Mathematics

Senior 2 Part II

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Actual time spent: 21 days

Introduction

Why this book?

Right after I finished my Senior 2 Part I, I started this book without hesitation. Unlike the last book, where I started the book with little to no idea about LaTEX, this time I am able to finish the book in less than a month without spending too much time on formatting.

Since all the topics in this textbook are fairly new to me, I decided to translate every single word from the textbook and include it in this book, which ended up costing me a lot of time. In short, this is an English version of the textbook. However, I decided not to include sample questions in this book as I do not think it is necessary.

Now that this book is finished, I can focus on SPM Mathematics and Additional Mathematics for the next few months. In any case, self-study is the best and most efficient way to learn.

The main reason I decided to write this is because school is too slow for me. I am able to finish the entire book in less than a month, but in school I can only do it in half a year, if not more. Since not everyone is as interested in math as I am, I can understand why school has to be slow. But for me, it's just too slow. If I' am able to finish the entire book in just a few weeks, why should I have to wait half a year to finish it?

Disclaimer

This book is just my own solution and notes to the textbook, all the solutions it is not guaranteed to be correct (trust me, there are a bunch of stupid mistakes), and there might be some missing stuff that I've forgotten to add into it. I am not responsible for any consequences caused by using this book. And please, don't copy my solutions, it's not going to help you in the long run.

Acknowledgements

I would like to thank myself for wasting my entire month of holiday to complete this book. I would also like to thank my parents for sponsoring the printing of this book. Special thanks to Singapore National Library for providing me a cozy place to study.

Contents

Chapter 18

Statistics

18.1 Basic Concepts

Statistics mainly study how to collect, organize, summarize, and interpret data. It is a branch of mathematics that deals with the collection, analysis, interpretation, and presentation of data. It is used to answer questions about the data and to make decisions based on the data.

Population and Sample

In statistics, a population is the entire group of individuals that we are studying, and the units that form a population are called individuals or elements. A sample is a subset of the population. The number of elements in a sample is called the sample size. For example: select 20 of the 4,000 senior high school mathematics UEC exam papers and record their scores:

72	80	96	20	42
75	60	92	18	53
82	77	53	29	34
57	79	82	90	41

Here, the population is the 4,000 scores, each of which is an element of the population. The sample is the 20 scores, the sample size is 20.

Census and Sample Survey

The way of surveying can be divided into two types: census and sample survey. A census is a survey in which every element of the population is included in the sample. For example: national census. The data collected in a census is more accurate and reliable, but it is very expensive and time-consuming.

A sample survey is a survey in which only a part of the population is included in the sample. Researchers can use a sample survey to estimate the characteristics of the population. For example: a light bulb manufacturer produces a lot of light bulbs, thus it is impossible to test every single light bulb. The manufacturer can randomly select a sample of light bulbs and test them.

18.2 Data Processing

Data that are collected must be processed before they can be analyzed.

Frequency Distribution

When the possible values of a dataset are not too many, we can use a frequency distribution table to organize the data. The frequency distribution table is a table that shows the frequency of each value in a dataset. The frequency of a value is the number of times that value appears in the dataset.

When there are too many possible values, we must group the values into classes. Before grouping the values, we must first determine the range of the values, aka the difference between the largest and smallest values, then determine the number of classes. The number of classes should be determined according to the purpose of the study and the identity of the data. After classifying the data, the range of each group is called the class interval. Typically, the class interval is the same for all classes, and must be greater than the number of classes divided by the range of the data. After the number and interval of the classes are determined, we can arrange the frequency of each class in a frequency distribution table.

Take 100 sample from a population of some kind of component, their weight (in g), are as below:

1.36	1.49	1.43	1.41	1.37	1.40
1.32	1.42	1.47	1.39	1.41	1.36
1.40	1.34	1.42	1.42	1.45	1.35
1.42	1.39	1.44	1.42	1.39	1.42
1.42	1.30	1.34	1.42	1.37	1.36
1.37	1.34	1.37	1.37	1.44	1.45
1.32	1.48	1.40	1.45	1.39	1.46
1.39	1.53	1.36	1.48	1.40	1.39
1.38	1.40	1.36	1.45	1.50	1.43
1.38	1.43	1.41	1.48	1.39	1.45

1.37	1.37	1.39	1.45	1.31	1.41
1.44	1.44	1.42	1.47	1.35	1.36
1.39	1.40	1.38	1.35	1.38	1.43
1.42	1.42	1.42	1.40	1.41	1.37
1.46	1.36	1.37	1.27	1.37	1.38
1.42	1.34	1.43	1.42	1.41	1.41
1.44	1.48	1.55	1.39		

In the dataset above, the minimum value is 1.27 and the maximum value is 1.55.

$\therefore$ The range of the data is $1.55 - 1.27 = 0.28$.

If we classify the data into 10 classes, then the class interval must be greater than $\frac{0.28}{10} = 0.028$. Thus, we can use a class interval of 0.03.

Let the lower limit of the first class be 1.27, then the lower limit of the second class is $1.27 + 0.03 = 1.30$.

Since all the values in the dataset are of 2 decimal places, the upper limit of the first class is should be 1.29. By the same logic, we can get all the classes: $1.27 - 1.29$, $1.30 - 1.32$, $\cdots$, $1.54 - 1.56$.

Now we can arrange the data into the frequency distribution table:

Weight $m(g)$	Frequency
$1.27 - 1.29$	1
$1.30 - 1.32$	4
$1.33 - 1.35$	7
$1.36 - 1.38$	22
$1.39 - 1.41$	24
$1.42 - 1.44$	24
$1.45 - 1.47$	10
$1.48 - 1.50$	6
$1.51 - 1.53$	1
$1.54 - 1.56$	1

In the example above, we assume that the weight of the components is accurate to 2 decimal places. Hence, if a component has a weight of $1.443g$, it is rounded to $1.44g$, thus it belongs to the class $1.42 - 1.44$. Hence, the actual range of the first class $1.27 - 1.29$ is $1.265 \le m < 1.295$, written as $1.265 - 1.295$, while 1.265 and 1.295 are the boundaries of the first class, 1.265 is the lower boundary and 1.295 is the upper boundary. The mean of the lower boundary and upper boundary of a class is called the class midpoint. For example, the class midpoint of the first class is $\frac{1.265+1.295}{2} = 1.28$.

When we are analyzing the data data that have been classified into classes, the midpoint of each class is used as the representative value of the class. Thus, we should try our best to make the data-intensive place the group midpoint when choosing the class interval and boundaries, so that the data can be analyzed more precisely.

The distribution of frequency can be represented by a histogram or a frequency polygon.

The histogram is a row of continuous bars, the bottom side of each bar on the x-axis. For unclassified data, the bottom side of each bar is marked with the values, while the height of each bar is the frequency of the corresponding value. For classified data, the bottom side of each bar is marked with the boundaries of the corresponding class, while the area of each bar must be proportional to the frequency of the corresponding class. When the class interval of each class is the same, we can use the frequency of each class as the height of the bar.

Histogram
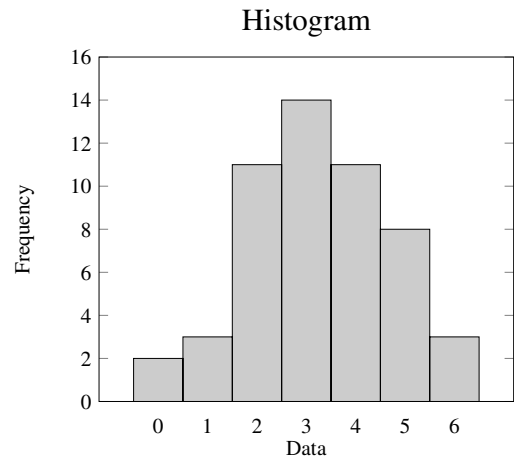
The frequency polygon is a continuous line graph, the x-axis is the midpoint of each class, and the y-axis is the frequency of each class. To draw a frequency polygon, we plot each point, including the point before the first class and the point after the last class that uses 0 as their frequency, and then connect the points with a continuous line.

Frequency Polygon

Histogram and Frequency Polygon
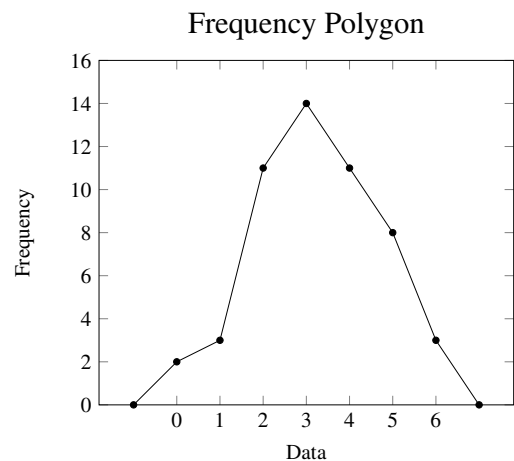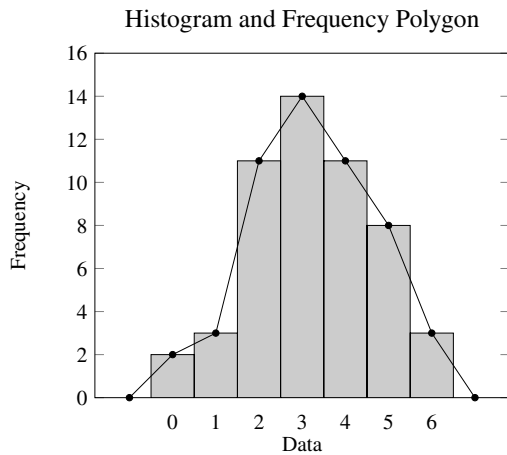
18.2.1 Practice 1

There are 105 students in a senior 3 art and commerce class. In a mock exam of UEC, their scores for Mathematics subject are as follows:

35	88	67	32	38	34	45
78	54	58	69	21	90	78
74	43	42	35	57	34	77
89	66	74	71	44	56	48
33	24	73	63	51	59	49
34	55	52	75	72	62	62
44	48	73	49	57	67	80
70	66	54	32	29	35	37
47	41	51	36	46	55	53
60	53	62	39	35	48	42
71	63	70	33	45	42	44
61	59	67	30	42	43	89
96	82	47	63	54	34	45
45	87	28	34	29	77	64
64	50	48	75	33	56	84

(a) Find the range of the data.

Sol.

$$\text{Max value} = 96$$
$$\text{Min value} = 21$$
$$\therefore \text{Range} = 96 - 21$$
$$= 75$$

(b) Group the data into 10 classes, draw a frequency distribution table, and find the upper and lower boundary and midpoint of each class.

Sol.

$$\text{Range} = 75$$
$$\text{Number of classes} = 10$$
$$\text{Class width} = \frac{75}{10}$$
$$= 7.5$$
$$\approx 8$$

Score	Lower	Upper	Mid	Freq.
21 - 28	20.5	28.5	24.5	3
29 - 36	28.5	36.5	32.5	18
37 - 44	36.5	44.5	40.5	13
45 - 52	44.5	52.5	48.5	17
53 - 60	52.5	60.5	56.5	15
61 - 68	60.5	68.5	64.5	14
69 - 76	68.5	76.5	72.5	12
77 - 84	76.5	84.5	80.5	7
85 - 92	84.5	92.5	88.5	5
93 - 100	92.5	100.5	96.5	1

(c) Construct a histogram and frequency polygon.

Sol.

Histogram and Frequency Polygon of Distribution of Mathematics Score
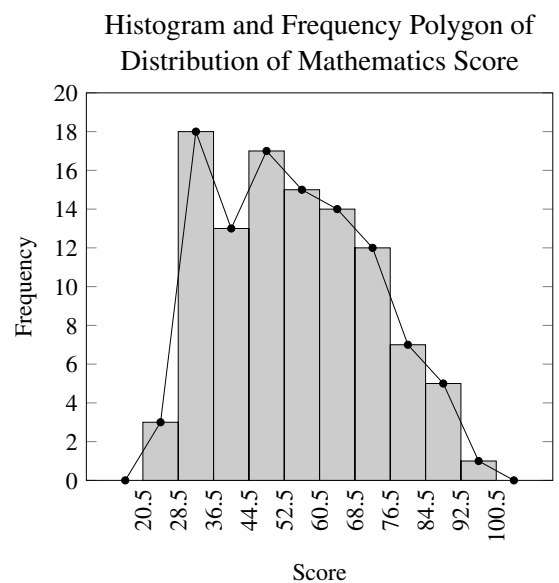
Cumulative Frequency Distribution

Summing up the frequency of each class, we obtain the cumulative frequency distribution. Use the upper boundary of each class as the x-axis, and the cumulative frequency as the y-axis, we can draw the cumulative frequency distribution by plotting each point including the point before the first class that uses 0 as its frequency and connect them together. If we split the x-axis and the higest point of the curve into 100 equal

parts, we get the percentage of the cumulative frequency distribution.

18.2.2 Practice 2

There are 155 students in a senior 3 art and commerce class, and the frequency distribution table of their average marks is shown below:

Average Mark	Frequency
50 - 55	3
55 - 60	8
60 - 65	25
65 - 70	38
70 - 75	46
75 - 80	19
80 - 85	12
85 - 90	4

(a) Make a cumulative frequency distribution table and draw a cumulative frequency polygon.

Sol.

Avg	Freq.	Lower Than	Cumm. Freq.
50 - 55	3	55	3
55 - 60	8	60	11
60 - 65	25	65	36
65 - 70	38	70	74
70 - 75	46	75	120
75 - 80	19	80	139
80 - 85	12	85	151
85 - 90	4	90	155

Cumulative Frequency Polygon of Distribution of Average Mark
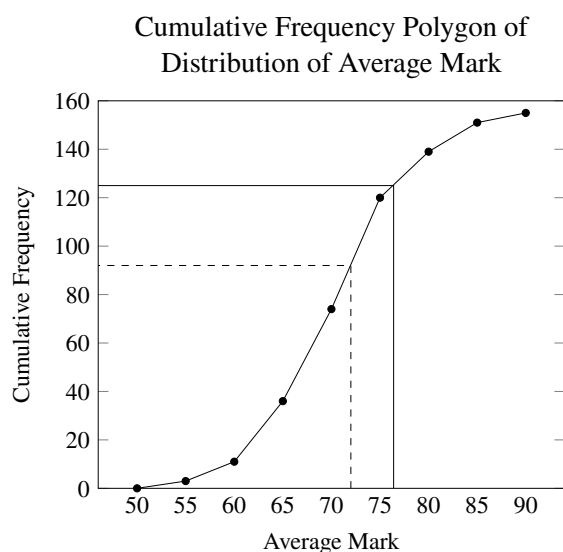
(b) If the average mark of a student is 72, find his rank in the class.

Sol.

In the graph above, we can see that there are approximately 92 students who have an average mark lower than 72. Therefore, the rank of the student is $155 - 92 = 63$.

(c) If the top 20% of the class are to be awarded a certificate, find the minimum average mark required for the certificate.

Sol.

$$\text{Top } 20\% = 20\% \cdot 155$$
$$= 31$$

Therefore, students with an average mark corresponding to cumulative frequency higher than 124 will be awarded a certificate.

In the graph above, The minimum average mark required for the certificate is 76.

18.2.3 Exercise 18.2

1. A company performed an ability test on 100 job seekers and the results are shown in the following table:

Score	8	7	6	5	4	3
Frequency	5	12	24	33	19	7

Construct a hustogram and a frequency polygon for the data above.

Sol.

Histogram and Frequency Polygon of Ability Test Score
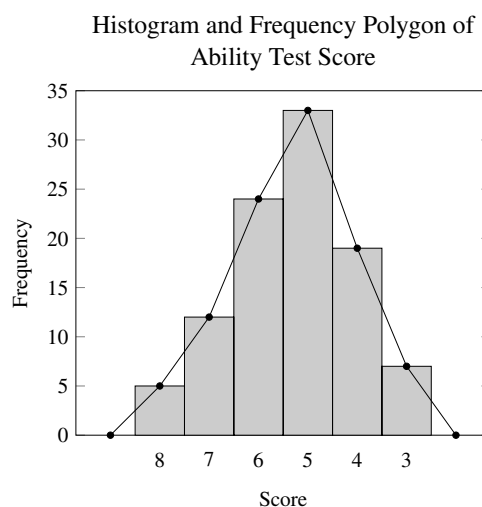
2. Take 120 ears of rice from a rice field, the length of each ear is measured (in *cm*) and the results are as fol-

lowing:

6.5	6.4	6.7	5.8	5.9	5.9
5.2	4.0	5.4	4.6	5.8	5.5
6.0	6.5	5.1	6.2	5.4	5.0
5.0	6.8	6.0	5.0	5.7	6.0
5.5	6.8	6.0	6.3	5.5	5.0
6.4	5.8	5.9	5.7	6.8	6.6
6.0	6.4	5.7	7.4	6.0	5.4
6.5	6.0	6.8	5.3	6.4	5.7
6.7	6.2	5.6	6.0	6.7	6.7
6.0	5.5	6.2	6.1	5.3	6.2
5.8	5.3	7.0	6.0	6.0	5.9
5.4	6.0	5.2	6.0	6.3	5.7
6.8	6.1	4.5	5.4	6.3	6.9
4.9	5.1	5.6	5.9	6.1	6.5
6.6	5.7	5.8	5.8	6.2	6.3
6.5	5.3	5.9	5.5	5.8	6.3
5.2	6.0	7.0	6.4	5.8	6.3
6.0	6.3	5.6	6.8	6.6	4.7
5.7	5.7	5.6	6.3	6.0	5.8
6.3	7.5	6.2	6.4	7.0	6.5

(a) Find the range of the dataset.

Sol.

$$\text{Min value} = 4.0$$
$$\text{Max value} = 7.5$$
$$\therefore \text{Range} = 7.5 - 4.0$$
$$= 3.5$$

(b) Group the data into 12 classes, make a frequency distribution table, find the upper and lower boundaries and midpoint of each class, and calculate the cumulative frequency.

Sol.

$$\text{Range} = 3.5$$
$$\text{Number of classes} = 12$$
$$\therefore \text{Class width} = \frac{3.5}{12}$$
$$= \frac{3.5}{12}$$
$$\approx 0.3$$

Weight	Lower	Upper	Mid	Freq.
4.0 - 4.2	3.95	4.25	4.10	1
4.3 - 4.5	4.25	4.55	4.40	1
4.6 - 4.8	4.55	4.85	4.70	2
4.9 - 5.1	4.85	5.15	5.00	7
5.2 - 5.4	5.15	5.45	5.30	12
5.5 - 5.7	5.45	5.75	5.60	17
5.8 - 6.0	5.75	6.05	5.90	31
6.1 - 6.3	6.05	6.35	6.20	18
6.4 - 6.6	6.35	6.65	6.50	15
6.7 - 6.9	6.65	6.95	6.80	11
7.0 - 7.2	6.95	7.25	7.10	3
7.3 - 7.5	7.25	7.55	7.40	2

Weight	Freq.	Lower Than	Cum. Freq.
4.0 - 4.3	1	4.3	1
4.3 - 4.6	1	4.6	2
4.6 - 4.9	2	4.9	4
4.9 - 5.2	7	5.2	11
5.2 - 5.5	12	5.5	23
5.5 - 5.8	17	5.8	40
5.8 - 6.1	31	6.1	71
6.1 - 6.4	18	6.4	89
6.4 - 6.7	15	6.7	104
6.7 - 7.0	11	7.0	115
7.0 - 7.3	3	7.3	118
7.3 - 7.6	2	7.6	120

(c) Construct a frequency polygon.

Sol.

Frequency Polygon of Distribution of Weight

(d) Construct a cumulative frequency polygon.

Sol.
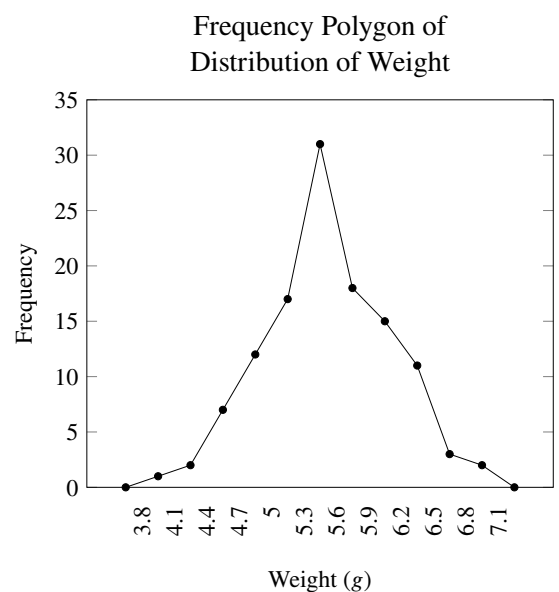
Cumulative Frequency Polygon of Distribution of Weight
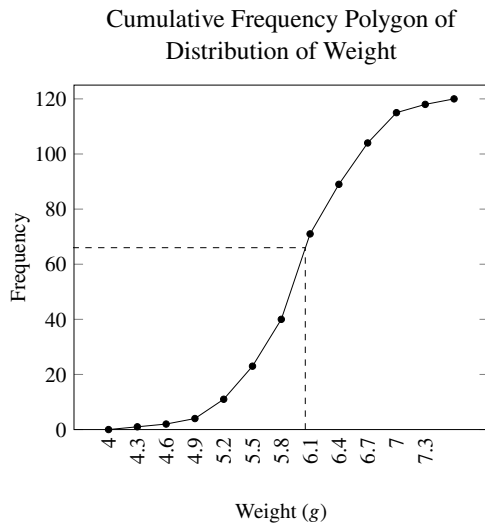
(e) Find the percentage of the ears of rice whose length is greater than $6cm$.

Sol.

In the diagram above, there are approximately $120 - 66 = 54$ ears of rice whose length is greater than $6cm$, which is about $\frac{54}{120} \cdot 100\% = 45\%$ of the total number of ears of rice.

3. The table below shows the weight distribution of 90 babies (in kg):

Weight	Frequency
1.5 - 2.0	2
2.0 - 2.5	4
2.5 - 3.0	13
3.0 - 3.5	32
3.5 - 4.0	28
4.0 - 4.5	10
4.5 - 5.0	1

(a) Make a cumulative frequency table.

Sol.

Weight	Freq.	Less than	Cum. Freq.
1.5 - 2.0	2	2.0	2
2.0 - 2.5	4	2.5	6
2.5 - 3.0	13	3.0	19
3.0 - 3.5	32	3.5	51
3.5 - 4.0	28	4.0	79
4.0 - 4.5	10	4.5	89
4.5 - 5.0	1	5.0	90

(b) Construct a cumulative frequency polygon.

Sol.

Cumulative Frequency Polygon of Distribution of Weight
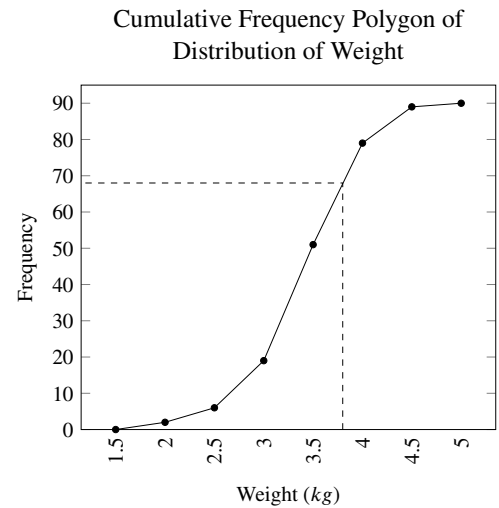
(c) Find the percentage of babies whose weight is greater than $3.8kg$.

Sol.

In the diagram above, there are approximately $90 - 68 = 22$ babies whose weight is greater than $3.8kg$, which is about $\frac{22}{90} \cdot 100\% = 24.44\%$ of the total number of babies.

4. The table below shows the average score distribution of 50 students in a class:

Average Score	Frequency
50.0 - 59.9	4
60.0 - 69.9	9
70.0 - 79.9	23
80.0 - 89.9	12
90.0 - 99.9	2

(a) Make a cumulative frequency table and draw a cumulative frequency polygon.

Sol.

