Mathematics

Senior 3 Part I

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Introduction

Why this book?

Disclaimer

Acknowledgements

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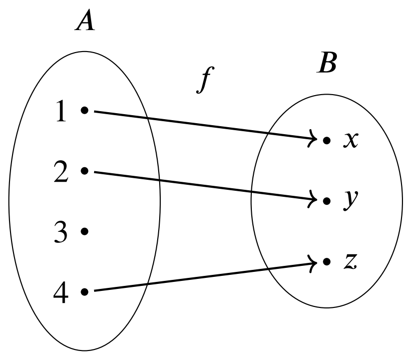
Chapter 22

Function

22.1 Definition of a Function

Mapping, Preimage and Image

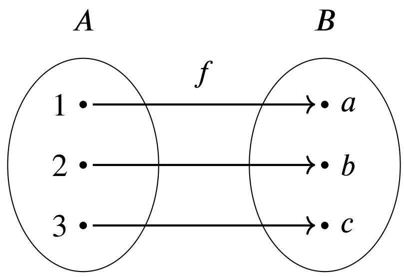
For two non-empty sets A and B, If an element a inside set A has a corresponding element b inside set B, denoted as $a \to b$, then we say that a is mapped to b or a and b are paired. The mapping between two sets is normally denoted as f, g, h, etc. The mapping shown in the diagram below can be denoted as $f : 1 \to x,\ 2 \to y,\ 4 \to z$.



Let $f : A \to B$ is a mapping, a is an element in A. If a is mapped to b under the mapping f, then b is said to be the image of a under the mapping f, denoted as $b = f(a)$; a is said to be the preimage of b under the mapping f. In the diagram above, under the mapping f, the image of 1, 2, and 4 are x, y, and z respectively, while the preimage of x, y, and z are 1, 2, and 4 respectively.

> Let A and B be two non-empty sets, f is a mapping from A to B such that for all elements in A, there is a unique corresponding element in B, then f is a function or a mapping from A to B, denoted as $f : A \to B$.

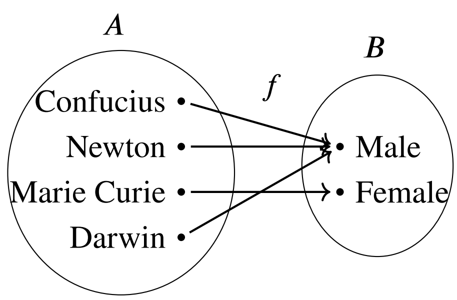
The mapping shown in the diagram below is a function.



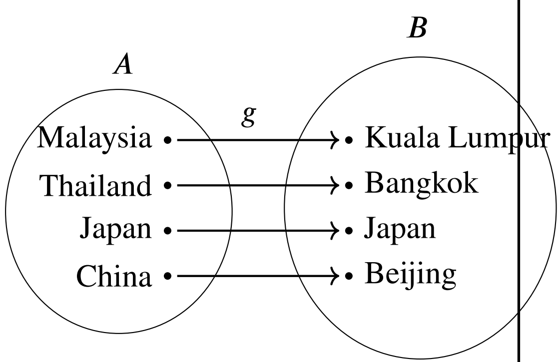
Practice 1

1. For the following mappings, list the image of each element in A and the preimage of each element in B, and determine whether the mapping is a function or not:

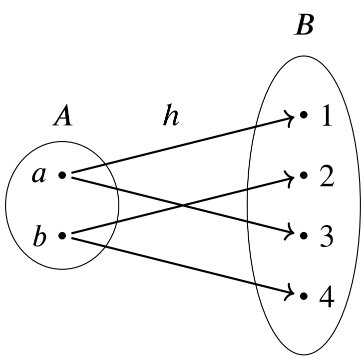
(a)



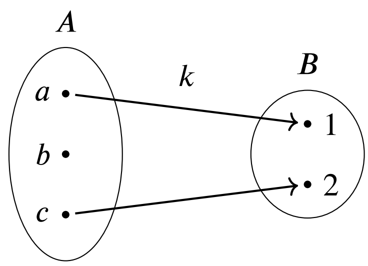
(b)



(c)



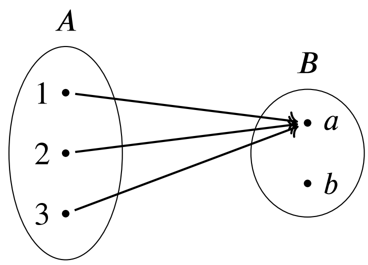
(d)



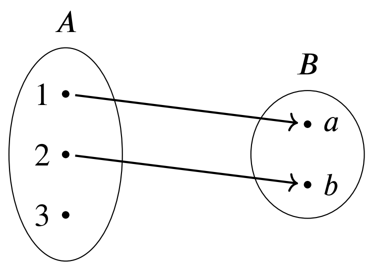
2. Given a mapping $g : x \to x + 3$, $x \in \{-2, -1, 0, 1, 2, 3\}$, find the image of each x.

3. Determine whether the following mappings are functions.

(a)



(b)



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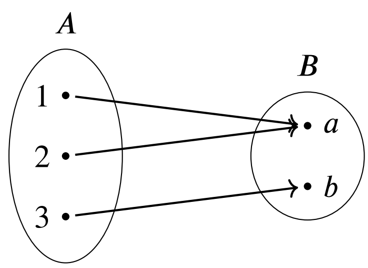
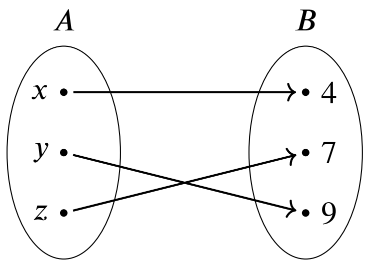
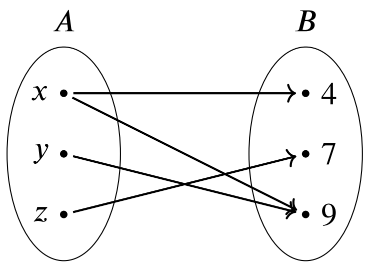


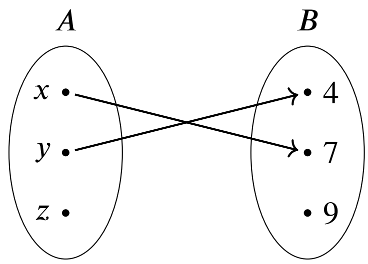
(d)



(e)



(f)



The function $f : A \to B$ can be written as $y = f(x)$, x is the element of A and y is the element of B. When x changes, y changes as well. x is called independent variable, while y is called dependent variable.

Keep in mind that $f(x)$ is NOT the product of f and x.

Representation of Functions

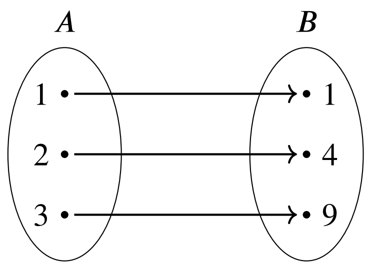
Generally speaking, there are a few ways to represent a function:

1. **Narrative Form**: express the function of two sets in words. For example, Let $A = \{1, 2, 3\}$ and $B = \{1, 4, 9\}$, f is a function from A to B, its definition is that for any element x in A, its corresponding element is x^2 in B.

2. **Arrow Method**: draw an arrow to connect the preimage and image of a function such that the preimage is corresponding to the image. To express the example above, we express it as $f : 1 \to 1,\ 2 \to 4,\ 3 \to 9$.

3. **Analytical Method**: express the function in the form of mathematical expression to represent the relationship between the independent variable and the dependent variable. For example, $f(x) = x^2, x \in A$.

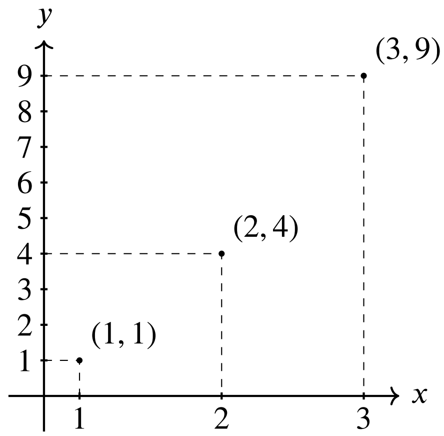
4. **Venn Diagram**: draw arrows between the venn diagram of two sets to represent the function, as shown below:



5. **Table Method**: express the function in the form of table, showing the relationship of the chosen value between independent variable x and the value of its corresponding dependent variable y, as shown below:

x	1	2	3
y	1	4	9

6. **Graphical Method**: draw a graph to represent the function of the two variables, as shown below:



Practice 2

Express the following functions using analytical method, venn diagram, table method and graphical method.

(a) f mapping each integers from -3 to 3 to its squares plus 4.

(b) g mapping each natural numbers from 1 to 4 to its cubes.

