

## 8.4 三角函数的积化和差

(选择题)

1.  $\sin 52\frac{1}{2}^\circ \cdot \cos 7\frac{1}{2}^\circ = ?$

解:

$$\begin{aligned}\sin 52\frac{1}{2}^\circ \cdot \cos 7\frac{1}{2}^\circ &= \frac{1}{2} \left[ \sin \left( 52\frac{1}{2}^\circ + 7\frac{1}{2}^\circ \right) + \sin \left( 52\frac{1}{2}^\circ - 7\frac{1}{2}^\circ \right) \right] \\&= \frac{1}{2} (\sin 60^\circ + \sin 45^\circ) \\&= \frac{1}{2} \left( \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \right) \\&= \frac{\sqrt{3} + \sqrt{2}}{4}\end{aligned}$$

(作答题)

1. 试证  $\sec \left( \frac{\pi}{4} + \theta \right) \sec \left( \frac{\pi}{4} - \theta \right) = 2 \sec 2\theta$ 。

解:

$$\begin{aligned}\sec \left( \frac{\pi}{4} + \theta \right) \sec \left( \frac{\pi}{4} - \theta \right) &= \frac{1}{\cos \left( \frac{\pi}{4} + \theta \right) \cos \left( \frac{\pi}{4} - \theta \right)} \\&= \frac{1}{\frac{\cos \left( \frac{\pi}{4} + \theta + \frac{\pi}{4} - \theta \right) + \cos \left( \frac{\pi}{4} + \theta - \frac{\pi}{4} + \theta \right)}{2}} \\&= \frac{2}{\cos \frac{\pi}{2} + \cos 2\theta} \\&= \frac{2}{0 + \cos 2\theta} \\&= 2 \sec 2\theta\end{aligned}$$

2. 试证  $8 \cos \theta \cos 2\theta \cos 3\theta - \frac{\sin 7\theta}{\sin \theta} = 1$ 。解:

$$\begin{aligned}8 \cos \theta \cos 2\theta \cos 3\theta - \frac{\sin 7\theta}{\sin \theta} &= \frac{8 \sin \theta \cos \theta \cos 2\theta \cos 3\theta - \sin 7\theta}{\sin \theta} \\&= \frac{4 \sin 2\theta \cos 2\theta \cos 3\theta - \sin 7\theta}{\sin \theta} \\&= \frac{2 \sin 4\theta \cos 3\theta - \sin 7\theta}{\sin \theta} \\&= \frac{\sin 7\theta + \sin \theta - \sin 7\theta}{\sin \theta} \\&= \frac{\sin \theta}{\sin \theta} \\&= 1\end{aligned}$$

3. (a) 证明  $\cot(\theta + 15^\circ) - \tan(\theta - 15^\circ) = \frac{4 \cos 2\theta}{1 + 2 \sin 2\theta}$ ;

解:

$$\begin{aligned}
 \cot(\theta + 15^\circ) - \tan(\theta - 15^\circ) &= \frac{\cos(\theta + 15^\circ)}{\sin(\theta + 15^\circ)} - \frac{\sin(\theta - 15^\circ)}{\cos(\theta - 15^\circ)} \\
 &= \frac{\cos(\theta + 15^\circ) \cos(\theta - 15^\circ) - \sin(\theta + 15^\circ) \sin(\theta - 15^\circ)}{\sin(\theta + 15^\circ) \cos(\theta - 15^\circ)} \\
 &= \frac{\frac{1}{2}(\cos 2\theta + \cos 30^\circ) + \frac{1}{2}(\cos 2\theta - \cos 30^\circ)}{\frac{1}{2}(\sin 2\theta + \sin 30^\circ)} \\
 &= \frac{\cos 2\theta + \cos 2\theta}{\sin 2\theta + \sin 30^\circ} \\
 &= \frac{2 \cos 2\theta}{2 \sin 2\theta + 1} \\
 &= \frac{4 \cos 2\theta}{1 + 2 \sin 2\theta}
 \end{aligned}$$

