Exercise 13e - Applications of Differential Equation

1. A the rate of change of a function y with respect to x is $\frac{1}{3}y$. When $x = -1$, $y = 4$, find the relationship between x and y.

   Sol.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{3}y \\
\frac{dy}{y} &= \frac{1}{3}dx \\
\ln|y| &= \frac{1}{3}x + C \\
y &= Ce^{\frac{1}{3}x} \\
y(-1) &= 4 \\
4 &= Ce^{-\frac{1}{3}} \\
C &= 4e^{\frac{1}{3}} \\
&\approx 5.58 \\
y &= 5.58e^{\frac{1}{3}x}
\end{aligned}$$

2. A the rate of change of a function y with respect to x is $2 - y$. When $x = 0$, $y = 8$, find the relationship between x and y.

   Sol.

$$\begin{aligned}
\frac{dy}{dx} &= 2 - y \\
\frac{dy}{y-2} &= -dx \\
\ln|y-2| &= -x + C \\
y - 2 &= Ce^{-x} \\
y &= 2 + Ce^{-x} \\
y(0) &= 8 \\
8 &= 2 + C \\
C &= 6 \\
y &= 2 + 6e^{-x}
\end{aligned}$$

3. The gradient of tangent of a curve at any point is xy, and the curve passes through $(0, 1)$, find the equation of the curve.

   Sol.

$$\begin{aligned}
\frac{dy}{dx} &= xy \\
\frac{dy}{y} &= xdx \\
\ln|y| &= \frac{1}{2}x^2 + C \\
y &= Ce^{\frac{1}{2}x^2} \\
y(0) &= 1 \\
C &= 1 \\
y &= e^{\frac{1}{2}x^2}
\end{aligned}$$

4. The decay of a substance is proportional to the amount of the substance present. When $t = 0$, the amount of the substance is 9g, it's also given that the substance had reduced by 1g after the first hour passed. Find the amount of time needed for the substance to be reduced by 5g.

   Sol.

$$\begin{aligned}
\frac{dR}{dt} &= -kR \\
\frac{dR}{R} &= -kdt \\
\ln|R| &= -kt + C \\
R &= Ce^{-kt} \\
R(0) &= 9 \\
C &= 9 \\
R &= 9e^{-kt} \\
R(1) &= 8 \\
8 &= 9e^{-k} \\
k &= \ln\frac{9}{8} \approx 0.1178 \\
R &= 9e^{-0.1178t} \\
4 &= 9e^{-0.1178t} \\
t &\approx 6.9
\end{aligned}$$

   $\therefore$ The amount of time needed for the substance to be reduced by 5g is about 6.9 hours.

5. Let the population growth rate of a country is directly proportional to the population of the country at that year. Given that in the year 1951, its population is 1.5 million, and in the year 1970, its population is 2 million, find

   (a) The population of the country in the year 2000.

   Sol.

$$\begin{aligned}
\frac{dP}{dt} &= kP \\
\frac{dP}{P} &= kdt \\
\ln|P| &= kt + C \\
P &= Ce^{kt} \\
Ce^{1950k} &= 1.5 \ \cdots (1) \\
Ce^{1970k} &= 2 \ \cdots (2) \\
(2) \div (1) \Rightarrow e^{20k} &= \frac{4}{3} \\
k &= \frac{1}{20}\ln\frac{4}{3} \\
C &= 2e^{-1970\times\frac{1}{20}\ln\frac{4}{3}} \\
P &= 2e^{-1970\times\frac{1}{20}\ln\frac{4}{3}}e^{\frac{1}{19}\ln\frac{4}{3}t} \\
&= 2e^{\frac{1}{20}\ln\frac{4}{3}(t-1970)} \\
P(2000) &= 2e^{\frac{1}{20}\ln\frac{4}{3}(2000-1970)} \\
&\approx 3.0792
\end{aligned}$$

   $\therefore$ The population of the country in the year 2000 is about 307.9 million.

(b) The year (approximate value) when the population of the country reaches 100 million.

