Mathematics

Senior 3 Part I

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Contents

25.9 Revision Exercise 25

1. Find the gradient of the tangent to the curve $y = 2x^2 + 1$ at the point where $x = 2$.

   Sol.

$$\begin{aligned}\Delta y &= 2(x+\Delta x)^2 + 1 - (2x^2+1)\\ &= 2x^2 + 4x\Delta x + 2\Delta x^2 + 1 - 2x^2 - 1\\ &= 4x\Delta x + 2\Delta x^2\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= 4x + 2\Delta x\\ \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0}(4x + 2\Delta x)\\ \frac{dy}{dx} &= 4x\end{aligned}$$

   $\therefore$ The gradient of the tangent to the curve $y = 2x^2 + 1$ at the point where $x = 2$ is $4(2) = 8$.

2. Find the gradient of the curve $y = 3x^2 - 1$ at the point $A(-1, 2)$.

   Sol.

$$\begin{aligned}\Delta y &= 3(x+\Delta x)^2 - 1 - (3x^2-1)\\ &= 3x^2 + 6x\Delta x + 3\Delta x^2 - 1 - 3x^2 + 1\\ &= 6x\Delta x + 3\Delta x^2\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= 6x + 3\Delta x\\ \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0}(6x + 3\Delta x)\\ \frac{dy}{dx} &= 6x\end{aligned}$$

   $\therefore$ The gradient of the curve $y = 3x^2 - 1$ at the point $A(-1, 2)$ is $6(-1) = -6$.

3. Find the gradient of the curve $y = 2x - x^3$ at the point $B(-1, -1)$.

Sol.

$$\begin{aligned}
\Delta y &= 2(x+\Delta x) - (x+\Delta x)^3 - (2x - x^3) \\
&= 2x + 2\Delta x - x^3 - 3x^2\Delta x - 3x\Delta x^2 - \Delta x^3 - 2x + x^3 \\
&= 2\Delta x - 3x^2\Delta x - 3x\Delta x^2 - \Delta x^3
\end{aligned}$$

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= 2 - 3x^2 - 3x\Delta x - \Delta x^2 \\
\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0}(2 - 3x^2 - 3x\Delta x - \Delta x^2) \\
\frac{dy}{dx} &= 2 - 3x^2
\end{aligned}$$

$\therefore$ The gradient of the curve $y = 2x - x^3$ at the point $B(-1,-1)$ is $2 - 3(-1)^2 = -1$.

4. Find the derivative of the following functions using the definition of the derivative, and find the value of the derivative at the point where $x = 1$:

   (a) $f(x) = x^2 + 2x$

   Sol.

$$\begin{aligned}
\Delta y &= (x+\Delta x)^2 + 2(x+\Delta x) - (x^2 + 2x) \\
&= x^2 + 2x\Delta x + \Delta x^2 + 2x + 2\Delta x - x^2 - 2x \\
&= 2x\Delta x + \Delta x^2 + 2\Delta x
\end{aligned}$$

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= 2x + \Delta x + 2 \\
\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0}(2x + \Delta x + 2) \\
f'(x) &= 2x + 2
\end{aligned}$$

   (b) $g(x) = x^3$

   Sol.

$$\begin{aligned}
\Delta y &= (x+\Delta x)^3 - x^3 \\
&= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - x^3 \\
&= 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3
\end{aligned}$$

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= 3x^2 + 3x\Delta x + \Delta x^2 \\
\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0}(3x^2 + 3x\Delta x + \Delta x^2) \\
g'(x) &= 3x^2
\end{aligned}$$

(c) $h(x) = \dfrac{5}{x}$

Sol.

$$\begin{aligned}
\Delta y &= \frac{5}{x+\Delta x} - \frac{5}{x} \\
&= \frac{5x - 5(x+\Delta x)}{x(x+\Delta x)} \\
&= \frac{5x - 5x - 5\Delta x}{x(x+\Delta x)} \\
&= \frac{-5\Delta x}{x(x+\Delta x)}
\end{aligned}$$