Exercise 22.1

1. Express the mapping from set A to set B, and determine which of the following mappings are functions.

	Set A	Set B	Mapping
(a)	{0, 3, 9, 12}	{0, 1, 2, 3}	Divide by 3
(b)	{-2, -1, 0, 1, 2}	{0, 1, 4, 9, 16}	Power of 4
(c)	{-2, -1, 0, 1, 2}	{0, 1, 4}	Square
(d)	{30°, 45°, 60°}	$\left\{\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right\}$	Sine
(e)	{-1, 0, 1, 2}	{-1, 0, 1}	Cube

2. Let function $f(x) = 3x^2 + 1$.

(a) Find the image of the following elements:

i. -3

ii. -2

iii. 0

iv. 2

v. 5

(b) Find the preimage of the following elements:

i. 13

ii. 28

iii. 1

iv. 0

v. 4

3. Let function $g(x) = 5x - 2$. Find:

(a) $g(-2)$

(b) $g(-1)$

(c) $g(0)$

4. Let function $f(x) = \begin{cases} 2x, & x \leq -1 \\ x - 1, & -1 \leq x < 3 \\ 4x + 2, & x \geq 3 \end{cases}$,

find

(a) $f(-5)$

(b) $f(-2)$

(c) $f(0)$

(d) $f(2)$

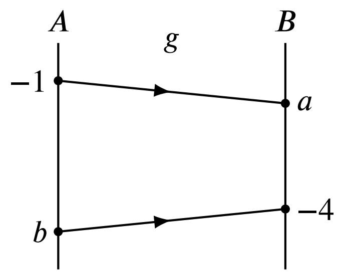
(e) $f(10)$

5. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$. Find the image of -1, 0, 1, and 2 under f.

6. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$. Find the preimage of 0, 1, and 4 under f.

In $\mathbb{R}$, which element does not have a preimage?

7. In the diagram below, given that function $g : A \to B$ is defined as $g : x \to 2x - 8$. Find the value of a and b.

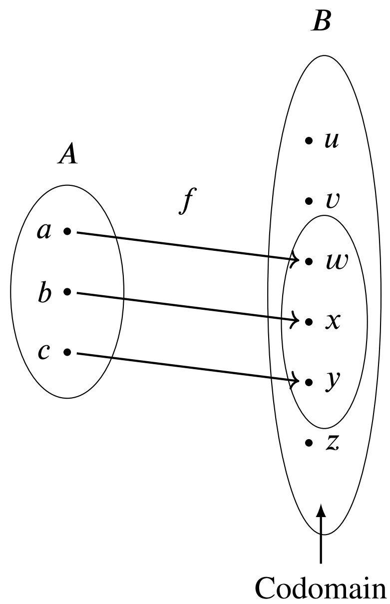


8. Using narrative form, arrow method, venn diagram, table method and graphical method, express the function $f(x) = 2x$, $x \in \{-2, -1, 0, 1, 2\}$.

22.2 Domain and Range

Let f is a function from set A to set B, then set A is called the domain of f, denoted by D_f; set B is called the codomain of f; the set of the images of all elements of A under f is called the range of f, denoted by R_f.

If the domain A and range B of function $f : A \to B$ are both subsets of real number set $\mathbb{R}$, then this function is called real valued function / real function. This book primarily discusses about real valued functions. When the domain of a real function is not mentioned and only the mapping rule is given, its domain is assumed to be the set of all real numbers that yield defined values $f(x)$. After the domain and the mapping rule are determined, the range of a function will then be determined.



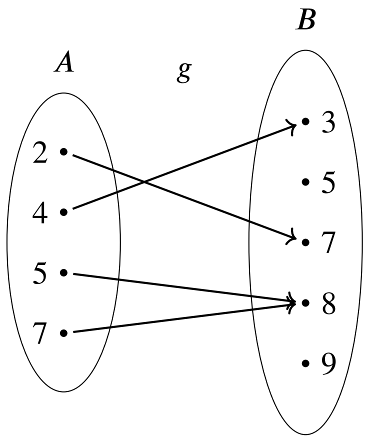
Interval Notation

Let a and b be two real number, $a < b$.

Intervals	Set Notations
(a, b)	$\{x \mid x \in \mathbb{R}, a < x < b\}$
$[a, b)$	$\{x \mid x \in \mathbb{R}, a \leq x < b\}$
$(a, b]$	$\{x \mid x \in \mathbb{R}, a < x \leq b\}$
$[a, b]$	$\{x \mid x \in \mathbb{R}, a \leq x \leq b\}$
(a, ∞)	$\{x \mid x \in \mathbb{R}, x > a\}$
$[a, \infty)$	$\{x \mid x \in \mathbb{R}, x \leq a\}$
$(-\infty, a)$	$\{x \mid x \in \mathbb{R}, x < a\}$
$(-\infty, a]$	$\{x \mid x \in \mathbb{R}, x \leq a\}$

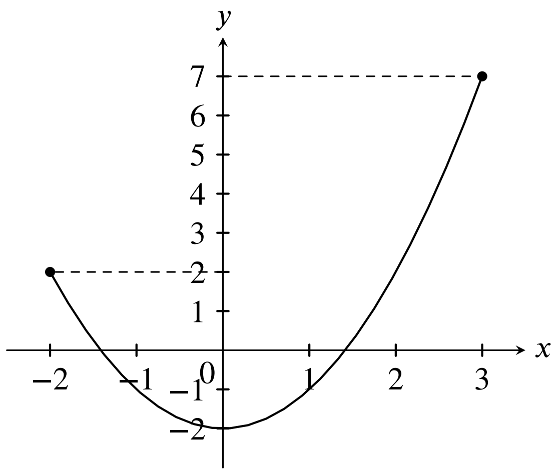
Practice 3

1. Let $A = \{2, 4, 5, 7\}$ and $B = \{3, 5, 7, 8, 9\}$, the definition of function g is given by the diagram below. Find the domain, codomain and range of function g.



2. Let $A = \{-2, -1, 0, 1, 2\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = x^2 - 1$. Find the domain and range of f.

3. The curve in the diagram below represents the function $y = f(x)$, $-2 \leq x \leq 3$. Find the domain and range of f.



4. Find the domain and range of the following functions:

   (a) $f(x) = -4x + 5$

   (b) $g(x) = x^2 - 1$

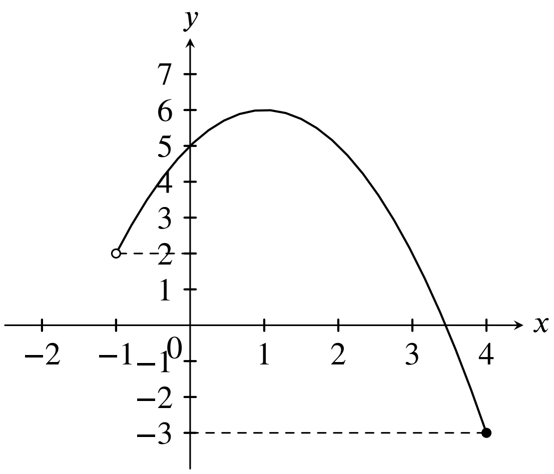
   (c) $h(x) = \dfrac{1}{4x + 7}$

   (d) $k(x) = \sqrt{6 - x}$

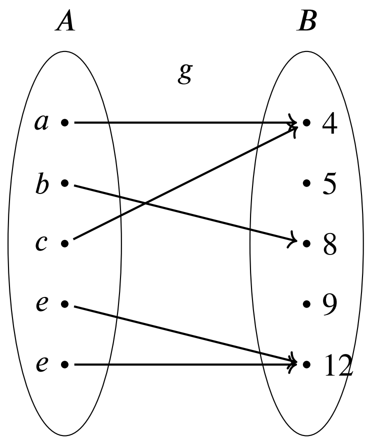
Exercise 22.2

1. Let $X = \{a, b, c, d\}$ and $Y = \{-1, 2, 9, 11\}$, function $f : X \to Y$ is defined by $f(a) = 2$, $f(b) = -1$, $f(c) = 2$, $f(d) = 9$. Find the domain and range of the f.

2. The curve in the diagram below represents the function $y = f(x)$, $-1 < x \leq 4$. Find the domain and range of f.



3. Let $A = \{a, b, c, d, e\}$ and $B = \{4, 5, 8, 9, 12\}$, the definition of function $g : A \to B$ is given by the digram below. Find the domain, codomain and range of function g.



4. Let $A = \{-1, 0, 1, 2\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = 3x^2 - 2$, find the domain and range of f.

5. Let $A = \{-1, 0, 2, 5, 11\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = x^2 - x - 2$, find the domain and range of f.

6. Find the domain and range of the following functions:

   (a) $f(x) = x^3$

   (b) $g(x) = \sqrt{1 - x^2}$

   (c) $h(x) = \dfrac{1}{2x + 3}$

   (d) $k(x) = x^2 - 2x + 4$

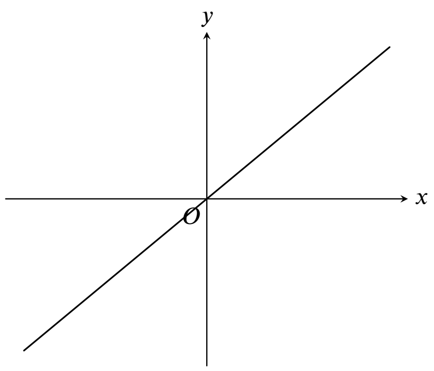
22.3 Graphs of Functions and Their Transformations

Graphs of Simple Functions

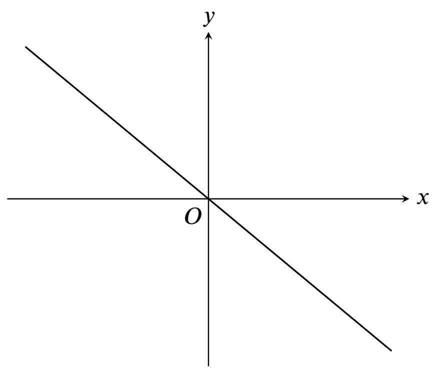
On a Cartesian plane, the graphs formed by all the point (x, y) that satisfied the equation $y = f(x)$ are called graphs of function f. Below are some examples of graphs of simple functions.

