8.4 三角函数的积化和差

(选择题)

1. $\sin 52\frac{1}{2}^\circ \cdot \cos 7\frac{1}{2}^\circ = ?$

   解:

$$\begin{aligned}\sin 52\frac{1}{2}^\circ \cdot \cos 7\frac{1}{2}^\circ &= \frac{1}{2}\left[\sin\left(52\frac{1}{2}^\circ + 7\frac{1}{2}^\circ\right) + \sin\left(52\frac{1}{2}^\circ - 7\frac{1}{2}^\circ\right)\right] \\ &= \frac{1}{2}(\sin 60^\circ + \sin 45^\circ) \\ &= \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{3}+\sqrt{2}}{4} \qquad \blacksquare\end{aligned}$$

(作答题)

1. 试证 $\sec\left(\frac{\pi}{4}+\theta\right)\sec\left(\frac{\pi}{4}-\theta\right) = 2\sec 2\theta$。

   解:

$$\begin{aligned}\sec\left(\frac{\pi}{4}+\theta\right)\sec\left(\frac{\pi}{4}-\theta\right) &= \frac{1}{\cos\left(\frac{\pi}{4}+\theta\right)\cos\left(\frac{\pi}{4}-\theta\right)} \\ &= \frac{1}{\frac{\cos\left(\frac{\pi}{4}+\theta+\frac{\pi}{4}-\theta\right)+\cos\left(\frac{\pi}{4}+\theta-\frac{\pi}{4}+\theta\right)}{2}} \\ &= \frac{2}{\cos\frac{\pi}{2}+\cos 2\theta} \\ &= \frac{2}{0+\cos 2\theta} \\ &= 2\sec 2\theta \qquad \blacksquare\end{aligned}$$

2. 试证 $8\cos\theta\cos 2\theta\cos 3\theta - \frac{\sin 7\theta}{\sin\theta} = 1$。**解:**

$$\begin{aligned}8\cos\theta\cos 2\theta\cos 3\theta - \frac{\sin 7\theta}{\sin\theta} &= \frac{8\sin\theta\cos\theta\cos 2\theta\cos 3\theta - \sin 7\theta}{\sin\theta} \\ &= \frac{4\sin 2\theta\cos 2\theta\cos 3\theta - \sin 7\theta}{\sin\theta} \\ &= \frac{2\sin 4\theta\cos 3\theta - \sin 7\theta}{\sin\theta} \\ &= \frac{\sin 7\theta + \sin\theta - \sin 7\theta}{\sin\theta} \\ &= \frac{\sin\theta}{\sin\theta} \\ &= 1 \qquad \blacksquare\end{aligned}$$

3. (a) 证明 $\cot(\theta+15^\circ)-\tan(\theta-15^\circ)=\dfrac{4\cos 2\theta}{1+2\sin 2\theta}$;

解:

$$\begin{aligned}
\cot(\theta+15^\circ)-\tan(\theta-15^\circ) &= \frac{\cos(\theta+15^\circ)}{\sin(\theta+15^\circ)}-\frac{\sin(\theta-15^\circ)}{\cos(\theta-15^\circ)} \\
&= \frac{\cos(\theta+15^\circ)\cos(\theta-15^\circ)-\sin(\theta+15^\circ)\sin(\theta-15^\circ)}{\sin(\theta+15^\circ)\cos(\theta-15^\circ)} \\
&= \frac{\frac{1}{2}(\cos 2\theta+\cos 30^\circ)+\frac{1}{2}(\cos 2\theta-\cos 30^\circ)}{\frac{1}{2}(\sin 2\theta+\sin 30^\circ)} \\
&= \frac{\cos 2\theta+\cos 2\theta}{\sin 2\theta+\sin 30^\circ} \\
&= \frac{2\cos 2\theta}{\frac{2\sin 2\theta+1}{2}} \\
&= \frac{4\cos 2\theta}{1+2\sin 2\theta} \qquad \blacksquare
\end{aligned}$$

