Mathematics

Senior 3 Part II

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Preface

Why this book?

Disclaimer

Acknowledgements

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Chapter 26

Applications of Differentiation

26.1 Revision Exercise 26

1. Find the equation of the tangent of the curve $y = x^3 - 3x$ at the point where $x = 3$.

   Sol.

$$\begin{aligned} y &= x^3 - 3x \\ \frac{dy}{dx} &= 3x^2 - 3 \end{aligned}$$

At $x = 3$, $y = (3)^3 - 3(3) = 18$.

$$\begin{aligned} \text{Gradient of tangent } \frac{dy}{dx} &= 3(3)^2 - 3 \\ &= 27 - 3 \\ &= 24 \\ \therefore \text{Equation of tangent is } y - 18 &= 24(x - 3) \\ y - 18 &= 24x - 72 \\ y &= 24x - 54 \end{aligned}$$

2. Find the equation of the normal of the curve $y = x(x-4)(x+1)$ at the points of intersection of the curve and the x-axis.

   Sol.

$$\begin{aligned} y &= x(x-4)(x+1) \\ &= x(x^2 - 3x - 4) \\ &= x^3 - 3x^2 - 4x \\ \frac{dy}{dx} &= 3x^2 - 6x - 4 \end{aligned}$$

When $x = 0$, $y = 0$

$$\begin{aligned} x(x-4)(x+1) &= 0 \\ x = 0 \text{ or } x &= 4 \text{ or } x = -1 \end{aligned}$$

When $x = 0$,

$$\begin{aligned} \because \text{Gradient of tangent } \frac{dy}{dx} &= 3(0)^2 - 6(0) - 4 = -4 \\ \therefore \text{Gradient of normal } &= \frac{1}{4} \\ \therefore \text{Equation of normal is } y - 0 &= \frac{1}{4}(x - 0) \\ y &= \frac{1}{4}x \\ x - 4y &= 0 \end{aligned}$$

When $x = 4$,

$$\begin{aligned} \because \text{Gradient of tangent } \frac{dy}{dx} &= 3(4)^2 - 6(4) - 4 = 20 \\ \therefore \text{Gradient of normal } &= -\frac{1}{20} \\ \therefore \text{Equation of normal is } y - 0 &= -\frac{1}{20}(x - 4) \\ x + 20y - 4 &= 0 \end{aligned}$$

When $x = -1$,

$$\begin{aligned} \because \text{Gradient of tangent } \frac{dy}{dx} &= 3(-1)^2 - 6(-1) - 4 = 5 \\ \therefore \text{Gradient of normal } &= -\frac{1}{5} \\ \therefore \text{Equation of normal is } y - 0 &= -\frac{1}{5}(x + 1) \\ x + 5y + 1 &= 0 \end{aligned}$$

Hence, the equations of the normals are $x - 4y = 0$, $x + 20y - 4 = 0$ and $x + 5y + 1 = 0$.

3. Given that the curve $y = ax^2 + bx - 10$ passes through the point $(2, 0)$, and that the gradient of the curve at the point is 3. Find the values of a and b.

Sol.

$$\begin{aligned} y &= ax^2 + bx - 10 \\ \frac{dy}{dx} &= 2ax + b \end{aligned}$$

Since the curve passes through $(2, 0)$,

$$\begin{aligned} 0 &= a(2)^2 + b(2) - 10 \\ 0 &= 4a + 2b - 10 \\ 4a &= 10 - 2b \\ a &= \frac{10 - 2b}{4} \\ &= \frac{5 - b}{2} \quad \cdots \ (1) \end{aligned}$$

Since the gradient of the curve at the point is 3,

$$\begin{aligned} 3 &= 2a(2) + b \\ 3 &= 4a + b \quad \cdots \ (2) \end{aligned}$$

Substituting (1) into (2),

$$\begin{aligned} 3 &= 4\left(\frac{5 - b}{2}\right) + b \\ 3 &= 2(5 - b) + b \\ 3 &= 10 - 2b + b \\ b &= 7 \end{aligned}$$

Substituting $b = 7$ into (1),

$$\begin{aligned} a &= \frac{5 - 7}{2} \\ &= -1 \end{aligned}$$

Hence, $a = -1$ and $b = 7$.

4. Find the equation of the normal of the curve $y = x + \frac{2}{x}$ at the point $(2, 3)$. If the normal line intersects with the x-axis and y-axis at A and B respectively, find the length of AB.

Sol.

$$\begin{aligned} y &= x + \frac{2}{x} \\ \frac{dy}{dx} &= 1 - \frac{2}{x^2} \end{aligned}$$

At $x = 2$,

$$\begin{aligned} \frac{dy}{dx} &= 1 - \frac{2}{2^2} \\ &= \frac{1}{2} \end{aligned}$$

Hence, the gradient of the normal at the point $(2, 3)$ is -2.

