Notes for Calculus III

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Contents

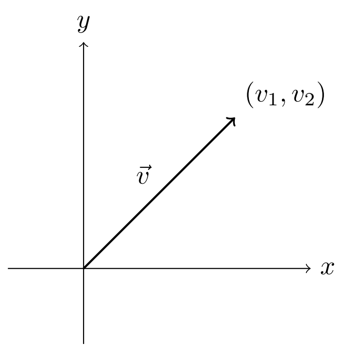
Chapter 1

Vector in Plane Geometry

If $\vec{v}$ is a vector whose initial point is $(0,0)$ and terminal point is (v_1, v_2), then

$$\vec{v} = \langle v_1, v_2 \rangle$$

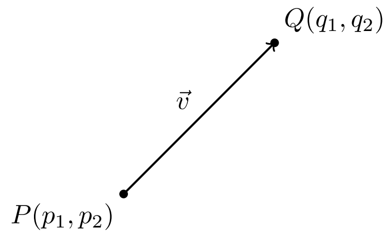
is the **component form** of $\vec{v}$. Here is the graph of the vector $\vec{v}$ in the cartesian plane.



Since the initial point is $(0,0)$, we say that $\vec{v}$ is in **standard position**.

> **Note:**
> The vector with initial point and terminal point $(0,0)$ is called the **zero vector** and is denoted by $\vec{0}$.

Consider



where P is the initial point and Q is the terminal point of the vector $\vec{v}$. $\vec{v}$ can be calculated by subtracting the coordinates of the terminal point from the coordinates of the initial point. That is,

$$\vec{v} = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle v_1, v_2 \rangle$$

Length / Norm / Magnitude of a Vector

The **length** or **norm** or **magnitude** of a vector $\vec{v}$ is denoted by $\|\vec{v}\|$ and is given by

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$$

If $\|\vec{v}\| = 1$, then $\vec{v}$ is called a **unit vector**.

Example 1. Find the length of the vector $\vec{v} = \langle 1, 2 \rangle$.

$$\begin{aligned} \|\vec{v}\| &= \sqrt{v_1^2 + v_2^2} \\ &= \sqrt{1^2 + 2^2} \\ &= \sqrt{5} \end{aligned}$$

Example 2. Calculate the component form of the vector that starts with the point $P(1, 2)$ and ends with the point $Q(5, 4)$.

$$\begin{aligned} \vec{v} &= \langle q_1 - p_1, q_2 - p_2 \rangle \\ &= \langle 5 - 1, 4 - 2 \rangle \\ &= \langle 4, 2 \rangle \end{aligned}$$

By calculating the component form of the vector, we are basically just translating the vector to the origin. Hence, we can conclude that the component form of the vector is the vector that starts from the origin with the same direction and magnitude as the original vector.

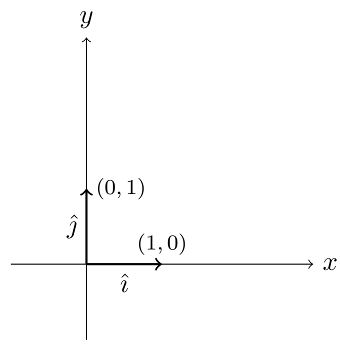
If two vectors of different initial and terminal points has the same component form, then the two vectors are said to be **equivalent**.

Unit Vectors

There are two unit vectors that are commonly used in the cartesian plane. They are the **standard unit vectors** $\hat{\imath}$ and $\hat{\jmath}$. That is,

$$\hat{\imath} = \langle 1, 0\rangle \text{ and } \hat{\jmath} = \langle 0, 1\rangle$$

Below is a graph of the standard unit vectors.



Given a vector $\vec{v} = \langle v_1, v_2\rangle$, we can split it into $\langle v_1, 0\rangle + \langle 0, v_2\rangle$. Then, factor out the scalars to get $v_1\langle 1, 0\rangle + v_2\langle 0, 1\rangle$. Hence, we can conclude that

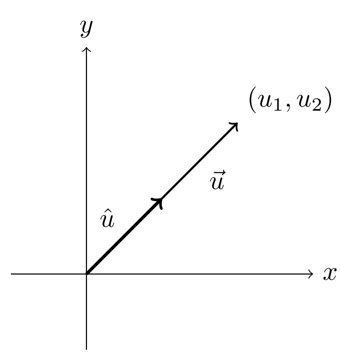
$$\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath}$$

where $\vec{v}$ known as the **linear combination** of $\hat{\imath}$ and $\hat{\jmath}$. The scalars v_1 and v_2 are known as the **horizontal component** and **vertical component** of $\vec{v}$ respectively.

Sometimes when we only care about the direction of the vector, we can convert any vector into a unit vector by dividing the vector by its length. That is,

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} \text{ where } \|\hat{u}\| = 1$$

A visual representation of this is shown below.



Example 3. Find the unit vector in the direction of $\vec{v} = \langle 1, 2 \rangle$.

$$\begin{aligned} \hat{v} &= \frac{\vec{v}}{\|\vec{v}\|} \\ &= \frac{\langle 1, 2 \rangle}{\sqrt{5}} \\ &= \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle \end{aligned}$$

> **Note:**
> If you are asked to normalize a vector, you are asked to find the unit vector in the direction of the vector.

Example 4. Find the magnitude of the vector $\vec{v} = 2\hat{\imath} + 3\hat{\jmath}$.

$$\begin{aligned} \|\vec{v}\| &= \sqrt{v_1^2 + v_2^2} \\ &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \end{aligned}$$

Example 5. Find the vector $\vec{u}$ with magnitude 4 and same direction as $\vec{v} = \langle 0, 3 \rangle$.

First, we find the unit vector in the direction of $\vec{v}$.

$$\begin{aligned} \hat{v} &= \frac{\vec{v}}{\|\vec{v}\|} \\ &= \frac{\langle 0, 3 \rangle}{\sqrt{0^2 + 3^2}} \\ &= \langle 0, 1 \rangle \end{aligned}$$

Then, we multiply the unit vector by the magnitude of the target vector.

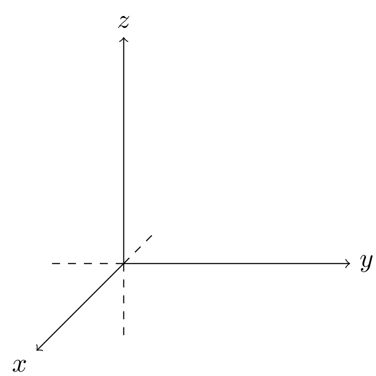
$$\begin{aligned} \vec{u} &= 4\hat{v} \\ &= 4\langle 0, 1 \rangle \\ &= \langle 0, 4 \rangle \end{aligned}$$

Chapter 2

Vector in Space

Space Coordinates

The cartesian plane that we are used to is a two-dimensional plane. However, if we extend the dimension further into the third dimension, we get the **three-dimensional space**, which is also known as **Euclidean space**. Below is a graph of the three-dimensional space.



A point in the three-dimensional space is represented by an ordered triple (x, y, z), where x is the horizontal component, y is the vertical component and z is the depth component.

Example 1. Find the magnitude of the vector $\vec{v} = \langle 3, -2, 1 \rangle$.

$$\begin{aligned} \|\vec{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ &= \sqrt{3^2 + (-2)^2 + 1^2} \\ &= \sqrt{14} \end{aligned}$$

Example 2. Find the magnitude of the vector $\vec{v} = -2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$.

$$\begin{aligned}
\|\vec{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\
&= \sqrt{(-2)^2 + 3^2 + 4^2} \\
&= \sqrt{29}
\end{aligned}$$

To find the magnitude of a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$, we can use the formula

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Note:
For vector in any dimension, the magnitude of the vector is given by

$$\|\vec{v}\| = \sqrt{\sum_{i=1}^{n} v_i^2}$$

where n is the dimension of the vector.

