How to Prove It: A Structured Approach, Second Edition

Exercises for Section 1.2

1. 1. Make truth tables for the following formulas:

   (a) $\neg P \lor Q$.

   Solution.

P	Q	$\neg P$	$\neg P \lor Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

■

   (b) $(S \lor G) \land (\neg S \lor \neg G)$.

   Solution.

S	G	$\neg S$	$\neg G$	$S \lor G$	$\neg S \lor \neg G$	$(S \lor G) \land (\neg S \lor \neg G)$
T	T	F	F	T	F	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	T	F

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2. Make truth tables for the following formulas:

   (a) $\neg[P \land (Q \lor \neg P)]$.

   Solution.

P	Q	$\neg P$	$Q \lor \neg P$	$P \land (Q \lor \neg P)$	$\neg[P \land (Q \lor \neg P)]$
T	T	F	T	T	F
T	F	F	F	F	T
F	T	T	T	F	T
F	F	T	T	F	T

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(b) $(P \vee Q) \wedge (\neg P \vee R)$.

Solution.

P	Q	R	$\neg P$	$P \vee Q$	$\neg P \vee R$	$(P \vee Q) \wedge (\neg P \vee R)$
T	T	T	F	T	T	T
T	T	F	F	T	F	F
T	F	T	F	T	T	T
T	F	F	F	T	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	F	T	F
F	F	F	T	F	T	F

■

3. In this exercise we will use the symbol $+$ to mean exclusive or. In other words, $P + Q$ means " P or Q, but not both."

(a) Make a truth table for $P + Q$.

Solution.

P	Q	$P + Q$
T	T	F
T	F	T
F	T	T
F	F	F

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(b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P + Q$. Justify your answer with a truth table.

Solution.

$$P + Q \equiv (P \vee Q) \wedge \neg(P \wedge Q)$$

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$P \vee Q$	$(P \vee Q) \wedge \neg(P \wedge Q)$	$P + Q$
T	T	T	F	T	F	F
T	F	F	T	T	T	T
F	T	F	T	T	T	T
F	F	F	T	F	F	F

■

4. Find a formula using only the connectives $\wedge$ and $\neg$ that is equivalent to $P \vee Q$. Justify your answer with a truth table.

Solution.

$$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$$

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$	$P \vee Q$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	T	F	F

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5. Some mathematicians use the symbol $\downarrow$ to mean nor. In other words, $P \downarrow Q$ means "neither P nor Q."

    (a) Make a truth table for $P \downarrow Q$.

    Solution.

P	Q	$P \downarrow Q$
T	T	F
T	F	F
F	T	F
F	F	T

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    (b) Find a formula using only the connectives $\wedge, \vee$, and $\neg$ that is equivalent to $P \downarrow Q$.

    Solution.

    $$P \downarrow Q \equiv \neg(P \vee Q)$$

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$P \downarrow Q$
T	T	T	F	F
T	F	T	F	F
F	T	T	F	F
F	F	F	T	T

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    (c) Find formulas using only the connective $\downarrow$ that are equivalent to $\neg P$, $P \vee Q$, and $P \wedge Q$.

    Solution.

    $$\neg P \equiv \neg(P \wedge P) \equiv P \downarrow P$$
    $$P \vee Q \equiv \neg(P \downarrow Q) \equiv (P \downarrow Q) \downarrow (P \downarrow Q)$$
    $$P \wedge Q \equiv \neg\neg(P \wedge Q) \equiv \neg(P \downarrow Q) \equiv (P \downarrow Q) \downarrow (P \downarrow Q)$$

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6. Some mathematicians write $P \mid Q$ to mean " P and Q are not both true." (This connective is called nand, and is used in the study of circuits in computer science.)

    (a) Make a truth table for $P \mid Q$.

    Solution.

P	Q	$P \mid Q$
T	T	F
T	F	T
F	T	T
F	F	T

    ■

(b) Find a formula using only the connectives $\wedge, \vee$, and $\neg$ that is equivalent to $P \mid Q$.

Solution.

$$P \mid Q \equiv \neg(P \wedge Q)$$

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(c) Find formulas using only the connective $\mid$ that are equivalent to $\neg P$, $P \vee Q$, and $P \wedge Q$.

Solution.

$$\neg P \equiv P \mid P$$
$$P \vee Q \equiv \neg P \mid \neg Q \equiv (P \mid P) \mid (Q \mid Q)$$
$$P \wedge Q \equiv \neg(P \mid Q) \equiv (P \mid Q) \mid (P \mid Q)$$

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7. Use truth tables to determine whether or not the arguments in exercise 7 of Section 1.1 are valid.

