Solution Book of Mathematic

Ssnior 2 Part I

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Contents

11.1 Revision Exercise 14

Calculate the following (Question 1 to 4):

1. $5\begin{pmatrix}-3 & -1\\ 3 & 4\end{pmatrix}+4\begin{pmatrix}6 & 2\\ 1 & -1\end{pmatrix}$

Sol.

$$\begin{aligned}&5\begin{pmatrix}-3 & -1\\ 3 & 4\end{pmatrix}+4\begin{pmatrix}6 & 2\\ 1 & -1\end{pmatrix}\\ &=\begin{pmatrix}-15 & -5\\ 15 & 20\end{pmatrix}+\begin{pmatrix}24 & 8\\ 4 & -4\end{pmatrix}\\ &=\begin{pmatrix}9 & 3\\ 19 & 16\end{pmatrix}\end{aligned}$$

2. $-4\begin{pmatrix}3 & 0\\ -1 & 5\end{pmatrix}-3\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}$

Sol.

$$\begin{aligned}&-4\begin{pmatrix}3 & 0\\ -1 & 5\end{pmatrix}-3\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}\\ &=\begin{pmatrix}-12 & 0\\ 4 & -20\end{pmatrix}-\begin{pmatrix}3 & 0\\ -3 & -3\end{pmatrix}\\ &=\begin{pmatrix}-15 & 0\\ 7 & 17\end{pmatrix}\end{aligned}$$

3. $\begin{pmatrix}2 & 6 & -1\\ 8 & -4 & 3\\ 5 & 7 & -2\end{pmatrix}+\begin{pmatrix}4 & -2 & 1\\ -2 & 1 & 0\\ -3 & 5 & -4\end{pmatrix}$

Sol.

$$\begin{aligned}&\begin{pmatrix}2 & 6 & -1\\ 8 & -4 & 3\\ 5 & 7 & -2\end{pmatrix}+\begin{pmatrix}4 & -2 & 1\\ -2 & 1 & 0\\ -3 & 5 & -4\end{pmatrix}\\ &=\begin{pmatrix}6 & 4 & 0\\ 6 & -3 & 3\\ 2 & 12 & -6\end{pmatrix}\end{aligned}$$

4. $2\begin{pmatrix}1 & -3 & 5\\ 7 & 2 & 0\\ 2 & 4 & -4\end{pmatrix}-\begin{pmatrix}-2 & -6 & 2\\ -5 & 1 & 1\\ 0 & 0 & -2\end{pmatrix}$

Sol.

$$\begin{aligned}&2\begin{pmatrix}1 & -3 & 5\\ 7 & 2 & 0\\ 2 & 4 & -4\end{pmatrix}-\begin{pmatrix}-2 & -6 & 2\\ -5 & 1 & 1\\ 0 & 0 & -2\end{pmatrix}\\ &=\begin{pmatrix}2 & -6 & 10\\ 14 & 4 & 0\\ 4 & 8 & -8\end{pmatrix}-\begin{pmatrix}-2 & -6 & 2\\ -5 & 1 & 1\\ 0 & 0 & -2\end{pmatrix}\\ &=\begin{pmatrix}4 & 0 & 8\\ 19 & 3 & -1\\ 4 & 8 & -6\end{pmatrix}\end{aligned}$$

5. Given that $\begin{pmatrix}2\\ -3\end{pmatrix}+3\begin{pmatrix}5\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$, find the value of x and y.

Sol.

$$\begin{aligned}\begin{pmatrix}2\\ -3\end{pmatrix}+3\begin{pmatrix}5\\ y\end{pmatrix}&=\begin{pmatrix}x\\ y\end{pmatrix}\\ \begin{pmatrix}2\\ -3\end{pmatrix}+\begin{pmatrix}15\\ 3y\end{pmatrix}&=\begin{pmatrix}x\\ y\end{pmatrix}\\ \begin{pmatrix}17\\ -3+3y\end{pmatrix}&=\begin{pmatrix}x\\ y\end{pmatrix}\\ x&=17\\ y&=-3+3y\\ 2y&=3\\ y&=\frac{3}{2}\\ \therefore x=17,\ y&=\frac{3}{2}\end{aligned}$$

6. Let $P=\begin{pmatrix}3 & -2 & 1\\ -1 & 2 & -3\\ 4 & 0 & -2\end{pmatrix}$, $Q=\begin{pmatrix}1 & -5 & -4\\ -2 & 0 & 6\\ 3 & 2 & 3\end{pmatrix}$ and $R=\begin{pmatrix}4 & 0 & 5\\ 1 & -2 & -7\\ 0 & -2 & 1\end{pmatrix}$. Find the following:

(a) $2Q+R'$

Sol.

$$\begin{aligned}2Q+R'&=2\begin{pmatrix}1 & -5 & -4\\ -2 & 0 & 6\\ 3 & 2 & 3\end{pmatrix}+\begin{pmatrix}4 & 0 & 5\\ 1 & -2 & -7\\ 0 & -2 & 1\end{pmatrix}'\\ &=\begin{pmatrix}2 & -10 & -8\\ -4 & 0 & 12\\ 6 & 4 & 6\end{pmatrix}+\begin{pmatrix}4 & 1 & 0\\ 0 & -2 & -2\\ 5 & -7 & 1\end{pmatrix}\\ &=\begin{pmatrix}6 & -9 & -8\\ -4 & -2 & 10\\ 11 & -3 & 7\end{pmatrix}\end{aligned}$$

