Mathematics

Senior 3 Part II

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Preface

Why this book?

Disclaimer

Acknowledgements

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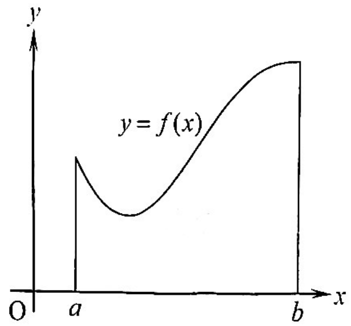
Chapter 28

Definite Integrals

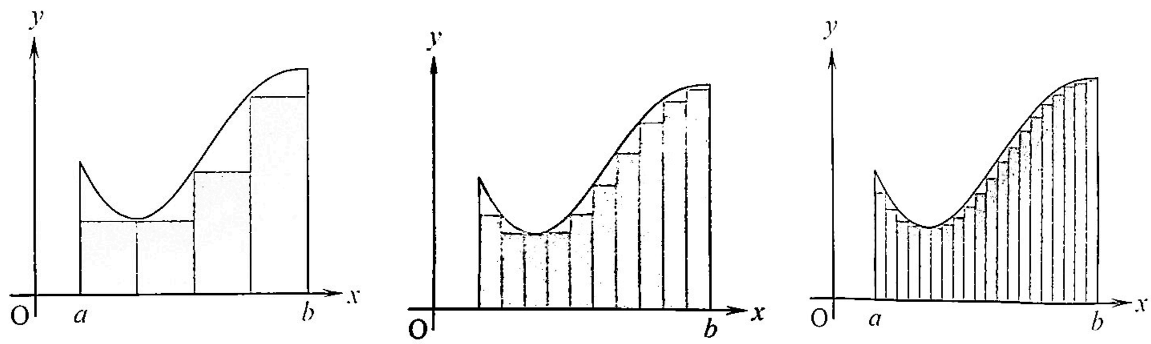
28.1 Concept of Definite Integrals and their Relationship with Indefinite Integrals

Concept of Definite Integrals

A lot of practical problems, for example finding area and volumes, can be reduced to finding the limit of a certain type of sum. Let's take finding area as an example to explain the method of solving this kind of problem, and hence introduce the concept of definite integrals.

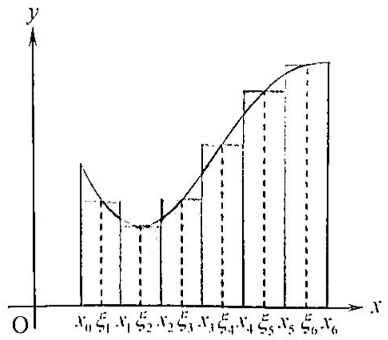


Shown in the diagram above (shaded area) is the area bounded by the line $x = a$, $x = b$, $y = 0$, and the curve $y = f(x)$ where $f(x) \geq 0$. This kind of graph is called the curved trapezoid.



As shown in the diagram above, in order to find the area of this curved trapezoid, we can split it into multiple small curved trapezoid, each of them being substituted by their respective rectangular shape. As such, an approximate value of the area of the curved trapezoid can be acquired by summing up of the area of each rectangle. As the curved trapezoid is being split into smaller and smaller pieces, the approximate value we get will get closer and closer to its actual area.

With this concept in mind, we can split the interval $[a, b]$ into n smaller interval $[x_0, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_{n-1}, x_n]$ where $x_0 = a$, $x_n = b$. From drawing lines that are perpendicular to the x-axis through the points x_1, x_2, $\cdots$, x_{n-1}, we can split the curved trapezoid into n smaller curved trapezoid.



Choose any point ξ_i in the i-th interval $[x_{i-1}, x_i]$, then the area ΔA of the i-th curved trapezoid can be approximated by the area of the rectangle with width $\Delta x_i = x_i - x_{i-1}$ and height $f(\xi_i)$, as shown in the diagram above, i.e.

$$\Delta A_i \approx f(\xi_i)\Delta x$$

And the approximated value of the area of the original curved trapezoid is the sum of the area of all the smaller rectangle, i.e.

$$A = \sum_{i=1}^{n} \Delta A_i \approx \sum_{i=1}^{n} f(\xi_i)\Delta x$$

As the number of smaller interval n increases and the width of each interval decreases, the approximated value of the area of the original curved trapezoid gets closer and closer to its actual area. To find the value of A, we split the interval $[a, b]$ into indefinitely many smaller interval such that $\Delta x \to 0$ (i.e. $n \to \infty$), hence the area of the original curved trapezoid can be defined as the limit of the sum of the area of all the smaller rectangle, i.e.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x$$

This limit is called the definite integral of $f(x)$ from a to b, and is denoted by the symbol $\int_a^b f(x)dx$, i.e.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x$$

where $f(x)$ is called the integrand, $[a, b]$ is called the interval of integration, a and b are called the lower and upper limits of integration respectively.

If $f(x) \geq 0$ in the interval $[a, b]$, we know from the above discussion that the value of the definite integral $\int_a^b f(x)dx$ is the area of the curved trapezoid bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, i.e. $A = \int_a^b f(x)dx$.

If $f(x) \leq 0$ in the interval $[a, b]$, as shown in the diagram above, $f(\xi_i)\Delta x$ is the negative value of the area of the i-th smaller rectangle. Hence, the definite integral $\int_a^b f(x)dx$ is negative, and its absolute value is the area of the curved trapezoid bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, i.e. $A = -\int_a^b f(x)dx$.

The Relationship between Definite Integrals and Indefinite Integrals

The definite integrals and the indefinite integrals has inseparable relationship between them. Consider the case of finding the area of curved trapezoid. Let $x_0 > a$ and $f(x) \geq 0$. The area of the curved trapezoid bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$, $x = x_0$ and $y = 0$ is $A(x_0) = \displaystyle\int_a^{x_0} f(x)dx$.

When x_0 changes, the area $A(x_0)$ also changes. From the diagram above, we know that

$$m\Delta x \leq A(x_0 + \Delta x) - A(x_0) \leq M\Delta x,$$

where m and M are the minimum and maximum values of $f(x)$ in the interval $[x_0, x_0 + \Delta x]$. Hence,

$$m \leq \frac{A(x_0 + \Delta x) - A(x_0)}{\Delta x} \leq M.$$

Apparently, as Δx approaches 0, both m and M approach $f(x_0)$. Besides, from the definition of derivative, $\lim_{\Delta x \to 0} \dfrac{A(x_0 + \Delta x) - A(x_0)}{\Delta x} = A'(x_0)$. Hence, we get $A'(x_0) = f(x_0)$. This relational expression is true for any $x_0 > a$. In other words, $A'(x) = f(x)$, i.e. the derivative of the area function $A(x)$ is the integrand $f(x)$.

Let $\displaystyle\int f(x)dx = F(x) + C$, i.e. $F(x)$ is the primitive of $f(x)$. Then, from $A'(x) = f(x) = F'(x)$, we get $A(x) = F(x) + C$, where C is a constant. When $A(a) = 0$, $x = a$, we get $C = -F(a)$. Hence, $C = -F(a)$, i.e. for any $x_0 > a$, we get $A(x_0) = F(x_0) - F(a)$.

Let $x_0 = b$, we get $A(b) = \displaystyle\int_a^b f(x)dx = F(b) - F(a)$. This relational expression is true for any continuous function $f(x)$. In order to make the relational expression true for any a and b, when $a > b$, the following definition is made.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Above all are the relationship between definite integrals and indefinite integrals, i.e.

$$\boxed{\text{If } \int f(x)dx = F(x) + C, \text{ then } \int_a^b f(x)dx = F(b) - F(a).}$$

This relationship is called the fundamental theorem of calculus. Generally, we express the expression as follows:

$$\int_a^b f(x)dx = \big[F(x)\big]_a^b = F(b) - F(a).$$

where $F(x)$ is any primitive of $f(x)$.

When finding the definite integral $\displaystyle\int_a^b f(x)dx$, we only have to find any primitive $F(x)$ of $f(x)$. The other primitive of $f(x)$ can be expressed as $F(x) + C$, where C is a constant. Hence, the definite integral $\displaystyle\int_a^b f(x)dx$ is independent of the choice of primitive of $f(x)$. But,

$$\begin{aligned}
&\big[F(x) + C\big]_a^b \\
&= [F(b) + C] - [F(a) + C] \\
&= F(b) - F(a)
\end{aligned}$$

Hence, the value of the definite integral has nothing to do with the constant C. The value of the definite integral remains the same no matter which primitive of $f(x)$ is chosen.

Note that $\big[F(x)\big]_a^b$ can also be written as $F(x)\Big|_a^b$.

28.1.1 Practice 2

1. $\int_{2}^{8} x dx$

Sol.

$$\begin{aligned} I &= \left[\frac{1}{2}x^2\right]_2^8 \\ &= \frac{1}{2}(64-4) \\ &= 30 \end{aligned}$$

2. $\int_{-2}^{4} x^3 dx$

Sol.

$$\begin{aligned} I &= \left[\frac{1}{4}x^4\right]_{-2}^4 \\ &= \frac{1}{4}(256-16) \\ &= 60 \end{aligned}$$

3. $\int_{-\pi}^{\pi} \cos x dx$

Sol.

$$\begin{aligned} I &= \Big[\sin x\Big]_{-\pi}^{\pi} \\ &= \sin\pi - \sin(-\pi) \\ &= 0-0 \\ &= 0 \end{aligned}$$

4. $\int_{0}^{\frac{\pi}{4}} \sec^2 x dx$

Sol.

$$\begin{aligned} I &= \Big[\tan x\Big]_0^{\frac{\pi}{4}} \\ &= \tan\frac{\pi}{4} - \tan 0 \\ &= 1-0 \\ &= 1 \end{aligned}$$

28.1.2 Exercise 28.1

1. $\int_{-2}^{4} x^2 dx$

Sol.

$$\begin{aligned} I &= \left[\frac{1}{3}x^3\right]_{-2}^4 \\ &= \frac{1}{3}(64+8) \\ &= 24 \end{aligned}$$

2. $\int_{1}^{4} \frac{1}{x^2} dx$

Sol.

$$\begin{aligned} I &= \left[-\frac{1}{x}\right]_1^4 \\ &= -\frac{1}{4}+1 \\ &= \frac{3}{4} \end{aligned}$$

3. $\int_{4}^{9} \sqrt{x} dx$

Sol.

