Mathematics

Senior 3 Part II

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Preface

Why this book?

Disclaimer

Acknowledgements

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Chapter 26

Applications of Differentiation

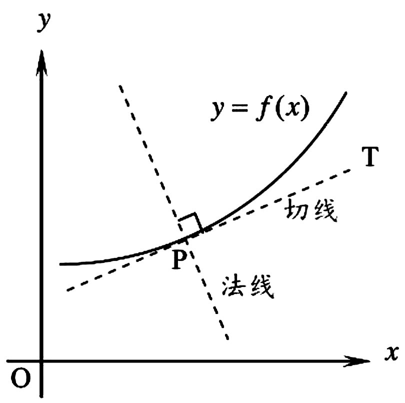
26.1 Tangent and Normal Lines

In the previous chapter, we have learnt that the gradient of the tangent to the curve $y = f(x)$ at point $P\big(x_0, f(x_0)\big)$ is

$$\begin{aligned} m &= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\ &= f'(x_0) \end{aligned}$$

Hence, in order to find the tangent to the curve $y = f(x)$ at $x = x_0$, we just have to find the derivative value $f'(x_0)$ of the curve $y = f(x)$ at $x = x_0$, then the equation of the tangent line can be acquired through the point gradient formula of linear equations.

As shown in the right figure, the line that passes through point P and perpendicular to the tangent line PT is known as the normal line to the curve $y = f(x)$. If two straight lines are perpendicular, the product of their gradients is -1. Since the gradient of the curve $y = f(x)$ at point $x = x_0$ is $f'(x_0)$, therefore when $f'(x_0) \neq 0$, the gradient of normal at point $x = x_0$ is $-\dfrac{1}{f'(x_0)}$.



26.1.1 Practice 1

1. Find the equations of tangent and normal to the curve $y = x^3$ where $x = 2$.
2. Given that the gradient of tangent to the curve $y = x^2 - 2x + 3$ at point Q is 4, find the coordinates of the point Q.
3. Find the equations of the tangent and normal to the curve $x^3 - 2xy + y^2 = 1$ at point $(1, 2)$.

26.1.2 Exercise 26.1

1. Find the equations of the tangent and normal to the curve $y = x^2 + 2$ at point $(2, 6)$.
2. Find the equation of the tangent to the curve $y = 2x^3 - 3x^2 - 12x + 8$ where $x = 0$.
3. Find the equation of the normal to the curve $y = 3x^3 - 4x + 7$ at point $(1, 6)$.
4. Find the equation of the tangent to the curve $y = \dfrac{1}{1 - x}$ where $x = -1$.

5. Find the equations of the tangent and normal to the curve $y = 1 - 2x^2$ where $x = -2$.

6. Find the equation of the tangent to the curve $y = (x + 1)(x - 1)(x - 3)$ where the curve intersects with the x-axis.

7. Find the equation of the tangent to the curve $y = 4x^3 - 27x + 7$ that is parallel to the x-axis.

8. Find the equation of the tangent to the curve $y = 11x - 3x^2$ that is parallel to the line $x + y - 2 = 0$.

9. Find the equations of the tangent and normal to the curve $y = \ln 2x - 1$ where $x = 1$.

10. If the straight line $y = 8x + k$ is the tangent to the curve $y = x^2 + 4x - 3$, find the value of k.

11. Given that the gradient of tangent to the curve $y = ax + bx^2$ at point $(1, 0)$ is $\frac{1}{2}$, find the value of a and b.

12. If $x + y + 2 = 0$ is the equation of tangent to the curve $y = ax^2 + bx$ where $x = 1$, find the value of a and b.

13. Given that the gradient of tangent to the curve $y = x^3 - 6x^2 + 10x - 5$ at point P is -2, find the coordinates of P.

14. Prove that the tangent lines to the curve $y = x^2 - 3x + 1$ and the curve $x(y + 3) = 4$ at point $(2, -1)$ are perpendicular to each other.

15. Find the equations of the tangent and normal to the curve $x^2 - x6 + y^2 = 7$ at point $(-2, 3)$.

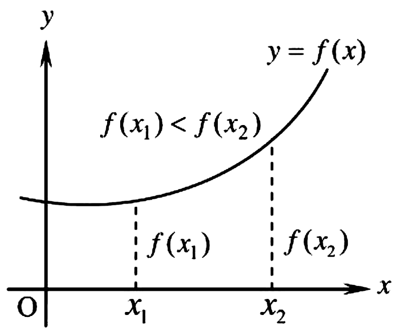
16. Find the equation of tangent to the curve $y^2 + y = 2 \sin x$ at point $(0, -1)$.

26.2 Increasing and Decreasing Functions

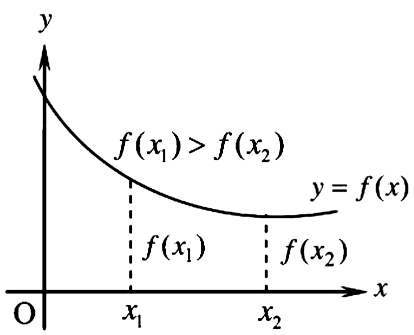
Monotonic Functions

For a function $f(x)$ being defined in the interval D,

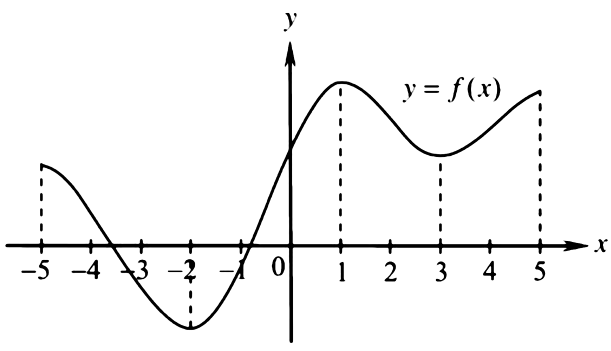
1. For any two numbers x_1 and x_2 in D, when $x_1 < x_2$, $f(x)$ is an increasing function in the interval D, as shown in the figure below.



2. For any two numbers x_1 and x_2 in D, when $x_1 > x_2$, $f(x)$ is a decreasing function in the interval D, as shown in the figure below.



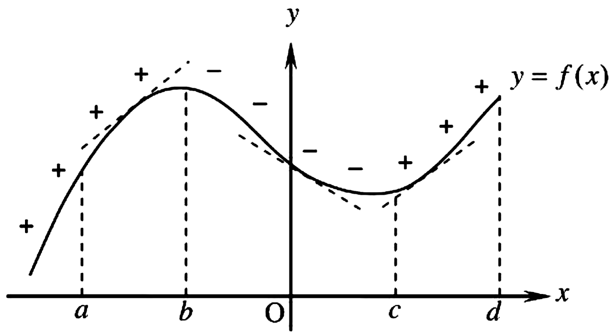
If $f(x)$ is an increasing function or a decreasing function in the interval D, we call $f(x)$ a monotonic function in the interval D.



