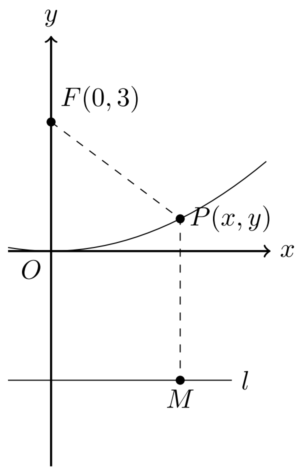
Exercise 5b

1. Find the equation of the parabola of which its vertex is at point $(0,0)$ and satisfies the following criteria:

   (a) The focus is at point $(0,3)$.

   Sol.

   Since the vertex is at $(0,0)$, the equation of the directrix is $y=-3$.
   Let the coordinates of the moving point P be (x,y).
   From the definition $|PF|=|PM|$,
   we have

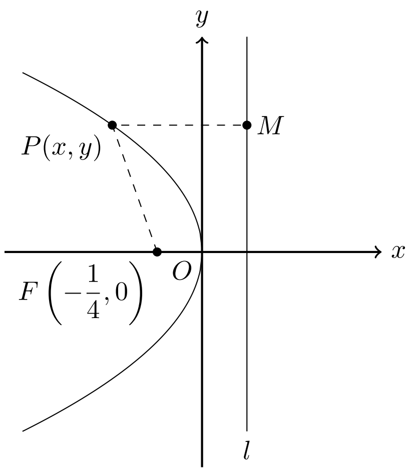


$$\begin{aligned}\sqrt{x^2+(y-3)^2} &= |y-(-3)| \\ x^2+(y-3)^2 &= (y+3)^2 \\ x^2+y^2-6y+9 &= y^2+6y+9 \\ x^2 &= 12y \qquad \square\end{aligned}$$

   (b) The directrix is $x-\dfrac{1}{4}=0$.

   Sol.

   Since the vertex is at $(0,0)$, the equation of the directrix is $x=\dfrac{1}{4}$
   we know that the focus is at $\left(-\dfrac{1}{4},0\right)$.
   Let the coordinates of the moving point P be (x,y).
   From the definition $|PF|=|PM|$,
   we have



$$\begin{aligned}\sqrt{\left(x+\frac{1}{4}\right)^2+y^2} &= \left|x-\frac{1}{4}\right| \\ \left(x+\frac{1}{4}\right)^2+y^2 &= \left(x-\frac{1}{4}\right)^2 \\ x^2+\frac{1}{2}x+\frac{1}{16}+y^2 &= x^2-\frac{1}{2}x+\frac{1}{16} \\ y^2 &= -x \qquad \square\end{aligned}$$

(c) Its focus is on the x-axis and passes through point $(5, -4)$.

Sol.

Since the focus is on the x-axis, we let the equation of the curve be $y^2 = 4ax$.

Substituting the coordinates of the point $(5, -4)$ into the equation, we have

$$\begin{aligned}(-4)^2 &= 4a(5)\\ 16 &= 20a\\ a &= \frac{4}{5}\end{aligned}$$

$\therefore$ the equation of the curve is $y^2 = \dfrac{16}{5}x$.

(d) Its focus is on the y-axis and passes through point $(-4, 2)$.

Sol.

Since the focus is on the y-axis, we let the equation of the curve be $x^2 = 4ay$.

Substituting the coordinates of the point $(-4, 2)$ into the equation, we have

$$\begin{aligned}(-4)^2 &= 4a(2)\\ 16 &= 8a\\ b &= 2\end{aligned}$$

$\therefore$ the equation of the curve is $x^2 = 8y$.

(e) The distance from the vertex to the directrix is 2.

Sol.

