Praktis 2
Differentiation

Melvin Chia

March 13, 2023

2.1 Limit and its Relation to Differentiation

1. Find the value of each of the following.

   (a) $\lim_{x\to 1}(x-1)$

   Sol.

$$\begin{aligned}\lim_{x\to 1}(x-1) &= 1-1\\ &= 0 \quad \square\end{aligned}$$

   (b) $\lim_{x\to 1}\dfrac{x^2-2}{x}$

   Sol.

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-2}{x} &= \frac{1^2-2}{1}\\ &= \frac{-1}{1}\\ &= -1 \quad \square\end{aligned}$$

   (c) $\lim_{x\to 0}\dfrac{2x-5}{x+3}$

   Sol.

$$\begin{aligned}\lim_{x\to 0}\frac{2x-5}{x+3} &= \frac{2(0)-5}{0+3}\\ &= \frac{-5}{3}\\ &= -\frac{5}{3} \quad \square\end{aligned}$$

   (d) $\lim_{x\to a}(x-a)$

   Sol.

$$\begin{aligned}\lim_{x\to a}(x-a) &= a-a\\ &= 0 \quad \square\end{aligned}$$

2. Calculate the value for each of the following.

   (a) $\lim_{x\to 0}\dfrac{2x^2-5x}{x}$

   Sol.

$$\begin{aligned}\lim_{x\to 0}\frac{2x^2-5x}{x} &= \lim_{x\to 0}\frac{x(2x-5)}{x}\\ &= \lim_{x\to 0}(2x-5)\\ &= 2(0)-5\\ &= -5 \quad \square\end{aligned}$$

   (b) $\lim_{x\to 2}\dfrac{x^2-4}{x-2}$

   Sol.

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-4}{x-2} &= \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}\\ &= \lim_{x\to 2}(x+2)\\ &= 2+2\\ &= 4 \quad \square\end{aligned}$$

   (c) $\lim_{x\to 5}\dfrac{x^2+4x-45}{x-5}$

   Sol.

$$\begin{aligned}\lim_{x\to 5}\frac{x^2+4x-45}{x-5} &= \lim_{x\to 5}\frac{(x-5)(x+9)}{x-5}\\ &= \lim_{x\to 5}(x+9)\\ &= 5+9\\ &= 14 \quad \square\end{aligned}$$

   (d) $\lim_{x\to 1}\dfrac{\log_{10}x^2}{\log_{10}x}$

   Sol.

$$\begin{aligned}\lim_{x\to 1}\frac{\log_{10}x^2}{\log_{10}x} &= \lim_{x\to 1}\frac{\log_{10}x^2}{\log_{10}x}\\ &= \lim_{x\to 1}\frac{2\log_{10}x}{\log_{10}x}\\ &= \lim_{x\to 1}2\\ &= 2 \quad \square\end{aligned}$$

3. Find the value for each of the following.

   (a) $\lim_{x\to 9}\dfrac{x-9}{\sqrt{x}-3}$

   Sol.

$$\begin{aligned}\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3} &= \lim_{x\to 9}\frac{(x-9)'}{(\sqrt{x}-3)'}\\ &= \lim_{x\to 9}\frac{1}{\frac{1}{2\sqrt{x}}}\\ &= \lim_{x\to 9}(2\sqrt{x})\\ &= 2\sqrt{9}\\ &= 2(3)\\ &= 6 \quad \square\end{aligned}$$

   (b) $\lim_{x\to -1}\dfrac{x+1}{\sqrt{x+5}-2}$

   Sol.

$$\begin{aligned}\lim_{x\to -1}\frac{x+1}{\sqrt{x+5}-2} &= \lim_{x\to -1}\frac{(x+1)'}{(\sqrt{x+5}-2)'}\\ &= \lim_{x\to -1}\frac{1}{\frac{1}{2\sqrt{x+5}}}\\ &= \lim_{x\to -1}(2\sqrt{x+5})\\ &= 2\sqrt{-1+5}\\ &= 2\sqrt{4}\\ &= 2(2)\\ &= 4 \quad \square\end{aligned}$$

(c) $\lim_{x\to 9} \dfrac{\sqrt{x+7}-4}{x-9}$

Sol.

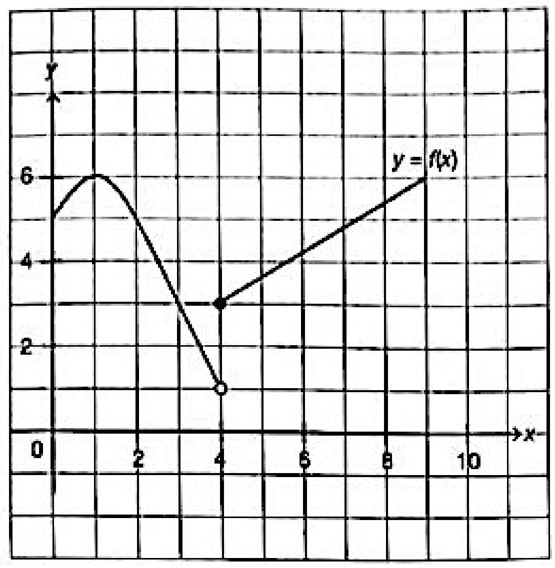
$$\begin{aligned}\lim_{x\to 9} \frac{\sqrt{x+7}-4}{x-9} &= \lim_{x\to 9} \frac{(\sqrt{x+7}-4)'}{(x-9)'} \\ &= \lim_{x\to 9} \frac{1}{2\sqrt{x+7}} \\ &= \frac{1}{2\sqrt{9+7}} \\ &= \frac{1}{8}\end{aligned}$$

(d) $\lim_{x\to 2} \dfrac{\sqrt{6-x}-2}{3-\sqrt{11-x}}$

Sol.

$$\begin{aligned}\lim_{x\to 2} \frac{\sqrt{6-x}-2}{3-\sqrt{11-x}} &= \lim_{x\to 2} \frac{(\sqrt{6-x}-2)'}{(3-\sqrt{11-x})'} \\ &= \lim_{x\to 2} \frac{\frac{1}{2\sqrt{6-x}}}{\frac{1}{-2\sqrt{11-x}}} \\ &= \lim_{x\to 2} \frac{-2\sqrt{11-x}}{2\sqrt{6-x}} \\ &= \lim_{x\to 2} \frac{-\sqrt{11-x}}{\sqrt{6-x}} \\ &= -\frac{\sqrt{11-2}}{\sqrt{6-2}} \\ &= -\frac{3}{2} \quad \square\end{aligned}$$

4. The following diagram shows part of a graph $y = f(x)$.



Based on this graph, find

(a) $f(4)$

Sol.

$$f(4) = 3$$

(b) $\lim_{x\to 4} f(x)$ and explain your answer.

Sol.

$$\begin{aligned}\lim_{x\to 4^-} f(x) &\neq 4 \\ \lim_{x\to 4^+} f(x) &= 4\end{aligned}$$

Since the left limit and right limit are different, $f(4)$ does not exist.

