

# **Solution Book of Mathematic**

*Senior 2 Part I*

MELVIN CHIA

Written on 9 October 2022

# Contents

<b>12 Sequence and Series</b>	<b>3</b>
12.1 Sequence and Series . . . . .	3
12.1.1 Practice 1 . . . . .	3
12.1.2 Practice 2 . . . . .	3
12.1.3 Exercise 12.1 . . . . .	3
12.2 Arithmetic Progression . . . . .	5
12.2.1 Practice 3 . . . . .	5
12.2.2 Practice 4 . . . . .	6
12.2.3 Practice 5 . . . . .	6
12.2.4 Exercise 12.2 . . . . .	7
12.3 Geometric Progression . . . . .	13
12.3.1 Practice 6 . . . . .	13
12.3.2 Practice 7 . . . . .	14
12.3.3 Practice 8 . . . . .	14
12.3.4 Practice 9 . . . . .	15
12.3.5 Exercise 12.3 . . . . .	16
12.4 Simple Summation of Special Series . . . . .	22
12.4.1 Practice 10 . . . . .	23
12.4.2 Exercise 12.4 . . . . .	24
12.5 Revision Exercise 12 . . . . .	26
<b>13 System of Equations</b>	<b>33</b>
13.1 System of Equations with Two Variables . . . . .	33
13.1.1 Practice 1 . . . . .	33
13.1.2 Exercise 13.1 . . . . .	33

13.2	System of Equations with Three Variables . . . . .	37
13.2.1	Practice 2 . . . . .	37
13.2.2	Exercise 13.2 . . . . .	37
13.3	Revision Exercise 13 . . . . .	39
<b>14</b>	<b>Marix and Determinant</b>	<b>43</b>
14.1	Matrix . . . . .	43
14.1.1	Exercise 14.1 . . . . .	44
14.2	Matrix Addition and Substraction . . . . .	44
14.2.1	Practice 1 . . . . .	45
14.2.2	Exercise 14.2 . . . . .	45
14.3	Scalar Product of Matrices . . . . .	46
14.3.1	Practice 2 . . . . .	46
14.3.2	Exercise 14.3 . . . . .	47
14.4	Multiplication of Matrices . . . . .	48
14.4.1	Practice 3 . . . . .	48
14.4.2	Exercise 14.4 . . . . .	48

# Chapter 12

## Sequence and Series

### 12.1 Sequence and Series

#### 12.1.1 Practice 1

1. Find the first 5 terms of the sequence  $a_n = \frac{2^n}{n+1}$ .

**sol.**  $a_1 = \frac{2}{2} = 1, a_2 = \frac{4}{3}, a_3 = \frac{8}{4}, a_4 = \frac{16}{5}, a_5 = \frac{32}{6}$

2. Write the general term of the sequence 1, 8, 27, 64, ...

**sol.**  $a_n = n^3$

#### 12.1.2 Practice 2

1. Express the series  $\sum_{n=1}^{10} n^2 + 1$  in the form of numbers.

**sol.**  $\sum_{n=1}^{10} n^2 + 1$   
 $= (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1)$   
 $+ (5^2 + 1) + (6^2 + 1) + (7^2 + 1)$   
 $+ (8^2 + 1) + (9^2 + 1) + (10^2 + 1)$   
 $= 2 + 5 + 10 + 17 + 26 + 37 + 50 + 65$   
 $+ 82 + 101$

2. Write the first term, last term and the number of terms of the series  $\sum_{n=1}^{10} (3^n - 2^n)$ .

**sol.** First term  $= (3^1 - 2^1) = 1$

Last term  $= (3^{10} - 2^{10}) = 59049$

Number of terms  $= 10$

3. Express the series  $2 \times 5 + 3 \times 7 + 4 \times 9 + \dots + 15 \times 31$  in the form of  $\sum$ .

**sol.**

$$a_1 = 2 \times 5 = 10$$

$$a_2 = 3 \times 7 = 21$$

$$a_3 = 4 \times 9 = 36$$

$$a_4 = 5 \times 11 = 55$$

$\vdots$

$$a_{15} = 15 \times 31 = 465$$

$$\therefore 2 \times 5 + 3 \times 7 + 4 \times 9 + \dots + 15 \times 31$$

$$= \sum_{n=1}^{15} a_n$$

#### 12.1.3 Exercise 12.1

1. Find the general term of the following sequences.

- (a) 5, 8, 11, 14, ...

**sol.**  $a_n = 3n + 2$

- (b) 2, 4, 8, 16, ...

**sol.**  $a_n = 2^n$

- (c)  $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots$

**sol.**  $a_n = \frac{n+1}{n}$

- (d)  $\frac{2}{5}, \frac{4}{7}, \frac{6}{9}, \frac{8}{11}, \dots$

**sol.**  $a_n = \frac{2n}{2n+1}$

2. Find the first 5 terms of the following sequences.

- (a)  $a_n = 2n + 3$

**sol.**  $a_1 = 2 \times 1 + 3 = 5, a_2 = 2 \times 2 + 3 = 7, a_3 = 2 \times 3 + 3 = 9, a_4 = 2 \times 4 + 3 = 11, a_5 = 2 \times 5 + 3 = 13$

- (b)  $a_n = n(n - 2)$

**sol.**  $a_1 = 1 \times (-1) = -1, a_2 = 2 \times 0 = 0, a_3 = 3 \times 1 = 3, a_4 = 4 \times 2 = 8, a_5 = 5 \times 3 = 15$

- (c)  $a_n = \frac{n}{2n+1}$

**sol.**  $a_1 = \frac{1}{2 \times 1 + 1} = \frac{1}{3}, a_2 = \frac{2}{2 \times 2 + 1} = \frac{2}{5}, a_3 = \frac{3}{2 \times 3 + 1} = \frac{3}{7}, a_4 = \frac{4}{2 \times 4 + 1} = \frac{4}{9}, a_5 = \frac{5}{2 \times 5 + 1} = \frac{5}{11}$

- (d)  $a_n = (-3)^n$

**sol.**  $a_1 = (-3)^1 = -3, a_2 = (-3)^2 = 9, a_3 = (-3)^3 = -27, a_4 = (-3)^4 = 81, a_5 = (-3)^5 = -243$

3. Express the following series in the form of numbers.

- (a)  $\sum_{n=1}^5 n(n + 3)$

$$\begin{aligned} \text{sol. } \sum_{n=1}^5 n(n+3) \\ &= (1 \times 4) + (2 \times 5) + (3 \times 6) + (4 \times 7) \\ &\quad + (5 \times 8) \\ &= 4 + 10 + 18 + 28 + 40 \end{aligned}$$

(b)  $\sum_{n=2}^6 \frac{1}{3^n}$

$$\begin{aligned} \text{sol. } \sum_{n=2}^6 \frac{1}{3^n} \\ &= \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6} \\ &= \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} \end{aligned}$$

(c)  $\sum_{n=1}^6 \frac{1}{n(2n+1)}$

$$\begin{aligned} \text{sol. } \sum_{n=1}^6 \frac{1}{n(2n+1)} \\ &= \frac{1}{1(2 \times 1 + 1)} + \frac{1}{2(2 \times 2 + 1)} \\ &\quad + \frac{1}{3(2 \times 3 + 1)} + \frac{1}{4(2 \times 4 + 1)} \\ &\quad + \frac{1}{5(2 \times 5 + 1)} + \frac{1}{6(2 \times 6 + 1)} \\ &= \frac{1}{3} + \frac{1}{10} + \frac{1}{21} + \frac{1}{36} + \frac{1}{55} + \frac{1}{78} \end{aligned}$$

(d)  $\sum_{n=2}^5 \frac{1}{n^2+2}$

$$\begin{aligned} \text{sol. } \sum_{n=2}^5 \frac{1}{n^2+2} \\ &= \frac{1}{4+2} + \frac{1}{9+2} + \frac{1}{16+2} + \frac{1}{25+2} \\ &= \frac{1}{6} + \frac{1}{11} + \frac{1}{18} + \frac{1}{27} \end{aligned}$$

4. Find the first term, last term and the number of terms of the following series.

(a)  $\sum_{n=3}^{10} 2^2$

**sol.**  $a_3 = 2^2 = 4, a_{10} = 2^2 = 4, n = 10 - 3 + 1 = 8$

(b)  $\sum_{n=1}^8 \frac{n+2}{n}$

**sol.**  $a_1 = \frac{1+2}{1} = \frac{3}{1} = 3, a_8 = \frac{8+2}{8} = \frac{10}{8} = \frac{5}{4}, n = 8 - 1 + 1 = 8$

(c)  $\sum_{n=1}^{10} 3n^2 - n$

**sol.**  $a_1 = 3 \times 1^2 - 1 = 2, a_{10} = 3 \times 10^2 - 10 = 290, n = 10 - 1 + 1 = 10$

(d)  $\sum_{n=9}^{14} n^2(n-7)$

**sol.**  $a_9 = 9^2(9-7) = 9^2 \times 2 = 162, a_{14} = 14^2(14-7) = 14^2 \times 7 = 2744, n = 14 - 9 + 1 = 6$

5. Express the following series in the form of  $\sum$ .

(a)  $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{30}$

**Sol.**

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \frac{1}{2} \\ a_3 &= \frac{1}{3} \\ &\vdots \\ a_{30} &= \frac{1}{30} \\ \therefore 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{30} &= \sum_{n=1}^{30} \frac{1}{n} \end{aligned}$$

(b)  $1^3 + 2^3 + 3^3 + \dots + 50^3$

**Sol.**

$$\begin{aligned} a_1 &= 1^3 \\ a_2 &= 2^3 \\ a_3 &= 3^3 \\ &\vdots \\ a_{50} &= 50^3 \\ \therefore 1^3 + 2^3 + 3^3 + \dots + 50^3 &= \sum_{n=1}^{50} n^3 \end{aligned}$$

(c)  $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16}$

**Sol.**

$$\begin{aligned}
 a_1 &= \left(-\frac{1}{2}\right)^{1-1} \\
 a_2 &= \left(-\frac{1}{2}\right)^{2-1} \\
 a_3 &= \left(-\frac{1}{2}\right)^{3-1} \\
 a_4 &= \left(-\frac{1}{2}\right)^{4-1} \\
 a_5 &= \left(-\frac{1}{2}\right)^{5-1} \\
 \therefore 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} \\
 &= \sum_{n=1}^5 \left(-\frac{1}{2}\right)^{n-1}
 \end{aligned}$$

(d)  $2 \times 4 + 4 \times 7 + 6 \times 10 + 8 \times 13 + 10 \times 16$

**Sol.**

$$\begin{aligned}
 a_1 &= 2 \times 1 \times (3 \times 1 + 1) \\
 a_2 &= 2 \times 2 \times (3 \times 2 + 1) \\
 a_3 &= 2 \times 3 \times (3 \times 3 + 1) \\
 a_4 &= 2 \times 4 \times (3 \times 4 + 1) \\
 a_5 &= 2 \times 5 \times (3 \times 5 + 1) \\
 \therefore 2 \times 4 + 4 \times 7 + 6 \times 10 + 8 \times 13 \\
 &+ 10 \times 16 = \sum_{n=1}^5 2n(3n + 1)
 \end{aligned}$$

## 12.2 Arithmetic Progression

General term of an Arithmetic Progression (AP) is given by

$$a_n = a_1 + (n - 1)d$$

where  $a_1$  is the first term,  $d$  is the common difference and  $n$  is the number of terms.

### 12.2.1 Practice 3

- Find the number of terms of the AP  $-4 - 2\frac{3}{4} - 1\frac{1}{2} - \frac{1}{4} + \dots + 16$ .

$$\begin{aligned}
 a_1 &= -4 \\
 a_n &= 16 \\
 d &= -2\frac{3}{4} - (-4) \\
 &= -2\frac{3}{4} + 4 \\
 &= \frac{5}{4} \\
 16 &= -4 + (n - 1)\frac{5}{4} \\
 20 &= \frac{5}{4}(n - 1) \\
 80 &= 5(n - 1) \\
 n - 1 &= 16 \\
 n &= 17
 \end{aligned}$$

- Given that  $a_2 = 4$  and  $a_6 = -8$ , find the 10th term of the AP.

**Sol.**

$$\begin{aligned}
 a_2 &= 4 \\
 a + (2 - 1)d &= 4 \\
 a_6 &= -8 \\
 a + (6 - 1)d &= -8
 \end{aligned}$$

$$\begin{cases} a + d = 4 & (1) \\ a + 5d = -8 & (2) \end{cases}$$

$$(2) - (1) : 4d = -12$$

$$d = -3$$

$$a + (-3) = 4$$

$$a = 7$$

$$\begin{aligned}
 \therefore a_{10} &= 7 + (10 - 1)(-3) \\
 &= 7 - 27 \\
 &= -20
 \end{aligned}$$

- How many multiples of 7 are there between 50 and 500?

**Sol.**

$$\begin{aligned}a_1 &= 56 \\a_n &= 497 \\d &= 7 \\497 &= 56 + (n - 1)7 \\441 &= 7(n - 1) \\n - 1 &= 63 \\n &= 64\end{aligned}$$

4. Find 5 numbers between 30 and 54 such that these numbers form an AP.

**Sol.**

$$\begin{aligned}a_1 &= 30 \\a_7 &= 54 \\54 &= 30 + (7 - 1)d \\24 &= 6d \\d &= 4\end{aligned}$$

$\therefore$  These 5 numbers are 34, 38, 42, 46, and 50.

### Arithmetic mean

If A is in between x and y, and x, A, y are in AP, then

$$A = \frac{x + y}{2}$$

#### 12.2.2 Practice 4

1. If 9, x, 17 are in AP, find x.

**Sol.**

$$\begin{aligned}x &= \frac{9 + 17}{2} \\&= \frac{26}{2} \\&= 13\end{aligned}$$

2. Find the arithmetic mean of 26 and -11.

**Sol.**

$$\begin{aligned}A &= \frac{26 - 11}{2} \\&= \frac{15}{2}\end{aligned}$$

3. Find x and y when 3, x, 12, y, 21 are in AP.

**Sol.**

$$\begin{aligned}x &= \frac{3 + 12}{2} \\&= \frac{15}{2} \\y &= \frac{12 + 21}{2} \\&= \frac{33}{2}\end{aligned}$$

### Summation of Arithmetic Progression

The summation formula for AP is given by

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

or

$$S_n = \frac{n}{2}(a_1 + a_n)$$

#### 12.2.3 Practice 5

1. Find the sum of the first 16 terms of the AP  $22 + 18 + 14 + 10 + \dots$

**Sol.**

$$\begin{aligned}a_1 &= 22 \\n &= 16 \\d &= -4 \\S_n &= \frac{16}{2}(2 \times 22 + (-4)(16 - 1)) \\&= \frac{16}{2}(44 + (-4)(15)) \\&= \frac{16}{2}(44 - 60) \\&= \frac{16}{2}(-16) \\&= -128\end{aligned}$$

2. If the sum of AP  $23 + 19 + 15 + \dots$  is 72, find the number of terms.

**Sol.**

$$\begin{aligned}a_1 &= 23 \\S_n &= 72 \\d &= -4 \\72 &= \frac{n}{2}(2 \times 23 + (-4)(n-1)) \\72 &= \frac{n}{2}(46 + (-4)(n-1)) \\144 &= n(46 + (-4)(n-1)) \\144 &= n(46 - 4n + 4) \\144 &= n(50 - 4n) \\144 &= 50n - 4n^2 \\72 &= 25n - 2n^2\end{aligned}$$

$$2n^2 - 25n + 72 = 0$$

$$(n-8)(2n-9) = 0$$

$$n = 8$$

3. Given that  $S_n = 2n + 3n^2$ , find the first term and the common difference of the AP.

**Sol.**

$$S_n = 2n + 3n^2$$

$$2n + 3n^2 = \frac{n}{2}(2a + (n-1)d)$$

$$4n + 6n^2 = n(2a + (n-1)d)$$

$$4n + 6n^2 = 2na + (n-1)nd$$

$$4n + 6n^2 = 2na + n^2d - nd$$

$$4n + 6n^2 = (2a - d)n + dn^2$$

*Comparing both sides,*

$$2a - d = 4$$

$$a = 6$$

$$d = 2$$

### 12.2.4 Exercise 12.2

1. Find the 10th terms of the AP 5, 13, 21, ...

**Sol.**

$$a_1 = 5$$

$$n = 10$$

$$d = 8$$

$$a_{10} = 5 + (10-1) \times 8$$

$$= 5 + 72$$

$$= 77$$

2. Find the 8th term of the AP  $5, 4\frac{1}{4}, 3\frac{1}{2}, 2\frac{3}{4}, \dots$

**Sol.**

$$a_1 = 5$$

$$n = 8$$

$$d = -\frac{3}{4}$$

$$a_8 = 5 + (8-1) \times -\frac{3}{4}$$

$$= 5 - \frac{3}{4} \times 7$$

$$= 5 - \frac{21}{4}$$

$$= -\frac{1}{4}$$

3. Find the number of terms of the following AP.

- (a) 4, 9, ..., 64

**Sol.**

$$a_1 = 4$$

$$a_n = 64$$

$$d = 5$$

$$64 = 4 + (n-1) \times 5$$

$$60 = 5(n-1)$$

$$12 = n-1$$

$$n = 13$$

- (b)  $4\frac{1}{3}, 3\frac{2}{3}, 3, \dots, -10\frac{1}{3}$



**Sol.**

$$\begin{aligned}a_1 &= 4\frac{1}{3} \\a_n &= -10\frac{1}{3} \\d &= -\frac{2}{3} \\-10\frac{1}{3} &= 4\frac{1}{3} + (n-1) \times -\frac{2}{3} \\-\frac{31}{3} &= \frac{13}{3} - \frac{1}{3}(n-1) \\-31 &= 13 - 2n + 2 \\-46 &= 2n \\n &= 23\end{aligned}$$

4. The 6th term of an AP is 43, and its 10th term is 75. Find the first term and common difference of this AP.

**Sol.**

$$\begin{aligned}a_6 &= 43 \\a_{10} &= 75 \\43 &= a + (6-1)d \\75 &= a + (10-1)d \\32 &= 4d \\d &= 8 \\43 &= a + 5 \times 8 \\43 &= a + 40 \\3 &= a \\a &= 3 \\\therefore a_1 &= 3, d = 8\end{aligned}$$

