Solution Book of Mathematic

Ssnior 2 Part I

MELVIN CHIA

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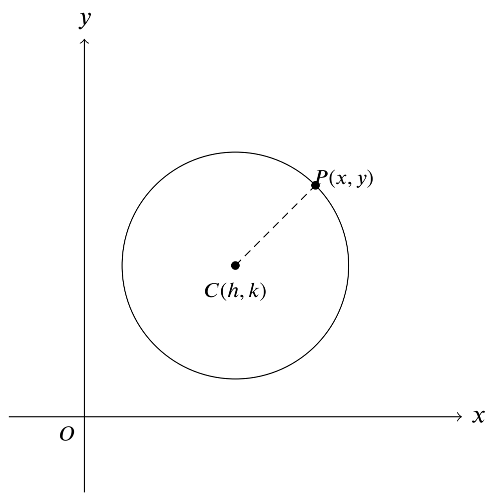
Contents

Chapter 15

Circle

15.1 Standard Equation of a Circle

The circle is a locus of points in a plane that are equidistant from a fixed point called the centre of the circle. The length from the centre to the points on the circle is called the radius of the circle.



The standard equation of a circle is given by

$$(x-h)^2 + (y-k)^2 = r^2$$

where (h, k) is the centre of the circle and r is the radius of the circle.

If the centre of the circle is at the origin, then the equation of the circle is

$$x^2 + y^2 = r^2 \quad (r > 0)$$

15.1.1 Practice 1

1. Find the equation of the circle with centre $(3, -1)$ and radius 2.

   Sol.

   $$\text{Equation}: (x-3)^2 + [y + (-1)]^2 = 2^2$$
   $$(x-3)^2 + (y+1)^2 = 4$$

2. Find the equation of the circle with centre $(-2, 9)$ and passing through the point $(1, 5)$.

   Sol.

   $$\begin{aligned} r &= \sqrt{(1-(-2))^2 + (5-9)^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

   $$\therefore \text{Equation}: [x-(-2)]^2 + (y-9)^2 = 5^2$$
   $$(x+2)^2 + (y-9)^2 = 25$$

15.1.2 Exercise 16.1

1. Find the equation of the circle with centre at the origin and radius 7.

   Sol.

   $$\text{Equation}: x^2 + y^2 = 7^2$$
   $$x^2 + y^2 = 49$$

2. Find the equation of circle of each of the following description:

   (a) Passing through the points $(5, -3)$ and centre at $(2, 1)$.

   Sol.

   $$\begin{aligned} r &= \sqrt{(5-2)^2 + (-3-1)^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \end{aligned}$$

   $$\therefore \text{Equation}: (x-2)^2 + (y-1)^2 = 5^2$$
   $$(x-2)^2 + (y-1)^2 = 25$$

   (b) Centre at $(3, 2)$ and radius 4.

   Sol.

   $$\text{Equation}: (x-3)^2 + (y-2)^2 = 4^2$$
   $$(x-3)^2 + (y-2)^2 = 16$$

   (c) Centre at (a, b) and radius $a + b$.

   Sol.

   $$\text{Equation}: (x-a)^2 + (y-b)^2 = (a+b)^2$$

3. Given that the coordinates of two points on the end of the diameter of a circle are $(5, -3)$ and $(3, 1)$, find the equation of the circle.

Sol.

$$\begin{aligned} C &= \left(\frac{5+3}{2}, \frac{-3+1}{2}\right) \\ &= \left(\frac{8}{2}, \frac{-2}{2}\right) \\ &= (4, -1) \end{aligned}$$

$$\begin{aligned} r &= \sqrt{(5-4)^2 + (-3-(-1))^2} \\ &= \sqrt{1+4} \\ &= \sqrt{5} \end{aligned}$$

$$\begin{aligned} \therefore \text{Equation} : (x-4)^2 + [y-(-1)]^2 &= (\sqrt{5})^2 \\ (x-4)^2 + (y+1)^2 &= 5 \end{aligned}$$

4. Find the equation of the circle with a diameter connected by the points $(-3, 4)$ and $(9, 2)$.

   Sol.

$$\begin{aligned} C &= \left(\frac{-3+9}{2}, \frac{4+2}{2}\right) \\ &= \left(\frac{6}{2}, \frac{6}{2}\right) \\ &= (3, 3) \end{aligned}$$

$$\begin{aligned} r &= \sqrt{(-3-3)^2 + (4-3)^2} \\ &= \sqrt{36+1} \\ &= \sqrt{37} \end{aligned}$$

$$\begin{aligned} \therefore \text{Equation} : (x-3)^2 + (y-3)^2 &= (\sqrt{37})^2 \\ (x-3)^2 + (y-3)^2 &= 37 \end{aligned}$$

5. Given two points $P(-2, 2)$ and $Q(4, 6)$, find the equation of the circle wuth line PQ as its diameter.

   Sol.

$$\begin{aligned} C &= \left(\frac{-2+4}{2}, \frac{2+6}{2}\right) \\ &= \left(\frac{2}{2}, \frac{8}{2}\right) \\ &= (1, 4) \end{aligned}$$

$$\begin{aligned} r &= \sqrt{(-2-1)^2 + (2-4)^2} \\ &= \sqrt{9+4} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} \therefore \text{Equation} : (x-1)^2 + (y-4)^2 &= (\sqrt{13})^2 \\ (x-1)^2 + (y-4)^2 &= 13 \end{aligned}$$

6. Turn the equation $x^2 + y^2 - 6x + 12y + 41 = 0$ into the standard form, and find the centre and radius of the circle.