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{-5}{x(x+\Delta x)} \\
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{-5}{x(x+\Delta x)} \\
h'(x) &= -\frac{5}{x^2}
\end{aligned}$$

(d) $k(x) = \sqrt{x+3}$

Sol.

$$\Delta y = \sqrt{x+3+\Delta x} - \sqrt{x+3}$$

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{\sqrt{x+3+\Delta x} - \sqrt{x+3}}{\Delta x} \\
&= \frac{\sqrt{x+3+\Delta x} - \sqrt{x+3}}{\Delta x} \cdot \frac{\sqrt{x+3+\Delta x} + \sqrt{x+3}}{\sqrt{x+3+\Delta x} + \sqrt{x+3}} \\
&= \frac{x+3+\Delta x - (x+3)}{\Delta x\left(\sqrt{x+3+\Delta x} + \sqrt{x+3}\right)} \\
&= \frac{\Delta x}{\Delta x\left(\sqrt{x+3+\Delta x} + \sqrt{x+3}\right)} \\
&= \frac{1}{\sqrt{x+3+\Delta x} + \sqrt{x+3}}
\end{aligned}$$

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+3+\Delta x} + \sqrt{x+3}} \\
k'(x) &= \frac{1}{2\sqrt{x+3}}
\end{aligned}$$

5. Find the derivative of the following functions:

(a) $y = 2x^4 - 3x^3 + 5x - 8$

Sol.

$$y' = 8x^3 - 9x^2 + 5$$

(b) $y = 2x + \dfrac{2}{x} - \dfrac{3}{x^2}$

Sol.

$$y' = 2 - \frac{2}{x^2} + \frac{6}{x^3}$$

(c) $y = \sqrt[3]{x} - \dfrac{1}{\sqrt{3x}}$

Sol.

$$\begin{aligned}
y' &= \left(x^{\frac{1}{3}} - (3x)^{-\frac{1}{2}}\right)' \\
&= \frac{1}{3}x^{-\frac{2}{3}} + \frac{3}{2}(3x)^{-\frac{3}{2}} \\
&= \frac{1}{3}x^{-\frac{2}{3}} + \frac{3}{2 \cdot 3\sqrt{3}}x^{-\frac{3}{2}} \\
&= \frac{1}{3}x^{-\frac{2}{3}} + \frac{3}{6\sqrt{3}}x^{-\frac{3}{2}} \\
&= \frac{1}{3}x^{-\frac{2}{3}} + \frac{3\sqrt{3}}{18}x^{-\frac{3}{2}} \\
&= \frac{1}{3}x^{-\frac{2}{3}} + \frac{\sqrt{3}}{6}x^{-\frac{3}{2}}
\end{aligned}$$

(d) $y = \left(x^3 - 4x\right)\left(x^2 + 3x - 1\right)$

Sol.

$$\begin{aligned}
y' &= (x^3 - 4x)'(x^2 + 3x - 1) + (x^3 - 4x)(x^2 + 3x - 1)' \\
&= (3x^2 - 4)(x^2 + 3x - 1) + (x^3 - 4x)(2x + 3) \\
&= 3x^4 + 9x^3 - 3x^2 - 4x^2 - 12x + 4 + 2x^4 + 3x^3 - 8x^2 - 12x \\
&= 5x^4 + 12x^3 - 15x^2 - 24x + 4
\end{aligned}$$

(e) $y = (x - 1)^5\sqrt{x + 2}$

Sol.

$$\begin{aligned}
y' &= 5(x-1)^4\sqrt{x+2} + (x-1)^5 \cdot \frac{1}{2\sqrt{x+2}} \\
&= (x-1)^4\left[5\sqrt{x+2} + \frac{(x-1)}{2\sqrt{x+2}}\right] \\
&= (x-1)^4\left[\frac{10(x+2)+x-1}{2\sqrt{x+2}}\right] \\
&= \frac{(x-1)^4(11x+19)}{2\sqrt{x+2}}
\end{aligned}$$

