

# Calculus

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# Chapter 1

## Limits

### 1.1 Arithmetic Properties of Limits

If  $\lim_{x \rightarrow x_0} f(x) = A$ ,  $\lim_{x \rightarrow x_0} g(x) = B$ , then:

(a)  $\lim_{x \rightarrow x_0} (f(x) \pm g(x)) = \lim_{x \rightarrow x_0} f(x) \pm \lim_{x \rightarrow x_0} g(x) = A \pm B$

(b)  $\lim_{x \rightarrow x_0} (f(x) \cdot g(x)) = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x) = A \cdot B$

(c)  $\lim_{x \rightarrow x_0} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)} = \frac{A}{B}, (B \neq 0)$

(d) If  $k$  is a constant, then  $\lim_{x \rightarrow x_0} k = k$

(e) If  $k$  is a constant, then  $\lim_{x \rightarrow x_0} k \cdot f(x) = k \lim_{x \rightarrow x_0} f(x) = kA$

(f) If  $n \in \mathbb{R}$ , and  $\lim_{x \rightarrow x_0} f(x) > 0$ , then  $\lim_{x \rightarrow x_0} [f(x)]^n = \left[ \lim_{x \rightarrow x_0} f(x) \right]^n = A^n$

(g) If  $\lim_{x \rightarrow x_0} f(x) = 0$ , then  $\lim_{x \rightarrow x_0} \frac{1}{f(x)} = \infty$

(h) L'Hopital's Rule: If  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)}$  is indeterminate, then  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}$

### Squeeze Theorem or Sandwich Rule

Near point  $x_0$ ,

$$\text{If } f(x) \leq g(x) \leq h(x)$$

$$\text{and } \lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} h(x) = A,$$

$$\text{then } \lim_{x \rightarrow x_0} g(x) = A.$$

1.  $\lim_{x \rightarrow 3} 3x$

**Sol.**

$$\begin{aligned} \lim_{x \rightarrow 3} 3x &= 3 \cdot 3 \\ &= 9 \quad \square \end{aligned}$$

2.  $\lim_{x \rightarrow -1} (x^2 + 4x)$

**Sol.**

$$\begin{aligned} \lim_{x \rightarrow -1} (x^2 + 4x) &= \lim_{x \rightarrow -1} x^2 + \lim_{x \rightarrow -1} 4x \\ &= (-1)^2 + 4(-1) \\ &= 1 - 4 \\ &= -3 \quad \square \end{aligned}$$

3.  $\lim_{x \rightarrow 3} (9 - x^2)$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 3} (9 - x^2) &= \lim_{x \rightarrow 3} 9 - \lim_{x \rightarrow 3} x^2 \\ &= 9 - 3^2 \\ &= 9 - 9 \\ &= 0 \quad \square\end{aligned}$$

4.  $\lim_{n \rightarrow -2} (x^2 - 2x + 1)$

**Sol.**

$$\begin{aligned}\lim_{n \rightarrow -2} (x^2 - 2x + 1) &= \lim_{n \rightarrow -2} (x - 1)^2 \\ &= (-3)^2 \\ &= 9 \quad \square\end{aligned}$$

5.  $\lim_{x \rightarrow -4} x^2(x + 2)$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow -4} x^2(x + 2) &= \lim_{x \rightarrow -4} x^2 \lim_{x \rightarrow -4} (x + 2) \\ &= (-4)^2 \cdot (-4 + 2) \\ &= 16 \cdot (-2) \\ &= -32 \quad \square\end{aligned}$$

6.  $\lim_{h \rightarrow 2} (h^2 - 4h + 4)$

**Sol.**

$$\begin{aligned}\lim_{h \rightarrow 2} (h^2 - 4h + 4) &= \lim_{h \rightarrow 2} (h - 2)^2 \\ &= (2 - 2)^2 \\ &= 0 \quad \square\end{aligned}$$

7.  $\lim_{a \rightarrow -1} (a + 3)(a - 4)$

**Sol.**

$$\begin{aligned}\lim_{a \rightarrow -1} (a + 3)(a - 4) &= \lim_{a \rightarrow -1} (a + 3) \lim_{a \rightarrow -1} (a - 4) \\ &= (-1 + 3) \cdot (-1 - 4) \\ &= 2 \cdot -5 \\ &= -10 \quad \square\end{aligned}$$

8.  $\lim_{x \rightarrow 3} \frac{x^2 - 5}{x + 2}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{x^2 - 5}{x + 2} &= \lim_{x \rightarrow 3} \frac{x^2 - 5}{x + 2} \\ &= \frac{3^2 - 5}{3 + 2} \\ &= \frac{4}{5} \quad \square\end{aligned}$$

9.  $\lim_{x \rightarrow -3} \frac{(x+5)(x+3)}{x+3}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow -3} \frac{(x+5)(x+3)}{x+3} &= \lim_{x \rightarrow -3} \frac{(x+5)(x+3)}{x+3} \\ &= \lim_{x \rightarrow -3} (x+5) \\ &= -3 + 5 \\ &= 2 \quad \square\end{aligned}$$