The curve shown in the diagram above is the graph of the function $f(x)$ in the interval $[-5, 5]$. From the graph, we can see that the function $f(x)$ is a decreasing function in the intervals $[-5, 2]$ and $[1, 3]$, and an increasing function in the interval $[-2, 1]$ and $[3, 5]$.

How to Judge the Increase or Decrease of Functions

As shown in the diagram below, when the function $f(x)$ is an increasing function in the interval $[a, b]$, the gradient of the tangent to the curve at any point in the interval (a, b) is positive, i.e. $f'(x) > 0$; when the function $f(x)$ is a decreasing function in the interval $[b, c]$, the gradient of the tangent to the curve at any point in the interval (b, c) is negative, i.e. $f'(x) < 0$.



Therefore, we can judge whether a function is an increasing function or a decreasing function in an interval by the sign of the derivative value of the function in the interval:

> Let $f(x)$ be a continuous function defined in the interval $[a, b]$, and can be differentiated in the interval (a, b).
>
> - In the interval (a, b), if $f'(x) > 0$, then $f(x)$ is an increasing function in the interval $[a, b]$.
> - In the interval (a, b), if $f'(x) < 0$, then $f(x)$ is a decreasing function in the interval $[a, b]$.

26.2.1 Practice 2

Determine which intervals the following functions is an increasing function or a decreasing function.

1. $f(x) = x^2 + 2x - 3$
2. $f(x) = x^3 - x^2 - x + 1$

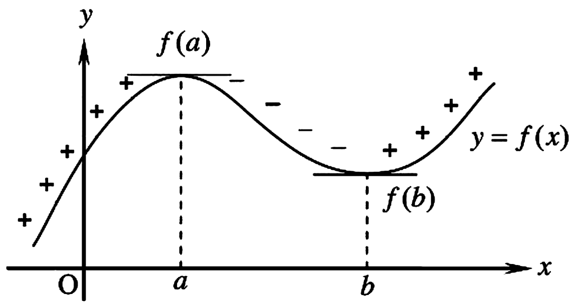
26.2.2 Exercise 26.2

Determine which intervals the following functions is an increasing function or a decreasing function.

1. $f(x) = x^2 - 2x + 4$
2. $f(x) = 2x^3 - 6x^2 + 7$
3. $f(x) = x^3 + x$
4. $f(x) = 2 + 3x - x^3$
5. $f(x) = x^2(x - 3)$
6. $f(x) = 3x^4 + 2x^3 - 3x^2 - 2$
7. $f(x) = \dfrac{x}{x^2 + 1}$
8. $f(x) = \cos 2x, 0 \leq x \leq \pi$

26.3 Relative Maximum and Minimum Values of Functions

As shown in the diagram below, the function value $f(a)$ at the point where $x = a$ is the maximum compared to its nearby points, we call $f(a)$ the relative maximum value; the function value $f(b)$ at the point where $x = b$ is the minimum compared to its nearby points, we call $f(b)$ the relative minimum value.



> Let $f(x)$ be a defined function near point $x = a$,
>
> - If the function value $f(a)$ is the maximum compared to its nearby points, we say that $f(x)$ has a relative maximum value $f(a)$ at point $x = a$, and the point $(a, f(a))$ is the relative maximum point of the function;
> - If the function value $f(a)$ is the minimum compared to its nearby points, we say that $f(x)$ has a relative minimum value $f(a)$ at point $x = a$, and the point $(a, f(a))$ is the relative minimum point of the function.

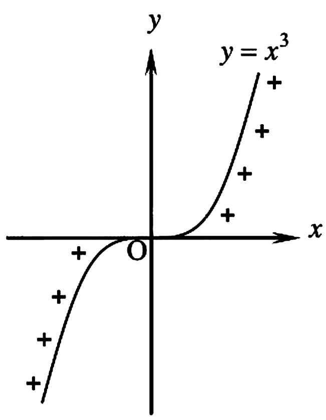
Relative maximum and minimum values are collectively known as the extreme values, and the points where the extreme values occur are collectively known as the extreme points.

From the diagram above, we can also see that the gradients of the tangent to the curve at the extreme points are zero. Hence, the following theorem can be obtained:

> If the function $f(x)$ is differentiable at point $x = a$, and $f(x)$ has a relative maximum or minimum value at point $x = a$, then $f'(a) = 0$.

The points that satisfy $f'(x) = 0$ are called the stationary points of the function.

If the function can be differentiated at point $x = a$, then the extreme point must be a stationary point. However, the stationary point is not necessarily an extreme point. For example, the derivative of the function $f(x) = x^3$ is $f'(x) = 3x^2$, and $f'(0) = 0$, i.e. $O(0, 0)$ is a stationary point of the function $f(x) = x^3$, but it is not an extreme point of the function, as shown in the right diagram.



How to Find the Extreme Values of Functions

1. **First Derivative Test**

   The curve has a positive gradient of tangent to the left of the relative maximum point, and a negative gradient of tangent to the right; the curve has a negative gradient of tangent to the left of the relative minimum point, and a positive gradient of tangent to the right. Hence, the following theorem is obtained:

> Let $f(x)$ be a function that is differentiable near point $x = a$, and $f'(a) = 0$,
>
> - If $f'(x) > 0$ to the left of $x = a$ and $f'(x) < 0$ to the right of $x = a$, then $f(a)$ is a relative maximum value;
> - If $f'(x) < 0$ to the left of $x = a$ and $f'(x) > 0$ to the right of $x = a$, then $f(a)$ is a relative minimum value.

2. **Second Derivative Test**

> Let $f(x)$ be a function that is second derivable near point $x = a$, and $f'(a) = 0$,
>
> - If $f''(a) < 0$, then $f(a)$ is a relative maximum value;
> - If $f''(a) > 0$, then $f(a)$ is a relative minimum value.

   Note that the second derivative test is invalid when $f''(a) = 0$, and the first derivative test should be used instead.

26.3.1 Practice 3

Find the extreme values of the following functions (Question 1 to 4):

1. $f(x) = x^2 + x - 6$

   Sol.

$$\begin{aligned} f'(x) &= 2x + 1 \\ 2x + 1 &= 0 \\ x &= -\frac{1}{2} \\ f''(x) &= 2 \\ f''\left(-\frac{1}{2}\right) &= 2 > 0 \end{aligned}$$

   When $x = -\frac{1}{2}$, $f(x)$ has a minimum value of $-\frac{25}{4}$.

