Chapter 12

Exponents and Logarithms

12.1 Exponents

Definition of Exponents and its Arithmetic Rules

Back in junior high, we learned the following definitions of exponents of rational numbers:

Exponents of Positive Integers $a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ times}}$ (where n is a positive integer)

Exponents of Zero $a^0 = 1$ (where $a \neq 0$)

Exponents of Negative Integers $a^{-n} = \dfrac{1}{a^n}$ (where n is a positive integer)

Exponents of Fractions $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ (where $a > 0, n > 1$ and m, n are positive integers)

Furthermore, there are the following arithmetic properties of exponents:

Arithmetic Properties of Exponents

$$a^x \times a^y = a^{x+y}$$

$$a^x \div a^y = a^{x-y} \quad (a \neq 0)$$

$$(a^x)^y = a^{xy}$$

$$(ab)^x = a^x b^x$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \quad (b \neq 0)$$

Example 1

Calculate $\left(\dfrac{4}{9}\right)^{\frac{1}{2}} + (-\pi)^0 - \left(2\dfrac{10}{27}\right)^{-\frac{2}{3}} + 0.125^{-\frac{1}{3}}$.

Solution:

$$\begin{aligned}
\left(\frac{4}{9}\right)^{\frac{1}{2}} + (-\pi)^0 - \left(2\frac{10}{27}\right)^{-\frac{2}{3}} + 0.125^{-\frac{1}{3}} &= \left(\left(\frac{2}{3}\right)^2\right)^{\frac{1}{2}} + 1 - \left(\frac{64}{27}\right)^{-\frac{2}{3}} + \frac{1}{(0.5^3)^{\frac{1}{3}}} \\
&= \frac{2}{3} + 1 - \left(\frac{3}{4}\right)^2 + 2 \\
&= 3\frac{5}{48}
\end{aligned}$$

Example 2

Simplify the following:

(a) $\dfrac{\left(a^{-2}b^{-3}\right)\left(-4a^{-1}b\right)}{12a^{-4}b^{-2}c}$ (b) $\dfrac{2\cdot 3^{2x+3}+3^{2x}}{9^x}$

Solution:

(a)
$$\begin{aligned}\frac{\left(a^{-2}b^{-3}\right)\left(-4a^{-1}b\right)}{12a^{-4}b^{-2}c} &= -\frac{1}{3}a^{-2+(-1)-(-4)}b^{-3+1-(-2)}c^{-1}\\ &= -\frac{1}{3}a^{1}b^{0}c^{-1}\\ &= -\frac{a}{3c}\end{aligned}$$

(b)
$$\begin{aligned}\frac{2\cdot 3^{2x+3}+3^{2x}}{9^x} &= \frac{2\cdot 3^3\cdot 3^{2x}+3^{2x}}{3^{2x}}\\ &= \frac{3^{2x}(2\times 27+1)}{3^{2x}}\\ &= 54+1\\ &= 55\end{aligned}$$

Practice 12.1a

1. Calculate $\left(\dfrac{81}{16}\right)^{-0.25}\times\left(\dfrac{8}{27}\right)^{-\frac{2}{3}}\times(0.25)^{-2.5}$.
2. Simplify the following:

   (a) $5a^{-2}b^{-3}\div 5^{-1}a^{2}b^{-3}\times 5^{-2}ab^{4}c$ (b) $\dfrac{25^{x+2}-5^{2x}}{4\cdot 5^{2x-1}}$

Graph of Exponential Functions and its Properties

In real life, we encounter many examples that involve exponentiation. For example: a cell divides into two, those two divide into four, and so on. Therefore, after x divisions, the total number of cells becomes $f(x)=2^x$.

Given $a>0, a\neq 1$, we call $f(x)=a^x, x\in \boldsymbol{R}$ the **exponential function**.

To study the graph and properties of exponential functions, we plot the graphs of $y=2^x$, $y=10^x$, and $y=\left(\frac{1}{2}\right)^x$.

Think about It:

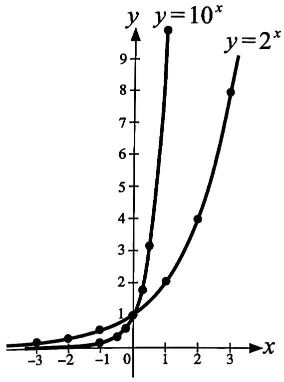
Why can't the base a of an exponential function be 1? And why can't it be negative?

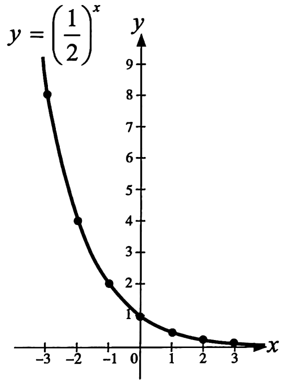
We first construct the following tables:

x	$\dots$	-3	-2	-1	0	1	2	3	$\dots$
2^x	$\dots$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	$\dots$

x	$\dots$	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	$\dots$
10^x	$\dots$	0.1	0.32	0.56	1	1.78	3.16	10	$\dots$

x	$\dots$	-3	-2	-1	0	1	2	3	$\dots$
$\left(\frac{1}{2}\right)^x$	$\dots$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\dots$

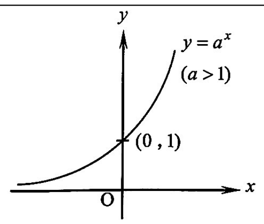
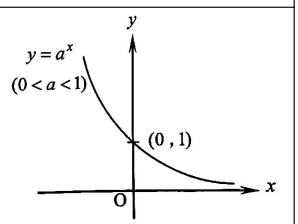




From the graphs, we can see that:

- The graphs of the functions $y = 2^x$, $y = 10^x$, and $y = \left(\frac{1}{2}\right)^x$ are all located above the x-axis. In fact, since $a > 0$, then $a^x > 0$, the values of the exponential function $y = a^x$ are always greater than 0.
- The -axis is the asymptote of the exponential function $y = a^x$.
- When $x = 0$, $y = a^0 = 1$. Therefore, all the graphs of the exponential functions $y = a^x$ pass through the point $(0, 1)$.
- For the functions $y = 2^x$ and $y = 10^x$, when $x > 0$, $y > 1$; when $x < 0$, $0 < y < 1$. As the value of x increases, the value of y also increases, meaning the functions are increasing on the interval $(-\infty, \infty)$.
- For the function $y = \left(\frac{1}{2}\right)^x$, when $x > 0$, $0 < y < 1$; when $x < 0$, $y > 1$. As the value of x increases, the value of y decreases, meaning the function is decreasing on the interval $(-\infty, \infty)$.

