Exercise 5a

1. Given that the coordinates of a fixed point are $(-1, 2)$, the equation of a fixed line is $x - y + 4 = 0$, the ratio of the distance between a moving point and the fixed point to the distance between the moving point and the fixed line is $\dfrac{\sqrt{2}}{2}$. Explain the graph of the locus of the moving point P. Hence, find its equation.

   Sol.

   From the definition of conic section, the locus of the moving point P is a conic section. Since the ratio is $\dfrac{\sqrt{2}}{2}$, the eccentricity is smaller than 1, so the conic section is an ellipse.

   Let the coordinates of the moving point P be (x, y), we have

$$\begin{aligned}
\frac{\sqrt{(x+1)^2+(y-2)^2}}{\left|\dfrac{x-y+4}{\sqrt{2}}\right|} &= \frac{\sqrt{2}}{2} \\
\frac{(x+1)^2+(y-2)^2}{\dfrac{(x-y+4)^2}{2}} &= \frac{1}{2} \\
4(x+1)^2+4(y-2)^2 &= (x-y+4)^2 \\
4(x^2+2x+1)+4(y^2-4y+4) &= x^2+y^2+16-2xy+8x-8y \\
4x^2+8x+4+4y^2-16y+16 &= x^2+y^2+16-2xy+8x-8y \\
3x^2+3y^2+2xy-8y+4 &= 0 \qquad \square
\end{aligned}$$

2. The distance from a moving point P and the point $A(0, 2)$ is equal to the distance between P and the x-axis. Find the equation of the locus of P.

   Sol.

$$\begin{aligned}
\sqrt{x^2+(y-2)^2} &= y \\
x^2+(y-2)^2 &= y^2 \\
x^2+y^2-4y+4 &= y^2 \\
x^2-4y+4 &= 0 \qquad \square
\end{aligned}$$

3. Given the coordinates of the focus point, the equation of the directrix, and the eccentricity of a conic section, find the equation of the conic section:

   (a) $(1, 1)$, $x - y - 1 = 0$, $e = 1$

   Sol.

$$\begin{aligned}
\frac{\sqrt{(x-1)^2+(y-1)^2}}{\left|\dfrac{x-y-1}{\sqrt{2}}\right|} &= 1 \\
\frac{(x-1)^2+(y-1)^2}{(x-y-1)^2} &= \frac{1}{2} \\
2(x-1)^2+2(y-1)^2 &= (x-y-1)^2 \\
2(x^2-2x+1)+2(y^2-2y+1) &= x^2+y^2+1-2xy-2x+2y \\
2x^2-4x+2+2y^2-4y+2 &= x^2+y^2+1-2xy-2x+2y \\
x^2+y^2+2xy-2x-6y+3 &= 0 \qquad \square
\end{aligned}$$

(b) $(1,0)$, $2x+y=0$, $e=1$

Sol.

$$
\begin{aligned}
\frac{\sqrt{(x-1)^2+y^2}}{\left|\dfrac{2x+y}{\sqrt{5}}\right|} &= 1 \\
\frac{(x-1)^2+y^2}{\dfrac{(2x+y)^2}{5}} &= 1 \\
5(x-1)^2+5y^2 &= (2x+y)^2 \\
5(x^2-2x+1)+5y^2 &= 4x^2+4xy+y^2 \\
5x^2-10x+5+5y^2 &= 4x^2+4xy+y^2 \\
x^2-y^2-4xy-10x+5 &= 0 \qquad \square
\end{aligned}
$$

(c) $(2,0)$, $x=0$, $e=1$

Sol.

$$
\begin{aligned}
\frac{\sqrt{(x-2)^2+y^2}}{\left|\dfrac{x}{\sqrt{2}}\right|} &= 1 \\
\frac{(x-2)^2+y^2}{\dfrac{x^2}{2}} &= 1 \\
2(x-2)^2+2y^2 &= x^2 \\
2(x^2-4x+4)+2y^2 &= x^2 \\
2x^2-8x+8+2y^2 &= x^2 \\
2y^2-8x+8 &= 0 \\
y^2-4x+4 &= 0 \qquad \square
\end{aligned}
$$

(d) $(0,0)$, $3x+3y+1=0$, $e=\dfrac{2}{3}$

Sol.

$$
\begin{aligned}
\frac{\sqrt{x^2+y^2}}{\left|\dfrac{3x+3y+1}{\sqrt{18}}\right|} &= \frac{2}{3} \\
\frac{x^2+y^2}{\dfrac{(3x+3y+1)^2}{18}} &= \frac{4}{9} \\
81(x^2+y^2) &= 2(3x+3y+1)^2 \\
81x^2+81y^2 &= 2(9x^2+18xy+9y^2+6x+6y+1) \\
81x^2+81y^2 &= 18x^2+36xy+18y^2+12x+12y+2 \\
63x^2+63y^2-36xy-12x-12y-2 &= 0 \qquad \square
\end{aligned}
$$

(e) $(-4, 0)$, $y + 3 = 0$, $e = \dfrac{1}{4}$

Sol.

$$\begin{aligned}
\frac{\sqrt{(x+4)^2 + y^2}}{\left|\dfrac{y+3}{\sqrt{1}}\right|} &= \frac{1}{4} \\
\frac{(x+4)^2 + y^2}{(y+3)^2} &= \frac{1}{16} \\
16(x+4)^2 + 16y^2 &= (y+3)^2 \\
16(x^2 + 8x + 16) + 16y^2 &= y^2 + 6y + 9 \\
16x^2 + 128x + 256 + 16y^2 &= y^2 + 6y + 9
\end{aligned}$$

$16x^2 + 15y^2 + 128x - 6y + 247 = 0 \qquad \square$

(f) $(1, 2)$, $x + y + 1 = 0$, $e = 2$

Sol.

$$\begin{aligned}
\frac{\sqrt{(x-1)^2 + (y-2)^2}}{\left|\dfrac{x+y+1}{\sqrt{2}}\right|} &= 2 \\
\frac{(x-1)^2 + (y-2)^2}{\dfrac{(x+y+1)^2}{2}} &= 4 \\
(x-1)^2 + (y-2)^2 &= 2(x+y+1)^2 \\
x^2 - 2x + 1 + y^2 - 4y + 4 &= 2(x^2 + y^2 + 1 + 2xy + 2x + 2y) \\
x^2 - 2x + 1 + y^2 - 4y + 4 &= 2x^2 + 2y^2 + 2 + 4xy + 4x + 4y
\end{aligned}$$

$x^2 + y^2 + 4xy + 6x + 8y - 3 = 0 \qquad \square$

(g) $(0, -1)$, $x + 2y - 1 = 0$, $e = \sqrt{3}$

Sol.

$$\begin{aligned}
\frac{\sqrt{x^2 + (y+1)^2}}{\left|\dfrac{x+2y-1}{\sqrt{5}}\right|} &= \sqrt{3} \\
\frac{x^2 + (y+1)^2}{\dfrac{(x+2y-1)^2}{5}} &= 3 \\
5x^2 + 5(y+1)^2 &= 3(x+2y-1)^2 \\
5x^2 + 5(y^2 + 2y + 1) &= 3(x^2 + 4xy - 2x + 4y^2 - 4y + 1) \\
5x^2 + 5y^2 + 10y + 5 &= 3x^2 + 12xy - 6x + 12y^2 - 12y + 3
\end{aligned}$$

$2x^2 - 7y^2 - 12xy + 6x + 22y + 2 = 0 \qquad \square$