Therefore, the equation of the normal is

$$\begin{aligned} y - 3 &= -2(x - 2) \\ y &= -2x + 7 \end{aligned}$$

When $y = 0$,

$$\begin{aligned} 0 &= -2x + 7 \\ x &= \frac{7}{2} \\ \therefore A &= \left(\frac{7}{2}, 0\right) \end{aligned}$$

When $x = 0$,

$$\begin{aligned} y &= -2(0) + 7 \\ y &= 7 \\ \therefore B &= (0, 7) \end{aligned}$$

$$\begin{aligned} AB &= \sqrt{\left(\frac{7}{2} - 0\right)^2 + (0 - 7)^2} \\ &= \sqrt{\frac{49}{4} + 49} \\ &= \sqrt{\frac{245}{4}} \\ &= \frac{\sqrt{245}}{2} \\ &= \frac{7\sqrt{5}}{2} \end{aligned}$$

Of the following functions, which intervals are the function increasing or decreasing? (Question 5 to 6)

5. $f(x) = 2x^2(6 - x)$

   Sol.

$$\begin{aligned} f(x) &= 2x^2(6-x) \\ &= 12x^2 - 2x^3 \\ f'(x) &= 24x - 6x^2 \\ f'(x) &= 0 \\ 24x - 6x^2 &= 0 \\ x(x-4) &= 0 \\ x &= 0 \text{ or } x = 4 \end{aligned}$$

   At the interval $(-\infty, 0)$, $f'(x) < 0$, hence $f(x)$ is decreasing at the interval $(-\infty, 0]$.

   At the interval $(0, 4)$, $f'(x) > 0$, hence $f(x)$ is increasing at the interval $[0, 4]$.

   At the interval $(4, \infty)$, $f'(x) < 0$, hence $f(x)$ is decreasing at the interval $[4, \infty)$.

6. $f(x) = 4x^3 - 3x^2 - 6x + 1$

   Sol.

$$\begin{aligned} f(x) &= 4x^3 - 3x^2 - 6x + 1 \\ f'(x) &= 12x^2 - 6x - 6 \\ f'(x) &= 0 \\ 12x^2 - 6x - 6 &= 0 \\ 2x^2 - x - 1 &= 0 \\ (2x+1)(x-1) &= 0 \\ x &= -\frac{1}{2} \text{ or } x = 1 \end{aligned}$$

   At the interval $\left(-\infty, -\frac{1}{2}\right)$, $f'(x) > 0$, hence $f(x)$ is increasing at the interval $\left(-\infty, -\frac{1}{2}\right]$.
   At the interval $\left(-\frac{1}{2}, 1\right)$, $f'(x) < 0$, hence $f(x)$ is decreasing at the interval $\left[-\frac{1}{2}, 1\right]$.
   At the interval $(1, \infty)$, $f'(x) > 0$, hence $f(x)$ is increasing at the interval $[1, \infty)$.

7. If $x - y = 3$, find the relative minimum value of x^2y.

Sol.

$$\begin{aligned} x - y &= 3 \\ y &= x - 3 \end{aligned}$$

Let $f(x) = x^2y$,

$$\begin{aligned} f(x) &= x^2y \\ &= x^2(x-3) \\ &= x^3 - 3x^2 \\ f'(x) &= 3x^2 - 6x \\ f'(x) &= 0 \\ 3x^2 - 6x &= 0 \\ x(x-2) &= 0 \\ x = 0 &\text{ or } x = 2 \end{aligned}$$

$f''(x) = 6x - 6$

$\because f''(0) = -6 < 0,\ f''(2) = 6 > 0$

$\therefore f(2) = -4$ is a relative minimum value.

8. If $2x^2 + y^2 = 6x$, find the relative maximum value of $x^2 + y^2 + 2x$.

Sol.

$$\begin{aligned} 2x^2 + y^2 &= 6x \\ y^2 &= 6x - 2x^2 \end{aligned}$$

Let $f(x) = x^2 + y^2 + 2x$,

$$\begin{aligned} f(x) &= x^2 + y^2 + 2x \\ &= x^2 + 6x - 2x^2 + 2x \\ &= -x^2 + 8x \\ f'(x) &= -2x + 8 \\ f'(x) &= 0 \\ -2x + 8 &= 0 \\ x - 4 &= 0 \\ x &= 4 \end{aligned}$$

$f''(x) = -2$

$\because f''(4) = -2 < 0$

$\therefore f(4) = 16$ is a relative maximum value.

9. Given that $y = 18x^2 + 12x + 7$ has a relative minimum value q and the point where $x = p$. Find the value of p and q.