10.  $\lim_{x \rightarrow 0} \frac{x^2 + 5x}{x}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x^2 + 5x}{x} &= \lim_{x \rightarrow 0} \frac{x^2 + 5x}{x} \\ &= \lim_{x \rightarrow 0} \frac{x(x+5)}{x} \\ &= \lim_{x \rightarrow 0} (x+5) \\ &= 0 + 5 \\ &= 5 \quad \square\end{aligned}$$

11.  $\lim_{x \rightarrow -2} \frac{x^3 + 8}{x + 2}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow -2} \frac{x^3 + 8}{x + 2} &= \lim_{x \rightarrow -2} \frac{(x+2)(x^2 - x + 4)}{x + 2} \\ &= \lim_{x \rightarrow -2} (x^2 - x + 4) \\ &= (-2)^2 - (-2) + 4 \\ &= 10 \quad \square\end{aligned}$$

12.  $\lim_{x \rightarrow 4} \frac{x^2 - 5x + 6}{x - 3}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{x^2 - 5x + 6}{x - 3} &= \lim_{x \rightarrow 4} \frac{(x-3)(x-2)}{x-3} \\ &= \lim_{x \rightarrow 4} (x-2) \\ &= 4 - 2 \\ &= 2 \quad \square\end{aligned}$$

13.  $\lim_{x \rightarrow 3} \frac{3x}{x+2}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{3x}{x+2} &= \lim_{x \rightarrow 3} \frac{3x}{x+2} \\ &= \frac{3(3)}{3+2} \\ &= \frac{9}{5} \quad \square\end{aligned}$$

14.  $\lim_{x \rightarrow 5} \frac{x-5}{2x^2-9x-5}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 5} \frac{x-5}{2x^2-9x-5} &= \lim_{x \rightarrow 5} \frac{x-5}{(2x+1)(x-5)} \\ &= \lim_{x \rightarrow 5} \frac{1}{2x+1} \\ &= \frac{1}{2(5)+1} \\ &= \frac{1}{11} \quad \square\end{aligned}$$

15.  $\lim_{x \rightarrow 1} \frac{x-1}{x^2+x-2}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x-1}{x^2+x-2} &= \lim_{x \rightarrow 1} \frac{x-1}{(x-1)(x+2)} \\ &= \lim_{x \rightarrow 1} \frac{1}{x+2} \\ &= \frac{1}{1+2} \\ &= \frac{1}{3} \quad \square\end{aligned}$$

16.  $\lim_{x \rightarrow 4} \frac{x-1}{x^2+x-2}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 4} \frac{x-1}{x^2+x-2} &= \lim_{x \rightarrow 4} \frac{x-1}{(x-1)(x+2)} \\ &= \lim_{x \rightarrow 4} \frac{1}{x+2} \\ &= \frac{1}{4+2} \\ &= \frac{1}{6} \quad \square\end{aligned}$$

17.  $\lim_{x \rightarrow -2} \frac{x-2}{x^2-4}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow -2} \frac{x-2}{x^2-4} &= \lim_{x \rightarrow -2} \frac{x-2}{(x+2)(x-2)} \\ &= \lim_{x \rightarrow -2} \frac{1}{x+2} \\ &= \infty \quad \square\end{aligned}$$

18.  $\lim_{h \rightarrow 0} \frac{2x^2h+3h}{h}$

**Sol.**

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{2x^2h+3h}{h} &= \lim_{h \rightarrow 0} \frac{h(2x^2+3)}{h} \\ &= \lim_{h \rightarrow 0} (2x^2+3) \\ &= 2x^2+3 \quad \square\end{aligned}$$

19.  $\lim_{h \rightarrow 0} \frac{(2+h)^2-4}{h}$

**Sol.**

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{(2+h)^2-4}{h} &= \lim_{h \rightarrow 0} \frac{[(2+h)+2][(2+h)-2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{(4+h)h}{h} \\ &= \lim_{h \rightarrow 0} (4+h) \\ &= 4+0 \\ &= 4 \quad \square\end{aligned}$$

20.  $\lim_{h \rightarrow 0} \frac{(1+h)^3-1}{h}$

**Sol.**

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{(1+h)^3-1}{h} &= \lim_{h \rightarrow 0} \frac{[(1+h)-1][(1+h)^2+(1+h)+1]}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(1+2h+h^2+1+h+1)}{h} \\ &= \lim_{h \rightarrow 0} (h^2+3h+3) \\ &= (0)^2+3(0)+3 \\ &= 3 \quad \square\end{aligned}$$

21.  $\lim_{x \rightarrow -1} 2x(x^2-4)$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow -1} 2x(x^2-4) &= \lim_{x \rightarrow -1} 2x(x^2-4) \\ &= -2(-1)^2[(-1)^2-4] \\ &= -2(-3) \\ &= 6 \quad \square\end{aligned}$$

