

Praktis 3 Integration

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Praktis Formatif

3.1 Integration as the Inverse of Differentiation

1. (a) Given $\frac{d}{dx}(2x^3 + 5x^2 - 7x) = 6x^2 + 10x - 7$, find $\int 6x^2 + 10x - 7 \, dx$.

Sol.

$$\int 6x^2 + 10x - 7 \, dx = 2x^3 + 5x^2 - 7x \quad \square$$

- (b) Given $\frac{d}{dx}(5x^4 + 3x^2 + x) = 20x^3 + 6x + 1$, find $\int 20x^3 + 6x + 1 \, dx$.

Sol.

$$\int 20x^3 + 6x + 1 \, dx = 5x^4 + 3x^2 + x \quad \square$$

2. (a) Given $\frac{d}{dx}(4x - 5x^2 + 2x^3) = 4 - 10x + 6x^2$, find $\int 2 - 5x + 3x^2 \, dx$.

Sol.

$$\begin{aligned} \int 2 - 5x + 3x^2 \, dx &= \frac{2}{2} \int 2 - 5x + 3x^2 \, dx \\ &= \frac{1}{2} \int 4 - 10x + 6x^2 \, dx \\ &= \frac{1}{2} (4x - 5x^2 + 2x^3) \\ &= 2x - \frac{5}{2}x^2 + x^3 \quad \square \end{aligned}$$

- (b) Given $\frac{d}{dx}\left(2x - \frac{3}{x^4}\right) = 2 + \frac{12}{x^5}$, find $\int 6 + \frac{36}{x^5} \, dx$.

Sol.

$$\begin{aligned} \int 6 + \frac{36}{x^5} \, dx &= 6 \int 1 + \frac{6}{x^5} \, dx \\ &= 3 \int 2 + \frac{12}{x^5} \, dx \\ &= 3 \left(2x - \frac{3}{x^4} \right) \\ &= 6x - \frac{9}{x^4} \quad \square \end{aligned}$$

- (c) Given $f(x) = \frac{d}{dx}[g(x)]$, find $\int 2f(x) \, dx$.

Sol.

$$\begin{aligned} \int 2f(x) \, dx &= 2 \int f(x) \, dx \\ &= 2g(x) \quad \square \end{aligned}$$

- (d) Differentiate $\frac{2x^2}{3x-1}$ with respect to x and hence, find $\int \frac{6x(3x-2)}{(3x-1)^2} \, dx$.

Sol.

$$\begin{aligned} \frac{d}{dx} \left(\frac{2x^2}{3x-1} \right) &= \frac{4x(3x-1) - 3(2x^2)}{(3x-1)^2} \\ &= \frac{12x^2 - 4x - 6x^2}{(3x-1)^2} \\ &= \frac{6x^2 - 4x}{(3x-1)^2} \\ &= \frac{2x(3x-2)}{(3x-1)^2} \\ \int \frac{6x(3x-2)}{(3x-1)^2} \, dx &= 3 \int \frac{2x(3x-2)}{(3x-1)^2} \, dx \\ &= 3 \left(\frac{2x^2}{3x-1} \right) \\ &= \frac{6x^2}{3x-1} \quad \square \end{aligned}$$

3. The daily production of bread of a bakery shop is given by the function $R(x) = -50(x^2 - 12x)$, where x represents the number of bakers who work in the shop with condition x is not more than 6.

- (a) Find the rate of daily production of bread in terms of x .

Sol.

$$R'(x) = -100x + 600 \quad \square$$

- (b) If the rate of daily production of bread becomes $300 - 50x$ on a particular day, calculate the revenue of the bakery shop if all the loaves of bread baked by three bakers on that day are sold out at a price of RM5.50 for each loaf.

Sol.

$$\begin{aligned} \int 300 - 50x \, dx &= \frac{1}{2} \int (600 - 100x) \, dx \\ &= \frac{1}{2} (-50x^2 + 600x) \\ &= -25x^2 + 300x \\ R(3) &= -25(3)^2 + 300(3) \\ &= -225 + 900 \\ &= 675 \end{aligned}$$

$$\begin{aligned} \text{Revenue} &= 675 \times 5.50 \\ &= \text{RM}3712.50 \quad \square \end{aligned}$$

4. Given $f(x) = x^4 - 2x^3$ and $f'(x) = 4x^3 - 6x^2$. Express $f'(x) \int f'(x) \, dx$ in factored form.

Sol.

$$\begin{aligned} f'(x) \int f'(x) \, dx &= (4x^3 - 6x^2)(x^4 - 2x^3) \\ &= 2x^5(2x-3)(x-2) \quad \square \end{aligned}$$

5. Given $y = \frac{2x - 6}{x}$.

(a) Find $\frac{dy}{dx}$.

Sol.

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x - 2x - 6}{x^2} \\ &= -\frac{6}{x^2} \quad \square\end{aligned}$$

(b) Solve $4 + \int \left(\frac{dy}{dx}\right) dx = 0$.

Sol.

$$\begin{aligned}4 + \int \left(\frac{dy}{dx}\right) dx &= 0 \\ 4 + \int \left(-\frac{6}{x^2}\right) dx &= 0 \\ 4 + \frac{2x - 6}{x} &= 0 \\ 4x + 2x - 6 &= 0 \\ 6x &= 6 \\ x &= 1 \quad \square\end{aligned}$$

6. Given $f'(x) = g(x)$. Find $\frac{3f(x)}{\int g(x)dx}$.

Sol.

$$\begin{aligned}f'(x) &= g(x) \\ f(x) &= \int g(x)dx \\ \frac{3f(x)}{\int g(x)dx} &= \frac{3f(x)}{f(x)} \\ &= 3 \quad \square\end{aligned}$$

7. The population of town A is given by a function $P(t) = \frac{5}{6}(2.72^{1.2t}) - t^2 + 1495$ and the population continues to increase at the rate of $2.72^{1.2t} - 2t$ people per year where t is the number of years. Given that the population of town B increases at twice the rate of the population of town A based on the same model, find, to the nearest integer,

(a) the rate of increase of the population of town B at $t = 5$ years.

Sol.

$$\begin{aligned}P'_B(5) &= 2[2.72^{1.2(5)} - 2(5)] \\ &= 2[404.96 - 10] \\ &= 2(394.96) \\ &= 789.92 \\ &= 790 \text{ people per year} \quad \square\end{aligned}$$

(b) the population of town B after 5 years.