   Sol.

$$\begin{aligned} x^2 + y^2 - 6x + 12y + 41 &= 0 \\ x^2 + y^2 - 6x + 12y &= -41 \\ (x^2 - 6x + 9) - 9 + (y^2 + 12y + 36) - 36 &= -41 \\ (x-3)^2 + (y-6)^2 &= 4 \end{aligned}$$

$$\therefore \text{Centre} : (3, 6), \ \text{Radius} : 2$$

15.2 General Equation of a Circle

Expand the standard equation of a circle, we get

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$$

Let $g = -h$, $f = -k$, $c = h^2 + k^2 - r^2$, we get the general equation of a circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

From $c = h^2 + k^2 - r^2$, we have $r^2 = h^2 + k^2 - c$

$$\begin{aligned} r &= \sqrt{h^2 + k^2 - c} \\ &= \sqrt{(-g)^2 + (-f)^2 - c} \\ &= \sqrt{g^2 + f^2 - c} \end{aligned}$$

Thus,

1. When $g^2 + f^2 - c > 0$, the image is a real circle with centre (g, f) and radius $\sqrt{g^2 + f^2 - c}$.
2. When $g^2 + f^2 - c = 0$, the image is point (g, f).
3. When $g^2 + f^2 - c < 0$, the image does not exist.

15.2.1 Practice 2

1. Find the centre and radius of the circle with equation $x^2 + y^2 - 6x - 8y + 21 = 0$.

   Sol.

$$\begin{aligned} x^2 + y^2 - 6x - 8y + 21 &= 0 \\ \therefore 2g = -6, \ 2f = -8, \ c &= 21 \\ g = -3, \ f = -4, \ c &= 21 \end{aligned}$$

$$\begin{aligned} \therefore C &= (3, 4) \\ r &= \sqrt{(-3)^2 + (-4)^2 - 21} \\ &= \sqrt{9 + 16 - 21} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

2. Find the equaton of the circle that passes through the following points:

(a) $A(0,0)$, $B(2,0)$, $C(0,-3)$.

Sol.

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases} 0+0+0g+0f+c=0 \\ 4+0+4g+0f+c=0 \\ 0+9+0g-6f+c=0 \end{cases}$$
$$\begin{cases} c=0 \\ 4+4g+c=0 \\ 9-6f+c=0 \end{cases}$$
$$\begin{cases} c=0 \\ 4+4g=0 \\ 9-6f=0 \end{cases}$$
$$\begin{cases} c=0 \\ 4g=-4 \\ -6f=-9 \end{cases}$$
$$\begin{cases} c=0 \\ g=-1 \\ f=\frac{3}{2} \end{cases}$$

$$\begin{aligned} \therefore \text{Equation}: x^2+y^2+2(-1)+2(\frac{3}{2})+0 &= 0 \\ x^2+y^2-2+3 &= 0 \\ x^2+y^2+1 &= 0 \end{aligned}$$

(b) $K(0,3)$, $L(1,2)$, $M(2,-1)$.

Sol.

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases} 0+9+0g+6f+c=0 \\ 1+4+2g+4f+c=0 \\ 4+1+4g-2f+c=0 \end{cases}$$
$$\begin{cases} 6f+c=-9 \\ 2g+4f+c=-5 \\ 4g-2f+c=-5 \end{cases}$$

$$g=3,\ f=1,\ c=-15$$

$$\begin{aligned} \therefore \text{Eq.}: x^2+y^2+2(3)x+2(1)y+(-15) &= 0 \\ x^2+y^2+6x+2y-15 &= 0 \end{aligned}$$

3. Given that the vertices of ΔABC are $(1,2)$, $(2,5)$ and $(-1,2)$, find the equation of the circumcircle of ΔABC.

Sol.

Let the equation of the circumcircle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases} 1+4+2g+4f+c=0 \\ 4+25+8g+10f+c=0 \\ 1+4-2g+4f+c=0 \end{cases}$$
$$\begin{cases} 2g+4f+c=-5 \\ 8g+10f+c=-29 \\ -2g+4f+c=-5 \end{cases}$$

$$g=0,\ f=-4,\ c=11$$

$$\begin{aligned} \therefore \text{Eq.}: x^2+y^2+2(0)x+2(-4)y+11 &= 0 \\ x^2+y^2-8y+11 &= 0 \end{aligned}$$

15.2.2 Exercise 16.2

1. Find the centre and radius of the circle with the following equation:

(a) $x^2+y^2-64=0$

Sol.

$$\begin{aligned} x^2+y^2-64 &= 0 \\ 2g=0,\ 2f=0,\ c &= -64 \\ g=0,\ f=0,\ c &= -64 \end{aligned}$$

$$\begin{aligned} \therefore C &= (0,0) \\ r &= \sqrt{0^2+0^2-(-64)} \\ &= 8 \end{aligned}$$

(b) $x^2+y^2-4x-8y=44$

Sol.

