1. (a) Given that $h(x) = \dfrac{27}{4+x}, x \neq -4$, find the value of

i. $h^2(-1)$,

Solution:

$$\begin{aligned}
h(-1) &= \frac{27}{4+(-1)} \\
&= \frac{27}{3} \\
&= 9 \\
h^2(-1) &= h(h(-1)) \\
&= h(9) \\
&= \frac{27}{4+9} \\
&= \frac{27}{13}
\end{aligned}$$

ii. $h^{-1}(3)$.

Solution:

Let $y = h^{-1}(x)$, then $x = h(y)$.

$$\begin{aligned}
x &= \frac{27}{4+y} \\
4x + xy &= 27 \\
xy &= 27 - 4x \\
y &= \frac{27-4x}{x} \\
h^{-1}(x) &= \frac{27-4x}{x} \\
h^{-1}(3) &= \frac{27-4(3)}{3} \\
&= \frac{27-12}{3} \\
&= 5
\end{aligned}$$

(b) Given the functions $fg(x) = 6x - 9$ and $g(x) = 3x + 2$, find $f(x)$.

Solution:

Let $y = g(x) = 3x + 2$, then $x = \dfrac{y-2}{3}$.

$$\begin{aligned}
f(y) &= 6\left(\frac{y-2}{3}\right) - 9 \\
&= 2(y-2) - 9 \\
&= 2y - 4 - 9 \\
&= 2y - 13 \\
f(x) &= 2x - 13
\end{aligned}$$

2. (a) Given that one of the roots of the quadratic equation $2x^2 - 6x + k = 0$ is three times the other root, find the value of k.

Solution:

Let the roots be p and $3p$, then

$$\begin{aligned} p + 3p &= \frac{6}{2} \\ 4p &= 3 \\ p &= \frac{3}{4} \\ 3p &= \frac{9}{4} \\ (x-p)(x-3p) &= 0 \\ (x - \frac{3}{4})(x - \frac{9}{4}) &= 0 \\ x^2 - \frac{12}{4}x + \frac{27}{16} &= 0 \\ 2x^2 - 6x + \frac{27}{8} &= 0 \\ k &= \frac{27}{8} \end{aligned}$$

(b) Given the quadratic function $h(x) = x^2 - 12x + 3p$, where p is a constant.

i. Express $h(x)$ in the form $h(x) = (x+m)^2 + n$, such that m and n are constants.

Solution:

$$\begin{aligned} h(x) &= x^2 - 12x + 3p \\ &= (x^2 - 12x + 36) - 36 + 3p \\ &= (x-6)^2 + 3p - 36 \end{aligned}$$

ii. Given that the minimum value of $h(x)$ is -15 , find the value of p.

Solution:

$$\begin{aligned} h(x) &= (x-6)^2 + 3p - 36 \\ -15 &= 3p - 36 \\ 3p &= 36 - 15 \\ 3p &= 21 \\ p &= 7 \end{aligned}$$

3. (a) Find the range of values of x for $6x^2 \geqslant 3 - 7x$.

Solution:

$$\begin{aligned} 6x^2 &\geqslant 3 - 7x \\ 6x^2 + 7x - 3 &\geqslant 0 \\ (3x-1)(2x+3) &\geqslant 0 \\ x \geqslant \frac{1}{3} \quad &\text{or} \quad x \leqslant -\frac{3}{2} \end{aligned}$$

(b) Sketch the graph of $y = \left|x^2 - 4\right|$ for $-3 \leqslant x \leqslant 3$.
Solution:
Lazy to draw the graph. =)