Sol.

$$\begin{aligned}P_B(5) &= 2 \left[\frac{5}{6}(2.72^{1.2 \cdot 5}) - (5)^2 + 1495 \right] \\ &= \frac{5}{3}(2.72^6) - 50 + 2990 \\ &= 3614.93 \\ &= 3615 \text{ people} \quad \square\end{aligned}$$

3.2 Indefinite Integral

8. By using the indefinite integral formula, find the integral of each of the following constants or algebraic functions.

(a) $\int 3 dx$

Sol.

$$\int 3 dx = 3x + C \quad \square$$

(b) $\int 24x dx$

Sol.

$$\int 24x dx = 12x^2 + C \quad \square$$

(c) $\int 6x^2 dx$

Sol.

$$\int 6x^2 dx = 2x^3 + C \quad \square$$

(d) $\int 3x^2 + 4x dx$

Sol.

$$\int 3x^2 + 4x dx = x^3 + 2x^2 + C \quad \square$$

(e) $\int \frac{2}{x^4} dx$

Sol.

$$\int \frac{2}{x^4} dx = -\frac{2}{x^3} + C \quad \square$$

(f) $\int x^2(x - 3) dx$

Sol.

$$\begin{aligned}\int x^2(x - 3) dx &= \int x^3 - 3x^2 dx \\ &= \frac{1}{4}x^4 - x^3 + C \quad \square\end{aligned}$$

(g) $\int (x + 2)(2x^4 - 1) dx$

Sol.

$$\begin{aligned}\int (x + 2)(2x^4 - 1) dx &= \int 2x^5 - x + 4x^4 - 2 \\ &= \frac{1}{3}x^6 + \frac{4}{5}x^5 - \frac{1}{2}x^2 - 2x + C \quad \square\end{aligned}$$

(h) $\int \frac{x^2 + 3x + 2}{x + 2} dx$

Sol.

$$\begin{aligned}\int \frac{x^2 + 3x + 2}{x + 2} dx &= \int \frac{(x + 2)(x + 1)}{x + 2} dx \\ &= \int x + 1 dx \\ &= \frac{1}{2}x^2 + x + C \quad \square\end{aligned}$$

9. Find the indefinite integral for each of the following by using

(a) the substitution method.

(b) the indefinite integral formula.

i. $\int \frac{2}{(x + 2)^5} dx$

Sol.

(a) Let $v = (x + 2)$.

$$\begin{aligned}\int \frac{2}{(x + 2)^5} dx &= \int \frac{2}{v^5} dv \\ &= \int 2v^{-5} dv \\ &= -\frac{1}{2}v^{-4} + C \\ &= -\frac{1}{2v^4} + C \\ &= -\frac{1}{2(x + 2)^4} + C \quad \square\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{2}{(x + 2)^5} dx &= \int 2(x + 2)^{-5} dx \\ &= 2 \int (x + 2)^{-5} dx \\ &= 2 \left[\frac{(x + 2)^{-4}}{-4} \right] + C \\ &= -\frac{1}{2(x + 2)^4} + C \quad \square\end{aligned}$$

ii. $\int \frac{3}{5}(3x + 2)^8 dx$

Sol.

(a) Let $v = 3x + 2$, $\frac{dv}{dx} = 3$.

$$\begin{aligned}\int \frac{3}{5}(3x + 2)^8 dx &= \int \frac{3}{5}v^8 dv \\ &= \int \frac{3}{5}v^8 \frac{dv}{3} \\ &= \int \frac{1}{5}v^8 dv \\ &= \frac{1}{45}v^9 + C \\ &= \frac{(3x + 2)^9}{45} + C \quad \square\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{3}{5}(3x + 2)^8 dx &= \frac{3}{5} \int (3x + 2)^8 dx \\ &= \frac{3}{5} \left[\frac{(3x + 2)^9}{27} \right] + C \\ &= \frac{(3x + 2)^9}{45} + C \quad \square\end{aligned}$$

10. Determine the equation of a curve based on the following information.

(a) The gradient function of the curve is $\frac{dy}{dx} = 3x^2 + x - 2$ and it passes through the point $p(2, 15)$.

Sol.

$$\begin{aligned}\frac{dy}{dx} &= 3x^2 + x - 2 \\ y &= \int 3x^2 + x - 2 dx \\ &= x^3 + \frac{x^2}{2} - 2x + C\end{aligned}$$

When $x = 2, y = 15$,

$$\begin{aligned}15 &= 2^3 + \frac{2^2}{2} - 2(2) + C \\ 15 &= 8 + 2 - 4 + C \\ 15 &= 6 + C \\ C &= 9\end{aligned}$$

Hence, the equation of the curve is $y = x^3 + \frac{x^2}{2} - 2x + 9$. \square

(b) The gradient function of the curve is $f'(x) = 2x + 9$ and $f(3) = 21$.

Sol.

$$\begin{aligned}f'(x) &= 2x + 9 \\ f(x) &= \int 2x + 9 dx \\ &= x^2 + 9x + C \\ f(3) &= 3^2 + 9(3) + C \\ 21 &= 9 + 27 + C \\ C &= -15\end{aligned}$$

Hence, the equation of the curve is $f(x) = x^2 + 9x - 15$. \square

(c) The gradient function of the curve is given by $g(t) = \frac{5t^2 - 6t + 1}{t^3(t - 1)}$ and it passes through the point $(1, 3)$.

Sol.

$$\begin{aligned}g(t) &= \frac{5t^2 - 6t + 1}{t^3(t-1)} \\&= \frac{(5t-1)(t-1)}{t^3(t-1)} \\&= \frac{5t-1}{t^3} \\&= \frac{5}{t^2} - \frac{1}{t^3} \\&= 5t^{-2} - t^{-3} \\f(t) &= \int 5t^{-2} - t^{-3} dt \\&= -\frac{5}{t} + \frac{1}{2t^2} + C\end{aligned}$$

When $t = 1$, $f(1) = 3$,

$$\begin{aligned}3 &= -5 + \frac{1}{2} + C \\3 &= -\frac{9}{2} + C \\C &= \frac{15}{2}\end{aligned}$$

Hence, the equation of the curve is $f(t) = -\frac{5}{t} + \frac{1}{2t^2} + \frac{15}{2}$. \square

11. Tommy moves in his roller skates at the rate of change in displacement, $\frac{ds}{dt} = t^2 + 9$ metres per second, where t is the time in seconds. At $t = 3$ seconds, Tommy is 4 metres away from his starting place. Find the displacement, s metres, when $t = 10$ seconds.