(b) $(P - R) + 2Q'$

Sol.

$$
\begin{aligned}
&(P - R) + 2Q' \\
&= \begin{pmatrix} 3 & -2 & 1 \\ -1 & 2 & -3 \\ 4 & 0 & -2 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 5 \\ 1 & -2 & -7 \\ 0 & -2 & 1 \end{pmatrix} \\
&\quad + 2\begin{pmatrix} 1 & -5 & -4 \\ -2 & 0 & 6 \\ 3 & 2 & 3 \end{pmatrix}' \\
&= \begin{pmatrix} -1 & -2 & -4 \\ -2 & 4 & 4 \\ 4 & 2 & -3 \end{pmatrix} + \begin{pmatrix} 2 & -4 & 6 \\ -10 & 0 & 4 \\ -8 & 12 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 6 & 2 \\ -12 & 4 & 8 \\ -4 & 14 & 3 \end{pmatrix}
\end{aligned}
$$

(c) $[2(Q - P)]'$

Sol.

$$
\begin{aligned}
&[2(Q - P)]' \\
&= \left\{ 2\left[\begin{pmatrix} 1 & -5 & -4 \\ -2 & 0 & 6 \\ 3 & 2 & 3 \end{pmatrix} - \begin{pmatrix} 3 & -2 & 1 \\ -1 & 2 & -3 \\ 4 & 0 & -2 \end{pmatrix} \right] \right\}' \\
&= \left[2\begin{pmatrix} -2 & -3 & -5 \\ -1 & -2 & 9 \\ -1 & 2 & 5 \end{pmatrix} \right]' \\
&= \begin{pmatrix} -4 & -6 & -10 \\ -2 & -4 & 18 \\ -2 & 4 & 10 \end{pmatrix}' \\
&= \begin{pmatrix} -4 & -2 & -2 \\ -6 & -4 & 4 \\ -10 & 18 & 10 \end{pmatrix}
\end{aligned}
$$

(d) $(R' - Q)'$

Sol.

$$
\begin{aligned}
(R' - Q)' &= \left[\begin{pmatrix} 4 & 0 & 5 \\ 1 & -2 & -7 \\ 0 & -2 & 1 \end{pmatrix}' - \begin{pmatrix} 1 & -5 & -4 \\ -2 & 0 & 6 \\ 3 & 2 & 3 \end{pmatrix} \right]' \\
&= \begin{pmatrix} 4 & 0 & 5 \\ 1 & -2 & -7 \\ 0 & -2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -5 & -4 \\ -2 & 0 & 6 \\ 3 & 2 & 3 \end{pmatrix}' \\
&= \begin{pmatrix} 4 & 0 & 5 \\ 1 & -2 & -7 \\ 0 & -2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 3 \\ -5 & 0 & 2 \\ -4 & 6 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 2 & 2 \\ 6 & -2 & -9 \\ 4 & -8 & -2 \end{pmatrix}
\end{aligned}
$$

7. Let $M = \begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix}$ and $N = \begin{pmatrix} 3 & 6 \\ 7 & -1 \\ -4 & 2 \end{pmatrix}$. Find the matrix X in the following equations:

(a) $2N - 3M = 2M - X$

Sol.

$$
\begin{aligned}
2N - 3M &= 2M - X \\
X &= 5M - 2N \\
&= 5\begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix} - 2\begin{pmatrix} 3 & 6 \\ 7 & -1 \\ -4 & 2 \end{pmatrix} \\
&= \begin{pmatrix} -5 & 0 \\ 20 & -15 \\ 10 & 20 \end{pmatrix} - \begin{pmatrix} 6 & 12 \\ 14 & -2 \\ -8 & 4 \end{pmatrix} \\
&= \begin{pmatrix} -11 & -12 \\ 6 & -13 \\ 18 & 16 \end{pmatrix}
\end{aligned}
$$

(b) $2(M - 2N) + X = M + N$

Sol.

$$\begin{aligned}2(M-2N)+X &= M+N\\ X &= M+N-2(M-2N)\\ &= M+N-2M+4N\\ &= -M+5N\\ &= -\begin{pmatrix}-1 & 0\\ 4 & -3\\ 2 & 4\end{pmatrix}+5\begin{pmatrix}3 & 6\\ 7 & -1\\ -4 & 2\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ -4 & 3\\ -2 & -4\end{pmatrix}+\begin{pmatrix}15 & 30\\ 35 & -5\\ -20 & 10\end{pmatrix}\\ &= \begin{pmatrix}16 & 30\\ 31 & -2\\ -22 & 6\end{pmatrix}\end{aligned}$$

(c) $(M+2N)' = X$

Sol.

$$\begin{aligned}(M+2N)' &= X\\ X &= (M+2N)'\\ &= \left[\begin{pmatrix}-1 & 0\\ 4 & -3\\ 2 & 4\end{pmatrix}+2\begin{pmatrix}3 & 6\\ 7 & -1\\ -4 & 2\end{pmatrix}\right]'\\ &= \left[\begin{pmatrix}-1 & 0\\ 4 & -3\\ 2 & 4\end{pmatrix}+\begin{pmatrix}6 & 12\\ 14 & -2\\ -8 & 4\end{pmatrix}\right]\\ &= \begin{pmatrix}5 & 12\\ 18 & -5\\ -6 & 8\end{pmatrix}'\\ &= \begin{pmatrix}5 & 18 & -6\\ 12 & -5 & 8\end{pmatrix}\end{aligned}$$