(b) 据此, 设 $\theta=60^\circ$, 证明 $\tan 75^\circ=2+\sqrt{3}$。**解**:

$$\begin{aligned}
\cot(60^\circ+15^\circ)-\tan(60^\circ-15^\circ) &= \frac{4\cos 120^\circ}{1+2\sin 120^\circ} \\
\cot 75^\circ-\tan 45^\circ &= \frac{4\left(-\frac{1}{2}\right)}{1+2\left(\frac{\sqrt{3}}{2}\right)} \\
\cot 75^\circ-1 &= \frac{-2}{1+\sqrt{3}} \\
\cot 75^\circ &= \frac{-2+1+\sqrt{3}}{1+\sqrt{3}} \\
\cot 75^\circ &= \frac{\sqrt{3}-1}{\sqrt{3}+1} \\
\tan 75^\circ &= \frac{\sqrt{3}+1}{\sqrt{3}-1} \\
&= \frac{3+1+2\sqrt{3}}{3-1} \\
&= 2+\sqrt{3} \qquad \blacksquare
\end{aligned}$$

8.5 三角函数的和差化积

(选择题)

1. $\cos^2\theta + \cos^2(\theta+1^\circ) + \cos^2(\theta+2^\circ) + \cdots + \cos^2(\theta+179^\circ)$ 的值是

   解:

$$\begin{aligned}
\cos^2(\theta+90^\circ) &= (-\sin\theta)^2 = \sin^2\theta \\
\cos^2(\theta+91^\circ) &= (-\sin(\theta+1^\circ))^2 = \sin^2(\theta+1^\circ) \\
&\vdots \\
\cos^2(\theta+179^\circ) &= (-\sin(\theta+89^\circ))^2 = \sin^2(\theta+89^\circ)
\end{aligned}$$

$$\begin{aligned}
\cos^2\theta + \cos^2(\theta+90^\circ) &= \cos^2\theta + \sin^2\theta = 1 \\
\cos^2(\theta+1^\circ) + \cos^2(\theta+91^\circ) &= \cos^2(\theta+1^\circ) + \sin^2(\theta+1^\circ) = 1 \\
&\vdots \\
\cos^2(\theta+89^\circ) + \cos^2(\theta+179^\circ) &= \cos^2(\theta+89^\circ) + \sin^2(\theta+89^\circ) = 1
\end{aligned}$$

   $\therefore$ 原式 $= 1\times 90 = 90$ ∎

2. $\cos^2 2x + 2\sin^2 x$ 的极小值是

   解:

$$\begin{aligned}
\frac{d}{dx}(\cos^2 2x + 2\sin^2 x) &= -4\sin 2x\cos 2x + 4\sin x\cos x \\
&= -2\sin 4x + 2\sin 2x \\
&= -2(\sin 4x - \sin 2x) \\
&= -4\cos 3x\sin x
\end{aligned}$$

$$\begin{aligned}
\cos 3x\sin x &= 0 \\
\cos 3x = 0 \quad &\text{or} \quad \sin x = 0 \\
3x = \frac{\pi}{2} + k\pi \quad &\text{or} \quad x = k\pi \\
x = \frac{\pi}{6} + \frac{k\pi}{3} \quad &\text{or} \quad x = k\pi
\end{aligned}$$

   When $k=0$, $x=\frac{\pi}{6}$ or $x=0$; When $k=1$, $x=\frac{\pi}{2}$ or $x=\pi$

$$\frac{d^2}{dx^2}(\cos^2 2x + 2\sin^2 x) = -8\cos 4x + 4\cos 2x$$

   When $x=\frac{\pi}{6}$, $\frac{d^2}{dx^2}=6$; When $x=0$, $\frac{d^2}{dx^2}=-4$; When $x=\frac{\pi}{2}$, $\frac{d^2}{dx^2}=-12$; When $x=\frac{\pi}{6}$, $\frac{d^2}{dx^2}=6$

   Hence, the minimum value is $\frac{3}{4}$ when $x=\frac{\pi}{6}$. ∎

3. 化简 $\cos\theta+\cos\left(\theta+\dfrac{2\pi}{3}\right)+\cos\left(\theta+\dfrac{4\pi}{3}\right)$。

