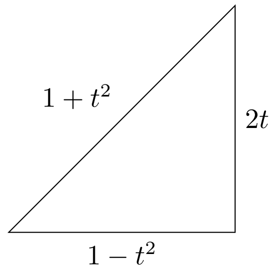
8.3 倍角及半角的三角函数

1. 试证公式 $\sin\theta = \dfrac{2t}{1+t^2}, \cos\theta = \dfrac{1-t^2}{1+t^2}$, 其中 $t = \tan\dfrac{\theta}{2}$。然后据此, 或用其他方法, 证明: 若 $0 \le \theta \le \dfrac{\pi}{2}$, 则 $\dfrac{1-\sin\theta}{1+\sin\theta} = \tan^2\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$ 及 $1 \le \dfrac{1+\sin\theta+\cos\theta}{1+\cos\theta} \le 2$。

解:

Let $t = \tan\dfrac{\theta}{2}$, then $\tan\theta = \dfrac{2t}{1-t^2}$,



$$\begin{aligned}\sin\theta &= \frac{2t}{1+t^2},\\ \cos\theta &= \frac{1-t^2}{1+t^2}\end{aligned} \quad \blacksquare$$

$$L.H.S. = \frac{1-\sin\theta}{1+\sin\theta} = \frac{1-\dfrac{2t}{1+t^2}}{1+\dfrac{2t}{1+t^2}} = \frac{1+t^2-2t}{1+t^2+2t} = \frac{(1-t)^2}{(1+t)^2}$$

$$R.H.S. = \tan^2\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \left(\frac{\tan\dfrac{\pi}{4} - \tan\dfrac{\theta}{2}}{1+\tan\dfrac{\pi}{4}\tan\dfrac{\theta}{2}}\right)^2 = \left(\frac{1-\tan\dfrac{\theta}{2}}{1+\tan\dfrac{\theta}{2}}\right)^2 = \frac{(1-t)^2}{(1+t)^2}$$

$\because L.H.S. = R.H.S.$, hence proved. $\blacksquare$

$$\begin{aligned}\frac{1+\sin\theta+\cos\theta}{1+\cos\theta} &= \frac{1+\dfrac{2t}{1+t^2}+\dfrac{1-t^2}{1+t^2}}{1+\dfrac{1-t^2}{1+t^2}}\\ &= \frac{1+t^2+2t+1-t^2}{1+t^2+1-t^2}\\ &= \frac{2+2t}{2}\\ &= 1+t\\ &= 1+\tan\frac{\theta}{2}\end{aligned}$$

$$0 \le \tan\frac{\theta}{2} \le 1 \Rightarrow 1 \le 1+\tan\frac{\theta}{2} \le 2 \quad \blacksquare$$

2. 从 $\tan\theta = a$ 及 $\cos 2\theta = b$ 二式中消去 θ, 写出 a 与 b 之间的关系式子。

解:

$$
\begin{aligned}
\tan\theta &= a \\
\frac{\sin\theta}{\cos\theta} &= a \\
\sin\theta &= a\cos\theta \\
\sin^2\theta &= a^2\cos^2\theta \\
1-\cos^2\theta &= a^2\cos^2\theta \\
1 &= a^2\cos^2\theta + \cos^2\theta \\
1 &= (a^2+1)\cos^2\theta \\
\cos^2\theta &= \frac{1}{a^2+1}
\end{aligned}
$$

$$
\begin{aligned}
\cos 2\theta &= b \\
\cos^2\theta - \sin^2\theta &= b \\
\cos^2\theta - a^2\cos^2\theta &= b \\
(1-a^2)\cos^2\theta &= b \\
\cos^2\theta &= \frac{b}{1-a^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{a^2+1} &= \frac{b}{1-a^2} \\
b &= \frac{1-a^2}{1+a^2}
\end{aligned}
$$

■

3. 已知 $t = \tan\frac{\theta}{2}$, 试证

(a) $\sin\theta = \frac{2t}{1+t^2}$;

(b) $\cos\theta = \frac{1-t^2}{1+t^2}$。

解:

Same as question 1. ■

4. 试证明 $\cos^2 x + \cos^2\left(x+\frac{2\pi}{3}\right) + \cos^2\left(x+\frac{4\pi}{3}\right) = \frac{3}{2}$。

解:

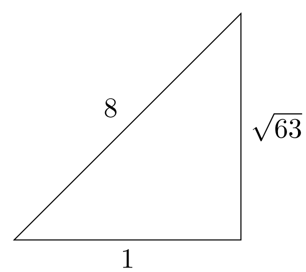
$$
\begin{aligned}
&\cos^2 x + \cos^2\left(x+\frac{2\pi}{3}\right) + \cos^2\left(x+\frac{4\pi}{3}\right) \\
&= \cos^2 x + \left[\cos x\cos\frac{2\pi}{3} - \sin x\sin\frac{2\pi}{3}\right]^2 + \left[\cos x\cos\frac{4\pi}{3} - \sin x\sin\frac{4\pi}{3}\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&= \cos^2 x + \left[-\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right]^2 + \left[-\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right]^2 \\
&= \cos^2 x + \frac{1}{4}\cos^2 x + \frac{3}{4}\sin^2 x + \frac{1}{4}\cos^2 x + \frac{3}{4}\sin^2 x \\
&= \frac{3}{2}\cos^2 x + \frac{3}{2}\sin^2 x \\
&= \frac{3}{2}(\cos^2 x + \sin^2 x) \\
&= \frac{3}{2}
\end{aligned}
$$

■

5. 已知 $\cos y = \frac{1}{8}$, 不许查表或用计算机, 求 (i) $\cos 2y$ 及 (ii) $\cos\frac{1}{2}y$ 之值。

解:



$$
\sin y = \frac{\sqrt{63}}{8} \qquad \cos y = \frac{1}{8}
$$

$$
\cos 2y = 2\cos^2 y - 1 = 2\left(\frac{1}{8}\right)^2 - 1 = -\frac{31}{32}
$$

$$
\cos\frac{1}{2}y = \pm\sqrt{\frac{1+\cos y}{2}} = \pm\sqrt{\frac{1+\frac{1}{8}}{2}} = \pm\sqrt{\frac{9}{16}} = \pm\frac{3}{4}
$$

■

6. 试证 $\sin 3\theta = 4\sin\theta\sin(60^\circ+\theta)\sin(60^\circ-\theta)$。

解:

$$
\begin{aligned}
L.H.S. = \sin 3\theta &= \sin(2\theta+\theta) \\
&= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \\
&= 2\sin\theta\cos^2\theta + (1-2\sin^2\theta)\sin\theta \\
&= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta \\
&= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta \\
&= 3\sin\theta - 4\sin^3\theta \\
R.H.S. = 4\sin\theta\sin(60^\circ+\theta)\sin(60^\circ-\theta) &= 4\sin\theta\left(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right)\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right) \\
&= 4\sin\theta\left(\frac{3}{4}\cos^2\theta - \frac{1}{4}\sin^2\theta\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 3\sin\theta\cos^2\theta - \sin^3\theta \\
&= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta \\
&= 3\sin\theta - 3\sin^3\theta - \sin^3\theta \\
&= 3\sin\theta - 4\sin^3\theta
\end{aligned}
$$

$\because L.H.S. = R.H.S.$, hence proved. ■

7. 试证 $4\sin^6\theta + 4\cos^6\theta + 3\sin^2 2\theta = 4$。

解:

$$
\begin{aligned}
&\ \ 4\sin^6\theta + 4\cos^6\theta + 3\sin^2 2\theta \\
&= 4(1-\cos^2\theta)^3 + 4\cos^6\theta + 12\sin^2\theta\cos^2\theta \\
&= 4(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta) + 4\cos^6\theta + 12\sin^2\theta(1-\sin^2\theta) \\
&= 4 - 12\cos^2\theta + 12\cos^4\theta - 4\cos^6\theta + 4\cos^6\theta + 12\sin^2\theta - 12\sin^4\theta \\
&= 4
\end{aligned}
$$
■

8. 不许应用对数表或计算机, 试求 $\tan 67\frac{1}{2}^\circ$ 之值。

解:

$$
\begin{aligned}
\tan 67\frac{1}{2}^\circ = \tan\left(\frac{135^\circ}{2}\right) = \sqrt{\frac{1-\cos 135^\circ}{1+\cos 135^\circ}} &= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}} \\
&= \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} = \sqrt{\frac{6+4\sqrt{2}}{2}} = \sqrt{3+2\sqrt{2}} \\
&= \sqrt{2+1+2\sqrt{2\cdot 1}} = \sqrt{(1+\sqrt{2})^2} = 1+\sqrt{2}
\end{aligned}
$$
■