Hence, when we are discussing the graphs and properties of the function $y = a^x$, we must discuss the cases where $a > 1$ and $0 < a < 1$ separately, as shown in the following table:

<table>
<tr><td></td><td>$a > 1$</td><td>$0 < a < 1$</td></tr>
<tr><td>Graph</td><td></td><td></td></tr>
<tr><td rowspan="4">Properties</td><td colspan="2">$y > 0$, i.e. range = $(0, \infty)$</td></tr>
<tr><td colspan="2">When $x = 0$, $y = 1$</td></tr>
<tr><td>When $x > 0$, $y > 1$;
When $x < 0$, $0 < y < 1$</td><td>When $x > 0$, $0 < y < 1$;
When $x < 0$, $y > 1$</td></tr>
<tr><td>It is an increasing function in the interval $(-\infty, +\infty)$</td><td>It is a decreasing function in the interval $(-\infty, +\infty)$</td></tr>
</table>

Exploration Activity 1

Aim: To explore the graph of the exponential function.

Tool: https://www.geogebra.org/m/rweqzcrt

Example 3

Without using a calculator, compare the two values in each of the following pairs:

(a) $1.5^{3.1}$ and $1.5^{3.5}$ (b) $0.7^{2.3}$ and $0.7^{3.2}$ (c) $0.2^{-5.1}$ and $0.2^{-4.1}$

Solution:

(a) Since the base number $1.5 > 1$, the exponential function $y = 1.5^x$ is increasing in the interval $(-\infty, \infty)$, the exponent $3.1 < 3.5$
$\therefore 1.5^{3.1} < 1.5^{3.5}$

(b) Since the base number $0 < 0.7 < 1$, the exponential function $y = 0.7^x$ is decreasing in the interval $(-\infty, \infty)$, the exponent $2.3 < 3.2$
$\therefore 0.7^{2.3} > 0.7^{3.2}$

(c) Since the base number $0 < 0.2 < 1$, the exponential function $y = 0.2^x$ is decreasing in the interval $(-\infty, \infty)$, the exponent $-5.1 < -4.1$
$\therefore 0.2^{-5.1} > 0.2^{-4.1}$

Example 4

Solve the inequality $\left(\dfrac{1}{2}\right)^x \leq \left(\dfrac{1}{2}\right)^{x^2}$.

Solution:

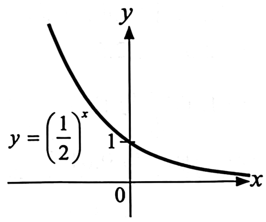
$\left(\dfrac{1}{2}\right)^x \leq \left(\dfrac{1}{2}\right)^{x^2}$

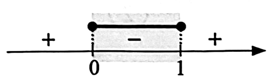
$\because y = \left(\dfrac{1}{2}\right)^x$ is a decreasing function,

$x \geq x^2$

$x(x-1) \leq 0$

$\therefore 0 \leq x \leq 1$





Practice 12.1b

1. Without using a calculator, compare the two values in each of the following pairs:

   (a) $\pi^{2.1}$ and $\pi^{3.5}$ (b) $0.15^{-2.3}$ and $0.15^{-3.8}$

2. Solve the inequality $3^{x^2-3x+5} \geq 3^{x+10}$.

Exercise 12.1

Calculate (Question 1 to 4):

1. $\left(2\frac{1}{4}\right)^{-\frac{3}{2}} + \left(1\frac{11}{25}\right)^{-\frac{1}{2}} - \left(2\frac{2}{3}\right)^{0}$
2. $12^{\frac{1}{3}} \times 6^{\frac{1}{3}} \div 27^{\frac{1}{6}} \div 3^{\frac{1}{6}}$
3. $\left(\frac{1}{2}\right)^{-2} + 125^{\frac{2}{3}} + 343^{\frac{1}{3}} - \left(\frac{1}{27}\right)^{-\frac{1}{3}}$
4. $\dfrac{\sqrt[5]{4}\cdot\sqrt{8}\left(\sqrt[3]{\sqrt[5]{4}}\right)^2}{\sqrt[3]{\sqrt{2}}}$

Simplify the following expressions (Question 5 to 8):

5. $\left(\frac{b}{2a^2}\right)^3 \div \left(\frac{2b^2}{3a}\right)^0 \times \left(-\frac{b}{a}\right)^{-3}$
6. $\left(x^{\frac{1}{4}} - y^{-\frac{1}{4}}\right)\left(x^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)\left(x^{\frac{1}{4}} + y^{-\frac{1}{4}}\right)$
7. $\dfrac{3\cdot 2^n - 4\cdot 2^{n-2}}{2^n - 2^{n-1}}$
8. $\dfrac{3^{n+6} - 6\cdot 3^{n+1}}{7\cdot 3^{n+2}}$
9. Given that $a^{2x} = \sqrt{2}+1$, find the value of $\dfrac{a^{3x}+a^{-3x}}{a^x+a^{-x}}$.
10. In the same Cartesian coordinates system, sketch the graphs for $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$.
11. Without using a calculator, compare the two values in each of the following pairs:
    (a) $2.5^{7.1}$ and $2.5^{8.5}$
    (b) $0.35^{6.5}$ and $0.35^{5.6}$
    (c) $1.03^{-2.1}$ and $1.03^{-3.2}$
    (d) $(\sqrt{2})^{\pi}$ and $(\sqrt{2})^{3.5}$
    (e) $0.01^{-\frac{1}{3}}$ and $0.01^{-\frac{1}{2}}$
    (f) $2.7^{\sqrt{20}}$ and $2.7^{\sqrt[3]{35}}$
12. Solve the inequality $\left(\frac{1}{2}\right)^{3x-x^2} < 1$.

12.2 Logarithms

Definition of Logarithms and its Properties

Let a be a positive integer and $a \neq 1$. If $a^n = x$, then we define $\log_a x = n$, read as "logarithm of x to the base a is n". In $\log_a x = n$, a is called the **base**, and x is called the **antilogarithm**.

Conversely, if $\log_a x = n$, then $a^n = x$. That is,

Definition of Logarithms

$$a^n = x \iff \log_a x = n \qquad \text{where } a > 0, a \neq 1, x > 0$$

From this, we can get

Properties of Logarithms

$$\log_a a^n = n$$

When $n = 1$ and $n = 0$, we have the following special cases:

Special Cases of Logarithms

$\log_a a = 1$

$\log_a 1 = 0$

The logarithm with base 10 is called the **common logarithm**, denoted as $\log_{10} x$, $\log x$, or $\lg x$.

There also exists a special base e in logarithms. It is an irrational number approximately equal to 2.71828. The logarithm with base e is called the **natural logarithm**, denoted as $\log_e x$, $\ln x$. It is widely used in the fields of natural science.

