

# Mathematics

## *Senior 3 Part I*

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# **Introduction**

**Why this book?**

**Disclaimer**

**Acknowledgements**

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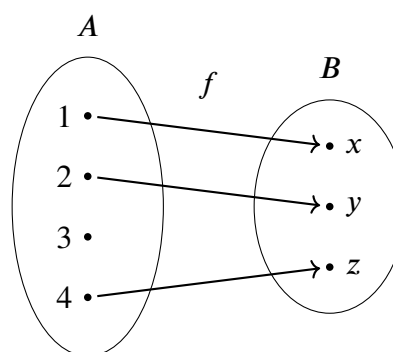
# Chapter 22

## Function

### 22.1 Definition of a Function

#### Mapping, Preimage and Image

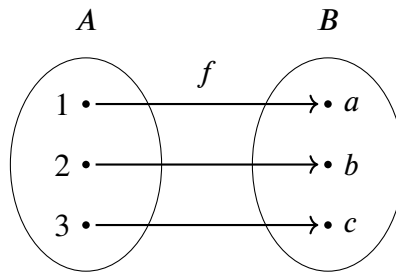
For two non-empty sets  $A$  and  $B$ , If an element  $a$  inside set  $A$  has a corresponding element  $b$  inside set  $B$ , denoted as  $a \rightarrow b$ , then we say that  $a$  is mapped to  $b$  or  $a$  and  $b$  are paired. The mapping between two sets is normally denoted as  $f, g, h$ , etc. The mapping shown in the diagram below can be denoted as  $f : 1 \rightarrow x, 2 \rightarrow y, 4 \rightarrow z$ .



Let  $f : A \rightarrow B$  is a mapping,  $a$  is an element in  $A$ . If  $a$  is mapped to  $b$  under the mapping  $f$ , then  $b$  is said to be the image of  $a$  under the mapping  $f$ , denoted as  $b = f(a)$ ;  $a$  is said to be the preimage of  $b$  under the mapping  $f$ . In the diagram above, under the mapping  $f$ , the image of 1, 2, and 4 are  $x$ ,  $y$ , and  $z$  respectively, while the preimage of  $x$ ,  $y$ , and  $z$  are 1, 2, and 4 respectively.

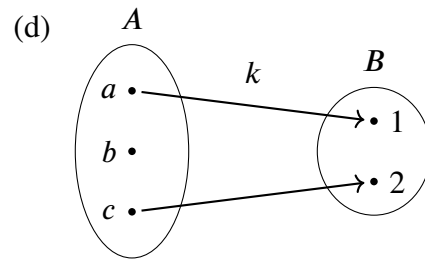
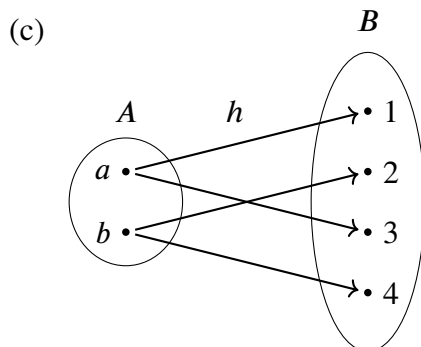
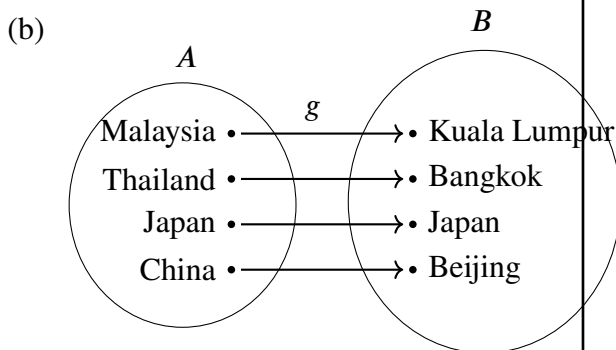
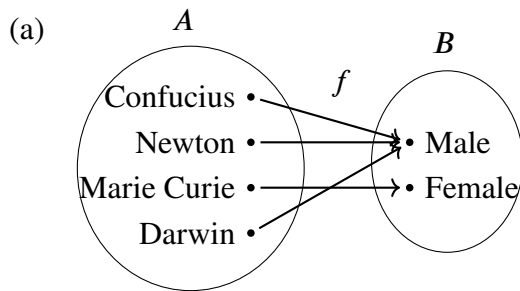
Let  $A$  and  $B$  be two non-empty sets,  $f$  is a mapping from  $A$  to  $B$  such that for all elements in  $A$ , there is a unique corresponding element in  $B$ , then  $f$  is a function or a mapping from  $A$  to  $B$ , denoted as  $f : A \rightarrow B$ .

The mapping shown in the diagram below is a function.

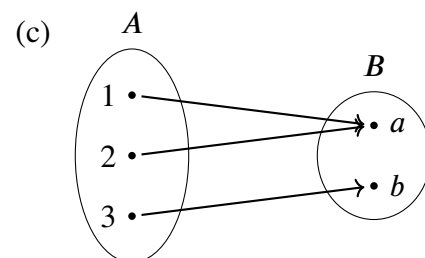
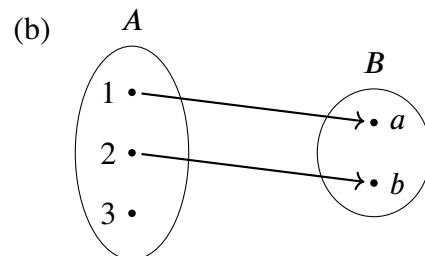
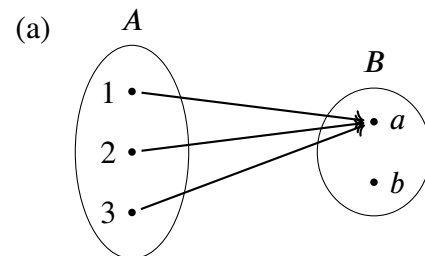


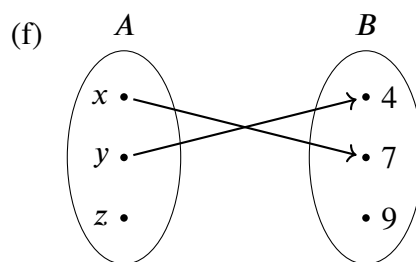
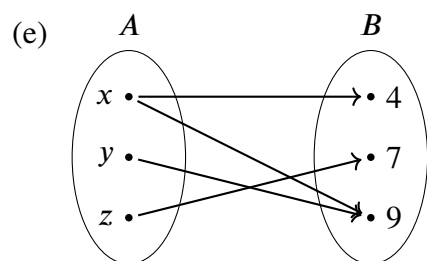
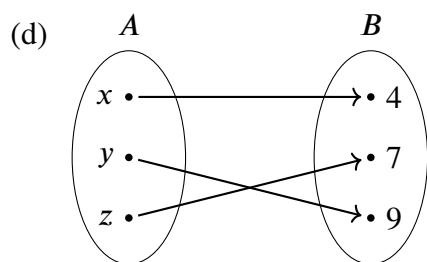
## Practice 1

1. For the following mappings, list the image of each element in  $A$  and the preimage of each element in  $B$ , and determine whether the mapping is a function or not:



2. Given a mapping  $g : x \rightarrow x + 3, x \in \{-2, -1, 0, 1, 2, 3\}$ , find the image of each  $x$ .
3. Determine whether the following mappings are functions.



