

# **Solution Book of Mathematic**

*Senior 2 Part I*

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## 11.1 Revision Exercise 14

Calculate the following (Question 1 to 4):

1.  $5 \begin{pmatrix} -3 & -1 \\ 3 & 4 \end{pmatrix} + 4 \begin{pmatrix} 6 & 2 \\ 1 & -1 \end{pmatrix}$

**Sol.**

$$\begin{aligned} & 5 \begin{pmatrix} -3 & -1 \\ 3 & 4 \end{pmatrix} + 4 \begin{pmatrix} 6 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -15 & -5 \\ 15 & 20 \end{pmatrix} + \begin{pmatrix} 24 & 8 \\ 4 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 9 & 3 \\ 19 & 16 \end{pmatrix} \end{aligned}$$

2.  $-4 \begin{pmatrix} 3 & 0 \\ -1 & 5 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$

**Sol.**

$$\begin{aligned} & -4 \begin{pmatrix} 3 & 0 \\ -1 & 5 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -12 & 0 \\ 4 & -20 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ -3 & -3 \end{pmatrix} \\ &= \begin{pmatrix} -15 & 0 \\ 7 & 17 \end{pmatrix} \end{aligned}$$

3.  $\begin{pmatrix} 2 & 6 & -1 \\ 8 & -4 & 3 \\ 5 & 7 & -2 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 1 \\ -2 & 1 & 0 \\ -3 & 5 & -4 \end{pmatrix}$

**Sol.**

$$\begin{aligned} & \begin{pmatrix} 2 & 6 & -1 \\ 8 & -4 & 3 \\ 5 & 7 & -2 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 1 \\ -2 & 1 & 0 \\ -3 & 5 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 4 & 0 \\ 6 & -3 & 3 \\ 2 & 12 & -6 \end{pmatrix} \end{aligned}$$

4.  $2 \begin{pmatrix} 1 & -3 & 5 \\ 7 & 2 & 0 \\ 2 & 4 & -4 \end{pmatrix} - \begin{pmatrix} -2 & -6 & 2 \\ -5 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix}$

**Sol.**

$$\begin{aligned} & 2 \begin{pmatrix} 1 & -3 & 5 \\ 7 & 2 & 0 \\ 2 & 4 & -4 \end{pmatrix} - \begin{pmatrix} -2 & -6 & 2 \\ -5 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -6 & 10 \\ 14 & 4 & 0 \\ 4 & 8 & -8 \end{pmatrix} - \begin{pmatrix} -2 & -6 & 2 \\ -5 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 & 8 \\ 19 & 3 & -1 \\ 4 & 8 & -6 \end{pmatrix} \end{aligned}$$

5. Given that  $\begin{pmatrix} 2 \\ -3 \end{pmatrix} + 3 \begin{pmatrix} 5 \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ , find the value of  $x$  and  $y$ .

**Sol.**

$$\begin{aligned} & \begin{pmatrix} 2 \\ -3 \end{pmatrix} + 3 \begin{pmatrix} 5 \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \\ & \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 15 \\ 3y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \\ & \begin{pmatrix} 17 \\ -3 + 3y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

$$x = 17$$

$$y = -3 + 3y$$

$$2y = 3$$

$$y = \frac{3}{2}$$

$$\therefore x = 17, y = \frac{3}{2}$$

6. Let  $P = \begin{pmatrix} 3 & -2 & 1 \\ -1 & 2 & -3 \\ 4 & 0 & -2 \end{pmatrix}$ ,  $Q = \begin{pmatrix} 1 & -5 & -4 \\ -2 & 0 & 6 \\ 3 & 2 & 3 \end{pmatrix}$  and  $R = \begin{pmatrix} 4 & 0 & 5 \\ 1 & -2 & -7 \\ 0 & -2 & 1 \end{pmatrix}$ . Find the following:

(a)  $2Q + R'$

**Sol.**

$$\begin{aligned} 2Q + R' &= 2 \begin{pmatrix} 1 & -5 & -4 \\ -2 & 0 & 6 \\ 3 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 5 \\ 1 & -2 & -7 \\ 0 & -2 & 1 \end{pmatrix}' \\ &= \begin{pmatrix} 2 & -10 & -8 \\ -4 & 0 & 12 \\ 6 & 4 & 6 \end{pmatrix} + \begin{pmatrix} 4 & 1 & 0 \\ 0 & -2 & -2 \\ 5 & -7 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 6 & -9 & -8 \\ -4 & -2 & 10 \\ 11 & -3 & 7 \end{pmatrix} \end{aligned}$$

(b)  $(P - R) + 2Q'$

**Sol.**

$$\begin{aligned} & (P - R) + 2Q' \\ &= \begin{pmatrix} 3 & -2 & 1 \\ -1 & 2 & -3 \\ 4 & 0 & -2 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 5 \\ 1 & -2 & -7 \\ 0 & -2 & 1 \end{pmatrix} \\ & \quad + 2 \begin{pmatrix} 1 & -5 & -4 \\ -2 & 0 & 6 \\ 3 & 2 & 3 \end{pmatrix}' \\ &= \begin{pmatrix} -1 & -2 & -4 \\ -2 & 4 & 4 \\ 4 & 2 & -3 \end{pmatrix} + \begin{pmatrix} 2 & -4 & 6 \\ -10 & 0 & 4 \\ -8 & 12 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 6 & 2 \\ -12 & 4 & 8 \\ -4 & 14 & 3 \end{pmatrix} \end{aligned}$$

(c)  $[2(Q - P)]'$

**Sol.**

$$\begin{aligned} & [2(Q - P)]' \\ &= \left\{ 2 \left[ \begin{pmatrix} 1 & -5 & -4 \\ -2 & 0 & 6 \\ 3 & 2 & 3 \end{pmatrix} - \begin{pmatrix} 3 & -2 & 1 \\ -1 & 2 & -3 \\ 4 & 0 & -2 \end{pmatrix} \right] \right\}' \\ &= \left[ 2 \begin{pmatrix} -2 & -3 & -5 \\ -1 & -2 & 9 \\ -1 & 2 & 5 \end{pmatrix} \right]' \\ &= \begin{pmatrix} -4 & -6 & -10 \\ -2 & -4 & 18 \\ -2 & 4 & 10 \end{pmatrix}' \\ &= \begin{pmatrix} -4 & -2 & -2 \\ -6 & -4 & 4 \\ -10 & 18 & 10 \end{pmatrix} \end{aligned}$$

(d)  $(R' - Q)'$

**Sol.**

$$\begin{aligned} (R' - Q)' &= \left[ \begin{pmatrix} 4 & 0 & 5 \\ 1 & -2 & -7 \\ 0 & -2 & 1 \end{pmatrix}' - \begin{pmatrix} 1 & -5 & -4 \\ -2 & 0 & 6 \\ 3 & 2 & 3 \end{pmatrix} \right]' \\ &= \begin{pmatrix} 4 & 0 & 5 \\ 1 & -2 & -7 \\ 0 & -2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -5 & -4 \\ -2 & 0 & 6 \\ 3 & 2 & 3 \end{pmatrix}' \\ &= \begin{pmatrix} 4 & 0 & 5 \\ 1 & -2 & -7 \\ 0 & -2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 3 \\ -5 & 0 & 2 \\ -4 & 6 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 2 & 2 \\ 6 & -2 & -9 \\ 4 & -8 & -2 \end{pmatrix} \end{aligned}$$

7. Let  $M = \begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix}$  and  $N = \begin{pmatrix} 3 & 6 \\ 7 & -1 \\ -4 & 2 \end{pmatrix}$ . Find the matrix  $X$  in the following equations:

(a)  $2N - 3M = 2M - X$

**Sol.**

$$\begin{aligned} 2N - 3M &= 2M - X \\ X &= 5M - 2N \\ &= 5 \begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix} - 2 \begin{pmatrix} 3 & 6 \\ 7 & -1 \\ -4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -5 & 0 \\ 20 & -15 \\ 10 & 20 \end{pmatrix} - \begin{pmatrix} 6 & 12 \\ 14 & -2 \\ -8 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -11 & -12 \\ 6 & -13 \\ 18 & 16 \end{pmatrix} \end{aligned}$$

