

Exercise 5c

1. Find the standard form of the equation of the ellipse that satisfies the given conditions:

- (a) Passes through point $P(-2\sqrt{2}, 0)$, $Q(0, \sqrt{5})$;

Sol.

Point P is on the x -axis, while point Q is on the y -axis.

$$|OP| = 2\sqrt{2}, |OQ| = \sqrt{5},$$

$$\therefore |OP| > |OQ|,$$

\therefore The major axis is along the x -axis.

\therefore The equation of the ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Substituting the coordinates of P and Q into the equation, we get

$$\frac{(-2\sqrt{2})^2}{a^2} + \frac{0^2}{b^2} = 1$$

$$\frac{0^2}{a^2} + \frac{(\sqrt{5})^2}{b^2} = 1$$

Simplifying, we get

$$\frac{8}{a^2} = 1$$

$$\frac{5}{b^2} = 1$$

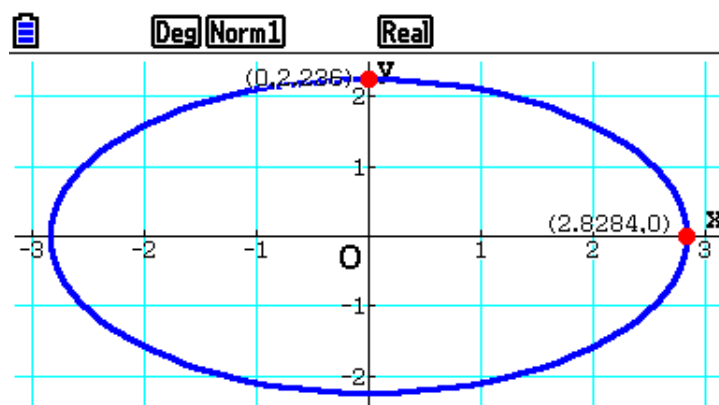
Solving for a and b , we get

$$a^2 = 8$$

$$b^2 = 5$$

\therefore The standard form of the equation of the ellipse is

$$\frac{x^2}{8} + \frac{y^2}{5} = 1 \quad \square$$



- (b) Coordinates of its foci are $(-2\sqrt{3}, 0)$ and $(2\sqrt{3}, 0)$, and it passes through the point $P(\sqrt{5}, -\sqrt{6})$;
The foci are on the x -axis, therefore let the equation of the ellipse be of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

From the coordinates of the foci, we have $ae = 2\sqrt{3}$, $a^2e^2 = 12$,

$$\begin{aligned}\therefore b^2 &= a^2 - a^2e^2 \\ &= a^2 - 12 \dots (1)\end{aligned}$$

Substituting the coordinates of P into the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

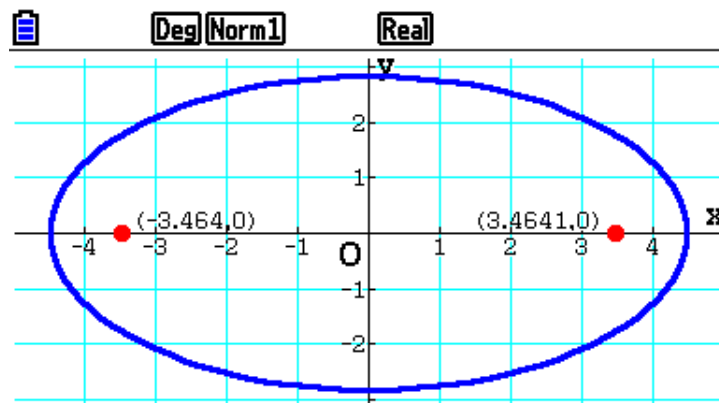
$$\begin{aligned}\frac{(\sqrt{5})^2}{a^2} + \frac{(-\sqrt{6})^2}{b^2} &= 1 \\ \frac{5}{a^2} + \frac{6}{b^2} &= 1\end{aligned}$$

Substituting (1) into the equation, we get

$$\begin{aligned}\frac{5}{a^2} + \frac{6}{a^2 - 12} &= 1 \\ 5(a^2 - 12) + 6a^2 &= a^2(a^2 - 12) \\ 5a^2 - 60 + 6a^2 &= a^4 - 12a^2 \\ a^4 - 23a^2 + 60 &= 0 \\ (a^2 - 20)(a^2 - 3) &= 0 \\ a^2 &= 20 \text{ or } a^2 = 3 \text{ (rejected, } b > 0)\end{aligned}$$

When $a^2 = 20$, $b^2 = 20 - 12 = 8$. \therefore The standard form of the equation of the ellipse is

$$\frac{x^2}{20} + \frac{y^2}{8} = 1 \quad \square$$



- (c) Equations of its directrices are $y \pm \frac{25}{3} = 0$, and it passes through the point $(4, 0)$;

Sol.

