# Revision Exercise 5

1. Conic curve equations are given as follows. Identify the type of curve and find their eccentricity $e$:

   (a) $2x^2 - y^2 + 6x + 2y + 3 = 0$

   **Sol.**

$$
\begin{aligned}
2x^2 + 6x - y^2 + 2y &= -3 \\
2(x^2 + 3x) - (y^2 - 2y) &= -3 \\
2\left(x^2 + 3x + \frac{9}{4}\right) - (y^2 - 2y + 1) &= -3 + 2\left(\frac{9}{4}\right) - 1 \\
2\left(x + \frac{3}{2}\right)^2 - (y-1)^2 &= \frac{1}{2} \\
4\left(x + \frac{3}{2}\right)^2 - 2(y-1)^2 &= 1
\end{aligned}
$$

   The equation is of the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

   Therefore, the curve is a **hyperbola** with $a^2 = \frac{1}{4}$ and $b^2 = \frac{1}{2}$.
   The eccentricity of the hyperbola is given by

$$
\begin{aligned}
e &= \frac{1}{\frac{1}{2}}\sqrt{\frac{1}{4} + \frac{1}{2}} \\
&= 2\sqrt{\frac{3}{4}} \\
&= \sqrt{3} \qquad \square
\end{aligned}
$$

   (b) $y^2 - 4y - 4x + 16 = 0$

   **Sol.**

$$
\begin{aligned}
y^2 - 4y - 4x + 16 &= 0 \\
y^2 - 4y + 16 &= 4x \\
(y-2)^2 - 4 + 16 &= 4x \\
(y-2)^2 &= 4x - 12 \\
(y-2)^2 &= 4(x-3)
\end{aligned}
$$

   The equation is of the form

$$(y-k)^2 = 4a(x-h)$$

   Therefore, the curve is a **parabola**.
   The eccentricity of the parabola is 1. $\square$

(c) $x^2 + 4y^2 + 8x - 16y - 17 = 0$

**Sol.**

$$\begin{aligned} x^2 + 8x + 4y^2 - 16y - 17 &= 0 \\ (x^2 + 8x + 16) - 16 + 4(y^2 - 4y + 4) - 16 - 17 &= 0 \\ (x+4)^2 + 4(y-2)^2 &= 49 \\ \frac{(x+4)^2}{49} + \frac{4(y-2)^2}{49} &= 1 \end{aligned}$$

The equation is of the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Therefore, the curve is an **ellipse** with $a^2 = 49$ and $b^2 = \frac{49}{4}$.
The eccentricity of the ellipse is given by

$$\begin{aligned} e &= \frac{1}{7}\sqrt{49 - \frac{49}{4}} \\ &= \frac{1}{7}\sqrt{\frac{147}{4}} \\ &= \frac{1}{7} \cdot \frac{\sqrt{147}}{2} \\ &= \frac{\sqrt{3}}{2} \qquad \square \end{aligned}$$

(d) $x^2 - y^2 - 4x + 2y - 1 = 0$

**Sol.**

$$\begin{aligned} x^2 - 4x - y^2 + 2y - 1 &= 0 \\ x^2 - 4x - y^2 + 2y &= 1 \\ (x^2 - 4x + 4) - 4 - (y^2 - 2y + 1) - 1 &= 1 \\ (x-2)^2 - (y-1)^2 &= 6 \end{aligned}$$

The equation is of the form

$$(x-h)^2 - (y-k)^2 = a^2$$

Therefore, the curve is a **rectangular hyperbola** with $a^2 = 6$.
The eccentricity of any rectangular hyperbola $\sqrt{2}$. $\square$

2. Discuss the conic curve represented by $kx^2 + 2y^2 - 8x = 0$ for the three cases of $k > 0$, $k = 0$, and $k < 0$.

**Sol.**

When $k > 0$,

$$\begin{aligned}
kx^2 + 2y^2 - 8x &= 0 \\
kx^2 - 8x + 2y^2 &= 0 \\
k\left(x^2 - \frac{8}{k}x\right) + 2y^2 &= 0 \\
k\left(x^2 - 2 \cdot \frac{4}{k}x + \left(\frac{4}{k}\right)^2 - \left(\frac{4}{k}\right)^2\right) + 2y^2 &= 0 \\
k\left(x^2 - 2 \cdot \frac{4}{k}x + \left(\frac{4}{k}\right)^2\right) - \frac{16}{k} + 2y^2 &= 0 \\
k\left(x - \frac{4}{k}\right)^2 + 2y^2 &= \frac{16}{k} \\
\frac{k\left(x - \frac{4}{k}\right)^2}{\frac{16}{k}} + \frac{2y^2}{\frac{16}{k}} &= 1 \\
\frac{k^2\left(x - \frac{4}{k}\right)^2}{16} + \frac{ky^2}{8} &= 1
\end{aligned}$$