(b)  $2(M - 2N) + X = M + N$

**Sol.**

$$\begin{aligned}
 2(M - 2N) + X &= M + N \\
 X &= M + N - 2(M - 2N) \\
 &= M + N - 2M + 4N \\
 &= -M + 5N \\
 &= -\begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix} + 5\begin{pmatrix} 3 & 6 \\ 7 & -1 \\ -4 & 2 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 \\ -4 & 3 \\ -2 & -4 \end{pmatrix} + \begin{pmatrix} 15 & 30 \\ 35 & -5 \\ -20 & 10 \end{pmatrix} \\
 &= \begin{pmatrix} 16 & 30 \\ 31 & -2 \\ -22 & 6 \end{pmatrix}
 \end{aligned}$$

(c)  $(M + 2N)' = X$

**Sol.**

$$\begin{aligned}
 (M + 2N)' &= X \\
 X &= (M + 2N)' \\
 &= \left[ \begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix} + 2\begin{pmatrix} 3 & 6 \\ 7 & -1 \\ -4 & 2 \end{pmatrix} \right]' \\
 &= \left[ \begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 6 & 12 \\ 14 & -2 \\ -8 & 4 \end{pmatrix} \right]' \\
 &= \begin{pmatrix} 5 & 12 \\ 18 & -5 \\ -6 & 8 \end{pmatrix}' \\
 &= \begin{pmatrix} 5 & 18 & -6 \\ 12 & -5 & 8 \end{pmatrix}
 \end{aligned}$$

(d)  $3N' - M' = 2X$

**Sol.**

$$\begin{aligned}
 3N' - M' &= 2X \\
 2X &= (3N - M)' \\
 &= \left[ 3\begin{pmatrix} 3 & 6 \\ 7 & -1 \\ -4 & 2 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix} \right]' \\
 &= \left[ \begin{pmatrix} 9 & 18 \\ 21 & -3 \\ -12 & 6 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix} \right]' \\
 &= \begin{pmatrix} 10 & 18 \\ 17 & 0 \\ -14 & 2 \end{pmatrix}' \\
 &= \begin{pmatrix} 10 & 17 & -14 \\ 18 & 0 & 2 \end{pmatrix} \\
 X &= \begin{pmatrix} 5 & \frac{17}{2} & -7 \\ 9 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

Of the following matrices, determine if  $AB$  and  $BA$  are defined. If any of them is defined, find the value of them (Question 8 to 11):

8.  $A = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

**Sol.**

$\therefore$  The number of columns of  $A$  is not equal to the number of rows of  $B$   
 $\therefore AB$  is not defined

$\therefore$  The number of columns of  $B$  is equal to the number of rows of  $A$   
 $\therefore BA$  is defined

$$\therefore BA = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ -2 \\ 0 \end{pmatrix}$$

9.  $A = \begin{pmatrix} -5 & 4 & 3 \\ 1 & 2 & 3 \end{pmatrix}, B = \begin{pmatrix} -2 & 1 \\ -3 & 0 \\ 1 & -4 \end{pmatrix}$

**Sol.**

∴ The number of columns of  $A$  is equal to the number of rows of  $B$   
 ∴  $AB$  is defined

$$\therefore AB = \begin{pmatrix} -5 & 4 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -3 & 0 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -17 \\ -5 & -11 \end{pmatrix}$$

∴ The number of columns of  $B$  is equal to the number of rows of  $A$   
 ∴  $BA$  is defined

$$\begin{aligned} \therefore BA &= \begin{pmatrix} -2 & 1 \\ -3 & 0 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} -5 & 4 & 3 \\ 1 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 11 & -6 & -3 \\ 13 & -12 & -9 \\ -9 & -4 & -9 \end{pmatrix} \end{aligned}$$

$$10. A = \begin{pmatrix} 3 & 2 & -1 \\ 5 & 4 & 0 \\ -2 & 6 & 1 \end{pmatrix}, B = \begin{pmatrix} 9 & 0 \\ -7 & 1 \\ 2 & 3 \end{pmatrix}$$