4. Solve the following equations:

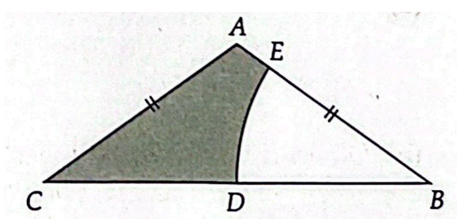
(a) $8^{\log_2 u} = 125$
Solution:

$$\begin{aligned} 8^{\log_2 u} &= 125 \\ (2^3)^{\log_2 u} &= 5^3 \\ 2^{3\log_2 u} &= 5^3 \\ 3\log_2 u &= 3\log_2 5 \\ \log_2 u &= \log_2 5 \\ u &= 5 \end{aligned}$$

(b) $\log_{81}\left[\log_2(3x - 10)\right] = \dfrac{1}{4}$
Solution:

$$\begin{aligned} \log_{81}\left[\log_2(3x - 10)\right] &= \frac{1}{4} \\ 81^{\frac{1}{4}} &= \log_2(3x - 10) \\ \log_2(3x - 10) &= 3 \\ 3x - 10 &= 2^3 \\ 3x &= 8 + 10 \\ x &= 6 \end{aligned}$$

5. In Diagram 1, ABC is an isosceles triangle such that $AB = AC = 16$ cm. DBE is a sector with centre B and a radius of 14 cm.



Given that A is vertically above D, find

(a) $\angle DBE$, in radians,
Solution:

$$\begin{aligned} \cos\angle DBE &= \frac{DB}{AB} = \frac{14}{16} = \frac{7}{8} \\ \angle DBE &= \cos^{-1}\left(\frac{7}{8}\right) \\ &= 0.5054 \text{ rad} \end{aligned}$$

(b) the area, in cm^2, of the shaded region.

Solution:

$$\begin{aligned}
\frac{AD}{AB} &= \sin 0.5054 \\
AD &= 16 \sin 0.5054 \\
&= 7.746 \text{ cm} \\
\text{Area of } \triangle ABC &= \frac{1}{2} \times 28 \times 7.746 \\
&= 108.4435 \text{ cm}^2 \\
\text{Area of sector } DBE &= \frac{1}{2} \times 14^2 \times 0.5054 \\
&= 49.52 \text{ cm}^2 \\
\text{Area of shaded region} &= 108.4435 - 49.52 \\
&= 58.92 \text{ cm}^2
\end{aligned}$$

6. (a) Sketch the graph of $y = 4 \sin \frac{3x}{2}$ for $0 \leqslant x \leqslant \pi$.

Solution:

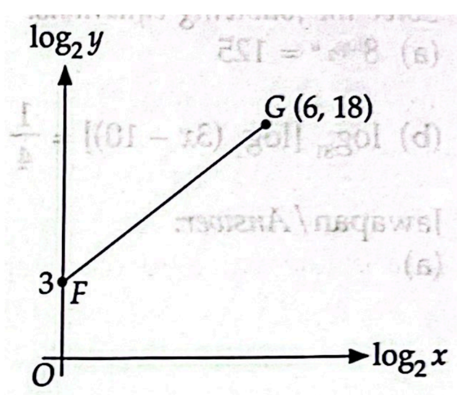
Lazy to draw the graph. =)

(b) Solve the equation $2 \sin 2x = \cos x$ for $0° \leqslant x \leqslant 360°$.

Solution:

$$\begin{aligned}
2 \sin 2x &= \cos x \\
4 \sin x \cos x &= \cos x \\
4 \sin x \cos x - \cos x &= 0 \\
\cos x(4 \sin x - 1) &= 0 \\
\cos x = 0 \text{ or } \sin x &= \frac{1}{4} \\
x = 90° \text{ or } x = 270° \text{ or } x &= 14.48° \text{ or } x = 165.52°
\end{aligned}$$

7. Diagram 2 shows part of the line of best fit obtained by plotting $\log_2 y$ against $\log_2 x$.



Given that $y = mx^n$, such that m and n are constants, find

(a) the value of m and of n,

Solution:

$$\begin{aligned}
\text{Gradient} &= \frac{18-3}{6-0} = \frac{5}{2} \\
\text{Intercept} &= 3 \\
\log_2 y &= \frac{5}{2}\log_2 x + 3 \\
2\log_2 y &= 5\log_2 x + 6 \\
\log_2 y^2 &= \log_2 x^5 + 6 \\
\log_2 y^2 - \log_2 x^5 &= 6 \\
\log_2 \frac{y^2}{x^5} &= 6 \\
\frac{y^2}{x^5} &= 64 \\
y^2 &= 64x^5 \\
y &= 8x^{\frac{5}{2}}
\end{aligned}$$

$\therefore m = 8$ and $n = \frac{5}{2}$.