22.  $\lim_{x \rightarrow 3} \frac{x^2+2}{x+1}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{x^2+2}{x+1} &= \frac{3^2+2}{3+1} \\ &= \frac{11}{4} \quad \square\end{aligned}$$

23.  $\lim_{x \rightarrow 2} (x^2-3x+5)$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 2} (x^2-3x+5) &= 2^2-3(2)+5 \\ &= 3 \quad \square\end{aligned}$$

24.  $\lim_{x \rightarrow 1} \frac{2x^2+1}{3x^2+4x-1}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{2x^2+1}{3x^2+4x-1} &= \frac{2(1)^2+1}{3(1)^2+4(1)-1} \\ &= \frac{3}{6} \\ &= \frac{1}{2} \quad \square\end{aligned}$$

25.  $\lim_{x \rightarrow 1} \frac{x^2-5x+6}{x^2-9}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^2-5x+6}{x^2-9} &= \frac{(x-3)(x-2)}{(x+3)(x-3)} \\ &= \frac{x-2}{x+3} \\ &= \frac{1-2}{1+3} \\ &= -\frac{1}{4} \quad \square\end{aligned}$$

26.  $\lim_{x \rightarrow -1} \frac{x^3+1}{x+1}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow -1} \frac{x^3+1}{x+1} &= \lim_{x \rightarrow -1} \frac{(x+1)(x^2-x+1)}{x+1} \\ &= \lim_{x \rightarrow -1} (x^2-x+1) \\ &= (-1)^2 - (-1) + 1 \\ &= 3 \quad \square\end{aligned}$$

27.  $\lim_{x \rightarrow 1} \frac{x^3-1}{x-1}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^3-1}{x-1} &= \lim_{x \rightarrow 1} \frac{(x-1)(x^2+x+1)}{x-1} \\ &= \lim_{x \rightarrow 1} (x^2+x+1) \\ &= 1^2 + 1 + 1 \\ &= 3 \quad \square\end{aligned}$$

28.  $\lim_{x \rightarrow 0} \frac{2x^3+3x^2}{x^3}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{2x^3+3x^2}{x^3} &= \lim_{x \rightarrow 0} \left( \frac{2x^3}{x^3} + \frac{3x^2}{x^3} \right) \\ &= \lim_{x \rightarrow 0} \left( 2 + \frac{3}{x} \right) \\ &= 2 \quad \square\end{aligned}$$

29.  $\lim_{k \rightarrow 0} \frac{(x-k)^2-2kx^3}{x(x+k)}$

**Sol.**

$$\begin{aligned}\lim_{k \rightarrow 0} \frac{(x-k)^2-2kx^3}{x(x+k)} &= \frac{(x-0)^2-2(0)(x^3)}{x(x+0)} \\ &= \frac{x^2}{x^2} \\ &= 1 \quad \square\end{aligned}$$

30.  $\lim_{x \rightarrow 1} \frac{x^2-2x+5}{x^2+7}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^2-2x+5}{x^2+7} &= \frac{(1)^2-2(1)+5}{(1)^2+7} \\ &= \frac{4}{8} \\ &= \frac{1}{2} \quad \square\end{aligned}$$

31.  $\lim_{x \rightarrow -2} \frac{x^4-16}{x^3-2}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow -2} \frac{x^4-16}{x^3-2} &= \frac{(-2)^4-16}{(-2)^3-2} \\ &= \frac{16-16}{-8-2} \\ &= \frac{0}{-10} \\ &= 0 \quad \square\end{aligned}$$

32.  $\lim_{x \rightarrow 1} \frac{x-1}{x^2-1}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x-1}{x^2-1} &= \lim_{x \rightarrow 1} \frac{x-1}{(x-1)(x+1)} \\ &= \lim_{x \rightarrow 1} \frac{1}{x+1} \\ &= \frac{1}{1+1} \\ &= \frac{1}{2} \quad \square\end{aligned}$$

33.  $\lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{x}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{1+x}-1}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)} \\ &= \lim_{x \rightarrow 0} \frac{1+x-1}{x(\sqrt{1+x}+1)} \\ &= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{1+x}+1)} \\ &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{1+x}+1} \\ &= \frac{1}{\sqrt{1+0}+1} \\ &= \frac{1}{2}\end{aligned}$$

34.  $\lim_{x \rightarrow 2} \frac{x^2+4}{x^2+1}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^2+4}{x^2+1} &= \frac{(2)^2+4}{(2)^2+1} \\ &= \frac{8}{5}\end{aligned}$$

35.  $\lim_{x \rightarrow 0} \frac{x^2+3x+2}{x^2+2}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{x^2+3x+2}{x^2+2} &= \frac{(0)^2+3(0)+2}{(0)^2+2} \\ &= \frac{2}{2} \\ &= 1 \quad \square\end{aligned}$$

