Mathematics

Senior 3 Part I

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Introduction

Why this book?

Disclaimer

Acknowledgements

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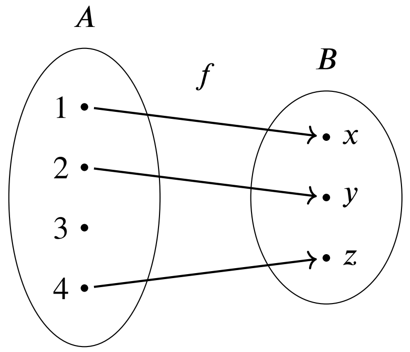
Chapter 1

Function

1.1 Definition of a Function

Mapping, Preimage and Image

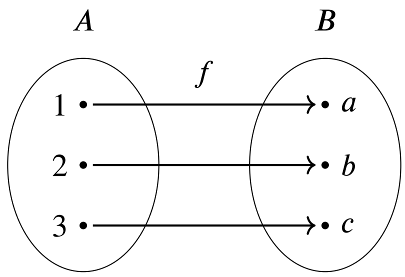
For two non-empty sets A and B, If an element a inside set A has a corresponding element b inside set B, denoted as $a \to b$, then we say that a is mapped to b or a and b are paired. The mapping between two sets is normally denoted as f, g, h, etc. The mapping shown in the diagram below can be denoted as $f : 1 \to x,\ 2 \to y,\ 4 \to z$.



Let $f : A \to B$ is a mapping, a is an element in A. If a is mapped to b under the mapping f, then b is said to be the image of a under the mapping f, denoted as $b = f(a)$; a is said to be the preimage of b under the mapping f. In the diagram above, under the mapping f, the image of 1, 2, and 4 are x, y, and z respectively, while the preimage of x, y, and z are 1, 2, and 4 respectively.

> Let A and B be two non-empty sets, f is a mapping from A to B such that for all elements in A, there is a unique corresponding element in B, then f is a function or a mapping from A to B, denoted as $f : A \to B$.

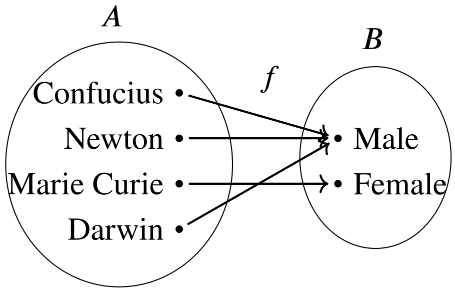
The mapping shown in the diagram below is a function.



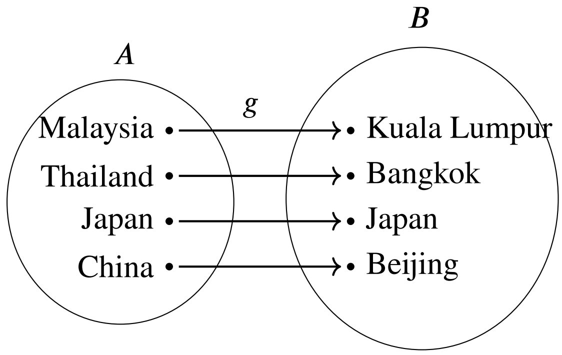
Practice 1

1. For the following mappings, list the image of each element in A and the preimage of each element in B, and determine whether the mapping is a function or not:

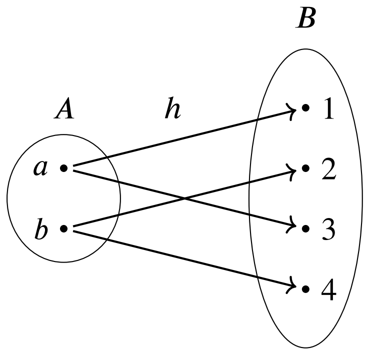
   (a)



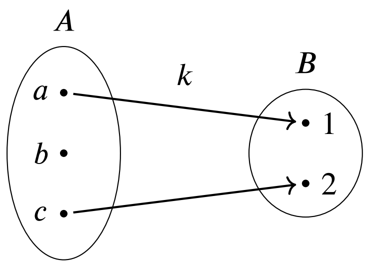
   (b)



   (c)



