Calculus

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Contents

Chapter 1

Limits

1.1 Arithmetic Properties of Limits

If $\lim_{x \to x_0} f(x) = A$, $\lim_{x \to x_0} g(x) = B$, then:

   (a) $\lim_{x \to x_0} (f(x) \pm g(x)) = \lim_{x \to x_0} f(x) \pm \lim_{x \to x_0} g(x) = A \pm B$

   (b) $\lim_{x \to x_0} (f(x) \cdot g(x)) = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x) = A \cdot B$

   (c) $\lim_{x \to x_0} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)} = \frac{A}{B}$, $(B \neq 0)$

   (d) If k is a constant, then $\lim_{x \to x_0} k = k$

   (e) If k is a constant, then $\lim_{x \to x_0} k \cdot f(x) = k \lim_{x \to x_0} f(x) = kA$

   (f) If $n \in \mathbb{R}$, and $\lim_{x \to x_0} f(x) > 0$, then $\lim_{x \to x_0} [f(x)]^n = \left[\lim_{x \to x_0} f(x) \right]^n = A^n$

   (g) If $\lim_{x \to x_0} f(x) = 0$, then $\lim_{x \to x_0} \frac{1}{f(x)} = \infty$

   (h) L'Hopital's Rule: If $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ is indeterminate, then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$

Squeeze Theorem or Sandwich Rule

     Near point x_0,

   If $f(x) \leq g(x) \leq h(x)$

  and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = A$,

then $\lim_{x \to x_0} g(x) = A$.

1. $\lim_{x \to 3} 3x$

   Sol.

$$\lim_{x \to 3} 3x = 3 \cdot 3$$
$$= 9 \quad \square$$

2. $\lim_{x \to -1} (x^2 + 4x)$

Sol.

$$\lim_{x \to -1} (x^2 + 4x) = \lim_{x \to -1} x^2 + \lim_{x \to -1} 4x$$
$$= (-1)^2 + 4(-1)$$
$$= 1 - 4$$
$$= -3 \quad \square$$

3. $\lim\limits_{x \to 3}(9 - x^2)$

   Sol.
   $$\lim_{x \to 3}(9 - x^2) = \lim_{x \to 3} 9 - \lim_{x \to 3} x^2$$
   $$= 9 - 3^2$$
   $$= 9 - 9$$
   $$= 0 \quad \square$$

4. $\lim\limits_{n \to -2}(x^2 - 2x + 1)$

   Sol.
   $$\lim_{n \to -2}(x^2 - 2x + 1) = \lim_{n \to -2}(x - 1)^2$$
   $$= (-3)^2$$
   $$= 9 \quad \square$$

5. $\lim\limits_{x \to -4} x^2(x + 2)$

   Sol.
   $$\lim_{x \to -4} x^2(x + 2) = \lim_{x \to -4} x^2 \lim_{x \to -4}(x + 2)$$
   $$= (-4)^2 \cdot (-4 + 2)$$
   $$= 16 \cdot (-2)$$
   $$= -32 \quad \square$$

6. $\lim\limits_{h \to 2}(h^2 - 4h + 4)$

   Sol.
   $$\lim_{h \to 2}(h^2 - 4h + 4) = \lim_{h \to 2}(h - 2)^2$$
   $$= (2 - 2)^2$$
   $$= 0 \quad \square$$

7. $\lim\limits_{a \to -1}(a + 3)(a - 4)$

   Sol.
   $$\lim_{a \to -1}(a + 3)(a - 4) = \lim_{a \to -1}(a + 3) \lim_{a \to -1}(a - 4)$$
   $$= (-1 + 3) \cdot (-1 - 4)$$
   $$= 2 \cdot -5$$
   $$= -10 \quad \square$$

8. $\lim\limits_{x \to 3} \frac{x^2 - 5}{x + 2}$

   Sol.
   $$\lim_{x \to 3} \frac{x^2 - 5}{x + 2} = \lim_{x \to 3} \frac{x^2 - 5}{x + 2}$$
   $$= \frac{3^2 - 5}{3 + 2}$$
   $$= \frac{4}{5} \quad \square$$

9. $\lim\limits_{x \to -3} \frac{(x+5)(x+3)}{x+3}$

   Sol.
   $$\lim_{x \to -3} \frac{(x + 5)(x + 3)}{x + 3} = \lim_{x \to -3} \frac{(x + 5)(x + 3)}{x + 3}$$
   $$= \lim_{x \to -3}(x + 5)$$
   $$= -3 + 5$$
   $$= 2 \quad \square$$

10. $\lim\limits_{x \to 0} \frac{x^2 + 5x}{x}$

    Sol.
    $$\lim_{x \to 0} \frac{x^2 + 5x}{x} = \lim_{x \to 0} \frac{x^2 + 5x}{x}$$
    $$= \lim_{x \to 0} \frac{x(x + 5)}{x}$$
    $$= \lim_{x \to 0}(x + 5)$$
    $$= 0 + 5$$
    $$= 5 \quad \square$$

