4 \cos^2 \theta - 2 + 4 \cos^3 \theta - 2 \cos \theta \\
 R.H.S. &= 4 \cos 2\theta \cos^2 \frac{1}{2}\theta = 4(2 \cos^2 \theta - 1) \left( \frac{1 + \cos \theta}{2} \right) \\
 &= 2(2 \cos^2 \theta - 1)(1 + \cos \theta) \\
 &= 2(2 \cos^2 \theta - 1 + 2 \cos^3 \theta - \cos \theta) \\
 &= 4 \cos^2 \theta - 2 + 4 \cos^3 \theta - 2 \cos \theta
 \end{aligned}$$

Since  $L.H.S. = R.H.S.$ , the equation is true.

$$\begin{aligned}
 \cos \theta + 2 \cos 2\theta + \cos 3\theta &= 0 \\
 4 \cos 2\theta \cos^2 \frac{1}{2}\theta &= 0 \\
 \cos 2\theta \cos^2 \frac{1}{2}\theta &= 0 \\
 \cos 2\theta = 0 \text{ or } \cos \frac{1}{2}\theta &= 0 \\
 2\theta = 2k\pi \pm \frac{\pi}{2} \text{ or } \frac{1}{2}\theta &= 2k\pi \pm \frac{\pi}{2} \\
 \theta = k\pi \pm \frac{\pi}{4} \text{ or } \theta &= 4k\pi \pm \pi
 \end{aligned}$$

When  $k = 0$ ,  $\theta = \frac{\pi}{4}$  or  $\theta = -\frac{\pi}{4}$  or  $\theta = \pi$  or  $\theta = -\pi$ .

When  $k = 1$ ,  $\theta = \frac{5\pi}{4}$  or  $\theta = \frac{3\pi}{4}$  or  $\theta = 3\pi$  or  $\theta = 5\pi$ .

When  $k = 2$ ,  $\theta = \frac{9\pi}{4}$  or  $\theta = \frac{7\pi}{4}$  or  $\theta = 7\pi$  or  $\theta = 9\pi$ .

Since  $0 \leq \theta \leq 2\pi$ , the solutions are  $\frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}$ . ■

22. 求  $\frac{1 - \tan x}{1 + \tan x} = \cos 2x$  的一般解。

解:

$$\begin{aligned}
 \frac{1 - \tan x}{1 + \tan x} &= \cos 2x \\
 1 - \frac{\sin x}{\cos x} &= \cos 2x \\
 \frac{\cos x - \sin x}{\cos x} &= \cos 2x \\
 \frac{\cos x - \sin x}{\cos x + \sin x} &= \cos 2x
 \end{aligned}$$

$$\frac{\cos x - \sin x}{\cos x + \sin x} = \frac{1 - 2 \sin^2 x}{1 + 2 \sin x \cos x}$$

$$\frac{(\cos x - \sin x)^2}{\cos^2 x - \sin^2 x} = \cos 2x$$

$$\frac{\cos^2 x - 2 \cos x \sin x + \sin^2 x}{\cos 2x} = \cos 2x$$

$$1 - \sin 2x = \cos^2 2x$$

$$1 - \sin 2x = 1 - \sin^2 2x$$

$$\sin^2 2x - \sin 2x = 0$$

$$\sin 2x(\sin 2x - 1) = 0$$

$$\sin 2x = 0 \text{ or } \sin 2x = 1$$

$$2x = k\pi \text{ or } 2x = \frac{\pi}{2} + 2k\pi$$

$$x = k\pi \text{ or } x = \frac{\pi}{4} + k\pi \quad \text{where } k \in \mathbb{Z} \quad \blacksquare$$

23. 将  $y = \cos x - \sqrt{3} \sin x$  表达成  $R \cos(x + \alpha)$  的形式, 其中  $R > 0$  及  $0^\circ < \alpha < 90^\circ$ 。据此或用其他方法, 求

解:

$$\cos x - \sqrt{3} \sin x = R \cos(x + \alpha) = R \cos x \cos \alpha - R \sin x \sin \alpha$$

$$\begin{cases} R \cos \alpha = 1 \cdots (1) \\ R \sin \alpha = \sqrt{3} \cdots (2) \end{cases}$$

$$(1)^2 + (2)^2 \Rightarrow R^2(\cos^2 \alpha + \sin^2 \alpha) = 1^2 + 3$$

$$R = 2$$

$$\frac{(2)}{(1)} \Rightarrow \tan \alpha = \sqrt{3}$$

$$\alpha = 60^\circ$$

$$y = 2 \cos(x + 60^\circ) \quad \blacksquare$$

(a)  $y$  的极大值与极小值; 解:

$y$  is maximum when  $\cos(x + 60^\circ) = 1$ , hence the maximum value is 2.