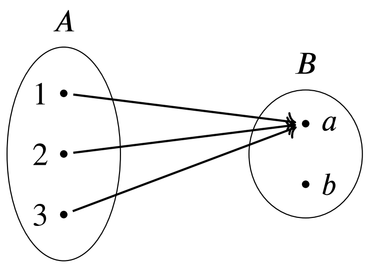
   (d)



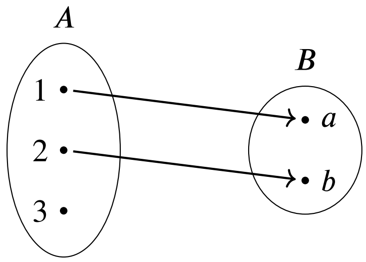
2. Given a mapping $g : x \to x + 3$, $x \in \{-2, -1, 0, 1, 2, 3\}$, find the image of each x.

3. Determine whether the following mappings are functions.

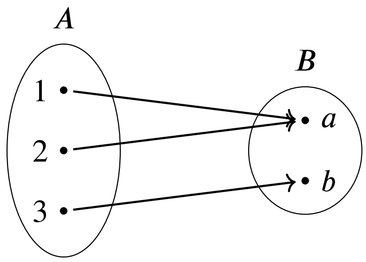
   (a)



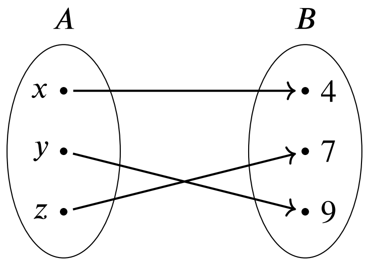
(b)



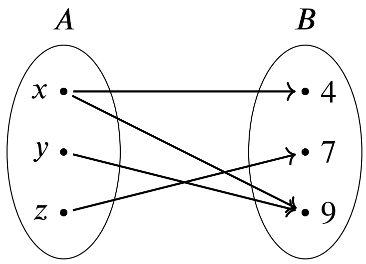
(c)



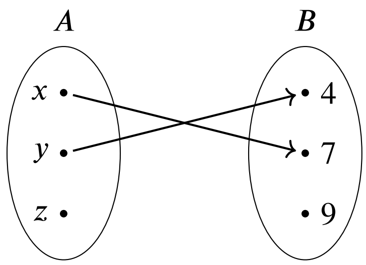
(d)



(e)



(f)



The function $f : A \to B$ can be written as $y = f(x)$, x is the element of A and y is the element of B. When x changes, y changes as well. x is called independent variable, while y is called dependent variable.

Keep in mind that $f(x)$ is NOT the product of f and x.

Representation of Functions

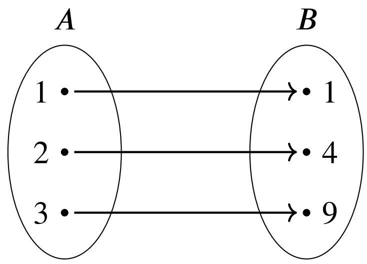
Generally speaking, there are a few ways to represent a function:

1. **Narrative Form**: express the function of two sets in words. For example, Let $A = \{1, 2, 3\}$ and $B = \{1, 4, 9\}$, f is a function from A to B, its definition is that for any element x in A, its corresponding element is x^2 in B.

2. **Arrow Method**: draw an arrow to connect the preimage and image of a function such that the preimage is corresponding to the image. To express the example above, we express it as $f : 1 \to 1,\ 2 \to 4,\ 3 \to 9$.

3. **Analytical Method**: express the function in the form of mathematical expression to represent the relationship between the independent variable and the dependent variable. For example, $f(x) = x^2, x \in A$.

