2015 STPM Mathematics (T)
Paper 2

Section A

1. Evaluate

(a) $\lim_{x \to 2} \frac{6(x-2)}{x^3-8}$

Sol.

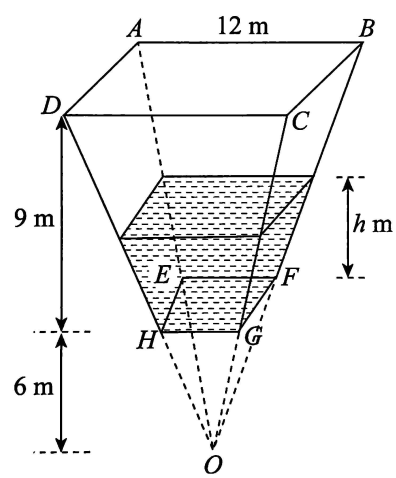
$$\begin{aligned}
\lim_{x \to 2} \frac{6(x-2)}{x^3-8} &= \lim_{x \to 2} \frac{6\cancel{(x-2)}}{\cancel{(x-2)}(x^2+2x+4)} \\
&= \lim_{x \to 2} \frac{6}{x^2+2x+4} \\
&= \frac{6}{2^2+2(2)+4} \\
&= \frac{6}{12} \\
&= \frac{1}{2}
\end{aligned}$$

(b) $\lim_{x \to 8} \frac{x-8}{\sqrt{6}-\sqrt{x-2}}$

Sol.

$$\begin{aligned}
\lim_{x \to 8} \frac{x-8}{\sqrt{6}-\sqrt{x-2}} &= \lim_{x \to 8} \frac{(x-8)'}{\left(\sqrt{6}-\sqrt{x-2}\right)'} \\
&= \lim_{x \to 8} \frac{1}{-\frac{1}{2\sqrt{x-2}}} \\
&= \lim_{x \to 8} \left(-2\sqrt{x-2}\right) \\
&= -2\sqrt{8-2} \\
&= -2\sqrt{6}
\end{aligned}$$

2. A water storage tank *ABCDEFGH* is a part of an inverted right square based pyramid, as shown in the diagram below.



The complete pyramid *OABCD* has a square base of sides 12m and height 15m. THe depth of the tank is 9m. Water is pumped into the tan at the constant rate of $\frac{1}{3}\text{m}^3\text{min}^{-1}$.

(a) Show that the volume of water $V\text{m}^3$ when the depth of water int he tank is hm is given by $V = \frac{16}{75}h(h^2 + 18h + 108)$.

Sol.

Let M be the midpoint of DC, M' be the midpoint of HG.

$$\begin{aligned}
\because\ & \triangle HOM' \sim \triangle DOM \\
\therefore\ & \frac{HM'}{DM} = \frac{OM'}{OM} \\
& \frac{HM'}{6} = \frac{6}{15} \\
& HM' = \frac{36}{15} \\
& HG = 2HM' = \frac{72}{15} = \frac{24}{5} \\
& V_{OEFGH} = \frac{1}{3} \times 6 \times \frac{576}{25} \\
& = \frac{1152}{25}
\end{aligned}$$

Let the edge of the water surface above $EFGH$ be $PQRS$. Let the mid point of RS be N.

$$\begin{aligned}
\because\ & \triangle SON \sim \triangle DOM \\
\therefore\ & \frac{SN}{DM} = \frac{ON}{OM} \\
& \frac{SN}{6} = \frac{6+h}{15} \\
& SN = \frac{36+6h}{15} \\
& = \frac{12+2h}{5} \\
& RS = 2SN \\
& = \frac{24+4h}{5} \\
& V_{OPQRS} = \frac{1}{3} \times (6+h) \times \left(\frac{24+4h}{5}\right)^2 \\
& = (6+h) \times \frac{576+192h+16h^2}{75} \\
& = \frac{16}{75}(6+h)(h^2+12h+36) \\
& = \frac{16}{75}(6h^2+72h+216+h^3+12h^2+36h) \\
& = \frac{16}{75}(h^3+18h^2+108h+216)
\end{aligned}$$

$$\begin{aligned}
\therefore\ V &= V_{OPQRS} - V_{OEFGH} \\
&= \frac{16}{75}(h^3+18h^2+108h+216) - \frac{1152}{25} \\
&= \frac{16}{75}(h^3+18h^2+108h) + \frac{3456}{75} - \frac{3456}{75} \\
&= \frac{16}{75}(h^3+18h^2+108h) \\
&= \frac{16}{75}h(h^2+18h+108) \quad \text{(shown)} \quad \blacksquare
\end{aligned}$$

(b) Find the rate at which the depth is increasing at the moment when the depth of water is 3m.

