Exercise 5c

1. Find the standard form of the equation of the ellipse that satisfies the given conditions:

   (a) Passes through point $P(-2\sqrt{2}, 0)$, $Q(0, \sqrt{5})$;

   Sol.

   Point P is on the x-axis, while point Q is on the y-axis.
   $|OP| = 2\sqrt{2}$, $|OQ| = \sqrt{5}$,
   $\because |OP| > |OQ|$,
   $\therefore$ The major axis is along the x-axis.
   $\therefore$ The equation of the ellipse is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

   Substituting the coordinates of P and Q into the equation, we get

$$\frac{(-2\sqrt{2})^2}{a^2} + \frac{0^2}{b^2} = 1$$
$$\frac{0^2}{a^2} + \frac{(\sqrt{5})^2}{b^2} = 1$$

   Simplifying, we get

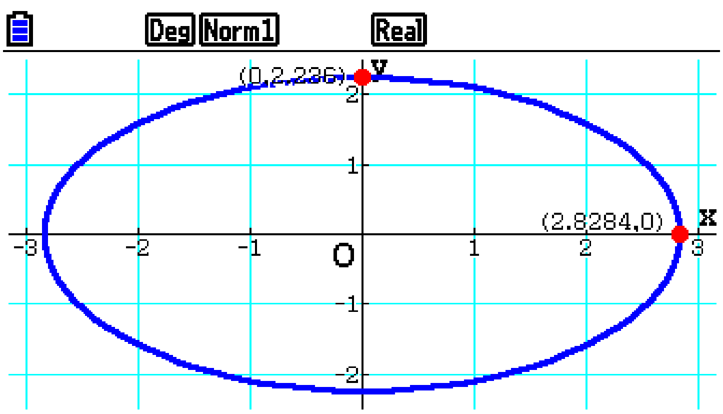
$$\frac{8}{a^2} = 1$$
$$\frac{5}{b^2} = 1$$

   Solving for a and b, we get

$$a^2 = 8$$
$$b^2 = 5$$

   $\therefore$ The standard form of the equation of the ellipse is

$$\frac{x^2}{8} + \frac{y^2}{5} = 1 \qquad \square$$



(b) Coordinates of its foci are $(-2\sqrt{3}, 0)$ and $(2\sqrt{3}, 0)$, and it passes through the point $P(\sqrt{5}, -\sqrt{6})$;

The foci are on the x-axis, therefore let the equation of the ellipse be of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

From the coordinates of the foci, we have $ae = 2\sqrt{3}$, $a^2e^2 = 12$,

$$\begin{aligned} \therefore\ b^2 &= a^2 - a^2e^2 \\ &= a^2 - 12 \ \cdots\ (1) \end{aligned}$$

Substituting the coordinates of P into the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

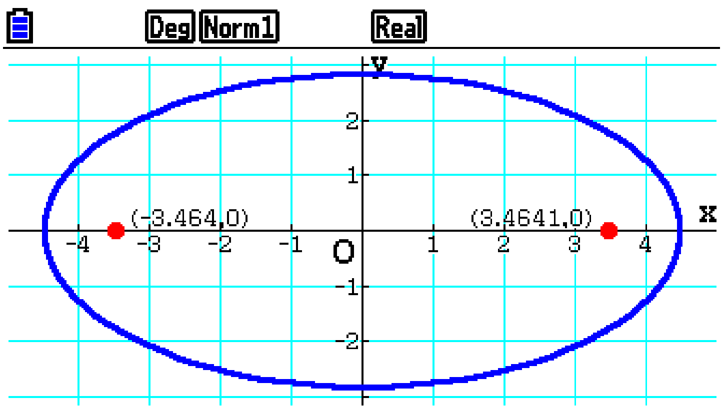
$$\begin{aligned} \frac{(\sqrt{5})^2}{a^2} + \frac{(-\sqrt{6})^2}{b^2} &= 1 \\ \frac{5}{a^2} + \frac{6}{b^2} &= 1 \end{aligned}$$

Substituting (1) into the equation, we get

$$\begin{aligned} \frac{5}{a^2} + \frac{6}{a^2 - 12} &= 1 \\ 5(a^2 - 12) + 6a^2 &= a^2(a^2 - 12) \\ 5a^2 - 60 + 6a^2 &= a^4 - 12a^2 \\ a^4 - 23a^2 + 60 &= 0 \\ (a^2 - 20)(a^2 - 3) &= 0 \\ a^2 = 20 \text{ or } a^2 &= 3 \text{ (rejected, } b > 0) \end{aligned}$$

When $a^2 = 20$, $b^2 = 20 - 12 = 8$. $\therefore$ The standard form of the equation of the ellipse is

$$\frac{x^2}{20} + \frac{y^2}{8} = 1 \qquad \square$$



(c) Equations of its directrices are $y \pm \frac{25}{3} = 0$, and it passes through the point $(4, 0)$;

Sol.