Sol.

$$\begin{aligned} y &= 18x^2 + 12x + 7 \\ y' &= 36x + 12 \\ y' &= 0 \\ 36x + 12 &= 0 \\ 3x + 1 &= 0 \\ p = x &= -\frac{1}{3} \end{aligned}$$

When $x = -\frac{1}{3},\ y = 5$

$y'' = 36 > 0$

$\therefore$ The relative minimum value is $q = 5$.

10. There's a rectangular field where one side of it is a wall and the other three sides are fenced. If the total length of the fence is $40m$, find the width and height of the field such that the area of the field is the maximum.

Sol. Let x be the length of the field and y be the width of the field.

$$\begin{aligned}
2x + y &= 40 \\
y &= 40 - 2x \\
A &= xy \\
&= x(40 - 2x) \\
&= 40x - 2x^2 \\
\frac{dA}{dx} &= 40 - 4x \\
\frac{dA}{dx} &= 0 \\
40 - 4x &= 0 \\
x &= 10
\end{aligned}$$

$\because \dfrac{d^2A}{dx^2} = -4 < 0$

$\therefore$ The area of the field is the maximum when $x = 10$.When $x = 10,\ y = 20$.

$\therefore$ The field has a width of $20m$ and a height of $10m$ when the area is the maximum.

11. One side of a rectangle with a perimeter of $18cm$ is revolved about one side to form a cylinder. If the volume of the cylinder is the maximum, find the dimensions of the rectangle and the maximum volume of the cylinder.

Sol.

Let the length of the rectangle be x and the width of the rectangle be y.

$$\begin{aligned}
2x + 2y &= 18 \\
x + y &= 9 \\
y &= 9 - x
\end{aligned}$$

$$\begin{aligned}
V &= \pi r^2 h \\
&= \pi x^2 y \\
&= \pi(9x^2 - x^3) \\
\frac{dV}{dx} &= \pi(18x - 3x^2) \\
\frac{dV}{dx} &= 0 \\
\pi(18x - 3x^2) &= 0 \\
x^2 - 6x &= 0 \\
x(x - 6) &= 0 \\
x &= 6,\ x = 0 \text{ (rejected, } x > 0)
\end{aligned}$$

$\because \dfrac{d^2V}{dx^2} = \pi(18 - 6x) = -18\pi < 0$

$\therefore$ The volume of the cylinder is the maximum when $x = 6$.When $x = 6,\ y = 3$.

$\therefore$ The rectangle has a length of $6cm$ and a width of $3cm$ when the volume is the maximum.

Also, the maximum volume of the cylinder is $V = \pi(6)^2(3) = 108\pi \text{ cm}^3$ when the volume is the maximum.

12. The cross section of a tunnel is a rectangle with a semicircle on top of it. If the area of the cross section is fixed, find the ratio of the radius of the semicircle to the height of the rectangle such that the perimeter of the cross section is the minimum.

Sol.

Let the radius of the semicircle be r and the height of the rectangle be h.

$$\begin{aligned}
A &= \frac{1}{2}\pi r^2 + 2rh \\
2rh &= A - \frac{1}{2}\pi r^2 \\
h &= \frac{A - \frac{1}{2}\pi r^2}{2r} \\
&= \frac{A}{2r} - \frac{1}{4}\pi r \\
P &= \pi r + 2h + 2r \\
&= (\pi + 2)r + \frac{A}{r} - \frac{1}{2}\pi r \\
\frac{dP}{dr} &= \pi + 2 - \frac{A}{r^2} - \frac{1}{2}\pi \\
&= \frac{1}{2}\pi + 2 - \frac{A}{r^2} \\
\frac{dP}{dr} &= 0 \\
\frac{1}{2}\pi + 2 - \frac{A}{r^2} &= 0 \\
\frac{1}{2}\pi + 2 - \left(\frac{1}{2}\pi r^2 + 2rh\right) \cdot \frac{1}{r^2} &= 0 \\
\frac{1}{2}\pi + 2 - \frac{1}{2}\pi - \frac{2}{r}h &= 0 \\
2 - \frac{2}{r}h &= 0 \\
2 &= \frac{2}{r}h \\
2r &= 2h \\
r &= h
\end{aligned}$$

Hence, the ratio of the radius of the semicircle to the height of the rectangle is 1 : 1.

13. Split 28 into two parts such that the sum of the squares of the one part and the cube of the other part is the minimum.

Sol.