解:

$$
\begin{aligned}
\cos\theta+\cos\left(\theta+\frac{2\pi}{3}\right)+\cos\left(\theta+\frac{4\pi}{3}\right) &= \cos\theta+2\cos\left(\frac{\theta+\frac{2\pi}{3}+\theta+\frac{4\pi}{3}}{2}\right)\cos\left(-\frac{\pi}{3}\right) \\
&= \cos\theta+\cos(\theta+\pi) \\
&= \cos\theta-\cos\theta \\
&= 0
\end{aligned}
$$

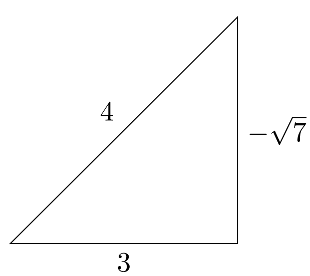
■

4. 已知 $\sin\alpha-\sin\beta=-\dfrac{1}{2},\cos\alpha-\cos\beta=\dfrac{1}{2}$, 且 α 与 β 都是锐角, 求 $\tan(\alpha-\beta)$ 的值。

解:

$$
\begin{aligned}
\sin\alpha-\sin\beta &= -\frac{1}{2} \\
\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta &= \frac{1}{4} \\
\cos\alpha-\cos\beta &= \frac{1}{2} \\
\cos^2\alpha+\cos^2\beta-2\cos\alpha\cos\beta &= \frac{1}{4} \\
2-2\sin\alpha\sin\beta-2\cos\alpha\cos\beta &= \frac{1}{2} \\
2-2\cos(\alpha-\beta) &= \frac{1}{2} \\
\cos(\alpha-\beta) &= \frac{3}{4}
\end{aligned}
$$

$\because \sin\alpha-\sin\beta<0, \therefore \alpha<\beta \implies \alpha-\beta<0 \implies \alpha-\beta$ is in the fourth quadrant.



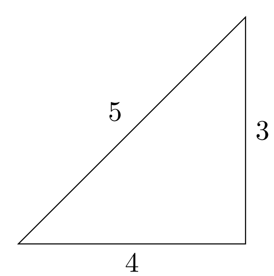
$$\tan(\alpha-\beta)=-\frac{\sqrt{7}}{3}$$

■

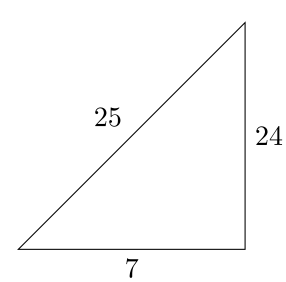
5. 已知 $\sin\alpha+\sin\beta=\dfrac{1}{4}, \cos\alpha+\cos\beta=\dfrac{1}{3}$, 求 $\tan(\alpha+\beta)$ 的值。

解:

$$\begin{aligned}
\sin\alpha+\sin\beta &= \frac{1}{4} \\
2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) &= \frac{1}{4} \\
\cos\alpha+\cos\beta &= \frac{1}{3} \\
2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) &= \frac{1}{3} \\
\frac{\sin\left(\dfrac{\alpha+\beta}{2}\right)}{\cos\left(\dfrac{\alpha+\beta}{2}\right)} &= \frac{3}{4} \\
\tan\left(\frac{\alpha+\beta}{2}\right) &= \frac{3}{4}
\end{aligned}$$



$$\begin{aligned}
\sin(\alpha+\beta) &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \\
&= 2\times\frac{3}{5}\times\frac{4}{5} \\
&= \frac{24}{25}
\end{aligned}$$



$$\tan(\alpha+\beta)=\frac{24}{7} \qquad \blacksquare$$

6. 若 $\alpha+\beta=\dfrac{\pi}{3}$ 且 $y=\cos^2\alpha+\cos^2\beta$, 则 y 的极大值是

解:

$$\begin{aligned}
y&=\cos^2\alpha+\cos^2\beta\\
&=(\cos\alpha+\cos\beta)^2-2\cos\alpha\cos\beta\\
&=\left(2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right)^2-\cos(\alpha+\beta)-\cos(\alpha-\beta)\\
&=\left(2\cos\frac{\pi}{6}\cos\frac{\alpha-\beta}{2}\right)^2-\cos\frac{\pi}{3}-\cos(\alpha-\beta)\\
&=\left(\sqrt{3}\cos\frac{\alpha-\beta}{2}\right)^2-\frac{1}{2}-\cos(\alpha-\beta)\\
&=3\cos^2\frac{\alpha-\beta}{2}-\frac{1}{2}-\cos(\alpha-\beta)\\
&=3\left(\frac{1+\cos(\alpha-\beta)}{2}\right)-\frac{1}{2}-\cos(\alpha-\beta)\\
&=\frac{3}{2}+\frac{3}{2}\cos(\alpha-\beta)-\frac{1}{2}-\cos(\alpha-\beta)\\
&=1+\frac{1}{2}\cos(\alpha-\beta)
\end{aligned}$$

Since $-1\le\cos(\alpha-\beta)\le 1$, the maximum value of y is $\dfrac{3}{2}$ when $\cos(\alpha-\beta)=1$. ■

7. 已知 $\cos 2\theta=5\sin\theta-2$, 求 $\cos 2\theta+\sin\theta$ 的值。

解:

$$\begin{aligned}
\cos 2\theta&=5\sin\theta-2\\
1-2\sin^2\theta&=5\sin\theta-2\\
2\sin^2\theta+5\sin\theta-3&=0\\
(2\sin\theta-1)(\sin\theta+3)&=0\\
\sin\theta=\frac{1}{2}\quad\text{or}\quad\sin\theta&=-3\ \text{(rejected)}\\
\cos 2\theta+\sin\theta&=5\sin\theta-2+\sin\theta\\
&=6\sin\theta-2=1
\end{aligned}$$

■

8. 化简 $\dfrac{\sin 13x+\sin 7x}{\cos 13x+\cos 7x}$。

解:

$$\begin{aligned}
\frac{\sin 13x+\sin 7x}{\cos 13x+\cos 7x}&=\frac{2\sin 10x\cos 3x}{2\cos 10x\cos 3x}\\
&=\frac{\sin 10x}{\cos 10x}\\
&=\tan 10x
\end{aligned}$$

■

(作答题)

1. 设 $A+B+C=180^\circ$, 证明 $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$。

   解:

$$
\begin{aligned}
A+B+C &= 180^\circ \\
A+B &= 180^\circ - C \\
\frac{A+B}{2} &= 90^\circ - \frac{C}{2} \\
\sin\frac{A+B}{2} &= \cos\frac{C}{2} \\
\sin A+\sin B+\sin C &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2} \\
&= 2\cos\frac{C}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2} \\
&= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \sin\frac{C}{2}\right) \\
&= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \sin\left(90^\circ - \frac{A+B}{2}\right)\right) \\
&= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\right) \\
&= 2\cos\frac{C}{2}\left(2\cos\frac{A}{2}\cos\frac{B}{2}\right) \\
&= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \qquad \blacksquare
\end{aligned}
$$

2. 若 $A+B+C=180^\circ$, 利用两角和正切公式, 试证: $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

   若 $\tan A=1, \tan B=2$, 求 $\angle C$。

   解:

$$
\begin{aligned}
A+B+C &= 180^\circ \\
A+B &= 180^\circ - C \\
\tan(A+B) &= \tan(180^\circ - C) \\
\tan(A+B) &= -\tan C \\
\frac{\tan A+\tan B}{1-\tan A\tan B} &= -\tan C \\
\tan A+\tan B &= -\tan C + \tan A\tan B\tan C \\
\tan A+\tan B+\tan C &= \tan A\tan B\tan C \qquad \blacksquare
\end{aligned}
$$

$$
\begin{aligned}
\tan A+\tan B+\tan C &= \tan A\tan B\tan C \\
1+2+\tan C &= 2\tan C \\
3 &= \tan C \\
\angle C &= \arctan 3 = 71.57^\circ \qquad \blacksquare
\end{aligned}
$$

3. 若 $A+B+C=180^\circ$, 试证 $\cos(B+C-A)+\cos(C+A-B)+\cos(A+B-C)=1+4\cos A\cos B\cos C$。