$$\begin{aligned} I &= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_4^9 \\ &= \frac{2}{3}(27-8) \\ &= \frac{38}{3} \end{aligned}$$

4. $\int_{1}^{27} \frac{1}{\sqrt[3]{x^5}} dx$

Sol.

$$\begin{aligned} I &= \left[-\frac{3}{2}x^{-\frac{2}{3}}\right]_1^{27} \\ &= -\frac{3}{2}\left(\frac{1}{9}-1\right) \\ &= \frac{4}{3} \end{aligned}$$

5. $\displaystyle\int_0^{\frac{\pi}{3}} \cos x dx$

Sol.

$$I = \Big[\sin x\Big]_0^{\frac{\pi}{3}}$$
$$= \sin\frac{\pi}{3} - \sin 0$$
$$= \frac{\sqrt{3}}{2} - 0$$
$$= \frac{\sqrt{3}}{2}$$

6. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x dx$

Sol.

$$I = \Big[-\cos x\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= -\cos\frac{\pi}{2} - (-\cos(-\frac{\pi}{4}))$$
$$= 0 + \frac{\sqrt{2}}{2}$$
$$= \frac{\sqrt{2}}{2}$$

7. $\displaystyle\int_0^{2} e^x dx$

Sol.

$$I = e^2 - e^0$$
$$= e^2 - 1$$

8. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \operatorname{cosec}^2 x dx$

Sol.

$$I = \Big[-\cot x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= -\cot\frac{\pi}{2} + \cot\frac{\pi}{4}$$
$$= 0 + 1$$
$$= 1$$

9. $\displaystyle\int_1^{2} \frac{1}{x} dx$

Sol.

$$I == \Big[\ln|x|\Big]_1^{2}$$
$$= \ln 2 - \ln 1$$
$$= \ln 2$$

10. $\displaystyle\int_0^{2} \frac{1}{x+1} dx$

Sol.

$$I = \Big[\ln|x+1|\Big]_0^{2}$$
$$= \ln 3 - \ln 1$$
$$= \ln 3$$

28.2 Properties and Calculations of Definite Integrals

Properties of Definite Integrals

The definite integrals have the following basic properties:

Property 1 $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

Property 2 $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Property 3 $\int_c^a f(x)dx + \int_b^c f(x)dx = \int_b^a f(x)dx$

28.2.1 Practice 3

1. Find the following definite integrals (Question 1 to 4):

(a) $\int_1^4 \frac{2x^2+3x+2}{x}dx$

Sol.

$$\begin{aligned} I &= \int_1^4 \left(2x+3+\frac{2}{x}\right)dx \\ &= \left[x^2+3x+2\ln|x|\right]_1^4 \\ &= 16+12+2\ln 4-1-3-2\ln 1 \\ &= 24+2\ln 4 \end{aligned}$$

(b) $\int_{-1}^1 \left(3e^{2x}-5x\right)dx$

Sol.

$$\begin{aligned} I &= \int_{-1}^1 \left(3e^{2x}-5x\right)dx \\ &= \left[\frac{3}{2}e^{2x}-\frac{5}{2}x^2\right]_{-1}^1 \\ &= \frac{3}{2}e^2-\frac{5}{2}-\frac{3}{2}e^{-2}+\frac{5}{2} \\ &= \frac{3}{2}(e^2-e^{-2}) \end{aligned}$$

(c) $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(2\cos x-3\sec^2 x\right)dx$

Sol.

$$\begin{aligned} I &= \left[2\sin x-3\tan x\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \\ &= 2\sin\frac{\pi}{3}-3\tan\frac{\pi}{3}-2\sin\left(-\frac{\pi}{3}\right)+3\tan\left(-\frac{\pi}{3}\right) \\ &= \sqrt{3}-3\sqrt{3}+\sqrt{3}-3\sqrt{3} \\ &= -4\sqrt{3} \end{aligned}$$

(d) $\int_{-2}^4 f(x)dx,\ f(x)=\begin{cases} x^2-2, & -2\le x<2 \\ 4-x, & 2\le x\le 4 \end{cases}$

Sol.

$$\begin{aligned} I &= \int_{-2}^2 (x^2-2)dx+\int_2^4 (4-x)dx \\ &= \left[\frac{1}{3}x^3-2x\right]_{-2}^2+\left[4x-\frac{1}{2}x^2\right]_2^4 \\ &= \frac{8}{3}-4+\frac{8}{3}-4+16-8-8+2 \\ &= -\frac{2}{3} \end{aligned}$$

2. Given that $\int_{-2}^{5} f(x)dx = 2$, $\int_{-2}^{3} f(x)dx = -1$, $\int_{3}^{4} g(x)dx = 3$ and $\int_{4}^{5} g(x)dx = 2$, find:

   (a) $\int_{3}^{5} f(x)dx$;

   Sol.

$$\begin{aligned}\int_{3}^{5} f(x)dx &= \int_{-2}^{5} f(x)dx - \int_{-2}^{3} f(x)dx \\ &= 2 - (-1) \\ &= 3\end{aligned}$$

   (b) $\int_{3}^{5} \left(\frac{1}{3}g(x) + \frac{1}{2}f(x)\right) dx$.

   Sol.

$$\begin{aligned}\int_{3}^{5} \left(\frac{1}{3}g(x) + \frac{1}{2}f(x)\right) dx &= \frac{1}{3}\int_{3}^{5} g(x)dx + \frac{1}{2}\int_{3}^{5} f(x)dx \\ &= \frac{1}{3}\left(\int_{3}^{4} g(x)dx + \int_{4}^{5} g(x)dx\right) + \frac{1}{2}\int_{3}^{5} f(x)dx \\ &= \frac{1}{3}(3+2) + \frac{1}{2}(3) \\ &= \frac{19}{6}\end{aligned}$$

3. Given that $f(x) = \sqrt{x^2+1}$, find $f'(x)$. Hence, find $\int_{0}^{1} \frac{x}{\sqrt{x^2+1}}dx$.

   Sol.

$$\begin{aligned}f'(x) &= \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x) \\ &= \frac{x}{\sqrt{x^2+1}}\end{aligned}$$

$$\begin{aligned}\int_{0}^{1} \frac{x}{\sqrt{x^2+1}}dx &= \left[\sqrt{x^2+1}\right]_{0}^{1} \\ &= \sqrt{2} - 1\end{aligned}$$

Integration by Substitution

In the last chapter, we have learned how to solve indefinite integrals using the method of integration by substitution: $\int f(g(x))g'(x)dx = \int f(u)du$ where $u = g(x)$. From the basic theorem of calculus, the method of integration by substitution of definite integrals can be derived:

$$\boxed{\int_{a}^{b} f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du}$$

The method of integration by substitution of definite integrals is similar to that of indefinite integrals. The only difference is that the limits of integration are changed accordingly.

28.2.2 Practice 4

1. $\int_0^3 4e^{2x}dx$

   Sol.

   Let $u = 2x$, $du = 2dx$.

   When $x = 0$, $u = 0$.

   When $x = 3$, $u = 6$.

$$\begin{aligned} I &= \int_0^6 2e^u du \\ &= \Big[2e^u\Big]_0^6 \\ &= 2e^6 - 2 \\ &= 2(e^6 - 1) \end{aligned}$$

2. $\int_1^3 \frac{x}{3x^2+5}dx$

   Sol.

   Let $u = 3x^2 + 5$, $du = 6xdx$.

   When $x = 1$, $u = 8$.

   When $x = 3$, $u = 32$.

$$\begin{aligned} I &= \frac{1}{6}\int_8^{32} \frac{1}{u}du \\ &= \frac{1}{6}\Big[\ln|u|\Big]_8^{32} \\ &= \frac{1}{6}\ln 32 - \frac{1}{6}\ln 8 \\ &= \frac{1}{6}\ln 4 \\ &= \frac{1}{3}\ln 2 \end{aligned}$$

3. $\int_0^4 \frac{x}{25-x^2}dx$

   Sol.

   Let $u = 25 - x^2$, $du = -2xdx$.

   When $x = 0$, $u = 25$.

   When $x = 4$, $u = 9$.

$$\begin{aligned} I &= -\frac{1}{2}\int_{25}^{9} \frac{1}{u}du \\ &= -\frac{1}{2}\Big[\ln|u|\Big]_{25}^{9} \\ &= -\frac{1}{2}\ln 9 + \frac{1}{2}\ln 25 \\ &= \frac{1}{2}\ln\frac{25}{9} \\ &= \ln\frac{5}{3} \end{aligned}$$

4. $\int_0^\pi 2\sin x\cos^2 x dx$

   Sol.

   Let $u = \cos x$, $du = -\sin x dx$.

   When $x = 0$, $u = 1$.

   When $x = \pi$, $u = -1$.

$$\begin{aligned} I &= 2\int_{-1}^{1} u^2 du \\ &= 2\left[\frac{1}{3}u^3\right]_{-1}^{1} \\ &= \frac{4}{3} \end{aligned}$$

28.2.3 Exercise 28.2

1. $\displaystyle\int_0^4 \left(x^2 - 2x\right) dx$

Sol.

$$\begin{aligned} I &= \left[\frac{1}{3}x^3 - x^2\right]_0^4 \\ &= \frac{64}{3} - 16 \\ &= \frac{16}{3} \end{aligned}$$

2. $\displaystyle\int_1^4 \frac{2x^2 - 3\sqrt{x} + 1}{x} dx$

Sol.

$$\begin{aligned} I &= \int_1^4 \left(2x - 3x^{-\frac{1}{2}} + x^{-1}\right) dx \\ &= \left[x^2 - 6\sqrt{x} + \ln|x|\right]_1^4 \\ &= 16 - 12 + \ln 4 - 1 + 6 - \ln 1 \\ &= 9 + \ln 4 \end{aligned}$$

3. $\displaystyle\int_{-3}^3 (x+3)^2 dx$

Sol.

Let $u = x + 3$, $du = dx$.

When $x = -3$, $u = 0$.