$y$  is minimum when  $\cos(x + 60^\circ) = -1$ , hence the minimum value is  $-2$ . \blacksquare

(b) 当  $y = 1$  时,  $x$  的一般解。

解:

$$2 \cos(x + 60^\circ) = 1$$

$$\cos(x + 60^\circ) = \frac{1}{2}$$

$$x + 60^\circ = 360^\circ k \pm 60^\circ$$

$$x = 360^\circ k - 60^\circ \pm 60^\circ$$

$$x = 360^\circ k \text{ or } x = 360^\circ k - 120^\circ \quad \text{where } k \in \mathbb{Z} \quad \blacksquare$$

24. 证明  $\sin 2x = \frac{2 \tan x}{1 + \tan^2 x}$ 。

据此, 或用其它方法, 解方程式  $\frac{1 + \tan x}{1 - \tan x} = 1 + \sin 2x$ , 式中  $2 < x < 12$ , 且  $x$  为弧度。

解:

$$\begin{aligned}\frac{2 \tan x}{1 + \tan^2 x} &= \frac{2 \left( \frac{\sin x}{\cos x} \right)}{\sec^2 x} \\ &= \frac{2 \sin x}{\cos x} \times \cos^2 x \\ &= 2 \sin x \cos x \\ &= \sin 2x\end{aligned}$$

■

$$\begin{aligned}\frac{1 + \tan x}{1 - \tan x} &= 1 + \sin 2x \\ \frac{1 + \tan x}{1 - \tan x} &= 1 + \frac{2 \tan x}{1 + \tan^2 x} \\ \frac{1 + \tan x}{1 - \tan x} &= \frac{1 + \tan^2 x + 2 \tan x}{1 + \tan^2 x} \\ \frac{(1 + \tan x)^2}{1 - \tan^2 x} &= \frac{(1 + \tan x)^2}{1 + \tan^2 x} \\ (1 + \tan x)^2 \left( \frac{1}{1 - \tan^2 x} - \frac{1}{1 + \tan^2 x} \right) &= 0 \\ (1 + \tan x)^2 \left( \frac{1 + \tan^2 x - 1 + \tan^2 x}{1 - \tan^4 x} \right) &= 0 \\ (1 + \tan x)^2 \left( \frac{2 \tan^2 x}{1 - \tan^4 x} \right) &= 0 \\ 1 + \tan x &= 0 \text{ or } \tan x = 0 \\ \tan x &= -1 \text{ or } \tan x = 0 \\ x &= k\pi - \frac{\pi}{4} \text{ or } x = k\pi\end{aligned}$$

When  $k = 0$ ,  $x = -\frac{\pi}{4}$  or  $x = 0$ .

When  $k = 1$ ,  $x = \frac{3\pi}{4}$  or  $x = \pi$ .

When  $k = 2$ ,  $x = \frac{7\pi}{4}$  or  $x = 2\pi$ .

When  $k = 3$ ,  $x = \frac{11\pi}{4}$  or  $x = 3\pi$ .

When  $k = 4$ ,  $x = \frac{15\pi}{4}$  or  $x = 4\pi$ .

Since  $2 < x < 12$ , the solutions are  $\frac{3\pi}{4}, \pi, \frac{7\pi}{4}, 2\pi, \frac{11\pi}{4}, 3\pi, \frac{15\pi}{4}$ .