Average Score	Freq.	Less than	Cum. Freq.
50.0 - 59.9	4	60	4
60.0 - 69.9	9	70	13
70.0 - 79.9	23	80	36
80.0 - 89.9	12	90	48
90.0 - 99.9	2	100	50

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Cumulative Frequency Polygon of
Distribution of Average Score
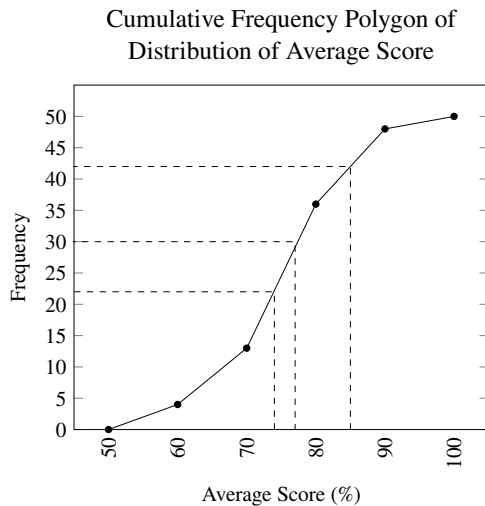
(b) A student get an average score of 74, find his rank in the class.

Sol.

In the diagram above, there are approximately 22 students whose average score is less than 74, which means that the student is ranked $50 - 22 = 28$.

(c) Find the average score of the student who is ranked 20.

Sol.

In the diagram above, the student who is ranked 20 has an average score of about 77.

(d) Find the percentage of students whose average score is greater than 85.

Sol.

In the diagram above, there are approximately $50 - 42 = 8$ students whose average score is greater than 85, which is about $\frac{8}{50} \cdot 100\% = 16\%$ of the total number of students.

5. The table below shows the score distribution of 1200 students in UEC accounting exam:

Score	Number of Students
10 - 19	20
20 - 29	60
30 - 39	95
40 - 49	130
50 - 59	340
60 - 69	310
70 - 79	135
80 - 89	80
90 - 99	30

Examinees are categorised into 4 groups based on their score: *Excellent*, *Good*, *Pass*, and *Fail*.

(a) Make a cumulative frequency table and draw a cumulative frequency polygon.

Sol.

Score	Freq.	Less than	Cum. Freq.
10 - 19	20	20	20
20 - 29	60	80	80
30 - 39	95	175	175
40 - 49	130	305	305
50 - 59	340	645	645
60 - 69	310	955	955
70 - 79	135	1090	1090
80 - 89	80	1170	1170
90 - 99	30	1200	1200

Cumulative Frequency Polygon of
Distribution of Score
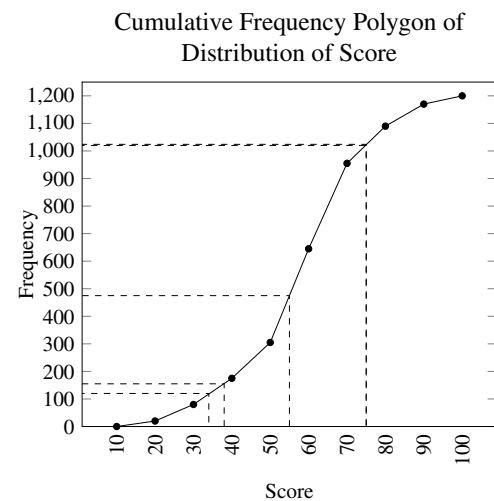
(b) If the passing score is 38, find the percentage of students who pass the exam.

Sol.

In the diagram above, there are approximately $1200 - 155 = 1045$ students whose score is greater or equal to 38, which is about $\frac{1045}{1200} \cdot 100\% = 86.67\%$ of the total number of students.

(c) Assume that the minimum score to be categorised as *Excellent* and *Good* is 75 and 55 respectively, find the percentage of students who are categorised as *Excellent* and *Good* respectively.

Sol.

In the diagram above, there are approximately $1200 - 1024 = 176$ students whose score is greater or equal to 75, which is about $\frac{176}{1200} \cdot 100\% = 14.67\%$ of the total number of students who are categorised as *Excellent*.

Also, there are approximately $1024 - 475 = 549$ students whose score is greater or equal to 55, which is about $\frac{549}{1200} \cdot 100\% = 45.75\%$ of the total number of students who are categorised as *Good*.

(d) Find the passing mark if the percentage of students who pass the exam is 90%.

Sol.

If the percentage of students who pass the exam is 90%, then the number of students who pass the exam is 90% of 1200 students, which is 1080 students. That means, there are $1200 - 1080 = 120$ students who fail the exam.

In the diagram above, the passing mark is about 34 given that there are 120 students who fail the exam.

(e) Find the minimum mark of a student who is categorised as *Excellent* if the percentage of students who are categorised as *Excellent* is 15%.

Sol.

If the percentage of students who are categorised as *Excellent* is 15%, then the number of students who are categorised as *Excellent* is 15% of 1200 students, which is 180 students. That means, there are $1200 - 180 = 1020$ students who are not categorised as *Excellent*.

In the diagram above, the minimum mark of a student who is categorised as *Excellent* is about 75 given that there are 1020 students who are not categorised as *Excellent*.

18.3   Central Tendency

Central tendency is a measure of the central position of a distribution, or a single value that attempts to describe a set of data. The most common measures of central tendency are the mean, median, and mode.

Mean

Mean is also known as arithmetic mean. For n values $x_1, x_2, \ldots, x_n$, the mean is defined as

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$
$$= \frac{\sum x_i}{n}$$

For data whose possible values are $x_1, x_2, \ldots, x_n$, and their respective frequencies are $f_1, f_2, \ldots, f_n$, the mean is defined as

$$\bar{x} = \frac{x_1 f_1 + x_2 f_2 + \cdots + x_n f_n}{f_1 + f_2 + \cdots + f_n}$$
$$= \frac{\sum f_i x_i}{\sum f_i}$$

For grouped data, we take the mean of each class as the representative value x_i of the class.

Weighted Mean

In some scenario, weighted mean is better than the mean to describe the data.

When calculating the arithmetic mean, each value is given equal weight. However, in some cases, each value in a dataset may not be equally important. For example, the importance of the mark of a student for each subject is weighted according to the number of classes of the subject in a week. Hence, when calculating the average mark of the student, each mark must be multiplied by a value that represents the importance of the subject, and that value is called the weight. The weighted mean is defined as

$$\bar{x} = \frac{w_1 x_1 + w_2 x_2 + \cdots + w_n x_n}{w_1 + w_2 + \cdots + w_n}$$
$$= \frac{\sum w_i x_i}{\sum w_i}$$

where x_i are the values and w_i are the weights of x_i.

18.3.1   Practice 3

1. Find the mean of 34, 50, 24, 32, 53, 30, 62, 27.

   Sol.

   $$\bar{x} = \frac{30 + 50 + 24 + 32 + 53 + 30 + 62 + 27}{8}$$
   $$= \frac{312}{8}$$
   $$= 39$$

2. There are three workshop A, B, and C in a factory. Workshop A has 10 workers, their wages are $35 per day, workshop B has 30 workers, their wages are $45 per day, and workshop C has 15 workers, their wages are $55 per day. Find the mean of the wages of the workers in the factory.

   Sol.

   Let the wages of workers be x_1, and the amount of workers be f_1.

x_1	f_1	$x_1 f_1$
35	10	350
45	30	1350
55	15	825
	$\sum f_i = 55$	$\sum f_i x_i = 2525$

$\therefore$ Average wages of workers in the factory is $\frac{2525}{55} = $ $45.91.

3. A school appoints students to participate in a Math competition. During the competition, candidates must answer 25 questions within an hour. The table below shows the distribution of frequency of the number of questions that those candidates answer correctly:

Answered Correctly	Frequency
1 - 5	3
6 - 10	12
11 - 15	7
16 - 20	8
21 - 25	5

Complete the following table, and find the mean of the number of questions that those candidates answer correctly.

Ans. Correctly	Freq. f_i	Midpoint x_i	$f_i x_i$
1 - 5			
6 - 10			
11 - 15			
16 - 20			
21 - 25			

Sol.

Ans. Correctly	Freq. f_i	Midpoint x_i	$f_i x_i$
1 - 5	3	3	9
6 - 10	12	8	96
11 - 15	7	13	91
16 - 20	8	18	144
21 - 25	5	23	115
	$\sum f_i = 35$	$\sum f_i x_i = 455$	

$\therefore$ The mean of the number of questions that those candidates answer correctly is $\frac{455}{35} = 13$.

18.3.2 Exercise 18.3a

1. Take a sample of 20 from a batch of machine parts, their weight (in *g*) are as follows:

210	208	200	205	202	218
206	214	215	207	195	207
218	192	202	216	185	227
187	215				

Find the mean weight of these machine parts.

Sol.

$$\bar{x} = \frac{210 + 208 + 200 + \cdots + 215}{20}$$
$$= \frac{4129}{20}$$
$$= 206.45$$

2. Given that the mean of a dataset 4, -3, 2, k, 5, 8 is 10, find the value of k.

Sol.

$$\frac{4 + (-3) + 2 + k + 5 + 8}{6} = 10$$
$$16 + k = 60$$
$$k = 44$$

3. Given that the mean of x_1, x_2, x_3, x_4, x_5 is 40, and the mean of y_1, y_2, y_3 is 15. Find the mean after combining these two datasets.

Sol.

$$\frac{x_1 + \cdots + x_5}{5} = 40$$
$$x_1 + \cdots + x_5 = 200$$

$$\frac{y_1 + y_2 + y_3}{3} = 15$$
$$y_1 + y_2 + y_3 = 45$$

$$\bar{xy} = \frac{x_1 + x_2 + \cdots + y_3}{8}$$
$$= \frac{245}{8}$$
$$= 30.63$$

4. A school have 2 senior 3 classes: *A* and *B*. In a Chinese language test, the average mark of 49 students in *A* class in 72, while the average mark for 45 students

in class B is 68. Find the average mark of all students in these two class combined.

Sol.

$$\begin{aligned}
\bar{x} &= \frac{72 \cdot 49 + 68 \cdot 45}{49 + 45} \\
&= \frac{3528 + 3060}{94} \\
&= \frac{6588}{94} \\
&= 70.09
\end{aligned}$$

5. Given that the mean for 8 values are 5. The mean increased by 1.4 after adding two values: x and $3x$. Find the value of x.

Sol.

$$\begin{aligned}
\frac{8 \cdot 5 + x + 3x}{8 + 2} &= 5 + 1.4 \\
\frac{40 + 4x}{10} &= 6.4 \\
40 + 4x &= 64 \\
4x &= 24 \\
x &= 6
\end{aligned}$$

6. Throwing 6 coin at the same time and record the number of heads. After throwing 100 times, we get the following frequency distribution table:

Number of Heads	Frequency
0	2
1	10
2	24
3	35
4	22
5	6
6	1

Find the mean of the number of heads for each throw.

Sol.

Let the number of heads be x_i and the frequency be f_i.

x_i	f_i	$x_i f_i$
0	2	0
1	10	10
2	24	48
3	35	105
4	22	88
5	6	30
6	1	6
	$\sum f_i = 100$	$\sum x_i f_i = 287$

∴ The mean of the number of heads for each throw is $\frac{287}{100} = 2.87$.

7. The table below shows the score distribution of 66 students in a Chinese language test:

Score	Frequency
31 - 40	6
41 - 50	12
51 - 60	15
61 - 70	15
71 - 80	8
81 - 90	6
91 - 100	4

Find their mark in average.

Score	Mid x_1	Freq. f_1	$x_1 f_1$
31 - 40	35.5	6	213
41 - 50	45.5	12	546
51 - 60	55.5	15	832.5
61 - 70	65.5	15	982.5
71 - 80	75.5	8	604
81 - 90	85.5	6	513
91 - 100	95.5	4	382
		$\sum f_1 = 66$	$\sum x_1 f_1 = 4073$

∴ The mark in average is $\frac{4073}{66} = 61.71$.

8. Below are the number of classes and marks for each subject of a junior student:

Subject	Number of Classes	Average Mark
Chinese	7	75
Malay	7	73
English	7	65
Mathematics	7	82
Science	5	86
History	3	73
Geography	3	87

(a) Find his mark in average.

Sol.

$$\begin{aligned}
\bar{x} &= \frac{75 + 73 + 65 + 82 + 86 + 73 + 87}{7} \\
&= \frac{541}{7} \\
&= 77.29
\end{aligned}$$

(b) Use the number of classes as the weight to find his average mark.

Sol.

$$\bar{x} = \frac{75 \cdot 7 + 73 \cdot 7 + \cdots + 87 \cdot 3}{7 + 7 + 7 + 7 + 5 + 3 + 3}$$

$$= \frac{525 + 511 + 455 + 574 + 430 + 219 + 261}{39}$$

$$= \frac{2975}{39}$$

$$= 76.28$$

9. The weight of 60 junior 2 students in a school are as follows:

Weight (kg)	Frequency
54 - 56	10
57 - 59	20
60 - 62	x
63 - 65	8
66 - 68	4
69 - 71	y

Given that the mean weight of these students is 60.1 kg, find the value of x and y.

Sol.

Total weight $= 60.1 \cdot 60 = 3606$

Wght (kg)	M. x_1	Freq. f_1	$x_1 f_1$
54 - 56	55	10	550
57 - 59	58	20	1160
60 - 62	61	x	$61x$
63 - 65	64	8	512
66 - 68	67	4	268
69 - 71	70	y	$70y$
		$\sum f_1 = 60$	$\sum x_1 f_1 = 3606$

$$\begin{cases} 10 + 20 + x + 8 + 4 + y = 60 & (1) \\ 550 + 1160 + 61x + 512 + 268 + 70y = 3606 & (2) \end{cases}$$

$$(1) : 42 + x + y = 60$$

$$x + y = 18$$

$$(2) : 61x + 70y = 1116$$

$$(1) \cdot 61 : 61x + 61y = 1098$$

$$(2) - (1) : 9y = 18$$

$$y = 2$$

From $(1) : x = 16$

Median

The median is the middle value of a sorted dataset. The number of values must be equal for both side of the median.

If the number of values is n, when n is odd, the median is the number in $\frac{n+1}{2}$ position.

When n is even, the median is the mean of the number in $\frac{n}{2}$ and $\frac{n}{2} + 1$ position.

For grouped data, we can make a cumulative frequency polygon, and the median is the value corresponding to 50% of the percentage of the cumulative frequency.

Let n be the number of values in the dataset, aka $\sum f_1$,

L_m be the lower boundaries of the group of the median,

C_m be the range of the group of the median,

f_m be the frequency of the group of the median,

F_m be the cum. frequency of the group of the median,
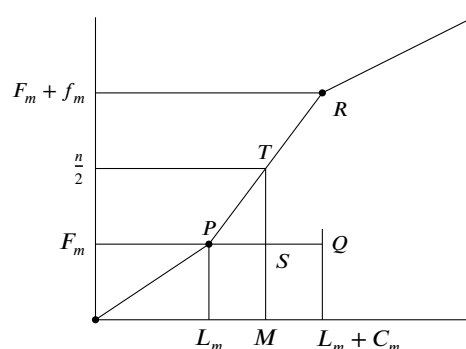
Diagram above shows a part of a cumulative frequency polygon, where R is the point corresponding to the group containing the median, P is the point corresponding to the group before the group containing the median, and M is the median. Since $\triangle PQR \sim \triangle PST$,

$$\therefore \frac{PS}{PQ} = \frac{ST}{QR}$$

That is, $\dfrac{M - L_m}{C_m} = \dfrac{\frac{n}{2} - F_m}{f_m}$

We get the following after simplifying the equation:

$$M = L_m + \left(\frac{\frac{n}{2} - F_m}{f_m} \right) C_m$$

18.3.3 Practice 4

1. 10 workers in a factory made the same type of product in a day, the number of products made are as follows:

$$15 \quad 17 \quad 14 \quad 10 \quad 15$$
$$19 \quad 17 \quad 16 \quad 14 \quad 12$$

Find the median of the number of products made by these 10 workers.

Sol.

Sort the dataset:

$$10 \quad 12 \quad 14 \quad 14 \quad 15 \quad 15 \quad 16 \quad 17 \quad 17 \quad 19$$

The median is the mean of the number in $\frac{10}{2} = 5$ and $\frac{10}{2} + 1 = 6$ position, which is $\frac{15+15}{2} = 15$.

2. The table below shows the result of a right eye vision test for 49 students in a class:

Vision	Number of Students
0.2	2
0.3	3
0.4	4
0.5	3
0.6	4
0.8	9
1.0	9
1.2	10
1.5	5

Find the median of the right eye vision of these students.

Sol.

Vision	Number of Students	Cum. Frequency
0.2	2	2
0.3	3	5
0.4	4	9
0.5	3	12
0.6	4	16
0.8	9	25
1.0	9	34
1.2	10	44
1.5	5	49

Since $n = 49$ is odd, the median is the number in the $\frac{49+1}{2} = 25$ position, which is 0.8.

3. The table below shows time distribution of 21 students browsing the Internet:

Time (hours)	Number of Students
1.1 - 1.3	4
1.4 - 1.6	3
1.7 - 1.9	5
2.0 - 2.2	4
2.3 - 2.5	5

Find the median of the time distribution of these students.

Sol.

Time	Freq.	Cum. Freq.
1.1 - 1.3	4	4
1.4 - 1.6	3	7
1.7 - 1.9	5	12
2.0 - 2.2	4	16
2.3 - 2.5	5	21

The median is the number in the $\frac{21}{2} = 10.5$ position, which is $1.7 - 1.9$. $C_m = 0.3$, $L_m = 1.65$, and $f_m = 5$, $F_m = 7$.

$$\therefore Mean = 1.65 + \frac{10.5 - 7}{5} \cdot 0.3 = 1.86$$

18.3.4 Exercise 18.3b

1. During a gymnastic competition, there are four judges scoring the performance of each contestant, and the median of these four scores are taken as the final score of the contestant. Given that the scores given by four judges are 9.5, 9.4, 9.8, and 9.4 respectively, find the final score of the contestant.

Sol.

Sort the scores:

$$9.4 \quad 9.4 \quad 9.5 \quad 9.8$$

The median is the mean of the number in $\frac{4}{2} = 2$ and $\frac{4}{2} + 1 = 3$ position, which is $\frac{9.4+9.5}{2} = 9.45$.

2. Following are the weight of 15 boys with same age:

$$36 \quad 35 \quad 33 \quad 37 \quad 35$$
$$42 \quad 40 \quad 38 \quad 38 \quad 39$$
$$40 \quad 41 \quad 36 \quad 38 \quad 37$$

(a) Find the median of these 15 boys.

Sol.

Sort the data:

33   35   35   36   36   37   37   38   38   38

39   40   41   42   43

The median is the mean of the number in $\frac{15+1}{2} = 8$ position, which is 38.

(b) Group the data using pattern $33-35, 35-37, \ldots, 41-43$. Then, find the median.

Sol.

Weight (kg)	Frequency	Cum. Frequency
33 - 35	1	1
35 - 37	4	5
37 - 39	5	10
39 - 41	2	12
41 - 43	2	14
43 - 45	1	15

The median is the number in the $\frac{15}{2} = 7.5$ position. $C_m = 2$, $L_m = 37$, $f_m = 5$, and $F_m = 5$.

$$\therefore Median = 37 + \frac{7.5 - 5}{5} \cdot 2 = 38$$

3. The table below shows the score distribution of a group of pupils in a minor test:

Score	Number of Pupils
5	4
10	2
15	3
20	x
25	4

Assume that the median is 15, find the possibility value of x.

Sol.

Score	Freq.	Cum. Freq.
5	4	4
10	2	6
15	3	9
20	x	9 + x
25	4	13 + x

$$\frac{13 + x + 1}{2} \leq 9$$
$$14 + x \leq 18$$
$$x \leq 4$$
$$\therefore 0 \leq x \leq 4$$

Therefore, the possibility values of x are 0, 1, 2, 3, and 4.

4. The following table shows the income of employees in a company:

Income ($)	Number of Employees
1000 - 2000	11
2000 - 3000	17
3000 - 4000	20
4000 - 5000	10
5000 - 6000	2

(a) Find the median of their income using cumulative frequency polygon.

Sol.

Income ($)	Freq.	Cum. Freq.
1000 - 2000	11	11
2000 - 3000	17	28
3000 - 4000	20	48
4000 - 5000	10	58
5000 - 6000	2	60

The median is the number in $\frac{60}{2} = 30$ position.

Cumulative Frequency Polygon of Distribution of Income
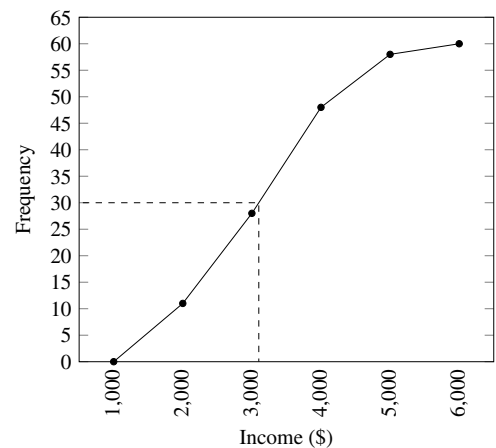
Therefore, the median of their income is $3100.

(b) Find the median of their income using formula and compare the result with (a).

Sol.

The median is the number in the $\frac{60}{2} = 30$ position, which is $3000 - 4000$. $C_m = 1000$, $L_m = 3000$, and $f_m = 20$, $F_m = 28$.

$$\therefore Median = 3000 + \frac{30 - 28}{20} \cdot 1000 = 3100$$

Therefore, the median of their income is $3100, which is the same as (a).

17

5. The table below shows the distribution of height of 20 students:

Height (cm)	Number of Students
120 - 130	3
130 - 140	4
140 - 150	x
150 - 160	5
160 - 170	6

Find:

(a) The value of x.

Sol.

$$x + 3 + 4 + 5 + 6 = 20$$
$$x = 20 - 18$$
$$= 2$$

(b) The median of their height.

Sol.

Height (cm)	Freq.	Cum. Freq.
120 - 130	3	3
130 - 140	4	7
140 - 150	2	9
150 - 160	5	14
160 - 170	6	20

The median is the number in $\frac{20}{2} = 10$ position, which is $150-160$. $C_m = 10$, $L_m = 150$, $f_m = 5$, and $F_m = 9$.

$$\therefore Median = 150 + \frac{10 - 9}{5} \cdot 10 = 152$$

Therefore, the median of their height is $152cm$.

6. The table below shows the distribution of wages of workers in a factory:

Wages $	Number of Workers
40 - 49	4
50 - 59	14
60 - 69	5
70 - 79	x
80 - 89	2

Given that the median is 63.5, find the value of x.

Sol.

Wages $	Freq.	Cum. Freq.
40 - 49	4	4
50 - 59	14	18
60 - 69	5	23
70 - 79	x	23 + x
80 - 89	2	25 + x

63.5 is in between $60 - 69$, which is in the $\frac{25+x}{2}$ position. $C_m = 10$, $L_m = 59.5$, $f_m = 5$, $F_m = 18$.

$$59.5 + \frac{\frac{25+x}{2} - 18}{5} \cdot 10 = 63.5$$
$$\frac{\frac{25+x}{2} - 18}{5} \cdot 10 = 4$$
$$\frac{\frac{25+x}{2} - 18}{5} = 0.4$$
$$\frac{25 + x}{2} - 18 = 2$$
$$\frac{25 + x}{2} = 20$$
$$25 + x = 40$$
$$x = 15$$

Mode

In a set of data, the mode is the value that occurs most frequently. There can be more than one mode in a set of data. If all the values in a dataset occur with the same frequency, then there is no mode for the data.

