Notes for Calculus III

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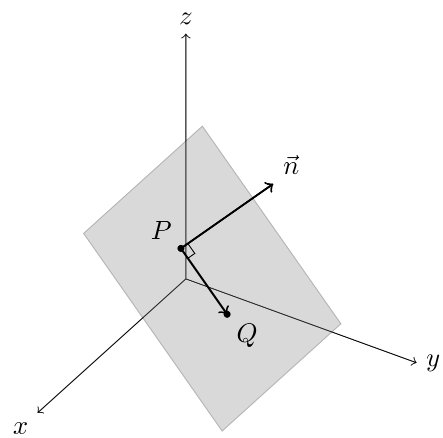
Contents

Chapter 9

Planes in Space



To find the equation of a plane in space, we need a point $P(x_1, y_1, z_1)$ on the plane and a vector $\vec{n} = \langle a, b, c \rangle$ that is orthogonal to the plane, called the **normal vector** of the plane.

For any point $Q(x, y, z)$ on the plane, the vector $\overrightarrow{PQ}$ is orthogonal to $\vec{n}$, that is,

$$\begin{aligned} \overrightarrow{PQ} \cdot \vec{n} &= 0 \\ \langle x - x_1, y - y_1, z - z_1 \rangle \cdot \langle a, b, c \rangle &= 0 \\ a(x - x_1) + b(y - y_1) + c(z - z_1) &= 0 \end{aligned}$$

This equation is called the **standard form** of the equation of the plane. Regrouping the terms, we obtain the **general form** of equation of the plane.

$$ax + by + cz + d = 0$$

where $d = -ax_1 - by_1 - cz_1$.

Example 1. Find the equation of the plane that passes through the point $P(4,5,-7)$ and is perpendicular to the vector $\vec{n} = \hat{\jmath}$.

$$\begin{aligned} \vec{n} &= \langle 0,1,0\rangle \\ 0(x-4)+1(y-5)+0(z+7) &= 0 \\ y-5 &= 0 \\ y &= 5 \end{aligned}$$

Example 2. Find the equation of the plane that passes through the point $P(0,7,0)$ and is perpendicular to the vector $\vec{n} = 3\hat{\imath} + 8\hat{k}$.

$$\begin{aligned} \vec{n} &= \langle 3,0,8\rangle \\ 3(x-0)+0(y-7)+8(z-0) &= 0 \\ 3x+8z &= 0 \end{aligned}$$

Example 3. Given three points $(0,0,0)$, $(2,0,7)$, and $(-2,-1,7)$ in space, find the equation of the plane that passes through these points.

$$\begin{aligned} \vec{u} &= \langle 2-0, 0-0, 7-0\rangle \\ &= \langle 2,0,7\rangle \\ \vec{v} &= \langle -2-0, -1-0, 7-0\rangle \\ &= \langle -2,-1,7\rangle \\ \vec{n} &= \vec{u}\times\vec{v} \\ &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 7 \\ -2 & -1 & 7 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 7 \\ -1 & 7 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 2 & 7 \\ -2 & 7 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 2 & 0 \\ -2 & -1 \end{vmatrix}\hat{k} \\ &= (0(7)-7(-1))\hat{\imath} - (2(7)-(-2)(7))\hat{\jmath} + (2(-1)-(-2)(0))\hat{k} \\ &= 7\hat{\imath} - 28\hat{\jmath} - 2\hat{k} \\ &= \langle 7,-28,-2\rangle \end{aligned}$$

$$\begin{aligned} 7(x-0)-28(y-0)-2(z-0) &= 0 \\ 7x-28y-2z &= 0 \end{aligned}$$

Example 4. Find the equation of the plane that passes through $(4, 2, 1)$, $(-1, 8, 8)$ and is parallel to z-axis.

$$\begin{aligned}
\vec{v} &= \langle -1-4, 8-2, 8-1\rangle \\
&= \langle -5, 6, 7\rangle \\
\vec{n} &= \vec{v} \times \hat{k} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -5 & 6 & 7 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 6 & 7 \\ 0 & 1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} -5 & 7 \\ 0 & 1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} -5 & 6 \\ 0 & 0 \end{vmatrix} \hat{k} \\
&= (6(1) - 7(0))\hat{\imath} - (-5(1) - 7(0))\hat{\jmath} + (-5(0) - 6(0))\hat{k} \\
&= 6\hat{\imath} + 5\hat{\jmath} \\
&= \langle 6, 5, 0\rangle
\end{aligned}$$

$$\begin{aligned}
6(x-4) + 5(y-2) + 0(z-1) &= 0 \\
6x + 5y - 34 &= 0
\end{aligned}$$

