Exercise 11g

Find the following indefinite integrals:

1. $\displaystyle\int \frac{1}{2\sin x - \cos x + 5}\,dx$

   Sol.

   Let $t = \tan\dfrac{x}{2}$, $dx = \dfrac{2}{1+t^2}\,dt$.

$$\begin{aligned}
\int \frac{1}{2\sin x - \cos x + 5}\,dx &= \int \frac{1}{2\left(\dfrac{2t}{1+t^2}\right) - \dfrac{1-t^2}{1+t^2} + 5} \cdot \frac{2}{1+t^2}\,dt \\
&= \int \frac{2}{\dfrac{4t - 1 + t^2 + 5(1+t^2)}{1+t^2} \cdot (1+t^2)}\,dt \\
&= \int \frac{2}{6t^2 + 4t + 4}\,dt \\
&= \int \frac{1}{3t^2 + 2t + 2}\,dt \\
&= \frac{1}{3}\int \frac{1}{t^2 + \dfrac{2}{3}t + \dfrac{2}{3}}\,dt \\
&= \frac{1}{3}\int \frac{1}{\left(t + \dfrac{1}{3}\right)^2 - \dfrac{1}{9} + \dfrac{2}{3}}\,dt \\
&= \frac{1}{3}\int \frac{1}{\left(t + \dfrac{1}{3}\right)^2 + \dfrac{5}{9}}\,dt \qquad \left(\text{Let } u = t + \frac{1}{3},\ du = dt\right) \\
&= \frac{1}{3}\int \frac{1}{u^2 + \left(\dfrac{\sqrt{5}}{3}\right)^2}\,du \\
&= \frac{1}{3} \cdot \frac{1}{\dfrac{\sqrt{5}}{3}} \cdot \tan^{-1}\frac{u}{\dfrac{\sqrt{5}}{3}} + C \\
&= \frac{1}{\sqrt{5}}\tan^{-1}\frac{3t+1}{\sqrt{5}} + C \\
&= \frac{1}{\sqrt{5}}\tan^{-1}\frac{3\tan\dfrac{x}{2} + 1}{\sqrt{5}} + C \qquad \square
\end{aligned}$$

2. $\displaystyle\int \frac{1}{4 + 5\cos x}\,dx$

   Sol.

   Let $t = \tan\dfrac{x}{2}$, $dx = \dfrac{2}{1+t^2}\,dt$.

$$\begin{aligned}
\int \frac{1}{4 + 5\cos x}\,dx &= \int \frac{1}{4 + 5\left(\dfrac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\,dt \\
&= \int \frac{2}{4 + 4t^2 + 5 - 5t^2}\,dt \\
&= \int \frac{2}{9 - t^2}\,dt
\end{aligned}$$

$$= \int \frac{2}{(3+t)(3-t)}\, dt \qquad \square$$

Let $\dfrac{2}{(3+t)(3-t)} = \dfrac{A}{3+t} + \dfrac{B}{3-t}$

$$\begin{aligned} 2 &= A(3-t) + B(3+t) \\ &= (-A+B)t + 3(A+B) \end{aligned}$$

Comparing coefficients,

$$\begin{aligned} -A+B &= 0 \quad \cdots\cdots \ (1) \\ 3(A+B) &= 2 \\ A+B &= \frac{2}{3} \quad \cdots\cdots \ (2) \\ (1)+(2) \implies 2B &= \frac{2}{3} \\ B &= \frac{1}{3} \\ A &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \therefore \int \frac{1}{4+5\cos x}\, dx &= \frac{1}{3}\int \frac{1}{3+t} + \frac{1}{3}\int \frac{1}{3-t}\, dt \\ &= \frac{1}{3}\left(\ln|3+t| - \ln|3-t|\right) + C \\ &= \frac{1}{3}\ln\left|\frac{3+t}{3-t}\right| + C \\ &= \frac{1}{3}\ln\left|\frac{3+\tan\frac{x}{2}}{3-\tan\frac{x}{2}}\right| + C \end{aligned} \qquad \square$$

3. $\displaystyle\int \frac{1}{5+4\sin x}\, dx$

Sol.