11. $\lim\limits_{x \to -2} \frac{x^3 + 8}{x + 2}$

    Sol.
    $$\lim_{x \to -2} \frac{x^3 + 8}{x + 2} = \lim_{x \to -2} \frac{(x + 2)(x^2 - x + 4)}{x + 2}$$
    $$= \lim_{x \to -2}(x^2 - x + 4)$$
    $$= (-2)^2 - (-2) + 4$$
    $$= 10 \quad \square$$

12. $\lim\limits_{x \to 4} \frac{x^2 - 5x + 6}{x - 3}$

    Sol.
    $$\lim_{x \to 4} \frac{x^2 - 5x + 6}{x - 3} = \lim_{x \to 4} \frac{(x - 3)(x - 2)}{x - 3}$$
    $$= \lim_{x \to 4}(x - 2)$$
    $$= 4 - 2$$
    $$= 2 \quad \square$$

13. $\lim\limits_{x \to 3} \frac{3x}{x + 2}$

    Sol.
    $$\lim_{x \to 3} \frac{3x}{x + 2} = \lim_{x \to 3} \frac{3x}{x + 2}$$
    $$= \frac{3(3)}{3 + 2}$$
    $$= \frac{9}{5} \quad \square$$

14. $\lim\limits_{x \to 5} \frac{x - 5}{2x^2 - 9x - 5}$

    Sol.
    $$\lim_{x \to 5} \frac{x - 5}{2x^2 - 9x - 5} = \lim_{x \to 5} \frac{x - 5}{(2x + 1)(x - 5)}$$
    $$= \lim_{x \to 5} \frac{1}{2x + 1}$$
    $$= \frac{1}{2(5) + 1}$$
    $$= \frac{1}{11} \quad \square$$

15. $\lim\limits_{x\to 1} \dfrac{x-1}{x^2+x-2}$

Sol.

$$\lim_{x\to 1}\frac{x-1}{x^2+x-2}=\lim_{x\to 1}\frac{x-1}{(x-1)(x+2)}$$
$$=\lim_{x\to 1}\frac{1}{x+2}$$
$$=\frac{1}{1+2}$$
$$=\frac{1}{3}\quad\square$$

16. $\lim\limits_{x\to 4} \dfrac{x-1}{x^2+x-2}$

Sol.

$$\lim_{x\to 4}\frac{x-1}{x^2+x-2}=\lim_{x\to 4}\frac{x-1}{(x-1)(x+2)}$$
$$=\lim_{x\to 4}\frac{1}{x+2}$$
$$=\frac{1}{4+2}$$
$$=\frac{1}{6}\quad\square$$

17. $\lim\limits_{x\to -2} \dfrac{x-2}{x^2-4}$

Sol.

$$\lim_{x\to -2}\frac{x-2}{x^2-4}=\lim_{x\to -2}\frac{x-2}{(x+2)(x-2)}$$
$$=\lim_{x\to -2}\frac{1}{x+2}$$
$$=\infty\quad\square$$

18. $\lim\limits_{h\to 0} \dfrac{2x^2h+3h}{h}$

Sol.

$$\lim_{h\to 0}\frac{2x^2h+3h}{h}=\lim_{h\to 0}\frac{h(2x^2+3)}{h}$$
$$=\lim_{h\to 0}(2x^2+3)$$
$$=2x^2+3\quad\square$$

19. $\lim\limits_{h\to 0} \dfrac{(2+h)^2-4}{h}$

Sol.

$$\lim_{h\to 0}\frac{(2+h)^2-4}{h}=\lim_{h\to 0}\frac{[(2+h)+2][(2+h)-2]}{h}$$
$$=\lim_{h\to 0}\frac{(4+h)h}{h}$$
$$=\lim_{h\to 0}(4+h)$$
$$=4+0$$
$$=4\quad\square$$

20. $\lim\limits_{h\to 0} \dfrac{(1+h)^3-1}{h}$

Sol.

$$\lim_{h\to 0}\frac{(1+h)^3-1}{h}$$
$$=\lim_{h\to 0}\frac{[(1+h)-1][(1+h)^2+(1+h)+1]}{h}$$
$$=\lim_{h\to 0}\frac{h(1+2h+h^2+1+h+1)}{h}$$
$$=\lim_{h\to 0}(h^2+3h+3)$$
$$=(0)^2+3(0)+3$$
$$=3\quad\square$$

21. $\lim\limits_{x\to -1} 2x(x^2-4)$

Sol.

$$\lim_{x\to -1}2x(x^2-4)=\lim_{x\to -1}2x(x^2-4)$$
$$=-2(-1)^2[(-1)^2-4]$$
$$=-2(-3)$$
$$=6\quad\square$$

22. $\lim\limits_{x\to 3} \dfrac{x^2+2}{x+1}$

Sol.

$$\lim_{x\to 3}\frac{x^2+2}{x+1}=\frac{3^2+2}{3+1}$$
$$=\frac{11}{4}\quad\square$$

23. $\lim\limits_{x\to 2} \left(x^2-3x+5\right)$

Sol.