The directrices are perpendicular to the y -axis, therefore let the equation of the ellipse be of the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

From the equation of the directrices, we have $\frac{a}{e} = \frac{25}{3}$, $\frac{a^2}{e^2} = \frac{625}{9}$, $e^2 = \frac{9}{625}a^2$,

$$\begin{aligned}\therefore b^2 &= a^2 - a^2 e^2 \\ &= a^2 - \frac{9}{625}a^4 \quad \dots (1)\end{aligned}$$

Substituting the point $(4, 0)$ into the equation $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, we get

$$\begin{aligned}\frac{(4)^2}{b^2} + \frac{0^2}{a^2} &= 1 \\ \frac{16}{b^2} &= 1\end{aligned}$$

Substituting (1) into the equation, we get

$$\begin{aligned}\frac{16}{a^2 - \frac{9}{625}a^4} &= 1 \\ 16 &= a^2 - \frac{9}{625}a^4 \\ 9a^4 - 625a^2 + 10000 &= 0 \\ (9a^2 - 400)(a^2 - 25) &= 0 \\ a^2 &= 25 \text{ or } a^2 = \frac{400}{9}\end{aligned}$$

When $a^2 = 25$, $b^2 = 25 - \frac{9}{625}(25)^2 = 25 - 9 = 16$.

When $a^2 = \frac{400}{9}$, $b^2 = \frac{400}{9} - \frac{9}{625}\left(\frac{400}{9}\right)^2 = 16$.

\therefore The standard form of the equations of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{25} = 1 \quad \text{or} \quad \frac{x^2}{16} + \frac{9y^2}{400} = 1 \quad \square$$

- (d) Its eccentricity is $\frac{4}{5}$ while the distance between its two foci is 8.

Sol.

Let the equation of the ellipse be of the forms

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

From the given information, we have

$$\sqrt{(2ae)^2 - 0^2} = 8$$

$$2ae = 8$$

$$ae = 4 \dots (1)$$

$$e = \frac{1}{a} \sqrt{a^2 - b^2} = \frac{4}{5} \dots (2)$$

Substituting (2) into (1), we get

$$\frac{4}{5}a = 4$$

$$a = 5$$

Substituting $a = 10$ into (2), we get

$$\frac{1}{5} \sqrt{5^2 - b^2} = \frac{4}{5}$$

$$\sqrt{25 - b^2} = 4$$

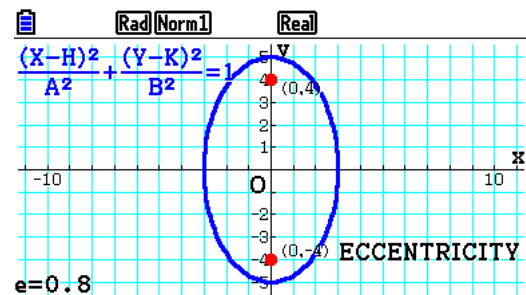
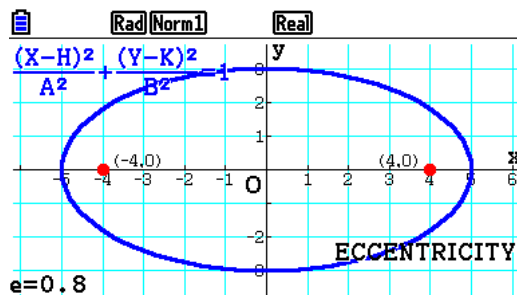
$$25 - b^2 = 16$$

$$b^2 = 9$$

$$b = 3 \ (b > 0)$$

\therefore The standard form of the equation of the ellipse is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \quad \text{or} \quad \frac{x^2}{9} + \frac{y^2}{25} = 1 \quad \square$$



2. The two vertices of one of the sides of the triangle $\triangle ABC$ is $B(0, 6)$ and $C(0, -6)$, while the product of the slopes of the other two sides is $-\frac{4}{9}$, find the equation of the locus of the point A .

Sol.

Let point A be (x, y) .

$$\text{Slope of } AB = \frac{y-6}{x-0} = \frac{y-6}{x}$$

$$\text{Slope of } AC = \frac{y+6}{x-0} = \frac{y+6}{x}$$

The product of the slopes of AB and AC is

$$\frac{y-6}{x} \cdot \frac{y+6}{x} = -\frac{4}{9}$$

$$\frac{y^2-36}{x^2} = -\frac{4}{9}$$

$$9y^2 - 324 = -4x^2$$

$$9y^2 + 4x^2 = 324$$

$$\frac{x^2}{81} + \frac{y^2}{36} = 1 \quad \square$$

3. The ratio between point M and the two foci of the ellipse $\frac{x^2}{13^2} + \frac{y^2}{12^2} = 1$ is $\frac{2}{3}$, find the equation of the locus of point M . Hence, sketch the graph.