2. $f(x) = 2 - x - x^2$

   Sol.

$$\begin{aligned} f'(x) &= -2x - 1 \\ -2x - 1 &= 0 \\ x &= -\frac{1}{2} \\ f''(x) &= -2 \\ f''\left(-\frac{1}{2}\right) &= -2 < 0 \end{aligned}$$

   When $x = -\frac{1}{2}$, $f(x)$ has a maximum value of $\frac{9}{4}$.

3. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 2$

Sol.

$$\begin{aligned} f'(x) &= x^2 - x - 2 \\ &= (x-2)(x+1) \\ f''(x) &= 2x - 1 \\ &= 2(x - \frac{1}{2}) \\ f''(-1) &= 2(-1 - \frac{1}{2}) \\ &= -\frac{3}{2} < 0 \\ f''(2) &= 2(2 - \frac{1}{2}) \\ &= \frac{7}{2} > 0 \end{aligned}$$

$f(x)$ has a maximum value of $\frac{19}{6}$.

$f(x)$ has a minimum value of $-\frac{4}{3}$.

4. $f(x) = 4x - 3x^3$

Sol.

$$\begin{aligned} f'(x) &= 4 - 9x^2 \\ 4 - 9x^2 &= 0 \\ x &= \pm\frac{2}{3} \\ f''(x) &= -18x \\ f''\left(\frac{2}{3}\right) &= -12 < 0 \\ f''\left(-\frac{2}{3}\right) &= 12 > 0 \end{aligned}$$

When $x = \frac{2}{3}$, $f(x)$ has a maximum value of $\frac{16}{9}$.

When $x = -\frac{2}{3}$, $f(x)$ has a minimum value of $-\frac{16}{9}$.

Find the coordinates of the extreme points of the following functions (Question 5 to 6):

5. $y = 2x^3 - 3x^2 - 12x - 7$

Sol.

$$\begin{aligned} y' &= 6x^2 - 6x - 12 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x &= 2 \text{ or } -1 \\ y'' &= 12x - 6 \\ y''(2) &= 18 > 0 \\ y''(-1) &= -18 < 0 \end{aligned}$$

$(2, -27)$ is a minimum point.

$(-1, 0)$ is a maximum point.

6. $y = x + \frac{1}{x}$

Sol.

$$\begin{aligned} y' &= 1 - \frac{1}{x^2} \\ 1 - \frac{1}{x^2} &= 0 \\ x &= \pm 1 \\ y'' &= \frac{2}{x^3} \\ y''(1) &= 2 > 0 \\ y''(-1) &= -2 < 0 \end{aligned}$$

$(1, 2)$ is a minimum point.

$(-1, -2)$ is a maximum point.

26.3.2 Exercise 26.3

Find the extreme values of the following functions (Question 1 to 6):

1. $f(x) = \frac{1}{2}x^2 - 3x$
2. $f(x) = 4 + 2x - x^2$
3. $f(x) = -2x^2 + 4x + 7$
4. $f(x) = 3x^2 - 2x + 1$
5. $f(x) = 2x^3 - 9x^2 - 24x - 12$

6. $f(x) = 15 + 9x - 3x^2 - x^3$

Find the coordinates of the extreme points of the following functions (Question 7 to 11):

7. $f(x) = x\left(x^2 - 12\right)$

8. $f(x) = 4x^3 - 3x^2 - 6x + 2$

9. $f(x) = x(x-8)(x-3)$

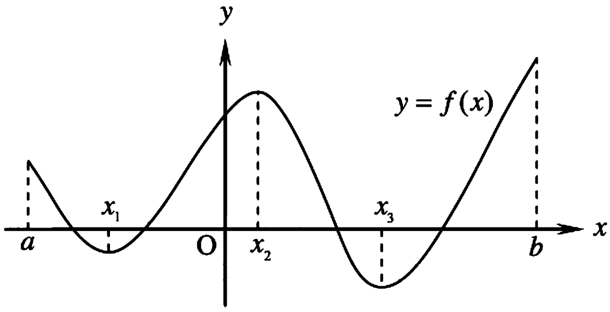
10. $f(x) = 4x^2 + \dfrac{1}{x}$

11. $f(x) = x - 2\sin x, \quad -\pi < x < \pi$

12. Find the stationery points of the function $f(x) = x^2(3 - x)$, and determine whether the stationery points are relative maximum points or relative minimum points.

26.4 Absolute Maximum and Minimum Values of Functions

Shown in the diagram below is the graph of the curve of the function $f(x)$ in the interval $[a, b]$. From the diagram, we know that $f(x_1)$ and $f(x_3)$ are the relative minimum value, while $f(x_2)$ is the relative maximum value. In solving practical problems, we are often concerned with the maximum and minimum values of the function in the entire domain. In the diagram below, the absolute maximum value of the function $f(x)$ is $f(b)$, and the absolute minimum value is $f(x_3)$.



> If the function $f(x)$ is continuous in the close interval $[a, b]$, then the function $f(x)$ must have the absolute maximum value and the absolute minimum value in the interval $[a, b]$.

The function $f(x)$ that is continuous in the open interval (a, b) may not have the absolute maximum value and the absolute minimum value. For example, the function $y = \dfrac{1}{x}$ is continuous in the interval $(0, \infty)$, but it does not have the absolute maximum value and the absolute minimum value. Also in the diagram above, if the domain is defined as the open interval (a, b), then the function $f(x)$ only has the relative minimum value but not the absolute minimum value.

From the diagram above, if the function is continuous in the interval $[a, b]$, we just have to make comparison between all the extreme points and vertices of the function to find the absolute maximum value and the absolute minimum value of the function.

26.4.1 Practice 4

Find the absolute maximum value and the absolute minimum value of the following functions (Question 1 to 2):

1. $f(x) = 3x^3 - 9x + 5, [-2, 2]$

2. $f(x) = x^4 - 2x^2 + 5, [-2, 3]$

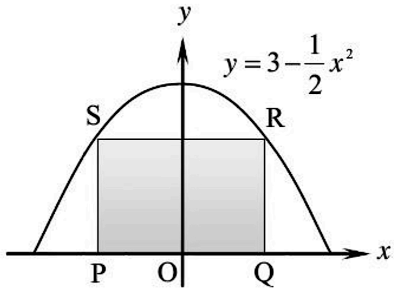
3. If $x + y = 8$, find the absolute minimum value of $x^2 + y^2$.

4. A metal wire with a length of 100cm is bent into a rectangle. Find the width and the length of the rectangle so that the area of the rectangle is the largest.

