

Mathematics

Senior 3 Part I

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25.9 Revision Exercise 25

1. Find the gradient of the tangent to the curve $y = 2x^2 + 1$ at the point where $x = 2$.

Sol.

$$\begin{aligned}\Delta y &= 2(x + \Delta x)^2 + 1 - (2x^2 + 1) \\ &= 2x^2 + 4x\Delta x + 2\Delta x^2 + 1 - 2x^2 - 1 \\ &= 4x\Delta x + 2\Delta x^2\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= 4x + 2\Delta x \\ \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \rightarrow 0} (4x + 2\Delta x) \\ \frac{dy}{dx} &= 4x\end{aligned}$$

\therefore The gradient of the tangent to the curve $y = 2x^2 + 1$ at the point where $x = 2$ is $4(2) = 8$.

2. Find the gradient of the curve $y = 3x^2 - 1$ at the point $A(-1, 2)$.

Sol.

$$\begin{aligned}\Delta y &= 3(x + \Delta x)^2 - 1 - (3x^2 - 1) \\ &= 3x^2 + 6x\Delta x + 3\Delta x^2 - 1 - 3x^2 + 1 \\ &= 6x\Delta x + 3\Delta x^2\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= 6x + 3\Delta x \\ \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \rightarrow 0} (6x + 3\Delta x) \\ \frac{dy}{dx} &= 6x\end{aligned}$$

\therefore The gradient of the curve $y = 3x^2 - 1$ at the point $A(-1, 2)$ is $6(-1) = -6$.

3. Find the gradient of the curve $y = 2x - x^3$ at the point $B(-1, -1)$.

Sol.

$$\begin{aligned}\Delta y &= 2(x + \Delta x) - (x + \Delta x)^3 - (2x - x^3) \\ &= 2x + 2\Delta x - x^3 - 3x^2\Delta x - 3x\Delta x^2 - \Delta x^3 - 2x + x^3 \\ &= 2\Delta x - 3x^2\Delta x - 3x\Delta x^2 - \Delta x^3\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= 2 - 3x^2 - 3x\Delta x - \Delta x^2 \\ \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \rightarrow 0} (2 - 3x^2 - 3x\Delta x - \Delta x^2) \\ \frac{dy}{dx} &= 2 - 3x^2\end{aligned}$$

\therefore The gradient of the curve $y = 2x - x^3$ at the point $B(-1, -1)$ is $2 - 3(-1)^2 = -1$.

4. Find the derivative of the following functions using the definition of the derivative, and find the value of the derivative at the point where $x = 1$:

(a) $f(x) = x^2 + 2x$

Sol.

$$\begin{aligned}\Delta y &= (x + \Delta x)^2 + 2(x + \Delta x) - (x^2 + 2x) \\ &= x^2 + 2x\Delta x + \Delta x^2 + 2x + 2\Delta x - x^2 - 2x \\ &= 2x\Delta x + \Delta x^2 + 2\Delta x\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= 2x + \Delta x + 2 \\ \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \rightarrow 0} (2x + \Delta x + 2) \\ f'(x) &= 2x + 2\end{aligned}$$

(b) $g(x) = x^3$

Sol.

$$\begin{aligned}\Delta y &= (x + \Delta x)^3 - x^3 \\ &= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - x^3 \\ &= 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= 3x^2 + 3x\Delta x + \Delta x^2 \\ \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \rightarrow 0} (3x^2 + 3x\Delta x + \Delta x^2) \\ g'(x) &= 3x^2\end{aligned}$$

(c) $h(x) = \frac{5}{x}$

Sol.

$$\begin{aligned}\Delta y &= \frac{5}{x + \Delta x} - \frac{5}{x} \\ &= \frac{5x - 5(x + \Delta x)}{x(x + \Delta x)} \\ &= \frac{5x - 5x - 5\Delta x}{x(x + \Delta x)} \\ &= \frac{-5\Delta x}{x(x + \Delta x)}\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{-5}{x(x + \Delta x)} \\ \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{-5}{x(x + \Delta x)} \\ h'(x) &= -\frac{5}{x^2}\end{aligned}$$

(d) $k(x) = \sqrt{x + 3}$

Sol.