$$\lim_{x\to 2}\left(x^2-3x+5\right)=2^2-3(2)+5$$
$$=3\quad\square$$

24. $\lim\limits_{x\to 1} \dfrac{2x^2+1}{3x^2+4x-1}$

Sol.

$$\lim_{x\to 1}\frac{2x^2+1}{3x^2+4x-1}=\frac{2(1)^2+1}{3(1)^2+4(1)-1}$$
$$=\frac{3}{6}$$
$$=\frac{1}{2}\quad\square$$

25. $\lim\limits_{x\to 1} \dfrac{x^2-5x+6}{x^2-9}$

Sol.

$$\lim_{x\to 1}\frac{x^2-5x+6}{x^2-9}=\frac{(x-3)(x-2)}{(x+3)(x-3)}$$
$$=\frac{x-2}{x+3}$$
$$=\frac{1-2}{1+3}$$
$$=-\frac{1}{4}\quad\square$$

26. $\lim\limits_{x \to -1} \frac{x^3+1}{x+1}$

Sol.

$$\lim_{x \to -1} \frac{x^3+1}{x+1} = \lim_{x \to -1} \frac{(x+1)(x^2-x+1)}{x+1}$$
$$= \lim_{x \to -1} (x^2 - x + 1)$$
$$= (-1)^2 - (-1) + 1$$
$$= 3 \quad \square$$

27. $\lim\limits_{x \to 1} \frac{x^3-1}{x-1}$

Sol.

$$\lim_{x \to 1} \frac{x^3-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{x-1}$$
$$= \lim_{x \to 1} (x^2 + x + 1)$$
$$= 1^2 + 1 + 1$$
$$= 3 \quad \square$$

28. $\lim\limits_{x \to 0} \frac{2x^3+3x^2}{x^3}$

Sol.

$$\lim_{x \to 0} \frac{2x^3+3x^2}{x^3} = \lim_{x \to 0} \left(\frac{2x^3}{x^3} + \frac{3x^2}{x^3} \right)$$
$$= \lim_{x \to 0} \left(2 + \frac{3}{x} \right)$$
$$= 2 \quad \square$$

29. $\lim\limits_{k \to 0} \frac{(x-k)^2 - 2kx^3}{x(x+k)}$

Sol.

$$\lim_{k \to 0} \frac{(x-k)^2 - 2kx^3}{x(x+k)} = \frac{(x-0)^2 - 2(0)(x^3)}{x(x+0)}$$
$$= \frac{x^2}{x^2}$$
$$= 1 \quad \square$$

30. $\lim\limits_{x \to 1} \frac{x^2-2x+5}{x^2+7}$

Sol.

$$\lim_{x \to 1} \frac{x^2-2x+5}{x^2+7} = \frac{(1)^2 - 2(1) + 5}{(1)^2 + 7}$$
$$= \frac{4}{8}$$
$$= \frac{1}{2} \quad \square$$

31. $\lim\limits_{x \to -2} \frac{x^4-16}{x^3-2}$

Sol.

$$\lim_{x \to -2} \frac{x^4-16}{x^3-2} = \frac{(-2)^4 - 16}{(-2)^3 - 2}$$
$$= \frac{16 - 16}{-8 - 2}$$
$$= \frac{0}{-10}$$
$$= 0 \quad \square$$

32. $\lim\limits_{x \to 1} \frac{x-1}{x^2-1}$

Sol.

$$\lim_{x \to 1} \frac{x-1}{x^2-1} = \lim_{x \to 1} \frac{x-1}{(x-1)(x+1)}$$
$$= \lim_{x \to 1} \frac{1}{x+1}$$
$$= \frac{1}{1+1}$$
$$= \frac{1}{2} \quad \square$$

33. $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-1}{x}$

Sol.

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x \to 0} \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)}$$
$$= \lim_{x \to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)}$$
$$= \lim_{x \to 0} \frac{x}{x(\sqrt{1+x}+1)}$$
$$= \lim_{x \to 0} \frac{1}{\sqrt{1+x}+1}$$
$$= \frac{1}{\sqrt{1+0}+1}$$
$$= \frac{1}{2}$$

34. $\lim\limits_{x \to 2} \frac{x^2+4}{x^2+1}$

Sol.

$$\lim_{x \to 2} \frac{x^2+4}{x^2+1} = \frac{(2)^2 + 4}{(2)^2 + 1}$$
$$= \frac{8}{5}$$

35. $\lim\limits_{x \to 0} \frac{x^2+3x+2}{x^2+2}$

Sol.

$$\lim_{x \to 0} \frac{x^2+3x+2}{x^2+2} = \frac{(0)^2 + 3(0) + 2}{(0)^2 + 2}$$
$$= \frac{2}{2}$$
$$= 1 \quad \square$$

36. $\lim\limits_{x \to 1} \frac{x^2-2x+1}{x^2-1}$

Sol.

$$\lim_{x \to 1} \frac{x^2-2x+1}{x^2-1} = \frac{(x-1)^2}{(x-1)(x+1)}$$
$$= \frac{x-1}{x+1}$$
$$= \frac{1-1}{1+1}$$
$$= 0 \quad \square$$

$$\mathbf{a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})}$$

Proof.