36.  $\lim_{x \rightarrow 1} \frac{x^2-2x+1}{x^2-1}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^2-2x+1}{x^2-1} &= \frac{(x-1)^2}{(x-1)(x+1)} \\ &= \frac{x-1}{x+1} \\ &= \frac{1-1}{1+1} \\ &= 0 \quad \square\end{aligned}$$

$$\mathbf{a}^n - \mathbf{b}^n = (\mathbf{a} - \mathbf{b})(\mathbf{a}^{n-1} + \mathbf{a}^{n-2}\mathbf{b} + \dots + \mathbf{a}\mathbf{b}^{n-2} + \mathbf{b}^{n-1})$$

**Proof.**

$$\begin{aligned} (a-b)(a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1}) &= (a-b) \left( \sum_{k=0}^{n-1} a^{n-k-1}b^k \right) \\ &= a \sum_{k=0}^{n-1} a^{n-k-1}b^k - b \sum_{k=0}^{n-1} a^{n-k-1}b^k \\ &= \sum_{k=0}^{n-1} a^{n-k}b^k - \sum_{k=0}^{n-1} a^{n-k-1}b^{k+1} \\ &= a^n + \sum_{k=1}^{n-1} a^{n-k}b^k - \sum_{l=0}^{n-2} a^{n-l-1}b^{l+1} - b^n \\ &= a^n + \sum_{k=1}^{n-1} a^{n-k}b^k - \sum_{k=1}^{n-1} a^{n-k}b^k - b^n \quad (l = k-1) \\ &= a^n - b^n \quad \square \end{aligned}$$

37.  $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} \quad (n \in \mathbb{N})$

**Sol.**

$$\begin{aligned} \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \lim_{x \rightarrow a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \dots + xa^{n-2} + a^{n-1})}{(x-a)} \\ &= \lim_{x \rightarrow a} (x^{n-1} + x^{n-2}a + \dots + xa^{n-2} + a^{n-1}) \\ &= a^{n-1} + a^{n-2}a + \dots + a^{n-2}a + a^{n-1} \\ &= a^{n-1} + a^{n-1} + \dots + a^{n-1} \\ &= na^{n-1} \quad \square \end{aligned}$$

38.  $\lim_{x \rightarrow 1} (3x^2 - 6x + 5)$

**Sol.**

$$\begin{aligned} \lim_{x \rightarrow 1} (3x^2 - 6x + 5) &= 3(1)^2 - 6(1) + 5 \\ &= 3 - 6 + 5 \\ &= 2 \quad \square \end{aligned}$$

39.  $\lim_{x \rightarrow 1} \frac{2x^2 - 1}{3x^3 - 6x^2 + 5}$

**Sol.**

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{2x^2 - 1}{3x^3 - 6x^2 + 5} &= \frac{2(1)^2 - 1}{3(1)^3 - 6(1)^2 + 5} \\ &= \frac{1}{2} \quad \square \end{aligned}$$

40.  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - 4x + 3}$

**Sol.**

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - 4x + 3} &= \lim_{x \rightarrow 3} \frac{(x+1)(x-1)}{(x-1)(x-3)} \\ &= \lim_{x \rightarrow 3} \frac{x+1}{x-3} \\ &= -1 \quad \square \end{aligned}$$

41.  $\lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{7x^2 - 22x + 3}$

**Sol.**

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^2 - 5x + 6}{7x^2 - 22x + 3} &= \lim_{x \rightarrow 3} \frac{(x-3)(x-2)}{(7x-1)(x-3)} \\ &= \lim_{x \rightarrow 3} \frac{x-2}{7x-1} \\ &= \frac{3-2}{7(3)-1} \\ &= \frac{1}{20} \quad \square \end{aligned}$$

42.  $\lim_{x \rightarrow 3} \frac{x^3 - 2x^2 - 2x + 4}{x^3 + x^2 - 10x + 8}$

**Sol.**

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^3 - 2x^2 - 2x + 4}{x^3 + x^2 - 10x + 8} &= \lim_{x \rightarrow 3} \frac{x^2(x-2) - 2(x-2)}{(x-2)(x+4)(x-1)} \\ &= \lim_{x \rightarrow 3} \frac{(x-2)(x^2-2)}{(x-2)(x+4)(x-1)} \\ &= \lim_{x \rightarrow 3} \frac{x^2-2}{(x+4)(x-1)} \\ &= \frac{3^2-2}{(3+4)(3-1)} \\ &= \frac{1}{2} \quad \square \end{aligned}$$

43.  $\lim_{x \rightarrow 1} \frac{x^4 + 2x^2 - 3}{x^2 - 3x + 2}$

**Sol.**

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{x^4 + 2x^2 - 3}{x^2 - 3x + 2} &= \lim_{x \rightarrow 3} \frac{(x-1)(x+1)(x^2+3)}{(x-2)(x-1)} \\ &= \lim_{x \rightarrow 3} \frac{(x+1)(x^2+3)}{x-2} \\ &= \frac{(1+1)(1^2+3)}{1-2} \\ &= \frac{8}{-1} \\ &= -8 \quad \square \end{aligned}$$