$$\begin{aligned} x^2+y^2-4x-8y &= 44 \\ x^2+y^2-4x-8y-44 &= 0 \\ 2g=-4,\ 2f=-8,\ c &= -44 \\ g=-2,\ f=-4,\ c &= -44 \end{aligned}$$

$$\begin{aligned} \therefore C &= (2,4) \\ r &= \sqrt{(-2)^2+(-4)^2-(-44)} \\ &= \sqrt{4+16+44} \\ &= \sqrt{64} \\ &= 8 \end{aligned}$$

(c) $x^2+y^2-8x=0$

Sol.

$$\begin{aligned} x^2+y^2-8x&=0\\ 2g=-8,\ 2f=0,\ c&=0\\ g=-4,\ f=0,\ c&=0 \end{aligned}$$

$$\therefore C=(4,0)$$

$$\begin{aligned} r&=\sqrt{(-4)^2+0^2-0}\\ &=4 \end{aligned}$$

(d) $9x^2+9y^2+2x-6y-6=0$
Sol.

$$\begin{aligned} 9x^2+9y^2+2x-6y-6&=0\\ x^2+y^2+\frac{2}{9}x-\frac{2}{3}y-\frac{2}{3}&=0\\ 2g=\frac{2}{9},\ 2f=-\frac{2}{3},\ c&=-\frac{2}{3}\\ g=\frac{1}{9},\ f=-\frac{1}{3},\ c&=-\frac{2}{3} \end{aligned}$$

$$\therefore C=\left(-\frac{1}{9},\frac{1}{3}\right)$$

$$\begin{aligned} r&=\sqrt{\left(\tfrac{1}{9}\right)^2+\left(-\tfrac{1}{3}\right)^2-\left(-\tfrac{2}{3}\right)}\\ &=\sqrt{\frac{64}{81}}\\ &=\frac{8}{9} \end{aligned}$$

2. Find the equation of the circle that passes through the following points:

(a) $A(1,1)$, $B(1,-1)$, $C(-2,1)$
Sol. Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases} 1+1+2g+2f+c=0\\ 1+1+2g-2f+c=0\\ 4+1-4g+2f+c=0 \end{cases}$$

$$\begin{cases} 2g+2f+c=-2\\ 2g-2f+c=-2\\ -4g+2f+c=-5 \end{cases}$$

$$g=\frac{1}{2},\ f=0,\ c=-3$$

$$\begin{aligned} \therefore \text{Eq}:\ x^2+y^2+2\left(\frac{1}{2}\right)x+2(0)y+(-3)&=0\\ x^2+y^2+x-3&=0 \end{aligned}$$

(b) $F(0,0)$, $G(3,-3)$, $H(-1,0)$

Sol. Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases} 0+0+0g+0f+c=0\\ 9+9+6g-6f+c=0\\ 1+0-2g+0f+c=0 \end{cases}$$

$$\begin{cases} c=0\\ 6g-6f=-18\\ -2g=-1 \end{cases}$$

$$\begin{cases} c=0\\ g-f=-3\\ g=\frac{1}{2} \end{cases}$$

$$\begin{cases} c=0\\ f=\frac{7}{2}\\ g=\frac{1}{2} \end{cases}$$

$$\begin{aligned} \therefore \text{Eq}:\ x^2+y^2+2\left(\frac{1}{2}\right)x+2\left(\frac{7}{2}\right)y+0&=0\\ x^2+y^2+x+7y&=0 \end{aligned}$$

(c) $P(1,0)$, $Q(0,-3)$, $R(3,4)$

Sol. Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$,

$$\begin{cases} 1+0+2g+0f+c=0\\ 0+9+0g-6f+c=0\\ 9+16+6g+8f+c=0 \end{cases}$$

$$\begin{cases} 2g+c=-1\\ -6f+c=-9\\ 6g+8f+c=-25 \end{cases}$$

$$g=-26,\ f=10,\ c=51$$

$$\begin{aligned} \therefore \text{Eq}:\ x^2+y^2+2(-26)x+2(10)y+51&=0\\ x^2+y^2-52x+20y+51&=0 \end{aligned}$$

3. A circle passes through point $A(2,2)$ and $B(5,3)$ while intersecting the line $x+y=4$ at y-axis. Find the equation of the circle.

Sol.

$$\begin{aligned} x+y&=4\\ \text{When } x=0,\ y&=4\\ \therefore \text{Another point: } &C(0,4) \end{aligned}$$

Let the equation of the circle be $x^2+y^2+2gx+2fy+$

$c = 0,$

$$\begin{cases} 4+4+4g+4f+c=0 \\ 25+9+10g+6f+c=0 \\ 0+16+0g+8f+c=0 \end{cases}$$

$$\begin{cases} 4g+4f+c=-8 \\ 10g+6f+c=-34 \\ 8f+c=-16 \end{cases}$$

$$g = -\frac{11}{4},\ f = -\frac{19}{4}, c = 22$$

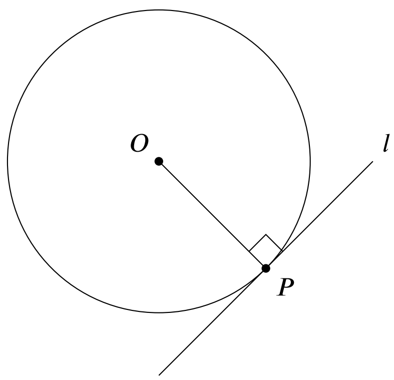
$$\therefore \text{Eq}: x^2+y^2+2\left(-\frac{11}{4}\right)x+2\left(-\frac{19}{4}\right)y+22=0$$