9. 试证 $\frac{1-\cos x}{1+\cos x} = \tan^2\frac{x}{2}$。
然后由上述结果推证 $\tan 15^\circ = 2-\sqrt{3}$。

解:

$$
\frac{1-\cos x}{1+\cos x} = \left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)^2 = \left(\tan\frac{x}{2}\right)^2 = \tan^2\frac{x}{2}
$$
■

$$
\begin{aligned}
\tan 15^\circ = \tan\frac{30^\circ}{2} = \sqrt{\frac{1-\cos 30^\circ}{1+\cos 30^\circ}} &= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{1+\frac{\sqrt{3}}{2}}} \\
&= \sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}} = \sqrt{\frac{(2-\sqrt{3})^2}{4-3}} = \sqrt{\frac{4-4\sqrt{3}+3}{1}} = \sqrt{7-4\sqrt{3}} \\
&= \sqrt{7-2\sqrt{12}} = \sqrt{4+3-2\sqrt{4\cdot 3}} = \sqrt{(2-\sqrt{3})^2} = 2-\sqrt{3}
\end{aligned}
$$
■

10. 已知 $\sin^2\theta + 6\sin\theta\cos\theta + 9\cos^2\theta \equiv a + b\sin 2\theta + c\cos 2\theta$，试求 a, b, c 之值。据此或其他方法，试求 $\sin^2 + 6\sin\theta\cos\theta + 9\cos^2\theta$ 的极大值和极小值。

解:

$$\begin{aligned}\sin^2\theta + 6\sin\theta\cos\theta + 9\cos^2\theta &= \frac{1-\cos 2\theta}{2} + 3\cdot 2\sin\theta\cos\theta + 9\left(\frac{1+\cos 2\theta}{2}\right)\\ &= \frac{1-\cos 2\theta + 9 + 9\cos 2\theta}{2} + 3\sin 2\theta\\ &= 5 + 3\sin 2\theta + 4\cos 2\theta\\ \therefore a = 5, b = 3, c = 4 \end{aligned}$$ ■

$$\begin{aligned}\frac{d}{d\theta}(5 + 3\sin 2\theta + 4\cos 2\theta) = 6\cos 2\theta - 8\sin 2\theta = 0\\ \tan 2\theta = \frac{3}{4}\\ 2\theta = k\pi + \arctan\frac{3}{4}\\ \theta = \frac{k\pi}{2} + \frac{1}{2}\arctan\frac{3}{4}\\ \frac{d^2}{d\theta^2}(5 + 3\sin 2\theta + 4\cos 2\theta) = -6\sin 2\theta - 8\cos 2\theta\end{aligned}$$

When $k = 0$, $\theta = \frac{1}{2}\arctan\frac{3}{4}$, $\frac{d^2}{d\theta^2} = -6\sin\arctan\frac{3}{4} - 8\cos\arctan\frac{3}{4} = -3.02 < 0$.

Hence, the max value is $5 + 3\sin 2\theta + 4\cos 2\theta = 5 + 3\sin\arctan\frac{3}{4} + 4\cos\arctan\frac{3}{4} = 10$. ■

When $k = 1$, $\theta = \frac{\pi}{2} + \frac{1}{2}\arctan\frac{3}{4}$, $\frac{d^2}{d\theta^2} = -6\sin\left(\pi + \arctan\frac{3}{4}\right) - 8\cos\left(\pi + \arctan\frac{3}{4}\right) = 3.02 > 0$.