5. The 7th term of an AP is -10, and the 12th term -25, find the 15th term of this AP.

**Sol.**

$$\begin{aligned}a_7 &= -10 \\a_{12} &= -25 \\-10 &= a + (7-1)d \\-25 &= a + (12-1)d \\-15 &= 5d \\d &= -3 \\-10 &= a + 6 \times -3 \\-10 &= a - 18 \\a &= 8 \\a_{15} &= 8 + (15-1) \times -3 \\&= 8 - 42 \\&= -34\end{aligned}$$

6. How many multiples of 7 are there between 100 and 200?

**Sol.**

$$\begin{aligned}a &= 105 \\d &= 7 \\a_n &= 196 \\196 &= 105 + (n-1) \times 7 \\91 &= 7(n-1) \\13 &= n-1 \\n &= 14\end{aligned}$$

7. Find the arithmetic mean of the following number pairs.

- (a) (8, 20)

**Sol.**

$$\frac{8+20}{2} = 14$$

- (b) (-9, 17)

**Sol.**

$$\frac{-9+17}{2} = 4$$

8. Find 5 numbers between 22 and 58 such that these 7 numbers are in AP.

**Sol.**

$$\begin{aligned}a_1 &= 22 \\a_7 &= 58 \\58 &= 22 + (7-1)d \\36 &= 6d \\d &= 6 \\\therefore \text{These 5 numbers are } 22, 28, 34, 40, 46\end{aligned}$$

9. Find the sum of first 20 terms of AP 12 + 15 + 18 + ...

**Sol.**

$$\begin{aligned}a_1 &= 12 \\n &= 20 \\d &= 3 \\S_{20} &= \frac{20}{2}(2 \times 12 + (20-1) \times 3) \\&= 10(24 + 57) \\&= 10(81) \\&= 810\end{aligned}$$

10. Find the sum of first 12 terms of the AP  $18 + 10 + 2 - 6 - \dots$

**Sol.**

$$\begin{aligned}a_1 &= 18 \\n &= 12 \\d &= -8 \\S_{12} &= \frac{12}{2}(2 \times 18 + (12 - 1) \times -8) \\&= 6(36 - 88) \\&= 6(-52) \\&= -312\end{aligned}$$

11. Find the sum of first 14 terms of the AP  $\frac{1}{6} + \frac{4}{3} + \frac{5}{2} + \dots$

**Sol.**

$$\begin{aligned}a_1 &= \frac{1}{6} \\n &= 14 \\d &= \frac{7}{6} \\S_{14} &= \frac{14}{2}(2 \times \frac{1}{6} + (14 - 1) \times \frac{7}{6}) \\&= 7(\frac{1}{3} + \frac{91}{6}) \\&= 7 \times \frac{93}{6} \\&= 7 \times \frac{31}{2} \\&= \frac{217}{2}\end{aligned}$$

12. Find the sum of all the multiples of 13 in between 200 and 800.

**Sol.**

$$\begin{aligned}a_1 &= 208 \\a_n &= 793 \\d &= 13 \\793 &= 208 + (n - 1) \times 13 \\585 &= 13(n - 1) \\45 &= n - 1 \\n &= 46 \\S_{46} &= \frac{46}{2}(2 \times 208 + (46 - 1) \times 13) \\&= 23(416 + 585) \\&= 23(1001) \\&= 23023\end{aligned}$$

13. If the sum of first  $n$  terms of the AP  $-3, -7, -11, \dots$  is  $-903$ , find the value of  $n$ .

**Sol.**

$$\begin{aligned}a_1 &= -3 \\d &= -4 \\-903 &= \frac{n}{2}(2 \times (-3) - 4(n - 1)) \\-1806 &= -2n - 4n^2 \\4n^2 + 2n - 1806 &= 0 \\2n^2 + n - 903 &= 0 \\(n - 21)(2n + 43) &= 0 \\n &= 21, -43(\text{invalid}) \\\therefore n &= 21\end{aligned}$$

14. Given that the first 3 terms of an AP are  $x, 3x - 4, 2x + 7$ , find:

- (a) The value of  $x$

**Sol.**

$$\begin{aligned}3x - 4 &= \frac{x + 2x + 7}{2} \\6x - 8 &= 3x + 7 \\3x &= 15 \\x &= 5\end{aligned}$$

- (b) The common difference

**Sol.**

$$\begin{aligned}a_1 &= x = 5 \\a_2 &= 3x - 4 = 3 \times 5 - 4 = 11 \\d &= 11 - 5 \\&= 6\end{aligned}$$

- (c) The sum of first 10 terms.

**Sol.**

$$\begin{aligned}a_1 &= x = 5 \\n &= 10 \\d &= 6 \\S_{10} &= \frac{10}{2}(2 \times 5 + (10 - 1) \times 6) \\&= 5(10 + 54) \\&= 5(64) \\&= 320\end{aligned}$$

15. Let the sum of the first  $n$  terms of an AP to be  $S_n = \frac{n(n+1)}{4}$ , find:

- (a) The first term

**Sol.**

$$\begin{aligned}\frac{n(n+1)}{4} &= \frac{n}{2}(2a + (n-1)d) \\ n(n+1) &= 2n(2a + dn - d) \\ n^2 + n &= 4na + 2dn^2 - 2nd \\ n^2 + n &= 2dn^2 + (4a - 2d)n\end{aligned}$$

Comparing both sides,

$$\begin{aligned}2d &= 1 \\ d &= \frac{1}{2} \\ 4a - 2d &= 1 \\ 4a - 1 &= 1 \\ 4a &= 2 \\ a &= \frac{1}{2}\end{aligned}$$

(b) The common difference

**Sol.**

$$d = \frac{1}{2}$$

gg

(c) The 6th terms

**Sol.**

$$\begin{aligned}a_1 &= \frac{1}{2} \\ n &= 6 \\ d &= \frac{1}{2} \\ a_6 &= \frac{1}{2} + (6-1) \times \frac{1}{2} \\ &= \frac{1}{2} + \frac{5}{2} \\ &= 3\end{aligned}$$

(d) The sum from 6th term to 10th term

**Sol.**

$$\begin{aligned}a &= \frac{1}{2} \\ d &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}S_{10} &= \frac{10}{2}(2 \times \frac{1}{2} + (10-1) \times \frac{1}{2}) \\ &= \frac{10}{2}(1 + \frac{9}{2}) \\ &= 5 \times \frac{11}{2} \\ &= \frac{55}{2}\end{aligned}$$

$$\begin{aligned}S_5 &= \frac{5}{2}(2 \times \frac{1}{2} + (5-1) \times \frac{1}{2}) \\ &= \frac{5}{2}(1 + 2) \\ &= \frac{15}{2}\end{aligned}$$

$$\begin{aligned}S_{10} - S_5 &= \frac{55}{2} - \frac{15}{2} \\ &= \frac{40}{2} \\ &= 20\end{aligned}$$

16. Given three numbers in an AP, the sum of these three numbers is 30, and the sum of square of these numbers is 318, find these three numbers.

**Sol.**

$$\begin{aligned}a_1 + a_2 + a_3 &= 30 \\a_1^2 + a_2^2 + a_3^2 &= 318 \\a_2 - a_1 &= a_3 - a_2 \\a_1 - 2a_2 + a_3 &= 0 \\3a_2 &= 30 \\a_2 &= 10 \\a_1 - 20 + a_3 &= 0 \\a_1 + a_3 &= 20 \\a_3 &= 20 - a_1 \\a_1^2 + 100 + (20 - a_1)^2 &= 318 \\a_1^2 + 100 + 400 + a_1^2 - 40a_1 &= 318 \\2a_1^2 - 40a_1 + 182 &= 0 \\a_1^2 - 20a_1 + 91 &= 0 \\(a_1 - 7)(a_1 - 13) &= 0 \\a_1 &= 7 \text{ or } a_1 = 13\end{aligned}$$

$\therefore$  These three numbers are 7, 10, and 13

17. Find the sum of all the numbers between 100 and 200 that are both the multiples of 2 and 3.

**Sol.**

$$\begin{aligned}a_1 &= 102 \\d &= 6 \\a_n &= 198 \\198 &= 102 + (n - 1) \times 6 \\96 &= 6(n - 1) \\6n - 6 &= 96 \\6n &= 102 \\n &= 17 \\S_{17} &= \frac{17}{2}(2 \times 102 + (17 - 1) \times 6) \\&= \frac{17}{2}(204 + 96) \\&= \frac{17}{2}(300) \\&= 150 \times 17 \\&= 2550\end{aligned}$$

18. Given an AP  $-100 - 96 - 92 - \dots$ :

- (a) Find the term where the number become positive.

**Sol.**

$$\begin{aligned}a_1 &= -100 \\d &= 4 \\a_n &= -100 + (n - 1) \times 4 > 0 \\-100 + 4n - 4 &> 0 \\4n &> 104 \\n &> 26 \\\therefore n &= 27\end{aligned}$$

- (b) Find the term where the sum of this AP becomes positive.

**Sol.**

$$\begin{aligned}S_n &= \frac{n}{2}(2(-100) + (n - 1) \times (4)) > 0 \\\frac{n}{2}(-200 + 4n - 4) &> 0 \\\frac{n}{2}(-204 + 4n) &> 0 \\n(2n - 102) &> 0 \\n(n - 51) &> 0 \\n &> 51 \\\therefore n &= 52\end{aligned}$$

19. Find the first negative term of the AP  $20, 19\frac{1}{5}, 18\frac{2}{5}, \dots$

**Sol.**

$$\begin{aligned}a_1 &= 20 \\d &= -\frac{4}{5} \\a_n &= 20 + (n - 1) \times (-\frac{4}{5}) < 0 \\100 - 4n + 4 &< 0 \\4n &> 104 \\n &> 26 \\\therefore n &= 27\end{aligned}$$

20. Given an AP  $10 + 9\frac{1}{5} + 8\frac{2}{5} + \dots$ , what is the first negative term? When will the sum of the terms become negative, and what's the value of it?

**Sol.**

$$a_n = 10 + (n-1) \times \left(-\frac{4}{5}\right) < 0$$

$$10 - \frac{4}{5}(n-1) < 0$$

$$50 - 4n + 4 < 0$$

$$-4n < -54$$

$$n > 13\frac{1}{2}$$

$$\therefore n = 14$$

$$S_n = \frac{n}{2}(2 \times 10 + (n-1) \times \left(-\frac{4}{5}\right)) < 0$$

$$\frac{n}{2}(20 - \frac{4}{5}(n-1)) < 0$$

$$20n - \frac{4}{5}(n^2 - n) < 0$$

$$100n - 4n^2 + 4n < 0$$

$$25n - n^2 + n < 0$$

$$26n - n^2 < 0$$

$$n(n-26) > 0$$

$$n > 26$$

$$\therefore n = 27$$

$$S_{27} = \frac{27}{2}(2 \times 10 + (27-1) \times \left(-\frac{4}{5}\right))$$

$$= \frac{27}{2}(20 - \frac{4}{5}(27-1))$$

$$= \frac{27}{2}(20 - \frac{4}{5}(26))$$

$$= \frac{27}{2} \times \left(-\frac{4}{5}\right)$$

$$= -\frac{54}{5}$$

*$\therefore$  The first negative term is the 14th term*

*$\therefore$  The first term where the sum of the terms becomes negative is the 27th term*

*$\therefore$  The value of the sum of the terms when it becomes negative is  $-\frac{54}{5}$*

21. Given a polygon which all their internal angles are in AP. The common difference of this AP is  $6^\circ$ , the largest angle is  $135^\circ$ . How many sides does this polygon have?

**Sol.**

$$a_1 = 135$$

$$d = -6$$

$$\frac{n}{2}(2 \times 135 + (n-1) \times (-6)) = 180(n-2)$$

$$n(270 - 6(n-1)) = 360(n-2)$$

$$n(276 - 6n) = 360n - 720$$

$$276n - 6n^2 = 360n - 720$$

$$46n - n^2 = 60n - 120$$

$$n^2 + 14n - 120 = 0$$

$$(n+20)(n-6) = 0$$

$$n = -20 \text{ (invalid)}$$

$$n = 6$$

*$\therefore$  The number of sides is 6*

22. Given an AP which its 5th term is 3 and the sum of its first 10 terms is  $26\frac{1}{4}$ . Which term in this AP is 0?

**Sol.**

$$a_5 = a + (5-1)d = 3$$

$$a + 4d = 3$$

$$S_{10} = \frac{10}{2}(2a + (10-1)d) = 26\frac{1}{4}$$

$$5(2a + 9d) = 26\frac{1}{4}$$

$$20(2a + 9d) = 105$$

$$4(2a + 9d) = 21$$

$$8a + 36d = 21$$

$$8a + 32d = 24$$

$$4d = -3$$

$$d = -\frac{3}{4}$$

$$a = 3 + \frac{3}{4} \times 4$$

$$= 6$$

$$a_n = 6 + (n-1) \times \left(-\frac{3}{4}\right) = 0$$

$$6 - \frac{3}{4}(n-1) = 0$$

$$24 - 3n + 3 = 0$$

$$3n = 27$$

$$n = 9$$

23. Given that the sum of the first 6 terms of an AP is 96, and the sum of the first 20 terms is 3 times the sum of the first 10 terms of this AP. Find the first term and the 10th term of it.

**Sol.**

$$\begin{aligned}
 S_6 &= \frac{6}{2}(2a + (6-1)d) = 96 \\
 3(2a + 5d) &= 96 \\
 2a + 5d &= 32 \\
 S_{20} &= 3S_{10} \\
 \frac{20}{2}(2a + (20-1)d) &= 3 \times \frac{10}{2}(2a + (10-1)d) \\
 10(2a + 19d) &= 15(2a + 9d) \\
 2(2a + 19d) &= 3(2a + 9d) \\
 4a + 38d &= 6a + 27d \\
 2a - 11d &= 0 \\
 16d &= 32 \\
 d &= 2 \\
 a &= \frac{11 \times 2}{2} \\
 &= 11 \\
 a_{10} &= 11 + (10-1) \times 2 \\
 &= 29
 \end{aligned}$$

24. Given that  $5^2 \times 5^4 \times 5^6 \times \dots \times 5^{2n} = (0.04)^{-28}$ , find the value of n.

**Sol.**

$$\begin{aligned}
 (0.04)^{-28} &= \frac{1}{25}^{-28} \\
 &= (5^{-2} - 2)^{-28} \\
 &= 5^{56} \\
 \therefore n^a \times n^b &= n^{a+b} \\
 2 + 4 + 6 + \dots + 2n &= 56 \\
 S_n &= \frac{n}{2}(2 \times 2 + (n-1) \times 2) = 56 \\
 n(4 + 2(n-1)) &= 112 \\
 n(2 + 2n) &= 112 \\
 2n^2 + 2n &= 112 \\
 n^2 + n - 56 &= 0 \\
 (n+8)(n-7) &= 0 \\
 n &= -8 \text{ (invalid)} \\
 n &= 7
 \end{aligned}$$

25. Given that the 9th term of an AP is double the 5th term of it. Find the ratio of the sum of first 9 terms and the sum of first 5 terms of the AP.

**Sol.**

$$\begin{aligned}
 a_9 &= 2a_5 \\
 a + (9-1)d &= 2(a + (5-1)d) \\
 a + 8d &= 2a + 8d \\
 a &= 0 \\
 S_9 : S_5 &= \frac{9}{2}(2a + a_9) : \frac{5}{2}(2a + a_5) \\
 &= \frac{9}{2}(2a + 2a_5) : \frac{5}{2}(2a + a_5) \\
 &= 9(a + a_5) : \frac{5}{2}(2a + a_5) \\
 \frac{S_9}{S_5} &= \frac{9(a + a_5)}{\frac{5}{2}(2a + a_5)} \\
 &= \frac{18(a + a_5)}{5(2a + a_5)} \\
 &= \frac{18 \times a_5}{5 \times a_5} \\
 &= \frac{18}{5} \\
 \therefore S_9 : S_5 &= 18 : 5
 \end{aligned}$$

## 12.3 Geometric Progression

The general formula of a geometric progression (GP) is given by

$$a_n = a_1 \times r^{n-1}$$

where  $a_1$  is the first term,  $r$  is the common ratio, and  $n$  is the number of terms.

### 12.3.1 Practice 6

- Find the 6th term of the GP 12, -18, 27, ...

**Sol.**

$$\begin{aligned}
 a_1 &= 12 \\
 r &= \frac{-18}{12} \\
 &= -\frac{3}{2} \\
 a_6 &= 12 \times \left(-\frac{3}{2}\right)^{6-1} \\
 &= 12 \times \left(-\frac{3}{2}\right)^5 \\
 &= 12 \times \left(-\frac{243}{32}\right) \\
 &= -\frac{729}{8}
 \end{aligned}$$

2. Find the number of terms of GP  $\frac{1}{64} - \frac{1}{32} + \frac{1}{16} - \frac{1}{8} + \dots - 512$

**Sol.**

$$\begin{aligned}
 a_1 &= \frac{1}{64} \\
 r &= \frac{-\frac{1}{32}}{\frac{1}{64}} \\
 &= -2 \\
 -512 &= \frac{1}{64}(-2)^{n-1} \\
 (-2)^9 &= \frac{1}{2^6}(-2)^{n-1} \\
 (-2)^{15} &= (-2)^{n-1} \\
 n-1 &= 15 \\
 n &= 16
 \end{aligned}$$

3. The 5th term of a GP is 3, and its 9th term is  $\frac{1}{27}$ , find the first term and the common ratio of this GP.

**Sol.**

$$\begin{aligned}
 a_5 &= ar^4 = 3 \\
 a_9 &= ar^8 = \frac{1}{27} \\
 r^4 &= \frac{1}{27} \times \frac{1}{3} \\
 &= \frac{1}{81} \\
 r &= \frac{1}{3} \\
 a_1 &= 3 \times 81 \\
 &= 243
 \end{aligned}$$

4. Find 5 numbers between  $\frac{1}{2}$  and

$\frac{1}{128}$  such that these 7 numbers are in GP.