The equation is of the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Therefore, the curve is an **ellipse**. $\square$

When $k = 0$,

$$\begin{aligned}
0x^2 + 2y^2 - 8x &= 0 \\
2y^2 - 8x &= 0 \\
2y^2 &= 8x \\
y^2 &= 4x
\end{aligned}$$

The equation is of the form

$$y^2 = 4ax$$

Therefore, the curve is a **parabola**. $\square$

When $k < 0$,

$$\begin{aligned}
-kx^2 + 2y^2 - 8x &= 0 \\
\frac{k^2\left(x + \frac{4}{k}\right)^2}{16} - \frac{ky^2}{8} &= 1
\end{aligned}$$

The equation is of the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Therefore, the curve is a **hyperbola**. $\square$

3. Find the equations of curves that satisfy the following conditions:

(a) An ellipse with center at $O'(-2, 1)$, semi-major axis length 10, foci on lines parallel to the $x$-axis, and the distance between the two foci is 12.

**Sol.**

Move the center of the ellipse to the origin by translating the axes such that

$$x' = x + 2 \qquad \text{and} \qquad y' = y - 1$$

Since the foci are on lines parallel to the $x$-axis, the equation of the ellipse is

$$\begin{aligned} \frac{(x')^2}{a^2} + \frac{(y')^2}{b^2} &= 1 \\ a &= 10 \\ 2ae &= 12 \\ ae &= 6 \\ b^2 &= 100 - 36 \\ &= 64 \end{aligned}$$

Therefore, the equation of the ellipse is

$$\begin{aligned} \frac{(x')^2}{100} + \frac{(y')^2}{64} &= 1 \\ \frac{(x+2)^2}{100} + \frac{(y-1)^2}{64} &= 1 \qquad \square \end{aligned}$$

(b) A hyperbola with imaginary axis length of 8, vertices at $\mathrm{A}(2, 1)$ and $\mathrm{A}'(2, -5)$.

**Sol.**

The center of the hyperbola is at the midpoint of the vertices, which is at $\left(2, \dfrac{1 + (-5)}{2}\right) = (2, -2)$.

Move the center of the hyperbola to the origin by translating the axes such that

$$x' = x - 2 \qquad \text{and} \qquad y' = y + 2$$

The vertices of the hyperbola are now at $(0, 3)$ and $(0, -3)$.

Since the vertices are on the $y$-axis, the equation of the hyperbola is of the form

$$\frac{(y')^2}{a^2} - \frac{(x')^2}{b^2} = 1$$

The imaginary axis length is $2b = 8$, so $b = 4$.

The vertices are at $(0, \pm 3)$, so $a = 3$. Therefore, the equation of the hyperbola is

$$\frac{(y+2)^2}{9} - \frac{(x-2)^2}{16} = 1 \qquad \square$$

(c) Foci at F$(3, -3)$, directrix equation is y = 1 for a parabola.

**Sol.**

Let any point on the parabola be P$(x, y)$.

$$
\begin{aligned}
(x-3)^2 + (y+3)^2 &= \left(\frac{y-1}{1}\right)^2 \\
(x-3)^2 + (y+3)^2 &= (y-1)^2 \\
x^2 - 6x + 9 + y^2 + 6y + 9 &= y^2 - 2y + 1 \\
x^2 - 6x + 17 + 8y &= 0 \\
(x-3)^2 + 8 + 8y &= 0 \\
(x-3)^2 &= -8y - 8 \\
(x-3)^2 &= -8(y+1) \qquad \square
\end{aligned}
$$

4. Lazy to do. =)

5. A straight line passing through the focus of the parabola $y^2 = 4ax$ intersects the parabola at two points with vertical coordinates $y_1$ and $y_2$. Prove that $y_1 \cdot y_2 = -4a^2$.

**Proof.**

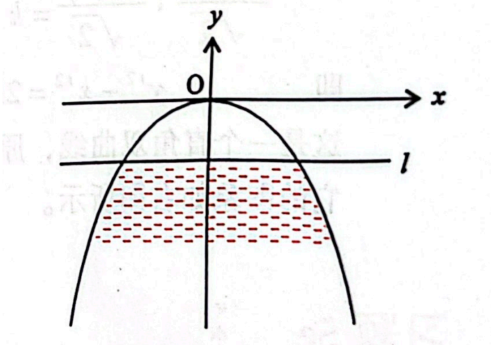
The focus of the parabola is at $(a, 0)$.