$$\Delta y = \sqrt{x + 3 + \Delta x} - \sqrt{x + 3}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{\sqrt{x + 3 + \Delta x} - \sqrt{x + 3}}{\Delta x} \\ &= \frac{\sqrt{x + 3 + \Delta x} - \sqrt{x + 3}}{\Delta x} \cdot \frac{\sqrt{x + 3 + \Delta x} + \sqrt{x + 3}}{\sqrt{x + 3 + \Delta x} + \sqrt{x + 3}} \\ &= \frac{x + 3 + \Delta x - (x + 3)}{\Delta x (\sqrt{x + 3 + \Delta x} + \sqrt{x + 3})} \\ &= \frac{\Delta x}{\Delta x (\sqrt{x + 3 + \Delta x} + \sqrt{x + 3})} \\ &= \frac{1}{\sqrt{x + 3 + \Delta x} + \sqrt{x + 3}}\end{aligned}$$

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x + 3 + \Delta x} + \sqrt{x + 3}} \\ k'(x) &= \frac{1}{2\sqrt{x + 3}}\end{aligned}$$

5. Find the derivative of the following functions:

(a) $y = 2x^4 - 3x^3 + 5x - 8$

Sol.

$$y' = 8x^3 - 9x^2 + 5$$

(b) $y = 2x + \frac{2}{x} - \frac{3}{x^2}$

Sol.

$$y' = 2 - \frac{2}{x^2} + \frac{6}{x^3}$$

(c) $y = \sqrt[3]{x} - \frac{1}{\sqrt{3x}}$

Sol.

$$\begin{aligned} y' &= \left(x^{\frac{1}{3}} - (3x)^{-\frac{1}{2}} \right)' \\ &= \frac{1}{3}x^{-\frac{2}{3}} + \frac{3}{2}(3x)^{-\frac{3}{2}} \\ &= \frac{1}{3}x^{-\frac{2}{3}} + \frac{3}{2 \cdot 3\sqrt{3}}x^{-\frac{3}{2}} \\ &= \frac{1}{3}x^{-\frac{2}{3}} + \frac{3}{6\sqrt{3}}x^{-\frac{3}{2}} \\ &= \frac{1}{3}x^{-\frac{2}{3}} + \frac{3\sqrt{3}}{18}x^{-\frac{3}{2}} \\ &= \frac{1}{3}x^{-\frac{2}{3}} + \frac{\sqrt{3}}{6}x^{-\frac{3}{2}} \end{aligned}$$

(d) $y = (x^3 - 4x)(x^2 + 3x - 1)$

Sol.

$$\begin{aligned} y' &= (x^3 - 4x)'(x^2 + 3x - 1) + (x^3 - 4x)(x^2 + 3x - 1)' \\ &= (3x^2 - 4)(x^2 + 3x - 1) + (x^3 - 4x)(2x + 3) \\ &= 3x^4 + 9x^3 - 3x^2 - 4x^2 - 12x + 4 + 2x^4 + 3x^3 - 8x^2 - 12x \\ &= 5x^4 + 12x^3 - 15x^2 - 24x + 4 \end{aligned}$$

(e) $y = (x - 1)^5 \sqrt{x + 2}$

Sol.

$$\begin{aligned}y' &= 5(x-1)^4 \sqrt{x+2} + (x-1)^5 \cdot \frac{1}{2\sqrt{x+2}} \\&= (x-1)^4 \left[5\sqrt{x+2} + \frac{(x-1)}{2\sqrt{x+2}} \right] \\&= (x-1)^4 \left[\frac{10(x+2) + x-1}{2\sqrt{x+2}} \right] \\&= \frac{(x-1)^4(11x+19)}{2\sqrt{x+2}}\end{aligned}$$

(f) $y = (2x+5)(x^2-2)(x^3-1)$

Sol.

$$\begin{aligned}y' &= [(2x^3-4x+5x^2-10)(x^3-1)]' \\&= (2x^3-4x+5x^2-10)'(x^3-1) + (2x^3-4x+5x^2-10)(x^3-1)' \\&= (6x^2-4+10x)(x^3-1) + (2x^3-4x+5x^2-10)(3x^2) \\&= 6x^5-4x^3+10x^4-6x^2+4-10x+6x^5-12x^3+15x^4-30x^2 \\&= 12x^5+25x^4-16x^3-36x^2-10x+4\end{aligned}$$

(g) $y = \frac{2x^3-3x^2+4}{x^2}$

Sol.