**Sol.**

$$\begin{aligned}
 a_1 &= \frac{1}{2} \\
 n &= 7 \\
 \frac{1}{128} &= \frac{1}{2}r^{7-1} \\
 r^6 &= \frac{1}{64} \\
 r &= \frac{1}{2}
 \end{aligned}$$

$\therefore$  These 5 numbers are  $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$

### Geometric Mean

The geometric mean G of two numbers x and y is given by

$$\begin{aligned}
 \frac{G}{x} &= \frac{G}{y} \\
 G^2 &= xy \\
 G &= \pm \sqrt[2]{xy}
 \end{aligned}$$

### 12.3.2 Practice 7

Find the geometric mean of  $\frac{27}{8}$  and  $\frac{2}{3}$ .

**Sol.**

$$\begin{aligned}
 G &= \pm \sqrt[2]{\frac{27}{8} \times \frac{2}{3}} \\
 &= \pm \sqrt[2]{\frac{9}{4}} \\
 &= \pm \frac{3}{2}
 \end{aligned}$$

### Summation of Geometric Progression

The sum of n terms of a GP is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad (r \neq 1)$$

### 12.3.3 Practice 8

1. Find the sum of the first 8 terms of GP  $3 + 6 + 12 + \dots$

**Sol.**

$$\begin{aligned}
 a_1 &= 3 \\
 r &= \frac{6}{3} \\
 &= 2 \\
 n &= 8 \\
 S_n &= \frac{3(1-2^8)}{1-2} \\
 &= \frac{3(1-256)}{1-2} \\
 &= 3 \times 255 \\
 &= 765
 \end{aligned}$$

2. Find the sum of the GP  $1 + \sqrt{3} + 3 + \dots + 81$

**Sol.**

$$\begin{aligned}
 a_1 &= 1 \\
 r &= \sqrt{3} \\
 81 &= 1 \times (\sqrt{3})^{n-1} \\
 3^4 &= (\sqrt{3})^{n-1} \\
 (\sqrt{3})^8 &= (\sqrt{3})^{n-1} \\
 n-1 &= 8 \\
 n &= 9 \\
 S_n &= \frac{1(1-(\sqrt{3})^9)}{1-\sqrt{3}} \\
 &= \frac{1-81\sqrt{3}}{1-\sqrt{3}} \\
 &= \frac{(1-81\sqrt{3})(1+\sqrt{3})}{-2} \\
 &= \frac{1-81\sqrt{3}+\sqrt{3}-243}{-2} \\
 &= \frac{-242-80\sqrt{3}}{-2} \\
 &= 121+40\sqrt{3}
 \end{aligned}$$

3. Given that the sum of the first  $n$  terms of GP  $4\frac{4}{5}, 1\frac{3}{5}, \frac{8}{15}, \dots$  is  $7\frac{145}{729}$ , find  $n$ .

**Sol.**

$$\begin{aligned}
 a_1 &= \frac{24}{5} \\
 r &= \frac{8}{5} \times \frac{5}{24} \\
 &= \frac{1}{3} \\
 S_n &= \frac{24}{5} \times \frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}} \\
 \frac{5248}{729} &= \frac{24}{5} \times \frac{1-(\frac{1}{3})^n}{\frac{2}{3}} \\
 \frac{5248}{729} \times \frac{5}{24} \times \frac{2}{3} &= 1-(\frac{1}{3})^n \\
 \frac{6560}{6561} &= 1-(\frac{1}{3})^n \\
 -\frac{1}{6561} &= -(\frac{1}{3})^n \\
 (\frac{1}{3})^8 &= (\frac{1}{3})^n \\
 n &= 8
 \end{aligned}$$

## Summation of Infinite Geometric Progression

The sum of infinite GP is given by

$$S_{\infty} = \frac{a_1}{1-r} \quad (-1 < r < 1)$$

### 12.3.4 Practice 9

1. Find the sum of the following infinite GP.

(a)  $16 + 8 + 4 + \dots$

**Sol.**

$$\begin{aligned}
 a_1 &= 16 \\
 r &= \frac{8}{16} \\
 &= \frac{1}{2} \\
 S_{\infty} &= \frac{16}{1-\frac{1}{2}} \\
 &= \frac{16}{\frac{1}{2}} \\
 &= 32
 \end{aligned}$$

(b)  $18 - 12 + 8 + \dots$



**Sol.**

$$\begin{aligned}a_1 &= 18 \\r &= \frac{8}{-12} \\&= -\frac{2}{3} \\S_\infty &= \frac{18}{1 + \frac{2}{3}} \\&= \frac{18}{\frac{5}{3}} \\&= \frac{54}{5}\end{aligned}$$

(c)  $1 + \frac{3}{4} + \frac{9}{16} + \dots$

**Sol.**

$$\begin{aligned}a_1 &= 1 \\r &= \frac{9}{16} \times \frac{16}{9} \\&= \frac{3}{4} \\S_\infty &= \frac{1}{1 - \frac{3}{4}} \\&= \frac{1}{\frac{1}{4}} \\&= 4\end{aligned}$$

(d)  $\sqrt{2} + 1 + \frac{1}{\sqrt{2}} + \dots$

**Sol.**

$$\begin{aligned}a_1 &= \sqrt{2} \\r &= \frac{1}{\sqrt{2}} \\S_\infty &= \frac{\sqrt{2}}{1 - \frac{1}{\sqrt{2}}} \\&= \frac{\sqrt{2}}{\frac{\sqrt{2}-1}{\sqrt{2}}} \\&= \frac{2}{\sqrt{2}-1} \\&= 2(\sqrt{2}+1)\end{aligned}$$

2. Convert the following recurring decimals to fraction using the summation of infinite geometric series.

(a)  $0.\overline{3}$

**Sol.**

$$\begin{aligned}a_1 &= 0.3 \\r &= 0.1 \\S_\infty &= \frac{0.3}{1 - 0.1} \\&= \frac{0.3}{0.9} \\&= \frac{1}{3}\end{aligned}$$

$$\therefore 0.\overline{3} = \frac{1}{3}$$

(b)  $0.5\overline{3}$

**Sol.**

$$\begin{aligned}a_1 &= 0.03 \\r &= 0.01 \\S_\infty &= \frac{0.03}{1 - 0.01} \\&= \frac{0.03}{0.99} \\&= \frac{3}{99}\end{aligned}$$

$$\begin{aligned}\therefore 0.5\overline{3} &= \frac{5}{10} + \frac{3}{99} \\&= \frac{53}{99}\end{aligned}$$

### 12.3.5 Exercise 12.3

1. Find the 10th term of the GP 2, 4, 8, ...

**Sol.**

$$\begin{aligned}a_1 &= 2 \\r &= \frac{4}{2} \\&= 2 \\a_{10} &= 2 \times 2^{10-1} \\&= 2 \times 512 \\&= 1024\end{aligned}$$

2. Find the 8th term of the GP 243, -162, 108, ...

**Sol.**

$$\begin{aligned}a_1 &= 243 \\r &= \frac{-162}{243} \\&= -\frac{2}{3} \\a_8 &= 243 \times \left(-\frac{2}{3}\right)^{8-1} \\&= 243 \times \left(-\frac{128}{2187}\right) \\&= -\frac{128}{9}\end{aligned}$$

3. Find the number of terms of the following GP.

(a)  $8, 4, 2, 1, \dots, \frac{1}{64}$

**Sol.**

$$\begin{aligned}a_1 &= 8 \\r &= \frac{4}{8} \\&= \frac{1}{2} \\\frac{1}{64} &= 8 \times \left(\frac{1}{2}\right)^{n-1} \\\frac{1}{512} &= \left(\frac{1}{2}\right)^{n-1} \\\frac{1}{2^9} &= \left(\frac{1}{2}\right)^{n-1} \\n-1 &= 9 \\n &= 10\end{aligned}$$

(b)  $6, -18, 54, \dots, -13122$

**Sol.**

$$\begin{aligned}a_1 &= 6 \\r &= \frac{-18}{6} \\&= -3 \\-13122 &= 6 \times (-3)^{n-1} \\-2187 &= (-3)^{n-1} \\(-3)^7 &= (-3)^{n-1} \\n-1 &= 7 \\n &= 8\end{aligned}$$

(c)  $54, 36, 24, \dots, 3\frac{13}{81}$

**Sol.**

$$\begin{aligned}a_1 &= 54 \\r &= \frac{36}{54} \\&= \frac{2}{3} \\\frac{256}{81} &= 54 \times \left(\frac{2}{3}\right)^{n-1} \\\frac{256}{81} \times \frac{1}{54} &= \left(\frac{2}{3}\right)^{n-1} \\\frac{128}{2187} &= \left(\frac{2}{3}\right)^{n-1} \\\left(\frac{2}{3}\right)^7 &= \left(\frac{2}{3}\right)^{n-1} \\n-1 &= 7 \\n &= 8\end{aligned}$$

4. Given that the 2nd term of a GP is 12, and its 4th term is 108, find the first term and the common ratio of it.

**Sol.**

$$\begin{aligned}a_2 &= ar = 12 \\a_4 &= ar^3 = 109 \\r^2 &= 9 \\r &= \pm 3 \\a_1 &= \pm 4 \\\therefore a_1 &= 4, r = 3 \text{ or } a_1 = -4, r = -3\end{aligned}$$

5. Given that the 3rd term of an GP is  $1\frac{1}{3}$ , and its 8th term is  $-10\frac{1}{8}$ . Find the 5th term of this AP.

**Sol.**

$$\begin{aligned}a_3 &= ar^2 = \frac{4}{3} \\a_8 &= ar^7 = -\frac{81}{8} \\r^5 &= -\frac{81}{8} \times \frac{3}{4} \\&= -\frac{243}{32} \\&= \left(-\frac{3}{2}\right)^5 \\r &= -\frac{3}{2} \\a &= \frac{4}{3} \times \frac{4}{9} \\&= \frac{16}{27} \\a_5 &= \frac{16}{27} \times \left(\frac{3}{2}\right)^4 \\&= \frac{16}{27} \times \frac{81}{16} \\&= 3\end{aligned}$$

6. Find the geometric mean of 2 and 18.

**Sol.**

$$\begin{aligned}G &= \pm \sqrt[2]{2 \times 18} \\&= \pm \sqrt[2]{36} \\&= \pm 6\end{aligned}$$

7. Given that  $x+12$ ,  $x+4$  and  $x-2$  are in GP, find the value of  $x$  and the common ratio of this GP.

**Sol.**

$$\begin{aligned}x+4 &= \pm \sqrt{(x+12)(x-2)} \\x^2 + 8x + 16 &= x^2 + 10x - 24 \\2x &= 40 \\x &= 20 \\a_1 &= 20 + 12 = 32 \\a_2 &= 20 + 4 = 24 \\r &= \frac{24}{32} \\&= \frac{3}{4}\end{aligned}$$

8. Find 3 numbers between 14 and 224 such that these 5 numbers are in GP.

**Sol.**

$$\begin{aligned}a_1 &= 14 \\a_5 &= 224 \\244 &= 14 \times r^4 \\16 &= r^4 \\(\pm 2)^4 &= r^4 \\r &= \pm 2\end{aligned}$$

$\therefore$  These 3 numbers are 28, 56, 112

or -28, 56, -112

9. Calculate the sum of the first 6 terms of the GP  $2 + 6 + 18 + \dots$

**Sol.**

$$\begin{aligned}a_1 &= 2 \\r &= \frac{6}{2} \\&= 3 \\S_6 &= \frac{2(1-3^6)}{1-3} \\&= \frac{2(1-729)}{-2} \\&= 728\end{aligned}$$

10. Calculate the sum of the first 8 terms of the GP  $32 - 16 + 8 - \dots$

**Sol.**

$$\begin{aligned}a_1 &= 32 \\r &= \frac{-16}{32} \\&= -\frac{1}{2} \\S_8 &= \frac{32(1-(\frac{1}{2})^8)}{1+\frac{1}{2}} \\&= \frac{32(1-\frac{1}{256})}{\frac{3}{2}} \\&= 32 \times \frac{255}{256} \times \frac{2}{3} \\&= \frac{85}{4}\end{aligned}$$

11. Find the sum of the GP  $14 - 28 + 56 - \dots + 3584$

**Sol.**

$$\begin{aligned}a_1 &= 14 \\r &= \frac{-28}{14} = -2 \\3584 &= 14 \times (-2)^{n-1} \\256 &= (-2)^{n-1} \\(-2)^8 &= (-2)^{n-1} \\n-1 &= 8 \\n &= 9 \\S_9 &= \frac{14(1 - (-2)^9)}{1 - (-2)} \\&= \frac{14(1 + 512)}{3} \\&= \frac{14 \times 513}{3} \\&= 2394\end{aligned}$$

12. If the first term of a GP is 7, its common ratio is 3, and the sum of its terms is 847, find the number of terms and the last term of this GP.

**Sol.**

$$\begin{aligned}a_1 &= 7 \\r &= 3 \\S_n &= \frac{7(1 - 3^n)}{1 - 3} = 847 \\7(1 - 3^n) &= -1694 \\1 - 3^n &= -242 \\3^n &= 243 \\3^n &= 3^5 \\n &= 5 \\a_5 &= 7 \times 3^4 = 567\end{aligned}$$

13. Find the sum of the following infinite GP.

(a)  $24 + 18 + 13\frac{1}{2} + \dots$

**Sol.**

$$\begin{aligned}a_1 &= 24 \\r &= \frac{18}{24} = \frac{3}{4} \\S_\infty &= \frac{24}{1 - \frac{3}{4}} \\&= \frac{24}{\frac{1}{4}} \\&= 96\end{aligned}$$

(b)  $27 - 9 + 3 - 1 + \dots$

**Sol.**

$$\begin{aligned}a_1 &= 27 \\r &= \frac{-9}{27} = -\frac{1}{3} \\S_\infty &= \frac{27}{1 + \frac{1}{3}} \\&= \frac{27}{\frac{4}{3}} \\&= \frac{81}{4}\end{aligned}$$

(c)  $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \dots$

**Sol.**

$$\begin{aligned}a_1 &= 2 \\r &= \frac{-\frac{1}{2}}{2} = -\frac{1}{4} \\S_\infty &= \frac{2}{1 + \frac{1}{4}} \\&= \frac{2}{\frac{5}{4}} \\&= \frac{8}{5}\end{aligned}$$

14. Given an infinite GP which has a sum of 24 and first term of 30, find the common difference.

**Sol.**

$$\begin{aligned}a_1 &= 30 \\S_\infty &= 24 \\24 &= \frac{30}{1 - r} \\24(1 - r) &= 30 \\24 - 24r &= 30 \\-24r &= 6 \\r &= -\frac{1}{4}\end{aligned}$$

15. Convert the following recurring decimals into fractions.

(a)  $0.\overline{45}$

**Sol.**

$$\begin{aligned}a_1 &= 0.45 \\r &= 0.01 \\S_\infty &= \frac{0.45}{1 - 0.01} \\&= \frac{0.45}{0.99} \\&= \frac{45}{99} \\&= \frac{5}{11}\end{aligned}$$

$$\therefore 0.\overline{45} = \frac{5}{11}$$

(b)  $0.\overline{037}$

**Sol.**

$$\begin{aligned}a_1 &= 0.037 \\r &= 0.001 \\S_\infty &= \frac{0.037}{1 - 0.001} \\&= \frac{0.037}{0.999} \\&= \frac{37}{999} \\&= \frac{1}{27}\end{aligned}$$

$$\therefore 0.\overline{037} = \frac{1}{27}$$

(c)  $0.\overline{218}$

**Sol.**

$$\begin{aligned}a_1 &= 0.018 \\r &= 0.01 \\S_\infty &= \frac{0.018}{1 - 0.01} \\&= \frac{0.018}{0.99} \\&= \frac{18}{990} \\&= \frac{1}{55}\end{aligned}$$

$$\begin{aligned}\therefore 0.\overline{218} &= \frac{1}{5} + \frac{1}{55} \\&= \frac{12}{55}\end{aligned}$$

(d)  $1.\overline{3}$

**Sol.**

$$\begin{aligned}a_1 &= 0.3 \\r &= 0.1 \\S_\infty &= \frac{0.3}{1 - 0.1} \\&= \frac{0.3}{0.9} \\&= \frac{1}{3}\end{aligned}$$

$$\begin{aligned}\therefore 1.\overline{3} &= 1 + \frac{1}{3} \\&= \frac{4}{3}\end{aligned}$$

16. Three integers are in GP, summing up to 42 while accumulating up to 512, find these three integers.

**Sol.**

$$\begin{aligned}a_1 + a_2 + a_3 &= 42 \\a_1 a_2 a_3 &= 512 \\a_2 &= \pm \sqrt{a_1 a_3} \\a_1 a_3 &= a_2^2 \\a_2^3 &= 512 \\a_2 &= \sqrt[3]{512} \\&= 8 \\a_1 a_3 &= 64 \\a_3 &= \frac{64}{a_1} \\a_1 + 8 + \frac{64}{a_1} &= 42 \\a_1 + \frac{64}{a_1} &= 34 \\a_1^2 + 64 &= 34a_1 \\a_1^2 - 34a_1 + 64 &= 0 \\(a_1 - 32)(a_1 - 2) &= 0 \\a_1 &= 32 \text{ or } a_1 = 2\end{aligned}$$

$\therefore$  These three integers are 2, 8, 32

17. The sum of first 6 term of a GP is 9 times the sum of first 3 terms. Find the common ratio.

**Sol.**

$$\begin{aligned}
 S_6 &= 9S_3 \\
 \frac{a(1-r^6)}{1-r} &= 9 \times \frac{a(1-r^3)}{1-r} \\
 a(1-r^6) &= 9a(1-r^3) \\
 1-r^6 &= 9(1-r^3) \\
 &= 9-9r^3 \\
 r^6-9r^3+8 &= 0 \\
 (r^3-8)(r^3-1) &= 0 \\
 r^3 &= 8 \text{ or } r^3 = 1 \\
 r &= 1 \text{ (invalid)} \\
 r &= 2
 \end{aligned}$$

18. Given a GP, its first term is 16, last term is  $\frac{1}{2}$  and its sum is  $31\frac{1}{2}$ , find its common ratio and number of terms.