Let the two points of intersection be A$(x_1, y_1)$ and B$(x_2, y_2)$.

Substituting $(x_1, y_1)$ and $(x_2, y_2)$ into the equation of the parabola, we get

$$
\begin{aligned}
y_1^2 &= 4ax_1 \\
x_1 &= \frac{y_1^2}{4a} \\
y_2^2 &= 4ax_2 \\
x_2 &= \frac{y_2^2}{4a} \\
\frac{0 - y_1}{a - x_1} &= \frac{y_2 - y_1}{x_2 - x_1} \\
\frac{-y_1}{a - \frac{y_1^2}{4a}} &= \frac{y_2 - y_1}{\frac{y_2^2}{4a} - \frac{y_1^2}{4a}} \\
\frac{-y_1}{\frac{4a^2 - y_1^2}{4a}} &= \frac{y_2 - y_1}{\frac{y_2^2 - y_1^2}{4a}} \\
\frac{-y_1}{4a^2 - y_1^2} &= \frac{y_2 - y_1}{y_2^2 - y_1^2} \\
\frac{-y_1}{4a^2 - y_1^2} &= \frac{y_2 - y_1}{(y_2 - y_1)(y_2 + y_1)} \\
\frac{-y_1}{4a^2 - y_1^2} &= \frac{1}{y_2 + y_1} \\
-y_1(y_2 + y_1) &= 4a^2 - y_1^2 \\
-y_1y_2 - y_1^2 &= 4a^2 - y_1^2 \\
-y_1y_2 &= 4a^2 \\
y_1y_2 &= -4a^2 \qquad \square
\end{aligned}
$$

6. In the figure, there is a parabolic arch bridge. When the water surface is at $l$, the height of the arch from the water surface is 2 m, and the width of the water surface is 4 m. After the water surface descends by 1 m, what is the new width of the water surface?




**Sol.**

Let the equation of the parabola be of the form

$$x^2 = 4ay$$

When $x = 2$, $y = -2$

$$\begin{aligned} 2^2 &= 4a(-2) \\ 4 &= -8a \\ a &= -\frac{1}{2} \end{aligned}$$

Therefore, the equation of the parabola is

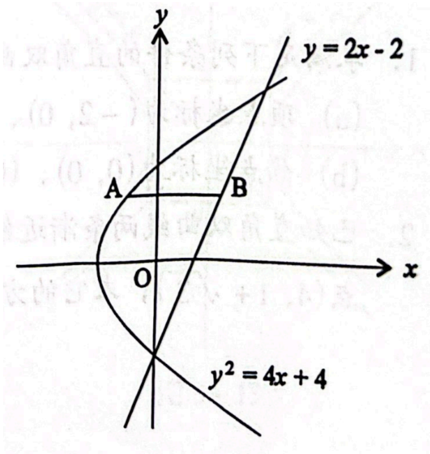
$$x^2 = -2y$$

When $y = -3$,

$$\begin{aligned} x^2 &= -2(-3) \\ x^2 &= 6 \\ x &= \pm\sqrt{6} \end{aligned}$$

Hence, the new width of the water surface is $2\sqrt{6} \approx 4.9$m. $\square$

7. The parabola $y^2 = 4x + 4$ intersects the line $y = 2x - 2$. A chord AB is parallel to the $x$-axis, and the $y$-coordinate of points on the chord is $t$. Find the range of $t$ and the length of the chord AB.




**Sol.**

Substituting $y = 2x - 2$ into the equation of the parabola, we get

$$\begin{aligned} (2x-2)^2 &= 4x+4 \\ 4x^2 - 8x + 4 &= 4x + 4 \\ 4x^2 - 12x &= 0 \\ 4x(x-3) &= 0 \\ x = 0 \quad &\text{or} \quad x = 3 \end{aligned}$$

When $x = 0$, $y = -2$, and when $x = 3$, $y = 4$. Therefore, the range of $t$ is $-2 \le t \le 4$. $\square$

When $y = t$,

$$A\left(\frac{t^2-4}{4}, t\right)$$
$$B\left(\frac{t+2}{2}, t\right)$$

The length of the chord $\overline{AB}$ is

$$\begin{aligned} |\overline{AB}| &= \sqrt{\left(\frac{t+2}{2} - \frac{t^2-4}{4}\right)^2 + (t-t)^2} \\ &= \sqrt{\left(\frac{2t+4-t^2+4}{4}\right)^2} \\ &= \sqrt{\left(\frac{-t^2+2t+8}{4}\right)^2} \\ &= \frac{1}{4}(t+2)(4-t) \qquad \square \end{aligned}$$

8. For a line segment $\overline{AB}$ with length 1, where the endpoints $A$ and $B$ move along the $x$-axis and $y$-axis respectively, let $P$ be a point on $\overline{AB}$ such that $|\overline{PA}| : |\overline{PB}| = m : n$. Find the equation of the locus of point $P$, and state what type of curve it is.

