1. Given that $f : x \to 3x + 4$, find

   (a) $f^{-1}(x)$,
   Solution:
   Let $y = f^{-1}(x)$, then $f(y) = x$.

$$\begin{aligned} x &= 3y + 4 \\ 3y &= x - 4 \\ y &= \frac{x-4}{3} \\ f^{-1}(x) &= \frac{x-4}{3} \end{aligned}$$

   (b) $f^{-1}(5)$,
   Solution:

$$\begin{aligned} f^{-1}(5) &= \frac{5-4}{3} \\ &= \frac{1}{3} \end{aligned}$$

   (c) the value of x such that $f(x) = 7$.
   Solution:

$$\begin{aligned} f(x) &= 7 \\ 3x + 4 &= 7 \\ 3x &= 3 \\ x &= 1 \end{aligned}$$

2. Given that the function $f : x \to x - 4$ and $g : x \to x^2 - 3x + 5$, find

   (a) $fg(2)$,
   Solution:

$$\begin{aligned} fg(2) &= f(g(2)) \\ &= f(2^2 - 3(2) + 5) \\ &= f(4 - 6 + 5) \\ &= f(3) \\ &= 3 - 4 \\ &= -1 \end{aligned}$$

(b) the values of x when $fg(x) = 5$.
Solution:

$$\begin{aligned}
fg(x) &= 5 \\
f(x^2 - 3x + 5) &= 5 \\
x^2 - 3x + 5 - 4 &= 5 \\
x^2 - 3x - 4 &= 0 \\
(x-4)(x+1) &= 0 \\
x &= -1 \text{ or } x = 4
\end{aligned}$$

3. (a) Form a quadratic equation which has the roots 4 and -3.
Solution:

$$\begin{aligned}
(x-4)(x+3) &= 0 \\
x^2 + 3x - 4x - 12 &= 0 \\
x^2 - x - 12 &= 0
\end{aligned}$$

(b) It is given that $x^2 - 4ax + 3b = 0$ has two equal real roots. Express a in terms of b.
Solution:

$$\begin{aligned}
b^2 - 4ac &= 0 \\
16a^2 - 12b &= 0 \\
16a^2 &= 12b \\
a^2 &= \frac{12b}{16} \\
&= \frac{3b}{4} \\
a &= \pm\sqrt{\frac{3b}{4}} = \pm\frac{1}{2}\sqrt{3b}
\end{aligned}$$

4. (a) Solve the equation:

$$3^{x-5} = 35^{x-3}$$

Solution:

$$\begin{aligned}
3^{x-5} &= 35^{x-3} \\
\log 3^{x-5} &= \log 35^{x-3} \\
(x-5)\log 3 &= (x-3)\log 35 \\
x\log 3 - 5\log 3 &= x\log 35 - 3\log 35 \\
x\log 3 - x\log 35 &= 5\log 3 - 3\log 35 \\
x(\log 3 - \log 35) &= 5\log 3 - 3\log 35 \\
x &= \frac{5\log 3 - 3\log 35}{\log 3 - \log 35} \\
&\approx 2.106
\end{aligned}$$

(b) Solve $\ln(5x-3)=7$.
Solution:

$$\begin{aligned} e^{\ln(5x-3)} &= e^7 \\ 5x-3 &= e^7 \\ 5x &= e^7+3 \\ x &= \frac{e^7+3}{5} \\ &\approx 219.93 \end{aligned}$$

5. (a) Find the sum of the first 25 terms of an arithmetic progression $3, 5, 7, \ldots$
Solution:
From the progression, we know that $a=3$, $d=5-3=2$, and $n=25$.

$$\begin{aligned} S_n &= \frac{n}{2}\left[2a+(n-1)d\right] \\ &= \frac{25}{2}\left[2(3)+(25-1)2\right] \\ &= 675 \end{aligned}$$

