Chapter 11

Logical Reasoning

11.1 Logic

Logic is a branch of science that studies the way we reason and its patterns. When we are reasoning, We need to use concepts, make judgments, and make inferences. Concept, judgment, and inference are the three basic elements of the thinking process. They are interconnected and follow a certain pattern.

The science of logic emerged over 2000 years ago. Ancient philosophers already started to study the formation of thinking and its patterns from a long time ago. Aristotle from ancient Greek was the first to systematically study logic, thus he is known as the father of classical logic. In the 17th century, Leibniz from Germany was the first to put forward the idea of using symbolic operations to study logic problems. This has led to the emergence of mathematical logic, a branch of logic that uses mathematical methods to study logic problems. In the 19th century, Boole, a British mathematician, created a fringe science that is in between algebra and logic, known as **Logical algebra** (also known as Boolean algebra). Since the 20th century, mathematical logic has received greater and deeper development. It describes and studies logic more accurately and mathematically. It has provided a meaningful tool and method for the study of the foundation of mathematics. Its research on computerizing, programming, and mechanizing the thinking process has also become the theoretical basis of computer science.

Human society is evolving into an age of information. The popularization of computers has led to the digitalization of science and technology, the mechanization of human thinking, and computerization is increasing day by day. Mathematics as a logically rigorous basic science, is playing an increasingly important role, and logic as one of the foundations of mathematics is also increasingly valued by people.

11.2 Proposition

Proposition is a declarative sentence that is used to express a certain judgment. For example,

(a) The sum of the interior angles of a triangle is $180°$.

(b) $\sqrt{2}$ is not a rational number.

(c) $(a+b)^2$ is equal to $a^2+2ab+b^2$. (That is, the equation $a^2+2ab+b^2=(a+b)^2$)

(d) Two lines that are perpendicular to the same plane are parallel to each other.

(e) The equation $x^2+4x+5=0$ has two real roots.

(f) $\sin^2 x - \cos^2 x = 2$.

Some of these sentences are true, some are false. The sentences (a), (b), (c), (d) above are true, while (e) and (f) are false.

The sentences that can be judged as true or false are called **propositions**. All the six sentences above are propositions.

Some sentences cannot be judged as true or false, these kinds of uncertain sentences are not propositions. For example,

(g) The two base angles of $\triangle ABC$ are equal.

(h) a is the smallest among the three numbers a, b, and c.

Since $\triangle ABC$ and the number a, b, and c are not specified, the sentence (g) and (h) cannot be judged as true or false.

The proposition that is true is called a **true proposition**, and the proposition that is false is called a **false proposition**. Generally, we use small letters p, q, r, s, $\cdots$ to represent propositions. For example, below are 4 propositions represented by p, q, r, and s respectively.

$$p:\ \sin^2 x + \cos^2 x = 1$$
$$q: \text{When } x \in \mathbb{R},\ x^2 \geq 0$$
$$r: 3\sin x = 4$$
$$s: \emptyset \in \{0\}$$

Among the propositions above, p and q are true propositions, while r and s are false propositions. The true or false of a proposition is called the **truth value** of the proposition. We stipulate that the truth value of a true proposition is 1, and the truth value of a false proposition is 0. For example, the propositions p and q above are true, denoted as $p = 1$ and $q = 1$, while the propositions r and s are false, denoted as $r = 0$ and $s = 0$.

Example 1 State whether the following sentences are propositions. If it is a proposition, state whether it is true or false and give the reason to your answer.

p : The square of any number is not less than zero.

q : The parabola $y = x^2 + 1$ has no point of intersection with the x-axis.

$r : x - y = 0$.

s : At least two interior angles of a triangle are acute angles.

t : For any real number x, $2x + 1 > x$.

Solution p is a true proposition. The square of any number is non-negative.

q is a true proposition. The parabola $y = x^2 + 1$ is on top of the x-axis and its vertex is $(0, 1)$.

r is not a proposition. For any x and y, we cannot tell whether $x - y$ is equal to 0.

s is a true proposition. The sum of the interior angles of a triangle is $180°$, there cannot be more two obtuse angle or straight angle at the same time.

t is a false proposition. For example, $2(-1) + 1 < -2$.

Example 2 Write down the truth value of the following propositions.

p : For any real number x, $x < x+1$.

q : For any real number a, if $a^3 > 0$, then $a > 0$.

r : The period of the function $y = \sin x$ is π.

s : The equation $2\sin x - \cos x = 4$ has no solution.

t : The line $3x - 4y + 1 = 0$ passes through the origin.

Solution $p = 1$ $q = 1$ $r = 0$ $s = 1$ $t = 0$

Exercise 11a

1. State whether the following sentences are propositions.

   p: The equation $x^2 - 5x + 6 = 0$ has two positive real roots.

   q: $x + 5 = y + 3$.

   r: THe line $y = 3x + b$ and the line $y = 3x - b$ ($b \neq 0$) are parallel to each other.

   s: $\triangle ABC \cong \triangle A'B'C'$.

   t: 5 is the greatest common factor of 25 and 30.

   u: The probability of a sure event is 1, and the probability of an impossible event is 0.