Example 3. Find the vector $\vec{u}$ with magnitude 6 and same direction as $\vec{v} = \langle -6, 4, 0 \rangle$.

$$\begin{aligned}
\|\vec{v}\| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\
&= \sqrt{(-6)^2 + 4^2 + 0^2} \\
&= \sqrt{52} \\
&= 2\sqrt{13}
\end{aligned}$$

$$\begin{aligned}
\hat{v} &= \frac{\vec{v}}{\|\vec{v}\|} \\
&= \frac{\langle -6, 4, 0 \rangle}{2\sqrt{13}} \\
&= \left\langle -\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}, 0 \right\rangle
\end{aligned}$$

$$\begin{aligned}
\vec{u} &= 6\hat{v} \\
&= \left\langle -\frac{18}{\sqrt{13}}, \frac{12}{\sqrt{13}}, 0 \right\rangle
\end{aligned}$$

Chapter 3

Dot Product

Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{u} = \langle u_1, u_2, u_3 \rangle$ be two vectors in $\mathbb{R}^3$ (three-dimensional space), then the dot product of $\vec{v}$ and $\vec{u}$ is given by

$$\vec{v} \cdot \vec{u} = v_1 u_1 + v_2 u_2 + v_3 u_3$$

The final result of the dot product is a number, hence the name **scalar product**. The dot product is also known as the **inner product** of two vectors.

Example 1. Calculate the dot product of the vectors $\vec{v} = \langle 1, 2 \rangle$ and $\vec{u} = \langle -2, 1 \rangle$.

$$\begin{aligned} \vec{v} \cdot \vec{u} &= v_1 u_1 + v_2 u_2 \\ &= 1(-2) + 2(1) \\ &= 0 \end{aligned}$$

Note:
If $\vec{v} \cdot \vec{u} = 0$, then $\vec{v}$ and $\vec{u}$ are perpendicular to each other, or they are **orthogonal**. That is,

$$\vec{v} \cdot \vec{u} = 0 \Longleftrightarrow \vec{v} \perp \vec{u}$$

Properties of Dot Product

1. $\vec{v} \cdot \vec{u} = \vec{u} \cdot \vec{v}$

2. $\vec{v} \cdot (\vec{u} + \vec{w}) = \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{w}$

3. $(c\vec{v}) \cdot \vec{u} = c(\vec{v} \cdot \vec{u})$, where c is a scalar

4. $\vec{0} \cdot \vec{v} = 0$, where $\vec{0} = \langle 0, 0 \rangle$

5. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

6. The angle θ between two vectors $\vec{v}$ and $\vec{u}$ is given by

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} \qquad \text{where} \qquad 0 \le \theta \le \pi$$

   From this, we can easily derive that

7. $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$

Example 1. Let $\vec{v} = \langle 1, 2 \rangle$, $\vec{u} = \langle 3, 4 \rangle$. Calculate $\vec{u} \cdot (5\vec{v})$.

$$\begin{aligned} \vec{u} \cdot (5\vec{v}) &= 5(\vec{u} \cdot \vec{v}) \\ &= 5(1 \cdot 3 + 2 \cdot 4) \\ &= 55 \end{aligned}$$

Example 2. Prove property 5.

Let $\vec{v} = \langle v_1, v_2 \rangle$, then

$$\begin{aligned} \vec{v} \cdot \vec{v} &= \langle v_1, v_2 \rangle \cdot \langle v_1, v_2 \rangle \\ &= v_1 v_1 + v_2 v_2 \\ &= v_1^2 + v_2^2 \\ &= \left(\sqrt{v_1^2 + v_2^2}\right)^2 \\ &= \|\vec{v}\|^2 \qquad \blacksquare \end{aligned}$$

Example 3. Calculate the dot product of the vectors $\vec{v} = \langle 1, 2, 4 \rangle$ and $\vec{u} = \langle -3, -2, 0 \rangle$.

$$\begin{aligned} \vec{v} \cdot \vec{v} &= \langle 1, 2, 4 \rangle \cdot \langle -3, -2, 0 \rangle \\ &= 1(-3) + 2(-2) + 4(0) \\ &= -7 \end{aligned}$$

Example 4. Calculate the dot product of the vectors $\vec{v} = 3i + 2j + 4k$ and $\vec{u} = -i + j$.

$$\begin{aligned}
\vec{v} \cdot \vec{u} &= (3i + 2j + 4k) \cdot (-i + j) \\
&= 3(-1) + 2(1) + 4(0) \\
&= -1
\end{aligned}$$

Example 5. Determine whether the following vectors are orthogonal.

$$\vec{v} = -2i + 3j + k \qquad \vec{u} = 2i + j + k$$

$$\begin{aligned}
\vec{v} \cdot \vec{u} &= (-2i + 3j + k) \cdot (2i + j + k) \\
&= -2(2) + 3(1) + 1(1) \\
&= 0
\end{aligned}$$

Since $\vec{v} \cdot \vec{u} = 0$, $\vec{v}$ and $\vec{u}$ are orthogonal.

Example 6. Find the angle between the vectors $\vec{v} = \langle 3, 1 \rangle$ and $\vec{u} = \langle 2, -1 \rangle$.

$$\begin{aligned}
\cos\theta &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} \\
&= \frac{3(2) + 1(-1)}{\sqrt{3^2 + 1^2}\sqrt{2^2 + (-1)^2}} \\
&= \frac{5}{\sqrt{10}\sqrt{5}} \\
&= \frac{1}{\sqrt{2}} \\
\theta &= \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \\
&= \frac{\pi}{4}
\end{aligned}$$

Example 7. Find the angle between the vectors $\vec{v} = i - 3j + k$ and $\vec{u} = 2i + 4j - k$.

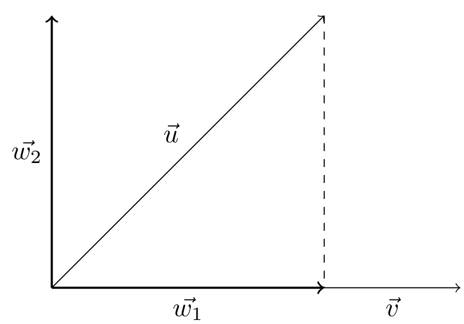
$$\begin{aligned}
\cos\theta &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} \\
&= \frac{1(2) + (-3)(4) + 1(-1)}{\sqrt{1^2 + (-3)^2 + 1^2}\sqrt{2^2 + 4^2 + (-1)^2}} \\
&= \frac{-11}{\sqrt{231}} \\
\theta &= 2.38 \text{ rad}
\end{aligned}$$

Chapter 4

Projections

Given two vectors $\vec{v}$ and $\vec{u}$. Construct a line from the terminal point of $\vec{u}$ perpendicular to $\vec{v}$. The vector that starts from the initial point of $\vec{u}$ and ends at the intersection of the line and $\vec{v}$ is called the **projection of $\vec{u}$ onto $\vec{v}$**, which is also known as the **vector component of $\vec{u}$ along $\vec{v}$**.

Construct another vector from the initial point of $\vec{u}$ that is orthogonal to $\vec{w_1}$, and the projection of $\vec{u}$ onto that vector is called the **vector component of $\vec{u}$ orthogonal to $\vec{v}$**. The visual representation is shown below.



From the diagram, it is not hard to see that

$$\vec{u} = \vec{w_1} + \vec{w_2}$$

Hence, the vector component of $\vec{u}$ orthogonal to $\vec{v}$ is given by

$$\vec{w_2} = \vec{u} - \vec{w_1}$$

The projection of $\vec{u}$ onto $\vec{v}$ is given by

$$\vec{w_1} = proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right)\vec{v}$$

Example 1. Find the projection of $\vec{u} = \langle 6, 7 \rangle$ onto $\vec{v} = \langle 1, 4 \rangle$. Hence, find the vector component of $\vec{u}$ orthogonal to $\vec{v}$.

$$\begin{aligned} proj_{\vec{v}}\vec{u} &= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v} \\ &= \left(\frac{6(1)+7(4)}{1^2+4^2}\right)\langle 1,4\rangle \\ &= \left(\frac{34}{17}\right)\langle 1,4\rangle \\ &= \langle 2,8\rangle \end{aligned}$$

$$\begin{aligned} \vec{w_2} &= \vec{u}-\vec{w_1} \\ &= \langle 6,7\rangle - \langle 2,8\rangle \\ &= \langle 4,-1\rangle \end{aligned}$$