(b) the value of y when $x = 4$.

Solution:

$$\begin{aligned}
y &= 8x^{\frac{5}{2}} \\
&= 8 \times 4^{\frac{5}{2}} \\
&= 8 \times 32 \\
&= 256
\end{aligned}$$

8. (a) Given that $y = x(3x+1)(2x-3)$, find $\frac{dy}{dx}$.

Solution:

$$\begin{aligned}
y &= x(3x+1)(2x-3) \\
&= x(6x^2 - 9x + 2x - 3) \\
&= 6x^3 - 7x^2 - 3x \\
\frac{dy}{dx} &= 18x^2 - 14x - 3
\end{aligned}$$

(b) The volume of a spherical balloon that is leaking decreases at a rate of $\frac{\pi}{2}$ cm^3 s^{-1}. Find the rate of change of its radius when the radius is 1 cm.

Solution:

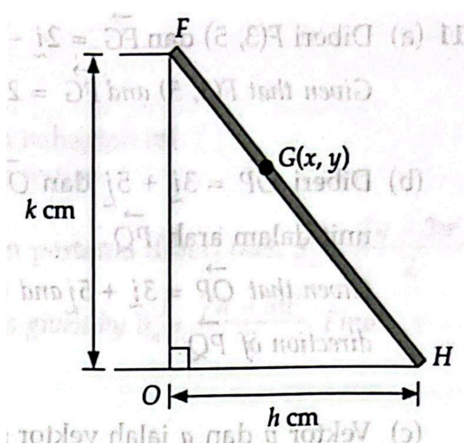
$$\begin{aligned}
V &= \frac{4}{3}\pi r^3 \\
\frac{dV}{dr} &= 4\pi r^2 \\
\frac{dV}{dt} &= -\frac{\pi}{2} \\
\frac{dr}{dt} &= \frac{dr}{dV} \times \frac{dV}{dt} \\
&= \frac{1}{4\pi r^2} \times -\frac{\pi}{2} \\
&= -\frac{1}{8r^2} \text{ cm s}^{-1}
\end{aligned}$$

When $r = 1$ cm,

$$\begin{aligned}
\frac{dr}{dt} &= -\frac{1}{8(1)^2} \\
&= -\frac{1}{8} \text{ cm s}^{-1}
\end{aligned}$$

$\therefore$ the rate of change of its radius is $-\frac{1}{8}$ cm s^{-1} when the radius is 1 cm.

9. Diagram 3 shows a rod FH of length 1.5 cm is leaning against a wall at point F and touches the floor at point H. Point G divides the rod in the ratio 1 : 2.



If the rod is sliding down, show that the equation of locus of the moving point G is $4x^2 + y^2 = 1$.

Solution:

$$\begin{aligned}
k^2 + h^2 &= 1.5^2 \\
k^2 + h^2 &= 2.25 \quad \cdots \ (1)
\end{aligned}$$

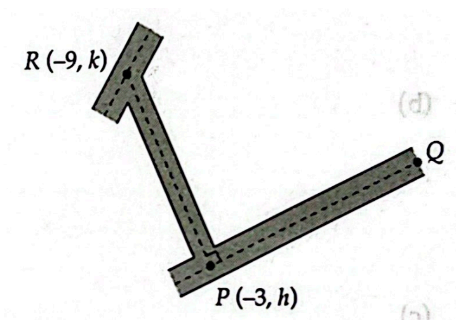
Let the coordinates of F be $(0, k)$ and the coordinates of H be $(h, 0)$.