$$x^2+y^2-\frac{11}{2}x-\frac{19}{2}y+22=0$$

15.3 Problems Related to Circles

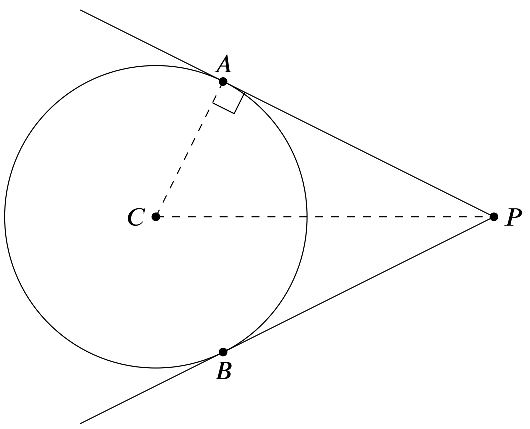
Tangent to a Circle

When a straight line l and a circle intersect at a point P, the line l is called a tangent to the circle, and the point P is called the point of contact. The tangent line is perpendicular to the radius at the point of contact. That is to say, when the length from the point of tangency to the centre of the circle is equal to the radius of the circle, the line is a tangent to the circle.



Length of a Tangent

According to the theorem of length of tangent, the lengths of tangents drawn from an external point to a circle are equal.



Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$, the external point P be (x_1, y_1). Connect PC and CA, $\angle CPA = 90°$, the coordinate of centre of the circle C be $(-g, -f)$.

$$\therefore CA = \sqrt{g^2+f^2-c},\ PC = \sqrt{(x_1+g)^2+(y_1+f)^2}$$

From the Pythagorean theorem,

$$\begin{aligned} PA^2 &= PC^2 - CA^2 \\ &= (x_1+g)^2+(y_1+f)^2-(g^2+f^2-c) \\ &= x_1^2+y_1^2+2gx_1+2fy_1+c \end{aligned}$$

Thus, the length of the tangent is given by

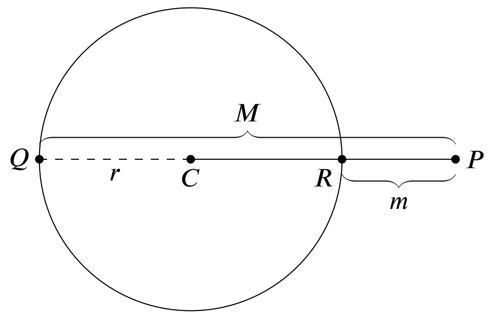
$$PA = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$

Note that the coefficient of x_1 and y_1 in the above equation must be 1.

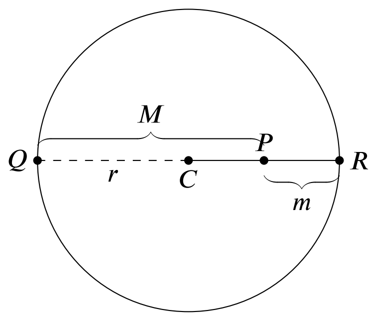
Maximum and Minimum Distance of a Point from a Circle

Given a circle with centre C and radius r and a point P anywhere on the plane,

When $PC > r$, point P is said to be outside the circle, the maximum distance of P from the circle is $M = PC - r$, and the minimum distance of P from the circle is $m = PC - r$.



When $PC < r$, point P is said to be inside the circle, the maximum distance of P from the circle is $M = PC + r$, and the minimum distance of P from the circle is $m = r - PC$.



15.3.1 Practice 3

1. Find the euqaiton of the circle with centre $(3, 4)$ and is tangent to the line $x + 2y - 6 = 0$.

Sol.

$$\begin{aligned} r &= \left|\frac{1(3)+2(4)-6}{\sqrt{1^2+2^2}}\right| \\ &= \left|\frac{3+8-6}{\sqrt{5}}\right| \\ &= \left|\frac{5}{\sqrt{5}}\right| \\ &= \frac{5\sqrt{5}}{5} \\ &= \sqrt{5} \end{aligned}$$

$$\begin{aligned} g = -3,\ f = -4,\ c &= (-3)^2 + (-4)^2 - (\sqrt{5})^2 \\ &= 9 + 16 - 5 \\ &= 20 \end{aligned}$$

$$\begin{aligned} \therefore \text{Eq} : x^2 + y^2 + 2(-3)x + 2(-4)y - 20 &= 0 \\ x^2 + y^2 - 6x - 8y - 20 &= 0 \end{aligned}$$

2. A circle passes through the points $(2, -3)$ and $(-2, -5)$, and its centre is on the line $x - 2y = 3$. Find the equation of the circle.

   Sol.