(a) $\neg(P \wedge R) \wedge (P \vee Q) \wedge R \Rightarrow Q$.

Solution.

P	Q	R	$P \wedge R$	$\neg(P \wedge R)$	$P \vee Q$	$\neg(P \wedge R) \wedge (P \vee Q) \wedge R$
T	T	T	T	F	T	F
T	T	F	F	T	T	F
T	F	T	T	F	T	F
T	F	F	F	T	T	F
F	T	T	F	T	T	T
F	T	F	F	T	T	F
F	F	T	F	T	F	F
F	F	F	F	T	F	F

where
P is the statement "Pete will win the math prize",
Q is the statement "Pete will win the chemistry prize",
R is the statement "Jane will win the math prize",

The result is true for all cases where the premises are true, hence the argument is valid.

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(b) $(P \vee \neg P) \wedge (Q \vee \neg Q) \wedge \neg(\neg P \wedge \neg Q) \Rightarrow \neg(P \wedge Q)$.

Solution.

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$	$\neg(P \wedge Q)$	$(P \vee \neg P) \wedge (Q \vee \neg Q) \wedge \neg(\neg P \wedge \neg Q)$
T	T	F	F	F	T	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	T	F	T	F

where P is the statement "The main course will be beef",
Q is the statement "The vegetable will be peas",

The conclusion is false but the premises are all true when P and Q are true, hence the argument is invalid.

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(c) $(P \vee Q) \wedge (\neg R \vee Q) \wedge (P \vee \neg R) \Rightarrow (P \vee \neg R)$.

Solution.

P	Q	R	$\neg R$	$P \vee Q$	$\neg R \vee Q$	$P \vee \neg R$	$(P \vee Q) \wedge (\neg R \vee Q) \wedge (P \vee \neg R)$
T	T	T	F	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	F	T	F
T	F	F	T	T	T	T	T
F	T	T	F	T	T	F	F
F	T	F	T	T	T	T	T
F	F	T	F	F	F	F	F
F	F	F	T	F	T	T	F

where P is the statement "John is telling the truth",
Q is the statement "Bill is telling the truth",
R is the statement "Sam is telling the truth",

The conclusion is true for all cases where the conjunction of premises are true, hence the argument is valid.

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(d) $(P \wedge R) \vee (Q \wedge \neg R) \Rightarrow \neg(P \wedge Q)$

Solution.

P	Q	R	$\neg R$	$P \wedge R$	$Q \wedge \neg R$	$\neg(P \wedge Q)$	$(P \wedge R) \vee (Q \wedge \neg R)$
T	T	T	F	T	F	F	T
T	T	F	T	F	T	F	T
T	F	T	F	T	F	T	T
T	F	F	T	F	F	T	F
F	T	T	F	F	F	T	F
F	T	F	T	F	T	T	T
F	F	T	F	F	F	T	F
F	F	F	T	F	F	T	F

where P is the statement "Sales will go up",
Q is the statement "Expenses will go up",
R is the statement "The boss will be happy",

there are cases where the conjunction of premises are true but the conclusion is false, hence the argument is invalid.

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8. Use truth tables to determine which of the following formulas are equivalent to each other:

(a) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.

Solution.

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg P \wedge \neg Q$	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$
T	T	F	F	T	F	T
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	F	T	T	F	T	T

(b) $\neg P \vee Q$.

Solution.

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(c) $(P \vee \neg Q) \wedge (Q \vee \neg P)$.

Solution.

P	Q	$\neg P$	$\neg Q$	$P \vee \neg Q$	$Q \vee \neg P$	$(P \vee \neg Q) \wedge (Q \vee \neg P)$
T	T	F	F	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	T	T	T

(d) $\neg(P \vee Q)$.

Solution.

P	Q	$P \vee Q$	$\neg(P \vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

(e) $(Q \wedge P) \vee \neg P$.

Solution.

P	Q	$\neg P$	$Q \wedge P$	$(Q \wedge P) \vee \neg P$
T	T	F	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	T

Hence, (a) and (c) are equivalent, (b) and (e) are equivalent.

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9. Use truth tables to determine which of these statements are tautologies, which are contradictions, and which are neither:

(a) $(P \vee Q) \wedge (\neg P \vee \neg Q)$.

Solution.

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg P \vee \neg Q$	$(P \vee Q) \wedge (\neg P \vee \neg Q)$
T	T	F	F	T	F	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	T	F

Hence, the statement is neither a tautology nor a contradiction.