Example 5

If $\log_3 x = 5$, find the value of x.

Solution:

$$\begin{aligned} \log_3 x &= 5 \\ x &= 3^5 \\ &= 243 \end{aligned}$$

Example 6

Find the value of the following:

(a) $\log_5 5$

(b) $\lg 1$

(c) $\log_7 343$

(d) $\ln \frac{1}{e^2}$

Solution:

(a) $\log_5 5 = 1$

(b) $\lg 1 = 0$

(c) $\log_7 343 = \log_7 7^3 = 3$

(d) $\ln \dfrac{1}{e^2} = \ln e^{-2} = -2$

Practice 12.2a

1. If $\log_4 x = -2$, find the value of x.
2. Find the value of the following:

   (a) $\log_2 \dfrac{1}{2}$

   (b) $\lg 10$

   (c) $\log_{\frac{1}{3}} 27$

   (d) $\log_4 \dfrac{1}{128}$

Arithmetic Rules of Logarithms

If a, x, and y are positive numbers, and $x = a^p$, $y = a^q$, then $\log_a x = p$ and $\log_a y = q$.

- $$xy = a^p \times a^q = a^{p+q}$$
  $$\begin{aligned} \therefore \log_a(xy) &= p + q \\ &= \log_a x + \log_a y \end{aligned}$$
- $$\frac{x}{y} = a^p \div a^q = a^{p-q}$$
  $$\begin{aligned} \therefore \log_a \frac{x}{y} &= p - q \\ &= \log_a x - \log_a y \end{aligned}$$
- $x = a^p \cdots (1)$, then $\log_a x = p \cdots (2)$
  Substituting (2) into (1), we get $x = a^{\log_a x}$

Summarizing the above, we have the following properties of logarithms:

Arithmetic Properties of Logarithms If $x > 0$, $y > 0$,

$$\begin{aligned} \log_a(xy) &= \log_a x + \log_a y \\ \log_a \frac{x}{y} &= \log_a x - \log_a y \\ \log_a x^m &= m \log_a x \\ a^{\log_a x} &= x \end{aligned}$$

Example 7

Let $a = \log_2$, express the following in terms of a:

(a) $\log 8^2$

(b) $(\log 8)^2$

Solution:

(a) $$\begin{aligned} \log 8^2 &= \log 2^6 \\ &= 6 \log 2 \\ &= 6a \end{aligned}$$

(b) $$\begin{aligned} (\log 8)^2 &= \left(\log 2^3\right)^2 \\ &= (3 \log 2)^2 \\ &= (3a)^2 \\ &= 9a^2 \end{aligned}$$

Example 8

Without using a calculator, find the value of the following:

(a) $100^{\log 3}$

(b) $2\log_3 15 - \log_3 50 + \log_3 6$

Solution:

(a)
$$\begin{aligned} 100^{\log 3} &= 10^{2\log 3} \\ &= 10^{\log 3^2} \\ &= 9 \end{aligned}$$

(b)
$$\begin{aligned} 2\log_3 15 - \log_3 50 + \log_3 6 &= \log_3 (15)^2 - \log_3 50 + \log_3 6 \\ &= \log_3 \frac{225 \times 6}{50} \\ &= \log_3 3^3 \\ &= 3 \end{aligned}$$

Example 9

Given that $\log_2 3 = a$, $\log_2 5 = b$. Express the following in terms of a and b:

(a) $\log_2 30$

(b) $\log_2 \sqrt{1.25}$

Solution:

(a)
$$\begin{aligned} \log_2 30 &= \log_2 (2 \times 3 \times 5) \\ &= \log_2 2 + \log_2 3 + \log_2 5 \\ &= 1 + a + b \end{aligned}$$

(b)
$$\begin{aligned} \log_2 \sqrt{1.25} &= \log_2 \left(\frac{5}{4}\right)^{\frac{1}{2}} \\ &= \frac{1}{2}\left(\log_2 5 - \log_2 2^2\right) \\ &= \frac{1}{2}(b - 2) \end{aligned}$$

Practice 12.2b

1. Calculate the following:

   (a) $2^{\log_2 7}$

   (b) $\log \frac{1}{35} + \log 70 - \log \frac{1}{2} + 2\log 5$

2. Given that $\log_3 2 = a$, $\log_3 5 = b$, express $\log_3 3.75$ in terms of a and b.

Change of Base Formula

Some commonly used calculator calculate logarithms using base 10 or e. The change of base formula allows for the conversion of logarithms with any base to those with base 10 or e, making it easy to calculate logarithms with bases other than 1.

The change of base formula also facilitates simplification of computations by converting logarithms with different bases to those with the same base.

If $\log_a b = n$, then

$$\begin{aligned} a^n &= b \\ \log_c a^n &= \log_c b \\ n \log_c a &= \log_c b \\ n &= \frac{\log_c b}{\log_c a} \end{aligned}$$

Change of Base Formula

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Using the change of base formula, we can convert logarithms with any base to a specific base. In the formula, let $c = b$, we get

Change of Base Formula (Special Case)

$$\log_a b = \frac{1}{\log_b a}$$

Example 10

Find the value of $\log_3 5$.

Solution:

$$\log_3 5 = \frac{\log_{10} 5}{\log_{10} 3} = 1.4650$$

Example 11

Without using a calculator, find the value of the following:

(a) $\log_8 32$

(b) $\log_{\sqrt{3}} \frac{1}{81}$

Solution:

(a) $\log_8 32 = \dfrac{\log_2 2^5}{\log_2 2^3} = \dfrac{5}{3}$

(b) $\log_{\sqrt{3}} \dfrac{1}{81} = \dfrac{\log_3 3^{-4}}{\log_3 3^{\frac{1}{2}}} = \dfrac{-4}{\frac{1}{2}} = -8$

Think about It:

In Example 11, can they be changed into logarithms with other bases?

Example 12

Given that $\log_2 3 = x$, $\log_3 5 = y$, express $\log_9 2.5$ in terms of x and y.

Solution:

$$\begin{aligned}\log_9 2.5 &= \frac{\log_3 \frac{5}{2}}{\log_3 3^2} \\ &= \frac{\log_3 5 - \log_3 2}{2} \\ &= \frac{1}{2}\left(y - \frac{1}{\log_2 3}\right) \\ &= \frac{1}{2}\left(y - \frac{1}{x}\right) \\ &= \frac{xy - 1}{2x}\end{aligned}$$

Practice 12.2c

1. Without using a calculator, find the value of $\dfrac{\log_4 27}{\log_2 3}$.
2. If $\log_2 3 = a$, $\log_5 3 = b$, express $\log 5$ in terms of a and b.