   Let the centre of the circle be $C(h, k)$, point (2, -3) be A and point (-2, -5) be B.

   $$\begin{aligned} \because C \text{ is on the line } x - 2y &= 3 \\ h - 2k &= 3 \qquad (1) \end{aligned}$$

   $$\begin{aligned} CA &= CB \\ \sqrt{(2-h)^2 + (-3-k)^2} &= \sqrt{(-2-h)^2 + (-5-k)^2} \\ h^2 - 4h + 4 + k^2 +6k + 9 &= h^2 + 4h + 4 + k^2 + 10k + 25 \\ -4h + 6k + 13 &= 4h + 10k + 29 \\ -8h - 4k &= 16 \\ 2h + k &= -4 \qquad (2) \end{aligned}$$

   Solving (1) and (2), $h = -1,\ k = -2$

   $$\begin{aligned} \therefore C = (-1, -2),\ r &= \sqrt{(2-(-1))^2 + (-3-(-2))^2} \\ &= \sqrt{3^2 + 1^2} \\ &= \sqrt{10} \end{aligned}$$

   $$\begin{aligned} g = 1,\ f = -2,\ c &= (-1)^2 + (-2)^2 - (\sqrt{10})^2 \\ &= 1 + 4 - 10 \\ &= -5 \end{aligned}$$

   $$\begin{aligned} \therefore \text{Eq} : x^2 + y^2 + 2(-1)x + 2(-2)y - 5 &= 0 \\ x^2 + y^2 - 2x - 4y - 5 &= 0 \end{aligned}$$

3. A circle with radius $\sqrt{5}$ are tangent with the line $x - 2y - 1 = 0$ at the point $(3, 1)$. Find the equation of the circle.

   Sol.

   Let the centre of the circle be $C(h, k)$, point (3, 1) be P.

   $$\begin{aligned} x - 2y - 1 &= 0 \\ 2y &= x - 1 \\ y &= \frac{1}{2}x - \frac{1}{2} \\ m &= \frac{1}{2} \end{aligned}$$

   Let the line that passes through P and is perpendicular to $x - 2y - 1 = 0$ be l.

   $$\begin{aligned} m_l \times m &= -1 \\ m_l &= -2 \\ l : y - 1 &= -2(x - 3) \\ y - 1 &= -2x + 6 \\ y &= -2x + 7 \end{aligned}$$

   $$\begin{aligned} &\because C(h, k) \text{ is on the line } l \\ &k = -2h + 7 \qquad (1) \end{aligned}$$

   $$\begin{aligned} \sqrt{(3-h)^2 + (1-k)^2} &= \sqrt{5} \\ h^2 - 6h + 9 + k^2 - 2k + 1 &= 5 \qquad (2) \end{aligned}$$

   $$\begin{aligned} &\text{Sub (1) in (2),} \\ h^2 - 6h + 9 + (-2h+7)^2 - 2(-2h+7) + 1 &= 5 \\ h^2 - 6h + 9 + 4h^2 - 28h + 49 + 4h - 14 + 1 &= 5 \\ 5h^2 - 30h + 40 &= 0 \\ h^2 - 6h + 8 &= 0 \\ (h-4)(h-2) &= 0 \\ h = 4 \text{ or } h &= 2 \end{aligned}$$

   Sub $h = 4$ in (1), $k = -2(4) + 7 = -1$
   Sub $h = 2$ in (1), $k = -2(2) + 7 = 3$

   $\therefore C = (4, -1)$ or $C = (2, 3)$

   When $C = (4, -1)$,

   $$\begin{aligned} g = -4,\ f = 1,\ c &= 4^2 + (-1)^2 - (\sqrt{5})^2 \\ &= 16 + 1 - 5 \\ &= 12 \end{aligned}$$

$$\therefore \text{Eq}: x^2+y^2+2(-4)x+2(1)y+12=0$$
$$x^2+y^2-8x+2y+12=0$$

When $C=(2,3)$,

$$g=-2,\ f=-3,\ c=2^2+3^2-(\sqrt{5})^2$$
$$=4+9-5$$
$$=8$$

$$\therefore \text{Eq}: x^2+y^2+2(-2)x+2(-3)y+8=0$$
$$x^2+y^2-4x-6y+8=0$$

4. Prove the following lines are tangent to the following circles:

   (a) $3x-y-5=0,\ x^2+y^2-16x+2y+25=0$

   Proof.

$$C=(8,-1)$$
$$r=\sqrt{(-8)^2+1^2-25}=2\sqrt{10}$$
$$d=\left|\frac{3(8)-1(-1)-5}{\sqrt{3^2+(-1)^2}}\right|$$
$$=\left|\frac{20}{\sqrt{10}}\right|$$
$$=\frac{20\sqrt{10}}{10}$$
$$=2\sqrt{10}$$

$\because d=r$

$\therefore$ The line $3x-y-5=0$ is tangent to the circle $x^2+y^2-16x+2y+25=0$

   (b) $2x-y-1=0,\ x^2+y^2+2x-4y=0$

   Proof.

$$C=(-1,2)$$
$$r=\sqrt{1^2+(-2)^2}=\sqrt{5}$$
$$d=\left|\frac{2(-1)-1(2)-1}{\sqrt{2^2+(-1)^2}}\right|$$
$$=\left|\frac{-5}{\sqrt{5}}\right|$$
$$=\frac{5\sqrt{5}}{5}$$
$$=\sqrt{5}$$

$\because d=r$

$\therefore$ The line $2x-y-1=0$ is tangent to the circle $x^2+y^2+2x-4y=0$

5. Find the length of the tangent from the point $P(8,3)$ to the circle $x^2+y^2-8=0$.

   Sol.

$$d=\sqrt{8^2+3^2+2(0)(8)+2(0)(3)-8}=\sqrt{65}$$

15.3.2 Exercise 16.3

1. Find the equation of the circle that passes through the points $(1,4)$ and $(0,-3)$, and its centre is on the line $x-2y=4$.

