MIT Integration Bee 2023 Regular Season

Problem 8

$$\int_0^\pi x\sin^4(x)dx$$

Let $x = \pi - u,\ u = \pi - x,\ dx = -du$

When $x = 0,\ u = \pi,$ and when $x = \pi,\ u = 0$

$$= \int_\pi^0 (\pi - u)\sin^4(\pi - u)(-du)$$

$$= \int_0^\pi (\pi - u)\sin^4(\pi - u)du$$

$$= \int_0^\pi (\pi - u)\sin^4(u)du$$

$$= \int_0^\pi \pi\sin^4(u)du - \int_0^\pi u\sin^4(u)du$$

$$= \int_0^\pi \pi\sin^4(u)du - I$$

$$= \frac{\pi}{2}\int_0^\pi \sin^4(u)du$$

$$= \frac{\pi}{2}\int_0^\pi \left(\frac{1-\cos 2u}{2}\right)^2 du$$

$$= \frac{\pi}{8}\int_0^\pi \left(1 - 2\cos 2u + \cos^2 2u\right)du$$

$$= \frac{\pi}{8}\left(\int_0^\pi du - 2\int_0^\pi \cos(2u)du + \int_0^\pi \cos^2(2u)du\right)$$

$$= \frac{\pi}{8}\left(\pi - 2\int_0^\pi \cos(2u)du + \int_0^\pi \frac{1+\cos(4u)}{2}du\right)$$

$$= \frac{\pi}{8}\left(\pi - 2\cdot\left[\frac{1}{2}\sin(2u)\right]_0^\pi + \frac{1}{2}\int_0^\pi \left[1+\cos(4u)\right]du\right)$$

$$= \frac{\pi}{8}\left(\pi - 2\cdot(0-0) + \frac{1}{2}\int_0^\pi du + \frac{1}{2}\int_0^\pi \cos(4u)du\right)$$

$$= \frac{\pi}{8}\left(\pi + \frac{1}{2}\pi + \frac{1}{2}\cdot\left[\frac{1}{4}\sin(4u)\right]_0^\pi\right)$$

$$= \frac{\pi}{8}\left(\frac{3}{2}\pi + \frac{1}{2}\cdot(0-0)\right)$$

$$= \frac{3}{16}\pi$$