Example 5. Find the equation of the plane such that the point $(2, 0, 1)$ and the line $\frac{x}{2} = \frac{y-4}{-1} = \frac{z}{1}$ is on the plane.
When $x = 0$, $y = 4$ and $z = 0$, hence the point $(0, 4, 0)$ is on the plane.

$$\begin{aligned}
\vec{v} &= \langle 2-0, 0-4, 1-0\rangle \\
&= \langle 2, -4, 1\rangle \\
\vec{n} &= \langle 2, -1, 1\rangle \times \langle 2, -4, 1\rangle \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 2 & -4 & 1 \end{vmatrix} \\
&= \begin{vmatrix} -1 & 1 \\ -4 & 1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 2 & -1 \\ 2 & -4 \end{vmatrix} \hat{k} \\
&= (-1(1) - 1(-4))\hat{\imath} - (2(1) - 2(1))\hat{\jmath} + (2(-4) - 2(-1))\hat{k} \\
&= -3\hat{\imath} + 0\hat{\jmath} + (-6)\hat{k} \\
&= \langle -3, 0, -6\rangle
\end{aligned}$$

$$\begin{aligned}
-3(x-2) + 0(y-0) - 6(z-1) &= 0 \\
-3x - 6z + 12 &= 0
\end{aligned}$$

Example 6. Find the equation of the plane that passes through the points $(3, 4, 1)$ and $(3, 1, -7)$ and is perpendicular to the plane $8x + 9y + 3z = 13$.

The normal vector of the plane is $\langle 8, 9, 3\rangle$. Since the target plane is perpendicular to the given plane, the normal vector of the given plane is parallel to the target plane.

$$\begin{aligned}
\vec{u} &= \langle 8, 9, 3\rangle \\
\vec{v} &= \langle 3-3, 1-4, -7-1\rangle \\
&= \langle 0, -3, -8\rangle \\
\vec{n} &= \vec{u} \times \vec{v} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 8 & 9 & 3 \\ 0 & -3 & -8 \end{vmatrix} \\
&= \begin{vmatrix} 9 & 3 \\ -3 & -8 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 8 & 3 \\ 0 & -8 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 8 & 9 \\ 0 & -3 \end{vmatrix} \hat{k} \\
&= (9(-8) - 3(-3))\hat{\imath} - (8(-8) - 3(0))\hat{\jmath} + (8(-3) - 9(0))\hat{k} \\
&= -63\hat{\imath} + 64\hat{\jmath} - 24\hat{k} \\
&= \langle -63, 64, -24\rangle
\end{aligned}$$

$$\begin{aligned}
-63(x-3) + 64(y-4) - 24(z-1) &= 0 \\
-63x + 64y - 24z - 43 &= 0
\end{aligned}$$

Selected Exercises

Source: Larson Calculus 11th Ed. Exercise 11.5

Checking Points in a Plane In Exercises 37 and 38, determine whether each point lies in the plane.

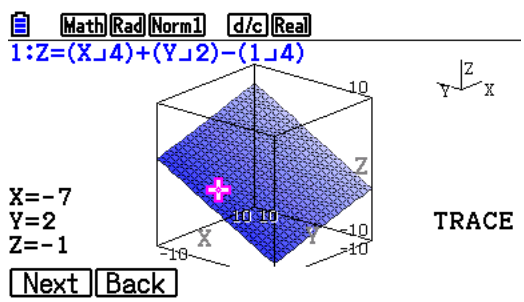
37. $x + 2y - 4z - 1 = 0$

(a) $(-7, 2, -1)$

Solution.

$$-7 + 2(2) - 4(-1) - 1 = 0$$

Therefore, $(-7, 2, -1)$ lies in the plane. ■

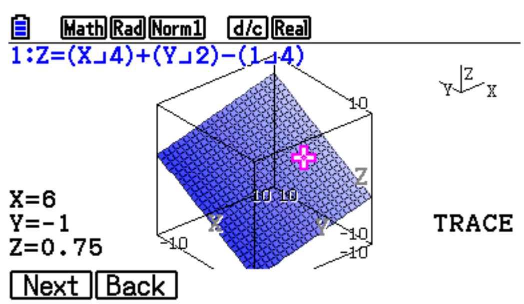


(c) $(-6, 1, -1)$

Solution.

$$-6 + 2(1) - 4(-1) - 1 = -1 \neq 0$$

Therefore, $(-6, 1, -1)$ does not lie in the plane. ■



Finding an Equation of a Plane In Exercises 39-44, find an equation of the plane that passes through the given point and is perpendicular to the given vector or line.