The directrices are perpendicular to the y-axis, therefore let the equation of the ellipse be of the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

From the equation of the directrices, we have $\frac{a}{e} = \frac{25}{3}$, $\frac{a^2}{e^2} = \frac{625}{9}$, $e^2 = \frac{9}{625}a^2$,

$$\begin{aligned}
\therefore\ b^2 &= a^2 - a^2e^2 \\
&= a^2 - \frac{9}{625}a^4 \quad \cdots \ (1)
\end{aligned}$$

Substituting the point $(4, 0)$ into the equation $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, we get

$$\begin{aligned}
\frac{(4)^2}{b^2} + \frac{0^2}{a^2} &= 1 \\
\frac{16}{b^2} &= 1
\end{aligned}$$

Substituting (1) into the equation, we get

$$\begin{aligned}
\frac{16}{a^2 - \frac{9}{625}a^4} &= 1 \\
16 &= a^2 - \frac{9}{625}a^4 \\
9a^4 - 625a^2 + 10000 &= 0 \\
(9a^2 - 400)(a^2 - 25) &= 0 \\
a^2 = 25 \text{ or } a^2 &= \frac{400}{9}
\end{aligned}$$

When $a^2 = 25$, $b^2 = 25 - \frac{9}{625}(25)^2 = 25 - 9 = 16$.

When $a^2 = \frac{400}{9}$, $b^2 = \frac{400}{9} - \frac{9}{625}\left(\frac{400}{9}\right)^2 = 16$.

$\therefore$ The standard form of the equations of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{25} = 1 \qquad \text{or} \qquad \frac{x^2}{16} + \frac{9y^2}{400} = 1 \qquad \square$$

(d) Its eccentricity is $\frac{4}{5}$ while the distance between its two foci is 8.

Sol.

Let the equation of the ellipse be of the forms

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{or} \qquad \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

From the given information, we have

$$\begin{aligned} \sqrt{(2ae)^2 - 0^2} &= 8 \\ 2ae &= 8 \\ ae &= 4 \ \cdots \ (1) \\ e = \frac{1}{a}\sqrt{a^2 - b^2} &= \frac{4}{5} \ \cdots \ (2) \end{aligned}$$

Substituting (2) into (1), we get

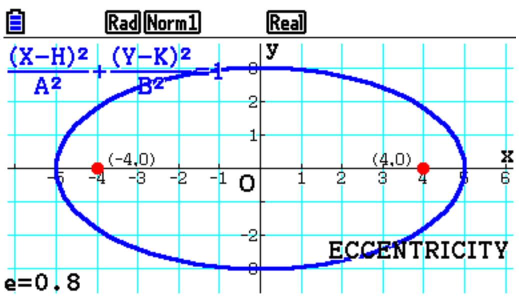
$$\begin{aligned} \frac{4}{5}a &= 4 \\ a &= 5 \end{aligned}$$

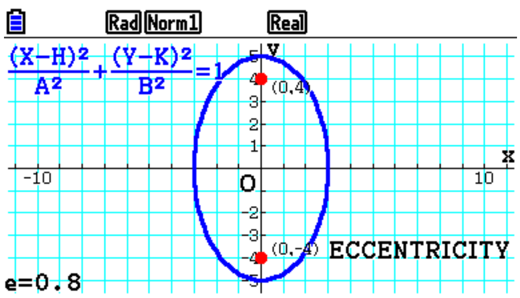
Substituting $a = 10$ into (2), we get

$$\begin{aligned} \frac{1}{5}\sqrt{5^2 - b^2} &= \frac{4}{5} \\ \sqrt{25 - b^2} &= 4 \\ 25 - b^2 &= 16 \\ b^2 &= 9 \\ b &= 3 \ (b > 0) \end{aligned}$$

$\therefore$ The standard form of the equation of the ellipse is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \qquad \text{or} \qquad \frac{x^2}{9} + \frac{y^2}{25} = 1 \qquad \square$$





2. The two vertices of one of the sides of the triangle $\triangle ABC$ is $B(0,6)$ and $C(0,-6)$, while the product of the slopes of the other two sides is $-\frac{4}{9}$, find the equation of the locus of the point A.

Sol.