2. Write down the truth value of the following propositions, and give a counterexample for the false propositions.

   p: All the even numbers are not prime numbers.

   q: When $x \in \mathbb{R}$, $x^2 + x + 1$ is always greater than 0.

   r: The maximum value of the function $y = ax^2 + bx + c$ ($a \neq 0$) is $\dfrac{4ac - b^2}{4a}$.

   s: The equation $\sin x < \sin 2x$ is true for any real number x.

   t: The solution set of the equation $\sin x = \cos x$ is $\left\{x | x = \dfrac{\pi}{4} + 2k\pi, k \in \mathbb{Z}\right\}$.

   u: The value of the function $y = 2^x$ is always greater than 0, where x is any real number.

11.3 Compound Propositions

Lets consider the following propositions:

p: 4 is a factor of 8.

q: 4 is a factor of 12.

r: 4 is not a factor of 8.

s: 4 is a factor of 8 and is a factor of 12.

t: 4 is a factor of 8 or is a factor of 12.

Among these propositions, some of them are true propositions, while some of them are false propositions. In the proposition r, s, and t, they contain the words ”not”, ”and”, and ”or” respectively. These words are called **logical connectives**.

Propositions that do not contain any logical connectives are called **simple propositions**. For example, the propositions p and q above are simple propositions.

Propositions that are formed by connecting simple propositions using logical connectives are called **compound propositions**. For example, the propositions r, s, and t above are compound propositions.

Inverse Proposition and its Truth Table

Let p be a proposition. Adding a logical connective ”not” to p gives us a new proposition, that is, the inverse proposition of p, denoted by $\neg p$, read as ”not p”.

The meaning of the inverse proposition $\neg p$ is the **negation** of the original proposition p.

Example 3 Write down the inverse proposition of the following propositions:

p: $2 + 2 = 4$.

q: 25 is a multiple of 5.

r: The equation $y = 3x^3 - x$ is an odd function.

s: The square of 15 is 235.

t: The base number a of $\log_a x$ can be a negative number.

Solution $\neg p$: $2 + 2 \neq 4$.

$\neg q$: 25 is not a multiple of 5.

$\neg r$: The equation $y = 3x^3 - x$ is not an odd function.

$\neg s$: The square of 15 is not 235.

$\neg t$: The base number a of $\log_a x$ cannot be a negative number.

Apparently, if p is a true proposition, then $\neg p$ is a false proposition. If p is a false proposition, then $\neg p$ is a true proposition. The truth value of p and $\neg p$ are opposite to each other, and their truth table is as follows:

p	$\neg p$
1	0
0	1

This kind of table consisting of true and false that is used to analyse a compound proposition is called a **truth table**.

Example 4 Write down the inverse proposition of the following propositions:

s: All integers are positive numbers.

t: THe square root of 4 must be 2.

u: All equilateral triangles are isosceles triangles.

v: All even numbers are divisible by 3.

Solution $\neg s$: There exists an integer that is not a positive number.

$\neg t$: The square root of 4 is not necessary 2.

$\neg u$: There exists an equilateral triangle that is not an isosceles triangle.

$\neg v$: All even numbers are not divisible by 3.

NOTE: In the example above, proposition s is a false proposition, expressing $\neg s$ as "There exists an integer that is not a positive number" is not accurate. Therefore caution must be taken that either the proposition or its inverse proposition must be true, but not both, and they cannot be both false either.

Exercise 11b

1. Write down the inverse proposition of the following propositions:

   p: The sum of two sides of a triangle is greater than the third side.

   q: The smallest natural number is 1.

   r: π belongs to the set of irrational numbers.

   s: The equation $x^2 + 2x + 2 = 0$ has real roots.

   t: Two lines with equal gradient are not necessary parallel to each other.

   u: All the numbers with the last digit 3 are divisible by 3.

   v: Regular polygons have equal sides.

2. Write down the truth value of the propositions and their inverse propositions in Question 1. If the proposition is true, give a reason. If the proposition

Conjunctive Proposition and its Truth Table

Let p and q be two propositions. Adding a logical connective "and" to p and q gives us a new proposition, that is, the conjunctive proposition of p and q, denoted by $p \wedge q$, read as "p and q". In the field of Mathematics, $p \wedge q$ is also read as the **conjunction** of p and q.

For example p: The weather is clear today.

q: Today is a warm day.

$p \wedge q$: The weather is clear and warm today.

The definition of the conjunctive proposition $p \wedge q$ is that when p and q are both true, then $p \wedge q$ is true, otherwise $p \wedge q$ is false.

The truth table of $p \wedge q$ is as follows:

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Example 5 Write down the conjunctive proposition of the following pairs of propositions:

(a) p: 9 is an odd number.

q: 9 is a composite number.

(b) p: A trapezium has two parallel sides.

q: A trapezium has at least two equal sides.