For grouped data, the mode is the class that has the highest frequency, and there can be more than one mode. Besides that, the mode can also be estimated using histogram. The method is as follows:
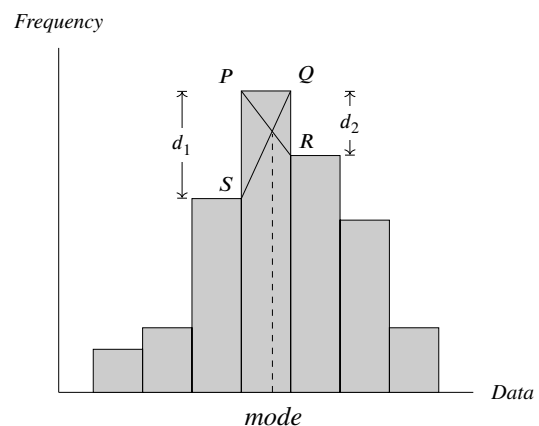
The diagram above shows a histogram of a set of data. The class corresponding to the highest rectangle is the mode of the data, and the mode is the x-value of the intersection point of PR and QS.

Unlike median, the formula of mode can be derived from similar triangles. Let:

L be the lower boundaries of the modal class

C be the range of the modal class

d_1 be the difference between the lower boundary of the modal class and the lower boundary of the class immediately before the modal class

d_2 be the difference between the lower boundary of the modal class and the lower boundary of the class immediately after the modal class

then

$$mode = L + \left(\frac{d_1}{d_1 + d_2} \right) C$$

18.3.5 Practice 5

The following table shows the distribution of the score of 36 students in a Mathematics exam:

Score	Number of Students
20 - 29	2
30 - 39	6
40 - 49	10
50 - 59	12
60 - 69	3
70 - 79	2
80 - 89	1

(a) Find the modal class.

Sol.

The modal class is $50 - 59$, which has the highest frequency of 12.

(b) Find the mode of score of the students using histogram.

Sol.

Histogram of
Distribution of Mathematics Score
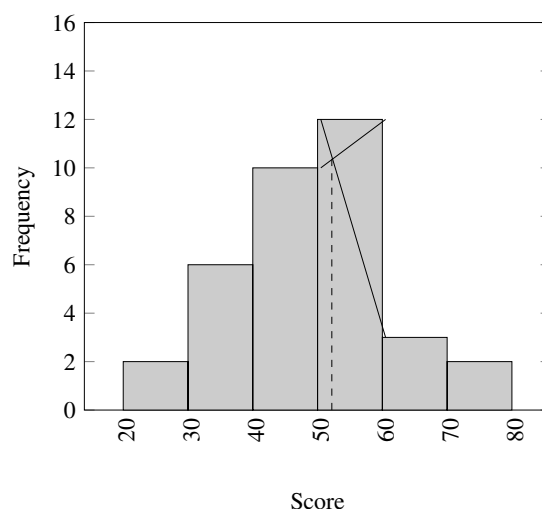
Score

The mode of score of the students is approximately 51.5.

(c) Find the mode of score of the students using formula.

Sol.

$L = 49.5, C = 10, d_1 = 12 - 10 = 2, d_2 = 12 - 3 = 9.$

$$\therefore Mode = 49.5 + \left(\frac{2}{2+9} \right) 10 = 51.32$$

Comparing mean, median and mode

Generally, the mean, median and mode of a set of data are all dirrent, and they are used to describe the data in different ways.

18.3.6 Exercise 18.3c

1. Find the mode of the following data:

   (a) 3   4   3   2   4   5   5   5   4   4
   Sol.
   The mode is 4, which has the highest frequency of 4.

   (b) 7   6   8   8   5   6   6   9   8   5
   Sol.
   The mode is 6 and 8, which has the highest frequency of 3.

   (c) 1.0   1.1   1.0   0.9   0.8   1.2   1.0   0.9   1.1   1.0
   Sol.
   The mode is 1.0, which has the highest frequency of 4.

2. In the sport competition of a high school, the scores of 17 athletes participating in men's high jump are as follows:

Scores (m)	Number of Athletes
1.50	2
1.60	3
1.65	2
1.70	3
1.75	4
1.80	1
1.85	1
1.90	1

Find the mean, median and mode of their scores.

Sol.

$$
\begin{aligned}
Mean &= \frac{1.50 \cdot 2 + 1.60 \cdot 3 + \cdots + 1.90 \cdot 1}{17} \\
&= \frac{3 + 4.8 + 3.3 + 5.1 + 7 + 1.8 + 1.85 + 1.9}{17} \\
&= \frac{28.75}{17} \\
&= 1.69m
\end{aligned}
$$

Scores (m)	No. of Athletes	Cum. Frequency
1.50	2	2
1.60	3	5
1.65	2	7
1.70	3	10
1.75	4	14
1.80	1	15
1.85	1	16
1.90	1	17

The median is the number at $\frac{17+1}{2} = $ 9th position, which is $1.70m$.

The mode is $1.75m$, which has the highest frequency of 4.

3. In a Mathematics competition, the scores and the number of students who obtained the scores are as follows:

Scores (%)	Number of Students
10 - 19	20
20 - 29	60
30 - 39	80
40 - 49	40
50 - 59	10

Find the modal class and the mode.

Sol.

The modal clsas is $30 - 39$, which has the highest frequency of 80.

$L = 29.5, C = 10, d_1 = 80 - 60 = 20, d_2 = 80 - 40 = 40$.

$$\therefore Mode = 29.5 + \left(\frac{20}{20 + 40} \right) 10 = 32.83$$

4. Given that the mean of a dataset 3, 5, 8, 6, 8, 10, 5, 3, x, y is 6,

    (a) Prove that $x + y = 12$

    Proof.

    $$
    \begin{aligned}
    \frac{3 + 5 + 8 + 6 + 8 + 10 + 5 + 3 + x + y}{10} &= 6 \\
    48 + x + y &= 60 \\
    x + y &= 12
    \end{aligned}
    $$

    (b) With that, if the following conditions are satisfied, find the mode of the dataset.

        i. $x = y$
        Sol.

        $$
        \begin{aligned}
        x &= y \\
        x + y &= 12 \\
        2x &= 12 \\
        x &= 6 \\
        y &= 6
        \end{aligned}
        $$

        $\therefore$ The dataset becomes $3, 5, 8, 6, 8, 10, 5, 3, 6, 6$.
        $\therefore$ The mode is 6, which has the highest frequency of 3.

        ii. $x < y$
        Sol.

        $$
        \begin{aligned}
        x &< y \\
        x + y &= 12 \\
        2x &< 12 \\
        x &< 6 \\
        y &> 6
        \end{aligned}
        $$

        When $x = 1$, $y = 11$, the dataset becomes $3, 5, 8, 6, 8, 10, 5, 3, 1, 11$.
        $\therefore$ The mode are $3, 5, 8$, which has the highest frequency of 2.
        When $x = 2$, $y = 10$, the dataset becomes $3, 5, 8, 6, 8, 10, 5, 3, 2, 10$.

∴ The mode are 3, 5, 8, 10, which has the highest frequency of 2.

When $x = 3$, $y = 9$, the dataset becomes $3, 5, 8, 6, 8, 10, 5, 3, 3, 9$.

∴ The mode is 3, which has the highest frequency of 3.

When $x = 4$, $y = 8$, the dataset becomes $3, 5, 8, 6, 8, 10, 5, 3, 4, 8$.

∴ The mode is 8, which has the highest frequency of 3.

When $x = 5$, $y = 7$, the dataset becomes $3, 5, 8, 6, 8, 10, 5, 3, 5, 7$.

∴ The mode is 5, which has the highest frequency of 3.

5. The mean of a set of data 13, 5, 5, n, 5, 10, 10, 11, 9, n^2 is 7.4,

(a) Find the possible values of n.

    Sol.

    $$\frac{13 + 5 + 5 + n + \cdots + n^2}{10} = 7.4$$

    $$68 + n + n^2 = 74$$

    $$n^2 + n - 6 = 0$$

    $$(n + 3)(n - 2) = 0$$

    $$n = -3 \text{ or } n = 2$$

(b) With that, if the following conditions are satisfied, find the median of the dataset.

    i. $n > 0$

    Sol.

    If $n > 0$, $n = 2$, the dataset becomes 13, 5, 5, 2, 5, 10, 10, 11, 9, 4. Rearranging the dataset, we get 2, 4, 5, 5, 5, 9, 10, 10, 11, 13. There are a total of 10 elements in the dataset, so the median is the average of the $\frac{10}{2} = $ 5th and $\frac{10}{2} + 1 = $ 6th elements, which is $\frac{5+9}{2} = 7$.

    ii. $n < 0$

    Sol.

    If $n < 0$, $n = -3$, the dataset becomes 13, 5, 5, -3, 5, 10, 10, 11, 9, 9. Rearranging the dataset, we get -3, 5, 5, 5, 9, 9, 10, 10, 11, 13. There are a total of 10 elements in the dataset, so the median is the average of the $\frac{10}{2} = $ 5th and $\frac{10}{2} + 1 = $ 6th elements, which is $\frac{9+9}{2} = 9$.

6. The following table shows the distribution of scores of a group of students in a competition:

Scores	Number of Students
0	3
1	x
2	4
3	6
4	2

(a) Assume that the mode is 1, find the minimum value of x.

    Sol.

    Given that the **only** mode is 1, $x > 6$. Therefore, the minimum value of x is 7.

(b) Assume that the median is 2, find the maximum value of x.

    Sol.

    Construct a cumulative frequency table:

Scores	Number of Students	Cum. Freq.
0	3	3
1	x	3 + x
2	4	7 + x
3	6	13 + x
4	2	15 + x

    $$\frac{15 + x}{2} > 3 + x$$

    $$15 + x > 6 + 2x$$

    $$x < 9$$

    $$x = 8$$

(c) Assume that the mean is 1.95, find the value of x.

    Sol.

x_i	f_i	$f_i x_i$
0	3	0
1	x	x
2	4	8
3	6	18
4	2	8
	$\sum f_i = 15 + x$	$\sum = f_i x_i\, 34 + x$

    $$\frac{34 + x}{15 + x} = 1.95$$

    $$34 + x = 29.25 + 1.95x$$

    $$0.95x = 4.75$$

    $$x = 5$$

7. Given thet the mode, median and mean of 5 positive integers are 9, 8, and 7.6 respectively, find these 5 numbers.

   Sol.

   Let the 5 numbers be a, b, c, d, e, arranged in ascending order.

   Since the median is 8, and c is the middle of these 5 numbers, $c = 8$.

   Since the mode is 9, there must be more than 1 number that is 9. Since $9 > 8$, $d = e = 9$.

   Since the mean is 7.6,

   $$\frac{a+b+c+d+e}{5} = 7.6$$
   $$\frac{a+b+8+9+9}{5} = 7.6$$
   $$a+b+26 = 38$$
   $$a+b = 12$$

   Since the numbers are arranged in ascending order, and there is only one mode, $a < b < c$, that is, $a < b < 8$.

   $$a+b = 12$$
   $$a = 12 - b$$

   $$a < b < 8$$
   $$12 - b < b < 8$$

   $$12 < 2b$$
   $$b < 8$$

   $$b > 6$$
   $$b < 8$$
   $$6 < b < 8$$

   $$\therefore b = 7$$
   $$a = 12 - 7 = 5$$

   Therefore, the 5 numbers are 5, 7, 8, 9, 9.

8. The following table shows the amount of sales of a brand of shoes in a month:

Shoes Number	Amount of Sales
5	4
6	10
7	11
8	18
9	2

   (a) Find the mean, median, and mode.

   Sol.

x_i	f_i	$f_i x_i$
5	4	20
6	10	60
7	11	77
8	18	144
9	2	18
	$\sum f_i = 45$	$\sum f_i x_i = 319$

   $$\text{Mean} = \frac{319}{45} = 7.09$$

Shoes Number	Sales	Cum. Freq.
5	4	4
6	10	14
7	11	25
8	18	43
9	2	45

   Since $n = 45$, the median is the number at the 23rd position.

   $$\text{Median} = 7$$

   Since the number with the highest frequency of 18 is 8,

   $$\text{Mode} = 8$$

   $\therefore$ mean = 7.09, median = 7, mode = 8.

   (b) Which of the following central tendency represents the data best? Why?

   Sol.

   The **mode** is the best central tendency to represent the data because it is the number with the highest frequency, which is 18, and it is the shoes number that most of the shoes are sold.

9. In between 54 examinees in an exam, 15 of them come from cities, 39 of them come from suburbs. Below are the frequency distribution table of their scores:

Scores	City	Suburb
12 - 23	0	1
23 - 34	0	0
34 - 45	0	5
45 - 56	1	6
56 - 67	3	5
67 - 78	4	13
78 - 89	6	4
89 - 100	1	5

(a) Find the mean, median, and mode of the scores of the examinees from cities and suburbs respectively.

Sol.

For the scores of the examinees from cities:

Score	Mid. x_i	Freq. f_i	$f_i x_i$
12 - 23	17.5	0	0
23 - 34	28.5	0	0
34 - 45	39.5	0	0
45 - 56	50.5	1	50.5
56 - 67	61.5	3	184.5
67 - 78	72.5	4	290
78 - 89	83.5	6	501
89 - 100	94.5	1	94.5
		$\sum f_i = 15$	1120.5

$$\text{Mean} = \frac{1120.5}{15} = 74.7$$

Score	Feq.	Lower Than	Cum. Freq.
12 - 23	0	23	0
23 - 34	0	34	0
34 - 45	0	45	0
45 - 56	1	56	1
56 - 67	3	67	4
67 - 78	4	78	8
78 - 89	6	89	14
89 - 100	1	100	15

$n = 15$, $\frac{n}{2} = 7.5$, the group of scores that contains the median is $67 - 78$, $C_m = 11$, $L_m = 67$, $f_m = 3$, $F_m = 4$,

$$\text{Median} = 67 + \frac{7.5 - 4}{4} \cdot 11 = 76.63$$

Since the modal class is $78 - 89$ with the highest frequency of 6, $C = 11$, $d_1 = 6 - 4 = 2$, $d_2 = 6 - 1 = 5$, $L_m = 78$,

$$\text{Mode} = 78 + \frac{2}{2 + 5} \cdot 11 = 81.14$$

∴ mean = 74.7, median = 76.63, mode = 81.14.

For the scores of the examinees from suburbs:

Score	Mid. x_i	Freq. f_i	$f_i x_i$
12 - 23	17.5	1	17.5
23 - 34	28.5	0	0
34 - 45	39.5	5	197.5
45 - 56	50.5	6	303
56 - 67	61.5	5	307.5
67 - 78	72.5	13	942.5
78 - 89	83.5	4	334
89 - 100	94.5	5	472.5
		$\sum f_i = 39$	2574.5

$$\text{Mean} = \frac{2574.5}{39} = 66.01$$

Score	Feq.	Lower Than	Cum. Freq.
12 - 23	1	23	1
23 - 34	0	34	1
34 - 45	5	45	6
45 - 56	6	56	12
56 - 67	5	67	17
67 - 78	13	78	30
78 - 89	4	89	34
89 - 100	5	100	39

$n = 39$, $\frac{n}{2} = 19.5$, the group of scores that contains the median is $67 - 78$, $C_m = 11$, $L_m = 67$, $f_m = 13$, $F_m = 17$,

$$\text{Median} = 67 + \frac{19.5 - 17}{13} \cdot 11 = 69.12$$

Since the modal class is $67 - 78$ with the highest frequency of 13, $C = 11$, $d_1 = 13 - 5 = 8$, $d_2 = 13 - 4 = 9$, $L = 67$,

$$\text{Mode} = 67 + \frac{8}{8 + 9} \cdot 11 = 72.18$$

∴ mean = 66.01, median = 69.12, mode = 72.18.

Therefore, the mean, median and mode of the scores are:

	City	Suburbs
Mean	74.7	66.01
Median	76.63	69.12
Mode	81.14	72.18

(b) Find the mean, median and mode of the scores of all the examinees.

Sol.

Score	Mid. x_i	Freq. f_i	f_ix_i
12 - 23	17.5	1	17.5
23 - 34	28.5	0	0
34 - 45	39.5	5	197.5
45 - 56	50.5	7	353.5
56 - 67	61.5	8	492
67 - 78	72.5	17	1232.5
78 - 89	83.5	10	835
89 - 100	94.5	6	567
		$\sum f_i = 54$	3695

$$\text{Mean} = \frac{3695}{54} = 68.43$$

Score	Feq.	Lower Than	Cum. Freq.
12 - 23	1	23	1
23 - 34	0	34	1
34 - 45	5	45	6
45 - 56	7	56	13
56 - 67	8	67	21
67 - 78	17	78	38
78 - 89	10	89	48
89 - 100	6	100	54

$n = 54$, $\frac{n}{2} = 27$, the group of scores that contains the median is $67 - 78$, $C_m = 11$, $L_m = 67$, $f_m = 17$, $F_m = 21$,

$$\text{Median} = 67 + \frac{27 - 21}{17} \cdot 11 = 70.88$$

Since the modal class is $67 - 78$ with the highest frequency of 17, $C = 11$, $d_1 = 17 - 8 = 9$, $d_2 = 17 - 10 = 7$, $L = 67$,

$$\text{Mode} = 67 + \frac{9}{9 + 7} \cdot 11 = 73.19$$

10. The following table shows the distribution of scores of a group of students in a Chinese language test:

Scores x	Number of Students
$40 < x \le 50$	12
$50 < x \le 60$	30
$60 < x \le 70$	35
$70 < x \le 80$	25
$80 < x \le 90$	10
$9 < x \le 100$	3

Find:

(a) Mean.

Sol.

Scores x	Mid. x_i	Freq. f_i	f_ix_i
$40 < x \le 50$	45	12	540
$50 < x \le 60$	55	30	1650
$60 < x \le 70$	65	35	2275
$70 < x \le 80$	75	25	1875
$80 < x \le 90$	85	10	850
$90 < x \le 100$	95	3	285
		$\sum f_i = 115$	7475

$$\text{Mean} = \frac{7475}{115} = 65$$

(b) Modal class and mode.

Sol.

The modal class is $60 < x \le 70$ with the highest frequency of 35, $C = 10$, $d_1 = 35 - 30 = 5$, $d_2 = 35 - 25 = 10$, $L = 60$,

$$\text{Mode} = 60 + \frac{5}{5 + 10} \cdot 10 = 63.33$$

(c) Median.

Sol.

Score	Freq.	Lower Than	Cum. Freq.
$40 < x \le 50$	12	50	12
$50 < x \le 60$	30	60	42
$60 < x \le 70$	35	70	77
$70 < x \le 80$	25	80	102
$80 < x \le 90$	10	90	112
$90 < x \le 100$	3	100	115

$n = 115$, $\frac{n}{2} = 57.5$, the group of scores that contains the median is $60 < x \le 70$, $C_m = 10$, $L_m = 60$, $f_m = 35$, $F_m = 42$,

$$\text{Median} = 60 + \frac{57.5 - 42}{35} \cdot 10 = 64.43$$

18.4 Measures of Dispersion

The measures of dispersion can be used to describe the spread of the data.

When we're describing a set of data, if we only use the mean, the information provided by the dataset is not enough. For example, given the mean, median, and mode of the average marks of four students in a Mathematics test are all 70 marks, we can't tell the difference between the four students. Their marks might be similar (e.g. 68, 72, 70, 70) or they might be very different (e.g. 100, 40, 70, 70). The latter case is obviously more spread out than the former case.

The most common measures of dispersion are range, interquartile range, quartile deviation, standard deviation, mean deviation, variance, and standard deviation.

Range

The range of a set of data is the difference between the largest and the smallest value in the dataset.

For grouped data, the range is the difference between the upper limit of the highest class and the lower limit of the lowest class.

Quartile, Interquartile Range, and Quartile Deviation

Quartiles are three value Q_1, Q_2, and Q_3 that divide a dataset into four equal parts. Q_2 is the median of the dataset. Q_1 and Q_3 are the medians of the two halves of the dataset, called the lower quartile and the upper quartile respectively.

Assume that the number of data in a sorted dataset is n. If n is odd, then

When n is even, split the dataset into two halves, with $n/2$ data in each half.

When n is odd, split the data into two halves after removing the median, with $(n-1)/2$ data in each half.

The median of the lower half is Q_1 and the median of the upper half is Q_3.

For grouped data, we can make a cumulative frequency polygon. In the percentage of the polygon,

25% of the data is below Q_1.

50% of the data is below Q_2.

75% of the data is below Q_3.

Using the same method of deriving the formula for median, we can derive the formula for upper and lower quartiles. Let

n be the number of data in the dataset, aka $\sum f_i$

L_k be the lower boundaries of the class of Q_k

C_k be the class range of the class of Q_k

f_k be the frequency of the class of Q_k

F_k be the cumulative frequency of the class of Q_k

then

$$Q_1 = L_1 + \left(\frac{\frac{n}{4} - F_1}{f_1} \right) C_1$$

$$Q_3 = L_3 + \left(\frac{\frac{3n}{4} - F_2}{f_2} \right) C_2$$

The difference between the upper and lower quartiles is called the interquartile range. That is,

$$Interquartile\ range = Q_3 - Q_1$$

The quartile deviation is the interquartile range divided by 2, written as $Q.D.$, that is,

$$Q.D. = \frac{Q_3 - Q_1}{2}$$

Since the interquartile range and the quartile deviation are not affected by the outliers, they are more robust than the range, and are more suitable for representing the spread of the data.

18.4.1   Practice 6

1. Find the range, quartiles and interquartile range of the following data:

   (a) 4   8   7   3   3   9   6   5   1   1   2
   Sol.
   Sorting the data, we get

   $$
   \begin{array}{ccccccccccc}
   1 & 1 & 2 & 3 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
   & & \uparrow & & & \uparrow & & & \uparrow & & \\
   & & Q_1 = 2 & & & Q_2 = 4 & & & Q_3 = 7 & &
   \end{array}
   $$

   The range is $9 - 1 = 8$.
   The interquartile range is $Q.D. = 7 - 2 = 5$.

   (b) 7   6   8   8   5   6   1   9   8
   Sol.
   Sorting the data, we get

   $$Q_2 = 7$$
   $$
   \begin{array}{ccccccccc}
   1 & 5 & 6 & 6 & 7 & 8 & 8 & 8 & 9 \\
   & \uparrow & & & & & \uparrow & & \\
   \end{array}
   $$
   $$Q_1 = \frac{5+6}{2} = 5.5 \qquad Q_3 = \frac{8+8}{2} = 8$$

   The range is $9 - 1 = 8$.
   The interquartile range is $8 - 5.5 = 2.5$.

   (c) 1.0   1.1   1.5   0.7   0.8   1.2   1.4   0.9
       1.6   1.3
   Sol.

Sorting the data, we get

$$Q_2 = \frac{1.1+1.2}{2} = 1.15$$

$$\downarrow$$

| 0.7 | 0.8 | 0.9 | 1.0 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 |

$$\uparrow \qquad\qquad\qquad\qquad \uparrow$$

$$Q_1 = 0.9 \qquad\qquad\qquad Q_3 = 1.4$$

The range is $1.6 - 0.7 = 0.9$.

The interquartile range is $1.4 - 0.9 = 0.5$.

(d) 3   4   7   2   4   6   5   8

Sol.

Sorting the data, we get

$$Q_2 = \frac{4+5}{2} = 4.5$$

$$\downarrow$$

| 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |

$$\uparrow \qquad\qquad\qquad \uparrow$$

$$Q_1 = \frac{3+4}{2} = 3.5 \qquad Q_2 = \frac{6+7}{2} = 6.5$$

The range is $8 - 2 = 6$.

The interquartile range is $6.5 - 3.5 = 3$.

2. The table below shows the cumulative frequency distribution table of the heights of 60 students:

Height (cm)	Cumulative Frequency
150-155	3
155-160	10
160-165	22
165-170	37
170-175	51
175-180	58
180-185	60

(a) Find the interquartile range of the heights of the students from the cumulative frequency polygon.

Sol.

Cumulative Frequency Polygon of Distribution of Height
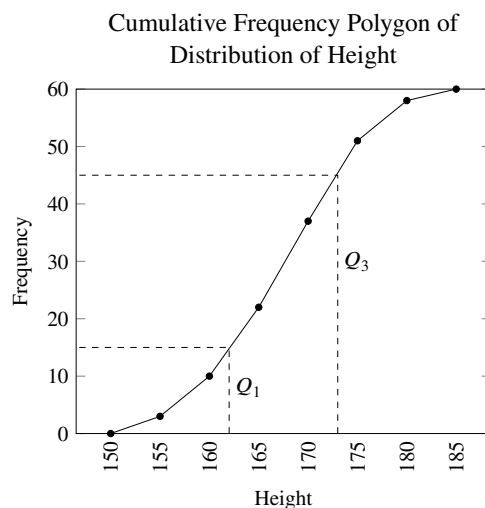
From the graph, Q_1 is approximately $162cm$ and Q_3 is approximately $173cm$. Hence,

$$Q.D. = \frac{173 - 162}{2} = 5.5$$

(b) Find the interquartile range of the heights of the students using formula.