When $x = 3$, $u = 6$.

$$\begin{aligned} I &= \int_0^6 u^2 du \\ &= \left[\frac{1}{3}u^3\right]_0^6 \\ &= 72 \end{aligned}$$

4. $\displaystyle\int_{-1}^1 (2+x)\left(2 - x^2\right) dx$

Sol.

$$\begin{aligned} I &= \int_{-1}^1 \left(4 - 2x^2 + 2x - x^3\right) dx \\ &= \left[4x - \frac{2}{3}x^3 + x^2 - \frac{1}{4}x^4\right]_{-1}^1 \\ &= 4 - \frac{2}{3} + 1 - \frac{1}{4} + 4 - \frac{2}{3} - 1 + \frac{1}{4} \\ &= \frac{20}{3} \end{aligned}$$

5. $\displaystyle\int_0^{\frac{\pi}{2}} (2\sin 3\theta - 3\cos 2\theta) d\theta$

Sol.

$$\begin{aligned} I &= \left[-\frac{2}{3}\cos 3\theta - \frac{3}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} \\ &= \frac{2}{3} + \frac{3}{2} + \frac{2}{3} \\ &= \frac{13}{6} \end{aligned}$$

6. $\displaystyle\int_0^{\frac{\pi}{3}} \tan\theta d\theta$

Sol.

$$I = \int_0^{\frac{\pi}{3}} \frac{\sin\theta}{\cos\theta} d\theta$$

Let $u = \cos\theta$, $du = -\sin\theta d\theta$.

When $\theta = 0$, $u = 1$.

When $\theta = \frac{\pi}{3}$, $u = \frac{1}{2}$.

$$\begin{aligned} I &= -\int_1^{\frac{1}{2}} \frac{1}{u} du \\ &= -\left[\ln|u|\right]_1^{\frac{1}{2}} \\ &= -\ln\frac{1}{2} + \ln 1 \\ &= \ln 2 \end{aligned}$$

7. $\displaystyle\int_0^1 (e^x-1)^2\,dx$

Sol.

$$
\begin{aligned}
I &= \int_0^1 \left(e^{2x}-2e^x+1\right)dx \\
&= \left[\frac{1}{2}e^{2x}-2e^x+x\right]_0^1 \\
&= \frac{1}{2}e^2-2e+1-\frac{1}{2}+2 \\
&= \frac{1}{2}e^2-2e+\frac{5}{2} \\
&= \frac{e^2-4e+5}{2}
\end{aligned}
$$

8. $\displaystyle\int_1^4 \frac{2}{4x-1}dx$

Sol.

Let $u=4x-1$, $du=4dx$.

When $x=1$, $u=3$.

When $x=4$, $u=15$.

$$
\begin{aligned}
I &= \frac{1}{2}\int_3^{15}\frac{1}{u}du \\
&= \frac{1}{2}\Big[\ln|u|\Big]_3^{15} \\
&= \frac{1}{2}(\ln 15-\ln 3) \\
&= \frac{1}{2}\ln 5
\end{aligned}
$$

9. $\displaystyle\int_{-1}^3 \sqrt{2x+3}dx$

Sol.

Let $u=2x+3$, $du=2dx$.

When $x=-1$, $u=1$.

When $x=3$, $u=9$.

$$
\begin{aligned}
I &= \frac{1}{2}\int_1^9\sqrt{u}du \\
&= \frac{1}{3}\left[u^{\frac{3}{2}}\right]_1^9 \\
&= \frac{1}{3}(27-1) \\
&= \frac{26}{3}
\end{aligned}
$$

10. $\displaystyle\int_1^2 \frac{1}{(2x-1)^3}dx$

Sol.

Let $u=2x-1$, $du=2dx$.

When $x=1$, $u=1$.

When $x=2$, $u=3$.

$$
\begin{aligned}
I &= \frac{1}{2}\int_1^3 u^{-3}du \\
&= -\frac{1}{4}\left[u^{-2}\right]_1^3 \\
&= -\frac{1}{4}\left(\frac{1}{9}-1\right) \\
&= -\frac{1}{4}\cdot\left(-\frac{8}{9}\right) \\
&= \frac{2}{9}
\end{aligned}
$$

11. $\displaystyle\int_{-1}^1 x^2\left(x^3-1\right)^4dx$

Sol.

Let $u=x^3-1$, $du=3x^2dx$.

When $x=-1$, $u=-2$.

When $x=1$, $u=0$.

$$
\begin{aligned}
I &= \frac{1}{3}\int_{-2}^0 u^4du \\
&= \frac{1}{15}\left[u^5\right]_{-2}^0 \\
&= \frac{32}{15}
\end{aligned}
$$

12. $\displaystyle\int_1^6 x\sqrt{3x-2}dx$

Sol.

Let $u=3x-2$, $du=3dx$, $x=\dfrac{u+2}{3}$.

When $x=1$, $u=1$.

When $x=6$, $u=16$.

$$
\begin{aligned}
I &= \frac{1}{3}\int_1^{16}\frac{u+2}{3}\sqrt{u}du \\
&= \frac{1}{9}\int_1^{16}\left(u^{\frac{3}{2}}+2u^{\frac{1}{2}}\right)du \\
&= \frac{1}{9}\left[\frac{2}{5}u^{\frac{5}{2}}+\frac{4}{3}u^{\frac{3}{2}}\right]_1^{16} \\
&= \frac{1}{9}\left(\frac{2048}{5}+\frac{256}{3}-\frac{2}{5}-\frac{4}{3}\right) \\
&= \frac{274}{5}
\end{aligned}
$$

13. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 \theta d\theta$

Sol.

$$
\begin{aligned}
I &= \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1-\cos 2\theta) d\theta \\
&= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
&= \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{4} - \frac{1}{2}\right) \\
&= \frac{\pi}{4} - \frac{1}{2} \\
&= \frac{\pi - 2}{4}
\end{aligned}
$$

14. $\displaystyle\int_3^5 \frac{1}{x^2 - x - 2} dx$

Sol.

$$I = \int_3^5 \frac{1}{(x-2)(x+1)} dx$$

Let $\dfrac{1}{(x-2)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+1}$.

$$
\begin{aligned}
Ax + A + Bx - 2B &= 1 \\
(A+B)x + (A-2B) &= 1
\end{aligned}
$$

$$
\begin{cases} A + B = 0 \\ A - 2B = 1 \end{cases} \Rightarrow \begin{cases} A = \dfrac{1}{3} \\ B = -\dfrac{1}{3} \end{cases}
$$

$$
\begin{aligned}
I &= \int_3^5 \left(\frac{1}{3(x-2)} - \frac{1}{3(x+1)}\right) dx \\
&= \frac{1}{3}\int_3^5 \left(\frac{1}{x-2} - \frac{1}{x+1}\right) dx \\
&= \frac{1}{3}\Big[\ln|x-2| - \ln|x+1|\Big]_3^5 \\
&= \frac{1}{3}(\ln 3 - \ln 6 - \ln 1 + \ln 4) \\
&= \frac{1}{3}\ln 2
\end{aligned}
$$

15. $\displaystyle\int_2^5 \frac{x}{x^3 - x^2 - x + 1} dx$

Sol.

$$
\begin{aligned}
I &= \int_2^5 \frac{x}{x^2(x-1) - (x-1)} dx \\
&= \int_2^5 \frac{x}{(x^2-1)(x-1)} dx \\
&= \int_2^5 \frac{x}{(x+1)(x-1)^2} dx
\end{aligned}
$$

Let $\dfrac{x}{(x+1)(x-1)^2} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$.

$$Ax^2 - 2Ax + A + Bx^2 - B + Cx + C = x$$

$$(A+B)x^2 + (-2A+C)x + (A-B+C) = x$$

$$
\begin{cases} A + B = 0 \\ -2A + C = 1 \\ A - B + C = 0 \end{cases}
\Rightarrow
\begin{cases} A = -\frac{1}{4} \\ B = \frac{1}{4} \\ C = \frac{1}{2} \end{cases}
$$

$$
\begin{aligned}
I &= \int_2^5 \left(-\frac{1}{4(x+1)} + \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} \right) dx \\
&= \left[-\frac{1}{4} \ln|x+1| + \frac{1}{4} \ln|x-1| - \frac{1}{2(x-1)} \right]_2^5 \\
&= -\frac{1}{4}\ln 6 + \frac{1}{4}\ln 4 - \frac{1}{8} + \frac{1}{4}\ln 3 - \frac{1}{4}\ln 1 + \frac{1}{2} \\
&= -\frac{1}{4}\ln 6 + \frac{1}{4}\ln 4 - \frac{1}{8} + \frac{1}{4}\ln 3 + \frac{1}{2} \\
&= \frac{1}{4}\ln 2 + \frac{3}{8}
\end{aligned}
$$

16. $\displaystyle\int_0^1 \frac{x^2}{x^2 + 2x + 1} dx$

Sol.

$$I = \int_0^1 \frac{x^2}{(x+1)^2} dx$$

Let $u = x + 1$, $du = dx$, $x = u - 1$.

When $x = 0$, $u = 1$.

When $x = 1$, $u = 2$.

$$
\begin{aligned}
I &= \int_1^2 \frac{(u-1)^2}{u^2} du \\
&= \int_1^2 \frac{u^2 - 2u + 1}{u^2} du \\
&= \int_1^2 \left(1 - \frac{2}{u} + \frac{1}{u^2} \right) du \\
&= \left[u - 2\ln|u| - \frac{1}{u} \right]_1^2 \\
&= 2 - 2\ln 2 - \frac{1}{2} - 1 + 2\ln 1 + 1 \\
&= \frac{3}{2} - 2\ln 2
\end{aligned}
$$

17. $\displaystyle\int_0^3 \frac{x}{\sqrt{25 - x^2}} dx$

Sol.

Let $u = 25 - x^2$, $du = -2x dx$.

When $x = 0$, $u = 25$.

When $x = 3$, $u = 16$.

$$
\begin{aligned}
I &= \frac{1}{2} \int_{16}^{25} \frac{1}{\sqrt{u}} du \\
&= \left[\sqrt{u} \right]_{16}^{25} \\
&= 5 - 4 \\
&= 1
\end{aligned}
$$

18. $\displaystyle\int_0^{\frac{\pi}{6}} \sin^2\theta \cos\theta d\theta$

Sol.

Let $u = \sin\theta$, $du = \cos\theta d\theta$.

When $\theta = 0$, $u = 0$.

When $\theta = \dfrac{\pi}{6}$, $u = \dfrac{1}{2}$.

$$
\begin{aligned}
I &= \int_0^{\frac{1}{2}} u^2 du \\
&= \left[\frac{1}{3} u^3 \right]_0^{\frac{1}{2}} \\
&= \frac{1}{24}
\end{aligned}
$$

19. Given that $f(x) = \begin{cases} 2x^2 - 1, & -2 \le x \le 2 \\ 3x + 1, & 2 < x \le 4 \end{cases}$, find $\int_{-2}^{4} f(x)dx$.

