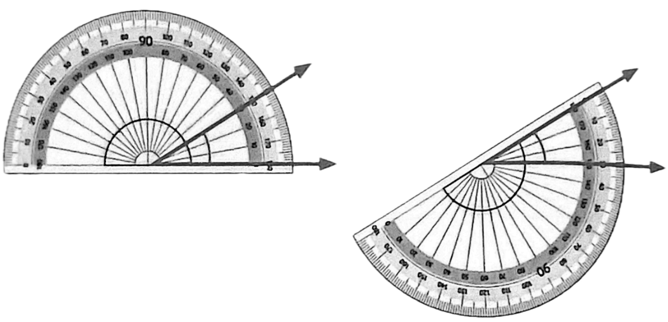
Chapter 8

Degree and Radian

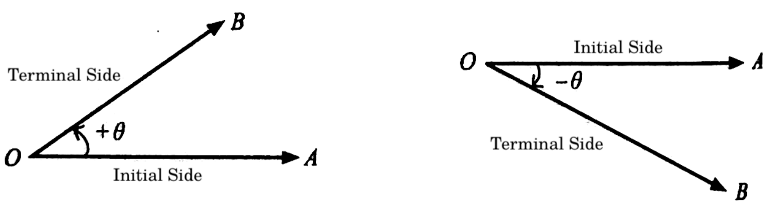
8.1 Concept and Measurement of Arbitrary Angle

Arbitrary Angle

Back in junior high, we learned that the angle is the geometric object formed by rays from the same endpoint. As shown in the figure below, whether it is from ray OA to ray OB, or from ray OA to ray OC, the angle can be measured as $30°$.



In fact, angle can be seen as a rotation of a ray about its endpoint in a plane. As shown in the figure below, the endpoint O of the ray is known as the **vertex** of the angle, the initial position OA of the rotation is known as the **initial side** of the angle, and the final position OB of the rotation is known as the **terminal side** of the angle, while the amount of rotation is $\angle AOB$. The rotation can be either clockwise or counterclockwise, as shown in the figure below, the angle formed counterclockwise is positive, while the angle formed clockwise is negative, $\theta > 0°$ in the figure below.



Exploration Activity 1

Aim: To understand the definition of arbitrary angle.

Materials Needed: A piece of paper, a protractor, a ruler

Steps:

1. Draw three parallel rays OA (as shown below) on the paper. Hence, with OA as the initial side of the angle, measure the following using a protractor:
   (a) The terminal side OB of the $160°$.
   (b) The terminal side OC of the $-45°$.
   (c) The terminal side OD of the $450°$.
2. Write down clearly the respective direction and the measurement of the rotation in the three figures above.

Tool: https://www.geogebra.org/m/u7a3zxfd

From exploration activity 1, this kind of directional rotational measurement that is not limited to $0°$ to $360°$ is known as the **arbitrary angle**. If the initial side doesn't rotate at all, the angle formed is called a **zero angle**.

Degree and Radian

Back in junior high, the angle unit that we learned is formed by dividing a circle into 360 equal parts, and the corresponding central angle of each part is known as 1 degree, denoted as $1°$. Dividing the arc corresponding to 1 degree angle into 60 equal parts, the corresponding central angle of each part is known as 1 minute, denoted as $1'$. Dividing the arc corresponding to 1 minute angle into 60 equal parts, the corresponding central angle of each part is known as 1 second, denoted as $1''$.

$$\begin{aligned} \therefore\ 1 \text{ full angle} &= 360° \\ 1° &= 60' \\ 1' &= 60'' \end{aligned}$$

This form of base 60 measurement of angles is known as the **degree measure**.

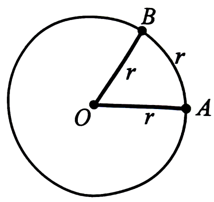
Exploration Activity 2

Aim: To inspect the relationship between arc length and radius, and to understand teh definition of radian measure.

Materials Needed: A compass, a protractor, a string of length 10 cm, a scrap paper.

Steps:

1. Form a group of 2 to 4 people, each group member is required to draw a circle with different radii. (Choose any arbitrary radius like 2cm, 3cm, 4cm, $\cdots$)
2. Each person uses a line to measure the arc AB with the same length as the radius on his own arc, then marks it on the string, as shown in the figure below.