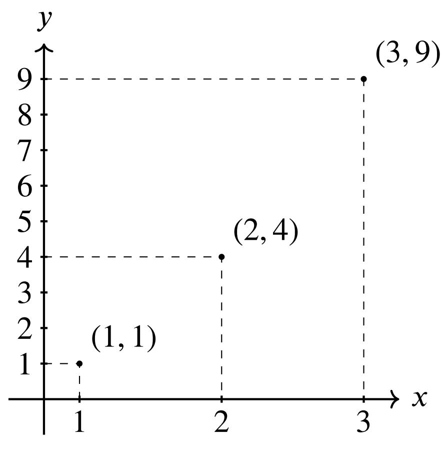
4. **Venn Diagram**: draw arrows between the venn diagram of two sets to represent the function, as shown below:



5. **Table Method**: express the function in the form of table, showing the relationship of the chosen value between independent variable x and the value of its corresponding dependent variable y, as shown below:

x	1	2	3
y	1	4	9

6. **Graphical Method**: draw a graph to represent the function of the two variables, as shown below:



Practice 2

Express the following functions using analytical method, venn diagram, table method and graphical method.

(a) f mapping each integers from -3 to 3 to its squares plus 4.

(b) g mapping each natural numbers from 1 to 4 to its cubes.

Exercise 22.1

1. Express the mapping from set A to set B, and determine which of the following mappings are functions.

	Set A	Set B	Mapping
(a)	{0, 3, 9, 12}	{0, 1, 2, 3}	Divide by 3
(b)	{-2, -1, 0, 1, 2}	{0, 1, 4, 9, 16}	Power of 4
(c)	{-2, -1, 0, 1, 2}	{0, 1, 4}	Square
(d)	{30°, 45°, 60°}	$\left\{\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right\}$	Sine
(e)	{-1, 0, 1, 2}	{-1, 0, 1}	Cube

2. Let function $f(x) = 3x^2 + 1$.

   (a) Find the image of the following elements:

      i. -3
      ii. -2
      iii. 0
      iv. 2
      v. 5

   (b) Find the preimage of the following elements:

      i. 13
      ii. 28
      iii. 1
      iv. 0
      v. 4

3. Let function $g(x) = 5x - 2$. Find:

   (a) $g(-2)$
   (b) $g(-1)$
   (c) $g(0)$

4. Let function $f(x) = \begin{cases} 2x, & x \leq -1 \\ x - 1, & -1 \leq x < 3 \\ 4x + 2, & x \geq 3 \end{cases}$, find

   (a) $f(-5)$
   (b) $f(-2)$
   (c) $f(0)$
   (d) $f(2)$

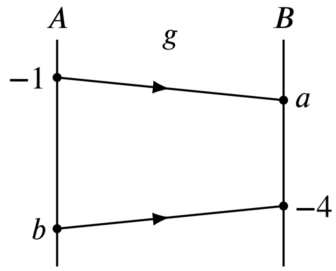
(e) $f(10)$

5. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$. Find the image of -1, 0, 1, and 2 under f.

6. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$. Find the preimage of 0, 1, and 4 under f.

   In $\mathbb{R}$, which element does not have a preimage?

7. In the diagram below, given that function $g : A \to B$ is defined as $g : x \to 2x - 8$. Find the value of a and b.

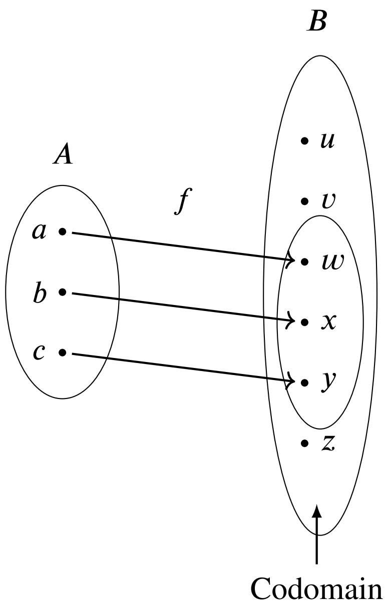


8. Using narrative form, arrow method, venn diagram, table method and graphical method, express the function $f(x) = 2x$, $x \in \{-2, -1, 0, 1, 2\}$.

1.2 Domain and Range

Let f is a function from set A to set B, then set A is called the domain of f, denoted by D_f; set B is called the codomain of f; the set of the images of all elements of A under f is called the range of f, denoted by R_f.

If the domain A and range B of function $f : A \to B$ are both subsets of real number set $\mathbb{R}$, then this function is called real valued function / real function. This book primarily discusses about real valued functions. When the domain of a real function is not mentioned and only the mapping rule is given, its domain is assumed to be the set of all real numbers that yield defined values $f(x)$. After the domain and the mapping rule are determined, the range of a function will then be determined.