Let the two parts be x and y.

$$
\begin{aligned}
x + y &= 28 \\
y &= 28 - x \\
S &= x^2 + y^3 \\
&= x^2 + (28 - x)^3 \\
\frac{dS}{dx} &= 2x - 3(28 - x)^2 \\
\frac{dS}{dx} &= 0
\end{aligned}
$$

$$
\begin{aligned}
2x - 3(28 - x)^2 &= 0 \\
2x - 3(784 - 56x + x^2) &= 0 \\
2x - 2352 + 168x - 3x^2 &= 0 \\
3x^2 - 170x + 2352 &= 0 \\
(3x - 98)(x - 24) &= 0 \\
x = 24 \text{ or } x &= \frac{98}{3}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2S}{dx^2} &= 2 + 6(28 - x) \\
&= 2 + 168 - 6x \\
&= -6x + 170
\end{aligned}
$$

When $x = 24$, $\frac{d^2S}{dx^2} = -6(24) + 170$

$= 26 > 0$

When $x = \frac{98}{3}$, $\frac{d^2S}{dx^2} = -6\left(\frac{98}{3}\right) + 170$

$= -26 < 0$

$\because$ When $x = 24$, $\frac{d^2S}{dx^2} > 0$,

$\therefore$ The sum of the squares of the one part and the cube of the other part is the minimum when $x = 24$.

$\because$ When $x = 24$, $y = 4$.

$\therefore$ The two parts are 24 and 4.

14. The capacity of a cylindrical can is fixed. If the material used to make the can is the minimum, what should be the ratio of the radius of the base to the height of the can?

Sol.

Let the radius of the base be r and the height of the can be h.

$$\begin{aligned}
V &= \pi r^2 h \\
h &= \frac{V}{\pi r^2} \\
A &= 2\pi r^2 + 2\pi r h \\
&= 2\pi r^2 + \frac{2V}{r} \\
\frac{dA}{dr} &= 4\pi r - \frac{2V}{r^2} \\
\frac{dA}{dr} &= 0 \\
4\pi r - \frac{2V}{r^2} &= 0 \\
2\pi r^3 - \pi r^2 h &= 0 \\
2r^3 - r^2 h &= 0 \\
2r - h &= 0 \\
2r &= h \\
\frac{r}{h} &= \frac{1}{2}
\end{aligned}$$

Hence, the ratio of the radius of the base to the height of the can is $1 : 2$.

Find the coordinate of the point of inflection of the following functions. (Question 15 to 16)

15. $y = x^3 - 2$

16. $3x + (2 - x)^3$

17. Given the function $y = \dfrac{x}{1 - x^2}$. Find the extreme values of the function, and determine the coordinates of the convex intervals and the point of inflection.

18. Given the function $y = \dfrac{x}{x^2 + 1}$.

    (a) Find the coordinates of the stationary points.

    (b) Determine which intervals the function is increasing or decreasing.

    (c) Find the coordinates of the convex intervals and the point of inflection.

Construct the graph of the following functions. (Question 19 to 20)

19. $y = x^3 - 5x^2 + 3x - 2$

20. $y = x^3 - 3x^2 + 4$

21. In a container, the relationship between the volume of water V (cm^3) and the depth of water x (cm) is given by the equation $V = 4x^2 + \frac{1}{6}x^3$. If the water is poured into the container at a rate of 6 cm^3 per second, find the rate of change of the depth of water when $x = 2$ cm.

22. The water is poured into a conical pool with a width and a base radius of 20m and 10m respectively at a rate of 5m^3/min. When the height of the water is 10cm, find

    (a) the rate of increasing of the height of the water.

(b) the rate of change of the radius of the water surface.

23. The radius of a spherical container decreases from 4cm to 3.95cm. Find the approximate amount of decrease in the volume and the surface area of the container.

24. The capacity of water of a spherical container is given by $V = \left[\frac{\pi h^2}{3}(15-h)\right]\text{cm}^3$, where h is the depth of the water. Find the approximate amount of increase in the capacity of the container when the depth of the water increases from 4cm to 4.01cm.

25. In a bowl, when the height of the water is $h$cm, the volume of the water is given by $V = \left(h^2 + 3h^2 + 11h\right)\text{cm}^3$. When the height of the water is $7cm$, pour an additional $\Delta V\text{cm}^3$ of water into the bowl. Find the approximate amount of increase in the height of the water.

26. If $y = \frac{1}{\sqrt[3]{[x]}}$, find $\frac{dy}{dx}$. Hence, find the approximate value of $\frac{1}{\sqrt[3]{[130}}$. (Correct to 3 decimal places)

Chapter 27

Indefinite Integrals

27.1 Revision Exercise 27

Find the following indefinite integral (Question 1 to 34):

1. $\int 2x^{\frac{1}{5}}dx$

Sol.

$$\int 2x^{\frac{1}{5}}dx = 2\cdot\frac{5}{6}x^{\frac{6}{5}} + C$$
$$= \frac{5}{3}x^{\frac{6}{5}} + C$$

2. $\int (2x-1)^3dx$

Sol.

$$\int (2x-1)^3dx = \frac{1}{2}\int (2x-1)^3d(2x-1)$$
$$= \frac{1}{2}\cdot\frac{1}{4}(2x-1)^4 + C$$
$$= \frac{1}{8}(2x-1)^4 + C$$

3. $\int (x+4)^{100}dx$

Sol.