   Sol.

   Let the centre of the circle be $C(h,k)$, point (1, 4) be A and point (0, -3) be B.

$\because C$ is on the line $x-2y=4$

$$h-2k=4 \qquad (1)$$

$$CA=CB$$
$$\sqrt{(1-h)^2+(4-k)^2}=\sqrt{(0-h)^2+(-3-k)^2}$$
$$h^2-2h+1+k^2-8k+16=h^2+k^2+6k+9$$
$$-2h+-14k=-8$$
$$h+7k=4 \qquad (2)$$

Solving (1) and (2), $h=4,\ k=0$

$$\therefore C=(4,0),\ r=\sqrt{(1-4)^2+(4-0)^2}$$
$$=\sqrt{(-3)^2+4^2}$$
$$=\sqrt{25}$$
$$=5$$
$$g=-4,\ f=0,\ c=(-4)^2+0^2-5^2$$
$$=16-25$$
$$=-9$$

$$\therefore \text{Eq}: x^2+y^2+2(-4)x+2(0)y-9=0$$
$$x^2+y^2-8x-9=0$$

2. Find the equation of the circle that passes through the points $(3,2)$ and $(-4,-5)$, and its centre is on the line $3x+y+6=0$.

   Sol.

   Let the centre of the circle be $C(h,k)$, point (3, 2) be A and point (-4, -5) be B.

$\because C$ is on the line $3x+y+6=0$

$$3h+k+6=0 \qquad (1)$$

$$CA = CB$$
$$\sqrt{(3-h)^2 + (2-k)^2} = \sqrt{(-4-h)^2 + (-5-k)^2}$$
$$h^2 - 6h + 9 + k^2 = h^2 + 8h + 16 + k^2$$
$$-4k + 4 \quad + 10k + 25$$
$$-14h - 14k = 28$$
$$h + k = -2 \qquad (2)$$

Solving (1) and (2), $h = -2,\ k = 0$

$$\therefore C = (-2, 0),\ r = \sqrt{[3-(-2)]^2 + (2-0)^2}$$
$$= \sqrt{5^2 + 2^2}$$
$$= \sqrt{29}$$
$$g = 2,\ f = 0,\ c = (-2)^2 + 0^2 - \sqrt{29}^2$$
$$= 4 - 29$$
$$= -25$$

$$\therefore \text{Eq}: x^2 + y^2 + 2(2)x + 2(0)y - 25 = 0$$
$$x^2 + y^2 + 4x - 25 = 0$$

3. Find the equation of the circle that passes through the points $A(5, 2)$ and $B(-3, 0)$, and its centre is on the y-axis.

Sol.

Let the centre of the circle be $C(0, k)$, point (5, 2) be A and point (-3, 0) be B.

$$CA = CB$$
$$\sqrt{(5-0)^2 + (2-k)^2} = \sqrt{(-3-0)^2 + (0-k)^2}$$
$$25 + k^2 - 4k + 4 = 9 + k^2$$
$$-4k = -20$$
$$k = 5$$

$$\therefore C = (0, 5),\ r = \sqrt{(5-0)^2 + (2-5)^2}$$
$$= \sqrt{36}$$
$$= 6$$
$$g = 0,\ f = -5,\ c = 0^2 + (-5)^2 - 6^2$$
$$= 25 - 36$$
$$= -9$$

$$\therefore \text{Eq}: x^2 + y^2 + 2(0)x + 2(-5)y - 9 = 0$$
$$x^2 + y^2 - 10y - 9 = 0$$

4. Find the equation of the circle with centre at the origin and is tangent to the line $3x - 4y + 20 = 0$.

Sol.

$$r = \left| \frac{3(-0) - 4(-0) + 20}{\sqrt{3^2 + (-4)^2}} \right|$$
$$= \frac{20}{5}$$
$$= 4$$
$$g = 0,\ f = 0,\ c = 0^2 + 0^2 - 4^2$$
$$= -16$$

$$\therefore \text{Eq}: x^2 + y^2 + 2(0)x + 2(0)y - 16 = 0$$
$$x^2 + y^2 - 16 = 0$$

5. Find the equation of the circle with centre $A(-5, 4)$, and is tangent to the x-axis.

Sol.

$$r = \left| \frac{0(-5) + 1(4) + 0}{\sqrt{0^2 + 1^2}} \right|$$
$$= \left| \frac{4}{1} \right|$$
$$= 4$$
$$g = 5,\ f = -4,\ c = 5^2 + (-4)^2 - 4^2$$
$$= 25$$

$$\therefore \text{Eq}: x^2 + y^2 + 2(5)x + 2(-4)y + 25 = 0$$
$$x^2 + y^2 + 10x - 8y + 25 = 0$$

6. Find the equation of the circle with centre $(-4, 2)$, and is tangent to the line $3x + 2y = 5$.

Sol.

$$r = \left| \frac{3(-4) + 2(2) - 5}{\sqrt{3^2 + 2^2}} \right|$$
$$= \left| \frac{-13}{\sqrt{13}} \right|$$
$$= \frac{13\sqrt{13}}{13}$$
$$= \sqrt{13}$$
$$g = 4,\ f = -2,\ c = 4^2 + (-2)^2 - \sqrt{13}^2$$
$$= 16 + 4 - 13$$
$$= 7$$

$$\therefore \text{Eq}: x^2 + y^2 + 2(4)x + 2(-2)y + 7 = 0$$
$$x^2 + y^2 + 8x - 4y + 7 = 0$$

7. Find the equation of the circle that passes through the

point (3, 0), and is tangent to the line $2x - 3y - 24 = 0$ at point $(3, -6)$.