Example 13

Given that $3\log_7 (xy^2) + \log_7 x = 4 + 2\log_7 y$, express y in terms of x.

Solution:

$$\begin{aligned}3\log_7 \left(xy^2\right) + \log_7 x &= 4 + 2\log_7 y \\ 3\left(\log_7 x + \log_7 y^2\right) + \log_7 x - 2\log_7 y &= 4 \\ 4\log_7 x + 4\log_7 y &= 4 \\ \log_7 x + \log_7 y &= 1 \\ \log_7 (xy) &= 1 \\ xy &= 7 \\ y &= \frac{7}{x}\end{aligned}$$

Exercise 12.2a

Find the value of x in the following (Question 1 to 2):

1. $\log_{125} x = \dfrac{1}{3}$
2. $\log_x 81 = -4$

Simplify the following expressions (Question 3 to 8):

3. $\log_2 4^x$
4. $\log_2 a^{\log_a 2}$
5. $3^{\log_3 x - \log_3 y}$
6. $\log_3 (9^x \times 27^y)$
7. $2^{-\log_8 x}$
8. $3 \log_4 2^x$

Without using a calculator, find the value of the following (Question 9 to 18):

9. $\log_7 \sqrt[3]{49}$
10. $49^{\log_7 3}$
11. $\left(\dfrac{1}{2}\right)^{\log_2 7}$
12. $\dfrac{\log \sqrt{3}}{\log \frac{1}{9}}$
13. $\log(0.1)^4 - \log \sqrt[3]{0.001}$
14. $\dfrac{\log 4 + \log 3}{1 + \log 0.4 + \frac{1}{2} \log 9}$
15. $\log_2 \dfrac{1}{25} \cdot \log_3 \dfrac{1}{8} \cdot \log_5 \dfrac{1}{9}$
16. $\log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8$
17. $\left(\log_2 3 + \log_2 \sqrt{3}\right) \log_{\sqrt{3}} 2$
18. $\dfrac{1}{2} \log \dfrac{81}{17} + 2 \log \dfrac{5}{3} - \log \dfrac{17}{4} + \dfrac{3}{2} \log 17$

19 Given that $\log_2 3 = a$ and $\log_2 5 = b$. Express $\log_4 15$ in terms of a and b.

20 Given that $\log_3 5 = m$ and $\log_5 6 = n$. Express $\log_{25} 54$ in terms of m and n.

21 Given that $2^a = 3$ and $3^b = 7$. Express $\log_{42} 14$ in terms of a and b.

22 Given that $\log_3 6 = x$. Express $\log_9 12$ in terms of x.

23 If $\log 24 = a$ and $\log 18 = b$, express $\log 1.35$ in terms of a and b.

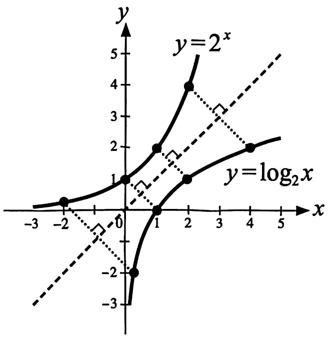
24 Given that $\log_{25}(2x - 1) = \log_5(x - 3) + \log_{25} 5$, prove $5x^2 - 32x + 46 = 0$.

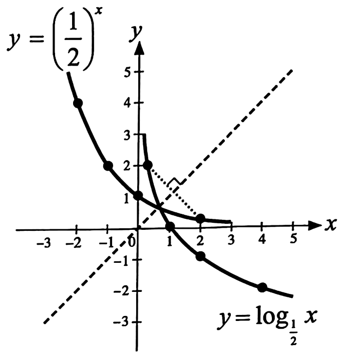
25 If a, b, x are positive numbers, and a, b are not equal to 1, prove $\dfrac{1}{\log_a x} + \dfrac{1}{\log_b x} = \dfrac{1}{\log_{ab} x}$.

Graph of Logarithmic Functions and its Properties

According to the definition of logarithms, if $y = 2^x$, then $x = \log_2 y$. Because $y = 2^x$ is a one-to-one function mapping from the set of real numbers to the set of positive real numbers, it has an inverse function. According to the concept of inverse functions, $y = \log_2 x$ is the inverse function of $y = 2^x$. The function $y = \log_a x$ is called a **logarithmic function**, where $a > 0$ and $a \neq 1$. Since the domain of the exponential function $y = a^x$ is the set of real numbers and the range is the set of positive real numbers, the domain of the logarithmic function $y = \log_a x$ is the set of positive real numbers, and the range is the set of real numbers.

Since the logarithmic function $y = \log_a x$ is the inverse function of the exponential function $y = a^x$, the graph of $y = \log_a x$ is symmetric to the graph of $y = a^x$ about the line $y = x$. We only need to draw the curve symmetric to the graph of $y = a^x$ about the line $y = x$ to obtain the graph of $y = \log_a x$. For example, in the figures below, the curves symmetric to the graph of the functions $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ about the line $y = x$ are the graphs of $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$, respectively.

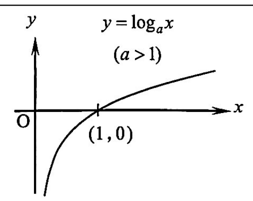
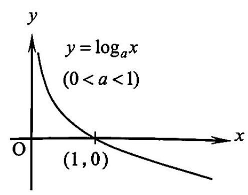




From the graphs, we can see that:

- The graphs of the functions $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ are located only to the right of the y-axis because their domains are $(0, \infty)$. The y-axis serves as the asymptote for the logarithmic function $y = \log_a x$.
- When $x = 1$, $y = \log_a 1 = 0$, so the graphs of the logarithmic functions $y = \log_a x$ all pass through the point $(1, 0)$.
- For the function $y = \log_2 x$, when $x > 1$, $y > 0$; when $0 < x < 1$, $y < 0$. As the value of x increases, the value of y also increases, meaning the function is increasing over the interval $(0, \infty)$.
- For the function $y = \log_{\frac{1}{2}} x$, when $x > 1$, $y < 0$; when $0 < x < 1$, $y > 0$. As the value of x increases, the value of y decreases, indicating the function is decreasing over the interval $(0, \infty)$.

From the above, we can see that when discussing the graphs and properties of the function $y = \log_a x$, we must discuss the cases where $a > 1$ and $0 < a < 1$ separately, as shown in the following table:

	$a > 1$	$0 < a < 1$
Graph		
Properties	$x > 0$, i.e. domain = $(0, \infty)$	
	When $x = 1$, $y = 0$	
	When $x > 1$, $y > 0$; When $0 < y < 1$, $y < 0$	When $x > 1$, $y < 0$; When $0 < x < 1$, $y > 0$
	It is an increasing function in the interval $(0, +\infty)$	It is a decreasing function in the interval $(0, +\infty)$

Exploration Activity 2

Aim: To explore the graph of the logarithmic function.

