Solution Book of Mathematic

Ssnior 2 Part I

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Contents

11.0.1 Exercise 14.5b

1. Given $\begin{vmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix} = 1$, Find the value of the following determinants.

(a) $\begin{vmatrix} 2 & 0 & -1 \\ -2 & -1 & 2 \\ 3 & -2 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 2 & 0 & -1 \\ -2 & -1 & 2 \\ 3 & -2 & 1 \end{vmatrix}$$
$$= \left|\begin{pmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{pmatrix}'\right|$$
$$= -1 \qquad \text{(Theorem 1)}$$

(b) $\begin{vmatrix} 2 & 0 & -1 \\ 3 & -2 & 1 \\ -2 & -1 & 2 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 2 & 0 & -1 \\ 3 & -2 & 1 \\ -2 & -1 & 2 \end{vmatrix}$$

Sol.

$$\begin{vmatrix} 2 & 0 & -1 \\ 3 & -2 & 1 \\ -2 & -1 & 2 \end{vmatrix}$$
$$= -\begin{vmatrix} 2 & 0 & -1 \\ -2 & -1 & 2 \\ 3 & -2 & 1 \end{vmatrix}$$
$$= -\left|\begin{pmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{pmatrix}'\right|$$
$$= 1 \qquad \text{(Theorem 1)}$$

(c) $\begin{vmatrix} 4 & -2 & 3 \\ 0 & -2 & -4 \\ -2 & 2 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 4 & -2 & 3 \\ 0 & -2 & -4 \\ -2 & 2 & 1 \end{vmatrix}$$
$$= 2 \times 2 \begin{vmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$
$$= -4$$

(d) $\begin{vmatrix} -3 & -2 & -2 \\ 2 & -1 & 0 \\ -1 & 2 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} -3 & -2 & -2 \\ 2 & -1 & 0 \\ -1 & 2 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} 3 & -2 & 2 \\ -2 & -1 & 0 \\ 1 & 2 & -1 \end{vmatrix}$$
$$= \begin{vmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$
$$= 1$$

(e) $\begin{vmatrix} 0 & 2 & 5 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 0 & 2 & 5 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} 2 + (2 \times (-1)) & -2 + (-2 \times 2) & 3 + (2 \times 1) \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$
$$= -1$$

(f) $\begin{vmatrix} 2 & 4 & 3 \\ 0 & -5 & -2 \\ -1 & 4 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 2 & 4 & 3 \\ 0 & -5 & -2 \\ -1 & 4 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} 2 & -2 + (2 \times 3) & 3 \\ 0 & -1 + (2 \times (-2)) & -2 \\ -1 & 2 + (2 \times 1) & 1 \end{vmatrix}$$
$$= -1$$

2. Prove the following equations using identities of determinants without expanding them.

(a) $\begin{vmatrix} 1 & 0 & 3 \\ 4 & 6 & 12 \\ 3 & 3 & 9 \end{vmatrix} = 0$

Proof.

$$\begin{aligned} L.H.S. &= \begin{vmatrix} 1 & 0 & 3 \\ 4 & 6 & 12 \\ 3 & 3 & 9 \end{vmatrix} \\ &= 2 \times 3 \begin{vmatrix} 1 & 0 & 3 \\ 2 & 3 & 6 \\ 1 & 1 & 3 \end{vmatrix} \\ &= 2 \times 3 \times 3 \begin{vmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix} \\ &= 0 = R.H.S. && \text{(Theorem 3)} \end{aligned}$$

(b) $\begin{vmatrix} 2 & -1 & 3 \\ -1 & 1 & 2 \\ 4 & -2 & 6 \end{vmatrix} = 0$

Proof.

$$\begin{aligned} L.H.S. &= \begin{vmatrix} 2 & -1 & 3 \\ -1 & 1 & 2 \\ 4 & -2 & 6 \end{vmatrix} \\ &= 2 \begin{vmatrix} 2 & -1 & 3 \\ -1 & 1 & 2 \\ 2 & -1 & 3 \end{vmatrix} \\ &= 0 = R.H.S. && \text{(Theorem 3)} \end{aligned}$$

(c) $\begin{vmatrix} 3 & 2 & 2 \\ 9 & 6 & 5 \\ 12 & 8 & 8 \end{vmatrix} = 0$

Proof.

$$\begin{aligned} L.H.S. &= \begin{vmatrix} 3 & 2 & 2 \\ 9 & 6 & 5 \\ 12 & 8 & 8 \end{vmatrix} \\ &= 4 \begin{vmatrix} 3 & 2 & 2 \\ 9 & 6 & 5 \\ 3 & 2 & 2 \end{vmatrix} \\ &= 0 = R.H.S. && \text{(Theorem 3)} \end{aligned}$$