26.4.2 Exercise 26.4

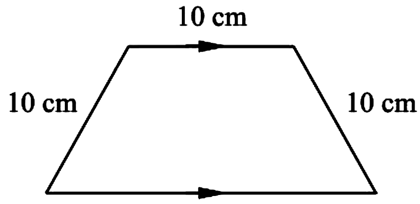
Find the absolute maximum value and the absolute minimum value of the following functions (Question 1 to 3):

1. $f(x) = 5 - 36x + 3x^2 + 4x^3$, $[-1, 2]$
2. $f(x) = 4x^2\left(x^2 - 2\right)$, $[-1, 3]$
3. $f(x) = x^5 - 5x^4 + 5x^3$, $[0, 4]$
4. A metal wire with a length of 60cm is bent into a rectangle. Find the width and the length of the rectangle so that the area of the rectangle is the largest.
5. A metal wire with a length of 100cm is cut into two sections. Each section is bent into a square. Find the length of these two sections of the wire so that the sum of the areas of the two squares is the smallest.
6. As shown in the diagram below, a trapezium has three sides of length 10cm.If the area of the trapezium is the largest, find the length of the fourth side. Hence, find the maximum area of the trapezium.



7. A right cone has a slant height of 9cm. Find the height of the cylinder such that the volume of the cylinder is the largest.
8. A cylinder shaped can with lid has a volume of $250\pi\text{cm}^3$. Find the bottom radius and the height of the can so that the material used is the least.
9. Split the number 20 into two parts such that one part is 4 times the reciprocal of another part, and the the sum of it with with 9 times the reciprocal of another part is the smallest.
10. A metal wire with a length of 150cm is split into two sections, and they are bent into a square and a circle respectively. Find the length of these two sections such that the sum of the area of the square and the circle is the smallest.
11. As shown in the diagram below, a window is formed by a rectangle and a semicircle. The perimeter of the entire window is 300cm. If the area of the window is the largest, find the length of the rectangle. Hence, find the maximum area of the window.

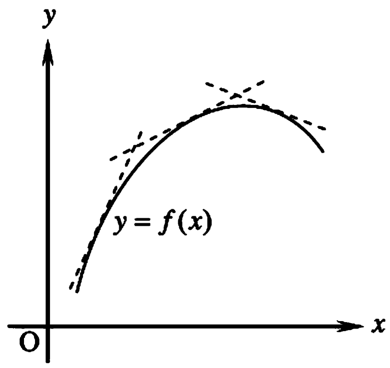
12. In the diagram below, $PQRS$ is a rectangle, the coordinates of P and Q are $(-k, 0)$ and $(k, 0)$ respectively, where $k > 0$, and the two points R and S are on the curve $y = 3 - \frac{1}{2}x^2$. Find the value of k such that the area of the rectangle is the largest.



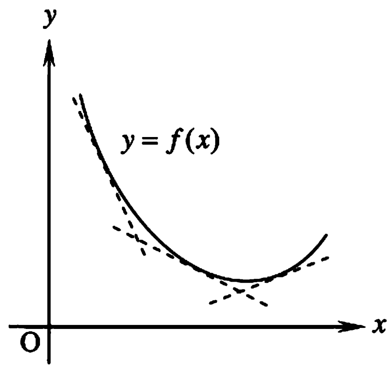
26.5 The Convexity and the Point of Inflection of Functions

For a curve $y = f(x)$

1. In a given interval, if the tangent line of the curve is always above the curve, then the curve convex up in the interval, as shown in the diagram below.

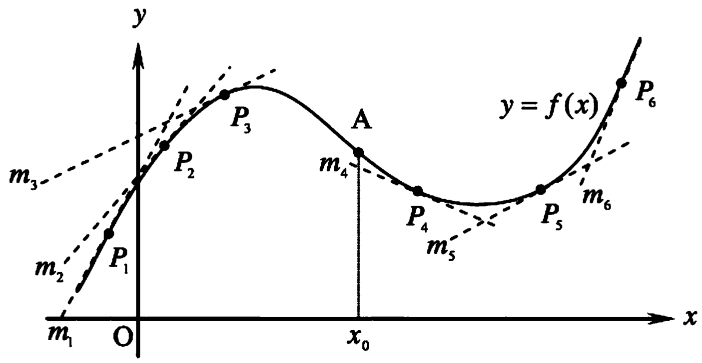


2. In a given interval, if the tangent line of the curve is always below the curve, then the curve is convex down in the interval, as shown in the diagram below.



If two sides of a point on a curve of a function $y = f(x)$ changes their concavity, then the demarcation point is called the point of inflection of the curve.

Now we discuss the way to determine the convexity and the point of inflection of a function. In the diagram below, the left side of the point $x = x_0$ convex down, and the right side convex up.



In the interval $(-\infty, x_0)$ that convex up, when the tangents to the curve cut the curve at P_1, P_2, and P_3 respectively from left to right, the gradients of the tangent lines m_1, m_2, and m_3 are decreasing, i.e. the gradient of the tangent $f'(x)$ is decreasing.

In the interval (x_0, ∞) that convex down, when the tangents to the curve cut the curve at P_4, P_5, and P_6 respectively from left to right, the gradients of the tangent lines m_4, m_5, and m_6 are increasing, i.e. the gradient of the tangent $f'(x)$ is increasing.

We have the following theorem:

Let function $f(x)$ has second derivative $f''(x)$.

- If $f''(x) > 0$ in a given interval, then the curve is convex up;
- If $f''(x) < 0$ in a given interval, then the curve is convex down.

26.5.1 Practice 5

Find the intervals where the following functions are convex up or convex down, and find the points of inflection of the functions:

1. $f(x) = 3x^2 - x^3$
2. $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 2$

26.5.2 Practice 26.5

Find the coordinates of the points of inflection of the following functions (Question 1 to 3):

1. $f(x) = x^3 - 6x + 4$
2. $f(x) = x^3(4 - x)$
3. $f(x) = x^{\frac{7}{3}}$

Find the intervals where the following functions are convex up or convex down, and find the points of inflection of the functions (Question 4 to 6):

4. $f(x) = -(x - 2)^3$
5. $f(x) = 2x^3 - 3x^2 - 36x + 25$
6. $f(x) = x^4 - 2x^3 + 1$

Find the extreme values, the coordinates of the points of inflection, and the intervals where the following functions are convex up or convex down (Question 7 to 8):

7. $f(x) = x(6 - 2x)^2$
8. $f(x) = -\dfrac{2}{1 + x^2}$

26.6 Curve Sketching

Having learnt the derivatives, we can use the concepts of the increasing and decreasing of a function, the convexity and the point of inflection of a function to sketch the curve of a function in a rather accurate way. Listed below are the steps to sketch the curve of a function:

1. Find the point of intersections of the curve with the axes;
2. Solve the equation $f'(x) = 0$, and determine the intervals where the function is increasing or decreasing and the extreme values;
3. Solve the equation $f''(x) = 0$, and determine the intervals where the curve is convex up or convex down and the points of inflection;
4. Draw the curve according to the above information.