$$\begin{aligned}y' &= \frac{(2x^3-3x^2+4)'(x^2) - (2x^3-3x^2+4)(x^2)'}{x^4} \\&= \frac{(6x^2-6x)(x^2) - (2x^3-3x^2+4)(2x)}{x^4} \\&= \frac{6x^4-6x^3-4x^4+6x^3-8x}{x^4} \\&= \frac{2x^4-8x}{x^4} \\&= \frac{2x^3-8}{x^3} \\&= 2 - \frac{8}{x^3}\end{aligned}$$

(h) $y = \frac{x^2+4}{x+1}$

Sol.

$$\begin{aligned}y' &= \frac{(x^2 + 4)'(x + 1) - (x^2 + 4)(x + 1)'}{(x + 1)^2} \\&= \frac{(2x)(x + 1) - (x^2 + 4)}{(x + 1)^2} \\&= \frac{2x^2 + 2x - x^2 - 4}{(x + 1)^2} \\&= \frac{x^2 + 2x - 4}{(x + 1)^2}\end{aligned}$$

(i) $y = \frac{x + 2}{x^2 + 5x + 6}$

Sol.

$$\begin{aligned}y' &= \frac{(x + 2)'(x^2 + 5x + 6) - (x + 2)(x^2 + 5x + 6)'}{(x^2 + 5x + 6)^2} \\&= \frac{(x^2 + 5x + 6) - (x + 2)(2x + 5)}{(x^2 + 5x + 6)^2} \\&= \frac{x^2 + 5x + 6 - 2x^2 - 9x - 10}{(x^2 + 5x + 6)^2} \\&= \frac{-x^2 - 4x - 4}{(x^2 + 5x + 6)^2} \\&= -\frac{(x + 2)^2}{(x + 2)^2(x + 3)^2} \\&= -\frac{1}{(x + 3)^2}\end{aligned}$$

(j) $y = \frac{x^2}{(x^2 - 1)^3}$

Sol.

$$\begin{aligned}y' &= \frac{(x^2)'(x^2 - 1)^3 - x^2 [(x^2 - 1)^3]'}{(x^2 - 1)^6} \\&= \frac{(2x)(x^2 - 1)^3 - x^2 [3(x^2 - 1)^2(2x)]}{(x^2 - 1)^6} \\&= \frac{2x(x^2 - 1)^3 - 6x^3(x^2 - 1)^2}{(x^2 - 1)^6} \\&= \frac{2x(x^2 - 1)^2(x^2 - 1 - 3x^2)}{(x^2 - 1)^6} \\&= \frac{2x(x^2 - 1)^2(-2x^2 - 1)}{(x^2 - 1)^6} \\&= \frac{-2x(2x^2 + 1)}{(x^2 - 1)^4} \\&= \frac{-4x^3 - 2x}{(x^2 - 1)^4}\end{aligned}$$

6. Find the derivative of the following functions:

(a) $y = (x^3 - 1)^4$

Sol.

$$\begin{aligned}y' &= 4(x^3 - 1)^3(x^3 - 1)' \\&= 4(x^3 - 1)^3(3x^2) \\&= 12x^2(x^3 - 1)^3\end{aligned}$$

(b) $y = (5x + 3)^6$

Sol.

$$\begin{aligned}y' &= 6(5x + 3)^5(5x + 3)' \\&= 6(5x + 3)^5(5) \\&= 30(5x + 3)^5\end{aligned}$$

(c) $y = (x^3 - 3x)^5$

Sol.

$$\begin{aligned}y' &= 5(x^3 - 3x)^4(x^3 - 3x)' \\&= 5(x^3 - 3x)^4(3x^2 - 3) \\&= 15(x^3 - 3x)^4(x^2 - 1)\end{aligned}$$

(d) $y = \sqrt{x^2 - 2x}$

Sol.

$$\begin{aligned}y' &= \left[(x^2 - 2x)^{\frac{1}{2}} \right]' \\&= -\frac{1}{2}(x^2 - 2x)^{-\frac{1}{2}}(x^2 - 2x)' \\&= -\frac{1}{2}(x^2 - 2x)^{-\frac{1}{2}}(2x - 2) \\&= -\frac{2x - 2}{2\sqrt{x^2 - 2x}} \\&= -\frac{x - 1}{\sqrt{x^2 - 2x}}\end{aligned}$$

