Calculus Question Bank

Trigonometric Substitution

Find the following indefinite integrals:

1. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}$

2. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}}$

3. $\displaystyle\int \frac{\sqrt{x^2-1}}{x}\,dx$

4. $\displaystyle\int \sqrt{a^2-x^2}\,dx$

5. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$

6. $\displaystyle\int \frac{x^2\,dx}{\sqrt{a^2-x^2}}$

7. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2-a^2}}$

8. $\displaystyle\int \frac{x\,dx}{\sqrt{4x-x^2}}$

9. $\displaystyle\int \frac{x\,dx}{\sqrt{ax-x^2}}$, $a>0$

10. $\displaystyle\int \frac{x^3\,dx}{(a^2+x^2)^{\frac{3}{2}}}$

11. $\displaystyle\int \frac{dx}{2-\sqrt{3x}}$

12. $\displaystyle\int \frac{dx}{(x^2+4)^2}$

13. $\displaystyle\int y\sqrt{2y-y^2}\,dy$

    Sol.

    Let $y=2\sin^2\theta$, then $dy=4\sin\theta\cos\theta\,d\theta$.

$$\begin{aligned}
y &= 2\sin^2\theta \\
\sin\theta &= \sqrt{\frac{y}{2}} \\
\theta &= \sin^{-1}\sqrt{\frac{y}{2}} \\
\cos\theta &= \sqrt{1-\frac{y}{2}}
\end{aligned}$$

$$\begin{aligned}
\int y\sqrt{2y-y^2}\,dy &= \int 2\sin^2\theta\sqrt{4\sin^2\theta-4\sin^4\theta}\cdot 4\sin\theta\cos\theta\,d\theta \\
&= \int 2\sin^2\theta\sqrt{4\sin^2\theta(1-\sin^2\theta)}\cdot 4\sin\theta\cos\theta\,d\theta \\
&= \int 2\sin^2\theta\sqrt{4\sin^2\theta\cos^2\theta}\cdot 4\sin\theta\cos\theta\,d\theta \\
&= \int 2\sin^2\theta\cdot 2\sin\theta\cos\theta\cdot 4\sin\theta\cos\theta\,d\theta
\end{aligned}$$

$$\begin{aligned}
&= \int 2\sin^2\theta \cdot \sin 2\theta \cdot 2\sin 2\theta \, d\theta \\
&= 4\int \sin^2\theta \cdot \sin^2 2\theta \, d\theta \\
&= 4\int \frac{1-\cos 2\theta}{2} \cdot \frac{1-\cos 4\theta}{2} \, d\theta \\
&= \int (1-\cos 2\theta - \cos 4\theta + \cos 2\theta \cos 4\theta) \, d\theta \\
&= \int \left[1-\cos 2\theta - \cos 4\theta + \frac{1}{2}(\cos 6\theta + \cos 2\theta)\right] d\theta \\
&= \int \left[1-\frac{1}{2}\cos 2\theta - \cos 4\theta + \frac{1}{2}\cos 6\theta\right] d\theta \\
&= \theta - \frac{1}{4}\sin 2\theta - \frac{1}{4}\sin 4\theta + \frac{1}{12}\sin 6\theta + C
\end{aligned}$$

14. $\int y^2\sqrt{a^2-y^2}\,dy$

Sol.

Let $y = a\sin\theta$, then $dy = a\cos\theta\,d\theta$.

$$\begin{aligned}
y &= a\sin\theta \\
\sin\theta &= \frac{y}{a} \\
\theta &= \sin^{-1}\frac{y}{a} \\
\cos\theta &= \frac{\sqrt{a^2-y^2}}{a}
\end{aligned}$$