We have two scenarios:

i. The focus is on the x-axis.
Since $|PF| = |PM|$, the foci are at $(\pm 1, 0)$, and the equations of the directrices are $x = \mp 1$.
Let the equations of the curves be $y^2 = 4ax$.
Since the latus rectum is $|4 \cdot (\pm 1)| = 4$, the curves pass through $(\pm 1, \pm 2)$.
Substituting the coordinates of the point $(\pm 1, \pm 2)$ into the equation, we have we have

$$\begin{aligned}(\pm 2)^2 &= 4a(\pm 1)\\ 4 &= \pm 4a\\ a &= \pm 1\end{aligned}$$

$\therefore$ the equation of the curve is $y^2 = \pm 4x$.

ii. The focus is on the y-axis.
Since $|PF| = |PM|$, the foci are at $(0, \pm 1)$, and the equations of the directrices are $y = \mp 1$.
Let the equations of the curves be $x^2 = 4ay$.
Since the latus rectum is $|4 \cdot (\pm 1)| = 4$, the curves pass through $(\pm 2, \pm 1)$.
Substituting the coordinates of the point $(\pm 2, \pm 1)$ into the equation, we have we have

$$\begin{aligned}(\pm 2)^2 &= 4a(\pm 1)\\ 4 &= \pm 4a\\ a &= \pm 1\end{aligned}$$

$\therefore$ the equation of the curve is $x^2 = \pm 4y$.

2. Find the coordinates of the focus point and the equation of the directrix of the following parabolas. Hence, sketch the graphs of the parabolas:

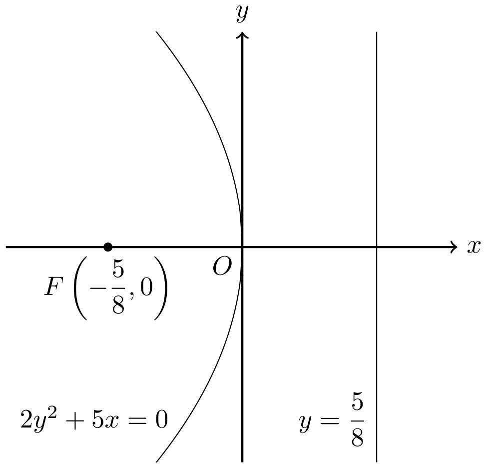
   (a) $2y^2 + 5x = 0$

   Sol.

$$\begin{aligned} 2y^2 + 5x &= 0 \\ y^2 &= -\frac{5}{2}x \\ a &= -\frac{5}{8} \end{aligned}$$

   The coordinates of the focus point are $\left(-\frac{5}{8}, 0\right)$,

   The equation of the directrices is $x = \frac{5}{8}$.



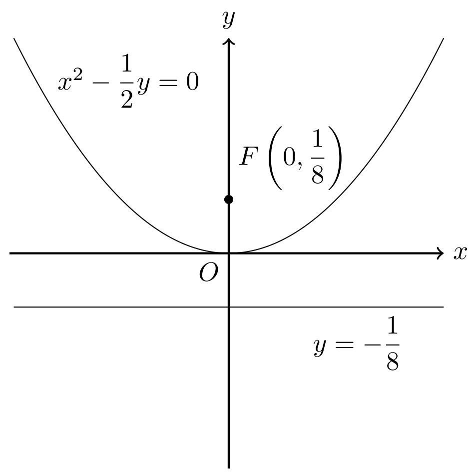
   (b) $x^2 - \frac{1}{2}y = 0$

   Sol.

$$\begin{aligned} x^2 - \frac{1}{2}y &= 0 \\ x^2 &= \frac{1}{2}y \\ a &= \frac{1}{8} \end{aligned}$$

   The coordinates of the focus point are $\left(0, \frac{1}{8}\right)$,

   The equation of the directrices is $y = -\frac{1}{8}$.



3. Find the coordinates of the vertex, the coordinates of the focus point, the axis of symmetry, the equation of the directrix, and the latus rectum of the following parabolas. Hence, sketch the graph.

   (a) $y^2 - 6x = 0$

   Sol.

$$\begin{aligned} y^2 - 6x &= 0 \\ y^2 &= 6x \\ a &= \frac{6}{4} = \frac{3}{2} \end{aligned}$$

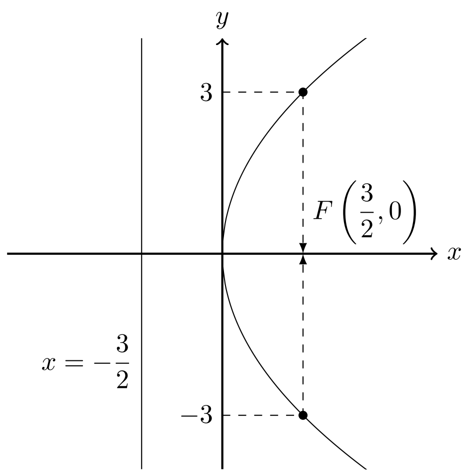
   The coordinates of the vertex are $(0, 0)$,