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(b) $(P \vee Q) \wedge (\neg P \wedge \neg Q)$.

Solution.

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg P \wedge \neg Q$	$(P \vee Q) \wedge (\neg P \wedge \neg Q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

Hence, the statement is a contradiction.

■

(c) $(P \vee Q) \vee (\neg P \vee \neg Q)$.

Solution.

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg P \vee \neg Q$	$(P \vee Q) \vee (\neg P \vee \neg Q)$
T	T	F	F	T	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	T	T

Hence, the statement is a tautology.

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(d) $[P \wedge (Q \vee \neg R)] \vee (\neg P \vee R)$.

Solution.

P	Q	R	$\neg P$	$\neg R$	$Q \vee \neg R$	$P \wedge (Q \vee \neg R)$	$[P \wedge (Q \vee \neg R)] \vee (\neg P \vee R)$
T	T	T	F	F	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	F	F	F	T
T	F	F	F	T	T	T	T
F	T	T	T	F	T	F	T
F	T	F	T	T	T	F	T
F	F	T	T	F	T	F	T
F	F	F	T	T	T	F	T

Hence, the statement is a tautology.

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10. Use truth tables to check these laws:

(a) The second DeMorgan's law. (The first was checked in the text.)

Solution.

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

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(b) The distributive laws.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

P	Q	R	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

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11. Use the laws stated in the text to find simpler formulas equivalent to these formulas. (See Examples 1.2.5 and 1.2.7.)

(a) $\neg(\neg P \wedge \neg Q)$.

Solution.

$$\begin{aligned} \neg(\neg P \wedge \neg Q) &\equiv \neg\neg P \vee \neg\neg Q && \text{(DeMorgan's law)} \\ &\equiv P \vee Q && \text{(double negation law)} \end{aligned}$$

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(b) $(P \wedge Q) \vee (P \wedge \neg Q)$.

Solution.

$$\begin{aligned} (P \wedge Q) \vee (P \wedge \neg Q) &\equiv P \wedge (Q \vee \neg Q) && \text{(distributive law)} \\ &\equiv P \wedge \top && \text{(complement law)} \\ &\equiv P && \text{(identity law)} \end{aligned}$$

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(c) $\neg(P \wedge \neg Q) \vee (\neg P \wedge Q)$.

Solution.

$$\begin{aligned}
\neg(P \wedge \neg Q) \vee (\neg P \wedge Q) &\equiv (\neg P \vee Q) \vee (\neg P \wedge Q) && \text{(DeMorgan's law)}\\
&\equiv [(\neg P \vee Q) \vee \neg P] \wedge [(\neg P \vee Q) \vee Q] && \text{(distributive law)}\\
&\equiv (\neg P \vee \neg P \vee Q) \wedge (\neg P \vee Q \vee Q) && \text{(associative law)}\\
&\equiv (\neg P \vee Q) \wedge (\neg P \vee Q) && \text{(idempotent law)}\\
&\equiv \neg P \vee Q && \text{(idempotent law)}
\end{aligned}$$

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12. Use the laws stated in the text to find simpler formulas equivalent to these formulas. (See Examples 1.2.5 and 1.2.7.)

(a) $\neg(\neg P \vee Q) \vee (P \wedge \neg R)$.

Solution.

$$\begin{aligned}
\neg(\neg P \vee Q) \vee (P \wedge \neg R) &\equiv (\neg\neg P \wedge \neg Q) \vee (P \wedge \neg R) && \text{(DeMorgan's law)}\\
&\equiv (P \wedge \neg Q) \vee (P \wedge \neg R) && \text{(double negation law)}\\
&\equiv P \wedge (\neg Q \vee \neg R) && \text{(distributive law)}\\
&\equiv P \wedge \neg(Q \wedge R) && \text{(DeMorgan's law)}
\end{aligned}$$

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(b) $\neg(\neg P \wedge Q) \vee (P \wedge \neg R)$.