(c) $\lim_{x\to 1} f(x)$

Sol.

$$\lim_{x\to 1} f(x) = 6$$

5. Find $\dfrac{dy}{dx}$ by using the first principle.

(a) $y = 3x + 5$

Sol.

$$\begin{aligned}y &= 3x+5 \quad (1)\\ y+\delta y &= 3(x+\delta x)+5 \\ y+\delta y &= 3x+3\delta x+5 \quad (2)\end{aligned}$$

$(2)-(1):$

$$\begin{aligned}\delta y &= 3\delta x \\ \frac{\delta y}{\delta x} &= 3 \\ \therefore\ \frac{dy}{dx} &= \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) \\ &= \lim_{\delta x\to 0} 3 \\ &= 3 \quad \square\end{aligned}$$

(b) $y = x^2 - 7$

Sol.

$$\begin{aligned}y &= x^2-7 \quad (1)\\ y+\delta y &= (x+\delta x)^2-7 \\ y+\delta y &= x^2+2x\delta x+(\delta x)^2-7 \quad (2)\end{aligned}$$

$(2)-(1):$

$$\begin{aligned}\delta y &= 2x\delta x+(\delta x)^2 \\ \frac{\delta y}{\delta x} &= 2x+2\delta x \\ \therefore\ \frac{dy}{dx} &= \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) \\ &= \lim_{\delta x\to 0} (2x+2\delta x) \\ &= 2x \quad \square\end{aligned}$$

(c) $y = x^2 + 2x + 1$

Sol.

$$\begin{aligned} y &= x^2+2x+1 && (1)\\ y+\delta y &= (x+\delta x)^2+2(x+\delta x)+1\\ y+\delta y &= x^2+2x\delta x+(\delta x)^2+2x+2\delta x\\ &\quad +1 && (2)\end{aligned}$$

$(2)-(1):$

$$\begin{aligned} \delta y &= 2x\delta x+(\delta x)^2+2\delta x\\ \frac{\delta y}{\delta x} &= 2x+\delta x+2\\ \therefore\ \frac{dy}{dx} &= \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)\\ &= \lim_{\delta x\to 0}(2x+\delta x+2)\\ &= 2x+2 \quad \square\end{aligned}$$

(d) $y=-x^3+9$

Sol.

$$\begin{aligned} y &= -x^3+9 && (1)\\ y+\delta y &= -(x+\delta x)^3+9\\ y+\delta y &= -x^3-3x^2\delta x-3x(\delta x)^2-\delta x^3\\ &\quad +9 && (2)\end{aligned}$$

$(2)-(1):$

$$\begin{aligned} \delta y &= -3x^2\delta x-3x(\delta x)^2-\delta x^3\\ \frac{\delta y}{\delta x} &= -3x^2-3x\delta x-(\delta x)^2\\ \therefore\ \frac{dy}{dx} &= \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)\\ &= \lim_{\delta x\to 0}\left[-3x^2-3x\delta x-(\delta x)^2\right]\\ &= -3x^2 \quad \square\end{aligned}$$

(e) $y=2-\dfrac{3}{x}$

Sol.

$$\begin{aligned} y &= 2-3x^{-1} && (1)\\ y+\delta y &= 2-3(x+\delta x)^{-1} && (2)\end{aligned}$$

$(2)-(1):$

$$\begin{aligned} \delta y &= -3(x+\delta x)^{-1}+3x^{-1}\\ &= -\frac{3}{x+\delta x}+\frac{3}{x}\\ &= \frac{-3x+3x+3\delta x}{x(x+\delta x)}\\ &= \frac{3\delta x}{x^2+x\delta x}\\ \frac{\delta y}{\delta x} &= \frac{3}{x^2+x\delta x}\\ \therefore\ \frac{dy}{dx} &= \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)\\ &= \lim_{\delta x\to 0}\left(\frac{3}{x^2+x\delta x}\right)\\ &= \frac{3}{x^2} \quad \square\end{aligned}$$

6. Given a curve $y=x^2-ax+b$

(a) By using the first principle, find the gradient function to the curve.

Sol.

$$\begin{aligned} y &= x^2-ax+b && (1)\\ y+\delta y &= (x+\delta x)^2-a(x+\delta x)+b\\ y+\delta y &= x^2+2x\delta x+(\delta x)^2-ax-a\delta x\\ &\quad +b && (2)\end{aligned}$$

$(2)-(1):$

$$\begin{aligned} \delta y &= 2x\delta x+(\delta x)^2-a\delta x\\ \frac{\delta y}{\delta x} &= 2x+\delta x-a\\ \therefore\ \frac{dy}{dx} &= \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)\\ &= \lim_{\delta x\to 0}(2x+\delta x-a)\\ &= 2x-a \quad \square\end{aligned}$$

(b) Given that the value of gradient of the curve at $(2,-3)$ is 2, find the value of a and b.

Sol.

$$\begin{aligned} \frac{dy}{dx} &= 2x-a\\ 2 &= 2(2)-a\\ \therefore\ a &= 2 \quad \square\end{aligned}$$

$$\begin{aligned} y &= x^2-2x+b\\ -3 &= (2)^2-2(2)+b\\ -3 &= 4-4+b\\ \therefore\ b &= -3 \quad \square\end{aligned}$$

2.2 The First Derivative

7. Find the first derivative for each of the following functions.

(a) $y=6x^2$

Sol.

$$\frac{dy}{dx}=12x \quad \square$$

(b) $y=-x^4$

Sol.

$$\frac{dy}{dx}=-4x^3 \quad \square$$

(c) $y=\sqrt[3]{x^4}$

Sol.

$$\begin{aligned} y &= x^{\frac{4}{3}}\\ \frac{dy}{dx} &= \frac{4}{3}x^{\frac{4}{3}-1}\\ &= \frac{4}{3}\sqrt[3]{x} \quad \square\end{aligned}$$

(d) $y = -\frac{2}{x^2}$

Sol.

$$\begin{aligned} y &= -2x^{-2} \\ \frac{dy}{dx} &= -2\left(-2x^{-3}\right) \\ &= 4x^{-3} \\ &= \frac{4}{x^3} \quad \square \end{aligned}$$

8. Find each of the following.

(a) $\frac{d}{dx}\left(2x^2 + 3x - 9\right)$

Sol.

$$\frac{d}{dx}\left(2x^2 + 3x - 9\right) = 4x + 3 \quad \square$$

(b) $\frac{d}{dx}\left(x^2 + \frac{2}{x}\right)$

Sol.

$$\begin{aligned} \frac{d}{dx}\left(x^2 + \frac{2}{x}\right) &= \frac{d}{dx}\left(x^2 + 2x^{-1}\right) \\ &= 2x - 2x^{-2} \\ &= 2x - \frac{2}{x^2} \quad \square \end{aligned}$$

(c) $\frac{d}{dx}\left(5x^3 + 2x^2 + 4x - 7 - \frac{1}{x} + \frac{3}{x^2}\right)$

Sol.

$$\begin{aligned} &\frac{d}{dx}\left(5x^3 + 2x^2 + 4x - 7 - \frac{1}{x} + \frac{3}{x^2}\right) \\ &= \frac{d}{dx}\left(5x^3 + 2x^2 + 4x - 7 - x^{-1} + 3x^{-2}\right) \\ &= 15x^2 + 4x + 4 + x^{-2} - 6x^{-3} \\ &= 15x^2 + 4x + 4 + \frac{1}{x^2} - \frac{6}{x^3} \quad \square \end{aligned}$$

9. Differentiate each of the following functions with respect to x.

(a) $f(x) = x\left(\frac{1}{2}x^4 - x^2 - 5x\right)$

Sol.

$$\begin{aligned} f(x) &= \frac{1}{2}x^5 - x^3 - 5x^2 \\ \frac{d}{dx} &= \frac{5}{2}x^4 - 3x^2 - 10x \quad \square \end{aligned}$$

(b) $f(x) = (x^2 - 5)(x + 3)$

Sol.