(f) $y = (2x+5)\left(x^2-2\right)\left(x^3-1\right)$

Sol.

$$\begin{aligned}
y' &= \left[(2x^3-4x+5x^2-10)(x^3-1)\right]' \\
&= (2x^3-4x+5x^2-10)'(x^3-1) + (2x^3-4x+5x^2-10)(x^3-1)' \\
&= (6x^2-4+10x)(x^3-1) + (2x^3-4x+5x^2-10)(3x^2) \\
&= 6x^5-4x^3+10x^4-6x^2+4-10x+6x^5-12x^3+15x^4-30x^2 \\
&= 12x^5+25x^4-16x^3-36x^2-10x+4
\end{aligned}$$

(g) $y = \dfrac{2x^3-3x^2+4}{x^2}$

Sol.

$$\begin{aligned}
y' &= \frac{(2x^3-3x^2+4)'(x^2)-(2x^3-3x^2+4)(x^2)'}{x^4} \\
&= \frac{(6x^2-6x)(x^2)-(2x^3-3x^2+4)(2x)}{x^4} \\
&= \frac{6x^4-6x^3-4x^4+6x^3-8x}{x^4} \\
&= \frac{2x^4-8x}{x^4} \\
&= \frac{2x^3-8}{x^3} \\
&= 2-\frac{8}{x^3}
\end{aligned}$$

(h) $y = \dfrac{x^2+4}{x+1}$

Sol.

$$\begin{aligned} y' &= \frac{(x^2+4)'(x+1)-(x^2+4)(x+1)'}{(x+1)^2} \\ &= \frac{(2x)(x+1)-(x^2+4)}{(x+1)^2} \\ &= \frac{2x^2+2x-x^2-4}{(x+1)^2} \\ &= \frac{x^2+2x-4}{(x+1)^2} \end{aligned}$$

(i) $y = \dfrac{x+2}{x^2+5x+6}$

Sol.

$$\begin{aligned} y' &= \frac{(x+2)'(x^2+5x+6)-(x+2)(x^2+5x+6)'}{(x^2+5x+6)^2} \\ &= \frac{(x^2+5x+6)-(x+2)(2x+5)}{(x^2+5x+6)^2} \\ &= \frac{x^2+5x+6-2x^2-9x-10}{(x^2+5x+6)^2} \\ &= \frac{-x^2-4x-4}{(x^2+5x+6)^2} \\ &= -\frac{(x+2)^2}{(x+2)^2(x+3)^2} \\ &= -\frac{1}{(x+3)^2} \end{aligned}$$

(j) $y = \dfrac{x^2}{\left(x^2-1\right)^3}$

Sol.

$$\begin{aligned} y' &= \frac{(x^2)'(x^2-1)^3-x^2\left[(x^2-1)^3\right]'}{(x^2-1)^6} \\ &= \frac{(2x)(x^2-1)^3-x^2\left[3(x^2-1)^2(2x)\right]}{(x^2-1)^6} \\ &= \frac{2x(x^2-1)^3-6x^3(x^2-1)^2}{(x^2-1)^6} \\ &= \frac{2x(x^2-1)^2(x^2-1-3x^2)}{(x^2-1)^6} \\ &= \frac{2x(x^2-1)^2(-2x^2-1)}{(x^2-1)^6} \\ &= \frac{-2x(2x^2+1)}{(x^2-1)^4} \\ &= \frac{-4x^3-2x}{(x^2-1)^4} \end{aligned}$$

6. Find the derivative of the following functions:

(a) $y = \left(x^3 - 1\right)^4$

Sol.

$$\begin{aligned} y' &= 4(x^3-1)^3(x^3-1)' \\ &= 4(x^3-1)^3(3x^2) \\ &= 12x^2(x^3-1)^3 \end{aligned}$$

(b) $y = (5x+3)^6$

Sol.

$$\begin{aligned} y' &= 6(5x+3)^5(5x+3)' \\ &= 6(5x+3)^5(5) \\ &= 30(5x+3)^5 \end{aligned}$$