(e) $y = \frac{1}{\sqrt[3]{2x^2 - 1}}$

Sol.

$$\begin{aligned}y' &= \left[(2x^2 - 1)^{-\frac{1}{3}} \right]' \\&= -\frac{1}{3} (2x^2 - 1)^{-\frac{4}{3}} (2x^2 - 1)' \\&= -\frac{1}{3} (2x^2 - 1)^{-\frac{4}{3}} (4x) \\&= -\frac{4x}{3\sqrt[3]{(2x^2 - 1)^4}}\end{aligned}$$

(f) $y = \frac{2x - 1}{\sqrt{1 - 2x}}$

Sol.

$$\begin{aligned}y' &= \frac{(2x - 1)' \sqrt{1 - 2x} - (2x - 1) (\sqrt{1 - 2x})'}{1 - 2x} \\&= \frac{2\sqrt{1 - 2x} - (2x - 1) \left[\frac{1}{2} (1 - 2x)^{-\frac{1}{2}} (1 - 2x)' \right]}{1 - 2x} \\&= \frac{2\sqrt{1 - 2x} - (2x - 1) \left[\frac{1}{2} (1 - 2x)^{-\frac{1}{2}} (-2) \right]}{1 - 2x} \\&= \frac{2\sqrt{1 - 2x} + (2x - 1)(1 - 2x)^{-\frac{1}{2}}}{1 - 2x} \\&= \frac{2\sqrt{1 - 2x} + \frac{2x - 1}{\sqrt{1 - 2x}}}{1 - 2x} \\&= \frac{\frac{2(1 - 2x) + 2x - 1}{\sqrt{1 - 2x}}}{1 - 2x} \\&= \frac{\frac{2 - 4x + 2x - 1}{\sqrt{1 - 2x}}}{1 - 2x} \\&= \frac{\frac{1 - 2x}{\sqrt{1 - 2x}}}{1 - 2x} \\&= \frac{1}{(1 - 2x)\sqrt{1 - 2x}} \\&= \frac{1}{\sqrt{1 - 2x}}\end{aligned}$$

7. Find the second derivative of the following functions:

(a) $y = x^2(3x - 4)$

Sol.

$$\begin{aligned}y' &= x^2(3x - 4)' + (3x - 4)(x^2)' \\&= x^2(3) + (3x - 4)(2x) \\&= 3x^2 + 6x^2 - 8x \\&= 9x^2 - 8x\end{aligned}$$

$$\begin{aligned}y'' &= (9x^2 - 8x)' \\&= 18x - 8\end{aligned}$$

(b) $y = 2x^5 - 6x^4 - 3x + 5$

Sol.

$$\begin{aligned}y' &= 2x^5 - 6x^4 - 3x + 5' \\&= 10x^4 - 24x^3 - 3\end{aligned}$$

$$\begin{aligned}y'' &= (10x^4 - 24x^3 - 3)' \\&= 40x^3 - 72x^2\end{aligned}$$

(c) $y = \frac{3}{x^5}$

Sol.

$$\begin{aligned}y' &= (3x^{-5})' \\&= -15x^{-6}\end{aligned}$$

$$\begin{aligned}y'' &= (-15x^{-6})' \\&= 90x^{-7}\end{aligned}$$

$$= \frac{90}{x^7}$$

(d) $y = \sqrt{2x + 1}$

Sol.

$$\begin{aligned}y' &= \left[(2x+1)^{\frac{1}{2}}\right]' \\&= \frac{1}{2}(2x+1)^{-\frac{1}{2}}(2x+1)' \\&= \frac{1}{2}(2x+1)^{-\frac{1}{2}}(2) \\&= \frac{1}{\sqrt{2x+1}}\end{aligned}$$

$$\begin{aligned}y'' &= \left[(2x+1)^{-\frac{1}{2}}\right]' \\&= -\frac{1}{2}(2x+1)^{-\frac{3}{2}}(2x+1)' \\&= -\frac{1}{2}(2x+1)^{-\frac{3}{2}}(2) \\&= -\frac{1}{\sqrt{(2x+1)^3}}\end{aligned}$$

8. If the function $y = \frac{x^3}{(x-1)^2}$, find y' and y'' .

Sol.