3. Use a protractor to measure the corresponding central angle of the arc AB.

Discuss and Discover:

1. Compare the diagram drawn by each group member, are the measured central angles the same?
2. Use a string to measure how many parts with the same length as the AB of the circumference can be divided.
3. Is it possible to derive an accurate answer for the question above using the formula of the circumference of a circle?

From the activities above, we have led out another form of angle measurement that is commonly used in higher mathematics and other fields of science, known as the **radian measure**. We call the central angle corresponding to the arc with the same length as the radius (i.e. $\angle AOB$ in the figure above) 1 radian, denoted as 1 rad. From discussion 2 and discussion 3, we know that the ratio between the circumference of a circle and its radius is 2π, i.e. 1 full angle $= 360^\circ = 2\pi$ rad, hence we can derive the following conversion formula between degree and radian:

Conversion between Degree and Radian

$$\begin{aligned} \pi \text{ rad} &= 180^\circ \\ 1 \text{ rad} &= \frac{180^\circ}{\pi} \\ 1^\circ &= \frac{\pi}{180} \text{ rad} \end{aligned}$$

Example 1

Convert the following angles from degree to radian:

(a) 218.84° (b) $48^\circ 36'$

Solution:

(a)

$$\begin{aligned} 218.84^\circ &= 218.84 \times \frac{\pi}{180} \\ &= 3.8195 \text{ rad} \end{aligned}$$

(b)

$$\begin{aligned} 48^\circ 36' &= \left(48 + \frac{36}{60}\right)^\circ \\ &= 48.6 \times \frac{\pi}{180} \\ &= 0.8482 \text{ rad} \end{aligned}$$

Example 2

Convert 1.5 radian to degree (accurate to minute).

Solution:

$$\begin{aligned} 1.5 \text{ rad} &= 1.5 \times \frac{180^\circ}{\pi} \\ &= 85^\circ 57' \end{aligned}$$

Example 3

Convert $\frac{\pi}{45}$ radian to degree.

Solution:

$$\begin{aligned} \frac{\pi}{45} \text{ rad} &= \frac{\pi}{45} \times \frac{180^\circ}{\pi} \\ &= 4^\circ \end{aligned}$$

It is worth nothing that when we use the radian measure, the unit ”radian” is usually omitted. We usually write $90^\circ = \frac{\pi}{2}$, but if we use degree measure to express the same angle, the unit ”degree” cannot be omitted.

Practice 8.1

1. Complete the following table, express the angle in *pi*:

Angle	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$	$270°$
Radian										

2. Convert the following angles from radian to degree (if the result is not an integer, round to 2 decimal places):
   (a) 0.5 rad
   (b) $\dfrac{7\pi}{6}$ rad

Exercise 8.1

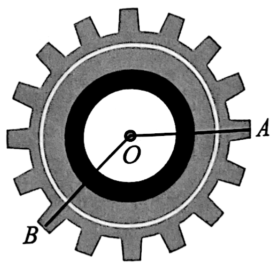
1. Using the horizontal ray OA in the figure below as the initial side, sketch the position of the terminal side OB of the following angles:

   (a) $\dfrac{5\pi}{4}$
   (b) $-\dfrac{6\pi}{7}$
   (c) $\dfrac{5\pi}{2}$

2. Convert the following angles from degree to radian (accurate to 4 decimal places):
   (a) $68°93'$
   (b) $139°12'$
   (c) $-502°46'$

3. Convert the following angles from radian to degree (if the result is not an integer, round to minute):
   (a) 0.89 rad
   (b) $-\dfrac{17\pi}{4}$ rad
   (c) 3 rad

4. If the gear in the figure below rotates counterclockwise about its origin O, how many radians does the gear rotate through such that



   (a) The line segment OB reaches the position of OA in the figure above for the first time?
   (b) The line segment OB reaches the position of OA in the figure above for the second time?