Sol.

$$\begin{aligned}
P &= 2e^{\frac{1}{20}\ln\frac{4}{3}(t-1970)} \\
10 &= 2e^{\frac{1}{20}\ln\frac{4}{3}(t-1970)} \\
e^{\frac{1}{20}\ln\frac{4}{3}(t-1970)} &= 5 \\
\frac{1}{20}\ln\frac{4}{3}(t-1970) &= \ln 5 \\
\ln\frac{4}{3}(t-1970) &= 20\ln 5 \\
t-1970 &= \frac{20\ln 5}{\ln\frac{4}{3}} \\
t &= 1970 + \frac{20\ln 5}{\ln\frac{4}{3}} \\
&\approx 2081.89
\end{aligned}$$

$\therefore$ The country will reach 100 million population in the year 2082.

6. The decaying speed of a mothball is that it's decayed to half its original volume every three weeks. At the beginning, its volume is 1cm^3. When its volume become 0.1cm^3, the mothball loss its effectiveness. Find the time of effectiveness of the mothball.

Sol.

$$\begin{aligned}
-\frac{dV}{dt} &= kV \\
\frac{dV}{V} &= -kdt \\
\ln|V| &= -kt + C \\
V &= Ce^{-kt} \\
V(0) &= 1 \\
C &= 1 \\
V &= e^{-kt} \\
V(3) &= 0.5 \\
0.5 &= e^{-3k} \\
k &= \frac{1}{3}\ln 2 \\
V &= e^{-\frac{1}{3}t\ln 2} \\
e^{-\frac{1}{3}t\ln 2} &= 0.1 \\
-\frac{1}{3}t\ln 2 &= \ln 0.1 \\
t &= \frac{3\ln 10}{\ln 2} \\
&\approx 9.96
\end{aligned}$$

$\therefore$ The mothball loss its effectiveness in about 10 weeks.

7. The atmospheric pressure p of each point above the earth is a function of its altitude h. Now it's given that when $h = 0$, $p = 1.033 \times 10^5$ N/m^2, and when $h = 3048$m, $p = 6.88 \times 10^4$ N/m^2. Find the atmosperic pressure p when $h = 2000$m.

   Sol.

$$\begin{aligned}
-\frac{dp}{dh} &= kh \\
\frac{dp}{p} &= -kdh \\
\ln|p| &= -kh + C \\
p &= Ce^{-kh} \\
p(0) &= 1.033 \times 10^5 \\
C &= 1.033 \times 10^5 \\
p &= 1.033 \times 10^5 e^{-kh} \\
p(3048) &= 6.88 \times 10^4 \\
1.033 \times 10^5 e^{-3048k} &= 6.88 \times 10^4 \\
e^{-3048k} &= \frac{688}{1033} \\
-3048k &= \ln\frac{688}{1033} \\
k &= -\frac{1}{3048}\ln\frac{688}{1033} \\
p &= 1.033 \times 10^5 e^{\frac{1}{3048}h\ln\frac{688}{1033}} \\
p(2000) &= 1.033 \times 10^5 e^{\frac{1}{3048}\times 2000\ln\frac{688}{1033}} \\
&\approx 79118.7 \\
&\approx 7.91 \times 10^4
\end{aligned}$$

   $\therefore$ The atmospheric pressure when $h = 2000$m is about 7.91×10^4 N/m^2.

8. A boat is sailing on still water. The the deceleration produced by the resistance of the water is directly proportional to the speed of the boat. Prove that the speed of the boat t seconds after its engine has stopped is $v = v_0 e^{-kt}$, where v_0 is the speed of the boat when the boat engine stop, and k is a constant of proportionality.

   Proof.

$$\begin{aligned}
-\frac{dv}{dt} &= kv \\
\frac{dv}{v} &= -kdt \\
\ln|v| &= -kt + C \\
v &= Ce^{-kt} \\
v(0) &= v_0 \\
C &= v_0 \\
v &= v_0 e^{-kt} \quad \blacksquare
\end{aligned}$$

9. A capacitor is releasing its charge, and the rate of change of V with respect to the time is proportional to V, while V decreases as the time increases. Now, it's given that the constant of proportionality is $k = 40$, if V is decreased to 10% of its original value, find t.