$$\begin{aligned}y' &= \frac{(x^3)'(x-1)^2 - x^3[(x-1)^2]'}{(x-1)^4} \\&= \frac{3x^2(x-1)^2 - x^3(2)(x-1)}{(x-1)^4} \\&= \frac{(3x^3 - 3x^2 - 2x^3)}{(x-1)^3} \\&= \frac{x^3 - 3x^2}{(x-1)^3}\end{aligned}$$

$$\begin{aligned}y'' &= \frac{(x^3 - 3x^2)'(x-1)^3 - (x^3 - 3x^2)[(x-1)^3]'}{(x-1)^6} \\&= \frac{(3x^2 - 6x)(x-1)^3 - (x^3 - 3x^2)(3)(x-1)^2}{(x-1)^6} \\&= \frac{(x-1)^2[(3x^2 - 6x)(x-1) - 3(x^3 - 3x^2)]}{(x-1)^6} \\&= \frac{3x^3 - 9x^2 + 6x - 3x^3 + 9x^2}{(x-1)^4} \\&= \frac{6x}{(x-1)^4}\end{aligned}$$

9. Given the function $y = 2x^3 + 3x^2 - 72x + 21$, find the value of x when $\frac{dy}{dx} = 0$.

Sol.

$$y' = 6x^2 + 6x - 72$$

$$0 = 6x^2 + 6x - 72$$

$$0 = x^2 + x - 12$$

$$0 = (x + 4)(x - 3)$$

$$x = -4 \text{ or } x = 3$$

10. Given the function $y = (2 - 3x^2)^4$, find the value of $\frac{d^2y}{dx^2}$ when $x = 1$.

Sol.

$$\begin{aligned}\frac{dy}{dx} &= 4(2 - 3x^2)^3(-6x) \\ &= -24x(2 - 3x^2)^3\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= -24(2 - 3x^2)^3 - 24x(3)(2 - 3x^2)^2(-6x) \\ &= -24(2 - 3x^2)^3 + 432x^2(2 - 3x^2)^2 \\ &= (2 - 3x^2)^2 [-24(2 - 3x^2) + 432x^2] \\ &= (2 - 3x^2)^2 [-48 + 72x^2 + 432x^2] \\ &= (2 - 3x^2)^2 (504x^2 - 48)\end{aligned}$$

When $x = 1$,

$$\begin{aligned}\frac{d^2y}{dx^2} &= (2 - 3)^2 (504 - 48) \\ &= (-1)^2 (456) \\ &= 456\end{aligned}$$

11. If the function $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + x + 1$, find

(a) $f'(1)$ and $f''(2)$

Sol.

$$f'(x) = x^2 - 5x + 1$$

$$f'(1) = 1 - 5 + 1$$

$$= -3$$

$$f''(x) = 2x - 5$$

$$f''(2) = 4 - 5$$

$$= -1$$

(b) the value of x when $f''(x) = 0$

Sol.

$$f''(x) = 0$$

$$2x - 5 = 0$$

$$2x = 5$$

$$x = \frac{5}{2}$$

12. If the function $f(x) = \sqrt{x\sqrt{x\sqrt{x}}}$, find $f'(1)$, $f''(1)$, $f'''(1)$ and $f^{(4)}(1)$.

Sol.

$$\begin{aligned}f(x) &= x^{\frac{1}{2}} \cdot x^{\frac{1}{4}} \cdot x^{\frac{1}{8}} \\&= x^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}} \\&= x^{\frac{7}{8}}\end{aligned}$$

$$\begin{aligned}f'(x) &= \frac{7}{8}x^{-\frac{1}{8}} \\f'(1) &= \frac{7}{8}\end{aligned}$$

$$\begin{aligned}f''(x) &= -\frac{7}{64}x^{-\frac{9}{8}} \\f''(1) &= -\frac{7}{64}\end{aligned}$$

$$\begin{aligned}f'''(x) &= \frac{63}{512}x^{-\frac{17}{8}} \\f'''(1) &= \frac{63}{512}\end{aligned}$$

$$\begin{aligned}f^{(4)}(x) &= -\frac{1071}{4096}x^{-\frac{25}{8}} \\f^{(4)}(1) &= -\frac{1071}{4096}\end{aligned}$$

13. Find the derivative $\frac{dy}{dx}$ of the following implicit functions:

(a) $x^2 + 2y = 2x + 3$

Sol.

$$\begin{aligned}x^2 + 2y &= 2x + 3 \\2x + 2y' &= 2 \\y' &= \frac{2 - 2x}{2} \\&= 1 - x\end{aligned}$$