(c) $y = \left(x^3 - 3x\right)^5$

Sol.

$$\begin{aligned} y' &= 5(x^3-3x)^4(x^3-3x)' \\ &= 5(x^3-3x)^4(3x^2-3) \\ &= 15(x^3-3x)^4(x^2-1) \end{aligned}$$

(d) $y = \sqrt{x^2 - 2x}$

Sol.

$$\begin{aligned} y' &= \left[\left(x^2-2x\right)^{\frac{1}{2}}\right]' \\ &= -\frac{1}{2}(x^2-2x)^{-\frac{1}{2}}(x^2-2x)' \\ &= -\frac{1}{2}(x^2-2x)^{-\frac{1}{2}}(2x-2) \\ &= -\frac{2x-2}{2\sqrt{x^2-2x}} \\ &= -\frac{x-1}{\sqrt{x^2-2x}} \end{aligned}$$

(e) $y = \dfrac{1}{\sqrt[3]{2x^2-1}}$

Sol.

$$\begin{aligned}
y' &= \left[\left(2x^2-1\right)^{-\frac{1}{3}}\right]' \\
&= -\frac{1}{3}\left(2x^2-1\right)^{-\frac{4}{3}}\left(2x^2-1\right)' \\
&= -\frac{1}{3}\left(2x^2-1\right)^{-\frac{4}{3}}(4x) \\
&= -\frac{4x}{3\sqrt[3]{(2x^2-1)^4}}
\end{aligned}$$

(f) $y = \dfrac{2x-1}{\sqrt{1-2x}}$

Sol.

$$\begin{aligned}
y' &= \frac{(2x-1)'\sqrt{1-2x}-(2x-1)\left(\sqrt{1-2x}\right)'}{1-2x} \\
&= \frac{2\sqrt{1-2x}-(2x-1)\left[\frac{1}{2}(1-2x)^{-\frac{1}{2}}(1-2x)'\right]}{1-2x} \\
&= \frac{2\sqrt{1-2x}-(2x-1)\left[\frac{1}{2}(1-2x)^{-\frac{1}{2}}(-2)\right]}{1-2x} \\
&= \frac{2\sqrt{1-2x}+(2x-1)(1-2x)^{-\frac{1}{2}}}{1-2x} \\
&= \frac{2\sqrt{1-2x}+\dfrac{2x-1}{\sqrt{1-2x}}}{1-2x} \\
&= \frac{\dfrac{2(1-2x)+2x-1}{\sqrt{1-2x}}}{1-2x} \\
&= \frac{\dfrac{2-4x+2x-1}{\sqrt{1-2x}}}{1-2x} \\
&= \frac{1-2x}{(1-2x)\sqrt{1-2x}} \\
&= \frac{1}{\sqrt{1-2x}}
\end{aligned}$$

7. Find the second derivative of the following functions:

(a) $y = x^2(3x-4)$

Sol.

$$\begin{aligned} y' &= x^2(3x-4)' + (3x-4)(x^2)' \\ &= x^2(3) + (3x-4)(2x) \\ &= 3x^2 + 6x^2 - 8x \\ &= 9x^2 - 8x \end{aligned}$$

$$\begin{aligned} y'' &= (9x^2 - 8x)' \\ &= 18x - 8 \end{aligned}$$

(b) $y = 2x^5 - 6x^4 - 3x + 5$

Sol.

$$\begin{aligned} y' &= 2x^5 - 6x^4 - 3x + 5' \\ &= 10x^4 - 24x^3 - 3 \end{aligned}$$

$$\begin{aligned} y'' &= (10x^4 - 24x^3 - 3)' \\ &= 40x^3 - 72x^2 \end{aligned}$$

(c) $y = \dfrac{3}{x^5}$

Sol.

$$\begin{aligned} y' &= (3x^{-5})' \\ &= -15x^{-6} \end{aligned}$$

$$\begin{aligned} y'' &= (-15x^{-6})' \\ &= 90x^{-7} \\ &= \frac{90}{x^7} \end{aligned}$$