Sol.

The eccentricity of the ellipse is

$$e = \frac{1}{13} \sqrt{13^2 - 12^2} = \frac{5}{13}$$

\therefore The foci of the ellipse are at $(\pm 5, 0)$.

Let point M be (x, y) .

$$\sqrt{(x-5)^2 + y^2} = \frac{2}{3} \sqrt{(x+5)^2 + y^2}$$

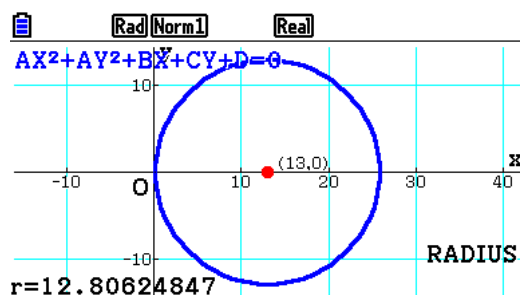
$$(x-5)^2 + y^2 = \frac{4}{9} [(x+5)^2 + y^2]$$

$$9(x-5)^2 + 9y^2 = 4(x+5)^2 + 4y^2$$

$$9x^2 - 90x + 225 + 9y^2 = 4x^2 + 40x + 100 + 4y^2$$

$$5x^2 - 130x + 125 + 5y^2 = 0$$

$$x^2 + y^2 - 26x + 25 = 0 \quad \square$$



4. Find the distance between a point $M\left(\frac{12}{5}, 4\right)$ on the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ and the foci of the ellipse.

Sol.

The eccentricity of the ellipse is $e = \frac{1}{5}\sqrt{25 - 16} = \frac{3}{5}$.

Hence, the foci F and F' of the ellipse are at $(0, 3)$ and $(0, -3)$ respectively.

$$|MF| = \sqrt{\left(\frac{12}{5} - 0\right)^2 + (4 - 3)^2} = \sqrt{\frac{144}{25} + 1} = \frac{13}{5} \quad \square$$

$$|MF'| = \sqrt{\left(\frac{12}{5} - 0\right)^2 + (4 - (-3))^2} = \sqrt{\frac{144}{25} + 49} = \frac{37}{5} \quad \square$$

5. Lazy to do this one. =)

6. Find the equation of the ellipse of which the length of its minor axis is 8 and its foci are $(1, 5)$ and $(4, 5)$.

Sol.

The center of the ellipse is at $\left(\frac{4+1}{2}, 5\right) = \left(\frac{5}{2}, 5\right)$.

Translate the coordinates system so that the center of the ellipse is at the origin, we obtain a new set of coordinates (x', y') such that

$$x' = x - \frac{5}{2} \quad y' = y - 5$$

Since the foci are on the x -axis, the equation of the ellipse is of the form $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$

The coordinates of the foci in the new coordinates system are $\left(-\frac{3}{2}, 0\right)$ and $\left(\frac{3}{2}, 0\right)$.

$$ae = \frac{3}{2} \implies e = \frac{3}{2a}$$

$$e = \frac{1}{a}\sqrt{a^2 - b^2}$$

$$\frac{3}{2a} = \frac{1}{a}\sqrt{a^2 - b^2}$$

$$2a\sqrt{a^2 - b^2} = 3a$$

$$4a^2(a^2 - b^2) = 9a^2$$

$$4a^4 - 64a^2 = 9a^2$$

$$a^2(4a^2 - 73) = 0$$

$$a^2 = 0 \text{ or } a^2 = \frac{73}{4}$$

$$a = \sqrt{\frac{73}{4}} \quad (a > 0)$$

\therefore The standard form of the equation of the ellipse is $\frac{4x'^2}{73} + \frac{y'^2}{16} = 1$.

Substituting $x' = x - \frac{5}{2}$ and $y' = y - 5$ into the equation, we get

$$\frac{4\left(x - \frac{5}{2}\right)^2}{73} + \frac{(y - 5)^2}{16} = 1 \quad \square$$

7. Find the equation of the ellipse of which the length of its minor axis is $2\sqrt{2}$ and the two ends of its major axis are at $(-2, 2)$ and $(8, 2)$.

Sol.

The center of the ellipse is at $\left(\frac{-2+8}{2}, 2\right) = (3, 2)$.