(b) It is given that the sum of the first ten terms of an arithmetic progression is 430 and the sum of the next ten terms is 630. Find the first term and the common difference of the progression.
Solution:
Let a be the first term and d be the common difference. Then

$$\begin{aligned} S_{10} &= \frac{10}{2}\left[2a+(10-1)d\right] = 430 \\ &= 5(2a+9d) = 430 \\ 2a+9d &= 86 \quad \cdots \; (1) \\ S_{20}-S_{10} &= 630 \\ \frac{20}{2}\left[2a+(20-1)d\right]-430 &= 630 \\ 10(2a+19d) &= 1060 \\ 2a+19d &= 106 \quad \cdots \; (2) \end{aligned}$$

$$\begin{aligned} (2)-(1) \Rightarrow 10d &= 20 \\ d &= 2 \\ 2a+9(2) &= 86 \\ 2a+18 &= 86 \\ 2a &= 68 \\ a &= 34 \end{aligned}$$

$\therefore$ the first term is 34 and the common difference is 2.

6. (a) Prove that $\dfrac{\cos(A+B)}{\sin A \cos B} = \cot A - \tan B$.

Proof:

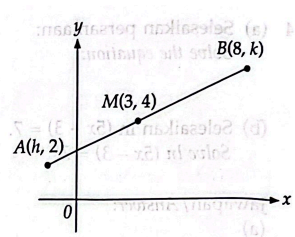
$$\begin{aligned}
\text{L.H.S.} &= \frac{\cos(A+B)}{\sin A \cos B} \\
&= \frac{\cos A \cos B - \sin A \sin B}{\sin A \cos B} \\
&= \frac{\cos A \cos B}{\sin A \cos B} - \frac{\sin A \sin B}{\sin A \cos B} \\
&= \frac{\cos A}{\sin A} - \frac{\sin B}{\cos B} \\
&= \cot A - \tan B \\
&= \text{R.H.S.}
\end{aligned}$$

(b) Solve the equation $2\cos 2x = 1 - \cos x$ for $0^\circ \leq x \leq 360^\circ$.

Solution:

$$\begin{aligned}
2\cos 2x &= 1 - \cos x \\
2(2\cos^2 x - 1) &= 1 - \cos x \\
4\cos^2 x - 2 &= 1 - \cos x \\
4\cos^2 x + \cos x - 3 &= 0 \\
(4\cos x - 3)(\cos x + 1) &= 0 \\
\cos x = \frac{3}{4} \quad &\text{or} \quad \cos x = -1 \\
x &= 41.41^\circ,\ 180^\circ,\ 318.59^\circ
\end{aligned}$$

7. Diagram 1 shows a straight line AB. It is given that M is the midpoint of the straight line AB and the coordinates of points A, M, and B are $(h, 2)$, $(3, 4)$, and $(8, k)$ respectively.



Find:

(a) the value of h and k,

Solution:

$$\frac{8+h}{2} = 3$$

$$8 + h = 6$$

$$h = -2$$

$$\frac{k+2}{2} = 4$$

$$k + 2 = 8 \qquad\qquad k = 6$$

(b) the gradient of the straight line AB, and

Solution:

$$\begin{aligned} m &= \frac{6-4}{8-3} \\ &= \frac{2}{5} \end{aligned}$$

(c) the equation fo the perpendicular bisector of AB.

Solution:

$$\begin{aligned} \text{Gradient of the perpendicular bisector} &= -\frac{1}{m} \\ &= -\frac{1}{\frac{2}{5}} \\ &= -\frac{5}{2} \end{aligned}$$

Therefore the equation of the perpendicular bisector is

$$\begin{aligned} y - 4 &= -\frac{5}{2}(x-3) \\ 2y - 8 &= -5x + 15 \\ 2y &= -5x + 23 \\ y &= -\frac{5}{2}x + \frac{23}{2} \end{aligned}$$

8. It is given that $y = 2x^2 - 4x + 6$. When $x = 2$, there is a small change of x of 3%. Find the corresponding percentage of change in y.