Note that any line that is parallel to the y-axis intersects the graph of a function at most once.

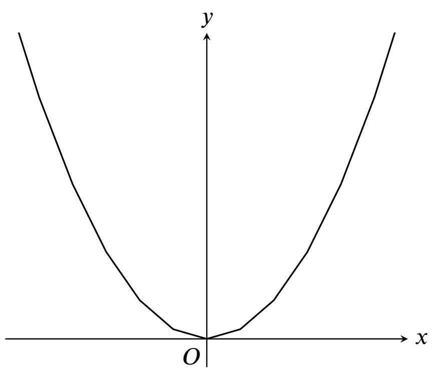
(a) $y = x$



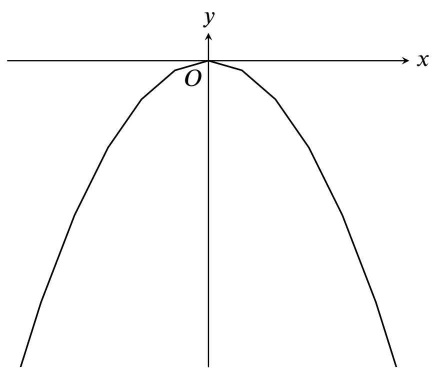
(b) $y = -x$



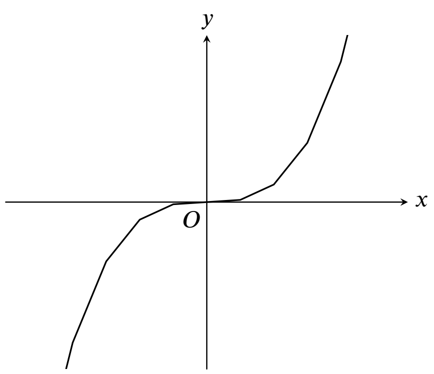
(c) $y = x^2$



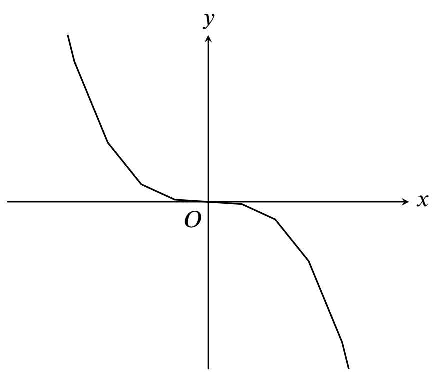
(d) $y = x^2$



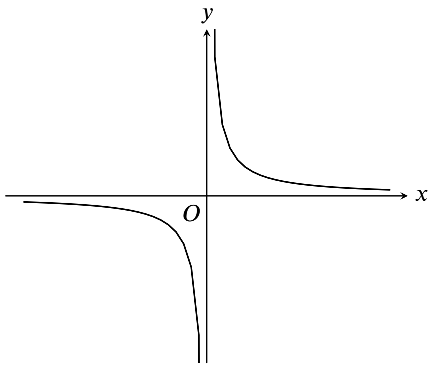
(e) $y = x^3$



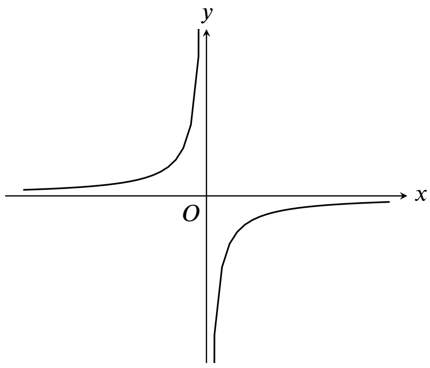
(f) $y = -x^3$



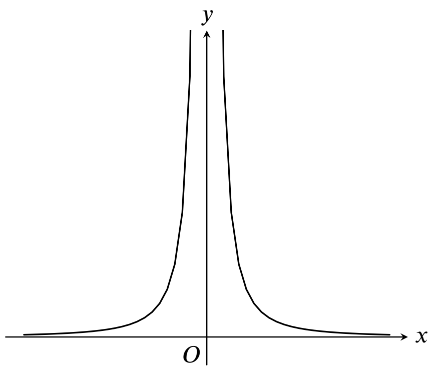
(g) $y = \frac{1}{x}$



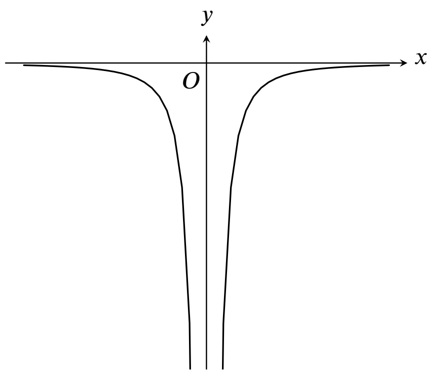
(h) $y = -\frac{1}{x}$



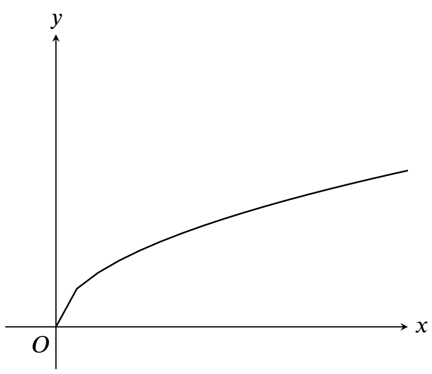
(i) $y = \frac{1}{x^2}$



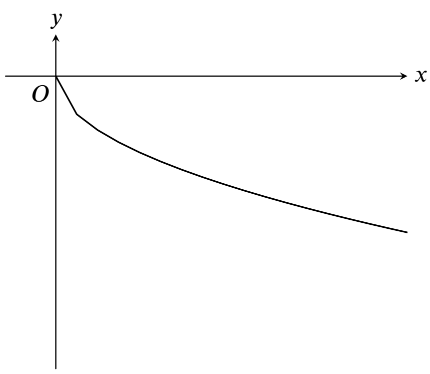
(j) $y = -\frac{1}{x^2}$



(k) $y = \sqrt{x}$



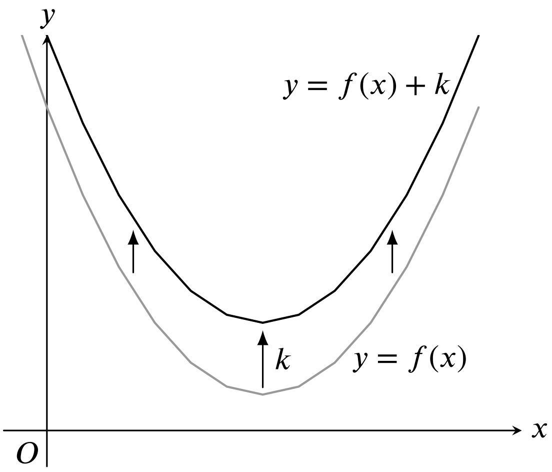
(l) $y = -\sqrt{x}$

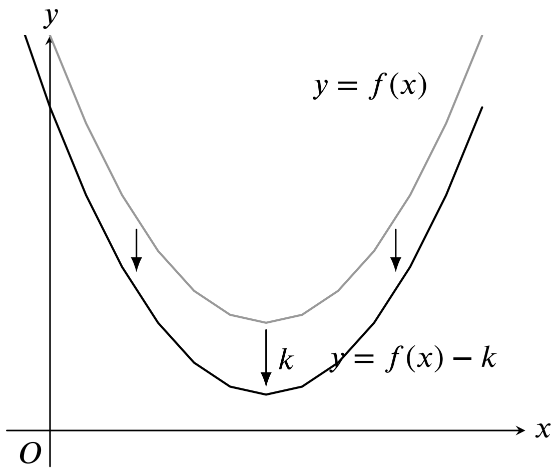


Transformations of Graphs

- If $k > 0$, translate the graph of $y = f(x)$ vertically upwards by k units, the graph of $y = f(x) + k$ is obtained.
- If $k > 0$, translate the graph of $y = f(x)$ vertically downwards by k units, the graph of $y = f(x) - k$

is obtained.





- If $h > 0$, translate the graph of $y = f(x)$ horizontally to the right by h units, the graph of $y = f(x+h)$ is obtained.
- If $h > 0$, translate the graph of $y = f(x)$ horizontally to the left by h units, the graph of $y = f(x-h)$ is obtained.

- If $k > 0$, reflect the graph of $y = f(x)$ about the x-axis, the graph of $y = -f(x)$ is obtained.
- If $k > 0$, reflect the graph of $y = f(x)$ about the y-axis, the graph of $y = f(-x)$ is obtained.