Translate the coordinates system so that the center of the ellipse is at the origin, we obtain a new set of coordinates (x', y') such that

$$x' = x - 3 \quad y' = y - 2$$

Since the major axis is along the x -axis, the equation of the ellipse is of the form $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$

The length of the major axis is $8 - (-2) = 10$, therefore $2a = 10$, $a = 5$.

The length of the minor axis is $2\sqrt{2}$, therefore $2b = 2\sqrt{2}$, $b = \sqrt{2}$.

\therefore The standard form of the equation of the ellipse is $\frac{x'^2}{25} + \frac{y'^2}{2} = 1$.

Substituting $x' = x - 3$ and $y' = y - 2$ into the equation, we get

$$\frac{(x-3)^2}{25} + \frac{(y-2)^2}{2} = 1 \quad \square$$

8. The orbit of the earth is an ellipse with half major axis of length $a = 1.50 \times 10^8$ km, eccentricity $e = 0.0192$, and the sun at one of its foci. Find the maximum and the minimum distance of the earth from the sun.

Sol.

$$ae = 1.50 \times 10^8 \times 0.0192 = 2.88 \times 10^6 \text{ km}$$

The maximum distance of the earth from the sun is $a + ae = 1.50 \times 10^8 + 2.88 \times 10^6 = 1.5288 \times 10^8$ km.

The minimum distance of the earth from the sun is $a - ae = 1.50 \times 10^8 - 2.88 \times 10^6 = 1.4712 \times 10^8$ km. \square

9. Find the eccentricity of the following ellipses:

- (a) The view angle from the focus to the two ends of the minor axis is 60° .

Sol.

The angle between the major axis and the line joining the foci is 30° .

$$\begin{aligned} \frac{b}{ae} &= \tan 30^\circ = \frac{1}{\sqrt{3}} \\ b &= \frac{ae}{\sqrt{3}} \\ a^2 e^2 &= 3b^2 \\ a^2 e^2 &= 3[a^2(1 - e^2)] \\ &= 3a^2 - 3a^2 e^2 \\ e^2 &= 3 - 3e^2 \\ 4e^2 &= 3 \\ e^2 &= \frac{3}{4} \\ e &= \frac{\sqrt{3}}{2} \quad (e > 0) \quad \square \end{aligned}$$

- (b) The viewing angle from one end of the minor axis to the foci is straight angle.

Sol.

Let the coordinates of the foci be $(\pm ae, 0)$, and the coordinates of one of the ends of the minor axis be $(0, b)$.

The slope of the lines joining the foci and the end of the minor axis is $\frac{b}{ae}$ and $\frac{-b}{ae}$.

Since the two lines are perpendicular, we have

$$\begin{aligned}\frac{b}{ae} \cdot \frac{-b}{ae} &= -1 \\ \frac{-b^2}{a^2e^2} &= -1 \\ -b^2 &= -a^2e^2 \\ a^2 - a^2e^2 &= a^2e^2 \\ 2e^2 &= 1 \\ e^2 &= \frac{1}{2} \\ e &= \frac{\sqrt{2}}{2} \quad (e > 0) \quad \square\end{aligned}$$

10. Calculate the side length of the inscribed square in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol.

Let one of the vertices of the square that is in the first quadrant be (m, m) .

Substituting the coordinates of the vertex into the equation of the ellipse, we get

$$\begin{aligned}\frac{m^2}{a^2} + \frac{m^2}{b^2} &= 1 \\ m^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) &= 1 \\ m^2 &= \frac{a^2b^2}{a^2 + b^2} \\ m &= \frac{ab}{\sqrt{a^2 + b^2}}\end{aligned}$$

Hence, the side length of the inscribed square is $\frac{2ab}{\sqrt{a^2 + b^2}} = \frac{2ab}{a^2 + b^2} \sqrt{a^2 + b^2}$. \square

11. Prove that the points of intersection of the two ellipses $b^2x^2 + a^2y^2 - a^2b^2 = 0$ and $a^2x^2 + b^2y^2 - a^2b^2 = 0$ ($a > b > 0$) are on the circumference of a circle with the center at the origin. Hence, find the equation of the circle.

Proof.

Adding the two equations, we get

$$\begin{aligned}b^2x^2 + a^2y^2 - a^2b^2 + a^2x^2 + b^2y^2 - a^2b^2 &= 0 \\ (a^2 + b^2)x^2 + (a^2 + b^2)y^2 - 2a^2b^2 &= 0 \\ x^2 + y^2 &= \frac{2a^2b^2}{a^2 + b^2} \quad \square\end{aligned}$$