(d) $\begin{vmatrix} 10 & 8 & 2 \\ 15 & 12 & 3 \\ 20 & 32 & 12 \end{vmatrix} = 0$

Proof.

$$\begin{aligned} L.H.S. &= \begin{vmatrix} 10 & 8 & 2 \\ 15 & 12 & 3 \\ 20 & 32 & 12 \end{vmatrix} \\ &= 2 \times 3 \times 4 \begin{vmatrix} 5 & 4 & 1 \\ 5 & 4 & 1 \\ 5 & 8 & 3 \end{vmatrix} \\ &= 2 \times 3 \times 4 \times 5 \times 4 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} \\ &= 0 = R.H.S. && \text{(Theorem 3)} \end{aligned}$$

(e) $\begin{vmatrix} 1 & -4 & 5 \\ 2 & 3 & -2 \\ 7 & 3 & 0 \end{vmatrix} = \begin{vmatrix} -4 & 3 & 3 \\ 5 & -2 & 0 \\ 1 & 2 & 7 \end{vmatrix}$

Proof.

$$\begin{aligned} L.H.S &= \begin{vmatrix} 1 & -4 & 5 \\ 2 & 3 & -2 \\ 7 & 3 & 0 \end{vmatrix} \\ &= - \begin{vmatrix} 5 & -4 & 1 \\ -2 & 3 & 2 \\ 0 & 3 & 7 \end{vmatrix} \\ &= \begin{vmatrix} -4 & 5 & 1 \\ 3 & -2 & 2 \\ 3 & 0 & 7 \end{vmatrix} && \text{(Theorem 2)} \\ &= \left| \begin{pmatrix} -4 & 5 & 1 \\ 3 & -2 & 2 \\ 3 & 0 & 7 \end{pmatrix}' \right| && \text{(Theorem 1)} \\ &= \begin{vmatrix} -4 & 3 & 3 \\ 5 & -2 & 0 \\ 1 & 2 & 7 \end{vmatrix} = R.H.S. \end{aligned}$$

(f) $\begin{vmatrix} -6 & 6 & 3 \\ 0 & -9 & -3 \\ 3 & -3 & -6 \end{vmatrix} = -27 \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ -2 & 2 & -1 \end{vmatrix}$

Proof.

$L.H.S.$

$$= \begin{vmatrix} -6 & 6 & 3 \\ 0 & -9 & -3 \\ 3 & -3 & -6 \end{vmatrix}$$

$$= 3 \times 3 \times 3 \begin{vmatrix} -2 & 2 & 1 \\ 0 & -3 & -1 \\ 1 & -1 & -2 \end{vmatrix}$$

$$= -27 \begin{vmatrix} 1 & -1 & -2 \\ 0 & -3 & -1 \\ -2 & 2 & 1 \end{vmatrix} \qquad \text{(Theorem 2)}$$

$$= 27 \begin{vmatrix} -1 & 1 & -2 \\ -3 & 0 & -1 \\ 2 & -2 & 1 \end{vmatrix} \qquad \text{(Theorem 2)}$$

$$= -27 \begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ -2 & 2 & -1 \end{vmatrix} = R.H.S. \qquad \text{(Theorem 4)}$$

(g) $\begin{vmatrix} 1 & 0 & -3 \\ 3 & -2 & 4 \\ 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ -1 & 2 & 4 \\ 7 & 3 & -2 \end{vmatrix}$

Proof.

$$L.H.S. = \begin{vmatrix} 1 & 0 & -3 \\ 3 & -2 & 4 \\ 1 & 3 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 + (2 \times 0) & 0 & -3 \\ 3 + (2 \times (-2)) & -2 & 4 \\ 1 + (2 \times 3) & 3 & -2 \end{vmatrix} \qquad \text{(Theorem 6)}$$

$$= \begin{vmatrix} 1 & 0 & -3 \\ -1 & 2 & 4 \\ 7 & 3 & -2 \end{vmatrix} = R.H.S.$$

(h) $\begin{vmatrix} 5 & 1 & -1 \\ 2 & -1 & -1 \\ 1 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 0 \\ 4 & -1 & -2 \\ 5 & -2 & 2 \end{vmatrix}$

Proof.

$L.H.S.$

$$= \begin{vmatrix} 5 & 1 & -1 \\ 2 & -1 & -1 \\ 1 & -2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 5 + (-2 \times 1) & 1 & -1 + 1 \\ 2 + (-2 \times (-1)) & -1 & -2 - 1 \\ 1 + (-2 \times (-2)) & -2 & 4 - 2 \end{vmatrix} \qquad \text{(Theorem 6)}$$

$$= \begin{vmatrix} 3 & 1 & 0 \\ 4 & -1 & -2 \\ 5 & -2 & 2 \end{vmatrix} = R.H.S.$$