$$\begin{aligned} x &= \frac{2(0)+h}{3} = \frac{h}{3} \\ h &= 3x \ \cdots \ (2) \\ y &= \frac{2(k)+0}{3} = \frac{2k}{3} \\ k &= \frac{3y}{2} \ \cdots \ (3) \end{aligned}$$

Substituting (2) and (3) into (1),

$$\begin{aligned} (3x)^2 + \left(\frac{3y}{2}\right)^2 &= 2.25 \\ 9x^2 + \frac{9y^2}{4} &= 2.25 \\ 36x^2 + 9y^2 &= 9 \\ 4x^2 + y^2 &= 1 \end{aligned}$$

10. Diagram 4 shows part of a plan of three straight roads.



The straight road PQ is represented by the equation $2y = x - 7$.
Find

(a) the value of h,
Solution: When $x = -3$,

$$\begin{aligned} 2y &= -3 - 7 \\ 2y &= -10 \\ y &= -5 \\ \therefore\ h &= -5 \end{aligned}$$

(b) the equation of the straight road PR,
Solution: The gradient of PQ is $\frac{1}{2}$, therefore the gradient of PR is -2.
Let the equation of PR is

$$\begin{aligned} y + 5 &= -2(x+3) \\ y &= -2x - 11 \end{aligned}$$

(c) the value of k. When $x = -9$,

$$\begin{aligned} y &= -2(-9) - 11 \\ &= 7 \\ \therefore\ k &= 7 \end{aligned}$$

11. (a) Given that $F(3,5)$ and $\overrightarrow{FG} = 2\vec{\imath} - 8\vec{\jmath}$, find the coordinates of G.

Solution:

$$\begin{aligned} \overrightarrow{FG} &= \overrightarrow{OG} - \overrightarrow{OF} \\ \overrightarrow{OG} &= \overrightarrow{OF} + \overrightarrow{FG} \\ &= 3\vec{\imath} + 5\vec{\jmath} + 2\vec{\imath} - 8\vec{\jmath} \\ &= 5\vec{\imath} - 3\vec{\jmath} \end{aligned}$$

Therefore, the coordinates of G are $(5, -3)$.

(b) Given that $\overrightarrow{OP} = 3\vec{\imath} + 5\vec{\jmath}$ and $\overrightarrow{OQ} = 7\vec{\imath} - 6\vec{\jmath}$, such that O is the origin, find the unit vector in the direction of $\overrightarrow{PQ}$.

Solution:

$$\begin{aligned} \overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= 7\vec{\imath} - 6\vec{\jmath} - 3\vec{\imath} - 5\vec{\jmath} \\ &= 4\vec{\imath} - 11\vec{\jmath} \\ \text{Magnitude of } \overrightarrow{PQ} &= \sqrt{4^2 + (-11)^2} \\ &= \sqrt{16 + 121} \\ &= \sqrt{137} \\ \text{Unit vector of } \overrightarrow{PQ} &= \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} \\ &= \frac{4\vec{\imath} - 11\vec{\jmath}}{\sqrt{137}} \end{aligned}$$

(c) Vectors $\vec{p}$ and $\vec{q}$ are parallel vectors. It is given that $|3a - b|\vec{p} = 4\vec{q}$, where a and b are constants. Express a in terms of b.

Solution:

Skipped cuz very sus.

12. Given that $y = \dfrac{48}{x^4}$

(a) find the value of $\dfrac{dy}{dx}$ when $x = 2$,

Solution:

$$\begin{aligned} y &= 48x^{-4} \\ \frac{dy}{dx} &= -192x^{-5} \end{aligned}$$

When $x = 2$,

$$\begin{aligned} \frac{dy}{dx} &= -192(2)^{-5} \\ &= -192 \times \frac{1}{32} \\ &= -6 \end{aligned}$$

(b) without using calculator, find the approximate value of $\dfrac{48}{1.996^4}$.