Solution:

$$\begin{aligned} y &= 2x^2 - 4x + 6 \\ \frac{dy}{dx} &= 4x - 4 \\ \frac{\delta y}{\delta x} &\approx \frac{dy}{dx} \\ \delta y &\approx \frac{dy}{dx}\delta x \end{aligned}$$

When $x = 2$, $\delta x = 3\%x = 0.03x = 0.06$, $\frac{dy}{dx} = 4(2) - 4 = 4$. Therefore

$$\begin{aligned} \delta y &\approx 4 \times 0.06 \\ &= 0.24 \end{aligned}$$

Therefore the percentage change in y is

$$\frac{\delta y}{y} \times 100 = \frac{0.24}{2(2)^2 - 4(2) + 6} \times 100$$

$$
\begin{aligned}
&= \frac{0.24}{6} \times 100 \\
&= 4\%
\end{aligned}
$$

9. It is given that the position vector of town A is $-8\vec{\imath} + 8\vec{\jmath}$ and the position vector of town B is $8\vec{\imath} - 9\vec{\jmath}$. THe towns A, B, and C are collinear such that the distance between town A and town C is twice the distance between town A and town B. THe distance between the towns is in km. Find

   (a) $\overrightarrow{AB}$

   Solution:

$$
\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\
&= 8\vec{\imath} - 9\vec{\jmath} - (-8\vec{\imath} + 8\vec{\jmath}) \\
&= 16\vec{\imath} - 17\vec{\jmath}
\end{aligned}
$$

   (b) the distance, in km, between town A and town B.

   Solution:

$$
\begin{aligned}
\text{Distance} &= |AB| \\
&= \sqrt{(16)^2 + (-17)^2} \\
&= \sqrt{256 + 289} \\
&= \sqrt{545} \\
&\approx 23.52 \text{ km}
\end{aligned}
$$

   (c) $\overrightarrow{OC}$

   Solution:

$$
\begin{aligned}
\overrightarrow{AC} &= 2\overrightarrow{AB} \\
&= 2(16\vec{\imath} - 17\vec{\jmath}) \\
&= 32\vec{\imath} - 34\vec{\jmath} \\
\overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA} \\
\overrightarrow{OC} &= \overrightarrow{AC} + \overrightarrow{OA} \\
&= 32\vec{\imath} - 34\vec{\jmath} + (-8\vec{\imath} + 8\vec{\jmath}) \\
&= 24\vec{\imath} - 26\vec{\jmath}
\end{aligned}
$$

10. Given that $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 6x - 8$ adn the gradient function of the curve is 2 when $x = 1$. Find the equation of the curve which passes through the point $(2, 3)$.

Solution:

$$\begin{aligned}\frac{d}{dx}\left(\frac{dy}{dx}\right) &= 6x - 8\\ \frac{dy}{dx} &= \int (6x - 8)dx\\ &= 3x^2 - 8x + c\end{aligned}$$

When $x = 1$, $\frac{dy}{dx} = 2$.

$$\begin{aligned}2 &= 3(1)^2 - 8(1) + c\\ c &= 2 - 3 + 8\\ &= 7\end{aligned}$$

Therefore $\frac{dy}{dx} = 3x^2 - 8x + 7$.

$$\begin{aligned}\frac{dy}{dx} &= 3x^2 - 8x + 7\\ \int dy &= \int (3x^2 - 8x + 7)dx\\ y &= x^3 - 4x^2 + 7x + c\end{aligned}$$

When $x = 2$, $y = 3$.

$$\begin{aligned}3 &= 2^3 - 4(2)^2 + 7(2) + c\\ c &= 3 - 8 + 16 - 14\\ &= -3\end{aligned}$$

Therefore the equation of the curve is $y = x^3 - 4x^2 + 7x - 3$.

11. (a) Five workers, A, B, C, D, and E, are to be arranged in a row. Find the number of possible arrangement if

    i. there are no conditions,
    Solution:
    The number of possible arrangements is ${}_5P_5 = 5! = 120$.

    ii. the workers A and B are always together.
    Solution:
    Since A and B are always together, we can treat them as one entity. Therefore the number of possible arrangements is ${}_4P_4 \times 2! = 48$.

(b) A class monitor wants to divide 10 students into three groups such that the groups consist of 2 members, 3 members and 5 members. Find the number of different ways to divide all the students.