44.  $\lim_{x \rightarrow 1} \frac{1 - \sqrt[4]{x}}{1 - \sqrt[3]{x}}$

**Sol.**

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{1 - \sqrt[4]{x}}{1 - \sqrt[3]{x}} &= \lim_{x \rightarrow 1} \frac{(x^{\frac{1}{4}} - 1)'}{(x^{\frac{1}{3}} - 1)'} \\ &= \lim_{x \rightarrow 1} \frac{\frac{1}{4}x^{-\frac{3}{4}}}{\frac{1}{3}x^{-\frac{2}{3}}} \\ &= \lim_{x \rightarrow 1} \frac{\frac{1}{4}x^{\frac{3}{4}}}{\frac{1}{3}x^{\frac{2}{3}}} \\ &= \lim_{x \rightarrow 1} \frac{3x^{\frac{2}{3}}}{4x^{\frac{3}{4}}} \\ &= \frac{3(1)^{\frac{2}{3}}}{4(1)^{\frac{3}{4}}} \\ &= \frac{3}{4} \quad \square \end{aligned}$$

45.  $\lim_{x \rightarrow 0} \frac{\sqrt[n]{1+x} - 1}{x} \quad (n \in \mathbb{W})$

**Sol.**

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt[n]{1+x} - 1}{x} &= \lim_{x \rightarrow 1} \frac{(\sqrt[n]{1+x} - 1)'}{x'} \\ &= \lim_{x \rightarrow 1} \frac{(\sqrt[n]{1+x})' - 1'}{1} \\ &= \lim_{x \rightarrow 1} \left[ (1+x)^{\frac{1}{n}} \right]' \\ &= \lim_{x \rightarrow 1} \frac{1}{n} \left[ (1+x)^{\frac{1-n}{n}} \right] \\ &= \frac{1}{n} (1+0)^{\frac{1-n}{n}} \\ &= \frac{1}{n} (1)^{\frac{1-n}{n}} \\ &= \frac{1}{n} \quad \square \end{aligned}$$

46.  $\lim_{x \rightarrow 1} \frac{2 - \sqrt{x+3}}{x^2 - 1}$

**Sol.**

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{2 - \sqrt{x+3}}{x^2 - 1} &= \lim_{x \rightarrow 1} \frac{(2 - \sqrt{x+3})'}{(x^2 - 1)'} \\ &= \lim_{x \rightarrow 1} \frac{2' - (\sqrt{x+3})'}{2x} \\ &= \lim_{x \rightarrow 1} \frac{-\left[(x+3)^{\frac{1}{2}}\right]'}{2x} \\ &= \lim_{x \rightarrow 1} \frac{-\frac{1}{2}(x+3)^{-\frac{1}{2}}}{2x} \\ &= \lim_{x \rightarrow 1} \frac{-\frac{1}{2\sqrt{x+3}}}{2x} \\ &= \frac{-\frac{1}{2\sqrt{1+3}}}{2(1)} \\ &= \frac{-\frac{1}{4}}{2} \\ &= -\frac{1}{8} \end{aligned}$$

47.  $\lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{\sqrt{x} - 4}$

**Sol.**

$$\begin{aligned} \lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{\sqrt{x} - 4} &= \lim_{x \rightarrow 16} \frac{(\sqrt[4]{x} - 2)'}{(\sqrt{x} - 4)'} \\ &= \lim_{x \rightarrow 16} \frac{(\sqrt[4]{x})' - 2'}{(\sqrt{x})' - 4'} \\ &= \lim_{x \rightarrow 16} \frac{\frac{1}{4}x^{-\frac{3}{4}}}{\frac{1}{2}x^{-\frac{1}{2}}} \\ &= \lim_{x \rightarrow 16} \frac{\frac{1}{4}x^{\frac{3}{4}}}{\frac{1}{2}x^{\frac{1}{2}}} \\ &= \lim_{x \rightarrow 16} \frac{2x^{\frac{3}{4}}}{x^{\frac{1}{2}}} \\ &= \frac{16^{\frac{1}{2}}}{2(16)^{\frac{3}{4}}} \\ &= \frac{4}{2(8)} \\ &= \frac{1}{4} \quad \square \end{aligned}$$

48.  $\lim_{x \rightarrow 2} (x^2 + 3x - 1)$

**Sol.**

$$\lim_{x \rightarrow 2} (x^2 + 3x - 1) = (2)^2 + 3(2) - 1 \quad (1)$$

$$= 4 + 6 - 1 \quad (2)$$

$$= 9 \quad \square \quad (3)$$

49.  $\lim_{x \rightarrow -1} \frac{x^2 + 2}{x^2 + x + 3}$



**Sol.**

$$\begin{aligned}\lim_{x \rightarrow -1} \frac{x^2 + 2}{x^2 + x + 3} &= \frac{(-1)^2 + 2}{(-1)^2 + (-1) + 3} \\ &= \frac{1 + 2}{1 - 1 + 3} \\ &= 1 \quad \square\end{aligned}$$