Sol.

Let the centre of the circle be $C(h, k)$, point (3, -6) be P.

$$\begin{aligned} 2x - 3y - 24 &= 0 \\ 3y &= 2x - 24 \\ y &= \frac{2}{3}x - 8 \\ m &= \frac{2}{3} \end{aligned}$$

Let the line that passes through P and is perpendicular to $2x - 3y - 24 = 0$ be l.

$$\begin{aligned} m_l \times m &= -1 \\ m_l &= -\frac{3}{2} \\ l : y + 6 &= -\frac{3}{2}(x - 3) \\ 2y + 12 &= -3x + 9 \\ 3x + 2y &= -3 \end{aligned}$$

$\because C(h, k)$ is on the line l

$$3h + 2k = -3 \qquad (1)$$

$$\begin{aligned} \sqrt{(3 - h)^2 + (0 - k)^2} &= \sqrt{(3 - h)^2 + (-6 - k)^2} \\ h^2 - 6h + 9 + k^2 &= h^2 - 6h + 9 + k^2 + 12k + 36 \\ 12k + 36 &= 0 \\ k &= -3 \\ \text{Sub } k &= -3 \text{ into (1),} \\ 3h &= 3 \text{ into (1),} \\ h &= 1 \end{aligned}$$

$$\begin{aligned} \therefore C = (1, -3),\ r &= \sqrt{(3 - 1)^2 + [0 - (-3)]^2} \\ &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} g = -1,\ f = 3,\ c &= (-1)^2 + 3^2 - \sqrt{13}^2 \\ &= 1 + 9 - 13 \\ &= -3 \end{aligned}$$

$$\begin{aligned} \therefore \text{Eq} : x^2 + y^2 + 2(-1)x + 2(3)y - 3 &= 0 \\ x^2 + y^2 - 2x + 6y - 3 &= 0 \end{aligned}$$

8. Given a circle C_1 and another circle $C_2 : x^2+y^2-4x-6y+8 = 0$ shares the same centre, and C_1 is tangent to the line $3x + 4y - 13 = 0$. Find the equation of the circle C_1.

Sol.

$$\begin{aligned} C_{C1} &= C_{C2} = (2, 3) \\ r_{C1} &= \left| \frac{3(2) + 4(3) - 13}{\sqrt{3^2 + 4^2}} \right| \\ &= \left| \frac{5}{\sqrt{25}} \right| \\ &= 1 \end{aligned}$$

$$\begin{aligned} g = -2,\ f = -3,\ c &= (-2)^2 + (-3)^2 - 1^2 \\ &= 4 + 9 - 1 \\ &= 12 \end{aligned}$$

$$\begin{aligned} \therefore \text{Eq} : x^2 + y^2 + 2(-2)x + 2(-3)y + 12 &= 0 \\ x^2 + y^2 - 4x - 6y + 12 &= 0 \end{aligned}$$

9. Prove the following lines are tangent to the following circles:

   (a) $6x + 5y - 31 = 0$, $x^2 + y^2 + 4x - 5y - 5 = 0$

   Sol.

$$\begin{aligned} C &= \left(-2, \frac{5}{2}\right) \\ r &= \sqrt{2^2 + \left(-\frac{5}{2}\right)^2 + 5} \\ &= \frac{\sqrt{61}}{2} \\ d &= \left| \frac{6(-2) + 5\left(\frac{5}{2}\right) - 31}{\sqrt{6^2 + 5^2}} \right| \\ &= \left| \frac{-12 + \frac{25}{2} - 31}{\sqrt{61}} \right| \\ &= \frac{61}{2\sqrt{61}} \\ &= \frac{61\sqrt{61}}{2 \times 61} \\ &= \frac{\sqrt{61}}{2} \end{aligned}$$

$\because d = r$

$\therefore$ The line $6x + 5y - 31 = 0$ is tangent to the circle $x^2 + y^2 + 4x - 5y - 5 = 0$

   (b) $3x + 1 = 0$, $9x^2 + 9y^2 + 3x + 6y + 1 = 0$

   Sol.

$$\begin{aligned} 9x^2 + 9y^2 + 3x + 6y + 1 &= 0 \\ x^2 + y^2 + \frac{1}{3}x + \frac{2}{3}y + \frac{1}{9} &= 0 \end{aligned}$$

$$C = \left(-\frac{1}{6}, -\frac{1}{3}\right)$$

$$\begin{aligned} r &= \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2 - \frac{1}{9}} \\ &= \frac{1}{6} \end{aligned}$$

$$\begin{aligned} d &= \left|\frac{3(-\frac{1}{6}) + 0\left(-\frac{1}{3}\right) + 1}{\sqrt{3^2 + 0^2}}\right| \\ &= \left|\frac{-\frac{1}{2} + 1}{3}\right| \\ &= \frac{1}{6} \end{aligned}$$