(d) $3N' - M' = 2X$

Sol.

$$\begin{aligned}3N'-M' &= 2X\\ 2X &= (3N-M)'\\ &= \left[3\begin{pmatrix}3 & 6\\ 7 & -1\\ -4 & 2\end{pmatrix}-\begin{pmatrix}-1 & 0\\ 4 & -3\\ 2 & 4\end{pmatrix}\right]'\\ &= \left[\begin{pmatrix}9 & 18\\ 21 & -3\\ -12 & 6\end{pmatrix}-\begin{pmatrix}-1 & 0\\ 4 & -3\\ 2 & 4\end{pmatrix}\right]'\\ &= \begin{pmatrix}10 & 18\\ 17 & 0\\ -14 & 2\end{pmatrix}'\\ &= \begin{pmatrix}10 & 17 & -14\\ 18 & 0 & 2\end{pmatrix}\\ X &= \begin{pmatrix}5 & \frac{17}{2} & -7\\ 9 & 0 & 1\end{pmatrix}\end{aligned}$$

Of the following matrices, determine if AB and BA are defined. If any of them is defined, find the value of them (Question 8 to 11):

8. $A = \begin{pmatrix}3\\ 1\\ 4\end{pmatrix}$, $B = \begin{pmatrix}1 & 0 & 2\\ -2 & 0 & 1\\ 1 & 1 & -1\end{pmatrix}$

Sol.

$\because$ The number of columns of A is not equal to the number of rows of B
$\therefore$ AB is not defined

$\because$ The number of columns of B is equal to the number of rows of A
$\therefore$ BA is defined

$$\therefore BA = \begin{pmatrix}1 & 0 & 2\\ -2 & 0 & 1\\ 1 & 1 & -1\end{pmatrix}\begin{pmatrix}3\\ 1\\ 4\end{pmatrix} = \begin{pmatrix}11\\ -2\\ 0\end{pmatrix}$$

9. $A = \begin{pmatrix}-5 & 4 & 3\\ 1 & 2 & 3\end{pmatrix}$, $B = \begin{pmatrix}-2 & 1\\ -3 & 0\\ 1 & -4\end{pmatrix}$

Sol.

$\because$ The number of columns of A is equal to the number of rows of B

$\therefore AB$ is defined

$$\therefore AB = \begin{pmatrix} -5 & 4 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -3 & 0 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -17 \\ -5 & -11 \end{pmatrix}$$

$\because$ The number of columns of B is equal to the number of rows of A

$\therefore BA$ is defined

$$\therefore BA = \begin{pmatrix} -2 & 1 \\ -3 & 0 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} -5 & 4 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -6 & -3 \\ 13 & -12 & -9 \\ -9 & -4 & -9 \end{pmatrix}$$

10. $A = \begin{pmatrix} 3 & 2 & -1 \\ 5 & 4 & 0 \\ -2 & 6 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 9 & 0 \\ -7 & 1 \\ 2 & 3 \end{pmatrix}$

Sol.

$\because$ The number of columns of A is equal to the number of rows of B

$\therefore AB$ is defined

$$\therefore AB = \begin{pmatrix} 3 & 2 & -1 \\ 5 & 4 & 0 \\ -2 & 6 & 1 \end{pmatrix} \begin{pmatrix} 9 & 0 \\ -7 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 11 & -1 \\ 17 & 4 \\ -58 & 9 \end{pmatrix}$$

$\because$ The number of columns of B is not equal to the number of rows of A

$\therefore BA$ is not defined

11. $A = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 5 & 2 \end{pmatrix}$

Sol.

$\because$ The number of columns of A is not equal to the number of rows of B

$\therefore AB$ is not defined

$\because$ The number of columns of B is equal to the number of rows of A

$\therefore BA$ is defined

$$\therefore BA = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 10 & 4 \\ 18 & 2 \end{pmatrix}$$

12. Given that $A = \begin{pmatrix} a & 3a \\ 2b & b \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $AB = \begin{pmatrix} 45 \\ 48 \end{pmatrix}$, find the value of a and b.

Sol.

$$\begin{aligned} AB = \begin{pmatrix} a & 3a \\ 2b & b \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} &= \begin{pmatrix} 45 \\ 48 \end{pmatrix} \\ \begin{pmatrix} 3a + 6a \\ 6b + 2b \end{pmatrix} &= \begin{pmatrix} 45 \\ 48 \end{pmatrix} \\ 9a &= 45 \\ 8b &= 48 \\ a &= 5 \\ b &= 6 \end{aligned}$$

13. Given that $A = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $A + B = AB$, find the value of a, b and c.

Sol.

$$\begin{aligned} A + B &= AB \\ \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} &= \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \\ \begin{pmatrix} 3 + a & b \\ 0 & 4 + c \end{pmatrix} &= \begin{pmatrix} 3a & 3b \\ 0 & 4c \end{pmatrix} \\ 3 + a &= 3a \\ 2a &= 3 \\ b &= 3b \\ 2b &= 0 \\ 4 + c &= 4c \\ 3c &= 4 \\ a = \frac{3}{2},\ b = 0,\ c &= \frac{4}{3} \end{aligned}$$

