Exercise 11j

Find the following indefinite integrals:

1. $\int x^2 e^{-x}\,dx$

   Sol.
   Let $u = x^2$, $du = 2x\,dx$.
   Let $dv = e^{-x}\,dx$, $v = -e^{-x}$.

$$\begin{aligned}\int x^2 e^{-x}\,dx &= -x^2e^{-x} + \int 2xe^{-x}\,dx \\ &= -x^2e^{-x} + 2\int xe^{-x}\,dx \\ &= -x^2e^{-x} + 2\left(-xe^{-x} + \int e^{-x}\,dx\right) \\ &= -x^2e^{-x} + 2\left(-xe^{-x} - e^{-x}\right) + C \\ &= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C \\ &= -(x^2+2x+2)e^{-x} + C \qquad \square\end{aligned}$$

2. $\int x^2 \cos x\,dx$

   Sol.
   Let $u = x^2$, $du = 2x\,dx$.
   Let $dv = \cos x\,dx$, $v = \sin x$.

$$\begin{aligned}\int x^2 \cos x\,dx &= x^2\sin x - \int 2x\sin x\,dx \\ &= x^2\sin x + 2\int x\sin x\,dx \\ &= x^2\sin x + 2\left(-x\cos x - \int \cos x\,dx\right) \\ &= x^2\sin x + 2\left(-x\cos x - \sin x\right) + C \\ &= x^2\sin x - 2x\cos x - 2\sin x + C \\ &= (x^2-2)\sin x - 2x\cos x + C \qquad \square\end{aligned}$$

3. $\int x^2 \cos^2 x\,dx$

   Sol.
   Let $u = x^2$, $du = 2x\,dx$.

$$\begin{aligned}\text{Let } dv = \cos^2 x\,dx,\ v &= \int \cos^2 x\,dx \\ &= \frac{1}{2}\int (1+\cos 2x)\,dx \\ &= \frac{1}{2}x + \frac{1}{4}\sin 2x\end{aligned}$$

$$\begin{aligned}\int x^2\cos^2 x\,dx &= x^2\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right) - \int 2x\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right)\,dx \\ &= \frac{1}{2}x^3 + \frac{1}{4}x^2\sin 2x - \int \left(x^2 + \frac{1}{2}x\sin 2x\right)\,dx \\ &= \frac{1}{2}x^3 + \frac{1}{4}x^2\sin 2x - \int x^2\,dx - \frac{1}{2}\int x\sin 2x\,dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}x^3 + \frac{1}{4}x^2 \sin 2x - \frac{1}{3}x^3 - \frac{1}{2}\left(-\frac{1}{2}x\cos 2x - \int -\frac{1}{2}\cos 2x\,dx\right) \\
&= \frac{1}{6}x^3 + \frac{1}{4}x^2 \sin 2x + \frac{1}{4}x\cos 2x - \frac{1}{4}\int \cos 2x\,dx \\
&= \frac{1}{6}x^3 + \frac{1}{4}x^2 \sin 2x + \frac{1}{4}x\cos 2x - \frac{1}{8}\sin 2x + C \qquad \square
\end{aligned}
$$

4. $\displaystyle\int x^5 \sin x^2\,dx$

Sol.

Let $u = x^4$, $du = 4x^3\,dx$.

Let $dv = x\sin x^2\,dx$, $v = -\frac{1}{2}\cos x^2$.

$$
\begin{aligned}
\int x^5 \sin x^2\,dx &= -\frac{1}{2}x^4\cos x^2 + \int 2x^3\cos x^2\,dx \\
&= -\frac{1}{2}x^4\cos x^2 + 2\int x^3\cos x^2\,dx
\end{aligned}
$$

Let $u = x^2$, $du = 2x\,dx$.

Let $dv = x\cos x^2\,dx$, $v = \frac{1}{2}\sin x^2$.

$$
\begin{aligned}
\int x^5 \sin x^2\,dx &= -\frac{1}{2}x^4\cos x^2 + 2\left(\frac{1}{2}x^2\sin x^2 - \int \frac{1}{2}\cdot 2x\sin x^2\,dx\right) \\
&= -\frac{1}{2}x^4\cos x^2 + x^2\sin x^2 - 2\int x\sin x^2\,dx \\
&= -\frac{1}{2}x^4\cos x^2 + x^2\sin x^2 + \cos x^2 + C \qquad \square
\end{aligned}
$$

5. $\displaystyle\int \sin(\ln x)\,dx$

Sol.

Let $t = \ln x$, $x = e^t$, $dx = e^t\,dt$.

$$
\begin{aligned}
\int \sin(\ln x)\,dx &= \int \sin t\cdot e^t\,dt \\
&= \int e^t\sin t\,dt
\end{aligned}
$$

Let $u = e^t$, $du = e^t\,dt$.

Let $dv = \sin t\,dt$, $v = -\cos t$.

$$
\begin{aligned}
\int \sin(\ln x)\,dx &= -e^t\cos t - \int -\cos t\cdot e^t\,dt \\
&= -e^t\cos t + \int e^t\cos t\,dt
\end{aligned}
$$

Let $u = e^t$, $du = e^t\,dt$.

Let $dv = \cos t\,dt$, $v = \sin t$.

$$
\begin{aligned}
\int \sin(\ln x)\,dx &= -e^t\cos t + e^t\sin t - \int e^t\sin t\,dt \\
&= \frac{1}{2}e^t(\sin t - \cos t) + C \\
&= \frac{1}{2}x\left[\sin(\ln x) - \cos(\ln x)\right] + C \qquad \square
\end{aligned}
$$

6. $\displaystyle\int e^x \cos x \, dx$

Sol.