$\because d = r$

$\therefore$ The line $3x + 1 = 0$ is tangent to the circle $x^2 + y^2 + \frac{1}{3}x + \frac{2}{3}y + \frac{1}{9} = 0$

10. Find the length of the tangent from the following circles to the following circles:
    (a) $(-2, 3)$, $x^2 + y^2 - 6x - 2y = 0$
    (b) $(-6, 0)$, $x^2 + y^2 - 6x + 2y + 8 = 0$
    (c) $(2, 2)$, $2x^2 + 2y^2 + 2x + 4y - 3 = 0$

11. If the following lines and circles are tengant to each other, find the value of k:
    (a) $4x + 3y - k = 0$, $x^2 + y^2 - 6x + 4y - 12 = 0$
    (b) $x + 3y + k = 0$, $2x^2 + 2y^2 + 12y + 13 = 0$

12. Find the maximum and minimum distance of the point $P(-2, 5)$ from the circle $x^2 + y^2 - 2x - 2y + 1 = 0$.

13. Find the maximum and minimum distance of the point $Q(0, 1)$ from the circle $x^2 + y^2 - 6x - 10y - 2 = 0$.

14. Assume that the maximum and minimum distance of the point $R(5, 2)$ from the circle $x^2+y^2-4x+4y-1 = 0$ are M and N respectively, find the product of M and N.

15.4 Revision Exercise 16

1. Find the euqation of the following circles:
    (a) A circle with centre $(1, -1)$ and radius 3.
    (b) A circle with centre $(2, -3)$ and radius 7.

2. Find the equation of the circle with centre at the origin and passes through the point $(2, -1)$.

3. Find the equation of the circle with centre at $(-5, 6)$ and passes through the point $(2, 3)$.

4. Find the equation of the circle with diameter connecting the points $(2, -5)$ and $(8, 1)$.

5. Find the centre and radius of the following circle:
    (a) $x^2 + y^2 - 6x + 14y + 50 = 0$
    (b) $x^2 + y^2 + 5x - 2y + 1 = 0$
    (c) $3x^2 + 3y^2 + 6x - 12y + 1 = 0$
    (d) $4x^2 + 4y^2 - 12x + 16y - 7 = 0$

6. Find the equation of the circle that passes through the following three points:
    (a) $(-1, -1)$, $(-3, 5)$, $(1, 3)$
    (b) $(2, 1)$, $(2, -4)$, $(3, -5)$
    (c) $(0, 0)$, $(0, a)$, $(b, 0)$

7. Given the radius of of the circle $x^2 + y^2 - 8x + 10y + c$ is 9, find the value of c.

8. Given two circles $x^2 + y^2 - 2x - 4y - 95 = 0$ and $x^2 + y^2 - 8x - 12y + 48 = 0$, find the distance between their centres.

9. Find the equation of the circle with centre at $(1, -1)$ and is tangent to the line $5x - 12y + 9 = 0$.

10. Find the equation of the circle that passes through the points $(1, -1)$ and $(1, 1)$, and is tangent to the line $x - 2 = 0$.

11. Find the equation of the circle that passes through the points $(6, 4)$ and $(1, 7)$, and its centre is on the line $2x - 3y = 6$.

12. Find the equation of the circle that passes through the points $(-1, 1)$ and $(1, 3)$, and its centre is on x-axis.

13. Find the equation of the circle that is tangent to the line $3x + 4y + 18 = 0$ at point $(-2, -3)$, and its centre is on the line $x - y = 0$.

14. If the following lines and circles are tangent to each other, find the value of k:
    (a) $2x - y + k = 0$, $x^2 + y^2 - 1 = 0$
    (b) $2x + 3y + 3\sqrt{13} = 0$, $x^2 + y^2 = k$
    (c) $y = x + k$, $x^2 + y^2 = 9$

15. If the circle $x^2 + y^2 - 6y - 4y + k = 0$ is tangent to the x-axis, find the value of k and the coordinates of the point of tangency.

16. Given the coordinates and equations of the following points and circles respectively, find the length of the tangent from the point to the circle:
    (a) $(1, 6)$, $x^2 + y^2 + 2x - 19 = 0$
    (b) $(2, 4)$, $x^2 + y^2 - 2x + 6y + 9 = 0$
    (c) $(3, 2)$, $2x^2 + 2y^2 + 10x + 11y - 52 = 0$
    (d) $(0, 0)$, $x^2 + y^2 - 2ax + 4ay + 4a^2 = 0$

17. Prove that the distance of the tangent from the point $A(3, -4)$ to the circle $C_1 : x^2 + y^2 - 10x - 7y + 13 = 0$ is equal to the distance of the tangent to the circle $C_2 : x^2 + y^2 - 10x - 7y + 26 = 0$.

18. Find the longest and the shortest distance of the point $P(-5, -12)$ to the circle $x^2 + y^2 + 8y - 6y = 0$.

19. Given the equation of circle $x^2 + y^2 + 8y - 6y = 0$.

    (a) Find the centre and radius of the circle.

    (b) Prove that $P(-2, 7)$ is on the circle.

    (c) Find the equation of chord of the circle that is split into two equal parts by the point $P(-2, 7)$.