Solution:

First, choose 2 students from 10, we have ${}_{10}C_2$. Then choose 3 students from remaining 8, we have ${}_8C_3$. The remaining 5 students form the last group. Therefore the number of different ways to divide all the students is

$$\begin{aligned} {}_{10}C_2 \times {}_8C_3 &= 45 \times 56 \\ &= 2520 \end{aligned}$$

12. During winter, the probability that it will snow on a particular day is 0.55. Find the probability that in a particular week, it will snow

(a) exactly 3 days,

Solution:

The probability that it will snow on a particular day is 0.55. Therefore the probability that it will not snow on a particular day is $1 - 0.55 = 0.45$. The probability that it will snow exactly 3 days in a week is

$${}_7C_3(0.55)^3(0.45)^4 = 0.2388$$

(b) more than 2 days.

Solution:

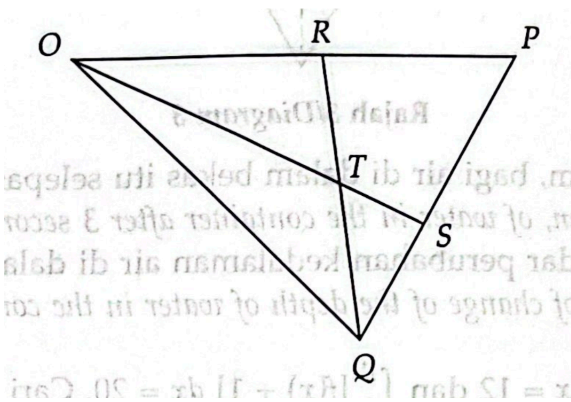
$$\begin{aligned} P(\text{more than 2 days}) &= 1 - P(\text{less than or equal to 2 days}) \\ &= 1 - \left[{}_7C_0(0.55)^0(0.45)^7 + {}_7C_1(0.55)^1(0.45)^6 + {}_7C_2(0.55)^2(0.45)^5\right] \\ &= 1 - [0.003736 + 0.031969 + 0.117221] \\ &= 1 - 0.152926 \\ &= 0.8471 \end{aligned}$$

(c) Given that a discrete random variable $X \sim B(160, 0.3)$, find the mean and the variance.

Solution:

$$\begin{aligned} \mu &= np = 160 \times 0.3 = 48 \\ \sigma^2 &= npq = 160 \times 0.3 \times 0.7 = 33.6 \end{aligned}$$

13. Diagram 2 shows the road in a residential area. It is given that $OP = \vec{u}$, $OQ = \vec{v}$, $3OR = 2OP$ and $PQ = 2PS$. Building R is situated in road OP, building S is situated in road PQ and building T is situated at the intersection of the roads QR and OS.



(a) Express the following vectors in terms of $\vec{u}$ and $\vec{v}$.

i. $\overrightarrow{QR}$

Solution:

$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{OR} - \overrightarrow{OQ}\\ &= \frac{2}{3}\overrightarrow{OP} - \overrightarrow{OQ}\\ &= \frac{2}{3}\vec{u} - \vec{v}\end{aligned}$$

ii. $\overrightarrow{OS}$

Solution:

$$\begin{aligned}\overrightarrow{OS} &= \overrightarrow{OQ} + \overrightarrow{QS}\\ &= \vec{v} - \frac{1}{2}\overrightarrow{PQ}\\ &= \vec{v} - \frac{1}{2}(\overrightarrow{PO} + \overrightarrow{OQ})\\ &= \vec{v} - \frac{1}{2}(-\vec{u} + \vec{v})\\ &= \frac{1}{2}\vec{v} + \frac{1}{2}\vec{u}\\ &= \frac{1}{2}(\vec{u} + \vec{v})\end{aligned}$$

(b) It is given that $OT = mOS$ and $QT = nQR$, where m and n are constants. Express $\overrightarrow{OT}$ in terms of

i. m, $\vec{u}$, and $\vec{v}$,

Solution:

$$\overrightarrow{OT} = m\overrightarrow{OS}$$

$$
\begin{aligned}
&= m\left[\frac{1}{2}(\vec{u}+\vec{v})\right] \\
&= \frac{m}{2}(\vec{u}+\vec{v})
\end{aligned}
$$

ii. n, $\vec{u}$, and $\vec{v}$.