Solution:

$$\begin{aligned} \frac{\delta y}{\delta x} &\approx \frac{dy}{dx} \\ \delta y &\approx \frac{dy}{dx} \times \delta x \end{aligned}$$

When $x = 2$, $\delta x = -0.004$, $y = \dfrac{48}{2^4} = 3$.

$$\begin{aligned} \delta y &\approx -6 \times -0.004 \\ &= 0.024 \\ \therefore y &\approx 3 + 0.024 \\ &= 3.024 \end{aligned}$$

13. (a) In an arithmetic progression, the sum of the first n terms is given by $S_n = \dfrac{7n - 3n^2}{2}$. Find

i. the 9^{th} term,

Solution:

$$\begin{aligned} T_9 &= S_9 - S_8 \\ &= \frac{7(9) - 3(9)^2}{2} - \frac{7(8) - 3(8)^2}{2} \\ &= \frac{63 - 243}{2} - \frac{56 - 192}{2} \\ &= -90 - (-68) \\ &= -22 \end{aligned}$$

ii. the n^{th} term.
Solution:

$$\begin{aligned}
T_1 &= S_1 \\
&= \frac{7(1) - 3(1)^2}{2} \\
&= 2 \\
T_2 &= S_2 - S_1 \\
&= \frac{7(2) - 3(2)^2}{2} - 2 \\
&= -1 \\
d &= -1 - 2 = -3 \\
T_n &= 2 + (-3)(n - 1) \\
&= 2 - 3n + 3 \\
&= 5 - 3n
\end{aligned}$$

(b) Given that $p, q, 72$ and 216 are the first four terms of a geometric progression, find

i. the value of p and of q,
Solution:

$$\begin{aligned}
\frac{72}{q} &= \frac{216}{72} \\
216q &= 72^2 \\
q &= 24
\end{aligned}$$

$$\begin{aligned}
\frac{24}{p} &= \frac{72}{24} \\
72p &= 24^2 \\
p &= 8
\end{aligned}$$

ii. the sum of the 7^{th} term to the 9^{th} term. **Solution:**

$$\begin{aligned}
&a = 8,\ r = 3 \\
S_n &= \frac{8(3^n - 1)}{3 - 1} \\
&= 4(3^n - 1) \\
S &= S_9 - S_6 \\
&= 4(3^9 - 1) - 4(3^6 - 1) \\
&= 75816
\end{aligned}$$

14. (a) Given that $\int_1^3 h(x)dx = m$, find in terms of m for

i. $\displaystyle\int_1^3 \frac{5h(x)}{2}dx$,

Solution:

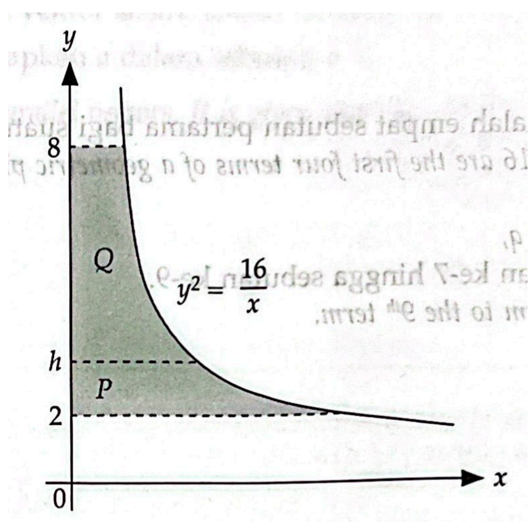
$$\begin{aligned}\int_1^3 \frac{5h(x)}{2}dx &= \frac{5}{2}\int_1^3 h(x)dx \\ &= \frac{5}{2}m\end{aligned}$$

ii. $\displaystyle\int_1^3 [6x + 2h(x)]dx$.