If $a > 0$, zooming (when $a > 1$) or shrinking (when $0 < a < 1$) the graph of $y = f(x)$ by a factor of a in the y-direction, the graph of $y = af(x)$ is obtained.

If $a > 0$, shrinking (when $a > 1$) or zooming (when $0 < a < 1$) the graph of $y = f(x)$ by a factor of $\frac{1}{a}$ in the x-direction, the graph of $y = f(ax)$ is obtained.

Practice 4

Find the line of symmetry and vertex of the following parabola, and sketch its graph. (Question 1 to 2):

1. $y = 2x^2 + 8x + 11$
2. $y = -3x^2 + 18x - 7$

Sketch the graph of the following functions. (Question 3 to 4):

3. $y = \dfrac{4}{(x+2)^2}$
4. $y = \sqrt{x-1} + 3$

Exercise 22.3

Find the line of symmetry and vertex of the following parabola, and sketch its graph.

1. $y = 2x^2 + 4x + 5$
2. $y = -3x^2 + 12x - 4$
3. $y = 4x^2 - 20x + 19$
4. $y = -3x^2 - 6x - 4$

Sketch the graph of the following functions.

5. $y = (x+2)^3 - 5$
6. $y = \sqrt{x-5}$
7. $y = \dfrac{1}{(x+2)^2}$
8. $y = -\dfrac{1}{2(x-1)^2}$
9. $y = 3\sqrt{x+1} - 4$
10. $y = \dfrac{4}{2x+3}$
11. $y = \begin{cases} 4x+9, & x \le 0 \\ 9-2x, & x > 0 \end{cases}$
12. $y = \begin{cases} x, & x < -1 \\ \sqrt{x+1}, & x \ge -1 \end{cases}$
13. Sketch the graph for the function $f(x) = x^2 - 6x + 12$, $-2 \le x \le 8$, and find its domain and range.
14. Sketch the graph for the function $g(x) = -x^2 - 4x - 7$, $-2 \le x \le 5$, and find its domain and range.
15. Sketch the graph for the function $f(x) = -x^2 + 2x + 10$, and find its domain and range.
16. Sketch the graph of the function $y = \sqrt{x}$, and transform it according the following

steps. Sketch the graph of each function after each step on the same diagram, and write down the corresponding function.

Step 1: Translate 4 units to the left;

Step 2: Scale up by a factor of 2 in the x-direction;

Step 3: Reflect about the y-axis;

Step 4: Translate 3 units downwards.

Step 5: Scale down by half in the y-direction.

22.4 Composite Functions

Let A, B, and C be three non-empty sets, $f : A \to B$ and $g : B \to C$ be two functions, an element x in set A is mapped to an element $f(x)$ in set B by function f, and $f(x)$ is mapped to an element $g\big(f(x)\big)$ in set C by function g. In other words, x in set A is mapped to an element $g\big(f(x)\big)$ in C after two mappings. That is:

$$x \xrightarrow{\ f\ } f(x) \xrightarrow{\ g\ } g\big(f(x)\big)$$

The combination of these two mappings are a function from set A to set C, this function is called the *composite function* of f and g, denoted by $g \circ f$. When defining the composite function $g \circ f$, the range of f must be a subset of the domain of g, that is, $R_f \subseteq D_g$.

Note that $D_{g \circ f} = D_f$, $R_{g \circ f} \subseteq R_g$.

$\forall n \in \mathbb{N}$, $f^{n+1} = f \circ f^n$.

Generally speaking, $g \circ f \neq f \circ g$.

If $f \circ (g \circ h)$ is defined, then $(f \circ g) \circ h$ is also defined, and $f \circ (g \circ h) = (f \circ g) \circ h$. Therefore, we can write $f \circ g \circ h$ without ambiguity.

Practice 5

1. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x+3$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 5 - x$. Find $(g \circ f)(x)$ and $(f \circ g)(x)$.

2. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 2x + 3$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 3x - 4$. Find

   (a) $g \circ f$ and $f \circ g$;

   (b) $g\big(f(2)\big)$, $f\big(g(2)\big)$, $(g \circ f)(2)$, and $(f \circ g)(2)$.

3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4 - x^2$ and $g : \{x | x \leq 4\} \to \mathbb{R}$, $g(x) = \sqrt{4 - x}$. Prove the existence of $f \circ g$ and $g \circ f$ respectively.

22.5 One to One Function, Onto Function and One to One Onto Function

One to One Function

Let $f : A \to B$ be a function, if there is at most one preimage in set A for each element in set B, then f is called a *one to one function*.

As shown in the diagram above, each element in the codomain B of the function $f : A \to B$ has at most one preimage in the domain A of the function, thus f is a one to one function; while the element b_2 in the codomain B of the function $g : A \to B$ has two preimages a_2 and a_3, thus g is not a one to one function.

A function $y = f(x)$ is a one to one function, if and only if any line parallel to the x-axis intersects the graph of the function at most once.

Onto Function

If each element in the codomain B of the function $f : A \to B$ has at least one preimage under the function f, then f is said to be an *onto function*.

As shown in the diagram above, each element in the codomain B of the function $f : A \to B$ has at least one preimage under the function f, therefore f is an onto function; while the element b_3 in the codomain B of the function $g : A \to B$ has no preimage under the function g, therefore g is not an onto function.

One to One Onto Function

If a function is both a one to one function and an onto function, then it is a *one to one onto function*, as shown in the diagram above.

Practice 7

Determine whether the following functions are one to one functions or onto functions.

Exercise 22.5

1. Let $A = \{1, 2, 3\}$, $f : A \to A$ is defined by $f : 1 \to 1, 2 \to 3, 3 \to 2$. Determine if f is a one to one function or an onto function.

2. Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$, $f : A \to B$ is defined by $f : a \to y, b \to x$, $c \to z$, $d \to y$. Determine if f is a one to one function or an onto function.

3. Let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 2x+1$. Determine if g is a one to one function or an onto function.

4. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x^3 - 3$. Determine if f is a one to one function or an onto function.

5. Let the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = \dfrac{1}{x}$. Determine if f is a one to one function or an onto function.

6. Let the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = \sqrt{x}$. Determine if f is a one to one function or an onto function.

7. Determine whether the following functions are one to one, onto or one to one onto functions.

   (a) $A = \{a, b, c\}$, $B = \{x, y, z\}$, $f : A \to B$, $f : a \to x, b \to x, c \to y$

   (b) $A = \{a, b, c\}$, $B = \{x, y, z\}$, $g : A \to B$, $g : a \to x, b \to y, c \to z$

   (c) $A = \{a, b, c\}$, $B = \{x, y\}$, $h : A \to B$, $h : a \to x, b \to y, c \to y$

   (d) $A = \{a, b, c\}$, $B = \{x, y\}$, $k : A \to B$, $k : a \to x, a \to y, c \to y$

   (e) $A = \{a, b, c\}$, $B = \{x, y\}$, $f : A \to B$, $f : a \to x, a \to y, b \to x, c \to y$

   (f) $A = \{a, b, c, d\}$, $B = \{u, v, x, y, z\}$, $g : A \to B$, $g : a \to u, b \to v, c \to x$, $d \to y$

8. Determine whether the following functions mapping A to B are one to one functions or onto functions.

22.6 Inverse Functions

If $f : A \to B$ is a one to one onto function, then there exist a function $g : B \to A$, such that if $y = f(x)$, then $g(y) = x$. The function g is called the *inverse function* of f, and is denoted by f^{-1}.

from the diagram above, we can conclude the following:

$$x \xrightarrow{f} y = f(x) \xrightarrow{f^{-1}} f^{-1}\big(f(x)\big) = f^{-1}(y)$$

or

$$y \xrightarrow{f^{-1}} x = f^{-1}(y) \xrightarrow{f} f\big(f^{-1}(y)\big) = f(x)$$

If both $(g \circ f)^{-}1$ and $f^{-1} \circ g^{-1}$ exist, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Practice 8

Exercise 22.5

1. Find the inverse function of the following functions:

   (a) $f : x \to 7x - 3$

   (b) $g : x \to \frac{1}{2}x + 9$

   (c) $h : x \to \frac{x+1}{x-8}$, $x \neq 8$

   (d) $k : x \to \frac{x-1}{2x}$, $x \neq 0$

2. Given the function $f : x \to 2x + 1$ and $g : x \to \frac{1}{x-4}$, $x \neq 4$. Find:

   (a) f^{-1}

   (b) g^{-1}

   (c) $f^{-1} \circ g^{-1}$

   (d) $g^{-1} \circ f^{-1}$

   (e) $(f \circ g)^{-1}$

   (f) $(g \circ f)^{-1}$

Graph of Inverse Functions

If f is a one to one function, then the graph of f^{-1} is the reflection of the graph of f about the line $y = x$.

Practice 9

Given the function $g : \mathbb{R}^+ \cup 0 \to \mathbb{R}^+ \cup 0$, $g : x \to x^2$. On the same set of axes, draw the graph of the function g and its inverse function g^{-1}.

Exercise 22.6

1. Find the inverse function of the following functions:

   (a) $f : x \to 2x - 7$

   (b) $g : x \to \dfrac{1}{x-2}$, $x \neq 2$

   (c) $h : x \to \dfrac{2x-5}{x-2}$, $x \neq 2$

   (d) $k : x \to \dfrac{3x}{x-4}$, $x \neq 4$

2. Given that $f : x \to \dfrac{160}{ax+b}$, $f(5) = 8$ and $f(9) = 10$. Find

   (a) the values of a and b;

   (b) $f^{-1}(16)$.