The steps above are not necessarily to be followed in the order listed, and can be adjusted according to the actual situation.

26.6.1 Practice 6

Sketch the graph of the function $f(x) = x^3 - 3x^2 + 2$.

26.6.2 Exercise 26.6

Sketch the graph of the following functions (Question 1 to 3):

1. $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x$
2. $f(x) = x^4 - 32x + 10$
3. $f(x) = (x-1)^3(x-2)$
4. Given the function $f(x) = x^3(4-x)$.
   (a) Find the extreme values and the intervals of increasing and decreasing of $f(x)$.
   (b) Find the points of inflection and the intervals of convexity and concavity of $f(x)$.
   (c) Hence, sketch the graph of $f(x)$.

26.7 Rate of Change and Related Rate of Change

The derivative of the function $y = f(x)$ at the point $x = x_0$ is known as the rate of change of the dependent value y with respect to the independent value x at the point $x = x_0$. For example, $\frac{dy}{dx} = 3$ means that when x increases by 1 unit, y increases by 3 units, i.e. y and x are both changing, and they are changing in the ratio of $3 : 1$.

The same can be said that the derivative of the area function $A = A(t)$ at the time $t = t_0$ is the rate of change of the area with respect to the time at the time $t = t_0$. When the rate of change of the area at $t = t_0$ is $\frac{dA}{dt} = 4\text{cm}^2/\text{s}$, it means that the area is increasing in the rate of 4 square centimetres per second; while $\frac{dA}{dt} = -4\text{cm}^2/\text{s}$ means that the area is decreasing in the rate of 4 square centimetres per second.

If multiple variables are correlated by a specific relationship, these variables are all changing with respect to time, then there must be some kind of bonds between their respective rate of changes. This relationship between the rate of changes is called the related rate of change. If $y = f(x)$ is the function of x, and x changes with respect to t, then since y is changing with respect to x, y is also changing with respect to t. In other words, y is also a function of the time t. HEnce, from the chain rule, we can obtain the following relationship:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

i.e. the rate of change of y is correlated to the rate of change of x.

26.7.1 Practice 7

1. A drop of ink gradually spread after being dropped onto a piece of paper. At t seconds, its area is $A = \left(3t^2 + \frac{1}{5}t + 2\right)\text{mm}^2$. Find the rate of change of the ink spread at $t = 2$ second.

   Sol.

   $$\frac{dA}{dt} = 6t + \frac{1}{5}$$

   When $t = 2$,

   $$\begin{aligned}\frac{dA}{dt} &= 6(2) + \frac{1}{5}\\ &= 12 + \frac{1}{5}\\ &= \frac{61}{5}\\ &= 12.2\text{mm}^2/\text{s}\end{aligned}$$

2. The radius of a sphere increases at a rate of 3cm/s. When the radius is 5cm, find the rate of change of the surface area of the sphere.

   Sol.

   $$\begin{aligned}\frac{dr}{dt} &= 3\text{cm/s}\\ A &= 4\pi r^2\\ \frac{dA}{dr} &= 8\pi r\\ \frac{dA}{dt} &= \frac{dA}{dr}\cdot\frac{dr}{dt}\\ &= 8\pi r \cdot 3\text{cm/s}\\ &= 24\pi r\text{cm}^2/\text{s}\end{aligned}$$

   When $r = 5$,

   $$\begin{aligned}\frac{dA}{dt} &= 24\pi(5)\text{cm}^2/\text{s}\\ &= 120\pi\text{cm}^2/\text{s}\end{aligned}$$

26.7.2 Exercise 26.7

1. Water is poured into a container, the relationship between the volume of the water and the time is $V = (2t^2 + 3t)\text{cm}^3$. WHen $t = 3$s, find the rate of change of the volume of the water.

2. One throws a piece of stone into the water. The radius of the ripple on the water surface caused by the stone is increasing at a rate of 0.1m/s. When the radius is 1m, find the rate of change of the area of the ripple.

3. The side length of a square is increasing at a rate of 3cm per second. When the side length is 15cm, find the rate of change of its area.

4. A cube expanded after being heated, the rate of change of its side length is 5cm/s. When the side length is 4cm, find the rate of change of its area.

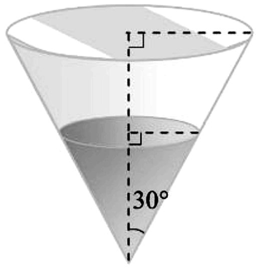
5. The radius of a sphere increases by 1cm per second. When the radius is 3cm, find the rate of change of its volume.

6. The area of a circle increases by 5cm^2 per minute. When the circumference of the circle is 40cm find the rate of change of its radius.

7. The volume of a sphere decreases at a rate of $12\pi\text{cm}^3$ per minute. When the radius of the sphere is 6cm, find the rate of change of its radius and surface area.

8. The surface area of a sphere increase at a rate of $10\text{cm}^2/\text{s}$. When its radius is 5cm, find the rate of change of its radius and volume.

9. Water is poured into the cone shaped container as shown in the diagram below, the rate of rising of the water surface is 1cm per second. When the depth of the water is $2m$, find the rate of change of the water volume.



10. The radius r of a solid cylinder decreases by 0.04cm per second, its height constantly equal to 20cm. When the radius is 2cm, find the rate of change of the surface area of the cylinder.

11. Given the function $y = x^3 + 10$. When the rate of change of y is 27 times the rate of change of x, find the value of x.

12. Water is poured ito a cone shaped container facing downwards with a height of 18m and a base radius of 24m. WHen the height of the water is 6m, find the rate of rising of the water surface.

26.8 Approximate Calculation

In the previous chapter, we have learnt that the derivative of the function $y = f(x)$ is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

From the definition of limit, we know that when Δx is small enough,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta x} \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

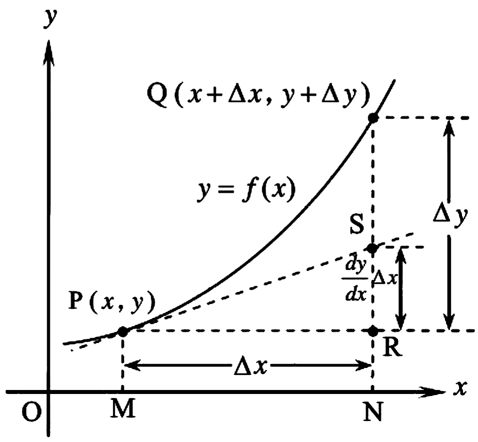
Hence,

$$\boxed{\Delta y \approx \frac{dy}{dx} \Delta x}$$

or $f(x + \Delta x) - f(x) \approx f'(x)\Delta x$, i.e.

$$\boxed{f(x + \Delta x) \approx f(x) + f'(x)\Delta x}$$

The expression above is a simple formula for approximate calculation that can be used to find the approximate value of a function.