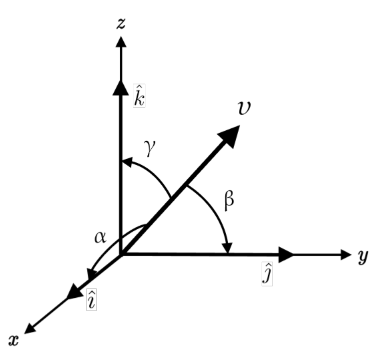
Example 2. Find the projection of $\vec{u} = 2i+3j$ onto $\vec{v} = 5i+j$. Hence, find the vector component of $\vec{u}$ orthogonal to $\vec{v}$.

$$\begin{aligned} proj_{\vec{v}}\vec{u} &= \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v} \\ &= \left(\frac{2(5)+3(1)}{5^2+1^2}\right)(5i+j) \\ &= \left(\frac{13}{26}\right)(5i+j) \\ &= \left(\frac{5}{2}\right)i+\left(\frac{1}{2}\right)j \end{aligned}$$

$$\begin{aligned} \vec{w_2} &= \vec{u}-\vec{w_1} \\ &= (2i+3j)-\left(\left(\frac{5}{2}\right)i+\left(\frac{1}{2}\right)j\right) \\ &= \left(-\frac{1}{2}\right)i+\left(\frac{5}{2}\right)j \end{aligned}$$

Chapter 5

Direction Cosines and Direction Angles



Direction angles are angles that a vector makes with the unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, denoted by α, β and γ respectively. In other words, *alpha*, β and γ are the angles between the vector and the x, y and z axis respectively.

Recall the formula for the dot product of two vectors $\vec{v}$ and $\vec{u}$

$$\vec{v} \cdot \vec{u} = \|\vec{v}\|\|\vec{u}\| \cos\theta$$

where θ is the angle between the two vectors.

Let's calculate the dot product of $\vec{v}$ and $\hat{\imath}$.

$$\begin{aligned} \vec{v} \cdot \hat{\imath} &= \|\vec{v}\|\|\hat{\imath}\| \cos\alpha \\ \langle v_1, v_2, v_3 \rangle \cdot \langle 1, 0, 0 \rangle &= \|\vec{v}\| \times 1 \times \cos\alpha \\ v_1 &= \|\vec{v}\| \cos\alpha \\ \cos\alpha &= \frac{v_1}{\|\vec{v}\|} \end{aligned}$$

Similarly, we can calculate the dot product of $\vec{v}$ and $\hat{\jmath}$ and $\hat{k}$ in the same way. Hence, we can conclude that

$$\cos\alpha = \frac{v_1}{\|\vec{v}\|} \qquad \cos\beta = \frac{v_2}{\|\vec{v}\|} \qquad \cos\gamma = \frac{v_3}{\|\vec{v}\|}$$

These are called the **direction cosines** of $\vec{v}$.

We can also express any unit vector $\hat{v}$ in terms of its direction cosines.

$$\begin{aligned}
\vec{v} &= v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k} \\
\frac{\vec{v}}{\|\vec{v}\|} &= \frac{v_1}{\|\vec{v}\|}\hat{\imath} + \frac{v_2}{\|\vec{v}\|}\hat{\jmath} + \frac{v_3}{\|\vec{v}\|}\hat{k} \\
\hat{v} &= \cos\alpha\hat{\imath} + \cos\beta\hat{\jmath} + \cos\gamma\hat{k}
\end{aligned}$$

If we take the magnitude of $\hat{v}$, we get

$$\|\hat{v}\| = \sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma} = 1$$

Squaring both sides, we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Example 1. Find the direction cosines and direction angles of the vector $\vec{u} = \hat{\imath} + 8\hat{\jmath} + 4\hat{k}$.

$$\begin{aligned}
\|\vec{u}\| &= \sqrt{1^2 + 8^2 + 4^2} \\
&= \sqrt{81} \\
&= 9
\end{aligned}$$

$$\begin{aligned}
\cos\alpha &= \frac{v_1}{\|\vec{v}\|} = \frac{1}{9} & \qquad \alpha &= \cos^{-1}\left(\frac{1}{9}\right) \approx 1.459 \text{ rad} \\
\cos\beta &= \frac{v_2}{\|\vec{v}\|} = \frac{8}{9} & \qquad \beta &= \cos^{-1}\left(\frac{8}{9}\right) \approx 0.476 \text{ rad} \\
\cos\gamma &= \frac{v_3}{\|\vec{v}\|} = \frac{4}{9} & \qquad \gamma &= \cos^{-1}\left(\frac{4}{9}\right) \approx 1.110 \text{ rad}
\end{aligned}$$

Chapter 6

Cross Product

Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{u} = \langle u_1, u_2, u_3 \rangle$ be two vectors in $\mathbb{R}^3$ (three-dimensional space), then the cross product of $\vec{v}$ and $\vec{u}$ is given by

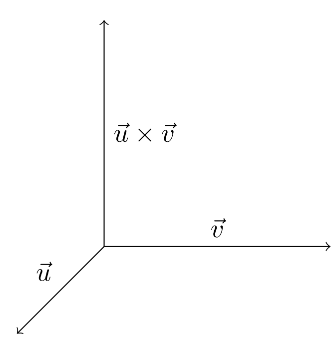
$$\vec{v} \times \vec{u} = (v_2u_3 - v_3u_2)\hat{\imath} + (v_3u_1 - v_1u_3)\hat{\jmath} + (v_1u_2 - v_2u_1)\hat{k}$$

The final result of the cross product is a vector, hence the name **vector product**.

It can also be calculated using the determinant of a 3×3 matrix as shown below.

$$\begin{aligned} \vec{v} \times \vec{u} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \\ &= \begin{vmatrix} v_2 & v_3 \\ u_2 & u_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} v_1 & v_3 \\ u_1 & u_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} v_1 & v_2 \\ u_1 & u_2 \end{vmatrix} \hat{k} \\ &= (v_2u_3 - v_3u_2)\hat{\imath} - (v_1u_3 - v_3u_1)\hat{\jmath} + (v_1u_2 - v_2u_1)\hat{k} \end{aligned}$$

Geometrically speaking, the cross product of two vectors $\vec{v}$ and $\vec{u}$ is a vector that is orthogonal to both $\vec{v}$ and $\vec{u}$, and its direction is given by the right-hand rule. That is, $\vec{v} \times \vec{u} \neq \vec{u} \times \vec{v}$.



Another property of the cross product is that the magnitude of the cross product of two vectors $\vec{v}$ and $\vec{u}$ is given by

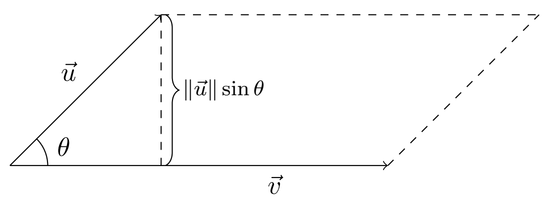
$$\|\vec{v} \times \vec{u}\| = \|\vec{v}\|\|\vec{u}\| \sin\theta$$

where θ is the angle between the two vectors.

The magnitude of the cross product of two vectors $\vec{v}$ and $\vec{u}$ is the area of the parallelogram formed by $\vec{v}$ and $\vec{u}$.

Proof. Note that $\cos\theta = \dfrac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|\|\vec{u}\|}$, hence

$$\begin{aligned}
\|\vec{v}\times\vec{u}\| &= \|\vec{v}\|\|\vec{u}\|\sin\theta \\
&= \|\vec{v}\|\|\vec{u}\|\sqrt{1-\cos^2\theta} \\
&= \|\vec{v}\|\|\vec{u}\|\sqrt{1-\left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|\|\vec{u}\|}\right)^2} \\
&= \sqrt{\|\vec{v}\|^2\|\vec{u}\|^2-(\vec{v}\cdot\vec{u})^2} \\
&= \sqrt{(v_1^2+v_2^2+v_3^2)(u_1^2+u_2^2+u_3^2)-(v_1u_1+v_2u_2+v_3u_3)^2} \\
&= \sqrt{(v_2u_3-v_3u_2)^2+(v_3u_1-v_1u_3)^2+(v_1u_2-v_2u_1)^2} \\
&= \|\vec{v}\times\vec{u}\|
\end{aligned}$$



Since the height of the parallelogram is $\|\vec{u}\|\sin\theta$ and the base of the parallelogram is $\|\vec{v}\|$, the area of the parallelogram is given By

$$\begin{aligned}
\text{Area} &= (\text{base})(\text{height}) \\
&= \|\vec{v}\|\|\vec{u}\|\sin\theta \\
&= \|\vec{v}\times\vec{u}\| \qquad \blacksquare
\end{aligned}$$

Example 1. Find the cross product of the vectors $\vec{v} = \langle 1, 2, 3\rangle$ and $\vec{u} = \langle -1, 0, 4\rangle$.

$$\begin{aligned}
\vec{v} \times \vec{u} &= \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -1 & 0 & 4 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} i - \begin{vmatrix} 1 & 3 \\ -1 & 4 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} k \\
&= (2(4) - 3(0))i - (1(4) - 3(-1))j + (1(0) - 2(-1))k \\
&= 8i - 7j + 2k
\end{aligned}$$

Example 2. Find the unit vector that is orthogonal to both $\vec{v} = 2i - 3j + k$ and $\vec{u} = i + 2j - k$.