**Sol.**

$$\begin{aligned}
 a_1 &= 16 \\
 \frac{1}{2} &= 16r^{n-1} \\
 \frac{1}{32} &= r^{n-1} \\
 &= r^n \times \frac{1}{r} \\
 r^n &= \frac{r}{32} \\
 \frac{63}{2} &= \frac{16(1-r^n)}{1-r} \\
 63(1-r) &= 32(1-r^n) \\
 63-63r &= 32-32r^n \\
 -31 &= 32r^n-63r \\
 -31 &= r-63r \\
 -31 &= -62r \\
 r &= \frac{1}{2} \\
 \left(\frac{1}{2}\right)^{n-1} &= \frac{1}{32} \\
 &= \left(\frac{1}{2}\right)^5 \\
 n-1 &= 5 \\
 n &= 6
 \end{aligned}$$

19. Given a GP, its 3rd term is 6 less than its 2nd term, and its 2nd term is 9 less than its 1st term. Find the 4th term and the sum of the first 4 terms.

**Sol.**

$$\begin{aligned}
 \text{Let } x &= a_2 \\
 a_3 &= x-6 \\
 a_1 &= x+9 \\
 x &= \pm\sqrt{(x-6)(x+9)} \\
 x^2 &= x^2+3x-54 \\
 3x-54 &= 0 \\
 x &= 18 \\
 a_2 &= 18 \\
 a_1 &= 27 \\
 r &= \frac{12}{18} \\
 &= \frac{2}{3} \\
 a_4 &= 27 \times \left(\frac{2}{3}\right)^3 \\
 &= 8 \\
 S_4 &= \frac{27(1-(\frac{16}{3})^4)}{1-\frac{2}{3}} \\
 &= \frac{27(1-\frac{8}{81})}{\frac{1}{3}} \\
 &= 81 \times \frac{65}{81} \\
 &= 65
 \end{aligned}$$

20. Given an infinite GP, its common ratio is positive and the sum of it is 9. The sum of the first two terms is 5, find the 4th term.

**Sol.**

$$\begin{aligned}
 S_{\infty} &= \frac{a}{1-r} = 9 \\
 a &= 9(1-r) \\
 &= 9 - 9r \\
 S_2 &= \frac{a(1-r^2)}{1-r} = 5 \\
 a - ar^2 &= 5 - 5r \\
 9 - 9r - (9 - 9r)r^2 &= 5 - 5r \\
 9 - 9r - 9r^2 + 9r^3 &= 5 - 5r \\
 4 - 4r - 9r^2 + 9r^3 &= 0 \\
 4(1-r) - 9r^2(1-r) &= 0 \\
 (4 - 9r^2)(1-r) &= 0 \\
 (9r^2 - 4)(r-1) &= 0 \\
 (3r^2 + 2)(3r^2 - 2)(r-1) &= 0 \\
 r &= 1 \text{ (invalid)} \\
 r &= -\frac{2}{3} \text{ (invalid)} \\
 r &= \frac{2}{3} \\
 a &= 9(1 - \frac{2}{3}) \\
 &= 3 \\
 a_4 &= 3(\frac{2}{3})^3 \\
 &= 3 \times \frac{8}{27} \\
 &= \frac{8}{9}
 \end{aligned}$$

21. If  $x+1, x-2, \frac{1}{2}x$  are the first three terms of an infinite GP, find:

- (a) The value of  $x$

**Sol.**

$$\begin{aligned}
 x-2 &= \pm \sqrt{(x+1)(\frac{1}{2}x)} \\
 x^2 - 4x + 4 &= \frac{1}{2}x(x+1) \\
 2x^2 - 8x + 8 &= x^2 + x \\
 x^2 - 9x + 8 &= 0 \\
 (x-8)(x-1) &= 0 \\
 x &= 8 \text{ or } x = 1
 \end{aligned}$$

- (b) The common ratio

**Sol.**

$$\begin{aligned}
 \text{When } x &= 8, \\
 r &= \frac{8-2}{8+1} \\
 &= \frac{6}{9} \\
 &= \frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x &= 1, \\
 r &= \frac{1-2}{1+1} \\
 &= -\frac{1}{2}
 \end{aligned}$$

- (c) The sum of the GP

**Sol.**

$$\begin{aligned}
 \text{When } x &= 8, \\
 S_{\infty} &= \frac{a}{1-r} \\
 &= \frac{9}{1-\frac{2}{3}} \\
 &= 9 \times 3 \\
 &= 27
 \end{aligned}$$

$$\begin{aligned}
 \text{When } x &= 1, \\
 S_{\infty} &= \frac{a}{1-r} \\
 &= \frac{2}{1+\frac{1}{2}} \\
 &= 2 \times \frac{2}{3} \\
 &= \frac{4}{3}
 \end{aligned}$$

## 12.4 Simple Summation of Special Series

Sum formula of natural number:

$$\sum_{i=1}^n k = \frac{n(n+1)}{2}$$

Sum formula of square of natural number:

$$\sum_{i=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum formula of cube of natural number:

$$\sum_{i=1}^n k^3 = \left[ \frac{n(n+1)}{2} \right]^2$$

### 12.4.1 Practice 10

1. Find the sum of the following series.

(a)  $\sum_{k=1}^8 3k$

**Sol.**

$$\begin{aligned} \sum_{k=1}^8 3k &= 3 \sum_{k=1}^8 k \\ &= 3 \times \frac{8(8+1)}{2} \\ &= 3 \times \frac{8 \times 9}{2} \\ &= 3 \times \frac{72}{2} \\ &= 3 \times 36 \\ &= 108 \end{aligned}$$

(b)  $\sum_{k=1}^{12} k^2$

**Sol.**

$$\begin{aligned} \sum_{k=1}^{12} k^2 &= \frac{12(12+1)(2 \times 12 + 1)}{6} \\ &= \frac{12 \times 13 \times 25}{6} \\ &= 650 \end{aligned}$$

(c)  $\sum_{k=3}^{10} (2k - 3)$

**Sol.**

$$\begin{aligned} &\sum_{k=3}^{10} (2k - 3) \\ &= 2 \sum_{k=3}^{10} k - \sum_{k=3}^{10} 3 \\ &= 2 \left[ \sum_{k=1}^{10} k - \sum_{k=1}^2 k \right] - (30 - 6) \\ &= 2 \left[ \frac{10(10+1)}{2} - \frac{2(2+1)}{2} \right] - 8 \\ &= 2(55 - 3) - 24 \\ &= 2 \times 52 - 24 \\ &= 104 - 24 \\ &= 80 \end{aligned}$$

(d)  $\sum_{k=7}^{13} 3k^2$

**Sol.**

$$\begin{aligned} &\sum_{k=7}^{13} 3k^2 \\ &= 3 \left[ \sum_{k=1}^{13} k^2 - \sum_{k=1}^6 k^2 \right] \\ &= 3 \times \left[ \frac{13(13+1)(2 \times 13 + 1)}{6} \right. \\ &\quad \left. - \frac{6(6+1)(2 \times 6 + 1)}{6} \right] \\ &= 3 \times \left[ \frac{13 \times 14 \times 27}{6} - \frac{6 \times 7 \times 13}{6} \right] \\ &= 3 \times \left[ \frac{4914}{6} - \frac{546}{6} \right] \\ &= 3 \times \frac{4368}{6} \\ &= 3 \times 728 \\ &= 2184 \end{aligned}$$

2. Given that the  $n$ th term of a series is  $n(n+3)$ , find the sum of the first 20 terms of the series.

**Sol.**

$$\begin{aligned} &\sum_{k=1}^{20} k(k+3) \\ &= \sum_{k=1}^{20} k^2 + 3k \\ &= \sum_{k=1}^{20} k^2 + 3 \sum_{k=1}^{20} k \\ &= \frac{20(20+1)(2 \times 20 + 1)}{6} + 3 \times \frac{20(20+1)}{2} \\ &= \frac{20 \times 21 \times 41}{6} + 3 \times \frac{20 \times 21}{2} \\ &= 2870 + 630 \\ &= 3500 \end{aligned}$$

3. Find the sum of series  $1 \times 3 + 2 \times 4 + 3 \times 5 + \dots + n(n+2)$ .



**Sol.**

$$\begin{aligned} & \sum_{k=1}^n k(k+2) \\ &= \sum_{k=1}^n k^2 + 2k \\ &= \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k \\ &= \frac{n(n+1)(2n+1)}{6} + 2 \times \frac{n(n+1)}{2} \\ &= \frac{n(n+1)(2n+1)}{6} + n(n+1) \\ &= \frac{n(n+1)(2n+1) + 6n(n+1)}{6} \\ &= \frac{n(n+1)(2n+7)}{6} \end{aligned}$$

### 12.4.2 Exercise 12.4

1. Find the sum of the following series.

(a)  $\sum k = 1^8 5k^2$

**Sol.**

$$\begin{aligned} \sum_{k=1}^8 5k^2 &= 5 \sum_{k=1}^8 k^2 \\ &= 5 \times \frac{8(8+1)(2 \times 8 + 1)}{6} \\ &= 5 \times \frac{8 \times 9 \times 17}{6} \\ &= 5 \times \frac{1368}{6} \\ &= 5 \times 204 \\ &= 1020 \end{aligned}$$

(b)  $\sum_{k=1}^9 k^3$

**Sol.**

$$\begin{aligned} \sum_{k=1}^9 k^3 &= \left[ \frac{9(9+1)}{2} \right]^2 \\ &= 45^2 \\ &= 2025 \end{aligned}$$

(c)  $\sum_{n=1}^{10} (3n-5)$

**Sol.**

$$\begin{aligned} \sum_{n=1}^{10} (3n-5) &= 3 \sum_{n=1}^{10} n - 5 \sum_{n=1}^{10} 1 \\ &= 3 \times \frac{10(10+1)}{2} - 5 \times 10 \\ &= 3 \times \frac{10 \times 11}{2} - 5 \times 10 \\ &= 3 \times 55 - 50 \\ &= 3 \times 5 - 50 \\ &= 165 - 50 \\ &= 115 \end{aligned}$$

(d)  $\sum_{k=3}^6 2k^3$

**Sol.**

$$\begin{aligned} \sum_{k=3}^6 2k^3 &= 2 \sum_{k=3}^6 k^3 \\ &= 2 \left( \sum_{k=1}^6 k^3 - \sum_{k=1}^2 k^3 \right) \\ &= 2 \left\{ \left[ \frac{6(6+1)}{2} \right]^2 - \left[ \frac{2(2+1)}{2} \right]^2 \right\} \\ &= 2(21^2 - 3^2) \\ &= 2(441 - 9) \\ &= 2 \times 432 \\ &= 864 \end{aligned}$$

(e)  $\sum_{k=6}^{10} (2k^2 + 3)$

**Sol.**

$$\begin{aligned}
 & \sum_{k=6}^{10} (2k^2 + 3) \\
 &= 2 \sum_{k=6}^{10} k^2 + 3 \sum_{k=6}^{10} 1 \\
 &= 2 \left( \sum_{k=1}^{10} k^2 - \sum_{k=1}^5 k^2 \right) \\
 &\quad + 3 \times (10 - 5) \\
 &= 2 \times \left[ \frac{10 \times 11 \times 21}{6} - \frac{5 \times 6 \times 11}{6} \right] \\
 &\quad + 3 \times 5 \\
 &= 2 \times \left[ \frac{2310}{6} - \frac{330}{6} \right] + 3 \times 5 \\
 &= 2 \times \frac{1980}{6} + 3 \times 5 \\
 &= 2 \times 330 + 3 \times 5 \\
 &= 660 + 15 \\
 &= 675
 \end{aligned}$$

(f)  $\sum_{n=11}^{15} (n^2 + 2n)$

**Sol.**

$$\begin{aligned}
 & \sum_{n=11}^{15} (n^2 + 2n) \\
 &= \sum_{n=11}^{15} n^2 + 2 \sum_{n=11}^{15} n \\
 &= \left[ \sum_{n=1}^{15} n^2 - \sum_{n=1}^{10} n^2 \right] \\
 &\quad + 2 \left[ \sum_{n=1}^{15} n - \sum_{n=1}^{10} n \right] \\
 &= \left[ \frac{15 \times 16 \times 31}{6} - \frac{10 \times 11 \times 21}{6} \right] \\
 &\quad + 2 \left[ \frac{15 \times 16}{2} - \frac{10 \times 11}{2} \right] \\
 &= 985
 \end{aligned}$$

(g)  $\sum_{n=2}^6 n(n^2 - n + 1)$

**Sol.**

$$\begin{aligned}
 & \sum_{n=2}^6 n(n^2 - n + 1) \\
 &= \sum_{n=2}^6 n^3 - \sum_{n=2}^6 n^2 + \sum_{n=2}^6 n \\
 &= \left[ \sum_{n=1}^6 n^3 - \sum_{n=1}^1 n^3 \right] - \left[ \sum_{n=1}^6 n^2 - \sum_{n=1}^1 n^2 \right] \\
 &\quad + \left[ \sum_{n=1}^6 n - \sum_{n=1}^1 n \right] \\
 &= \left[ \left( \frac{6 \times 7}{2} \right)^2 - \left( \frac{1 \times 2}{2} \right)^2 \right] \\
 &\quad - \left( \frac{6 \times 7 \times 13}{6} - \frac{1 \times 2 \times 3}{6} \right) \\
 &\quad + \left( \frac{6 \times 7}{2} - \frac{1 \times 2}{2} \right) \\
 &= 21^2 - 1^2 - (7 \times 13 - 1) + (3 \times 7 - 1) \\
 &= 440 - 90 + 20 \\
 &= 370
 \end{aligned}$$

2. Given that the  $n$ th term of a series is  $3n^2 + n$ , find the sum of the first 10 terms of the series.

**Sol.**

$$\begin{aligned}
 \sum_{n=1}^{10} 3n^2 + n &= 3 \sum_{n=1}^{10} n^2 + \sum_{n=1}^{10} n \\
 &= 3 \left( \frac{10 \times 11 \times 21}{6} \right) + \left( \frac{10 \times 11}{2} \right) \\
 &= 3 \times \frac{2310}{6} + \frac{110}{2} \\
 &= 3 \times 385 + 55 \\
 &= 1210
 \end{aligned}$$

3. Find the sum of first  $n$ th term of series  $1 \times 3 + 2 \times 7 + 3 \times 11 + \dots$

**Sol.**

$$\begin{aligned}
 & \sum_{n=1}^n n \times (4n - 1) \\
 &= 4 \sum_{n=1}^n n^2 - \sum_{n=1}^n n \\
 &= 4 \left( \frac{n(n+1)(2n+1)}{6} \right) - \left( \frac{n(n+1)}{2} \right) \\
 &= \frac{4n(n+1)(2n+1) - 3n(n+1)}{6} \\
 &= \frac{n(n+1)(8n+1)}{6}
 \end{aligned}$$

4. Find the sum for the series  $1^2 + 3^2 + 5^2 + \dots + 15^2$

**Sol.**

$$\begin{aligned}
 \sum_{n=1}^8 (2n-1)^2 &= \sum_{n=1}^8 (4n^2 - 4n + 1) \\
 &= 4 \sum_{n=1}^8 n^2 - 4 \sum_{n=1}^8 n + \sum_{n=1}^8 1 \\
 &= 4 \left( \frac{8 \times 9 \times 17}{6} \right) - 4 \left( \frac{8 \times 9}{2} \right) + 8 \\
 &= 4 \times 204 - 4 \times 36 + 8 \\
 &= 816 - 144 + 8 \\
 &= 680
 \end{aligned}$$

## 12.5 Revision Exercise 12

1. Express the following series in form of  $\sum$ .

(a)  $\frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \dots + \frac{49}{50}$

**Sol.**

$$\begin{aligned}
 a_1 &= \frac{2 \times 1 - 1}{2 \times 1} \\
 a_2 &= \frac{2 \times 2 - 1}{2 \times 2} \\
 a_3 &= \frac{2 \times 3 - 1}{2 \times 3} \\
 &\vdots \\
 a_{25} &= \frac{2 \times 25 - 1}{2 \times 25} \\
 \therefore \frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \dots + \frac{49}{50} &= \sum_{n=1}^{25} \frac{2n-1}{2n}
 \end{aligned}$$

(b)  $6 - 7 + 8 - 9 + \dots$

**Sol.**

$$\begin{aligned}
 a_1 &= (-1)^6 \times 6 \\
 a_2 &= (-1)^7 \times 7 \\
 a_3 &= (-1)^8 \times 8 \\
 &\vdots \\
 a_n &= (-1)^n n \therefore 6 - 7 + 8 - 9 + \dots = \sum_{n=1}^{\infty} (-1)^n n
 \end{aligned}$$

(c)  $2 \times 5 + 3 \times 7 + 4 \times 9 + \dots + 15 \times 31$

**Sol.**

$$\begin{aligned}
 a_1 &= (1+1)(2 \times 1 + 3) \\
 a_2 &= (2+1)(2 \times 2 + 3) \\
 a_3 &= (3+1)(2 \times 3 + 3) \\
 &\vdots \\
 a_{14} &= (14+1)(2 \times 14 + 3) \\
 \therefore 2 \times 5 + 3 \times 7 + 4 \times 9 + \dots + 15 \times 31 \\
 &= \sum_{n=1}^{14} (n+1)(2n+3)
 \end{aligned}$$

2. Given a general formula  $a_n = \frac{3^n}{2n-3}$ , state the first 5 terms of the sequence.

**Sol.**

$$\begin{aligned}
 a_1 &= \frac{3^1}{2 \times 1 - 3} = -3 \\
 a_2 &= \frac{3^2}{2 \times 2 - 3} = 9 \\
 a_3 &= \frac{3^3}{2 \times 3 - 3} = 9 \\
 a_4 &= \frac{3^4}{2 \times 4 - 3} = \frac{81}{5} \\
 a_5 &= \frac{3^5}{2 \times 5 - 3} = \frac{243}{7}
 \end{aligned}$$

3. Express the series  $\sum_{k=1}^{10} (2k^2 - 3)$

**Sol.**

$$\begin{aligned} & \sum_{k=1}^{10} (2k^2 - 3) \\ &= (2 \times 1^2 - 3) + (2 \times 2^2 - 3) + (2 \times 3^2 - 3) \\ & \quad + (2 \times 4^2 - 3) + (2 \times 5^2 - 3) + (2 \times 6^2 - 3) \\ & \quad + (2 \times 7^2 - 3) + (2 \times 8^2 - 3) + (2 \times 9^2 - 3) \\ & \quad + (2 \times 10^2 - 3) \\ &= -1 + 5 + 15 + 29 + 47 + 69 + 95 + 125 \\ & \quad + 159 + 197 \end{aligned}$$