(c) p: The equation $x^2 - 2x + 1 = 0$ has two distinct roots.

q: The two roots of the equation $x^2 - 2x + 1 = 0$ are both negative numbers.

(d) p: The probability of a sure event is 1.

q: The probability of an impossible event is 0.

Solution $p \wedge q$: 9 is an odd number and is a composite number.

$p \wedge q$: A trapezium has two parallel sides and has at least two equal sides.

$p \wedge q$: The equation $x^2 - 2x + 1 = 0$ has two distinct roots and the two roots are both negative numbers.

$p \wedge q$: The probability of a sure event is 1 and the probability of an impossible event is 0.

Example 6 Write down the truth value of the pairs of propositions and their conjunctive propositions in Question 5.

Solution $p = 1 \quad q = 1 \quad p \wedge q = 1$

$p = 1 \quad q = 0 \quad p \wedge q = 0$

$p = 0 \quad q = 0 \quad p \wedge q = 0$

$p = 0 \quad q = 0 \quad p \wedge q = 0$

Exercise 11c

1. Write down the conjunctive proposition of the following pairs of propositions:

   (a) p: Four sides of a square are equal.

   q: Four angles of a square are equal.

   (b) p: There exists the smallest element in the set of natural numbers.

   q: There exists the largest element in the set of natural numbers.

   (c) p: The sum of two sides of a triangle is greater than the third side.

   q: The difference between two sides of a triangle is less than the third side.

   (d) p: The sign of the two roots of the equation $x^2 - 9 = 0$ are different.

   q: The absolute values of the two roots of the equation $x^2 - 9 = 0$ are different.

   (e) p: The solution set of the inequality $x^2 - 4x + 5 > 0$ is a set of positive real numbers.

   q: The solution set of the inequality $x^2 - 4x + 5 < 0$ is $\emptyset$.

2. Write down the truth value of the pairs of propositions and their conjunctive propositions in Question 1.

3. Given that $p = 1$, $q = 0$, $r = 0$, and $s = 1$, write down the truth value of $(\neg p) \wedge (\neg q)$, $(\neg r) \wedge s$, and $(\neg q) \wedge (\neg r)$, $(\neg p) \wedge (\neg s)$.

Disjunctive Proposition and its Truth Table

Let p and q be two propositions. Adding a logical connective ”or” to p and q gives us a new proposition, that is, the disjunctive proposition of p and q, denoted by $p \vee q$, read as ”p or q”. In the field of Mathematics, $p \vee q$ is also read as the **disjunction** of p and q.

The definition of the disjunctive proposition $p \vee q$ is that when either p or q is true, then $p \vee q$ is true. The only case when $p \vee q$ is false is when p and q are both false.

The truth table of $p \vee q$ is as follows:

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

For example p: I will play basketball tomorrow.

q: I will go swimming tomorrow.

$p \vee q$: I will play basketball or go swimming tomorrow.

If I said to you that I will play basketball or go swimming tomorrow, and I actually were to do one of them, then $p \vee q$ is undoubtedly true. If I were to play basketball but not go swimming, then you can’t say I am lying to you, hence $p \vee q$ is still true. Only if I were to do neither of them, then you can say I am lying to you, hence $p \vee q$ is false. This matches what are being shown in the truth table.

NOTE: In our daily life, sometimes the word “or” might have different meaning than the one in the truth table. Take the word “or” in the restaurant menu as an example, it means that only one of the two choices can be chosen, “coffee or tea” means that if you choose coffee, you cannot choose tea, and vice versa. All the propositions that will be mentioned in the later part of this chapter are using the meaning of “or” in the truth table.

Example 7 Write down the disjunctive proposition of the following pairs of propositions:

(a) p: 2 is a natural number.
q: 2 is an even number.

(b) p: The equation $x^2 + 4x + 3 = 0$ has two distinct real roots.
q: The equation $x^2 + 4x + 3 = 0$ has two equal real roots.

(c) p: The function $y = x^2 + 4x + 3$ has a maximum value.
q: The function $y = x^2 + 4x + 3$ has a minimum value.

(d) p: The sum of interior angles of a quadrilateral is $180°$.
q: The sum of exterior angles of a quadrilateral is $540°$.

Solution $p \vee q$: 2 is either a natural number, or an even number.

$p \vee q$: The equation $x^2 + 4x + 3 = 0$ has either two distinct real roots, or two equal real roots.

$p \vee q$: The function $y = x^2 + 4x + 3$ has either a maximum value or a minimum value.

$p \vee q$: The sum of interior angles of a quadrilateral is either $180°$, or $540°$.

Example 8 Write down the truth value of the pairs of propositions and their disjunctive propositions in Question 7.

Solution $p = 1 \quad q = 1 \quad p \vee q = 1$

$p = 1 \quad q = 0 \quad p \vee q = 1$

$p = 0 \quad q = 1 \quad p \vee q = 1$

$p = 0 \quad q = 0 \quad p \vee q = 0$

Exercise 11d

1. Write down the disjunctive proposition of the following pairs of propositions:

   (a) p: The diagonals of a rhombus are perpendicular to each other.
   q: The diagonals of a rhombus bisect each other.