Sol.

$$\begin{aligned}\frac{ds}{dt} &= t^2 + 9 \\s &= \int t^2 + 9 dt \\&= \frac{t^3}{3} + 9t + C\end{aligned}$$

When $t = 3$, $s = 4$,

$$\begin{aligned}4 &= \frac{3^3}{3} + 9(3) + C \\4 &= 9 + 27 + C \\4 &= 36 + C \\C &= -32 \\s &= \frac{t^3}{3} + 9t - 32\end{aligned}$$

When $t = 10$,

$$\begin{aligned}s &= \frac{10^3}{3} + 9(10) - 32 \\&= 333 + 90 - 32 \\&= 391\frac{1}{3} \text{ m} \quad \square\end{aligned}$$

12. Given the gradient function of a curve is $\frac{dy}{dx} = kx^2 + 2x$ where k is a constant. The curve passes through point $A(1, 6)$ and point $B(-2, 0)$. Determine the equation of the curve.

Sol.

$$\begin{aligned}\frac{dy}{dx} &= kx^2 + 2x \\y &= \int kx^2 + 2x dx \\&= \frac{kx^3}{3} + x^2 + C\end{aligned}$$

When $x = 1$, $y = 6$,

$$\begin{aligned}6 &= \frac{k}{3} + 1 + C \\k + 3C &= 15 \quad (1)\end{aligned}$$

When $x = -2$, $y = 0$,

$$\begin{aligned}0 &= -\frac{8k}{3} + 4 + C \\8k - 3C &= 12 \quad (2)\end{aligned}$$

$$\begin{aligned}(2) + (1) \cdot 8 : 9k &= 27 \\k &= 3 \\C &= 4\end{aligned}$$

Hence, the equation of the curve is $y = x^3 + x^2 + 4$. \square

3.3 Definite Integral

13. Calculate each of the following.

(a) $\int_2^1 \left(\sqrt{x} + \frac{1}{x} \right)$

Sol.

$$\begin{aligned}\int_2^1 \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) &= \int_1^2 \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}} \right) \\&= \left[\frac{2\sqrt{x^3}}{3} + 2\sqrt{x} \right]_1^2 \\&= \left[\frac{4\sqrt{2}}{3} + 2\sqrt{2} \right] - \left[\frac{2}{3} + 2 \right] \\&= \frac{10\sqrt{2}}{3} - \frac{8}{3} \\&= \frac{10\sqrt{2} - 8}{3} \\&\approx 2.0474 \quad \square\end{aligned}$$

(b) $\int_0^3 \left(\frac{x^4 + 3x}{x} \right) dx$

Sol.

$$\begin{aligned}\int_0^3 \left(\frac{x^4 + 3x}{x} \right) dx &= \int_0^3 (x^3 + 3) dx \\&= \left[\frac{1}{4}x^4 + 3x \right]_0^3 \\&= \left[\frac{1}{4}(3^4) + 3 \cdot 3 \right] - 0 \\&= \frac{81}{4} + 9 \\&= \frac{117}{4} \\&= 29.25 \quad \square\end{aligned}$$

(c) $\int_{-2}^{-1} \left(\frac{(4-x)(3-x)}{x^5} \right) dx$

Sol.

$$\begin{aligned}\int_{-2}^{-1} \left(\frac{(4-x)(3-x)}{x^5} \right) dx &= \int_{-2}^{-1} \left(\frac{x^2 - 7x + 12}{x^5} \right) dx \\&= \int_{-2}^{-1} \left(\frac{1}{x^3} - \frac{7}{x^4} + \frac{12}{x^5} \right) dx \\&= \left[-\frac{1}{2x^2} + \frac{7}{3x^3} - \frac{3}{x^4} \right]_{-2}^{-1} \\&= \left[-\frac{1}{2} - \frac{7}{3} - 3 \right] - \left[-\frac{1}{8} - \frac{7}{24} - \frac{3}{16} \right] \\&= -\frac{35}{6} + \frac{29}{48} \\&= -5\frac{11}{48} \quad \square\end{aligned}$$

14. Given $\int_a^b f(x) dx = 5$, $\int_b^c f(x) dx = 8$ and

$\int_a^b g(x) dx = 2$. Find each of the following.

[answer can be in terms of a and/or b .]

(a) $\int_a^b 3f(x) dx$

Sol.

$$\begin{aligned}\int_a^b 3f(x) dx &= 3 \int_a^b f(x) dx \\&= 3 \cdot 5 \\&= 15 \quad \square\end{aligned}$$

(b) $\int_a^c f(x) dx$

Sol.

$$\begin{aligned}\int_a^c f(x) dx &= \int_a^b f(x) dx + \int_b^c f(x) dx \\&= 5 + 8 \\&= 13 \quad \square\end{aligned}$$

(c) $\int_a^b [f(x) + g(x)] dx$

Sol.

$$\begin{aligned}\int_a^b [f(x) + g(x)] dx &= \int_a^b f(x) dx - \int_a^b g(x) dx \\&= 5 - 2 \\&= 3 \quad \square\end{aligned}$$

(d) $\int_c^a f(x) dx$

Sol.

$$\begin{aligned}\int_c^a f(x) dx &= - \int_a^c f(x) dx \\&= -13 \quad \square\end{aligned}$$

(e) $\int_a^b [g(x) + 3] dx$

Sol.

$$\begin{aligned}\int_a^b [g(x) + 3] dx &= \int_a^b g(x) dx + \int_a^b 3 dx \\&= 2 + 3(b-a) \\&= 3b - 3a + 2 \quad \square\end{aligned}$$

(f) $\int_a^a f(x) dx$

Sol.

$$\int_a^a f(x) dx = 0 \quad \square$$

(g) The value of k such that $\int_b^a [f(x) + kx] dx = 25$ if $a = 1$ and $b = 4$.