4. State the first term, last term and the number of terms of the series  $\sum_{k=3}^7 (3^k - 2^k - k)$

**Sol.**

$$\begin{aligned} a_3 &= 3^3 - 2^3 - 3 = 27 - 8 - 3 = 16 \\ a_7 &= 3^7 - 2^7 - 7 = 2187 - 128 - 7 = 2052 \\ n &= 5 \end{aligned}$$

5. Find the number of terms of the AP  $-4 - 2\frac{3}{4} - 112 - \frac{1}{4} + \dots + 16$

**Sol.**

$$\begin{aligned} a &= -4 \\ d &= \frac{5}{4} \\ 16 &= -4 + (n-1)\frac{5}{4} \\ 20 &= \frac{5}{4}(n-1) \\ 5n - 5 &= 80 \\ 5n &= 85 \\ n &= 17 \end{aligned}$$

6. If  $x+1$ ,  $2x+1$ ,  $x-3$  are the first 3 terms of AP, find:

- (a) The value of  $x$

**Sol.**

$$\begin{aligned} 2x + 1 &= \frac{x + 1 + x - 3}{2} \\ 4x + 2 &= 2x - 2 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

- (b) Sum from the 10th term to the 20th term

**Sol.**

$$\begin{aligned} a_1 &= -1 \\ a_2 &= -3 \\ r &= -2 \\ S &= S_{20} - S_9 \\ &= \frac{20}{2}(-2 + (20-1)(-2)) \\ & \quad - \frac{9}{2}(-2 + (9-1)(-2)) \\ &= 10 \times (-40) - 9 \times (-9) \\ &= -400 + 81 \\ &= -319 \end{aligned}$$

7. Find 4 numbers between 28 and -12 such that these 6 numbers form an AP.

**Sol.**

$$\begin{aligned} a_1 &= 28 \\ a_n &= -12 \\ n &= 6 \\ -12 &= 28 + 5d \\ 5d &= 40 \\ d &= 8 \end{aligned}$$

$\therefore$  These 4 numbers are  $-4, 4, 12, 20$

8. Find the sum of the following AP.

- (a)  $7 + 11 + 15 + \dots$  up to the 10th term

**Sol.**

$$\begin{aligned} a_1 &= 7 \\ d &= 4 \\ n &= 10 \\ S_{10} &= \frac{10}{2}(2 \times 7 + (10-1)4) \\ &= 5(14 + 36) \\ &= 250 \end{aligned}$$

- (b)  $20 + 18\frac{1}{2} + 17 + \dots$  up to the 16th term

**Sol.**

$$\begin{aligned}a_1 &= 20 \\d &= -\frac{3}{2} \\n &= 16 \\S_{16} &= \frac{16}{2}(2 \times 20 + (16 - 1)(-\frac{3}{2})) \\&= 8(40 - \frac{45}{2}) \\&= 8 \times \frac{35}{2} \\&= 140\end{aligned}$$

(c)  $2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} + \dots + 13\sqrt{2}$

**Sol.**

$$\begin{aligned}a_1 &= 2\sqrt{2} \\d &= \sqrt{2} \\n &= 12 \\S_{12} &= \frac{12}{2}(2 \times 2\sqrt{2} + (12 - 1)\sqrt{2}) \\&= 6(4\sqrt{2} + 11\sqrt{2}) \\&= 6 \times 15\sqrt{2} \\&= 90\sqrt{2}\end{aligned}$$

9. Given an AP which the sum of the first  $n$  terms  $S_n = n(1 + 2n)$ , find:

(a) First term

**Sol.**

$$\begin{aligned}\frac{n}{2}(2a + (n - 1)d) &= n(1 + 2n) \\n(2a + (n - 1)d) &= 2n(1 + 2n) \\2an + dn^2 - dn &= 2n - 4n^2 \\(2a - d)n + dn^2 &= 2n - 4n^2\end{aligned}$$

Comparing both sides,

$$\begin{aligned}a &= 3 \\d &= 4\end{aligned}$$

(b) Common Difference

**Sol.**

According to the sol. of (a),  
 $d = 4$

(c) Sum of the first 20 terms.

**Sol.**

According to the sol. of (a),

$$\begin{aligned}a &= 3 \\d &= 4 \\n &= 20 \\S_{20} &= \frac{20}{2}(2 \times 3 + (20 - 1)4) \\&= 10(6 + 76) \\&= 10 \times 82 \\&= 820\end{aligned}$$

10. Given an AP  $33 + 27 + 21 + \dots$

(a) If the first sum of the first  $n$  terms is 105, find the value of  $n$ .

**Sol.**

$$\begin{aligned}a_1 &= 33 \\d &= -6 \\105 &= \frac{n}{2}(2 \times 33 + (n - 1)(-6)) \\210 &= n(66 - (n - 1)6) \\35 &= 11n - n^2 + n \\n^2 - 12n + 35 &= 0 \\(n - 7)(n - 5) &= 0 \\n &= 7 \text{ or } n = 5\end{aligned}$$

(b) If the sum of the first  $n$  terms is negative value, find the minimum value of  $n$ .

**Sol.**

$$\begin{aligned}a_1 &= 33 \\d &= -6 \\\frac{n}{2}(2 \times 33 + (n - 1)(-6)) &< 0 \\n(66 - 6n + 6) &< 0 \\12n - n^2 &< 0 \\n(12 - n) &< 0 \\n &> 12\end{aligned}$$

$\therefore$  The minimum value of  $n$  is 13

11. Find the sum of the numbers between 150 and 300 that are multiple of both 5 and 3.

**Sol.**

$$a_1 = 165$$

$$a_n = 285$$

$$d = 15$$

$$285 = 165 + (n - 1) \times 15$$

$$8 = n - 1$$

$$n = 9$$

$$S_9 = \frac{9}{2}(2 \times 165 + (9 - 1) \times 15)$$

$$= \frac{9}{2} \times 450$$

$$= 2025$$

**Sol.**

$$a_1 = 102$$

$$a_n = 198$$

When  $d = 2$ ,

$$198 = 102 + (n - 1) \times 2$$

$$48 = n - 1$$

$$n = 49$$

$$S_{49} = \frac{49}{2}(2 \times 102 + (49 - 1) \times 2)$$

$$= \frac{49}{2} \times (204 + 96)$$

$$= 7350$$

When  $d = 3$ ,

$$198 = 102 + (n - 1) \times 3$$

$$32 = n - 1$$

$$n = 33$$

$$S_{33} = \frac{33}{2}(2 \times 102 + (33 - 1) \times 3)$$

$$= \frac{33}{2} \times (204 + 96)$$

$$= 4950$$

When  $d = 6$ ,

$$198 = 102 + (n - 1) \times 6$$

$$16 = n - 1$$

$$n = 17$$

$$S_{17} = \frac{17}{2}(2 \times 102 + (17 - 1) \times 6)$$

$$= \frac{17}{2} \times (204 + 96)$$

$$= 2550$$

$$\therefore S = 7350 + 4950 - 2550$$

$$= 9750$$

12. Find the sum of all the numbers between 100 and 200 that can be divided by 2 or 3.

13. Find the sum of the numbers between 50 and 100 that cannot be divided by 5.

**Sol.**

When  $d = 1$ ,

$$a_1 = 51$$

$$a_n = 99$$

$$99 = 51 + (n - 1) \times 1$$

$$48 = n - 1$$

$$n = 49$$

$$S_{49} = \frac{49}{2}(2 \times 51 + (49 - 1) \times 1)$$

$$= \frac{49}{2} \times (102 + 48)$$

$$= 3675$$

When  $d = 5$ ,

$$a_1 = 55$$

$$a_n = 95$$

$$95 = 55 + (n - 1) \times 5$$

$$8 = n - 1$$

$$n = 9$$

$$S_9 = \frac{9}{2}(2 \times 55 + (9 - 1) \times 5)$$

$$= \frac{9}{2} \times (110 + 40)$$

$$= 675$$

$$\therefore S = 3675 - 675$$

$$= 3000$$

14. Which term is the first negative term of the AP  $20 + 16\frac{1}{4} + 12\frac{1}{2} + \dots$ ?

**Sol.**

$$a_1 = 20$$

$$d = -\frac{15}{4}$$

$$a_n = 20 - (n - 1) \times \frac{15}{4} < 0$$

$$80 - 15(n - 1) < 0$$

$$16 - 3n + 3 < 0$$

$$3n > 19$$

$$n > 6\frac{1}{3}$$

$\therefore$  The first negative term is 7

15. Three numbers are in AP, their sum is 15 while the sum of the square of these numbers is 83. Find these three

numbers.

**Sol.**

$$a_1 + a_2 + a_3 = 15$$

$$a_1^2 + a_2^2 + a_3^2 = 83$$

$$a_2 - a_1 = a_3 - a_2$$

$$a_1 + a_3 = 2a_2$$

$$3a_2 = 15$$

$$a_2 = 5$$

$$a_3 = 10 - a_1$$

$$a_1^2 + a_3^2 = 83 - 25$$

$$= 58$$

$$a_1^2 + (10 - a_1)^2 = 58$$

$$a_1^2 + 100 - 20a_1 + a_1^2 = 58$$

$$2a_1^2 - 20a_1 + 100 = 58$$

$$2a_1^2 - 20a_1 + 42 = 0$$

$$a_1^2 - 10a_1 + 21 = 0$$

$$(a_1 - 7)(a_1 - 3) = 0$$

$$a_1 = 7 \text{ or } a_1 = 3$$

$\therefore$  The three numbers are 7, 5, 3

16. Find the sum of the series  $18^2 - 17^2 + 16^2 - 15^2 + 14^2 - 13^2 + \dots + 2^2 - 1^2$

**Sol.**

$$18^2 - 17^2 + 16^2 - 15^2 + \dots + 2^2 - 1^2$$

$$= (18^2 - 17^2) + (16^2 - 15^2) + \dots + (2^2 - 1^2)$$

$$= ((2 \times 9)^2 - (2 \times 9 - 1)^2) + ((2 \times 8)^2 - (2 \times 8 - 1)^2)$$

$$+ \dots + ((2 \times 1)^2 - (2 \times 1 - 1)^2)$$

$$= \sum_{n=1}^9 [(2n)^2 - (2n - 1)^2]$$

$$= \sum_{n=1}^9 (4n - 1)$$

$$= 4 \sum_{n=1}^9 n - \sum_{n=1}^9 1$$

$$= 4 \times \frac{9 \times 10}{2} - 9$$

$$= 180 - 9$$

$$= 171$$

17. State the general formula of the series  $20, -10, 5, -2\frac{1}{2}, \dots$

**Sol.**

$$a_1 = 20$$

$$r = -\frac{1}{2}$$

$$a_n = 20\left(-\frac{1}{2}\right)^{n-1}$$

18. Given three integers  $x-3$ ,  $x+1$ ,  $4x-2$  that are in GP. If the sum of this GP is  $S$ , common ratio is  $r$ , find the value of  $S+r$ .

**Sol.**

$$x+1 = \pm\sqrt{(x-3)(4x-2)}$$

$$x^2 + 2x + 1 = 4x^2 - 14x + 6$$

$$3x^2 - 16x + 5 = 0$$

$$(3x-1)(x-5) = 0$$

$$x = 5 \text{ or } x = \frac{1}{3}$$

$$a_1 = x - 3 = 5 - 3 = 2$$

$$a_2 = x + 1 = 5 + 1 = 6$$

$$a_3 = 4x - 2 = 4(5) - 2 = 18$$

$$S = a_1 + a_2 + a_3$$

$$= 2 + 6 + 18$$

$$= 26$$

$$r = \frac{a_3}{a_2} = \frac{18}{6} = 3$$

$$\therefore S + r = 26 + 3$$

$$= 29$$

19. Find the geometric mean of  $\frac{1}{3}$  and  $\frac{1}{5}$

**Sol.**

$$G = \pm\sqrt{\frac{1}{3} \times \frac{1}{5}}$$

$$= \pm\sqrt{\frac{1}{15}}$$

$$= \pm\frac{1}{\sqrt{15}}$$

$$= \pm\frac{\sqrt{15}}{15}$$

20. Find 5 numbers between  $-\frac{1}{4}$  and  $-\frac{1}{256}$  such that these 7 numbers form a GP.

**Sol.**

$$a_1 = -\frac{1}{4}$$

$$n = 7$$

$$-\frac{1}{256} = -\frac{1}{4}r^6$$

$$\frac{1}{64} = r^6$$

$$\left(\pm\frac{1}{2}\right)^6 = r^6$$

$$r = \pm\frac{1}{2}$$

$$\text{When } r = \frac{1}{2},$$

These 5 numbers are

$$\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$$

$$\text{When } r = -\frac{1}{2},$$

These 5 numbers are

$$\frac{1}{8}, -\frac{1}{16}, \frac{1}{32}, -\frac{1}{64}, \frac{1}{128}$$

21. Find the sum of the series  $\sum_{n=5}^{15} n^2(3n+1)$



**Sol.**

$$\begin{aligned}
 \sum_{n=5}^{15} n^2(3n+1) &= \sum_{n=5}^{15} n^3 + \sum_{n=5}^{15} 3n^2 \\
 &= 3 \sum_{n=5}^{15} n^3 + \sum_{n=5}^{15} n^2 \\
 &= 3 \left[ \sum_{n=1}^{15} n^3 - \sum_{n=1}^4 n^3 \right] \\
 &\quad + \left[ \sum_{n=1}^{15} n^2 - \sum_{n=1}^4 n^2 \right] \\
 &= 3 \left[ \left( \frac{15 \times 16}{2} \right)^2 - \left( \frac{4 \times 5}{2} \right)^2 \right] \\
 &\quad + \left[ \frac{15 \times 16 \times 31}{6} - \frac{4 \times 5 \times 9}{6} \right] \\
 &= 3 [(15 \times 8)^2 - (2 \times 5)^2] \\
 &\quad + 1240 - 30 \\
 &= 3(14400 - 100) + 1210 \\
 &= 42900 + 1210 \\
 &= 44110
 \end{aligned}$$

**Sol.**

$$\begin{aligned}
 &\sum_{n=1}^n (n+1)3n^2 \\
 &= \sum_{n=1}^n 3n^3 + \sum_{n=1}^n 3n^2 \\
 &= 3 \left[ \sum_{n=1}^n n^3 + \sum_{n=1}^n n^2 \right] \\
 &= 3 \left[ \left( \frac{n(n+1)}{2} \right)^2 + \frac{n(n+1)(2n+1)}{6} \right] \\
 &= 3 \left[ \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} \right] \\
 &= 3 \left[ \frac{3n^2(n+1)^2 + 2n(n+1)(2n+1)}{12} \right] \\
 &= \frac{n(n+1) [3n^2 + 3n + 4n + 2]}{4} \\
 &= \frac{n(n+1) [3n^2 + 7n + 2]}{4} \\
 &= \frac{n(n+1)(n+2)(3n+1)}{4}
 \end{aligned}$$

22. Find the sum of the series  $5^2 + 7^2 + 9^2 + \dots + 25^2$

**Sol.**

$$\begin{aligned}
 &\sum_{n=1}^{11} (2n+3)^2 \\
 &= \sum_{n=1}^{11} 4n^2 + 12n + 9 \\
 &= 4 \sum_{n=1}^{11} n^2 + 12 \sum_{n=1}^{11} n + 11 \\
 &= 4 \left[ \frac{11 \times 12 \times 23}{6} \right] + 12 \left[ \frac{11 \times 12}{2} \right] + 99 \\
 &= 2024 + 792 + 99 \\
 &= 2915
 \end{aligned}$$

23. Find the sum of the series  $2 \times 3 + 3 \times 12 + 4 \times 27 + \dots + (n+1) \times 3n^2$

## Chapter 13

# System of Equations

### 13.1 System of Equations with Two Variables

#### 13.1.1 Practice 1

Solve the following system of equations.

1.

$$\begin{cases} 2x - 3y = 11 \\ xy = -5 \end{cases}$$

**Sol.**

$$\begin{cases} 2x - 3y = 11 & (1) \\ xy = -5 & (2) \end{cases}$$

$$(2) \Rightarrow y = -\frac{5}{x} \quad (3)$$

$$\text{Sub (3) into (1)} \Rightarrow 2x - \frac{15}{x} = 11$$

$$2x^2 - 15 = 11x$$

$$2x^2 - 11x - 15 = 0$$

$$(2x - 5)(x + 3) = 0$$

$$x = \frac{5}{2} \text{ or } x = -3$$

$$\text{Sub } x = \frac{5}{2} \text{ into (2)} \Rightarrow y = -\frac{5}{\frac{5}{2}} = -2$$

$$\text{Sub } x = -3 \text{ into (2)} \Rightarrow y = -\frac{5}{-3} = \frac{5}{3}$$

$$\Rightarrow y = \frac{5}{3}$$

$$\Rightarrow y = -1$$

$$\therefore \begin{cases} x = \frac{5}{2} \\ y = -2 \end{cases} \text{ or } \begin{cases} x = -3 \\ y = \frac{5}{3} \end{cases}$$

2.

$$\begin{cases} 3x + y = 5 \\ x^2 - 2xy = 8 \end{cases}$$

**Sol.**

$$\begin{cases} 3x + y = 5 & (1) \\ x^2 - 2xy = 8 & (2) \end{cases}$$

$$3(1) \Rightarrow y = 5 - 3x \quad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow x^2 - 2x(5 - 3x) = 8$$

$$x^2 - 10x + 6x^2 = 8$$

$$7x^2 - 10x + 8 = 0$$

$$(7x + 4)(x - 2) = 0$$

$$x = -\frac{4}{7} \text{ or } x = 2$$

$$\text{Sub } x = -\frac{4}{7} \text{ into (1)} \Rightarrow y = 5 - 3\left(-\frac{4}{7}\right)$$

$$\Rightarrow y = \frac{47}{7}$$

$$\text{Sub } x = 2 \text{ into (1)} \Rightarrow y = -1$$

$$\therefore \begin{cases} x = -\frac{4}{7} \\ y = \frac{47}{7} \end{cases} \text{ or } \begin{cases} x = 2 \\ y = -1 \end{cases}$$

#### 13.1.2 Exercise 13.1

Solve the following system of equations.