$$\begin{aligned} f(x) &= x^3 + 3x^2 - 5x - 15 \\ \frac{d}{dx} &= 3x^2 + 6x - 5 \quad \square \end{aligned}$$

(c) $f(x) = \frac{(x^3 - x + 4)}{x}$

Sol.

$$\begin{aligned} f(x) &= \frac{x^2}{x} - 1 + \frac{4}{x} \\ &= x^2 - 1 + 4x^{-1} \\ \frac{d}{dx} &= 2x - 4x^{-2} \\ &= 2x - \frac{4}{x^2} \quad \square \end{aligned}$$

(d) $f(x) = \frac{(x^2 - x - 2)}{(x - 2)}$

Sol.

$$\begin{aligned} f(x) &= \frac{(x-2)(x+1)}{x-2} \\ &= x + 1 \\ \frac{d}{dx} &= 1 \quad \square \end{aligned}$$

10. Find $f'(x)$ for each of the following functions.

(a) $f(x) = (3x - 5)^4$

Sol.

$$\begin{aligned} f'(x) &= 4(3x-5)^3 \cdot \frac{d}{dx}(3x - 5) \\ &= 4(3x-5)^3 \cdot 3 \\ &= 12(3x-5)^2 \quad \square \end{aligned}$$

(b) $f(x) = 5(x^3 + 4x)^3$

Sol.

$$\begin{aligned} f'(x) &= 5(x^3 + 4x)^3 \cdot \frac{d}{dx}(x^3 + 4x) \\ &= 5(x^3 + 4x)^3 \cdot \left(3x^2 + 4\right) \\ &= 15\left(3x^2 + 4\right)\left(x^3 + 4x\right)^2 \quad \square \end{aligned}$$

(c) $f(x) = \frac{2}{(5x^2 - 3x)^{10}}$

Sol.

$$\begin{aligned} f(x) &= \frac{-20 \cdot \frac{d}{dx}(5x^2 - 3x)}{(5x^2 - 3x)^{11}} \\ &= \frac{-20(10x - 3)}{(5x^2 - 3x)^{11}} \end{aligned}$$

11. Find the first derivative for each of the following functions by using the product rule.

(a) $y = 6x^2(x + 5x^2)^3$

Sol.

$$\begin{aligned} y &= 6x^2[x(1+5x)]^3 \\ &= 6x^2(x^3)(1+5x)^3 \\ &= 6x^5(1+5x)^3 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= 6x^5\frac{d}{dx}(1+5x)^3 + (1+5x)^3\frac{d}{dx}6x^5 \\ &= 6x^5\cdot 5\cdot 3(1+5x)^2 + 30x^4(1+5x)^3 \\ &= 90x^5\cdot(1+5x)^2 + 30x^4(1+5x)^3 \\ &= 30x^4(1+5x)^2(1+5x+3x) \\ &= 30x^4(5x+1)^2(8x+1) \end{aligned}$$

(b) $y = x(7x+3)^5$

Sol.

$$\begin{aligned} \frac{dy}{dx} &= x\frac{d}{dx}(7x+3)^5 + (7x+3)^5\frac{d}{dx}x \\ &= x\cdot 5(7x+3)^4\cdot 7 + (7x+3)^5\cdot 1 \\ &= 35x(7x+3)^4 + (7x+3)^5 \\ &= (7x+3)^4(35x+7x+3) \\ &= (7x+3)^4(42x+3) \quad \square \end{aligned}$$

(c) $y = (4x^2-3x)(1-2x^2)^{10}$

Sol.

$$\begin{aligned} \frac{dy}{dx} &= (4x^2-3x)\frac{d}{dx}(1-2x^2)^{10} + (1-2x^2)^{10} \\ &\quad \frac{d}{dx}(4x^2-3x) \\ &= (4x^2-3x)\cdot 10(1-2x^2)^9\cdot(-4x) + (1 \\ &\quad -2x^2)^{10}(8x-3) \\ &= (1-2x^2)^9[(-40x)(4x^2-3x) + (1-2x^2) \\ &\quad (8x-3)] \\ &= (1-2x^2)^9[-160x^3+120x^2+8x-3 \\ &\quad -16x^3+6x^2] \\ &= (1-2x^2)^9[-176x^3+126x^2+8x-3] \quad \square \end{aligned}$$

12. Find $\frac{dy}{dx}$ for each of the following functions by using the quotient rule.

(a) $y = \frac{x-2}{2x+1}$

Sol.

$$\begin{aligned} \frac{dy}{dx} &= \frac{(2x+1)\frac{d}{dx}(x-2) - (x-2)\frac{d}{dx}(2x+1)}{(2x+1)^2} \\ &= \frac{2x+1-2(x-2)}{(2x+1)^2} \\ &= \frac{5}{(2x+1)^2} \quad \square \end{aligned}$$

(b) $y = \frac{x^2+3x-4}{x-1}$

Sol.

$$\begin{aligned} y &= \frac{(x+4)(x-1)}{x-1} \\ &= x+4 \end{aligned}$$

$$\frac{dy}{dx} = 1 \quad \square$$

(c) $y = \frac{x^3}{(2x-1)^2}$

Sol.

$$\begin{aligned} \frac{dy}{dx} &= \frac{(2x-1)^2\frac{d}{dx}(x^3) - (x^3)\frac{d}{dx}(2x-1)^2}{(2x-1)^4} \\ &= \frac{(2x-1)^2\cdot 3x^2 - x^3\cdot 4(2x-1)}{(2x-1)^4} \\ &= \frac{(2x-1)[3x^2(2x-1)-4x^3]}{(2x-1)^4} \\ &= \frac{6x^3-3x^2-4x^3}{(2x-1)^3} \\ &= \frac{2x^3-3x^2}{(2x-1)^3} \\ &= \frac{x^2(2x-3)}{(2x-1)^3} \quad \square \end{aligned}$$

13. Find the gradient function to the curve $y = \sqrt{x}(4x+1)$. Hence, find the value of the gradient of the curve at $x = 4$.

Sol.

$$\begin{aligned} y &= \sqrt{x}(4x+1) \\ &= 4x\sqrt{x} + \sqrt{x} \\ &= 4x\cdot x^{\frac{1}{2}} + x^{\frac{1}{2}} \\ &= 4x^{\frac{3}{2}} + x^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= 4\left(\frac{3}{2}x^{\frac{1}{2}}\right) + \frac{1}{2}x^{-\frac{1}{2}} \\ &= 6\sqrt{x} + \frac{1}{2\sqrt{x}} \quad \square \end{aligned}$$

When $x = 4$,

$$\begin{aligned} \frac{dy}{dx} &= 6\sqrt{4} + \frac{1}{2\sqrt{4}} \\ &= 12 + \frac{1}{4} \\ &= \frac{49}{4} \quad \square \end{aligned}$$

14. Given $x^2y = 5$, find $\frac{dy}{dx}$ when $x = 2$.

Sol.

$$
\begin{aligned}
x^2y &= 5 \\
y &= \frac{5}{x^2} \\
&= 5x^{-2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dy}{dx} &= 5(-2x^{-3}) \\
&= -10x^{-3}
\end{aligned}
$$

When $x = 2$,

$$
\begin{aligned}
\frac{dy}{dx} &= -10(2)^{-3} \\
&= -10\left(\frac{1}{8}\right) \\
&= -\frac{5}{4} \quad \square
\end{aligned}
$$

15. Given $y = 5x^m$ and $\frac{dy}{dx} = x^n$, find the value of m and n.

Sol.

$$
\begin{aligned}
\frac{dy}{dx} &= 5m \cdot x^{m-1} \\
x^n &= 5mx^{m-1}
\end{aligned}
$$

Comparing both sides,

$$
\begin{aligned}
5m &= 1 \\
m &= \frac{1}{5} \quad \square \\
m - 1 &= n \\
\frac{1}{5} - 1 &= n \\
n &= -\frac{4}{5} \quad \square
\end{aligned}
$$

16. Given $f(x) = ax^3 - bx^2 + 9x + 5$ where $a, b > 0$. Show that $f'(x)$ is always positive for all the values of x when $b^2 < 27a$.