(b) 据此, 设  $\theta = 60^\circ$ , 证明  $\tan 75^\circ = 2 + \sqrt{3}$ 。解:

$$\begin{aligned}
 \cot(60^\circ + 15^\circ) - \tan(60^\circ - 15^\circ) &= \frac{4 \cos 120^\circ}{1 + 2 \sin 120^\circ} \\
 \cot 75^\circ - \tan 45^\circ &= \frac{4 \left(-\frac{1}{2}\right)}{1 + 2 \left(\frac{\sqrt{3}}{2}\right)} \\
 \cot 75^\circ - 1 &= \frac{-2}{1 + \sqrt{3}} \\
 \cot 75^\circ &= \frac{-2 + 1 + \sqrt{3}}{1 + \sqrt{3}} \\
 \cot 75^\circ &= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \\
 \tan 75^\circ &= \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \\
 &= \frac{3 + 1 + 2\sqrt{3}}{3 - 1} \\
 &= 2 + \sqrt{3}
 \end{aligned}$$

## 8.5 三角函数的和差化积

(选择题)

1.  $\cos^2 \theta + \cos^2 (\theta + 1^\circ) + \cos^2 (\theta + 2^\circ) + \cdots + \cos^2 (\theta + 179^\circ)$  的值是

解:

$$\cos^2(\theta + 90^\circ) = (-\sin \theta)^2 = \sin^2 \theta$$

$$\cos^2(\theta + 91^\circ) = (-\sin(\theta + 1^\circ))^2 = \sin^2(\theta + 1^\circ)$$

$$\vdots$$

$$\cos^2(\theta + 179^\circ) = (-\sin(\theta + 89^\circ))^2 = \sin^2(\theta + 89^\circ)$$

$$\cos^2 \theta + \cos^2(\theta + 90^\circ) = \cos^2 \theta + \sin^2 \theta = 1$$

$$\cos^2(\theta + 1^\circ) + \cos^2(\theta + 91^\circ) = \cos^2(\theta + 1^\circ) + \sin^2(\theta + 1^\circ) = 1$$

$$\vdots$$

$$\cos^2(\theta + 89^\circ) + \cos^2(\theta + 179^\circ) = \cos^2(\theta + 89^\circ) + \sin^2(\theta + 89^\circ) = 1$$

$\therefore$  原式  $= 1 \times 90 = 90$

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2.  $\cos^2 2x + 2 \sin^2 x$  的极小值是

解:

$$\frac{d}{dx}(\cos^2 2x + 2 \sin^2 x) = -4 \sin 2x \cos 2x + 4 \sin x \cos x$$

$$= -2 \sin 4x + 2 \sin 2x$$

$$= -2(\sin 4x - \sin 2x)$$

$$= -4 \cos 3x \sin x$$

$$\cos 3x \sin x = 0$$

$$\cos 3x = 0 \quad \text{or} \quad \sin x = 0$$

$$3x = \frac{\pi}{2} + k\pi \quad \text{or} \quad x = k\pi$$

$$x = \frac{\pi}{6} + \frac{k\pi}{3} \quad \text{or} \quad x = k\pi$$

When  $k = 0$ ,  $x = \frac{\pi}{6}$  or  $x = 0$ ; When  $k = 1$ ,  $x = \frac{\pi}{2}$  or  $x = \pi$

$$\frac{d^2}{dx^2}(\cos^2 2x + 2 \sin^2 x) = -8 \cos 4x + 4 \cos 2x$$

When  $x = \frac{\pi}{6}$ ,  $\frac{d^2}{dx^2} = 6$ ; When  $x = 0$ ,  $\frac{d^2}{dx^2} = -4$ ; When  $x = \frac{\pi}{2}$ ,  $\frac{d^2}{dx^2} = -12$ ; When  $x = \pi$ ,  $\frac{d^2}{dx^2} = 6$

Hence, the minimum value is  $\frac{3}{4}$  when  $x = \frac{\pi}{6}$ .