Find the value of the following determinants (Question 14 to 22):

14. $\begin{vmatrix} 20 & 15 \\ 8 & 6 \end{vmatrix}$

Sol.

$$\begin{aligned} \begin{vmatrix} 20 & 15 \\ 8 & 6 \end{vmatrix} &= 20 \cdot 6 - 15 \cdot 8 \\ &= 120 - 120 \\ &= 0 \end{aligned}$$

15. $\begin{vmatrix} 6 & -7 \\ 15 & -2 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 6 & -7 \\ 15 & -2 \end{vmatrix} = 6 \cdot -2 - (-7) \cdot 15$$
$$= -12 + 105$$
$$= 93$$

16. $\begin{vmatrix} -4 & -10 \\ 12 & 7 \end{vmatrix}$

Sol.

$$\begin{vmatrix} -4 & -10 \\ 12 & 7 \end{vmatrix} = -4 \cdot 7 - (-10) \cdot 12$$
$$= -28 + 120$$
$$= 92$$

17. $\begin{vmatrix} 3 & 3 & -6 \\ 2 & -2 & 0 \\ 3 & 0 & -3 \end{vmatrix}$

$$\begin{array}{ccc|cc} + & + & + & & \\ 3 & 3 & -6 & 3 & 3 \\ 2 & -2 & 0 & 2 & -2 \\ 3 & 0 & -3 & 3 & 0 \\ - & - & - & & \end{array}$$

Sol.

$$\begin{vmatrix} 3 & 3 & -6 \\ 2 & -2 & 0 \\ 3 & 0 & -3 \end{vmatrix} = 18 + 0 + 0 - 36 - 0 + 18$$
$$= 0$$

18. $\begin{vmatrix} 5 & 7 & 1 \\ -3 & 6 & 9 \\ 4 & 7 & 3 \end{vmatrix}$

$$\begin{array}{ccc|cc} + & + & + & & \\ 5 & 7 & 1 & 5 & 7 \\ -3 & 6 & 9 & -3 & 6 \\ 4 & 7 & 3 & 4 & 7 \\ - & - & - & & \end{array}$$

Sol.

$$\begin{vmatrix} 5 & 7 & 1 \\ -3 & 6 & 9 \\ 4 & 7 & 3 \end{vmatrix} = 90 + 252 - 21 - 24 - 315 + 63$$
$$= 45$$

19. $\begin{vmatrix} -2 & 7 & -4 \\ 3 & -5 & 2 \\ -1 & 0 & -3 \end{vmatrix}$

$$\begin{array}{ccc|cc} + & + & + & & \\ -2 & 7 & -4 & -2 & 7 \\ 3 & -5 & 2 & 3 & -5 \\ -1 & 0 & -3 & -1 & 0 \\ - & - & - & & \end{array}$$

Sol.

$$\begin{vmatrix} -2 & 7 & -4 \\ 3 & -5 & 2 \\ -1 & 0 & -3 \end{vmatrix} = -30 - 14 - 0 + 20 + 0 + 63$$
$$= -39$$

20. $\begin{vmatrix} 1 & 0 & -1 \\ 3 & -2 & 5 \\ -1 & 1 & 3 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 1 & 0 & -1 \\ 3 & -2 & 5 \\ -1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -2 & 5 \\ 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ -2 & 5 \end{vmatrix}$$
$$= -11 - 3 + 2 = -12$$

21. $\begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 1 \\ -2 & -6 & 5 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 1 \\ -2 & -6 & 5 \end{vmatrix} = 3\begin{vmatrix} 2 & 2 & 4 \\ 1 & 1 & 1 \\ -2 & -2 & 5 \end{vmatrix}$$
$$= 0 \quad \text{(col 1 and 2 are the same)}$$

22. $\begin{vmatrix} 10 & 8 & -2 \\ 15 & 16 & -3 \\ -5 & -4 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 10 & 8 & -2 \\ 15 & 16 & -3 \\ -5 & -4 & 1 \end{vmatrix} = -5\begin{vmatrix} 2 & 8 & 2 \\ 3 & 16 & 3 \\ -1 & -4 & -1 \end{vmatrix}$$
$$= 0 \quad \text{(col 1 and 3 are the same)}$$

Using the identities of determinant, prove the following equations (Question 23 to 24):

23. $\begin{vmatrix} bc & 1 & bc(b+c) \\ ca & 1 & ca(c+a) \\ ab & 1 & ab(a+b) \end{vmatrix} = 0$

Proof.