8.2 Arc Length and Sector Area

Formula of Arc Length

Back in junior high, we learned that in a circle with radius r and central angle θ,

(1) if the angle is expressed in degree, the arc length l is given by

$$l = \frac{\theta}{360^\circ} \times 2\pi r \qquad \text{(where } \theta \text{ is in degree)}$$

(2) if the angle is expressed in radian, i.e. $360^\circ = 2\pi$, the arc length l is given by

$$l = \frac{\theta}{2\pi} \times 2\pi r = \theta r \qquad \text{(where } \theta \text{ is in radian)}$$

From the definition of radians, we can see that if the central angle corresponding to the arc AB is θ radian, then the arc length AB is θ times the length of radius r. That is,

Formula of Arc Length

$$\theta = \frac{l}{r}$$

$$r = \frac{l}{\theta}$$

Example 4

In a circle with radius 5 cm, find

(a) The arc length corresponding to the central angle of 1.5 rad.

(b) The central angle corresponding to the arc length of 17 cm.

Solution:

(a) The corresponding arc length $= 1.5 \times 5$

$$= 7.5 \text{ cm}$$

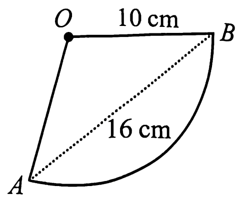
(b) Let θ be the central angle, then

$$5\theta = 17$$

$$\theta = 3.4$$

The central angle corresponding to the arc length of 17 cm is 3.4 rad.

Example 5



The figure above shows a sector with center O and radius of 10 cm. If the length of chord AB is 16 cm, find

(a) $\angle AOB$ (in radian);

(b) The length of arc AB.

Solution:

(a) Let M be the midpoint of chord AB,

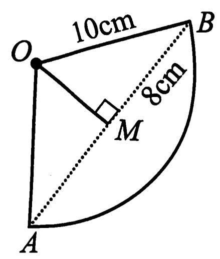
$\therefore OM \perp AB$ and $\angle MOB = \angle MOA$

In $\triangle OMB$,

$$\begin{aligned} \sin \angle BOM &= \frac{MB}{OB} \\ &= \frac{8}{10} \\ \angle BOM &= 0.9273 \end{aligned}$$

Hence,

$$\begin{aligned} \angle AOB &= 2 \times 0.9273 \\ &= 1.8546 \text{ rad} \end{aligned}$$



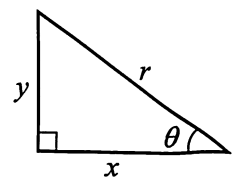
(b)

$$\begin{aligned} \text{Arc length } AB &= 1.8546 \times 10 \\ &= 18.546 \text{ cm} \end{aligned}$$

Knowledge Review

The trigonometric functions that we learned in junior high are:

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}$$



Adding appropriate auxiliary lines to form a right angle triangle, we can use the trigonometric functions to find the relationship between the sides and the angles of the triangle.

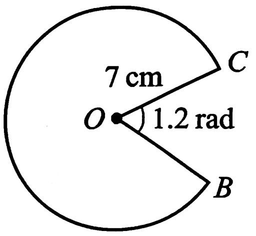
Think about It:

Can we use degree to find the arc length AB?

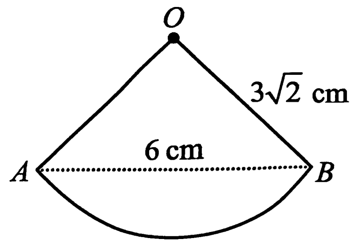
Practice 8.2a

Find the circumference of the following sectors:

(a)



(b)



Formula of Sector Area

Exploration Activity 3

Aim: To strengthen the understanding of the relationship between the ratio of radian and arc length, and the ratio of radian and sector area.

Tool: https://www.geogebra.org/m/a4brp8yg

Steps:

1. Without moving point B, move the radius of the circle by moving the slider L, and inspect the changes of the following three ratios by the coloured section of the sector:
   (1) $\angle AOB$ and 2π rad (i.e. $360°$).
   (2) The arc length AB and the circumference of the circle.
   (3) The area of sector OAB and the area of the circle.
2. Without changing the radius of the circle, move point B to change the central angle, and inspect the changes of the same three ratios above.

From the inspection above, Have you found any identical values?