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3. 化简  $\cos \theta + \cos \left( \theta + \frac{2\pi}{3} \right) + \cos \left( \theta + \frac{4\pi}{3} \right)$ 。

解:

$$\begin{aligned}\cos \theta + \cos \left( \theta + \frac{2\pi}{3} \right) + \cos \left( \theta + \frac{4\pi}{3} \right) &= \cos \theta + 2 \cos \left( \frac{\theta + \frac{2\pi}{3} + \theta + \frac{4\pi}{3}}{2} \right) \cos \left( -\frac{\pi}{3} \right) \\ &= \cos \theta + \cos (\theta + \pi) \\ &= \cos \theta - \cos \theta \\ &= 0\end{aligned}$$

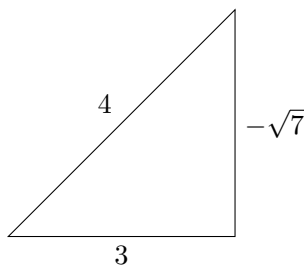
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4. 已知  $\sin \alpha - \sin \beta = -\frac{1}{2}$ ,  $\cos \alpha - \cos \beta = \frac{1}{2}$ , 且  $\alpha$  与  $\beta$  都是锐角, 求  $\tan(\alpha - \beta)$  的值。

解:

$$\begin{aligned}\sin \alpha - \sin \beta &= -\frac{1}{2} \\ \sin^2 \alpha + \sin^2 \beta - 2 \sin \alpha \sin \beta &= \frac{1}{4} \\ \cos \alpha - \cos \beta &= \frac{1}{2} \\ \cos^2 \alpha + \cos^2 \beta - 2 \cos \alpha \cos \beta &= \frac{1}{4} \\ 2 - 2 \sin \alpha \sin \beta - 2 \cos \alpha \cos \beta &= \frac{1}{2} \\ 2 - 2 \cos(\alpha - \beta) &= \frac{1}{2} \\ \cos(\alpha - \beta) &= \frac{3}{4}\end{aligned}$$

$\therefore \sin \alpha - \sin \beta < 0, \therefore \alpha < \beta \implies \alpha - \beta < 0 \implies \alpha - \beta$  is in the fourth quadrant.



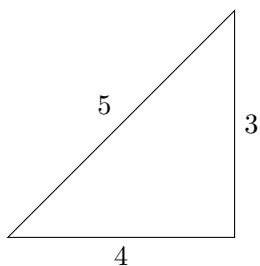
$$\tan(\alpha - \beta) = -\frac{\sqrt{7}}{3}$$

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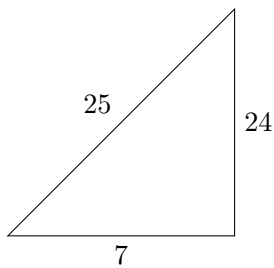
5. 已知  $\sin \alpha + \sin \beta = \frac{1}{4}$ ,  $\cos \alpha + \cos \beta = \frac{1}{3}$ , 求  $\tan(\alpha + \beta)$  的值。

解:

$$\begin{aligned}\sin \alpha + \sin \beta &= \frac{1}{4} \\ 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right) &= \frac{1}{4} \\ \cos \alpha + \cos \beta &= \frac{1}{3} \\ 2 \cos \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right) &= \frac{1}{3} \\ \frac{\sin \left( \frac{\alpha + \beta}{2} \right)}{\cos \left( \frac{\alpha + \beta}{2} \right)} &= \frac{3}{4} \\ \tan \left( \frac{\alpha + \beta}{2} \right) &= \frac{3}{4}\end{aligned}$$



$$\begin{aligned}\sin(\alpha + \beta) &= 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha + \beta}{2} \right) \\ &= 2 \times \frac{3}{5} \times \frac{4}{5} \\ &= \frac{24}{25}\end{aligned}$$



$$\tan(\alpha + \beta) = \frac{24}{7}$$



6. 若  $\alpha + \beta = \frac{\pi}{3}$  且  $y = \cos^2 \alpha + \cos^2 \beta$ , 则  $y$  的极大值是