1.

$$\begin{cases} x - y = 1 \\ xy = 6 \end{cases}$$

**Sol.**

$$\begin{cases} x - y = 1 & (1) \\ xy = 6 & (2) \end{cases}$$

$$(1) \Rightarrow y = x - 1$$

$$\text{Sub (3) into (2)} \Rightarrow x(x - 1) = 6$$

$$x^2 - x = 6$$

$$x^2 - x - 6 = 0$$

$$(x + 2)(x - 3) = 0$$

$$x = -2 \text{ or } x = 3$$

$$\text{Sub } x = -2 \text{ into (1)} \Rightarrow y = -2 - 1$$

$$\Rightarrow y = -3$$

$$\text{Sub } x = 3 \text{ into (1)} \Rightarrow y = 3 - 1$$

$$\Rightarrow y = 2$$

$$\therefore \begin{cases} x = -2 \\ y = -3 \end{cases} \text{ or } \begin{cases} x = 3 \\ y = 2 \end{cases}$$

2.

$$\begin{cases} 3x - y = 4 \\ xy = 4 \end{cases}$$

**Sol.**

$$\begin{cases} 3x - y = 4 \\ xy = 4 \end{cases}$$

(1)

(2)

$$(1) \Rightarrow y = 3x - 4$$

$$\text{Sub (3) into (2)} \Rightarrow x(3x - 4) = 4$$

$$3x^2 - 4x = 4$$

$$3x^2 - 4x - 4 = 0$$

$$(3x + 2)(x - 2) = 0$$

$$x = -\frac{2}{3} \text{ or } x = 2$$

$$\text{Sub } x = -\frac{2}{3} \text{ into (1)} \Rightarrow y = 3\left(-\frac{2}{3}\right) - 4$$

$$\Rightarrow y = -6$$

$$\text{Sub } x = 2 \text{ into (1)} \Rightarrow y = 3(2) - 4$$

$$\Rightarrow y = 2$$

$$\therefore \begin{cases} x = -\frac{2}{3} \\ y = -6 \end{cases} \text{ or } \begin{cases} x = 2 \\ y = 2 \end{cases}$$

3.

$$\begin{cases} 3x + 4y = -39 \\ xy = 30 \end{cases}$$

(3)

**Sol.**

$$\begin{cases} 3x + 4y = -39 \\ xy = 30 \end{cases} \quad (1)$$

(2)

$$(2) \Rightarrow y = \frac{30}{x} \quad (3)$$

$$\text{Sub (3) into (1)} \Rightarrow 3x + 4\frac{30}{x} = -39$$

$$3x^2 + 120 = -39x$$

$$3x^2 + 39x + 120 = 0$$

$$x^2 + 13x + 40 = 0$$

$$(x + 5)(x + 8) = 0$$

$$x = -5 \text{ or } x = -8$$

$$\text{Sub } x = -5 \text{ into (1)} \Rightarrow y = \frac{30}{-5} - 39$$

$$\Rightarrow y = -6$$

$$\text{Sub } x = -8 \text{ into (1)} \Rightarrow y = \frac{30}{-8} - 39$$

$$\Rightarrow y = -\frac{15}{4}$$

$$\therefore \begin{cases} x = -5 \\ y = -6 \end{cases} \text{ or } \begin{cases} x = -8 \\ y = -\frac{15}{4} \end{cases}$$

4.

$$\begin{cases} y = 2x + 3 \\ y = x^2 - 2x + 1 \end{cases}$$

**Sol.**

$$\begin{cases} y = 2x + 3 \\ y = x^2 \end{cases} \quad (1)$$

(2)

$$(1) = (2) \Rightarrow 2x + 3 = x^2$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow (x + 1)(x - 3) = 0$$

$$\Rightarrow x = -1 \text{ or } x = 3$$

$$\text{Sub } x = -1 \text{ into (1)} \Rightarrow y = 2(-1) + 3$$

$$\Rightarrow y = 1$$

$$\text{Sub } x = 3 \text{ into (1)} \Rightarrow y = 2(3) + 3$$

$$\Rightarrow y = 9$$

$$\therefore \begin{cases} x = -1 \\ y = 1 \end{cases} \text{ or } \begin{cases} x = 3 \\ y = 9 \end{cases}$$

5.

$$\begin{cases} x - y = 1 \\ x^2 + y^2 = 25 \end{cases}$$

**Sol.**

$$\begin{cases} x - y = 1 & (1) \\ x^2 + y^2 = 25 & (2) \end{cases}$$

$$(1) \Rightarrow x = y + 1 \quad (3)$$

$$\begin{aligned} \text{Sub (3) into (2)} &\Rightarrow (y + 1)^2 + y^2 = 25 \\ &\Rightarrow y^2 + 2y + 1 + y^2 = 25 \\ &\Rightarrow 2y^2 + 2y = 24 \\ &\Rightarrow y^2 + y = 12 \\ &\Rightarrow y^2 + y - 12 = 0 \\ &\Rightarrow (y + 4)(y - 3) = 0 \\ &\Rightarrow y = -4 \text{ or } y = 3 \end{aligned}$$

$$\begin{aligned} \text{Sub } y = -4 \text{ into (1)} &\Rightarrow x = -4 + 1 \\ &\Rightarrow x = -3 \end{aligned}$$

$$\begin{aligned} \text{Sub } y = 3 \text{ into (1)} &\Rightarrow x = 3 + 1 \\ &\Rightarrow x = 4 \end{aligned}$$

$$\therefore \begin{cases} x = -3 \\ y = -4 \end{cases} \text{ or } \begin{cases} x = 4 \\ y = 3 \end{cases}$$

6.

$$\begin{cases} 5x - y = 3 \\ y^2 - 6x^2 = 25 \end{cases}$$

**Sol.**

$$\begin{cases} 5x - y = 3 & (1) \\ y^2 - 6x^2 = 25 & (2) \end{cases}$$

$$(1) \Rightarrow y = 5x - 3 \quad (3)$$

$$\begin{aligned} \text{Sub (3) into (2)} &\Rightarrow (5x - 3)^2 - 6x^2 = 25 \\ &\Rightarrow 25x^2 - 30x + 9 \\ &\quad - 6x^2 = 25 \\ &\Rightarrow 19x^2 - 30x + 16 = 0 \\ &\Rightarrow (19x + 8)(x - 2) = 0 \\ &\Rightarrow x = -\frac{8}{19} \text{ or } x = 2 \end{aligned}$$

$$\begin{aligned} \text{Sub } x = -\frac{8}{19} \text{ into (1)} &\Rightarrow y = 5\left(-\frac{8}{19}\right) - 3 \\ &\Rightarrow y = -\frac{97}{19} \end{aligned}$$

$$\text{Sub } x = 2 \text{ into (1)} \Rightarrow y = 7$$

$$\therefore \begin{cases} x = -\frac{8}{19} \\ y = -\frac{97}{19} \end{cases} \text{ or } \begin{cases} x = 2 \\ y = 7 \end{cases}$$

7.

$$\begin{cases} x + y = 3 \\ (x + 2)(y + 3) = 12 \end{cases}$$

**Sol.**

$$\begin{cases} x + y = 3 & (1) \\ (x + 2)(y + 3) = 12 & (2) \end{cases}$$

$$(1) \Rightarrow x = 3 - y \quad (3)$$

$$\begin{aligned} \text{Sub (3) into (2)} &\Rightarrow (3 - y + 2)(y + 3) = 12 \\ &\Rightarrow (5 - y)(y + 3) = 12 \\ &\Rightarrow 5y + 15 - y^2 - 3y = 12 \\ &\Rightarrow 2y - y^2 = -3 \\ &\Rightarrow y^2 - 2y - 3 = 0 \\ &\Rightarrow (y + 1)(y - 3) = 0 \\ &\Rightarrow y = -1 \text{ or } y = 3 \end{aligned}$$

$$\text{Sub } y = -1 \text{ into (1)} \Rightarrow x = 4$$

$$\text{Sub } y = 3 \text{ into (1)} \Rightarrow x = 0$$

$$\therefore \begin{cases} x = 4 \\ y = -1 \end{cases} \text{ or } \begin{cases} x = 0 \\ y = 3 \end{cases}$$

8.

$$\begin{cases} 5x - 6y = -1 \\ 25x^2 + 36y^2 = 61 \end{cases}$$

**Sol.**

$$\begin{cases} 5x - 6y = -1 & (1) \\ 25x^2 + 36y^2 = 61 & (2) \end{cases}$$

$$(1) \Rightarrow y = \frac{5x+1}{6} \quad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow 25x^2 + 36\left(\frac{5x+1}{6}\right)^2 = 61$$

$$\Rightarrow 25x^2 + 36\left(\frac{5x+1}{6}\right)^2 + 36 = 61$$

$$\Rightarrow 25x^2 + 25x^2 + 10x + 1 = 61$$

$$\Rightarrow 50x^2 + 10x = 60$$

$$\Rightarrow 5x^2 + x - 6 = 0$$

$$\Rightarrow (5x+6)(x-1) = 0$$

$$\Rightarrow x = -\frac{6}{5} \text{ or } x = 1$$

$$\text{Sub } x = -\frac{6}{5} \text{ into (1)} \Rightarrow y = \frac{5(-\frac{6}{5})+1}{6}$$

$$\Rightarrow y = -\frac{5}{6}$$

$$\text{Sub } x = 1 \text{ into (1)} \Rightarrow y = \frac{5(1)+1}{6}$$

$$\Rightarrow y = \frac{6}{6}$$

$$\Rightarrow y = 1$$

$$\therefore \begin{cases} x = -\frac{6}{5} \\ y = -\frac{5}{6} \end{cases} \text{ or } \begin{cases} x = 1 \\ y = 1 \end{cases}$$

9.

$$\begin{cases} x + 4y = 5 \\ 2x^2 + 21xy + 27y^2 = 0 \end{cases}$$

**Sol.**

$$\begin{cases} x + 4y = 5 & (1) \\ 2x^2 + 21xy + 27y^2 = 0 & (2) \end{cases}$$

$$(1) \Rightarrow x = 5 - 4y \quad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow 2(5-4y)^2 + 21(5-4y)y + 27y^2 = 0$$

$$\Rightarrow 2(25 - 40y + 16y^2)$$

$$+ 105y - 84y^2 + 27y^2 = 0$$

$$\Rightarrow 50 - 80y + 32y^2 + 105y$$

$$- 57y^2 = 0$$

$$\Rightarrow 25y^2 - 25y - 50 = 0$$

$$\Rightarrow y^2 - y - 2 = 0$$

$$\Rightarrow (y+1)(y-2) = 0$$

$$\Rightarrow y = -1 \text{ or } y = 2$$

$$\text{Sub } y = -1 \text{ into (1)} \Rightarrow x = 5 - 4(-1) = 9$$

$$\text{Sub } y = 2 \text{ into (1)} \Rightarrow x = 5 - 4(2) = -3$$

$$\therefore \begin{cases} x = 9 \\ y = -1 \end{cases} \text{ or } \begin{cases} x = -3 \\ y = 2 \end{cases}$$

10.

$$\begin{cases} \frac{x}{3} - \frac{y}{10} = \frac{5}{6} \\ x(y-2) = 2y+3 \end{cases}$$

**Sol.**

$$\begin{cases} \frac{x}{3} - \frac{y}{10} = \frac{5}{6} & (1) \\ x(y-2) = 2y+3 & (2) \end{cases}$$

$$(1) \Rightarrow 10x - 3y = 25 \quad (3)$$

$$(2) \Rightarrow x = \frac{2y+3}{y-2} \quad (4)$$

$$\text{Sub (4) into (3)} \Rightarrow 10 \left( \frac{2y+3}{y-2} \right) - 3y = 25$$

$$\Rightarrow 10(2y+3) - 3y(y-2) = 25(y-2)$$

$$\Rightarrow 20y + 30 - 3y^2 + 6y = 25y - 50$$

$$\Rightarrow 3y^2 - y - 80 = 0$$

$$\Rightarrow (y+5)(3y-16) = 0$$

$$\Rightarrow y = -5 \text{ or } y = \frac{16}{3}$$

$$\text{Sub } y = -5 \text{ into (1)} \Rightarrow 10x - 3(-5) = 25$$

$$\Rightarrow 10x + 15 = 25$$

$$\Rightarrow 10x = 10$$

$$\Rightarrow x = 1$$

$$\text{Sub } y = \frac{16}{3} \text{ into (1)} \Rightarrow 10x - 3 \left( \frac{16}{3} \right) = 25$$

$$\Rightarrow 10x = 41$$

$$\Rightarrow x = \frac{41}{10}$$

$$\therefore \begin{cases} x = 1 \\ y = -5 \end{cases} \text{ or } \begin{cases} x = \frac{41}{10} \\ y = \frac{16}{3} \end{cases}$$

## 13.2 System of Equations with Three Variables

### 13.2.1 Practice 2

Solve the system of equation

$$\begin{cases} x + 2y - z = -5 \\ 2x - y + z = 6 \\ x - y - 3z = -3 \end{cases}$$

**Sol.**

$$\begin{cases} x + 2y - z = -5 & (1) \\ 2x - y + z = 6 & (2) \\ x - y - 3z = -3 & (3) \end{cases}$$

$$(1) \times 3 \Rightarrow 3x + 6y - 3z = -15 \quad (4)$$

$$(2) \times 3 \Rightarrow 6x - 3y + 3z = 18 \quad (5)$$

$$(3) + (5) \Rightarrow 7x - 4y = 15 \quad (6)$$

$$(4) + (5) \Rightarrow 9x + 3y = 3 \quad (7)$$

$$(6) \times 3 \Rightarrow 21x - 12y = 45 \quad (8)$$

$$(7) \times 4 \Rightarrow 36x + 12y = 12 \quad (9)$$

$$(8) + (9) \Rightarrow 57x = 57 \quad (10)$$

$$\Rightarrow x = 1$$

$$\text{Sub } x = 1 \text{ into (7)} \Rightarrow -4y = 8$$

$$\Rightarrow y = -2$$

$$\text{Sub } y = -2 \text{ and } x = 1 \text{ into (1)} \Rightarrow -z = -2$$

$$\Rightarrow z = 2$$

$$\therefore x = 1, y = -2, z = 2$$

### 13.2.2 Exercise 13.2

Solve the following system of equations.

1.

$$\begin{cases} x + y - z = 1 \\ 2x - 3y + z = 0 \\ 2x + y + 2z = 5 \end{cases}$$

**Sol.**

$$\begin{cases} x + y - z = 1 & (1) \\ 2x - 3y + z = 0 & (2) \\ 2x + y + 2z = 5 & (3) \end{cases}$$

$$(1) \times 2 \Rightarrow 2x + 2y - 2z = 2 \quad (4)$$

$$(4) - (3) \Rightarrow y - 4z = -3 \quad (5)$$

$$(3) - (2) \Rightarrow 4y + z = 5 \quad (6)$$

$$(5) \times 4 \Rightarrow 4y - 16z = -12 \quad (7)$$

$$(6) - (7) \Rightarrow 17z = 17$$

$$\Rightarrow z = 1$$

$$\text{Sub } z = 1 \text{ into (5)} \Rightarrow y = 1$$

$$\text{Sub } y = 1 \text{ and } z = 1 \text{ into (1)} \Rightarrow x = 1$$

$$\therefore x = 1, y = 1, z = 1$$

2.

$$\begin{cases} x - 2y = 5 \\ 2x + y - 3z = 8 \\ x + 4y - z = 0 \end{cases}$$

**Sol.**

$$\begin{cases} x - 2y = 5 & (1) \\ 2x + y - 3z = 8 & (2) \\ x + 4y - z = 0 & (3) \end{cases}$$

$$(3) \times 3 \Rightarrow 3x + 12y - 3z = 0 \quad (4)$$

$$(4) - (2) \Rightarrow x + 11y = -8 \quad (5)$$

$$(5) - (1) \Rightarrow 13y = -13$$

$$\Rightarrow y = -1$$

$$\text{Sub } y = -1 \text{ into (1)} \Rightarrow x + 2 = 5$$

$$\Rightarrow x = 3$$

$$\text{Sub } x = 3$$

$$\text{and } y = -1 \text{ into (2)} \Rightarrow -3z = 3$$

$$\Rightarrow z = -1$$

$$\therefore x = 3, y = -1, z = -1$$

3.

$$\begin{cases} x + y = z - 5 \\ y + z = x - 3 \\ z + x = y + 1 \end{cases}$$

**Sol.**

$$\begin{cases} x + y = z - 5 & (1) \\ y + z = x - 3 & (2) \\ z + x = y + 1 & (3) \end{cases}$$

$$(1) \Rightarrow x + y - z = -5 \quad (4)$$

$$(2) \Rightarrow -x + y + z = -3 \quad (5)$$

$$(3) \Rightarrow x - y + z = 1 \quad (6)$$

$$(4) + (5) \Rightarrow 2y = -8$$

$$\Rightarrow y = -4$$

$$(5) + (6) \Rightarrow 2z = -2$$

$$\Rightarrow z = -1$$

$$\text{Sub } y = -4$$

$$\text{and } z = -1 \text{ into (2)} \Rightarrow x - 3 = -5$$

$$\Rightarrow x = -2$$

$$\therefore x = -2, y = -4, z = -1$$

4.

$$\begin{cases} x + 4y + 2z = 4 \\ 2x - 2y + z = 4 \\ x - 2y + 3z = 3 \end{cases}$$

**Sol.**

$$\begin{cases} x + 4y + 2z = 4 & (1) \\ 2x - 2y + z = 4 & (2) \\ x - 2y + 3z = 3 & (3) \end{cases}$$

$$(1) \times 2 \Rightarrow 2x + 8y + 4z = 8 \quad (4)$$

$$(3) \times 2 \Rightarrow 2x - 4y + 6z = 6 \quad (5)$$

$$(4) - (2) \Rightarrow 10y + 3z = 4 \quad (6)$$

$$(5) - (4) \Rightarrow -12y + 2z = -2 \quad (7)$$

$$(6) \times 2 \Rightarrow 20y + 6z = 8 \quad (8)$$

$$(7) \times 3 \Rightarrow -36y + 6z = -6 \quad (9)$$

$$(8) - (9) \Rightarrow 56y = 14$$

$$\Rightarrow y = \frac{1}{4}$$

$$\text{Sub } y = \frac{1}{4} \text{ into (6)} \Rightarrow 6z = 3$$

$$\Rightarrow z = \frac{1}{2}$$

$$\text{Sub } y = \frac{1}{4} \text{ and } z = \frac{1}{2} \text{ into (1)} \Rightarrow x + 1 + 1 = 4$$

$$\Rightarrow x = 2$$

$$\therefore x = 2, y = \frac{1}{4}, z = \frac{1}{2}$$

5.