   (b) p: The diagonals of a rectangle are equal.
   q: The diagonals of a rectangle are perpendicular to each other.

   (c) p: The square of an even number is an odd number.
   q: The square of an even number is an even number.

2. Write down the truth value of the pairs of propositions and their conjunctive propositions in Question 1.

3. When $p \vee q = 0$, what is the truth value of $p \wedge q$? Explain your answer.

4. When $p \wedge q = 0$, can the truth value of $p \vee q$ be determined? Explain your answer.

11.4 Truth Table and Logical Equivalence

Truth Table

Regardless of the complexity of a proposition, its truth value in any case can always be calculated using a truth table.

Example 9 Construct a truth table for $\neg(p \wedge \neg q)$.

Solution

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
1	1	0	0	1
1	0	1	1	0
0	1	0	0	1
0	0	1	0	1

The steps of constructing the table are as follows:

(1) The propositions p and q has two possibilities each, so when combining them, there are $2 \times 2 = 4$ possibilities.

(2) Find the truth value of $\neg q$, it is the opposite of q.

(3) Find the conjunction of p and $\neg q$, i.e. $p \wedge \neg q$. According to the truth table of $p \wedge q$, when p and q are both true, $p \wedge q$ is true, otherwise $p \wedge q$ is false.

(4) Find the truth value of $\neg(p \wedge \neg q)$, it is the opposite of $p \wedge \neg q$.

Example 10 Construct a truth table for $(p \vee \neg q) \wedge q$.

Solution

p	q	$\neg q$	$p \vee \neg q$	$(p \vee \neg q) \wedge q$
1	1	0	1	1
1	0	1	1	0
0	1	0	0	0
0	0	1	1	0

Steps:

(1) Write down the combinations of the truth values of p and q.

(2) Find the truth value of $\neg q$.

(3) Find the truth value of $p \vee \neg q$. According to the truth table of $p \vee q$, When both p and q are false, $p \vee q$ is false, otherwise true.

(4) Find the truth value of $(p \vee \neg q) \wedge q$.

If a compound proposition is true in all cases, then this proposition is called a **tautology**. On the other hand, if a compound proposition is false in all cases, then this proposition is called a **contradiction**. For example,

$p \vee \neg p$ is a tautology, its truth table is as follows:

p	$\neg p$	$p \vee \neg p$
1	0	1
0	1	1

$p \wedge \neg p$ is a contradiction, its truth table is as follows:

p	$\neg p$	$p \wedge \neg p$
1	0	0
0	1	0

Exercise 11e

1. Construct a truth table for each of the following propositions:

   (a) $\neg p \wedge q$

   (b) $\neg(p \vee q)$

   (c) $\neg(p \vee \neg q)$

   (d) $(\neg p \vee q) \wedge p$

2. Determine whether the following propositions are tautologies or contradictions:

   (a) $p \wedge \neg q$

   (b) $p \vee \neg(p \wedge q)$

   (c) $(p \wedge q) \wedge \neg(p \vee q)$

   (d) $(p \wedge \neg p) \wedge q$

3. Construct a truth table for each of the following propositions:

   (a) $p \wedge (q \vee r)$

   (b) $(\mathrm{p} \wedge q) \vee (p \wedge r)$

Logical Equivalence

Let P and Q be two compound propositions, if P and Q have the same truth table, that is, when the same truth values are being chosen for the simple propositions in P and Q, P and Q have the same truth value, then P and Q are called **logically equivalent**, denoted by $P \equiv Q$.

For example, the truth table of $\neg(p \wedge q)$ and $\neg p \vee \neg q$ is as follows:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

From the truth table above, we can see that in all cases, $\neg(p \wedge q)$ and $\neg p \vee \neg q$ have the same truth value, hence $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent, i.e.

$$\boxed{\neg(p \wedge q) \equiv \neg p \vee \neg q}$$

The meaning of this expression is that the negation of the conjunction of p and q is logically equivalent to the disjunction of the negation of $\neg p$ and $\neg q$.

Similarly, we can list down the following truth table:

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

From the truth table above, we can see that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent, i.e.

$$\boxed{\neg(p \vee q) \equiv \neg p \wedge \neg q}$$

The meaning of this expression is that the negation of the disjunction of p and q is logically equivalent to the conjunction of the negation of $\neg p$ and $\neg q$.

The two formulas above are known as the **De Morgan's Laws**.

Example 11 Use the truth table to prove that $\neg(\neg p) \equiv p$.

Solution

p	$\neg p$
1	0
0	1

From the truth table above, we know that $\neg(\neg p) \equiv p$.

That means, negating the original proposition two times, we get the original proposition itself.

$\neg(\neg p) \equiv p$ is known as the **double negation law**.