Sol.

$$
\begin{aligned}
f'(x) &= 3ax^2 - 2bx + 9 \\
&= 3a\left(x^2 - \frac{2b}{3a}x\right) + 9 \\
&= 3a\left[\left(x^2 - \frac{2b}{3a}x + \frac{b^2}{9a^2}\right) - \frac{b^2}{9a^2}\right] + 9 \\
&= 3a\left(x - \frac{b}{3a}\right)^2 - \frac{b^2}{3a} + 9
\end{aligned}
$$

$f'(x)$ is always positive when $-\frac{b^2}{3a} + 9 > 0$.

$$
\begin{aligned}
-\frac{b^2}{3a} + 9 &> 0 \\
\frac{b^2}{3a} &< 9 \\
b^2 &< 27a \quad \text{(shown)} \quad \square
\end{aligned}
$$

17. Given $\frac{d}{dx}(ax^m + bx^n) = 12x^s + 9x^t$ where $a, b > 0$.

(a) Find $\frac{s}{t}$ in terms of a and b.

Sol.

$$
\begin{aligned}
\frac{d}{dx}(ax^m + bx^n) &= max^{m-1} + nbx^{n-1} \\
12x^s + 9x^t &= max^{m-1} + nbx^{n-1}
\end{aligned}
$$

Comparing both sides,

$$
\begin{aligned}
ma &= 12 \\
m &= \frac{12}{a} \\
nb &= 9 \\
n &= \frac{9}{b} \\
s &= m - 1 \\
&= \frac{12}{a} - 1 \\
&= \frac{12 - a}{a} \\
t &= n - 1 \\
&= \frac{9}{b} - 1 \\
&= \frac{9 - b}{b} \\
\frac{s}{t} &= \frac{12 - a}{a} \cdot \frac{b}{9 - b} \\
&= \frac{b(12 - a)}{a(9 - b)} \quad \square
\end{aligned}
$$

(b) Find the values of a and b if $3s = 5t$ and $\frac{m}{n} = \frac{3}{2}$.

Sol.

$$
\begin{aligned}
3s &= 5t \\
3\left(\frac{12 - a}{a}\right) &= 5\left(\frac{9 - n}{b}\right) \\
\frac{36 - 3a}{a} &= \frac{45 - 5b}{b} \\
36b - 3ab &= 45a - 5ab \\
45a - 36b - 2ab &= 0 \qquad (1) \\
\frac{m}{n} &= \frac{3}{2} \\
\frac{12}{a} \cdot \frac{b}{9} &= \frac{3}{2} \\
\frac{12b}{9a} &= \frac{3}{2} \\
\frac{4b}{3a} &= \frac{3}{2} \\
8b &= 9a \\
a &= \frac{8b}{9} \qquad (2)
\end{aligned}
$$

Substituting (2) in (1),

$$
\begin{aligned}
45\left(\frac{8b}{9}\right) - 36b - 2\left(\frac{8b}{9}\right)b &= 0 \\
40b - 36b - \frac{16b^2}{9} &= 0 \\
4b - \frac{16b^2}{9} &= 0 \\
16b^2 - 36b &= 0 \\
b(4b-9) &= 0 \\
b = 0 \quad \text{or} \quad b &= \frac{9}{4} \\
\because\ b > 0, b = 0 &\text{ is not possible} \\
\therefore\ b &= \frac{9}{4} \quad □
\end{aligned}
$$

Substituting $b = \frac{9}{4}$ in (2),

$$
\begin{aligned}
a &= \frac{8b}{9} \\
&= \frac{8 \cdot \frac{9}{4}}{9} \\
&= \frac{18}{9} \\
&= 2 \quad □
\end{aligned}
$$

(c) Hence, or otherwise, find the values of m, n, s, and t.

Sol.

Substituting $a = 2$ in $m = \dfrac{12}{a}$,

$$
\begin{aligned}
m &= \frac{12}{a} \\
&= \frac{12}{2} \\
&= 6 \quad □
\end{aligned}
$$

Substituting $b = \dfrac{9}{4}$ in $n = \dfrac{9}{b}$,

$$
\begin{aligned}
n &= \frac{9}{b} \\
&= \frac{9}{\frac{9}{4}} \\
&= 4 \quad □
\end{aligned}
$$

Substituting $a = 2$ in $s = m - 1$,

$$
\begin{aligned}
s &= m - 1 \\
&= 6 - 1 \\
&= 5 \quad □
\end{aligned}
$$

Substituting $b = \dfrac{9}{4}$ in $t = n - 1$,

$$
\begin{aligned}
t &= n - 1 \\
&= 4 - 1 \\
&= 3 \quad □
\end{aligned}
$$

2.3 The Second Derivative

18. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for each of the following.

(a) $y = 4x^3 + 7x^{-1}$

Sol.

$$
\begin{aligned}
\frac{dy}{dx} &= 12x^2 - 7x^{-2} \quad □ \\
\frac{d^2y}{dx^2} &= 24x + 14x^{-3} \quad □
\end{aligned}
$$

(b) $y = (2x^3-3)^5$

Sol.

$$
\begin{aligned}
\frac{dy}{dx} &= 5(2x^3-3)^4(6x^2) \\
&= 30x^2(2x^3-3)^4 \quad □ \\
\frac{d^2y}{dx^2} &= 60x(2x^3-3)^4 + 720x^4(2x^3-3)^3 \\
&= 60x(2x^3-3)^3(2x^3-3+12x^3) \\
&= 60x(2x^3-3)^3(14x^3-3) \quad □
\end{aligned}
$$

(c) $y = \dfrac{4}{3}\pi x^3$

Sol.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{4}{3}\pi(3x^2) \\
&= 4\pi x^2 \quad □ \\
\frac{d^2y}{dx^2} &= \frac{4}{3}\pi(6x) \\
&= 8\pi x \quad □
\end{aligned}
$$

(d) $y = \dfrac{3}{(x^2+1)^2}$

Sol.