50.  $\lim_{x \rightarrow -1} \frac{x^3 + 1}{x^2 - 1}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow -1} \frac{x^3 + 1}{x^2 - 1} &= \frac{(x + 1)(x^2 - x + 1)}{(x + 1)(x - 1)} \\ &= \frac{x^2 - x + 1}{x - 1} \\ &= \frac{(-1)^2 - (-1) + 1}{-1 - 1} \\ &= -\frac{3}{2} \quad \square\end{aligned}$$

51.  $\lim_{x \rightarrow 1} \frac{x^5 - x^4}{x^3 - x}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^5 - x^4}{x^3 - x} &= \lim_{x \rightarrow 1} \frac{(x^5 - x^4)'}{(x^3 - x)'} \\ &= \lim_{x \rightarrow 1} \frac{5x^4 - 4x^3}{3x^2 - 1} \\ &= \frac{5(1)^4 - 4(1)^3}{3(1)^2 - 1} \\ &= \frac{1}{2} \quad \square\end{aligned}$$

52.  $\lim_{x \rightarrow a} \frac{x^2 + ax - 2a^2}{x^2 - a^2}, a \neq 0$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow a} \frac{x^2 + ax - 2a^2}{x^2 - a^2} &= \lim_{x \rightarrow a} \frac{(x + 2a)(x - a)}{(x + a)(x - a)} \\ &= \lim_{x \rightarrow a} \frac{x + 2a}{x + a} \\ &= \frac{a + 2a}{a + a} \\ &= \frac{3}{2} \quad \square\end{aligned}$$

53.  $\lim_{x \rightarrow a} \frac{\sqrt{3x - a} - \sqrt{x + a}}{x - a}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow a} \frac{\sqrt{3x - a} - \sqrt{x + a}}{x - a} &= \lim_{x \rightarrow a} \frac{(\sqrt{3x - a} - \sqrt{x + a})'}{(x - a)'} \\ &= \lim_{x \rightarrow a} [(\sqrt{3x - a})' - (\sqrt{x + a})'] \\ &= \lim_{x \rightarrow a} [(\sqrt{3x - a})' - (\sqrt{x + a})'] \\ &= \lim_{x \rightarrow a} \left[ \frac{1}{2\sqrt{3x - a}}(3x - a)' - \frac{1}{2\sqrt{x + a}} \right] \\ &= \lim_{x \rightarrow a} \left[ \frac{3}{2\sqrt{3x - a}} - \frac{1}{2\sqrt{x + a}} \right] \\ &= \frac{3}{2\sqrt{3a - a}} - \frac{1}{2\sqrt{a + a}} \\ &= \frac{2}{2\sqrt{2a}} \\ &= \frac{1}{\sqrt{2a}}\end{aligned}$$

54. Given that  $f(x) = x^2 - 3x$ , find  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

**Sol.**

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - 3(x+h) - x^2 + 3x}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2) - 3x - 3h - x^2 + 3x}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2 - 3h}{h} \\ &= \lim_{h \rightarrow 0} (2x + 2h - 3) \\ &= 2x - 3 \quad \square\end{aligned}$$

55.  $\lim_{x \rightarrow 2} \sqrt{2x^2 + 1}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 2} \sqrt{2x^2 + 1} &= \sqrt{2(2)^2 + 1} \\ &= \sqrt{9} \\ &= 3 \quad \square\end{aligned}$$

56.  $\lim_{x \rightarrow 7} \frac{x^2 \sqrt{x+2}}{x^2 + 14}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 7} \frac{x^2 \sqrt{x+2}}{x^2 + 14} &= \frac{(7)^2 \sqrt{7+2}}{(7)^2 + 14} \\ &= \frac{49(3)}{49 + 14} \\ &= \frac{7}{3}\end{aligned}$$

57.  $\lim_{x \rightarrow 0} \frac{\sqrt{3x+4} - 2}{x}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{3x+4}-2}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{3x+4})' - 2'}{x'} \\ &= \lim_{x \rightarrow 0} \left[ \frac{1}{2\sqrt{3x+4}} (3x+4)' \right] \\ &= \lim_{x \rightarrow 0} \left( \frac{3}{2\sqrt{3x+4}} \right) \\ &= \frac{3}{2\sqrt{3(0)+4}} \\ &= \frac{3}{4} \quad \square\end{aligned}$$

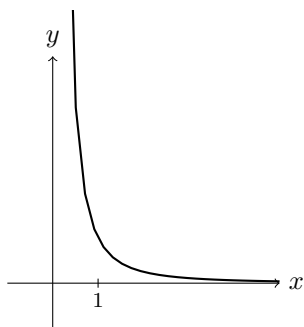
58.  $\lim_{x \rightarrow 0} \frac{1}{x^2}$

**Sol.**

$$\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty \quad \square$$

59.  $\lim_{x \rightarrow 1} \frac{1}{x-1}$

**Sol.**



$$\lim_{x \rightarrow 1} \frac{1}{x-1} = \infty$$

60.  $\lim_{x \rightarrow 1} \frac{4x-3}{x^2-5x+4}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{4x-3}{x^2-5x+4} &= \lim_{x \rightarrow 1} \frac{(4x-3)'}{(x^2-5x+4)'} \\ &= \lim_{x \rightarrow 1} \frac{4}{2x-5} \\ &= \frac{4}{2(1)-5} \\ &= -\frac{4}{3}\end{aligned}$$