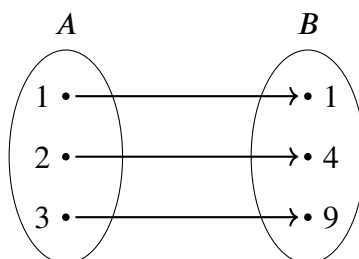


The function  $f : A \rightarrow B$  can be written as  $y = f(x)$ ,  $x$  is the element of  $A$  and  $y$  is the element of  $B$ . When  $x$  changes,  $y$  changes as well.  $x$  is called independent variable, while  $y$  is called dependent variable. Keep in mind that  $f(x)$  is NOT the product of  $f$  and  $x$ .

## Representation of Functions

Generally speaking, there are a few ways to represent a function:

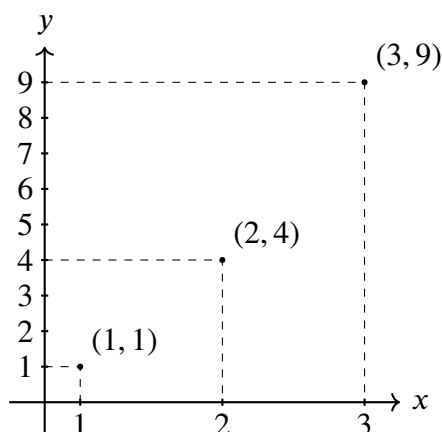
1. **Narrative Form:** express the function of two sets in words. For example, Let  $A = \{1, 2, 3\}$  and  $B = \{1, 4, 9\}$ ,  $f$  is a function from  $A$  to  $B$ , its definition is that for any element  $x$  in  $A$ , its corresponding element is  $x^2$  in  $B$ .
2. **Arrow Method:** draw an arrow to connect the preimage and image of a function such that the preimage is corresponding to the image. To express the example above, we express it as  $f : 1 \rightarrow 1, 2 \rightarrow 4, 3 \rightarrow 9$ .
3. **Analytical Method:** express the function in the form of mathematical expression to represent the relationship between the independent variable and the dependent variable. For example,  $f(x) = x^2, x \in A$ .
4. **Venn Diagram:** draw arrows between the venn diagram of two sets to represent the function, as shown below:



5. **Table Method:** express the function in the form of table, showing the relationship of the chosen value between independent variable  $x$  and the value of its corresponding dependent variable  $y$ , as shown below:

$x$	1	2	3
$y$	1	4	9

6. **Graphical Method:** draw a graph to represent the function of the two variables, as shown below:



## Practice 2

Express the following functions using analytical method, venn diagram, table method and graphical method.

- (a)  $f$  mapping each integers from  $-3$  to  $3$  to its

squares plus 4.

- (b)  $g$  mapping each natural numbers from 1 to 4 to its cubes.

## Exercise 22.1

1. Express the mapping from set  $A$  to set  $B$ , and determine which of the following mappings are functions.

	Set $A$	Set $B$	Mapping
(a)	$\{0, 3, 9, 12\}$	$\{0, 1, 2, 3\}$	Divide by 3
(b)	$\{-2, -1, 0, 1, 2\}$	$\{0, 1, 4, 9, 16\}$	Power of 4
(c)	$\{-2, -1, 0, 1, 2\}$	$\{0, 1, 4\}$	Square
(d)	$\{30^\circ, 45^\circ, 60^\circ\}$	$\left\{\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right\}$	Sine
(e)	$\{-1, 0, 1, 2\}$	$\{-1, 0, 1\}$	Cube

2. Let function  $f(x) = 3x^2 + 1$ .



(a) Find the image of the following elements:

- i. -3
- ii. -2
- iii. 0
- iv. 2
- v. 5

(b) Find the preimage of the following elements:

- i. 13
- ii. 28
- iii. 1
- iv. 0
- v. 4

3. Let function  $g(x) = 5x - 2$ . Find:

- (a)  $g(-2)$
- (b)  $g(-1)$
- (c)  $g(0)$

4. Let function  $f(x) = \begin{cases} 2x, & x \leq -1 \\ x - 1, & -1 \leq x < 3 \\ 4x + 2, & x \geq 3 \end{cases}$ ,

find

- (a)  $f(-5)$

(b)  $f(-2)$

(c)  $f(0)$

(d)  $f(2)$

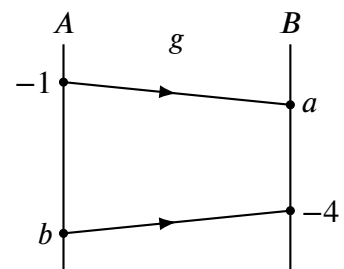
(e)  $f(10)$

5. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^4$ . Find the image of  $-1, 0, 1$ , and  $2$  under  $f$ .

6. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^4$ . Find the preimage of  $0, 1$ , and  $4$  under  $f$ .

In  $\mathbb{R}$ , which element does not have a preimage?

7. In the diagram below, given that function  $g : A \rightarrow B$  is defined as  $g : x \rightarrow 2x - 8$ . Find the value of  $a$  and  $b$ .

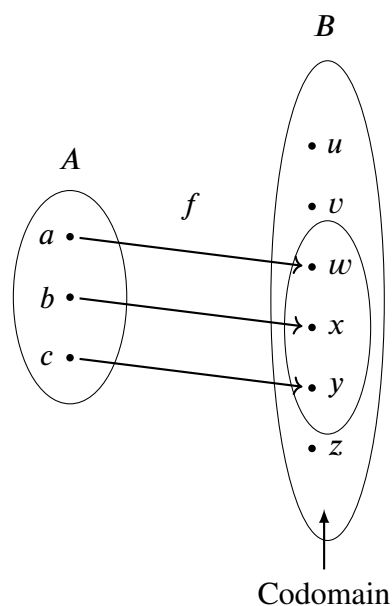


8. Using narrative form, arrow method, venn diagram, table method and graphical method, express the function  $f(x) = 2x$ ,  $x \in \{-2, -1, 0, 1, 2\}$ .

## 22.2 Domain and Range

Let  $f$  is a function from set  $A$  to set  $B$ , then set  $A$  is called the domain of  $f$ , denoted by  $D_f$ ; set  $B$  is called the codomain of  $f$ ; the set of the images of all elements of  $A$  under  $f$  is called the range of  $f$ , denoted by  $R_f$ .

If the domain  $A$  and range  $B$  of function  $f : A \rightarrow B$  are both subsets of real number set  $\mathbb{R}$ , then this function is called real valued function / real function. This book primarily discusses about real valued functions. When the domain of a real function is not mentioned and only the mapping rule is given, its domain is assumed to be the set of all real numbers that yield defined values  $f(x)$ . After the domain and the mapping rule are determined, the range of a function will then be determined.



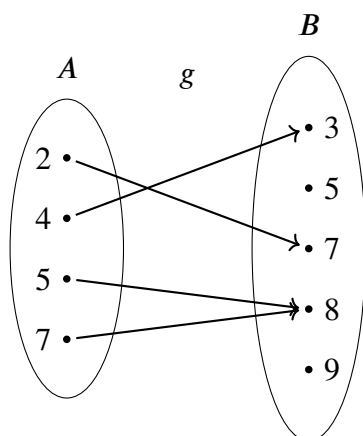
### Interval Notation

Let  $a$  and  $b$  be two real number,  $a < b$ .