Let $t = \tan\dfrac{x}{2}$, $dx = \dfrac{2}{1+t^2}\, dt$.

$$\begin{aligned} \int \frac{1}{5+4\sin x}\, dx &= \int \frac{1}{5+4\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\, dt \\ &= \int \frac{2}{5+5t^2+8t}\, dt \\ &= \frac{2}{5}\int \frac{1}{t^2+\frac{8}{5}t+1}\, dt \\ &= \frac{2}{5}\int \frac{1}{\left(t+\frac{4}{5}\right)^2 - \frac{16}{25} + 1}\, dt \\ &= \frac{2}{5}\int \frac{1}{\left(t+\frac{4}{5}\right)^2 + \frac{9}{25}}\, dt \\ &= \frac{2}{5}\int \frac{1}{\left(\frac{3}{5}\right)^2 + \left(t+\frac{4}{5}\right)^2}\, dt \\ &= \frac{2}{5}\cdot\frac{5}{3}\cdot\tan^{-1}\frac{t+\frac{4}{5}}{\frac{3}{5}} + C \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{3}\tan^{-1}\frac{5t+4}{3} + C \\
&= \frac{2}{3}\tan^{-1}\frac{5\tan\frac{x}{2}+4}{3} + C \qquad \square
\end{aligned}
$$

4. $\displaystyle\int \frac{1+\sin x}{\sin x(1+\cos x)}\,dx$

Sol.

Let $t = \tan\frac{x}{2}$, $dx = \dfrac{2}{1+t^2}\,dt$.

$$
\begin{aligned}
\int \frac{1+\sin x}{\sin x(1+\cos x)}\,dx &= \int \frac{1+\frac{2t}{1+t^2}}{\frac{2t}{1+t^2}(1+\frac{1-t^2}{1+t^2})}\cdot\frac{2}{1+t^2}\,dt \\
&= \int \frac{1+t^2+2t}{t(1+t^2+1-t^2)}\,dt \\
&= \int \frac{1+t^2+2t}{2t}\,dt \\
&= \frac{1}{2}\int\left(t+2+\frac{1}{t}\right)\,dt \\
&= \frac{1}{4}t^2+t+\frac{1}{2}\ln|t|+C \\
&= \frac{1}{4}\tan^2\frac{x}{2}+\tan\frac{x}{2}+\frac{1}{2}\ln\left|\tan\frac{x}{2}\right|+C \qquad \square
\end{aligned}
$$

5. $\displaystyle\int \frac{\cot x}{\sin x+\cos x-1}\,dx$

Sol.

Let $t = \tan\frac{x}{2}$, $dx = \dfrac{2}{1+t^2}\,dt$.

$$
\begin{aligned}
\int \frac{\cot x}{\sin x+\cos x-1}\,dx &= \int \frac{\cos x}{\sin x(\sin x+\cos x-1)}\,dx \\
&= \int \frac{\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}\left(\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}-1\right)}\cdot\frac{2}{1+t^2}\,dt \\
&= \int \frac{2(1-t^2)}{2t\left(\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}-1\right)}\cdot\frac{1}{1+t^2}\,dt \\
&= \int \frac{1-t^2}{t\,(2t+1-t^2-1-t^2)}\,dt \\
&= \int \frac{1-t^2}{t\,(2t-2t^2)}\,dt \\
&= \int \frac{(1+t)(1-t)}{2t^2\,(1-t)}\,dt \\
&= \int \frac{1+t}{2t^2}\,dt \\
&= \frac{1}{2}\int\frac{1}{t^2}\,dt+\frac{1}{2}\int\frac{1}{t}\,dt \\
&= -\frac{1}{2t}+\frac{1}{2}\ln|t|+C \\
&= -\frac{1}{2}\cot\frac{x}{2}+\frac{1}{2}\ln\left|\tan\frac{x}{2}\right|+C \qquad \square
\end{aligned}
$$

6. $\displaystyle\int \frac{1}{5\sec x - 3}\,dx$

Sol.