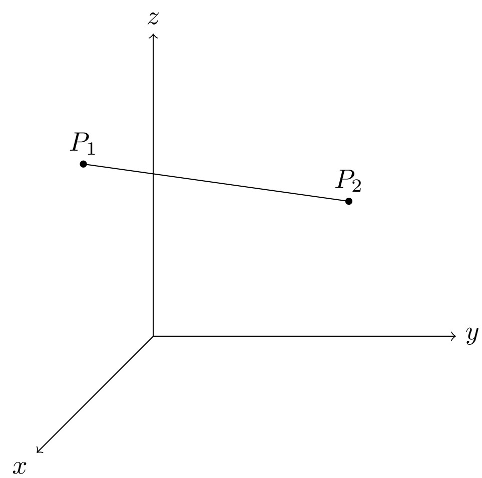
$$\begin{aligned}
\vec{v} \times \vec{u} &= \begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ 1 & 2 & -1 \end{vmatrix} \\
&= \begin{vmatrix} -3 & 1 \\ 2 & -1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} j + \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} k \\
&= (-3(-1) - 1(2))i - (2(-1) - 1(1))j + (2(2) - 1(-3))k \\
&= i + 3j + 7k
\end{aligned}$$

$$\begin{aligned}
\|\vec{v} \times \vec{u}\| &= \sqrt{1^2 + 3^2 + 7^2} \\
&= \sqrt{59}
\end{aligned}$$

$$\begin{aligned}
\text{Unit vector} &= \frac{\vec{v} \times \vec{u}}{\|\vec{v} \times \vec{u}\|} \\
&= \frac{1}{\sqrt{59}} i + \frac{3}{\sqrt{59}} j + \frac{7}{\sqrt{59}} k
\end{aligned}$$

Chapter 7

Distance in Space



The distance between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in space is essentially the magnitude of the vector $\overrightarrow{P_1P_2}$, which is given by

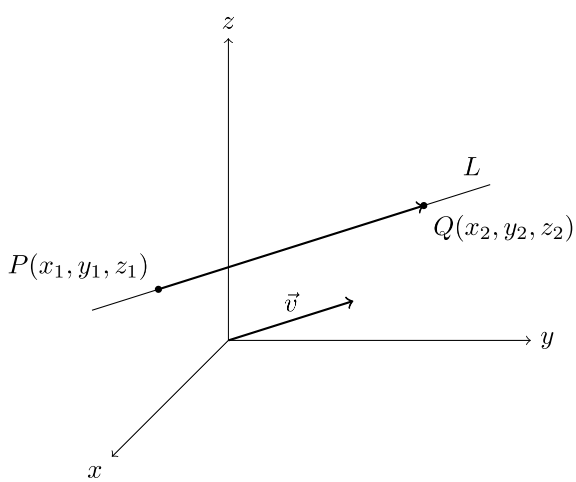
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 1. Find the distance between the points $P_1(-3, 2, 5)$ and $P_2(4, 0, 8)$ in space.

$$\begin{aligned} d &= \sqrt{(4 - (-3))^2 + (0 - 2)^2 + (8 - 5)^2} \\ &= \sqrt{7^2 + (-2)^2 + 3^2} \\ &= \sqrt{62} \end{aligned}$$

Chapter 8

Lines in Space



To find the equation of a line in space, we need a point $P(x_1, y_1, z_1)$ on the line and a vector $\vec{v} = \langle a, b, c\rangle$ that is parallel to the line. The vector $\vec{v}$ is called the **direction vector** of the line, while a, b and c are called the **direction numbers**.

Since $\vec{v}$ is parallel to L, the vector $\overrightarrow{PQ}$ is also parallel to L, where $Q(x, y, z)$ is any point on L. Hence, $\overrightarrow{PQ}$ is a scalar multiple of $\vec{v}$, that is,

$$\begin{aligned} \overrightarrow{PQ} &= t\vec{v} \\ \langle x - x_1, y - y_1, z - z_1\rangle &= t\langle a, b, c\rangle \\ &= \langle at, bt, ct\rangle \end{aligned}$$

Comparing both sides, we get

$$x - x_1 = at \qquad y - y_1 = bt \qquad z - z_1 = ct$$

Rearranging the equations, we get the **parametric equations** of the line L.

$$x = x_1 + at \qquad y = y_1 + bt \qquad z = z_1 + ct$$

If $a, b, c \neq 0$, we can solve for t for each of the parametric equations to get the **symmetric equations** of the line L.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Example 1. Find the equation of the line that passes through the point $P(-1, 4, 5)$ and is parallel to the vector $\vec{v} = 4i - j$.

$$x = -1 + 4t$$
$$y = 4 - t$$
$$z = 5$$

Note that it is impossible to find the symmetric equation of the line since $c = 0$.

Example 2. L passes through the point $P(2, 7, 1)$ and is parallel to the vector $\vec{v} = \langle -2, -4, 6 \rangle$. Find the parametric and symmetric equations of L.

$$x = 2 - 2t$$
$$y = 7 - 4t$$
$$z = 1 + 6t$$

$$\frac{x - 2}{-2} = \frac{y - 7}{-4} = \frac{z - 1}{6}$$

Example 3. Find the equation of the line passing through the point $(1, 0, 1)$ and parallel to the line given by the parametric equations

$$x = 3 + 3t \qquad y = 5 - 2t \qquad z = -7 + t$$

The line is parallel to the vector $\vec{v} = \langle 3, -2, 1 \rangle$. Hence, the equation of the line is given by

$$x = 1 + 3t \qquad y = -2t \qquad z = 1 + t$$

Also, the symmetric equations of the line is given by

$$\frac{x - 1}{3} = \frac{y}{-2} = \frac{z - 1}{1}$$

Example 4. Find the equation of the line passing through points $(7, -2, 6)$ and $(-3, 0, 6)$.

$$\begin{aligned}\vec{v} &= \langle -3 - 7, 0 - (-2), 6 - 6 \rangle \\ &= \langle -10, 2, 0 \rangle\end{aligned}$$

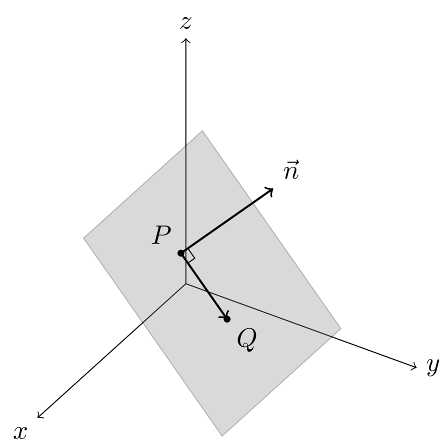
The equation of the line is given by

$$x = 7 - 10t \qquad y = -2 + 2t \qquad z = 6$$

There is no symmetric equation of the line since $c = 0$.

Chapter 9

Planes in Space



To find the equation of a plane in space, we need a point $P(x_1, y_1, z_1)$ on the plane and a vector $\vec{n} = \langle a, b, c\rangle$ that is orthogonal to the plane, called the **normal vector** of the plane.

For any point $Q(x, y, z)$ on the plane, the vector $\overrightarrow{PQ}$ is orthogonal to $\vec{n}$, that is,

$$\begin{aligned} \overrightarrow{PQ} \cdot \vec{n} &= 0 \\ \langle x - x_1, y - y_1, z - z_1\rangle \cdot \langle a, b, c\rangle &= 0 \\ a(x - x_1) + b(y - y_1) + c(z - z_1) &= 0 \end{aligned}$$

This equation is called the **standard form** of the equation of the plane. Regrouping the terms, we obtain the **general form** of equation of the plane.

$$ax + by + cz + d = 0$$

where $d = -ax_1 - by_1 - cz_1$.