40. Point $(0, -1, 4)$, perpendicular to $n = k$

Solution. The normal vector of the plane is $\langle 0, 0, 1 \rangle$.

Therefore, the equation of the plane is

$$\begin{aligned} 0(x-0) + 0(y+1) + 1(z-4) &= 0 \\ z - 4 &= 0 \\ z &= 4 \end{aligned}$$

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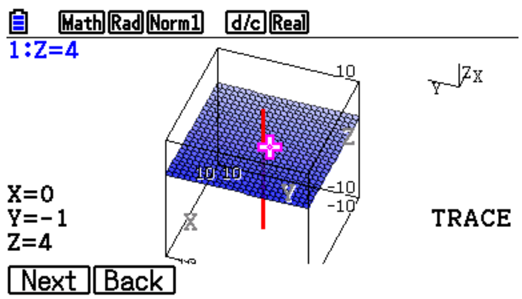

41. Point $(3, 2, 2)$, perpendicular to $n = 2i + 3j - k$

Solution. The normal vector of the plane is $\langle 2, 3, -1 \rangle$.

Therefore, the equation of the plane is

$$\begin{aligned} 2(x-3) + 3(y-2) - 1(z-2) &= 0 \\ 2x + 3y - z - 10 &= 0 \end{aligned}$$

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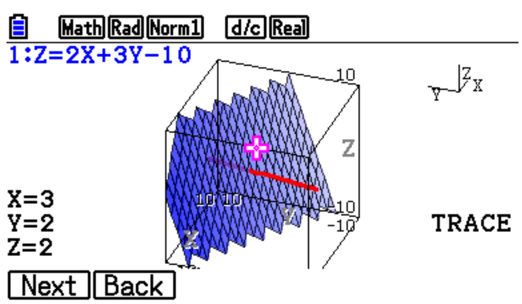

43. Point $(-1, 4, 0)$, perpendicular to $x = -1 + 2t, y = 5 - t, z = 3 - 2t$

Solution. The normal vector of the plane is $\langle 2, -1, -2 \rangle$.

Therefore, the equation of the plane is

$$\begin{aligned} 2(x+1) - 1(y-4) - 2(z-0) &= 0 \\ 2x - y - 2z + 6 &= 0 \end{aligned}$$

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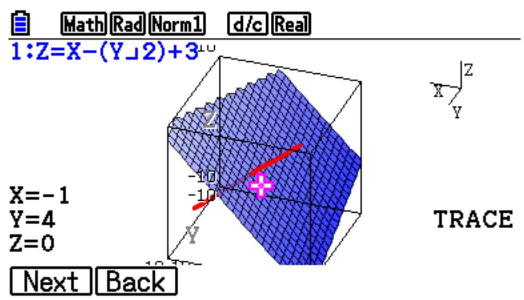

44. Point $(3, 2, 2)$, perpendicular to line $\dfrac{x-1}{4} = y + 2 = \dfrac{z+3}{-3}$

Solution. The normal vector of the plane is $\langle 4, 1, -3 \rangle$.

Therefore, the equation of the plane is

$$\begin{aligned} 4(x-3) + 1(y-2) - 3(z-2) &= 0 \\ 4x + y - 3z - 8 &= 0 \end{aligned}$$

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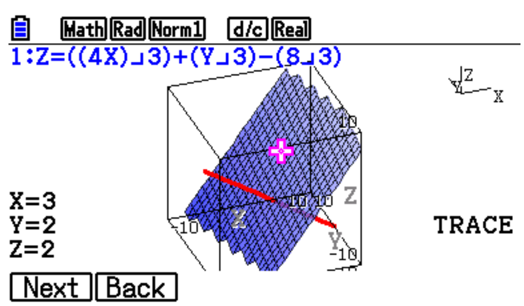

Finding an Equation of a Plane In Exercises 45-56, find an equation of the plane with the given characteristics.