Sol.

$$\begin{aligned} I &= \int_{-2}^{2}(2x^2 - 1)dx + \int_{2}^{4}(3x + 1)dx \\ &= \left[\frac{2}{3}x^3 - x\right]_{-2}^{2} + \left[\frac{3}{2}x^2 + x\right]_{2}^{4} \\ &= \frac{16}{3} - 2 + \frac{16}{3} - 2 + 24 + 4 - 6 - 2 \\ &= \frac{80}{3} \end{aligned}$$

20. Given that $\int_{3}^{5} f(x)dx = 6$, $\int_{5}^{9} f(x)dx = 18$, $\int_{1}^{4} g(x)dx = 4$ and $\int_{3}^{4} g(x)dx = -4$. Find:

(a) $\int_{1}^{3} g(x)dx$;

Sol.

$$\begin{aligned} \int_{1}^{3} g(x)dx &= \int_{1}^{4} g(x)dx - \int_{3}^{4} g(x)dx \\ &= 4 - (-4) \\ &= 8 \end{aligned}$$

(b) $\int_{1}^{3} f(3x)dx$;

Sol.

Let $u = 3x$, $du = 3dx$.

When $x = 1$, $u = 3$.

When $x = 3$, $u = 9$.

$$\begin{aligned} \int_{1}^{3} f(3x)dx &= \frac{1}{3}\int_{3}^{9} f(u)du \\ &= \frac{1}{3}\int_{3}^{5} f(u)du + \frac{1}{3}\int_{5}^{9} f(u)du \\ &= \frac{1}{3}(6 + 18) \\ &= 8 \end{aligned}$$

(c) $\int_{1}^{3} [f(3x) - 3g(x)]dx$.

Sol.

$$\begin{aligned} \int_{1}^{3} [f(3x) - 3g(x)]dx &= \int_{1}^{3} f(3x)dx - 3\int_{1}^{3} g(x)dx \\ &= 8 - 3 \cdot 8 \\ &= -16 \end{aligned}$$

21. Given the function $y = x\sqrt{x+1}$, find $\frac{dy}{dx}$. Hence, find $\int_3^8 \frac{3x+2}{\sqrt{x+1}}dx$.

Sol.

$$\begin{aligned}\frac{dy}{dx} &= \sqrt{x+1} + \frac{x}{2\sqrt{x+1}} \\ &= \frac{2x+x+2}{2\sqrt{x+1}} \\ &= \frac{3x+2}{2\sqrt{x+1}}\end{aligned}$$

$$\begin{aligned}\int_3^8 \frac{3x+2}{\sqrt{x+1}}dx &= 2\int_3^8 \frac{3x+2}{2\sqrt{x+1}}dx \\ &= 2\int_3^8 \frac{dy}{dx}dx \\ &= 2\left[x\sqrt{x+1}\right]_3^8 \\ &= 2\,(24-6) \\ &= 36\end{aligned}$$

22. Given the function $y = \frac{x^2-1}{2x+1}$, find $\frac{dy}{dx}$. Hence, find $\int_0^2 \frac{x^2+x+1}{4x^2+4x+1}dx$.

Sol.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2x+1)(2x)-(x^2-1)(2)}{(2x+1)^2} \\ &= \frac{4x^2+2x-2x^2+2}{(2x+1)^2} \\ &= \frac{2(x^2+x+1)}{(2x+1)^2}\end{aligned}$$

$$\begin{aligned}\int_0^2 \frac{x^2+x+1}{4x^2+4x+1}dx &= \frac{1}{2}\int_0^2 \frac{2(x^2+x+1)}{(2x+1)^2}dx \\ &= \frac{1}{2}\int_0^2 \frac{dy}{dx}dx \\ &= \frac{1}{2}\left[\frac{x^2-1}{2x+1}\right]_0^2 \\ &= \frac{1}{2}\left(\frac{3}{5}-\frac{-1}{1}\right) \\ &= \frac{4}{5}\end{aligned}$$

23. Given the function $y = xe^x - e^x$, find $\frac{dy}{dx}$. Hence, find $\int_1^4 2xe^x dx$.

Sol.

$$\begin{aligned}\frac{dy}{dx} &= xe^x + e^x - e^x \\ &= xe^x\end{aligned}$$

$$\begin{aligned}\int_1^4 2xe^x dx &= 2\int_1^4 xe^x dx \\ &= 2\int_1^4 \frac{dy}{dx}dx \\ &= 2\left[xe^x - e^x\right]_1^4 \\ &= 2\left(4e^4 - e^4 - e + e\right) \\ &= 6e^4\end{aligned}$$

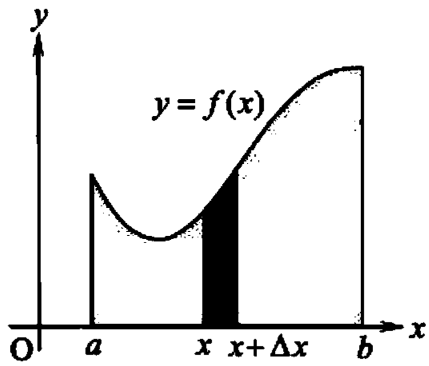
28.3 Area

When we first introduced the concept of definite integrals, we have studied that, if $f(x) \geq 0$ in the interval $a \leq x \leq b$, then the area of the curved trapezoid bounded by the curve $y = f(x)$, the lines $x = a$ and $x = b$, the x-axis, and the y-axis is given by

$$A = \int_a^b f(x)dx$$

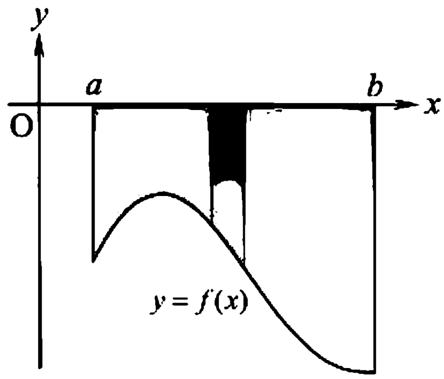
The applications of definite integrals are not limited to finding area of curved trapezoid. In fact, it is widely used in different fields of technologies. To solve real-life problems, the basic mindset is the steps described in the definition: splitting, approximating, finding sum, and taking limit. After understanding the concept, part of the steps can be skipped, and the related definite integrals can be written out straight away.

Let's take finding the area of the curved trapezoid bounded by the line $x = a$, $x = b$, the x-axis, and the curve $y = f(x)$ as an example and do some further elaboration. Since both sides of the curved trapezoid are the straight lines $x = a$ and $x = b$, we can split the interval $a \leq x \leq b$ into multiple smaller intervals, hence the original curved trapezoid is split into multiple smaller curved trapezoid. As shown in the diagram below, take any one of the smaller intervals, and express it as $[x, x + \Delta x]$, its corresponding smaller curved trapezoid can be approximated by a rectangle with width of Δx and height of the function value $f(x)$ at the point x. As such, the area of the smaller curved trapezoid is $\Delta A \approx f(x)\Delta x$.



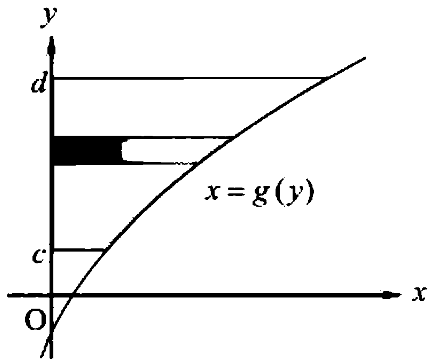
Summing up the approximated area of all the smaller curved trapezoid, then find the limit of the sum when $\Delta x \to 0$, we can get the area of the original curved trapezoid, i.e.

$$\sum \Delta A \approx \sum f(x)\Delta x \xrightarrow{\Delta x \to 0} \int_a^b f(x)dx$$



The same can be applied if $f(x) \leq 0$ in the interval $a \leq x \leq b$, as shown in the diagram above. The area of the curved trapezoid bounded by the curve $y = f(x)$, the lines $x = a$ and $x = b$, the x-axis, and the y-axis is given by

$$A = -\int_a^b f(x)dx$$



If the target area A is bounded by the lines $y = c$, $y = d$, the y-axis, and the curve $x = f(y)$, where $f(y) \geq 0$ in the interval $c \leq y \leq d$, as shown in the diagram above, using the same concept, we can get the area of the region

$$A = \int_c^d f(y)dy$$

28.3.1 Practice 5

1. Find the area of the region bounded by the curve $y = x^2 - 2x + 8$ and the x-axis.

$(x-4)(x+2) = 0$

$x = 4$ or $x = -2$

$$\begin{aligned} A &= -\int_{-2}^{4}(x^2 - 2x - 8)dx \\ &= -\left[\frac{1}{3}x^3 - x^2 - 8x\right]_{-2}^{4} \\ &= -\left[\frac{64}{3} - 16 - 32 + \frac{8}{3} + 4 - 16\right] = -36 \end{aligned}$$

2. Find the area of the region bounded by the curve $x = \sqrt{y}$, the lines $y = 1$, $y = 4$, and the y-axis.

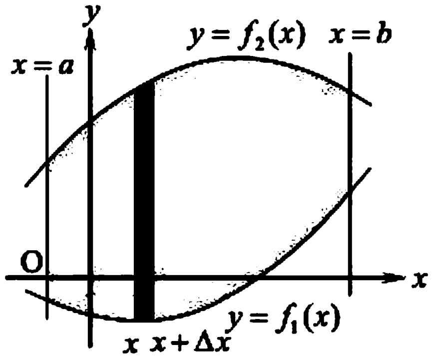
$$\begin{aligned} A &= \int_1^4 \sqrt{y}dy \\ &= \left[\frac{2}{3}y^{\frac{3}{2}}\right]_1^4 \\ &= \frac{16}{3} - \frac{2}{3} \\ &= \frac{14}{3} \end{aligned}$$

3. Find the area of the region bounded by the curve $x = y(y+2)(y-2)$ and the y-axis.

Sol.

$$\begin{aligned} A &= \int_{-2}^{0} y(y+2)(y-2)dy - \int_0^2 y(y+2)(y-2)dy \\ &= \int_{-2}^{0}(y^3 - 4y)dy - \int_0^2 (y^3 - 4y)dy \\ &= \left[\frac{1}{4}y^4 - 2y^2\right]_{-2}^{0} - \left[\frac{1}{4}y^4 - 2y^2\right]_0^2 \\ &= -(4-8) - (4-8) = 8 \end{aligned}$$

If, in the internal $a \le x \le b$, $f_2(x) \ge f_1(x) \ge 0$, then the area of the region bounded by the curve $y = f_1(x)$, $y = f_2(x)$, the lines $x = a$ and $x = b$ can be found using definite integrals.