Sol.

$$\begin{aligned}
\frac{dV}{dt} &= -40V \\
\frac{dV}{V} &= -40dt \\
\ln|V| &= -40t + C \\
V &= Ce^{-40t} \\
V(0) &= 1 \\
C &= 1 \\
V &= e^{-40t} \\
e^{-40t} &= 0.1 \\
-40t &= \ln 0.1 \\
t &= \frac{\ln 10}{40} \\
&\approx 0.058
\end{aligned}$$

10. The decay of a radioactive substance has been found to be proportional to the amount of the substance present R. The experiments show that the substance is reduced to half of its original amount R_0 in 1,600 years. Find the relationship between the amount of the substance R and the time t.

Sol.

$$\begin{aligned}
\frac{dR}{dt} &= -kR \\
\frac{dR}{R} &= -kdt \\
\ln|R| &= -kt + C \\
R &= Ce^{-kt} \\
R(0) &= R_0 \\
C &= R_0 \\
R &= R_0e^{-kt} \\
R(1600) &= \frac{1}{2}R_0 \\
R_0e^{-1600k} &= \frac{1}{2}R_0 \\
e^{-1600k} &= \frac{1}{2} \\
-1600k &= \ln\frac{1}{2} \\
k &= \frac{\ln 2}{1600} \\
R &= R_0e^{-\frac{\ln 2}{1600}t}
\end{aligned}$$

11. The Newton's law of cooling stated that the rate of change of the temperature of a body is proportional to the difference between its temperature and the temperature of the surrounding medium. If the temperature of a body decreases from 80°C to 60°C in 20 minutes, Find the temperature of the body after 40 minutes (assume that the temperature of the surrounding medium is 20°C).

    Sol.

$$
\begin{aligned}
-\frac{dT}{dt} &= k(T - T_s) \\
\frac{dT}{T - T_s} &= -kdt \\
\ln|T - T_s| &= -kt + C \\
T - T_s &= e^{-kt+C} \\
T &= T_s + Ce^{-kt} \\
T(0) &= T_s + C = T_0 \\
C &= T_0 - T_s \\
T &= T_s + (T_0 - T_s)e^{-kt} \\
T(20) &= 20 + (80 - 20)e^{-k(20)} \\
60 &= 20 + 60e^{-20k} \\
e^{-20k} &= \frac{2}{3} \\
-20k &= \ln\frac{2}{3} \\
k &= -\frac{1}{20}\ln\frac{2}{3} \\
T &= T_s + (T_0 - T_s)e^{\frac{1}{20}\ln\frac{2}{3}t} \\
T(40) &= 20 + (80 - 20)e^{\frac{1}{20}\ln\frac{2}{3}40} \\
&= 46.7^\circ\mathrm{C}
\end{aligned}
$$

    The temperature of the body after 40 minutes is 46.7°C.

12. It's been proven in the experiments that the pressure p of the gas in the cylinder of a diesel engine decreases as its volume increases, while the rate of change of p with respect to V is proportional to p and inversely proportional to V. Find the relationship between the pressure p and the volume V.

    Sol.

$$
\begin{aligned}
-\frac{dp}{dV} &= \frac{kp}{V} \\
\frac{dp}{p} &= -\frac{k}{V}dV \\
\ln p &= -k\ln V + C \\
p &= e^{\ln V^{-k} + C} \\
&= e^{\ln V^{-k}}e^{C} \\
&= \frac{C}{V^k}
\end{aligned}
$$

    $\therefore$ The relationship between the pressure p and the volume V is $p = \dfrac{C}{V^k}$.

13. A submarine with its mass m starts descending from still water surface, its resistance is directly proportional to its descending speed (the constant of proportionality is k). Find the relationship between the depth x and the time t.

Sol.