**Sol.**

Since the length of $\overline{AB}$ is 1,

$$x^2 + y^2 = 1$$

Let the coordinates of $A$ and $B$ be $(x, 0)$ and $(0, y)$ respectively, and the coordinates of $P$ be $(s, t)$.

$$\begin{aligned}
(s,t) &= \left(\frac{nx}{m+n}, \frac{my}{m+n}\right) \\
s &= \frac{nx}{m+n} \\
x &= \frac{(m+n)s}{n} \\
t &= \frac{my}{m+n} \\
y &= \frac{(m+n)t}{m}
\end{aligned}$$

Substituting the equations of $x$ and $y$ into the equation of the circle, we get

$$\begin{aligned}
\left(\frac{(m+n)s}{n}\right)^2 + \left(\frac{(m+n)t}{m}\right)^2 &= 1 \\
\frac{(m+n)^2 s^2}{n^2} + \frac{(m+n)^2 t^2}{m^2} &= 1
\end{aligned}$$

Hence, the equation of the locus of point $P$ is

$$\frac{(m+n)^2 x^2}{n^2} + \frac{(m+n)^2 y^2}{m^2} = 1 \qquad \square$$

And the type of curve is an **ellipse**. $\square$

9. Find a point on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ such that the angle formed by the lines connecting it to the two foci of the ellipse is a right angle.

**Sol.**

The foci of the ellipse are at $(\pm 4, 0)$.

Let the point on the ellipse be $\mathrm{P}(x, y)$.

$$\begin{aligned}
\frac{x^2}{25} + \frac{y^2}{9} &= 1 \\
9x^2 + 25y^2 &= 225 \\
25y^2 &= 225 - 9x^2 \\
y^2 &= 9 - \frac{9}{25}x^2
\end{aligned}$$

$$\begin{aligned}
\frac{y-0}{x-4} \cdot \frac{y-0}{x+4} &= -1 \\
\frac{y^2}{x^2 - 16} &= -1 \\
y^2 &= -x^2 + 16 \\
9 - \frac{9}{25}x^2 &= -x^2 + 16 \\
\frac{16}{25}x^2 &= 7 \\
x^2 &= \frac{175}{16} \\
x &= \pm\frac{5\sqrt{7}}{4} \\
y^2 &= 9 - \frac{9}{25} \times \frac{175}{16} \\
y &= \pm\frac{9}{4}
\end{aligned}$$

Therefore, the points on the ellipse are $\left(\frac{5\sqrt{7}}{4}, \frac{9}{4}\right)$, $\left(\frac{5\sqrt{7}}{4}, -\frac{9}{4}\right)$, $\left(-\frac{5\sqrt{7}}{4}, \frac{9}{4}\right)$, and $\left(-\frac{5\sqrt{7}}{4}, -\frac{9}{4}\right)$. $\square$

10. Given that the vertex of a parabola lies at the center of the ellipse $\frac{x^2}{100} + \frac{y^2}{64} = 1$ and one focus lies on the right focus of the ellipse, find the intersection points of the ellipse and the parabola.

**Sol.**

The center of the ellipse is at the origin, and the foci are at $(\pm 6, 0)$.

Let the equation of the parabola be of the form

$$y^2 = 4ax$$

The focus of the parabola is at $(6, 0)$, therefore $a = 6$. The equation of the parabola is

$$y^2 = 24x$$

Substituting the equation of the parabola into the equation of the ellipse, we get

$$\begin{aligned}
\frac{x^2}{100} + \frac{24x}{64} &= 1 \\
\frac{x^2}{100} + \frac{3x}{8} &= 1 \\
8x^2 + 300x - 800 &= 0 \\
2x^2 + 75x - 200 &= 0 \\
(x + 40)(2x - 5) &= 0 \\
x = -40 \text{ (rejected)} \quad \text{or} \quad x &= \frac{5}{2} \\
y^2 &= 24\left(\frac{5}{2}\right) \\
y^2 &= 60 \\
y &= \pm 2\sqrt{15}
\end{aligned}$$

Therefore, the intersection points of the ellipse and the parabola are at $\left(\frac{5}{2}, 2\sqrt{15}\right)$ and $\left(\frac{5}{2}, -2\sqrt{15}\right)$. □

11. Find the equation of the hyperbola with eccentricity $e = \frac{5}{4}$ that shares a common focus with the ellipse $\frac{x^2}{49} + \frac{y^2}{24} = 1$.