解:

$$\begin{aligned} y &= \cos^2 \alpha + \cos^2 \beta \\ &= (\cos \alpha + \cos \beta)^2 - 2 \cos \alpha \cos \beta \\ &= \left( 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \right)^2 - \cos(\alpha + \beta) - \cos(\alpha - \beta) \\ &= \left( 2 \cos \frac{\pi}{6} \cos \frac{\alpha - \beta}{2} \right)^2 - \cos \frac{\pi}{3} - \cos(\alpha - \beta) \\ &= \left( \sqrt{3} \cos \frac{\alpha - \beta}{2} \right)^2 - \frac{1}{2} - \cos(\alpha - \beta) \\ &= 3 \cos^2 \frac{\alpha - \beta}{2} - \frac{1}{2} - \cos(\alpha - \beta) \\ &= 3 \left( \frac{1 + \cos(\alpha - \beta)}{2} \right) - \frac{1}{2} - \cos(\alpha - \beta) \\ &= \frac{3}{2} + \frac{3}{2} \cos(\alpha - \beta) - \frac{1}{2} - \cos(\alpha - \beta) \\ &= 1 + \frac{1}{2} \cos(\alpha - \beta) \end{aligned}$$

Since  $-1 \leq \cos(\alpha - \beta) \leq 1$ , the maximum value of  $y$  is  $\frac{3}{2}$  when  $\cos(\alpha - \beta) = 1$ . ■

7. 已知  $\cos 2\theta = 5 \sin \theta - 2$ , 求  $\cos 2\theta + \sin \theta$  的值。

解:

$$\begin{aligned} \cos 2\theta &= 5 \sin \theta - 2 \\ 1 - 2 \sin^2 \theta &= 5 \sin \theta - 2 \\ 2 \sin^2 \theta + 5 \sin \theta - 3 &= 0 \\ (2 \sin \theta - 1)(\sin \theta + 3) &= 0 \\ \sin \theta &= \frac{1}{2} \quad \text{or} \quad \sin \theta = -3 \text{ (rejected)} \\ \cos 2\theta + \sin \theta &= 5 \sin \theta - 2 + \sin \theta \\ &= 6 \sin \theta - 2 = 1 \end{aligned} \quad \blacksquare$$

8. 化简  $\frac{\sin 13x + \sin 7x}{\cos 13x + \cos 7x}$ 。

解:

$$\begin{aligned} \frac{\sin 13x + \sin 7x}{\cos 13x + \cos 7x} &= \frac{2 \sin 10x \cos 3x}{2 \cos 10x \cos 3x} \\ &= \frac{\sin 10x}{\cos 10x} \\ &= \tan 10x \end{aligned} \quad \blacksquare$$

(作答题)

1. 设  $A + B + C = 180^\circ$ , 证明  $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$ 。

解:

$$\begin{aligned} A + B + C &= 180^\circ \\ A + B &= 180^\circ - C \\ \frac{A + B}{2} &= 90^\circ - \frac{C}{2} \\ \sin \frac{A + B}{2} &= \cos \frac{C}{2} \\ \sin A + \sin B + \sin C &= 2 \sin \frac{A + B}{2} \cos \frac{A - B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2} \\ &= 2 \cos \frac{C}{2} \cos \frac{A - B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2} \\ &= 2 \cos \frac{C}{2} \left( \cos \frac{A - B}{2} + \sin \frac{C}{2} \right) \\ &= 2 \cos \frac{C}{2} \left( \cos \frac{A - B}{2} + \sin \left( 90^\circ - \frac{A + B}{2} \right) \right) \\ &= 2 \cos \frac{C}{2} \left( \cos \frac{A - B}{2} + \cos \frac{A + B}{2} \right) \\ &= 2 \cos \frac{C}{2} \left( 2 \cos \frac{A}{2} \cos \frac{B}{2} \right) \\ &= 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \quad \blacksquare \end{aligned}$$

2. 若  $A + B + C = 180^\circ$ , 利用两角和正切公式, 试证:  $\tan A + \tan B + \tan C = \tan A \tan B \tan C$   
若  $\tan A = 1, \tan B = 2$ , 求  $\angle C$ 。

解:

$$\begin{aligned} A + B + C &= 180^\circ \\ A + B &= 180^\circ - C \\ \tan(A + B) &= \tan(180^\circ - C) \\ \tan(A + B) &= -\tan C \\ \frac{\tan A + \tan B}{1 - \tan A \tan B} &= -\tan C \\ \tan A + \tan B &= -\tan C + \tan A \tan B \tan C \\ \tan A + \tan B + \tan C &= \tan A \tan B \tan C \quad \blacksquare \end{aligned}$$

$$\begin{aligned} \tan A + \tan B + \tan C &= \tan A \tan B \tan C \\ 1 + 2 + \tan C &= 2 \tan C \\ 3 &= \tan C \\ \angle C &= \arctan 3 = 71.57^\circ \quad \blacksquare \end{aligned}$$