(d) $y = \sqrt{2x+1}$

Sol.

$$\begin{aligned}
y' &= \left[(2x+1)^{\frac{1}{2}}\right]' \\
&= \frac{1}{2}(2x+1)^{-\frac{1}{2}}(2x+1)' \\
&= \frac{1}{2}(2x+1)^{-\frac{1}{2}}(2) \\
&= \frac{1}{\sqrt{2x+1}}
\end{aligned}$$

$$\begin{aligned}
y'' &= \left[(2x+1)^{-\frac{1}{2}}\right]' \\
&= -\frac{1}{2}(2x+1)^{-\frac{3}{2}}(2x+1)' \\
&= -\frac{1}{2}(2x+1)^{-\frac{3}{2}}(2) \\
&= -\frac{1}{\sqrt{(2x+1)^3}}
\end{aligned}$$

8. If the function $y = \dfrac{x^3}{(x-1)^2}$, find y' and y''.

Sol.

$$\begin{aligned}
y' &= \frac{(x^3)'(x-1)^2 - x^3\left[(x-1)^2\right]'}{(x-1)^4} \\
&= \frac{3x^2(x-1)^2 - x^3(2)(x-1)}{(x-1)^4} \\
&= \frac{(3x^3 - 3x^2 - 2x^3)}{(x-1)^3} \\
&= \frac{x^3 - 3x^2}{(x-1)^3}
\end{aligned}$$

$$\begin{aligned}
y'' &= \frac{(x^3-3x^2)'(x-1)^3 - (x^3-3x^2)\left[(x-1)^3\right]'}{(x-1)^6} \\
&= \frac{(3x^2-6x)(x-1)^3 - (x^3-3x^2)(3)(x-1)^2}{(x-1)^6} \\
&= \frac{(x-1)^2\left[(3x^2-6x)(x-1) - 3(x^3-3x^2)\right]}{(x-1)^6} \\
&= \frac{3x^3 - 9x^2 + 6x - 3x^3 + 9x^2}{(x-1)^4} \\
&= \frac{6x}{(x-1)^4}
\end{aligned}$$

9. Given the function $y = 2x^3 + 3x^2 - 72x + 21$, find the value of x when $\frac{dy}{dx} = 0$.

Sol.

$$\begin{aligned} y' &= 6x^2 + 6x - 72 \\ 0 &= 6x^2 + 6x - 72 \\ 0 &= x^2 + x - 12 \\ 0 &= (x+4)(x-3) \end{aligned}$$

$$x = -4 \text{ or } x = 3$$

10. Given the function $y = (2 - 3x^2)^4$, find the value of $\frac{d^2y}{dx^2}$ when $x = 1$.

Sol.

$$\begin{aligned} \frac{dy}{dx} &= 4(2 - 3x^2)^3(-6x) \\ &= -24x(2 - 3x^2)^3 \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= -24(2 - 3x^2)^3 - 24x(3)(2 - 3x^2)^2(-6x) \\ &= -24(2 - 3x^2)^3 + 432x^2(2 - 3x^2)^2 \\ &= (2 - 3x^2)^2\left[-24(2 - 3x^2) + 432x^2\right] \\ &= (2 - 3x^2)^2\left[-48 + 72x^2 + 432x^2\right] \\ &= (2 - 3x^2)^2\left(504x^2 - 48\right) \end{aligned}$$

When $x = 1$,

$$\begin{aligned} \frac{d^2y}{dx^2} &= (2 - 3)^2\,(504 - 48) \\ &= (-1)^2(456) \\ &= 456 \end{aligned}$$

11. If the function $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + x + 1$, find

(a) $f'(1)$ and $f''(2)$

Sol.

$$\begin{aligned} f'(x) &= x^2 - 5x + 1 \\ f'(1) &= 1 - 5 + 1 \\ &= -3 \end{aligned}$$

$$\begin{aligned} f''(x) &= 2x - 5 \\ f''(2) &= 4 - 5 \\ &= -1 \end{aligned}$$

(b) the value of x when $f''(x) = 0$

Sol.