46. The plane passes through $(3,-1,2),(2,1,5)$, and $(1,-2,-2)$.
Solution. Let $u=\langle 3-2,-1-1,2-5\rangle=\langle 1,-2,-3\rangle$ and $v=\langle 1-2,-2-1,-2-5\rangle=\langle -1,-3,-7\rangle$. The normal vector of the plane is

$$\begin{aligned} n &= u\times v\\ &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k}\\ 1 & -2 & -3\\ -1 & -3 & -7 \end{vmatrix}\\ &= \begin{vmatrix} -2 & -3\\ -3 & -7 \end{vmatrix}\hat{\imath}-\begin{vmatrix} 1 & -3\\ -1 & -7 \end{vmatrix}\hat{\jmath}+\begin{vmatrix} 1 & -2\\ -1 & -3 \end{vmatrix}\hat{k}\\ &= 5\hat{\imath}+10\hat{\jmath}-5\hat{k} \end{aligned}$$

Hence, the equation of the plane is

$$\begin{aligned} 5(x-3)+10(y+1)-5(z-2)&=0\\ x-3+2(y+1)-(z-2)&=0\\ x+2y-z+1&=0 \end{aligned}$$

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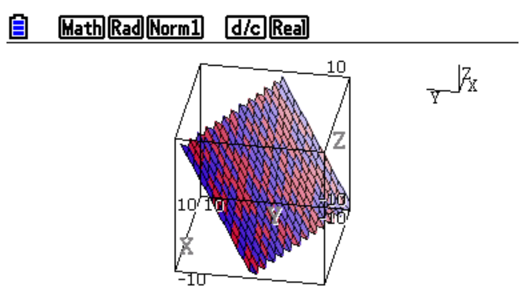

48. The plane passes through the point $(1, 2, 3)$ and is parallel to the yz-plane.

Solution. The normal vector of the plane is $\langle 1, 0, 0\rangle$.

Hence, the equation of the plane is

$$\begin{aligned} 1(x-1) + 0(y-2) + 0(z-3) &= 0 \\ x - 1 &= 0 \\ x &= 1 \end{aligned}$$

■

50. The plane contains the y-axis and makes an angle of $\pi/6$ with the positive x-axis.

Solution. The line $u = \langle 0, 1, 0\rangle$ and $v = \left\langle \cos\dfrac{\pi}{6}, 0, \sin\dfrac{\pi}{6}\right\rangle = \left\langle \dfrac{\sqrt{3}}{2}, 0, \dfrac{1}{2}\right\rangle$ are parallel to the plane. The normal vector of the plane is

$$\begin{aligned} n &= u \times v \\ &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 0 \\ \dfrac{\sqrt{3}}{2} & 0 & \dfrac{1}{2} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ 0 & \dfrac{1}{2} \end{vmatrix}\hat{\imath} - \begin{vmatrix} 0 & 0 \\ \dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 0 & 1 \\ \dfrac{\sqrt{3}}{2} & 0 \end{vmatrix}\hat{k} \\ &= \frac{1}{2}\hat{\imath} - \frac{\sqrt{3}}{2}\hat{k} \end{aligned}$$

Hence, the equation of the plane is

$$\begin{aligned} \frac{1}{2}(x-0) + 0(y-0) - \frac{\sqrt{3}}{2}(z-0) &= 0 \\ \frac{1}{2}x - \frac{\sqrt{3}}{2}z &= 0 \end{aligned}$$

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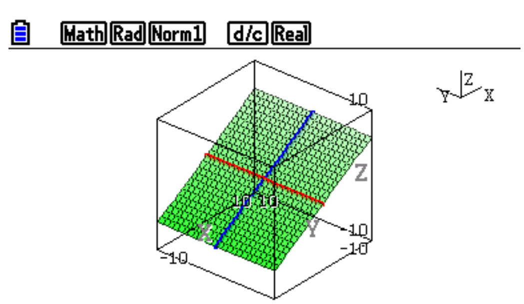

51. The plane contains the lines given by $\dfrac{x-1}{-2} = y-4 = z$ and $\dfrac{x-2}{-3} = \dfrac{y-1}{4} = \dfrac{z-2}{-1}$

Solution. The vector $\vec{u} = \langle -2, 1, 1\rangle$ and $\vec{v} = \langle -3, 4, -1\rangle$ are parallel to the plane. The normal vector of the plane is

$$\begin{aligned}
n &= \vec{u} \times \vec{v} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} -2 & 1 \\ -3 & -1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} -2 & 1 \\ -3 & 4 \end{vmatrix} \hat{k} \\
&= (-1-4)\hat{\imath} - [2-(-3)]\hat{\jmath} + [-8-(-3)]\hat{k} \\
&= -5\hat{\imath} - 5\hat{\jmath} - 5\hat{k}
\end{aligned}$$