Intervals	Set Notations
$(a, b)$	$\{x x \in \mathbb{R}, a < x < b\}$
$[a, b)$	$\{x x \in \mathbb{R}, a \leq x < b\}$
$(a, b]$	$\{x x \in \mathbb{R}, a < x \leq b\}$
$[a, b]$	$\{x x \in \mathbb{R}, a \leq x \leq b\}$
$(a, \infty)$	$\{x x \in \mathbb{R}, x > a\}$
$[a, \infty)$	$\{x x \in \mathbb{R}, x \geq a\}$
$(-\infty, a)$	$\{x x \in \mathbb{R}, x < a\}$
$(-\infty, a]$	$\{x x \in \mathbb{R}, x \leq a\}$

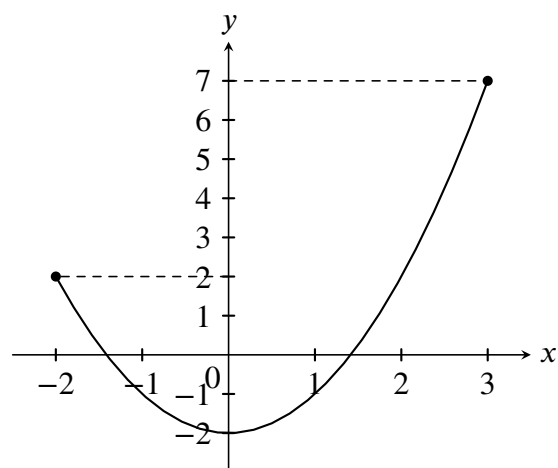
### Practice 3

1. Let  $A = \{2, 4, 5, 7\}$  and  $B = \{3, 5, 7, 8, 9\}$ , the definition of function  $g$  is given by the diagram below. Find the domain, codomain and range of function  $g$ .



2. Let  $A = \{-2, -1, 0, 1, 2\}$ , function  $f : A \rightarrow \mathbb{R}$  is defined by  $f(x) = x^2 - 1$ . Find the domain and range of  $f$ .
3. The curve in the diagram below represents the function  $y = f(x)$ ,  $-2 \leq x \leq 3$ . Find

the domain and range of  $f$ .

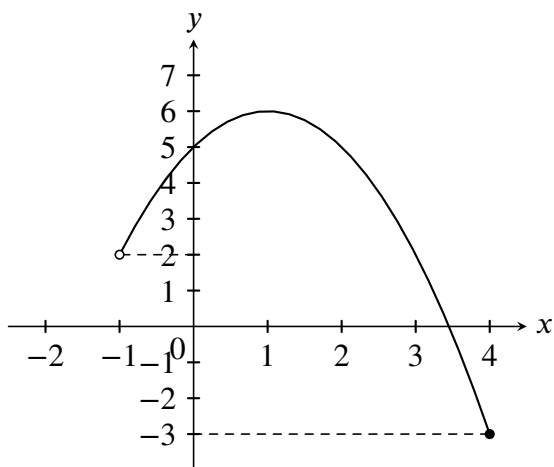


4. Find the domain and range of the following functions:
- $f(x) = -4x + 5$
  - $g(x) = x^2 - 1$
  - $h(x) = \frac{1}{4x + 7}$
  - $k(x) = \sqrt{6 - x}$

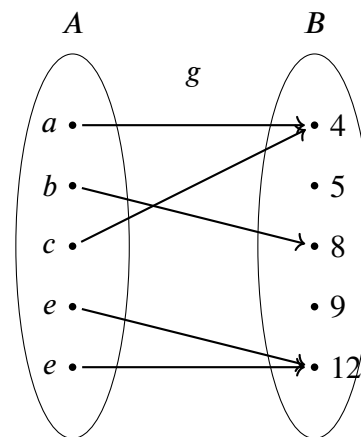
### Exercise 22.2

1. Let  $X = \{a, b, c, d\}$  and  $Y = \{-1, 2, 9, 11\}$ , function  $f : X \rightarrow Y$  is defined by  $f(a) = 2$ ,  $f(b) = -1$ ,  $f(c) = 2$ ,  $f(d) = 9$ . Find the domain and range of the  $f$ .

2. The curve in the diagram below represents the function  $y = f(x)$ ,  $-1 < x \leq 4$ . Find the domain and range of  $f$ .



3. Let  $A = \{a, b, c, d, e\}$  and  $B = \{4, 5, 8, 9, 12\}$ , the definition of function  $g : A \rightarrow B$  is given by the diagram below. Find the domain, codomain and range of function  $g$ .



4. Let  $A = \{-1, 0, 1, 2\}$ , function  $f : A \rightarrow \mathbb{R}$  is defined by  $f(x) = 3x^2 - 2$ , find the domain and range of  $f$ .
5. Let  $A = \{-1, 0, 2, 5, 11\}$ , function  $f : A \rightarrow \mathbb{R}$  is defined by  $f(x) = x^2 - x - 2$ , find the domain and range of  $f$ .
6. Find the domain and range of the following functions:

- (a)  $f(x) = x^3$
- (b)  $g(x) = \sqrt{1 - x^2}$
- (c)  $h(x) = \frac{1}{2x + 3}$
- (d)  $k(x) = x^2 - 2x + 4$

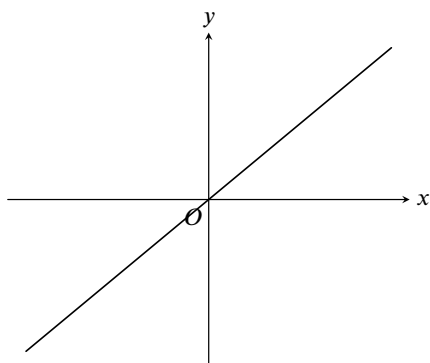
## 22.3 Graphs of Functions and Their Transformations

### Graphs of Simple Functions

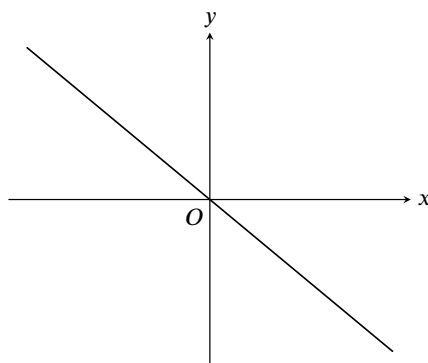
On a Cartesian plane, the graphs formed by all the point  $(x, y)$  that satisfied the equation  $y = f(x)$  are called graphs of function  $f$ . Below are some examples of graphs of simple functions.

Note that any line that is parallel to the  $y$ -axis intersects the graph of a function at most once.