Hence, the min value is $5 + 3\sin 2\theta + 4\cos 2\theta = 5 + 3\sin\left(\pi + \arctan\frac{3}{4}\right) + 4\cos\left(\pi + \arctan\frac{3}{4}\right) = 0$. ■

11. 证 $\operatorname{cosec}\theta - \cot\theta = \tan\frac{1}{2}\theta$。据此或其他方法，证 $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$。

解:

$$\operatorname{cosec}\theta - \cot\theta = \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \frac{1-\cos\theta}{\sin\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$ ■

$$\tan 22\frac{1}{2}^\circ = \tan\frac{45^\circ}{2} = \operatorname{cosec} 45^\circ - \cot 45^\circ = \sqrt{2} - 1$$ ■

12. 试证 $8\left(\sin^6\theta + \cos^6\theta\right) = 5 + 3\cos 4\theta$。

解:

$$\begin{aligned}L.H.S. &= 8\left(\sin^6\theta + \cos^6\theta\right)\\ &= 8\left((1-\cos^2\theta)^3 + \cos^6\theta\right)\end{aligned}$$

$$
\begin{aligned}
&= 8(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta + \cos^6\theta) \\
&= 8(1 - 3\cos^2\theta + 3\cos^4\theta) \\
R.H.S. &= 5 + 3\cos 4\theta \\
&= 5 + 3(2\cos^2 2\theta - 1) \\
&= 2 + 6\cos^2 2\theta \\
&= 2 + 6(2\cos^2\theta - 1)^2 \\
&= 2 + 6(4\cos^4\theta - 4\cos^2\theta + 1) \\
&= 2 + 24\cos^4\theta - 24\cos^2\theta + 6 \\
&= 8(1 - 3\cos^2\theta + 3\cos^4\theta)
\end{aligned}
$$

$\because L.H.S. = R.H.S.$, hence proved. ■

13. 已知 $\tan^2\alpha = 2\tan^2\beta + 1$, 试证 $\cos 2\alpha + \sin^2\beta = 0$。

解:

$$
\begin{aligned}
\tan^2\alpha &= 2\tan^2\beta + 1 \\
\frac{\sin^2\alpha}{\cos^2\alpha} &= 2\left(\frac{\sin^2\beta}{\cos^2\beta}\right) + 1 \\
\frac{1-\cos^2\alpha}{\cos^2\alpha} &= 2\left(\frac{1-\cos^2\beta}{\cos^2\beta}\right) + 1 \\
\frac{1}{\cos^2\alpha} - 1 &= 2\left(\frac{1}{\cos^2\beta} - 1\right) + 1 \\
\frac{1}{\cos^2\alpha} &= \frac{2}{\cos^2\beta} \\
\cos^2\beta &= 2\cos^2\alpha
\end{aligned}
$$

$$
\begin{aligned}
\cos 2\alpha + \sin^2\beta &= \cos 2\alpha + 1 - \cos^2\beta \\
&= 2\cos^2\alpha - 1 + 1 - 2\cos^2\alpha \\
&= 0 \qquad ■
\end{aligned}
$$

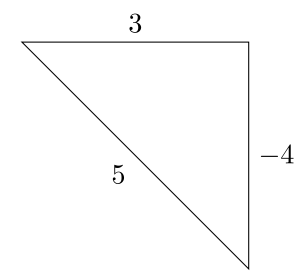
14. 已知 $x = \cos\theta$, 式中 $\frac{3}{2}\pi < \theta < 2\pi$, 且 $2\cos\theta - \sin\theta = 2$, 证明 $\sqrt{1-x^2} = 2(1-x)$。据此据此或其他方法, 求 x 的值并推证 $\tan 2\theta = \frac{24}{7}$。

解:

$$
\begin{aligned}
2\cos\theta - \sin\theta &= 2 \\
2\cos\theta - \sqrt{1-\cos^2\theta} &= 2 \\
2x - \sqrt{1-x^2} &= 2 \\
\sqrt{1-x^2} &= 2(1-x) \qquad ■
\end{aligned}
$$

$$
\sqrt{1-x^2} = 2(1-x)
$$

$$
\begin{aligned}
1-x^2 &= 4(1-x)^2 \\
1-x^2 &= 4(1-2x+x^2) \\
1-x^2 &= 4-8x+4x^2 \\
5x^2-8x+3 &= 0 \\
(5x-3)(x-1) &= 0 \\
x = 1 \text{ or } x &= \frac{3}{5} \\
\because x = \cos\theta,\ \theta &\in \left(\frac{3}{2}\pi, 2\pi\right) \\
\therefore\ x > 0 \Rightarrow x &= \frac{3}{5} \qquad \blacksquare
\end{aligned}
$$



$$
\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\cdot\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2} = \frac{24}{7} \qquad \blacksquare
$$

15. 已知 $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$, 求 $32\cos^4 15^\circ - 32\cos^2 15^\circ + 4$ 的值。