**Sol.**

∴ The number of columns of  $A$  is equal to the number of rows of  $B$   
 ∴  $AB$  is defined

$$\therefore AB = \begin{pmatrix} 3 & 2 & -1 \\ 5 & 4 & 0 \\ -2 & 6 & 1 \end{pmatrix} \begin{pmatrix} 9 & 0 \\ -7 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 11 & -1 \\ 17 & 4 \\ -58 & 9 \end{pmatrix}$$

∴ The number of columns of  $B$  is not equal to the number of rows of  $A$   
 ∴  $BA$  is not defined

$$11. A = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 5 & 2 \end{pmatrix}$$

**Sol.**

∴ The number of columns of  $A$  is not equal to the number of rows of  $B$   
 ∴  $AB$  is not defined

∴ The number of columns of  $B$  is equal to the number of rows of  $A$   
 ∴  $BA$  is defined

$$\therefore BA = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 10 & 4 \\ 18 & 2 \end{pmatrix}$$

12. Given that  $A = \begin{pmatrix} a & 3a \\ 2b & b \end{pmatrix}$ ,  $B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ ,  $AB = \begin{pmatrix} 45 \\ 48 \end{pmatrix}$ , find the value of  $a$  and  $b$ .

**Sol.**

$$\begin{aligned} AB &= \begin{pmatrix} a & 3a \\ 2b & b \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 45 \\ 48 \end{pmatrix} \\ \begin{pmatrix} 3a + 6a \\ 6b + 2b \end{pmatrix} &= \begin{pmatrix} 45 \\ 48 \end{pmatrix} \\ 9a &= 45 \\ 8b &= 48 \\ a &= 5 \\ b &= 6 \end{aligned}$$

13. Given that  $A = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ ,  $B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ ,  $A + B = AB$ , find the value of  $a$ ,  $b$  and  $c$ .

**Sol.**

$$\begin{aligned} A + B &= AB \\ \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \\ \begin{pmatrix} 3+a & b \\ 0 & 4+c \end{pmatrix} &= \begin{pmatrix} 3a & 3b \\ 0 & 4c \end{pmatrix} \\ 3+a &= 3a \\ 2a &= 3 \\ b &= 3b \\ 2b &= 0 \\ 4+c &= 4c \\ 3c &= 4 \\ a = \frac{3}{2}, b &= 0, c = \frac{4}{3} \end{aligned}$$

Find the value of the following determinants (Question 14 to 22):

$$14. \begin{vmatrix} 20 & 15 \\ 8 & 6 \end{vmatrix}$$

**Sol.**

$$\begin{aligned} \begin{vmatrix} 20 & 15 \\ 8 & 6 \end{vmatrix} &= 20 \cdot 6 - 15 \cdot 8 \\ &= 120 - 120 \\ &= 0 \end{aligned}$$

$$15. \begin{vmatrix} 6 & -7 \\ 15 & -2 \end{vmatrix}$$

**Sol.**

$$\begin{vmatrix} 6 & -7 \\ 15 & -2 \end{vmatrix} = 6 \cdot -2 - (-7) \cdot 15 \\ = -12 + 105 \\ = 93$$

16.  $\begin{vmatrix} -4 & -10 \\ 12 & 7 \end{vmatrix}$

**Sol.**

$$\begin{vmatrix} -4 & -10 \\ 12 & 7 \end{vmatrix} = -4 \cdot 7 - (-10) \cdot 12 \\ = -28 + 120 \\ = 92$$

17.  $\begin{vmatrix} 3 & 3 & -6 \\ 2 & -2 & 0 \\ 3 & 0 & -3 \end{vmatrix}$

$$\begin{array}{ccccc} + & + & + & & \\ 3 & 3 & -6 & 3 & 3 \\ \diagdown & & \diagup & \diagdown & \diagup \\ 2 & -2 & 0 & 2 & -2 \\ \diagup & & \diagdown & \diagup & \diagdown \\ 3 & 0 & -3 & 3 & 0 \\ - & - & - & - & - \end{array}$$

**Sol.**

$$\begin{vmatrix} 3 & 3 & -6 \\ 2 & -2 & 0 \\ 3 & 0 & -3 \end{vmatrix} = 18 + 0 + 0 - 36 - 0 + 18 \\ = 0$$