$$\begin{vmatrix} bc & 1 & bc(b+c) \\ ca & 1 & ca(c+a) \\ ab & 1 & ab(a+b) \end{vmatrix}$$

$$= a^2b^2c^2 \begin{vmatrix} 1 & \frac{1}{bc} & b+c \\ 1 & \frac{1}{ca} & c+a \\ 1 & \frac{1}{ab} & a+b \end{vmatrix}$$

$$= a^2b^2c^2 \begin{vmatrix} 1 & \frac{1}{bc} & -a \\ 1 & \frac{1}{ca} & -b \\ 1 & \frac{1}{ab} & -c \end{vmatrix} \qquad C_3 \to C_3 + (a+b+c)C_1$$

$$= a^2b^2c^2 \begin{vmatrix} 1 & a & -a \\ 1 & b & -b \\ 1 & c & -c \end{vmatrix} \qquad C_2 \to C_2 + abcC_1$$

$$= -a^2b^2c^2 \begin{vmatrix} 1 & a & a \\ 1 & b & b \\ 1 & c & c \end{vmatrix}$$

$$= 0 \qquad C_2 = C_3$$

24. $\begin{vmatrix} a & 1 & a^2(b+c) \\ b & 1 & b^2(c+a) \\ c & 1 & c^2(a+b) \end{vmatrix} = 0$

Proof.

$$\begin{vmatrix} a & 1 & a^2(b+c) \\ b & 1 & b^2(c+a) \\ c & 1 & c^2(a+b) \end{vmatrix}$$

$$= \begin{vmatrix} a & 1 & a^2(b+c) \\ b-a & 0 & b^2(c+a)-a^2(b+c) \\ c-a & 0 & c^2(a+b)-a^2(b+c) \end{vmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$= \begin{vmatrix} b-a & b^2(c+a)-a^2(b+c) \\ c-a & c^2(a+b)-a^2(b+c) \end{vmatrix}$$

$$= (b-a)[c^2(a+b)-a^2(b+c)]$$
$$- (c-a)[b^2(c+a)-a^2(b+c)]$$
$$= c^2(b-a)(b+a) - a^2(b+c)(c-a)$$
$$- b^2(c-a)(c+a) + a^2(b+c)(c-a)$$
$$= c^2(b^2-a^2) - b^2(c^2-a^2)$$
$$= b^2c^2 - a^2c^2 - b^2c^2 + a^2c^2$$
$$= 0$$

Find the value of x in the following expressions (Question 25 to 26):

25. $\begin{vmatrix} 2x & 3 & x+5 \\ -3 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 5x - 1$

Sol.

$$\begin{vmatrix} 2x & 3 & x+5 \\ -3 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 5x - 1$$
$$x+5 \begin{vmatrix} -3 & 2 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 2x & 3 \\ 2 & 1 \end{vmatrix} = 5x - 1$$
$$-7(x+5) - (2x-6) = 5x - 1$$
$$-7x - 35 - 2x + 6 = 5x - 1$$
$$-14x = 28$$
$$x = -2$$

26. $\begin{vmatrix} x+3 & 1 & 0 \\ x & 3 & 0 \\ 1 & 0 & -x-2 \end{vmatrix} = x + 6$

Sol.

$$\begin{vmatrix} x+3 & 1 & 0 \\ x & 3 & 0 \\ 1 & 0 & -x-2 \end{vmatrix} = x + 6$$
$$-x-2 \begin{vmatrix} x+3 & 1 \\ x & 3 \end{vmatrix} = x + 6$$
$$-(x+2)(3x+9-x) = x + 6$$
$$(x+2)(2x+9) = -x - 6$$
$$2x^2 + 13x + 18 = -x - 6$$
$$2x^2 + 14x + 24 = 0$$
$$x^2 + 7x + 12 = 0$$
$$(x+4)(x+3) = 0$$
$$x = -4 \text{ or } x = -3$$

27. Given an identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $(2I + J)^{-1} = rI + sJ$, find the value of r and s.

Sol.

$$\begin{aligned}(2I+J)^{-1} &= \left[2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]^{-1}\\ &= \left[\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}+\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right]^{-1}\\ &= \begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}^{-1}\\ &= \frac{1}{5}\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}\\ &= \begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix}\\ rI+sJ &= r\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+s\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\\ &= \begin{pmatrix}r & 0\\ 0 & r\end{pmatrix}+\begin{pmatrix}0 & s\\ -s & 0\end{pmatrix}\\ &= \begin{pmatrix}r & s\\ -s & r\end{pmatrix}\end{aligned}$$

$$(2I+J)^{-1} = rI+sJ$$

$$\begin{pmatrix}\frac{2}{5} & -\frac{1}{5}\\ \frac{1}{5} & \frac{2}{5}\end{pmatrix} = \begin{pmatrix}r & s\\ -s & r\end{pmatrix}$$

$$\therefore r=\frac{2}{5},\ s=-\frac{1}{5}$$

Find the value of a in the following matrices if they are non-inversible (Question 28 to 31):

28. $\begin{pmatrix}3 & a\\ -2 & 6\end{pmatrix}$

Sol.

$$\begin{aligned}\begin{vmatrix}3 & a\\ -2 & 6\end{vmatrix} &= 0\\ 18+2a &= 0\\ 2a &= -18\\ a &= -9\end{aligned}$$

29. $\begin{pmatrix}5a+2 & 4\\ 6 & a\end{pmatrix}$

Sol.

$$\begin{aligned}\begin{vmatrix}5a+2 & 4\\ 6 & a\end{vmatrix} &= 0\\ 5a^2+2a-24 &= 0\\ (x-2)(5x+12) &= 0\\ x=2 \text{ or } x &= \frac{12}{5}\end{aligned}$$

30. $\begin{pmatrix}-7 & a & 3\\ 2 & -3 & 1\\ 0 & -a & 4\end{pmatrix}$

Sol.