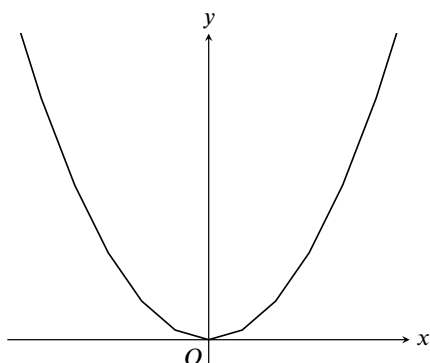
(a)  $y = x$



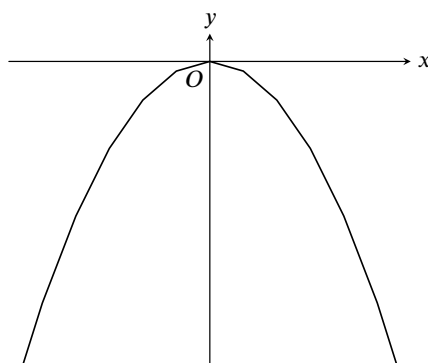
(b)  $y = -x$



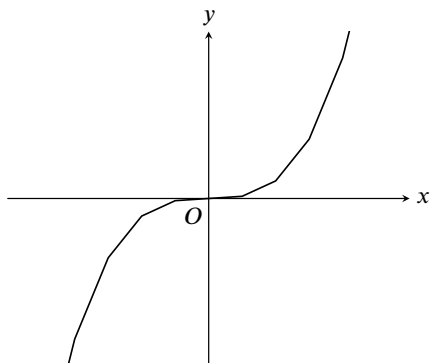
(c)  $y = x^2$



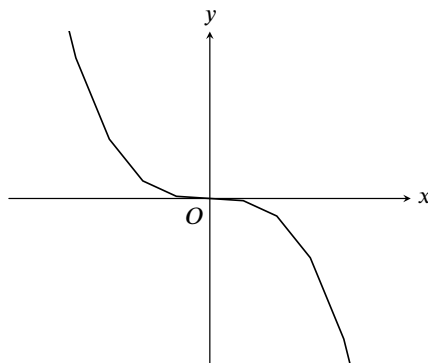
(d)  $y = x^2$



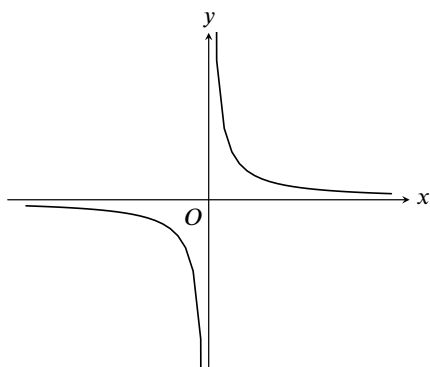
(e)  $y = x^3$



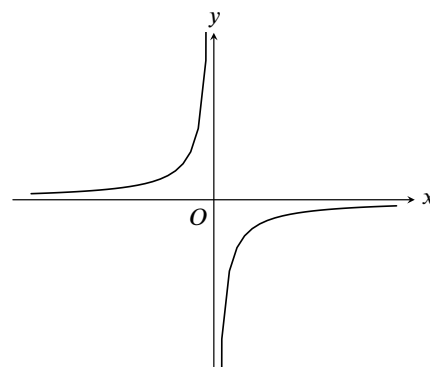
(f)  $y = -x^3$



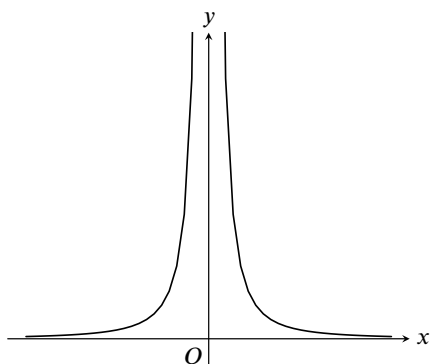
(g)  $y = \frac{1}{x}$



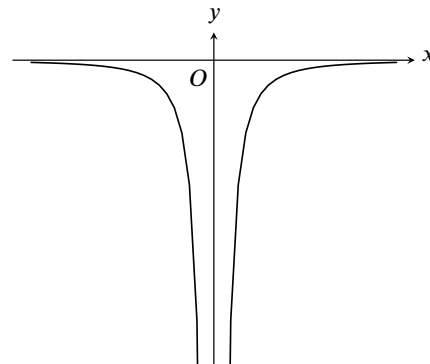
(h)  $y = -\frac{1}{x}$



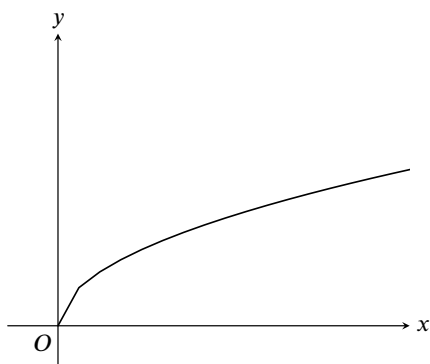
(i)  $y = \frac{1}{x^2}$



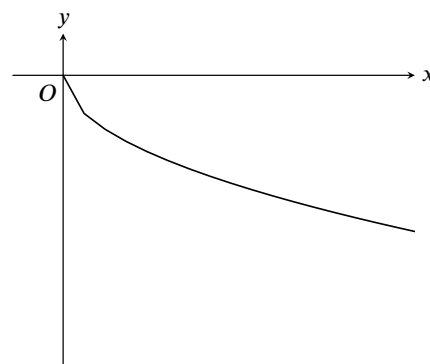
(j)  $y = -\frac{1}{x^2}$



(k)  $y = \sqrt{x}$



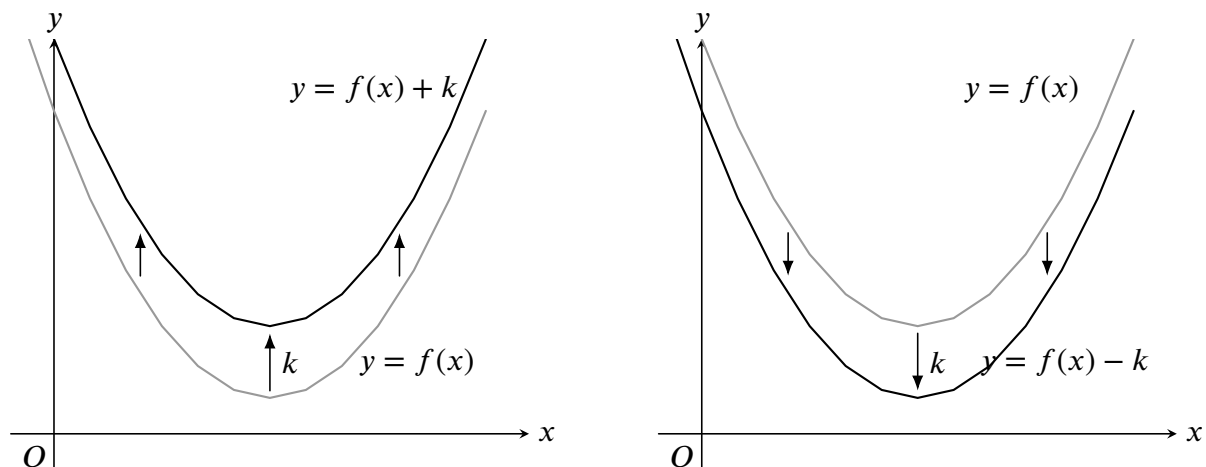
(l)  $y = -\sqrt{x}$



## Transformations of Graphs

- If  $k > 0$ , translate the graph of  $y = f(x)$  vertically upwards by  $k$  units, the graph of  $y = f(x) + k$  is obtained.
- If  $k > 0$ , translate the graph of  $y = f(x)$  vertically downwards by  $k$  units, the graph of  $y = f(x) - k$

is obtained.