3. Given that $f : x \to \dfrac{a}{x+b}$, $f(3) = -1$, and $f(-9) = 3$. Find

   (a) the values of a and b;

   (b) the value of x such that $f(x) = f^{-1}(x)$.

4. Given the function $g : x \to \dfrac{6}{x} - 3$, $x \neq 0$. Find

   (a) g^{-1};

   (b) the value of x such that $g^{-1}(x) = x - 2$.

5. Given the function $f : x \to ax + b$ and $f^2 : x \to 4x + 12$. If $a > 0$, find

   (a) the values of a and b;

   (b) $f^{-1}(3)$.

6. Given the function $f : x \to 3x - 2$ and $g : x \to \dfrac{x}{x+4}$, $x \neq -4$. Find

   (a) f^{-1}

   (b) g^{-1}

   (c) $f^{-1} \circ g^{-1}$

   (d) $g^{-1} \circ f^{-1}$

   (e) $(f \circ g)^{-1}$

   (f) $(g \circ f)^{-1}$

7. Given the function $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{0\}$, $f : x \to \dfrac{1}{x-2}$.

   (a) Find f^{-1}.

   (b) On the same set of axes, draw the graph of f and f^{-1}.

8. Given the function $f : x \to 2\sqrt{x+4}$, $x \geq -4$,

(a) Find f^{-1}.

(b) On the same set of axes, draw the graph of f and f^{-1}.

Revision Exercise 22

1. Determine whether the following mappings from set $A = \{1,2,3,4\}$ to set $B = \{a,b,c,d\}$ are functions or not.

   (a) $1 \to a, 2 \to c, 4 \to b$

   (b) $1 \to a, 2 \to d, 3 \to b, 4 \to a$

   (c) $1 \to c, 2 \to c, 3 \to b, 4 \to b$

   (d) $1 \to a, 2 \to c, 2 \to b, 4 \to d$

   (e) $1 \to c, 2 \to b, 3 \to d, 4 \to c, 4 \to a$

2. Given the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 3x-2, & x < -3 \\ 2x^2+4, & -3 \leq x < 2 \\ -2x+9, & x \geq 2 \end{cases}$, find

   (a) $f(-4)$

   (b) $f(0)$

   (c) $f(2)$

   (d) $f(3)$

3. Find the domain and range of the following functions:

   (a) $f : 1 \to 3, 2 \to 5, 4 \to 8$

   (b) $g : 2 \to 4, 4 \to 5, 5 \to 7, 6 \to 9$

   (c) $h : 1 \to 3, 2 \to 5, 3 \to 6, 4 \to 8$

4. The table below shows a function f:

x	-3	-2	-1	0	1
f(x)	-22	-3	4	5	6

   (a) Find the domain and range of the function;

   (b) Sketch the graph of the function.

   (c) Determine if the inverse function of f exists.

5. As shown in the diagram below, let a function $f : x \to ax + b$. Find the value of $f(4)$ and $f^{-1}(5)$.

6. Given the function $f : x \to x^2 - x + 1$, $-1 \leq x \leq 3$, find its range.

7. Let function $f : x \to 2x^2 - 4x + 3$.

   (a) If $D_f = \mathbb{R}$, find the range of f;

   (b) If $D_f = \{x | x \geq 3\}$, find the range of f.

8. Find the domain and range of the following functions:

   (a) $f(x) = \dfrac{1}{x}$

   (b) $f(x) = \sqrt{2x-5}$

   (c) $f(x) = x^2 + 4x + 7$

   (d) $f(x) = \dfrac{1}{x^2+4}$

9. Find the domain of the following functions:

(a) $f(x)=\dfrac{2x}{x-3}$

(b) $f(x)=\sqrt{4-x^2}$

(c) $f(x)=\dfrac{x-2}{2x^2-5x+2}$

(d) $f(x)=\dfrac{x-3}{\sqrt{x^2-9}}$

10. Sketch the graph for the following functions:

(a) $f(x)=2x^2-5x+9$

(b) $f(x)=-3x^2+6x+11$

(c) $f(x)=3x^2+12x+10$

(d) $f(x)=-5x^2+6x+11$

(e) $f(x)=2x^3-7$

(f) $f(x)=\sqrt{3x-9}$

(g) $f(x)=\dfrac{4}{2x+11}$

(h) $f(x)=\dfrac{2x+7}{x-1}$

(i) $f(x)=2\sqrt{x+5}-4$

(j) $f(x)=\dfrac{1}{(2x-3)^2}$

(k) $f(x)=\begin{cases} 2x+1, & x<0 \\ x^2, & x\geq 0 \end{cases}$

(l) $f(x)=\begin{cases} 1-x^2, & x\leq 1 \\ x^2+2x-3, & x>1 \end{cases}$

11. Given the function $f : x \to 2x^2$ and $g : x \to 3x-4$. Find the value of m such that $(f\circ g)(m)=(g\circ f)(m)$.

12. Given the function $f : x \to x^2+2x-3$ and $g : x \to 3x-4$. If $(f\circ g)(k)=(g\circ f)(k)$, find the value of k.

13. Given that $f(x) = 3x+1$, $x \neq 0$. If $(f\circ g)(x)=6x^2-9x+4$, find $g(x)$.

14. Given that $f(x) = \dfrac{x+1}{x}$, $x \neq 0$. IF $(f\circ g)(x)=x$, find $g(x)$.

15. A function f is defined by $f : x \to x-3$. Find another function g such that $g\circ f : x \to 4x^2-20x+25$.

16. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} -2, & x\leq -3 \\ |x|-2x, & -3<x<3 \\ 2x-1, & x\geq 3 \end{cases}$. Find $(f\circ f\circ f)(-1000)$.

17. Let function $f : A \to \mathbb{R}$ be defined by $f : x \to 2x^2$. Determine if f is one to one function when A is the following sets.

(a) $A=\{x|0\leq x<6\}$

(b) $A=\{x|x<0\}$

(c) $A=\{x|-2\leq x<2\}$

(d) $A=\{x|x>3\}$

18. Determine whether the following functions are one to one functions or onto functions.

(a) $f : \mathbb{R}^+ \to \mathbb{R}$, $f : x \to |x|-2$

(b) $f : \mathbb{R}\setminus\{2\} \to \mathbb{R}\setminus\{1\}$, $f : x \to \dfrac{x}{x-2}$

(c) $f : \mathbb{R} \to \mathbb{R}^+\cup\{0\}$, $f : x \to |x|$

19. Let $A=\mathbb{R}\setminus\left\{-\frac{1}{2}\right\}$ and $A=\mathbb{R}\setminus\left\{\frac{1}{2}\right\}$, function $f : A \to B$ is defined by $f(x)=\dfrac{x-3}{2x+1}$. Find

(a) $f^{-1}(-2)$

(b) $f^{-1}(0)$

(c) $f^{-1}(3)$

20. Let function $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = x^2 + 2x + 1$. Find $f^{-1}(4)$ and $f^{-1}(9)$.

21. A function f is defined by $f : x \to \frac{x}{2} + 1$. If $g \circ f^{-1} : x \to 4x^2 - 8x + 7$, find the function g.

22. Given the functioon $f : x \to 3x^2 + 5x + 9$, $x \leq a$. Find the maximum value of a such that the inverse function of f exists.

23. Let the function f and g be defined as $f : x \to 5x + 3$ and $g : x \to 2x - 7$ respectively. Find

    (a) $f \circ g$

    (b) f^{-1}

    (c) g^{-1}

24. Given the function $f : x \to 2x + 3$ and $g : x \to 3 - x2x + 5, x \neq -\frac{5}{2}$. Find

    (a) $f \circ g$

    (b) f^{-1}

    (c) g^{-1}

    Show that $g^{-1} \circ f^{-1} = (f \circ g)^{-1}$.

25. Given the function $f : x \to \sqrt{x}, x \neq 0$ and $g : x \to x^3$. Find

    (a) $g \circ f$

    (b) f^{-1}

    (c) g^{-1}

    (d) $(g \circ f)^{-1}$

    (e) $g^{-1} \circ f^{-1}$

26. Given the function $f : x \to 2\sqrt{x - 4} + 3$, $x \geq 4$.

    (a) Find the range of f.

    (b) Find the inverse function f^{-1} of f.

    (c) On the same diagram, sketch the graphs of f and f^{-1}.

Chapter 23

Exponents and Logarithms

23.1 Exponents

Definition and Properties of Exponents

Back in Junior 1, we have learnt the following definitions of exponents:

Positive exponent	$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$
Zero exponent	$a^0 = 1$
Negative exponent	$a^{-n} = \dfrac{1}{a^n}$ $(a \neq 0, n \in \mathbb{Z}^+)$
Fractional exponent	$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ $(a \geq 0, n > 1, m, n \in \mathbb{Z}^+)$

The exponent of rational numbers have the following properties:

1. $a^m \times a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^n = a^n b^n$
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ $(b \neq 0)$

Practice 1

Without using the calculator, find the value of the following expressions (Question 1 to 2):

1. $2^{-2} + 2{-}5 - (-2)^{-3}$

2. $\left(3\frac{6}{25}\right)^{-\frac{1}{2}}$

3. Simplify $a^{-}4 \div a^{-5} \times (b^{-3})^{-4}$

Exponential Functions and Graphs

Let a is a constant that is bigger than zero and not equal to 1, then the function being expressed in the form of $y = a^x$ is called an *exponential function*. The domain of an exponential function is $\mathbb{R}$.