Let $u = e^x$, $du = e^x \, dx$.

Let $dv = \cos x \, dx$, $v = \sin x$.

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Let $u = e^x$, $du = e^x \, dx$.

Let $dv = \sin x \, dx$, $v = -\cos x$.

$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx \\ &= \frac{1}{2} e^x (\sin x + \cos x) + C \qquad \square\end{aligned}$$

7. $\displaystyle\int x^5 e^{x^2} \, dx$

Sol.

Let $u = x^4$, $du = 4x^3 \, dx$.

Let $dv = xe^{x^2} \, dx$, $v = \dfrac{1}{2} e^{x^2}$.

$$\begin{aligned}\int x^5 e^{x^2} \, dx &= \frac{1}{2} x^4 e^{x^2} - \int 2x^3 e^{x^2} \, dx \\ &= \frac{1}{2} x^4 e^{x^2} - 2 \int x^3 e^{x^2} \, dx\end{aligned}$$

Let $u = x^2$, $du = 2x \, dx$.

Let $dv = xe^{x^2} \, dx$, $v = \dfrac{1}{2} e^{x^2}$.

$$\begin{aligned}\int x^5 e^{x^2} \, dx &= \frac{1}{2} x^4 e^{x^2} - 2 \left(\frac{1}{2} x^2 e^{x^2} - \int \frac{1}{2} \cdot 2xe^{x^2} \, dx \right) \\ &= \frac{1}{2} x^4 e^{x^2} - x^2 e^{x^2} + 2 \int xe^{x^2} \, dx \\ &= \frac{1}{2} x^4 e^{x^2} - x^2 e^{x^2} + e^{x^2} + C \\ &= \frac{1}{2} e^{x^2} (x^4 - 2x^2 + 2) + C \qquad \square\end{aligned}$$

8. $\displaystyle\int e^{2x} \cos x \, dx$

Sol.

Let $u = e^{2x}$, $du = 2e^{2x} \, dx$.

Let $dv = \cos x \, dx$, $v = \sin x$.

$$\int e^{2x} \cos x \, dx = e^{2x} \sin x - 2 \int e^{2x} \sin x \, dx$$

Let $u = e^{2x}$, $du = 2e^{2x} \, dx$.

Let $dv = \sin x \, dx$, $v = -\cos x$.

$$\begin{aligned}\int e^{2x} \cos x \, dx &= e^{2x} \sin x - 2 \left(-e^{2x} \cos x - 2 \int -e^{2x} \cos x \, dx \right) \\ &= e^{2x} \sin x + 2e^{2x} \cos x - 4 \int e^{2x} \cos x \, dx \\ &= \frac{1}{5} e^{2x} (\sin x + 2 \cos x) + C \qquad \square\end{aligned}$$

9. $\int (x^2+7x-5)\cos 2x\,dx$

Sol.

Let $u = x^2+7x-5,\ du = (2x+7)\,dx$.

Let $dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x$.

$$\begin{aligned}\int (x^2+7x-5)\cos 2x\,dx &= \frac{1}{2}(x^2+7x-5)\sin 2x - \int \frac{1}{2}\sin 2x(2x+7)\,dx \\ &= \frac{1}{2}(x^2+7x-5)\sin 2x - \frac{1}{2}\int 2x\sin 2x\,dx - \frac{7}{2}\int \sin 2x\,dx \\ &= \frac{1}{2}(x^2+7x-5)\sin 2x - \int x\sin 2x\,dx + \frac{7}{4}\cos 2x\end{aligned}$$

Let $u = x,\ du = dx$.

Let $dv = \sin 2x\,dx,\ v = -\frac{1}{2}\cos 2x$.

$$\begin{aligned}\int (x^2+7x-5)\cos 2x\,dx &= \frac{1}{2}(x^2+7x-5)\sin 2x - \left(-\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\,dx\right) + \frac{7}{4}\cos 2x \\ &= \frac{1}{2}(x^2+7x-5)\sin 2x + \frac{1}{2}x\cos 2x - \frac{1}{2}\int \cos 2x\,dx + \frac{7}{4}\cos 2x \\ &= \frac{1}{2}(x^2+7x-5)\sin 2x + \frac{2}{4}x\cos 2x - \frac{1}{4}\sin 2x + \frac{7}{4}\cos 2x + C \\ &= \frac{1}{2}(x^2+7x-5)\sin 2x + \frac{1}{4}(2x+7)\cos 2x - \frac{1}{4}\sin 2x + C \qquad \square\end{aligned}$$

10. $\int e^{ax}\sin bx\,dx$

Sol.

Let $u = e^{ax},\ du = ae^{ax}\,dx$.

Let $dv = \sin bx\,dx,\ v = -\frac{1}{b}\cos bx$.

$$\int e^{ax}\sin bx\,dx = -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx$$

Let $u = e^{ax},\ du = ae^{ax}\,dx$.

Let $dv = \cos bx\,dx,\ v = \frac{1}{b}\sin bx$.

$$\begin{aligned}\int e^{ax}\sin bx\,dx &= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\left(\frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx\right) \\ &= -\frac{1}{b}e^{ax}\cos bx + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx \\ &= \frac{b^2}{b^2+a^2}\left(\frac{a}{b^2}e^{ax}\sin bx - \frac{1}{b}e^{ax}\cos bx\right) + C \\ &= \frac{b^2}{a^2+b^2}\cdot\frac{e^{ax}}{b^2}\left(a\sin bx - b\cos bx\right) + C \\ &= \frac{e^{ax}}{b^2+a^2}\left(a\sin bx - b\cos bx\right) + C \qquad \square\end{aligned}$$