$$\begin{aligned} f''(x) &= 0 \\ 2x - 5 &= 0 \\ 2x &= 5 \\ x &= \frac{5}{2} \end{aligned}$$

12. If the function $f(x) = \sqrt{x\sqrt{x\sqrt{x}}}$, find $f'(1)$, $f''(1)$, $f'''(1)$ and $f^{(4)}(1)$.

Sol.

$$\begin{aligned} f(x) &= x^{\frac{1}{2}} \cdot x^{\frac{1}{4}} \cdot x^{\frac{1}{8}} \\ &= x^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} \\ &= x^{\frac{7}{8}} \end{aligned}$$

$$\begin{aligned} f'(x) &= \frac{7}{8}x^{-\frac{1}{8}} \\ f'(1) &= \frac{7}{8} \end{aligned}$$

$$\begin{aligned} f''(x) &= -\frac{7}{64}x^{-\frac{9}{8}} \\ f''(1) &= -\frac{7}{64} \end{aligned}$$

$$\begin{aligned} f'''(x) &= \frac{63}{512}x^{-\frac{17}{8}} \\ f'''(1) &= \frac{63}{512} \end{aligned}$$

$$\begin{aligned} f^{(4)}(x) &= -\frac{1071}{4096}x^{-\frac{25}{8}} \\ f^{(4)}(1) &= -\frac{1071}{4096} \end{aligned}$$

13. Find the derivative $\frac{dy}{dx}$ of the following implicit functions:

(a) $x^2 + 2y = 2x + 3$

Sol.

$$\begin{aligned} x^2 + 2y &= 2x + 3 \\ 2x + 2y' &= 2 \\ y' &= \frac{2 - 2x}{2} \\ &= 1 - x \end{aligned}$$

(b) $x^2 + 3x = y^2 - 5y$

Sol.

$$
\begin{aligned}
x^2 + 3x &= y^2 - 5y \\
2x + 3 &= 2yy' - 5y' \\
(2y - 5)y' &= 2x + 3 \\
y' &= \frac{2x + 3}{2y - 5}
\end{aligned}
$$

(c) $3x^2 + 7xy - 9y^2 = 2$

Sol.

$$
\begin{aligned}
3x^2 + 7xy - 9y^2 &= 2 \\
6x + 7y + 7xy' - 18yy' &= 0 \\
7xy' - 18yy' &= -6x - 7y \\
y'(7x - 18y) &= -6x - 7y \\
y' &= -\frac{6x + 7y}{7x - 18y}
\end{aligned}
$$

(d) $x^3y + xy^3 = 3xy$

Sol.

$$
\begin{aligned}
3x^2y + x^3y' + y^3 + 3xy^2y' &= 3y + 3xy' \\
x^3y' + 3xy^2y' - 3xy' &= 3y - 3x^2y - y^3 \\
y'(x^3 + 3xy^2 - 3x) &= 3y - 3x^2y - y^3 \\
y' &= \frac{3y - 3x^2y - y^3}{x^3 + 3xy^2 - 3x} \\
&= \frac{y(3 - 3x^2 - y^2)}{x(x^2 + 3y^2 - 3)}
\end{aligned}
$$

14. Find the gradient of the tangent to the curve $x^2 + xy + y^2 = 4$ at the point $A(2, -2)$.

Sol.

$$
\begin{aligned}
x^2 + xy + y^2 &= 4 \\
2x + y + xy' + 2yy' &= 0 \\
xy' + 2yy' &= -2x - y \\
(x + 2y)y' &= -2x - y \\
y' &= -\frac{2x + y}{x + 2y}
\end{aligned}
$$

When $x = 2$ and $y = -2$,

$$\begin{aligned} y' &= -\frac{2(2) + (-2)}{2 + 2(-2)} \\ &= -\frac{2}{-2} \\ &= 1 \end{aligned}$$