Tool: https://www.geogebra.org/m/ks3qqrua

Example 14

Without using a calculator, compare the two values in each of the following pairs:

(a) $\log_3 5$ and $\log_3 7$

(b) $\log_{\frac{1}{2}} 5$ and $\log_{\frac{1}{2}} 9$

Solution:

(a) Since the base $3 > 1$, the logarithmic function $y = \log_3 x$ is an increasing function in the interval $(0, \infty)$, the antilogarithm $5 < 7$
$\therefore \log_3 5 < \log_3 7$

(b) Since the base $0 < \dfrac{1}{2} < 1$, the logarithmic function $y = \log_{\frac{1}{2}} x$ is a decreasing function in the interval $(0, \infty)$, the antilogarithm $5 < 9$
$\therefore \log_{\frac{1}{2}} 5 > \log_{\frac{1}{2}} 9$

Example 15

Find the domain of the following functions:

(a) $f(x) = \log_a(x - 4)$

(b) $y = \log_2 \left(x^2 + 5\right)$

(c) $y = \log_3 \dfrac{x+3}{2-x}$

(d) $f(x) = \sqrt{\log_{\frac{1}{2}}(3 - x)}$

Solution:

(a) From $x - 4 > 0$, we get $x > 4$.
$\therefore$ the domain of the function $f(x) = \log_a(x - 4)$ is $x \mid x \in \mathbb{R}, x > 4$.

(b) For all $x \in \mathbb{R}$, $x^2 + 5 > 0$.
$\therefore$ the domain of the function $y = \log_2 \left(x^2 + 5\right)$ is $\mathbb{R}$.

(c) From $2 - x \neq 0$ and $\dfrac{x+3}{2-x} > 0$, i.e. $\dfrac{x+3}{x-2} < 0$, we get $-3 < x < 2$.
$\therefore$ the domain of the function $y = \log_3 \dfrac{x+3}{2-x}$ is $x \mid x \in \mathbb{R}, -3 < x < 2$.

(d) x must satisfy both $3 - x > 0$ and $\log_{\frac{1}{2}}(3 - x) \geq 0$,
i.e. $\begin{cases} x < 3 \\ 3 - x \leq 1 \end{cases} \Rightarrow 2 \leq x < 3.$
$\therefore$ the domain of the function $f(x) = \sqrt{\log_{\frac{1}{2}}(x - 3)}$ is $[2, 3)$.

▸ Example 16

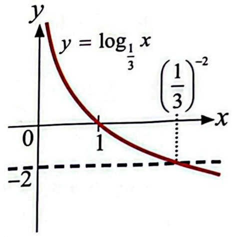
Solve the inequality $\log_{\frac{1}{3}}(2x-1)+2>0$.

Solution:

$\log_{\frac{1}{3}}(2x-1)+2>0$

$\log_{\frac{1}{3}}(2x-1)>-2$

From the graph of $y=\log_{\frac{1}{3}}x$, we get:



$0<2x-1<\left(\frac{1}{3}\right)^{-2}$

$0<2x-1<9$

$\therefore \frac{1}{2}<x<5$

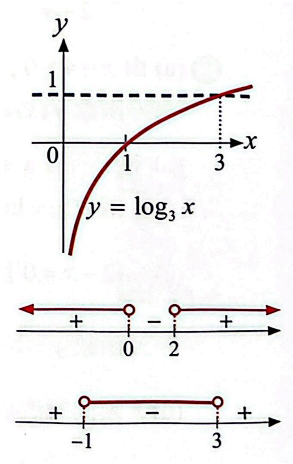
▸ Example 17

Solve the inequality $\log_3\left(x^2-2x\right)<1$.

Solution:

$\log_3\left(x^2-2x\right)<1$

From the graph of $y=\log_3 x$, we know that $0<x^2-2x<3$.



Hence, we get the following system of inequalities:

$$\begin{cases} x^2-2x>0 \cdots (1) \\ x^2-2x<3 \cdots (2) \end{cases}$$

From (1), we get $x(x-2)>0$

$x<0$ or $x>2$

From (2), we get $(x+1)(x-3)<0$

$-1<x<3$

$\therefore$ the solution to the inequality $\log_3\left(x^2-2x\right)<1$ is $-1<x<0$ or $2<x<3$.

Practice 12.2d

1. Without using a calculator, compare the two values in each of the following pairs:

   (a) $\log_{0.3} 5$ and $\log_{0.3} 7$

   (b) $\log_{\frac{1}{2}} 0.4$ and $\log_2 0.7$

2. Find the domain of the following functions:

   (a) $f(x) = \log_2 \left(x^2 + 8\right)$

   (b) $f(x) = \log_3 \dfrac{1-2x}{x+2}$

3. Solve the inequality $\log_3 \left(x^2 - 2x\right) \geq 1$.

Applications of Logarithms

Example 18

In 1935, American scientist C. F. Richter proposed the Richter scale for measuring the magnitude of earthquakes, which is commonly denoted as M, with the calculation formula $M = \log_{10} A - \log_{10} A_0$, where A is the maximum amplitude of the earthquake and A_0 is the amplitude of a "standard earthquake."

(a) Suppose in an earthquake, a seismometer located 100 km from the epicenter records a maximum amplitude of 30 mm, and the amplitude of the standard earthquake (A_0) is 0.001 mm. Calculate the magnitude of this earthquake (accurate to 0.1).

(b) How many times greater is the maximum amplitude of a 7.2 magnitude earthquake compared to that of a 5 magnitude earthquake?

Solution:

(a) Magnitude
$$\begin{aligned} M &= \lg 30 - \lg 0.001 \\ &= \lg \frac{30}{0.001} \\ &= \lg 30000 \\ &= 4.5 \end{aligned}$$

(b) Since $M = \lg A - \lg A_0$, we get
$$\begin{aligned} M &= \lg \frac{A}{A_0} \\ \frac{A}{A_0} &= 10^M \\ A &= A_0 \times 10^M \end{aligned}$$

When $M = 7.2$, $A = 10^{7.2}$

When $M = 5$, $A_2 = A_0 \times 10^5$

$$\frac{A_1}{A_2} = \frac{A_0 \times 10^7.2}{A_0 \times 10^5} = 10^{2.2} = 158$$

$\therefore$ the maximum amplitude of a 7.2 magnitude earthquake is 158 times greater than that of a 5 magnitude earthquake. Clearly, although the difference in earthquake magnitude is only 2.2 levels, the amplitude is 158 times larger. Therefore, the destructive power of a 7 magnitude earthquake is much greater than that of a 5 magnitude earthquake.