18.  $\begin{vmatrix} 5 & 7 & 1 \\ -3 & 6 & 9 \\ 4 & 7 & 3 \end{vmatrix}$

$$\begin{array}{ccccc} + & + & + & & \\ 5 & 7 & 1 & 5 & 7 \\ \diagdown & & \diagup & \diagdown & \diagup \\ -3 & 6 & 9 & -3 & 6 \\ \diagup & & \diagdown & \diagup & \diagdown \\ 4 & 7 & 3 & 4 & 7 \\ - & - & - & - & - \end{array}$$

**Sol.**

$$\begin{vmatrix} 5 & 7 & 1 \\ -3 & 6 & 9 \\ 4 & 7 & 3 \end{vmatrix} = 90 + 252 - 21 - 24 - 315 + 63 \\ = 45$$

19.  $\begin{vmatrix} -2 & 7 & -4 \\ 3 & -5 & 2 \\ -1 & 0 & -3 \end{vmatrix}$

$$\begin{array}{ccccc} + & + & + & & \\ -2 & 7 & -4 & -2 & 7 \\ \diagdown & & \diagup & \diagdown & \diagup \\ 3 & -5 & 2 & 3 & -5 \\ \diagup & & \diagdown & \diagup & \diagdown \\ -1 & 0 & -3 & -1 & 0 \\ - & - & - & - & - \end{array}$$

**Sol.**

$$\begin{vmatrix} -2 & 7 & -4 \\ 3 & -5 & 2 \\ -1 & 0 & -3 \end{vmatrix} = -30 - 14 - 0 + 20 + 0 + 63 \\ = -39$$

20.  $\begin{vmatrix} 1 & 0 & -1 \\ 3 & -2 & 5 \\ -1 & 1 & 3 \end{vmatrix}$

**Sol.**

$$\begin{vmatrix} 1 & 0 & -1 \\ 3 & -2 & 5 \\ -1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -2 & 5 \\ 1 & 3 \end{vmatrix} - 3 \begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ -2 & 5 \end{vmatrix} \\ = -11 - 3 + 2 = -12$$

21.  $\begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 1 \\ -2 & -6 & 5 \end{vmatrix}$

**Sol.**

$$\begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 1 \\ -2 & -6 & 5 \end{vmatrix} = 3 \begin{vmatrix} 2 & 2 \\ 1 & 1 \\ -2 & -2 \end{vmatrix} \\ = 0 \quad (\text{col 1 and 2 are the same})$$

22.  $\begin{vmatrix} 10 & 8 & -2 \\ 15 & 16 & -3 \\ -5 & -4 & 1 \end{vmatrix}$

**Sol.**

$$\begin{vmatrix} 10 & 8 & -2 \\ 15 & 16 & -3 \\ -5 & -4 & 1 \end{vmatrix} = -5 \begin{vmatrix} 2 & 8 & 2 \\ 3 & 16 & 3 \\ -1 & -4 & -1 \end{vmatrix} \\ = 0 \quad (\text{col 1 and 3 are the same})$$

Using the identities of determinant, prove the following equations (Question 23 to 24):

$$23. \begin{vmatrix} bc & 1 & bc(b+c) \\ ca & 1 & ca(c+a) \\ ab & 1 & ab(a+b) \end{vmatrix} = 0$$

**Proof.**

$$\begin{aligned} & \begin{vmatrix} bc & 1 & bc(b+c) \\ ca & 1 & ca(c+a) \\ ab & 1 & ab(a+b) \end{vmatrix} \\ &= a^2 b^2 c^2 \begin{vmatrix} 1 & \frac{1}{bc} & b+c \\ 1 & \frac{1}{ca} & c+a \\ 1 & \frac{1}{ab} & a+b \end{vmatrix} \\ &= a^2 b^2 c^2 \begin{vmatrix} 1 & \frac{1}{bc} & -a \\ 1 & \frac{1}{ca} & -b \\ 1 & \frac{1}{ab} & -c \end{vmatrix} & C_3 \rightarrow C_3 + (a+b+c)C_1 \\ &= a^2 b^2 c^2 \begin{vmatrix} 1 & a & -a \\ 1 & b & -b \\ 1 & c & -c \end{vmatrix} & C_2 \rightarrow C_2 + abcC_1 \\ &= -a^2 b^2 c^2 \begin{vmatrix} 1 & a & a \\ 1 & b & b \\ 1 & c & c \end{vmatrix} \\ &= 0 & C_2 = C_3 \end{aligned}$$