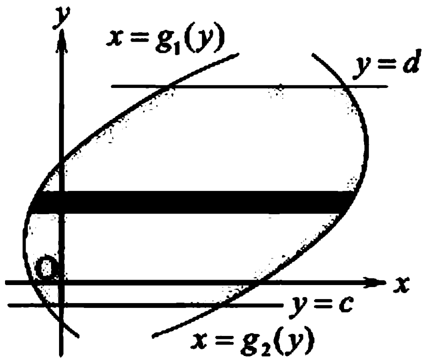


As shown in the diagram above, split the interval $a \le x \le b$ into multiple smaller intervals. Take any one of the smaller intervals, and express it as $[x, x + \Delta x]$. The region in this smaller interval can be approximated by a rectangle with width of Δx and height of the function value $f_2(x) - f_1(x)$, i.e.

$$\Delta A \approx [f_2(x) - f_1(x)]\Delta x$$

Summing up the approximated area of all the smaller intervals, then find the limit of the sum when $\Delta x \to 0$, we can get the area of the region bounded by the curve $y = f_1(x)$, $y = f_2(x)$, the lines $x = a$ and $x = b$, i.e.

$$A = \int_a^b [f_2(x) - f_1(x)]dx$$



Similarly, if, in the internal $c \le y \le d$, $g_2(y) \ge g_1(y) \ge 0$, then the area of the region bounded by the curve $x = g_1(y)$, $x = g_2(y)$, the lines $y = c$ and $y = d$ is given by

$$A = \int_c^d [g_2(y) - g_1(y)]dy$$

Note that when using these formulas, the function $f_2(x)$ must always be greater than or equal to $f_1(x)$ in the interval $[a, b]$.

28.3.2 Practice 6

1. Find the area of the region bounded by the curve $y = x^2 - 4x + 5$ and the line $y = x + 1$.

   Sol.

$$\begin{aligned} x^2 - 5x + 4 &= 0 \\ (x-4)(x-1) &= 0 \\ x = 4 &\text{ or } x = 1 \end{aligned}$$

In the interval $1 \le x \le 4$, $x + 1 \ge x^2 - 4x + 5$

$$\begin{aligned} A &= \int_1^4 \left[(x+1) - (x^2 - 4x + 5)\right] dx \\ &= \int_1^4 (-x^2 + 5x - 4)dx \\ &= \left[-\frac{1}{3}x^3 + \frac{5}{2}x^2 - 4x\right]_1^4 \\ &= -\frac{64}{3} + 40 - 16 + \frac{1}{3} - \frac{5}{2} + 4 \\ &= \frac{9}{2} \end{aligned}$$

2. Find the area of the region bounded by the curve $x = 4y - y^2$ and the line $x - 2y + 3 = 0$.

   Sol.

$$\begin{aligned} 4y - y^2 - 2y + 3 &= 0 \\ -y^2 + 2y + 3 &= 0 \\ (y-3)(y+1) &= 0 \end{aligned}$$

In the interval $-1 \le y \le 3$, $4y - y^2 \ge 2y - 3$.

$$\begin{aligned} A &= \int_{-1}^3 \left[(4y - y^2) - (2y - 3)\right] dy \\ &= \int_{-1}^3 (-y^2 + 2y + 3)dy \\ &= \left[-\frac{1}{3}y^3 + y^2 + 3y\right]_{-1}^3 \\ &= -9 + 9 + 9 - \frac{1}{3} - 1 + 3 \\ &= \frac{32}{3} \end{aligned}$$

28.3.3 Exercise 28.3

Find the area of the region bounded by the following curves and lines:

1. $y = 3x^2$, $x = 2$, $x = 5$, and x-axis

   Sol.

$$\begin{aligned} A &= \int_2^5 3x^2 dx \\ &= \left[x^3\right]_2^5 \\ &= 125 - 8 = 117 \end{aligned}$$

2. $y = (x-1)^2$, $x = 4$, x-axis, and y-axis

   Sol.

$$\begin{aligned} A &= \int_0^4 (x-1)^2 dx \\ &= \left[\frac{1}{3}(x-1)^3\right]_0^4 \\ &= \frac{1}{3}(3^3 + 1) = \frac{28}{3} \end{aligned}$$

3. $y = x^2 + 4x - 21$, and x-axis

   Sol.

$$\begin{aligned} (x+7)(x-3) &= 0 \\ x = -7 &\text{ or } x = 3 \end{aligned}$$

In the interval $-7 \le x \le 3$, $x^2 + 4x - 21 \le 0$.

$$\begin{aligned} A &= \left|\int_{-7}^3 (x^2 + 4x - 21)dx\right| \\ &= \left|\left[\frac{1}{3}x^3 + 2x^2 - 21x\right]_{-7}^3\right| \\ &= \left|9 + 18 - 63 - \left(-\frac{343}{3} + 98 + 147\right)\right| = \frac{500}{3} \end{aligned}$$

4. $y = e^{2x}$, $x = 0$, $x = 4$, and x-axis

   Sol.

$$\begin{aligned} A &= \int_0^4 e^{2x} dx \\ &= \left[\frac{1}{2}e^{2x}\right]_0^4 \\ &= \frac{1}{2}(e^8 - 1) \end{aligned}$$

5. $y = \sin\frac{x}{2}, 0 \le x \le 2\pi$, and x-axis

Sol.

$$\begin{aligned} A &= \int_0^{2\pi} \sin\frac{x}{2}dx \\ &= \left[-2\cos\frac{x}{2}\right]_0^{2\pi} \\ &= 4 \end{aligned}$$

6. $y = \cos x, x = 2\pi$, x-axis, and y-axis

Sol.

$$\begin{aligned} \cos x &= 0 \\ x &= \frac{\pi}{2}, \frac{3\pi}{2} \end{aligned}$$

In the interval $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$, $\cos x \le 0$.

$$\begin{aligned} A &= \int_0^{\frac{\pi}{2}} \cos x dx + \left|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x dx\right| + \int_{\frac{3\pi}{2}}^{2\pi} \cos x dx \\ &= \left[\sin x\right]_0^{\frac{\pi}{2}} + \left|\left[\sin x\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right| + \left[\sin x\right]_{\frac{3\pi}{2}}^{2\pi} \\ &= 1 + 1 + 1 + 1 = 4 \end{aligned}$$

7. $x = y^2, y = 3$, and y-axis

Sol.

$$\begin{aligned} A &= \int_0^3 y^2 dy \\ &= \left[\frac{1}{3}y^3\right]_0^3 \\ &= 9 \end{aligned}$$

8. $x = 9y - y^3$, and y-axis

Sol.

$$\begin{aligned} 9y - y^3 &= 0 \\ y(9 - y^2) &= 0 \\ y &= 0 \text{ or } y = \pm 3 \end{aligned}$$

In the interval $-3 \le y \le 0$, $9y - y^3 \le 0$.

$$\begin{aligned} A &= \left|\int_{-3}^{0} (9y - y^3) dy\right| + \int_0^3 (9y - y^3) dy \\ &= \left|\left[\frac{9}{2}y^2 - \frac{1}{4}y^4\right]_{-3}^{0}\right| + \left[\frac{9}{2}y^2 - \frac{1}{4}y^4\right]_0^3 \\ &= \left|-\frac{81}{2} + \frac{81}{4}\right| + \frac{81}{2} - \frac{81}{4} \\ &= \frac{81}{2} \end{aligned}$$

9. $y = \frac{1}{x}, y = \frac{1}{2}, y = 2$, and y-axis

Sol.

$$\begin{aligned} A &= \int_{\frac{1}{2}}^{2} \frac{1}{x} dx \\ &= \left[\ln x\right]_{\frac{1}{2}}^{2} \\ &= \ln 2 - \ln\frac{1}{2} \\ &= 2\ln 2 \end{aligned}$$

10. $y^2 = x, x = 4$, and $x = 16$

Sol.

$$\begin{aligned} A &= 2\int_4^{16} \sqrt{x} dx \\ &= 2\left[\frac{2}{3}x^{\frac{3}{2}}\right]_4^{16} \\ &= 2\left(\frac{128}{3} - \frac{16}{3}\right) \\ &= \frac{224}{3} \end{aligned}$$

11. Find the area of the region bounded by the curve $y = x^2 - 4$ and the line $y = 3x$.

Sol.

$$\begin{aligned} x^2 - 4 &= 3x \\ x^2 - 3x - 4 &= 0 \\ (x-4)(x+1) &= 0 \\ x &= 4 \text{ or } x = -1 \end{aligned}$$

In the interval $-1 \le x \le 4$, $x^2 - 4 \le 3x$.

$$\begin{aligned} A &= \int_{-1}^{4} (3x - x^2 + 4)dx \\ &= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3 + 4x\right]_{-1}^{4} \\ &= 24 - \frac{64}{3} + 16 - \frac{3}{2} - \frac{1}{3} + 4 \\ &= \frac{125}{6} \end{aligned}$$

12. Find the area of the region bounded by the curve $y = x^2 + 2x$ and the curve $y = 12 + 4x - x^2$.

$$\begin{aligned} x^2 + 2x &= 12 + 4x - x^2 \\ 2x^2 - 2x - 12 &= 0 \\ (x-3)(2x+4) &= 0 \\ x &= 3 \text{ or } x = -2 \end{aligned}$$

In the interval $-2 \le x \le 3$, $x^2 + 2x \le 12 + 4x - x^2$.

$$\begin{aligned} A &= \int_{-2}^{3} (12 + 4x - x^2 - x^2 - 2x)dx \\ &= \int_{-2}^{3} (12 + 2x - 2x^2)dx \\ &= \left[12x + x^2 - \frac{2}{3}x^3\right]_{-2}^{3} \\ &= 36 + 9 - 18 + 24 - 4 - \frac{16}{3} \\ &= \frac{125}{3} \end{aligned}$$

13. Find the area of the region bounded by the curve $y = \sin x$ and $y = -2\sin x$ in the interval $0 \le x \le \pi$.

Sol.

$$\begin{aligned} \sin x &= -2\sin x \\ \sin x &= 0 \\ x &= 0, \pi \end{aligned}$$

In the interval $0 \le x \le \pi$, $-2\sin x \le \sin x$.

$$\begin{aligned} A &= \int_{0}^{\pi} (3\sin x)dx \\ &= \Big[-3\cos x\Big]_{0}^{\pi} \\ &= 3 - (-3) = 6 \end{aligned}$$

14. Find the area of the region bounded by the curve $y = e^x$, $y = e^{-2x}$ and the lines $x = -2$ and $x = 4$.

Sol.

$$\begin{aligned} e^x &= e^{-2x} \\ x &= 0 \end{aligned}$$

In the interval $-2 \le x \le 0$, $e^x \le e^{-2x}$.