- If  $h > 0$ , translate the graph of  $y = f(x)$  horizontally to the right by  $h$  units, the graph of  $y = f(x+h)$  is obtained.
- If  $h > 0$ , translate the graph of  $y = f(x)$  horizontally to the left by  $h$  units, the graph of  $y = f(x-h)$  is obtained.

- If  $k > 0$ , reflect the graph of  $y = f(x)$  about the  $x$ -axis, the graph of  $y = -f(x)$  is obtained.
- If  $k > 0$ , reflect the graph of  $y = f(x)$  about the  $y$ -axis, the graph of  $y = f(-x)$  is obtained.

If  $a > 0$ , zooming (when  $a > 1$ ) or shrinking (when  $0 < a < 1$ ) the graph of  $y = f(x)$  by a factor of  $a$  in the  $y$ -direction, the graph of  $y = af(x)$  is obtained.

If  $a > 0$ , shrinking (when  $a > 1$ ) or zooming (when  $0 < a < 1$ ) the graph of  $y = f(x)$  by a factor of  $\frac{1}{a}$  in the  $x$ -direction, the graph of  $y = f(ax)$  is obtained.

## Practice 4

Find the line of symmetry and vertex of the following parabola, and sketch its graph. (Question 1 to 2):

1.  $y = 2x^2 + 8x + 11$

2.  $y = -3x^2 + 18x - 7$

Sketch the graph of the following functions.

(Question 3 to 4):

$$3. y = \frac{4}{(x+2)^2}$$

$$4. y = \sqrt{x-1} + 3$$

### Exercise 22.3

Find the line of symmetry and vertex of the following parabola, and sketch its graph.

$$1. y = 2x^2 + 4x + 5$$

$$2. y = -3x^2 + 12x - 4$$

$$3. y = 4x^2 - 20x + 19$$

$$4. y = -3x^2 - 6x - 4$$

Sketch the graph of the following functions.

$$5. y = (x+2)^3 - 5$$

$$6. y = \sqrt{x-5}$$

$$7. y = \frac{1}{(x+2)^2}$$

$$8. y = -\frac{1}{2(x-1)^2}$$

$$9. y = 3\sqrt{x+1} - 4$$

$$10. y = \frac{4}{2x+3}$$

$$11. y = \begin{cases} 4x+9, & x \leq 0 \\ 9-2x, & x > 0 \end{cases}$$

$$12. y = \begin{cases} x, & x < -1 \\ \sqrt{x+1}, & x \geq -1 \end{cases}$$

13. Sketch the graph for the function  $f(x) = x^2 - 6x + 12$ ,  $-2 \leq x \leq 8$ , and find its domain and range.

14. Sketch the graph for the function  $g(x) = -x^2 - 4x - 7$ ,  $-2 \leq x \leq 5$ , and find its domain and range.

15. Sketch the graph for the function  $f(x) = -x^2 + 2x + 10$ , and find its domain and range.

16. Sketch the graph of the function  $y = \sqrt{x}$ , and transform it according the following steps. Sketch the graph of each function after each step on the same diagram, and write down the corresponding function.

Step 1: Translate 4 units to the left;

Step 2: Scale up by a factor of 2 in the  $x$ -direction;

Step 3: Reflect about the  $y$ -axis;

Step 4: Translate 3 units downwards.

Step 5: Scale down by half in the  $y$ -direction.

## 22.4 Composite Functions

Let  $A$ ,  $B$ , and  $C$  be three non-empty sets,  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be two functions, an element  $x$  in set  $A$  is mapped to an element  $f(x)$  in set  $B$  by function  $f$ , and  $f(x)$  is mapped to an element  $g(f(x))$  in



set  $C$  by function  $g$ . In other words,  $x$  in set  $A$  is mapped to an element  $g(f(x))$  in  $C$  after two mappings. That is:

$$x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x))$$

The combination of these two mappings are a function from set  $A$  to set  $C$ , this function is called the *composite function* of  $f$  and  $g$ , denoted by  $g \circ f$ . When defining the composite function  $g \circ f$ , the range of  $f$  must be a subset of the domain of  $g$ , that is,  $R_f \subseteq D_g$ .

Note that  $D_{g \circ f} = D_f$ ,  $R_{g \circ f} \subseteq R_g$ .

$$\forall n \in \mathbb{N}, f^{n+1} = f \circ f^n.$$

Generally speaking,  $g \circ f \neq f \circ g$ .

If  $f \circ (g \circ h)$  is defined, then  $(f \circ g) \circ h$  is also defined, and  $f \circ (g \circ h) = (f \circ g) \circ h$ . Therefore, we can write  $f \circ g \circ h$  without ambiguity.

## Practice 5

- |   |   |
|---|---|
| <p>1. Let <math>f : \mathbb{R} \rightarrow \mathbb{R}</math>, <math>f(x) = 2x + 3</math> and <math>g : \mathbb{R} \rightarrow \mathbb{R}</math>, <math>g(x) = 5 - x</math>. Find <math>(g \circ f)(x)</math> and <math>(f \circ g)(x)</math>.</p> <p>2. Let <math>f : \mathbb{R} \rightarrow \mathbb{R}</math>, <math>f(x) = x^2 - 2x + 3</math> and <math>g : \mathbb{R} \rightarrow \mathbb{R}</math>, <math>g(x) = 3x - 4</math>. Find</p> <p>(a) <math>g \circ f</math> and <math>f \circ g</math>;</p> | <p>(b) <math>g(f(2))</math>, <math>f(g(2))</math>, <math>(g \circ f)(2)</math>, and <math>(f \circ g)(2)</math>.</p> <p>3. Let <math>f : \mathbb{R} \rightarrow \mathbb{R}</math>, <math>f(x) = 4 - x^2</math> and <math>g : \{x   x \leq 4\} \rightarrow \mathbb{R}</math>, <math>g(x) = \sqrt{4 - x}</math>. Prove the existence of <math>f \circ g</math> and <math>g \circ f</math> respectively.</p> |
|---|---|

## 22.5 One to One Function, Onto Function and One to One Onto Function

### One to One Function

Let  $f : A \rightarrow B$  be a function, if there is at most one preimage in set  $A$  for each element in set  $B$ , then  $f$  is called a *one to one function*.

As shown in the diagram above, each element in the codomain  $B$  of the function  $f : A \rightarrow B$  has at most one preimage in the domain  $A$  of the function, thus  $f$  is a one to one function; while the element  $b_2$  in the codomain  $B$  of the function  $g : A \rightarrow B$  has two preimages  $a_2$  and  $a_3$ , thus  $g$  is not a one to one function.

A function  $y = f(x)$  is a one to one function, if and only if any line parallel to the  $x$ -axis intersects the graph of the function at most once.

## Onto Function

If each element in the codomain  $B$  of the function  $f : A \rightarrow B$  has at least one preimage under the function  $f$ , then  $f$  is said to be an *onto function*.