15. Find the limit of the following:

(a) $\lim_{x \to 0} \frac{\sin 4x}{x}$

Sol.

$$\begin{aligned} \lim_{x \to 0} \frac{\sin 4x}{x} &= \lim_{x \to 0} \frac{4 \sin 4x}{4x} \\ &= 4 \lim_{x \to 0} \frac{\sin 4x}{4x} \\ &= 4(1) \\ &= 4 \end{aligned}$$

(b) $\lim_{x \to 0} \frac{\tan 2x}{\tan 5x}$

Sol.

$$\begin{aligned} \lim_{x \to 0} \frac{\tan 2x}{\tan 5x} &= \lim_{x \to 0} \frac{2 \tan 2x}{2 \tan 5x} \\ &= \lim_{x \to 0} \left(\frac{\sin 2x}{\cos 2x} \cdot \frac{\cos 5x}{\sin 5x} \right) \\ &= \lim_{x \to 0} \left(\frac{\sin 2x}{2x} \cdot \frac{5x}{\sin 5x} \cdot \frac{\cos 5x}{\cos 2x} \cdot \frac{2}{5} \right) \\ &= \frac{2}{5} \lim_{x \to 0} \frac{\cos 5x}{\cos 2x} \\ &= \frac{2}{5} \end{aligned}$$

(c) $\lim_{x \to 0} \frac{x^2}{\tan^2 3x}$

Sol.

$$\begin{aligned} \lim_{x \to 0} \frac{x^2}{\tan^2 3x} &= \lim_{x \to 0} \left(\frac{x}{\tan 3x} \right)^2 \\ &= \lim_{x \to 0} \left(\frac{x \cos 3x}{\sin 3x} \right)^2 \\ &= \lim_{x \to 0} \left(\frac{3x}{3 \sin 3x} \cdot \cos 3x \right)^2 \\ &= \frac{1}{9} \lim_{x \to 0} \left(\frac{3x}{\sin 3x} \right)^2 \\ &= \frac{1}{9} \end{aligned}$$

(d) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{4x+1}$

Sol.

$$\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{4x+1} &= \lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{4x}\cdot\left(1+\frac{1}{x}\right)\right] \\
&= \lim_{x\to\infty}\left[\left(1+\frac{1}{x}\right)^{x}\right]^{4} \\
&= \left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\right]^{4} \\
&= e^{4}
\end{aligned}$$

(e) $\lim_{x\to 0}(1-3x)^{\frac{2}{x}}$

Sol.

$$\begin{aligned}
\lim_{x\to 0}(1-3x)^{\frac{2}{x}} &= \lim_{x\to 0}\left[(1-3x)^{\frac{1}{3x}}\right]^{-6} \\
&= \left[\lim_{x\to 0}(1-3x)^{\frac{1}{3x}}\right]^{-6} \\
&= e^{-6}
\end{aligned}$$

(f) $\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x}$

Sol.

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x} &= \lim_{x\to\infty}\left(1+\frac{2}{x+1}\right)^{x} \\
&= \lim_{x\to\infty}\left\{\left[\left(1+\frac{2}{x+1}\right)^{\frac{x+1}{2}}\right]^{2} - \left(1+\frac{2}{x+1}\right)\right\} \\
&= \lim_{x\to\infty}\left[\left(1+\frac{2}{x+1}\right)^{\frac{x+1}{2}}\right]^{2} \\
&= e^{2}
\end{aligned}$$

16. Find the derivative of the following functions:

(a) $y = \tan^2 3x$

Sol.

$$\begin{aligned}
y' &= 2\tan 3x\sec^2 3x\cdot 3 \\
&= 6\tan 3x\sec^2 3x
\end{aligned}$$

(b) $y = \cos^4 2x$

Sol.

$$\begin{aligned} y' &= 4\cos^3 2x \cdot (-\sin 2x) \cdot 2 \\ &= -8\cos^3 2x \sin 2x \end{aligned}$$

(c) $y = \sec^3 2x$

Sol.