Sol.

$$\begin{aligned}\int_b^a [f(x) + kx] dx &= \int_b^a f(x) dx + \int_b^a kx dx \\&= -5 + \int_b^a kx dx\end{aligned}$$

$$-5 + \int_1^4 kx dx = 25$$

$$\int_1^4 kx dx = 30$$

$$k \left[\frac{x^2}{2} \right]_1^4 = 30$$

$$k \left(\frac{1}{2} - \frac{16}{2} \right) = 30$$

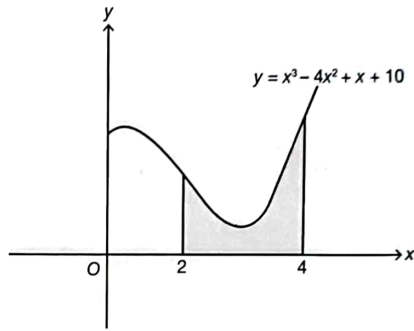
$$-\frac{15k}{2} = 30$$

$$-15k = 60$$

$$k = -4 \quad \square$$

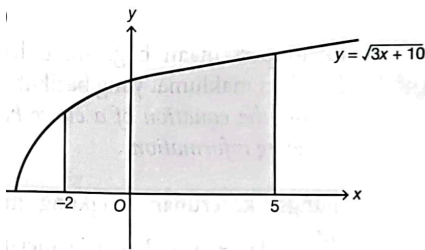
15. Find the area of the shaded region for each of the following diagrams.

(a)

**Sol.**

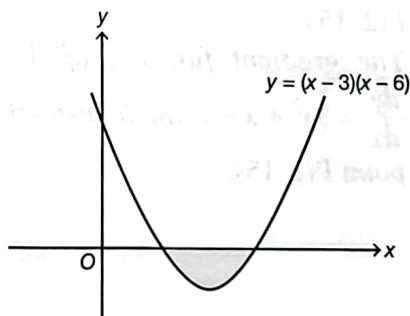
$$\begin{aligned}
 A &= \int_2^4 (x^3 - 4x^2 + x + 10) dx \\
 &= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{1}{2}x^2 + 10x \right]_2^4 \\
 &= \left(64 - \frac{256}{3} + 8 + 40 \right) - \left(4 - \frac{32}{3} + 2 + 20 \right) \\
 &= \frac{80}{3} - \frac{46}{3} \\
 &= 11\frac{1}{3} \text{ units}^2 \quad \square
 \end{aligned}$$

(b)

**Sol.**

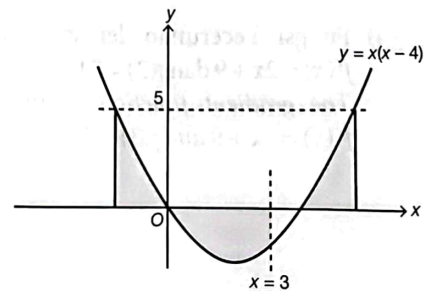
$$\begin{aligned}
 A &= \int_{-2}^5 \sqrt{3x+10} dx \\
 &= \int_{-2}^5 (3x+10)^{\frac{1}{2}} dx \\
 &= \left[\frac{2(3x+10)^{\frac{3}{2}}}{9} \right]_{-2}^5 \\
 &= \frac{2(25)^{\frac{3}{2}}}{9} - \frac{2(4)^{\frac{3}{2}}}{9} \\
 &= \frac{250}{9} - \frac{16}{9} \\
 &= \frac{234}{9} \\
 &= 26 \text{ units}^2 \quad \square
 \end{aligned}$$

(c)

**Sol.**

$$\begin{aligned}
 A &= \left| \int_3^6 (x-3)(x-6) dx \right| \\
 &= \left| \int_3^6 (x^2 - 9x + 18) dx \right| \\
 &= \left| \left[\frac{1}{3}x^3 - \frac{9}{2}x^2 + 18x \right]_3^6 \right| \\
 &= \left| (72 - 162 + 108) - \left(9 - \frac{81}{2} + 54 \right) \right| \\
 &= \left| 18 - \frac{45}{2} \right| \\
 &= 4.5 \text{ units}^2 \quad \square
 \end{aligned}$$

(d)

**Sol.**When $y = 5$,

$$\begin{aligned}
 x(x-4) &= 5 \\
 x^2 - 4x - 5 &= 0 \\
 (x-5)(x+1) &= 0 \\
 x &= -1 \text{ or } x = 5
 \end{aligned}$$

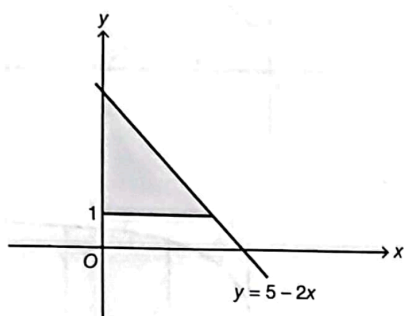
When $y = 0$,

$$\begin{aligned}
 x(x-4) &= 0 \\
 x &= 0 \text{ or } x = 4
 \end{aligned}$$

$$\begin{aligned}
 A &= \int_{-1}^0 x(x-4) dx + \left| \int_0^3 x(x-4) dx \right| \\
 &\quad + \int_4^5 x(x-4) dx \\
 &= \int_{-1}^0 (x^2 - 4x) dx + \left| \int_0^3 (x^2 - 4x) dx \right| \\
 &\quad + \int_4^5 (x^2 - 4x) dx \\
 &= \left[\frac{1}{3}x^3 - 2x^2 \right]_{-1}^0 + \left| \left[\frac{1}{3}x^3 - 2x^2 \right]_0^3 \right| \\
 &\quad + \left[\frac{1}{3}x^3 - 2x^2 \right]_4^5 \\
 &= 0 - \left(-\frac{1}{3} - 2 \right) + |(9 - 18) - 0| \\
 &\quad + \left(\frac{125}{3} - 50 \right) - \left(\frac{64}{3} - 32 \right) \\
 &= 13\frac{2}{3} \text{ units}^2 \quad \square
 \end{aligned}$$

16. Determine the area bounded by the curve, the horizontal line(s) and the y-axis.

(a)