As shown in the diagram above, each element in the codomain  $B$  of the function  $f : A \rightarrow B$  has at least one preimage under the function  $f$ , therefore  $f$  is an onto function; while the element  $b_3$  in the codomain  $B$  of the function  $g : A \rightarrow B$  has no preimage under the function  $g$ , therefore  $g$  is not an onto function.

## One to One Onto Function

If a function is both a one to one function and an onto function, then it is a *one to one onto function*, as shown in the diagram above.

## Practice 7

Determine whether the following functions are one to one functions or onto functions.

## Exercise 22.5

1. Let  $A = \{1, 2, 3\}$ ,  $f : A \rightarrow A$  is defined by  $f : 1 \rightarrow 1, 2 \rightarrow 3, 3 \rightarrow 2$ . Determine if  $f$  is a one to one function or an onto function.
2. Let  $A = \{a, b, c, d\}$  and  $B = \{x, y, z\}$ ,  $f : A \rightarrow B$  is defined by  $f : a \rightarrow y, b \rightarrow x, c \rightarrow z, d \rightarrow y$ . Determine if  $f$  is a one to one function or an onto function.
3. Let the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $g(x) = 2x + 1$ . Determine if  $g$  is a one to one function or an onto function.
4. Let the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$f(x) = 2x^3 - 3$ . Determine if  $f$  is a one to one function or an onto function.

5. Let the function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be defined by  $f(x) = \frac{1}{x}$ . Determine if  $f$  is a one to one function or an onto function.
6. Let the function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be defined by  $f(x) = \sqrt{x}$ . Determine if  $f$  is a one to one function or an onto function.
7. Determine whether the following functions are one to one, onto or one to one onto functions.

- (a)  $A = \{a, b, c\}$ ,  $B = \{x, y, z\}$ ,  $f : A \rightarrow B$ ,  $f : a \rightarrow x, b \rightarrow x, c \rightarrow y$
- (b)  $A = \{a, b, c\}$ ,  $B = \{x, y, z\}$ ,  $g : A \rightarrow B$ ,  $g : a \rightarrow x, b \rightarrow y, c \rightarrow z$
- (c)  $A = \{a, b, c\}$ ,  $B = \{x, y\}$ ,  $h : A \rightarrow B$ ,  $h : a \rightarrow x, b \rightarrow y, c \rightarrow y$
- (d)  $A = \{a, b, c\}$ ,  $B = \{x, y\}$ ,  $k : A \rightarrow B$ ,  $k : a \rightarrow x, a \rightarrow y, c \rightarrow y$

- (e)  $A = \{a, b, c\}$ ,  $B = \{x, y\}$ ,  $f : A \rightarrow B$ ,  $f : a \rightarrow x, a \rightarrow y, b \rightarrow x, c \rightarrow y$
- (f)  $A = \{a, b, c, d\}$ ,  $B = \{u, v, x, y, z\}$ ,  $g : A \rightarrow B$ ,  $g : a \rightarrow u, b \rightarrow v, c \rightarrow x, d \rightarrow y$

8. Determine whether the following functions mapping  $A$  to  $B$  are one to one functions or onto functions.

## 22.6 Inverse Functions

If  $f : A \rightarrow B$  is a one to one onto function, then there exist a function  $g : B \rightarrow A$ , such that if  $y = f(x)$ , then  $g(y) = x$ . The function  $g$  is called the *inverse function* of  $f$ , and is denoted by  $f^{-1}$ .

from the diagram above, we can conclude the following:

$$x \xrightarrow{f} y = f(x) \xrightarrow{f^{-1}} f^{-1}(f(x)) = f^{-1}(y)$$

or

$$y \xrightarrow{f^{-1}} x = f^{-1}(y) \xrightarrow{f} f(f^{-1}(y)) = f(x)$$

If both  $(g \circ f)^{-1}$  and  $f^{-1} \circ g^{-1}$  exist, then  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ .

### Practice 8

#### Exercise 22.5

- Find the inverse function of the following functions:

(a)  $f : x \rightarrow 7x - 3$

(b)  $g : x \rightarrow \frac{1}{2}x + 9$

(c)  $h : x \rightarrow \frac{x+1}{x-8}, x \neq 8$

(d)  $k : x \rightarrow \frac{x-1}{2x}, x \neq 0$

- Given the function  $f : x \rightarrow 2x + 1$  and

$$g : x \rightarrow \frac{1}{x-4}, x \neq 4. \text{ Find:}$$

(a)  $f^{-1}$

(b)  $g^{-1}$

(c)  $f^{-1} \circ g^{-1}$

(d)  $g^{-1} \circ f^{-1}$

(e)  $(f \circ g)^{-1}$

(f)  $(g \circ f)^{-1}$

### Graph of Inverse Functions

If  $f$  is a one to one function, then the graph of  $f^{-1}$  is the reflection of the graph of  $f$  about the line  $y = x$ .

### Practice 9

Given the function  $g : \mathbb{R}^+ \cup 0 \rightarrow \mathbb{R}^+ \cup 0, g : x \rightarrow x^2$ . On the same set of axes, draw the graph of the function  $g$  and its inverse function  $g^{-1}$ .

## Exercise 22.6

1. Find the inverse function of the following functions:

(a)  $f : x \rightarrow 2x - 7$

(b)  $g : x \rightarrow \frac{1}{x-2}, x \neq 2$

(c)  $h : x \rightarrow \frac{2x-5}{x-2}, x \neq 2$

(d)  $k : x \rightarrow \frac{3x}{x-4}, x \neq 4$

2. Given that  $f : x \rightarrow \frac{160}{ax+b}, f(5) = 8$  and  $f(9) = 10$ . Find

(a) the values of  $a$  and  $b$ ;

(b)  $f^{-1}(16)$ .

3. Given that  $f : x \rightarrow \frac{a}{x+b}, f(3) = -1$ , and  $f(-9) = 3$ . Find

(a) the values of  $a$  and  $b$ ;

(b) the value of  $x$  such that  $f(x) = f^{-1}(x)$ .

4. Given the function  $g : x \rightarrow \frac{6}{x} - 3, x \neq 0$ . Find

(a)  $g^{-1}$ ;

(b) the value of  $x$  such that  $g^{-1}(x) = x - 2$ .

5. Given the function  $f : x \rightarrow ax + b$  and  $f^2 : x \rightarrow 4x + 12$ . If  $a > 0$ , find

(a) the values of  $a$  and  $b$ ;

(b)  $f^{-1}(3)$ .

6. Given the function  $f : x \rightarrow 3x - 2$  and  $g : x \rightarrow \frac{x}{x+4}, x \neq -4$ . Find

(a)  $f^{-1}$

(b)  $g^{-1}$

(c)  $f^{-1} \circ g^{-1}$

(d)  $g^{-1} \circ f^{-1}$

(e)  $(f \circ g)^{-1}$

(f)  $(g \circ f)^{-1}$

7. Given the function  $f : \mathbb{R} \setminus \{2\} \rightarrow \mathbb{R} \setminus \{0\}$ ,  $f : x \rightarrow \frac{1}{x-2}$ .

(a) Find  $f^{-1}$ .

(b) On the same set of axes, draw the graph of  $f$  and  $f^{-1}$ .

