Exercise 11i

Find the following indefinite integrals:

1. $\displaystyle\int x\cos x\,dx$

   Sol.
   Let $u = x$, $du = dx$
   Let $dv = \cos x\,dx$, $v = \sin x$.

   $$\begin{aligned}\int x\cos x\,dx &= x\sin x - \int \sin x\,dx\\ &= x\sin x + \cos x + C \qquad \square\end{aligned}$$

2. $\displaystyle\int (x+1)\sin x\,dx$

   Sol.
   Let $u = x+1$, $du = dx$
   Let $dv = \sin x\,dx$, $v = -\cos x$.

   $$\begin{aligned}\int (x+1)\sin x\,dx &= -(x+1)\cos x - \int -\cos x\,dx\\ &= -(x+1)\cos x + \sin x + C \qquad \square\end{aligned}$$

3. $\displaystyle\int x\sin(x+1)\,dx$

   Sol.
   Let $u = x$, $du = dx$
   Let $dv = \sin(x+1)\,dx$, $v = -\cos(x+1)$.

   $$\begin{aligned}\int x\sin(x+1)\,dx &= -x\cos(x+1) - \int -\cos(x+1)\,dx\\ &= -x\cos(x+1) + \sin(x+1) + C \qquad \square\end{aligned}$$

4. $\displaystyle\int x\cos 3x\,dx$

   Sol.
   Let $u = x$, $du = dx$
   Let $dv = \cos 3x\,dx$, $v = \dfrac{1}{3}\sin 3x$.

   $$\begin{aligned}\int x\cos 3x\,dx &= \frac{1}{3}x\sin 3x - \int \frac{1}{3}\sin 3x\,dx\\ &= \frac{1}{3}x\sin 3x - \frac{1}{9}\cos 3x + C \qquad \square\end{aligned}$$

5. $\displaystyle\int x\ln x\,dx$

   Sol.
   Let $u = \ln x$, $du = \dfrac{1}{x}\,dx$
   Let $dv = x\,dx$, $v = \dfrac{1}{2}x^2$.

   $$\begin{aligned}\int x\ln x\,dx &= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2\cdot\frac{1}{x}\,dx\\ &= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C \qquad \square\end{aligned}$$

6. $\displaystyle\int xe^{-x}\,dx$

Sol.

Let $u = x$, $du = dx$

Let $dv = e^{-x}\,dx$, $v = -e^{-x}$.

$$\begin{aligned}\int xe^{-x}\,dx &= -xe^{-x} - \int -e^{-x}\,dx \\ &= -xe^{-x} + e^{-x} + C \qquad \square\end{aligned}$$

7. $\displaystyle\int x\sin x\cos x\,dx$

Sol.

Let $u = x$, $du = dx$

$$\begin{aligned}\text{Let } dv = \sin x\cos x\,dx,\ v &= \int \sin x\cos x\,dx \\ &= \frac{1}{2}\int \sin 2x\,dx \\ &= -\frac{1}{4}\cos 2x\end{aligned}$$

$$\begin{aligned}\int x\sin x\cos x\,dx &= -\frac{1}{4}x\cos 2x + \frac{1}{4}\int \cos 2x\,dx \\ &= -\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C \qquad \square\end{aligned}$$

8. $\displaystyle\int \frac{\ln x}{\sqrt{x}}\,dx$

Sol.

Let $u = \ln x$, $du = \dfrac{1}{x}\,dx$

Let $dv = \dfrac{1}{\sqrt{x}}\,dx$, $v = 2\sqrt{x}$.

$$\begin{aligned}\int \frac{\ln x}{\sqrt{x}}\,dx &= 2\sqrt{x}\ln x - \int 2\sqrt{x}\cdot\frac{1}{x}\,dx \\ &= 2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}}\,dx \\ &= 2\sqrt{x}\ln x - 4\sqrt{x} + C \qquad \square\end{aligned}$$

9. $\displaystyle\int \ln(1+x^2)\,dx$

Sol.

Let $u = \ln(1+x^2)$, $du = \dfrac{2x}{1+x^2}\,dx$

Let $dv = dx$, $v = x$.

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx$$

Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$

$$\begin{aligned}\int \frac{2x^2}{1+x^2}\,dx &= \int \frac{2\tan^2\theta}{1+\tan^2\theta}\cdot\sec^2\theta\,d\theta \\ &= \int \frac{2\tan^2\theta}{\sec^2\theta}\cdot\sec^2\theta\,d\theta\end{aligned}$$

$$
\begin{aligned}
&= 2\int \tan^2\theta\, d\theta \\
&= 2\int (\sec^2\theta - 1)\, d\theta \\
&= 2\tan\theta - 2\theta + C \\
&= 2x - 2\arctan x + C
\end{aligned}
$$

$$\therefore \int \ln(1+x^2)\, dx = x\ln(1+x^2) - 2x + 2\arctan x + C \qquad \square$$

10. $\displaystyle\int x^2 \tan^{-1} x\, dx$

Sol.