Example 19

The acidity or alkalinity of a solution is measured by its pH value, calculated using the formula $\text{pH} = -\lg\,[\text{H}^+]$, where $[\text{H}^+]$ represents the concentration of hydrogen ions in the solution (measured in moles per liter).

(a) According to the pH calculation formula, explain the relationship between acidity/alkalinity and the concentration of hydrogen ions.

(b) If the concentration of hydrogen ions in an acidic solution is 2.5×10^{-5} moles per liter, calculate the pH value of this solution.

Solution:

(a) From the formula $\text{pH} = -\lg\,[\text{H}^+]$,
we know that the pH value decreases as the concentration of $[\text{H}^+]$ increases. Therefore, as the concentration of hydrogen ions in a solution increases, the pH value decreases, indicating a stronger acidity of the solution.

(b) When $[\text{H}^+] = 2.5 \times 10^{-5}$, $\text{pH} = -\lg\left(2.5 \times 10^{-5}\right)$

$$= 4.602$$

Exercise 12.2b

1. Plot the graphs of $y = \log_3 x$ and $y = \log_{\frac{1}{3}} x$ on the same Cartesian coordinate system.

2. Without using a computer, compare the sizes of the following pairs of values:

   (a) $\log_{1.5} 1.4$ and $\log_{1.5} 1.6$

   (b) $\log_{0.4} \sqrt{2}$ and $\log_{0.4} \sqrt{3}$

   (c) $\log_{\frac{1}{2}} 3$ and $\log_{\frac{1}{3}} \frac{1}{4}$

   (d) $\log_{\frac{1}{2}} \frac{1}{3}$ and $\log_{\frac{1}{3}} \frac{1}{2}$

3. Find the domain of the following functions:

   (a) $f(x) = \log_3\left(9 - 16x^2\right)$

   (b) $f(x) = \log_9 \dfrac{1}{x-2}$

   (c) $f(x) = \dfrac{1}{\log_3(7x-5)}$

   (d) $f(x) = \sqrt{\lg(2x-1)}$

4. Solve the following inequalities:

   (a) $\log_5(2x-3) \le 2$

   (b) $\log_{\frac{1}{6}}(3x-4) + 1 < 0$

   (c) $\log_2 x \ge \log_2(2x-1)$

   (d) $2\log_{\frac{1}{4}}\left(3x^2 - x\right) + 1 \ge 0$

12.3 Exponential and Logarithmic Equations

Exponential Equations

An equation with the unknown variable in the position of an exponent is called an exponential equation. For example:

- $2^x = 4^{x+1}$
- $3^x + 3^{-x} = 8$ are both exponential equations.

Example 20

Solve the equation $4^x = 3$. (Correct to 4 decimal places)

Solution:

Method 1:

$$\begin{aligned} 4^x &= 3 \\ x &= \log_4 3 \\ &= \frac{\lg 3}{\lg 4} \\ &= 0.7925 \end{aligned}$$

Method 2:

$$\begin{aligned} 4^x &= 3 \\ \lg 4^x &= \lg 3 \\ x \lg 4 &= \lg 3 \\ x &= \frac{\lg 3}{\lg 4} \\ &= 0.7925 \end{aligned}$$

From Example 20, we can see that if a and b are positive numbers, the solution to the equation $a^x = b$ is $x = \frac{\log b}{\log a}$.

Example 21

Solve the equation $3 \cdot 5^{x+1} = 2^{2x-1}$. (Correct to 2 decimal places)

Solution:

$$3 \cdot 5^{x+1} = 2^{2x-1}$$

Taking logarithm of both sides

$$\begin{aligned} \lg 3 + (x+1) \lg 5 &= (2x-1) \lg 2 \\ x(\lg 5 - 2 \lg 2) &= -\lg 2 - \lg 3 - \lg 5 \\ x &= -\frac{\lg 2 + \lg 3 + \lg 5}{\lg 5 - 2 \lg 2} \\ &= -15.24 \end{aligned}$$

Practice 12.3a

Solve the following equations: (Correct to 4 decimal places)

1. $2 \cdot 5^{x+1} = 9$
2. $7^{x+3} = 5^{3x+1}$

Example 22

Solve the equation $5^{x+1} - 3 \cdot 5^{x-1} = 20$. (Correct to 4 decimal places)

Solution:

$$\begin{aligned}
5^{x+1} - 3 \cdot 5^{x-1} &= 20 \\
5 \cdot 5^x - \frac{3}{5} \cdot 5^x &= 20 \\
5^x \left(5 - \frac{3}{5}\right) &= 20 \\
5^x &= \frac{50}{11} \\
x &= \frac{\lg \frac{50}{11}}{\lg 5} \\
&= 0.9408
\end{aligned}$$

Example 23

Solve the equation $2^{x+1} - 3 \cdot 2^{-x} + 5 = 0$.

Solution:

$$\begin{aligned}
2^{x+1} - 3 \cdot 2^{-x} + 5 &= 0 \\
2\left(2^x\right) - 3\left(\frac{1}{2^x}\right) + 5 &= 0
\end{aligned}$$

Let $y = 2^x$, then the equation becomes

$$\begin{aligned}
2y - \frac{3}{y} + 5 &= 0 \\
2y^2 + 5y - 3 &= 0 \\
(2y - 1)(y + 3) &= 0 \\
y &= \frac{1}{2} \text{ or } y = -3
\end{aligned}$$

When $y = \frac{1}{2}, 2^x = \frac{1}{2} = 2^{-1}$, we get $x = -1$;
When $y = -3, 2^x = -3$, since 2^x must be greater than 0, there is no solution.
$\therefore$ The solution to the original equation is $x = -1$.

Practice 12.3b

Solve the following equations:

1. $3^{x+1} + 9^x - 18 = 0$
2. $2^{x+2} + 3\left(2^{1-x}\right) - 14 = 0$

Example 24

The given formula for the compound amount after n years is $p\left(1+\frac{r}{100}\right)^n$, where p is the principal amount and r is the annual interest rate. If Mr. Tan deposits RM $100,000$ into the bank with an annual interest rate of 3.5%, after how many years does it take for the compound amount exceed twice the principal amount?

Solution:

$$\begin{aligned}
p\left(1+\frac{3.5}{100}\right)^n &> 2p \\
1.035^n &> 2 \\
\log 1.035^n &> \log 2 \\
n\log 1.035 &> \log 2 \\
n &> \frac{\log 2}{\log 1.035} = 20.15
\end{aligned}$$

$\therefore$ It requires at least 21 years for the compound amount to exceed twice the principal amount.