**Sol.**

The foci of the ellipse are at $(\pm 5, 0)$.

Since the foci of the hyperbola are at $(\pm ae, 0)$, the equation of the hyperbola is of the form

$$\begin{aligned}
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \\
ae &= 5 \\
a &= \frac{5}{\frac{5}{4}} = 4 \\
b^2 &= a^2e^2 - a^2 = 25 - 16 = 9
\end{aligned}$$

Therefore, the equation of the hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \qquad \square$$

12. The cross-section of an oil tank on a tanker truck is an ellipse with a major axis length of 1.5 m and a minor axis length of 1 m. If the length of the tank is 4 m, find its volume. Given the formula for the area of an ellipse $S = \pi ab$ (where $a$ and $b$ are the lengths of the semi-major and semi-minor axes respectively), and assuming the density of gasoline is $0.7\,\mathrm{g/cm}^3$, find the maximum weight of gasoline the tank can hold in kilograms.

    **Sol.**

    Let the equation of the cross section of the ellipse be

    $$\frac{x^2}{(0.75)^2} + \frac{y^2}{(0.5)^2} = 1$$

    Hence, the volume of the tank is approximately

    $$\begin{aligned} V &= \pi abh \\ &= \pi (0.75)(0.5)\,4 \\ &= 1.5\pi \\ &\approx 4.71\,\mathrm{m}^3 \qquad \square \end{aligned}$$

    The maximum weight of gasoline the tank can hold is approximately

    $$\begin{aligned} W &= \text{density} \times \text{volume} \\ &= \frac{0.7}{1000} \times 4.71 \times 1000000 \\ &= 3297\,\mathrm{kg} \qquad \square \end{aligned}$$

13. Given the equations of two ellipses $\mathcal{C}_1 : x^2 + 9y^2 - 45 = 0$ and $\mathcal{C}_2 : x^2 + 9y^2 - 6x - 27 = 0$:

    (a) Find the coordinates of the centers and foci of these two ellipses.

    **Sol.**

    For $\mathcal{C}_1$:

    $$\begin{aligned} x^2 + 9y^2 - 45 &= 0 \\ x^2 + 9y^2 &= 45 \\ \frac{x^2}{45} + \frac{y^2}{5} &= 1 \end{aligned}$$

    The center of the ellipse is at the origin, and the foci are at $(\pm 2\sqrt{10}, 0)$ $\square$

    For $\mathcal{C}_2$:

    $$\begin{aligned} x^2 + 9y^2 - 6x - 27 &= 0 \\ x^2 - 6x + 9y^2 - 27 &= 0 \\ (x-3)^2 - 9 + 9y^2 - 27 &= 0 \\ (x-3)^2 + 9y^2 &= 36 \\ \frac{(x-3)^2}{36} + \frac{y^2}{4} &= 1 \end{aligned}$$

    The center of the ellipse is at $(3, 0)$, and the foci are at $(3 \pm 4\sqrt{2}, 0)$ $\square$

(b) Find the equation of a circle passing through the intersection points of these two ellipses and tangent to the line $x - 2y + 11 = 0$.

**Sol.**

$$\begin{aligned}
x^2 + 9y^2 - 45 &= 0 \\
9y^2 &= 45 - x^2 \\
y^2 &= 5 - \frac{1}{9}x^2 \\
x^2 + 9y^2 - 6x - 27 &= 0 \\
x^2 + 45 - x^2 - 6x - 27 &= 0 \\
6x &= 18 \\
x &= 3 \\
y^2 &= 5 - \frac{1}{9}(3)^2 \\
y^2 &= 4 \\
y &= \pm 2
\end{aligned}$$

The intersection points are at $(3, 2)$ and $(3, -2)$.
Let the equation of the circle be of the form

$$(x - h)^2 + (y - k)^2 = r^2$$

Substituting the intersection points into the equation of the circle, we get

$$\begin{aligned}
(3 - h)^2 + (2 - k)^2 &= r^2 \\
(3 - h)^2 + (-2 - k)^2 &= r^2
\end{aligned}$$

Since the circle is tangent to the line $x - 2y + 11 = 0$, the distance from the center of the circle to the line is equal to the radius of the circle. Therefore,

$$\begin{aligned}
\frac{|1(h) - 2(k) + 11|}{\sqrt{1^2 + 2^2}} &= r \\
|h - 2k + 11| &= \sqrt{5}r \\
5r^2 &= (h - 2k + 11)^2 \\
r^2 &= \frac{(h - 2k + 11)^2}{5}
\end{aligned}$$

Substituting the value of $r^2$ into the equation of the circle, we get

$$\begin{aligned}
(3 - h)^2 + (2 - k)^2 &= \frac{(h - 2k + 11)^2}{5} \\
(3 - h)^2 + (-2 - k)^2 &= \frac{(h - 2k + 11)^2}{5}
\end{aligned}$$

Solving the two equations, we get $h = -1$, $k = 0$, or $h = 14$, $k = 0$.

