Exercise 5d

1. Find the standard equations of hyperbolas that satisfy the following conditions:

   (a) $a = 4, b = 3$, with foci on the x-axis.

   Sol.
   Since the foci are on the x-axis, the hyperbola is of the form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.
   Substituting $a = 4, b = 3$, we get $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$. □

   (b) $a = 2\sqrt{5}$, passing through the point A$(2, -5)$, with foci on the y-axis.

   Sol.
   Since the foci are on the y-axis, the hyperbola is of the form $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$.
   Substituting $a = 2\sqrt{5}$, and the point A$(2, -5)$, we get $\dfrac{y^2}{20} - \dfrac{x^2}{b^2} = 1$.
   Substituting the point A$(2, -5)$, we get

$$\begin{aligned}
\frac{(-5)^2}{20} - \frac{2^2}{b^2} &= 1 \\
\frac{25}{20} - \frac{4}{b^2} &= 1 \\
\frac{5}{4} - \frac{4}{b^2} &= 1 \\
\frac{1}{4} &= \frac{4}{b^2} \\
b^2 &= 16 \\
b &= 4
\end{aligned}$$

   Hence, the equation of the hyperbola is $\dfrac{y^2}{20} - \dfrac{x^2}{16} = 1$. □

   (c) Foci coordinates are $(0, -5), (0, 5)$, eccentricity is $\dfrac{5}{4}$.

   Sol.
   Since the foci are on the y-axis, the hyperbola is of the form $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$.

$$\begin{aligned}
ae &= 5 \\
\frac{5}{4}a &= 5 \\
a &= 4 \\
a^2e^2 &= 25 \\
b^2 &= a^2e^2 - a^2 \\
&= 25 - 16 \\
&= 9 \\
b &= 3
\end{aligned}$$

   Hence, the equation of the hyperbola is $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$. □

(d) Foci on the x-axis, passing through points $\mathrm{A}(-8,2),\ \mathrm{B}(4\sqrt{3},-\sqrt{2})$.

Sol.

Since the foci are on the x-axis, the hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Substituting the points $\mathrm{A}(-8,2),\ \mathrm{B}(4\sqrt{3},-\sqrt{2})$, we get

$$\begin{aligned}
\frac{(-8)^2}{a^2}-\frac{2^2}{b^2}&=1\\
\frac{64}{a^2}-\frac{4}{b^2}&=1\\
64b^2-a^2b^2&=4a^2\\
(64-a^2)b^2&=4a^2\\
b^2&=\frac{4a^2}{64-a^2} && (1)\\
\frac{(4\sqrt{3})^2}{a^2}-\frac{(-\sqrt{2})^2}{b^2}&=1 && (2)
\end{aligned}$$

Substituting equation (1) into equation (2), we get

$$\begin{aligned}
\frac{48}{a^2}-\frac{2}{\frac{4a^2}{64-a^2}}&=1\\
\frac{48}{a^2}-\frac{64-a^2}{2a^2}&=1\\
96-64+a^2&=2a^2\\
a^2&=32\\
b^2&=\frac{4\times 32}{64-32}\\
b^2&=4
\end{aligned}$$

Hence, the equation of the hyperbola is $\frac{x^2}{32}-\frac{y^2}{4}=1$. □

2. Find the equation of the hyperbola with foci $(-2,0),(6,0)$, and eccentricity of 2.

Sol.

The center of the hyperbola is the midpoint of the foci, which is $(2,0)$.

Move the center of the hyperbola to the origin and form a new coordinate system such that

$$x'=x-2 \qquad y'=y$$

Since the foci are on the x-axis, the hyperbola is of the form $\frac{x'^2}{a^2}-\frac{y'^2}{b^2}=1$.

The foci are now at $(-4,0),(4,0)$, and the eccentricity is 2.

$$\begin{aligned}
ae&=4\\
a&=2\\
b^2&=a^2e^2-a^2=12
\end{aligned}$$

Hence, the equation of the hyperbola is $\frac{x'^2}{4}-\frac{y'^2}{12}=1$.

Substituting $x'=x-2$, we get $\frac{(x-2)^2}{4}-\frac{y^2}{12}=1$. □

3. Find the equation of the hyperbola that has common foci with the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, and eccentricity of $\frac{\sqrt{5}}{2}$.