3. 若  $A + B + C = 180^\circ$ , 试证  $\cos(B + C - A) + \cos(C + A - B) + \cos(A + B - C) = 1 + 4 \cos A \cos B \cos C$ 。

解:

$$\begin{aligned}
 & \cos(B + C - A) + \cos(C + A - B) + \cos(A + B - C) \\
 &= \cos(180^\circ - A - A) + \cos(180^\circ - B - B) + \cos(180^\circ - C - C) \\
 &= -\cos 2A - \cos 2B - \cos 2C \\
 &= -(2 \cos^2 A - 1) - 2 \cos(B + C) \cos(B - C) \\
 &= -2 \cos^2 A + 1 - 2 \cos(180^\circ - A) \cos(B - C) \\
 &= -2 \cos^2 A + 1 + 2 \cos A \cos(B - C) \\
 &= 1 - 2 \cos A [\cos A - \cos(B + C)] \\
 &= 1 - 2 \cos A [\cos(180^\circ - (B + C)) - \cos(B + C)] \\
 &= 1 + 2 \cos A [\cos(B + C) + \cos(B + C)] \\
 &= 1 + 4 \cos A \cos B \cos C
 \end{aligned}$$

■

4. 证明  $\sin \theta + \sin 2\theta + \sin 4\theta - \sin 7\theta = 4 \sin \frac{3\theta}{2} \sin 3\theta$ 。(Faulty Problem Statement)

解:

$$\begin{aligned}
 \sin \theta + \sin 2\theta + \sin 4\theta - \sin 7\theta &= \sin \theta - 7 \sin \theta + \sin 4\theta + \sin 2\theta \\
 &= -2 \cos 4\theta \sin 3\theta + 2 \sin 3\theta \cos \theta \\
 &= -2 \sin 3\theta (\cos 4\theta - \cos \theta) \\
 &= 4 \sin 3\theta \sin \frac{5\theta}{2} \sin \frac{3\theta}{2}
 \end{aligned}$$

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5. 如果  $A + B + C = 180^\circ$ , 试证  $\frac{1 + \cos A - \cos B + \cos C}{1 + \cos A + \cos B - \cos C} = \tan \frac{B}{2} \cot \frac{C}{2}$ 。

解:

$$\begin{aligned}
 A + B + C &= 180^\circ \\
 A + B &= 180^\circ - C \\
 \cos(A + B) &= \cos(180^\circ - C) \\
 \cos(A + B) &= -\cos C \\
 \frac{1 + \cos A - \cos B + \cos C}{1 + \cos A + \cos B - \cos C} &= \frac{2 \sin^2 \frac{B}{2} + 2 \cos \frac{A + C}{2} \cos \frac{A - C}{2}}{2 \sin^2 \frac{C}{2} + 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}} \\
 &= \frac{2 \sin^2 \frac{B}{2} + 2 \cos \left(90^\circ - \frac{B}{2}\right) \cos \left(\frac{A - C}{2}\right)}{2 \sin^2 \frac{C}{2} + 2 \cos \left(90^\circ - \frac{C}{2}\right) \cos \left(\frac{A - B}{2}\right)} \\
 &= \frac{2 \sin^2 \frac{B}{2} + 2 \sin \frac{B}{2} \cos \left(\frac{A - C}{2}\right)}{2 \sin^2 \frac{C}{2} + 2 \sin \frac{C}{2} \cos \left(\frac{A - B}{2}\right)}
 \end{aligned}$$

