

# Notes for Calculus III

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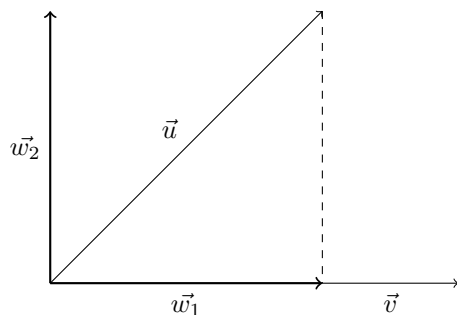
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## Chapter 4

# Projections

Given two vectors  $\vec{v}$  and  $\vec{u}$ . Construct a vector  $\vec{w}_1$  from the terminal point of  $\vec{u}$  perpendicular to  $\vec{v}$ . The vector that starts from the initial point of  $\vec{u}$  and ends at the intersection of the line and  $\vec{v}$  is called the **projection of  $\vec{u}$  onto  $\vec{v}$** , which is also known as the **vector component of  $\vec{u}$  along  $\vec{v}$** .



From the diagram, it is not hard to see that  $\vec{u} = \vec{w}_1 + \vec{w}_2$

Hence, the vector component of  $\vec{u}$  orthogonal to  $\vec{v}$  is given by  $\vec{w}_2 = \vec{u} - \vec{w}_1$

Let  $\vec{w}_1 = t\vec{v}$ , for some scalar  $t$ .

Then  $\vec{w}_2 = \vec{u} - t\vec{v}$  is orthogonal to  $\vec{v}$ , which implies that  $\vec{w}_2 \cdot \vec{v} = 0$ .

$$\vec{w}_2 \cdot \vec{v} = (\vec{u} - t\vec{v}) \cdot \vec{v} = 0$$

$$\vec{u} \cdot \vec{v} - t\vec{v} \cdot \vec{v} = 0$$

$$t = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$$

Therefore, The projection of  $\vec{u}$  onto  $\vec{v}$  is given by

$$proj_{\vec{v}} \vec{u} = \vec{w}_1 = t\vec{v} = \left( \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v} = \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v} = \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \right) \frac{\vec{v}}{\|\vec{v}\|}$$

where  $\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$  is the **scalar projection** of  $\vec{u}$  onto  $\vec{v}$ , denoted by  $comp_{\vec{v}} \vec{u}$ .

**Example 1.** Find the projection of  $\vec{u} = \langle 6, 7 \rangle$  onto  $\vec{v} = \langle 1, 4 \rangle$ . Hence, find the vector component of  $\vec{u}$  orthogonal to  $\vec{v}$ .

$$\begin{aligned} \text{proj}_{\vec{v}} \vec{u} &= \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v} \\ &= \left( \frac{6(1) + 7(4)}{1^2 + 4^2} \right) \langle 1, 4 \rangle \\ &= \left( \frac{34}{17} \right) \langle 1, 4 \rangle \\ &= \langle 2, 8 \rangle \end{aligned}$$

$$\begin{aligned} \vec{w}_2 &= \vec{u} - \vec{w}_1 \\ &= \langle 6, 7 \rangle - \langle 2, 8 \rangle \\ &= \langle 4, -1 \rangle \end{aligned}$$

**Example 2.** Find the projection of  $\vec{u} = 2i + 3j$  onto  $\vec{v} = 5i + j$ . Hence, find the vector component of  $\vec{u}$  orthogonal to  $\vec{v}$ .

$$\begin{aligned} \text{proj}_{\vec{v}} \vec{u} &= \left( \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v} \\ &= \left( \frac{2(5) + 3(1)}{5^2 + 1^2} \right) (5i + j) \\ &= \left( \frac{13}{26} \right) (5i + j) \\ &= \left( \frac{5}{2} \right) i + \left( \frac{1}{2} \right) j \end{aligned}$$

$$\begin{aligned} \vec{w}_2 &= \vec{u} - \vec{w}_1 \\ &= (2i + 3j) - \left( \left( \frac{5}{2} \right) i + \left( \frac{1}{2} \right) j \right) \\ &= \left( -\frac{1}{2} \right) i + \left( \frac{5}{2} \right) j \end{aligned}$$

**Example 3.** Find the scalar projection of the force  $\vec{F} = 4i - 2j + 3k$  in the direction of the vector  $v = i - j + 2k$ .

**Solution.**

$$\begin{aligned} \text{comp}_{\vec{v}} \vec{F} &= \frac{\vec{F} \cdot \vec{v}}{\|\vec{v}\|} \\ &= \frac{4(1) + (-2)(-1) + 3(2)}{\sqrt{1^2 + (-1)^2 + 2^2}} \\ &= \frac{4 + 2 + 6}{\sqrt{6}} \\ &= \frac{12}{\sqrt{6}} \\ &= 2\sqrt{6} \end{aligned}$$

**Notes:** Selected exercises are mixed in the exercises of the previous chapters.