Let $t = \tan\dfrac{x}{2}$, $dx = \dfrac{2}{1+t^2}\,dt$.

$$\begin{aligned}
\int \frac{1}{5\sec x - 3}\,dx &= \int \frac{1}{5\left(\dfrac{1+t^2}{1-t^2}\right) - 3} \cdot \frac{2}{1+t^2}\,dt \\
&= \int \frac{2}{\dfrac{(5+5t^2-3+3t^2)(1+t^2)}{1-t^2}}\,dt \\
&= \int \frac{2(1-t^2)}{(8t^2+2)(1+t^2)}\,dt \\
&= \int \frac{2(1-t^2)}{2(4t^2+1)(1+t^2)}\,dt \\
&= \int \frac{1-t^2}{(4t^2+1)(1+t^2)}\,dt
\end{aligned}$$

Let $\dfrac{1-t^2}{(4t^2+1)(1+t^2)} = \dfrac{A}{4t^2+1} + \dfrac{B}{1+t^2}$.

$$\begin{aligned}
1-t^2 &= A(1+t^2) + B(4t^2+1) \\
&= A + At^2 + 4Bt^2 + B \\
&= (A+4B)t^2 + (A+B)
\end{aligned}$$

Comparing coefficients,

$$\begin{aligned}
A + 4B &= -1 \quad \cdots\cdots \ (1) \\
A + B &= 1 \quad \cdots\cdots \ (2) \\
(1)-(2) \implies 3B &= -2 \\
B &= -\frac{2}{3} \\
A &= 1 - B \\
&= 1 + \frac{2}{3} \\
&= \frac{5}{3}
\end{aligned}$$

$$\begin{aligned}
\therefore \int \frac{1}{5\sec x - 3}\,dx &= \frac{5}{3}\int \frac{1}{1+4t^2}\,dt - \frac{2}{3}\int \frac{1}{1+t^2}\,dt \\
&= \frac{5}{6}\tan^{-1}(2t) - \frac{2}{3}\tan^{-1} t + C \\
&= \frac{5}{6}\tan^{-1}\left(2\tan\frac{x}{2}\right) - \frac{2}{3}\tan^{-1}\left(\tan\frac{x}{2}\right) + C \\
&= \frac{5}{6}\tan^{-1}\left(2\tan\frac{x}{2}\right) - \frac{2}{3}\cdot\frac{x}{2} + C \\
&= \frac{5}{6}\tan^{-1}\left(2\tan\frac{x}{2}\right) - \frac{1}{3}x + C \qquad \square
\end{aligned}$$

7. $\displaystyle\int \frac{\cos x}{1-\cos x}\,dx$

Sol.

Let $t = \tan\frac{x}{2}$, $dx = \frac{2}{1+t^2}\,dt$.

$$\begin{aligned}
\int \frac{\cos x}{1-\cos x}\,dx &= \int \frac{\frac{1-t^2}{1+t^2}}{1-\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt \\
&= \int \frac{2(1-t^2)}{1+t^2-1+t^2} \cdot \frac{1}{1+t^2}\,dt \\
&= \int \frac{2(1-t^2)}{2t^2(1+t^2)}\,dt \\
&= \int \frac{1-t^2}{t^2(1+t^2)}\,dt
\end{aligned}$$

Let $\dfrac{1-t^2}{t^2(1+t^2)} = \dfrac{A}{t^2} + \dfrac{B}{1+t^2}$.

$$\begin{aligned}
1-t^2 &= A(1+t^2) + Bt^2 \\
&= A + At^2 + Bt^2 \\
&= (A+B)t^2 + A
\end{aligned}$$

Comparing coefficients,

$$\begin{aligned}
A+B &= -1 \quad \cdots\cdots \ (1) \\
A &= 1 \quad \cdots\cdots \ (2)
\end{aligned}$$