Sol.

$n = 60$, $\frac{n}{4} = 15$, the class that contains Q_1 is $160 - 165$, $C_1 = 5$, $L_1 = 160$, $f_1 = 11$, $F_1 = 10$,

$$Q_1 = 160 + \frac{15 - 10}{12} \cdot 5 = 162.083$$

$n = 60$, $\frac{3n}{4} = 45$, the class that contains Q_3 is $170 - 175$, $C_3 = 5$, $L_3 = 170$, $f_3 = 14$, $F_3 = 37$,

$$Q_3 = 170 + \frac{45 - 37}{14} \cdot 5 = 172.857$$

Hence,

$$Q.D. = \frac{172.857 - 162.083}{2} = 5.39$$

18.4.2 Exercise 18.4a

1. Following are the sales of televisions of a shop in 11 days:

4 9 0 1 3 4 2 5 7 2 3

Find:

(a) The range.

Sol.

Rearranging the data,

$$0 \quad 1 \quad 2 \quad 2 \quad 3 \quad 3 \quad 4 \quad 4 \quad 5 \quad 7 \quad 9$$

Hence, the range is $9 - 0 = 9$.

(b) The quartiles and interquartile range.

Sol.

Rearranging the data,

$$0 \quad 1 \quad \underset{\uparrow}{2} \quad 2 \quad 3 \quad \underset{\uparrow}{3} \quad 4 \quad 4 \quad \underset{\uparrow}{5} \quad 7 \quad 9$$
$$Q_1 = 2 \qquad Q_2 = 3 \qquad Q_3 = 5$$

Hence, the interquartile range is $5 - 2 = 3$.

2. Given a set of data: 1.2, 1.0, 1.1, 1.3, 1.5, 1.7, 1.2, 1.0. Find:

(a) The range.

Sol.

Rearranging the data,

$$1.0 \quad 1.0 \quad 1.1 \quad 1.2 \quad 1.2 \quad 1.3 \quad 1.5 \quad 1.7$$

Hence, the range is $1.7 - 1.0 = 0.7$.

(b) The quartiles and interquartile deviation.

Sol.

Rearranging the data,

$$Q_2 = \frac{1.2 + 1.2}{2} = 1.2$$
$$\downarrow$$
$$1.0 \quad 1.0 \quad \underset{\uparrow}{1.1} \quad 1.2 \quad 1.2 \quad \underset{\uparrow}{1.3} \quad 1.5 \quad 1.7$$
$$Q_1 = \frac{1.0}{1.1} = 1.05 \qquad Q_3 = \frac{1.3 + 1.5}{2} = 1.4$$

Hence, the interquartile deviation is $\frac{1.4 - 1.05}{2} = 0.18$.

3. The distribution of scores of Mathematics test of 100 senior 1 students from a high school are as follows:

Scores	Number of Students
30 - 40	3
40 - 50	4
50 - 60	13
60 - 70	22
70 - 80	30
80 - 90	23
90 - 100	5

Find the interquartile deviation of the scores.

Sol.

Scores	Freq.	Lower Than	Cumulative Freq.
30 - 40	3	40.5	3
40 - 50	4	50.5	7
50 - 60	13	60.5	20
60 - 70	22	70.5	42
70 - 80	30	80.5	72
80 - 90	23	90.5	95
90 - 100	5	100.5	100

$n = 100$, $\frac{n}{4} = 25$, The class that contains the lower quartile is $60 - 70$, $C_1 = 10$, $L_1 = 60.5$, $f_1 = 22$, $F_1 = 20$,

$$Q_1 = 60.5 + \frac{25 - 20}{22} \cdot 10 = 62.773$$

$n = 100$, $\frac{3n}{4} = 75$, The class that contains the upper quartile is $80 - 90$, $C_2 = 10$, $L_2 = 80.5$, $f_2 = 23$, $F_2 = 72$,

$$Q_3 = 80.5 + \frac{75 - 72}{23} \cdot 10 = 81.804$$

Hence,

$$Q.D. = \frac{81.804 - 62.773}{2} = 9.52$$

Mean Deviation

Let the mean of a set of data $x_1, x_2, \ldots, x_n$ be $\bar{x}$, $|x_i - \bar{x}|$ is the difference between the ith data and the mean, the mean of these n differences are called the mean deviation, and can be used to calculate the measure of dispersion of the data. That is,

$$Mean\ Deviation = \frac{\sum |x_i - \bar{x}|}{n}$$

If the possible value given data are $x_1, x_2, \ldots, x_n$, their frequencies are $f_1, f_2, \ldots, f_n$, respectively, then the mean deviation can be calculated as follows:

$$Mean\ Deviation = \frac{\sum |x_i - \bar{x}| f_i}{\sum f_i}$$

For grouped data, we take the midpoints of the classes as the representative value x_i.

18.4.3 Practice 7

Complete the following table, and find the mean and mean deviation of the data.

Lim.	f_i	Mid. x_i	$f_i x_i$	$\|x_i - \bar{x}\|$	$\|x_i - \bar{x}\|f_i$
50 - 54	2				
55 - 59	3				
60 - 64	6				
65 - 69	9				

Sol.

Lim.	f_i	Mid. x_i	$f_i x_i$	$\|x_i - \bar{x}\|$	$\|x_i - \bar{x}\|f_i$
50 - 54	2	52	104	10.5	21
55 - 59	3	57	171	5.5	16.5
60 - 64	6	62	372	0.5	3
65 - 69	9	67	603	4.5	40.5
	20		1250		81

$$\bar{x} = \frac{1250}{20} = 62.5$$
$$Mean\ Deviation = \frac{81}{20} = 4.05$$

18.4.4 Exercise 18.4b

1. Find the mean deviation of the following dataset:

   (a) 7   10   9   12   4   11   3

   Sol.

   $$\bar{x} = \frac{7 + 10 + 9 + 12 + 4 + 11 + 3}{7} = 8$$
   $$Mean\ Dev. = \frac{1}{7}(|7 - 8| + \cdots + |3 - 8|)$$
   $$= 2.86$$

   (b) 58   65   38   76   43

   Sol.

   $$\bar{x} = \frac{58 + 65 + 38 + 76 + 43}{5} = 56$$
   $$Mean\ Dev. = \frac{1}{5}(|58 - 56| + \cdots + |43 - 56|)$$
   $$= 12.4$$

   (c) 45.0   46.5   47.0   48.0   48.7   48.9   49.5
   50.4

   Sol.

   $$\bar{x} = \frac{45.0 + 46.5 + \cdots + 50.4}{8} = 48$$
   $$M\ Dev. = \frac{1}{8}(|45.0 - 48.3| + \cdots + |50.4 - 48.3|)$$
   $$= 1.38$$

2. The table below shows the frequency of the number of questions answered correctly by 26 students in a Mathematics minor test:

Num. of Corr. Ans. Ques.	Num. of Stud.
1	0
2	1
3	1
4	1
5	6
6	8
7	6
8	1
9	1
10	1

Find the mean deviation fo the number of questions answered correctly.

Sol.

x_i	f_i	$f_i x_i$	$\|x_i - \bar{x}\|$	$\|x_i - \bar{x}\|f_i$
1	0	0	5	0
2	1	2	4	4
3	1	3	3	3
4	1	4	2	2
5	6	30	1	6
6	8	48	0	0
7	6	42	1	6
8	1	8	2	2
9	1	9	3	3
10	1	10	4	4
	26			30

$$\bar{x} = \frac{156}{26} = 6$$
$$Mean\ Deviation = \frac{30}{26} = 1.15$$

3. Following are the test scores of 36 students:

   77   60   52   73   60   50   70   60   52

   68   59   50   72   59   48   66   58   46

   60   48   34   61   55   40   62   55   42

   63   55   43   65   56   45   65   57   46

   (a) Group the dataset above according to the pattern $[34 - 38)$, $[38 - 42)$, $[42 - 46)$, ..., then make a frequency distribution table.

   Sol.

Range	Frequency
$34 \le x < 38$	1
$38 \le x < 42$	1
$42 \le x < 46$	3
$46 \le x < 50$	4
$50 \le x < 54$	4
$54 \le x < 58$	5
$58 \le x < 62$	8
$62 \le x < 66$	4
$66 \le x < 70$	2
$70 \le x < 74$	3
$74 \le x < 78$	1

(b) Find the mean from the frequency distribution table.

Sol.

Range	Mid x_i	Freq. f_i	$f_i x_i$
$34 \le x < 38$	36	1	36
$38 \le x < 42$	40	1	40
$42 \le x < 46$	44	3	132
$46 \le x < 50$	48	4	192
$50 \le x < 54$	52	4	208
$54 \le x < 58$	56	5	280
$58 \le x < 62$	60	8	480
$62 \le x < 66$	64	4	256
$66 \le x < 70$	68	2	136
$70 \le x < 74$	72	3	216
$74 \le x < 78$	76	1	76
		36	2052

$$\bar{x} = \frac{2052}{36} = 57$$

(c) Find the mean deviation from the frequency distribution table.

Sol.

x_i	f_i	$|x_i - \bar{x}|$	$f_i |x_i - \bar{x}|$
36	1	21	21
40	1	17	17
44	3	13	39
48	4	9	36
52	4	5	20
56	5	1	5
60	8	3	24
64	4	7	28
68	2	11	22
72	3	15	45
76	1	19	19
	36	107	276

$$\text{Mean Deviation} = \frac{276}{36} = 7.67$$

Variance and Standard Deviation

Let the mean of a set of data $x_1, x_2, ..., x_n$ be $\bar{x}$, $(x_1 - \bar{x})^2$ be the square of the difference between the ith data and the mean, the square of the mean of these n differences are called the variance, written as σ^2, that is,

$$\sigma^2 = \frac{\sum(x_i - \bar{x})^2}{n}$$

The square root of the variance is called the standard deviation, written as σ, that is,

$$\sigma = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n}}$$

If the possible values of given data are $x_1, x_2, ..., x_n$, their frequencies are $f_1, f_2, ..., f_n$, respectively, then

$$\sigma^2 = \frac{\sum(x_i - \bar{x})^2 f_i}{\sum f_i}$$

$$\sigma = \sqrt{\frac{\sum(x_i - \bar{x})^2 f_i}{\sum f_i}}$$

For grouped data, we take the midpoints of the classes as the representative value x_i.

The above formula are a bit complicated, so we can simplify the formula:

$$\begin{aligned}
\sigma^2 &= \frac{\sum(x_i - \bar{x})^2 f_i}{\sum f_i} \\
&= \frac{\sum x_i^2 f_i - 2\bar{x} \sum x_i f_i + \sum \bar{x}^2 f_i}{\sum f_i} \\
&= \frac{\sum x_i^2 f_i}{\sum f_i} - 2\bar{x}^2 + \bar{x}^2 \\
&= \frac{\sum x_i^2 f_i}{\sum f_i} - \bar{x}^2
\end{aligned}$$

Hence, when the frequency of value x_i is f_i, Then

$$\sigma^2 = \frac{\sum x_i^2 f_i}{\sum f_i} - \bar{x}^2$$

$$\sigma = \sqrt{\frac{\sum x_i^2 f_i}{\sum f_i} - \bar{x}^2}$$

When all the frequencies f_i are equal to 1, then

$$sigma^2 = \frac{\sum x_i^2}{n} - \bar{x}^2$$

$$\sigma = \sqrt{\frac{\sum x_i^2}{n} - \bar{x}^2}$$

Compared to mean deviation, the variance and standard deviation do not contain absolute value, so it is more convenient to use them. Furthermore, the variance and standard deviation are more sensitive to the difference between the data and the mean, so they are more commonly used in daily life.

18.4.5 Practice 8

1. Measuring the height of 10 plant seedlings (in cm) in a lab, we get the following data:

   12   6   15   3   12   6   21   15   18   12

   Find the standard deviation of the height of the plant seedlings.

   Sol.

   $$\bar{x} = \frac{12 + 6 + \cdots + 12}{10} = 12$$
   $$\sigma^2 = \frac{12^2 + 6^2 + \cdots + 12^2}{10} - 12^2$$
   $$= \frac{1728}{10} - 144$$
   $$= 28.8$$
   $$\sigma = \sqrt{28.8} = 5.37$$

2. Complete the following table, then find the standard deviation.

   (a)

x_i	f_i	$x_i f_i$	$x_i^2 f_i$
3	30		
5	35		
7	28		

   Sol.

x_i	f_i	$x_i f_i$	$x_i^2 f_i$
3	30	90	270
5	35	175	875
7	28	196	1372
	$\sum f_i = 93$	$\sum x_i f_i = 461$	$\sum x_i^2 f_i = 2517$

   $$\bar{x} = \frac{461}{93} = 4.96$$
   $$\sigma^2 = \frac{2517}{93} - 4.96^2$$
   $$= 27.06 - 24.60$$
   $$= 2.46$$
   $$\sigma = \sqrt{2.46} = 1.57$$

(b)

Limit	f_i	$Mid.x_i$	$x_i f_i$	$x_i^2 f_i$
150 - 154	5			
155 - 159	8			
160 - 164	10			
165 - 169	7			
170 - 174	6			
175 - 179	4			

Sol.

Limit	f_i	$Mid.x_i$	$x_i f_i$	$x_i^2 f_i$
150 - 154	5	152	760	115520
155 - 159	8	157	1256	197192
160 - 164	10	162	1620	262440
165 - 169	7	167	1169	195223
170 - 174	6	172	1032	177504
175 - 179	4	177	708	125316
	$\sum f_i = 40$		$\sum x_i f_i = 6545$	$\sum x_i^2 f_i = 1073195$

$$\bar{x} = \frac{6545}{40} = 163.625$$
$$\sigma^2 = \frac{1073195}{40} - 163.88^2$$
$$= 26829.88 - 26773.14$$
$$= 56.74$$
$$\sigma = \sqrt{55.74} = 7.53$$

18.4.6 Exercise 18.4c

1. Find the variance and standard deviation of the following dataset:

   (a) 3   6   3   8

   Sol.

   $$\bar{x} = \frac{3 + 6 + 3 + 8}{4} = 5$$
   $$\sigma^2 = \frac{3^2 + 6^2 + 3^2 + 8^2}{4} - 5^2$$
   $$= 29.5 - 25$$
   $$= 4.5$$
   $$\sigma = \sqrt{4.5} = 2.12$$

   (b) 3   3   4   5   10

   Sol.

   $$\bar{x} = \frac{3 + 3 + 4 + 5 + 10}{5} = 5$$
   $$\sigma^2 = \frac{3^2 + 3^2 + 4^2 + 5^2 + 10^2}{5} - 5^2$$
   $$= 31.8 - 25$$
   $$= 6.8$$
   $$\sigma = \sqrt{6.8} = 2.61$$

(c) 2   9   10   10   12   2   10   9

Sol.

$$\bar{x} = \frac{2 + 9 + \cdots + 10 + 9}{8} = 8$$

$$\sigma^2 = \frac{2^2 + 9^2 + \cdots + 9^2}{8} - 8^2$$

$$= 76.75 - 64$$

$$= 12.75$$

$$\sigma = \sqrt{12.75} = 3.57$$

2. Find the variance and standard deviation of the data:

(a)

Values	Frequency
6	35
5	36
4	30

Sol.

x_i	f_i	$x_i f_i$	$x_i^2 f_i$
6	35	210	1260
5	36	180	900
4	30	120	480
	$\sum f_i = 101$	$\sum x_i f_i = 510$	$\sum x_i^2 f_i = 2640$

$$\bar{x} = \frac{510}{101} = 5.05$$

$$\sigma^2 = \frac{2640}{101} - 5.05^2$$

$$= 26.14 - 25.5$$

$$= 0.64$$

$$\sigma = \sqrt{0.64} = 0.80$$

(b)

Values	Frequency
60	4
70	6
80	2
90	5
100	1

Sol.

x_i	f_i	$x_i f_i$	$x_i^2 f_i$
60	4	240	14400
70	6	420	29400
80	2	160	12800
90	5	450	40500
100	1	100	10000
	$\sum f_i = 18$	$\sum x_i f_i = 1370$	$\sum x_i^2 f_i = 107100$

$$\bar{x} = \frac{1370}{18} = 76.11$$

$$\sigma^2 = \frac{107100}{18} - 76.11^2$$

$$= 5950 - 5792.90$$

$$= 157.10$$

$$\sigma = \sqrt{157.10} = 12.53$$

3. Given two sets of data:

A	B
9.9	10.3
10.3	10
9.8	9.5
10.1	10.4
10.4	10.5
10	9.4
9.8	9.8
9.7	10.1

Find the mean and variance of these two sets of data respectively, and state which set of data is more spread out.

Sol.

For data A,

$$\bar{x} = \frac{9.9 + 10.3 + \cdots + 9.7}{8} = 10$$

$$\sigma^2 = \frac{9.9^2 + 10.3^2 + \cdots + 9.7^2}{8} - 10^2$$

$$= 100.055 - 100$$

$$= 0.06$$

For data B,

$$\bar{x} = \frac{10.3 + 10 + \cdots + 10.1}{8} = 10.0$$

$$\sigma^2 = \frac{10.3^2 + 10^2 + \cdots + 10.1^2}{8} - 10.05^2$$

$$= 100.145 - 100$$

$$= 0.15$$

Since σ^2 of data B is larger than that of data A, data B is more spread out.

4. Given the Chinese language test scores of two groups of students are as follows:

Group A	Group B
76	82
90	84
84	85
86	89
81	79
87	80
86	91
82	89
85	79
83	74

Find the mean and standard deviation of these two sets of data respectively, and state which set of data is more centered.

Sol.

For Group A,

$$\bar{x} = \frac{76 + 90 + \cdots + 83}{10} = 84$$

$$\sigma^2 = \frac{76^2 + 90^2 + \cdots + 83^2}{10} - 84^2$$

$$= 7069.2 - 7056$$

$$= 13.2$$

$$\sigma = \sqrt{13.2} = 3.63$$

For Group B,

$$\bar{x} = \frac{82 + 84 + \cdots + 74}{10} = 83.2$$

$$\sigma^2 = \frac{82^2 + 84^2 + \cdots + 74^2}{10} - 83.2^2$$

$$= 6948.6 - 6922.24$$

$$= 26.36$$

$$\sigma = \sqrt{26.36} = 5.13$$

Since σ of Group A is smaller than that of Group B, Group A is more centered.

5. The table below shows the height distribution of all students of the same grade:

Height (cm)	Frequency
145 - 149	10
150 - 154	36
155 - 159	193
160 - 164	205
165 - 169	240
170 - 174	83
175 - 179	33

Find the mean and standard deviation of the height of all students of the same grade.

Sol.

Range	x_i	f_i	$x_i f_i$	$x_i^2 f_i$
145 - 149	147	10	1470	216090
150 - 154	152	36	5472	831744
155 - 159	157	193	30301	4757257
160 - 164	162	205	33210	5380020
165 - 169	167	240	40080	6693360
170 - 174	172	83	14276	2455472
175 - 179	177	33	5841	1033857
		$\sum f_i = 800$	$\sum f_i x_i = 130650$	$\sum f_i x_i^2 = 21367800$

$$\bar{x} = \frac{130650}{800} = 163.31 cm$$

$$\sigma^2 = \frac{21367800}{800} - 163.31^2$$

$$= 26709.75 - 26670.16$$

$$= 38.78 cm$$

$$\sigma = \sqrt{38.78} = 6.23 cm$$

6. Following are teh weight distribution of 100 students in a school:

Weight (kg)	Number of Students
45 - 47	3
48 - 50	16
51 - 53	20
54 - 56	32
57 - 59	15
60 - 62	10
63 - 65	4

Find the variance and standard deviation.

Sol.

Range	x_i	f_i	$x_i f_i$	$x_i^2 f_i$
45 - 47	46	3	138	6348
48 - 50	49	16	784	38416
51 - 53	52	20	1040	54080
54 - 56	55	32	1760	96800
57 - 59	58	15	870	50460
60 - 62	61	10	610	37210
63 - 65	64	4	256	16384
		$\sum f_i = 100$	$\sum f_i x_i = 5458$	$\sum f_i x_i^2 = 299698$

$$\bar{x} = \frac{5458}{100} = 54.58 kg$$

$$\sigma^2 = \frac{299698}{100} - 54.58^2$$

$$= 2996.98 - 2978.98$$

$$= 18.00 kg$$

$$\sigma = \sqrt{18.00} = 4.24 kg$$

7. Given the sum of 10 values is 400, and the sum of their square is 16400. Find the mean and variance of these 10 values.

   Sol.

   $$\bar{x} = \frac{400}{10} = 40$$

   $$\sigma^2 = \frac{16400}{10} - 40^2$$

   $$= 1640 - 1600$$

   $$= 40$$

8. Given 30 values x_1, x_2, $\cdots$, x_{30}, the mean of these values is 5, and the standard deviation is 2. Find $\sum_{i=1}^{30} x_i$ and $\sum_{i=1}^{30} x_i^2$.

   Sol.

   $$\bar{x} = 5$$

   $$\frac{\sum_{i=1}^{30} x_i}{30} = 5$$

   $$\sum_{i=1}^{30} x_i = 150$$

   $$\sigma = 2$$

   $$\sigma^2 = 4$$

   $$\frac{\sum_{i=1}^{30} x_i^2}{30} - 5^2 = 4$$

   $$\frac{\sum_{i=1}^{30} x_i^2}{30} = 29$$

   $$\sum_{i=1}^{30} x_i^2 = 870$$

9. The mean of 5 values is 10, and it remains the same after adding p to dataset.

   (a) Find the value of p.

      Sol.

Let the dataset be x.

$$\bar{x} = 10$$

$$\frac{\sum_{i=1}^{5} x_i}{5} = 10$$

$$\sum_{i=1}^{5} x_i = 50$$

$$\frac{\sum_{i=1}^{5} x_i + p}{6} = 10$$

$$50 + p = 60$$

$$p = 10$$

   (b) If the sum of square of 5 original values is 558, find the variance of the 6 values after adding p.

      Sol.

      $$\sigma^2 = \frac{\sum_{i=1}^{5} x_i^2 + p^2}{6} - 10^2$$

      $$= \frac{558 + 100}{6} - 100$$

      $$= 109.67 - 100$$

      $$= 9.67$$

10. Given that the mean of 3, 6, 7, 8, 9, 12, 14, 15, x, y is 13, standard deviation is $\sqrt{102}$, find the value of x and y.

   Sol.

   $$\bar{x} = \frac{3 + 6 + \cdots + 15 + x + y}{10} = 13$$

   $$\frac{74 + x + y}{10} = 13$$

   $$74 + x + y = 130$$

   $$x + y = 56$$

   $$\sigma = \sqrt{102}$$

   $$\sigma^2 = 102$$

   $$\frac{3^2 + 6^2 + \cdots + 15^2 + x^2 + y^2}{10} - 13^2 = 102$$

   $$\frac{804 + x^2 + y^2}{10} - 169 = 102$$

   $$\frac{804 + x^2 + y^2}{10} = 271$$

   $$804 + x^2 + y^2 = 2710$$

   $$x^2 + y^2 = 1906$$

$$\begin{cases} x + y = 56 & (1) \\ x^2 + y^2 = 1906 & (2) \end{cases}$$

$$(1) \Rightarrow y = 56 - x \qquad (1)$$

Sub (1) into (2) $\Rightarrow x^2 + (56 - x)^2 = 1906$

$$x^2 + x^2 - 112x + 3136 = 1906$$

$$2x^2 - 112x + 1230 = 0$$

$$x^2 - 56x + 615 = 0$$

$$(x - 15)(x - 41) = 0$$

$$x = 15 \text{ or } x = 41$$

When $x = 15, y = 56 - 15 = 41$

When $x = 41, y = 56 - 41 = 15$

$$\therefore \begin{cases} x = 15 \\ y = 41 \end{cases} \text{ or } \begin{cases} x = 41 \\ y = 15 \end{cases}$$

18.5 Coefficient of Variation

Generally speaking, when we want to compare the variability of two or more sets of data, only comparing the standard deviation of each group is not enough. If the properties or the units of the data are different, the standard deviation of each group must not be comparable. For example, if we want to know whether the deviation of the height of students in a class is larger than that of the weight of students in the same class, we need a relative metric as the standard of comparison, and the coefficient of variation is such a metric. For a non-negative set of value, the definition of coefficient of variation is as follows:

$$v = \frac{\sigma}{\bar{x}} \cdot 100\%$$

From the definition, we can see that the coefficient of variation the standard deviation when the mean is 1. Thus, when the coefficient of variation is large, it means that the variability of the data is large, and vice versa.