8. Given the function  $f : x \rightarrow 2\sqrt{x+4}, x \geq -4$ ,

(a) Find  $f^{-1}$ .

(b) On the same set of axes, draw the graph of  $f$  and  $f^{-1}$ .

## Revision Exercise 22

1. Determine whether the following mappings from set  $A = \{1, 2, 3, 4\}$  to set  $B = \{a, b, c, d\}$  are functions or not.

(a)  $1 \rightarrow a, 2 \rightarrow c, 4 \rightarrow b$

(b)  $1 \rightarrow a, 2 \rightarrow d, 3 \rightarrow b, 4 \rightarrow a$

(c)  $1 \rightarrow c, 2 \rightarrow c, 3 \rightarrow b, 4 \rightarrow b$

(d)  $1 \rightarrow a, 2 \rightarrow c, 2 \rightarrow b, 4 \rightarrow d$

(e)  $1 \rightarrow c, 2 \rightarrow b, 3 \rightarrow d, 4 \rightarrow c, 4 \rightarrow a$

2. Given the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined

$$\text{by } f(x) = \begin{cases} 3x - 2, & x < -3 \\ 2x^2 + 4, & -3 \leq x < 2 \\ -2x + 9, & x \geq 2 \end{cases}, \text{ find}$$

(a)  $f(-4)$

(b)  $f(0)$

(c)  $f(2)$

(d)  $f(3)$

3. Find the domain and range of the following functions:

(a)  $f : 1 \rightarrow 3, 2 \rightarrow 5, 4 \rightarrow 8$

(b)  $g : 2 \rightarrow 4, 4 \rightarrow 5, 5 \rightarrow 7, 6 \rightarrow 9$

(c)  $h : 1 \rightarrow 3, 2 \rightarrow 5, 3 \rightarrow 6, 4 \rightarrow 8$

4. The table below shows a function  $f$ :

x	-3	-2	-1	0	1
f(x)	-22	-3	4	5	6

(a) Find the domain and range of the function;

(b) Sketch the graph of the function.

(c) Determine if the inverse function of  $f$  exists.

5. As shown in the diagram below, let a function  $f : x \rightarrow ax + b$ . Find the value of  $f(4)$  and  $f^{-1}(5)$ .

6. Given the function  $f : x \rightarrow x^2 - x + 1$ ,  $-1 \leq x \leq 3$ , find its range.

7. Let function  $f : x \rightarrow 2x^2 - 4x + 3$ .

(a) If  $D_f = \mathbb{R}$ , find the range of  $f$ ;

(b) If  $D_f = \{x|x \geq 3\}$ , find the range of  $f$ .

8. Find the domain and range of the following functions:

(a)  $f(x) = \frac{1}{x}$

(b)  $f(x) = \sqrt{2x - 5}$

(c)  $f(x) = x^2 + 4x + 7$

(d)  $f(x) = \frac{1}{x^2 + 4}$

9. Find the domain of the following functions:

(a)  $f(x) = \frac{2x}{x - 3}$

(b)  $f(x) = \sqrt{4 - x^2}$

(c)  $f(x) = \frac{x - 2}{2x^2 - 5x + 2}$

(d)  $f(x) = \frac{x - 3}{\sqrt{x^2 - 9}}$

10. Sketch the graph for the following functions:

(a)  $f(x) = 2x^2 - 5x + 9$

(b)  $f(x) = -3x^2 + 6x + 11$

(c)  $f(x) = 3x^2 + 12x + 10$

(d)  $f(x) = -5x^2 + 6x + 11$

(e)  $f(x) = 2x^3 - 7$

(f)  $f(x) = \sqrt{3x - 9}$

(g)  $f(x) = \frac{4}{2x + 11}$

(h)  $f(x) = \frac{2x + 7}{x - 1}$

(i)  $f(x) = 2\sqrt{x + 5} - 4$

(j)  $f(x) = \frac{1}{(2x - 3)^2}$

(k)  $f(x) = \begin{cases} 2x + 1, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

$$(l) f(x) = \begin{cases} 1 - x^2, & x \leq 1 \\ x^2 + 2x - 3, & x > 1 \end{cases}$$

11. Given the function  $f : x \rightarrow 2x^2$  and  $g : x \rightarrow 3x - 4$ . Find the value of  $m$  such that  $(f \circ g)(m) = (g \circ f)(m)$ .

12. Given the function  $f : x \rightarrow x^2 + 2x - 3$  and  $g : x \rightarrow 3x - 4$ . If  $(f \circ g)(k) = (g \circ f)(k)$ , find the value of  $k$ .

13. Given that  $f(x) = 3x + 1$ ,  $x \neq 0$ . If  $(f \circ g)(x) = 6x^2 - 9x + 4$ , find  $g(x)$ .

14. Given that  $f(x) = \frac{x+1}{x}$ ,  $x \neq 0$ . IF  $(f \circ g)(x) = x$ , find  $g(x)$ .

15. A function  $f$  is defined by  $f : x \rightarrow x - 3$ . Find another function  $g$  such that  $g \circ f : x \rightarrow 4x^2 - 20x + 25$ .

16. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} -2, & x \leq -3 \\ |x| - 2x, & -3 < x < 3 \\ 2x - 1, & x \geq 3 \end{cases}$$

Find  $(f \circ f \circ f)(-1000)$ .

17. Let function  $f : A \rightarrow \mathbb{R}$  be defined by  $f : x \rightarrow 2x^2$ . Determine if  $f$  is one to one function when  $A$  is the following sets.

(a)  $A = \{x | 0 \leq x < 6\}$

(b)  $A = \{x | x < 0\}$

(c)  $A = \{x | -2 \leq x < 2\}$

(d)  $A = \{x | x > 3\}$

18. Determine whether the following functions are one to one functions or onto functions.

(a)  $f : \mathbb{R}^+ \rightarrow \mathbb{R}, f : x \rightarrow |x| - 2$

(b)  $f : \mathbb{R} \setminus \{2\} \rightarrow \mathbb{R} \setminus \{1\}, f : x \rightarrow \frac{x}{x-2}$

(c)  $f : \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{0\}, f : x \rightarrow |x|$

19. Let  $A = \mathbb{R} \setminus \left\{-\frac{1}{2}\right\}$  and  $B = \mathbb{R} \setminus \left\{\frac{1}{2}\right\}$ , function  $f : A \rightarrow B$  is defined by  $f(x) = \frac{x-3}{2x+1}$ . Find

(a)  $f^{-1}(-2)$

(b)  $f^{-1}(0)$

(c)  $f^{-1}(3)$

20. Let function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be defined by  $f(x) = x^2 + 2x + 1$ . Find  $f^{-1}(4)$  and  $f^{-1}(9)$ .

21. A function  $f$  is defined by  $f : x \rightarrow \frac{x}{2} + 1$ . If  $g \circ f^{-1} : x \rightarrow 4x^2 - 8x + 7$ , find the function  $g$ .

22. Given the function  $f : x \rightarrow 3x^2 + 5x + 9$ ,  $x \leq a$ . Find the maximum value of  $a$  such that the inverse function of  $f$  exists.