   The coordinates of the focus point are $\left(\frac{3}{2}, 0\right)$,

   The axis of symmetry is $y = 0$,

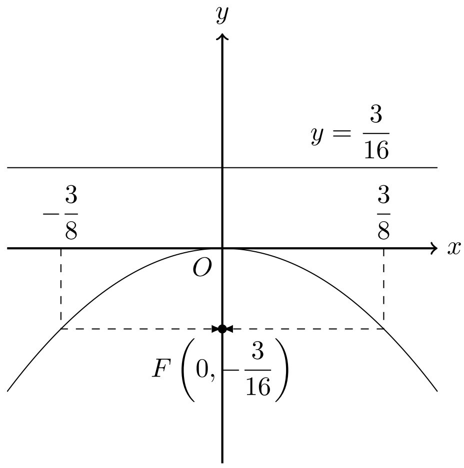
   The equation of the directrices is $x = -\frac{3}{2}$,

   The latus rectum is $\left|4 \cdot \frac{3}{2}\right| = 6$.



(b) $4x^2 + 3y = 0$

Sol.

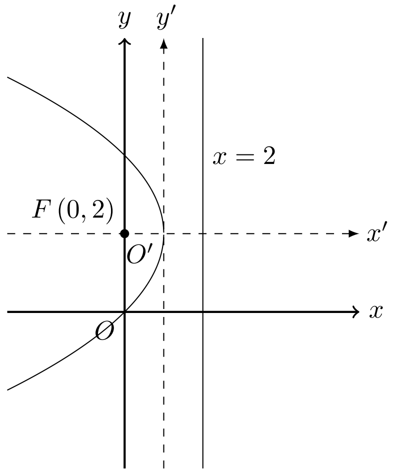


$$
\begin{aligned}
4x^2 + 3y &= 0 \\
x^2 &= -\frac{3}{4}y \\
a &= -\frac{3}{16}
\end{aligned}
$$

The coordinates of the vertex are $(0,0)$,
The coordinates of the focus point are $\left(0, -\frac{3}{16}\right)$,
The axis of symmetry is $x = 0$,
The equation of the directrices is $y = \frac{3}{16}$,
The latus rectum is $\left|4 \cdot \left(-\frac{3}{16}\right)\right| = \frac{3}{4}$.

(c) $y^2 + 4x - 4y = 0$

Sol.



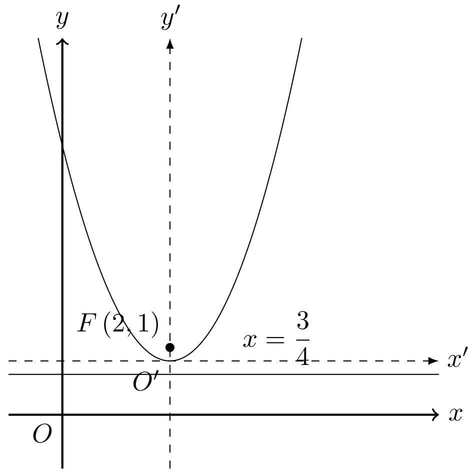
$$
\begin{aligned}
y^2 + 4x - 4y &= 0 \\
y^2 - 4y &= -4x \\
y^2 - 4y + 4 &= -4x + 4 \\
(y-2)^2 &= -4(x-1)
\end{aligned}
$$

Let $y' = y - 2$, $x' = x - 1$, we have $(y')^2 = -4x'$.
The coordinates of the vertex are $(0,0)$,
The coordinates of the focus point are $(-1,0)$,
The axis of symmetry is $y' = 0$,
The equation of the directrices is $y' = 1$,

In the original equation,
The coordinates of the vertex are $\begin{cases} 0 = x - 1 \\ 0 = y - 2 \end{cases}$, i.e. $(1,2)$,
The coordinates of the focus point are $\begin{cases} -1 = x - 1 \\ 0 = y - 2 \end{cases}$, i.e. $(0,2)$,
The axis of symmetry is $0 = y - 2$, i.e. $y = 2$,
The equation of the directrices is $1 = x - 1$, i.e. $x = 2$,
The latus rectum is $|4 \cdot (-1)| = 4$.