$$24. \begin{vmatrix} a & 1 & a^2(b+c) \\ b & 1 & b^2(c+a) \\ c & 1 & c^2(a+b) \end{vmatrix} = 0$$

**Proof.**

$$\begin{aligned} & \begin{vmatrix} a & 1 & a^2(b+c) \\ b & 1 & b^2(c+a) \\ c & 1 & c^2(a+b) \end{vmatrix} \\ &= \begin{vmatrix} a & 1 & a^2(b+c) \\ b-a & 0 & b^2(c+a) - a^2(b+c) \\ c-a & 0 & c^2(a+b) - a^2(b+c) \end{vmatrix} \\ &R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1 \\ &= \begin{vmatrix} b-a & b^2(c+a) - a^2(b+c) \\ c-a & c^2(a+b) - a^2(b+c) \end{vmatrix} \\ &= (b-a)[c^2(a+b) - a^2(b+c)] \\ &\quad - (c-a)[b^2(c+a) - a^2(b+c)] \\ &= c^2(b-a)(b+a) - a^2(b+c)(c-a) \\ &\quad - b^2(c-a)(c+a) + a^2(b+c)(c-a) \\ &= c^2(b^2 - a^2) - b^2(c^2 - a^2) \\ &= b^2 c^2 - a^2 c^2 - b^2 c^2 + a^2 c^2 \\ &= 0 \end{aligned}$$

Find the value of  $x$  in the following expressions (Question 25 to 26):

$$25. \begin{vmatrix} 2x & 3 & x+5 \\ -3 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 5x - 1$$

**Sol.**

$$\begin{aligned} & \begin{vmatrix} 2x & 3 & x+5 \\ -3 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 5x - 1 \\ & x+5 \begin{vmatrix} -3 & 2 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 2x & 3 \\ 2 & 1 \end{vmatrix} = 5x - 1 \\ & -7(x+5) - (2x-6) = 5x - 1 \\ & -7x - 35 - 2x + 6 = 5x - 1 \\ & -14x = 28 \\ & x = -2 \end{aligned}$$

$$26. \begin{vmatrix} x+3 & 1 & 0 \\ x & 3 & 0 \\ 1 & 0 & -x-2 \end{vmatrix} = x+6$$

**Sol.**

$$\begin{aligned} & \begin{vmatrix} x+3 & 1 & 0 \\ x & 3 & 0 \\ 1 & 0 & -x-2 \end{vmatrix} = x+6 \\ & -x-2 \begin{vmatrix} x+3 & 1 \\ x & 3 \end{vmatrix} = x+6 \\ & -(x+2)(3x+9-x) = x+6 \\ & (x+2)(2x+9) = -x-6 \\ & 2x^2 + 13x + 18 = -x-6 \\ & 2x^2 + 14x + 24 = 0 \\ & x^2 + 7x + 12 = 0 \\ & (x+4)(x+3) = 0 \\ & x = -4 \text{ or } x = -3 \end{aligned}$$

27. Given an identity matrix  $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Let  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ,  $(2I + J)^{-1} = rI + sJ$ , find the value of  $r$  and  $s$ .