Example 25

Some chemical elements naturally decay, and after a period of time T, their quantity reduces to half of the original. This period T is known as the half-life of the element. If the quantity of the element at time $t = 0$ is N_0, then after time T, its quantity becomes $\frac{1}{2}N_0$. Therefore, after time $2T$, its quantity becomes $\frac{1}{2}\times\frac{1}{2}N_0 = \left(\frac{1}{2}\right)^2 N_0$, and so on. Hence, after time t, its quantity is $N = \left(\frac{1}{2}\right)^{\frac{t}{T}} N_0$. Given that the half-life of Carbon-14 is 5730 years, how many years does a certain amount of Carbon-14 need to pass through to have less than 90% of its original quantity?

Solution:

$$\begin{aligned}
\left(\frac{1}{2}\right)^{\frac{t}{5730}} N_0 &< \frac{90}{100}N_0 \\
\left(\frac{1}{2}\right)^{\frac{t}{5730}} &< 0.9 \\
\frac{t}{5730} &> \log_{\frac{1}{2}} 0.9 \\
t &> 5730\times\frac{\log 0.9}{\log 0.5} \\
t &> 870.98
\end{aligned}$$

$\therefore$ It requires at least 870.98 years.

Think about It:

Why is the direction of the inequality changed from the second step to the third step?

Exercise 12.3a

Solve the following equations (Question 1 to 15):

1. $8^{x-3} = \frac{1}{256}$
2. $3^{2^x+1} = 243$
3. $10^{x^2-4} = 1$
4. $3^{x^2+3} = 27^{x+7}$
5. $4^{x^2} = 2^{5x+7}$
6. $\left(\frac{9}{16}\right)^x = \left(\frac{4}{3}\right)^{x-6}$
7. $3^{x+1} = 4^{x-1}$
8. $7^{5-3x} = 2\left(5^{x+2}\right)$
9. $2^{x^2-1} = 3^{x+1}$
10. $\left(\frac{1}{3}\right)^x - \left(\frac{1}{3}\right)^{-x} = \frac{80}{9}$
11. $25^x - 23 \cdot 5^x - 50 = 0$
12. $3^{x-1} + 3^{3-x} - 10 = 0$
13. $3^x - 5^{x+2} = 3^{x+1} - 5^{x+3}$
14. $4^x + 2^{1-x} = 3$
15. $25\left(4^x\right) + 46\left(10^x\right) = 8\left(25^x\right)$

16 Given that the formula for the compound interest after t years is $p\left(1 + \frac{r}{100}\right)^t$, where p is the principal amount and r is the annual interest rate. Mr. Tan deposited RM $150,000$ into a bank at the beginning of the year with an annual interest rate of 3.2%. How many years does it take for his deposit in the bank to exceed RM $200,000$?

17 Given that the half-life of radium-226 is 1600 years, how many years at least is required for a certain amount of radium-226 to decay to less than 10% of its original quantity?

Logarithmic Equations

An equation in which the base or argument of a logarithm contains an unknown variable is called a logarithmic equation. For example:

- $\lg(x-1) = 3$
- $\log_x 2 = 4$

When solving logarithmic equations, it is necessary to verify the roots obtained.

Example 26

Solve the equation $2\log_5 x = 1$.

Solution:

$$
\begin{aligned}
2\log_5 x &= 1\\
\log_5 x &= \frac{1}{2}\\
x &= \sqrt{5}
\end{aligned}
$$

$\therefore x = \sqrt{5}$ is the solution to the original equation.

Think about It:

Compare with Example 26. If we solve the equation $\log_5 x^2 = 1$, will we obtain the same answer?

Example 27

Solve the equation $\log_{0.1}\left(6x^2 - 5x - 3\right) = 0$.

Solution:

$$\begin{aligned}\log_{0.1}\left(6x^2 - 5x - 3\right) &= 0\\ 6x^2 - 5x - 3 &= 1\\ 6x^2 - 5x - 4 &= 0\\ (2x+1)(3x-4) &= 0\\ x = -\frac{1}{2} \text{ □ } x &= \frac{4}{3}\end{aligned}$$

After verifying, $x = -\frac{1}{2}$ and $x = \frac{4}{3}$ are both solutions to the original equation.

Example 28

Solve the equation $\lg(5-x) = 2\lg(x+1)$.

Solution:

$$\begin{aligned}\lg(5-x) &= 2\lg(x+1)\\ \lg(5-x) &= \lg(x+1)^2\\ 5 - x &= x^2 + 2x + 1\\ x^2 + 3x - 4 &= 0\\ (x+4)(x-1) &= 0\\ x = -4 \text{ or } x &= 1\end{aligned}$$

After verifying, $x = -4$ is an extraneous root, and $x = 1$ is the solution to the original equation.

Practice 12.3c

Solve the following equations:

1. $\log_2\left(x^2 - 2x\right) = 3$ □
2. $\log_2 x + \log_2(3-x) = 1$ □

Example 29

Solve the equation $\log_3^2 x - \log_3 x^3 = 0$.

$$\begin{aligned} \log_3{}^2 x - \log_3 x^3 &= 0 \\ (\log_3 x)^2 - 3\log_3 x &= 0 \\ (\log_3 x)(\log_3 x - 3) &= 0 \\ \log_3 x = 0 \text{ or } \log_3 x &= 3 \\ x = 1 \qquad\qquad x &= 27 \end{aligned}$$

After verifying, $x = 1$ and $x = 27$ are both solutions to the original equation.

Keep in mind

- $(\log_3 x)^2$ can be written as $\log_3{}^2 x$.
- $\log_3 x^2$ means $\log_3 \left(x^2\right)$, but it is not equivalent to $(\log_3 x)^2$.

Example 30

Solve the equation $\log_5 x - 6\log_x 5 = 1$.

Solution:

$$\begin{aligned} \log_5 x - 6\log_x 5 &= 1 \\ \log_5 x - \frac{6}{\log_5 x} &= 1 \end{aligned}$$

Let $y = \log_5 x$, then

$$\begin{aligned} y - \frac{6}{y} &= 1 \\ y^2 - y - 6 &= 0 \\ (y+2)(y-3) &= 0 \end{aligned}$$

$$\begin{aligned} y &= -2 & &\text{or} & y &= 3 \\ \log_5 x &= -2 & & & \log_5 x &= 3 \\ x &= \frac{1}{25} & & & x &= 125 \end{aligned}$$

After verifying, $x = \frac{1}{25}$ and $x = 125$ are both solutions to the original equation.

Example 31

Solve the equation $x^{1+\lg x} = 100$.