61.  $\lim_{x \rightarrow \infty} \frac{3x^3-4x^2+2}{7x^3+5x^2-3}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{3x^3-4x^2+2}{7x^3+5x^2-3} &= \lim_{x \rightarrow \infty} \frac{\frac{3x^3}{x^3} - \frac{4x^2}{x^3} + \frac{2}{x^3}}{\frac{7x^3}{x^3} + \frac{5x^2}{x^3} - \frac{3}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{3 - \frac{4}{x} + \frac{2}{x^3}}{7 + \frac{5}{x} - \frac{3}{x^3}} \\ &= \frac{3-0+0}{7+0-0} \\ &= \frac{3}{7} \quad \square\end{aligned}$$

62.  $\lim_{n \rightarrow \infty} x^2$

**Sol.**

$$\lim_{n \rightarrow \infty} x^2 = \infty \quad \square$$

63.  $\lim_{x \rightarrow \infty} \frac{3x^2-2x-1}{2x^3-x^2+5}$

**Sol.**

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{3x^2-2x-1}{2x^3-x^2+5} &= \lim_{x \rightarrow \infty} \frac{\frac{3x^2}{x^3} - \frac{2x}{x^3} - \frac{1}{x^3}}{\frac{2x^3}{x^3} - \frac{x^2}{x^3} + \frac{5}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{3}{x} - \frac{2}{x^2} - \frac{1}{x^3}}{2 - \frac{1}{x} + \frac{5}{x^3}} \\ &= \frac{0-0-0}{2-0+0} \\ &= 0 \quad \square\end{aligned}$$

64.  $\lim_{n \rightarrow \infty} \frac{1}{n^2} (1+2+3+\dots+n)$

**Sol.**

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{1}{n^2} (1+2+3+\dots+n) &= \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n k \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^2} \left[ \frac{n(n+1)}{2} \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left( \frac{n+1}{2} \right) \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{2n} \\ &= \lim_{n \rightarrow \infty} \frac{\frac{n}{n} + \frac{1}{n}}{\frac{2n}{n}} \\ &= \frac{1+0}{2} \\ &= \frac{1}{2} \quad \square\end{aligned}$$

65.  $\lim_{n \rightarrow \infty} \left[ \frac{1+2+3+\dots+n}{n+2} - \frac{n}{2} \right]$

**Sol.**

$$\begin{aligned}\lim_{n \rightarrow \infty} \left[ \frac{1+2+3+\dots+n}{n+2} - \frac{n}{2} \right] &= \lim_{n \rightarrow \infty} \left[ \frac{\sum_{k=1}^n k}{n+2} - \frac{n}{2} \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{n(n+1)}{2(n+2)} - \frac{n}{2} \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{n(n+1) - n(n+2)}{2(n+2)} \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{n(n+1-n-2)}{2(n+2)} \right] \\ &= \lim_{n \rightarrow \infty} \left[ -\frac{\frac{n}{n}}{\frac{2n}{n} + \frac{4}{n}} \right] \\ &= -\frac{1}{2+0} \\ &= -\frac{1}{2} \quad \square\end{aligned}$$

$$66. \lim_{n \rightarrow \infty} \left[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} \right]$$

**Sol.**

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} \right] \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k(k+1)} \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{k+1} \right) \\ &= \lim_{n \rightarrow \infty} \left[ \sum_{k=1}^n \left( \frac{1}{k} \right) - \sum_{k=1}^n \left( \frac{1}{k+1} \right) \right] \\ &= \lim_{n \rightarrow \infty} \left[ \sum_{k=1}^n \left( \frac{1}{k} \right) - \sum_{k=2}^{n+1} \left( \frac{1}{k} \right) \right] \\ &= \lim_{n \rightarrow \infty} \left[ 1 + \sum_{k=2}^n \left( \frac{1}{k} \right) - \sum_{k=2}^n \left( \frac{1}{k} \right) - \frac{1}{n+1} \right] \\ &= \lim_{n \rightarrow \infty} \left[ 1 - \frac{1}{n+1} \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{n+1-1}{n+1} \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{n}{n+1} \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} \right] \\ &= \frac{1}{1+0} \\ &= 1 \quad \square \end{aligned}$$

$$67. \lim_{x \rightarrow \infty} \frac{5x^3 + 4x^2 - 6x + 2}{8x^3 - 7x^2 + 4x - 1}$$