$(1)-(2) \implies B = -2$

$$\begin{aligned}
\therefore \int \frac{\cos x}{1-\cos x}\,dx &= \int \frac{1}{t^2}\,dt - 2\int \frac{1}{1+t^2}\,dt \\
&= -\frac{1}{t} - 2\tan^{-1} t + C \\
&= -\frac{1}{\tan\frac{x}{2}} - 2\tan^{-1}\left(\tan\frac{x}{2}\right) + C \\
&= -\frac{2}{\tan x} - 2\cdot\frac{x}{2} + C \\
&= -\frac{2}{\tan x} - x + C \qquad \square
\end{aligned}$$

8. $\displaystyle\int \frac{1}{\sin x + \tan x}\,dx$

Sol.

Let $t = \tan\frac{x}{2}$, $dx = \frac{2}{1+t^2}\,dt$.

$$\begin{aligned}
\int \frac{1}{\sin x + \tan x}\,dx &= \int \frac{1}{\sin x + \frac{\sin x}{\cos x}}\,dx \\
&= \int \frac{\cos x}{\sin x(\cos x + 1)}\,dx \\
&= \int \frac{\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}\left(\frac{1-t^2}{1+t^2}+1\right)} \cdot \frac{2}{1+t^2}\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \int \frac{1-t^2}{t\left(\dfrac{1-t^2}{1+t^2}+1\right)} \cdot \frac{1}{1+t^2}\, dt \\
&= \int \frac{1-t^2}{t\,(1-t^2+1+t^2)}\, dt \\
&= \int \frac{1-t^2}{2t}\, dt \\
&= \frac{1}{2}\int \frac{1}{t}\, dt - \frac{1}{2}\int t\, dt \\
&= \frac{1}{2}\ln|t| - \frac{1}{4}t^2 + C \\
&= \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{4}\tan^2\frac{x}{2} + C \qquad \square
\end{aligned}
$$

9. $\displaystyle\int \frac{1}{1-\sin x}\, dx$

Sol.

Let $t = \tan\dfrac{x}{2}$, $dx = \dfrac{2}{1+t^2}\, dt$.

$$
\begin{aligned}
\int \frac{1}{1-\sin x}\, dx &= \int \frac{1}{1-\dfrac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt \\
&= \int \frac{2}{1+t^2-2t}\, dt \\
&= \int \frac{2}{(t-1)^2}\, dt \\
&= -\frac{2}{t-1} + C \\
&= -\frac{2}{\tan\dfrac{x}{2}-1} + C
\end{aligned}
$$

10. $\displaystyle\int \frac{\sec x}{2\tan x + \sec x - 1}\, dx$

Sol.

Let $t = \tan\dfrac{x}{2}$, $dx = \dfrac{2}{1+t^2}\, dt$.

$$
\begin{aligned}
\int \frac{\sec x}{2\tan x + \sec x - 1}\, dx &= \int \frac{\sec x}{2\cdot\dfrac{\sin x}{\cos x} + \dfrac{1}{\cos x} - 1}\, dx \\
&= \int \frac{\sec x}{\dfrac{2\sin x + 1 - \cos x}{\cos x}}\, dx \\
&= \int \frac{\sec x}{\sec x(2\sin x + 1 - \cos x)}\, dx \\
&= \int \frac{1}{2\sin x + 1 - \cos x}\, dx \\
&= \int \frac{1}{2\cdot\dfrac{2t}{1+t^2} + 1 - \dfrac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt \\
&= \int \frac{2}{4t+1+t^2-1+t^2}\, dt \\
&= \int \frac{2}{2t^2+4t}\, dt \\
&= \int \frac{1}{t(t+2)}\, dt
\end{aligned}
$$

Let $\dfrac{1}{t(t+2)} = \dfrac{A}{t} + \dfrac{B}{t+2}$.