Solution:

$$\begin{aligned} x^{1+\lg x} &= 100 \\ \lg x^{1+\lg x} &= \lg 100 \\ (1+\lg x)(\lg x) &= 2 \end{aligned}$$

Let $y = \lg x$, then

$$\begin{aligned} y + y^2 &= 2 \\ y^2 + y - 2 &= 0 \\ (y+2)(y-1) &= 0 \end{aligned}$$

$$\begin{aligned} y &= -2 & &\text{or} & y &= 1 \\ \lg x &= -2 & & & \lg x &= 1 \\ x &= \frac{1}{100} & & & x &= 10 \end{aligned}$$

After verifying, $x = \dfrac{1}{100}$ and $x = 10$ are both solutions to the original equation.

Practice 12.3d

Solve the following equations:

1. $\log_2 x - \log_x 8 = 2$
2. $x^{\lg x} = 100x$

Example 32

Solve the equation $2\ln x = \ln(x-2) + 3$.

Solution:

$$\begin{aligned} 2\ln x &= \ln(x-2) + 3 \\ \ln x^2 &= \ln(x-2) + \ln e^3 \\ \ln x^2 &= \ln\left[e^3(x-2)\right] \\ x^2 &= e^3x - 2e^3 \\ x^2 - e^3x + 2e^3 &= 0 \\ x &= \frac{e^3 \pm \sqrt{e^6 - 4\left(2e^3\right)}}{2} \\ x &= 17.83 \text{ or } x = 2.25 \end{aligned}$$

After verifying, $x = 17.83$ and $x = 2.25$ are both solutions to the original equation.

Practice 12.3e

Solve the equation $\ln(x+1) - \ln(x-1) = 1$.

Exercise 12.3b

Solve the following equations:

1. $\log_{\sqrt{3}} x = -2$
2. $\log_4(3-x) = -\dfrac{1}{2}$
3. $\log_2 x^4 = 8$
4. $\log_8\left(x^2 - 3x - 2\right) = \dfrac{1}{3}$
5. $\log x + \log(x-3) = 1$
6. $\log_6 x + \log_6\left(x^2 - 7\right) = 1$
7. $2\log_3{}^2 x + \log_3 x - 1 = 0$
8. $2\log_{25} x - 3\log_x 25 = 1$
9. $2\log_x 4 + 2\log_4 x = 5$
10. $x^{2\ln x} = ex$
11. $\log_3 \log_2 \log_x 25 = 0$
12. $\log_2 x + \log_8 x = 2\log_2 x \log_8 x$
13. $\ln\left(x^2 - x - 2\right) - \ln(x+1) = 0$
14. $\log(x+6) - \dfrac{1}{2}\log(2x-3) = 2 - \log 25$

Revision Exercise 12

Without using a calculator, simplify the following expressions (Question 1 to 4):

1. $\dfrac{3^{n+2} - 2 \cdot 3^n}{5 \cdot 3^{n+1} + 4 \cdot 3^{n-1}}$
2. $\log_4 \cos \dfrac{\pi}{4} + \log_4 \sin \dfrac{\pi}{6}$
3. $(\log_2 3 + \log_4 9)(\log_3 4 + \log_9 2)$
4. $\log_8\left(\log_2 \sqrt{8 + 4\sqrt{3}} + \log_2 \sqrt{8 - 4\sqrt{3}}\right)$
5. Given that $3^{\log x} = 2^{\log 3}$, without using a calculator, find the value of x.
6. If a and x are positive numbers, and $a \neq 1$, prove that $\log_{a^n} x = \dfrac{1}{n}\log_a x$.
7. Given that $\log_2 3 = m$ and $\log_5 2 = n$, express $\log_6 10$ in terms of m and n.
8. Given that $f(x) = 4 - \left(\dfrac{1}{2}\right)^x$ and $g(x) = \sqrt{4 - \left(\dfrac{1}{2}\right)^x}$.
   (a) Sketch the graph of $f(x)$ and find its domain;
   (b) Find the domain of $g(x)$.
9. Find the domain of the function $f(x) = \dfrac{\log_3(2-x)}{\log_3(2+x)}$.
10. Find the domain of the function $f(x) = \sqrt{\log_{\frac{1}{5}}\left(2x^2 - x\right)}$.
11. Find the domain of the function $f(x) = \dfrac{1}{\sqrt{1 - \log x}}$.

12. Given that a, x, and y are positive numbers and $x \neq 1$, $x + y \neq 1$. If $\frac{1}{x} + \frac{1}{y} = 1$, prove that $\frac{1}{\log_{x+y} a} = \frac{1}{\log_x a} + \frac{1}{\log_y a}$.
13. If a, b, c, and x are positive numbers and a, b, c are not equal to 1, prove that $\log_a x \log_b x + \log_b x \log_c x + \log_c x \log_a x = \frac{\log_a x \log_b x \log_c x}{\log_{abc} x}$.

Solve the following equations (Question 14 to 23):

14. $7^x - 7^{x-1} = 6$
15. $3^{2x+1} = 10\left(3^x\right) - 3$
16. $2^{x+3} = 2^{-x} - 7$
17. $4\left(9^x\right) - 9\left(4^x\right) = 5\left(6^x\right)$
18. $\log x + \log(x+3) = \log(x+8)$
19. $\log_2{}^2 x = \log_2 x + 6$
20. $4^{\log x} = 2^{3\log x + 2}$
21. $3\log_8 x + 2 = 2\log_2 x$
22. $\log_3 x + 6\log_x 3 = 5$
23. $\log_4(x+4) + 1 = \log_2(x+1)$

Solve the following inequalities (Question 24 to 27):

24. $\left(\frac{1}{2}\right)^{3x-x^2} < 1$
25. $3^{x+2} > 5\left(2^{2x-1}\right)$
26. $\log_2\left(5x - x^2\right) < 2$
27. $\log_{\frac{1}{5}}\left(x^2 - 4x\right) + 1 \geq 0$

28. In the application of artificial intelligence, there's a commonly used function called the logistic function $f(x) = \frac{A}{1 + e^{-k(x-a)}}$, where a, k, and A are constants. Find the expression for $f^{-1}(x)$.
29. The price of a machine is RM 500, 000. If the machine depreciates in value by 4% each year, such that its value after n years is $A(1 - r\%)^n$, where A is the original price and r is the depreciation percentage. How many years does it take for its price to fall below RM 300, 000?
30. Given that in the absence of air resistance, the function relating the maximum speed of a rocket (v km/s), the mass of the fuel (M kg), and the mass of the rocket excluding fuel (m kg) is $\frac{1}{2}v = \ln\left(1 + \frac{M}{m}\right)$, how many times the mass of the fuel compared to the mass of the rocket is required for the maximum speed of the rocket to reach 10 km/s?