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{0 - 3(2)(x^2+1)(2x)}{(x^2+1)^4} \\
&= \frac{-12x(x^2+1)}{(x^2+1)^4} \\
&= \frac{-12x}{(x^2+1)^3} \quad □ \\
\frac{d^2y}{dx^2} &= \frac{-12(x^2+1)^3 + 12x(3)(x^2+1)^2(2x)}{(x^2+1)^6} \\
&= \frac{-12(x^2+1)^3 + 72x^2(x^2+1)^2}{(x^2+1)^6} \\
&= \frac{-12(x^2+1)^2(x^2+1-6x^2)}{(x^2+1)^6} \\
&= \frac{-12(1-5x^2)}{(x^2+1)^4} \quad □
\end{aligned}
$$

19. Given a curve $y = 4x^3 - 2x^2 + 5$. Find the first and the second derivatives for the curve y when $x = 2$.

Sol.

$$\frac{dy}{dx} = 12x^2 - 4x$$
$$\frac{d^2y}{dx^2} = 24x - 4$$

Substituting $x = 2$ in the above,

$$\begin{aligned}\frac{dy}{dx} &= 12(2)^2 - 4(2)\\ &= 48 - 8\\ &= 40 \quad \square\end{aligned}$$
$$\begin{aligned}\frac{d^2y}{dx^2} &= 24(2) - 4\\ &= 48 - 4\\ &= 44 \quad \square\end{aligned}$$

20. Given $y = \frac{1}{x}$. Prove that $y + \frac{d^2y}{dx^2} = y^3(x^2 + 2)$.

Proof.

$$\begin{aligned}y &= x^{-1}\\ \frac{dy}{dx} &= -x^{-2}\\ \frac{d^2y}{dx^2} &= 2x^{-3}\\ &= \frac{2}{x^3}\\ L.H.S. = y + \frac{d^2y}{dx^2} &= \frac{1}{x} + \frac{2}{x^3}\\ &= \frac{x^2+2}{x^3}\\ R.H.S. = y^3(x^2+2) &= \frac{1}{x^3}(x^2+2)\\ &= \frac{x^2+2}{x^3}\end{aligned}$$

$\because L.H.S. = R.H.S.$

$\therefore y + \frac{d^2y}{dx^2} = y^3(x^2 + 2) \quad \square$

21. Prove that for all values, of x,
$\frac{d^2}{dx^2}\left(\frac{x^4}{12} - x^3 + \frac{9}{2}x^2 + 6x - 3\right)$ is never negative.

Proof.

$$\begin{aligned}\frac{d}{dx} &= \frac{1}{3}x^3 - 3x^2 + 9x + 6\\ \frac{d^2}{dx^2} &= x^2 - 6x + 9\\ &= (x-3)^2\end{aligned}$$

$\forall x \in \mathbb{R}$,

$\because\ (x-3)^2 \geq 0$

$\therefore\ \frac{d^2}{dx^2}\left(\frac{x^4}{12} - x^3 + \frac{9}{2}x^2 + 6x - 3\right) \geq 0 \quad \square$

22. Given $h(x) = 3x^3 + mx^2 + x - 1$. Find the value of m if $h''(1) = 10$.

Sol.

$$\begin{aligned}h'(x) &= 9x^2 + 2mx + 1\\ h''(x) &= 18x + 2m\\ h''(1) &= 10\\ 18 + 2m &= 10\\ 2m &= -8\\ m &= -4 \quad \square\end{aligned}$$

23. Given $f(x) = \frac{1}{2}x^4 + px^3 + \frac{3}{2}x^2 - 16x$. Determine the range of values for p such that the equation $f''(x) = 0$ has at least one real solution.

Sol.

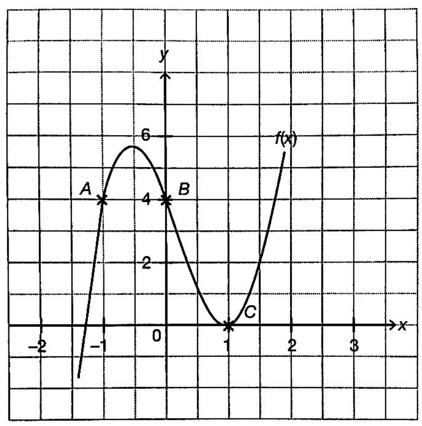
$$\begin{aligned}f'(x) &= 2x^3 + 3px^2 + 3x - 16\\ f''(x) &= 6x^2 + 6px + 3\\ f''(x) &= 0\\ 6x^2 + 6px + 3 &= 0\\ 2x^2 + 2px + 1 &= 0\end{aligned}$$

When $f''(x) = 0$ has at least one real solution,

$$\begin{aligned}4p^2 - 8 &\geq 0\\ p^2 &\geq 2\end{aligned}$$
$$p \leq -\sqrt{2} \text{ or } p \geq \sqrt{2} \quad \square$$

2.4 Application of Differentiation

24. The following diagram shows the graph of part of the curve $f(x) = 3x^3 - 2x^2 - 5x + 4$. The points $A(-1, 4)$, $B(0, 4)$, and $C(1, 0)$ lie on the curve.



(a) Find the gradient function of the tangent to the curve $f(x)$.

Sol.

$$f'(x) = 9x^2 - 4x - 5 \quad \square$$

(b) i. Find the values of gradient of the tangents to the curve at points A, B, and C.

Sol.

$$\begin{aligned}
m_A &= f'(-1) \\
&= 9(-1)^2 - 4(-1) - 5 \\
&= 9 + 4 - 5 \\
&= 8 \quad \square \\
m_B &= f'(0) \\
&= 9(0)^2 - 4(0) - 5 \\
&= -5 \quad \square \\
m_C &= f'(1) \\
&= 9(1)^2 - 4(1) - 5 \\
&= 9 - 4 - 5 \\
&= 0 \quad \square
\end{aligned}$$

ii. Hence, elaborate the situations of the tangents at points A, B, and C based on the values of the gradient obtained in (i).

Sol.

The gradient of the tangent at point A is positive, hence the tangent is rising.

The gradient of the tangent at point B is negative, hence the tangent is falling.

The gradient of the tangent at point C is zero, hence the tangent is horizontal.

25. Find the gradient of the tangent for each of the following furves at the given point P.

(a) $y = 4x - \dfrac{8}{x}; P(4, 14)$

Sol.

$$\begin{aligned}
y' &= 4 - \frac{8}{x^2} \\
&= 4 + \frac{8}{x^2} \\
&= 4 + \frac{8}{(4)^2} \\
&= 4 + \frac{1}{2} \\
&= 4.5 \quad \square
\end{aligned}$$

(b) $y = \dfrac{4 - 3x^2}{3 - 2x}; P(2, 8)$

Sol.

$$\begin{aligned}
y' &= \frac{(3-2x)(-6x) - (4-3x^2)(-2)}{(3-2x)^2} \\
&= \frac{-18x + 12x^2 + 8 - 6x^2}{(3-2x)^2} \\
&= \frac{6x^2 - 18x + 8}{(3-2x)^2} \\
&= \frac{2(3x^2 - 9x + 4)}{(3-2x)^2} \\
&= \frac{2\left[3(2)^2 - 9(2) + 4\right]}{(3-2(2))^2} \\
&= -4 \quad \square
\end{aligned}$$

26. (a) Find the value of gradient of the tangent to the curve $y = 2x^3 - 3x^2$ when $x = 1$.

Sol.

$$\begin{aligned}
\frac{dy}{dx} &= 6x^2 - 6x \\
&= 6(1) - 6(1) \\
&= 0 \quad \square
\end{aligned}$$

(b) Find the coordinates of points to the curve $y = \dfrac{x^3}{3} + x^2 - 1$ such that the gradient to the curve at the points is 8.

Sol.

$$\begin{aligned}
\frac{dy}{dx} &= x^2 + 2x \\
8 &= x^2 + 2x \\
x^2 + 2x - 8 &= 0 \\
(x+4)(x-2) &= 0 \\
x &= -4 \text{ or } x = 2
\end{aligned}$$

When $x = -4$,

$$\begin{aligned}
y &= \frac{(-4)^3}{3} + (-4)^2 - 1 \\
&= -\frac{64}{3} + 16 - 1 \\
&= -\frac{19}{3}
\end{aligned}$$

When $x = 2$,

$$\begin{aligned}
y &= \frac{2^3}{3} + 2^2 - 1 \\
&= \frac{8}{3} + 4 - 1 \\
&= \frac{17}{3}
\end{aligned}$$

Therefore, the coordinates of the points are $(-4, -\frac{19}{3})$ and $(2, \frac{17}{3})$. $\square$

(c) Given the curve $y = ax^2 + bx + 3$ has the gradient 5 when $x = 2$ and the gradient 0 when $x = -3$. Determine the values of a and b.