Solution.

$$\begin{aligned}
\neg(\neg P \wedge Q) \vee (P \wedge \neg R) &\equiv (\neg\neg P \vee \neg Q) \vee (P \wedge \neg R) && \text{(DeMorgan's law)}\\
&\equiv (P \vee \neg Q) \vee (P \wedge \neg R) && \text{(double negation law)}\\
&\equiv [(P \vee \neg Q) \vee P] \wedge [(P \vee \neg Q) \vee \neg R] && \text{(distributive law)}\\
&\equiv (P \vee \neg Q \vee P) \wedge (P \vee \neg Q \vee \neg R) && \text{(associative law)}\\
&\equiv (P \vee P \vee \neg Q) \wedge (P \vee \neg Q \vee \neg R) && \text{(commutative law)}\\
&\equiv (P \vee \neg Q) \wedge (P \vee \neg Q \vee \neg R) && \text{(idempotent law)}\\
&\equiv P \vee \neg Q && \text{(absorption law)}
\end{aligned}$$

■

(c) $(P \wedge R) \vee [\neg R \wedge (P \vee Q)]$.

Solution.

$$\begin{aligned}
(P \wedge R) \vee [\neg R \wedge (P \vee Q)] &\equiv (P \wedge R) \vee (\neg R \wedge P) \vee (\neg R \wedge Q) && \text{(distributive law)}\\
&\equiv P \wedge (R \vee \neg R) \vee (\neg R \wedge Q) && \text{(distributive law)}\\
&\equiv P \wedge \top \vee (\neg R \wedge Q) && \text{(complement law)}\\
&\equiv P \vee (\neg R \wedge Q) && \text{(identity law)}
\end{aligned}$$

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13. Use the first DeMorgan's law and the double negation law to derive the second DeMorgan's law.

Solution.

$$\begin{aligned}
\neg(P \vee Q) &\equiv \neg(\neg\neg P \vee \neg\neg Q) && \text{(double negation law)} \\
&\equiv \neg\neg(\neg P \wedge \neg Q) && \text{(first DeMorgan's law)} \\
&\equiv \neg P \wedge \neg Q && \text{(double negation law)}
\end{aligned}$$

■

14. Note that the associative laws say only that parentheses are unnecessary when combining three statements with $\wedge$ or $\vee$. In fact, these laws can be used to justify leaving parentheses out when more than three statements are combined. Use associative laws to show that $[P \wedge (Q \wedge R)] \wedge S$ is equivalent to $(P \wedge Q) \wedge (R \wedge S)$.

Solution.

$$\begin{aligned}
[P \wedge (Q \wedge R)] \wedge S &\equiv [(P \wedge Q) \wedge R] \wedge S \\
&\equiv (P \wedge Q) \wedge (R \wedge S)
\end{aligned}$$

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15. How many lines will there be in the truth table for a statement containing n letters?

Solution.

According to permutation and combination that will be one of the topic in the syllabus of my final year exam tomorrow ;-;, the number of permutation for two letters T and F when they can be repeated every time is 2^n. Hence, the number of lines will be 2^n.

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16. Find a formula involving the connectives $\wedge$, $\vee$, and $\neg$ that has the following truth table:

P	Q	???
F	F	T
F	T	F
T	F	T
T	T	T

Solution.

Take the disjunction of all the cases where the result is true.

$$\begin{aligned}
(\neg P \wedge \neg Q) \vee (P \wedge \neg Q) \vee (P \wedge Q) &\equiv \neg Q \wedge (\neg P \vee P) \vee (P \wedge Q) && \text{(distributive law)} \\
&\equiv \neg Q \wedge \top \vee (P \wedge Q) && \text{(complement law)} \\
&\equiv \neg Q \vee (P \wedge Q) && \text{(identity law)} \\
&\equiv (\neg Q \vee P) \wedge (\neg Q \vee Q) && \text{(distributive law)} \\
&\equiv (\neg Q \vee P) \wedge \top && \text{(complement law)} \\
&\equiv \neg Q \vee P && \text{(identity law)}
\end{aligned}$$

■

17. Find a formula involving the connectives $\wedge$, $\vee$, and $\neg$ that has the following truth table:

P	Q	???
F	F	F
F	T	T
T	F	T
T	T	F

Solution.

Take the conjunction of all the cases where the result is true

$$
\begin{aligned}
(\neg P \wedge Q) \vee (P \wedge \neg Q) &\equiv [(\neg P \wedge Q) \vee P] \wedge [(\neg P \wedge Q) \vee \neg Q] && \text{(distributive law)} \\
&\equiv [(\neg P \vee P) \wedge (Q \vee P)] \wedge [(\neg P \vee \neg Q) \wedge (Q \vee \neg Q)] && \text{(distributive law)} \\
&\equiv [\top \wedge (Q \vee P)] \wedge [(\neg P \vee \neg Q) \wedge \top] && \text{(complement law)} \\
&\equiv (P \vee Q) \wedge (\neg P \vee \neg Q) && \text{(identity law)}
\end{aligned}
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