18.5.1 Practice 9

In a minor test, the full mark of Chinese language test for senior 2 students is 100, its average mark is 70, and the standard deviation is 10, while the full mark of Mathematics test is 70, its average mark is 40, and the standard deviation is 8. Compare the variability of the two tests.

Sol.

$$v_{\text{Chinese}} = \frac{10}{70} \cdot 100\% = 14.29\%$$

$$v_{\text{Math}} = \frac{8}{40} \cdot 100\% = 20\%$$

$\because v_{\text{Math}} > v_{\text{Chinese}}$

$\therefore$ Mathematics test is more variable than Chinese language test.

18.5.2 Exercise 18.5

1. The statistics of the height and weight if grade 1 students in a primary school are as follows:

	Mean	Standard Deviation
Height (cm)	115.87	4.86
Weight (cm)	19.39	2.16

   Compare the variability of the height and weight of the students.

   Sol.

   $$v_{\text{Height}} = \frac{4.86}{115.87} \cdot 100\% = 4.19\%$$

   $$v_{\text{Weight}} = \frac{2.16}{19.39} \cdot 100\% = 11.14\%$$

   $\because v_{\text{Weight}} > v_{\text{Height}}$

   $\therefore$ The weight of the students is more variable than their height.

2. The table below shows the first semester Mathematics exam average mark and standard deviation of five junior 1 classes in a school:

Class	Average Mark	Standard Deviation
A	62	11
B	74	9
C	65	10
D	70	7
E	53	8

   Which class has the smallest coefficient of variation?

   Sol.

   $$v_{\text{A}} = \frac{11}{62} \cdot 100\% = 17.74\%$$

   $$v_{\text{B}} = \frac{9}{74} \cdot 100\% = 12.16\%$$

   $$v_{\text{C}} = \frac{10}{65} \cdot 100\% = 15.38\%$$

   $$v_{\text{D}} = \frac{7}{70} \cdot 100\% = 10\%$$

   $$v_{\text{E}} = \frac{8}{53} \cdot 100\% = 15.09\%$$

$\because v_D < v_B < v_A < v_C < v_E$

$\therefore$ Class D has the smallest coefficient of variation.

3. The table below shows the Mathematics exam results of two groups of students A and B:

Group	Marks				
A	60	98	76	84	52
B	88	58	90	69	78

(a) Find the average mark of each group.

Sol.

$$\bar{x}_A = \frac{60 + 98 + 76 + 84 + 52}{5} = 74$$

$$\bar{x}_B = \frac{88 + 58 + 90 + 69 + 78}{5} = 76.6$$

(b) Find the standard deviation of each group.

Sol.

$$\sigma_A^2 = \frac{60^2 + \cdots + 52^2}{5} - 74^2$$

$$= 5748 - 5476$$

$$= 272$$

$$\sigma_A = \sqrt{272} = 16.49$$

$$\sigma_B^2 = \frac{88^2 + \cdots + 78^2}{5} - 76.6^2$$

$$= 6010.6 - 5867.56$$

$$= 143.04$$

$$\sigma_B = \sqrt{143.04} = 11.96$$

(c) Find the coefficient of variation of each group.

Sol.

$$v_A = \frac{16.49}{74} \cdot 100\% = 22.28\%$$

$$v_B = \frac{11.96}{76.6} \cdot 100\% = 15.61\%$$

4. The table below shows the price of of papayas and grapes per kilogram in the first half of the year (in \$):

Month	Papaya	Grapes
January	3.50	20.00
February	3.00	22.00
March	2.50	24.00
April	3.20	23.00
May	3.60	18.00
June	2.80	21.00

(a) Find the average price and standard deviation of papayas and grapes respectively in the first half of the year.

Sol.

$$\bar{x}_{\text{Papaya}} = \frac{3.50 + \cdots + 2.80}{6} = 3.10$$

$$\bar{x}_{\text{Grapes}} = \frac{20 + \cdots + 21}{6} = 21.33$$

$$\sigma_{\text{Papaya}}^2 = \frac{3.50^2 + \cdots + 2.80^2}{6} - 3.10^2$$

$$= 9.77 - 9.61$$

$$= 0.15$$

$$\sigma_{\text{Papaya}} = \sqrt{0.15} = 0.38$$

$$\sigma_{\text{Grapes}}^2 = \frac{20^2 + \cdots + 21^2}{6} - 21.33^2$$

$$= 459 - 455.11$$

$$= 3.89$$

$$\sigma_{\text{Grapes}} = \sqrt{3.89} = 1.97$$

(b) Which fruit has greater variability in price?

$\because \sigma_{\text{Papaya}} > \sigma_{\text{Grapes}}$

$\therefore$ Papaya has greater variability in price.

5. The table below shows the distribution of annual average marks of two classes of students A and B:

Marks Range	Class A	Class B
40 - 49	3	4
50 - 59	4	10
60 - 69	10	17
70 - 79	16	14
80 - 89	12	1

Find the coefficient of variation of annual average marks of each class respectively.

Sol.

For class A,

Range	x_i	f_i	$f_i x_i$	$f_i x_i^2$
40 - 49	44.5	3	133.5	5940.75
50 - 59	54.5	4	218	11881
60 - 69	64.5	10	645	41602.5
70 - 79	74.5	16	1192	88804
80 - 89	84.5	12	1014	85683
		$\sum f_i = 45$	$\sum f_i x_i = 3202.5$	$\sum f_i x_i^2 = 233911.25$

$$\bar{x}_A = \frac{3202.5}{45} = 71.17$$

$$\sigma_A^2 = \frac{233911.25}{45} - 71.17^2$$

$$= 5198.03 - 5064.69$$

$$= 133.34$$

$$\sigma_A = \sqrt{133.34} = 11.55$$

$$v_A = \frac{11.55}{71.17} \cdot 100\% = 16.23\%$$

For class B,

Range	x_i	f_i	$f_i x_i$	$f_i x_i^2$
40 - 49	44.5	4	178	7921
50 - 59	54.5	10	545	29702.5
60 - 69	64.5	17	1096.5	70724.25
70 - 79	74.5	14	1043	77703.5
80 - 89	84.5	1	84.5	7140.25
		$\sum f_i = 46$	$\sum f_i x_i = 2947$	$\sum f_i x_i^2 = 4193191.5$

$$\bar{x}_B = \frac{2947}{46} = 64.07$$

$$\sigma_B^2 = \frac{193191.5}{46} - 64.07^2$$

$$= 4199.82 - 4104.96$$

$$= 95.46$$

$$\sigma_B = \sqrt{95.46} = 9.77$$

$$v_B = \frac{9.77}{64.07} \cdot 100\% = 15.25\%$$

18.6 Correlation and Correlation Coefficient

Correlation

In statistics, correlation is a statistical measure of the degree to which two or more variables move in relation to each other. For example, the correlation between the height and weight of a person, the correlation between the price of a stock and the volume of the stock traded.

Scatter Plot

A scatter plot is a type of mathematical diagram to show the relationship between two variables. Let two groups of data be $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$, respectively. The scatter plot of the two groups of data is a graph of the points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$.

Linear Correlation

If the scatter plot of two groups of data can be approximated by a straight line, then the two groups of data are said to be linearly correlated. According to the trend of the two groups of data, the correlation can be positive, negative, or zero. For example, the weight of a higher person is usually larger, so the correlation between the weight and height of a person is positive. The sales of a product are usually lower when the price of the product is higher, so the correlation between the price of a product and the volume of the product sold is negative. If there is no relationship between the two groups of data, then it is considered zero correlation.

Below are the possible cases of linear correlation:
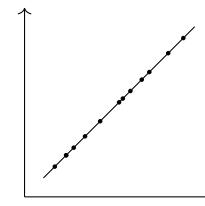
Perfect Positive Correlation
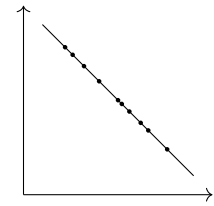
Perfect Negative Correlation
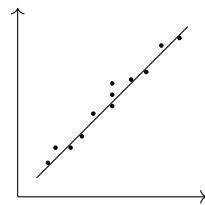
Strong Positive Correlation
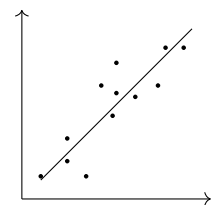
Moderate Positive Correlation
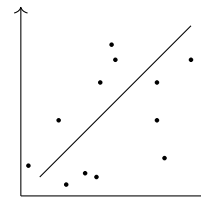
Weak Positive Correlation
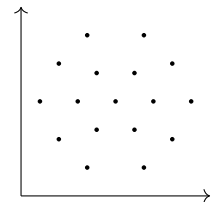
Zero Correlation

1. If every single point in the scatter plot is on the line of best fit, then it's a perfect positive correlation. If the slope of the line of best fit is positive, then it's a positive correlation. If the slope of the line of best fit is negative, then it's a negative correlation.

2. If the points in the scatter plot are scattered around the line of best fit with non-zero slope, then the closer the points are to the line of best fit, the stronger the correlation is.

3. If the points in the scatter plot are scattered evenly around the whole plot with no obvious pattern, then there is no correlation between the two variables, aka zero correlation.

Correlation Coefficient

Telling the correlation between two variables by looking at the scatter plot is not a very accurate way. To accurately measure the correlation between two sets of data, we need to use a coefficient that can distinguish the strength of the correlation.
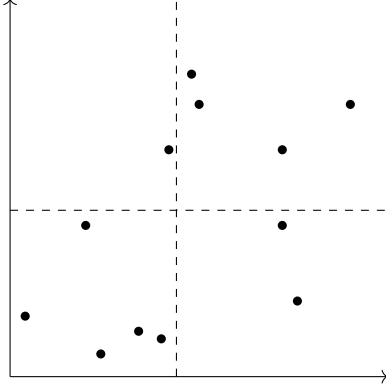
Let the mean value of two sets of data be $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be $\bar{x}$ and $\bar{y}$ respectively. Draw two lines $x = \bar{x}$ and $y = \bar{y}$ on the scatter plot of the two sets of data, splitting the plot into four quadrants, as shown in the figure above. Now the origin of the plot is at $(\bar{x}, \bar{y})$. If a point (x_i, y_i) is in the first or the third quadrant, then $(x_i - \bar{x})(y_i - \bar{y})$ is positive. As discussed in the previous section, if the correlation is positive, the points are scattering around the line of best fit with positive slope. Therefore, the points are more likely to be in the first or the third quadrant. That means, there are more positive value of $(x_i - \bar{x})(y_i - \bar{y})$ than negative value, therefore the value of $\sum (x_i - \bar{x})(y_i - \bar{y})$ is positive. The higher the correlation is, the more points are in the first or the third quadrant, the higher the positive value of $\sum (x_i - \bar{x})(y_i - \bar{y})$ is.

On the other hand, if a point (x_i, y_i) is in the second or the fourth quadrant, then $(x_i - \bar{x})(y_i - \bar{y})$ is negative, which means there are more negative value of $(x_i - \bar{x})(y_i - \bar{y})$ than positive value, therefore the value of $\sum (x_i - \bar{x})(y_i - \bar{y})$ is negative. Similarly, he higher the correlation is, the lower the negative value of $\sum (x_i - \bar{x})(y_i - \bar{y})$ is.

Hence, the value and the sign of $\sum (x_i - \bar{x})(y_i - \bar{y})$ can be used to measure the correlation between two sets of data. The value of $\sum (x_i - \bar{x})(y_i - \bar{y})$ will be affected by the measurement unit of the data. To make the value of $\sum (x_i - \bar{x})(y_i - \bar{y})$ independent of the measurement unit, we define the correlation coefficient of two sets of data $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ as:

$$r = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}}$$

The value of r is always between -1 and 1. If $r = 0$, then there is no correlation between the two sets of data. If $r > 0$, then the correlation is positive. If $r < 0$, then the correlation is negative. The absolute value of r is the strength of the correlation, and is generally divided as follows:

1. $|r| = 1$: perfect correlation

2. $0 < |r| < 0.3$: weak correlation

3. $0.3 \leq |r| < 0.7$: moderate correlation

4. $0.7 \leq |r| \leq 1$: strong correlation

Dividing both the denominator and the numerator of the formula of r by the number of data points n, then the numerator is the mean value of $(x_i - \bar{x})(y_i - \bar{y})$, and the denominator is the product of the standard deviation of $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. Similar to the standard deviation, there is an easier way to calculate the correlation coefficient:

$$r = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}}$$

$$= \frac{\sum (x_i y_i - x_i \bar{y} - \bar{x} y_i + \bar{x}\bar{y})}{\sqrt{\sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2) \sum (y_i^2 - 2y_i\bar{y} + \bar{y}^2)}}$$

$$= \frac{\sum x_i y_i - \bar{y}\sum x_i - y_i \sum \bar{x} + \sum \bar{x}\bar{y}}{\sqrt{\sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2) \sum (y_i^2 - 2y_i\bar{y} + \bar{y}^2)}}$$

$$= \frac{\frac{\sum x_i y_i - \bar{y}\sum x_i - y_i \sum \bar{x} + \sum \bar{x}\bar{y}}{n}}{\frac{\sqrt{\sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2) \sum (y_i^2 - 2y_i\bar{y} + \bar{y}^2)}}{n}}$$

$$= \frac{\frac{\sum x_i y_i}{n} - \frac{\bar{y}\sum x_i}{n} - \frac{y_i \sum \bar{x}}{n} + \frac{\sum \bar{x}\bar{y}}{n}}{\sqrt{\frac{\sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2) \sum (y_i^2 - 2y_i\bar{y} + \bar{y}^2)}{n \cdot n}}}$$

$$= \frac{\frac{\sum x_i y_i}{n} - \frac{\bar{y}\sum x_i}{n} - \frac{\bar{x}\sum y_i}{n} + \frac{n\bar{x}\bar{y}}{n}}{\sqrt{\left(\frac{\sum x_i^2}{n} - \frac{2\bar{x}\sum x_i}{n} + \frac{\sum \bar{x}^2}{n}\right)\left(\frac{\sum y_i^2}{n} - \frac{2\bar{y}\sum y_i}{n} + \frac{\sum \bar{y}^2}{n}\right)}}$$

$$= \frac{\frac{\sum x_i y_i}{n} - \bar{x}\bar{y} - \bar{x}\bar{y} + \bar{x}\bar{y}}{\sqrt{\left(\frac{\sum x_i^2}{n} - 2\bar{x}^2 + \frac{n\bar{x}^2}{n}\right)\left(\frac{\sum y_i^2}{n} - 2\bar{y}^2 + \frac{n\bar{y}^2}{n}\right)}}$$

$$= \frac{\frac{\sum x_i y_i}{n} - \bar{x}\bar{y}}{\sqrt{\left(\frac{\sum x_i^2}{n} - 2\bar{x}^2 + \bar{x}^2\right)\left(\frac{\sum y_i^2}{n} - 2\bar{y}^2 + \bar{y}^2\right)}}$$

$$= \frac{\frac{\sum x_i y_i}{n} - \bar{x}\bar{y}}{\sqrt{\left(\frac{\sum x_i^2}{n} - \bar{x}^2\right)\left(\frac{\sum y_i^2}{n} - \bar{y}^2\right)}}$$

18.6.1 Practice 10

1. The table below shows the height (in *cm*) and weight (in *kg*) of 15 10-year-old children:

Height	Weight
126	41
130	42
110	38
123	36
118	33
130	45
127	34
124	35
116	30
112	32
113	31
121	40
115	34
120	35
118	33

Calculate the correlation coefficient of the height and the weight of the 15 children, and determine on the strength of the correlation.

Sol.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
126	41	15876	1681	5166
130	42	16900	1764	5460
110	38	12100	1444	4180
123	36	15129	1296	4428
118	33	13924	1089	3894
130	45	16900	2025	5850
127	34	16129	1156	4318
124	35	15376	1225	4340
116	30	13456	900	3480
112	32	12544	1024	3584
113	31	12769	961	3503
121	40	14641	1600	4840
115	34	13225	1156	3910
120	35	14400	1225	4200
118	33	13924	1089	3894
$\sum x_i$	$\sum y_i$	$\sum x_i^2$	$\sum y_i^2$	$\sum x_i y_i$
1803	539	217293	19635	65047

$$\bar{x} = \frac{1803}{15} = 120.2$$

$$\bar{y} = \frac{539}{15} = 35.93$$

$$r = \frac{\frac{65047}{15} - 120.2 \cdot 35.93}{\sqrt{\left(\frac{217293}{15} - 120.2^2\right)\left(\frac{19635}{15} - 35.93^2\right)}}$$

$$= 0.6631$$

According to the result of the calculation, the height and the weight of the 15 children are positively and moderetely correlated.

2. In order to study the relationship between the systolic blood pressure (in *mmHg*) and the age (in *year*) of human, a medical school collected the data of 13 male patients:

Age	Systolic Blood Pressure
51	130
22	141
23	124
31	126
33	117
49	135
58	143
53	138
44	132
55	143
42	133
45	115
25	147

(a) Construct a scatter diagram of the data.

Sol.

Scattering Plot of Systolic Blood Pressure and Age
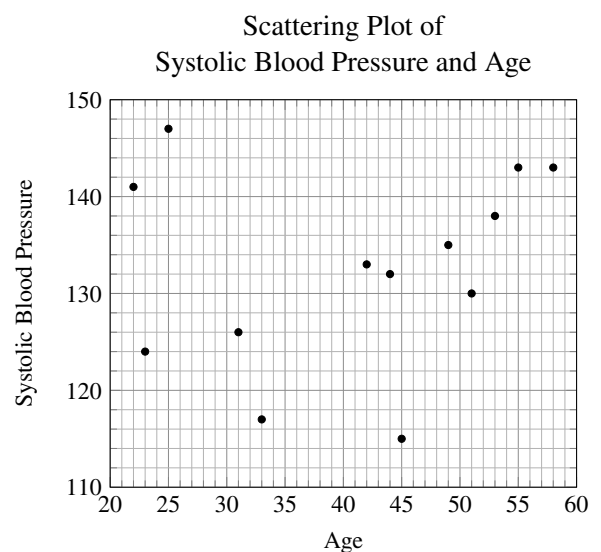
(b) Calculate the correlation coefficient of the age and the systolic blood pressure of the 13 patients, and determine on the strength of the correlation.

Sol.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
51	130	2601	16900	6630
22	141	484	19881	3102
23	124	529	15376	2852
31	126	961	15876	3906
33	117	1089	13689	3861
49	135	2401	18225	6615
58	143	3364	20449	8294
53	138	2809	19044	7314
44	132	1936	17424	5808
55	143	3025	20449	7865
42	133	1764	17689	5586
45	155	2025	24025	6975
25	147	625	21609	3675
$\sum x_i$	$\sum y_i$	$\sum x_i^2$	$\sum y_i^2$	$\sum x_i y_i$
531	1764	23613	240636	72483

$$\bar{x} = \frac{531}{13} = 40.85$$

$$\bar{y} = \frac{1764}{13} = 135.69$$

$$r = \frac{\frac{72483}{13} - 40.85 \cdot 135.69}{\sqrt{\left(\frac{23613}{13} - 40.85^2\right)\left(\frac{240636}{13} - 135.69^2\right)}}$$

$$= 0.2748$$

According to the result of the calculation, the height and the weight of the 15 children are positively and weakly correlated.

18.6.2 Exercise 18.6

1. The table below shows the value of fixed assets and total assets (in 10 thousand dollar) of 10 enterprises of the same industry:

No. of Enterprise	Fixed Assets	Total Assets
1	200	638
2	314	605
3	318	524
4	409	815
5	415	913
6	502	928
7	910	1019
8	1022	1219
9	1210	1516
10	1225	1624

(a) Construct a scatter diagram of the data.

Sol.
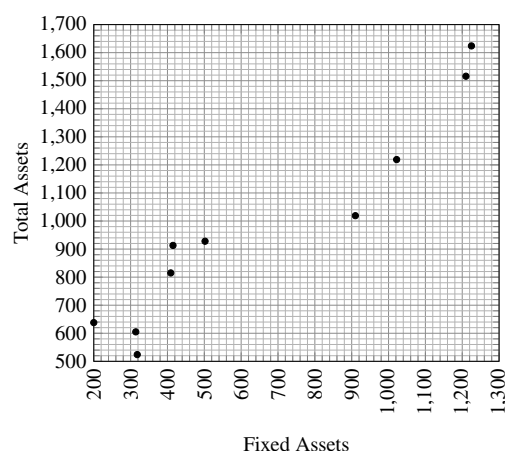
(b) Calculate the mean value of fixed assets and total assets respectively.

Sol.

Mean of fixed assets $\bar{x} = \dfrac{200 + 314 + \cdots + 1225}{10}$

$$= \frac{6525}{10}$$

$$= 652.5$$

$$= \$652,500,000$$

Mean of total assets $\bar{y} = \dfrac{638 + 605 + \cdots + 1624}{10}$

$$= \frac{9801}{10}$$

$$= 980.1$$

$$= \$980,100,000$$

(c) Find the correlation coefficient of the fixed assets and the total assets, and determine on the strength of the correlation.

Sol.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
200	638	40000	407044	127600
314	605	98596	366025	189970
318	524	101124	274576	166632
409	815	167281	664225	333335
415	913	172225	833569	378895
502	928	252004	861184	465856
910	1019	828100	1038361	927290
1022	1219	1044484	1485961	1245818
1210	1516	1464100	2298256	1834360
1225	1624	1500625	2637376	1989400
$\sum x_i$	$\sum y_i$	$\sum x_i^2$	$\sum y_i^2$	$\sum x_i y_i$
6525	9801	5668539	10866577	7659156

$$r = \frac{\frac{7659156}{13} - 652.5 \cdot 980.1}{\sqrt{\left(\frac{5668539}{13} - 652.5^2\right)\left(\frac{10866577}{13} - 980.1^2\right)}}$$

$$= 0.9478$$

According to the result of the calculation, According to the result of the calculation, the value of fixed assets and total assets of theese 10 enterprises are positively and strongly correlated.

2. The table shows the marks of Mathematics and Ecomoics of 15 students:

Mathematics	Economics
83	79
50	61
62	70
90	86
68	69
61	68
58	62
62	80
71	70
63	74
72	77
54	54
64	77
48	50
81	92

(a) Construct a scatter diagram of the data.

Sol.
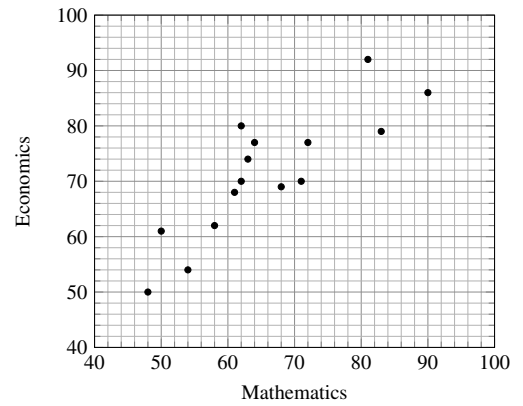
(b) Find the correlation coefficient of the marks of Mathematics and the Economics, and determine on the strength of the correlation.