解:

$$\begin{aligned}
&\cos(B+C-A)+\cos(C+A-B)+\cos(A+B-C)\\
&=\cos(180^\circ-A-A)+\cos(180^\circ-B-B)+\cos(180^\circ-C-C)\\
&=-\cos 2A-\cos 2B-\cos 2C\\
&=-(2\cos^2 A-1)-2\cos(B+C)\cos(B-C)\\
&=-2\cos^2 A+1-2\cos(180^\circ-A)\cos(B-C)\\
&=-2\cos^2 A+1+2\cos A\cos(B-C)\\
&=1-2\cos A[\cos A-\cos(B+C)]\\
&=1-2\cos A[\cos(180^\circ-(B+C))-\cos(B+C)]\\
&=1+2\cos A[\cos(B+C)+\cos(B+C)]\\
&=1+4\cos A\cos B\cos C
\end{aligned}$$

■

4. 证明 $\sin\theta+\sin 2\theta+\sin 4\theta-\sin 7\theta=4\sin\dfrac{3\theta}{2}\sin 3\theta$。(Faulty Problem Statement)

解:

$$\begin{aligned}
\sin\theta+\sin 2\theta+\sin 4\theta-\sin 7\theta&=\sin\theta-7\sin\theta+\sin 4\theta+\sin 2\theta\\
&=-2\cos 4\theta\sin 3\theta+2\sin 3\theta\cos\theta\\
&=-2\sin 3\theta(\cos 4\theta-\cos\theta)\\
&=4\sin 3\theta\sin\frac{5\theta}{2}\sin\frac{3\theta}{2}
\end{aligned}$$

■

5. 如果 $A+B+C=180^\circ$, 试证 $\dfrac{1+\cos A-\cos B+\cos C}{1+\cos A+\cos B-\cos C}=\tan\dfrac{B}{2}\cot\dfrac{C}{2}$。

解:

$$\begin{aligned}
A+B+C&=180^\circ\\
A+B&=180^\circ-C\\
\cos(A+B)&=\cos(180^\circ-C)\\
\cos(A+B)&=-\cos C
\end{aligned}$$

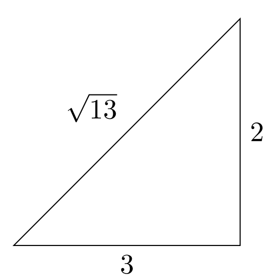
$$\begin{aligned}
\frac{1+\cos A-\cos B+\cos C}{1+\cos A+\cos B-\cos C}&=\frac{2\sin^2\frac{B}{2}+2\cos\frac{A+C}{2}\cos\frac{A-C}{2}}{2\sin^2\frac{C}{2}+2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}\\
&=\frac{2\sin^2\frac{B}{2}+2\cos\left(90^\circ-\frac{B}{2}\right)\cos\left(\frac{A-C}{2}\right)}{2\sin^2\frac{C}{2}+2\cos\left(90^\circ-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)}\\
&=\frac{2\sin^2\frac{B}{2}+2\sin\frac{B}{2}\cos\left(\frac{A-C}{2}\right)}{2\sin^2\frac{C}{2}+2\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sin\dfrac{B}{2}\left(\sin\dfrac{B}{2}+\cos\left(\dfrac{A-C}{2}\right)\right)}{\sin\dfrac{C}{2}\left(\sin\dfrac{C}{2}+\cos\left(\dfrac{A-B}{2}\right)\right)} \\
&= \frac{\sin\dfrac{B}{2}\left(\sin\left(90^\circ-\dfrac{A+C}{2}\right)+\cos\left(\dfrac{A-C}{2}\right)\right)}{\sin\dfrac{C}{2}\left(\sin\left(90^\circ-\dfrac{A+B}{2}\right)+\cos\left(\dfrac{A-B}{2}\right)\right)} \\
&= \frac{\sin\dfrac{B}{2}\left(\cos\left(\dfrac{A+C}{2}\right)+\cos\left(\dfrac{A-C}{2}\right)\right)}{\sin\dfrac{C}{2}\left(\cos\left(\dfrac{A+B}{2}\right)+\cos\left(\dfrac{A-B}{2}\right)\right)} \\
&= \frac{\sin\dfrac{B}{2}\left(2\cos\dfrac{A}{2}\cos\dfrac{C}{2}\right)}{\sin\dfrac{C}{2}\left(2\cos\dfrac{A}{2}\cos\dfrac{B}{2}\right)} \\
&= \frac{\sin\dfrac{B}{2}\cos\dfrac{C}{2}}{\sin\dfrac{C}{2}\cos\dfrac{B}{2}} \\
&= \tan\frac{B}{2}\cot\frac{C}{2}
\end{aligned}
$$