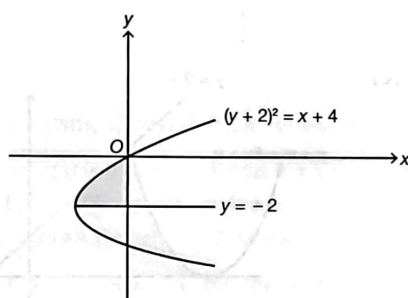
Sol. When $x = 0$,

$$\begin{aligned} y &= 5 - 2x \\ y &= 5 - 2(0) \\ &= 5 \end{aligned}$$

$$\begin{aligned} y &= 5 - 2x \\ 2x &= 5 - y \\ x &= \frac{5 - y}{2} \end{aligned}$$

$$\begin{aligned} A &= \int_1^5 \frac{5 - y}{2} dy \\ &= \int_1^5 \left(\frac{5}{2} - \frac{1}{2}y \right) dy \\ &= \left[\frac{5}{2}y - \frac{1}{4}y^2 \right]_1^5 \\ &= \left(\frac{25}{2} - \frac{25}{4} \right) - \left(\frac{5}{2} - \frac{1}{4} \right) \\ &= \frac{25}{4} - \frac{9}{4} \\ &= 4 \text{ units}^2 \quad \square \end{aligned}$$

(b)

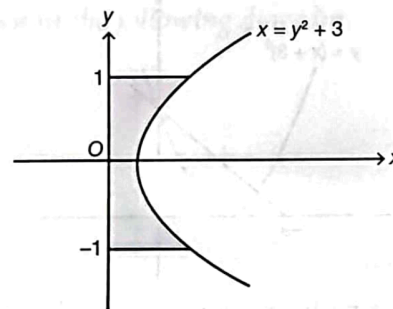


Sol.

$$\begin{aligned} (y + 2)^2 &= x + 4 \\ x &= (y + 2)^2 - 4 \\ &= y^2 + 4y + 4 - 4 \\ &= y^2 + 4y \end{aligned}$$

$$\begin{aligned} A &= \int_{-2}^0 (y^2 + 4y) dy \\ &= \left[\frac{1}{3}y^3 + 2y^2 \right]_{-2}^0 \\ &= 0 - \left(-\frac{8}{3} + 8 \right) \\ &= 5\frac{1}{3} \text{ units}^2 \quad \square \end{aligned}$$

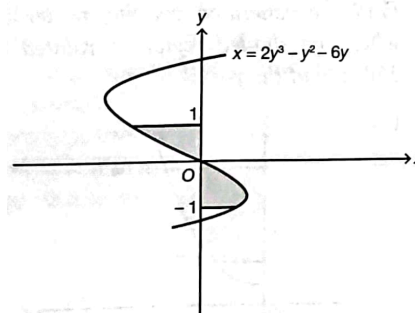
(c)



Sol.

$$\begin{aligned} A &= \int_{-1}^1 (y^2 + 3) dy \\ &= \left[\frac{1}{3}y^3 + 3y \right]_{-1}^1 \\ &= \left(\frac{1}{3} + 3 \right) - \left(-\frac{1}{3} - 3 \right) \\ &= 6\frac{2}{3} \text{ units}^2 \quad \square \end{aligned}$$

(d)

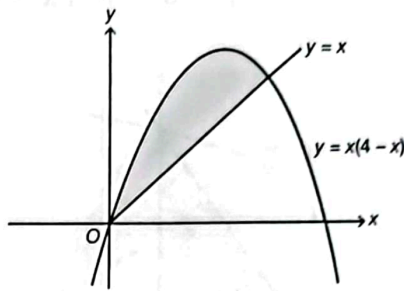


Sol.

$$\begin{aligned} A &= \int_{-1}^0 (2y^3 - y^2 - 6y) dy + \left| \int_0^1 (2y^3 - y^2 - 6y) dy \right| \\ &= \left[\frac{1}{2}y^4 - \frac{1}{3}y^3 - 3y^2 \right]_{-1}^0 + \left| \left[\frac{1}{2}y^4 - \frac{1}{3}y^3 - 3y^2 \right]_0^1 \right| \\ &= 0 - \left(\frac{1}{2} + \frac{1}{3} - 3 \right) + \left| \left(\frac{1}{2} - \frac{1}{3} - 3 \right) - 0 \right| \\ &= \frac{13}{6} + \frac{17}{6} \\ &= 5 \text{ units}^2 \quad \square \end{aligned}$$

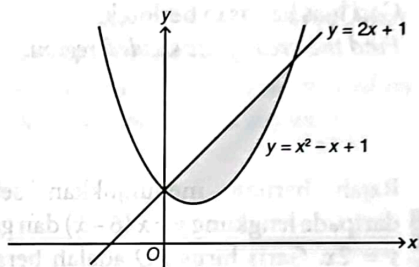
17. Find the area of the shaded region for each of the following.

(a)

**Sol.**

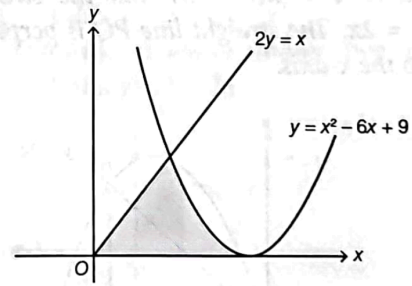
$$\begin{aligned}
 x &= x(4-x) \\
 x &= 4x - x^2 \\
 x^2 - 3x &= 0 \\
 x(x-3) &= 0 \\
 x &= 0 \text{ or } x = 3 \\
 A &= \int_0^3 [x(4-x) - x] dx \\
 &= \int_0^3 [4x - x^2 - x] dx \\
 &= \int_0^3 [3x - x^2] dx \\
 &= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3 \right]_0^3 \\
 &= \left(\frac{27}{2} - 9 \right) - 0 \\
 &= 4.5 \text{ units}^2 \quad \square
 \end{aligned}$$

(b)