$$\int (x+4)^{100}dx = \int (x+4)^{100}d(x+4)$$
$$= \frac{1}{101}(x+4)^{101} + C$$

4. $\int \left(\frac{5}{x^2} + 2x^{\frac{1}{2}} + 3\right)dx$

Sol.

$$\int \left(\frac{5}{x^2} + 2x^{\frac{1}{2}} + 3\right)dx$$
$$= 5\int x^{-2} + 2\int x^{\frac{1}{2}} + 3\int dx$$
$$= -\frac{5}{x} + \frac{4}{3}x^{\frac{3}{2}} + 3x + C$$

5. $\int \left(3x^2 + \frac{1}{x^2} - \sin x\right)dx$

Sol.

$$\int \left(3x^2 + \frac{1}{x^2} - \sin x\right)dx$$
$$= 3\int x^2 + \int x^{-2} - \int \sin x dx$$
$$= x^3 - x^{-1} + \cos x + C$$
$$= x^3 + \frac{1}{x} + \cos x + C$$

6. $\int \left(4\cos x + \frac{1}{x} + x^3\right)dx$

Sol.

$$\int \left(4\cos x + \frac{1}{x} + x^3\right)dx$$
$$= 4\int \cos x + \int x^{-1} + \int x^3dx$$
$$= 4\sin x + \ln|x| + \frac{1}{4}x^4 + C$$

7. $\int \frac{3x^3 - 2x^2 + x^{-1}}{x^2}dx$

Sol.

$$\int \frac{3x^3 - 2x^2 + x^{-1}}{x^2}dx$$
$$= \int (3x - 2 + x^{-3})dx$$
$$= \frac{3}{2}x^2 - 2x - \frac{1}{2}x^{-2} + C$$

8. $\int (2x-1)(x+2)dx$

Sol.

$$\int (2x-1)(x+2)dx$$
$$= \int (2x^2 + 3x - 2)dx$$
$$= \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x + C$$

9. $\displaystyle\int \left(x - \frac{1}{x^2}\right)^2 dx$

Sol.

$$\int \left(x - \frac{1}{x^2}\right)^2 dx$$
$$= \int (x^2 - 2x^{-1} + x^{-4}) dx$$
$$= \frac{1}{3}x^3 - 2\ln|x| - \frac{1}{3}x^{-3} + C$$

10. $\displaystyle\int \left(x + \frac{1}{x}\right)^3 dx$

Sol.

$$\int \left(x + \frac{1}{x}\right)^3 dx$$
$$= \int (x^3 + 3x + 3x^{-1} + x^{-3}) dx$$
$$= \frac{1}{4}x^4 + \frac{3}{2}x^2 + 3\ln|x| - \frac{1}{2}x^{-2} + C$$

11. $\displaystyle\int 10^{-x} dx$

Sol.

$$\int 10^{-x} dx$$
$$= -\int 10^{-x} d(-x)$$
$$= -\frac{10^{-x}}{\ln 10} + C$$
$$= -\frac{1}{10^x \ln 10} + C$$

12. $\displaystyle\int (e^x - e^{-x})^2 dx$

Sol.

$$\int (e^x - e^{-x})^2 dx$$
$$= \int (e^{2x} - 2 + e^{-2x}) dx$$
$$= \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + C$$

13. $\displaystyle\int 2x(x^2 - 1)^4 dx$

Sol.

$$\int 2x(x^2 - 1)^4 dx$$
$$= \int (x^2 - 1)^4 d(x^2 - 1)$$
$$= \frac{1}{5}(x^2 - 1)^5 + C$$

14. $\displaystyle\int 3x^2(x^3 + 1)^4 dx$

Sol.

$$\int 3x^2(x^3 + 1)^4 dx$$
$$= \int (x^3 + 1)^4 d(x^3 + 1)$$
$$= \frac{1}{5}(x^3 + 1)^5 + C$$

15. $\displaystyle\int \frac{x + 1}{(x^2 + 2x + 5)^3} dx$

Sol.

$$\int \frac{x + 1}{(x^2 + 2x + 5)^3} dx$$
$$= \frac{1}{2}\int \frac{1}{(x^2 + 2x + 5)^3} d(x^2 + 2x + 5)$$
$$= -\frac{1}{4(x^2 + 2x + 5)^2} + C$$

16. $\displaystyle\int \frac{2x}{\sqrt{x^2 - 4}} dx$

Sol.

$$\int \frac{2x}{\sqrt{x^2 - 4}} dx$$
$$= \int \frac{1}{\sqrt{x^2 - 4}} d(x^2 - 4)$$
$$= 2\sqrt{x^2 - 4} + C$$

17. $\int \frac{x-2}{\sqrt{(x-1)(x-3)}}dx$

Sol.