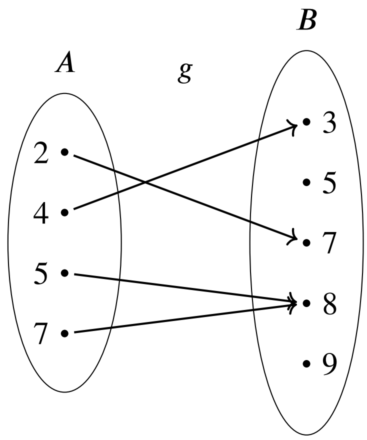
Interval Notation

Let a and b be two real number, $a < b$.

Intervals	Set Notations
(a, b)	$x\|x \in \mathbb{R}, a < x < b$
$[a, b)$	$x\|x \in \mathbb{R}, a \leq x < b$
$(a, b]$	$x\|x \in \mathbb{R}, a < x \leq b$
$[a, b]$	$x\|x \in \mathbb{R}, a \leq x \leq b$
(a, ∞)	$x\|x \in \mathbb{R}, x > a$
$[a, \infty)$	$x\|x \in \mathbb{R}, x \leq a$
$(-\infty, a)$	$x\|x \in \mathbb{R}, x < a$
$(-\infty, a]$	$x\|x \in \mathbb{R}, x \leq a$

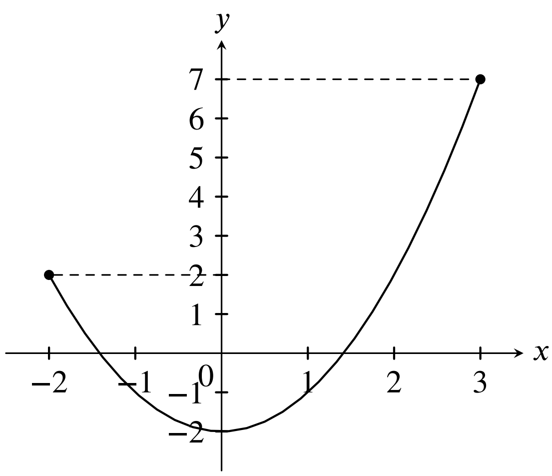
Practice 3

1. Let $A = \{2, 4, 5, 7\}$ and $B = \{3, 5, 7, 8, 9\}$, the definition of function g is given by the diagram below. Find the domain, codomain and range of function g.



2. Let $A = \{-2, -1, 0, 1, 2\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = x^2 - 1$. Find the domain and range of f.

3. The curve in the diagram below represents the function $y = f(x)$, $-2 \leq x \leq 3$. Find the domain and range of f.



4. Find the domain and range of the following functions:

   (a) $f(x) = -4x + 5$

   (b) $g(x) = x^2 - 1$

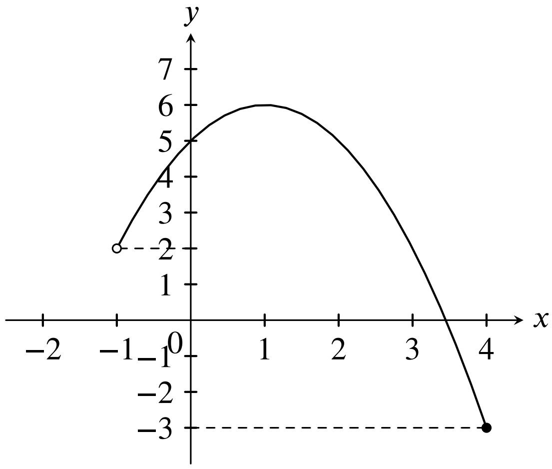
   (c) $h(x) = \dfrac{1}{4x + 7}$

   (d) $k(x) = \sqrt{6 - x}$

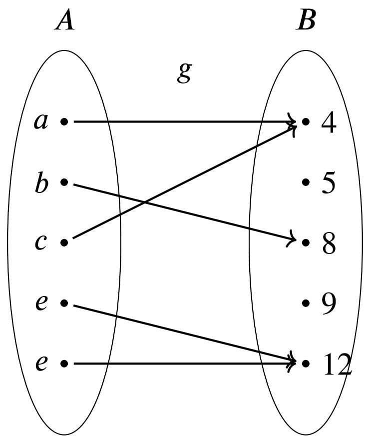
Exercise 22.2

1. Let $X = \{a, b, c, d\}$ and $Y = \{-1, 2, 9, 11\}$, function $f : X \to Y$ is defined by $f(a) = 2$, $f(b) = -1$, $f(c) = 2$, $f(d) = 9$. Find the domain and range of the f.