Let the descending speed of the submarine v_0 at time t be affected by two forces: the gravity mg and the resistance kv. Hence the forces on the submarine are

$$F = ma == mg - kv$$

From Newton's second law, the equation of motion of the submarine is

$$\begin{aligned}
m\frac{dv}{dt} &== mg - kv \\
\frac{dv}{dt} &= g - \frac{k}{m}v \\
\frac{dv}{dt} + \frac{k}{m}v &= g \\
p(t) &= \frac{k}{m} \\
\mu(t) &= e^{\int p(t)dt} \\
&= e^{\int \frac{k}{m}dt} \\
&= e^{\frac{k}{m}t} \\
e^{\frac{k}{m}t}\frac{dv}{dt} + e^{\frac{k}{m}t}\frac{k}{m}v &= e^{\frac{k}{m}t}g \\
\frac{d}{dt}\left(e^{\frac{k}{m}t}v\right) &= e^{\frac{k}{m}t}g \\
e^{\frac{k}{m}t}v &= \frac{mg}{k}e^{\frac{k}{m}t} + C \\
v &= \frac{mg}{k} + Ce^{-\frac{k}{m}t}
\end{aligned}$$

$$\begin{aligned}
v(0) &= \frac{mg}{k} + C \\
0 &= \frac{mg}{k} + C \\
C &= -\frac{mg}{k} \\
v &= \frac{mg}{k}(1 - e^{-\frac{k}{m}t}) \\
x &= \int vdt \\
&= \int \frac{mg}{k}(1 - e^{-\frac{k}{m}t})dt \\
&= \frac{mg}{k}\int (1 - e^{-\frac{k}{m}t})dt \\
&= \frac{mg}{k}\left(t + \frac{m}{k}e^{-\frac{k}{m}t} + C\right) \\
&= \frac{mg}{k}\left(t - \frac{mg}{k} + \frac{m}{k}e^{-\frac{k}{m}t}\right)
\end{aligned}$$

$\therefore$ The relationship between the depth x and the time t is $x = \dfrac{mg}{k}\left(t - \dfrac{mg}{k} + \dfrac{m}{k}e^{-\frac{k}{m}t}\right)$.

14. In an electric circuit, let the resistance be R ohms, the inductance L henrys, and the electromotive force E volts (R, L, and E are constants). It's been proven in the physics lab that the relationship between the current intensity i and the electromotive force E is given by the equation $E = Ri + L\frac{di}{dt}$. Find the general solution and the particular solution when $i(0) = 0$.

Sol.

$$
\begin{aligned}
E &= Ri + L\frac{di}{dt} \\
\frac{di}{dt} + \frac{R}{L}i &= \frac{E}{L} \\
p(t) &= \frac{R}{L} \\
\mu(t) &= e^{\int p(t)dt} \\
&= e^{\int \frac{R}{L}dt} \\
&= e^{\frac{R}{L}t} \\
e^{\frac{R}{L}t}\frac{di}{dt} + e^{\frac{R}{L}t}\frac{R}{L}i &= e^{\frac{R}{L}t}\frac{E}{L} \\
\frac{d}{dt}\left(e^{\frac{R}{L}t}i\right) &= e^{\frac{R}{L}t}\frac{E}{L} \\
e^{\frac{R}{L}t}i &= \int e^{\frac{R}{L}t}\frac{E}{L}dt \\
&= \frac{E}{L}\int e^{\frac{R}{L}t}dt \\
&= \frac{E}{L}\frac{L}{R}e^{\frac{R}{L}t} + C \\
&= \frac{E}{R}e^{\frac{R}{L}t} + C \\
i &= \frac{E}{R} + Ce^{-\frac{R}{L}t}
\end{aligned}
$$

$$
\begin{aligned}
i(0) &= \frac{E}{R} + C \\
0 &= \frac{E}{R} + C \\
C &= -\frac{E}{R} \\
i &= \frac{E}{R} - \frac{E}{R}e^{-\frac{R}{L}t} \\
&= \frac{E}{R}(1 - e^{-\frac{R}{L}t})
\end{aligned}
$$

$\therefore$ The general solution is $i = \frac{E}{R} + Ce^{-\frac{R}{L}t}$, and the particular solution is $i = \frac{E}{R}(1 - e^{-\frac{R}{L}t})$ when $i(0) = 0$.