Consider the following: a cell divides into two cells, and then each of the two cells divides into two cells again, and so on. If we let x be the number of divisions, the number of cells after the divisions be y, then the functional relationship between x and y is $y = 2^x$, which is an exponential function.

In order to look into the graph and its properties of an exponential function $y = a^x$, we sketch the graph of some exponential functions, the graph of $y = 2^x$, $y = 10^x$, and $y = \left(\frac{1}{2}\right)^x$ are shown in the diagram below.

From the diagram above, we can see that:

(1) The graph of the function $y = 2^x$, $y = 10^x$, and $y = \left(\frac{1}{2}\right)^x$ are only at the top of the x-axis. Actually, when $a > 0$, $a^x > 0$. Therefore, the value of the exponential function $y = a^x$ is always positive.

(2) When $x = 0$, $y = 1$. Hence, the graph of exponential functions $y = a^x$ always passes through the point $(0, 1)$.

(3) For the function $y = 2^x$, when $x > 0$ and $y = 10^x$, when $x < 0$, $y < 1$; when $x > 0$, $y > 1$. When the value of x increases, the value of y increases, that is, the function is an increasing function in the interval $(-\infty, +\infty)$.

(4) For the function $y = \left(\frac{1}{2}\right)^x$, when $x > 0$, $y > 1$; when $x < 0$, $y < 1$. When the value of x increases, the value of y decreases, that is, the function is a decreasing function in the interval $(-\infty, +\infty)$.

When we are discussing about the graph and its properties of an exponential function $y = a^x$, the following two cases are considered:

Practice 2

1. Without using the calculator, compare the value of the following expressions:

   (a) $\pi^{2.1}$ and $\pi^{3.5}$

   (b) $0.5^{-2.3}$ and $0.5^{-3.8}$

2. Given the exponential functions $f(x) = 3^{x^2-3x+5}$ and $g(x) = 3^{x+10}$. Find the value of x such that $f(x) = g(x)$.

Exercise 23.1

Without using the calculator, find the value of the following expressions (Question 1 to 10):

1. $\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^0 - \left(-\frac{1}{3}\right)^{-2}$
2. $\left(\dfrac{3^{-5} \cdot 3^2}{3^{-3}}\right)^{-2}$
3. $6^{-8} \div 6^{-5} + 3^{-3}$
4. $12^{\frac{1}{3}} \times 6^{\frac{1}{3}} \div 27^{\frac{1}{6}} \div 3^{\frac{1}{6}}$
5. $(0.2)^{-2} \times (0.125)^{\frac{2}{3}}$
6. $(0.3)^{-\frac{1}{3}} \times (0.0081)^{\frac{1}{3}} + (0.064)^{\frac{1}{3}}$
7. $\left(\frac{81}{16}\right)^{-0.25} \times \left(\frac{8}{27}\right)^{-\frac{2}{3}} \times (0.25)^{-2.5}$
8. $\left(\frac{1}{2}\right)^{-2} + 125^{\frac{2}{3}} + 343^{\frac{1}{3}} - \left(\frac{1}{27}\right)^{-\frac{1}{3}}$
9. $\left(2\frac{1}{4}\right)^{-\frac{3}{2}} + \left(1\frac{11}{25}\right)^{-1} - \left(2\frac{2}{3}\right)^0$
10. $\dfrac{5\sqrt{4}\sqrt{8}\left(\sqrt[3]{\sqrt[5]{4}}\right)^2}{\sqrt[3]{\sqrt{2}}}$

Simplify the follwoing expressions (Question 11 to 24):

11. $a^{\frac{1}{2}} \cdot a^{\frac{1}{4}} \cdot a^{-\frac{1}{8}} \cdot a^{\frac{1}{6}}$
12. $\left(9a^2b^{-2}c^4\right)^{-1}$
13. $\left(x^4y^{-5}\right)\left(x^{-2}y^2\right)^2$
14. $3a^{-2}b^{-3} \div \left(-3^{-1}a^2b^{-3}\right)$
15. $\sqrt[3]{\dfrac{a^2b^{-1}}{a^{\frac{1}{2}}b^5}}$
16. $5a^{-2}b^{-3} \div \left(5^{-1}a^2b^{-3}\right) \times 5^{-2}ab^4c$
17. $\dfrac{a^{-2} - b^{-2}}{a^{-2} + b^{-2}}$
18. $\left(a^{-1} + b^{-1}\right)(a+b)^{-1}$
19. $\left(x + x^{-1}\right)\left(x - x^{-1}\right)$
20. $\left(-2x^{\frac{1}{4}}y^{-\frac{1}{3}}\right)\left(3x^{-\frac{1}{2}}y^{\frac{2}{3}}\right)\left(-4x^{\frac{1}{4}}y^{\frac{2}{3}}\right)$
21. $2x^{-\frac{1}{3}}\left(\frac{1}{2}x^{\frac{1}{3}} - 2x^{-\frac{2}{3}}\right)$
22. $\left(\sqrt{x^3} \cdot \sqrt{y}\right)^2 \cdot \left(\sqrt{y} \cdot \sqrt{x^3}\right)^3$
23. $\dfrac{3 \times 2^n - 4 \times 2^{n-2}}{2^n - 2^{n-1}}$
24. $\left(3^{n+6} - 5 \times 3^{n+1}\right) \div \left(7 \times 3^{n+2}\right)$
25. Sketch the graph of the following functions on the same diagram:

    (a) $y = 3^x$

    (b) $y = \left(\frac{1}{3}\right)^x$
26. Without using the calculator, compare the value of the following expressions:

(a) $2.5^{7.1}$ and $2.5^{8.5}$

(b) $0.35^{6.5}$ and $0.35^{5.6}$

(c) $1.03^{-2.1}$ and $1.03^{-3.2}$

(d) $\left(\sqrt{2}\right)^{\pi}$ and $\left(\sqrt{2}\right)^{\pi-3.5}$

(e) $0.01^{-\frac{1}{3}}$ and $0.01^{-\frac{1}{2}}$

(f) $2.7^{\sqrt{20}}$ and $2.7^{\sqrt[3]{35}}$

27. Given that $f_1 : x \to 2^{3x}$ and $f_2 : x \to 2^{x^2+2}$. Find the value of x such that $f_1(x) = f_2(x)$.

28. Given the function $f(x) = (0.4)^{x^2-x+1}$ and $g(x) = (0.4)^{6x+19}$. Find the value of x such that $f(x) = g(x)$.

23.2 Logarithms

Definition of Logarithms

If $a_n = x$, where $a > 0$ and $a \neq 1$, then we define $\log_a x = n$, and we say that n is the logarithm of x to the base a. In $\log_a x$, a is called the base, x is called the antilogarithm.

On the other hand, if $\log_a x = n$, then $a_n = x$. This is the inversible relationship between exponents and logarithms. That is,

$$\log_a x = n \iff a^n = x \qquad a > 0,\ a \neq 1,\ x > 0$$

Logarithms with base 10 are called common logarithms, and are usually written as $\log a$.

Another common logarithm is the natural logarithm, which has base e ($e \approx 2.71828182846$), and is usually written as $\ln x$.

Practice 3

Find the value of x in the following equations:

1. $\log x = 3$
2. $\log_x 27 = \dfrac{3}{2}$
3. $2\log_x(3\sqrt{3}) = 1$
4. $\log_2(16\sqrt{2}) = x$

Logarithmic Functions and Graphs

From the definition of logarithms, we can see that if $y = a^x$, then $x = \log_a y$. From the concept of inverse functions, we know that $y = \log_a x$ is the inverse function of $y = a^x$. Function $y = \log_a x$ is called the logarithmic function, where $a > 0$ and $a \neq 1$. Since the domain of $y = a^x$ is $\mathbb{R}$, and its range is $\mathbb{R}^+$, so the domain of $y = \log_a x$ is $\mathbb{R}^+$, and its range is $\mathbb{R}$.

Since the logarithmic function $y = \log_a x$ is the inverse function of the exponential function $y = a^x$, so the graph of $y = \log_a x$ is the reflection of the graph of $y = a^x$ about the line $y = x$. If we draw a curve of $y = a^x$, then reflect it about the line $y = x$, we can get the graph of $y = \log_a x$. For example, in the diagram below, the curves that are the reflection of the graphs of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ about the line $y = x$ are the graphs of $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ respectively.

From the diagram above, we can see that:

(1) Since the domain of $y = \log_a x$ is $x > 0$, so the graph of the function $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ are only at the right side of the y-axis.

(2) When $x = 1$, $y = 0$. Hence, the graph of logarithmic functions $y = a^x$ always passes through the point $(1, 0)$.

(3) For the function $y = \log_2 x$, when $x > 1$, $y > 0$; when $0 < x < 1$, $y < 0$. When the value of x increases, the value of y increases, that is, the function is an increasing function in the interval $(0, +\infty)$.