Sol.

$$\begin{aligned} \frac{dy}{dx} &= 2ax + b \\ 5 &= 2a(2) + b \\ 4a + b &= 5 \qquad (1) \\ 0 &= 2a(-3) + b \\ -6a + b &= 0 \qquad (2) \end{aligned}$$

$(1) - (2): \ 10a = 5$

$$a = \frac{1}{2} \quad \square$$

Substituting $a = \frac{1}{2}$ into (1),

$$\begin{aligned} 4\left(\frac{1}{2}\right) + b &= 5 \\ b &= 3 \quad \square \end{aligned}$$

27. Find the equations of tangent and normal to the curve $y = 8 - 2x - x^2$ at each of the following points.

Sol.

$$\frac{dy}{dx} = -2 - 2x$$

(a) $A(1,5)$

Sol.

At point $A(1,5)$, the gradient of the tangent is $-2 - 2(1) = -4$.

Hence, the equation of the tangent is

$$\begin{aligned} y - 5 &= -4(x - 1) \\ y - 5 &= -4x + 4 \\ y &= -4x + 9 \quad \square \end{aligned}$$

At point $A(1,5)$, the gradient of the normal is $\frac{1}{4}$.

Hence, the equation of the normal is

$$\begin{aligned} y - 5 &= \frac{1}{4}(x - 1) \\ 4y - 20 &= x - 1 \\ x - 4y + 19 &= 0 \quad \square \end{aligned}$$

(b) $C(-1,9)$

Sol.

At point $C(-1,9)$, the gradient of the tangent is $-2 - 2(-1) = 0$.

Hence, the equation of the tangent is

$$\begin{aligned} y - 9 &= 0(x + 1) \\ y - 9 &= 0 \\ y &= 9 \quad \square \end{aligned}$$

At point $C(-1,9)$, the gradient of the normal is *undefined*.

Hence, the equation of the normal is

$$\begin{aligned} x + 1 &= 0 \\ x &= -1 \quad \square \end{aligned}$$

28. (a) Find the equation of normal to the curve $y = 3x^2 + 8x - 7$ at point $(-2,6)$.

Sol.

$$\frac{dy}{dx} = 6x + 8$$

The gradient of the tangent is $6(-2) + 8 = -4$.

The gradient of the normal is $\frac{1}{4}$.

Hence, the equation of the normal is

$$\begin{aligned} y - 6 &= \frac{1}{4}(x + 2) \\ 4y - 24 &= x + 2 \\ x - 4y + 26 &= 0 \quad \square \end{aligned}$$

(b) Given the tangent to the curve $y = ax^2 + bx$ at the point $P(4,8)$ is perpendicular to the straight line that passes through the point $A(4,1)$ and the point $B(12,0)$. Find the values of a and b.

Sol.

$$m_{AB} = -\frac{1}{8}$$

Since the tangent at point $P(4,8)$ is perpendicular to the straight line AB,

The gradient of the tangent at point $P(4,8)$ is 8.

$$\frac{dy}{dx} = 2ax + b$$

The gradient of the tangent at point $P(4,8)$ is $2a(4) + b = 8a + b$.

$$8a + b = 8 \quad \cdots \quad (1)$$

At point $P(4,8)$

$$\begin{aligned} a(4)^2 + b(4) &= 8 \\ 16a + 4b &= 8 \\ 4a + b &= 2 \quad \cdots \quad (2) \end{aligned}$$

$$\begin{aligned} (1) - (2): \ 4a &= 6 \\ a &= \frac{3}{2} \quad \square \end{aligned}$$

Substituting $a = \frac{3}{2}$ into (2),

$$\begin{aligned} b &= 2 - 4\left(\frac{3}{2}\right) \\ &= 2 - 6 \\ &= -4 \quad \square \end{aligned}$$

29. Find the coordinates of the turning points for each of the following curves. Hence, determine the nature of the turning points.

(a) $y = 5x^2 - 2x + 1$

Sol.

$$\begin{aligned} \frac{dy}{dx} &= 10x - 2 \\ 10x - 2 &= 0 \\ 10x &= 2 \\ x &= \frac{1}{5} \quad □ \end{aligned}$$

When $x = \frac{1}{5}$,

$$\begin{aligned} y &= 5\left(\frac{1}{5}\right)^2 - 2\left(\frac{1}{5}\right) + 1 \\ &= \frac{1}{5} - \frac{2}{5} + 1 \\ &= \frac{4}{5} \end{aligned}$$

Hence, the coordinates of the turning point are $\left(\frac{1}{5}, \frac{4}{5}\right)$. □

$$\frac{d^2y}{dx^2} = 10 > 0$$

Hence, $\left(\frac{1}{5}, \frac{4}{5}\right)$ is a *maximum point*. □

(b) $y = \dfrac{x^2}{x+1}$

Sol.

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x+1)(2x) - x^2}{(x+1)^2} \\ &= \frac{2x^2 + 2x - x^2}{(x+1)^2} \\ &= \frac{x(x+2)}{(x+1)^2} \end{aligned}$$

$$\begin{aligned} \frac{x(x+2)}{(x+1)^2} &= 0 \\ x(x+2) &= 0 \\ x = 0 \text{ or } x &= -2 \end{aligned}$$

When $x = 0$,

$$\begin{aligned} y &= \frac{0^2}{0+1} \\ &= 0 \end{aligned}$$

When $x = -2$,

$$\begin{aligned} y &= \frac{(-2)^2}{(-2)+1} \\ &= \frac{4}{-1} \\ &= -4 \end{aligned}$$

Hence, the coordinates of the turning points are $(0, 0)$ and $(-2, -4)$. □

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2 + 2x}{(x+1)^2} \\ \frac{d^2y}{dx^2} &= \frac{(2x+2)(x+1)^2 - 2(x+1)(x^2+2x)}{(x+1)^4} \\ &= \frac{(x+1)[(2x+2)(x+1) - 2(x^2+2x)]}{(x+1)^4} \\ &= \frac{2x^2 + 4x + 2 - 2x^2 - 4x}{(x+1)^3} \\ &= \frac{2}{(x+1)^3} \end{aligned}$$

When $x = 0$,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{2}{(0+1)^3} \\ &= \frac{2}{1^3} \\ &= 2 > 0 \end{aligned}$$

Hence, $(0, 0)$ is a *maximum point*. □

When $x = -2$,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{2}{(-2+1)^3} \\ &= \frac{2}{(-1)^3} \\ &= -2 < 0 \end{aligned}$$

Hence, $(-2, -4)$ is a *minimum point*. □

(c) $y = 7 - x^3$

Sol.

$$\begin{aligned} \frac{dy}{dx} &= -3x^2 \\ -3x^2 &= 0 \\ x &= 0 \end{aligned}$$

When $x = 0$,

$$\begin{aligned} y &= 7 - 0^3 \\ &= 7 \end{aligned}$$

Hence, the coord. of the turning point is $(0, 7)$. □

$$\frac{d^2y}{dx^2} = -6x$$

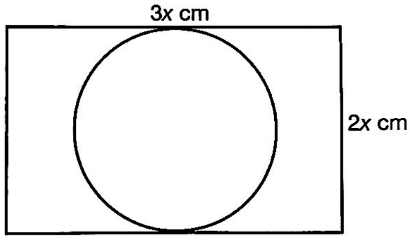
When $x = 0$,

$$\begin{aligned} \frac{d^2y}{dx^2} &= -6(0) \\ &= 0 \end{aligned}$$

Hence, $(0, 7)$ is a *inflection point*. □

30. Solve the following problems related to stationary points.

(a) The following diagram shows the plan of a cuboid in which its centre in the shape of a cylinder is taken out. The cuboid measures $3xcm \times 2xcm \times (45 - 5x)cm$.