■



25. 试证明  $\operatorname{cosec} \theta - \cot \theta = \tan \frac{1}{2} \theta$ 。

解:

$$\begin{aligned}\operatorname{cosec} \theta - \cot \theta &= \frac{1}{\sin \theta} - \frac{\cos \theta}{\sin \theta} \\ &= \frac{1 - \cos \theta}{\sin \theta} \\ &= \tan \frac{1}{2} \theta\end{aligned}$$

据此或用其他方法,

(a) 解方程式  $\operatorname{cosec} 3\theta - \cot 3\theta = \sqrt{3}$ , 式中  $0 \leq \theta \leq \pi$ 。

解:

$$\begin{aligned}\operatorname{cosec} 3\theta - \cot 3\theta &= \sqrt{3} \\ \tan \frac{3\theta}{2} &= \sqrt{3} \\ \frac{3\theta}{2} &= \frac{\pi}{3} + k\pi \\ \theta &= \frac{2\pi}{9} + \frac{2k\pi}{3}\end{aligned}$$

When  $k = 0$ ,  $\theta = \frac{2\pi}{9}$ .

When  $k = 1$ ,  $\theta = \frac{8\pi}{9}$ .

Since  $0 \leq \theta \leq \pi$ , the solutions are  $\frac{2\pi}{9}, \frac{8\pi}{9}$ .

(b) 求  $\tan \frac{3}{8}\pi$  的值, 答案以根式表示。

解:

$$\begin{aligned}\tan \frac{3}{8}\pi &= \tan \left( \frac{1}{2} \cdot \frac{3}{4}\pi \right) \\ &= \operatorname{cosec} \frac{3}{4}\pi - \cot \frac{3}{4}\pi \\ &= \sqrt{2} - 1\end{aligned}$$

26. 求  $\sin^2 x - \cos^2 x = 1 + \frac{1}{2} \sin 2x$  的一般解。

解:

$$\begin{aligned}\sin^2 x - \cos^2 x &= 1 + \frac{1}{2} \sin 2x \\ -\cos 2x &= 1 + \frac{1}{2} \sin 2x \\ -2 \cos 2x &= 2 + \sin 2x\end{aligned}$$

Let  $u = 2x$ .

$$-2 \cos u = 2 + \sin u$$

$$\sin u + 2 \cos u = -2$$

Let  $\tan \frac{u}{2} = t$ , then  $\cos u = \frac{1-t^2}{1+t^2}$  and  $\sin u = \frac{2t}{1+t^2}$ .

$$\frac{2t}{1+t^2} + 2 \left( \frac{1-t^2}{1+t^2} \right) = -2$$

$$2t = -4$$

$$t = -2$$

$$\tan \frac{u}{2} = -2$$

$$\tan x = -2$$

$$x = k\pi - \arctan 2 \quad \text{where } k \in \mathbb{Z}$$

27. (a) 将  $12 \cos \theta - 5 \sin \theta$  表达成  $R \cos(\theta + \alpha)$  的形成, 式中  $R > 0$  及  $0^\circ < \alpha < 90^\circ$ 。

解:

$$12 \cos \theta - 5 \sin \theta = R \cos(\theta + \alpha) = R \cos \theta \cos \alpha - R \sin \theta \sin \alpha$$

$$\begin{cases} R \cos \alpha = 12 \cdots (1) \\ R \sin \alpha = 5 \cdots (2) \end{cases}$$

$$(1)^2 + (2)^2 \Rightarrow R^2(\cos^2 \alpha + \sin^2 \alpha) = 12^2 + 5^2$$

$$R = 13$$

$$\frac{(2)}{(1)} \Rightarrow \tan \alpha = \frac{5}{12}$$

$$\alpha = 22.62^\circ$$

$$12 \cos \theta - 5 \sin \theta = 13 \cos(\theta + 22.62^\circ) \quad \blacksquare$$

(b) 函数  $g(x)$  定义成  $g(x) = 27 \cos^2 x - 10 \sin x \cos x + 3 \sin^2 x$ 。试将  $g(x)$  表达成  $a \cos 2x + b \sin 2x + c$  的形式, 式中  $a, b$  及  $c$  都是待定的常数。

解:

$$\begin{aligned} g(x) &= 27 \cos^2 x - 10 \sin x \cos x + 3 \sin^2 x \\ &= 27 \left( \frac{1 + \cos 2x}{2} \right) - 5(2 \sin x \cos x) + 3 \left( \frac{1 - \cos 2x}{2} \right) \\ &= \frac{27 + 27 \cos 2x + 3 - 3 \cos 2x}{2} - 5 \sin 2x \\ &= \frac{30 + 24 \cos 2x}{2} - 5 \sin 2x \\ &= 12 \cos 2x - 5 \sin 2x + 15 \quad \blacksquare \end{aligned}$$

(c) 根据 (a) 和 (b) 的结果, 或用其他方法, 求

i  $g(x)$  的极大值与极小值;

解:

$$\begin{aligned}g(x) &= 12 \cos 2x - 5 \sin 2x + 15 \\&= 13 \cos(2x + 22.62^\circ) + 15\end{aligned}$$

$g(x)$  is maximum when  $\cos(2x - 22.62^\circ) = 1$ , hence the maximum value is 28.