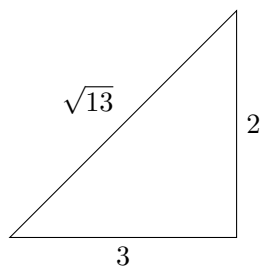


$$\begin{aligned}
&= \frac{\sin \frac{B}{2} \left( \sin \frac{B}{2} + \cos \left( \frac{A-C}{2} \right) \right)}{\sin \frac{C}{2} \left( \sin \frac{C}{2} + \cos \left( \frac{A-B}{2} \right) \right)} \\
&= \frac{\sin \frac{B}{2} \left( \sin \left( 90^\circ - \frac{A+C}{2} \right) + \cos \left( \frac{A-C}{2} \right) \right)}{\sin \frac{C}{2} \left( \sin \left( 90^\circ - \frac{A+B}{2} \right) + \cos \left( \frac{A-B}{2} \right) \right)} \\
&= \frac{\sin \frac{B}{2} \left( \cos \left( \frac{A+C}{2} \right) + \cos \left( \frac{A-C}{2} \right) \right)}{\sin \frac{C}{2} \left( \cos \left( \frac{A+B}{2} \right) + \cos \left( \frac{A-B}{2} \right) \right)} \\
&= \frac{\sin \frac{B}{2} \left( 2 \cos \frac{A}{2} \cos \frac{C}{2} \right)}{\sin \frac{C}{2} \left( 2 \cos \frac{A}{2} \cos \frac{B}{2} \right)} \\
&= \frac{\sin \frac{B}{2} \cos \frac{C}{2}}{\sin \frac{C}{2} \cos \frac{B}{2}} \\
&= \tan \frac{B}{2} \cot \frac{C}{2}
\end{aligned}$$

■

6. 已知  $\sin A - \sin B = \frac{1}{2}$  及  $\cos A - \cos B = -\frac{1}{3}$ , 试求  $\sin(A+B)$ 。解:

$$\begin{aligned}
&\sin A - \sin B = \frac{1}{2} \\
2 \cos \left( \frac{A+B}{2} \right) \sin \left( \frac{A-B}{2} \right) &= \frac{1}{2} \\
\cos A - \cos B &= -\frac{1}{3} \\
-2 \sin \left( \frac{A+B}{2} \right) \sin \left( \frac{A-B}{2} \right) &= -\frac{1}{3} \\
\frac{\sin \left( \frac{A+B}{2} \right)}{\cos \left( \frac{A+B}{2} \right)} &= \frac{2}{3} \\
\tan \left( \frac{A+B}{2} \right) &= \frac{2}{3}
\end{aligned}$$



$$\begin{aligned}\sin(A+B) &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A+B}{2}\right) \\ &= 2 \times \frac{2}{\sqrt{13}} \times \frac{3}{\sqrt{13}} \\ &= \frac{12}{13}\end{aligned}$$

7. 证  $\frac{\sin \alpha + \sin 3\alpha + \sin 5\alpha + \sin 7\alpha}{\cos \alpha + \cos 3\alpha + \cos 5\alpha + \cos 7\alpha} = \tan 4\alpha$ 。

解:

$$\begin{aligned}\frac{\sin \alpha + \sin 3\alpha + \sin 5\alpha + \sin 7\alpha}{\cos \alpha + \cos 3\alpha + \cos 5\alpha + \cos 7\alpha} &= \frac{2\sin 4\alpha \cos 3\alpha + 2\sin 4\alpha \cos \alpha}{2\cos 4\alpha \cos 3\alpha + 2\cos 4\alpha \cos \alpha} \\ &= \frac{2\sin 4\alpha(\cos 3\alpha + \cos \alpha)}{2\cos 4\alpha(\cos 3\alpha + \cos \alpha)} \\ &= \tan 4\alpha\end{aligned}$$

8. 试证  $2\sin^2 3\theta - 2\sin^2 \theta = \cos 2\theta - \cos 6\theta$ 。

以  $\theta = \frac{\pi}{10}$  代入上式, 证明  $\sin \frac{3\pi}{10} - \sin \frac{\pi}{10} = \frac{1}{2}$ 。

[ 提示: 用  $\cos x = \sin\left(\frac{\pi}{2} - x\right)$  ]

解:

$$\begin{aligned}2\sin^2 3\theta - 2\sin^2 \theta &= 2\left(\frac{1 - \cos 6\theta}{2}\right) - 2\left(\frac{1 - \cos 2\theta}{2}\right) \\ &= 1 - \cos 6\theta - 1 + \cos 2\theta \\ &= \cos 2\theta - \cos 6\theta\end{aligned}$$

9. 在  $\triangle ABC$  中, 试证明  $\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C}$ 。