Let $u = \tan^{-1} x,\ du = \dfrac{1}{1+x^2}\, dx$

Let $dv = x^2\, dx,\ v = \dfrac{1}{3}x^3.$

$$
\begin{aligned}
\int x^2 \tan^{-1} x\, dx &= \frac{1}{3}x^3 \tan^{-1} x - \int \frac{1}{3}x^3 \cdot \frac{1}{1+x^2}\, dx \\
&= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\int \frac{x^3}{1+x^2}\, dx
\end{aligned}
$$

Let $x = \tan\theta,\ dx = \sec^2\theta\, d\theta$

$$
\begin{aligned}
\int \frac{x^3}{1+x^2}\, dx &= \int \frac{\tan^3\theta}{1+\tan^2\theta} \cdot \sec^2\theta\, d\theta \\
&= \int \frac{\tan^3\theta}{\sec^2\theta} \cdot \sec^2\theta\, d\theta \\
&= \int \tan^3\theta\, d\theta \\
&= \int (\sec^2\theta - 1)\tan\theta\, d\theta \\
&= \int \sec^2\theta \tan\theta\, d\theta - \int \tan\theta\, d\theta \qquad (\text{Let } u = \tan\theta,\ du = \sec^2\theta\, d\theta) \\
&= \frac{1}{2}\tan^2\theta - \ln|\sec\theta| + C \\
&= \frac{1}{2}x^2 - \ln\left|\sqrt{1+x^2}\right| + C
\end{aligned}
$$

$$
\begin{aligned}
\therefore \int x^2 \tan^{-1} x\, dx &= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{3}\ln\left|\sqrt{1+x^2}\right| + C \\
&= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6}\ln\left|1+x^2\right| + C \qquad \square
\end{aligned}
$$

11. $\displaystyle\int \cos^{-1} x\, dx$

Sol.

Let $u = \cos^{-1} x,\ du = -\dfrac{1}{\sqrt{1-x^2}}\, dx$

Let $dv = dx,\ v = x.$

$$
\begin{aligned}
\int \cos^{-1} x\, dx &= x\cos^{-1} x - \int x \cdot \left(-\frac{1}{\sqrt{1-x^2}}\right)\, dx \\
&= x\cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}}\, dx
\end{aligned}
$$

Let $x = \sin\theta$, $dx = \cos\theta\, d\theta$

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}}\, dx &= \int \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\, d\theta \\ &= \int \frac{\sin\theta}{\sqrt{\cos^2\theta}} \cdot \cos\theta\, d\theta \\ &= \int \sin\theta\, d\theta \\ &= -\cos\theta + C \\ &= -\sqrt{1-x^2} + C\end{aligned}$$

$$\therefore \int \cos^{-1} x\, dx = x\cos^{-1} x - \sqrt{1-x^2} + C \qquad \square$$

12. $\int \ln x^2\, dx$

Sol.

Let $u = \ln x^2$, $du = \frac{2}{x}\, dx$

Let $dv = dx$, $v = x$.

$$\begin{aligned}\int \ln x^2\, dx &= x\ln x^2 - \int x \cdot \frac{2}{x}\, dx \\ &= x\ln x^2 - 2\int dx \\ &= x\ln x^2 - 2x + C\end{aligned}$$

13. $\int x a^{2x}\, dx \quad (a > 0)$

Sol.

Let $u = x$, $du = dx$

Let $dv = a^{2x}\, dx$, $v = \frac{a^{2x}}{2\ln a}$

$$\begin{aligned}\int x a^{2x}\, dx &= \frac{x a^{2x}}{2\ln a} - \int \frac{a^{2x}}{2\ln a}\, dx \\ &= \frac{x a^{2x}}{2\ln a} - \frac{1}{2\ln a}\int a^{2x}\, dx \\ &= \frac{x a^{2x}}{2\ln a} - \frac{a^{2x}}{4\ln^2 a} + C \\ &= \frac{a^{2x}}{2\ln a}\left(x - \frac{1}{2\ln a}\right) + C \qquad \square\end{aligned}$$

14. $\int x\tan^2 x\, dx$

Sol.

Let $u = x$, $du = dx$

$$\begin{aligned}\text{Let } dv = \tan^2 x dx,\ v &= \int \tan^2 x\, dx \\ &= \int (\sec^2 x - 1)\, dx \\ &= \tan x - x\end{aligned}$$

$$\begin{aligned}\int x\tan^2 x\,dx &= x(\tan x - x) - \int (\tan x - x)\,dx \\ &= x\tan x - x^2 + \ln|\cos x| + \frac{1}{2}x^2 + C \\ &= x\tan x + \ln|\cos x| - \frac{1}{2}x^2 + C \qquad \square\end{aligned}$$

15. $\int e^{\sqrt{x}}\,dx \quad (\text{Let } t = \sqrt{x})$

Sol.

Let $t = \sqrt{x}$, $dt = \dfrac{1}{2\sqrt{x}}\,dx$

$$\begin{aligned}\int e^{\sqrt{x}}\,dx &= \int e^t \cdot 2t\,dt \\ &= 2\int te^t\,dt\end{aligned}$$

Let $u = t$, $du = dt$

Let $dv = e^t\,dt$, $v = e^t$

$$\begin{aligned}\int e^{\sqrt{x}}\,dx &= 2\left(te^t - \int e^t\,dt\right) \\ &= 2\left(te^t - e^t\right) \\ &= 2e^t(t-1) \\ &= 2e^{\sqrt{x}}(\sqrt{x}-1) + C \qquad \square\end{aligned}$$