$$\int \frac{x-2}{\sqrt{(x-1)(x-3)}}dx$$
$$= \int \frac{x-2}{\sqrt{x^2-4x+3}}dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{x^2-4x+3}}d(x^2-4x+3)$$
$$= \frac{1}{2}\cdot 2\sqrt{x^2-4x+3}+C$$
$$= \sqrt{x^2-4x+3}+C$$

18. $\int \frac{7}{2x^2+5x-3}dx$

Sol.

$$\int \frac{7}{2x^2+5x-3}dx = \int \frac{7}{(2x-1)(x+3)}dx$$

Let $\frac{7}{(2x-1)(x+3)} = \frac{A}{2x-1}+\frac{B}{x+3}$

$$A(x+3)+B(2x-1)=7$$
$$(A+2B)x+(3A-B)=7$$

Comparing coefficients,

$$A+2B=0$$
$$3A-B=7$$
$$A=2,\ B=-1$$

$$\therefore \int \frac{7}{2x^2+5x-3}dx$$
$$= \int \left(\frac{2}{2x-1}-\frac{1}{x+3}\right)dx$$
$$= \int \frac{2}{2x-1}dx - \int \frac{1}{x+3}dx$$
$$= \ln|2x-1| - \ln|x+3| + C$$
$$= \ln\left|\frac{2x-1}{x+3}\right| + C$$

19. $\int \frac{8-7x}{2+x-3x^2}dx$

Sol.

$$\int \frac{8+7x}{2+x-3x^2}dx = \int \frac{-7x-8}{3x^2-x-2}dx$$
$$= -\int \frac{7x+8}{(3x+2)(x-1)}dx$$

Let $\frac{7x+8}{(3x+2)(x-1)} = \frac{A}{3x+2}+\frac{B}{x-1}$

$$A(x-1)+B(3x+2)=7x+8$$
$$(A+3B)x+(-A+2B)=7x+8$$

Comparing coefficients,

$$A+3B=7$$
$$-A+2B=8$$
$$A=-2,\ B=3$$

$$\therefore \int \frac{8-7x}{2+x-3x^2}dx$$
$$= -\int \left(-\frac{2}{3x+2}+\frac{3}{x-1}\right)dx$$
$$= -\int -\frac{2}{3x+2}dx - \int \frac{3}{x-1}dx$$
$$= \frac{2}{3}\ln|3x+2| - 3\ln|x-1| + C$$

20. $\int \frac{x+1}{(3x+2)(5x+3)}dx$

Sol.

$$\text{Let } \frac{x+1}{(3x+2)(5x+3)} = \frac{A}{3x+2} + \frac{B}{5x+3}$$

$$A(5x+3) + B(3x+2) = x+1$$

$$(5A+3B)x + (3A+2B) = x+1$$

Comparing coefficients,

$$5A + 3B = 1$$

$$3A + 2B = 1$$

$$A = -1, \; B = 2$$

$$\therefore \int \frac{x+1}{(3x+2)(5x+3)}dx$$

$$= \int \left(\frac{-1}{3x+2} + \frac{2}{5x+3} \right) dx$$

$$= -\int \frac{1}{3x+2}dx + \int \frac{2}{5x+3}dx$$

$$= -\frac{1}{3}\ln|3x+2| + \frac{2}{5}\ln|5x+3| + C$$

21. $\int \frac{2x^2+5x-2}{2x^2+x-3}dx$

Sol.

$$\int \frac{2x^2+5x-2}{2x^2+x-3}dx$$

$$= \int \left(1 + \frac{4x+1}{2x^2+x-3}\right) dx$$

$$= \int \left[1 + \frac{(2x^2+x-3)'}{2x^2+x-3}\right] dx$$

$$= x + \ln|2x^2+x-3| + C$$

22. $\int \left(\frac{x+1}{x-1}\right)^2 dx$

Sol.

$$\int \left(\frac{x+1}{x-1}\right)^2 dx$$

$$= \int \left(1 + \frac{2}{x-1}\right)^2 dx$$

$$= \int \left[1 + \frac{4}{x-1} + \frac{4}{(x-1)^2}\right] dx$$

$$= x + 4\ln|x-1| - \frac{4}{x-1} + C$$

23. $\int \frac{(x-1)^3}{(x-2)^2}dx$

Sol.

Let $u = x - 2$

$\therefore x = u + 2$

$dx = du$

$$\int \frac{(x-1)^3}{(x-2)^2}dx$$

$$= \int \frac{(u+1)^3}{u^2}du$$

$$= \int \left(\frac{u^3+3u^2+3u+1}{u^2}\right) du$$

$$= \int \left(u + 3 + \frac{3}{u} + \frac{1}{u^2}\right) du$$

$$= \frac{u^2}{2} + 3u + 3\ln|u| - \frac{1}{u} + C$$

$$= \frac{(x-2)^2}{2} + 3(x-2) + 3\ln|x-2| - \frac{1}{x-2} + C$$

$$= \frac{x^2-4x+4+6x-12}{2} + 3\ln|x-2| - \frac{1}{x-2} + C$$

$$= \frac{x^2+2x-8}{2} + 3\ln|x-2| - \frac{1}{x-2} + C$$

$$= \frac{1}{2}x^2 + x + 3\ln|x-2| - \frac{1}{x-2} + C$$

24. $\displaystyle\int \frac{x^2}{(x+2)^3}dx$

Sol.