Solution:

$$\begin{aligned}\int_1^3 [6x + 2h(x)]dx &= \int_1^3 6xdx + \int_1^3 2h(x)dx \\ &= 6\int_1^3 xdx + 2\int_1^3 h(x)dx \\ &= 6\left[\frac{x^2}{2}\right]_1^3 + 2m \\ &= 6\left[\frac{9}{2} - \frac{1}{2}\right] + 2m \\ &= 24 + 2m\end{aligned}$$

(b) Diagram 5 shows a curve $y^2 = \dfrac{16}{x}$.



Given that the area of the shaded region P = area of the shaded region Q, find the value of h.

Solution:

$$\begin{aligned}
y^2 &= \frac{16}{x} \\
x &= \frac{16}{y^2} \\
\int_2^h \frac{16}{y^2}dy &= \int_h^8 \frac{16}{y^2}dy \\
\left[-\frac{16}{y}\right]_2^h &= \left[-\frac{16}{y}\right]_h^8 \\
-\frac{16}{h} + 8 &= -2 + \frac{16}{h} \\
10 &= \frac{32}{h} \\
h &= 3.2
\end{aligned}$$

15. (a) Five students are to be chosen from a group of four boys and six girls to represent a school in a Mathematics quiz competition. Calculate the number of teams that can be formed if each team consists of

    i. one boy and four girls,
    Solution:
    First, choose 1 boy from 4 boys, there are ${}_4C_1$ ways to do this.
    Then, choose 4 girls from 6 girls, there are ${}_6C_4$ ways to do this.
    Hence, the number of teams that can be formed is ${}_4C_1 \times {}_6C_4 = 4 \times 15 = 60$.

    ii. at least two boys.
    Solution:
    The number of teams that can be formed is

$$\begin{aligned}
{}_{10}C_5 - {}_4C_0 \times {}_6C_5 - {}_4C_1 \times {}_6C_4 &= 252 - 6 - 60 \\
&= 186
\end{aligned}$$

(b) The masses of several bags of flour, in kg, are normally distributed with a mean of μ and a standard deviation of σ. It is given that 3.45% of bags of flour have masses more than 35 kg and 24.2% have masses less than 20 kg. Find the the value of μ and of σ.

Solution:

Let X be the mass of a bag of flour, then $Z \sim \dfrac{X-\mu}{\sigma}$ is a standard normal distribution.

$$\begin{aligned}
P(X > 35) &= 0.0345 \\
P\left(Z > \frac{35-\mu}{\sigma}\right) &= 0.0345 \\
\frac{35-\mu}{\sigma} &= 1.82 \\
\sigma &= \frac{35-\mu}{1.82} \quad \cdots \text{ (1)}
\end{aligned}$$

$$P(X < 20) = 0.242$$

$$
\begin{aligned}
P\left(Z < \frac{20-\mu}{\sigma}\right) &= 0.242 \\
P\left(Z > \frac{\mu-20}{\sigma}\right) &= 0.242 \\
\frac{\mu-20}{\sigma} &= 0.70 \\
\sigma &= \frac{\mu-20}{0.70} \quad \cdots \ (2)
\end{aligned}
$$

Equating (1) and (2),

$$
\begin{aligned}
\frac{35-\mu}{1.82} &= \frac{\mu-20}{0.70} \\
35-\mu &= 1.82\left(\frac{\mu-20}{0.70}\right) \\
35-\mu &= 2.6(\mu-20) \\
35-\mu &= 2.6\mu-52 \\
52+35 &= 2.6\mu+\mu \\
87 &= 3.6\mu \\
\mu &= 24.17 \text{ kg}
\end{aligned}
$$

Substituting $\mu = 24.17$ into (1),

$$
\begin{aligned}
\sigma &= \frac{35-\frac{87}{3.6}}{1.82} \\
&= 5.952 \text{ kg}
\end{aligned}
$$

Therefore, $\mu = 24.17$ kg and $\sigma = 5.952$ kg.