$$(a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) = (a - b)\left(\sum_{k=0}^{n-1} a^{n-k-1}b^k\right)$$

$$= a\sum_{k=0}^{n-1} a^{n-k-1}b^k - b\sum_{k=0}^{n-1} a^{n-k-1}b^k$$

$$= \sum_{k=0}^{n-1} a^{n-k}b^k - \sum_{k=0}^{n-1} a^{n-k-1}b^{k+1}$$

$$= a^n + \sum_{k=1}^{n-1} a^{n-k}b^k - \sum_{l=0}^{n-2} a^{n-l-1}b^{l+1} - b^n$$

$$= a^n + \sum_{k=1}^{n-1} a^{n-k}b^k - \sum_{k=1}^{n-1} a^{n-k}b^k - b^n \quad (l = k - 1)$$

$$= a^n - b^n \quad \blacksquare$$

37. $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} \quad (n \in \mathbb{N})$

Sol.

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{(x - a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{(x - a)}$$

$$= \lim_{x \to a}(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$$

$$= a^{n-1} + a^{n-2}a + \cdots + a^{n-2}a + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + \cdots + a^{n-1}$$

$$= na^{n-1} \quad \blacksquare$$

38. $\lim\limits_{x \to 1}(3x^2 - 6x + 5)$

Sol.

$$\lim_{x \to 1}(3x^2 - 6x + 5) = 3(1)^2 - 6(1) + 5$$

$$= 3 - 6 + 5$$

$$= 2 \quad \blacksquare$$

39. $\lim\limits_{x \to 1} \frac{2x^2 - 1}{3x^3 - 6x^2 + 5}$

Sol.

$$\lim_{x \to 1} \frac{2x^2 - 1}{3x^3 - 6x^2 + 5} = \frac{2(1)^2 - 1}{3(1)^3 - 6(1)^2 + 5}$$

$$= \frac{1}{2} \quad \blacksquare$$

40. $\lim\limits_{x \to 1} \frac{x^2 - 1}{x^2 - 4x + 3}$

Sol.

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - 4x + 3} = \lim_{x \to 3} \frac{(x + 1)(x - 1)}{(x - 1)(x - 3)}$$

$$= \lim_{x \to 3} \frac{x + 1}{x - 3}$$

$$= -1 \quad \blacksquare$$

41. $\lim\limits_{x \to 3} \frac{x^2 - 5x + 6}{7x^2 - 22x + 3}$

Sol.

$$\lim_{x \to 3} \frac{x^2 - 5x + 6}{7x^2 - 22x + 3} = \lim_{x \to 3} \frac{(x - 3)(x - 2)}{(7x - 1)(x - 3)}$$

$$= \lim_{x \to 3} \frac{x - 2}{7x - 1}$$

$$= \frac{3 - 2}{7(3) - 1}$$

$$= \frac{1}{20} \quad \blacksquare$$

42. $\lim\limits_{x \to 3} \frac{x^3 - 2x^2 - 2x + 4}{x^3 + x^2 - 10x + 8}$

Sol.

$$\lim_{x \to 3} \frac{x^3 - 2x^2 - 2x + 4}{x^3 + x^2 - 10x + 8} = \lim_{x \to 3} \frac{x^2(x - 2) - 2(x - 2)}{(x - 2)(x + 4)(x - 1)}$$

$$= \lim_{x \to 3} \frac{(x - 2)(x^2 - 2)}{(x - 2)(x + 4)(x - 1)}$$

$$= \lim_{x \to 3} \frac{x^2 - 2}{(x + 4)(x - 1)}$$

$$= \frac{3^2 - 2}{(3 + 4)(3 - 1)}$$

$$= \frac{1}{2} \quad \blacksquare$$

43. $\lim\limits_{x\to 1}\frac{x^4+2x^2-3}{x^2-3x+2}$

   Sol.

   $$\lim_{x\to 1}\frac{x^4+2x^2-3}{x^2-3x+2}=\lim_{x\to 3}\frac{(x-1)(x+1)(x^2+3)}{(x-2)(x-1)}$$
   $$=\lim_{x\to 3}\frac{(x+1)(x^2+3)}{x-2}$$
   $$=\frac{(1+1)(1^2+3)}{1-2}$$
   $$=\frac{8}{-1}$$
   $$=-8\quad\square$$

44. $\lim\limits_{x\to 1}\frac{1-\sqrt[4]{x}}{1-\sqrt[3]{x}}$

   Sol.

   $$\lim_{x\to 1}\frac{1-\sqrt[4]{x}}{1-\sqrt[3]{x}}=\lim_{x\to 1}\frac{\left(x^{\frac14}-1\right)'}{\left(x^{\frac13}-1\right)'}$$
   $$=\lim_{x\to 1}\frac{\frac14 x^{-\frac34}}{\frac13 x^{-\frac23}}$$
   $$=\lim_{x\to 1}\frac{\frac{1}{4x^{\frac34}}}{\frac{1}{3x^{\frac23}}}$$
   $$=\lim_{x\to 1}\frac{3x^{\frac23}}{4x^{\frac34}}$$
   $$=\frac{3(1)^{\frac23}}{4(1)^{\frac34}}$$
   $$=\frac34\quad\square$$