Example 1. Find the equation of the plane that passes through the point $P(4,5,-7)$ and is perpendicular to the vector $\vec{n} = \hat{\jmath}$.

$$\begin{aligned}
\vec{n} &= \langle 0,1,0\rangle \\
0(x-4)+1(y-5)+0(z+7) &= 0 \\
y-5 &= 0 \\
y &= 5
\end{aligned}$$

Example 2. Find the equation of the plane that passes through the point $P(0,7,0)$ and is perpendicular to the vector $\vec{n} = 3\hat{\imath} + 8\hat{k}$.

$$\begin{aligned}
\vec{n} &= \langle 3,0,8\rangle \\
3(x-0)+0(y-7)+8(z-0) &= 0 \\
3x+8z &= 0
\end{aligned}$$

Example 3. Given three points $(0,0,0)$, $(2,0,7)$, and $(-2,-1,7)$ in space, find the equation of the plane that passes through these points.

$$\begin{aligned}
\vec{u} &= \langle 2-0, 0-0, 7-0\rangle \\
&= \langle 2,0,7\rangle \\
\vec{v} &= \langle -2-0, -1-0, 7-0\rangle \\
&= \langle -2,-1,7\rangle \\
\vec{n} &= \vec{u}\times\vec{v} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & 7 \\ -2 & -1 & 7 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 7 \\ -1 & 7 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 2 & 7 \\ -2 & 7 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 2 & 0 \\ -2 & -1 \end{vmatrix}\hat{k} \\
&= (0(7)-7(-1))\hat{\imath} - (2(7)-(-2)(7))\hat{\jmath} + (2(-1)-(-2)(0))\hat{k} \\
&= 7\hat{\imath} - 28\hat{\jmath} - 2\hat{k} \\
&= \langle 7,-28,-2\rangle
\end{aligned}$$

$$\begin{aligned}
7(x-0)-28(y-0)-2(z-0) &= 0 \\
7x-28y-2z &= 0
\end{aligned}$$

Example 4. Find the equation of the plane that passes through $(4,2,1)$, $(-1,8,8)$ and is parallel to z-axis.

$$\begin{aligned}
\vec{v} &= \langle -1-4, 8-2, 8-1\rangle \\
&= \langle -5, 6, 7\rangle \\
\vec{n} &= \vec{v} \times \hat{k} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -5 & 6 & 7 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \begin{vmatrix} 6 & 7 \\ 0 & 1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} -5 & 7 \\ 0 & 1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} -5 & 6 \\ 0 & 0 \end{vmatrix} \hat{k} \\
&= (6(1) - 7(0))\hat{\imath} - (-5(1) - 7(0))\hat{\jmath} + (-5(0) - 6(0))\hat{k} \\
&= 6\hat{\imath} + 5\hat{\jmath} \\
&= \langle 6, 5, 0\rangle
\end{aligned}$$

$$6(x-4) + 5(y-2) + 0(z-1) = 0$$
$$6x + 5y - 34 = 0$$

Example 5. Find the equation of the plane such that the point $(2,0,1)$ and the line $\frac{x}{2} = \frac{y-4}{-1} = \frac{z}{1}$ is on the plane.
When $x = 0$, $y = 4$ and $z = 0$, hence the point $(0,4,0)$ is on the plane.

$$\begin{aligned}
\vec{v} &= \langle 2-0, 0-4, 1-0\rangle \\
&= \langle 2, -4, 1\rangle \\
\vec{n} &= \langle 2, -1, 1\rangle \times \langle 2, -4, 1\rangle \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 1 \\ 2 & -4 & 1 \end{vmatrix} \\
&= \begin{vmatrix} -1 & 1 \\ -4 & 1 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 2 & -1 \\ 2 & -4 \end{vmatrix} \hat{k} \\
&= (-1(1) - 1(-4))\hat{\imath} - (2(1) - 2(1))\hat{\jmath} + (2(-4) - 2(-1))\hat{k} \\
&= -3\hat{\imath} + 0\hat{\jmath} + (-6)\hat{k} \\
&= \langle -3, 0, -6\rangle
\end{aligned}$$

$$-3(x-2) + 0(y-0) - 6(z-1) = 0$$
$$-3x - 6z + 12 = 0$$

Example 6. Find the equation of the plane that passes through the points $(3, 4, 1)$ and $(3, 1, -7)$ and is perpendicular to the plane $8x + 9y + 3z = 13$.

The normal vector of the plane is $\langle 8, 9, 3\rangle$. Since the target plane is perpendicular to the given plane, the normal vector of the given plane is parallel to the target plane.

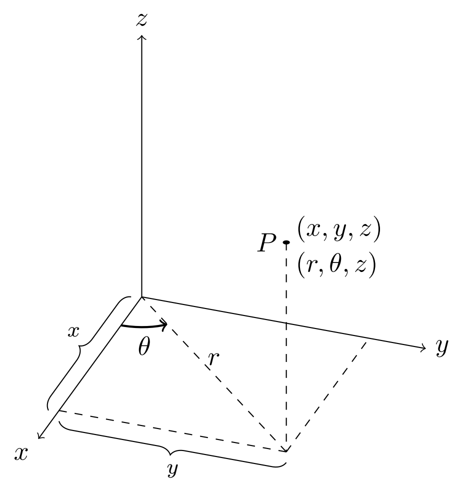
$$\begin{aligned}
\vec{u} &= \langle 8, 9, 3\rangle \\
\vec{v} &= \langle 3-3, 1-4, -7-1\rangle \\
&= \langle 0, -3, -8\rangle \\
\vec{n} &= \vec{u} \times \vec{v} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 8 & 9 & 3 \\ 0 & -3 & -8 \end{vmatrix} \\
&= \begin{vmatrix} 9 & 3 \\ -3 & -8 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 8 & 3 \\ 0 & -8 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 8 & 9 \\ 0 & -3 \end{vmatrix} \hat{k} \\
&= (9(-8) - 3(-3))\hat{\imath} - (8(-8) - 3(0))\hat{\jmath} + (8(-3) - 9(0))\hat{k} \\
&= -63\hat{\imath} + 64\hat{\jmath} - 24\hat{k} \\
&= \langle -63, 64, -24\rangle
\end{aligned}$$

$$\begin{aligned}
-63(x-3) + 64(y-4) - 24(z-1) &= 0 \\
-63x + 64y - 24z - 43 &= 0
\end{aligned}$$

Chapter 10

Cylindrical Coordinates

Cylindrical coordinates are an extension of polar coordinates into three dimensions. A point $P(x, y, z)$ in space is represented by the ordered triple (r, θ, z), called the **cylindrical coordinates** of P. Here, (r, θ) are the polar representation of the projection of P in the xy-plane, and z is the direct distance from the (r, θ) to P.



To convert from Cartesian coordinates to cylindrical coordinates, we use the following equations.

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

To convert from cylindrical coordinates to Cartesian coordinates, we use the following equations.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Example 1. Convert $\left(4, \dfrac{5\pi}{6}, 3\right)$ from cylindrical coordinates to Cartesian coordinates.

$$\begin{aligned} x &= r\cos\theta \\ &= 4\cos\left(\frac{5\pi}{6}\right) \\ &= -2\sqrt{3} \\ y &= r\sin\theta \\ &= 4\sin\left(\frac{5\pi}{6}\right) \\ &= 2 \\ z &= z \\ &= 3 \end{aligned}$$

Hence, $\left(-2\sqrt{3}, 2, 3\right)$ is the Cartesian coordinates of $\left(4, \dfrac{5\pi}{6}, 3\right)$.

Example 2. Convert $\left(1, \sqrt{3}, 2\right)$ from Cartesian coordinates to cylindrical coordinates.