(4) For the function $y = \log_{\frac{1}{2}} x$, when $x > 1$, $y < 0$; when $0 < x < 1$, $y > 0$. When the value of x increases, the value of y decreases, that is, the function is a decreasing function in the interval $(0, +\infty)$.

When we are discussing about the graph and properties of a logarithmic function $y = \log_a x$, the following two cases are considered:

Practice 4

1. Without using the calculator, Compare the value of the following expressions:
2. (a) $\log 6$ and $\log 9$
   (b) $\log_{0.5} 4.2$ and $\log_{0.5} 3.9$
   (c) $\log_2 1.8$ abd $\log_4 5.8$.
3. Find the domain of the following functions:
   (a) $y = \log_a(x+2)$
   (b) $y = \log_2(x^2 - 9)$
   (c) $y = \log_7 \dfrac{2}{3-2x}$
   (d) $y - \sqrt{\log_5(2-x)}$

Exercise 23.2

1. Find the value of x for the following expression:
   (a) $\log_2 x = 4$
   (b) $\log_{125} x = \dfrac{1}{3}$
   (c) $\log_{16}(2\sqrt{2}) = x$
   (d) $\log_{\frac{1}{3}} 81 = x$
   (e) $\log_x 81 = 4$
   (f) $\log_x 49 = -2$
2. Sketch the graph of the following functions on the same set of axes:
   (a) $y = \log_5 x$
   (b) $y = \log_{\frac{1}{5}} x$
3. Without using the calculator, compare the value of the following expressions:
   (a) $\log_3 5$ and $\log_3 6$
   (b) $\log 1.51.4$ and $\log_{1.5} 1.6$
   (c) $\log_{\sqrt{3}} 4.8$ and $\log_{\sqrt{3}} 5.8$
   (d) $\log_{2.3} \pi$ and $\log_{2.3}(\pi - 3)$
   (e) $\log_{0.4} \sqrt{2}$ and $\log_{0.4} \sqrt{3}$
   (f) $\log_{\frac{1}{2}} 3$ and $\log_{\frac{1}{2}} \dfrac{1}{4}$
4. Find the domain of the following functions:
   (a) $y = \log_2(3-2x)$
   (b) $y = \log(x^2+1)$
   (c) $y = \log_5(9-16x^2)$
   (d) $y = \log_9 \dfrac{1}{x-2}$
   (e) $y = \log_8 \sqrt{2x^2 - x - 3}$
   (f) $y = \dfrac{1}{\log_3(7x-5)}$

23.3 Arithmetic Properties of Logarithms and Base Changing Formula

Identties and Arithmetic Properties of Logarithms

Logarithms have the following identities:

$$\log_a 1 = 0$$
$$\log_a a = 1$$

Logarithms have the following arithmetic properties:

$$\begin{aligned}
&\log_a(xy) = \log_a x + \log_a y \quad (x > 0, y > 0)\\
&\log_a \frac{x}{y} = \log_a x - \log_a y \quad (x > 0, y > 0)\\
&\log_a x^n = n\log_a x \quad (x > 0)\\
&a^{\log_a x} = x
\end{aligned}$$

Base Changing Formula

The base of a logarithm can be changed from one to another. Let $\log_a x = n$, then $a^n = x$. Change both sides of the equation to logarithm with base b, we have

$$\begin{aligned}
\log_b a^n &= \log_b x\\
n\log_b a &= \log_b x\\
\because a \neq 1,\ \therefore\ \log_b a &\neq 0\\
n &= \frac{\log_b x}{\log_b a}\\
\therefore\ \log_a x &= \frac{\log_b x}{\log_b a}
\end{aligned}$$

The expression above is called the *base changing formula*.

When $x = b$, we have $\log_a b = \dfrac{1}{\log_b a}$.
In this book, the value of logarithms are rounded to 4 decimal places.

Practice 5

Exercise 23.2

Without using the calculator, compare the value of the following expressions (Question 1 to 4):

1. $5^{2\log_5 4}$
2. $4^{3\log_2 \sqrt{2}}$
3. $2\log 5 - \log \dfrac{1}{3} + \dfrac{1}{2}\log \dfrac{16}{9}$
4. $\dfrac{\log_4 27}{\log_2 3}$
5. Given that $\log_2 4 = a$ and $\log_2 5 = b$. Express the following expressions in terms of a and b:
   (a) $\log_2 90$
   (b) $\log_3 270$
   (c) $\log_9 1.8$
6. Given that $\log_{16} y = \dfrac{1}{2} + \log_4 x$. Express x in terms of y.

Exercise 23.3

Exercise 23.2

Simplify the following expressions (Question 1 to 6):

1. $\log_2 4^x$
2. $\log_2 a^{\log_a 2}$
3. $3^{\log_3 x - \log_3 y}$
4. $\log_3 (9^x \times 27^y)$
5. $2^{-\log_8 x}$
6. $3\log_4 2^x$

Without using the calculator, evaluate the following expressions (Question 7 to 22):

7. $\log_7 \sqrt[3]{49}$
8. $49^{\log_7 3}$
9. $2^{2\log_2 7} + \left(\dfrac{1}{2}\right)^{-\log_2 7}$
10. $\log_3 5 - \log_3 15$
11. $\dfrac{\log \sqrt{3}}{\log \dfrac{1}{9}}$
12. $\log_5 \dfrac{1}{5} + \log_5 \sqrt[3]{5} - \log_5 25$
13. $\log_3 \sqrt[3]{27\sqrt[4]{81}}$
14. $\log (0.1)^4 - \log \sqrt[3]{0.001}$

15. $\dfrac{\log 4 + \log 3}{1 + \log 0.4 + \frac{1}{2}\log 9}$

16. $\log_{36} 6 - \log_6 36 - \log_6 \frac{1}{6}$

17. $\log_2 \frac{1}{25} \log_3 \frac{1}{8} \cdot \log_5 \frac{1}{9}$

18. $\log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8$

19. $\log_4 8 - \log_{\frac{1}{9}} 3 - \log_{\sqrt{2}} 4$

20. $\left(\log_2 3 + \log_2 \sqrt{3}\right) \log_{\sqrt{3}} 2$

21. $\dfrac{\log_5 \sqrt{2} \cdot \log_7 9}{\log_7 \sqrt[3]{4} \cdot \log_5 \frac{1}{3}}$

22. $\frac{1}{2}\log \frac{81}{17} + 2\log \frac{5}{3} - \log \frac{17}{4} + \frac{3}{2}\log 17$

23. Given that $\log_2 3 = a$ and $\log_2 5 = b$. Express a and b in terms of $\log_4 15$.

24. Given that $\log_3 5 = m$ and $\log_5 6 = n$. Express m and n in terms of $\log_2 554$.

25. Given that $\log_2 3 = a$ and $\log_3 7 = b$. Express a and b in terms of $\log_4 214$.

26. Given that $\log_3 6 = x$. Epress x in terms of $\log_9 12$.

27. Given that $\log_3 y - \log_9 \sqrt[3]{x} = 1 + \log_2 7x$. Exress x in terms of y.

28. Given that $\log_2 5(2x - 1) = \log_5(x - 3) + \log_2 55$, prove that $5x^2 - 32x + 46 = 0$.

29. If $a > 0$, $b > 0$, and $a^2 + b^2 = 7ab$, prove that $2\log_3(a + b) = 2 + \log_3 a + \log_3 b$.

30. If $x > 0$, $y > 0$, and $x^2 + 9y^2 = 10xy$, prove that $2\log(x + 3y) - 4\log 2 = \log x + \log y$.

23.4 Exponential Equations

All the equations with terms that contain the variable in the exponent are called exponential equations. For example, $9^x = 3^{x-1}$, $3^x = 5$, and $2^{x-1} + 2^x - 2 = 0$ are all exponential equations.

Practice 6

Solve the following exponential equations:

1. $3^{2x} = -\frac{1}{9}$

2. $2^{x^2+4x} = \frac{1}{8}$

3. $6^x = 5^{x-1}$

4. $4^{x-1} + 2^{x-1} = 20$

Exercise 23.4

Solve the following exponential equations:

1. $8^{x-3} = \frac{1}{256}$

2. $3^{2x+1} = 243$

3. $10^{x^2-4} = 1$

4. $3^{x^2+3} = 27^{x+7}$

5. $4^{x^2} = 2^{5x+7}$

6. $5^{2x^2+x} = 25^{1+x-2x^2}$

7. $\left(\frac{9}{16}\right)^x = \left(\frac{4}{3}\right)^{x-6}$

8. $5^{2^x+1} = 5^{4^x+1}$

9. $2^{2x+3} \cdot 4^{x+6} = (8^x)^x$

10. $\frac{5^{x^2}}{5} = 7^{(x+1)(x-1)}$

11. $3^{x+1} = 4^{x-1}$

12. $7^{5-3x} = 5^{x+2}$

13. $13^{2x+5} = 14^{x+7}$

14. $2^{x^2-1} = 3^{x+1}$

15. $\left(\frac{1}{3}\right)^x - \left(\frac{1}{3}\right)^{-x} = \frac{80}{9}$

16. $3^{x+1} + 9^x - 18 = 0$

17. $25^x - 23 \cdot 5^x - 50 = 0$

18. $3^{x-1} + 3^{3-x} - 10 = 0$

19. $3^{2x} - 3^{x+1} + 2 = 0$

20. $2^{x+2} + 3\left(2^{1-x}\right) - 14 = 0$

21. $2^{2x-1} - 3 \cdot 2^{x-1} + 1 = 0$

22. $3^x - 5^{x+2} = 3^{x+1} - 5^{x+3}$

23.5 Logarithmic Equations

All the equations with logarithmic terms which contains variable in the base or in the argument are called logarithmic equations. For example, $\log(x-1) = 3$, $\log_x 9 = 2$, and $2\log_3 x + \log_9 x = 1$ are all logarithmic equations. The results acquired when solving logarithmic equations need to be checked.