$$\begin{aligned}
\int y^2\sqrt{a^2-y^2}\,dy &= \int a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta \\
&= \int a^2\sin^2\theta\sqrt{a^2\cos^2\theta}\cdot a\cos\theta\,d\theta \\
&= \int a^2\sin^2\theta\cdot a\cos\theta\cdot a\cos\theta\,d\theta \\
&= a^4\int \sin^2\theta\cos^2\theta\,d\theta \\
&= a^4\int \frac{1-\cos 2\theta}{2}\cdot\frac{1+\cos 2\theta}{2}\,d\theta \\
&= \frac{1}{4}a^4\int (1-\cos^2 2\theta)\,d\theta \\
&= \frac{1}{4}a^4\theta - \frac{1}{4}a^4\int \frac{1+\cos 4\theta}{2}\,d\theta \\
&= \frac{1}{4}a^4\theta - \frac{1}{8}a^4\int (1+\cos 4\theta)\,d\theta \\
&= \frac{1}{4}a^4\theta - \frac{1}{8}a^4\theta\frac{1}{8}a^4\int \cos 4\theta\,d\theta \\
&= \frac{1}{8}a^4\theta - \frac{1}{8}a^4\int \cos 4\theta\,d\theta \\
&= \frac{1}{8}a^4\theta - \frac{1}{32}a^4\sin 4\theta + C \\
&= \frac{1}{8}a^4\theta - \frac{1}{16}a^4\sin 2\theta\cos 2\theta + C \\
&= \frac{1}{8}a^4\theta - \frac{1}{8}a^4\sin\theta\cos\theta(\cos^2 - \sin^2\theta) + C \\
&= \frac{1}{8}a^4\theta - \frac{1}{8}a^4\sin\theta\cos^3\theta - \frac{1}{8}a^4\sin^3\theta\cos\theta + C
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}a^4 \sin^{-1}\frac{y}{a} - \frac{1}{8}a^4 \cdot \frac{y}{a} \cdot \frac{(a^2-y^2)^{\frac{3}{2}}}{a^3} + \frac{1}{8}a^4 \cdot \frac{y^3}{a^3} \cdot \frac{\sqrt{a^2-y^2}}{a} + C \\
&= \frac{1}{8}a^4 \sin^{-1}\frac{y}{a} - \frac{1}{8}y(a^2-y^2)^{\frac{3}{2}} + \frac{1}{8}y^3\sqrt{a^2-y^2} + C \qquad \square
\end{aligned}
$$

15. $\displaystyle\int \sqrt{\frac{x}{1-x}}\,dx$

Sol.

Let $x = \sin^2\theta$, then $dx = 2\sin\theta\cos\theta\,d\theta$.

$$
\begin{aligned}
x &= \sin^2\theta \\
\sqrt{x} &= \sin\theta \\
\theta &= \arcsin\sqrt{x}
\end{aligned}
$$

$$
\begin{aligned}
\cos^2\theta &= 1 - \sin^2\theta \\
&= 1 - x \\
\cos\theta &= \sqrt{1-x}
\end{aligned}
$$

$$
\begin{aligned}
\int \sqrt{\frac{x}{1-x}}\,dx &= \int \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}} \cdot 2\sin\theta\cos\theta\,d\theta \\
&= \int \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} \cdot 2\sin\theta\cos\theta\,d\theta \\
&= \int \frac{\sin\theta}{\cos\theta} \cdot 2\sin\theta\cos\theta\,d\theta \\
&= \int 2\sin^2\theta\,d\theta \\
&= \int 2 \cdot \frac{1-\cos 2\theta}{2}\,d\theta \\
&= \int (1-\cos 2\theta)\,d\theta \\
&= \int d\theta - \int \cos 2\theta\,d\theta \\
&= \theta - \frac{1}{2}\sin 2\theta + C \\
&= \theta - \sin\theta\cos\theta + C \\
&= \arcsin\sqrt{x} - \sqrt{x}\sqrt{1-x} + C \\
&= \arcsin\sqrt{x} - \sqrt{x(1-x)} + C \qquad \square
\end{aligned}
$$

16. $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2+1}}$

Sol.