$$\begin{aligned}\begin{vmatrix}-7 & a & 3\\ 2 & -3 & 1\\ 0 & -a & 4\end{vmatrix} &= 0\\ -7\begin{vmatrix}-3 & 1\\ -a & 4\end{vmatrix}-2\begin{vmatrix}a & 3\\ -a & 4\end{vmatrix} &= 0\\ -7(-12+a)-2(4a+3a) &= 0\\ 84-7a-14a &= 0\\ 21a &= 84\\ a &= 4\end{aligned}$$

31. $\begin{pmatrix}a & -1 & 0\\ 4 & 0 & -2\\ a+4 & a & -8\end{pmatrix}$

Sol.

$$\begin{aligned}\begin{vmatrix}a & -1 & 0\\ 4 & 0 & -2\\ a+4 & a & -8\end{vmatrix} &= 0\\ a\begin{vmatrix}0 & -2\\ a & -8\end{vmatrix}+\begin{vmatrix}4 & -2\\ a+4 & -8\end{vmatrix}\\ 2a^2-32+2a+8 &= 0\\ 2a^2+2a-24 &= 0\\ a^2+a-12 &= 0\\ (a+4)(a-3) &= 0\\ a=-4 \text{ or } a &= 3\end{aligned}$$

Find the inverse of the following matrices (Question 32 to 37):

32. $\begin{pmatrix}2 & 5\\ 2 & 3\end{pmatrix}$

Sol.

$$\begin{aligned}\begin{pmatrix}2 & 5\\ 2 & 3\end{pmatrix}^{-1} &= -\frac{1}{4}\begin{pmatrix}3 & -5\\ -2 & 2\end{pmatrix}\\ &= \begin{pmatrix}-\frac{3}{4} & \frac{5}{4}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}\end{aligned}$$

33. $\begin{pmatrix}-2 & -1\\ 4 & 6\end{pmatrix}$

Sol.

$$\begin{pmatrix}-2 & -1\\ 4 & 6\end{pmatrix}^{-1} = -\frac{1}{8}\begin{pmatrix}6 & 1\\ -4 & -2\end{pmatrix}$$
$$= \begin{pmatrix}-\frac{3}{8} & -\frac{1}{8}\\ \frac{1}{2} & \frac{1}{4}\end{pmatrix}$$

34. $\begin{pmatrix}1 & 0 & 3\\ 3 & 1 & 9\\ -2 & 2 & -4\end{pmatrix}$

Sol.

$$\begin{vmatrix}1 & 0 & 3\\ 3 & 1 & 9\\ -2 & 2 & -4\end{vmatrix} = 2$$

$$\operatorname{adj}\begin{pmatrix}1 & 0 & 3\\ 3 & 1 & 9\\ -2 & 2 & -4\end{pmatrix}$$

$$= \begin{pmatrix}\begin{vmatrix}1 & 9\\ 2 & -4\end{vmatrix} & -\begin{vmatrix}3 & 9\\ -2 & -4\end{vmatrix} & \begin{vmatrix}3 & 1\\ -2 & 2\end{vmatrix}\\ -\begin{vmatrix}0 & 3\\ 2 & -4\end{vmatrix} & \begin{vmatrix}1 & 3\\ -2 & -4\end{vmatrix} & -\begin{vmatrix}1 & 0\\ -2 & 2\end{vmatrix}\\ \begin{vmatrix}0 & 3\\ 1 & 9\end{vmatrix} & -\begin{vmatrix}1 & 3\\ 3 & 9\end{vmatrix} & \begin{vmatrix}1 & 0\\ 3 & 1\end{vmatrix}\end{pmatrix}'$$

$$= \begin{pmatrix}-22 & -6 & 8\\ 6 & 2 & -2\\ -3 & 0 & 1\end{pmatrix}'$$

$$= \begin{pmatrix}-22 & 6 & -3\\ -6 & 2 & 0\\ 8 & -2 & 1\end{pmatrix}$$

$$\therefore \begin{pmatrix}1 & 0 & 3\\ 3 & 1 & 9\\ -2 & 2 & -4\end{pmatrix}^{-1}$$

$$= \frac{1}{2}\begin{pmatrix}-22 & 6 & -3\\ -6 & 2 & 0\\ 8 & -2 & 1\end{pmatrix}$$

$$= \begin{pmatrix}-11 & 3 & -\frac{3}{2}\\ -3 & 1 & 0\\ 4 & -1 & \frac{1}{2}\end{pmatrix}$$

35. $\begin{pmatrix}1 & 2 & -3\\ 2 & 1 & -4\\ -2 & 5 & 1\end{pmatrix}$

Sol.