Practice 7

Solve the following logarithmic equations:

1. $\log_3 x = 5$

2. $\log_5(x-2) = 0$

3. $\log(x^2 + 2x - 3) - \log(x+3) = 0$

4. $\log_3(3x+1) + 1 = \log_3(2x-1) + \log_3 5$

5. $\log_x 3 + \log_x 81 = 5$

6. $3\log_2^2 x + 5\log_2 x - 2 = 0$

7. $\log_2 x - \log_x 8 = 2$

8. $x^{\log x} = 100x$

23.5.1 Exercise 23.5

Solve the following logarithmic equations:

1. $\log_{\sqrt{3}} x = -2$

2. $\log_2 x^4 = 4$

3. $\log \frac{x-2}{x+2} = \log \frac{1}{x-1}$

4. $2\log x + \log 7 = \log 14$
5. $\log x + \log(x-3) = 1$
6. $\log(x+6) - \log(x-3) = 1$
7. $\log_6 x + \log_6(x^2-7) = 1$
8. $\log_{1,2}(15x^2 - 2x - 12) = 0$
9. $\log_8(x^2 - 3x - 2) = \frac{1}{3}$
10. $\log_2(x^2 - x - 2) - \log_2(x+1) = 0$
11. $\log_3(2x-3) + \log_3(3x+2) = \log_3(2x-1)$
12. $\frac{1}{2}(\log x - \log 5) = \log 2 - \frac{1}{2}\log(9-x)$
13. $\log(x+6) - \frac{1}{2}\log(2x-3) = 2 - \log 25$
14. $\log_2 x = \log_8 x + 1$
15. $3^{\log x} = 2^{\log 3}$
16. $4x^{\log_2 x} = x^3$
17. $2\left(\log_3 x\right)^2 + \log_3 x - 1 = 0$
18. $\log_4^2 x - 5\log_4 x + 6 = 0$
19. $6\log^2 x + \log x^3 - 3 = 0$
20. $(\log x)^2 = 2\log x$
21. $\log_x 25 - \log_{25} x = 0$
22. $2\log_4 x - 3\log_x 4 + 5 = 0$
23. $2\log_x 10 - \log x + 1 = 0$
24. $\log_5\left[\log_2\left(\log_x 5\right)\right] = 0$

23.6 Compound Interest and Annuity

Simple interest and compound interest are two different methods of calculating interest. Simple interest is calculated on the principal amount of a loan only. Compound interest is calculated on the principal amount and also on the accumulated interest of previous periods, and can thus be regarded as interest on interest.

For example, a fund amounted to RMp is deposited into a bank account with a yearly interest rate of $r\%$.

$$\text{Principal amount} = \text{RM}p$$

When $t = 1$,

$$\begin{aligned}\text{Interest earned} &= p \times r\% = \frac{pr}{100}\\ \text{Accumulated amount} &= p + \frac{pr}{100} = p\left(1 + \frac{r}{100}\right)\end{aligned}$$

When $t = 2$,

$$\begin{aligned}
\text{Interest earned} &= \left(p + \frac{pr}{100}\right) \times r\% = \frac{pr}{100}\left(1 + \frac{r}{100}\right) \\
\text{Accumulated amount} &= p\left(1 + \frac{r}{100}\right) + \frac{pr}{100}\left(1 + \frac{r}{100}\right) \\
&= p\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right) \\
&= p\left(1 + \frac{r}{100}\right)^2
\end{aligned}$$

When $t = 3$,

$$\begin{aligned}
\text{Interest earned} &= \left(p\left(1 + \frac{r}{100}\right)^2\right) \times r\% = \frac{pr}{100}\left(1 + \frac{r}{100}\right)^2 \\
\text{Accumulated amount} &= p\left(1 + \frac{r}{100}\right)^2 + \frac{pr}{100}\left(1 + \frac{r}{100}\right)^2 \\
&= p\left(1 + \frac{r}{100}\right)^2\left(1 + \frac{r}{100}\right) \\
&= p\left(1 + \frac{r}{100}\right)^3
\end{aligned}$$

In general, the accumulated amount after t years is given by

$$\boxed{A = p\left(1 + \frac{r}{100}\right)^t}$$

where p is called the *present value* of A.

If the interest is compounded m times per year, then the accumulated amount is given by

$$\boxed{A = p\left(1 + \frac{r}{100m}\right)^{mt}}$$

Annuity and Present Value of Annuity

An annuity is a series of equal payments made at equal intervals of time according to some kind of contract, standing order or the amount received. For example, all sorts of insurance premiums, house rent, car loan, etc. are annuities. In this book, we will only consider annuities with equal payments made or received at equal intervals of time.

Note that the annuity is not limited to once a year.

We can compare which payment plan is better by comparing the present values of the annuities. From the formula $A = p(1 + r\%)^t$, we can know that the present value $p = \dfrac{A}{(1 + r\%)^t}$. If the yearly interest rate is $r\%$, the annuity is RMA, the payment is made once per year, then the present value of the amount paid after a year is $A(1 + r\%)^{-1}$, the present value of the amount paid after two years is $A(1 + r\%)^{-2}$, and so on. The present value of the amount paid after n years is $A(1 + r\%)^{-n}$. Hence, the sum of the present values of the amount paid after n years is

$$
\begin{aligned}
&\frac{A}{1 + r\%} + \frac{A}{(1 + r\%)^2} + \cdots + \frac{A}{(1 + r\%)^n} \\
&= A\left[\frac{1}{1 + r\%} + \frac{1}{(1 + r\%)^2} + \cdots + \frac{1}{(1 + r\%)^n}\right] \\
&= A\left[\frac{1 - \dfrac{1}{(1 + r\%)^n}}{1 - \dfrac{1}{1 + r\%}}\right] \\
&= \frac{A}{r\%}\left(1 - \frac{1}{(1 + r\%)^n}\right)
\end{aligned}
$$

Annuity that is paid indefinitely is called *perpetuity*, $n \to \infty$, $\dfrac{1}{(1 + r\%)^n} \to 0$. From that, we can know that the present value of perpetuity is $\dfrac{A}{r\%}$.

Practice 8

1. Given that the principal amount is RM75,000, interest rate is 4.5%. Using composite interest method, find the accumulated amount after 10 years.

2. A person has deposited RM40,000 into a bank account. The bank pays 8% interest per annum compounded half yearly. Using the compound interest method, find the amount in the account after 3 years.

3. Given that the interest rate is 6%, the interest is compounded half yearly. Using the compound interest method, the accumulated amount after 5 years is RM4031.75, find the principal amount.

4. Given that the interest rate is 4%, the annuity is RM3,500, the payment is made once per year. The payment has since been made for 15 years continuously. Find the present value. Hence, find the present value of the perpetuity.

Exercise 23.6

1. Given that the principal amount is RM90,000, the interest rate is 5%. Compounding the interest once per year, find the accumulated amount after 10 years.

2. A person has deposited a fund into a bank account. The bank pays 8% interest per annum compounded yearly. The amount in the account after 3 years has increased by RM779.14. Find the amount of the fund deposited.

3. RM80,000 was deposited into a financial institution. The interest rate is 8% per annum compounded once per three months. Find the amount in the account after 5 years.

4. Prove that the accumulated amount after being compounded with an interest of 5 for 15 years will exceed twice the principal amount.

5. Given that the principal amount is RM15,000, the interest rate is 6% being compouned once per year. How long does it take for the accumulated amount to be more than RM30,000?

6. Given that the principal amount is RM120,000, the interest rate is 5.5% being compounded half yearly. How long does it take for the accumulated amount to be more than RM200,000?

7. A person deposited RM2,500 into his bank account at the beginning of the yaar, the interest rate is 4.5% compounded once per year. Find the amount in the account after 15 years.

8. If the present value is RM15,443.46, the interest rate is 5%, find the annuity if the payment is made for 10 years.

9. Given that the annuity is RM5,000, the interest rate is 5%, the payment is made once per year for 25 years. Find the present value. Hence, find the present value of the perpetuity.

10. Given that the annuity is RM2,500, the interest rate is 4.5%, the payment is made once per year. How many years does it take for the present value to exceed RM30,000?

11. If a bank has introduced an annuity scheme, the investors can receive RM1,000 per year for life after paying RM20,000. If the annuity plan is considered approximately to be a perpetuity, find the interest rate.

Chapter 24

Limits

24.1 Concept of Limits

24.2 Limits of Functions

24.3 Arithmetic Properties of Limits of Functions

Chapter 25

Differentiation

25.1 Gradient of Tangent Line on a Curve

25.2 Gradient of Tangent Line and Derivative

25.3 Law of Differentiation

25.4 Chain Rule - Differentiation of Composite Functions

25.5 Higher Order Derivatives

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25.7 Two Basic Limits

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25.10 Derivatives of Logarithmic Functions