Example 12 Find the inverse proposition of the following propositions:

(a) The opposite sides of a parallelogram are parallel and equal to each other.

(b) All the odd numbers are divisible by 3 or 5.

Solution (a) p: The opposite sides of a parallelogram are parallel to each other.

q: The opposite sides of a parallelogram are equal to each other.

$p \wedge q$: The opposite sides of a parallelogram are parallel and equal to each other.

The target inverse proposition is $\neg(p \wedge q)$.

And $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

$\neg p$: The opposite sides of a parallelogram are not parallel to each other.

$\neg q$: The opposite sides of a parallelogram are not equal to each other.

$\neg p \vee \neg q \equiv \neg(p \wedge q)$: The opposite sides of a parallelogram are not parallel to each other or are not equal to each other.

(b) p: All the odd numbers are divisible by 3.

q: All the odd numbers are divisible by 5.

$p \vee q$: All the odd numbers are divisible by 3 or 5.

And $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

$\neg p$: Not all the odd numbers are divisible by 3.

$\neg q$: Not all the odd numbers are divisible by 5.

$\therefore \neg p \wedge \neg q$: Not all the odd numbers are divisible by 3 and 5.

Example 13 Given that $p = 0$, $q = 1$, find the truth value of $\neg(p \wedge q)$ and $\neg(p \vee q)$.

Sol. 1 $\because p = 0, q = 1$

$\therefore p \wedge q = 0, p \vee q = 1$

$\therefore \neg(p \wedge q) = 1, \neg(p \vee q) = 0$

Sol. 2 $\because p = 0, q = 1$

$\therefore \neg p = 1, \neg q = 0$

$\neg(p \wedge q) \equiv \neg p \vee \neg q = 1$

$\neg(p \vee q) \equiv \neg p \wedge \neg q = 0$

Exercise 11f

1. Prove the following equivalences using truth tables:

   (a) (a) $p \vee q \equiv \sim (\sim p \wedge \sim q)$

   (b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

   (c) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

   (d) $\sim (p \vee \sim q) \equiv \sim p \wedge q$

2. Find the inverse proposition of the following propositions and their truth values:

(a) The three center lines of a triangle intersect at one point, and the three altitude lines of the triangle also intersects at this point.

(b) The odd number $2n$ is divisible by 2 or 4.

3. (a) Given $p = 0, q = 1$, find the truth values of $\sim (p \vee q)$ and $\sim (p \wedge q)$.

   (b) Given $p = 1, q = 0$, find the truth values of $\sim (p \vee q)$ and $\sim (p \wedge q)$.

   (c) Given $p = 1, q = 1$, find the truth values of $\sim (p \vee q)$ and $\sim (p \wedge q)$.

4. Simplify the following propositions using De Morgan's Laws and double negation law:

   (a) $\sim (\sim p \wedge q)$

   (b) $\sim (\sim p \vee \sim q)$

   (c) $\sim (\sim p \vee q)$

11.5 Implication

Let p and q be two propositions, another proposition can be formed using the form "if p then q", known as the **implication** of p and q, denoted by $p \to q$, read as "p implies q". For example,

p: $x = 3$

q: $x^2 = 9$

$p \to q$: If $x = 3$, then $x^2 = 9$.

In the implication $p \to q$, p is called the **hypothesis** or **antecedent**, and q is called the **conclusion** or **consequent**. The truth value of $p \to q$ is as follows:

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

In the field of Mathematics, deriving $p \to q = 1$ from $p = 1$ and $q = 0$ is the most common way to prove the truthfulness of a proposition.

When $p = 1$, $q = 0$, $p \to q = 0$. This case can be easily understood.

When $p = 0$, $q = 1$, $p \to q = 1$. This case can be understood as follows: Although the hypothesis p is not satisfied, the conclusion q still stands, hence, $p \to q = 1$. Take a look at this example: If it rains tonight, I will stay at home. Whe p is false but q is true, that means: It doesn't rain tonight, but I still stay at home. Since I only told that I will stay at home if it rains tonight, but I didn't mention anything about the case when it doesn't rain tonight, therefore this sentence is logically correct. Hence, when p is false and q is true, $p \to q$ is true.

When $p = 0$, $q = 0$, $p \to q = 1$. This case can be understood using the following example.

p : You can jump 100m.

q : I can jump 200m.

Obviously, the proposition p and q are both false propositions. However, the proposition formed by p and q seems to be a logical joke, i.e.

$p \to q$: If you can jump 100m, then I can jump 200m.

The reality is though, since you will never be able to jump 100m, I need not have to fulfil the matter of jumping 200m. Hence, $p \to q$ is true.

Example 14 According to each of the following pairs of p and q, write down the proposition $p \to q$. Hence, determine their truth values.

(a) p: $3^2 + 4^2 = 5^2$

q: The triangle with side lengths of 3, 4 and 5 respectively is a right-angled triangle.

(b) p: $\dfrac{1}{3}$ is a repeating decimal.

q: $\dfrac{1}{3}$ is an irrational number.