$\blacksquare$

6. 已知 $\sin A - \sin B = \dfrac{1}{2}$ 及 $\cos A - \cos B = -\dfrac{1}{3}$, 试求 $\sin(A+B)$。解:

$$
\begin{aligned}
\sin A - \sin B &= \frac{1}{2} \\
2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) &= \frac{1}{2} \\
\cos A - \cos B &= -\frac{1}{3} \\
-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) &= -\frac{1}{3} \\
\frac{\sin\left(\dfrac{A+B}{2}\right)}{\cos\left(\dfrac{A+B}{2}\right)} &= \frac{2}{3} \\
\tan\left(\frac{A+B}{2}\right) &= \frac{2}{3}
\end{aligned}
$$



$$
\begin{aligned}
\sin(A+B) &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A+B}{2}\right) \\
&= 2\times\frac{2}{\sqrt{13}}\times\frac{3}{\sqrt{13}} \\
&= \frac{12}{13}
\end{aligned}
\qquad \blacksquare
$$

7. 证 $\dfrac{\sin\alpha+\sin 3\alpha+\sin 5\alpha+\sin 7\alpha}{\cos\alpha+\cos 3\alpha+\cos 5\alpha+\cos 7\alpha}=\tan 4\alpha$。

   解:

$$
\begin{aligned}
\frac{\sin\alpha+\sin 3\alpha+\sin 5\alpha+\sin 7\alpha}{\cos\alpha+\cos 3\alpha+\cos 5\alpha+\cos 7\alpha} &= \frac{2\sin 4\alpha\cos 3\alpha+2\sin 4\alpha\cos\alpha}{2\cos 4\alpha\cos 3\alpha+2\cos 4\alpha\cos\alpha} \\
&= \frac{2\sin 4\alpha(\cos 3\alpha+\cos\alpha)}{2\cos 4\alpha(\cos 3\alpha+\cos\alpha)} \\
&= \tan 4\alpha
\end{aligned}
\qquad \blacksquare
$$

8. 试证 $2\sin^2 3\theta-2\sin^2\theta=\cos 2\theta-\cos 6\theta$。

   以 $\theta=\dfrac{\pi}{10}$ 代入上式, 证明 $\sin\dfrac{3\pi}{10}-\sin\dfrac{\pi}{10}=\dfrac{1}{2}$。

   [提示: 用 $\cos x=\sin\left(\dfrac{\pi}{2}-x\right)$]

   解:

$$
\begin{aligned}
2\sin^2\frac{3\pi}{10}-2\sin^2\frac{\pi}{10} &= \cos\frac{\pi}{5}-\cos\frac{3\pi}{5} \\
2\left(\sin^2\frac{3\pi}{10}-\sin^2\frac{\pi}{10}\right) &= 2\sin\frac{2\pi}{5}\sin\frac{\pi}{5} \\
\sin^2\frac{3\pi}{10}-\sin^2\frac{\pi}{10} &= \sin\frac{2\pi}{5}\sin\frac{\pi}{5} \\
\left(\sin\frac{3\pi}{10}-\sin\frac{\pi}{10}\right)\left(\sin\frac{3\pi}{10}+\sin\frac{\pi}{10}\right) &= \sin\frac{2\pi}{5}\sin\frac{\pi}{5} \\
\left(\sin\frac{3\pi}{10}-\sin\frac{\pi}{10}\right)\left(2\sin\frac{\pi}{5}\cos\frac{\pi}{10}\right) &= \sin\frac{2\pi}{5}\sin\frac{\pi}{5} \\
\left(\sin\frac{3\pi}{10}-\sin\frac{\pi}{10}\right)\left(2\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\right) &= \sin\frac{2\pi}{5}\sin\frac{\pi}{5} \\
2\left(\sin\frac{3\pi}{10}-\sin\frac{\pi}{10}\right) &= 1 \\
\sin\frac{3\pi}{10}-\sin\frac{\pi}{10} &= \frac{1}{2}
\end{aligned}
\qquad \blacksquare
$$