Hence, the equation of the plane is

$$\begin{aligned}
-5(x-1) - 5(y-4) - 5(z-0) &= 0 \\
x - 1 + y - 4 + z - 0 &= 0 \\
x + y + z - 5 &= 0
\end{aligned}$$

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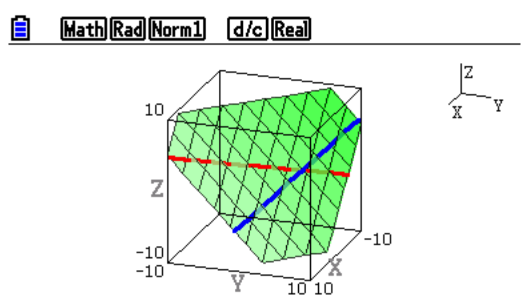

52. The plane passes through the point $(2,2,1)$ and contains the line given by $\frac{x}{2} = \frac{y-4}{-1} = z$

Solution. The vector $\vec{u} = \langle 2,-1,1\rangle$ and $\vec{v} = \langle 2-0, 2-4, 1-0\rangle = \langle 2,-2,1\rangle$ are parallel to the plane. The normal vector of the plane is

$$\begin{aligned}
n &= \vec{u} \times \vec{v} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 2 & -2 & 1 \end{vmatrix} \\
&= \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} \hat{k} \\
&= [-1-(-2)]\hat{\imath} - [2-2]\hat{\jmath} + [-4-(-2)]\hat{k} \\
&= \hat{\imath} - 2\hat{k}
\end{aligned}$$

Hence, the equation of the plane is

$$\begin{aligned}
1(x-2) + 0(y-2) - 2(z-1) &= 0 \\
x - 2 - 2z + 2 &= 0 \\
x - 2z &= 0
\end{aligned}$$

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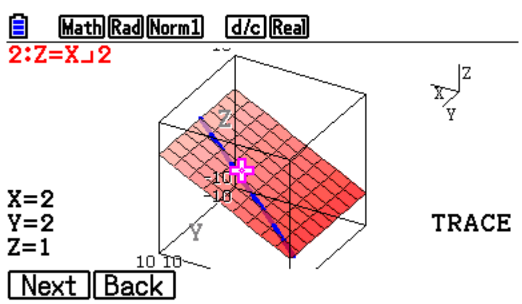

54. The plane passes through the points $(3,2,1)$ and $(3,1,-5)$ and is perpendicular to the plane $6x+7y+2z=10$

Solution. The normal vector of the given plane is $\langle 6,7,2\rangle$, which is parallel to the target plane.

Let $u=\langle 3-3,1-2,-5-1\rangle=\langle 0,-1,-6\rangle$ and $v=\langle 6,7,2\rangle$. The normal vector of the target plane is

$$\begin{aligned} n &= u\times v \\ &= \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ 0 & -1 & -6 \\ 6 & 7 & 2 \end{vmatrix} \\ &= \begin{vmatrix} -1 & -6 \\ 7 & 2 \end{vmatrix}\hat\imath - \begin{vmatrix} 0 & -6 \\ 6 & 2 \end{vmatrix}\hat\jmath + \begin{vmatrix} 0 & -1 \\ 6 & 7 \end{vmatrix}\hat k \\ &= (-2-(-42))\hat\imath - [0-(-36)]\hat\jmath + [0-(-6)]\hat k \\ &= 40\hat\imath - 36\hat\jmath + 6\hat k \end{aligned}$$

Hence, the equation of the plane is

$$\begin{aligned} 40(x-3)-36(y-2)+6(z-1)&=0 \\ 20(x-3)-18(y-2)+3(z-1)&=0 \\ 20x-18y+3z-27&=0 \end{aligned}$$

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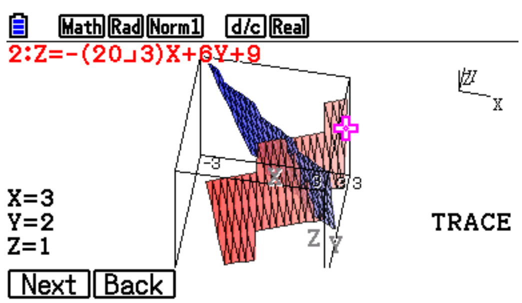