(c) p: $5 < 3$

q: $-5 < -3$

(d) p: $\sin^2 x + \cos^2 x = 2$

q: $\tan x \cot x = 2$

Solution (a) $p \to q$: If $3^2 + 4^2 = 5^2$, then the triangle with side lengths of 3, 4 and 5 respectively is a right-angled triangle.

$\because p = 1, q = 1$

$\therefore p \to q = 1$

(b) $p \to q$: If $\dfrac{1}{3}$ is a repeating decimal, then $\dfrac{1}{3}$ is an irrational number.

$\because p = 1, q = 0$

$\therefore p \to q = 0$

(c) $p \to q$: If $5 < 3$, then $-5 < -3$.

$\because p = 0, q = 1$

$\therefore p \to q = 1$

(d) $p \to q$: If $\sin^2 x + \cos^2 x = 2$, then $\tan x \cot x = 2$.

$\because p = 0, q = 0$

$\therefore p \to q = 1$

Example 15 Prove that $p \to q \equiv \neg p \vee q$.

Solution

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
1	1	1	0	1
1	0	0	0	0
0	1	1	1	1
0	0	1	1	1

From the truth table above, we can see that $p \to q \equiv \neg p \vee q$.

Four Forms of Implication

First let's take a look at an example.

$$\text{Let } p: \ x = 3$$
$$q: \ x^2 = 9$$

From p and q, we can form the following four propositions:

- $p \to q$: If $x = 3$, then $x^2 = 9$ (true)
- $q \to p$: If $x^2 = 9$, then $x = 3$. (false)
- $\neg p \to \neg q$: If $x \neq 3$, then $x^2 \neq 9$. (false)
- $\neg q \to \neg p$: If $x^2 \neq 9$, then $x \neq 3$. (true)

We call $q \to p$ the **converse proposition** of $p \to q$, $\neg p \to \neg q$ the **inverse proposition** of $p \to q$ (note that it is different from the inverse proposition in section 11.2), and $\neg q \to \neg p$ the **contrapositive proposition** of $p \to q$.

The truth table of $p \to q$, and its contrapositive proposition is as follows:

p	q	$p \to q$	$\neg q \to \neg p$
1	1	1	1
1	0	0	0
0	1	1	1
0	0	1	1

From the table above, we can see that $p \to q$ and $\neg q \to \neg p$ have the same truth value, i.e.

$$\boxed{p \to q \equiv \neg q \to \neg p}$$

That is to say, switching the hypothesis and conclusion of an implication while negating them at the same time, the truth value of the implication will not change. This is the origin of the technique of proving by contradiction, as saying goes, "If you want to prove a proposition, you can try to prove its contrapositive proposition instead."

Example 16 Write down four types of implications of the following pairs of p and q. Hence, determine their truth values.

(a) p: $x = y$
q: $x^2 = y^2$

(b) p: In $\triangle ABC$, $\angle B = \angle C$.
q: In $\triangle ABC$, $AB = AC$.

Solution (a) $p \to q$: If $x = y$, then $x^2 = y^2$.
$p \to q = 1$
$q \to p$: If $x^2 = y^2$, then $x = y$.
$q \to p = 0$
$\neg p \to \neg q$: If $x \neq y$, then $x^2 \neq y^2$.
$\neg p \to \neg q = 0$
$\neg q \to \neg p$: If $x^2 \neq y^2$, then $x \neq y$.
$\neg q \to \neg p = 1$

(b) $p \to q$: In $\triangle ABC$, if $\angle B = \angle C$, then $AB = AC$.

$p \to q = 1$

$q \to p$: In $\triangle ABC$, if $AB = AC$, then $\angle B = \angle C$.

$q \to p = 1$

$\neg p \to \neg q$: In $\triangle ABC$, if $\angle B \neq \angle C$, then $AB \neq AC$.

$\neg p \to \neg q = 1$

$\neg q \to \neg p$: In $\triangle ABC$, if $AB \neq AC$, then $\angle B \neq \angle C$.

$\neg q \to \neg p = 1$

Exercise 11g

1. According to the following pairs of p and q, write down the proposition $p \to q$. Hence, determine its truth value.

   (a) p: The sum of two sides of a triangle is larger than the third side.
   q: The difference between two sides of a triangle is smaller than the third side.

   (b) p: $a^2 + b^2 \leq 2ab$
   q: $a > 0, b > 0$

   (c) p: Any exterior angle of a triangle is obtuse angle.
   q: Any exterior angle is equal to the sum of the two interior opposite angles.

   (d) p: For any real number x, $|x| < x$ is true.
   q: For any real number x, $|x| > 0$ is true.

2. According to the following pairs of p and q, write down four types of implications formed by p and q. Hence, determine their truth values.

   (a) p: All sides of $\triangle ABC$ are equal.
   q: All interior angles of $\triangle ABC$ are equal.

   (b) p: Quadrilateral $ABCD$ is a square.
   q: All four sides of quadrilateral $ABCD$ are equal.