Sol.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
83	79	6889	6241	6557
50	61	2500	3721	3050
62	70	3844	4900	4340
90	86	8100	7396	7740
68	69	4624	4761	4692
61	68	3721	4624	4148
58	62	3364	3844	3596
62	80	3844	6400	4960
71	70	5041	4900	4970
63	74	3969	5476	4662
72	77	5184	5929	5544
54	54	2916	2916	2916
64	77	4096	5929	4928
48	50	2304	2500	2400
81	92	6561	8464	7452
$\sum x_i$	$\sum y_i$	$\sum x_i^2$	$\sum y_i^2$	$\sum x_i y_i$
987	1069	66957	78001	71955

$$\bar{x} = \frac{987}{15} = 65.8$$

$$\bar{y} = \frac{1069}{15} = 71.27$$

$$r = \frac{\frac{71955}{15} - 65.8 \cdot 71.27}{\sqrt{\left(\frac{66957}{15} - 65.8^2\right)\left(\frac{78001}{15} - 71.27^2\right)}}$$

$$= 0.8445$$

According to the result of the calculation, the marks of Mathematics and the Economics are positively and strongly correlated.

3. The table below shows the marks of 16 students in the Chinese language minor test. The paper was split into two sections: Vernacular and Classical Chinese and their full marks were 60 and 40 respectively.

Vernacular Chinese	Classical Chinese
43	30
50	21
38	20
45	19
58	15
47	18
32	30
36	28
38	26
51	30
44	29
28	29
49	22
42	32
46	33
35	25

(a) Construct a scatter diagram of the data.

Sol.
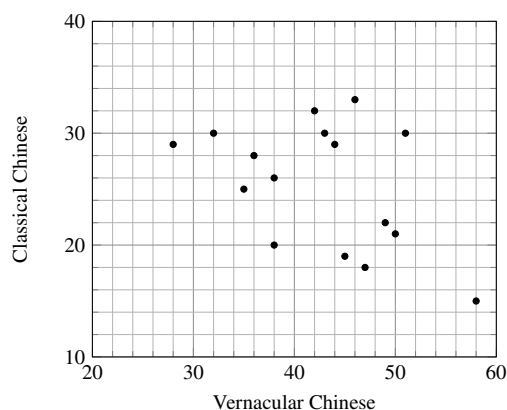
(b) Find the correlation coefficient of the marks of Vernacular Chinese and the Classical Chinese, and determine on the strength of the correlation.

Sol.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
43	30	1849	900	1290
50	21	2500	441	1050
38	20	1444	400	760
45	19	2025	361	855
58	15	3364	225	870
47	18	2209	324	846
32	30	1024	900	960
36	28	1296	784	1008
38	26	1444	676	988
51	30	2601	900	1530
44	29	1936	841	1276
28	29	784	841	812
49	22	2401	484	1078
42	32	1764	1024	1344
46	33	2116	1089	1518
35	25	1225	625	875
$\sum x_i$	$\sum y_i$	$\sum x_i^2$	$\sum y_i^2$	$\sum x_i y_i$
682	407	29982	10815	17060

$$\bar{x} = \frac{682}{16} = 42.62$$

$$\bar{y} = \frac{407}{16} = 25.44$$

$$r = \frac{\frac{17060}{16} - 42.62 \cdot 25.44}{\sqrt{\left(\frac{29982}{16} - 42.62^2\right)\left(\frac{10815}{16} - 25.44^2\right)}}$$

$$= -0.4444$$

According to the result of the calculation, the marks of Vernacular Chinese and Classical Chinese are negatively and moderately correlated.

4. Below shows the the service costs and values of properties sold by a property broker in 5 trades:

Service Costs (in $100)	Value of Prop. (in $10k)
16.5	3.9
17.4	4.2
16.8	4.1
17.9	4.5
18.4	4.8

Find the correlation coefficient of the service costs and the values of properties in these 5 trades, and determine on the strength of the correlation.

Sol.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
16.5	3.9	272.25	15.21	64.35
17.4	4.2	302.76	17.64	73.08
16.8	4.1	282.24	16.81	68.88
17.9	4.5	320.41	20.25	80.55
18.4	4.8	338.56	23.04	88.32
$\sum x_i$	$\sum y_i$	$\sum x_i^2$	$\sum y_i^2$	$\sum x_i y_i$
87.0	21.5	1516.22	92.95	375.18

$$\bar{x} = \frac{87.0}{5} = 17.4$$

$$\bar{y} = \frac{21.5}{5} = 4.3$$

$$r = \frac{\frac{375.18}{5} - 17.4 \cdot 4.3}{\sqrt{\left(\frac{1516.22}{5} - 17.4^2\right)\left(\frac{92.95}{5} - 4.3^2\right)}}$$

$$= 0.9818$$

According to the result of the calculation, the service costs and the values of properties are positively and strongly correlated.

5. The table below shows the degree of labor mechanization and labor productivity:

Mechanization Degree (%)	Productivity ($/pax)
40	800
45	880
50	1010
55	1034
60	980
65	1030
70	1077
75	1344
80	1460

(a) Construct a scatter diagram of the data.

   Sol.
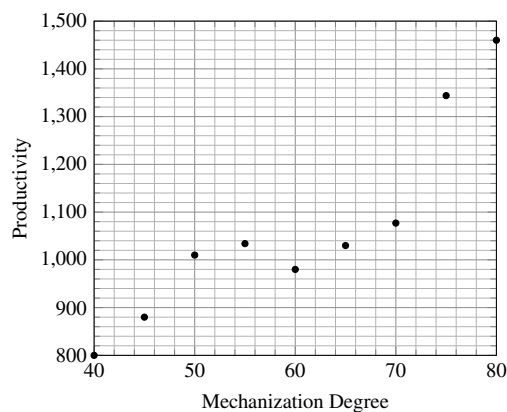
(b) Find the correlation coefficient of the degree of labor mechanization and the labor productivity, and determine on the strength of the correlation.

   Sol.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
40.0	800.0	1600.0	640000.0	32000.0
45.0	880.0	2025.0	774400.0	39600.0
50.0	1010.0	2500.0	1020100.0	50500.0
55.0	1034.0	3025.0	1069156.0	56870.0
60.0	980.0	3600.0	960400.0	58800.0
65.0	1030.0	4225.0	1060900.0	66950.0
70.0	1077.0	4900.0	1159929.0	75390.0
75.0	1344.0	5625.0	1806336.0	100800.0
80.0	1460.0	6400.0	2131600.0	116800.0
$\sum x_i$	$\sum y_i$	$\sum x_i^2$	$\sum y_i^2$	$\sum x_i y_i$
540.0	9615.0	33900.0	10622821.0	597710.0

$$\bar{x} = \frac{540.0}{9} = 60.0$$

$$\bar{y} = \frac{9615.0}{9} = 1068.33$$

$$r = \frac{\frac{597710.0}{9} - 60.0 \cdot 1068.33}{\sqrt{\left(\frac{33900.0}{9} - 60.0^2\right)\left(\frac{10622821.0}{9} - 1068.33^2\right)}}$$

$$= 0.9072$$

According to the result of the calculation, the degree of labor mechanization and the labor productivity are positively and strongly correlated.

6. Below are the sales (in million) and the net profit rate (%) of 10 department store:

Company	Sales	Net Profit Rate
A	18.4	15.3
B	16.5	14.8
C	14.6	13.6
D	23.3	14.3
E	35.6	12.9
F	24.2	14.6
G	33.6	13.8
H	44.5	13.7
I	26.8	13.5
J	31.9	14.2

(a) Construct a scatter diagram of the data.

   Sol.
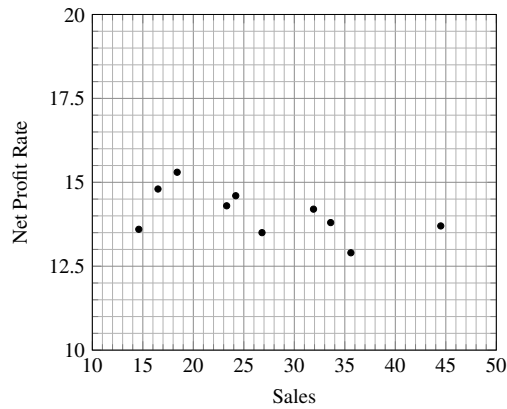
(b) Find the correlation coefficient of the sales and the net profit rate, and determine on the strength of the correlation.

Sol.

x_i	y_i	x_i^2	y_i^2	x_iy_i
18.4	15.3	338.56	234.09	281.52
16.5	14.8	272.25	219.04	244.2
14.6	13.6	213.16	184.96	198.56
23.3	14.3	542.89	204.49	333.19
35.6	12.9	1267.36	166.41	459.24
24.2	14.6	585.64	213.16	353.32
33.6	13.8	1128.96	190.44	463.68
44.5	13.7	1980.25	187.69	609.65
26.8	13.5	718.24	182.25	361.8
31.9	14.2	1017.61	201.64	452.98
$\sum x_i$	$\sum y_i$	$\sum x_i^2$	$\sum y_i^2$	$\sum x_iy_i$
269.4	140.7	8064.92	1984.17	3758.14

$$\bar{x} = \frac{269.4}{10} = 26.94$$

$$\bar{y} = \frac{140.7}{10} = 14.07$$

$$r = \frac{\frac{3758.14}{10} - 26.94 \cdot 14.07}{\sqrt{\left(\frac{8064.92}{10} - 26.94^2\right)\left(\frac{1984.17}{10} - 14.07^2\right)}}$$

$$= -0.5350$$

According to the result of the calculation, the sales and the net profit rate are negatively and moderately correlated.

18.7   Statistical Index

Index

In statistics, an index is a number that measures the changes in a figure from one point in time to another. There is a wide range of applications of an index, such as the price index which represents the changes in prices, the production index which represents the changes in production, and the wage index which represents the changes in salaries and wages. There are also other indices such as the living index, foreign exchange index, population index, stock market index, etc.

The index is a kind of relative number. The standard period that is used for comparison when calculating the index is called the base period. As the case may be, the base period can be a year or a month. The index of the base period is usually a number that is easier to be remembered and compared, such as 100, 500, or 1000, and the chosen number must be able to represent the changes in the figure. We will use 100 as our base period index. The period that is used for comparison to the base period is called the current period. Let Q_0 be the base period index and Q_1 be the current period index. The index of the current period is calculated by the following formula:

$$I = \frac{Q_1}{Q_0} \cdot 100$$

where 100 is the base period index.

Relative price

The relative price is a simple index that compares the prices of products in different periods. Let P_0 be the price of a product in the base period and P_1 be the price of the same product in the current period. The price relative of the current period is calculated by the following formula:

$$\text{P.R.} = \frac{P_1}{P_0} \cdot 100$$

18.7.1   Practice 11

The table below shows the net profits (in million) of a company from 2010 to 2014. Use the year 2010 as the base period and calculate the index of the net profit of the company in each year.

Year	2010	2011	2012	2013	2014
Net Profit	700	621	584.1	720.5	800

Sol.

$$\text{P.R.}_{2011} = \frac{621}{700} \cdot 100 = 88.71$$

$$\text{P.R.}_{2012} = \frac{584.1}{700} \cdot 100 = 83.44$$

$$\text{P.R.}_{2013} = \frac{720.5}{700} \cdot 100 = 102.93$$

$$\text{P.R.}_{2014} = \frac{800}{700} \cdot 100 = 114.29$$

18.7.2 Exercise 18.7a

1. The prices of white sugar in 2011, 2012, and 2013 are \$2.10, \$2.30, and \$2.50 respectively. Use the year 2011 and 2012 as the base period and calculate the relative price of the year 2013.

Sol.

$$P.R. \text{ with respect to } 2011 = \frac{2.50}{2.10} \cdot 100 = 119.05$$

$$P.R. \text{ with respect to } 2012 = \frac{2.50}{2.30} \cdot 100 = 108.70$$

2. The prices of a food product in 2011, 2013, and 2015 are \$3.40, \$3.75, and \$3.90 respectively. Use the year 2011 as the base period and calculate the relative price of the year 2013 and 2015.

Sol.

$$P.R. \text{ of } 2013 = \frac{3.75}{3.40} \cdot 100 = 110.29$$

$$P.R. \text{ of } 2015 = \frac{3.90}{3.40} \cdot 100 = 114.71$$

3. The number of new students of a school from 2011 to 2015 are as follows:

Year	2011	2012	2013	2014	2015
New Stud.	182	150	120	104	94

Use the year 2011 as the base period and calculate the index of the number of new students in each year.

Sol.

$$\text{New Stud. Index of } 2012 = \frac{150}{182} \cdot 100 = 82.42$$

$$\text{New Stud. Index of } 2013 = \frac{120}{182} \cdot 100 = 65.93$$

$$\text{New Stud. Index of } 2014 = \frac{104}{182} \cdot 100 = 57.14$$

$$\text{New Stud. Index of } 2015 = \frac{94}{182} \cdot 100 = 51.64$$

4. The table below shows the price of terraced houses (in \$10k) of a place in from 2019 to 2014:

Year	2010	2011	2012	2013	2014
Price	32.0	35.5	43.4	51.0	60.0

Use the year 2010 as the base period and calculate the relative price of terraced houses in each year.

Sol.

$$P.R. \text{ of } 2011 = \frac{35.5}{32.0} \cdot 100 = 110.94$$

$$P.R. \text{ of } 2012 = \frac{43.4}{32.0} \cdot 100 = 135.63$$

$$P.R. \text{ of } 2013 = \frac{51.0}{32.0} \cdot 100 = 159.38$$

$$P.R. \text{ of } 2014 = \frac{60.0}{32.0} \cdot 100 = 187.50$$

5. The table below shows the relative price of three products A, B, and C when using different years as the base period and the current period:

Current Period	Base Period	A	B	C
2010	2005	160	x	170
2015	2005	140	190	y
2015	2010	z	210	150

Find the value of x, y, and z.

Sol.

Let the price of A in 2005, 2010, and 2015 be $a = 100$, b, and c respectively. From the statement above, we have

$$\frac{b}{a} \cdot 100 = \frac{b}{100} \cdot 100$$

$$b = 160$$

$$\frac{c}{b} \cdot 100 = \frac{c}{160} \cdot 100$$

$$c = 140$$

$$z = \frac{c}{b} \cdot 100$$

$$= \frac{140}{160} \cdot 100$$

$$= 87.5$$

Let the price of B in 2005, 2010, and 2015 be a, b, and $c = 100$ respectively. From the statement above, we

have

$$\frac{c}{a} \cdot 100 = \frac{100}{a} \cdot 100$$

$$\frac{100}{a} \cdot 100 = 190$$

$$1.9a = 100$$

$$a = \frac{100}{1.9}$$

$$\frac{c}{b} \cdot 100 = \frac{100}{b} \cdot 100$$

$$\frac{100}{b} \cdot 100 = 210$$

$$2.1b = 100$$

$$b = \frac{100}{2.1}$$

$$x = \frac{b}{a} \cdot 100$$

$$= \frac{100}{2.1} \cdot \frac{1.9}{100} \cdot 100$$

$$= \frac{1.9}{2.1} \cdot 100$$

$$= 90.48$$

Let the price of C in 2005, 2010, and 2015 be a, $b = 100$, and c respectively. From the statement above, we have

$$\frac{b}{a} \cdot 100 = \frac{100}{a} \cdot 100$$

$$\frac{100}{a} \cdot 100 = 170$$

$$1.7a = 100$$

$$a = \frac{100}{1.7}$$

$$\frac{c}{b} \cdot 100 = \frac{c}{100} \cdot 100$$

$$c = 150$$

$$y = \frac{c}{a} \cdot 100$$

$$= 150 \cdot \frac{1.7}{100} \cdot 100$$

$$= 255$$

$$\therefore x = 90.48, \ y = 255, \ z = 87.5$$

Composite Index

The composite index is the mean value of indices of different figures. Since the importance of each figure might be different, the weight of each index is used to represent the importance of each figure, and the acquired weighted mean is called the composite index.

Let the simple index of n figures of the same base period

and the same current period be $x_1, x_2, \ldots, x_n$, and their respective weights be $w_1, w_2, \ldots, w_n$. The composite index is calculated by the following formula:

$$\text{Composite Index} = \frac{w_1 x_1 + w_2 x_2 + \cdots + w_n x_n}{w_1 + w_2 + \cdots + w_n}$$

$$= \frac{\sum w_i x_i}{\sum w_i}$$

If the study object is some product, where x_i is the relative price to the ith product, then its weighted mean is called the price index. If the study object is the daily living expenses, then its weighted mean is called the living consumption index.

18.7.3   Practice 12

The table below shows the prices and weights of sneakers of three brands in 2012 and 2015:

Sneakers	Unit Price		Weight
	2012	2015	
A	230	233	5
B	225	228	3
C	215	221	2

1. Use the year 2012 as the base period and calculate the relative price of each brand in 2015.

   Sol.

$$P.R. \text{ of } A = \frac{233}{230} \cdot 100 = 101.30$$

$$P.R. \text{ of } B = \frac{228}{225} \cdot 100 = 101.33$$

$$P.R. \text{ of } C = \frac{221}{215} \cdot 100 = 102.79$$

2. Use the year 2012 as the base period and calculate the price index of sneakers in 2015.

   Sol.

Sneakers	P.R. x_i	Weight w_i	$w_i x_i$
A	101.30	5	506.5
B	101.33	3	303.99
C	102.79	2	205.58
		$\sum w_i$	$\sum w_i x_i$
		10	1016.07

$$\text{Price Index} = \frac{1016.07}{10}$$

$$= 101.61$$

18.7.4 Exercise 18.7b

1. Using 2012 as the base period, the relative prices of foods, gases and clothes in 2014 are 111, 105, and 106 respectively, and their weights are 5, 1, and 2 respectively. Calculate the composite index of the three consumer items in 2014.

   Sol.

Item	P.R. x_i	Weight w_i	$w_i x_i$
Food	111	5	555
Gas	105	1	105
Cloth	106	2	212
		$\sum w_i$	$\sum w_i x_i$
		8	872

   $$\text{Composite Index} = \frac{872}{8}$$
   $$= 109$$

2. The table below shows the price of each primary food in 2015 (with 2005 as the base period). Find the price index in 2015.

Food	Relative price	Weight
Meat	130	15
Fish	150	14
Vegetable	200	10
Rice	110	20
Cooking Oil	120	8
Beverage	150	7
Fruit	160	6

   Sol.

Food	P.R. x_i	Weight w_i	$w_i x_i$
Meat	130	15	1950
Fish	150	14	2100
Vegetable	200	10	2000
Rice	110	20	2200
Cooking Oil	120	8	960
Beverage	150	7	1050
Fruit	160	6	960
		$\sum w_i$	$\sum w_i x_i$
		80	11200

   $$\text{Price Index} = \frac{11200}{80}$$
   $$= 140.25$$

3. The weight and unit price of 3 kind of materials bought by a factory are as follows:

Material	Weight (ton)	Unit Price ($) 2010	Unit Price ($) 2014
A	20	0.62	0.71
B	50	2.05	2.09
C	60	0.80	0.85

   Using 2010 as the base period, 2014 as the current period,

   (a) Find the composite index of the unit prices of the three materials without considering the weights (i.e. the weights are all 1).

   Sol.

   $$P.R. \text{ of } A = \frac{0.71}{0.62} \cdot 100$$
   $$= 114.52$$
   $$P.R. \text{ of } B = \frac{2.09}{2.05} \cdot 100$$
   $$= 101.95$$
   $$P.R. \text{ of } C = \frac{0.85}{0.80} \cdot 100$$
   $$= 106.25$$

Material	P.R. x_i	Weight w_i	$w_i x_i$
A	114.52	1	114.52
B	101.95	1	101.95
C	106.25	1	106.25
		$\sum w_i$	$\sum w_i x_i$
		3	322.72

   $$\text{Composite Index} = \frac{322.72}{3}$$
   $$= 107.57$$

   (b) Using the weight of each material as the weight, find the composite index of the unit prices of the three materials.

   Sol.

Material	P.R. x_i	Weight w_i	$w_i x_i$
A	114.52	20	2290.4
B	101.95	50	5097.5
C	106.25	60	6375
		$\sum w_i$	$\sum w_i x_i$
		130	13762.9

$$\text{Composite Index} = \frac{13762.9}{130}$$
$$= 105.87$$

4. The table below shows three indices and their weights. If their composite index is 103, find the value of x.

Index	90	11x	120
Weight	x	4	6

Sol.

$$\frac{x \cdot 90 + 4 \cdot 11 + 6 \cdot 120}{x + 4 + 6} = 103$$
$$\frac{90x + 44x + 720}{x + 10} = 103$$
$$134x + 720 = 103x + 1030$$
$$31x = 310$$
$$x = 10$$

5. The table below shows the relative price and weight of three products with 2013 as the base period and 2015 as the current period. Given that the price of item A in 2013 and 2015 are \$20 and \$25 respectively, the price relative of item B is twice the price of item A.

Item	Relative price	Weight
A	r	2
B	t	1
C	120	3

(a) Find the value of r adn t.

$$r = \frac{25}{20} \cdot 100$$
$$= 125$$
$$t = 2 \cdot 125$$
$$= 250$$

(b) Using 2013 as the base period, find the price index in 2015.

Sol.

Item	P.R. x_i	Weight w_i	$w_i x_i$
A	125	2	250
B	250	1	250
C	120	3	360
		$\sum w_i$	$\sum w_i x_i$
		6	860

$$\text{Price Index} = \frac{860}{6}$$
$$= 143.33$$

6. The table below shows the relative price and weight of 5 products with 2012 as the base period and 2014 as the current period:

Item	Relative price	Weight
A	125	2
B	120	3x
C	110	2
D	130	x
E	115	2

Given that the price index in 2014 is 120,

(a) Find the value of x.

Sol.

$$\frac{2 \cdot 125 + 3x \cdot 120 + \cdots + 2 \cdot 115}{2 + 3x + 2 + x + 2} = 120$$
$$\frac{250 + 360x + 220 + 130x + 230}{6 + 4x} = 120$$
$$\frac{125 + 180x + 110 + 65x + 115}{3 + 2x} = 120$$
$$350 + 245x = 360 + 240x$$
$$5x = 10$$
$$x = 2$$

(b) Assume that the price of item A in 2014 is \$30, find the price of the item in 2012.

Sol.

Let the price of item A in 2012 be x.

$$\frac{30}{x} \cdot 100 = 125$$
$$125x = 3000$$
$$x = 24$$

Therefore, the price of item A in 2012 is \$24.

7. The table below shows the price, relative price and weight of 4 products in 2012 and 2014:

Item	Price (\$)		Relative price	Weight
	2012	2014		
A	12	y	150	1
B	x	24	120	2
C	14	28	z	3
D	10	13	130	4

where the base period of the relative price is 2012, and the current period is 2014.

(a) Find the value of x, y and z.

Sol.

$$\frac{y}{12} \cdot 100 = 150$$

$$y = \frac{150 \cdot 12}{100}$$

$$= 18$$

$$\frac{24}{x} \cdot 100 = 120$$

$$120x = 2400$$

$$x = 20$$

$$z = \frac{28}{14} \cdot 100$$

$$= 200$$

$$\therefore x = 20, \ y = 18, \ z = 200$$

(b) Using 2012 as the base period, find the price index in 2014.

Sol.

Item	P.R. x_i	Weight w_i	$w_i x_i$
A	150	1	150
B	120	2	240
C	200	3	600
D	130	4	520
		$\sum w_i$	$\sum w_i x_i$
		10	1510

$$\text{Price Index} = \frac{1510}{10}$$

$$= 151$$

8. The table below shows the price of two products in 2005 and 2015:

Item	Price ($)		Relative price
	2005	2015	
A	30	x	2
B	50	$x + 10$	3

Using 2005 as base period,

(a) Assume that the price relative of these two items are the same in 2015, find the value of x.

Sol.

$$\frac{x}{30} \cdot 100 = \frac{x + 10}{50} \cdot 100$$

$$\frac{x}{30} = \frac{x + 10}{50}$$

$$50x = 30x + 300$$

$$20x = 300$$

$$x = 15$$

(b) Find the price index of these two items in 2015.