(d) $x^2 - 4x - y + 5 = 0$

Sol.



$$\begin{aligned} x^2 - 4x - y + 5 &= 0 \\ x^2 - 4x + 4 - y + 5 &= 4 \\ (x-2)^2 &= y - 1 \end{aligned}$$

Let $x' = x - 2$, $y' = y - 1$, we have $(x')^2 = y'$.
The coordinates of the vertex are $(0, 0)$,
The coordinates of the focus point are $\left(0, \frac{1}{4}\right)$,
The axis of symmetry is $x' = 0$,
The equation of the directrices is $x' = -\frac{1}{4}$,

In the original equation,
The coordinates of the vertex are $\begin{cases} 0 = x - 2 \\ 0 = y - 1 \end{cases}$, i.e. $(2, 1)$,
The coordinates of the focus point are $\begin{cases} 0 = x - 2 \\ \frac{1}{4} = y - 1 \end{cases}$, i.e. $\left(2, \frac{5}{4}\right)$,
The axis of symmetry is $0 = x - 2$, i.e. $x = 2$,
The equation of the directrices is $-\frac{1}{4} = y - 1$, i.e. $y = \frac{3}{4}$,
The latus rectum is $\left|4 \cdot \frac{1}{4}\right| = 1$.

4. The distance between a point on the parabola $y^2 = 12x$ and its focus point is 9, find the coordinates of this point.

Sol.
The vertex of the parabola is $(0, 0)$, the focus point is $(3, 0)$. Let the point be $\left(\frac{y^2}{12}, y\right)$, then

$$\begin{aligned} \sqrt{\left(\frac{y^2}{12} - 3\right)^2 + y^2} &= 9 \\ \frac{y^4}{144} - \frac{y^2}{2} + 9 + y^2 &= 81 \\ \frac{y^4}{144} + \frac{y^2}{2} &= 72 \\ y^4 + 72y^2 - 10368 &= 0 \\ (y^2 + 144)(y^2 - 72) &= 0 \\ y^2 = 72 \text{ or } y^2 &= -144 (\text{ rejected}) \\ y &= \pm 6\sqrt{2} \\ x &= \frac{\left(\pm 6\sqrt{2}\right)^2}{12} = 6 \end{aligned}$$

Therefore, the coordinates of the point are $\left(6, \pm 6\sqrt{2}\right)$.

5. Find the equation of the parabola that has the vertex of $(3, -2)$ and the directrix of $y + 8 = 0$. Hence, sketch the graph of the parabola.

Sol.

Let the equation of the parabola be $(x-3)^2 = 4a(y+2)$.

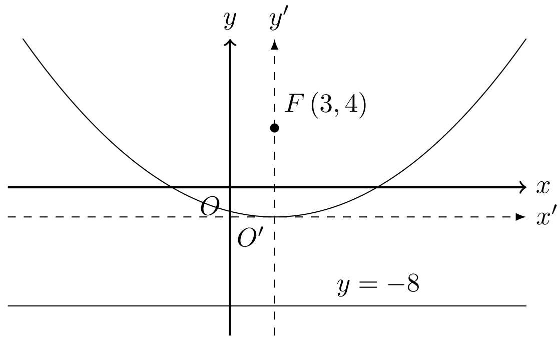
Let $x' = x - 3$, $y' = y + 2$, we have $(x')^2 = 4ay'$.

The equation of the directrix is $y = -8 \ \cdots \ (1)$,

$y + 2 = y'$, $y = y' - 2 \ \cdots \ (2)$

Substituting (1) into (2),

$$y' - 2 = -8$$
$$y' + 6 = 0$$



Hence, $a = 6$

Therefore, the equation of the parabola is

$$\begin{aligned}(x-3)^2 &= 24(y+2)\\ x^2 - 6x + 9 &= 24y + 48\\ x^2 - 6x - 24y - 39 &= 0 \quad \square\end{aligned}$$

The coordinates of the focus point are $\begin{cases} 0 = x - 3 \\ 6 = y + 2 \end{cases}$,

i.e. $(3, 4)$,