Substituting the values of $h$ and $k$ into $r^2 = \dfrac{(h-2k+11)^2}{5}$, we get $r^2 = 20$ or $r^2 = 125$. Hence, the equation of the circle is

$$\begin{aligned}(x+1)^2 + y^2 &= 20\\ x^2 + 2x + 1 + y^2 &= 20\\ x^2 + y^2 + 2x - 19 &= 0 \qquad \square\end{aligned}$$

or

$$\begin{aligned}(x-14)^2 + y^2 &= 125\\ x^2 - 28x + 196 + y^2 &= 125\\ x^2 + y^2 - 28x + 71 &= 0 \qquad \square\end{aligned}$$

14. If the eccentricity of a hyperbola is 2, find the angle between its two asymptotes.

**Sol.**

$$\begin{aligned}e &= \frac{c}{a}\\ \frac{c^2}{a^2} &= 4\\ \frac{a^2+b^2}{a^2} &= 4\\ 1 + \frac{b^2}{a^2} &= 4\\ \frac{b^2}{a^2} &= 3\\ \frac{b}{a} &= \pm\sqrt{3}\\ \tan\theta &= \pm\sqrt{3}\\ \theta &= 60^\circ \quad \text{or} \quad 120^\circ (\text{ rejected})\end{aligned}$$

Hence, the angle between the asymptotes of the hyperbola is $180 - 2 \times 60^\circ = 60^\circ$. $\square$

15. Given the point $P$ on the hyperbola $\frac{x^2}{64} - \frac{y^2}{36} = 1$ such that its distance to the right focus is 8, find its distance to the left directrix.

**Sol.**

The foci of the hyperbola are at $(\pm 10, 0)$.

Let the point $P$ be at $(x, y)$.

$$\begin{aligned}
\frac{x^2}{64} - \frac{y^2}{36} &= 1 \\
\frac{x^2}{64} - 1 &= \frac{y^2}{36} \\
y^2 &= \frac{9}{16}x^2 - 36
\end{aligned}$$

$$\begin{aligned}
\sqrt{(x-10)^2 + y^2} &= 8 \\
(x-10)^2 + y^2 &= 64 \\
x^2 - 20x + 100 + \frac{9}{16}x^2 - 36 &= 64 \\
x^2 - 20x + \frac{9}{16}x^2 &= 0 \\
\frac{25}{16}x^2 - 20x &= 0 \\
25x^2 - 320x &= 0 \\
x^2 - 12.8x &= 0 \\
x(x - 12.8) &= 0 \\
x = 0 \text{ (rejected)} \quad &\text{or} \quad x = 12.8 \\
y &= \frac{6\sqrt{39}}{5}
\end{aligned}$$

The equation left directrix is

$$\begin{aligned}
x + \frac{64}{10} &= 0 \\
x &= -\frac{32}{5}
\end{aligned}$$

The distance from the point $P$ to the left directrix is $12.8 + \frac{32}{5} = \frac{96}{5}$. $\square$

16. A hyperbola has its asymptotes given by $3x \pm 4y = 0$, one of its directrix given by $5y + 3\sqrt{3} = 0$. Find the equation of the hyperbola and its eccentricity.

**Sol.**

Let the equation of the hyperbola be of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Then

$$\begin{aligned} 3x \pm 4y &= 0 \\ 3x &= \pm 4y \\ y &= \pm\frac{3}{4}x \\ \pm\frac{ax}{b} &= \pm\frac{3}{4}x \\ \frac{a}{b} &= \frac{3}{4} \\ b &= \frac{4}{3}a \end{aligned}$$

$$\begin{aligned} 5y + 3\sqrt{3} &= 0 \\ y + \frac{3\sqrt{3}}{5} &= 0 \\ \frac{a^2}{c} &= \frac{3\sqrt{3}}{5} \\ c &= \frac{5a^2}{3\sqrt{3}} \end{aligned}$$

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + \frac{16}{9}a^2 &= \frac{25a^4}{27} \\ \frac{25}{9}a^2 &= \frac{25a^4}{27} \\ 3a^2 &= a^4 \\ a^2 &= 3 \\ b^2 &= \frac{16}{9}a^2 \\ b^2 &= \frac{16}{3} \end{aligned}$$