Find the value of x that makes the volume of the cylinder taken out a maximum.

Sol.

$$\begin{aligned} r &= x \\ V &= \pi r^2 h \\ &= \pi x^2(45-5x) \\ &= 45\pi x^2 - 5\pi x^3 \end{aligned}$$

V is maximum when

$$\begin{aligned} \frac{dV}{dx} &= 0 \\ 90\pi x - 15\pi x^2 &= 0 \\ 6x - x^2 &= 0 \\ x(x-6) &= 0 \\ x = 0 &\text{ or } x = 6 \\ x \neq 0,\ & x = 6 \\ x = 6, \frac{d^2V}{dx^2} &= 90\pi - 30\pi(6) \\ &= 90\pi - 180\pi \\ &= -90\pi < 0 \end{aligned}$$

Hence, $x = 6$ is the value of x that makes the volume of the cylinder taken out a maximum. ▯

(b) Given $A = bh$ where $b^2 + h^2 = 40$ and $b, h > 0$. Find the values of b and h so that A becomes a stationary point and show that the value of A is maximum.

Sol.

$$\begin{aligned} b^2 + h^2 &= 40 \\ b^2 &= 40 - h^2 \\ b &= \sqrt{40-h^2} \\ &= (40-h^2)^{\frac{1}{2}} \\ A &= bh \\ &= (40-h^2)^{\frac{1}{2}} h \end{aligned}$$

A is stationary when

$$\begin{aligned} \frac{dA}{dh} &= 0 \\ -\frac{2h^2}{2\sqrt{40-h^2}} + \sqrt{40-h^2} &= 0 \\ -\frac{2h^2}{2\sqrt{40-h^2}} + \sqrt{40-h^2} &= 0 \\ \frac{-h^2+40-h^2}{\sqrt{40-h^2}} &= 0 \\ -2h^2 + 40 &= 0 \\ h^2 &= 20 \\ h &= \sqrt{20}\ (h>0) \quad ▯ \\ b &= \sqrt{40 - \left(\sqrt{20}\right)^2} \\ &= \sqrt{20} \quad ▯ \end{aligned}$$

$$A = \sqrt{20} \cdot \sqrt{20} = 20$$

h	4	$\sqrt{20}$	5
$\frac{dA}{dh}$	1.63	0	-2.58
Tangent Sketch	/	−	\
Graph Sketch	/ − \		

Hence, $A = 20$ is maximum when $h = \sqrt{20}$. ▯

(c) A piece of wire with a length of $120cm$ is divided into two parts where is each is bent to form an equilateral triangle with an edge of xcm and a square with an edge of ycm respectively. Express y in terms of x. Hence, show that the total area of both shapes, Acm^2 is given by

$$A = \frac{9(40-x)^2 + 4\sqrt{3}x^2}{16}$$

Calculate the value of x so that A has a stationary value. Determine whether this value of x makes A a maximum of a minimum.

Sol.

$$\begin{aligned} C_{triangle} &= 3x \\ C_{square} &= 4y \\ 3x + 4y &= 120 \\ y &= \frac{120-3x}{4} \end{aligned}$$

$$\begin{aligned}
A_{triangle} &= \frac{1}{2}bh \\
&= \frac{1}{2}x\sqrt{x^2 - \left(\frac{x}{2}\right)^2} \\
&= \frac{1}{2}x\sqrt{x^2 - \frac{x^2}{4}} \\
&= \frac{1}{2}x\sqrt{\frac{3x^2}{4}} \\
&= \frac{1}{2}x\left(\frac{1}{2}x\sqrt{3}\right) \\
&= \frac{\sqrt{3}x^2}{4} \\
A_{square} &= y^2 \\
&= \left(\frac{120 - 3x}{4}\right)^2 \\
&= \frac{(120 - 3x)^2}{16} \\
&= \frac{[3(40 - x)]^2}{16} \\
&= \frac{9(40 - x)^2}{16} \\
A &= A_{square} + A_{triangle} \\
&= \frac{9(40 - x)^2}{16} + \frac{\sqrt{3}x^2}{4} \\
&= \frac{9(40 - x)^2 + 4\sqrt{3}x^2}{16} \quad \text{(shown)} \quad \square
\end{aligned}$$

A is stationary when $\dfrac{dA}{dx} = 0$.

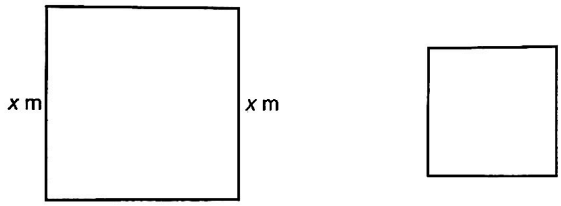
$$\begin{aligned}
\frac{dA}{dx} &= \frac{d}{dx}\left[\frac{9(40 - x)^2 + 4\sqrt{3}x^2}{16}\right] \\
&= \frac{d}{dx}\left[\frac{9(40 - x)^2}{16} + \frac{\sqrt{3}x^2}{4}\right] \\
&= \frac{-18(40 - x)}{16} + \frac{2\sqrt{3}x}{4} \\
&= -\frac{9(40 - x)}{8} + \frac{4\sqrt{3}x}{8} \\
&= -\frac{9(40 - x) - 4\sqrt{3}x}{8}
\end{aligned}$$

$$\begin{aligned}
-\frac{9(40 - x) - 4\sqrt{3}x}{8} &= 0 \\
360 - 9x - 4\sqrt{3}x &= 0 \\
9x + 4\sqrt{3}x &= 360 \\
\left(9 + 4\sqrt{3}\right)x &= 360 \\
x &= \frac{360}{9 + 4\sqrt{3}} \\
&\approx 22.6cm \quad \square
\end{aligned}$$

$$\frac{d^2A}{dx^2} = \frac{9 + 4\sqrt{3}}{8} > 0$$

Hence, A is a minimum when $x = 22.6cm$. $\square$

31. Chan wants to build two separate pens by using a fence of $100m$. Both pens are square in shape.



If the edge of the larger pen is xm,

(a) find the length of the side of the smaller pen in terms of x.

Sol. Let the length of the side of the smaller pen be ym.

$$\begin{aligned}
C_{larger} &= 4x \\
C_{smaller} &= 4y \\
C_{fence} &= C_{larger} + C_{smaller} \\
100 &= 4x + 4y \\
25 &= x + y \\
y &= (25 - x)m \quad \square
\end{aligned}$$

(b) find the value of x such that the total area of both pens is minimum.