In the interval $0 \le x \le 4$, $e^{-2x} \le e^x$.

$$\begin{aligned} A &= \int_{-2}^{0} (e^{-2x} - e^x)dx + \int_{0}^{4} (e^x - e^{-2x})dx \\ &= \left[-\frac{1}{2}e^{-2x} - e^x\right]_{-2}^{0} + \left[e^x + \frac{1}{2}e^{-2x}\right]_{0}^{4} \\ &= -\frac{1}{2} - 1 + \frac{1}{2}e^4 + e^{-2} + e^4 + \frac{1}{2}e^{-8} - 1 - \frac{1}{2} \\ &= \frac{3}{2}e^4 + e^{-2} + \frac{1}{2}e^{-8} - 3 \end{aligned}$$

15. Find the area of the region bounded by the curve $y = x^3 - 10x^2 + 28x$ and the line $y = 4x$.

Sol.

$$
\begin{aligned}
x^3 - 10x^2 + 28x &= 4x \\
x^3 - 10x^2 + 24x &= 0 \\
x(x^2 - 10x + 24) &= 0 \\
x = 0 \text{ or } x &= 4 \text{ or } x = 6
\end{aligned}
$$

In the interval $0 \leq x \leq 4$, $x^3 - 10x^2 + 28x \geq 4x$.

In the interval $4 \leq x \leq 6$, $x^3 - 10x^2 + 28x \leq 4x$.

$$
\begin{aligned}
A &= \int_0^4 (x^3 - 10x^2 + 28x - 4x)dx + \int_4^6 (4x - x^3 + 10x^2 - 28x)dx \\
&= \int_0^4 (x^3 - 10x^2 + 24x)dx + \int_4^6 (-x^3 + 10x^2 - 24x)dx \\
&= \left[\frac{1}{4}x^4 - \frac{10}{3}x^3 + 12x^2\right]_0^4 + \left[-\frac{1}{4}x^4 + \frac{10}{3}x^3 - 12x^2\right]_4^6 \\
&= 64 - \frac{640}{3} + 192 - 324 + 720 - 432 + 64 - \frac{640}{3} + 192 \\
&= \frac{148}{3}
\end{aligned}
$$

16. Find the area of the region bounded by the curve $x = 8y - y^2$, $x = 16y - y^2 - 48$, and y-axis.

Sol.

$$
\begin{aligned}
8y - y^2 &= 16y - y^2 - 48 \\
8y &= 48 \\
y &= 6
\end{aligned}
$$

$$
\begin{aligned}
8y - y^2 &= 0 \\
y(8 - y) &= 0 \\
y = 0 \text{ or } y &= 8
\end{aligned}
$$

$$
\begin{aligned}
16y - y^2 - 48 &= 0 \\
y^2 - 16y + 48 &= 0 \\
(y - 12)(y - 4) &= 0 \\
y = 12 \text{ or } y &= 4
\end{aligned}
$$

$$
\begin{aligned}
A &= \int_4^6 (16y - y^2 - 48)dy + \int_6^8 (8y - y^2)dy \\
&= \left[8y^2 - \frac{1}{3}y^3 - 48y\right]_4^6 + \left[4y^2 - \frac{1}{3}y^3\right]_6^8 \\
&= 288 - 72 - 288 - 128 + \frac{64}{3} + 192 + 256 - \frac{512}{3} - 144 + 72 \\
&= \frac{80}{3}
\end{aligned}
$$

17. Find the area of the region bounded by the curve $x = 2y^2 - 8y + 10$ and $x = y^2 - y$.

Sol.

$$
\begin{aligned}
2y^2 - 8y + 10 &= y^2 - y \\
y^2 - 7y + 10 &= 0 \\
(y-5)(y-2) &= 0 \\
y = 5 \text{ or } y &= 2
\end{aligned}
$$

In the interval $2 \le y \le 5$, $y^2 - y \ge 2y^2 - 8y + 10$.

$$
\begin{aligned}
A &= \int_2^5 (y^2 - y - 2y^2 + 8y - 10)dy \\
&= \int_2^5 (-y^2 + 7y - 10)dy \\
&= \left[-\frac{1}{3}y^3 + \frac{7}{2}y^2 - 10y\right]_2^5 \\
&= -\frac{125}{3} + \frac{175}{2} - 50 + \frac{8}{3} - 14 + 20 \\
&= \frac{9}{2}
\end{aligned}
$$

18. Given that the curve $y = f(x)$ passes through point $(1, 0)$, and the gradient of any point on the curve (x, y) is $3x^2 - 3$. Find the area of the region bounded by the curve, $x = 2$ and x.

Sol.

$$
\begin{aligned}
\frac{dy}{dx} &= 3x^2 - 3 \\
dy &= (3x^2 - 3)dx \\
\int dy &= \int (3x^2 - 3)dx \\
y &= x^3 - 3x + C
\end{aligned}
$$

When $x = 1$, $y = 0$.

$$
\begin{aligned}
0 &= 1 - 3 + C \\
C &= 2
\end{aligned}
$$

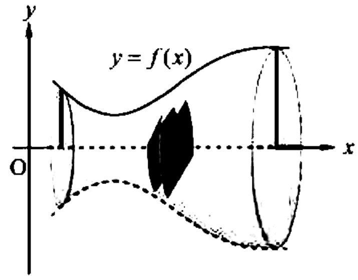
$\therefore$ The equation of the curve is $y = x^3 - 3x + 2$.

$$
\begin{aligned}
x^3 - 3x + 2 &= 0 \\
(x-1)(x^2 + x - 2) &= 0 \\
(x-1)(x+2)(x-1) &= 0 \\
(x-1)^2(x+2) &= 0 \\
x = 1 \text{ or } x &= -2
\end{aligned}
$$

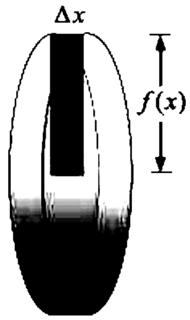
$$
\begin{aligned}
A &= \int_{-2}^2 (x^3 - 3x + 2)dx \\
&= \left[\frac{1}{4}x^4 - \frac{3}{2}x^2 + 2x\right]_{-2}^2 \\
&= 4 - 6 + 4 - 4 + 6 + 4 \\
&= 8
\end{aligned}
$$

28.4 Volume of Revolution

The solid formed by rotating a flat surface around a certain straight line on the surface is called the **solid of revolution**, for example the right cylinder, right cone, and sphere.



The diagram above shows a solid of revolution formed by rotating the area bounded by the curve $y = f(x)$, the lines $x = a$, $x = b$, and the x-axis around the x-axis. We can utilize the methods discussed in the last section to find the volume of this solid of revolution.

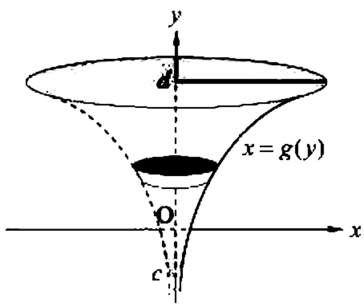


Split the interval $a \leq x \leq b$ into multiple smaller intervals. Take any one of the smaller intervals, and express it as $[x, x + \Delta x]$. The region in this smaller interval can be approximated by a rectangle with width of Δx and height of the function value $f(x)$. Rotate this rectangle around the x-axis, we get a cylinder with radius $f(x)$ and height Δx, as shown in the diagram above. Hence, the approximated volume of the solid of revolution in this smaller interval is

$$\Delta V \approx \pi[f(x)]^2 \Delta x$$

Summing up the approximated volume of all the smaller intervals, then find the limit of the sum when $\Delta x \to 0$, we can get the volume of the solid of revolution, i.e.

$$V = \pi \int_a^b [f(x)]^2 dx$$



Similarly, if the solid of revolution is formed by rotating the area bounded by the curve $x = g(y)$, the lines $y = c$, $y = d$, and the y-axis around the y-axis, then the volume of the solid of revolution is given by

$$V = \pi \int_c^d [g(y)]^2 dy$$

28.4.1 Practice 7

1. Find the volume of a cone with radius r and height h using definite integrals.

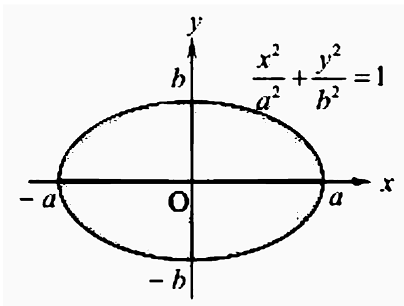
   Sol.

   Let y be the height of any cross section of the cone, and x be the radius of the cross section.

   $$\frac{y}{h} = \frac{x}{r}$$
   $$y = \frac{h}{r}x$$

   $$\begin{aligned} V &= \int_0^h \pi y^2 dx \\ &= \int_0^h \pi \left(\frac{h}{r}x\right)^2 dx \\ &= \pi \frac{h^2}{r^2} \int_0^h x^2 dx \\ &= \pi \frac{h^2}{r^2} \left[\frac{1}{3}x^3\right]_0^h \\ &= \pi \frac{h^2}{r^2} \cdot \frac{1}{3}h^3 \\ &= \frac{1}{3}\pi r^2 h \end{aligned}$$

2. Shown in the diagram below is the shaded region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0$ and $b > 0$. If the volume of the solid of revolution formed by rotating this region about the x-axis and the y-axis is V_x and V_y respectively,



   (a) find V_x and V_y.