$$\begin{vmatrix}1 & 2 & -3\\ 2 & 1 & -4\\ -2 & 5 & 1\end{vmatrix} = -3$$

$$\operatorname{adj}\begin{pmatrix}1 & 2 & -3\\ 2 & 1 & -4\\ -2 & 5 & 1\end{pmatrix}$$

$$= \begin{pmatrix}\begin{vmatrix}1 & -4\\ 5 & 1\end{vmatrix} & -\begin{vmatrix}2 & -4\\ -2 & 1\end{vmatrix} & \begin{vmatrix}2 & 1\\ -2 & 5\end{vmatrix}\\ -\begin{vmatrix}2 & -3\\ 5 & 1\end{vmatrix} & \begin{vmatrix}1 & -3\\ -2 & 1\end{vmatrix} & -\begin{vmatrix}1 & 2\\ -2 & 5\end{vmatrix}\\ \begin{vmatrix}2 & -3\\ 1 & -4\end{vmatrix} & -\begin{vmatrix}1 & -3\\ 2 & -4\end{vmatrix} & \begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix}\end{pmatrix}'$$

$$= \begin{pmatrix}21 & 6 & 12\\ -17 & -5 & -9\\ -5 & -2 & -3\end{pmatrix}'$$

$$= \begin{pmatrix}21 & -17 & -5\\ 6 & -5 & -2\\ 12 & -9 & -3\end{pmatrix}'$$

$$\therefore \begin{pmatrix}1 & 2 & -3\\ 2 & 1 & -4\\ -2 & 5 & 1\end{pmatrix}^{-1}$$

$$= -\frac{1}{3}\begin{pmatrix}21 & -17 & -5\\ 6 & -5 & -2\\ 12 & -9 & -3\end{pmatrix}$$

$$= \begin{pmatrix}-7 & -\frac{17}{3} & -\frac{5}{3}\\ -2 & -\frac{5}{3} & -\frac{2}{3}\\ -4 & 3 & 1\end{pmatrix}$$

36. $\begin{pmatrix}4 & 1 & 0\\ 0 & -2 & 3\\ 0 & -4 & 4\end{pmatrix}$

Sol.

$$\begin{vmatrix} 4 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & -4 & 4 \end{vmatrix} = 16$$

$$\operatorname{adj}\begin{pmatrix} 4 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & -4 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{vmatrix} -2 & 3 \\ -4 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 0 & -2 \\ 0 & -4 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ -4 & 4 \end{vmatrix} & \begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 0 & -4 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ -2 & 3 \end{vmatrix} & -\begin{vmatrix} 4 & 0 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 0 & -2 \end{vmatrix} \end{pmatrix}'$$

$$= \begin{pmatrix} 4 & 0 & 0 \\ -4 & 16 & 16 \\ 3 & -12 & -8 \end{pmatrix}'$$

$$= \begin{pmatrix} 4 & -4 & 0 \\ 0 & 16 & -12 \\ 9 & 16 & -8 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 4 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & -4 & 4 \end{pmatrix}^{-1}$$

$$= \frac{1}{16}\begin{pmatrix} 4 & -4 & 3 \\ 0 & 16 & -12 \\ 0 & 16 & -8 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{3}{16} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}$$

37. $\begin{pmatrix} 3 & 2 & 4 \\ -1 & -3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$

Sol.

$$\begin{vmatrix} 3 & 2 & 4 \\ -1 & -3 & 0 \\ 1 & 0 & 3 \end{vmatrix} = -9$$

$$\operatorname{adj}\begin{pmatrix} 3 & 2 & 4 \\ -1 & -3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{vmatrix} -3 & 0 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} -1 & 0 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} -1 & -3 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & 4 \\ -3 & 0 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ -1 & -3 \end{vmatrix} \end{pmatrix}'$$

$$= \begin{pmatrix} -9 & 3 & 3 \\ -6 & 5 & 2 \\ 12 & -4 & -7 \end{pmatrix}'$$

$$= \begin{pmatrix} -9 & -6 & 12 \\ 3 & 5 & -4 \\ 3 & 2 & -7 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 3 & 2 & 4 \\ -1 & -3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

$$= \frac{1}{-9}\begin{pmatrix} -9 & -6 & 12 \\ 3 & 5 & -4 \\ 3 & 2 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & \frac{2}{3} & -\frac{4}{3} \\ -\frac{1}{3} & -\frac{5}{9} & \frac{4}{9} \\ -\frac{1}{3} & -\frac{2}{9} & \frac{7}{9} \end{pmatrix}$$

Solve the following system of equations using the method of Gauss elimination (Question 38 to 41):

38. $\begin{cases} 2x - y + 4z = 5 \\ 2x + 3y - 4z = -7 \\ x + y + z = 2 \end{cases}$

Sol.

$$\left(\begin{array}{ccc|c} 2 & -1 & 4 & 5 \\ 2 & 3 & -4 & -7 \\ 1 & 1 & 1 & 2 \end{array}\right)$$

$$\xrightarrow[R_1\to R_1+R_3]{R_2\to R_2+R_1}\left(\begin{array}{ccc|c} 3 & 0 & 5 & 7 \\ 4 & 2 & 0 & -2 \\ 1 & 1 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_2\to \frac{1}{2}R_2}\left(\begin{array}{ccc|c} 3 & 0 & 5 & 7 \\ 2 & 1 & 0 & -1 \\ 1 & 1 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_3\to R_3-R_2}\left(\begin{array}{ccc|c} 3 & 0 & 5 & 7 \\ 2 & 1 & 0 & -1 \\ -1 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_1\to R_1+3R_3}\left(\begin{array}{ccc|c} 0 & 0 & 8 & 16 \\ 2 & 1 & 0 & -1 \\ -1 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_1\to \frac{1}{8}R_1}\left(\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & -1 \\ -1 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_3\to R_3-R_1}\left(\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3\to -R3]{R_2\to R_2+2R_3}\left(\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \end{array}\right)$$

$$\xrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\therefore x=-1,\ y=1,\ z=2$$