Sol.

$$\text{Price index} = \frac{15}{30} \cdot 100$$

$$= 50$$

Item	P.R. x_i	Weight w_i	$w_i x_i$
A	50	2	100
B	50	3	150
		$\sum w_i$	$\sum w_i x_i$
		5	250

$$\text{Price Index} = \frac{250}{5}$$

$$= 50$$

18.8 Revision Exercise 18

1. The length of 60 cotten fibers (in *mm*) in a laboratory are as follows:

82	202	352	321	25	293	293	86
28	206	323	355	357	33	325	113
233	294	50	296	115	236	357	326
52	301	140	328	238	358	58	255
143	360	340	302	370	343	260	303
59	146	60	263	170	175	348	305
380	346	61	305	264	383	62	306
195	350	265	385				

(a) Using 21*mm* as the lower limit and 40*mm* as the class range, construct a frequency distribution table.

Sol.

Length	Frequency
21 - 60	8
61 - 100	4
101 - 140	3
141 - 180	4
181 - 220	3
221 - 260	5
261 - 300	7
301 - 340	12
341 - 380	12
381 - 420	2

(b) Construct a histogram and a frequency polygon.

Sol.

Histogram and Frequency Polygon of Distribution of Length of Cotton Fibers
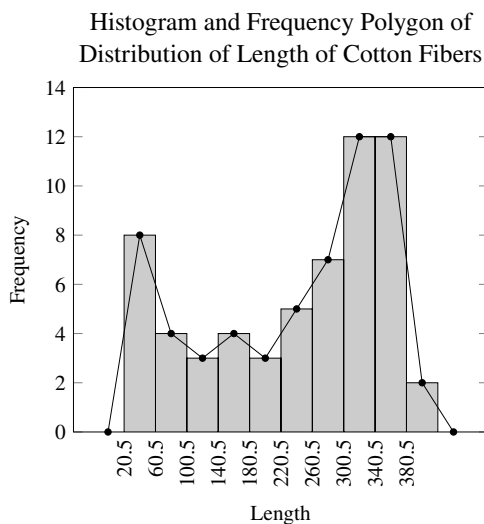
(c) Construct a cumulative frequency table and a cumulative frequency polygon.

Sol.

Range	Freq.	Lower Than	Cum. Freq.
21 - 60	8	60.5	8
61 - 100	4	100.5	12
101 - 140	3	140.5	15
141 - 180	4	180.5	19
181 - 220	3	220.5	22
221 - 260	5	260.5	27
261 - 300	7	300.5	34
301 - 340	12	340.5	46
341 - 380	12	380.5	58
381 - 420	2	420.5	60

Cumulative Frequency Polygon of Distribution of Length of Cotton Fibers
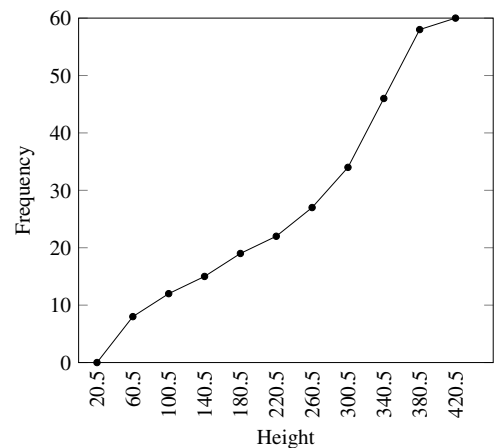
(d) Using the cumulative frequency polygon, find the percentage of fibers whose length is greater than 150mm.

Sol.

Cumulative Frequency Polygon of Distribution of Length of Cotton Fibers
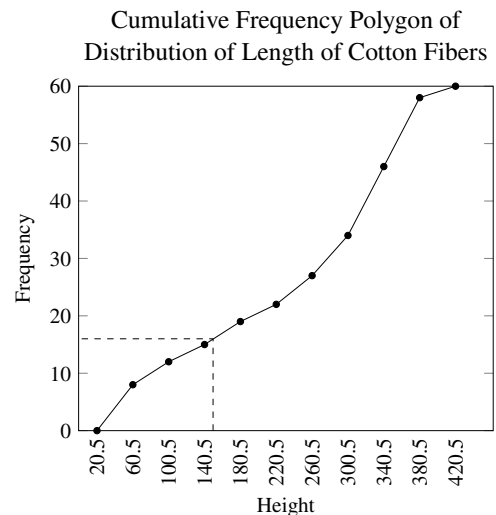
From the cumulative frequency polygon, there are 16 fibers whose length is less than 150mm, that is, there are $60 - 16 = 44$ fibers whose length is greater than 150mm. Therefore, the percentage of fibers whose length is greater than 150mm is $\frac{44}{60} \cdot 100 = 73.33\%$.

(e) Find the interquartile range.

Sol.

$n = 60$, $\frac{n}{4} = 15$, the class that contains Q_1 is $141 - 180$, $C_1 = 40$, $L_1 = 140.5$, $F_1 = 15$, $f_1 = 4$,

$$Q_1 = 140.5 + \frac{15 - 12}{3} \cdot 40 = 140.5$$

$n = 60$, $\frac{3n}{4} = 45$, the class that contains Q_3 is $301 - 340$, $C_3 = 40$, $L_3 = 300.5$, $F_3 = 34$,

$f_3 = 12,$

$$Q_3 = 300.5 + \frac{45 - 34}{12} \cdot 40 = 337.17$$

Therefore, the interquartile range is $Q_3 - Q_1 = 337.17 - 140.5 = 196.67$.

2. Find the mean, median, range, quartile deviation, and mean deviation of the data $8, 10, 9, 12, 4, 4, 2$.

Sol.

$$\text{Mean} = \frac{8 + 10 + 9 + 12 + 4 + 4 + 2}{7}$$
$$= \frac{49}{7}$$
$$= 7$$

Rearranging the data, we get

$$Q_2 = 8$$
$$\downarrow$$
$$2 \quad 4 \quad 4 \quad 8 \quad 9 \quad 10 \quad 12$$
$$\uparrow \qquad\qquad \uparrow$$
$$Q_1 = 4 \qquad\quad Q_3 = 10$$

The range is $12 - 2 = 10$, the quartile deviation is $\frac{10-4}{2} = \frac{6}{2} = 3$.

$$\text{Mean Dev.} = \frac{|8 - 7| + |10 - 7| + \cdots + |2 - 7|}{7}$$
$$= \frac{1 + 3 + 2 + 5 + 3 + 3 + 5}{7}$$
$$= \frac{22}{7}$$
$$= 3.14$$

Therefore, mean $= 7$, median $= 8$, range $= 10$, quartile deviation $= 3$, and mean deviation $= 3.14$.

3. The weight (in kg) if 16 babies are as follows:

$$8 \quad 9 \quad 10 \quad 9 \quad 8 \quad 7 \quad 9 \quad 10$$
$$9 \quad 8 \quad 8 \quad 9 \quad 10 \quad 9 \quad 8 \quad 7$$

Find the mean, meidan, mode, range, quartile deviation, mean deviation, and standard deviation of their weights.

Sol.

$$\text{Mean} = \frac{8 + 9 + \cdots + 7}{16}$$
$$= \frac{138}{16}$$
$$= 8.63$$

Rearranging the data, we get

$$7 \quad 7 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 9$$
$$9 \quad 9 \quad 9 \quad 9 \quad 9 \quad 10 \quad 10 \quad 10$$

$n = 16$, median is the average of the value at the $\frac{16}{2} = $ 8th position the $\frac{16}{2} + 1 = $ 9th position, that is, $\frac{9+9}{2} = 18$.

The mode is 9, which occurs at the highest frequency of 6.

The range is $10 - 7 = 3$.

Q_1 is the average of the value at the $\frac{16}{4} = $ 4th position and the $\frac{16}{4} + 1 = $ 5th position, that is, $\frac{8+8}{2} = 8$.

Q_3 is the average of the value at the $\frac{3 \cdot 16}{4} = $ 12th position and the $\frac{3 \cdot 16}{4} + 1 = $ 13th position, that is, $\frac{9+9}{2} = 9$.

The quartile deviation is $\frac{9-8}{2} = \frac{1}{2} = 0.5$.

$$\text{Mean Dev.} = \frac{|7 - 8.63| + \cdots + |10 - 8.63|}{16}$$
$$= \frac{1.63 + 1.63 + \cdots + 1.37}{16}$$
$$= \frac{12.74}{16}$$
$$= 0.80$$

$$\sigma^2 = \frac{7^2 + 7^2 + \cdots + 10^2}{16} - \left(\frac{138}{16}\right)^2$$
$$= 0.8594$$
$$\sigma = \sqrt{0.8594}$$
$$= 0.93$$

4. The table below shows the score distribution of business study minor test of senior 3 students in a high school:

Marks	No. of Students
0-9	7
10-19	21
20-29	32
30-39	27
40-49	13

(a) Construct a cumulative frequency distribution table.

Sol.

Range	Freq.	Lower Than	Cum. Freq.
0-9	7	9.5	7
10-19	21	19.5	28
20-29	32	29.5	60
30-39	27	39.5	87
40-49	13	49.5	100

(b) Construct a cumulative frequency polygon.

Sol.

Cumulative Frequency Polygon of Distribution of Marks of
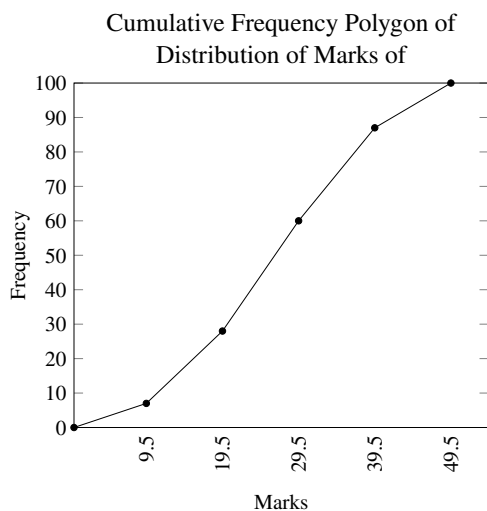
(c) Find the median and the interquartile range from the cumulative frequency polygon.

Sol.

Cumulative Frequency Polygon of Distribution of Marks
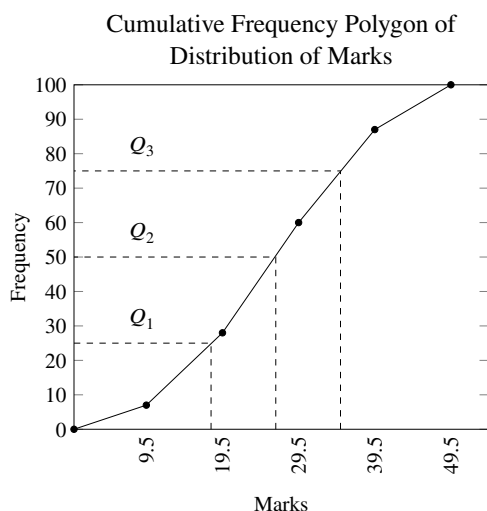
(d) Find the percentage of students who scored higher or equal to 45 marks.

Sol.

Cumulative Frequency Polygon of Distribution of Marks
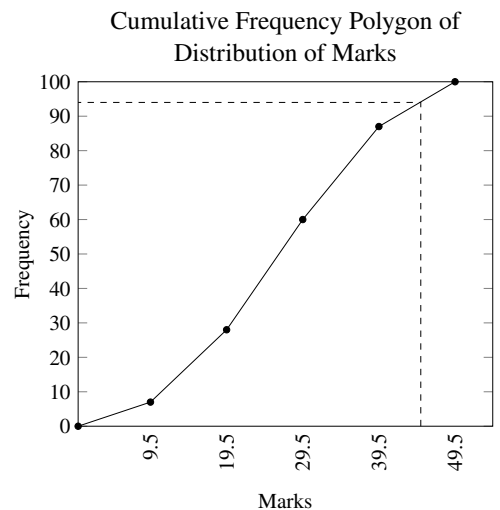
From the diagram, we can see that there are 6 students who scored 45 marks or above.

(e) Assume that the passing score is 15 marks. Find the percentage of students who failed the test.

Cumulative Frequency Polygon of Distribution of Marks
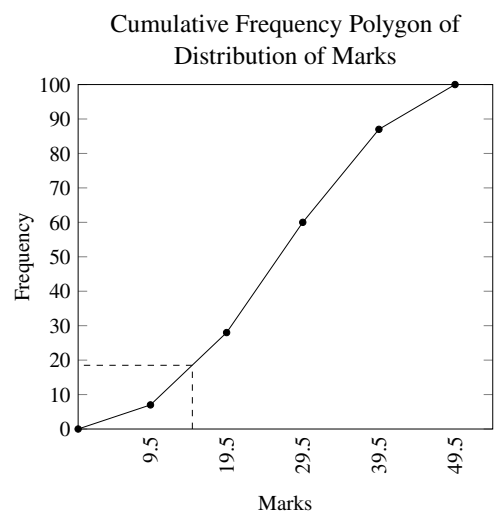
From the diagram above, there are about 18.5% of students who failed the test.

5. The burning time (in s) of 10 rocket boosters are as follows:

50.7 54.9 54.3 44.8 42.2

69.0 55.4 66.1 48.1 34.5

Find the range, variance and standard deviation of the burning time.

Sol.

$$\text{Range} = 69.0 - 34.5 = 34.5$$

$$\bar{x} = \frac{50.7 + 54.9 + \cdots + 34.5}{10}$$

$$= \frac{520}{10}$$

$$= 52$$

$$\sigma^2 = \frac{50.7^2 + 54.9^2 + \cdots + 34.5^2}{10} - 52^2$$

$$= \frac{28024.1}{10} - 52^2$$

$$= 98.41$$

$$\sigma = \sqrt{98.41}$$

$$= 9.92$$

6. The table below shows the scores of 30 rounds of game scored by someone:

Score	0	1	2	3	4
Times	5	3	4	$x+1$	7

Find:

(a) The value of x.

Sol.

$$5 + 3 + 4 + (x+1) + 7 = 30$$

$$20 + x = 30$$

$$x = 10$$

(b) The mean and standard deviation of the scores.

Sol.

$$\bar{x} = \frac{0 \cdot 5 + 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 10 + 4 \cdot 7}{30}$$

$$= \frac{0 + 3 + 8 + 33 + 28}{30}$$

$$= \frac{72}{30}$$

$$= 2.4$$

$$\sigma^2 = \frac{3 + 16 + 99 + 112}{30} - 2.4^2$$

$$= \frac{230}{30} - 5.76$$

$$= 1.91$$

$$\sigma = \sqrt{1.91}$$

$$= 1.38$$

7. The table below shows the distribution of scores of a minor test of students in a class:

Score	No. of Students
$0 < x \leq 5$	8
$5 < x \leq 10$	1
$10 < x \leq 15$	9
$15 < x \leq 20$	7
$20 < x \leq 25$	11
$25 < x \leq 30$	4

Find:

(a) Range

Sol.

$$\text{Range} = 30 - 0 = 30$$

(b) Median

Sol.

Range	Freq	Mid	Lw. Th.	Cum. Freq.
$0 < x \leq 5$	8	3	5	8
$5 < x \leq 10$	1	8	10	9
$10 < x \leq 15$	9	13	15	18
$15 < x \leq 20$	7	18	20	25
$20 < x \leq 25$	11	23	25	36
$25 < x \leq 30$	4	28	30	40

$n = 40$, $\frac{n}{2} = 20$, the class containing the median is $15 < x \leq 20$, $C_m = 4$, $L_m = 15$, $F_m = 18$, $f_m = 7$.

$$\text{Median} = 15 + \frac{20 - 18}{7} \cdot 5$$

$$= 16.43$$

(c) Mode

Sol.

The modal class is $20 < x \leq 25$ with the highest frequency of 11, $C = 5$, $L = 20$, $d_1 = 11 - 7 = 4$, $d_2 = 11 - 4 = 7$,

$$\text{Mode} = 20 + \frac{4}{4 + 7} \cdot 5$$

$$= 21.82$$

8. Below are the distribution of scores of business study exam of 40 students in a class:

Score	No. of Students
46 - 54	4
54 - 62	9
62 - 70	10
70 - 78	8
78 - 86	6
86 - 94	3

Find:

(a) Mean

Sol.

Score	Med. x_i	Freq x_i	$x_i f_i$
46 - 54	50	4	200
54 - 62	58	9	522
62 - 70	66	10	660
70 - 78	74	8	592
78 - 86	82	6	492
86 - 94	90	3	270
		$\sum x_i f_i$	$\sum x_i f_i$
		40	2736

$$\text{Mean} = \frac{2736}{40}$$
$$= 68.4$$

(b) Median

Sol. $\frac{n}{2} = 20$, the class containing the median is $62 - 70$, $C_m = 8$, $L_m = 62$, $F_m = 13$, $f_m = 10$.

$$\text{Median} = 62 + \frac{20 - 13}{10} \cdot 8$$
$$= 67.6$$

(c) Mode

Sol. The modal class is $62 - 70$ with the highest frequency of 10, $C = 8$, $L = 62$, $d_1 = 10 - 9 = 1$, $d_2 = 10 - 8 = 2$,

$$\text{Mode} = 62 + \frac{1}{1 + 2} \cdot 8$$
$$= 64.67$$

(d) Variance

Sol.

Mid. x_i	Freq. f_i	$x_i^2 f_i$
50	4	10000
58	9	30276
66	10	43560
74	8	43808
82	6	40344
90	3	24300
	$\sum f_i$	$\sum x_i^2 f_i$
	40	192288

$$\sigma^2 = \frac{192288}{40} - 68.4^2$$
$$= 128.64$$

9. The table below shows the frequency distribution of the life of 500 light bulbs:

Life (in hr)	No. of Bulbs
800 - 850	35
850 - 900	127
900 - 950	185
950 - 1000	103
1000 - 1050	42
1050 - 1100	8

Find:

(a) The mean and standard deviation of the life of the light bulbs.

Sol.

Life (in hr)	Mid. x_i	Freq. f_i	$x_i f_i$	$x_i^2 f_i$
800 - 850	825	35	28875	23821875
850 - 900	875	127	111125	97234375
900 - 950	925	185	171125	158290625
950 - 1000	975	103	100425	97914375
1000 - 1050	1025	42	43050	44126250
1050 - 1100	1075	8	8600	9245000
		$\sum f_i$	$\sum x_i f_i$	$\sum x_i^2 f_i$
		500	463200	430632500

$$\bar{x} = \frac{463200}{500}$$
$$= 926.4$$
$$\sigma^2 = \frac{430632500}{500} - 926.4^2$$
$$= 3048.04$$
$$\sigma = \sqrt{3048.04}$$
$$= 55.21$$

(b) Mean deviation.

Sol.

Mid x_i	Freq. f_i	$\|x_i - \bar{x}\|$	$\|x_i - \bar{x}\|f_i$
825	35	101.4	3549
875	127	51.4	6527.8
925	185	1.4	259
975	103	48.6	5005.8
1025	42	98.6	4141.2
1075	8	148.6	1188.8
	$\sum f_i$		$\sum \|x_i - \bar{x}\|f_i$
	500		20671.6

$$\text{Mean Dev.} = \frac{20671.6}{500}$$
$$= 41.34$$

(c) Median

Sol.

Range	Freq.	Lower Than	Cum. Freq.
800 - 850	35	850	35
850 - 900	127	900	162
900 - 950	185	950	347
950 - 1000	103	1000	450
1000 - 1050	42	1050	492
1050 - 1100	8	1100	500

$n = 500$, $\frac{n}{2} = 250$, the class containing the median is $900 - 950$, $C_m = 50$, $L_m = 900$, $F_m = 162$, $f_m = 185$,

$$\text{Median} = 900 + \frac{250 - 162}{185} \cdot 50$$
$$= 900 + \frac{88}{185} \cdot 50$$
$$= 900 + 23.78$$
$$= 923.78$$

(d) Quartile deviation.

Sol. $n = 500$, $\frac{n}{4} = 125$, the class containing Q_1 is $850 - 900$, $C_1 = 50$, $L_1 = 850$, $F_1 = 35$, $f_1 = 127$,

$$Q_1 = 850 + \frac{125 - 35}{127} \cdot 50$$
$$= 850 + \frac{90}{127} \cdot 50$$
$$= 850 + 35.43$$
$$= 885.43$$

$n = 500$, $\frac{3n}{4} = 375$, the class containing Q_3 is $950 - 1000$, $C_3 = 50$, $L_3 = 950$, $F_3 = 347$,

$f_3 = 103$,

$$Q_3 = 950 + \frac{375 - 347}{103} \cdot 50$$
$$= 950 + \frac{28}{103} \cdot 50$$
$$= 950 + 13.59$$
$$= 963.59$$

$$\text{Quartile Dev.} = \frac{963.59 - 885.43}{2}$$
$$= 39.08$$

10. Assume that the mean value of data $2, x + 1, 5, 2x + 1, 8, 2x - 3$ is 4,

   (a) Find the value of x.

   Sol.

   $$\frac{2 + x + 1 + 5 + 2x + 1 + 8 + 2x - 3}{6} = 4$$
   $$\frac{14 + 5x}{6} = 4$$
   $$14 + 5x = 24$$
   $$5x = 10$$
   $$x = 2$$

   (b) With that, find the standard deviation of the data.

   Sol.

   The data become $2, 3, 5, 5, 8, 1$

   $$\sigma^2 = \frac{4 + 9 + 25 + 25 + 64 + 1}{6} - 4^2$$
   $$= 5.33$$
   $$\sigma = \sqrt{5.33}$$
   $$= 2.31$$

11. The mean and mode of a set of data $2, 5, 3, 11, 9, 3, 11, p, q$ are 6 and 3 respectively, $p > q$. Find

   (a) The value of p and q

   Sol.

   $$\frac{2 + 5 + 3 + 11 + 9 + 3 + 11 + p + q}{9} = 6$$
   $$\frac{44 + p + q}{9} = 6$$
   $$44 + p + q = 54$$
   $$p + q = 10$$

∵ mode $= 3$, $p > q$,

∴ $p = 3$

$q = 7$

(b) Median

Sol.

Rearrange the data,

$$2, 3, 3, 3, 5, 7, 9, 11, 11$$

$n = 9$, the median is the value at the $\frac{9+1}{2} = $ 5th position, which is 5.

(c) Standard deviation

Sol.

$$\sigma^2 = \frac{4 + 9 + \cdots + 121}{9} - 6^2$$

$$= 11.56$$

$$\sigma = \sqrt{11.56}$$

$$= 3.40$$

12. Given that the mean value of $x, x+1, 2x-3, 5, y, 8$ is 6. After eliminating y, the mean value of the remaining data is 3.8.

   (a) Find the value of x and y.

   Sol.

   $$\frac{x + x + 1 + 2x - 3 + 5 + 8}{5} = 3.8$$

   $$\frac{4x + 11}{5} = 3.8$$

   $$4x + 11 = 19$$

   $$4x = 8$$

   $$x = 2$$

   $$\frac{x + x + 1 + 2x - 3 + 5 + y + 8}{6} = 6$$

   $$\frac{4x + y + 11}{6} = 6$$

   $$4x + y + 11 = 36$$

   $$4x + y = 25$$

   $$8 + y = 25$$

   $$y = 17$$

   $$\therefore x = 2, \; y = 17$$

   (b) With that, find the variance of the original 6 data.

Sol. The data become $2, 3, 1, 5, 17, 8$

$$\sigma^2 = \frac{4 + 9 + 1 + 25 + 289 + 64}{6} - 6^2$$

$$= 29.33$$

13. Given the sum of the square of 10 numbers is 400, and their mean value is 5. If a number 8 is eliminated from the data set, find the mean value and variance of the remaining data.

   Sol.

   Before eliminating 8,

   $$\sum x = 10 \cdot 5$$

   $$= 50$$

   $$\sum x^2 = 400$$

   After eliminating 8,

   $$\sum x = 50 - 8$$

   $$= 42$$

   $$\bar{x} = \frac{42}{9}$$

   $$= 4.67$$

   $$\sum x^2 = 400 - 8^2$$

   $$= 3$$

   $$\sigma^2 = \frac{336}{9} - 4.67^2$$

   $$= 15.52$$

14. There are two female chorus groups A and B, each of which has 5 members. Their heights (in cm) are as follows:

| Group A | 170 | 162 | 159 | 160 | 155 |
| Group B | 180 | 165 | 150 | 154 | 160 |

   (a) Find the mean and standard deviation of the heights of the members of the two groups.