9. 在 $\triangle ABC$ 中, 试证明 $\dfrac{a+b}{c}=\dfrac{\sin A+\sin B}{\sin C}$。

   解:

$$
\begin{gathered}
\frac{c}{\sin C}=\frac{a}{\sin A}=\frac{b}{\sin B}=R \\
a=R\sin A \qquad b=R\sin B \qquad c=R\sin C \\
\frac{a+b}{c}=\frac{R\sin A+R\sin B}{R\sin C}=\frac{\sin A+\sin B}{\sin C}
\end{gathered}
\qquad \blacksquare
$$

若在这三角形中, $a+b=2c$ 及 $A-B=90^\circ$,

(a) 试证明 $\sin\frac{C}{2}=\frac{1}{2\sqrt{2}}$;

解:

$$\begin{aligned}
\frac{a+b}{c} &= \frac{\sin A+\sin B}{\sin C}\\
\frac{2c}{c} &= \frac{\sin A+\sin B}{\sin C}\\
2 &= \frac{\sin A+\sin B}{\sin C}\\
2\sin C &= \sin A+\sin B\\
2\sin C &= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}\\
2\sin C &= 2\cos\frac{180^\circ-(A+B)}{2}\cos 45^\circ\\
2\sin C &= \sqrt{2}\cos\frac{C}{2}\\
4\sin\frac{C}{2}\cos\frac{C}{2} &= \sqrt{2}\cos\frac{C}{2}\\
4\sin\frac{C}{2} &= \sqrt{2}\\
\sin\frac{C}{2} &= \frac{\sqrt{2}}{4}=\frac{1}{2\sqrt{2}} \qquad \blacksquare
\end{aligned}$$

(b) 求 A, B 及 C 的值。

解:

$$\begin{aligned}
\sin\frac{C}{2} &= \frac{1}{2\sqrt{2}}\\
\frac{C}{2} &= \arcsin\frac{1}{2\sqrt{2}}\\
C &= 2\arcsin\frac{1}{2\sqrt{2}} \qquad \blacksquare\\
&= 41.41^\circ\\
A+B+C &= 180^\circ\\
A+B &= 180^\circ-41.41^\circ\\
&= 138.59^\circ\\
A-B &= 90^\circ\\
2A &= 228.59^\circ\\
A &= 114.30^\circ \qquad \blacksquare\\
B &= 24.30^\circ \qquad \blacksquare
\end{aligned}$$

10. 若 $\sin 6x = \dfrac{1}{\sin x}$, 证明 $\sin 5x \sin 4x - \sin 3x \sin 2x - \sin 8x \sin x = 1$。

解:

$$
\begin{aligned}
& \sin 5x \sin 4x - \sin 3x \sin 2x - \sin 8x \sin x \\
= & \frac{1}{2}\left[-\cos 9x + \cos x + \cos 5x - \cos x + \cos 9x - \cos 7x\right] \\
= & -\frac{1}{2}(\cos 7x - \cos 5x) \\
= & \sin 6x \sin x \\
= & \frac{1}{\sin x} \cdot \sin x \\
= & 1
\end{aligned}
\qquad \blacksquare
$$

11. 在 $\triangle ABC$ 中, 试证 $4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \cos A + \cos B + \cos C - 1$。