Sol.

$$\begin{aligned}
\frac{dV}{dt} &= \frac{1}{3} \\
\frac{dV}{dh} &= \frac{16(3h^2 + 36h + 108)}{75} \\
\frac{dh}{dt} &= \frac{dh}{dV} \cdot \frac{dV}{dt} \\
\frac{dV}{dh} \cdot \frac{dh}{dt} &= \frac{dV}{dt} \\
\frac{16(3h^2 + 36h + 108)}{75} \cdot \frac{dh}{dt} &= \frac{1}{3} \\
\frac{dh}{dt} &= \frac{1}{3} \cdot \frac{75}{16(3h^2 + 36h + 108)} \\
&= \frac{25}{16(3h^2 + 36h + 108)}
\end{aligned}$$

When $h = 3$,

$$\begin{aligned}
\frac{dh}{dt} &= \frac{25}{16\,[3(3)^2 + 36(3) + 108]} \\
&= \frac{25}{16(27 + 108 + 108)} \\
&= \frac{25}{16(243)} \\
&= \frac{25}{3888} \text{m min}^{-1}
\end{aligned}$$

(c) Calculate the time taken to fill up the tank if initially the tank is empty.

Sol.

When $h = 9$,

$$\begin{aligned}
V &= \frac{16}{75}(9)(9^2 + 18(9) + 108) \\
&= 673.92\text{m}^3 \\
\frac{dV}{dt} &= \frac{1}{3} \\
V &= \int \frac{1}{3} dt \\
&= \frac{t}{3} + C
\end{aligned}$$

When $t = 0$, $V = 0$,

$$\begin{aligned}
\therefore\ C &= 0 \\
V &= \frac{t}{3} \\
673.92 &= \frac{t}{3} \\
t &= 2021.76 \text{ mins} \\
&= 33.696 \text{ hours}
\end{aligned}$$

3. Show that $\int_0^1 x^2 \cos^{-1} x dx = \frac{2}{9}$.

Sol.

Let $u = \cos^{-1} x$, $du = -\frac{1}{\sqrt{1-x^2}}dx$. Let $dv = x^2 dx$, $v = \frac{x^3}{3}$.

$$\begin{aligned}
\int_0^1 x^2 \cos^{-1} x dx &= \left[\frac{x^3}{3}\cos^{-1} x\right]_0^1 + \int_0^1 \frac{x^3}{3}\cdot\frac{1}{\sqrt{1-x^2}}dx \\
&= \frac{1}{3}\cos^{-1} 1 - \frac{0}{3}\cos^{-1} 0 + \frac{1}{3}\int_0^1 \frac{x^3}{\sqrt{1-x^2}}dx \\
&= \frac{1}{3}\int_0^1 \frac{x^3}{\sqrt{1-x^2}}dx \\
&= \frac{1}{3}\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\cdot x dx
\end{aligned}$$

Let $u = 1 - x^2$, $du = -2x dx$, $x^2 = 1 - u$.

When x = 0, $u = 1$, when x = 1, $u = 0$.

$$\begin{aligned}
\int_0^1 x^2 \cos^{-1} x dx &= -\frac{1}{6}\int_1^0 \frac{1-u}{\sqrt{u}}du \\
&= \frac{1}{6}\int_0^1 (u^{-\frac{1}{2}} - u^{\frac{1}{2}})du \\
&= \frac{1}{6}\left[2u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_0^1 \\
&= \frac{1}{6}\left(2 - \frac{2}{3}\right) \\
&= \frac{1}{6}\cdot\frac{4}{3} \\
&= \frac{2}{9} \quad \text{(shown)} \quad \blacksquare
\end{aligned}$$

4. Find the solution of the differential equation $x\frac{dy}{dx} - y = 2$ which satisfies the condition $y = 0$ when $x = 1$.

Sol.

$$\begin{aligned}
x\frac{dy}{dx} - y &= 2 \\
\frac{dy}{dx} - \frac{y}{x} &= \frac{2}{x} \\
\mu(x) &= e^{\int -\frac{1}{x}dx} \\
&= e^{-\ln x} \\
&= \frac{1}{x} \\
\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} &= \frac{2}{x^2} \\
\frac{d}{dx}\left(\frac{y}{x}\right) &= \frac{2}{x^2}
\end{aligned}$$

$$\begin{aligned}
\frac{y}{x} &= \int \frac{2}{x^2}dx \\
&= -\frac{2}{x} + C \\
y &= -2 + Cx
\end{aligned}$$

When $x = 1$, $y = 0$,

$$\begin{aligned}
0 &= -2 + C \\
C &= 2
\end{aligned}$$

Hence, $y = 2x - 2$.