Sol.

$$e = \frac{1}{3}\sqrt{3^2 - 2^2} = \frac{\sqrt{5}}{3}$$

The foci of the ellipse are at $(\pm\sqrt{5}, 0)$.

$$\begin{aligned} ae &= \sqrt{5} \\ \frac{\sqrt{5}}{2}a &= \sqrt{5} \\ a &= 2 \\ b &= a^2e^2 - a^2 \\ &= 5 - 4 \\ &= 1 \end{aligned}$$

Hence, the equation of the hyperbola is $\frac{x^2}{4} - \frac{y^2}{1} = 1$. $\square$

4. Find the asymptotes of the following hyperbolas:

(a) $\frac{x^2}{25} - \frac{y^2}{144} = 1$

Sol.

$$\begin{aligned} \frac{x^2}{25} - \frac{y^2}{144} &= 0 \\ \frac{x^2}{25} &= \frac{y^2}{144} \\ 25y^2 &= 144x^2 \\ y^2 &= \frac{144}{25}x^2 \\ y &= \pm\frac{12}{5}x \\ 5y &= \pm 12x \\ 5y + 12x = 0 \text{ or } 5y - 12x &= 0 \qquad \square \end{aligned}$$

(b) $\frac{y^2}{9} - \frac{x^2}{16} = 1$

Sol.

$$\begin{aligned} \frac{y^2}{9} - \frac{x^2}{16} &= 0 \\ \frac{y^2}{9} &= \frac{x^2}{16} \\ 16y^2 &= 9x^2 \\ y^2 &= \frac{9}{16}x^2 \\ y &= \pm\frac{3}{4}x \\ 4y &= \pm 3x \\ 4y + 3x = 0 \text{ or } 4y - 3x &= 0 \qquad \square \end{aligned}$$

(c) $4x^2 - 9y^2 + 16x - 18y - 29 = 0$

Sol.

$$\begin{aligned}
4x^2 - 9y^2 + 16x - 18y - 29 &= 0 \\
4(x^2 + 4x) - 9(y^2 - 2y) - 29 &= 0 \\
4(x^2 + 4x + 4) - 9(y^2 - 2y + 1) &= 29 + 16 - 9 \\
4(x+2)^2 - 9(y-1)^2 &= 36 \\
\frac{(x+2)^2}{9} - \frac{(y-1)^2}{4} &= 1 \\
\frac{(x+2)^2}{9} - \frac{(y-1)^2}{4} &= 0 \\
\left(\frac{x+2}{3} + \frac{y-1}{2}\right)\left(\frac{x+2}{3} - \frac{y-1}{2}\right) &= 0 \\
\frac{x+2}{3} + \frac{y-1}{2} = 0 \text{ or } \frac{x+2}{3} - \frac{y-1}{2} &= 0 \\
2x + 4 + 3y - 3 = 0 \text{ or } 2x + 4 - 3y + 3 &= 0 \\
2x + 3y + 1 = 0 \text{ or } 2x - 3y + 7 &= 0 \qquad \square
\end{aligned}$$

(d) $\dfrac{(x-2)^2}{3} - \dfrac{(y-1)^2}{12} = 1$

Sol.

$$\begin{aligned}
\frac{(x-2)^2}{3} - \frac{(y-1)^2}{12} &= 0 \\
\frac{(x-2)^2}{3} &= \frac{(y-1)^2}{12} \\
12(x-2)^2 &= 3(y-1)^2 \\
4(x-2)^2 &= (y-1)^2 \\
2(x-2) &= \pm(y-1) \\
2x - 4 &= \pm(y-1) \\
2x - 4 + y - 1 = 0 \text{ or } 2x - 4 - y + 1 &= 0 \\
2x + y - 3 = 0 \text{ or } 2x - y + 5 &= 0 \qquad \square
\end{aligned}$$

5. I am lazy to do. =)

6. Find the equation of hyperbolas satisfying the following conditions:

(a) Center at the origin, foci on the coordinate axes, eccentricity $e = \sqrt{2}$, passing through point M$(-5, 3)$.

Sol.

Let the hyperbola be of the form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

$$\begin{aligned}
b^2 &= a^2e^2 - a^2 \\
&= 2a^2 - a^2 \\
&= a^2
\end{aligned}$$

Substituting the point M$(-5,3)$ and $a^2=b$ into the equation of the hyperbola, we get

$$\begin{aligned}\frac{(-5)^2}{a^2}-\frac{3^2}{a^2}&=1\\ \frac{25}{a^2}-\frac{9}{a^2}&=1\\ 25-9&=a^2\\ a^2&=16\\ a&=4\\ b&=4\end{aligned}$$

Hence, the equation of the hyperbola is $\dfrac{x^2}{16}-\dfrac{y^2}{16}=1$. $\square$

(b) Imaginary axis length is 4, foci at $(0,2),(0,-10)$.