23. Let the function  $f$  and  $g$  be defined as  $f : x \rightarrow 5x + 3$  and  $g : x \rightarrow 2x - 7$  respectively. Find

(a)  $f \circ g$

(b)  $f^{-1}$

(c)  $g^{-1}$

24. Given the function  $f : x \rightarrow 2x + 3$  and  $g : x \rightarrow 3 - x^2 + 5, x \neq -\frac{5}{2}$ . Find

(a)  $f \circ g$

(b)  $f^{-1}$

(c)  $g^{-1}$

Show that  $g^{-1} \circ f^{-1} = (f \circ g)^{-1}$ .

25. Given the function  $f : x \rightarrow \sqrt{x}$ ,  $x \neq 0$  and  $g : x \rightarrow x^3$ . Find

(a)  $g \circ f$

(b)  $f^{-1}$

(c)  $g^{-1}$

(d)  $(g \circ f)^{-1}$

(e)  $g^{-1} \circ f^{-1}$

26. Given the function  $f : x \rightarrow 2\sqrt{x-4} + 3$ ,  $x \geq 4$ .

(a) Find the range of  $f$ .

(b) Find the inverse function  $f^{-1}$  of  $f$ .

(c) On the same diagram, sketch the graphs of  $f$  and  $f^{-1}$ .



# Chapter 23

## Exponents and Logarithms

### 23.1 Exponents

#### Definition and Properties of Exponents

Back in Junior 1, we have learnt the following definitions of exponents:

**Positive exponent**      $a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$

**Zero exponent**      $a^0 = 1$

**Negative exponent**      $a^{-n} = \frac{1}{a^n} \quad (a \neq 0, n \in \mathbb{Z}^+)$

**Fractional exponent**      $a^{\frac{m}{n}} = \left( \sqrt[n]{a} \right)^m = \sqrt[n]{a^m} \quad (a \geq 0, n > 1, m, n \in \mathbb{Z}^+)$

The exponent of rational numbers have the following properties:

1.  $a^m \times a^n = a^{m+n}$
2.  $\frac{a^m}{a^n} = a^{m-n}$
3.  $(a^m)^n = a^{mn}$
4.  $(ab)^n = a^n b^n$
5.  $\left( \frac{a}{b} \right)^n = \frac{a^n}{b^n} \quad (b \neq 0)$

#### Practice 1

Without using the calculator, find the value of the following expressions (Question 1 to 2):

1.  $2^{-2} + 2 - 5 - (-2)^{-3}$
2.  $\left(3\frac{6}{25}\right)^{-\frac{1}{2}}$
3. Simplify  $a^{-4} \div a^{-5} \times (b^{-3})^{-4}$

## Exponential Functions and Graphs

Let  $a$  is a constant that is bigger than zero and not equal to 1, then the function being expressed in the form of  $y = a^x$  is called an *exponential function*. The domain of an exponential function is  $\mathbb{R}$ .

Consider the following: a cell divides into two cells, and then each of the two cells divides into two cells again, and so on. If we let  $x$  be the number of divisions, the number of cells after the divisions be  $y$ , then the functional relationship between  $x$  and  $y$  is  $y = 2^x$ , which is an exponential function.

In order to look into the graph and its properties of an exponential function  $y = a^x$ , we sketch the graph of some exponential functions, the graph of  $y = 2^x$ ,  $y = 10^x$ , and  $y = \left(\frac{1}{2}\right)^x$  are shown in the diagram below.

## 23.2 Logarithms

From the diagram above, we can see that:

- (1) Since the domain of  $y = \log_a x$  is  $x > 0$ , so the graph of the function  $y = \log_2 x$  and  $y = \log_{\frac{1}{2}} x$  are only at the right side of the  $y$ -axis.
- (2) When  $x = 1$ ,  $y = 0$ . Hence, the graph of logarithmic functions  $y = a^x$  passes through the point  $(1, 0)$ .
- (3) For the function  $y = \log_2 x$ , when  $x > 1$ ,  $y > 0$ ; when  $0 < x < 1$ ,  $y < 0$ . When the value of  $x$  increases, the value of  $y$  increases, that is, the function is an increasing function in the interval  $(0, +\infty)$ .
- (4) For the function  $y = \log_{\frac{1}{2}} x$ , when  $x > 1$ ,  $y < 0$ ; when  $0 < x < 1$ ,  $y > 0$ . When the value of  $x$  increases, the value of  $y$  decreases, that is, the function is a decreasing function in the interval  $(0, +\infty)$ .

When we are discussing about the graph and properties of an exponential function  $y = a^x$ , the following two cases are considered:

## Practice 4

- Without using the calculator, Compare the value of the following expressions:
  - $\log 6$  and  $\log 9$
  - $\log_{0.5} 4.2$  and  $\log_{0.5} 3.9$
  - $\log_2 1.8$  and  $\log_4 5.8$ .
- Find the domain of the following functions:
  - $y = \log_a(x + 2)$
  - $y = \log_2(x^2 - 9)$
  - $y = \log_7 \frac{2}{3 - 2x}$
  - $y = \sqrt{\log_5(2 - x)}$

## Exercise 23.2

- Find the value of  $x$  for the following expression:
  - $\log_2 x = 4$
  - $\log_{125} x = \frac{1}{3}$
  - $\log_{16}(2\sqrt{2}) = x$
  - $\log_{\frac{1}{3}} 81 = x$
  - $\log_x 81 = 4$
  - $\log_x 49 = -2$
- Sketch the graph of the following functions on the same set of axes:
  - $y = \log_5 x$
  - $y = \log_{\frac{1}{5}} x$
- Without using the calculator, compare the value of the following expressions:
  - $\log_3 5$  and  $\log_3 6$
  - $\log 1.51.4$  and  $\log_{1.5} 1.6$
  - $\log_{\sqrt{3}} 4.8$  and  $\log_{\sqrt{3}} 5.8$
  - $\log_{2.3} \pi$  and  $\log_{2.3}(\pi - 3)$
  - $\log_{0.4} \sqrt{2}$  and  $\log_{0.4} \sqrt{3}$
  - $\log_{\frac{1}{2}} 3$  and  $\log_{\frac{1}{2}} \frac{1}{4}$
- Find the domain of the following functions:
  - $y = \log_2(3 - 2x)$
  - $y = \log(x^2 + 1)$
  - $y = \log_5(9 - 16x^2)$
  - $y = \log_9 \frac{1}{x - 2}$
  - $y = \log_8 \sqrt{2x^2 - x - 3}$
  - $y = \frac{1}{\log_3(7x - 5)}$

### **23.3 Arithmetic Properties of Logarithms and Base Changing Formula**

### **23.4 Exponential Equations**

### **23.5 Logarithmic Equations**

### **23.6 Compound Interest and Annuity**

# **Chapter 24**

## **Limits**

### **24.1 Concept of Limits**

### **24.2 Limits of Functions**

### **24.3 Arithmetic Properties of Limits of Functions**

# **Chapter 25**

## **Differentiation**

**25.1 Gradient of Tangent Line on a Curve**

**25.2 Gradient of Tangent Line and Derivative**

**25.3 Law of Differentiation**

**25.4 Chain Rule - Differentiation of Composite Functions**

**25.5 Higher Order Derivatives**

**25.6 Implicit Differentiation**

**25.7 Two Basic Limits**

**25.8 Derivatives of Trigonometric Functions**

**25.9 Derivatives of Exponential Functions**

**25.10 Derivatives of Logarithmic Functions**