# Exercise 11h

Find the following indefinite integrals:

1. $\displaystyle\int \sqrt{1-4x^2}\,dx$

   **Sol.**

   Let $x = \dfrac{1}{2}\sin\theta$, then $\theta = \sin^{-1} 2x$ and $dx = \dfrac{1}{2}\cos\theta\,d\theta$.

$$\begin{aligned}
\int \sqrt{1-4x^2}\,dx &= \int \sqrt{1-\sin^2\theta}\cdot\frac{1}{2}\cos\theta\,d\theta \\
&= \frac{1}{2}\int \cos^2\theta\,d\theta \\
&= \frac{1}{2}\int \frac{1+\cos 2\theta}{2}\,d\theta \\
&= \frac{1}{4}\int (1+\cos 2\theta)\,d\theta \\
&= \frac{1}{4}\theta + \frac{1}{8}\sin 2\theta + C \\
&= \frac{1}{4}\sin^{-1} 2x + \frac{1}{8}(2\sin\theta\cos\theta) + C \\
&= \frac{1}{4}\sin^{-1} 2x + \frac{1}{2}\sin\theta\frac{1}{2}\cos\theta + C \\
&= \frac{1}{4}\sin^{-1} 2x + \frac{1}{2}x\sqrt{1-4x^2} + C \qquad \square
\end{aligned}$$

2. $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx$

   **Sol.**

   Let $x = a\tan\theta$, then $dx = a\sec^2\theta\,d\theta$.

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+a^2}}\,dx &= \int \frac{1}{\sqrt{a^2\tan^2\theta + a^2}}\cdot a\sec^2\theta\,d\theta \\
&= \int \frac{1}{\sqrt{a^2\sec^2\theta}}\cdot a\sec^2\theta\,d\theta \\
&= \int \sec\theta\,d\theta \\
&= \ln|\sec\theta + \tan\theta| + C' \\
&= \ln\left|\sqrt{1+\frac{x^2}{a^2}} + \frac{x}{a}\right| + C' \\
&= \ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right| + C' \\
&= \ln\left|\sqrt{x^2+a^2}+x\right| - \ln|a| + C' \\
&= \ln\left|\sqrt{x^2+a^2}+x\right| + C \qquad (\text{where } C = -\ln|a| + C') \qquad \square
\end{aligned}$$

3. $\displaystyle\int \frac{1}{\sqrt{49-5x^2}}\,dx$

   **Sol.**

   Let $x = \dfrac{7}{\sqrt{5}}\sin\theta$, then $\theta = \sin^{-1}\dfrac{\sqrt{5}}{7}x$ and $dx = \dfrac{7}{\sqrt{5}}\cos\theta\,d\theta$.

$$\int \frac{1}{\sqrt{49-5x^2}}\,dx = \int \frac{1}{\sqrt{49-49\sin^2\theta}}\cdot\frac{7}{\sqrt{5}}\cos\theta\,d\theta$$

$$
\begin{aligned}
&= \int \frac{1}{\sqrt{49\cos^2\theta}} \cdot \frac{7}{\sqrt{5}} \cos\theta \, d\theta \\
&= \int \frac{1}{7\cos\theta} \cdot \frac{7}{\sqrt{5}} \cos\theta \, d\theta \\
&= \int \frac{1}{\sqrt{5}} \, d\theta \\
&= \frac{1}{\sqrt{5}}\theta + C' \\
&= \frac{1}{\sqrt{5}} \sin^{-1} \frac{\sqrt{5}}{7} x + C \qquad \square
\end{aligned}
$$

4. $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}} \, dx$

   **Sol.**

   Let $x = 3\sin\theta$, then $dx = 3\cos\theta \, d\theta$.

$$
\begin{aligned}
\int \frac{x^2}{\sqrt{9-x^2}} \, dx &= \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}} \cdot 3\cos\theta \, d\theta \\
&= \int \frac{9\sin^2\theta}{\sqrt{9\cos^2\theta}} \cdot 3\cos\theta \, d\theta \\
&= \int \frac{9\sin^2\theta}{3\cos\theta} \cdot 3\cos\theta \, d\theta \\
&= \int 9\sin^2\theta \, d\theta \\
&= \int \frac{9}{2}(1-\cos 2\theta) \, d\theta \\
&= \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + C \\
&= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{9}{2}\sin\theta\cos\theta + C \\
&= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{9}{2} \cdot \frac{x}{3} \cdot \sqrt{1-\frac{x^2}{9}} + C \\
&= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{3}{2}x \cdot \frac{1}{3}\sqrt{9-x^2} + C \\
&= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{1}{2}x\sqrt{9-x^2} + C \qquad \square
\end{aligned}
$$