   (c) p: For all real numbers a, b, c, there exists $ac > bc$.
   q: For all real numbers a, b, there exists $a > b$.

   (d) p: $\sin \alpha = 1$
   q: $\alpha = \dfrac{\pi}{2}$

3. For the proposition "opposite angles are equal", it can be written as the implication "if two angles are opposite angles, then they are equal". Take this implication as an example, try to rewrite the following proposition in the form of implication:

   (a) The sum of the interior angles of a triangle is $180°$.

   (b) The absolute value of a positive number is equal to itself.

   (c) The corresponding angles of two similar polygons are equal.

4. Find the truth values of the following propositions:

   (a) $(p \to q) \to (p \wedge q)$

(b) $\neg p \to (q \to p)$

5. Prove that $(p \wedge q) \to (p \vee q)$ is a tautology.

6. Prove that $p \to (q \wedge r) \equiv (p \to q) \wedge (p \to r)$.

7. If $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$, construct a truth table for $p \leftrightarrow q$.

11.6 Arguments

Arguments

An **argument** is a thinking process of using some existing judgements (established propositions) $P_1, P_2, \cdots, P_n$ to derive a new judgement Q. Here, these existing judgements are called the **premises** of the argument, and the new judgement is called the **conclusion** of the argument, denoted by $P_1, P_2, \cdots, P_n;\ \therefore Q$.

Example 17 (a) If it is a parallelogram, then its diagonals bisect each other.
Rectangle is a parallelogram.
Therefore the diagonals of rectangle bisect each other.

(b) Going to the cinema or going to the park on Sunday.
Going to the cinema on Sunday.
Therefore not going to the park on Sunday.

(c) Committing crime is an act that damages the society.
Committing crime is an act that is against the law.
Therefore committing crime is an act that damages the society and is against the law.

The examples above are all judgements (conclusion) derived from a set of existing judgements (premises), albeit different in content. Hence, they are all arguments. These arguments can be expressed in logical symbols 'p', 'q', '$\wedge$', '$\vee$', '$\to$', etc.

Take Example 17(a) as an example, let p be the proposition "It is a parallelogram.", q be the proposition "Its diagonals bisect each other.", then the argument can be expressed as follows:

$$\begin{array}{c} p \to q \\ p \\ \hline \therefore q \\ \hline \end{array}$$

The argument that is expressed in logical symbols are called **formal arguments**. The argument above can also be written as $p \to q, p;\ \therefore q$.

Now take a look at Example 17(b), if

p : Going to the cinema on Sunday.

q : Going to the park on Sunday.

then the argument can be expressed as follows:

$$\begin{array}{c} p \vee q \\ p \\ \hline \therefore \neg q \\ \hline \end{array}$$

denoted as $p \vee q, p;\ \therefore \neg q$.

for Example 17(c), if

p : Committing crime is an act that damages the society.

q : Committing crime is an act that is against the law.

then the argument can be expressed as follows:

$$\begin{array}{c} p \\ q \\ \hline \therefore\ p \wedge q \\ \hline \end{array}$$

denoted as $p, q;\ \therefore p \wedge q$.

Validity of Arguments

When all the premises of an argument $P_1, P_2, \cdots, P_n$ are true, and the conclusion Q is also true, then we say that the argument is a **valid argument**.

Example 18 Check the validity of the argument $p \to q;\ \therefore q$.

Solution Truth table constructed for premises and conclusion is as follows:

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

where in the first line, two premises p and q as well as the conclusion $p \to q$ are all true. Hence, the argument is valid.

Example 19 Check the validity of the argument $p \vee q, p;\ \therefore \neg q$.

Solution

p	q	$p \vee q$	$\neg q$
1	1	1	0
1	0	1	1
0	1	1	0
0	0	0	1

where in the first and the second lines, two premises p and q are both true, but the conclusion $\neg q$ is false in the first line. Hence, the argument is invalid.

Due to the fact that when compound proposition $P_1 \wedge P_2 \wedge \cdots \wedge P_n$ is true, every term P_1, P_2, $\cdots$, P_n must also be true, therefore, the argument $P_1, P_2, \cdots, P_n;\ \therefore Q$ is valid if and only if $P_1 \wedge P_2 \wedge \cdots \wedge P_n \to Q$ is true. In other words, when $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \to Q$ is a tautology, the argument $P_1, P_2, \cdots, P_n;\ \therefore Q$ is valid.

Example 20 Check the validity of the argument $p \to q, q \to r;\ \therefore p \to r$.

Solution Construct the truth table of $[(p \to q) \wedge (q \to r)] \to (p \to r)$ as follows:

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
1	1	1	1	1	1	1	1
1	1	0	1	0	0	0	1
1	0	1	0	1	0	1	1
1	0	0	0	1	0	0	1
0	1	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	0	1	1	1	1	1	1
0	0	0	1	1	1	1	1

Since $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology,

$\therefore$ the argument $p \to q, q \to r;\ \therefore p \to r$ is valid.