**Sol.**

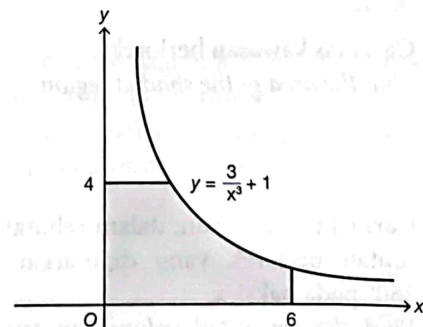
$$\begin{aligned}
 2x + 1 &= x^2 - x + 1 \\
 x^2 - 3x &= 0 \\
 x(x-3) &= 0 \\
 x &= 0 \text{ or } x = 3 \\
 A &= \int_0^3 [2x + 1 - x^2 + x - 1] dx \\
 &= \int_0^3 [-x^2 + 3x] dx \\
 &= \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2 \right]_0^3 \\
 &= \left(-9 + \frac{27}{2} \right) - 0 \\
 &= 4.5 \text{ units}^2 \quad \square
 \end{aligned}$$

(c)

**Sol.**

$$\begin{aligned}
 2y &= x \\
 y &= \frac{1}{2}x \\
 \frac{1}{2}x &= x^2 - 6x + 9 \\
 x &= 2x^2 - 12x + 18 \\
 2x^2 - 13x + 18 &= 0 \\
 (2x-9)(x-2) &= 0 \\
 x &= 2 \text{ or } x = 4.5 \\
 x^2 - 6x + 9 &= 0 \\
 (x-3)^2 &= 0 \\
 x &= 3 \\
 A &= \int_0^2 \frac{x}{2} dx + \int_2^3 (x^2 - 6x + 9) dx \\
 &= \left[\frac{1}{4}x^2 \right]_0^2 + \left[\frac{1}{3}x^3 - 3x^2 + 9x \right]_2^3 \\
 &= 1 - 0 + (9 - 27 + 27) - \left(\frac{8}{3} - 12 + 18 \right) \\
 &= 10 - \frac{26}{3} \\
 &= 1\frac{1}{3} \text{ units}^2 \quad \square
 \end{aligned}$$

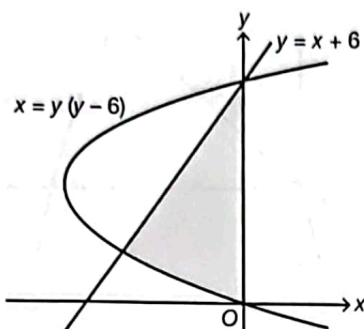
(d)

**Sol.**

$$\begin{aligned}
 \frac{3}{x^3} + 1 &= 4 \\
 \frac{3}{x^3} &= 3 \\
 x^3 &= 1 \\
 x &= 1
 \end{aligned}$$

$$\begin{aligned}
 A &= 1 \cdot 4 + \int_1^6 \left(\frac{3}{x^3} + 1 \right) dx \\
 &= 4 + \left[-\frac{3}{2x^2} + x \right]_1^6 \\
 &= 4 + \left(-\frac{1}{24} + 6 \right) - \left(-\frac{3}{2} + 6 \right) \\
 &= 4 + \frac{143}{24} + \frac{9}{2} \\
 &= 14\frac{11}{24} \text{ units}^2 \quad \square
 \end{aligned}$$

18. The following diagram shows a part of the curve $x = y(y - 6)$ and the straight line $y = x + 6$.



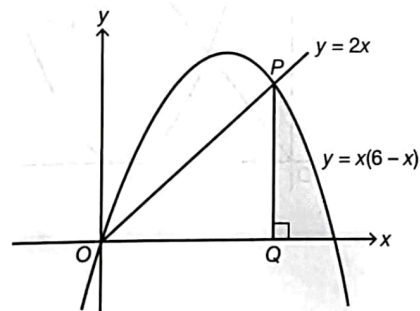
Find the area of the shaded region.

Sol.

$$\begin{aligned}
 y &= x + 6 \\
 x &= y - 6 \\
 y - 6 &= y(y - 6) \\
 &= y^2 - 6y \\
 y^2 - 7y + 6 &= 0 \\
 (y - 6)(y - 1) &= 0 \\
 y &= 6 \text{ or } y = 1 \\
 1 &= x + 6 \\
 x &= -5
 \end{aligned}$$

$$\begin{aligned}
 A &= \left| \int_0^1 y(y - 6) dy \right| + \frac{1}{2}(5)(5) \\
 &= \left| \int_0^1 (y^2 - 6y) dy \right| + \frac{25}{2} \\
 &= \left[\frac{1}{3}y^3 - 3y^2 \right]_0^1 + \frac{25}{2} \\
 &= \left| \frac{1}{3} - 3 - 0 \right| + \frac{25}{2} \\
 &= \frac{8}{3} + \frac{25}{2} \\
 &= 15\frac{1}{6} \text{ units}^2 \quad \square
 \end{aligned}$$

19. The following diagram shows a part of the curve $y = x(6 - x)$ and a straight line $y = 2x$. The straight line PQ is perpendicular to the x -axis.



Find the area of the shaded region.

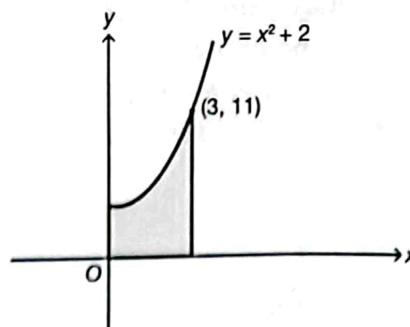
Sol.

$$\begin{aligned}
 x(6 - x) &= 0 \\
 x &= 0 \text{ or } x = 6 \\
 2x &= x(6 - x) \\
 2x &= 6x - x^2 \\
 x^2 - 4x &= 0 \\
 x(x - 4) &= 0 \\
 x &= 0 \text{ or } x = 4
 \end{aligned}$$

$$\begin{aligned}
 A &= \int_4^6 x(6 - x) dx \\
 &= \int_4^6 (6x - x^2) dx \\
 &= \left[3x^2 - \frac{1}{3}x^3 \right]_4^6 \\
 &= (108 - 72) - \left(48 - \frac{64}{3} \right) \\
 &= 36 - \frac{80}{3} \\
 &= 9\frac{1}{3} \text{ units}^2 \quad \square
 \end{aligned}$$

20. Find the generated volume, in terms of π , when the shaded region is rotated through 360° about the x -axis.

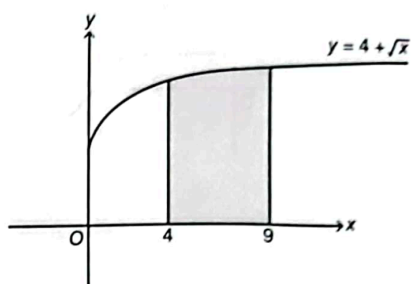
(a)