5. $\displaystyle\int \frac{1}{\sqrt{(x^2+1)^3}} \, dx$

   **Sol.**

   Let $x = \tan\theta$, then $dx = \sec^2\theta \, d\theta$.

$$
\begin{aligned}
\int \frac{1}{\sqrt{(x^2+1)^3}} \, dx &= \int \frac{1}{(x^2+1)\sqrt{x^2+1}} \, dx \\
&= \int \frac{1}{(\tan^2\theta+1)\sqrt{\tan^2\theta+1}} \cdot \sec^2\theta \, d\theta \\
&= \int \frac{1}{\sec^2\theta\sqrt{\sec^2\theta}} \cdot \sec^2\theta \, d\theta \\
&= \int \frac{1}{\sec\theta} \, d\theta \\
&= \int \cos\theta \, d\theta \\
&= \sin\theta + C \\
&= \frac{x}{\sqrt{x^2+1}} + C \qquad \square
\end{aligned}
$$

6. $\displaystyle\int \frac{1}{\sqrt{9+4x^2}}\,dx$

**Sol.**

Let $x = \dfrac{3}{2}\tan\theta$, then $dx = \dfrac{3}{2}\sec^2\theta\,d\theta$.

$$\begin{aligned}
\int \frac{1}{\sqrt{9+4x^2}}\,dx &= \int \frac{1}{\sqrt{9+9\tan^2\theta}}\cdot\frac{3}{2}\sec^2\theta\,d\theta \\
&= \int \frac{1}{\sqrt{9\sec^2\theta}}\cdot\frac{3}{2}\sec^2\theta\,d\theta \\
&= \int \frac{1}{3\sec\theta}\cdot\frac{3}{2}\sec^2\theta\,d\theta \\
&= \int \frac{1}{2}\sec\theta\,d\theta \\
&= \frac{1}{2}\ln|\sec\theta+\tan\theta| + C' \\
&= \frac{1}{2}\ln\left|\sqrt{\frac{4x^2}{9}+1}+\frac{2}{3}x\right| + C' \\
&= \frac{1}{2}\ln\left|\frac{\sqrt{4x^2+9}+2x}{3}\right| + C' \\
&= \frac{1}{2}\ln\left|\sqrt{4x^2+9}+2x\right| - \frac{1}{2}\ln 3 + C' \\
&= \frac{1}{2}\ln\left|\sqrt{4x^2+9}+2x\right| + C \qquad (\text{where } C = C' - \frac{1}{2}\ln 3) \qquad \square
\end{aligned}$$

7. $\displaystyle\int \frac{x}{a^2+x^2}\,dx$

**Sol.**

Let $x = a\tan\theta$, then $dx = a\sec^2\theta\,d\theta$.

$$\begin{aligned}
\int \frac{x}{a^2+x^2}\,dx &= \int \frac{a\tan\theta}{a^2+a^2\tan^2\theta}\cdot a\sec^2\theta\,d\theta \\
&= \int \frac{a\tan\theta}{a^2\sec^2\theta}\cdot a\sec^2\theta\,d\theta \\
&= \int \tan\theta\,d\theta \\
&= \ln|\sec\theta| + C' \\
&= \ln\left|\frac{\sqrt{a^2+x^2}}{a}\right| + C' \\
&= \ln\left|\sqrt{a^2+x^2}\right| - \ln|a| + C' \\
&= \frac{1}{2}\ln\left|a^2+x^2\right| + C \qquad (\text{where } C = C' - \ln|a|) \qquad \square
\end{aligned}$$

8. $\displaystyle\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$

**Sol.**

Let $x = \sin\theta$, then $dx = \cos\theta\,d\theta$.

$$\begin{aligned}
\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx &= \int \frac{1}{(1-\sin^2\theta)^{\frac{3}{2}}}\cdot\cos\theta\,d\theta \\
&= \int \frac{1}{\cos^3\theta}\cdot\cos\theta\,d\theta \\
&= \int \sec^2\theta\,d\theta
\end{aligned}$$

$$= \tan\theta + C$$
$$= \frac{x}{\sqrt{1-x^2}} + C \qquad \square$$

9. $\displaystyle\int \frac{x}{\sqrt{x^2-4}}\,dx$

**Sol.**

Let $x = 2\sec\theta$, then $dx = 2\sec\theta\tan\theta\,d\theta$.

$$\begin{aligned}
\int \frac{x}{\sqrt{x^2-4}}\,dx &= \int \frac{2\sec\theta}{\sqrt{4\sec^2\theta-4}} \cdot 2\sec\theta\tan\theta\,d\theta \\
&= \int \frac{2\sec\theta}{\sqrt{4\tan^2\theta}} \cdot 2\sec\theta\tan\theta\,d\theta \\
&= \int \frac{2\sec\theta}{2\tan\theta} \cdot 2\sec\theta\tan\theta\,d\theta \\
&= \int 2\sec^2\theta\,d\theta \\
&= 2\tan\theta + C \\
&= 2\cdot\frac{\sqrt{x^2-4}}{2} + C \\
&= \sqrt{x^2-4} + C \qquad \square
\end{aligned}$$