**Sol.**

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{5x^3 + 4x^2 - 6x + 2}{8x^3 - 7x^2 + 4x - 1} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{5x^3}{x^3} + \frac{4x^2}{x^3} - \frac{6x}{x^3} + \frac{2}{x^3}}{\frac{8x^3}{x^3} - \frac{7x^2}{x^3} + \frac{4x}{x^3} - \frac{1}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{5 + \frac{4}{x} - \frac{6}{x^2} + \frac{2}{x^3}}{8 - \frac{7}{x} + \frac{4}{x^2} - \frac{1}{x^3}} \\ &= \frac{5+0-0+0}{8-0+0-0} \\ &= \frac{5}{8} \quad \square \end{aligned}$$

$$68. \lim_{x \rightarrow \infty} \frac{x^4 - 2x^3 + x^2 + 3}{x^5 - x^4 + 1}$$

**Sol.**

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{x^4 - 2x^3 + x^2 + 3}{x^5 - x^4 + 1} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{x^4}{x^5} - \frac{2x^3}{x^5} + \frac{x^2}{x^5} + \frac{3}{x^5}}{\frac{x^5}{x^5} - \frac{x^4}{x^5} + \frac{1}{x^5}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x^3} + \frac{3}{x^5}}{1 - \frac{1}{x} + \frac{1}{x^5}} \\ &= \frac{0-0+0+0}{1-0+0} \\ &= 0 \quad \square \end{aligned}$$

$$69. \lim_{x \rightarrow \infty} \frac{x^3 - 8x^2 + 4x - 1}{x^2 - 6x + 3}$$

**Sol.**

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{x^3 - 8x^2 + 4x - 1}{x^2 - 6x + 3} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{x^3}{x^3} - \frac{8x^2}{x^3} + \frac{4x}{x^3} - \frac{1}{x^3}}{\frac{x^2}{x^3} - \frac{6x}{x^3} + \frac{3}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{1 - \frac{8x}{x} + \frac{4}{x^2} - \frac{1}{x^3}}{\frac{1}{x} - \frac{6}{x^2} + \frac{3}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{1 - \frac{8x}{x} + \frac{4}{x^2} - \frac{1}{x^3}}{\frac{1}{x} - \frac{6}{x^2} + \frac{3}{x^3}} \\ &= \frac{1-0+0-0}{0-0+0} \\ &= \infty \quad \square \end{aligned}$$

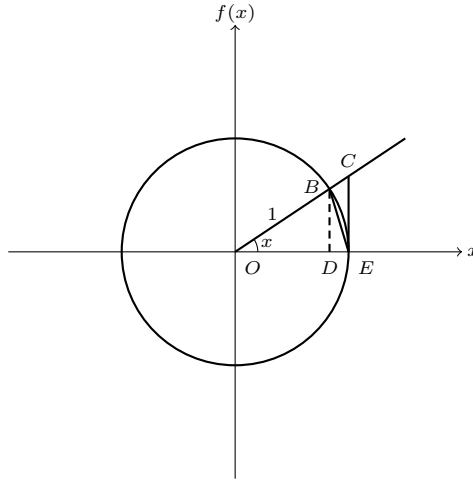
$$70. \lim_{x \rightarrow \infty} (\sqrt{x^4 + 1} - x^2)$$

**Sol.**

$$\begin{aligned} \lim_{x \rightarrow \infty} (\sqrt{x^4 + 1} - x^2) &= \lim_{x \rightarrow \infty} \sqrt{x^4 + 1} - x^2 \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x^4 + 1} + x^2} \\ &= \frac{\lim_{x \rightarrow \infty} 1}{\lim_{x \rightarrow \infty} (\sqrt{x^4 + 1} + x^2)} \\ &= \frac{1}{\infty} \\ &= 0 \quad \square \end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

**Proof.**



From the diagram above, we can see that:

$$BD = \sin x$$

$$\widehat{BD} = x$$

$$EC = \tan x$$

Also, the area of sector  $EOB$  is greater than that of  $\triangle EOB$ , but less than that of  $\triangle EOC$ . That is,

$$(\text{area of sector } EOB) < (\text{area of } \triangle EOB) < (\text{area of } \triangle EOC)$$

$$\text{Hence } \frac{1}{2} \times 1 \times \sin x < \frac{1}{2} \times 1^2 < \frac{1}{2} \times 1 \times \tan x$$

$$\therefore \sin x < x < \tan x$$

Dividing by  $\sin x$  ( $\sin x \neq 0$ ), we get

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$\text{But } \lim_{x \rightarrow 0^+} \frac{1}{\cos x} = \frac{x}{\lim_{x \rightarrow 0^+} \sin x} = 1$$

Hence,  $\lim_{x \rightarrow 0^+} \frac{x}{\sin x}$  is in between 1 and  $\lim_{x \rightarrow 0^+} \frac{1}{\cos x}$ , which is in between 1 and 1. Therefore,  $\lim_{x \rightarrow 0^+} \frac{x}{\sin x}$  must be equal to 1 (Squeeze Theorem).

$$\text{That is } \lim_{x \rightarrow 0^+} \frac{x}{\sin x} = 1,$$

$$\therefore \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = \lim_{x \rightarrow 0^+} \frac{1}{\frac{x}{\sin x}} = \lim_{x \rightarrow 0^+} \frac{1}{\lim_{x \rightarrow 0^+} \frac{x}{\sin x}} = 1 \quad \square$$