Let $u = x + 2$

$\therefore x = u - 2$

$dx = du$

$$\int \frac{x^2}{(x+2)^3}dx$$
$$= \int \frac{(u-2)^2}{u^3}du$$
$$= \int \left(\frac{u^2-4u+4}{u^3}\right) du$$
$$= \int \left(\frac{1}{u} - \frac{4}{u^2} + \frac{4}{u^3}\right) du$$
$$= \ln|u| + \frac{4}{u} - \frac{2}{u^2} + C$$
$$= \ln|x+2| + \frac{4}{x+2} - \frac{2}{(x+2)^2} + C$$

25. $\displaystyle\int \left(3\sin 2x - 4e^{3x}\right) dx$

Sol.

$$\int \left(3\sin 2x - 4e^{3x}\right) dx$$
$$= \frac{3}{2}\int \sin 2x d(2x) - \int 4e^{3x}dx$$
$$= -\frac{3}{2}\cos 2x - \frac{4}{3}e^{3x} + C$$

26. $\displaystyle\int \sin(5x-6)dx$

Sol.

$$\int \sin(5x-6)dx$$
$$= \frac{1}{5}\int \sin(5x-6)d(5x-6)$$
$$= -\frac{1}{5}\cos(5x-6) + C$$

27. $\displaystyle\int \left(\cos 6x + \sec^2 4x\right) dx$

Sol.

$$\int \left(\cos 6x + \sec^2 4x\right) dx$$
$$= \frac{1}{6}\int \cos 6x d(6x) + \frac{1}{4}\int \sec^2 4x d(4x)$$
$$= \frac{1}{6}\sin 6x + \frac{1}{4}\tan 4x + C$$

28. $\displaystyle\int \left(\sin\frac{x}{2} + \cos 2x - \cos\frac{x}{7}\right) dx$

Sol.

$$\int \left(\sin\frac{x}{2} + \cos 2x - \cos\frac{x}{7}\right) dx$$
$$= 2\int \sin\frac{x}{2}d\left(\frac{x}{2}\right) + \int \cos 2x d(2x)$$
$$-7\int \cos\frac{x}{7}d\left(\frac{x}{7}\right)$$
$$= -2\cos\frac{x}{2} + \frac{1}{2}\sin 2x - 7\sin\frac{x}{7} + C$$

29. $\displaystyle\int \tan^2 3x dx$

Sol.

$$\int \tan^2 3x dx$$
$$= \int \left(\sec^2 3x - 1\right) dx$$
$$= \frac{1}{3}\int \sec^2 3x d(3x) - \int dx$$
$$= \frac{1}{3}\tan 3x - x + C$$

30. $\int \tan x \sec^2 x dx$

Sol.

$$\int \tan x \sec^2 x dx$$
$$= \int \tan x d(\tan x)$$
$$= \frac{1}{2} \tan^2 x + C$$

31. $\int \frac{3 \sin x}{\cos 2x + 1} dx$

Sol.

$$\int \frac{3 \sin x}{\cos 2x + 1} dx$$
$$= \int \frac{3 \sin x}{2 \cos^2 x} dx$$
$$= \frac{3}{2} \int \frac{\sin x}{\cos^2 x} dx$$
$$= \frac{3}{2} \int \sec x \tan x dx$$
$$= \frac{3}{2} \sec x + C$$

32. $\int \frac{\sec^2 x}{\tan x + 2} dx$

Sol.

$$\int \frac{\sec^2 x}{\tan x + 2} dx$$
$$= \int \frac{1}{\tan x + 2} d(\tan x)$$
$$= \ln |\tan x + 2| + C$$

33. $\int \cot 2x \csc^3 2x dx$

Sol.

$$\int \cot 2x \csc^3 2x dx$$
$$= \int \csc 2x \cot 2x \csc^2 2x dx$$
$$= \frac{1}{2} \int \csc^2 2x d(\csc 2x)$$
$$= -\frac{1}{6} \csc^3 2x + C$$

34. $\int \tan^3 x \sec^3 x dx$

Sol.

$$\int \tan^3 x \sec^3 x dx$$
$$= \int \tan^2 x \sec^2 x \sec x \tan x dx$$
$$= \int \left(\sec^2 x - 1\right) \sec^2 x \sec x \tan x dx$$
$$= \int \left(\sec^4 x - \sec^2 x\right) d(\sec x)$$
$$= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$$