NOTE: $p \to q, q \to r;\ \therefore p \to r$ is called the **law of syllogism**.

Example 21 Check the validity of the argument $p \to q, \neg p;\ \therefore \neg q$.

Solution Construct a truth table for $[(p \to q) \wedge \neg p] \to \neg q$:

p	q	$\neg p$	$p \to q$	$\neg q$	$(p \to q) \wedge \neg p$	$[(p \to q) \wedge \neg p] \to \neg q$
1	1	0	1	0	0	1
1	0	0	0	1	0	1
0	1	1	1	0	1	0
0	0	1	1	1	1	1

Since $[(p \to q) \wedge \neg p] \to \neg q$ is not a tautology,

$\therefore$ the argument $p \to q, \neg p;\ \therefore \neg q$ is invalid.

Now take a look at this example,

If earth has wings, then it can fly.

Earth has wings.

$\therefore$ Earth can fly.

Its formal argument is $p \to q, p;\ \therefore\ q$ which is a valid argument. However, the premises "Earth has wings." and the conclusion "Earth can fly" are both false. Therefore, a valid argument does not necessarily mean that the conclusion is true. In order to construct an argument with a true conclusion, the following criteria must be met:

1. It is a valid argument. That is, the premises and the conclusion are connected together by a valid logical relationship.

2. All the premises are true.

Considering the fact that the validity of premises must be resolved by other fields of knowledge, therefore, traditional and modern studies of logic only focus on the validity of arguments.

Example 22 Check the validity of the following arguments:

(a) If two angles are corresponding angles, then they are equal.
This two angles are equal.
Therefore this two angles are corresponding angles.

(b) This student is interested in entertainment or sports.
This student is not interested in entertainment.
Therefore this student is interested in sports.

Solution (a) Use p, q to express "This two angles are corresponding angles.", "This two angles are equal." respectively. Then the argument can be expressed as follows:

$$\begin{array}{c} p \to q \\ q \\ \hline \therefore p \\ \hline \end{array}$$

Construct a truth table for $[(p \to q) \wedge q] \to p$:

p	q	$p \to q$	q	$(p \to q) \wedge q$	$[(p \to q) \wedge q] \to p$
1	1	1	1	1	1
1	0	0	0	0	1
0	1	1	1	1	0
0	0	1	0	0	1

Since $[(p \to q) \wedge q] \to p$ is not a tautology, the argument is invalid.

(b) Use p, q to express "This student is interested in entertainment.", "This student is interested in sports." respectively.

Then the argument can be expressed as follows:

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline \therefore q \\ \hline \end{array}$$

Construct a truth table for $[(p \vee q) \wedge \neg p] \to q$:

p	q	$\neg p$	$p \vee q$	$(p \vee q) \wedge \neg p$	$[(p \vee q) \wedge \neg p] \to q$
1	1	0	1	0	1
1	0	0	1	0	1
0	1	1	1	1	1
0	0	1	0	0	1

Since $[(p \vee q) \wedge \neg p] \to q$ is a tautology, the argument is valid.

Exercise 11h

1. Check the validity of the following arguments:

   (a) $p \to q, \sim q;\ \therefore \sim p$

   (b) $\sim p \to q, p;\ \therefore \sim q$

2. Prove that the following arguments are valid:

   (a) $p \to \sim q, r \to q, r; \quad \therefore \sim p$

   (b) $\sim p \to \sim q, q; \therefore \sim p$

3. Determine whether the following arguments are valid:

   (a) If it is a square, then its diagonals bisect each other.
   It is a square.
   Therefore its diagonals bisect each other.

   (b) If it is a square, then its diagonals bisect each other.
   Its diagonals bisect each other.
   Therefore it is a square.

   (c) If it is a square, then its diagonals bisect each other.
   It is not a square.
   Therefore its diagonals do not bisect each other.

   (d) If it is a square, then its diagonals bisect each other.
   Its diagonals do not bisect each other.
   Therefore it is not a square.

4. Determine the validity of the following arguments:

(a) If 6 is not an even number, then 5 is not a prime number.

6 is an even number.

Therefore 5 is a prime number.

(b) If being infected by pneumonia. then having fever.

He is not infected by pneumonia.

Therefore he does not have fever.

(c) If a number can be divided by 6, then the number can be divided by 2.

The number can be divided by 6.

Therefore the number can be divided by 2.

(d) If you are a good student, then you will not be absence from class.

If you are a good student, then you will not fight with others.

You are absence from class or you fight with others.

Therefore you are not a good student.

5. Determine the validity of the following arguments. Hence, state whether the conclusion of each valid arguments is true:

(a) If $x = 4$, then $x^2 = 16$.

$x = 4$.

Therefore $x^2 = 16$.

(b) If 7 is less than 4, then 7 is not a prime number.

7 is not less than 4.

Therefore 7 is a prime number.

(c) If 5 is a prime number, then 5 cannot divide 15.

5 can divide 15.

Therefore 5 is not a prime number.