$$\begin{cases} x - y - z = 0 \\ 3x + 2y = 13 \\ y - 3z = -1 \end{cases}$$

**Sol.**

$$\begin{cases} x - y - z = 0 & (1) \\ 3x + 2y = 13 & (2) \\ y - 3z = -1 & (3) \end{cases}$$

$$\begin{aligned}
 (3) &\Rightarrow y = 3z - 1 & (4) \\
 \text{Sub (4) into (1)} &\Rightarrow x - (3z - 1) - z = 0 \\
 &\Rightarrow x - 4z = -1 & (5) \\
 \text{Sub (4) into (2)} &\Rightarrow 3x + 2(3z - 1) = 13 \\
 &\Rightarrow 3x + 6z = 15 & (6) \\
 (5) \times 3 &\Rightarrow 3x - 12z = -3 & (7) \\
 (6) - (7) &\Rightarrow 18z = 18 \\
 &\Rightarrow z = 1 \\
 \text{Sub } z = 1 \text{ into (4)} &\Rightarrow y = 2 \\
 \text{Sub } z = 1 \text{ into (5)} &\Rightarrow x - 4 = -1 \\
 &\Rightarrow x = 3 \\
 \therefore x = 3, y = 2, z = 1
 \end{aligned}$$

6.

$$\begin{cases} 2x + 2y - z = -1 \\ x + 3y + z = -8 \\ 3x - 2y + 3z = 9 \end{cases}$$

**Sol.**

$$\begin{cases} 2x + 2y - z = -1 & (1) \\ x + 3y + z = -8 & (2) \\ 3x - 2y + 3z = 9 & (3) \end{cases}$$

$$\begin{aligned}
 (1) \times 3 &\Rightarrow 6x + 6y - 3z = -3 & (4) \\
 (2) \times 3 &\Rightarrow 3x + 9y + 3z = -24 & (5) \\
 (3) + (4) &\Rightarrow 9x + 4y = 6 & (6) \\
 (4) + (5) &\Rightarrow 9x + 15y = -27 & (7) \\
 (7) - (6) &\Rightarrow 11y = -33 \\
 &\Rightarrow y = -3 \\
 \text{Sub } y = -3 \text{ into (6)} &\Rightarrow 9x = 18 \\
 &\Rightarrow x = 2 \\
 \text{Sub } x = 2 & \\
 \text{and } y = -3 \text{ into (2)} &\Rightarrow -7 + z = -8 \\
 &\Rightarrow z = -1 \\
 \therefore x = 2, y = -3, z = -1
 \end{aligned}$$

7.

$$\begin{cases} \frac{3}{x} + \frac{1}{y} + \frac{4}{z} = 0 \\ \frac{1}{x} + \frac{4}{y} - \frac{2}{z} = 4 \\ \frac{2}{x} - \frac{3}{y} - \frac{1}{z} = -11 \end{cases}$$

**Sol.**

$$\begin{cases} \frac{3}{x} + \frac{1}{y} + \frac{4}{z} = 0 & (1) \\ \frac{1}{x} + \frac{4}{y} - \frac{2}{z} = 4 & (2) \\ \frac{2}{x} - \frac{3}{y} - \frac{1}{z} = -11 & (3) \end{cases}$$

$$\text{Let } u = \frac{1}{x}, v = \frac{1}{y}, w = \frac{1}{z}$$

$$(1) \Rightarrow 3u + v + 4w = 0 \quad (4)$$

$$(2) \Rightarrow u + 4v - 2w = 4 \quad (5)$$

$$(3) \Rightarrow 2u - 3v - w = -11 \quad (6)$$

$$(5) \times 2 \Rightarrow 2u + 8v - 4w = 8 \quad (7)$$

$$(6) \times 4 \Rightarrow 8u - 12v - 4w = -44 \quad (8)$$

$$(4) + (7) \Rightarrow 5u + 9v = 8 \quad (9)$$

$$(4) + (8) \Rightarrow 11u - 11v = -44$$

$$\Rightarrow u - v = -4 \quad (10)$$

$$(10) \times 5 \Rightarrow 5u - 5v = -20 \quad (11)$$

$$(9) - (11) \Rightarrow 14v = 28 \quad (12)$$

$$\Rightarrow v = 2$$

$$\text{Sub } v = 2 \text{ into (10)} \Rightarrow u = -2$$

$$\text{Sub } u = -2$$

$$\text{and } v = 2 \text{ into (4)} \Rightarrow -4 + 4w = 0$$

$$\Rightarrow w = 1$$

$$\therefore u = -2, v = 2, w = 1$$

$$\therefore x = -\frac{1}{2}, y = \frac{1}{2}, z = 1$$

### 13.3 Revision Exercise 13

Solve the following system of equations.

1.

$$\begin{cases} 3x + 4y = 24 \\ xy = 12 \end{cases}$$

**Sol.**

$$\begin{cases} 3x + 4y = 24 & (1) \\ xy = 12 & (2) \end{cases}$$



$$(2) \Rightarrow y = \frac{12}{x} \quad (3)$$

$$\text{Sub (3) into (1)} \Rightarrow 3x + 4\left(\frac{12}{x}\right) = 24$$

$$\Rightarrow 3x^2 + 48 = 24x$$

$$\Rightarrow x^2 - 8x + 16 = 0$$

$$\Rightarrow (x - 4)^2 = 0$$

$$\Rightarrow x = 4, x = -4$$

$$\text{Sub } x = 4 \text{ into (3)} \Rightarrow y = \frac{12}{4} = 3$$

$$\text{Sub } x = -4 \text{ into (3)} \Rightarrow y = \frac{12}{-4} = -3$$

$$\therefore \begin{cases} x = 4 \\ y = 3 \end{cases} \quad \text{or} \quad \begin{cases} x = -4 \\ y = -3 \end{cases}$$

2.

$$\begin{cases} x + 2y = 5 \\ 5x^2 + 4y^2 + 12x = 29 \end{cases}$$

**Sol.**

$$\begin{cases} x + 2y = 5 \\ 5x^2 + 4y^2 + 12x = 29 \end{cases} \quad (1)$$

$$(1) \Rightarrow x = 5 - 2y \quad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow 5(5 - 2y)^2 + 4y^2 + 12(5 - 2y) = 29$$

$$\Rightarrow 5(25 - 20y + 4y^2) + 4y^2 + 60 - 24y = 29$$

$$\Rightarrow 125 - 100y + 20y^2 + 4y^2 + 60 - 24y = 29$$

$$\Rightarrow 125 - 100y + 20y^2 + 4y^2 + 60 - 24y = 29$$

$$\Rightarrow 24y^2 + 124y + 156 = 0$$

$$\Rightarrow 6y^2 + 31y + 39 = 0$$

$$\Rightarrow (y - 3)(6y - 13) = 0$$

$$\Rightarrow y = 3, y = \frac{13}{6}$$

$$\text{Sub } y = 3 \text{ into (1)} \Rightarrow x = 5 - 2(3) = -1$$

$$\text{Sub } y = \frac{13}{6} \text{ into (1)} \Rightarrow x = 5 - 2\left(\frac{13}{6}\right) = \frac{2}{3}$$

$$\therefore \begin{cases} x = -1 \\ y = 3 \end{cases} \quad \text{or} \quad \begin{cases} x = \frac{2}{3} \\ y = \frac{13}{6} \end{cases}$$

3.

$$\begin{cases} 2x + y = 7 \\ x^2 - xy + y^2 = 7 \end{cases}$$

**Sol.**

$$\begin{cases} 2x + y = 7 \\ x^2 - xy + y^2 = 7 \end{cases} \quad (1)$$

$$(2)$$

$$(1) \Rightarrow y = 7 - 2x \quad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow x^2 - x(7 - 2x) + (7 - 2x)^2 = 7$$

$$\Rightarrow x^2 - 7x + 2x^2 - 28x + 49 + 4x^2 = 7$$

$$\Rightarrow 7x^2 - 35x + 42 = 0$$

$$\Rightarrow x^2 - 5x + 6 = 0$$

$$\Rightarrow (x - 2)(x - 3) = 0$$

$$\Rightarrow x = 2, x = 3$$

$$\text{Sub } x = 2 \text{ into (3)} \Rightarrow y = 7 - 2(2) = 3$$

$$\text{Sub } x = 3 \text{ into (3)} \Rightarrow y = 7 - 2(3) = 1$$

$$\therefore \begin{cases} x = 2 \\ y = 3 \end{cases} \quad \text{or} \quad \begin{cases} x = 3 \\ y = 1 \end{cases}$$

4.

$$\begin{cases} 2x + 3y = 7 \\ x^2 + xy + y^2 = 7 \end{cases}$$

**Sol.**

$$\begin{cases} 2x + 3y = 7 \\ x^2 + xy + y^2 = 7 \end{cases} \quad (1)$$

$$(2)$$

$$(1) \Rightarrow y = \frac{7-2x}{3} \quad (3)$$

$$\begin{aligned} \text{Sub (3) into (2)} &\Rightarrow x^2 + x\left(\frac{7-2x}{3}\right) \\ &\quad + \left(\frac{7-2x}{3}\right)^2 = 7 \\ &\Rightarrow x^2 + \frac{7x-2x^2}{3} \\ &\quad + \frac{49-28x+4x^2}{9} = 7 \end{aligned}$$

$$\begin{aligned} &\Rightarrow 9x^2 + 21x - 6x^2 + 49 \\ &\quad - 28x + 4x^2 = 63 \\ &\Rightarrow 7x^2 - 7x - 14 = 0 \\ &\Rightarrow x^2 - x - 2 = 0 \\ &\Rightarrow (x+1)(x-2) = 0 \\ &\Rightarrow x = -1, x = 2 \end{aligned}$$

$$\text{Sub } x = -1 \text{ into (3)} \Rightarrow y = \frac{7-2(-1)}{3} = 3$$

$$\text{Sub } x = 2 \text{ into (3)} \Rightarrow y = \frac{7-2(2)}{3} = 1$$

$$\therefore \begin{cases} x = -1 \\ y = 3 \end{cases} \quad \text{or} \quad \begin{cases} x = 2 \\ y = 1 \end{cases}$$

5.

$$\begin{cases} 4x - 3y + 2 = 0 \\ 2y + 5z - 19 = 0 \\ 5x - 7z + 16 = 0 \end{cases}$$

**Sol.**

$$\begin{cases} 4x - 3y + 2 = 0 & (1) \\ 2y + 5z - 19 = 0 & (2) \\ 5x - 7z + 16 = 0 & (3) \end{cases}$$

$$(1) \times 2 \Rightarrow 8x - 6y + 4 = 0 \quad (4)$$

$$(2) \times 3 \Rightarrow 6y + 15z - 57 = 0 \quad (5)$$

$$(4) + (5) \Rightarrow 8x + 15z - 53 = 0 \quad (6)$$

$$(3) \times 8 \Rightarrow 40x - 56z + 128 = 0 \quad (7)$$

$$(6) \times 5 \Rightarrow 40x + 75z - 265 = 0 \quad (8)$$

$$(7) - (8) \Rightarrow -131z + 393 = 0 \quad (9)$$

$$\Rightarrow 131z = 393$$

$$\Rightarrow z = 3$$

$$\text{Sub } z = 3 \text{ into (8)} \Rightarrow 40x + 75(3) - 265 = 0$$

$$\Rightarrow 40x + 225 - 265 = 0$$

$$\Rightarrow 40x - 40 = 0$$

$$\Rightarrow x = 1$$

$$\text{Sub } z = 3 \text{ into (2)} \Rightarrow 6y - 12 = 0$$

$$\Rightarrow y = 2$$

$$\therefore x = 1, y = 2, z = 3$$

6.

$$\begin{cases} x + y + z = 9 \\ 3x + y - 2z = 1 \\ x - 2y + z = 0 \end{cases}$$

**Sol.**

$$\begin{cases} x + y + z = 9 & (1) \\ 3x + y - 2z = 1 & (2) \\ x - 2y + z = 0 & (3) \end{cases}$$

$$(1) \Rightarrow x + z = 9 - y \quad (4)$$

$$\text{Sub (4) into (3)} \Rightarrow 9 - y - 2y = 0$$

$$\Rightarrow 3y = 9$$

$$\Rightarrow y = 3$$

$$\text{Sub } y = 3 \text{ into (2)} \Rightarrow 3x - 2z = -2 \quad (5)$$

$$\text{Sub } y = 3 \text{ into (3)} \Rightarrow x + z = 6 \quad (6)$$

$$(6) \times 2 \Rightarrow 2x + 2z = 12 \quad (7)$$

$$(5) + (7) \Rightarrow 5x = 10$$

$$\Rightarrow x = 2$$

$$\text{Sub } x = 2 \text{ into (6)} \Rightarrow z = 4$$

$$\therefore x = 2, y = 3, z = 4$$

7.

$$\begin{cases} 2x - 3y - z = 4 \\ 4x + y + 2z = 3 \\ x - 4y - 3z = 2 \end{cases}$$

**Sol.**

$$\begin{cases} 2x - 3y - z = 4 & (1) \\ 4x + y + 2z = 3 & (2) \\ x - 4y - 3z = 2 & (3) \end{cases}$$

$$(1) \times 2 \Rightarrow 4x - 6y - 2z = 8 \quad (4)$$

$$(3) \times 4 \Rightarrow 4x - 16y - 12z = 8 \quad (5)$$

$$(2) - (4) \Rightarrow 7y + 4z = -5 \quad (6)$$

$$(4) - (5) \Rightarrow 10y + 10z = 0$$

$$\Rightarrow y + z = 0$$

$$\Rightarrow y = -z$$

$$\text{Sub } y = -z \text{ into (6)} \Rightarrow 7(-z) + 4z = -5$$

$$\Rightarrow 3z = 5$$

$$\Rightarrow z = \frac{5}{3}$$

$$y = -z \Rightarrow y = -\frac{5}{3}$$

$$\text{Sub } y = -\frac{5}{3}$$

$$\text{and } z = \frac{5}{3} \text{ into (1)} \Rightarrow 2x - 3(-\frac{5}{3}) - \frac{5}{3} = 4$$

$$\Rightarrow 2x - \frac{5}{3} = -1$$

$$\Rightarrow 2x = \frac{2}{3}$$

$$\Rightarrow x = \frac{1}{3}$$

$$\therefore x = \frac{1}{3}, y = -\frac{5}{3}, z = \frac{5}{3}$$

8.

$$\begin{cases} \frac{3}{x+1} - \frac{1}{y+2} + \frac{1}{z-1} = 2 \\ \frac{2}{x+1} - \frac{3}{y+2} - \frac{1}{z-1} = 7 \\ \frac{1}{x+1} + \frac{1}{y+2} - \frac{4}{z-1} = 8 \end{cases}$$

**Sol.**

$$\begin{cases} \frac{3}{x+1} - \frac{1}{y+2} + \frac{1}{z-1} = 2 & (1) \end{cases}$$

$$\begin{cases} \frac{2}{x+1} - \frac{3}{y+2} - \frac{1}{z-1} = 7 & (2) \end{cases}$$

$$\begin{cases} \frac{1}{x+1} + \frac{1}{y+2} - \frac{4}{z-1} = 8 & (3) \end{cases}$$

$$\text{Let } u = \frac{1}{x+1}, v = \frac{1}{y+2}, w = \frac{1}{z-1}$$

$$(1) \Rightarrow 3u - v + w = 2 \quad (4)$$

$$(2) \Rightarrow 2u - 3v - w = 7 \quad (5)$$

$$(3) \Rightarrow u + v - 4w = 8 \quad (6)$$

$$(4) \times 3 \Rightarrow 9u - 3v + 3w = 6 \quad (7)$$

$$(6) \times 3 \Rightarrow 3u + 3v - 12w = 24 \quad (8)$$

$$(5) + (8) \Rightarrow 5u - 13w = 31 \quad (9)$$

$$(7) + (8) \Rightarrow 12u - 9w = 30$$

$$\Rightarrow 4u - 3w = 10 \quad (10)$$

$$(9) \times 4 \Rightarrow 20u - 52w = 124 \quad (11)$$

$$(10) \times 5 \Rightarrow 20u - 15w = 50 \quad (12)$$

$$(12) - (11) \Rightarrow 37w = -74 \quad (13)$$

$$\Rightarrow w = -2$$

$$\text{Sub } w = -2 \text{ into (10)} \Rightarrow 4u = 4$$

$$\Rightarrow u = 1$$

$$\text{Sub } u = 1$$

$$\text{and } w = -2 \text{ into (6)} \Rightarrow 9 + v = 8$$

$$\Rightarrow v = -1$$

$$\therefore u = 1, v = -1, w = -2$$

$$\therefore x = 0, y = -3, z = \frac{1}{2}$$

## Chapter 14

# Matrix and Determinant

### 14.1 Matrix

#### Definition of Matrix

A matrix is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. It is generally denoted as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where  $m$  is the number of rows and  $n$  is the number of columns.

Each number in the matrix is called *an entry of the matrix*, the number in the  $i^{th}$  row and  $j^{th}$  column is denoted as  $a_{ij}$ . Thus, a matrix can also be denoted as  $A = (a_{ij})$ , or  $A = (a_{ij})_{mn}$  where  $m$  is the number of rows and  $n$  is the number of columns.

A matrix with  $m$  rows and  $n$  columns is called an  $m \times n$  matrix, where  $m \times n$  is called the *order of the matrix*. For

example, the following matrix is a  $3 \times 4$  matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

When  $m = n$ , the matrix is called a *square matrix*. For example, the following matrix is a **third-order square matrix**:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

When  $m = 1$ , the matrix is called a *row matrix*. For example, the following matrix is a **row matrix**:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

When  $n = 1$ , the matrix is called a *column matrix*. For example, the following matrix is a **column matrix**:

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

#### Equal Matrices

Two matrices  $A$  and  $B$  are equal if they have the same order and the same entries. That is,  $A = B$  if and only if  $A_{ij} = B_{ij}$  for all  $i$  and  $j$ .

## Zero Matrix

The matrix with all entries equal to zero is called the *zero matrix* and is denoted as  $O$ . Zero matrix can be in any order.