$$1 = A(t+2) + Bt$$
$$1 = (A+B)t + 2A$$

Comparing coefficients,

$$\begin{aligned} A+B &= 0 \\ 2A &= 1 \\ A &= \frac{1}{2} \\ B &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \int \frac{1}{t(t+2)}\,dt &= \frac{1}{2}\int \frac{1}{t}\,dt - \frac{1}{2}\int \frac{1}{t+2}\,dt \\ &= \frac{1}{2}\ln|t| - \frac{1}{2}\ln|t+2| + C \\ &= \frac{1}{2}\ln\left|\frac{t}{t+2}\right| + C \\ &= \frac{1}{2}\ln\left|\frac{\tan\frac{x}{2}}{\tan\frac{x}{2}+2}\right| + C \qquad \square \end{aligned}$$

11. Let $\tan\theta = t$, find $\displaystyle\int \frac{1}{2\cos^2\theta - 1}\,dx$

Sol.
Let $t = \tan\theta$, $dx = \dfrac{1}{1+t^2}\,dt$.

$$\begin{aligned} \int \frac{1}{2\cos^2\theta - 1}\,dx &= \int \frac{1}{2\left(\frac{1}{\sec^2\theta}\right) - 1}\,dx \\ &= \int \frac{1}{2\left(\frac{1}{1+t^2}\right) - 1} \cdot \frac{1}{1+t^2}\,dt \\ &= \int \frac{1}{2 - 1 - t^2}\,dt \\ &= \int \frac{1}{1-t^2}\,dt \\ &= \int \frac{1}{(1+t)(1-t)}\,dt \end{aligned}$$

Let $\dfrac{1}{(1+t)(1-t)} = \dfrac{A}{1+t} + \dfrac{B}{1-t}$.

$$1 = A(1-t) + B(1+t)$$
$$1 = A + At - B + Bt$$
$$1 = A - B + t(A+B)$$

Comparing coefficients,

$$\begin{aligned} A+B &= 0 \\ A-B &= 1 \\ A &= \frac{1}{2} \end{aligned}$$

$$B = -\frac{1}{2}$$

$$\begin{aligned}
\int \frac{1}{(1+t)(1-t)}\,dt &= \frac{1}{2}\int \frac{1}{1+t}\,dt + \frac{1}{2}\int \frac{1}{1-t}\,dt \\
&= \frac{1}{2}\ln|1+t| + \frac{1}{2}\ln|1-t| + C \\
&= \frac{1}{2}\ln|(1+t)(1-t)| + C \\
&= \ln\sqrt{1-t^2} + C \\
&= \ln\sqrt{1-\tan^2\theta} + C \qquad \square
\end{aligned}$$

12. Let $\tan x = t$, find $\displaystyle\int \frac{\tan x}{3-4\sin^2 x}\,dx$

Sol.

Let $t = \tan x$, $x = \tan^{-1} t$, $dx = \dfrac{1}{1+t^2}\,dt$.

Sol.

$$\begin{aligned}
\int \frac{\tan x}{3-4\sin^2 x}\,dx &= \int \frac{\tan x}{3-4\cdot\dfrac{1}{\operatorname{cosec}^2 x}}\,dx \\
&= \int \frac{\tan x}{3-\dfrac{4}{\cot^2 x+1}}\,dx \\
&= \int \frac{t}{3-\dfrac{4}{\dfrac{1}{t^2}+1}}\cdot\frac{1}{1+t^2}\,dt \\
&= \int \frac{t}{3-\dfrac{4t^2}{1+t^2}}\cdot\frac{1}{1+t^2}\,dt \\
&= \int \frac{t}{3+3t^2-4t^2}\,dt \\
&= \int \frac{t}{3-t^2}\,dt \qquad (\text{Let } u = 3-t^2,\ du = -2t\,dt) \\
&= -\frac{1}{2}\int \frac{1}{u}\,du \\
&= -\frac{1}{2}\ln|u| + C \\
&= -\frac{1}{2}\ln|3-t^2| + C \\
&= -\frac{1}{2}\ln|3-\tan^2 x| + C \qquad \square
\end{aligned}$$