Sol.

$$\begin{aligned}
 V_x &= \int_0^3 \pi y^2 dx \\
 &= \pi \int_0^3 (x^2 + 2)^2 dx \\
 &= \pi \int_0^3 (x^4 + 4x^2 + 4) dx \\
 &= \pi \left[\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x \right]_0^3 \\
 &= \left[\left(\frac{243}{5} + 36 + 12 \right) - 0 \right] \pi \\
 &= 96.6\pi \text{ units}^3 \quad \square
 \end{aligned}$$

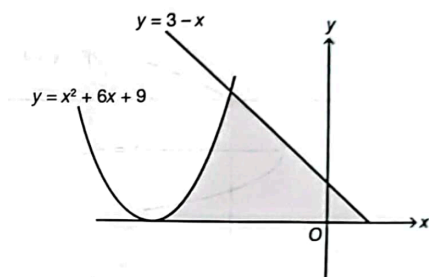
(b)



Sol.

$$\begin{aligned}
 V_x &= \int_4^9 \pi (4 + \sqrt{x})^2 dx \\
 &= \pi \int_4^9 (16 + 8\sqrt{x} + x) dx \\
 &= \pi \left[16x + \frac{16x^{\frac{3}{2}}}{3} + \frac{1}{2}x^2 \right]_4^9 \\
 &= \pi \left[\left(144 + 144 + \frac{81}{2} \right) - \left(64 + \frac{128}{3} + 8 \right) \right] \\
 &= 213.83\pi \text{ units}^3 \quad \square
 \end{aligned}$$

(c)

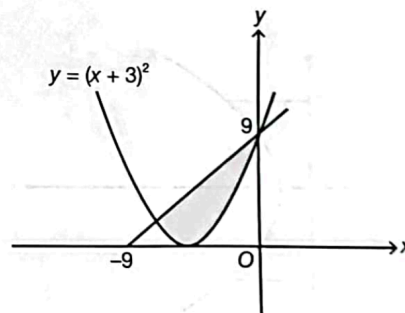


Sol.

$$\begin{aligned}
 x^2 + 6x + 9 &= 3 - x \\
 x^2 + 7x + 6 &= 0 \\
 (x + 6)(x + 1) &= 0 \\
 x &= -6 \text{ or } x = -1 \\
 x^2 + 6x + 9 &= 0 \\
 (x + 3)^2 &= 0 \\
 x &= -3 \\
 0 &= 3 - x \\
 x &= 3
 \end{aligned}$$

$$\begin{aligned}
 V_x &= \int_{-3}^{-1} \pi (x + 3)^4 dx + \int_{-1}^3 \pi (3 - x)^2 dx \\
 &= \pi \int_{-3}^{-1} (x + 3)^4 dx + \pi \int_{-1}^3 (3 - x)^2 dx \\
 &= \pi \left\{ \left[\frac{(x + 3)^5}{5} \right]_{-3}^{-1} + \left[-\frac{(3 - x)^3}{3} \right]_{-1}^3 \right\} \\
 &= \pi \left[\left(\frac{32}{5} - 0 \right) + \left(0 + \frac{64}{3} \right) \right] \\
 &= 27\frac{11}{15}\pi \text{ units}^3 \quad \square
 \end{aligned}$$

(d)



Sol.

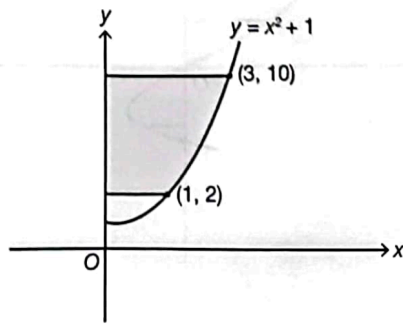
Let the line be l . Since l passes through $(-9, 0)$ and $(0, 9)$,

$$\begin{aligned}
 m_l &= \frac{9}{9} = 1 \\
 y - 9 &= 1(x - 0) \\
 y &= x + 9 \\
 x + 9 &= (x + 3)^2 \\
 &= x^2 + 6x + 9 \\
 x^2 + 5x &= 0 \\
 x(x + 5) &= 0 \\
 x &= 0 \text{ or } x = -5
 \end{aligned}$$

$$\begin{aligned}
 V_x &= \int_{-5}^0 \pi (x + 9)^2 dx - \int_{-5}^0 \pi (x + 3)^4 dx \\
 &= \pi \int_{-5}^0 (x + 9)^2 dx - \pi \int_{-5}^0 (x + 3)^4 dx \\
 &= \pi \left\{ \left[\frac{(x + 9)^3}{3} \right]_{-5}^0 - \left[\frac{(x + 3)^5}{5} \right]_{-5}^0 \right\} \\
 &= \pi \left[\left(243 - \frac{64}{3} \right) - \left(\frac{243}{5} + \frac{32}{5} \right) \right] \\
 &= 116\frac{2}{3}\pi \text{ units}^3 \quad \square
 \end{aligned}$$

21. Find the generated volume, in terms of π , when the shaded region is rotated through 360° about the y -axis.

(a)

**Sol.**

$$y = x^2 + 1$$

$$x^2 = y - 1$$

$$x = \sqrt{y - 1} = 0$$

$$V_y = \int_2^{10} \pi(\sqrt{y-1})^2 dy$$

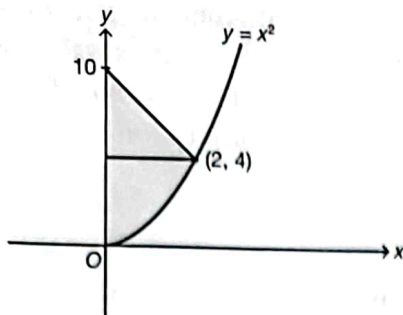
$$= \pi \int_2^{10} (y-1) dy$$

$$= \pi \left[\frac{y^2}{2} - y \right]_2^{10}$$

$$= \pi [(50 - 10) - (2 - 2)]$$

$$= 40\pi \text{ units}^3 \quad \square$$

(b)