45. $\lim\limits_{x\to 0}\frac{\sqrt[n]{1+x}-1}{x}\quad(n\in\mathbb{W})$

   Sol.

   $$\lim_{x\to 0}\frac{\sqrt[n]{1+x}-1}{x}=\lim_{x\to 1}\frac{\left(\sqrt[n]{1+x}-1\right)'}{x'}$$
   $$=\lim_{x\to 1}\frac{\left(\sqrt[n]{1+x}\right)'-1'}{1}$$
   $$=\lim_{x\to 1}\left[(1+x)^{\frac1n}\right]'$$
   $$=\lim_{x\to 1}\frac1n\left[(1+x)^{\frac{1-n}{n}}\right]$$
   $$=\frac1n(1+0)^{\frac{1-n}{n}}$$
   $$=\frac1n(1)^{\frac{1-n}{n}}$$
   $$=\frac1n\quad\square$$

46. $\lim\limits_{x\to 1}\frac{2-\sqrt{x+3}}{x^2-1}$

Sol.

$$\lim_{x\to 1}\frac{2-\sqrt{x+3}}{x^2-1}=\lim_{x\to 1}\frac{\left(2-\sqrt{x+3}\right)'}{\left(x^2-1\right)'}$$
$$=\lim_{x\to 1}\frac{2'-\left(\sqrt{x+3}\right)'}{2x}$$
$$=\lim_{x\to 1}\frac{-\left[(x+3)^{\frac12}\right]'}{2x}$$
$$=\lim_{x\to 1}\frac{-\frac12(x+3)^{-\frac12}}{2x}$$
$$=\lim_{x\to 1}\frac{-\frac{1}{2\sqrt{x+3}}}{2x}$$
$$=\frac{-\frac{1}{2\sqrt{1+3}}}{2(1)}$$
$$=\frac{-\frac14}{2}$$
$$=-\frac18$$

47. $\lim\limits_{x\to 16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4}$

   Sol.

   $$\lim_{x\to 16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4}=\lim_{x\to 16}\frac{\left(\sqrt[4]{x}-2\right)'}{\left(\sqrt{x}-4\right)'}$$
   $$=\lim_{x\to 16}\frac{\left(\sqrt[4]{x}\right)'-2'}{\left(\sqrt{x}\right)'-4'}$$
   $$=\lim_{x\to 16}\frac{\frac14 x^{-\frac34}}{\frac12 x^{-\frac12}}$$
   $$=\lim_{x\to 16}\frac{\frac{1}{4x^{\frac34}}}{\frac{1}{2x^{\frac12}}}$$
   $$=\lim_{x\to 16}\frac{x^{\frac12}}{2x^{\frac34}}$$
   $$=\frac{16^{\frac12}}{2(16)^{\frac34}}$$
   $$=\frac{4}{2(8)}$$
   $$=\frac14\quad\square$$

48. $\lim\limits_{x\to 2}(x^2+3x-1)$

   Sol.

   $$\lim_{x\to 2}(x^2+3x-1)=(2)^2+3(2)-1$$
   $$=4+6-1$$
   $$=9\quad\square$$

49. $\lim\limits_{x\to -1}\frac{x^2+2}{x^2+x+3}$

Sol.

$$\lim_{x \to -1} \frac{x^2 + 2}{x^2 + x + 3} = \frac{(-1)^2 + 2}{(-1)^2 + (-1) + 3}$$
$$= \frac{1 + 2}{1 - 1 + 3}$$
$$= 1 \quad \square$$

50. $\lim_{x \to -1} \frac{x^3 + 1}{x^2 - 1}$

Sol.

$$\lim_{x \to -1} \frac{x^3 + 1}{x^2 - 1} = \frac{(x + 1)(x^2 - x + 1)}{(x + 1)(x - 1)}$$
$$= \frac{x^2 - x + 1}{x - 1}$$
$$= \frac{(-1)^2 - (-1) + 1}{-1 - 1}$$
$$= -\frac{3}{2} \quad \square$$

51. $\lim_{x \to 1} \frac{x^5 - x^4}{x^3 - x}$

Sol.

$$\lim_{x \to 1} \frac{x^5 - x^4}{x^3 - x} = \lim_{x \to 1} \frac{(x^5 - x^4)'}{(x^3 - x)'}$$
$$= \lim_{x \to 1} \frac{5x^4 - 4x^3}{3x^2 - 1}$$
$$= \frac{5(1)^4 - 4(1)^3}{3(1)^2 - 1}$$
$$= \frac{1}{2} \quad \square$$

52. $\lim_{x \to a} \frac{x^2 + ax - 2a^2}{x^2 - a^2}, a \neq 0$

Sol.