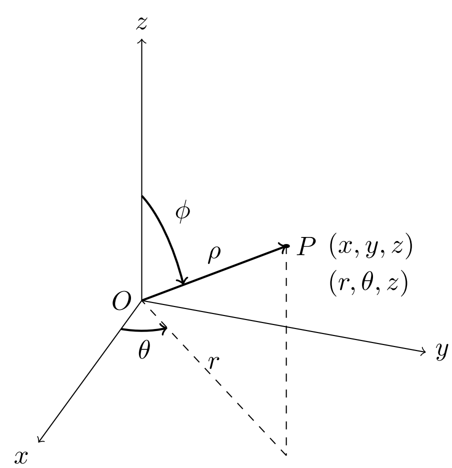
$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + \sqrt{3}^2} \\ &= 2 \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \\ &= \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) \\ &= \frac{\pi}{3} \\ z &= z \\ &= 2 \end{aligned}$$

Hence, $\left(2, \dfrac{\pi}{3}, 2\right)$ is the cylindrical coordinates of $\left(1, \sqrt{3}, 2\right)$.

Chapter 11

Spherical Coordinates

Spherical coordinates are an ordered triple $P = (\rho, \theta, \phi)$, where ρ is the distance between the origin and P ($\rho \geq 0$ since it is a distance), θ is the angle from cylindrical coordinates where $r \geq 0$ and ϕ is the angle between the positive z-axis and the line segment $\overrightarrow{OP}$ ($0 \leq \phi \leq \pi$).



To convert from Cartesian coordinates to spherical coordinates, we use the following equations.

$$\rho^2 = x^2 + y^2 + z^2 \qquad \tan\theta = \frac{y}{x} \qquad \cos\phi = \frac{z}{\rho}$$

To convert from spherical coordinates to Cartesian coordinates, we use the following equations.

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$$

Example 1. Convert $\left(10, \frac{\pi}{6}, \frac{\pi}{4}\right)$ from spherical coordinates to Cartesian coordinates.

$$\begin{aligned}
x &= \rho \sin\phi \cos\theta \\
&= 10 \sin\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{6}\right) \\
&= \frac{5\sqrt{6}}{2} \\
y &= \rho \sin\phi \sin\theta \\
&= 10 \sin\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{6}\right) \\
&= \frac{5\sqrt{2}}{2} \\
z &= \rho \cos\phi \\
&= 10 \cos\left(\frac{\pi}{4}\right) \\
&= 5\sqrt{2}
\end{aligned}$$

Hence, $\left(\frac{5\sqrt{6}}{2}, \frac{5\sqrt{2}}{2}, 5\sqrt{2}\right)$ is the Cartesian coordinates of $\left(10, \frac{\pi}{6}, \frac{\pi}{4}\right)$.

Example 2. Convert $\left(-8, -8, \sqrt{19}\right)$ from Cartesian coordinates to spherical coordinates.

$$\begin{aligned}
\rho &= \sqrt{x^2 + y^2 + z^2} \\
&= \sqrt{(-8)^2 + (-8)^2 + \left(\sqrt{19}\right)^2} \\
&= 7\sqrt{3} \\
\theta &= \tan^{-1}\left(\frac{y}{x}\right) \\
&= \tan^{-1}\left(\frac{-8}{-8}\right) \\
&= \frac{5\pi}{4} \text{ (3^{rd} quadrant since x and y are negative)} \\
\phi &= \arccos\left(\frac{z}{\rho}\right) \\
&= \arccos\left(\frac{\sqrt{19}}{7\sqrt{3}}\right)
\end{aligned}$$

Hence, $\left(7\sqrt{3}, \frac{5\pi}{4}, \arccos\left(\frac{\sqrt{19}}{7\sqrt{3}}\right)\right)$ is the spherical coordinates of $\left(-8, -8, \sqrt{19}\right)$.

Chapter 12

Vector-valued Functions

A vector valued-function is a function that maps a real number to a vector. In the plane, a vector-valued function is given by

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

and in space, a vector-valued function is given by

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

Note that different functions can give the same curve. For example,

$$\vec{r}(t) = \langle \cos t, \sin t \rangle \quad t \in [0, 2\pi]$$

and

$$\vec{r}(t) = \langle \cos 2t, \sin 2t \rangle \quad t \in [0, 2\pi]$$

both give the unit circle.

The domain of a vector-valued function $\vec{r}$ is the intersection of the domains of the component functions $x(t)$, $y(t)$ and $z(t)$. For example, given a vector-valued function $\vec{r}(t) = \frac{1}{t}\hat{\imath} + \frac{1}{t-1}\hat{\jmath} + \frac{1}{\cos t}\hat{k}$ is $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$. The domain of each component function is

$$D_{x(t)} = (-\infty, 0) \cup (0, \infty) \qquad D_{y(t)} = (-\infty, 1) \cup (1, \infty) \qquad D_{z(t)} = \mathbb{R}$$

Combining the domains of the component functions, we get the domain of the vector-valued function $\vec{r}(t)$

$$(D_{x(t)} \cap D_{y(t)} \cap D_{z(t)}) = (-\infty, 0) \cup (0, 1) \cup (1, \infty)$$

Example 1. Sketch the vector-valued function $\vec{r}(t) = \langle 2\cos t, -3\sin t\rangle$.

$$\begin{aligned}
x &= 2\cos t \\
\cos t &= \frac{x}{2} \\
y &= -3\sin t \\
\sin t &= -\frac{y}{3} \\
\cos^2 t + \sin^2 t &= 1 \\
\left(\frac{x}{2}\right)^2 + \left(-\frac{y}{3}\right)^2 &= 1 \\
\frac{x^2}{4} + \frac{y^2}{9} &= 1
\end{aligned}$$

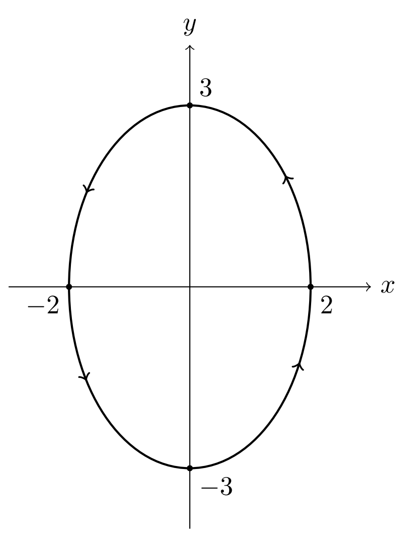
Hence, the graph of the vector-valued function $\vec{r}(t) = \langle 2\cos t, -3\sin t\rangle$ is an ellipse with major axis of length 6 and minor axis of length 4.

To find the orientation of the function, we can plot points in increasing values of t.

When $t = 0$, $\vec{r}(0) = \langle 2\cos 0, -3\sin 0\rangle = \langle 2, 0\rangle$.

When $t = \frac{\pi}{2}$, $\vec{r}\left(\frac{\pi}{2}\right) = \langle 2\cos\frac{\pi}{2}, -3\sin\frac{\pi}{2}\rangle = \langle 0, -3\rangle$.

Hence, the orientation of the function is clockwise.



Example 2. Sketch the vector-valued function $\vec{r}(t) = \frac{t}{8}\hat{\imath} + (t-1)\hat{\jmath}$.

$$x = \frac{t}{8}$$
$$t = 8x$$

$$y = t - 1$$
$$t = y + 1$$

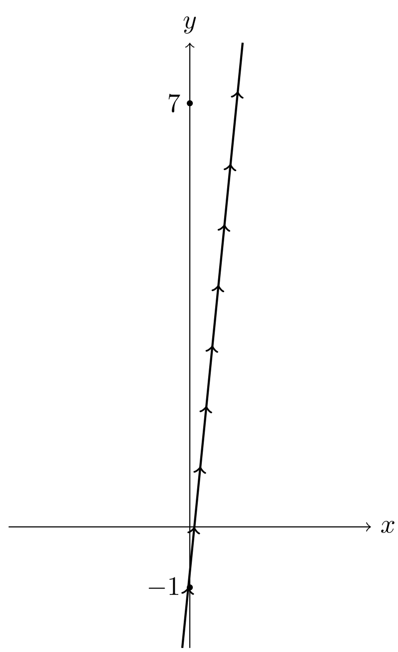
$$8x = y + 1$$
$$y = 8x - 1$$

Hence, the graph of the vector-valued function $\vec{r}(t) = \frac{t}{8}\hat{\imath} + (t-1)\hat{\jmath}$ is a line with y-intercept of -1 and slope of 8.