26.8.1 Practice 8

1. The side length of a cube increases from 1cm to 1.01cm, how much does its surface area increase approximately?
2. After a metal ball was heated, its radius had increased form 4cm to 4.01cm. Find the approximate increment of its volume and surface area.
3. Find the approximate value of $\sqrt{15}$.

26.8.2 Exercise 26.8

1. The side length of a cube increased from 4cm to 4.001cm, how much does its area increase approximately?
2. The radius of a circle increases from 3cm to 3.01cm, find the approximate increment of its area.
3. The radius of a sphere decrease from 3cm to 2.98cm, find the approximate decrement of its volume,
4. Let $y = 3x^5$. If x is decreased by 0.2%, how many percent does y decrease approximately?
5. If the side length of a cube increases ny 1%, how many percent does its volume increase approximately?
6. Let the surface area of a solid right cylinder with a height of 16cm and a radius of $r$cm be A.
   (a) Prove that $\frac{dA}{dr} = 4\pi(r+8)$.
   (b) If the height of the right cylinder remains the same, using the result obtained from (a), find, when the radius of the right cylinder increases from 4cm to 4.02cm, the approximate increment of its surface area.
7. If $y = \frac{1}{\sqrt{x}}$, find $\frac{dy}{dx}$. Hence, find the approximate value of $\frac{1}{\sqrt{99.4}}$. (Correct your answer to 4 decimal paces)
8. If $y = x^{\frac{1}{4}}$, find $\frac{dy}{dx}$. Hence, find the approximate value of $16.05^{\frac{1}{4}}$. (Correct your answer to 4 decimal places)

26.9 Revision Exercise 26

1. Find the equation of the tangent of the curve $y = x^3 - 3x$ at the point where $x = 3$.

Sol.

$$y = x^3 - 3x$$
$$\frac{dy}{dx} = 3x^2 - 3$$

At $x = 3$, $y = (3)^3 - 3(3) = 18$.

$$\begin{aligned}\text{Gradient of tangent } \frac{dy}{dx} &= 3(3)^2 - 3 \\ &= 27 - 3 \\ &= 24\end{aligned}$$

$$\begin{aligned}\therefore \text{Equation of tangent is } y - 18 &= 24(x - 3) \\ y - 18 &= 24x - 72 \\ y &= 24x - 54\end{aligned}$$

2. Find the equation of the normal of the curve $y = x(x-4)(x+1)$ at the points of intersection of the curve and the x-axis.

Sol.

$$\begin{aligned}y &= x(x-4)(x+1) \\ &= x(x^2 - 3x - 4) \\ &= x^3 - 3x^2 - 4x\end{aligned}$$
$$\frac{dy}{dx} = 3x^2 - 6x - 4$$

When $x = 0$, $y = 0$

$$x(x-4)(x+1) = 0$$
$$x = 0 \text{ or } x = 4 \text{ or } x = -1$$

When $x = 0$,

$$\because \text{Gradient of tangent } \frac{dy}{dx} = 3(0)^2 - 6(0) - 4 = -4$$
$$\therefore \text{Gradient of normal } = \frac{1}{4}$$
$$\begin{aligned}\therefore \text{Equation of normal is } y - 0 &= \frac{1}{4}(x - 0) \\ y &= \frac{1}{4}x \\ x - 4y &= 0\end{aligned}$$

When $x = 4$,

$$\because \text{Gradient of tangent } \frac{dy}{dx} = 3(4)^2 - 6(4) - 4 = 20$$
$$\therefore \text{Gradient of normal } = -\frac{1}{20}$$
$$\begin{aligned}\therefore \text{Equation of normal is } y - 0 &= -\frac{1}{20}(x - 4) \\ x + 20y - 4 &= 0\end{aligned}$$

When $x = -1$,

$$\because \text{Gradient of tangent } \frac{dy}{dx} = 3(-1)^2 - 6(-1) - 4 = 5$$
$$\therefore \text{Gradient of normal } = -\frac{1}{5}$$
$$\begin{aligned}\therefore \text{Equation of normal is } y - 0 &= -\frac{1}{5}(x + 1) \\ x + 5y + 1 &= 0\end{aligned}$$

Hence, the equations of the normals are $x - 4y = 0$, $x + 20y - 4 = 0$ and $x + 5y + 1 = 0$.

3. Given that the curve $y = ax^2 + bx - 10$ passes through the point $(2, 0)$, and that the gradient of the curve at the point is 3. Find the values of a and b.

Sol.

$$y = ax^2 + bx - 10$$
$$\frac{dy}{dx} = 2ax + b$$

Since the curve passes through $(2, 0)$,

$$\begin{aligned} 0 &= a(2)^2 + b(2) - 10 \\ 0 &= 4a + 2b - 10 \\ 4a &= 10 - 2b \\ a &= \frac{10 - 2b}{4} \\ &= \frac{5 - b}{2} \quad \cdots \ (1) \end{aligned}$$

Since the gradient of the curve at the point is 3,

$$\begin{aligned} 3 &= 2a(2) + b \\ 3 &= 4a + b \quad \cdots \ (2) \end{aligned}$$

Substituting (1) into (2),

$$\begin{aligned} 3 &= 4\left(\frac{5 - b}{2}\right) + b \\ 3 &= 2(5 - b) + b \\ 3 &= 10 - 2b + b \\ b &= 7 \end{aligned}$$

Substituting $b = 7$ into (1),

$$\begin{aligned} a &= \frac{5 - 7}{2} \\ &= -1 \end{aligned}$$

Hence, $a = -1$ and $b = 7$.

4. Find the equation of the normal of the curve $y = x + \frac{2}{x}$ at the point $(2, 3)$. If the normal line intersects with the x-axis and y-axis at A and B respectively, find the length of AB.

Sol.

$$y = x + \frac{2}{x}$$
$$\frac{dy}{dx} = 1 - \frac{2}{x^2}$$

At $x = 2$,

$$\begin{aligned} \frac{dy}{dx} &= 1 - \frac{2}{2^2} \\ &= \frac{1}{2} \end{aligned}$$

Hence, the gradient of the normal at the point $(2, 3)$ is -2.