若在这三角形中,  $a+b=2c$  及  $A-B=90^\circ$ ,

(a) 试证明  $\sin \frac{C}{2} = \frac{1}{2\sqrt{2}}$ ;

(b) 求  $A, B$  及  $C$  的值。

10. 若  $\sin 6x = \frac{1}{\sin x}$ , 证明  $\sin 5x \sin 4x - \sin 3x \sin 2x - \sin 8x \sin x = 1$ 。

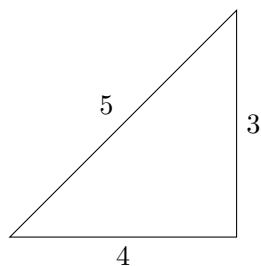
11. 在  $\triangle ABC$  中, 试证  $4\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \cos A + \cos B + \cos C - 1$ 。

12. 已知  $\sin \alpha + \sin \beta = \frac{1}{4}$ ,  $\cos \alpha + \cos \beta = \frac{1}{3}$ , 求  $\tan(\alpha + \beta)$  的值。

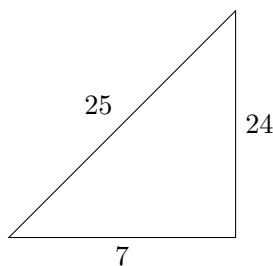
解:

$$\begin{aligned}\sin \alpha + \sin \beta &= \frac{1}{4} \\ 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) &= \frac{1}{4}\end{aligned}$$

$$\begin{aligned}\cos \alpha + \cos \beta &= \frac{1}{3} \\ 2 \cos \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha - \beta}{2} \right) &= \frac{1}{3} \\ \frac{\sin \left( \frac{\alpha + \beta}{2} \right)}{\cos \left( \frac{\alpha + \beta}{2} \right)} &= \frac{3}{4} \\ \tan \left( \frac{\alpha + \beta}{2} \right) &= \frac{3}{4}\end{aligned}$$



$$\begin{aligned}\sin(\alpha + \beta) &= 2 \sin \left( \frac{\alpha + \beta}{2} \right) \cos \left( \frac{\alpha + \beta}{2} \right) \\ &= 2 \times \frac{3}{5} \times \frac{4}{5} \\ &= \frac{24}{25}\end{aligned}$$



$$\tan(\alpha + \beta) = \frac{24}{7}$$

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13. 在  $\triangle ABC$  中, 如果  $\sin A$ 、 $\sin B$  及  $\sin C$  成等差数列, 试证  $\cot \frac{A}{2} \cot \frac{C}{2} = 3$ 。

解:

$$\begin{aligned}\sin B &= \frac{\sin A + \sin C}{2} \\ \sin[180^\circ - (A + C)] &= \frac{\sin A + \sin C}{2} \\ \sin(A + C) &= \frac{\sin A + \sin C}{2} \\ \sin(A + C) &= \sin \frac{A + C}{2} \cos \frac{A - C}{2} \\ 2 \sin \frac{A + C}{2} \cos \frac{A - C}{2} &= \sin \frac{A + C}{2} \cos \frac{A - C}{2}\end{aligned}$$

$$\begin{aligned}
2 \cos \frac{A+C}{2} &= \cos \frac{A-C}{2} \\
\cot \frac{A}{2} \cot \frac{C}{2} &= \frac{\cos \frac{A}{2} \cos \frac{C}{2}}{\sin \frac{A}{2} \sin \frac{C}{2}} = \frac{\cos \frac{A+C}{2} + \cos \frac{A-C}{2}}{\cos \frac{A-C}{2} - \cos \frac{A+C}{2}} \\
&= \frac{\cos \frac{A+C}{2} + 2 \cos \frac{A+C}{2}}{2 \cos \frac{A+C}{2} - \cos \frac{A+C}{2}} \\
&= \frac{3 \cos \frac{A+C}{2}}{\cos \frac{A+C}{2}} = 3
\end{aligned}$$

■

14. 若  $A + B + C = 180^\circ$  且  $A \neq 90^\circ$ , 证明  $\frac{\sin B}{\sin C} = \frac{\sin A \cos B - \sin C}{\sin A \cos C - \sin B}$ 。