Sol.

The center of the hyperbola is the midpoint of the foci, which is $(0,-4)$.

Move the center of the hyperbola to the origin and form a new coordinate system such that

$$x'=x \qquad y'=y+4$$

Since the foci are on the y-axis, the hyperbola is of the form $\dfrac{y'^2}{a^2}-\dfrac{x'^2}{b^2}=1$.

The foci are now at $(0,6),(0,-6)$, and the imaginary axis length is 4.

$$\begin{aligned}ae&=6\\ b^2&=a^2e^2-a^2\\ \left(\frac{4}{2}\right)^2&=36-a^2\\ a^2&=32\\ b^2&=36-32\\ &=4\end{aligned}$$

Hence, the equation of the hyperbola is $\dfrac{y'^2}{32}-\dfrac{x'^2}{4}=1$.

Substituting $y'=y+4$, we get $\dfrac{(y+4)^2}{32}-\dfrac{x^2}{4}=1$. $\square$

(c) Imaginary axis length is 6, vertices at $(3,2),(-5,2)$.

Sol. The center of the hyperbola is the midpoint of the vertices, which is $(-1,2)$.

Move the center of the hyperbola to the origin and form a new coordinate system such that

$$x'=x+1 \qquad y'=y-2$$

Since the vertices are on the x-axis, the hyperbola is of the form $\dfrac{x'^2}{a^2}-\dfrac{y'^2}{b^2}=1$.

The vertices are now at $(4,0),(-4,0)$, and the imaginary axis length is 6.

$$a=4 \qquad b=3$$

Hence, the equation of the hyperbola is $\dfrac{(x+1)^2}{16}-\dfrac{(y-2)^2}{9}=1$. $\square$

7. Find the equation of the hyperbola with vertices of the ellipse $\frac{x^2}{8}+\frac{y^2}{5}=1$ as its foci, and with the foci of the ellipse as its vertices.

Sol.

$$e=\frac{1}{8}\sqrt{8-5}=\frac{1}{8}\sqrt{3}$$

The foci of the ellipse are at $(\pm\sqrt{3},0)$, and the vertices are at $(\pm 2\sqrt{2},0)$.

The foci of the hyperbola are at $(\pm 2\sqrt{2},0)$, and the vertices are at $(\pm\sqrt{3},0)$.

Since the foci are on the x-axis, the hyperbola is of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

$$\begin{aligned}ae&=2\sqrt{2}\\ b^2&=a^2e^2-a^2\\ &=8-3\\ &=5\\ a^2&=3\end{aligned}$$

Hence, the equation of the hyperbola is $\frac{x^2}{3}-\frac{y^2}{5}=1$. $\square$

8. A hyperbola passes through the point $(0,3)$, and its asymptotes have equations $2x-y-3=0$ and $2x+y-5=0$. Find its equation.

Sol.

$$\begin{aligned}2x-y-3=0 \text{ or } 2x+y-5&=0\\ 2x-4&=y+3 \text{ or } 2x=-y+5\\ 2x-4&=y+3-4 \text{ or } 2x-4=-(y-5)-4\\ 2x-4&=y-1 \text{ or } 2x-4=-(y-1)\\ 2x-4&=\pm(y-1)\\ (2x-4)^2&=(y-1)^2\\ (2x-4)^2-(y-1)^2&=k\end{aligned}$$

Substituting the point $(0,3)$, we get

$$\begin{aligned}(2\times 0-4)^2-(3-1)^2&=k\\ 16-4&=k\\ k&=12\\ (2x-4)^2-(y-1)^2&=12\\ 4x^2-16x+16-y^2+2y-1&=12\\ 4x^2-y^2-16x+2y+3&=0 \qquad \square\end{aligned}$$

9. Find the equation of the hyperbola with foci $(-2,-2)$ and $(8,-2)$, and one asymptote with slope $\frac{3}{4}$.

Sol.

The center of the hyperbola is the midpoint of the foci, which is $(3,-2)$.

Move the center of the hyperbola to the origin and form a new coordinate system such that

$$x' = x - 3 \qquad y' = y + 2$$

Since the foci are on the x-axis, the hyperbola is of the form $\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1$.