$g(x)$  is minimum when  $\cos(2x - 22.62^\circ) = -1$ , hence the minimum value is 2. ■

ii 方程式  $g(x) = 2$  的一般解。

解:

$$\begin{aligned}13 \cos(2x + 22.62^\circ) + 15 &= 2 \\ \cos(2x + 22.62^\circ) &= -1 \\ 2x + 22.62^\circ &= 360^\circ k + 180^\circ \\ 2x &= 360^\circ k + 157.38^\circ \\ x &= 180^\circ k + 78.69^\circ \quad \text{where } k \in \mathbb{Z}\end{aligned}$$
 ■

28. 求方程式  $5 \sin^2 x + \sin 2x - 3 \cos^2 x = 2$  的一般解。

解:

$$\begin{aligned}5 \sin^2 x + \sin 2x - 3 \cos^2 x &= 2 \\ 5 \sin^2 x + 2 \sin x \cos x - 3 \cos^2 x &= 2 \sin^2 x + 2 \cos^2 x \\ 3 \sin^2 x + 2 \sin x \cos x - 5 \cos^2 x &= 0 \\ 3 \tan^2 x + 2 \tan x - 5 &= 0 \\ (3 \tan x + 5)(\tan x - 1) &= 0 \\ \tan x &= \frac{-5}{3} \text{ or } \tan x = 1 \\ x &= k\pi - \arctan \frac{5}{3} \text{ or } x = k\pi + \frac{\pi}{4} \quad \text{where } k \in \mathbb{Z}\end{aligned}$$
 ■

29. 求三角方程式  $\sin 4\theta = \cos 5\theta$  的一般解。

解:

$$\begin{aligned}\sin 4\theta &= \cos 5\theta \\ \cos 5\theta &= \cos\left(\frac{\pi}{2} - 4\theta\right) \\ 5\theta &= 2k\pi \pm \left(\frac{\pi}{2} - 4\theta\right) \\ 5\theta &= 2k\pi + \frac{\pi}{2} - 4\theta \text{ or } 5\theta = 2k\pi - \frac{\pi}{2} + 4\theta \\ 9\theta &= 2k\pi + \frac{\pi}{2} \text{ or } \theta = 2k\pi - \frac{\pi}{2} \\ \theta &= \frac{2k\pi}{9} + \frac{\pi}{18} \text{ or } \theta = 2k\pi - \frac{\pi}{2} \\ \theta &= \frac{\pi}{18}(4k+1) \text{ or } \theta = \frac{\pi}{2}(4k-1) \quad \text{where } k \in \mathbb{Z}\end{aligned}$$
 ■

30. 解方程式  $\sin^2 \theta + \sin^2 2\theta = \sin^2 3\theta$ 。

解:

$$\begin{aligned}
 \sin^2 \theta + \sin^2 2\theta &= \sin^2 3\theta \\
 \sin^2 \theta + 4 \sin^2 \theta \cos^2 \theta &= (\sin 2\theta \cos \theta + \cos 2\theta \sin \theta)^2 \\
 \sin^2 \theta + 4 \sin^2 \theta \cos^2 \theta &= [(2 \sin \theta \cos \theta) \cos \theta + (1 - 2 \sin^2 \theta) \sin \theta]^2 \\
 \sin^2 \theta + 4 \sin^2 \theta \cos^2 \theta &= [2 \sin \theta (1 - \sin^2 \theta) + \sin \theta - 2 \sin^3 \theta]^2 \\
 \sin^2 \theta + 4 \sin^2 \theta \cos^2 \theta &= (3 \sin \theta - 4 \sin^3 \theta)^2 \\
 \sin^2 \theta + 4 \sin^2 \theta \cos^2 \theta &= 9 \sin^2 \theta - 24 \sin^4 \theta + 16 \sin^6 \theta \\
 \sin^2 \theta + 4 \sin^2 \theta \cos^2 \theta - 9 \sin^2 \theta + 24 \sin^4 \theta - 16 \sin^6 \theta &= 0 \\
 \sin^2 \theta (-8 + 4 \cos^2 \theta + 24 \sin^2 \theta - 16 \sin^4 \theta) &= 0 \\
 \sin^2 \theta (2 - \cos^2 \theta - 6 \sin^2 \theta + 4 \sin^4 \theta) &= 0 \\
 \sin^2 \theta (2 - \cos^2 \theta - \sin^2 \theta - 5 \sin^2 \theta + 4 \sin^4 \theta) &= 0 \\
 \sin^2 \theta (1 - 5 \sin^2 \theta + 4 \sin^4 \theta) &= 0 \\
 \sin^2 \theta (\sin^2 \theta - 1)(4 \sin^2 \theta - 1) &= 0 \\
 \sin^2 \theta (-\cos^2 \theta)(4 \sin^2 \theta - 1) &= 0 \\
 \sin \theta = 0 \text{ or } \cos \theta = 0 \text{ or } \sin \theta = \pm \frac{1}{2} \\
 \theta = k\pi \text{ or } \theta = \frac{\pi}{2} + k\pi \text{ or } \theta = k\pi \pm \frac{\pi}{6} \\
 \theta = \frac{k\pi}{2} \text{ or } \theta = \frac{k\pi}{3} + \frac{\pi}{6} \\
 \theta = \frac{k\pi}{2} \text{ or } \theta = \frac{1}{6}(2k+1)\pi \quad \text{where } k \in \mathbb{Z} \quad \blacksquare
 \end{aligned}$$

31. 已知  $4 \cos \theta + 3 \sin \theta = R \cos(\theta - \alpha)$ , 式中  $R > 0$  且  $\alpha$  为锐角, 试求  $R$  与  $\alpha$  的值。据之解方程式  $4 \cos \theta + 3 \sin \theta = 3, 0^\circ < \theta < 360^\circ$ 。

解:

$$4 \cos \theta + 3 \sin \theta = R \cos(\theta - \alpha) = R \cos \theta \cos \alpha + R \sin \theta \sin \alpha$$

$$\begin{cases} R \cos \alpha = 4 \cdots (1) \\ R \sin \alpha = 3 \cdots (2) \end{cases}$$

$$(1)^2 + (2)^2 \Rightarrow R^2(\cos^2 \alpha + \sin^2 \alpha) = 16 + 9$$

$$R = 5 \quad \blacksquare$$

$$\frac{(2)}{(1)} \Rightarrow \tan \alpha = \frac{3}{4}$$

$$\alpha = 36.87^\circ \quad \blacksquare$$

$$\begin{aligned}
4 \cos \theta + 3 \sin \theta &= 3 \\
5 \cos(\theta - 36.87^\circ) &= 3 \\
\cos(\theta - 36.87^\circ) &= \frac{3}{5} \\
\theta - 36.87^\circ &= 360^\circ k \pm 53.13^\circ \\
\theta &= 360^\circ k + 36.87^\circ + 53.13^\circ \text{ or } \theta = 360^\circ k + 36.87^\circ - 53.13^\circ \\
\theta &= 360^\circ k + 90^\circ \text{ or } \theta = 360^\circ k - 16.26^\circ
\end{aligned}$$

When  $k = 0$ ,  $\theta = 90^\circ$  or  $\theta = -16.26^\circ$ .

When  $k = 1$ ,  $\theta = 450^\circ$  or  $\theta = 343.74^\circ$ .

Since  $0^\circ < \theta < 360^\circ$ , the solutions are  $90^\circ, 343.74^\circ$ . ■

32. 已知函数  $f(x) = \cos 2x + 4 \sin^2 x - \cos x - 2$ 。

(a) 解方程式  $f(x) = 0$ 。

解:

$$\begin{aligned}
\cos 2x + 4 \sin^2 x - \cos x - 2 &= 0 \\
2 \cos^2 x - 1 + 4(1 - \cos^2 x) - \cos x - 2 &= 0 \\
2 \cos^2 x - 1 + 4 - 4 \cos^2 x - \cos x - 2 &= 0 \\
-2 \cos^2 x - \cos x + 1 &= 0 \\
2 \cos^2 x + \cos x - 1 &= 0 \\
(2 \cos x - 1)(\cos x + 1) &= 0 \\
\cos x &= \frac{1}{2} \text{ or } \cos x = -1 \\
x &= 2k\pi \pm \frac{\pi}{3} \text{ or } x = 2k\pi + \pi \\
x &= \frac{\pi}{3} + \frac{2k\pi}{3} \\
x &= \frac{\pi}{3}(2k+1) \quad \text{where } k \in \mathbb{Z}
\end{aligned}$$
■