(b) $x^2 + 3x = y^2 - 5y$

Sol.

$$\begin{aligned}x^2 + 3x &= y^2 - 5y \\2x + 3 &= 2yy' - 5y' \\(2y - 5)y' &= 2x + 3 \\y' &= \frac{2x + 3}{2y - 5}\end{aligned}$$

(c) $3x^2 + 7xy - 9y^2 = 2$

Sol.

$$\begin{aligned}3x^2 + 7xy - 9y^2 &= 2 \\6x + 7y + 7xy' - 18yy' &= 0 \\7xy' - 18yy' &= -6x - 7y \\y'(7x - 18y) &= -6x - 7y \\y' &= -\frac{6x + 7y}{7x - 18y}\end{aligned}$$

(d) $x^3y + xy^3 = 3xy$

Sol.

$$\begin{aligned}3x^2y + x^3y' + y^3 + 3xy^2y' &= 3y + 3xy' \\x^3y' + 3xy^2y' - 3xy' &= 3y - 3x^2y - y^3 \\y'(x^3 + 3xy^2 - 3x) &= 3y - 3x^2y - y^3 \\y' &= \frac{3y - 3x^2y - y^3}{x^3 + 3xy^2 - 3x} \\&= \frac{y(3 - 3x^2 - y^2)}{x(x^2 + 3y^2 - 3)}\end{aligned}$$

14. Find the gradient of the tangent to the curve $x^2 + xy + y^2 = 4$ at the point $A(2, -2)$.

Sol.

$$\begin{aligned}x^2 + xy + y^2 &= 4 \\2x + y + xy' + 2yy' &= 0 \\xy' + 2yy' &= -2x - y \\(x + 2y)y' &= -2x - y \\y' &= -\frac{2x + y}{x + 2y}\end{aligned}$$

When $x = 2$ and $y = -2$,

$$\begin{aligned}y' &= -\frac{2(2) + (-2)}{2 + 2(-2)} \\&= -\frac{2}{-2} \\&= 1\end{aligned}$$

15. Find the limit of the following:

- (a) $\lim_{x \rightarrow 0} \frac{\sin 4x}{x}$
- (b) $\lim_{x \rightarrow 0} \frac{\tan 2x}{\tan 5x}$
- (c) $\lim_{x \rightarrow 0} \frac{x^2}{\tan^2 3x}$
- (d) $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^{4x+1}$
- (e) $\lim_{x \rightarrow 0} (1 - 3x)^{\frac{2}{x}}$
- (f) $\lim_{x \rightarrow \infty} \left(\frac{x+3}{x+1}\right)^x$

16. Find the derivative of the following functions:

- (a) $y = \tan^2 3x$
- (b) $y = \cos^4 2x$
- (c) $y = \sec^3 2x$
- (d) $y = \sec^2(3x + 5)$
- (e) $y = (1 + \sin x)^3$
- (f) $y = \sin(\cos x)$
- (g) $y = \sin 2x \cos 2x$
- (h) $y = \frac{1}{\sin x + \cos x}$
- (i) $y = \frac{\cos 5x}{\sin 3x}$
- (j) $y = \frac{1 + \cos x}{\sin x}$

17. Find the derivative of the following functions:

- (a) $y = 5^{3x-2}$
- (b) $y = 3e^{2x^2}$
- (c) $y = a^{3x} + e^{-3x}$
- (d) $y = \frac{e^{3x} - e^{2x} + e^{5x}}{e^{2x}}$

(e) $y = x^a - 2a^x$

(f) $y = e^{2x} \csc 2x$

18. If the function $y = \frac{\sin 2x}{e^x}$, prove that $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$.

19. Find the derivative of the following functions:

(a) $y = \ln(x^2 + 5)$

(b) $y = \ln(3x^2 + 6x)$

(c) $y = \ln(e^x + 2)$

(d) $y = \ln(\sin^2 4x)$

(e) $y = \log(x^3 + 3x - 4)$

(f) $y = \log_5(3x + 7)$

(g) $y = \log_2 \frac{x}{x+3}$

(h) $y = \frac{1 + \log x}{1 + \ln x}$

20. If the function $y = \ln(x + 1)$, find the value of $\frac{d^2y}{dx^2}$ when $x = 1$.