解:

$$
\begin{aligned}
32\cos^4 15^\circ - 32\cos^2 15^\circ + 4 &= 4(8\cos^4 15^\circ - 8\cos^2 15^\circ + 1) \\
&= 4\cos 60^\circ \\
&= 2 \qquad \blacksquare
\end{aligned}
$$

16. 试证明 $\sec x + \tan x = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$。据此, 求 $\tan\frac{3\pi}{8}$ 的值, 答案以根式表示。

解:

$$
L.H.S. = \sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \frac{1+\sin x}{\cos x}
$$

$$
R.H.S. = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}} = \frac{1+\tan\frac{x}{2}}{1-1\tan\frac{x}{2}} = \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}
$$

$$
= \frac{1+\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{1-\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}} = \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}} = \frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}} = \frac{1+\sin x}{\cos x}
$$

$\because L.H.S. = R.H.S.$, hence proved. $\blacksquare$

17. 证明 $\dfrac{1+\cos 2\theta+\sin 2\theta}{1-\cos 2\theta+\sin 2\theta}=\cot\theta$。

解:

$$\begin{aligned}
\frac{1+\cos 2\theta+\sin 2\theta}{1-\cos 2\theta+\sin 2\theta} &= \frac{1+2\cos^2\theta-1+2\sin\theta\cos\theta}{1-\cos 2\theta+2\sin\theta\cos\theta} \\
&= \frac{2\cos^2\theta+2\sin\theta\cos\theta}{2\times\dfrac{1-\cos^2\theta}{2}+2\sin\theta\cos\theta} \\
&= \frac{\cos^2\theta+\sin\theta\cos\theta}{\sin^2\theta+\sin\theta\cos\theta} \\
&= \frac{\cos\theta(\cos\theta+\sin\theta)}{\sin\theta(\sin\theta+\cos\theta)} \\
&= \frac{\cos\theta}{\sin\theta}=\cot\theta
\end{aligned}$$

■

18. 化简 $\cos^2\theta+\cos^2\left(\dfrac{\pi}{3}+\theta\right)+\cos^2\left(\dfrac{\pi}{3}-\theta\right)$。

解:

$$\begin{aligned}
&\cos^2\theta+\cos^2\left(\frac{\pi}{3}+\theta\right)+\cos^2\left(\frac{\pi}{3}-\theta\right) \\
&=\cos^2\theta+\left[\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}\right]^2+\left[\cos\theta\cos\frac{\pi}{3}+\sin\theta\sin\frac{\pi}{3}\right]^2 \\
&=\cos^2\theta+\left[\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\right]^2+\left[\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right]^2 \\
&=\cos^2\theta+\frac{1}{4}\cos^2\theta-\frac{\sqrt{3}}{2}\cos\theta\sin\theta+\frac{3}{4}\sin^2\theta+\frac{1}{4}\cos^2\theta+\frac{\sqrt{3}}{2}\cos\theta\sin\theta+\frac{3}{4}\sin^2\theta \\
&=\frac{3}{2}\sin^2\theta+\frac{3}{2}\cos^2\theta \\
&=\frac{3}{2}
\end{aligned}$$

■

19. 设 θ 满足 $\sin\theta+\cos\theta=\dfrac{4}{3}$, 且 $\sin 2\theta=\dfrac{p}{q}$, 式中 p 与 q 为互质的正整数, 求值。

解:

$$\begin{aligned}
\sin\theta+\cos\theta &= \frac{4}{3} \\
\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta &= \frac{16}{9} \\
1+2\sin\theta\cos\theta &= \frac{16}{9} \\
\sin 2\theta &= \frac{7}{9} \\
\therefore\ p=7,\ q&=9
\end{aligned}$$

■

20. 证明 $\dfrac{1-\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}=\tan\dfrac{\theta}{2}\tan\left(\dfrac{\theta}{2}-\dfrac{\pi}{4}\right)$。