(b) 在  $0 \leq x \leq 2\pi$  的条件下, 解不等式  $f(x) > 0$ 。

解:

$$\begin{aligned}
\cos 2x + 4 \sin^2 x - \cos x - 2 &> 0 \\
2 \cos^2 x - 1 + 4(1 - \cos^2 x) - \cos x - 2 &> 0 \\
2 \cos^2 x - 1 + 4 - 4 \cos^2 x - \cos x - 2 &> 0 \\
-2 \cos^2 x - \cos x + 1 &> 0 \\
2 \cos^2 x + \cos x - 1 &< 0 \\
(2 \cos x - 1)(\cos x + 1) &< 0 \\
-1 &< \cos x < \frac{1}{2}
\end{aligned}$$
■

33. 如果  $\sin^4 \theta + \sin^2 \theta \cos^2 \theta + \cos^4 \theta \leq \frac{3}{4}$ , 式中  $0 \leq \theta \leq 2\pi$ , 求  $\theta$  的值。

解:

$$\sin^4 \theta + \sin^2 \theta \cos^2 \theta + \cos^4 \theta \leq \frac{3}{4}$$

$$\sin^2 \theta (\sin^2 \theta + \cos^2 \theta) + \cos^4 \theta \leq \frac{3}{4}$$

$$\sin^2 \theta + \cos^4 \theta \leq \frac{3}{4}$$

$$1 - \cos^2 \theta + \cos^4 \theta \leq \frac{3}{4}$$

$$\cos^4 \theta - \cos^2 \theta + \frac{1}{4} \leq 0$$

$$\left( \cos^2 \theta - \frac{1}{2} \right)^2 \leq 0$$

$$\cos^2 \theta = \frac{1}{2}$$

$$\theta = k\pi \pm \frac{\pi}{4}$$

When  $k = 0$ ,  $\theta = \frac{\pi}{4}$  or  $\theta = -\frac{\pi}{4}$ .

When  $k = 1$ ,  $\theta = \frac{5\pi}{4}$  or  $\theta = \frac{3\pi}{4}$ .

When  $k = 2$ ,  $\theta = \frac{9\pi}{4}$  or  $\theta = \frac{7\pi}{4}$ .

Since  $0 \leq \theta \leq 2\pi$ , the solutions are  $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ . ■

34. 解方程式  $\cos^3 x \sin x - \sin^3 x \cos x = \frac{\sqrt{3}}{8}$ , 其中  $x$  以弧度为单位且  $0 < x < \frac{\pi}{2}$ 。

解:

$$\cos^3 x \sin x - \sin^3 x \cos x = \frac{\sqrt{3}}{8}$$

$$\sin x \cos x (\cos^2 x - \sin^2 x) = \frac{\sqrt{3}}{8}$$

$$2 \sin x \cos x \cos 2x = \frac{\sqrt{3}}{4}$$

$$\sin 2x \cos 2x = \frac{\sqrt{3}}{4}$$

$$2 \sin 2x \cos 2x = \frac{\sqrt{3}}{2}$$

$$\sin 4x = \frac{\sqrt{3}}{2}$$

$$4x = k\pi + (-1)^k \frac{\pi}{3}$$

$$x = \frac{k\pi}{4} + (-1)^k \frac{\pi}{12}$$

When  $k = 0$ ,  $x = \frac{\pi}{12}$ .

When  $k = 1$ ,  $x = \frac{\pi}{6}$ .