Hence, the equation of the hyperbola is

$$\frac{y^2}{3} - \frac{x^2}{\frac{16}{3}} = 1 \implies \frac{y^2}{3} - \frac{3x^2}{16} = 1 \qquad \square$$

The eccentricity of the hyperbola is given by

$$e = \frac{1}{\sqrt{3}}\sqrt{3 + \frac{16}{3}} = \frac{1}{\sqrt{3}}\sqrt{\frac{25}{3}} = \frac{5}{3} \qquad \square$$

17. On one branch of the hyperbola $\dfrac{y^2}{12} - \dfrac{x^2}{13} = 1$, there exist three distinct points $A(x_1, y_1)$, $B(\sqrt{26}, 6)$, and $C(x_2, y_2)$ such that the distances from each of these points to the focus $F(0, 5)$ form an arithmetic progression.

(a) Find $y_1 + y_2$.

**Sol.**

The eccentricity of the hyperbola is given by

$$\begin{aligned} e &= \frac{1}{\sqrt{12}}\sqrt{12+13} \\ &= \frac{5}{\sqrt{12}} \end{aligned}$$

The directrix of the hyperbola is given by

$$x = \frac{12}{5}$$

The distance from the points to the directrix is also an arithmetic progression.

$$\begin{aligned} \left(y_1 - \frac{12}{5}\right) + \left(y_2 - \frac{12}{5}\right) &= 2\left(6 - \frac{12}{5}\right) \\ y_1 + y_2 - \frac{24}{5} &= \frac{36}{5} \\ y_1 + y_2 &= 12 \end{aligned}$$

(b) Prove that the perpendicular bisector of segment $AC$ passes through a fixed point. Hence, find the coordinates of the fixed point.

**Sol.**

The midpoint of segment $AC$ is $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right) = \left(\dfrac{x_1+x_2}{2}, 6\right)$.

$$\begin{aligned} \frac{y_1^2}{12} - \frac{x_1^2}{13} &= 1 \\ \frac{y_2^2}{12} - \frac{x_2^2}{13} &= 1 \\ \frac{(y_1^2 - y_2^2)}{12} - \frac{(x_1^2 - x_2^2)}{13} &= 0 \\ \frac{(y_1+y_2)(y_1-y_2)}{12} - \frac{(x_1+x_2)(x_1-x_2)}{13} &= 0 \\ \frac{(y_1+y_2)(y_1-y_2)}{12} &= \frac{(x_1+x_2)(x_1-x_2)}{13} \\ 13(y_1+y_2)(y_1-y_2) &= 12(x_1+x_2)(x_1-x_2) \\ (y_1 - y_2) &= \frac{12(x_1+x_2)(x_1-x_2)}{13(y_1+y_2)} \end{aligned}$$

The gradient of the line $AC$ is given by

$$\begin{aligned} k &= \frac{y_1-y_2}{x_1-x_2} = \frac{12(x_1+x_2)(x_1-x_2)}{13(y_1+y_2)} \times \frac{1}{x_1-x_2} = \frac{12(x_1+x_2)}{13(y_1+y_2)} = \frac{12(x_1+x_2)}{13(12)} \\ &= \frac{x_1+x_2}{13} = \frac{2 \times \dfrac{x_1+x_2}{2}}{13} = \frac{2x_M}{13} \\ \frac{x_M}{k} &= \frac{13}{2} \end{aligned}$$

The equation of the perpendicular bisector of segment $AC$ is given by

$$\begin{aligned} y - y_M &= -\frac{1}{k}(x - x_M) \\ y - 6 &= -\frac{x}{k} + \frac{13}{2} \\ y &= -\frac{x}{k} + \frac{25}{2} \end{aligned}$$

When $x = 0$, $y = \dfrac{25}{2}$.

Hence, the perpendicular bisector of segment $AC$ passes through the point $\left(0, \dfrac{25}{2}\right)$, which is its y-intercept. $\square$

18. Given that the center of a hyperbola is at the origin, its foci lie on the $x$-axis, and a line passing through the right focus of the hyperbola with a slope of $\sqrt{\dfrac{3}{5}}$ intersects the hyperbola at points $P$ and $Q$ such that $OP \perp OQ$ and $|PQ| = 4$. Find the equation of the hyperbola.