$$\lim_{x \to a} \frac{x^2 + ax - 2a^2}{x^2 - a^2} = \lim_{x \to a} \frac{(x + 2a)(x - a)}{(x + a)(x - a)}$$
$$= \lim_{x \to a} \frac{x + 2a}{x + a}$$
$$= \frac{a + 2a}{a + a}$$
$$= \frac{3}{2} \quad \square$$

53. $\lim_{x \to a} \frac{\sqrt{3x - a} - \sqrt{x + a}}{x - a}$

Sol.

$$\lim_{x \to a} \frac{\sqrt{3x - a} - \sqrt{x + a}}{x - a}$$
$$= \lim_{x \to a} \frac{(\sqrt{3x - a} - \sqrt{x + a})'}{(x - a)'}$$
$$= \lim_{x \to a} \left[(\sqrt{3x - a})' - (\sqrt{x + a})'\right]$$
$$= \lim_{x \to a} \left[(\sqrt{3x - a})' - (\sqrt{x + a})'\right]$$
$$= \lim_{x \to a} \left[\frac{1}{2\sqrt{3x - a}}(3x - a)' - \frac{1}{2\sqrt{x + a}}\right]$$
$$= \lim_{x \to a} \left[\frac{3}{2\sqrt{3x - a}} - \frac{1}{2\sqrt{x + a}}\right]$$
$$= \frac{3}{2\sqrt{3a - a}} - \frac{1}{2\sqrt{a + a}}$$
$$= \frac{2}{2\sqrt{2a}}$$
$$= \frac{1}{\sqrt{2a}}$$

54. Given that $f(x) = x^2 - 3x$, find $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Sol.

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x + h)^2 - 3(x + h) - x^2 + 3x}{h}$$
$$= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - 3x - 3h - x^2 + 3x}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h}$$
$$= \lim_{h \to 0} (2x + 2h - 3)$$
$$= 2x - 3 \quad \square$$

55. $\lim_{x \to 2} \sqrt{2x^2 + 1}$

Sol.

$$\lim_{x \to 2} \sqrt{2x^2 + 1} = \sqrt{2(2)^2 + 1}$$
$$= \sqrt{9}$$
$$= 3 \quad \square$$

56. $\lim_{x \to 7} \frac{x^2 \sqrt{x + 2}}{x^2 + 14}$

Sol.

$$\lim_{x \to 7} \frac{x^2 \sqrt{x + 2}}{x^2 + 14} = \frac{(7)^2 \sqrt{7 + 2}}{(7)^2 + 14}$$
$$= \frac{49(3)}{49 + 14}$$
$$= \frac{7}{3}$$

57. $\lim_{x \to 0} \frac{\sqrt{3x + 4} - 2}{x}$

Sol.

$$\lim_{x\to 0}\frac{\sqrt{3x+4}-2}{x}=\lim_{x\to 0}\frac{\left(\sqrt{3x+4}\right)'-2'}{x'}$$

$$=\lim_{x\to 0}\left[\frac{1}{2\sqrt{3x+4}}(3x+4)'\right]$$

$$=\lim_{x\to 0}\left(\frac{3}{2\sqrt{3x+4}}\right)$$
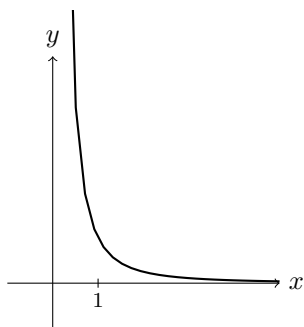
$$=\frac{3}{2\sqrt{3(0)+4}}$$

$$=\frac{3}{4}\quad\square$$

58. $\lim_{x\to 0}\frac{1}{x^2}$

Sol.

$$\lim_{x\to 0}\frac{1}{x^2}=\infty\quad\square$$

59. $\lim_{x\to 1}\frac{1}{x-1}$

Sol.

$$\lim_{x\to 1}\frac{1}{x-1}=\infty$$

60. $\lim_{x\to 1}\frac{4x-3}{x^2-5x+4}$

Sol.

$$\lim_{x\to 1}\frac{4x-3}{x^2-5x+4}=\lim_{x\to 1}\frac{(4x-3)'}{(x^2-5x+4)'}$$

$$=\lim_{x\to 1}\frac{4}{2x-5}$$

$$=\frac{4}{2(1)-5}$$

$$=-\frac{4}{3}$$

61. $\lim_{x\to\infty}\frac{3x^3-4x^2+2}{7x^3+5x^2-3}$

Sol.

$$\lim_{x\to\infty}\frac{3x^3-4x^2+2}{7x^3+5x^2-3}=\lim_{x\to\infty}\frac{\frac{3x^3}{x^3}-\frac{4x^2}{x^3}+\frac{2}{x^3}}{\frac{7x^3}{x^3}+\frac{5x^2}{x^3}-\frac{3}{x^3}}$$

$$=\lim_{x\to\infty}\frac{3-\frac{4}{x}+\frac{2}{x^3}}{7+\frac{5}{x}-\frac{3}{x^3}}$$

$$=\frac{3-0+0}{7+0-0}$$

$$=\frac{3}{7}\quad\square$$

62. $\lim_{n\to\infty}x^2$

Sol.