Let $x = \tan\theta$, then $dx = \sec^2\theta\,d\theta$.

$$
\begin{aligned}
\int \frac{x^2\,dx}{\sqrt{x^2+1}} &= \int \frac{\tan^2\theta\sec^2\theta\,d\theta}{\sqrt{\tan^2\theta+1}} \\
&= \int \frac{\tan^2\theta\sec^2\theta\,d\theta}{\sec\theta} \\
&= \int \tan^2\theta\sec\theta\,d\theta \\
&= \int (\sec^2\theta - 1)\sec\theta\,d\theta \\
&= \int \sec^3\theta\,d\theta - \int \sec\theta\,d\theta
\end{aligned}
$$

Let $u = \sec\theta$, then $du = \sec\theta\tan\theta\, d\theta$.
Let $dv = \sec^2\theta\, d\theta$, then $v = \tan\theta$.

$$\begin{aligned}
\int \sec^3\theta\, d\theta &= \int \sec^2\theta \sec\theta\, d\theta \\
&= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta \\
&= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta \\
\int \sec^3\theta\, d\theta &= \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta \\
2\int \sec^3\theta\, d\theta &= \sec\theta\tan\theta + \int \sec\theta\, d\theta \\
\int \sec^3\theta\, d\theta &= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta\, d\theta \\
&= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C
\end{aligned}$$

$$\begin{aligned}
\therefore\ \int \frac{x^2\, dx}{\sqrt{x^2+1}} &= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| - \ln|\sec\theta + \tan\theta| + C \\
&= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C \\
&= \frac{1}{2}x\sqrt{x^2+1} - \frac{1}{2}\ln|x + \sqrt{x^2+1}| + C \qquad \square
\end{aligned}$$

17. $\displaystyle\int \frac{t\, dt}{1-\sqrt{t}}$

Sol.
Let $\sqrt{t} = \sin^2\theta$, then $t = \sin^4\theta$, $dt = 4\sin^3\theta\cos\theta\, d\theta$.

$$\begin{aligned}
\int \frac{t\, dt}{1-\sqrt{t}} &= \int \frac{\sin^4\theta \cdot 4\sin^3\theta\cos\theta\, d\theta}{1-\sin^2\theta} \\
&= \int \frac{\sin^4\theta \cdot 4\sin^3\theta\cos\theta\, d\theta}{\cos^2\theta} \\
&= \int \frac{4\sin^7\theta}{\cos\theta}\, d\theta \\
&= 4\int \frac{(1-\cos^2\theta)\sin^5\theta}{\cos\theta}\, d\theta \\
&= 4\int \frac{\sin^5\theta}{\cos\theta}\, d\theta - 4\int \frac{\cos^2\theta\sin^5\theta}{\cos\theta}\, d\theta \\
&= 4\int \frac{\sin^3\theta(1-\cos^2\theta)}{\cos\theta}\, d\theta - 4\int \cos\theta\sin^5\theta\, d\theta \\
&= 4\int \frac{\sin^3\theta}{\cos\theta}\, d\theta - 4\int \sin^3\theta\cos\theta\, d\theta - 4\int \cos\theta\sin^5\theta\, d\theta \\
&= 4\int \frac{\sin\theta(1-\cos^2\theta)}{\cos\theta}\, d\theta - 4\int \sin^3\theta\cos\theta\, d\theta - 4\int \cos\theta\sin^5\theta\, d\theta \\
&= 4\int \frac{\sin\theta}{\cos\theta}\, d\theta - 4\int \sin\theta\cos\theta\, d\theta - 4\int \sin^3\theta\cos\theta\, d\theta - 4\int \cos\theta\sin^5\theta\, d\theta \\
&= -4\ln|\cos\theta| - 2\sin^2\theta - \sin^4\theta + \frac{2}{3}\sin^6\theta + C
\end{aligned}$$

$$\begin{aligned}
\sqrt{t} &= \sin^2\theta \\
&= 1 - \cos^2\theta
\end{aligned}$$

$$\cos^2\theta = 1 - \sqrt{t}$$

$$\cos\theta = \sqrt{1 - \sqrt{t}}$$

$$\begin{aligned}
\int \frac{t\,dt}{1-\sqrt{t}} &= -4\ln\left|\sqrt{1-\sqrt{t}}\right| - 2\sqrt{t} - t + \frac{2}{3}t\sqrt{t} + C \\
&= -\frac{1}{2}\ln\left|1-\sqrt{t}\right| - 2\sqrt{t} - t + \frac{2}{3}t\sqrt{t} + C \qquad \square
\end{aligned}$$