Sol.

$$\begin{aligned}
A_{larger} &= x^2 \\
A_{smaller} &= y^2 \\
&= (25 - x)^2 \\
A &= A_{larger} + A_{smaller} \\
&= x^2 + (25 - x)^2
\end{aligned}$$

A_{total} is stationary when

$$\begin{aligned}
\frac{dA}{dx} &= 0 \\
2x - 2(25 - x) &= 0 \\
2x - 50 + 2x &= 0 \\
4x &= 50 \\
x &= 12.5m \quad \square
\end{aligned}$$

$$\frac{d^2A}{dx^2} = 4 > 0$$

Hence, A is a minimum when $x = 12.5m$. $\square$

32. Solve the following problems related to the rates of change.

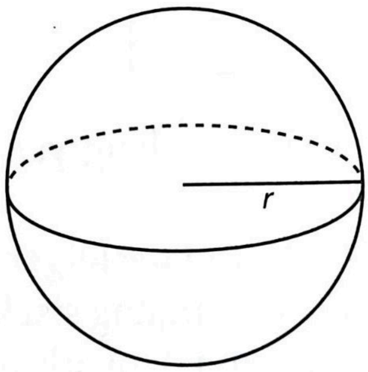
(a) The total surface area, Acm^2, of a metal solid which consists of a cone and a cylinder with a common radius, rcm is given by $A = 2\pi\left(\frac{18}{r} + \frac{r^2}{3}\right)$. When it is heated, its total surface area changes at the rate of

$2.1\pi cm^2s^{-1}$. Find the rate of change of the radius, in cms^{-1}, at the instant $r = 6cm$.

Sol.

$$\begin{aligned}
A &= 2\pi\left(\frac{18}{r} + \frac{r^2}{3}\right) \\
\frac{dA}{dr} &= 2\pi\left(-\frac{18}{r^2} + \frac{2r}{3}\right) \\
\frac{dA}{dt} &= 2.1\pi \\
\frac{dr}{dt} &= \frac{dr}{dA} \cdot \frac{dA}{dt} \\
\frac{dA}{dt} &= \frac{dA}{dr} \cdot \frac{dr}{dt} \\
2.1\pi &= 2\pi\left(-\frac{18}{r^2} + \frac{2r}{3}\right) \cdot \frac{dr}{dt} \\
\frac{dr}{dt} &= \frac{2.1}{2\left(-\frac{18}{r^2} + \frac{2r}{3}\right)} \\
&= \frac{2.1}{2\left(-\frac{18}{6^2} + \frac{2(6)}{3}\right)} \\
&= 0.3cms^{-1}
\end{aligned}$$

(b) A spherical balloon experiences a constant rate of increase of $6cm^2s^{-1}$.



At the instant when the radius is $5cm$, find

i. the rate of increase, in cms^{-1}, of the radius.

Sol.

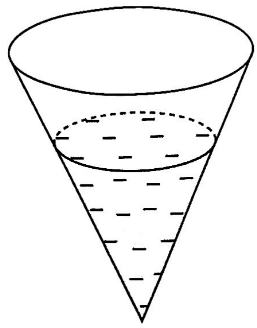
$$\begin{aligned}
S &= 4\pi r^2 \\
\frac{dS}{dr} &= 8\pi r \\
\frac{dS}{dt} &= 6 \\
\frac{dr}{dt} &= \frac{dr}{dS} \cdot \frac{dS}{dt} \\
\frac{dS}{dt} &= \frac{dr}{dt} \cdot \frac{dS}{dr} \\
6 &= \frac{dr}{dt} \cdot 8\pi r \\
\frac{dr}{dt} &= \frac{3}{4\pi r} \\
&= \frac{3}{20\pi} cms^{-1}
\end{aligned}$$

ii. the rate of increase if volume, in cm^3s^{-1}, of the sphere.

Sol.

$$\begin{aligned}
V &= \frac{4}{3}\pi r^3 \\
\frac{dV}{dr} &= 4\pi r^2 \\
\frac{dV}{dt} &= \frac{dV}{dr} \cdot \frac{dr}{dt} \\
&= 4\pi r^2 \cdot \frac{3}{4\pi r} \\
&= 3r \\
&= 3(5) \\
&= 15cm^3s^{-1}
\end{aligned}$$

(c) The following diagram shows a container in the shape of a cone. Given its height is equal to its base radius. Water is poured into the container at the rate of $80cm^3s^{-1}$. The volume of the water in the container is $\frac{1}{3}\pi x^3 cm^3$, when the depth of the water is xcm.



Calculate, at the instant when the depth of the water is $10cm$,

i. the rate of increase of the depth, in cms^{-1}, of the water.

Sol.

At time t, let $V =$ volume of water

$$\begin{aligned}
\frac{dV}{dt} &= 80 \\
\frac{dV}{dx} &= \pi x^2 \\
\frac{dx}{dt} &= \frac{dx}{dV} \cdot \frac{dV}{dt} \\
\frac{dV}{dt} &= \frac{dx}{dt} \cdot \frac{dV}{dx} \\
80 &= \frac{dx}{dt} \cdot \pi x^2 \\
\frac{dx}{dt} &= \frac{80}{\pi x^2} \\
&= \frac{80}{\pi (10)^2} \\
&= \frac{4}{5\pi} cms^{-1}
\end{aligned}$$

ii. the rate of increase of the horizontal surface area, in cm^2s^{-1}, of the water.

At time t, let

$A =$ horizontal surface area of water

$r =$ radius of the water surface
$R =$ radius of the base of the container
$h =$ height of the container

$$\begin{aligned}
R &= h \quad \text{(given)} \\
\frac{r}{R} &= \frac{x}{h} \\
\frac{r}{h} &= \frac{x}{h} \\
r &= x \\
A &= \pi r x \\
&= \pi x^2 \\
\frac{dA}{dx} &= 2\pi x \\
\frac{dA}{dt} &= \frac{dA}{dx} \cdot \frac{dx}{dt} \\
&= 2\pi x \cdot \frac{80}{\pi x^2} \\
&= \frac{160}{x} \\
&= \frac{160}{10} \\
&= 16 cm^2 s^{-1}
\end{aligned}$$

33. Solve the following problems related to the small changes and aproximations.

(a) Given that $y = 2x^3 - 5x^2 + x - 1$, find the value of $\frac{dy}{dx}$ when $x = 1$. Hence, find the small changes in y when x increases from 1 to 1.02.

(b) Given the equation of a curve is $y = \frac{9}{(2x-5)^2}$, find, in terms of p, where p is a small value, the approximate change in

i. y when x increases from 3 to $3 + p$.

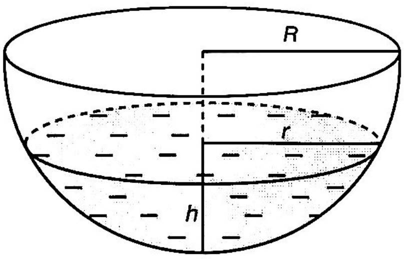
ii. x when y decreases from 1 to $1 - p$.

(c) Given $y = x^4$, by using the calculus method, find the approximate value of

i. 2.03^4.

ii. 1.99^4.

34. A hemispherical bowl of radius $R cm$ is filled with water to a depth of $h cm$.



The volume of the water in the bowl is given by $V = \frac{\pi}{3}(3Rh^2 - h^3)$.

(a) Show that the radius of the water surface, r, is given by $r = \sqrt{2Rh - h^2}$.

(b) Water is poured into the bown at a constand rate of $300 cm^3 s^{-1}$. Find, in terms of R, the rate of increase of the surface area, in $cm^2 min^{-1}$, of the water when $2h = R$.