Since  $0 < x < \frac{\pi}{2}$ , the solutions are  $\frac{\pi}{12}, \frac{\pi}{6}$ . ■

35. 证明  $\frac{1 - \cos \alpha}{1 + \cos \alpha} = \tan^2 \frac{\alpha}{2}$ 。

据此, 或用其他方法, 证明若  $\alpha \in (\pi, 2\pi)$ , 则  $\sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} + \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}} = -\frac{2}{\sin \alpha}$ 。

解:

$$\begin{aligned} \frac{1 - \cos \alpha}{1 + \cos \alpha} &= \frac{1 - \cos \alpha}{2} \div \frac{1 + \cos \alpha}{2} \\ &= \left( \pm \sqrt{\frac{1 - \cos \alpha}{2}} \right)^2 \div \left( \pm \sqrt{\frac{1 + \cos \alpha}{2}} \right)^2 \\ &= \frac{\sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2}} \\ &= \tan^2 \frac{\alpha}{2} \end{aligned}$$

$$\begin{aligned} \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} + \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}} &= \tan \frac{\alpha}{2} + \cot \frac{\alpha}{2} \\ &= \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} + \frac{\cos \frac{\alpha}{2}}{\sin \frac{\alpha}{2}} \\ &= \frac{\sin^2 \frac{\alpha}{2} + \cos^2 \frac{\alpha}{2}}{\sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} \\ &= \frac{1}{\sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} \\ &= \frac{2}{\sin \alpha} \end{aligned}$$

$\because \alpha \in (\pi, 2\pi), \therefore \sin \alpha < 0$ .

$$\therefore \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} + \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}} = -\frac{2}{\sin \alpha}.$$

■

36. 证明  $\cos \theta + 2 \cos 2\theta + \cos 3\theta = 4 \cos 2\theta \cos^2 \frac{\theta}{2}$ 。

据此, 或用其它方法, 解方程式  $\cos \theta + 2 \cos 2\theta + \cos 3\theta = 0$ , 其中  $0 \leq \theta \leq \pi$ 。

解:

Same as question 21.

The solutions are  $\frac{\pi}{4}, \frac{3\pi}{4}, \pi$ .

■

37. 求方程式  $\cos^2 x + 3 \cos^2 2x = \cos^2 3x$  的一般解。

解:

$$\begin{aligned}\cos^2 x + 3 \cos^2 2x &= \cos^2 3x \\ \cos^2 x + 3(2 \cos^2 x - 1)^2 &= (4 \cos^3 x - 3 \cos x)^2 \\ \cos^2 x + 3(4 \cos^4 x - 4 \cos^2 x + 1) &= 16 \cos^6 x - 24 \cos^4 x + 9 \cos^2 x \\ \cos^2 x + 12 \cos^4 x - 12 \cos^2 x + 3 &= 16 \cos^6 x - 24 \cos^4 x + 9 \cos^2 x \\ 16 \cos^6 x - 36 \cos^4 x + 20 \cos^2 x - 3 &= 0\end{aligned}$$

Let  $u = \cos^2 x$ .

$$\begin{aligned}16u^3 - 36u^2 + 20u - 3 &= 0 \\ (2u - 3)(2u - 1)(4u - 1) &= 0 \\ u = \frac{3}{2} \text{ or } u = \frac{1}{2} \text{ or } u = \frac{1}{4} \\ \cos^2 x = \frac{3}{2} \text{ or } \cos^2 x = \frac{1}{2} \text{ or } \cos^2 x = \frac{1}{4} \\ x = 2k\pi \pm \frac{\pi}{3} \text{ or } x = 2k\pi \pm \frac{\pi}{4} \text{ where } k \in \mathbb{Z}\end{aligned}$$

■

38. 证明  $\sin^4 x + \cos^4 x = 1 - \frac{1}{2} \sin^2 2x$ 。

据此, 求三角方程式  $2 \sin^4 x + 2 \cos^4 x = \sin 2x$  的一般解。

解:

$$\begin{aligned}\sin^4 x + \cos^4 x &= (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x \\ &= 1 - \frac{1}{2} (2 \sin x \cos x)^2 \\ &= 1 - \frac{1}{2} \sin^2 2x\end{aligned}$$

■

$$\begin{aligned}2 \sin^4 x + 2 \cos^4 x &= \sin 2x \\ 2 - \sin^2 2x &= \sin 2x \\ \sin^2 2x + \sin 2x - 2 &= 0 \\ (\sin 2x + 2)(\sin 2x - 1) &= 0\end{aligned}$$

$$\begin{aligned}\sin 2x &= -2 \text{ or } \sin 2x = 1 \\ 2x &= 2k\pi + \frac{\pi}{2} \\ x &= k\pi + \frac{\pi}{4} \text{ where } k \in \mathbb{Z}\end{aligned}$$

■