**Sol.**

$$y = x^2$$

$$x = \sqrt{y}$$

$$V_y = \int_0^4 \pi(\sqrt{y})^2 dy + \frac{1}{3}\pi \cdot 4 \cdot 6$$

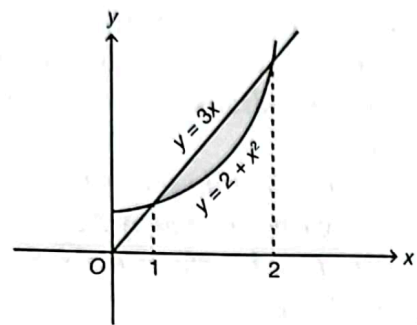
$$= \pi \int_0^4 y dy + 8\pi$$

$$= \pi \left[\frac{y^2}{2} \right]_0^4 + 8\pi$$

$$= 8\pi + 8\pi$$

$$= 16\pi \text{ units}^3 \quad \square$$

(c)

**Sol.**

$$y = 3x$$

$$x = \frac{y}{3}$$

$$y = 2 + x^2$$

$$x^2 = y - 2$$

$$x = \sqrt{y - 2}$$

$$x = 1, y = 3(1) = 3$$

$$x = 2, y = 3(2) = 6$$

$$V_y = \int_3^6 \pi(\sqrt{y-2})^2 dy - \int_3^6 \pi\left(\frac{y}{3}\right)^2 dy$$

$$= \pi \int_3^6 (y-2) dy - \pi \int_3^6 \frac{y^2}{9} dy$$

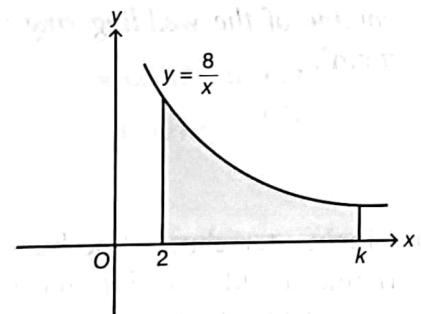
$$= \pi \left[\frac{y^2}{2} - 2y \right]_3^6 - \pi \left[\frac{y^3}{27} \right]_3^6$$

$$= \pi \left\{ \left[(18 - 12) - \left(\frac{9}{2} - 6 \right) \right] - \left(8 - 1 \right) \right\}$$

$$= \pi \left(\frac{15}{2} - 7 \right)$$

$$= \frac{1}{2}\pi \text{ units}^3 \quad \square$$

22. The region bounded by the curve $y = \frac{8}{x}$, the x-axis, and the straight line $x = 2$ and $x = k$ is rotated through 360° about the x-axis as shown in the following diagram.



Express the volume generated by the region in terms of k . If the value of k becomes extremely large, deduce

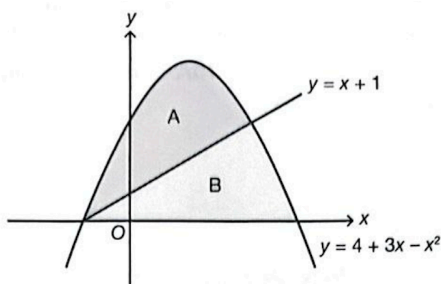
the nearest value of volume.

$$\begin{aligned} V_x &= \int_2^k \pi \left(\frac{8}{x} \right)^2 dx \\ &= \pi \int_2^k \frac{64}{x^2} dx \\ &= \pi \left[-\frac{64}{x} \right]_2^k \\ &= -\frac{64\pi}{k} + 32\pi \quad \square \end{aligned}$$

$$k \rightarrow \infty \Rightarrow \frac{1}{k} \approx 0$$

$$\therefore V_x \approx 32 \text{ units}^3 \quad \square$$

23. The following diagram shows a part of the curve $y = 4 + 3x - x^2$ and the straight line $y = x + 1$.



Find the ratio of the area of the shaded region A to the area of the shaded region B.

Sol.

$$\begin{aligned} x + 1 &= 4 + 3x - x^2 \\ -x^2 + 2x + 3 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x - 3)(x + 1) &= 0 \\ x &= 3 \text{ or } x = -1 \\ x &= 3, y = 3 + 1 = 4 \end{aligned}$$

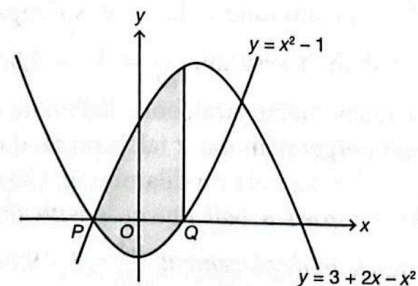
$$\begin{aligned} A_A &= \int_{-1}^3 (4 + 3x - x^2) dx - \frac{1}{2} \cdot 4 \cdot 4 \\ &= \left[4x + \frac{3}{2}x^2 - \frac{1}{3}x^3 \right]_{-1}^3 - 8 \\ &= \left(12 + \frac{27}{2} - 9 \right) - \left(-4 + \frac{3}{2} + \frac{1}{3} \right) - 8 \\ &= \frac{33}{2} + \frac{13}{6} - 8 \\ &= \frac{32}{3} \text{ units}^2 \end{aligned}$$

$$\begin{aligned} 4 + 3x - x^2 &= 0 \\ x^2 - 3x - 4 &= 0 \\ (x - 4)(x + 1) &= 0 \\ x &= 4 \text{ or } x = -1 \end{aligned}$$

$$\begin{aligned} A_B &= \int_3^4 (4 + 3x - x^2) dx + \frac{1}{2} \cdot 4 \cdot 4 \\ &= \left[4x + \frac{3}{2}x^2 - \frac{1}{3}x^3 \right]_3^4 + 8 \\ &= \left(16 + 24 - \frac{64}{3} \right) - \left(12 + \frac{27}{2} - 9 \right) + 8 \\ &= \frac{56}{3} - \frac{33}{2} + 8 \\ &= \frac{61}{6} \text{ units}^2 \end{aligned}$$

$$\begin{aligned} \therefore A_A : A_B &= \frac{32}{3} : \frac{61}{6} \\ &= \frac{64}{61} \\ &= 64 : 61 \quad \square \end{aligned}$$

24. The following diagram shows two curves $y = x^2 - 1$ and $y = 3 + 2x - x^2$.



Find the coordinate of the points P and Q. Hence, calculate the area of the shaded region.