10. $\displaystyle\int x\sqrt{4-x^2}\,dx$

**Sol.**

Let $x = 2\sin\theta$, then $dx = 2\cos\theta\,d\theta$.

$$\begin{aligned}
\int x\sqrt{4-x^2}\,dx &= \int 2\sin\theta\sqrt{4-4\sin^2\theta} \cdot 2\cos\theta\,d\theta \\
&= \int 2\sin\theta\sqrt{4\cos^2\theta} \cdot 2\cos\theta\,d\theta \\
&= \int 2\sin\theta \cdot 2\cos\theta \cdot 2\cos\theta\,d\theta \\
&= 8\int \sin\theta\cos^2\theta\,d\theta \qquad (\text{Let } u = \cos\theta,\ du = -\sin\theta\,d\theta) \\
&= -8\int u^2\,du \\
&= -\frac{8}{3}u^3 + C \\
&= -\frac{8}{3}\cos^3\theta + C \\
&= -\frac{8}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + C \\
&= -\frac{1}{3}\left(4-x^2\right)^{\frac{3}{2}} + C \qquad \square
\end{aligned}$$

11. $\displaystyle\int \frac{1}{x\sqrt{81+x^2}}\,dx$

**Sol.**

Let $x = 9\tan\theta$, then $dx = 9\sec^2\theta\,d\theta$.

$$\begin{aligned}
\int \frac{1}{x\sqrt{81+x^2}}\,dx &= \int \frac{1}{9\tan\theta\sqrt{81+81\tan^2\theta}} \cdot 9\sec^2\theta\,d\theta \\
&= \int \frac{1}{9\tan\theta\sqrt{81\sec^2\theta}} \cdot 9\sec^2\theta\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{9}\int \frac{\sec x}{\tan x}\,dx \\
&= \frac{1}{9}\int \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}\,dx \\
&= \frac{1}{9}\int \frac{1}{\sin x}\,dx \\
&= \frac{1}{9}\int \frac{\sin x}{\sin^2 x}\,dx \\
&= \frac{1}{9}\int \frac{\sin x}{1-\cos^2 x}\,dx \qquad (\text{Let } u=\cos x,\ du=-\sin x\,dx) \\
&= \frac{1}{9}\int \frac{1}{1-u^2}\,du \\
&= \frac{1}{18}\ln\left|\frac{u+1}{u-1}\right| + C \\
&= \frac{1}{18}\ln\left|\frac{\cos x+1}{\cos x-1}\right| + C \\
&= \frac{1}{9}\ln\left|\tan\frac{x}{2}\right| + C \\
&= \frac{1}{9}\ln\left|\frac{1-\cos x}{\sin x}\right| + C \\
&= \frac{1}{9}\ln\left|\frac{1-\frac{9}{\sqrt{81+x^2}}}{\frac{x}{\sqrt{81+x^2}}}\right| + C \\
&= \frac{1}{9}\ln\left|\frac{\sqrt{81+x^2}-9}{x}\right| + C \qquad \square
\end{aligned}
$$

12. $\displaystyle\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$

**Sol.**

Let $x=\sec\theta$, then $\theta=\cos^{-1}\frac{1}{x}$, $dx=\sec\theta\tan\theta\,d\theta$.

$$
\begin{aligned}
\int \frac{1}{x^3\sqrt{x^2-1}}\,dx &= \int \frac{1}{\sec^3\theta\sqrt{\sec^2\theta-1}}\cdot\sec\theta\tan\theta\,d\theta \\
&= \int \frac{1}{\sec^3\theta\sqrt{\tan^2\theta}}\cdot\sec\theta\tan\theta\,d\theta \\
&= \int \frac{1}{\sec^2\theta}\,d\theta \\
&= \int \cos^2\theta\,d\theta \\
&= \frac{1}{2}\int (1+\cos 2\theta)\,d\theta \\
&= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C \\
&= \frac{1}{2}\cos^{-1}\frac{1}{x} + \frac{1}{2}\sin\theta\cos\theta + C \\
&= \frac{1}{2}\cos^{-1}\frac{1}{x} + \frac{1}{2}\cdot\frac{\sqrt{x^2-1}}{x}\cdot\frac{1}{x} + C \\
&= \frac{1}{2}\cos^{-1}\frac{1}{x} + \frac{\sqrt{x^2-1}}{2x^2} + C \qquad \square
\end{aligned}
$$