Back in junior, we learned that the area of a sector is proportional to the central angle of the sector. Hence, we can derive that

$$\text{Area of sector } = \frac{\theta}{360^\circ} \times \pi r^2 \qquad \text{(where } \theta \text{ is in degree)}$$

Since $360^\circ = 2\pi$,

$$\therefore \text{ Area of sector } S = \frac{\theta}{2\pi} \times \pi r^2 \qquad \text{(where } \theta \text{ is in radian)}$$

That is,

Formula of Sector Area

$$S = \frac{1}{2} r^2 \theta$$

Because arc length $l = r\theta$, $\therefore$ the area formula of a sector can also be expressed as

Formula of Sector Area

$$S = \frac{1}{2} \times \pi r \times r$$
$$S = \frac{1}{2} rl$$

Example 6

There is a sector with radius 3cm and area 15 cm^2. Find the central and the circumference of the sector.

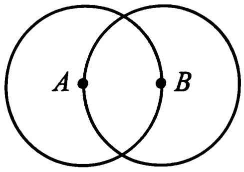
Solution:

$$S = \frac{1}{2}\pi r^2 \qquad S = \frac{1}{2} lr$$
$$15 = \frac{1}{2}\pi (3)^2 \qquad 15 = \frac{1}{2} l \times 3$$
$$\theta = \frac{10}{3} \qquad l = 10$$

$$\therefore \text{ The central angle } = \frac{10}{3} \text{ rad,}$$

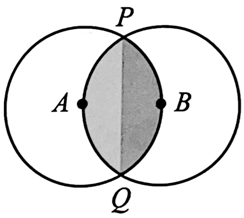
$$\begin{aligned} \text{The circumference } &= l + 2r \\ &= 10 + 2 \times 3 \\ &= 16 \text{ cm} \end{aligned}$$

Example 7



The figure above shows two circles with the same radius r and the center being A and B respectively. Prove that the area of the overlapping region of these two circles is $\frac{2}{3}\pi r^2 - \frac{\sqrt{3}}{2}r^2$.

Solution:



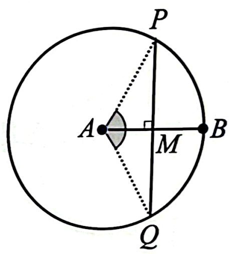
Let the two circles intersect at point P and Q,

Since the two circles are of the same size, the overlapping area is formed by two identical arches.

$\because \triangle ABP$ and $\triangle ABQ$ are equilateral triangles,

$\therefore \angle PAB = \angle BAQ = \frac{\pi}{3}$

Let PQ and AB intersect at point M, M happens to be the midpoint of line segment PQ and AB.



$\therefore AM = \frac{1}{2}AB = \frac{1}{2}r$

In $\triangle AMP$, $\sin\frac{\pi}{3} = \frac{MP}{AP}$

$$MP = \frac{\sqrt{3}}{2}r$$

Area of sector APQ

$$= \frac{1}{2}r^2\left(\frac{2\pi}{3}\right)$$

$$= \frac{\pi r^2}{3}$$

Area of $\triangle APQ$

$$= 2 \times \frac{1}{2}\left(\frac{1}{2}r\right)\left(\frac{\sqrt{3}}{2}r\right)$$

$$= \frac{\sqrt{3}}{4}r^2$$

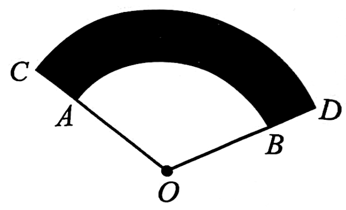
Hence, the area of the overlapping region of the two circles $= 2 \times$ (Area of sector APQ − Area of $\triangle APQ$)

$$= 2 \times \left[\frac{1}{3}\pi r^2 - \frac{\sqrt{3}}{4}r^2\right]$$

$$= \frac{2}{3}\pi r^2 - \frac{\sqrt{3}}{2}r^2$$

Example 8

As shown in the figure to the right, both the centers of the circles with arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ is O. Given that $AC = d$, $\overset{\frown}{AB} = l'$, and $\overset{\frown}{CD} = l$. Prove that:



(a) $\angle AOB = \dfrac{l - l'}{d}$ radian;

(b) The shaded area $= \dfrac{1}{2}(l + l')d$.