   Sol.

   $$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\ \frac{y^2}{b^2} &= 1 - \frac{x^2}{a^2} \\ y^2 &= b^2\left(1 - \frac{x^2}{a^2}\right) \end{aligned}$$

   $$\begin{aligned} V_x &= \int_{-a}^a \pi y^2 dx \\ &= \pi b^2 \int_{-a}^a \left(1 - \frac{x^2}{a^2}\right) dx \\ &= \pi b^2 \left[x - \frac{x^3}{3a^2}\right]_{-a}^a \\ &= \pi b^2 \left[a - \frac{a^3}{3a^2} - (-a) + \frac{(-a)^3}{3a^2}\right] \\ &= \pi b^2 \left[a - \frac{a}{3} + a - \frac{a}{3}\right] \\ &= \frac{4}{3}\pi a b^2 \end{aligned}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
$$\frac{x^2}{a^2}=1-\frac{y^2}{b^2}$$
$$x^2=a^2\left(1-\frac{y^2}{b^2}\right)$$

$$\begin{aligned}
V_y &= \int_{-b}^{b}\pi x^2 dy \\
&= \pi a^2\int_{-b}^{b}\left(1-\frac{y^2}{b^2}\right)dy \\
&= \pi a^2\left[y-\frac{y^3}{3b^2}\right]_{-b}^{b} \\
&= \pi a^2\left[b-\frac{b^3}{3b^2}-(-b)+\frac{(-b)^3}{3b^2}\right] \\
&= \pi a^2\left[b-\frac{b}{3}+b-\frac{b}{3}\right] \\
&= \frac{4}{3}\pi a^2 b
\end{aligned}$$

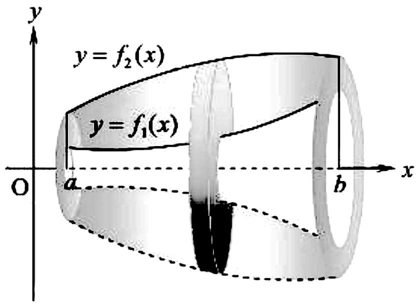
(b) if $V_x = 3V_y$, find the value of $a : b$.

Sol.

$$\begin{aligned}
\frac{4}{3}\pi ab^2 &= 3\cdot\frac{4}{3}\pi a^2 b \\
4\pi ab^2 &= 12\pi a^2 b \\
b &= 3a \\
a : b &= 1 : 3
\end{aligned}$$

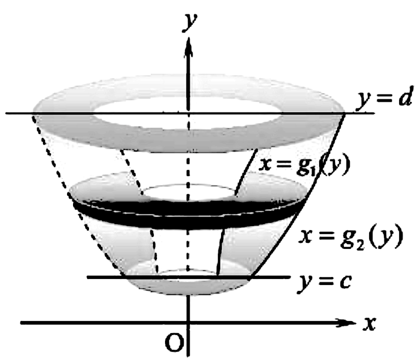
If a region is bounded by the curve $y = f_1(x)$, $y = f_2(x)$, the lines $x = a$, $x = b$, and the x-axis, and $f_2 \geq f_1 \geq 0$, then the volume of the solid of revolution formed by rotating this region around the x-axis, as shown in the diagram below, is given by

$$V=\pi\int_a^b[f_2(x)]^2dx-\pi\int_a^b[f_1(x)]^2dx=\pi\int_a^b\left\{[f_2(x)]^2-[f_1(x)]^2\right\}dx$$



Similarly, if a region is bounded by the curve $x = g_1(y)$, $x = g_2(y)$, the lines $y = c$, $y = d$, and the y-axis, and $g_2 \geq g_1 \geq 0$, then the volume of the solid of revolution formed by rotating this region around the y-axis, as shown in the diagram below, is given by

$$V=\pi\int_c^d[g_2(y)]^2dy-\pi\int_c^d[g_1(y)]^2dy=\pi\int_c^d\left\{[g_2(y)]^2-[g_1(y)]^2\right\}dy$$



28.4.2 Practice 8

1. Given that the line $x+y=a$ splits the circle $x^2+y^2=a^2$ into two parts, find the volume of the solid of revolution formed by rotating the smaller part of the circle about the x-axis.

   Sol.

   Substituting $y=a-x$ into $x^2+y^2=a^2$,

$$\begin{aligned}
x^2+(a-x)^2 &= a^2 \\
x^2+a^2-2ax+x^2 &= a^2 \\
2x^2-2ax &= 0 \\
x(x-a) &= 0 \\
x=0 \text{ or } x &= a
\end{aligned}$$

   In the interval $0 \le x \le a$, $x^2+y^2=a^2 \ge x+y=a$.

$$\begin{aligned}
V &= \int_0^a \pi(a^2-x^2)dx - \int_0^a \pi(a-x)^2 dx \\
&= \int_0^a \pi(a^2-x^2)dx - \int_0^a \pi(x^2-2ax+a^2)dx \\
&= \pi\left[a^2x-\frac{1}{3}x^3\right]_0^a - \pi\left[\frac{1}{3}x^3-ax^2+a^2x\right]_0^a \\
&= \pi\left[a^3-\frac{1}{3}a^3\right] - \pi\left[\frac{1}{3}a^3-a^3+a^3\right] \\
&= \frac{2}{3}\pi a^3 - \frac{1}{3}\pi a^3 \\
&= \frac{1}{3}\pi a^3
\end{aligned}$$

2. Given that a region is bounded by the curve $y^2=8-x$ and $y^2=x-4$. Find the volume of the solid of revolution formed by rotating this region about the x-axis and the y-axis respectively.

   Sol.

$$\begin{aligned}
V_x &= \int_4^6 \pi(x-4)dx + \int_6^8 \pi(8-x)dx \\
&= \pi\left[\frac{1}{2}x^2-4x\right]_4^6 + \pi\left[8x-\frac{1}{2}x^2\right]_6^8 \\
&= \pi[18-24-8+16] + \pi[64-32-48+18] \\
&= 2\pi+2\pi \\
&= 4\pi
\end{aligned}$$

$$\begin{aligned}
V_y &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi\left(8-y^2\right)^2 dy - \int_{-\sqrt{2}}^{\sqrt{2}} \pi\left(y^2+4\right)^2 dy \\
&= \pi\int_{-\sqrt{2}}^{\sqrt{2}} \left(64-16y^2+y^4\right) dy - \pi\int_{-\sqrt{2}}^{\sqrt{2}} \left(y^4+8y^2+16\right) dy \\
&= \pi\left[64y-\frac{16}{3}y^3+\frac{1}{5}y^5\right]_{-\sqrt{2}}^{\sqrt{2}} - \pi\left[\frac{1}{5}y^5+\frac{8}{3}y^3+16y\right]_{-\sqrt{2}}^{\sqrt{2}} \\
&= \pi\left[64\sqrt{2}-\frac{32}{3}\sqrt{2}+\frac{4}{5}\sqrt{2}+64\sqrt{2}-\frac{32}{3}\sqrt{2}+\frac{4}{5}\sqrt{2}\right] - \pi\left[\frac{4}{5}\sqrt{2}+\frac{16}{3}\sqrt{2}+16\sqrt{2}+\frac{4}{5}\sqrt{2}+\frac{16}{3}\sqrt{2}+16\sqrt{2}\right] \\
&= \frac{1624\sqrt{2}}{15}\pi - \frac{664\sqrt{2}}{15}\pi = 64\sqrt{2}\pi
\end{aligned}$$

28.4.3 Exercise 28.4

Find the volume of the solid of revolution formed by rotating the regions bounded by the following curves and lines about the x-axis (Question 1 to 7):

1. $y = \sqrt{x}, x = 4, x = 9$, and x-axis

   Sol.

$$\begin{aligned} V_x &= \int_4^9 \pi \left(\sqrt{x}\right)^2 dx \\ &= \pi \int_4^9 x dx \\ &= \pi \left[\frac{x^2}{2}\right]_4^9 \\ &= \pi \left(\frac{81}{2} - 8\right) \\ &= \frac{65\pi}{2} \end{aligned}$$

2. $y = 3x, x = 4$, and x-axis

   Sol.

$$\begin{aligned} V_x &= \int_0^4 \pi (3x)^2 dx \\ &= \pi \int_0^4 9x^2 dx \\ &= \pi \left[3x^3\right]_0^4 \\ &= 192\pi \end{aligned}$$

3. $y = x(x-2)$, and x-axis

   Sol.

$$\begin{aligned} V_x &= \pi \int_0^2 x^2(x-2)^2 dx \\ &= \pi \int_0^2 x^2(x^2 - 4x + 4) dx \\ &= \pi \int_0^2 x^4 - 4x^3 + 4x^2 dx \\ &= \pi \left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3}\right]_0^2 \\ &= \pi \left[\frac{32}{5} - 16 + \frac{32}{3}\right] \\ &= \frac{16\pi}{15} \end{aligned}$$

4. $x^2 + y^2 = 4, x = 0$, and $x = 2$

   Sol.

$$\begin{aligned} V_x &= \pi \int_0^2 \left(4 - x^2\right) dx \\ &= \pi \left[4x - \frac{x^3}{3}\right]_0^2 \\ &= \frac{16\pi}{3} \end{aligned}$$

5. $y = \sin x, x = 0, x = \pi$, and x-axis

   Sol.

$$\begin{aligned} V_x &= \pi \int_0^\pi \sin^2 x dx \\ &= \pi \int_0^\pi \frac{1 - \cos 2x}{2} dx \\ &= \frac{\pi}{2} \int_0^\pi (1 - \cos 2x) dx \\ &= \frac{\pi}{2} \left[x - \frac{\sin 2x}{2}\right]_0^\pi \\ &= \frac{\pi}{2} (\pi - 0) \\ &= \frac{\pi^2}{2} \end{aligned}$$

6. $y = e^x, x = -1, x = 1$, and x-axis

   Sol.

$$\begin{aligned} V_x &= \pi \int_{-1}^1 e^{2x} dx \\ &= \pi \left[\frac{e^{2x}}{2}\right]_{-1}^1 \\ &= \pi \left(\frac{e^2}{2} - \frac{e^{-2}}{2}\right) \\ &= \frac{\pi}{2} \left(e^2 - e^{-2}\right) \end{aligned}$$

7. $y = x^3 + x^2 - 2x$, and x-axis

Sol.

$$
\begin{aligned}
V_x &= \pi \int_{-2}^{1} \left(x^3 + x^2 - 2x\right)^2 dx \\
&= \pi \int_{-2}^{1} (x^6 + x^4 + 4x^2 + 2x^5 - 4x^3 - 4x^4) dx \\
&= \pi \int_{-2}^{1} (x^6 + 2x^5 - 3x^4 - 4x^3 + 4x^2) dx \\
&= \pi \left[\frac{x^7}{7} + \frac{x^6}{3} - \frac{3}{5}x^5 - x^4 + \frac{4x^3}{3}\right]_{-2}^{1} \\
&= \pi \left(\frac{1}{7} + \frac{1}{3} - \frac{3}{5} - 1 + \frac{4}{3} + \frac{128}{7} - \frac{64}{3} - \frac{96}{5} + 16 + \frac{32}{3}\right) \\
&= \frac{162}{35}\pi
\end{aligned}
$$