35. If the function $y = \ln x - \frac{3}{x}$, find $\frac{dy}{dx}$. Hence, find $\int \frac{3 + x}{3x^2} dx$.

Sol.

$$\begin{aligned} y &= \ln x - \frac{3}{x} \\ \frac{dy}{dx} &= \frac{1}{x} + \frac{3}{x^2} \\ &= \frac{x+3}{x^2} \end{aligned}$$

$$\begin{aligned} \int \frac{3 + x}{3x^2} dx &= \frac{1}{3} \int \frac{x+3}{x^2} d(x^3) \\ &= \frac{1}{3} \ln x - \frac{1}{x} + C \end{aligned}$$

36. If the function $y = \dfrac{1}{\sqrt{4x^2-1}}$, find $\dfrac{dy}{dx}$. Hence, find $\displaystyle\int \frac{x}{\sqrt{\left(4x^2-1\right)^3}}$.

Sol.

$$\begin{aligned}
y &= \left[\left(4x^2-1\right)^{-\frac{1}{2}}\right]' \\
&= -\frac{1}{2}\left(4x^2-1\right)^{-\frac{3}{2}} \cdot 8x \\
&= -\frac{4x}{\sqrt{\left(4x^2-1\right)^3}}
\end{aligned}$$

$$\begin{aligned}
\int \frac{x}{\sqrt{\left(4x^2-1\right)^3}} dx &= -\frac{1}{4}\int -\frac{4x}{\sqrt{\left(4x^2-1\right)^3}} d(4x^2) \\
&= -\frac{1}{4\sqrt{4x^2-1}} + C
\end{aligned}$$

37. Given the function $y = \dfrac{x^2+3}{1-x}$, and $\dfrac{dy}{dx} = \dfrac{1}{2}f(x)$, find $\displaystyle\int \left[3 - x^2 - f(x)\right] dx$.

Sol.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x^2+3}{1-x}\right) &= \frac{1}{2}f(x) \\
f(x) &= 2\left[\frac{d}{dx}\left(\frac{x^2+3}{1-x}\right)\right]
\end{aligned}$$

$$\begin{aligned}
\int \left[3 - x^2 - f(x)\right] dx &= \int 3dx - \int x^2 dx - f(x)dx \\
&= \int 3dx - \int x^2 dx - \int 2\left[\frac{d}{dx}\left(\frac{x^2+3}{1-x}\right)\right] dx \\
&= 3x - \frac{x^3}{3} - \frac{2(x^2+3)}{1-x} + C
\end{aligned}$$

38. Given the function $\dfrac{d}{dx}(x\ln x) = g(x)$, find $\displaystyle\int [g(x) - 2x]\, dx$.

Sol.

$$\begin{aligned}
\int [g(x) - 2x]\, dx &= \int g(x)dx - \int 2xdx \\
&= \int \frac{d}{dx}(x\ln x)dx - \int 2xdx \\
&= x\ln x - x^2 + C
\end{aligned}$$

39. The gradient of the tangent at any point on a curve is 3 times the x-coordinate of the point, and the curve passes through $(-2, 5)$. Find the equation of the curve.

Sol.

$$\begin{aligned}
\frac{dy}{dx} &= 3x \\
dy &= 3x dx \\
y &= \int 3x dx \\
&= \frac{3}{2}x^2 + C
\end{aligned}$$

Given that the curve passes through $(-2, 5)$,

$$\begin{aligned}
5 &= \frac{3(-2)^2}{2} + C \\
C &= 1
\end{aligned}$$

Therefore, the equation of the curve is $y = \frac{3}{2}x^2 + 1$.

40. The gradient of the tangent at any point on a curve is $\frac{dy}{dx} = 3x^2 - 8x + 1$, and the curve intersect with x-axis at point $(2, 0)$, find the other point of intersection of the curve and the x-axis.

Sol.

$$\begin{aligned}
\frac{dy}{dx} &= 3x^2 - 8x + 1 \\
y &= \int (3x^2 - 8x + 1) dx \\
&= x^3 - 4x^2 + x + C
\end{aligned}$$

Given that the curve passes through $(2, 0)$,

$$\begin{aligned}
0 &= 2^3 - 4(2)^2 + 2 + C \\
C &= -6
\end{aligned}$$

Therefore, the equation of the curve is $y = x^3 - 4x^2 + x - 6$.

When the curve intersects with the x-axis, $y = 0$,

$$\begin{aligned}
x^3 - 4x^2 + x + 6 &= 0 \\
(x-2)(x^2 - 2x - 3) &= 0 \\
(x-2)(x-3)(x+1) &= 0
\end{aligned}$$

Therefore, the curve also intersects with the x-axis at $(-1, 0)$ and $(3, 0)$.

Chapter 28

Definite Integrals