**Sol.**

Since the foci lie on the $x$-axis, the equation of the hyperbola is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The right focus is at $(c, 0)$, where $c = \sqrt{a^2 + b^2}$, so the slope of the line passing through the right focus is given by

$$\begin{aligned} y - 0 &= \sqrt{\frac{3}{5}}(x - c) \\ y &= \sqrt{\frac{3}{5}}(x - c) \end{aligned}$$

Substituting the equation of the line into the equation of the hyperbola, we get

$$\begin{aligned} \frac{x^2}{a^2} - \frac{\frac{3}{5}(x-c)^2}{b^2} &= 1 \\ b^2x^2 - \frac{3}{5}a^2(x-c)^2 &= a^2b^2 \\ 5b^2x^2 - 3a^2(x^2 - 2cx + c^2) &= 5a^2b^2 \\ 5b^2x^2 - 3a^2x^2 + 6a^2cx - 3a^2c^2 &= 5a^2b^2 \\ (5b^2 - 3a^2)x^2 + 6a^2cx - (3a^2c^2 + 5a^2b^2) &= 0 \end{aligned}$$

Let the two roots of the equation be $x_1$ and $x_2$.

$$\begin{aligned} x_1 + x_2 &= -\frac{6a^2c}{5b^2 - 3a^2} \\ x_1x_2 &= -\frac{3a^2c^2 + 5a^2b^2}{5b^2 - 3a^2} \end{aligned}$$

Since $P$ and $Q$ are on the hyperbola,

$$P\left(x_1, \sqrt{\frac{3}{5}}(x_1 - c)\right) \quad \text{and} \quad Q\left(x_2, \sqrt{\frac{3}{5}}(x_2 - c)\right)$$

Since $OP \perp OQ$, the product of the slopes of $OP$ and $OQ$ is $-1$.

$$\begin{aligned}
\frac{\sqrt{\frac{3}{5}}(x_1-c)}{x_1}\cdot\frac{\sqrt{\frac{3}{5}}(x_2-c)}{x_2} &= -1\\
\frac{3(x_1-c)(x_2-c)}{5x_1x_2} &= -1\\
\frac{3(x_1x_2-x_1c-x_2c+c^2)}{5x_1x_2} &= -1\\
3x_1x_2-3x_1c-3x_2c+3c^2 &= -5x_1x_2\\
8x_1x_2-3x_1c-3x_2c+3c^2 &= 0\\
3c(x_1+x_2)-8x_1x_2-3c^2 &= 0
\end{aligned}$$

Substituting the values of $x_1+x_2$ and $x_1x_2$ together with $c^2=a^2+b^2$ into the equation, we get

$$\begin{aligned}
3c\left(-\frac{6a^2c}{5b^2-3a^2}\right)-8\left(-\frac{3a^2c^2+5a^2b^2}{5b^2-3a^2}\right)-3(a^2+b^2) &= 0\\
-18a^2c^2+24a^2c^2+40a^2b^2-3(5b^2-3a^2)(a^2+b^2) &= 0\\
-18a^2c^2+24a^2c^2+40a^2b^2-6b^2a^2+9a^4-15b^4 &= 0\\
6a^2c^2+34a^2b^2+9a^4-15b^4 &= 0\\
6a^2(a^2+b^2)+34a^2b^2+9a^4-15b^4 &= 0\\
15a^4+40a^2b^2-15b^4 &= 0\\
3a^4+8a^2b^2-3b^4 &= 0\\
(3a^2-b^2)(a^2+3b^2) &= 0\\
3a^2 = b^2 \quad \text{or} \quad a^2 &= -3b^2 \text{ (rejected)}
\end{aligned}$$

From $|PQ|=4$, we get

$$\begin{aligned}
(x_1-x_2)^2+\left(\sqrt{\frac{3}{5}}(x_1-c)-\sqrt{\frac{3}{5}}(x_2-c)\right)^2 &= 16\\
(x_1-x_2)^2+\frac{3}{5}(x_1-x_2)^2 &= 16\\
\frac{8}{5}(x_1-x_2)^2 &= 16\\
(x_1-x_2)^2 &= 10\\
(x_1+x_2)^2-4x_1x_2 &= 10
\end{aligned}$$

Substituting the values of $x_1+x_2$ and $x_1x_2$ together with $b^2=3a^2$ into the equation, we get

$$\begin{aligned}
\left(-\frac{6a^2c}{5b^2-3a^2}\right)^2-4\left(-\frac{3a^2c^2+5a^2b^2}{5b^2-3a^2}\right) &= 10\\
\left(-\frac{12a^3}{15a^2-3a^2}\right)^2-4\left(-\frac{12a^4+15a^4}{15a^2-3a^2}\right) &= 10\\
10a^2 &= 10\\
a^2 &= 1\\
b^2 &= 3a^2 = 3
\end{aligned}$$

Therefore, the equation of the hyperbola is $x^2-\dfrac{y^2}{3}=1$ $\square$