$$\begin{aligned} y' &= 3\sec^2 2x \cdot \sec 2x \tan 2x \cdot 2 \\ &= 6\sec^3 2x \tan 2x \end{aligned}$$

(d) $y = \sec^2(3x+5)$

Sol.

$$\begin{aligned} y' &= 2\sec(3x+5) \cdot \sec(3x+5)\tan(3x+5) \cdot 3 \\ &= 6\sec^2(3x+5)\tan(3x+5) \end{aligned}$$

(e) $y = (1+\sin x)^3$

Sol.

$$\begin{aligned} y' &= 3(1+\sin x)^2 \cdot \cos x \\ &= 3\cos x(1+\sin x)^2 \end{aligned}$$

(f) $y = \sin(\cos x)$

Sol.

$$\begin{aligned} y' &= \cos(\cos x) \cdot (-\sin x) \\ &= -\sin x \cos(\cos x) \end{aligned}$$

(g) $y = \sin 2x \cos 2x$

Sol.

$$\begin{aligned} y' &= \cos 2x \cos 2x \cdot 2 + \sin 2x(-\sin 2x) \cdot 2 \\ &= 2\cos^2 2x - 2\sin^2 2x \\ &= 2\left(\cos^2 2x - \sin^2 2x\right) \\ &= 2\cos 4x \end{aligned}$$

(h) $y = \dfrac{1}{\sin x + \cos x}$

Sol.

$$\begin{aligned} y' &= \frac{-(\sin x + \cos x)'}{(\sin x + \cos x)^2} \\ &= \frac{-(\cos x - \sin x)}{(\sin x + \cos x)^2} \\ &= \frac{\sin x - \cos x}{(\sin x + \cos x)^2} \end{aligned}$$

(i) $y = \dfrac{\cos 5x}{\sin 3x}$

Sol.

$$\begin{aligned} y' &= \frac{(\cos 5x)' \cdot \sin 3x - \cos 5x \cdot (\sin 3x)'}{\sin^2 3x} \\ &= \frac{-5 \sin 5x \cdot \sin 3x - \cos 5x \cdot 3 \cos 3x}{\sin^2 3x} \\ &= \frac{-5 \sin 3x \sin 5x - 3 \cos 3x \cos 5x}{\sin^2 3x} \end{aligned}$$

(j) $y = \dfrac{1 + \cos x}{\sin x}$

Sol.

$$\begin{aligned} y' &= \frac{(\sin x)(1 + \cos x)' - (\sin x)'(1 + \cos x)}{(\sin x)^2} \\ &= \frac{\sin x(-\sin x) - \cos x(1 + \cos x)}{(\sin x)^2} \\ &= \frac{-\sin^2 x - \cos x - \cos^2 x}{(\sin x)^2} \\ &= \frac{-1 - \cos x}{\sin^2 x} \end{aligned}$$

17. Find the derivative of the following functions:

    (a) $y = 5^{3x-2}$

    (b) $y = 3e^{2x^2}$

    (c) $y = a^{3x} + e^{-3x}$

    (d) $y = \dfrac{e^{3x} - e^{2x} + e^{5x}}{e^{2x}}$

    (e) $y = x^a - 2a^x$

    (f) $y = e^{2x} \csc 2x$

18. If the function $y = \dfrac{\sin 2x}{e^x}$, prove that $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0$.

19. Find the derivative of the following functions:

    (a) $y = \ln\left(x^2 + 5\right)$

(b) $y = \ln\left(3x^2 + 6x\right)$

(c) $y = \ln\left(e^x + 2\right)$

(d) $y = \ln\left(\sin^2 4x\right)$

(e) $y = \log\left(x^3 + 3x - 4\right)$

(f) $y = \log_5 (3x + 7)$

(g) $y = \log_2 \dfrac{x}{x+3}$

(h) $y = \dfrac{1 + \log x}{1 + \ln x}$

20. If the function $y = \ln(x + 1)$, find the value of $\dfrac{d^2y}{dx^2}$ when $x = 1$.