解:

$$\begin{aligned}
\tan\frac{\theta}{2}\tan\left(\frac{\theta}{2}-\frac{\pi}{4}\right) &= \frac{1-\cos\theta}{\sin\theta}\cdot\frac{\tan\frac{\theta}{2}-1}{1+\tan\frac{\theta}{2}} \\
&= \frac{1-\cos\theta}{\sin\theta}\cdot\frac{\frac{\sin\theta}{1+\cos\theta}-1}{1+\frac{\sin\theta}{1+\cos\theta}} \\
&= \frac{1-\cos\theta}{\sin\theta}\cdot\frac{\sin\theta-1-\cos\theta}{1+\sin\theta+\cos\theta} \\
&= -\frac{1-\cos\theta}{\sin\theta}\cdot\frac{1-\sin\theta+\cos\theta}{1+\sin\theta+\cos\theta} \\
&= -\frac{(1-\cos\theta)(1-\sin\theta+\cos\theta)}{\sin\theta(1+\sin\theta+\cos\theta)} \\
&= -\frac{(1-\sin\theta+\cos\theta-\cos\theta+\sin\theta\cos\theta-\cos^2\theta)}{\sin\theta(1+\sin\theta+\cos\theta)} \\
&= -\frac{1-\sin\theta+\sin\theta\cos\theta-\cos^2\theta}{\sin\theta(1+\sin\theta+\cos\theta)} \\
&= -\frac{1-\sin\theta+\sin\theta\cos\theta-1+\sin^2\theta}{\sin\theta(1+\sin\theta+\cos\theta)} \\
&= -\frac{-\sin\theta+\sin\theta\cos\theta+\sin^2\theta}{\sin\theta(1+\sin\theta+\cos\theta)} \\
&= -\frac{-1+\cos\theta+\sin\theta}{1+\sin\theta+\cos\theta} \\
&= \frac{1-\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} \qquad \blacksquare
\end{aligned}$$

21. 若 $\tan x+\cot x=\dfrac{5}{2}$, 求 $\sin 2x$ 的值。

解:

$$\begin{aligned}
\tan x+\cot x &= \frac{5}{2} \\
\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x} &= \frac{5}{2} \\
\frac{\sin^2 x+\cos^2 x}{\sin x\cos x} &= \frac{5}{2} \\
\frac{1}{\sin x\cos x} &= \frac{5}{2} \\
\frac{2}{\sin 2x} &= \frac{5}{2} \\
\sin 2x &= \frac{4}{5} \qquad \blacksquare
\end{aligned}$$

22. 证明 $\dfrac{1-\cos 2\theta+\sin 2\theta}{1+\cos 2\theta+\sin 2\theta}=\tan\theta$。

解:

$$\begin{aligned}\frac{1-\cos 2\theta+\sin 2\theta}{1+\cos 2\theta+\sin 2\theta}&=\frac{2\times\dfrac{1-\cos 2\theta}{2}+2\sin\theta\cos\theta}{1+2\cos^2\theta-1+2\sin\theta\cos\theta}\\&=\frac{2\sin^2\theta+2\sin\theta\cos\theta}{2\sin\theta\cos\theta+2\cos^2\theta}\\&=\frac{2\sin\theta(\sin\theta+\cos\theta)}{2\cos\theta(\sin\theta+\cos\theta)}\\&=\frac{\sin\theta}{\cos\theta}=\tan\theta\end{aligned}$$ ■

23. (i) 证明 $\sin 3A=3\sin A-4\sin^3 A$。

解:

$$\begin{aligned}\sin 3A&=\sin(2A+A)\\&=\sin 2A\cos A+\cos 2A\sin A\\&=2\sin A\cos^2 A+(1-2\sin^2 A)\sin A\\&=2\sin A(1-\sin^2 A)+\sin A-2\sin^3 A\\&=2\sin A-2\sin^3 A+\sin A-2\sin^3 A\\&=3\sin A-4\sin^3 A\end{aligned}$$ ■

(ii) 证明 $\theta=54^\circ$ 满足方程式 $\sin 3\theta=-\cos 2\theta$。据此, 或其他方法, 求 $\sin 54^\circ$ 的值。

解:

$$\begin{aligned}\sin 3\theta=-\cos 2\theta&\implies\sin 3\theta+\cos 2\theta=0\\\sin 3\theta+\sin(90^\circ-2\theta)&=0\\2\sin\frac{3\theta+90^\circ-2\theta}{2}\cos\frac{3\theta-90^\circ+2\theta}{2}&=0\\2\sin\frac{90^\circ+\theta}{2}\cos\frac{5\theta-90^\circ}{2}&=0\end{aligned}$$