Therefore, the equation of the normal is

$$\begin{aligned} y - 3 &= -2(x - 2) \\ y &= -2x + 7 \end{aligned}$$

When $y = 0$,

$$\begin{aligned} 0 &= -2x + 7 \\ x &= \frac{7}{2} \\ \therefore A &= \left(\frac{7}{2}, 0\right) \end{aligned}$$

When $x = 0$,

$$\begin{aligned} y &= -2(0) + 7 \\ y &= 7 \\ \therefore B &= (0, 7) \end{aligned}$$

$$\begin{aligned} AB &= \sqrt{\left(\frac{7}{2} - 0\right)^2 + (0 - 7)^2} \\ &= \sqrt{\frac{49}{4} + 49} \\ &= \sqrt{\frac{245}{4}} \\ &= \frac{\sqrt{245}}{2} \\ &= \frac{7\sqrt{5}}{2} \end{aligned}$$

Of the following functions, which intervals are the function increasing or decreasing? (Question 5 to 6)

5. $f(x) = 2x^2(6-x)$

   Sol.

$$
\begin{aligned}
f(x) &= 2x^2(6-x) \\
&= 12x^2 - 2x^3 \\
f'(x) &= 24x - 6x^2 \\
f'(x) &= 0 \\
24x - 6x^2 &= 0 \\
x(x-4) &= 0 \\
x &= 0 \text{ or } x = 4
\end{aligned}
$$

   At the interval $(-\infty, 0)$, $f'(x) < 0$, hence $f(x)$ is decreasing at the interval $(-\infty, 0]$.

   At the interval $(0, 4)$, $f'(x) > 0$, hence $f(x)$ is increasing at the interval $[0, 4]$.

   At the interval $(4, \infty)$, $f'(x) < 0$, hence $f(x)$ is decreasing at the interval $[4, \infty)$.

6. $f(x) = 4x^3 - 3x^2 - 6x + 1$

   Sol.

$$
\begin{aligned}
f(x) &= 4x^3 - 3x^2 - 6x + 1 \\
f'(x) &= 12x^2 - 6x - 6 \\
f'(x) &= 0 \\
12x^2 - 6x - 6 &= 0 \\
2x^2 - x - 1 &= 0 \\
(2x+1)(x-1) &= 0 \\
x &= -\frac{1}{2} \text{ or } x = 1
\end{aligned}
$$

   At the interval $\left(-\infty, -\frac{1}{2}\right)$, $f'(x) > 0$, hence $f(x)$ is increasing at the interval $\left(-\infty, -\frac{1}{2}\right]$.
   At the interval $\left(-\frac{1}{2}, 1\right)$, $f'(x) < 0$, hence $f(x)$ is decreasing at the interval $\left[-\frac{1}{2}, 1\right]$.
   At the interval $(1, \infty)$, $f'(x) > 0$, hence $f(x)$ is increasing at the interval $[1, \infty)$.

7. If $x - y = 3$, find the relative minimum value of x^2y.

Sol.

$$x - y = 3$$
$$y = x - 3$$

Let $f(x) = x^2y$,

$$\begin{aligned} f(x) &= x^2y \\ &= x^2(x-3) \\ &= x^3 - 3x^2 \\ f'(x) &= 3x^2 - 6x \\ f'(x) &= 0 \end{aligned}$$
$$3x^2 - 6x = 0$$
$$x(x-2) = 0$$
$$x = 0 \text{ or } x = 2$$

$f''(x) = 6x - 6$

$\because f''(0) = -6 < 0,\ f''(2) = 6 > 0$

$\therefore f(2) = -4$ is a relative minimum value.

8. If $2x^2 + y^2 = 6x$, find the relative maximum value of $x^2 + y^2 + 2x$.

Sol.

$$2x^2 + y^2 = 6x$$
$$y^2 = 6x - 2x^2$$

Let $f(x) = x^2 + y^2 + 2x$,

$$\begin{aligned} f(x) &= x^2 + y^2 + 2x \\ &= x^2 + 6x - 2x^2 + 2x \\ &= -x^2 + 8x \\ f'(x) &= -2x + 8 \\ f'(x) &= 0 \end{aligned}$$
$$-2x + 8 = 0$$
$$x - 4 = 0$$
$$x = 4$$

$f''(x) = -2$

$\because f''(4) = -2 < 0$

$\therefore f(4) = 16$ is a relative maximum value.

9. Given that $y = 18x^2 + 12x + 7$ has a relative minimum value q and the point where $x = p$. Find the value of p and q.

Sol.

$$\begin{aligned} y &= 18x^2 + 12x + 7 \\ y' &= 36x + 12 \\ y' &= 0 \end{aligned}$$
$$36x + 12 = 0$$
$$3x + 1 = 0$$
$$p = x = -\frac{1}{3}$$

When $x = -\frac{1}{3},\ y = 5$

$y'' = 36 > 0$

$\therefore$ The relative minimum value is $q = 5$.

10. There's a rectangular field where one side of it is a wall and the other three sides are fenced. If the total length of the fence is $40m$, find the width and height of the field such that the area of the field is the maximum.

Sol. Let x be the length of the field and y be the width of the field.

$$\begin{aligned}
2x + y &= 40 \\
y &= 40 - 2x \\
A &= xy \\
&= x(40 - 2x) \\
&= 40x - 2x^2 \\
\frac{dA}{dx} &= 40 - 4x \\
\frac{dA}{dx} &= 0 \\
40 - 4x &= 0 \\
x &= 10
\end{aligned}$$

$\because \dfrac{d^2A}{dx^2} = -4 < 0$

$\therefore$ The area of the field is the maximum when $x = 10$.When $x = 10,\ y = 20$.

$\therefore$ The field has a width of $20m$ and a height of $10m$ when the area is the maximum.

11. One side of a rectangle with a perimeter of $18cm$ is revolved about one side to form a cylinder. If the volume of the cylinder is the maximum, find the dimensions of the rectangle and the maximum volume of the cylinder.

Sol.

Let the length of the rectangle be x and the width of the rectangle be y.

$$\begin{aligned}
2x + 2y &= 18 \\
x + y &= 9 \\
y &= 9 - x
\end{aligned}$$

$$\begin{aligned}
V &= \pi r^2 h \\
&= \pi x^2 y \\
&= \pi(9x^2 - x^3) \\
\frac{dV}{dx} &= \pi(18x - 3x^2) \\
\frac{dV}{dx} &= 0 \\
\pi(18x - 3x^2) &= 0 \\
x^2 - 6x &= 0 \\
x(x - 6) &= 0 \\
x &= 6,\ x = 0 \text{ (rejected, } x > 0)
\end{aligned}$$

$\because \dfrac{d^2V}{dx^2} = \pi(18 - 6x) = -18\pi < 0$

$\therefore$ The volume of the cylinder is the maximum when $x = 6$.When $x = 6,\ y = 3$.