Let point A be (x,y).

$$\text{Slope of } AB = \frac{y-6}{x-0} = \frac{y-6}{x}$$
$$\text{Slope of } AC = \frac{y+6}{x-0} = \frac{y+6}{x}$$

The product of the slopes of AB and AC is

$$\begin{aligned}\frac{y-6}{x}\cdot\frac{y+6}{x} &= -\frac{4}{9}\\ \frac{y^2-36}{x^2} &= -\frac{4}{9}\\ 9y^2-324 &= -4x^2\\ 9y^2+4x^2 &= 324\\ \frac{x^2}{81}+\frac{y^2}{36} &= 1 \qquad \square\end{aligned}$$

3. The ratio between point M and the two foci of the ellipse $\frac{x^2}{13^2}+\frac{y^2}{12^2}=1$ is $\frac{2}{3}$, find the equation of the locus of point M. Hence, sketch the graph.

Sol.

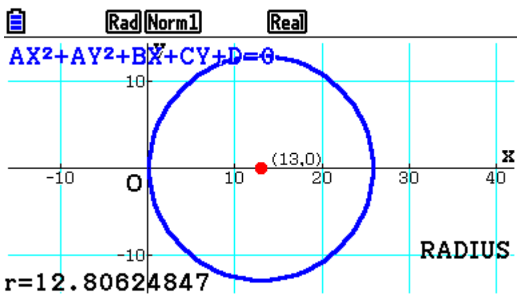
The eccentricity of the ellipse is

$$e = \frac{1}{13}\sqrt{13^2-12^2} = \frac{5}{13}$$

$\therefore$ The foci of the ellipse are at $(\pm 5, 0)$.

Let point M be (x,y).

$$\begin{aligned}\sqrt{(x-5)^2+y^2} &= \frac{2}{3}\sqrt{(x+5)^2+y^2}\\ (x-5)^2+y^2 &= \frac{4}{9}[(x+5)^2+y^2]\\ 9(x-5)^2+9y^2 &= 4(x+5)^2+4y^2\\ 9x^2-90x+225+9y^2 &= 4x^2+40x+100+4y^2\\ 5x^2-130x+125+5y^2 &= 0\\ x^2+y^2-26x+25 &= 0 \qquad \square\end{aligned}$$



4. Find the distance between a point $M\left(\frac{12}{5}, 4\right)$ on the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ and the foci of the ellipse.

Sol.

The eccentricity of the ellipse is $e = \frac{1}{5}\sqrt{25-16} = \frac{3}{5}$.

Hence, the foci F and F' of the ellipse are at $(0,3)$ and $(0,-3)$ respectively.

$$|MF| = \sqrt{\left(\frac{12}{5}-0\right)^2 + (4-3)^2} = \sqrt{\frac{144}{25}+1} = \frac{13}{5} \qquad \square$$

$$|MF'| = \sqrt{\left(\frac{12}{5}-0\right)^2 + (4-(-3))^2} = \sqrt{\frac{144}{25}+49} = \frac{37}{5} \qquad \square$$

5. Lazy to do this one. =)

6. Find the equation of the ellipse of which the length of its minor axis is 8 and its foci are $(1,5)$ and $(4,5)$.

Sol.

The center of the ellipse is at $\left(\frac{4+1}{2}, 5\right) = \left(\frac{5}{2}, 5\right)$.

Translate the coordinates system so that the center of the ellipse is at the origin, we obtain a new set of coordinates (x', y') such that

$$x' = x - \frac{5}{2} \qquad y' = y - 5$$

Since the foci are on the x-axis, the equation of the ellipse is of the form $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$

The coordinates of the loci in the new coordinates system are $\left(-\frac{3}{2}, 0\right)$ and $\left(\frac{3}{2}, 0\right)$.

$$\begin{aligned}
ae = \frac{3}{2} &\implies e = \frac{3}{2a} \\
e &= \frac{1}{a}\sqrt{a^2-16} \\
\frac{3}{2a} &= \frac{1}{a}\sqrt{a^2-16} \\
2a\sqrt{a^2-16} &= 3a \\
4a^2(a^2-16) &= 9a^2 \\
4a^4 - 64a^2 &= 9a^2 \\
a^2(4a^2-73) &= 0 \\
a^2 = 0 \text{ or } a^2 &= \frac{73}{4} \\
a &= \sqrt{\frac{73}{4}} \ (a > 0)
\end{aligned}$$

$\therefore$ The standard form of the equation of the ellipse is $\frac{4x'^2}{73} + \frac{y'^2}{16} = 1$.

Substituting $x' = x - \frac{5}{2}$ and $y' = y - 5$ into the equation, we get

$$\frac{4(x-\frac{5}{2})^2}{73} + \frac{(y-5)^2}{16} = 1 \qquad \square$$

7. Find the equation of the ellipse of which the length of its minor axis is $2\sqrt{2}$ and the two ends of its major axis are at $(-2, 2)$ and $(8, 2)$.

Sol.

The center of the ellipse is at $\left(\dfrac{-2+8}{2}, 2\right) = (3, 2)$.