$\because$ When $\theta=54^\circ$, $\cos\dfrac{5\theta-90^\circ}{2}=\cos 90=0$, hence $2\sin\dfrac{90^\circ+\theta}{2}\cos\dfrac{5\theta-90^\circ}{2}=0$,
i.e. $\sin 3\theta+\cos 2\theta=0\implies\sin 3\theta=-\cos 2\theta$,
$\therefore\theta=54^\circ$ satisfies the equation. ■

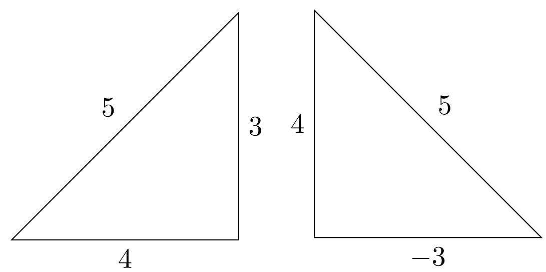
$$\begin{aligned}\text{Let }\theta=18^\circ&\implies 5\theta=90^\circ\\3\theta+2\theta=90^\circ&\implies 3\theta=90^\circ-2\theta\\\sin 3\theta&=\sin(90^\circ-2\theta)=\cos 2\theta\\3\sin\theta-4\sin^3\theta&=1-2\sin^2\theta\\4\sin^3\theta-2\sin^2\theta-3\sin\theta+1&=0\\(\sin\theta-1)(4\sin^2\theta+2\sin\theta-1)&=0\\\sin\theta=1\text{ (rejected) or }\sin\theta&=\frac{-1\pm\sqrt{5}}{4}\end{aligned}$$

$\because \theta = 18^\circ$ lies in the first quadrant, hence $\sin 18^\circ = \dfrac{-1+\sqrt{5}}{4}$

$$\begin{aligned}\sin 54^\circ &= 3\sin 18^\circ - 4\sin^3 18^\circ \\ &= 3\left(\frac{-1+\sqrt{5}}{4}\right) - 4\left(\frac{-1+\sqrt{5}}{4}\right)^3 \\ &= \frac{1}{4}(1+\sqrt{5})\end{aligned}$$ ■

24. (a) 若 $\sin \mathrm{A} = \dfrac{3}{5}, \sin \mathrm{B} = \dfrac{4}{5}$, 其中 A 和 B 分别是锐角及钝角。不许用计算求 $\cos(A+2B)$ 的值。

解:



$$\begin{aligned}\cos(A+2B) &= \cos A\cos 2B - \sin A\sin 2B \\ &= \cos A(1-2\sin^2 B) - \sin A(2\sin B\cos B) \\ &= \frac{4}{5}\left(1-2\left(\frac{4}{5}\right)^2\right) - \frac{3}{5}\left(2\cdot\frac{4}{5}\cdot\left(-\frac{3}{5}\right)\right) \\ &= \frac{44}{125}\end{aligned}$$ ■

(b) 不许用计算机, 求 $\cos^2\dfrac{\pi}{8} - \cos^2\dfrac{3\pi}{8}$ 的值。**解**:

$$\begin{aligned}\cos^2\frac{\pi}{8} - \cos^2\frac{3\pi}{8} &= \cos^2\frac{\pi}{8} - \sin^2\frac{\pi}{8} \\ &= \cos^2\left(\frac{1}{2}\cdot\frac{\pi}{4}\right) - \sin^2\left(\frac{1}{2}\cdot\frac{\pi}{4}\right) \\ &= \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}\end{aligned}$$ ■

25. 已知 $\cos\theta + \sin\theta = a$ 及 $\cos 2\theta = b$, 证明 $a^4 - 2a^2 + b^2 = 0$。

解:

$$\begin{aligned}a &= \cos\theta + \sin\theta \\ a^2 &= \cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta \\ &= 1 + \sin 2\theta \\ a^4 - 2a^2 + b^2 &= (1+\sin 2\theta)^2 - 2(1+\sin 2\theta) + \cos^2 2\theta \\ &= 1 + 2\sin 2\theta + \sin^2 2\theta - 2 - 2\sin 2\theta + \cos^2 2\theta \\ &= \sin^2 2\theta + \cos^2 2\theta - 1 \\ &= 1 - 1 = 0\end{aligned}$$ ■