Exercise 11e

Find the following indefinite integrals:

1. $\displaystyle\int \tan^4 x \sec^2 x\, dx$

   Sol.
   Let $u = \tan x,\ du = \sec^2 x\, dx$.

   $$\begin{aligned}\int \tan^4 x \sec^2 x\, dx &= \int u^4 du \\ &= \frac{1}{5}u^5 + C \\ &= \frac{1}{5}\tan^5 x + C\end{aligned}$$

2. $\displaystyle\int 3\cot^3 3x \operatorname{cosec}^2 3x\, dx$

   Sol.
   Let $u = \cot 3x,\ du = -3\operatorname{cosec}^2 3x\, dx$.

   $$\begin{aligned}\int 3\cot^3 3x \operatorname{cosec}^2 3x\, dx &= -\int u^3 du \\ &= -\frac{u^4}{4} + C \\ &= -\frac{1}{4}\cot^4 3x + C\end{aligned}$$

3. $\displaystyle\int (\sec x + \tan x)^2\, dx$

   Sol.

   $$\begin{aligned}\int (\sec x + \tan x)^2\, dx &= \int (\sec^2 x + 2\sec x \tan x + \tan^2 x)\, dx \\ &= \int \sec^2 x\, dx + 2\int \sec x \tan x\, dx + \int \tan^2 x\, dx \\ &= \tan x + 2\sec x + \int (\sec^2 x - 1)\, dx + C' \\ &= \tan x + 2\sec x + \tan x - x + C' \\ &= 2\tan x + 2\sec x - x + C\end{aligned}$$

4. $\displaystyle\int \tan^3 3x\, dx$

   Sol.

   $$\begin{aligned}\int \tan^3 3x\, dx &= \int \tan^2 3x \tan 3x\, dx \\ &= \int (\sec^2 3x - 1)\tan 3x\, dx \\ &= \int \sec^2 3x \tan 3x\, dx - \int \tan 3x\, dx \\ &= \frac{1}{3}\int u du - \frac{1}{3}|\sec 3x| + C' \qquad (\text{Let } u = \sec 3x,\ du = 3\sec 3x \tan 3x\, dx) \\ &= \frac{1}{6}\sec^2 3x - \frac{1}{3}|\sec 3x| + C \text{ or } \frac{1}{6}\sec^2 3x + \frac{1}{3}|\cos 3x| + C\end{aligned}$$

5. $\displaystyle\int \tan^4 x \sec^6 x \, dx$

Sol.

$$\begin{aligned}
\int \tan^4 x \sec^6 x \, dx &= \int \tan^4 x \sec^4 x \sec^2 x \, dx \\
&= \int \tan^4 x (\tan^2 x + 1)^2 \sec^2 x \, dx \qquad (\text{Let } u = \tan x,\ du = \sec^2 x \, dx) \\
&= \int u^4 (u^2+1)^2 du \\
&= \int u^4 (u^4 + 2u^2 + 1) du \\
&= \int (u^8 + 2u^6 + u^4) du \\
&= \frac{1}{9}u^9 + \frac{2}{7}u^7 + \frac{1}{5}u^5 + C \\
&= \frac{1}{9}\tan^9 x + \frac{2}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C
\end{aligned}$$

6. $\displaystyle\int \tan^4 \frac{x}{2} \, dx$

Sol.

$$\begin{aligned}
\int \tan^4 \frac{x}{2} \, dx &= \int \tan^2 \frac{x}{2} \tan^2 \frac{x}{2} \, dx \\
&= \int (\sec^2 \frac{x}{2} - 1) \tan^2 \frac{x}{2} \, dx \\
&= \int \sec^2 \frac{x}{2} \tan^2 \frac{x}{2} \, dx - \int \tan^2 \frac{x}{2} \, dx \qquad (\text{Let } u = \tan \frac{x}{2},\ du = \frac{1}{2}\sec^2 \frac{x}{2} \, dx) \\
&= 2\int u^2 du - \int \left(\sec^2 \frac{x}{2} - 1\right) \, dx \\
&= \frac{2}{3}u^3 - \int \sec^2 \frac{x}{2} \, dx + \int \, dx + C' \\
&= \frac{2}{3}\tan^3 \frac{x}{2} - 2\tan \frac{x}{2} + x + C
\end{aligned}$$

7. $\displaystyle\int \operatorname{cosec}^4 x \, dx$

Sol.

$$\begin{aligned}
\int \operatorname{cosec}^4 x \, dx &= \int \operatorname{cosec}^2 x \operatorname{cosec}^2 x \, dx \\
&= \int \operatorname{cosec}^2 x (1 + \cot^2 x) \, dx \qquad (\text{Let } u = \cot x,\ du = -\operatorname{cosec}^2 x \, dx) \\
&= -\int (1+u^2) du \\
&= -u - \frac{1}{3}u^3 + C \\
&= -\cot x - \frac{1}{3}\cot^3 x + C
\end{aligned}$$

8. $\displaystyle\int (1+\tan^2 x)(1-\tan^2 x)\,dx$

Sol.

$$\begin{aligned}
\int (1+\tan^2 x)(1-\tan^2 x)\,dx &= \int (1-\tan^4 x)\,dx \\
&= \int dx - \int \tan^2 x \tan^2 x\,dx \\
&= x - \int (\sec^2 x - 1)\tan^2 x\,dx + C' \\
&= x - \int \sec^2 x \tan^2 x\,dx + \int \tan^2 x\,dx + C' \qquad (\text{Let } u = \tan x,\ du = \sec^2 x\,dx) \\
&= x - \int u^2 du + \int (\sec^2 x - 1)\,dx + C' \\
&= x - \frac{1}{3}u^3 + \tan x - x + C \\
&= \tan x - \frac{1}{3}\tan^3 x + C
\end{aligned}$$

9. $\displaystyle\int \tan x \sec^5 x\,dx$

Sol.

$$\begin{aligned}
\int \tan x \sec^5 x\,dx &= \int \tan x \sec^4 x \sec x\,dx \qquad (\text{Let } u = \sec x,\ du = \sec x \tan x\,dx) \\
&= \int u^4 du \\
&= \frac{1}{5}u^5 + C \\
&= \frac{1}{5}\sec^5 x + C
\end{aligned}$$

10. $\displaystyle\int \sec^4 \frac{x}{3}\,dx$

Sol.

$$\begin{aligned}
\int \sec^4 \frac{x}{3}\,dx &= \int \sec^2 \frac{x}{3} \sec^2 \frac{x}{3}\,dx \\
&= \int (1+\tan^2 \frac{x}{3}) \sec^2 \frac{x}{3}\,dx \\
&= \int \sec^2 \frac{x}{3}\,dx + \int \tan^2 \frac{x}{3} \sec^2 \frac{x}{3}\,dx \qquad (\text{Let } u = \tan \frac{x}{3},\ du = \frac{1}{3}\sec^2 \frac{x}{3}\,dx) \\
&= 3\tan \frac{x}{3} + 3\int u^2 du \\
&= 3\tan \frac{x}{3} + u^3 + C \\
&= \tan^3 \frac{x}{3} + 3\tan \frac{x}{3} + C
\end{aligned}$$