Translate the coordinates system so that the center of the ellipse is at the origin, we obtain a new set of coordinates (x', y') such that

$$x' = x - 3 \qquad y' = y - 2$$

Since the major axis is along the x-axis, the equation of the ellipse is of the form $\dfrac{x'^2}{a^2} + \dfrac{y'^2}{b^2} = 1$

The length of the major axis is $8 - (-2) = 10$, therefore $2a = 10$, $a = 5$.

The length of the minor axis is $2\sqrt{2}$, therefore $2b = 2\sqrt{2}$, $b = \sqrt{2}$.

$\therefore$ The standard form of the equation of the ellipse is $\dfrac{x'^2}{25} + \dfrac{y'^2}{2} = 1$.

Substituting $x' = x - 3$ and $y' = y - 2$ into the equation, we get

$$\frac{(x-3)^2}{25} + \frac{(y-2)^2}{2} = 1 \qquad \square$$

8. The orbit of the earth is an ellipse with half major axis of length $a = 1.50 \times 10^8$ km, eccentricity $e = 0.0192$, and the sun at one of its foci. Find the maximum and the minimum distance of the earth from the sun.

Sol.

$$ae = 1.50 \times 10^8 \times 0.0192 = 2.88 \times 10^6 \text{ km}$$

The maximum distance of the earth from the sun is $a + ae = 1.50 \times 10^8 + 2.88 \times 10^6 = 1.5288 \times 10^8$ km.

The minimum distance of the earth from the sun is $a - ae = 1.50 \times 10^8 - 2.88 \times 10^6 = 1.4712 \times 10^8$ km. $\square$

9. Find the eccentricity of the following ellipses:

   (a) The view angle from the focus to the two ends of the minor axis is $60°$.

   Sol.

   The angle between the major axis and the line joining the foci is $30°$.

$$\begin{aligned}
\frac{b}{ae} &= \tan 30° = \frac{1}{\sqrt{3}} \\
b &= \frac{ae}{\sqrt{3}} \\
a^2e^2 &= 3b^2 \\
a^2e^2 &= 3[a^2(1-e^2)] \\
&= 3a^2 - 3a^2e^2 \\
e^2 &= 3 - 3e^2 \\
4e^2 &= 3 \\
e^2 &= \frac{3}{4} \\
e &= \frac{\sqrt{3}}{2} \ (e > 0) \qquad \square
\end{aligned}$$

(b) The viewing angle from one end of the minor axis to the foci is straight angle.

Sol.

Let the coordinates of the foci be $(\pm ae, 0)$, and the coordinates of one of the ends of the minor axis be $(0, b)$. The slope of the lines joining the foci and the end of the minor axis is $\frac{b}{ae}$ and $\frac{-b}{ae}$.
Since the two lines are perpendicular, we have

$$\begin{aligned}
\frac{b}{ae} \cdot \frac{-b}{ae} &= -1 \\
\frac{-b^2}{a^2e^2} &= -1 \\
-b^2 &= -a^2e^2 \\
a^2 - a^2e^2 &= a^2e^2 \\
2e^2 &= 1 \\
e^2 &= \frac{1}{2} \\
e &= \frac{\sqrt{2}}{2} \ (e > 0) \qquad \square
\end{aligned}$$

10. Calculate the side length of the inscribed square in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol.

Let one of the vertices of the square that is in the first quadrant be (m, m).

Substituting the coordinates of the vertex into the equation of the ellipse, we get

$$\begin{aligned}
\frac{m^2}{a^2} + \frac{m^2}{b^2} &= 1 \\
m^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) &= 1 \\
m^2 &= \frac{a^2b^2}{a^2+b^2} \\
m &= \frac{ab}{\sqrt{a^2+b^2}}
\end{aligned}$$

Hence, the side length of the inscribed square is $\frac{2ab}{\sqrt{a^2+b^2}} = \frac{2ab}{a^2+b^2}\sqrt{a^2+b^2}$. $\square$

11. Prove that the points of intersection of the two ellipses $b^2x^2 + a^2y^2 - a^2b^2 = 0$ and $a^2x^2 + b^2y^2 - a^2b^2 = 0$ $(a > b > 0)$ are on the circumference of a circle with the center at the origin. Hence, find the equation of the circle.

Proof.

Adding the two equations, we get

$$\begin{aligned}
b^2x^2 + a^2y^2 - a^2b^2 + a^2x^2 + b^2y^2 - a^2b^2 &= 0 \\
(a^2+b^2)x^2 + (a^2+b^2)y^2 - 2a^2b^2 &= 0 \\
x^2 + y^2 &= \frac{2a^2b^2}{a^2+b^2} \qquad \square
\end{aligned}$$