For example, the matrix below is a  $2 \times 2$  **zero matrix** or a **second-order square zero matrix**:

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

## Identity Matrix

The matrix with all entries equal to zero except the entries on the main diagonal, which are equal to one, is called the *identity matrix* and is denoted as  $I$ . Identity matrix can be in any order. The form of an identity matrix is:

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

## Transpose Matrix

The transpose of a matrix  $A$  is denoted as  $A'$ ,  $A^t$  or  $A^T$  and is obtained by interchanging the rows and columns of  $A$ . For example, given the matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

The transpose of  $A$  is:

$$A' = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Thus, we know that the transpose matrix of  $m \times n$  matrix is a  $n \times m$  matrix.

### 14.1.1 Exercise 14.1

1. State the order of the following matrices.

(a)  $A = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

**Sol.**  $A$  is a matrix with order  $3 \times 1$ .

(b)  $B = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 4 & 2 & 1 & 5 \end{bmatrix}$

**Sol.**  $B$  is a matrix with order  $2 \times 4$ .

(c)  $C = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 3 & 0 \\ 2 & 1 & -1 \end{bmatrix}$

**Sol.**  $C$  is a matrix with order  $3 \times 3$ .

2. Given  $A = \begin{bmatrix} 1 & 5 & -2 & 4 \\ 2 & -4 & 3 & 1 \\ 0 & 6 & 4 & 7 \end{bmatrix}$ , what is  $a_{23}$  and  $a_{34}$ ?

**Sol.**  $a_{23} = 3$  and  $a_{34} = 7$ .

3. If  $\begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & x \end{bmatrix}$ , what is  $x$ ?

**Sol.**  $x = -4$ .

## 14.2 Matrix Addition and Subtraction

Given two matrices  $A$  and  $B$  of the same order, the sum of  $A$  and  $B$  is defined as the matrix  $A + B$  whose  $(i, j)$ -th entry is the sum of the  $(i, j)$ -th entries of  $A$  and  $B$ . That is:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

The difference of  $A$  and  $B$  is defined as the matrix  $A - B$  whose  $(i, j)$ -th entry is the difference of the  $(i, j)$ -th entries of  $A$  and  $B$ . That is:

$$A - B = \begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdots & a_{2n} - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \cdots & a_{mn} - b_{mn} \end{bmatrix}$$

Note that the order of  $A$  and  $B$  must be the same. For example, the following matrices cannot be added or subtracted:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

The addition of matrices has the following properties:

- Commutative:  $A + B = B + A$ .
- Associative:  $(A + B) + C = A + (B + C)$ .
- Identity:  $A \pm O = A$ .
- Inverse:  $A + (-A) = O$ .
- Transpose:  $(A \pm B)' = A' \pm B'$ .

where  $A, B, C$  are matrices of the same order and  $O$  is the zero matrix of the same order as  $A$ .

Given a matrix  $A$ , if  $A = A'$ , then  $A$  is called a *symmetric matrix*. If  $A = -A'$ , then  $A$  is called an *antisymmetric matrix*.

For any given matrix  $A$ ,  $A + A'$  is symmetric, and  $A - A'$  is antisymmetric.

### 14.2.1 Practice 1

Let  $A = \begin{bmatrix} -4 & 2 & -7 \\ 5 & 4 & 0 \\ 3 & -2 & -3 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & -5 \\ 4 & -1 & 1 \end{bmatrix}$ . Find the following:

1.  $A + B'$ .

**Sol.**

$$B' = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 1 & -1 \\ 2 & -5 & 1 \end{bmatrix}$$

$$A + B' = \begin{bmatrix} -3 & 5 & -3 \\ 8 & 5 & -1 \\ 5 & -7 & -2 \end{bmatrix}$$

2.  $(A - B)'$

**Sol.**

$$A - B = \begin{bmatrix} -5 & -1 & -9 \\ 2 & 3 & 5 \\ -1 & -1 & -4 \end{bmatrix}$$

$$(A - B)' = \begin{bmatrix} -5 & 2 & -1 \\ -1 & 3 & -1 \\ -9 & 5 & -4 \end{bmatrix}$$

### 14.2.2 Exercise 14.2

Let  $P = \begin{bmatrix} -5 & 4 & 2 \\ 6 & -4 & 3 \\ -2 & 1 & 6 \end{bmatrix}$  and  $Q = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ . Evaluate the following:

1.  $(P + Q)'$

**Sol.**

$$P + Q = \begin{bmatrix} -4 & 2 & 2 \\ 9 & -2 & 4 \\ -2 & 1 & 10 \end{bmatrix}$$

$$\therefore (P + Q)' = \begin{bmatrix} -4 & 9 & -2 \\ 2 & -2 & 1 \\ 2 & 4 & 10 \end{bmatrix}$$

2.  $Q' - P'$

**Sol.**

$$Q - P = \begin{bmatrix} 6 & -6 & 2 \\ -3 & 6 & -2 \\ 2 & -1 & -2 \end{bmatrix}$$

$$\therefore Q' - P' = (Q - P)' = \begin{bmatrix} 6 & -3 & 2 \\ -6 & 6 & -1 \\ 2 & -2 & -2 \end{bmatrix}$$

3.  $(P' - Q)'$

**Sol.**

$$P' = \begin{bmatrix} -5 & 6 & -2 \\ 4 & -4 & 1 \\ 2 & 3 & 6 \end{bmatrix}$$

$$P' - Q = \begin{bmatrix} -6 & 8 & -2 \\ 1 & -6 & 0 \\ 2 & 3 & 2 \end{bmatrix}$$

$$\therefore (P' - Q)' = \begin{bmatrix} -6 & 1 & 2 \\ 8 & -6 & 3 \\ -2 & 0 & 2 \end{bmatrix}$$

4.  $P' - (I - Q)'$

**Sol.**

$$\begin{aligned} P' - (I - Q)' &= P' - I' + Q' \\ &= (P + Q)' - I' \\ &= (P + Q - I)' \end{aligned}$$

$$\begin{aligned} P + Q - I &= \begin{bmatrix} -4 & 2 & 2 \\ 9 & -2 & 4 \\ -2 & 1 & 10 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 2 & 2 \\ 9 & -3 & 4 \\ -2 & 1 & 9 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \therefore P' - (I - Q)' &= (P + Q - I)' \\ &= \begin{bmatrix} -5 & 9 & -2 \\ 2 & -3 & 1 \\ 2 & 4 & 9 \end{bmatrix} \end{aligned}$$

## 14.3 Scalar Product of Matrices

Let  $A = (a_{ij})_{m \times n}$  be an  $m \times n$  matrix,  $k$  be any real number, then  $kA = (ka_{ij})_{m \times n}$ . This is called scalar product of a matrix  $A$  and scalar  $k$ . For example:

$$k \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} k & 2k & 3k \\ 4k & 5k & 6k \end{bmatrix}$$

The scalar product of a matrix has the following properties:

- $r(A + B) = rA + sB$
- $(r + s)A = rA + sA$
- $(rs)A = r(sA)$

### 14.3.1 Practice 2

Let  $A = \begin{bmatrix} 2 & 0 \\ -3 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & -1 \\ 2 & -4 \end{bmatrix}$ . Evaluate the following:

1.  $3A + B$

**Sol.**

$$\begin{aligned} 3A + B &= \begin{bmatrix} 6 & 0 \\ -9 & 15 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & -4 \end{bmatrix} \\ &= \begin{bmatrix} 7 & -1 \\ -7 & 11 \end{bmatrix} \end{aligned}$$

2.  $2A - 3B$

**Sol.**

$$\begin{aligned} 2A - 3B &= \begin{bmatrix} 4 & 0 \\ -6 & 10 \end{bmatrix} - \begin{bmatrix} 3 & -3 \\ 6 & -12 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ -12 & 22 \end{bmatrix} \end{aligned}$$

3.  $4B - 2A$

**Sol.**

$$\begin{aligned} 4B - 2A &= 2(2B - A) \\ &= 2 \left( \begin{bmatrix} 2 & -2 \\ 4 & -8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ -3 & 5 \end{bmatrix} \right) \\ &= 2 \begin{bmatrix} 0 & -2 \\ 7 & -13 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -4 \\ 14 & -26 \end{bmatrix} \end{aligned}$$

4.  $A' - 2B'$

**Sol.**

$$\begin{aligned}A' - 2B' &= (A - 2B)' \\&= \left( \begin{bmatrix} 2 & 0 \\ -3 & 5 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 4 & -8 \end{bmatrix} \right)' \\&= \left( \begin{bmatrix} 0 & 2 \\ -7 & 13 \end{bmatrix} \right)' \\&= \begin{bmatrix} 0 & -7 \\ 2 & 13 \end{bmatrix}\end{aligned}$$

### 14.3.2 Exercise 14.3

1. Let  $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$ ,  $C = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ , Calculate the following:

(a)  $2A - 3B + 4C$

**Sol.**

$$\begin{aligned}2A - 3B + 4C &= \begin{bmatrix} 4 & 6 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 18 & 3 \\ 9 & 6 \end{bmatrix} + \begin{bmatrix} 12 & 4 \\ 4 & 0 \end{bmatrix} \\&= \begin{bmatrix} 16 & 10 \\ 6 & 0 \end{bmatrix} - \begin{bmatrix} 18 & 3 \\ 9 & 6 \end{bmatrix} \\&= \begin{bmatrix} -2 & 7 \\ -3 & -6 \end{bmatrix}\end{aligned}$$

(b)  $4A' - (C + B)'$

**Sol.**

$$\begin{aligned}4A' - (C + B)' &= 4 \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} - \left( \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 3 \\ 1 & 2 \end{bmatrix} \right) \\&= \begin{bmatrix} 8 & 4 \\ 12 & 0 \end{bmatrix} - \begin{bmatrix} 9 & 4 \\ 2 & 2 \end{bmatrix} \\&= \begin{bmatrix} -1 & 0 \\ 10 & -2 \end{bmatrix}\end{aligned}$$

2. Let  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{bmatrix}$ ,  $C = \begin{bmatrix} 4 & -1 \\ 3 & 1 \\ 2 & -3 \end{bmatrix}$ , evaluate the following:

(a)  $3A + B - 2C$

**Sol.**

$$\begin{aligned}3A + B - 2C &= \begin{bmatrix} 3 & 6 \\ 0 & 3 \\ 9 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 8 & -2 \\ 6 & 2 \\ 4 & -6 \end{bmatrix} \\&= \begin{bmatrix} 5 & 7 \\ 4 & 6 \\ 10 & 3 \end{bmatrix} - \begin{bmatrix} 8 & -2 \\ 6 & 2 \\ 4 & -6 \end{bmatrix} \\&= \begin{bmatrix} -3 & 9 \\ -2 & 4 \\ 6 & -3 \end{bmatrix}\end{aligned}$$

(b)  $3(A + C)' - B'$

**Sol.**

$$\begin{aligned}3(A + C)' - B' &= 3 \left( \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ 3 & 1 \\ 2 & -3 \end{bmatrix} \right)' - \begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{bmatrix}' \\&= 3 \left( \begin{bmatrix} 5 & 1 \\ 3 & 2 \\ 5 & -2 \end{bmatrix} \right)' - \begin{bmatrix} 2 & 4 & 1 \\ 1 & 3 & 0 \end{bmatrix} \\&= \begin{bmatrix} 15 & 9 & 15 \\ 3 & 6 & -6 \end{bmatrix} - \begin{bmatrix} 2 & 4 & 1 \\ 1 & 3 & 0 \end{bmatrix} \\&= \begin{bmatrix} 13 & 5 & 14 \\ 2 & 3 & -6 \end{bmatrix}\end{aligned}$$

3. Given  $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$ , Find the matrix X in the following expression:

(a)  $X + 4A = 3(X + B) - A$

**Sol.**

$$\begin{aligned}X + 4A &= 3(X + B) - A \\&= 3X + 3B - A \\2X &= 5A - 3B \\2X &= 5 \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} - 3 \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix} \\&= \begin{bmatrix} 5 & 10 & 15 \\ 0 & 5 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 3 & 9 \\ 3 & 6 & 0 \end{bmatrix} \\&= \begin{bmatrix} -1 & 7 & 6 \\ -3 & -1 & 5 \end{bmatrix} \\x &= \begin{bmatrix} -\frac{1}{2} & \frac{7}{2} & 3 \\ -\frac{3}{2} & -\frac{1}{2} & \frac{5}{2} \end{bmatrix}\end{aligned}$$

(b)  $2A - B + X' = B$



**Sol.**

$$\begin{aligned}2A - B + X' &= B \\X' &= -2A + 2B \\&= -2(A - B) \\&= -2 \left( \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \right) \\&\quad - \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix} \\&= -2 \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \\&= \begin{bmatrix} 2 & -2 & 0 \\ 2 & 2 & -2 \end{bmatrix} \\X &= \begin{bmatrix} 2 & 2 \\ -2 & 2 \\ 0 & -2 \end{bmatrix}\end{aligned}$$

## 14.4 Multiplication of Matrices

Let  $A$  and  $B$  be matrices of order  $m \times n$  and  $n \times p$  respectively. Then the product of  $A$  and  $B$  is defined as the matrix  $AB$  of order  $m \times p$  such that the  $(i, j)^{th}$  element of  $AB$  is given by

$$(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$$

The multiplication of matrices has the following properties:

- Associative:  $A(BC) = (AB)C$
- Distributive:  $A(B + C) = AB + AC$  and  $(B + C)A = BA + CA$
- $k(AB) \neq (kA)B$  for  $k \neq 0$

### 14.4.1 Practice 3

Evaluate the following:

1.  $\begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

**Sol.**

$$\begin{aligned}&\begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \\&= \begin{bmatrix} -1(-1) + (-1)(2) & -1(2) + (-1)(1) \\ 2(2) + 3(-1) & 2(1) + 3(2) \end{bmatrix} \\&= \begin{bmatrix} 1 & -3 \\ 1 & 8 \end{bmatrix}\end{aligned}$$

2.  $\begin{bmatrix} 3 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 3 & -3 & 2 \end{bmatrix}$

**Sol.**

$$\begin{aligned}&\begin{bmatrix} 3 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 3 & -3 & 2 \end{bmatrix} \\&= \begin{bmatrix} 3(0) + 4(3) & 3(1) + 4(-3) & 3(2) + 4(2) \\ -1(0) + 1(3) & -1(1) + 1(-3) & -1(2) + 1(2) \end{bmatrix} \\&= \begin{bmatrix} 12 & -9 & 14 \\ 3 & -4 & 0 \end{bmatrix} \\3. &\begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 6 & 1 & 5 \\ -2 & 3 & 2 \end{bmatrix}\end{aligned}$$

**Sol.**

$$\begin{aligned}&\begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} 6 & 1 & 5 \\ -2 & 3 & 2 \end{bmatrix} \\&= \begin{bmatrix} 1(6) + 0(-2) & 1(1) + 0(3) & 1(5) + 0(2) \\ 2(6) + 4(-2) & 2(1) + 4(3) & 2(5) + 4(2) \\ 3(6) + (-5)(-2) & 3(1) + (-5)(3) & 3(5) + (-5)(2) \end{bmatrix} \\&= \begin{bmatrix} 6 & 1 & 5 \\ 4 & 14 & 18 \\ 28 & -12 & 5 \end{bmatrix}\end{aligned}$$

4.  $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$

**Sol.**

$$\begin{aligned}&\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ -1 & 3 \end{bmatrix} \\&= \begin{bmatrix} 1(1) + 3(2) + 2(-1) & 1(3) + 3(1) + 2(3) \\ 0(1) + 1(2) + 5(-1) & 0(3) + 1(1) + 5(3) \end{bmatrix} \\&= \begin{bmatrix} 5 & 12 \\ -3 & 16 \end{bmatrix}\end{aligned}$$

### 14.4.2 Exercise 14.4

Calculate the following products (Question 1 to 8):

1.  $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

**Sol.**

$$\begin{aligned}&\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\&= [1(1) + 2(2) + 3(3)] \\&= [14]\end{aligned}$$

$$2. \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

**Sol.**

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \\ = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$$3. \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Sol.**

$$\begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} 2(1) + (-3)(0) & 2(0) + (-3)(1) \\ 1(1) + 5(0) & 1(0) + 5(1) \end{bmatrix} \\ = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$4. \begin{bmatrix} -6 & -4 & 2 \\ 7 & 8 & -5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

**Sol.**

$$\begin{bmatrix} -6 & -4 & 2 \\ 7 & 8 & -5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \\ = \begin{bmatrix} -6(1) + (-4)(2) + 2(3) \\ 7(1) + 8(2) + (-5)(3) \end{bmatrix} \\ = \begin{bmatrix} -8 \\ 8 \end{bmatrix}$$

$$5. \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 4 & 2 \end{bmatrix}$$

**Sol.**

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 4 & 2 \end{bmatrix} \\ = \begin{bmatrix} 2(2) + 3(3) + 4(4) & 2(0) + 3(1) + 4(2) \\ 0(2) + 1(3) + 2(4) & 0(0) + 1(1) + 2(2) \end{bmatrix} \\ = \begin{bmatrix} 27 & 11 \\ 11 & 5 \end{bmatrix}$$

$$6. \begin{bmatrix} 6 & 4 & 2 \\ 5 & -2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$$

**Sol.**

$$\begin{bmatrix} 6 & 4 & 2 \\ 5 & -2 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} \\ = \begin{bmatrix} 6(5) + 4(2) + 2(3) \\ 5(5) + (-2)(2) + 0(3) \\ 0(5) + 3(2) + 1(3) \end{bmatrix} \\ = \begin{bmatrix} 44 \\ 21 \\ 9 \end{bmatrix}$$

$$7. \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Sol.**

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} 0(0)+1(1)+0(0) & 0(1)+1(0)+0(0) & 0(0)+1(0)+0(1) \\ 1(0)+0(1)+0(0) & 1(1)+0(0)+0(0) & 1(0)+0(0)+0(1) \\ 0(0)+0(1)+1(0) & 0(1)+0(0)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$8. \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

**Sol.**

$$\begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix} \\ = \begin{bmatrix} 1+(-2)+1 & 2+(-4)+2 & 3+(-6)+3 \\ (-3)+4+(-1) & (-6)+8+(-2) & (-9)+12+(-3) \\ (-2)+2+0 & (-4)+4+0 & (-6)+6+0 \end{bmatrix} \\ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$