Solution:

(a) Let $\angle AOB = \theta$,

$$\begin{aligned} l' &= (OA)\theta \\ l &= (OC)\theta \\ l - l' &= (OC - OA)\theta \\ &= d\theta \\ \theta &= \frac{l - l'}{d} \end{aligned}$$

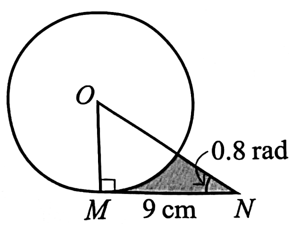
(b) Area of shaded region = Area of sectorOCD − Area of sectorOAB

$$\begin{aligned} &= \frac{1}{2}(OC)^2\theta - \frac{1}{2}(OA)^2\theta \\ &= \frac{1}{2}\theta\left[(OC)^2 - (OA)^2\right] \\ &= \frac{1}{2}\,[OC + OA]\,[OC - OA]\,\theta \qquad \because l + l' = (OC + OA)\theta \\ &= \frac{1}{2}d(l + l') \end{aligned}$$

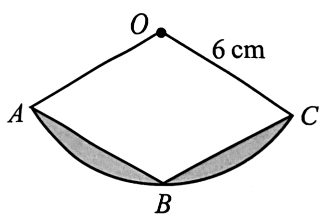
Practice 8.2b

Find the area of the shaded regions in the following figures:

(a)

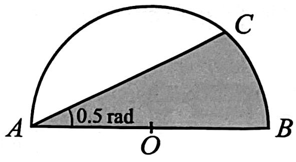


(b) OAC is a sector, $OABC$ is a rhombus



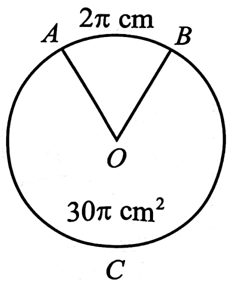
✎ Exercise 8.2a

1. The right figure shows a semicircle with center O and diameter 8 cm. C is a point on arc AB. Given that $\angle CAB = 0.5$ rad, find the length of arc BC and the area of the shaded region.



2. Given that the arc length of a sector is $8\pi\mathrm{cm}$ and the area is $48\pi\mathrm{cm}^2$, determine the radius and the central angle of the sector.

3. The circle in the right figure is formed by a major sector $OACB$ and a minor sector OAB. If the area of the major sector $OACB$ is $30\pi\mathrm{cm}^2$, and the length of the minor arc AB is $2\pi\mathrm{cm}$, find the radius of the circle and the central angles of the two sectors.

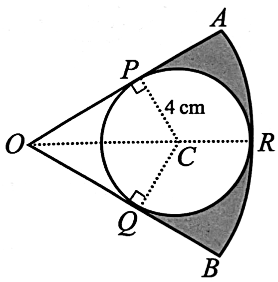


4. As shown in the figure, a circle with center C is tangent to the sector OAB at points P, Q, and R. Given that $CP = 4\,\mathrm{cm}$ and $\angle AOB = 60°$, without using a computer, find:

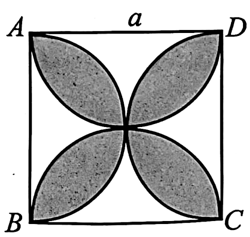
   (a) the length of OA;

   (b) the area of the major sector $CPRQ$;

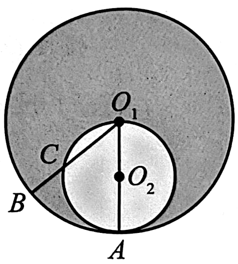
   (c) the area of the shaded region.



5. In the right figure, $ABCD$ is a square with a side length of a. Four semicircles are drawn with the sides of the square as their diameters. These semicircles enclose the shaded region in the figure. Find the area of the shaded region.



6. In the right figure, O_1 and O_2 are the centers of the large and small circles respectively. The two circles are tangent to each other at point A, line O_1CB intersects the large and small circles at points B and C respectively. Prove that $\overset{\frown}{AB} = \overset{\frown}{AC}$.



Problems related to Arc Length and Sector