Exercise 5c

1. Find the standard form of the equation of the ellipse that satisfies the given conditions:

   (a) Passes through point $P(-2\sqrt{2}, 0)$, $Q(0, \sqrt{5})$;

   Sol.

   Point P is on the x-axis, while point Q is on the y-axis.

   $|OP| = 2\sqrt{2}$, $|OQ| = \sqrt{5}$,

   $\because |OP| > |OQ|$,

   $\therefore$ The major axis is along the x-axis.

   $\therefore$ The equation of the ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

   Substituting the coordinates of P and Q into the equation, we get

$$\frac{(-2\sqrt{2})^2}{a^2} + \frac{0^2}{b^2} = 1$$
$$\frac{0^2}{a^2} + \frac{(\sqrt{5})^2}{b^2} = 1$$

   Simplifying, we get

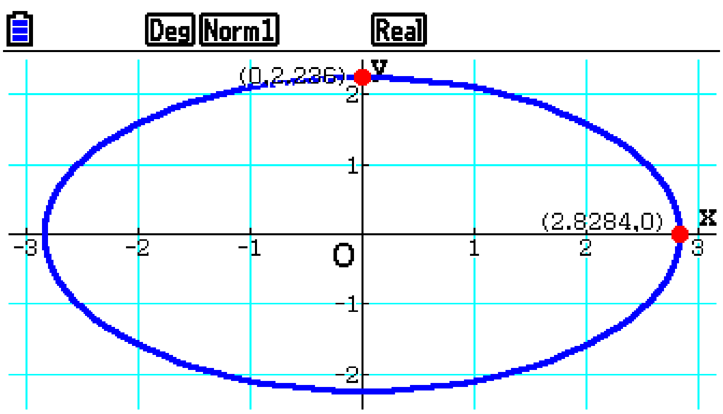
$$\frac{8}{a^2} = 1$$
$$\frac{5}{b^2} = 1$$

   Solving for a and b, we get

$$a^2 = 8$$
$$b^2 = 5$$

   $\therefore$ The standard form of the equation of the ellipse is

$$\frac{x^2}{8} + \frac{y^2}{5} = 1 \qquad \square$$



(b) Coordinates of its foci are $(-2\sqrt{3}, 0)$ and $(2\sqrt{3}, 0)$, and it passes through the point $P(\sqrt{5}, -\sqrt{6})$;

The foci are on the x-axis, therefore let the equation of the ellipse be of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

From the coordinates of the foci, we have $ae = 2\sqrt{3}$, $a^2e^2 = 12$,

$$\begin{aligned} \therefore\ b^2 &= a^2 - a^2e^2 \\ &= a^2 - 12 \ \cdots\ (1) \end{aligned}$$

Substituting the coordinates of P into the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

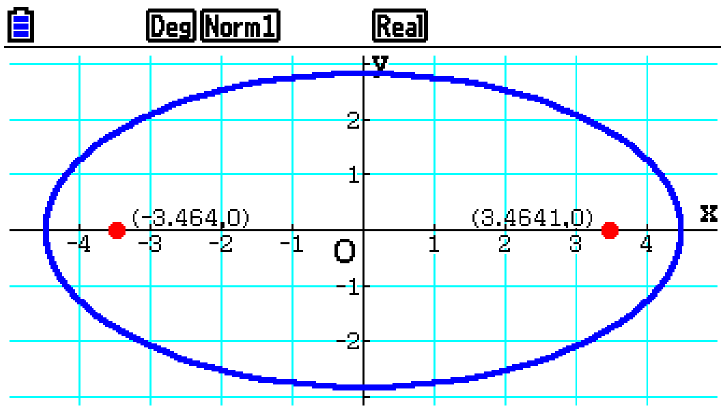
$$\begin{aligned} \frac{(\sqrt{5})^2}{a^2} + \frac{(-\sqrt{6})^2}{b^2} &= 1 \\ \frac{5}{a^2} + \frac{6}{b^2} &= 1 \end{aligned}$$

Substituting (1) into the equation, we get

$$\begin{aligned} \frac{5}{a^2} + \frac{6}{a^2 - 12} &= 1 \\ 5(a^2 - 12) + 6a^2 &= a^2(a^2 - 12) \\ 5a^2 - 60 + 6a^2 &= a^4 - 12a^2 \\ a^4 - 23a^2 + 60 &= 0 \\ (a^2 - 20)(a^2 - 3) &= 0 \\ a^2 = 20 \text{ or } a^2 &= 3 \text{ (rejected, } b > 0) \end{aligned}$$

When $a^2 = 20$, $b^2 = 20 - 12 = 8$. $\therefore$ The standard form of the equation of the ellipse is

$$\frac{x^2}{20} + \frac{y^2}{8} = 1 \qquad \square$$



(c) Equations of its directrices are $y \pm \dfrac{25}{3} = 0$, and it passes through the point $(4, 0)$;

Sol.

The directrices are perpendicular to the y-axis, therefore let the equation of the ellipse be of the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

From the equation of the directrices, we have $\dfrac{a}{e} = \dfrac{25}{3}$, $\dfrac{a^2}{e^2} = \dfrac{625}{9}$, $e^2 = \dfrac{9}{625}a^2$,

$$\begin{aligned}\therefore\ b^2 &= a^2 - a^2e^2 \\ &= a^2 - \frac{9}{625}a^4 \ \cdots\ (1)\end{aligned}$$

Substituting the point $(4, 0)$ into the equation $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, we get

$$\begin{aligned}\frac{(4)^2}{b^2} + \frac{0^2}{a^2} &= 1 \\ \frac{16}{b^2} &= 1\end{aligned}$$

Substituting (1) into the equation, we get

$$\begin{aligned}\frac{16}{a^2 - \frac{9}{625}a^4} &= 1 \\ 16 &= a^2 - \frac{9}{625}a^4 \\ 9a^4 - 625a^2 + 10000 &= 0 \\ (9a^2 - 400)(a^2 - 25) &= 0 \\ a^2 = 25 \text{ or } a^2 &= \frac{400}{9}\end{aligned}$$

When $a^2 = 25$, $b^2 = 25 - \dfrac{9}{625}(25)^2 = 25 - 9 = 16$.

When $a^2 = \dfrac{400}{9}$, $b^2 = \dfrac{400}{9} - \dfrac{9}{625}\left(\dfrac{400}{9}\right)^2 = 16$.

$\therefore$ The standard form of the equations of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{25} = 1 \qquad \text{or} \qquad \frac{x^2}{16} + \frac{9y^2}{400} = 1 \qquad \square$$