$$\lim_{n\to\infty}x^2=\infty\quad\square$$

63. $\lim_{x\to\infty}\frac{3x^2-2x-1}{2x^3-x^2+5}$

Sol.

$$\lim_{x\to\infty}\frac{3x^2-2x-1}{2x^3-x^2+5}=\lim_{x\to\infty}\frac{\frac{3x^2}{x^3}-\frac{2x}{x^3}-\frac{1}{x^3}}{\frac{2x^3}{x^3}-\frac{x^2}{x^3}+\frac{5}{x^3}}$$

$$=\lim_{x\to\infty}\frac{\frac{3}{x}-\frac{2}{x^2}-\frac{1}{x^3}}{2-\frac{1}{x}+\frac{5}{x^3}}$$

$$=\frac{0-0-0}{2-0+0}$$

$$=0\quad\square$$

64. $\lim_{n\to\infty}\frac{1}{n^2}(1+2+3+\cdots+n)$

Sol.

$$\lim_{n\to\infty}\frac{1}{n^2}(1+2+3+\cdots+n)=\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}k$$

$$=\lim_{n\to\infty}\frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{n+1}{2}\right)$$

$$=\lim_{n\to\infty}\frac{n+1}{2n}$$

$$=\lim_{n\to\infty}\frac{\frac{n}{n}+\frac{1}{n}}{\frac{2n}{n}}$$

$$=\frac{1+0}{2}$$

$$=\frac{1}{2}\quad\square$$

65. $\lim_{n\to\infty}\left[\frac{1+2+3+\cdots+n}{n+2}-\frac{n}{2}\right]$

Sol.

$$\lim_{n\to\infty}\left[\frac{1+2+3+\cdots+n}{n+2}-\frac{n}{2}\right]$$

$$=\lim_{n\to\infty}\left[\frac{\sum_{k=1}^{n}k}{n+2}-\frac{n}{2}\right]$$

$$=\lim_{n\to\infty}\left[\frac{n(n+1)}{2(n+2)}-\frac{n}{2}\right]$$

$$=\lim_{n\to\infty}\left[\frac{n(n+1)-n(n+2)}{2(n+2)}\right]$$

$$=\lim_{n\to\infty}\left[\frac{n(n+1-n-2)}{2(n+2)}\right]$$

$$=\lim_{n\to\infty}\left[-\frac{\frac{n}{n}}{\frac{2n}{n}+\frac{4}{n}}\right]$$

$$=-\frac{1}{2+0}$$

$$=-\frac{1}{2}\quad\square$$

66. $\lim_{n \to \infty} \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} \right]$

Sol.

$$\lim_{n \to \infty} \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} \right]$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k(k+1)}$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \lim_{n \to \infty} \left[\sum_{k=1}^{n} \left(\frac{1}{k} \right) - \sum_{k=1}^{n} \left(\frac{1}{k+1} \right) \right]$$

$$= \lim_{n \to \infty} \left[\sum_{k=1}^{n} \left(\frac{1}{k} \right) - \sum_{k=2}^{n+1} \left(\frac{1}{k} \right) \right]$$

$$= \lim_{n \to \infty} \left[1 + \sum_{k=2}^{n} \left(\frac{1}{k} \right) - \sum_{k=2}^{n} \left(\frac{1}{k} \right) - \frac{1}{n+1} \right]$$

$$= \lim_{n \to \infty} \left[1 - \frac{1}{n+1} \right]$$

$$= \lim_{n \to \infty} \left[\frac{n+1-1}{n+1} \right]$$

$$= \lim_{n \to \infty} \left[\frac{n}{n+1} \right]$$

$$= \lim_{n \to \infty} \left[\frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} \right]$$

$$= \frac{1}{1+0}$$

$$= 1 \quad \square$$

67. $\lim_{x \to \infty} \frac{5x^3 + 4x^2 - 6x + 2}{8x^3 - 7x^2 + 4x - 1}$

Sol.

$$\lim_{x \to \infty} \frac{5x^3 + 4x^2 - 6x + 2}{8x^3 - 7x^2 + 4x - 1}$$

$$= \lim_{x \to \infty} \frac{\frac{5x^3}{x^3} + \frac{4x^2}{x^3} - \frac{6x}{x^3} + \frac{2}{x^3}}{\frac{8x^3}{x^3} - \frac{7x^2}{x^3} + \frac{4x}{x^3} - \frac{1}{x^3}}$$

$$= \lim_{x \to \infty} \frac{5 + \frac{4}{x} - \frac{6}{x^2} + \frac{2}{x^3}}{8 - \frac{7}{x} + \frac{4}{x^2} - \frac{1}{x^3}}$$

$$= \frac{5 + 0 - 0 + 0}{8 - 0 + 0 - 0}$$

$$= \frac{5}{8} \quad \square$$

68. $\lim_{x \to \infty} \frac{x^4 - 2x^3 + x^2 + 3}{x^5 - x^4 + 1}$

Sol.