39. $\begin{cases} x-2y-3z=-4 \\ 3x+y-4z=-5 \\ 2x+4y-z=-5 \end{cases}$

40. $\begin{cases} x-2y-z=3 \\ 4x-y+2z=1 \\ x+3y=5 \end{cases}$

41. $\begin{cases} 2x-y-z=0 \\ 4x-3y+2z=1 \\ 3x-2y-4z=-1 \end{cases}$

Solve the following system of equations using the Cramer's rule (Question 42 to 45):

42. $\begin{cases} x-3y-2z=1 \\ 7x+4y-5z=0 \\ 3x+9y+z=-1 \end{cases}$

Sol.

$$\Delta=\begin{vmatrix} 1 & -3 & -2 \\ 7 & 4 & -5 \\ 3 & 9 & 1 \end{vmatrix}=13$$

$$\Delta_x=\begin{vmatrix} 1 & -3 & -2 \\ 0 & 4 & -5 \\ -1 & 9 & 1 \end{vmatrix}=26$$

$$\Delta_y=\begin{vmatrix} 1 & 1 & -2 \\ 7 & 0 & -5 \\ 3 & -1 & 1 \end{vmatrix}=-13$$

$$\Delta_z=\begin{vmatrix} 1 & -3 & 1 \\ 7 & 4 & 0 \\ 3 & 9 & -1 \end{vmatrix}=26$$

$$\therefore x=\frac{26}{13}=2,\ y=\frac{-13}{13}=-1,\ z=\frac{26}{13}=2$$

43. $\begin{cases} x-2y+3z=6 \\ 2x+3y-4z=20 \\ 3x-2y-5z=6 \end{cases}$

Sol.

$$\Delta=\begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -4 \\ 3 & -2 & -5 \end{vmatrix}=-58$$

$$\Delta_x=\begin{vmatrix} 6 & -2 & 3 \\ 20 & 3 & -4 \\ 6 & -2 & -5 \end{vmatrix}=-464$$

$$\Delta_y=\begin{vmatrix} 1 & 6 & 3 \\ 2 & 20 & -4 \\ 3 & 6 & -5 \end{vmatrix}=-232$$

$$\Delta_z=\begin{vmatrix} 1 & -2 & 6 \\ 2 & 3 & 20 \\ 3 & -2 & 6 \end{vmatrix}=-116$$

$$\therefore x=\frac{-464}{-58}=8,\ y=\frac{-232}{-58}=4,\ z=\frac{-116}{-58}=2$$

44. $\begin{cases} 2x-2y-4z+3=0 \\ 2x+3y+4z-2=0 \\ 7x+3y-2z-2=0 \end{cases}$

Sol.

$$\begin{cases} 2x - 2y - 4z = -3 \\ 2x + 3y + 4z = 2 \\ 7x + 3y - 2z = 2 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & -2 & -4 \\ 2 & 3 & 4 \\ 7 & 3 & -2 \end{vmatrix} = -40$$

$$\Delta_x = \begin{vmatrix} -3 & -2 & -4 \\ 2 & 3 & 4 \\ 2 & 3 & -2 \end{vmatrix} = 30$$

$$\Delta_y = \begin{vmatrix} 2 & -3 & -4 \\ 2 & 2 & 4 \\ 7 & 2 & -2 \end{vmatrix} = -80$$

$$\Delta_z = \begin{vmatrix} 2 & -2 & -3 \\ 2 & 3 & 2 \\ 7 & 3 & 2 \end{vmatrix} = 25$$

$$\therefore x = \frac{30}{-40} = -\frac{3}{4},\ y = \frac{-80}{-40} = 2,\ z = \frac{25}{-40} = -\frac{5}{8}$$

45. $\begin{cases} \frac{2}{x} - \frac{5}{y} + \frac{4}{z} = -3 \\ \frac{4}{x} + \frac{1}{y} - \frac{2}{z} = 7 \\ \frac{7}{x} - \frac{3}{z} = 4 \end{cases}$

Sol.

Let $a = \dfrac{1}{x},\ b = \dfrac{1}{y},\ c = \dfrac{1}{z}$

$$\begin{cases} 2a - 5b + 4c = -3 \\ 4a + b - 2c = 7 \\ 7a - 3c = 4 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & -5 & 4 \\ 4 & 1 & -2 \\ 7 & 0 & -3 \end{vmatrix} = -24$$

$$\Delta_a = \begin{vmatrix} -3 & -5 & 4 \\ 7 & 1 & -2 \\ 4 & 0 & -3 \end{vmatrix} = -72$$

$$\Delta_b = \begin{vmatrix} 2 & -3 & 4 \\ 4 & 7 & -2 \\ 7 & 4 & -3 \end{vmatrix} = -152$$

$$\Delta_c = \begin{vmatrix} 2 & -5 & -3 \\ 4 & 1 & 7 \\ 7 & 0 & 4 \end{vmatrix} = -136$$

$$\therefore a = \frac{-72}{-24} = 3,\ b = \frac{-152}{-24} = \frac{19}{3},\ c = \frac{-136}{-24} = \frac{17}{3}$$

$$\therefore x = \frac{1}{3},\ y = \frac{3}{19},\ z = \frac{3}{17}$$