When $t = 0$, $\vec{r}(0) = \frac{0}{8}\hat{\imath} + (0-1)\hat{\jmath} =< 0, -1 >$.

When $t = 8$, $\vec{r}(8) = \frac{8}{8}\hat{\imath} + (8-1)\hat{\jmath} =< 1, 7 >$.

Hence, the orientation of the function is going up.



Example 3. Represent $y = x + 9$ as vector-valued function.

$$\begin{aligned} x &= t \\ y &= t + 9 \\ \vec{r}(t) &= x(t)\hat{\imath} + y(t)\hat{\jmath} \\ &= t\hat{\imath} + (t+9)\hat{\jmath} \end{aligned}$$

Example 4. Represent $x^2 + y^2 = 64$ as vector-valued function.

$$\begin{aligned} x &= 8\cos t \\ y &= 8\sin t \\ \vec{r}(t) &= x(t)\hat{\imath} + y(t)\hat{\jmath} \\ &= 8\cos t\hat{\imath} + 8\sin t\hat{\jmath} \end{aligned}$$

Example 5. Represent $(x-2)^2 + (y+1)^2 = 4$ as vector-valued function.

$$\begin{aligned} 2\cos t &= x - 2 \\ x &= 2\cos t + 2 \\ 2\sin t &= y + 1 \\ y &= 2\sin t - 1 \\ \vec{r}(t) &= x(t)\hat{\imath} + y(t)\hat{\jmath} \\ &= (2\cos t + 2)\hat{\imath} + (2\sin t - 1)\hat{\jmath} \end{aligned}$$

Hence, we can conclude that the vector-valued function of a circle with radius r and centre (h, k) is

$$\vec{r}(t) = (h + r\cos t)\hat{\imath} + (k + r\sin t)\hat{\jmath}$$

Example 6. Represent $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ as vector-valued function.

$$\begin{aligned} x &= 3\cos t \\ y &= 2\sin t \\ \vec{r}(t) &= x(t)\hat{\imath} + y(t)\hat{\jmath} \\ &= 3\cos t\hat{\imath} + 2\sin t\hat{\jmath} \end{aligned}$$

Example 7. Represent $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{25} = 1$ as vector-valued function.

$$\begin{aligned} x - 1 &= 2\cos t \\ x &= 2\cos t + 1 \\ y + 2 &= 5\sin t \\ y &= 5\sin t - 2 \\ \vec{r}(t) &= x(t)\hat{\imath} + y(t)\hat{\jmath} \\ &= (2\cos t + 1)\hat{\imath} + (5\sin t - 2)\hat{\jmath} \end{aligned}$$

Hence, we can conclude that the vector-valued function of an ellipse with major axis of length $2a$ and minor axis of length $2b$ is

$$\vec{r}(t) = (h + a\cos t)\hat{\imath} + (k + b\sin t)\hat{\jmath}$$

Example 8. Represent $\frac{x^2}{25} - \frac{y^2}{16} = 1$ as vector-valued function.

$$\begin{aligned} x &= 5\cosh t \\ y &= 4\sinh t \\ \vec{r}(t) &= x(t)\hat{\imath} + y(t)\hat{\jmath} \\ &= 5\cosh t\hat{\imath} + 4\sinh t\hat{\jmath} \end{aligned}$$

Example 9. Represent $\frac{(x-1)^2}{4} - \frac{(y+7)^2}{9} = 1$ as vector-valued function.

$$\begin{aligned} x - 1 &= 2\cosh t \\ x &= 2\cosh t + 1 \\ y + 7 &= 3\sinh t \\ y &= 3\sinh t - 7 \\ \vec{r}(t) &= x(t)\hat{\imath} + y(t)\hat{\jmath} \\ &= (2\cosh t + 1)\hat{\imath} + (3\sinh t - 7)\hat{\jmath} \end{aligned}$$

Hence, we can conclude that the vector-valued function of a hyperbola with centre (h, k) is

$$\vec{r}(t) = (h + a\cosh t)\hat{\imath} + (k + b\sinh t)\hat{\jmath}$$

Chapter 13

Limits of Vector-valued Functions

The limit of a vector-valued function $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ as t approaches a is

$$\begin{aligned}\lim_{t\to a}\vec{r}(t) &= \langle \lim_{t\to a}x(t), \lim_{t\to a}y(t), \lim_{t\to a}z(t)\rangle \\ &= \lim_{t\to a}x(t)\hat{\imath} + \lim_{t\to a}y(t)\hat{\jmath} + \lim_{t\to a}z(t)\hat{k}\end{aligned}$$

Example 1. Find $\lim_{t\to\pi}(t\hat{\imath} + \cos t\hat{\jmath} + \sin t\hat{k})$.

$$\begin{aligned}\lim_{t\to\pi}(t\hat{\imath} + \cos t\hat{\jmath} + \sin t\hat{k}) &= \lim_{t\to\pi}t\hat{\imath} + \lim_{t\to\pi}\cos t\hat{\jmath} + \lim_{t\to\pi}\sin t\hat{k} \\ &= \pi\hat{\imath} + \cos\pi\hat{\jmath} + \sin\pi\hat{k} \\ &= \pi\hat{\imath} - \hat{\jmath}\end{aligned}$$

Example 2. Find $\lim_{t\to 0}(e^{6t}\hat{\imath} + \frac{\sin 2t}{2t}\hat{\jmath} + e^{-5t}\hat{k})$.

$$\begin{aligned}\lim_{t\to 0}(e^{6t}\hat{\imath} + \frac{\sin 2t}{2t}\hat{\jmath} + e^{-5t}\hat{k}) &= \lim_{t\to 0}e^{6t}\hat{\imath} + \lim_{t\to 0}\frac{\sin 2t}{2t}\hat{\jmath} + \lim_{t\to 0}e^{-5t}\hat{k} \\ &= 1\hat{\imath} + 1\hat{\jmath} + 1\hat{k} \\ &= \hat{\imath} + \hat{\jmath} + \hat{k}\end{aligned}$$

Example 3. Find $\lim_{t\to\infty}(e^{-t}\hat{\imath} + \frac{1}{t^2+7}\hat{\jmath} + e^{\arctan t}\hat{k})$.

$$\begin{aligned}\lim_{t\to\infty}(e^{-t}\hat{\imath} + \frac{1}{t^2+7}\hat{\jmath} + e^{\arctan t}\hat{k}) &= \lim_{t\to\infty}e^{-t}\hat{\imath} + \lim_{t\to\infty}\frac{1}{t^2+7}\hat{\jmath} + \lim_{t\to\infty}e^{\arctan t}\hat{k} \\ &= 0\hat{\imath} + 0\hat{\jmath} + e^{\frac{\pi}{2}}\hat{k} \\ &= e^{\frac{\pi}{2}}\hat{k}\end{aligned}$$

Chapter 14

Derivatives and Integration of Vector-valued Functions

The derivative of a vector-valued function $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ is

$$\vec{r}'(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}$$

The indefinite integral of a vector-valued function $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ is

$$\int \vec{r}(t)\ dt = \int x(t)\hat{\imath} + \int y(t)\hat{\jmath} + \int z(t)\hat{k} + C$$

The definite integral of a vector-valued function $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ from a to b is

$$\int_a^b \vec{r}(t)\ dt = \int_a^b x(t)\hat{\imath} + \int_a^b y(t)\hat{\jmath} + \int_a^b z(t)\hat{k}$$

Since this topic is relatively straightforward, there will be no examples. :)

Chapter 15

Velocity, Speed and Acceleration

If $x(t)$, $y(t)$ and $z(t)$ are differentiable functions, then the velocity of a vector-valued function $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ is

$$\vec{v}(t) = (\vec{r})'(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}$$

The speed of a vector-valued function $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ is

$$|\vec{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

The acceleration of a vector-valued function $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ is

$$\vec{a}(t) = (\vec{v})'(t) = x''(t)\hat{\imath} + y''(t)\hat{\jmath} + z''(t)\hat{k}$$

The path of a projectile launched from an initial height h with initial speed v_0 at an angle of elevation θ is given by

$$\vec{r}(t) = v_0 \cos\theta t\hat{\imath} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\hat{\jmath}$$

Since this topic is relatively straightforward also, there will be no examples. :)

Application Exercises

Source: Larson Calculus 11th Ed. Exercise 12.3

Projectile Motion

In Exercises 27-32, use the model for projectile motion, assuming there is no air resistance and $g = 9.8$ meters per second per second.