$$\lim_{x \to \infty} \frac{x^4 - 2x^3 + x^2 + 3}{x^5 - x^4 + 1}$$

$$= \lim_{x \to \infty} \frac{\frac{x^4}{x^5} - \frac{2x^3}{x^5} + \frac{x^2}{x^5} + \frac{3}{x^5}}{\frac{x^5}{x^5} - \frac{x^4}{x^5} + \frac{1}{x^5}}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{x} - \frac{2}{x^2} + \frac{1}{x^3} + \frac{3}{x^5}}{1 - \frac{1}{x} + \frac{1}{x^5}}$$

$$= \frac{0 - 0 + 0 + 0}{1 - 0 + 0}$$

$$= 0 \quad \square$$

69. $\lim_{x \to \infty} \frac{x^3 - 8x^2 + 4x - 1}{x^2 - 6x + 3}$

Sol.

$$\lim_{x \to \infty} \frac{x^3 - 8x^2 + 4x - 1}{x^2 - 6x + 3}$$

$$= \lim_{x \to \infty} \frac{\frac{x^3}{x^3} - \frac{8x^2}{x^3} + \frac{4x}{x^3} - \frac{1}{x^3}}{\frac{x^2}{x^3} - \frac{6x}{x^3} + \frac{3}{x^3}}$$

$$= \lim_{x \to \infty} \frac{1 - \frac{8x}{x} + \frac{4}{x^2} - \frac{1}{x^3}}{\frac{1}{x} - \frac{6}{x^2} + \frac{3}{x^3}}$$

$$= \lim_{x \to \infty} \frac{1 - \frac{8x}{x} + \frac{4}{x^2} - \frac{1}{x^3}}{\frac{1}{x} - \frac{6}{x^2} + \frac{3}{x^3}}$$

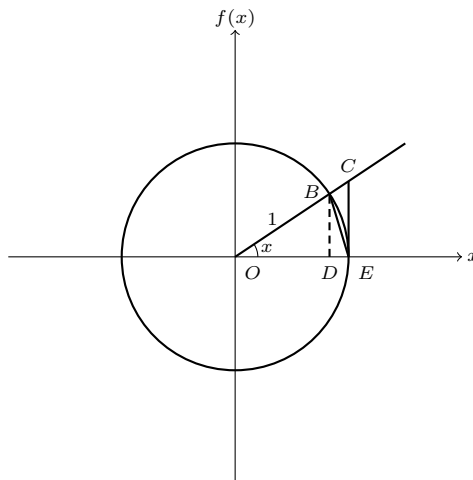$$= \frac{1 - 0 + 0 - 0}{0 - 0 + 0}$$

$$= \infty \quad \square$$

70. $\lim_{x \to \infty} \left(\sqrt{x^4 + 1} - x^2 \right)$

Sol.

$$\lim_{x \to \infty} \left(\sqrt{x^4 + 1} - x^2 \right) = \lim_{x \to \infty} \sqrt{x^4 + 1} - x^2$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{x^4 + 1} + x^2}$$

$$= \frac{\lim_{x \to \infty} 1}{\lim_{x \to \infty} \left(\sqrt{x^4 + 1} + x^2 \right)}$$

$$= \frac{1}{\infty}$$

$$= 0 \quad \square$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Proof.

From the diagram above, we can see that:

$$BD = \sin x$$
$$\overset{\frown}{BD} = x$$
$$EC = \tan x$$

Also, the area of sector EOB is greater than that of $\triangle EOB$, but less than that of $\triangle EOC$. That is,

$$(\text{area of sector } EOB) < (\text{area of } \triangle EOB) < (\text{area of } \triangle EOC)$$

Hence $\quad \dfrac{1}{2} \times 1 \times \sin x < \dfrac{1}{2} \times 1^2 < \dfrac{1}{2} \times 1 \times \tan x$

$\therefore \quad \sin x < x < \tan x$

Dividing by $\sin x$ ($\sin x \neq 0$), we get

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

But $\quad \displaystyle\lim_{x \to 0^+} \frac{1}{\cos x} = \frac{x}{\lim_{x \to 0^+} \sin x} = 1$

Hence, $\displaystyle\lim_{x \to 0^+} \frac{x}{\sin x}$ is in between 1 and $\displaystyle\lim_{x \to 0^+} \frac{1}{\cos x}$, which is in between 1 and 1. Therefore, $\displaystyle\lim_{x \to 0^+} \frac{x}{\sin x}$ must be equal to 1 (Squeeze Theorem).

That is $\quad \displaystyle\lim_{x \to 0^+} \frac{x}{\sin x} = 1,$

$\therefore \quad \displaystyle\lim_{x \to 0^+} \frac{\sin x}{x} = \lim_{x \to 0^+} \frac{1}{\frac{x}{\sin x}} = \lim_{x \to 0^+} \frac{1}{\lim_{x \to 0^+} \frac{x}{\sin x}} = 1$ $\quad \square$