2. The curve in the diagram below represents the function $y = f(x)$, $-1 < x \leq 4$. Find the domain and range of f.



3. Let $A = \{a, b, c, d, e\}$ and $B = \{4, 5, 8, 9, 12\}$, the definition of function $g : A \to B$ is given by the digram below. Find the domain, codomain and range of function g.



4. Let $A = \{-1, 0, 1, 2\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = 3x^2 - 2$, find the domain and range of f.

5. Let $A = \{-1, 0, 2, 5, 11\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = x^2 - x - 2$, find the domain and range of f.

6. Find the domain and range of the following functions:

   (a) $f(x) = x^3$

   (b) $g(x) = \sqrt{1 - x^2}$

   (c) $h(x) = \dfrac{1}{2x + 3}$

   (d) $k(x) = x^2 - 2x + 4$

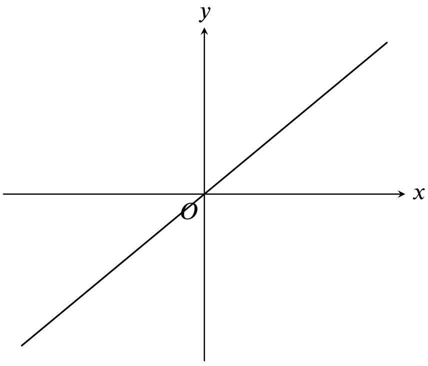
1.3 Graphs of Functions and Their Transformations

Graphs of Simple Functions

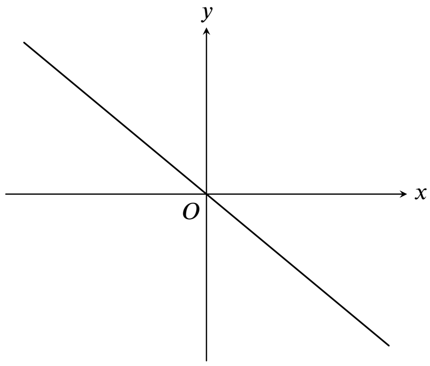
On a Cartesian plane, the graphs formed by all the point (x, y) that satisfied the equation $y = f(x)$ are called graphs of function f. Below are some examples of graphs of simple functions.

Note that any line that is parallel to the y-axis intersects the graph of a function at most once.