56. The plane passes through the points $(4, 2, 1)$ and $(-3, 5, 7)$ and is parallel to the z-axis.

Solution. Let $u = \langle 4-(-3), 2-5, 1-7\rangle = \langle 7, -3, -6\rangle$ and $v = \langle 0, 0, 1\rangle$. The normal vector of the plane is

$$\begin{aligned} n &= u \times v \\ &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 7 & -3 & -6 \\ 0 & 0 & 1 \end{vmatrix} \\ &= \begin{vmatrix} -3 & -6 \\ 0 & 1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 7 & -6 \\ 0 & 1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 7 & -3 \\ 0 & 0 \end{vmatrix} \hat{k} \\ &= -3\hat{\imath} - 7\hat{\jmath} \end{aligned}$$

Hence, the equation of the plane is

$$\begin{aligned} -3(x-4) - 7(y-2) + 0(z-1) &= 0 \\ -3x + 12 - 7y + 14 &= 0 \\ -3x - 7y + 26 &= 0 \end{aligned}$$

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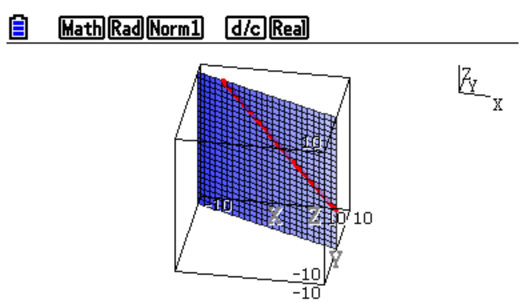

Finding an Equation of a Plane In Exercises 57-60, find an equation of the plane that contains all the points that are equidistant from the given points.

59. $(-3, 1, 2)$, $(6, -2, 4)$

Solution. The midpoint of the line segment joining the two points is $M = \left(\frac{-3+6}{2}, \frac{1-2}{2}, \frac{2+4}{2}\right) = (1.5, -0.5, 3)$. The normal vector of the plane is $\langle 6-(-3), -2-1, 4-2\rangle = \langle 9, -3, 2\rangle$.

Hence, the equation of the plane is

$$\begin{aligned} 9(x-1.5) - 3(y+0.5) + 2(z-3) &= 0 \\ 9x - 13.5 - 3y - 1.5 + 2z - 6 &= 0 \\ 9x - 3y + 2z - 21 &= 0 \end{aligned}$$

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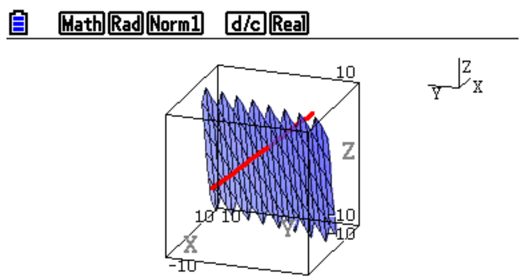

Intersection of Planes In Exercises 65-68, (a) find the angle between the two planes and (b) find a set of parametric equations for the line of intersection of the planes.

66. $-2x + y + z = 2$

$6x - 3y + 2z = 4$

Solution.

(a) The normal vector of the the two planes are $\langle -2, 1, 1\rangle$ and $\langle 6, -3, 2\rangle$. The angle between the two planes is

$$\begin{aligned} \cos\theta &= \frac{\langle -2,1,1\rangle \cdot \langle 6,-3,2\rangle}{\|\langle -2,1,1\rangle\| \, \|\langle 6,-3,2\rangle\|} \\ &= \frac{-12-3+2}{\sqrt{6}\sqrt{49}} \\ &= -\frac{13}{7\sqrt{6}} \\ \theta &= \arccos\left(-\frac{13}{7\sqrt{6}}\right) \\ &\approx 139.3^\circ \end{aligned}$$

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(b) Solving the two equations simultaneously,

$$\begin{cases} -2x + y + z = 2 \ \cdots \ (1) \\ 6x - 3y + 2z = 4 \ \cdots \ (2) \end{cases}$$

$$\begin{aligned} (1) \times 2 : & \ -4x + 2y + 2z = 4 \ \cdots \ (3) \\ (2) - (3) : & \ 10x - 5y = 0 \ \cdots \ (4) \\ & 2x - y = 0 \\ & y = 2x \end{aligned}$$

Substituting $y = 2x$ into (1),

$$\begin{aligned} -2x + 2x + z &= 2 \\ z &= 2 \end{aligned}$$

Let $x = t$, then $y = 2t$ and $z = 2$. Hence, the parametric equations of the line of intersection of the two planes are

$$\begin{aligned} x &= t \\ y &= 2t \\ z &= 2 \end{aligned}$$

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