Solution:

$$
\begin{aligned}
\overrightarrow{OT} &= \overrightarrow{OQ} + \overrightarrow{QT} \\
&= \vec{v} + n\overrightarrow{QR} \\
&= \vec{v} + n\left[\frac{2}{3}\vec{u} - \vec{v}\right] \\
&= \vec{v} + \frac{2}{3}n\vec{u} - n\vec{v} \\
&= \frac{2}{3}n\vec{u} + (1-n)\vec{v}
\end{aligned}
$$

(c) Hence, find the value of m and n.

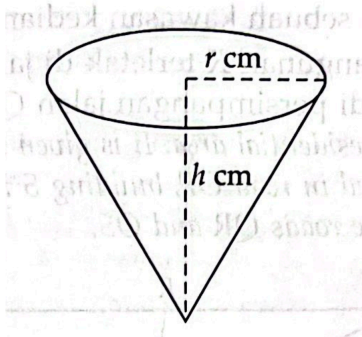
Solution:

$$
\begin{aligned}
\frac{m}{2}(\vec{u}+\vec{v}) &= \frac{2}{3}n\vec{u} + (1-n)\vec{v} \\
3m\vec{u} + 3m\vec{v} &= 4n\vec{u} + 6(1-n)\vec{v}
\end{aligned}
$$

Equating the coefficients of $\vec{u}$ and $\vec{v}$, we have

$$
\begin{aligned}
3m &= 4n \\
3m &= 6 - 6n \\
4n &= 6 - 6n \\
10n &= 6 \\
n &= \frac{3}{5} \\
m &= \frac{4}{5}
\end{aligned}
$$

14. Diagram 4 shows a conical container with a radius of r cm and a height of h cm. Water is poured into the container at a constant rate of 20 cm^3s^{-1}. it is given that the height of the container is three times the radius.



(a) i. Find the radius, in cm, of the water in the container after 3 seconds.

Solution:

The volume of the water in the container after 3 seconds is $20 \times 3 = 60$ cm^3.

$$\begin{aligned}
V &= \frac{1}{3}\pi r^2 h \\
60 &= \frac{1}{3}\pi r^2 (3r) \\
60 &= \pi r^3 \\
r^3 &= \frac{60}{\pi} \\
r &= \sqrt[3]{\frac{60}{\pi}} \\
r &\approx 2.67 \text{ cm}
\end{aligned}$$

ii. Hence, find the rate of change of the depth of water in the container at that moment.

Solution:

$$\begin{aligned}
h &= 3r \\
r &= \frac{h}{3} \\
V &= \frac{1}{3}\pi r^2 h \\
&= \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h \\
&= \frac{1}{27}\pi h^3 \\
\frac{dV}{dh} &= \frac{1}{9}\pi h^2 \\
\frac{dV}{dt} &= 20 \\
\frac{dh}{dt} &= \frac{dh}{dV} \cdot \frac{dV}{dt}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\frac{dV}{dh}} \cdot 20 \\
&= \frac{1}{\frac{1}{9}\pi h^2} \cdot 20 \\
&= \frac{9}{\pi h^2} \cdot 20 \\
&= \frac{180}{\pi h^2}
\end{aligned}
$$

When $r = \sqrt[3]{\frac{60}{\pi}}$,

$$
\begin{aligned}
60 &= \frac{1}{3}\pi r^2 h \\
180 &= \pi \left(\sqrt[3]{\frac{60}{\pi}}\right)^2 h \\
h &= \frac{180}{\pi \left(\sqrt[3]{\frac{60}{\pi}}\right)^2}
\end{aligned}
$$

Therefore the rate of change of the depth of water in the container at that moment is

$$
\begin{aligned}
\frac{dh}{dt} &= \frac{180}{\pi \left(\frac{180}{\pi \left(\sqrt[3]{\frac{60}{\pi}}\right)^2}\right)^2} \\
&\approx 0.891 \text{ cm s}^{-1}
\end{aligned}
$$

(b) It is given that $\int_3^p f(x)dx = 12$ and $\int_3^p [f(x)+1]dx = 20$. Find the value of p.

Solution:

$$
\begin{aligned}
\int_3^p [f(x)+1]dx &= 20 \\
\int_3^p f(x)dx + \int_3^p 1dx &= 20 \\
12 + p - 3 &= 20 \\
p &= 11
\end{aligned}
$$

15. Skip cuz I lazy to do yay =)