1. A baseball is hit from a height of 1 meter above the ground with an initial speed of 40 feet per second and at an angle of 22° above the horizontal. Find the maximum height reached by the baseball. Determine whether it will clear a 3-meters-high fence located 105 meters from home plate.

   Solution Given that $h = 1$, $v_0 = 40$, $\theta = 22°$ and $g = 9.8$,

$$\begin{aligned}\vec{r}(t) &= v_0 \cos\theta t\hat{\imath} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\hat{\jmath} \\ &= 40\cos 22° t\hat{\imath} + \left[1 + (40\sin 22°)t - \frac{1}{2}(9.8)t^2\right]\hat{\jmath}\end{aligned}$$

   The velocity vector is

$$\vec{v}(t) = (\vec{r})'(t) = 40\cos 22°\hat{\imath} + (40\sin 22° - 9.8t)\hat{\jmath}$$

   The maximum height is reached when the vertical component of the velocity is 0.

$$40\sin 22° - 9.8t = 0 \implies t = \frac{40\sin 22°}{9.8} \approx 1.53 \text{ seconds}$$

   The maximum height is

$$\begin{aligned}y &= 1 + (40\sin 22°)\left(\frac{40\sin 22°}{9.8}\right) - \frac{1}{2}(9.8)\left(\frac{40\sin 22°}{9.8}\right)^2 \\ &\approx 12.46 \text{ meters}\end{aligned}$$

   The ball is 105 meters from home plate when $x(t) = 105$.

$$40\cos 22° t = 105 \implies t = \frac{105}{40\cos 22°} \approx 2.83 \text{ seconds}$$

   At this time, the height of the ball is

$$\begin{aligned}y &= 1 + (40\sin 22°)\left(\frac{105}{40\cos 22°}\right) - \frac{1}{2}(9.8)\left(\frac{105}{40\cos 22°}\right)^2 \\ &\approx 4.15 \text{ meters}\end{aligned}$$

   Hence, the ball will clear the fence. ■

2. Determine the maximum height and range of a projectile fired at a height of 2 meters above the ground with an initial speed of 300 meters per second and at angle of 45° above the horizontal.

   Solution Given that $h = 2$, $v_0 = 300$, $\theta = 45°$ and $g = 9.8$,

$$\begin{aligned}\vec{r}(t) &= v_0 \cos\theta t\hat{\imath} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\hat{\jmath} \\ &= 300\cos 45° t\hat{\imath} + \left[2 + (300\sin 45°)t - \frac{1}{2}(9.8)t^2\right]\hat{\jmath} \\ &= 150\sqrt{2}t\hat{\imath} + \left[2 + 150\sqrt{2}t - 4.9t^2\right]\hat{\jmath}\end{aligned}$$

   The velocity vector is

$$\begin{aligned}\vec{v}(t) &= (\vec{r})'(t) \\ &= 150\sqrt{2}\hat{\imath} + (150\sqrt{2} - 9.8t)\hat{\jmath}\end{aligned}$$

   The maximum height is reached when the vertical component of the velocity is 0.

$$\begin{aligned}150\sqrt{2} - 9.8t &= 0 \\ t &= \frac{150\sqrt{2}}{9.8} \\ &\approx 21.65 \text{ seconds}\end{aligned}$$

   The maximum height is

$$\begin{aligned}y &= 2 + (150\sqrt{2})\left(\frac{150\sqrt{2}}{9.8}\right) - 4.9\left(\frac{150\sqrt{2}}{9.8}\right)^2 \\ &\approx 2,297.92 \text{ meters}\end{aligned}$$

   The projectile hits the ground when $y(t) = 0$.

$$\begin{aligned}2 + (150\sqrt{2})t - 4.9t^2 &= 0 \\ t &= \frac{-150\sqrt{2} - \sqrt{150^2(2) - 4(-4.9)(2)}}{2(-4.9)} \\ &\approx 43.302 \text{ seconds}\end{aligned}$$

   Hence, the range is

$$\begin{aligned}x &= 150\sqrt{2}\left(\frac{-150\sqrt{2} - \sqrt{150^2(2) - 4(-4.9)(2)}}{2(-4.9)}\right) \\ &\approx 9185.67 \text{ meters}\end{aligned}$$

■

3. A baseball, hit 1 meter above the ground, leaves the bat at an angle of 45° and is caught by an outfielder 1 feet above the ground and 100 feet from home plate. What is the initial speed of the ball, and how high does it rise?

Solution Given that $h = 1$, $\theta = 45°$ and $g = 9.8$,

$$\begin{aligned}\vec{r}(t) &= v_0 \cos\theta t\hat{\imath} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\hat{\jmath} \\ &= v_0 \cos 45° t\hat{\imath} + \left[1 + (v_0 \sin 45°)t - \frac{1}{2}(9.8)t^2\right]\hat{\jmath} \\ &= \frac{v_0}{\sqrt{2}}t\hat{\imath} + \left[1 + \frac{v_0}{\sqrt{2}}t - 4.9t^2\right]\hat{\jmath}\end{aligned}$$

When the ball is caught, $x(t) = 100$ and $y(t) = 1$.

$$\frac{v_0}{\sqrt{2}}t = 100 \implies t = \frac{100\sqrt{2}}{v_0} \quad \cdots \ (1)$$

$$\begin{aligned}1 + \frac{v_0}{\sqrt{2}}t - 4.9t^2 &= 1 \\ 4.9t^2 - \frac{v_0}{\sqrt{2}}t &= 0 \\ t(4.9t - \frac{v_0}{\sqrt{2}}) &= 0 \\ t &= \frac{v_0}{4.9\sqrt{2}} \quad \cdots \ (2)\end{aligned}$$

Equating (1) and (2),

$$\begin{aligned}\frac{100\sqrt{2}}{v_0} &= \frac{v_0}{4.9\sqrt{2}} \\ v_0^2 &= 980 \\ v_0 &= 14\sqrt{5} \text{ meters per second } (v_0 > 0)\end{aligned}$$

Hence,

$$\begin{aligned}\vec{r}(t) &= \frac{14\sqrt{5}}{\sqrt{2}}t\hat{\imath} + \left[1 + \frac{14\sqrt{5}}{\sqrt{2}}t - 4.9t^2\right]\hat{\jmath} \\ &= 7\sqrt{10}t\hat{\imath} + \left[1 + 7\sqrt{10}t - 4.9t^2\right]\hat{\jmath}\end{aligned}$$

The maximum height is reached when the vertical component of the velocity is 0.

$$\begin{aligned}7\sqrt{10} - 9.8t &= 0 \\ t &= \frac{7\sqrt{10}}{9.8} \\ &\approx 2.26 \text{ seconds}\end{aligned}$$

The maximum height is

$$\begin{aligned} y &= 1 + (7\sqrt{10})\left(\frac{7\sqrt{10}}{9.8}\right) - 4.9\left(\frac{7\sqrt{10}}{9.8}\right)^2 \\ &= 26 \text{ meters} \end{aligned}$$

■

4. A baseball player at second base throws a ball 90 feet to the player at first base. The ball is released at a point 5 feet above the ground with an initial speed of 50 miles per hour and at an angle of 15° above the horizontal. At what height does the player at first base catch the ball?

5. Eliminate the parameter t from the position vector for the motion of a projectile to show that the rectangular equation is

$$y = -\frac{g\sec^2\theta}{2v_0^2}x^2 + (\tan\theta)x + h$$

6. The path of a ball is given by the rectangular equation

$$y = x - 0.005x^2.$$

Use the result of Exercise 33 to find the position vector. Then find the speed and direction of the ball at the point at which it has traveled 60 feet horizontally.

Chapter 16

Tangent Vectors and Normal Vectors

Let C be a smooth curve represented by $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ on an open interval I, the unit tangent vector at $t = a$ is a vector that is tangent to the curve at $t = a$ and has a magnitude of 1. The unit tangent vector is given by

$$\vec{T}(t) = \frac{(\vec{r})'(t)}{\|(\vec{r})'(t)\|} = \frac{x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}}{\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}}, \quad (\vec{r})'(t) \neq \vec{0}$$