Find the value of  $a$  in the following matrices if they are non-inversible (Question 28 to 31):

$$28. \begin{pmatrix} 3 & a \\ -2 & 6 \end{pmatrix}$$



**Sol.**

$$\begin{vmatrix} 3 & a \\ -2 & 6 \end{vmatrix} = 0$$

$$18 + 2a = 0$$

$$2a = -18$$

$$a = -9$$

$$29. \begin{pmatrix} 5a+2 & 4 \\ 6 & a \end{pmatrix}$$

**Sol.**

$$\begin{vmatrix} 5a+2 & 4 \\ 6 & a \end{vmatrix} = 0$$

$$5a^2 + 2a - 24 = 0$$

$$(x-2)(5x+12) = 0$$

$$x = 2 \text{ or } x = \frac{12}{5}$$

$$30. \begin{pmatrix} -7 & a & 3 \\ 2 & -3 & 1 \\ 0 & -a & 4 \end{pmatrix}$$

**Sol.**

$$\begin{vmatrix} -7 & a & 3 \\ 2 & -3 & 1 \\ 0 & -a & 4 \end{vmatrix} = 0$$

$$-7 \begin{vmatrix} -3 & 1 \\ -a & 4 \end{vmatrix} - 2 \begin{vmatrix} a & 3 \\ -a & 4 \end{vmatrix} = 0$$

$$-7(-12 + a) - 2(4a + 3a) = 0$$

$$84 - 7a - 14a = 0$$

$$21a = 84$$

$$a = 4$$

$$31. \begin{pmatrix} a & -1 & 0 \\ 4 & 0 & -2 \\ a+4 & a & -8 \end{pmatrix}$$

**Sol.**

$$\begin{vmatrix} a & -1 & 0 \\ 4 & 0 & -2 \\ a+4 & a & -8 \end{vmatrix} = 0$$

$$a \begin{vmatrix} 0 & -2 \\ a & -8 \end{vmatrix} + \begin{vmatrix} 4 & -2 \\ a+4 & -8 \end{vmatrix}$$

$$2a^2 - 32 + 2a + 8 = 0$$

$$2a^2 + 2a - 24 = 0$$

$$a^2 + a - 12 = 0$$

$$(a+4)(a-3) = 0$$

$$a = -4 \text{ or } a = 3$$

Find the inverse of the following matrices (Question 32 to 37):

$$32. \begin{pmatrix} 2 & 5 \\ 2 & 3 \end{pmatrix}$$

$$33. \begin{pmatrix} -2 & -1 \\ 4 & 6 \end{pmatrix}$$

$$34. \begin{pmatrix} 1 & 0 & 3 \\ 3 & 1 & 9 \\ -2 & 2 & -4 \end{pmatrix}$$

$$35. \begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & -2 \\ -2 & 5 & 1 \end{pmatrix}$$

$$36. \begin{pmatrix} 4 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & -4 & 4 \end{pmatrix}$$

$$37. \begin{pmatrix} 3 & 2 & 3 \\ -1 & -3 & \\ 1 & 0 & 3 \end{pmatrix}$$

Solve the following system of equations using the method of Gauss elimination (Question 38 to 41):

$$38. \begin{cases} 2x - y + 4z = 5 \\ 2x + 3y - 4z = -7 \\ x + y + z = 2 \end{cases}$$

$$39. \begin{cases} x - 2y - 3z = -4 \\ 3x + y - 4z = -5 \\ 2x + 4y - z = -5 \end{cases}$$

$$40. \begin{cases} x - 2y - z = 3 \\ 4x - y + 2z = 1 \\ x + 3y = 5 \end{cases}$$

$$41. \begin{cases} 2x - y - z = 0 \\ 4x - 3y + 2z = 1 \\ 3x - 2y - 4z = -1 \end{cases}$$

Solve the following system of equations using the Cramer's rule (Question 42 to 45):

$$42. \begin{cases} x - 3y - 2z = 1 \\ 7x + 4y - 5z = 0 \\ 3x + 9y + z = -1 \end{cases}$$

$$43. \begin{cases} x - 2y + 3z = 6 \\ 2x + 3y - 4z = 20 \\ 3x - 2y - 5z = 6 \end{cases}$$

$$44. \begin{cases} 2x - 2x - 4z + 3 = 0 \\ 2x + 3y + 4z - 2 = 0 \\ 7x + 3y - 2z - 2 = 0 \end{cases}$$

$$45. \begin{cases} \frac{2}{x} - \frac{5}{y} + \frac{4}{z} = -3 \\ \frac{x}{4} + \frac{1}{y} - \frac{2}{z} = 7 \\ \frac{7}{x} - \frac{3}{z} = 4 \end{cases}$$