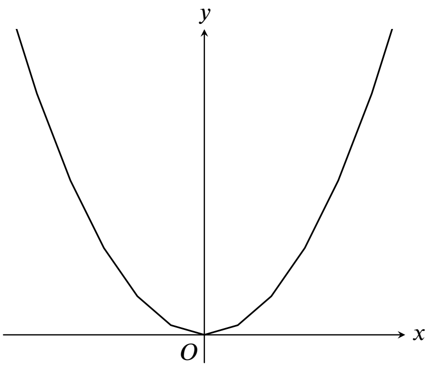
(a) $y = x$



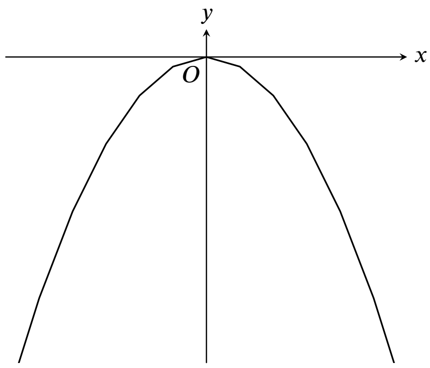
(b) $y = -x$



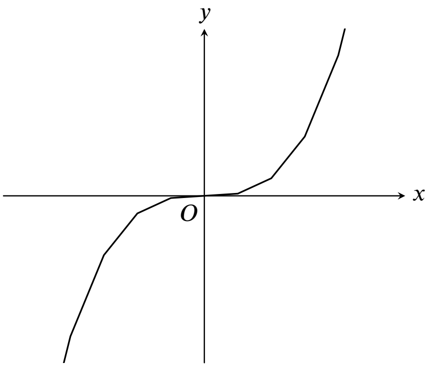
(c) $y = x^2$



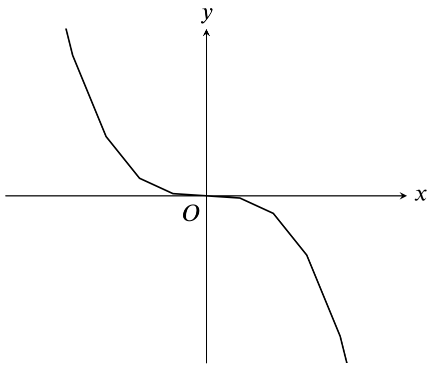
(d) $y = x^2$



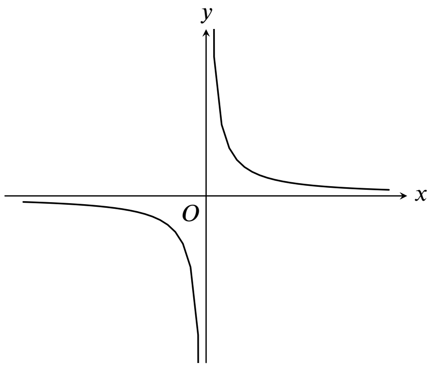
(e) $y = x^3$



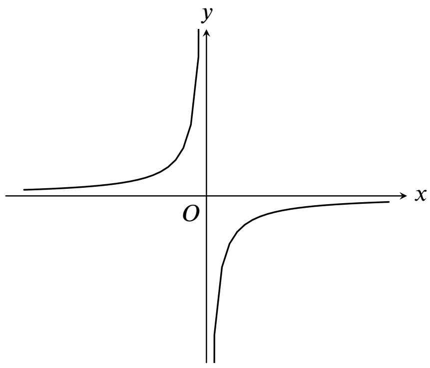
(f) $y = -x^3$



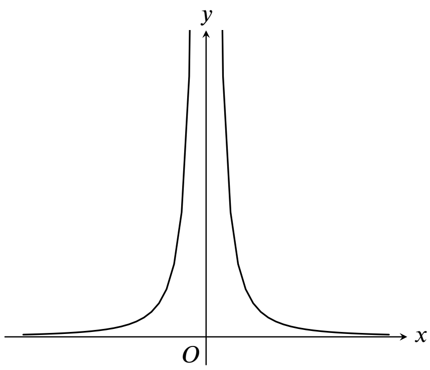
(g) $y = \frac{1}{x}$



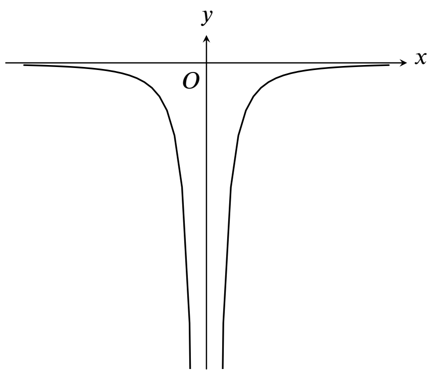
(h) $y = -\frac{1}{x}$



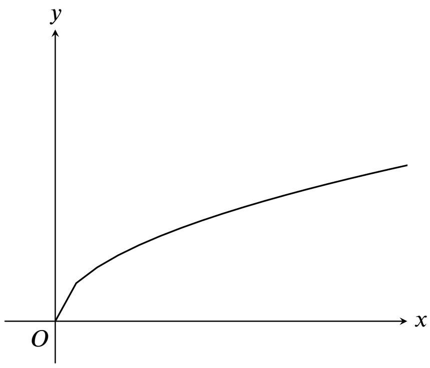
(i) $y = \frac{1}{x^2}$



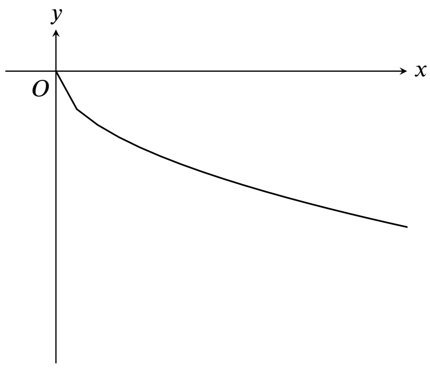
(j) $y = -\frac{1}{x^2}$



(k) $y = \sqrt{x}$

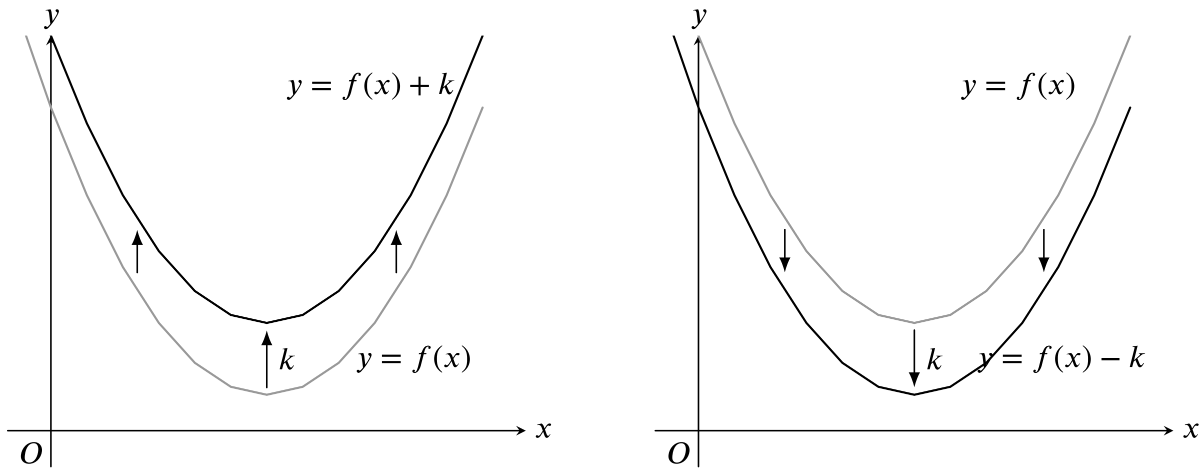


(l) $y = -\sqrt{x}$



Transformations of Graphs

- If $k > 0$, translate the graph of $y = f(x)$ vertically upwards by k units, the graph of $y = f(x) + k$ is obtained.
- If $k > 0$, translate the graph of $y = f(x)$ vertically downwards by k units, the graph of $y = f(x) - k$ is obtained.



- If $h > 0$, translate the graph of $y = f(x)$ horizontally to the right by h units, the graph of $y = f(x+h)$ is obtained.
- If $h > 0$, translate the graph of $y = f(x)$ horizontally to the left by h units, the graph of $y = f(x-h)$ is obtained.

- If $k > 0$, reflect the graph of $y = f(x)$ about the x-axis, the graph of $y = -f(x)$ is obtained.
- If $k > 0$, reflect the graph of $y = f(x)$ about the y-axis, the graph of $y = f(-x)$ is obtained.

If $a > 0$, zooming (when $a > 1$) or shrinking (when $0 < a < 1$) the graph of $y = f(x)$ by a factor of a in the y-direction, the graph of $y = af(x)$ is obtained.

If $a > 0$, shrinking (when $a > 1$) or zooming (when $0 < a < 1$) the graph of $y = f(x)$ by a factor of $\frac{1}{a}$ in the x-direction, the graph of $y = f(ax)$ is obtained.

1.4 Composite Functions

1.5 One to One Function, Onto Function and One to One Onto Function

1.6 Inverse Functions

Chapter 2

Exponents and Logarithms

2.1 Exponents

2.2 Logarithms

2.3 Arithmetic Properties of Logarithms and Base Changing Formula

2.4 Exponential Equations

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2.6 Compound Interest and Annuity

Chapter 3

Limits

3.1 Concept of Limits

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3.3 Arithmetic Properties of Limits of Functions

Chapter 4

Differentiation

4.1 Gradient of Tangent Line on a Curve

4.2 Gradient of Tangent Line and Derivative

4.3 Law of Differentiation

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4.5 Higher Order Derivatives

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