Find the volume of the solid of revolution formed by rotating the regions bounded by the following curves and lines about the y-axis (Question 8 to 14):

8. $y = x^3$, $y = 8$, and y-axis

Sol.

$$
\begin{aligned}
V_y &= \pi \int_{0}^{8} \left(\sqrt[3]{y}\right)^2 dy \\
&= \pi \int_{0}^{8} y^{\frac{2}{3}} dy \\
&= \pi \left[\frac{3}{5} y^{\frac{5}{3}}\right]_0^8 \\
&= \frac{3\pi}{5}\left(8^{\frac{5}{3}} - 0\right) \\
&= \frac{96\pi}{5}
\end{aligned}
$$

9. $x = \sqrt{y-1}$, $y = 4$, and y-axis

Sol.

$$
\begin{aligned}
V_y &= \pi \int_{1}^{4} \left(\sqrt{y-1}\right)^2 dy \\
&= \pi \int_{1}^{4} (y - 1) dy \\
&= \pi \left[\frac{y^2}{2} - y\right]_1^4 \\
&= \pi \left(8 - 4 - \frac{1}{2} + 1\right) \\
&= \frac{9\pi}{2}
\end{aligned}
$$

10. $y^2 = x + 3$, $y = 2$, x-axis, and y-axis

Sol.

$$
\begin{aligned}
V_y &= \pi \int_{0}^{2} \left(y^2 - 3\right)^2 dy \\
&= \pi \int_{0}^{2} (y^4 - 6y^2 + 9) dy \\
&= \pi \left[\frac{y^5}{5} - 2y^3 + 9y\right]_0^2 \\
&= \pi \left(\frac{32}{5} - 16 + 18\right) \\
&= \frac{42\pi}{5}
\end{aligned}
$$

11. $y^2 = x + 1$, and y-axis

Sol.

$$
\begin{aligned}
V_y &= \pi \int_{-1}^{1} \left(y^2 - 1\right)^2 dy \\
&= \pi \int_{-1}^{1} (y^4 - 2y^2 + 1) dy \\
&= \pi \left[\frac{y^5}{5} - \frac{2y^3}{3} + y\right]_{-1}^{1} \\
&= \pi \left(\frac{1}{5} - \frac{2}{3} + 1 + \frac{1}{5} - \frac{2}{3} + 1\right) \\
&= \frac{16\pi}{15}
\end{aligned}
$$

12. $x^2 - y^2 = 4, y = 3$, and x-axis

    Sol.

$$
\begin{aligned}
V_x &= \pi \int_0^3 (4 + y^2) dy \\
&= \pi \left[4y + \frac{y^3}{3} \right]_0^3 \\
&= \pi (12 + 9) \\
&= 21\pi
\end{aligned}
$$

13. $y = 1 - \sqrt{x}$, x-axis, and y-axis

    Sol.

$$
\begin{aligned}
V_x &= \pi \int_0^1 (1 - y)^4 \, dy \\
&= \pi \int_0^1 (1 - 4y + 6y^2 - 4y^3 + y^4) dy \\
&= \pi \left[y - 2y^2 + 2y^3 - y^4 + \frac{y^5}{5} \right]_0^1 \\
&= \pi \left(1 - 2 + 2 - 1 + \frac{1}{5} \right) \\
&= \frac{\pi}{5}
\end{aligned}
$$

14. $y = \dfrac{1}{x} - 1, y = 1$, x-axis, and y-axis

    Sol.

$$
V_x = \pi \int_0^1 \frac{1}{(y + 1)^2} dy
$$

Let $u = y + 1$, then $du = dy$. When $y = 0, u = 1$, when $y = 1, u = 2$.

$$
\begin{aligned}
V_x &= \pi \int_1^2 \frac{1}{u^2} du \\
&= \pi \left[-\frac{1}{u} \right]_1^2 \\
&= \pi \left(-\frac{1}{2} + 1 \right) \\
&= \frac{\pi}{2}
\end{aligned}
$$

15. Given that a region is bounded by the curve $y = 4 - x^2$ and the x-axis. Find the volume of the solid of revolution formed by rotating this region about the x-axis and the y-axis respectively.

16. Given that a region is bounded by the curve $y = 5 - \sqrt{x}$, x-axis, and y-axis. Find the volume of the solid of revolution formed by rotating this region about the x-axis and the y-axis respectively.

17. Find the volume of the solid of revolution formed by rotating the region bounded by the curve $y^2 = 9x$, the line $y = 6$, and the y-axis about the the y-axis.

18. Given that a region is bounded by the curve $y = x^2 + 1$, the line $x = -2, x = 2$, and x-axis. Find the volume of the solid of revolution formed by rotating this region about the x-axis and the y-axis respectively.

19. Find the volume of the solid of revolution formed by rotating the region bounded by the curve $y = x(6 - x)$ and the line $y = 3x$ about the x-axis.

20. Find the volume of the solid of revolution formed by rotating the region bounded by the curve $y = x^2$, the lines $x = 1$ and $y = 9$ about the y-axis.

21. Given that a region is bounded by the curve $y^2 = 8x$ and the line $y = 2x$. Find the volume of the solid of revolution formed by rotating this region about the x-axis and the y-axis respectively.

22. Given that a region is bounded by the curve $y^2 = 8x$ and $y = 8x^2$. Find the volume of the solid of revolution formed by rotating this region about the x-axis and the y-axis respectively.

23. Given that a region is bounded by the curve $y^2 = 2x$ and $y^2 = 12 - 4x$. Find the volume of the solid of revolution formed by rotating this region about the x-axis and the y-axis respectively.

24. Shown in the diagram below is the shaded region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$, where $a>0$ and $b>0$. If the volume of the solid of revolution formed by rotating this region about the x-axis and the y-axis is V_x and V_y respectively,

(a) find V_x and V_y.

(b) if $V_x=2V_y$, find the value of $a:b$.

28.5 Revision Exercise 28

1. $\int_0^a \left(2x^2-3x+2\right)dx$

2. $\int_1^3 \left(x^2+\frac{1}{x^3}\right)dx$

3. $\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} (3\sin\theta-2\cos 2\theta)d\theta$

4. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(3\sec^2\theta+\tan^2\theta\right)d\theta$

5. $\int_0^{\ln 2} e^{3x}dx$

6. $\int_1^3 \frac{2}{3x-1}dx$

7. $\int_1^{16} \frac{2x+3}{\sqrt{x}}dx$

8. $\int_1^4 \frac{(\sqrt{x}-1)^2}{x}dx$

9. $\int_1^2 \left(x+\frac{4}{x^2}\right)^2dx$

10. $\int_0^1 \frac{x+1}{x^2+2x+3}dx$

11. $\int_{-1}^2 \frac{5x}{\left(1+x^2\right)^4}dx$

12. $\int_0^2 \frac{x}{\sqrt{25-4x^2}}dx$

13. $\int_2^4 \frac{3x-2}{(2x-3)^2}dx$

14. $\int_2^4 \frac{2}{x^3-x}dx$

15. $\int_1^3 \frac{1}{x^3+2x^2+x}dx$

16. $\int_0^{\pi} (\sin\theta+\cos\theta)^2d\theta$

17. $\int_0^{\frac{\pi}{3}} \sec^2\theta \tan\theta d\theta$

18. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \cos\theta d\theta$

19. $\int_0^1 \frac{e^x}{e^x+1} dx$

20. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sec^2\theta}{\tan\theta} d\theta$

21. Given that $\int_0^4 f(x)dx = 2$, $\int_0^3 g(x)dx = 4$, and $\int_3^8 g(x)dx = 12$. Find the value of $\int_0^8 \left[f\left(\frac{x}{2}\right) - 2g(x)\right] dx$.

22. Given the function $y = (x+3)\sqrt{2x-3}$, find $\frac{dy}{dx}$. Hence, find $\int_2^6 \frac{x}{\sqrt{2x-3}} dx$.

23. Given the function $y = x\ln x$, find $\frac{dy}{dx}$. Hence, find the following definite integrals:

    (a) $\int_1^4 \ln x dx$

    (b) $\int_1^4 \ln(2x) dx$

24. Find the area of the region bounded by the curve $y = \frac{1}{x+1}$, the lines $x = 1, x = 7$, and the x-axis.

25. Find the area of the region bounded by the curve $y = \frac{3}{x}$ and the line $y = 4 - x$.

26. Find the area of the region bounded by the curve $x = y^2 - 5y$ and the line $x + 7y = 24$.

27. Find the area of the region bounded by the curves $y = x^2$ $y^3 = x$.

28. Shown in the diagram below is the shaded region bounded by the curves $y = \ln x$, $y = \ln(2x-1)$, and the line $y = 3$. Find the area of this region.

29. Find the area of the region bounded by the curves $x = y^3 - y$ $x = y - y^2$.

30. Shown in the diagram below is the shaded region bounded by the curves $y = \sin x$ $y = \sin 2x$ in the interval $0 \leq x \leq \pi$. Find the area of this region.

31. Find the volume of the solid of revolution formed by rotating the region bounded by the curve $y = \frac{1}{x+2}$, the line $x = 2$, and two axes about the x-axis.

32. Find the volume of the solid of revolution formed by rotating the region bounded by the curve $y = e^x - 3$ and the two axes about the x-axis.

33. Find the volume of the solid of revolution formed by rotating the region bounded by the curve $x = y^2 - 3y$ and the y-axis about the y-axis.

34. Find the volume of the solid of revolution formed by rotating the region bounded by the curve $y = x^2$ and the line $y = x+2$ about the x-axis.

35. Find the volume of the solid of revolution formed by rotating the region bounded by the curve $y^2 = x + 9$ and the line $y = x + 3$ about the y-axis.

36. Given that a region is bounded by the curve $y^2 = 8x$ and $y = x^2$. Find the volume of the solid of revolution formed by rotating this region about the x-axis and the y-axis respectively.

37. SHown in the diagram below is the shaded region bounded by the curve $y = 2\cos \pi x$, the line $y = 3x$, and the y-axis.

   (a) Prove that the x-coordinate of point A is $\frac{1}{3}$.

   (b) Find the volume of the solid of revolution formed by rotating this region about the x-axis.

38. Given that a region is bounded by the curve $xy = 12$, the line $x = 4$, and $y = 6$. Find the volume of the solid of revolution formed by rotating this region about the x-axis and the y-axis respectively.