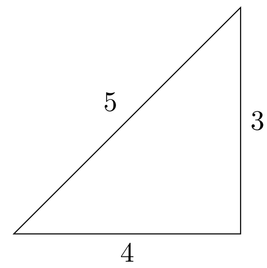
解:

$$
\begin{aligned}
\cos A + \cos B + \cos C &= 2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2} \\
&= 2\cos\frac{180^\circ - C}{2}\cos\frac{A-B}{2} - 2\sin^2\frac{C}{2} + 1 \\
&= 2\sin\frac{C}{2}\cos\frac{A-B}{2} - 2\sin^2\frac{C}{2} + 1 \\
&= 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \sin\frac{C}{2}\right) + 1 \\
&= 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \sin\frac{180^\circ - (A+B)}{2}\right) + 1 \\
&= 2\sin\frac{C}{2}\left(\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\right) + 1 \\
&= 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + 1 \\
4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} &= \cos A + \cos B + \cos C - 1
\end{aligned}
\qquad \blacksquare
$$

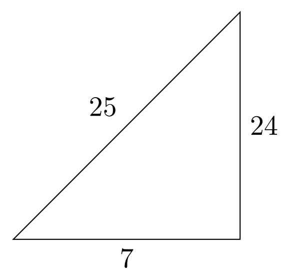
12. 已知 $\sin\alpha + \sin\beta = \dfrac{1}{4}, \cos\alpha + \cos\beta = \dfrac{1}{3}$, 求 $\tan(\alpha+\beta)$ 的值。

解:

$$
\begin{aligned}
\sin\alpha + \sin\beta &= \frac{1}{4} \\
2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) &= \frac{1}{4} \\
\cos\alpha + \cos\beta &= \frac{1}{3} \\
2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) &= \frac{1}{3} \\
\tan\left(\frac{\alpha+\beta}{2}\right) &= \frac{3}{4}
\end{aligned}
$$



$$\begin{aligned}\sin(\alpha+\beta) &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\\ &= 2\times\frac{3}{5}\times\frac{4}{5}\\ &= \frac{24}{25}\end{aligned}$$



$$\tan(\alpha+\beta) = \frac{24}{7}$$ ■

13. 在 $\triangle ABC$ 中, 如果 $\sin A$、$\sin B$ 及 $\sin C$ 成等差数列, 试证 $\cot\frac{A}{2}\cot\frac{C}{2} = 3$。

解:

$$\begin{aligned}\sin B &= \frac{\sin A+\sin C}{2}\\ \sin[180^\circ-(A+C)] &= \frac{\sin A+\sin C}{2}\\ \sin(A+C) &= \frac{\sin A+\sin C}{2}\\ \sin(A+C) &= \sin\frac{A+C}{2}\cos\frac{A-C}{2}\\ 2\sin\frac{A+C}{2}\cos\frac{A+C}{2} &= \sin\frac{A+C}{2}\cos\frac{A-C}{2}\\ 2\cos\frac{A+C}{2} &= \cos\frac{A-C}{2}\\ \cot\frac{A}{2}\cot\frac{C}{2} &= \frac{\cos\frac{A}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{C}{2}} = \frac{\cos\frac{A+C}{2}+\cos\frac{A-C}{2}}{\cos\frac{A-C}{2}-\cos\frac{A+C}{2}}\\ &= \frac{\cos\frac{A+C}{2}+2\cos\frac{A+C}{2}}{2\cos\frac{A+C}{2}-\cos\frac{A+C}{2}}\\ &= \frac{3\cos\frac{A+C}{2}}{\cos\frac{A+C}{2}} = 3\end{aligned}$$ ■

14. 若 $A+B+C=180^\circ$ 且 $A\neq 90^\circ$, 证明 $\dfrac{\sin B}{\sin C}=\dfrac{\sin A\cos B-\sin C}{\sin A\cos C-\sin B}$

解:

$$\begin{aligned}
\frac{\sin A\cos B-\sin C}{\sin A\cos C-\sin B} &= \frac{\sin A\cos B-\sin(180^\circ-(A+B))}{\sin A\cos C-\sin(180^\circ-(A+C))}\\
&= \frac{\sin A\cos B-\sin(A+B)}{\sin A\cos C-\sin(A+C)}\\
&= \frac{\sin A\cos B-\sin A\cos B-\cos A\sin B}{\sin A\cos C-\sin A\cos C-\cos A\sin C}\\
&= \frac{-\cos A\sin B}{-\cos A\sin C}\\
&= \frac{\sin B}{\sin C}
\end{aligned}$$

■