$\therefore$ The rectangle has a length of $6cm$ and a width of $3cm$ when the volume is the maximum.

Also, the maximum volume of the cylinder is $V = \pi(6)^2(3) = 108\pi \text{ cm}^3$ when the volume is the maximum.

12. The cross section of a tunnel is a rectangle with a semicircle on top of it. If the area of the cross section is fixed, find the ratio of the radius of the semicircle to the height of the rectangle such that the perimeter of the cross section is the minimum.

Sol.

Let the radius of the semicircle be r and the height of the rectangle be h.

$$\begin{aligned}
A &= \frac{1}{2}\pi r^2 + 2rh \\
2rh &= A - \frac{1}{2}\pi r^2 \\
h &= \frac{A - \frac{1}{2}\pi r^2}{2r} \\
&= \frac{A}{2r} - \frac{1}{4}\pi r \\
P &= \pi r + 2h + 2r \\
&= (\pi + 2)r + \frac{A}{r} - \frac{1}{2}\pi r \\
\frac{dP}{dr} &= \pi + 2 - \frac{A}{r^2} - \frac{1}{2}\pi \\
&= \frac{1}{2}\pi + 2 - \frac{A}{r^2} \\
\frac{dP}{dr} &= 0 \\
\frac{1}{2}\pi + 2 - \frac{A}{r^2} &= 0 \\
\frac{1}{2}\pi + 2 - \left(\frac{1}{2}\pi r^2 + 2rh\right) \cdot \frac{1}{r^2} &= 0 \\
\frac{1}{2}\pi + 2 - \frac{1}{2}\pi - \frac{2}{r}h &= 0 \\
2 - \frac{2}{r}h &= 0 \\
2 &= \frac{2}{r}h \\
2r &= 2h \\
r &= h
\end{aligned}$$

Hence, the ratio of the radius of the semicircle to the height of the rectangle is $1 : 1$.

13. Split 28 into two parts such that the sum of the squares of the one part and the cube of the other part is the minimum.

Sol.

Let the two parts be x and y.

$$\begin{aligned}
x + y &= 28 \\
y &= 28 - x \\
S &= x^2 + y^3 \\
&= x^2 + (28 - x)^3 \\
\frac{dS}{dx} &= 2x - 3(28 - x)^2 \\
\frac{dS}{dx} &= 0
\end{aligned}$$

$$\begin{aligned}
2x - 3(28 - x)^2 &= 0 \\
2x - 3(784 - 56x + x^2) &= 0 \\
2x - 2352 + 168x - 3x^2 &= 0 \\
3x^2 - 170x + 2352 &= 0 \\
(3x - 98)(x - 24) &= 0
\end{aligned}$$

$$x = 24 \text{ or } x = \frac{98}{3}$$

$$\begin{aligned}
\frac{d^2S}{dx^2} &= 2 + 6(28 - x) \\
&= 2 + 168 - 6x \\
&= -6x + 170
\end{aligned}$$

When $x = 24$, $\frac{d^2S}{dx^2} = -6(24) + 170$

$= 26 > 0$

When $x = \frac{98}{3}$, $\frac{d^2S}{dx^2} = -6\left(\frac{98}{3}\right) + 170$

$= -26 < 0$

$\because$ When $x = 24$, $\frac{d^2S}{dx^2} > 0$,

$\therefore$ The sum of the squares of the one part and the cube of the other part is the minimum when $x = 24$.

$\because$ When $x = 24$, $y = 4$.

$\therefore$ The two parts are 24 and 4.

14. The capacity of a cylindrical can is fixed. If the material used to make the can is the minimum, what should be the ratio of the radius of the base to the height of the can?

Sol.

Let the radius of the base be r and the height of the can be h.

$$\begin{aligned}
V &= \pi r^2 h \\
h &= \frac{V}{\pi r^2} \\
A &= 2\pi r^2 + 2\pi r h \\
&= 2\pi r^2 + \frac{2V}{r} \\
\frac{dA}{dr} &= 4\pi r - \frac{2V}{r^2} \\
\frac{dA}{dr} &= 0 \\
4\pi r - \frac{2V}{r^2} &= 0 \\
2\pi r^3 - \pi r^2 h &= 0 \\
2r^3 - r^2 h &= 0 \\
2r - h &= 0 \\
2r &= h \\
\frac{r}{h} &= \frac{1}{2}
\end{aligned}$$

Hence, the ratio of the radius of the base to the height of the can is $1 : 2$.

Find the coordinate of the point of inflection of the following functions. (Question 15 to 16)

15. $y = x^3 - 2$

16. $3x + (2 - x)^3$

17. Given the function $y = \dfrac{x}{1 - x^2}$. Find the extreme values of the function, and determine the coordinates of the convex intervals and the point of inflection.

18. Given the function $y = \dfrac{x}{x^2 + 1}$.

    (a) Find the coordinates of the stationary points.

    (b) Determine which intervals the function is increasing or decreasing.

    (c) Find the coordinates of the convex intervals and the point of inflection.

Construct the graph of the following functions. (Question 19 to 20)

19. $y = x^3 - 5x^2 + 3x - 2$

20. $y = x^3 - 3x^2 + 4$

21. In a container, the relationship between the volume of water V (cm^3) and the depth of water x (cm) is given by the equation $V = 4x^2 + \frac{1}{6}x^3$. If the water is poured into the container at a rate of 6 cm^3 per second, find the rate of change of the depth of water when $x = 2$ cm.

22. The water is poured into a conical pool with a width and a base radius of 20m and 10m respectively at a rate of 5m^3/min. When the height of the water is 10cm, find

    (a) the rate of increasing of the height of the water.

(b) the rate of change of the radius of the water surface.

23. The radius of a spherical container decreases from 4cm to 3.95cm. Find the approximate amount of decrease in the volume and the surface area of the container.

24. The capacity of water of a spherical container is given by $V = \left[\frac{\pi h^2}{3}(15-h)\right]\mathrm{cm}^3$, where h is the depth of the water. Find the approximate amount of increase in the capacity of the container when the depth of the water increases from 4cm to 4.01cm.

25. In a bowl, when the height of the water is $h$cm, the volume of the water is given by $V = \left(h^2 + 3h^2 + 11h\right)\mathrm{cm}^3$. When the height of the water is $7cm$, pour an additional $\Delta V\mathrm{cm}^3$ of water into the bowl. Find the approximate amount of increase in the height of the water.

26. If $y = \frac{1}{\sqrt[3]{[x]}}$, find $\frac{dy}{dx}$. Hence, find the approximate value of $\frac{1}{\sqrt[3]{[130}}$. (Correct to 3 decimal places)