The foci are now at $(-5,0),(5,0)$, and the slope of the asymptote is $\frac{3}{4}$.

$$\begin{aligned}
\frac{b}{a} &= \frac{3}{4} \\
b &= \frac{3}{4}a \\
e &= \frac{1}{a}\sqrt{a^2+b^2} \\
ae &= \sqrt{a^2+b^2} \\
(ae)^2 &= a^2+b^2 \\
25 &= a^2 + \frac{9}{16}a^2 \\
25 &= \frac{25}{16}a^2 \\
a^2 &= 16 \\
b^2 &= 25 - 16 \\
&= 9
\end{aligned}$$

Hence, the equation of the hyperbola is $\frac{x'^2}{16} - \frac{y'^2}{9} = 1$.

Substituting $x' = x-3, y' = y+2$, we get $\frac{(x-3)^2}{16} - \frac{(y+2)^2}{9} = 1$. □

10. (a) Given that a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has midpoint (α, β), prove that the equation of this chord is $y - \beta = \frac{b^2\alpha}{a^2\beta}(x - \alpha)$.

Proof.

Let the two points on the chord be (x_1, y_1) and (x_2, y_2).

$$\begin{aligned} \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} &= 1 \\ b^2x_1^2 - a^2y_1^2 &= a^2b^2 \\ \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} &= 1 \\ b^2x_2^2 - a^2y_2^2 &= a^2b^2 \end{aligned}$$

Subtracting the second equation from the first, we get

$$\begin{aligned} b^2(x_1^2 - x_2^2) - a^2(y_1^2 - y_2^2) = 0 & \\ a^2(y_1^2 - y_2^2) &= b^2(x_1^2 - x_2^2) \\ \frac{y_1^2 - y_2^2}{x_1^2 - x_2^2} &= \frac{2b^2}{a^2} \\ \frac{(y_1 + y_2)(y_1 - y_2)}{(x_1 + x_2)(x_1 - x_2)} &= \frac{2b^2}{a^2} \end{aligned}$$

Since the midpoint of the chord is (α, β), we have

$$\begin{aligned} x_1 + x_2 &= 2\alpha \\ y_1 + y_2 &= 2\beta \end{aligned}$$

Substituting, we get

$$\begin{aligned} \frac{2\beta(y_1 - y_2)}{2\alpha(x_1 - x_2)} &= \frac{b^2}{a^2} \\ \frac{\beta(y_1 - y_2)}{\alpha(x_1 - x_2)} &= \frac{b^2}{a^2} \\ \frac{y_1 - y_2}{x_1 - x_2} &= \frac{\alpha b^2}{\beta a^2} \end{aligned}$$

Since (x_1, y_1) and (x_2, y_2) are on the chord, $m = \frac{y_1 - y_2}{x_1 - x_2}$.

Since (α, β) is on the chord,the equation of the chord is $y - \beta = \frac{b^2\alpha}{a^2\beta}(x - \alpha)$. $\square$

(b) Given that a moving chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the fixed point (h, k), prove that the locus of the midpoints of these chords is a hyperbola, and its center is $\left(\frac{h}{2}, \frac{k}{2}\right)$.

Proof.

Let the midpoint of the chord be (α, β).

From part (a), the equation of the chord is $y - \beta = \frac{b^2\alpha}{a^2\beta}(x - \alpha)$.

Substituting (h, k), we get

$$\begin{aligned} k - \beta &= \frac{b^2\alpha}{a^2\beta}(h - \alpha) \\ \beta &= k - \frac{b^2\alpha}{a^2\beta}(h - \alpha) \end{aligned}$$

Let $\beta = y$ and $\alpha = x$.

$$\begin{aligned} y &= k - \frac{b^2x}{a^2y}(h - x) \\ a^2y^2 &= a^2ky - b^2hx + b^2x^2 \\ b^2x^2 - b^2hx - a^2y^2 + a^2ky + &= 0 \\ b^2(x^2 - hx) - a^2(y^2 - ky) &= 0 \\ b^2\left(x^2 - 2\left(\frac{h}{2}\right)x + \left(\frac{h}{2}\right)^2\right) - a^2\left(y^2 - 2\left(\frac{k}{2}\right)y + \left(\frac{k}{2}\right)^2\right) &= \frac{b^2h^2 - a^2k^2}{4} \\ b^2\left(x - \frac{h}{2}\right)^2 - a^2\left(y - \frac{k}{2}\right)^2 &= \frac{b^2h^2 - a^2k^2}{4} \\ \frac{4b^2\left(x - \frac{h}{2}\right)^2}{b^2h^2 - a^2k^2} - \frac{4a^2\left(y - \frac{k}{2}\right)^2}{b^2h^2 - a^2k^2} &= 1 \end{aligned}$$

$\because$ the equation of the locus is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore$ the locus is a hyperbola, and its center is $\left(\frac{h}{2}, \frac{k}{2}\right)$. $\square$