Sol.

$$\bar{A} = \frac{170 + 162 + 159 + 160 + 155}{5}$$
$$= 161.2$$
$$\bar{B} = \frac{180 + 165 + 150 + 154 + 160}{5}$$
$$= 161.8$$
$$\sigma_A = \sqrt{\frac{170^2 + 162^2 + \cdots + 155^2}{5} - 161.2^2}$$
$$= \sqrt{24.56}$$
$$= 4.96$$
$$\sigma_B = \sqrt{\frac{180^2 + 165^2 + \cdots + 160^2}{5} - 161.8^2}$$
$$= \sqrt{108.96}$$
$$= 10.44$$

(b) Which group has a smaller height variance?

Sol.

$\because \sigma_A < \sigma_B,$

$\therefore$ Group A has a smaller height variance.

15. The table below shows scores of maths exam of three classes:

Class	Avg. Marks	Std. Deviation	No. of Stud.
A	36.8	5.2	32
B	30.3	12.4	36
C	38.8	10.3	32

(a) In between class A, B and C, which class has the most consistent performance? Why?

Sol.

Class A has the most consistent performance because its standard deviation is the lowest among the three classes.

(b) Find the average marks and standard deviation of these three classes combined.

Sol.

$$\overline{ABC} = \frac{36.8 \cdot 32 + 30.3 \cdot 36 + 38.8 \cdot 32}{100}$$
$$= 35.1$$

$$\sum a_i^2 = (5.2^2 + 36.8^2) \cdot 32$$
$$= 44200.96$$
$$\sum b_i^2 = (12.4^2 + 30.3^2) \cdot 36$$
$$= 38586.6$$
$$\sum c_i^2 = (10.3^2 + 38.8^2) \cdot 32$$
$$= 51568.96$$
$$\sigma_{ABC} = \sqrt{\frac{\sum a_i^2 + \sum b_i^2 + \sum c_i^2}{100} - 35.1^2}$$
$$= \sqrt{111.56}$$
$$= 10.56$$

16. The score given by six judges to a gymnast are as follows:

$$7 \quad 5 \quad 9 \quad 7 \quad 8 \quad 6$$

Find the following of the gymnast:

(a) Mean

Sol.

$$\bar{x} = \frac{7 + 5 + 9 + 7 + 8 + 6}{6}$$
$$= \frac{42}{6}$$
$$= 7$$

(b) Standard deviation

Sol.

$$\sigma = \sqrt{\frac{7^2 + 5^2 + \cdots + 6^2}{6} - 7^2}$$
$$= \sqrt{1.67}$$
$$= 1.29$$

(c) Coefficient of variation

Sol.

$$v = \frac{\sigma}{\bar{x}} \cdot 100\%$$
$$= \frac{1.29}{7} \cdot 100\%$$
$$= 18.44\%$$

17. In an IQ test, the average score of 10 students is 114, and the scores of 9 of them are as follows:

$$101 \quad 125 \quad 118 \quad 128 \quad 106$$
$$115 \quad 99 \quad 118 \quad 109$$

Find:

(a) The IQ of the 10th student.

Sol.

$$x_{10} = 114 \cdot 10 - 101 - 125 - \cdots - 109$$
$$= 1140 - 101 - 125 - \cdots - 109$$
$$= 121$$

(b) The correlation coefficient of the IQ of the 10 students.

Sol.

$$\sigma = \sqrt{\frac{101^2 + 125^2 + \cdots + 109^2}{10} - 114^2}$$
$$= \sqrt{88.2}$$
$$= 9.39$$
$$v = \frac{9.39}{114} \cdot 100\%$$
$$= 8.24\%$$

18. Given that the data of the weight of two groups of girls (in kg) are as follows:

	Mean	Standard Deviation
1 years old	10.90	1.24
5 years old	19.00	2.11

Compare the variation of the weight of these girls.

Sol.

$$\text{1 year old} \, v_1 = \frac{1.24}{10.90} \cdot 100\% = 11.38\%$$
$$\text{5 year old} \, v_2 = \frac{2.11}{19.00} \cdot 100\% = 11.11\%$$

The 1 year old group has a bigger variation than the 5 year old group.

19. The prodution output and production cost of a factory in the first half of this year are as follows:

Month	1	2	3	4	5	6
Output (in $1k \, tons$)	2	3	1	4	3	5
Cost (in $\$1k$)	9	11	7	13	11	15

Sol.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
2	9	4	81	18
3	11	9	121	33
1	7	1	49	7
4	13	16	169	52
3	11	9	121	33
5	15	25	225	75
$\sum x_i$	$\sum y_i$	$\sum x_i^2$	$\sum y_i^2$	$\sum x_i y_i$
18	66	64	766	218

$$\bar{x} = \frac{18}{6} = 3$$
$$\bar{y} = \frac{66}{6} = 11$$
$$r = \frac{\frac{218}{6} - 3 \cdot 11}{\sqrt{\left(\frac{64}{6} - 3^2\right)\left(\frac{766}{6} - 11^2\right)}}$$
$$= 1$$

According to the result of the calculation, the production output and production cost of the factory are positively and prefectly correlated.

20. The marks of Chinese exam and Maths exam of 16 senior students in a school are as follows:

Chinese	Maths
82	59
79	63
76	99
63	67
56	61
67	88
69	82
81	77
77	75
73	74
58	67
64	79
68	75
72	65
75	64
80	66

(a) Construct a scatter diagram of the data.

Sol.
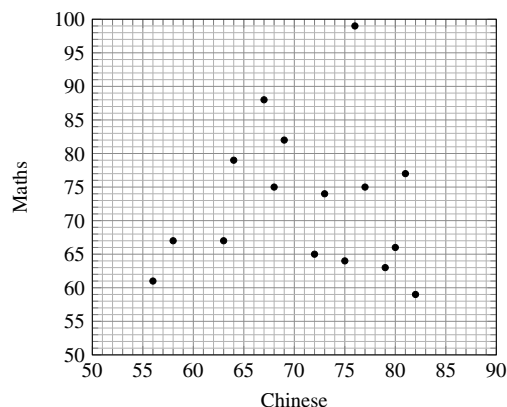
(b) Find the correlation coefficient of the two exams, and determine the strength of correlation.

Sol.

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
82	59	6724	3481	4838
79	63	6241	3969	4977
76	99	5776	9801	7524
63	67	3969	4489	4221
56	61	3136	3721	3416
67	88	4489	7744	5896
69	82	4761	6724	5658
81	77	6561	5929	6237
77	75	5929	5625	5775
73	74	5329	5476	5402
58	67	3364	4489	3886
64	79	4096	6241	5056
68	75	4624	5625	5100
72	65	5184	4225	4680
75	64	5625	4096	4800
80	66	6400	4356	5280
$\sum x_i$	$\sum y_i$	$\sum x_i^2$	$\sum y_i^2$	$\sum x_i y_i$
1140	1161	82208	85991	82746

$$\bar{x} = \frac{1140}{16} = 71.25$$

$$\bar{y} = \frac{1161}{16} = 72.56$$

$$r = \frac{\frac{82746}{16} - 71.25 \cdot 72.56}{\sqrt{\left(\frac{82208}{16} - 71.25^2\right)\left(\frac{85991}{16} - 72.56^2\right)}}$$

$$= 0.0189$$

According to the result of the calculation, the marks of Chinese exam and Maths exam of the 16 senior students are positively and weakly correlated.

21. The table below shows the prices of a product (in $) in

2005, 2010, and 2015:

Year	2005	2010	2015
Price	4	6	x

(a) Assume that the percentage of price increase from 2005 to 2010 is the same as that from 2010 to 2015, find the value of x.

Sol.

$$\frac{6}{4} \cdot 100\% = \frac{x}{6} \cdot 100\%$$

$$\frac{6}{4} = \frac{x}{6}$$

$$4x = 36$$

$$x = 9$$

(b) Find the relative price in 2015 with respect to 2005.

Sol.

$$P.R. = \frac{9}{4} \cdot 100\% = 225\%$$

22. The price data of primary food of a city with 2013 as base period and 2014 as current period are as follows:

Food	Relative price	Weight
Meat	105	8
Fish	111	7
Vegetables	98	5
Rice & Noodles	103	10
Cooking Oil	100	3
Beverage	107	2
Fruits	99	2

Find the price index in 2014.

Sol.

$$\text{Price Index} = \frac{105 \cdot 8 + 111 \cdot 7 + \cdots + 99 \cdot 2}{37}$$

$$= \frac{3849}{37}$$

$$= 104.03$$

23. The relative price of daily expenses of people in a place with respect to last year and their relative consumption are as follows:

Daily Expenses	Relative price	Relative Consumption
Clothing	120	23
Food	117	40
Housing	132	19
Transportation	130	18

Using the relative consumption as weight, find the composite index of daily expenses.

Sol.

$$\text{Composite Index} = \frac{120 \cdot 23 + \cdots + 130 \cdot 18}{100}$$
$$= \frac{12288}{100}$$
$$= 122.88$$

24. The table below shows the spending of a company in 4 different projects in 3 consecutive years:

Project	Year			A	B
	2012	2013	2014		
Salaries	x	20,000	30,000	150	P
Stationery	5,000	y	7,000	120	140
Repair	4,000	5000	z	125	150
Miscellaneous	8,000	Q	15,000	R	R

Given that A is the index where 2012 is the base period and 2013 is the current period; B is the index where 2012 is the base period and 2014 is the current period. Find the value of x, y, z, P, Q, and R.

Sol.

$$\frac{20000}{x} \cdot 100 = 150$$
$$\frac{20000}{x} = 1.5$$
$$1.5x = 20000$$
$$x = 13333.33$$

$$P = \frac{30000}{13333.33} \cdot 100$$
$$= 225$$

$$\frac{y}{5000} \cdot 100 = 120$$
$$\frac{y}{5000} = 1.2$$
$$y = 6000$$

$$\frac{z}{4000} \cdot 100 = 150$$
$$\frac{z}{4000} = 1.5$$
$$z = 6000$$

$$\frac{Q}{8000} \cdot 100 = \frac{15000}{8000} \cdot 100 = R$$
$$\frac{Q}{8000} = 1.875 = \frac{R}{100}$$
$$Q = 15000$$
$$R = 187.5$$

$\therefore x = 13333.33,\ y = 6000,\ z = 6000,$
$P = 225,\ Q = 15000,\ R = 187.5$

Chapter 19

Permutations and Combinations

Permutations and combinations are the foundation of probability and statistics. In our daily life, we often need to calculate the number of ways of completing a task. These calculations are based on two basic rules: addition rule and multiplication rule.

19.1 Addition and Multiplication rules

Theorem 1. *Addition rule*

If there are n methods of doing a task, the first method can be done in m_1 ways, the second method can be done in m_2 ways, $\cdots$, the nth methods can be done in m_n ways, and they are mutually exclusive, which means the task can be done in whatever way using whatever method, then the total number of ways of doing the task is

$$m_1 + m_2 + \cdots + m_n$$

Theorem 2. *Multiplication rule*

If there are n steps in doing a task, the first step can be done in m_1 ways, the second step can be done in m_2 ways, $\cdots$, the nth steps can be done in m_n ways, then the total number of ways of doing the task is

$$m_1 \times m_2 \times \cdots \times m_n$$

19.1.1 Practice 1

1. There are 2 Math reference books, 3 novels, and 4 storybooks of idioms. Xiao Hua wants to choose one book from each category. How many ways can he choose?

    Sol.

To choose one book from each category, there are 3 steps:

- First step, choose a Math reference book, which can be done in 2 ways.
- Second step, choose a novel, which can be done in 3 ways.
- Third step, choose a storybook of idioms, which can be done in 4 ways.

All these steps must be done in order to complete the task. Therefore, applying the multiplication rule, there are $2 \cdot 3 \cdot 4 = 24$ ways to choose one book from each category.

2. Travelling from A to B can be done by bus or train. There are 4 buses and 3 trains. How many ways are there to travel from A to B?

    Sol.

To travel from A to B, there are 2 method:

- First method, travel by bus, which can be done in 4 ways.
- Second method, travel by train, which can be done in 3 ways.

Any way in any method can be used to travel from A to B. Therefore, applying the addition rule, there are $4 + 3 = 7$ ways to travel from A to B.

19.1.2 Practice 19.1

1. During the eve of a festival, there are 3 trains, 4 buses, and 4 trains from Johor Bahru to Pinang. How many ways are there to travel from Johor Bahru to Pinang during the day?

    Sol.

To travel from Johor Bahru to Pinang, there are 3 methods:

- First method, travel by train, which can be done in 3 ways.
- Second method, travel by bus, which can be done in 4 ways.
- Third method, travel by train, which can be done in 4 ways.

Any way in any method can be used to travel from Johor Bahru to Pinang. Therefore, applying the addition rule, there are $3 + 4 + 4 = 11$ ways to travel from Johor Bahru to Pinang.

2. One has 13 shirts and 6 pants, how many ways can he dress up?

   Sol.

   To dress up, there are 2 steps:

   - First step, choose a shirt, which can be done in 13 ways.

   - Second step, choose a pants, which can be done in 6 ways.

   All these steps must be done in order to complete the task. Therefore, applying the multiplication rule, there are $13 \cdot 6 = 78$ ways to dress up.

3. There are 4 airlines A, B, C, and D that provide flights from Kuala Lumpur to Bangkok: A provdies 3 flights per day, B provides 2 flights per day, C and D provides 1 flight per day. How many choices are there to travel from Kuala Lumpur to Bangkok?

   Sol.

   To travel from Kuala Lumpur to Bangkok, there are 4 methods:

   - First method, travel by A, which can be done in 3 ways.

   - Second method, travel by B, which can be done in 2 ways.

   - Third method, travel by C, which can be done in 1 way.

   - Fourth method, travel by D, which can be done in 1 way.

   Any way in any method can be used to travel from Kuala Lumpur to Bangkok. Therefore, applying the addition rule, there are $3 + 2 + 1 + 1 = 7$ ways to travel from Kuala Lumpur to Bangkok.

4. How many set meal combinations are there if there are 6 type of main dishes, 5 type of drinks, and 2 type of desserts?

   Sol.

   To make a set meal, there are 3 steps:

   - First step, choose a main dish, which can be done in 6 ways.

   - Second step, choose a drink, which can be done in 5 ways.

   - Third step, choose a dessert, which can be done in 2 ways.

All these steps must be done in order to complete the task. Therefore, applying the multiplication rule, there are $6 \cdot 5 \cdot 2 = 60$ ways to make a set meal.

5. There are 4 doors in a classroom, student A and student B can enter the classroom through any door. How many ways are there for student A and student B to enter the classroom?

   Sol.

   To enter the classroom, there are 2 steps:

   - First step, student A choose a door, which can be done in 4 ways.

   - Second step, student B choose a door, which can be done in 4 ways.

   All these steps must be done in order to complete the task. Therefore, applying the multiplication rule, there are $4 \cdot 4 = 16$ ways for student A and student B to enter the classroom.

6. A friendly match is held between 2 ping pong teams, each team has to send 3 players, and each player has to play games with all the other players on the other team. How many games have to be played?

   Sol.

   To play a game betwen two teams: A and B, there are 3 steps:

   - First step, choose a player from the team A, which can be done in 3 ways.

   - Second step, choose a player from team B, which can be done in 3 ways.

   To match between all players from team A and team B, there are $3 \cdot 3 = 9$ games have to be played.

7. Matching 8 clothes of different colors with 5 different skirts, how many ways are there to dress up? If the above dresses are paired with 4 pairs of shoes of different colors, how many ways are there to dress up?

   Sol.

   To dress up, there are 2 steps:

   - First step, choose a clothes, which can be done in 8 ways.

   - Second step, choose a skirt, which can be done in 5 ways.

All these steps must be done in order to complete the task. Therefore, applying the multiplication rule, there are $8 \cdot 5 = 40$ combinations of clothes and skirts.

To pair the clothes and skirts with shoes, there are 2 steps:

- First step, choose a clothes and a skirt, which can be done in 40 ways.

- Second step, choose a shoes, which can be done in 4 ways.

All these steps must be done in order to complete the task. Therefore, applying the multiplication rule, there are $40 \cdot 4 = 160$ combinations of clothes, skirts, and shoes.

19.2 Permutations and Permutation Formula

19.2.1 Practice 2

How many ways are there to arrange the numbers 1, 2, 3, 4 into a two digit number with no repeated digits?

Sol.

First step: choose one of the 4 numbers as the first digit, there are 4 ways to do so.

Second step: choose one of the remaining 3 numbers as the second digit, there are 3 ways to do so.

According to the multiplication rule, the total number of ways to arrange the numbers 1, 2, 3, 4 into a two digit number with no repeated digits is $4 \cdot 3 = 12$.

If there are n elements, we want to pick r elements from them and arrange them in a sequence, how many ways are there to do so? This question can be treated as there are r empty boxes, which means this requires r steps to complete.
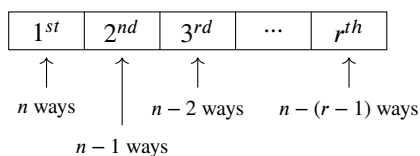
First step: Choose one element from n elements and put it in the first box, then there are n ways to do so.

Second step: Choose one element from $n-1$ elements and put it in the second box, then there are $n-1$ ways to do so.

Third step: Choose one element from $n-2$ elements and put it in the third box, then there are $n-2$ ways to do so.

So on and so forth, when $r-1$ boxes are filled, the last box r can only be filled with one of the remaining $n-(r-1)$ elements, so there are $n-(r-1)$ ways to do so. According to the multiplication rule, the total number of ways to fill in r boxes is

$$n(n-1)(n-2)\cdots(n-r+1)$$

Threrefore, there are $n(n-1)(n-2)\cdots(n-r+1)$ ways to arrange r elements, and this denoted as $_nP_r$, $_nP_r$, or P_r^n.

$$_nP_r = n(n-1)(n-2)\cdots(n-r+1)$$

Where $r \leq n$, $n \in N$, $r = 0, 1, 2, \cdots, n$. This formula is called the permutation formula.

When $r = n$, aka a full permutation, the formula becomes

$$_nP_n = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

Therefore, the permutation of all n elements is equal to the products of natural numbers from 1 to n. This is called the factorial of n, denoted as $n!$.

$$\begin{aligned} n! &= {}_nP_n \\ &= n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 \end{aligned}$$

Using factorial, the permutation formula can be transform into the following:

$$\begin{aligned} _nP_r &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)\cdots 3 \cdot 2 \cdot 1}{(n-r)\cdots 3 \cdot 2 \cdot 1} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Hence, the permutation formula can be written as

$$_nP_r = \frac{n!}{(n-r)!}$$

Note: 0! is defined as 1 to make the formula work when $n = r$.

19.2.2 Practice 3

1. Find the value of $_7P_3$ and 5!.

   Sol.

$$\begin{aligned} _7P_3 &= 7 \cdot 6 \cdot 5 \\ &= 210 \\ 5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ &= 120 \end{aligned}$$

2. Calculate $_{10}P_3 + {}_8P_4$.

Sol.

$$\begin{aligned}
_{10}P_3 + {}_8P_4 &= 10 \cdot 9 \cdot 8 + 8 \cdot 7 \cdot 6 \cdot 5 \\
&= 720 + 1680 \\
&= 2400
\end{aligned}$$

3. If $100(_nP_2) = {}_{2n}P_3$, find the value of n.

Sol.

$$100(_nP_2) = {}_{2n}P_3$$
$$100(n)(n-1) = (2n)(2n-1)(2n-2)$$
$$100(n)(n-1) = 4(n)(2n-1)(n-1)$$
$$100(n)(n-1) - 4(n)(2n-1)(n-1) = 0$$
$$4(n)(n-1)[25 - (2n-1)] = 0$$
$$(n)(n-1)(25 - 2n + 1) = 0$$
$$(n)(n-1)(26 - 2n) = 0$$
$$-2(n)(n-1)(n-13) = 0$$
$$(n)(n-1)(n-13) = 0$$
$$\therefore n = 0 \text{ or } n = 1 \text{ or } n = 13$$
$$\because n \in \mathbb{N}, \ n \geq 3$$
$$\therefore n = 13$$

19.2.3 Exercise 19.2a

1. Write down all the permutations of 3 elements in 4 elements A, B, C, D.

Sol.

$ABC, ABD, ACB, ACD, ADB, ADC, BAC, BAD,$
$BCA, BCD, BDA, BDC, CAB, CAD, CBA, CBD,$
$CDA, CDB, DAB, DAC, DBA, DBC, DCA, DCB$

2. Calculate:

(a) $_{15}P_4$

Sol.

$$\begin{aligned}
_{15}P_4 &= 15 \cdot 14 \cdot 13 \cdot 12 \\
&= 32760
\end{aligned}$$

(b) $_{100}P_3$

Sol.

$$\begin{aligned}
_{100}P_3 &= 100 \cdot 99 \cdot 98 \\
&= 970200
\end{aligned}$$

(c) $7!$

Sol.

$$\begin{aligned}
7! &= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
&= 5040
\end{aligned}$$

(d) $\frac{8!}{5!}$

Sol.

$$\begin{aligned}
\frac{8!}{5!} &= \frac{8!}{(8-3)!} \\
&= {}_8P_3 \\
&= 8 \cdot 7 \cdot 6 \\
&= 336
\end{aligned}$$

3. Calculate the following:

(a) $\frac{11! - 10!}{10! - 9!}$

Sol.

$$\begin{aligned}
\frac{11! - 10!}{10! - 9!} &= \frac{11 \cdot 10! - 10!}{10 \cdot 9! - 9!} \\
&= \frac{10! \cdot 10}{9! \cdot 9} \\
&= \frac{10 \cdot 10 \cdot 9!}{9! \cdot 9} \\
&= \frac{10 \cdot 10}{9} \\
&= \frac{100}{9}
\end{aligned}$$

(b) $\frac{7! - 6! - 5!}{5!}$

Sol.

$$\begin{aligned}
\frac{7! - 6! - 5!}{5!} &= \frac{7 \cdot 6 \cdot 5! - 6 \cdot 5! - 5!}{5!} \\
&= \frac{5!(7 \cdot 6 - 6 - 1)}{5!} \\
&= 42 - 7 \\
&= 35
\end{aligned}$$

(c) $\frac{13! - 12!}{(12)^2 10!}$

Sol.

$$\begin{aligned}
\frac{13! - 12!}{(12)^2 10!} &= \frac{13 \cdot 12! - 12!}{(12)^2 10!} \\
&= \frac{12!(12)}{(12)^2 10!} \\
&= \frac{12!}{12 \cdot 10!} \\
&= \frac{12 \cdot 11 \cdot 10!}{12 \cdot 10!} \\
&= 11
\end{aligned}$$

(d) $\frac{5(_8P_3)}{2(_6P_2)}$

Sol.

$$\frac{5(_8P_3)}{2(_6P_2)} = \frac{5 \cdot (8 \cdot 7 \cdot 6)}{2 \cdot (6 \cdot 5)}$$
$$= \frac{1680}{60}$$
$$= 28$$

(e) $\frac{_9P_5 + _9P_4}{_9P_3}$

Sol.

$$\frac{_9P_5 + _9P_4}{_9P_3} = \frac{\frac{9!}{(9-5)!} + \frac{9!}{(9-4)!}}{\frac{9!}{(9-3)!}}$$
$$= \frac{\frac{9!}{4!} + \frac{9!}{5!}}{\frac{9!}{6!}}$$
$$= \frac{9!(4! + 5!)}{4!5!} \cdot \frac{6!}{9!}$$
$$= \frac{6!(4! + 5!)}{4!5!}$$
$$= \frac{6 \cdot 5 \cdot 4!(4! + 5!)}{4!5!}$$
$$= \frac{6 \cdot 5(4! + 5!)}{5!}$$
$$= \frac{6 \cdot 5(4! + 5!)}{5 \cdot 4!}$$
$$= \frac{6(4! + 5!)}{4!}$$
$$= \frac{3!(4! + 5!)}{4 \cdot 3!}$$
$$= \frac{4! + 5!}{4}$$
$$= \frac{4 \cdot 3! + 5 \cdot 4 \cdot 3!}{4}$$
$$= \frac{4 \cdot 3! \cdot 6}{4}$$
$$= 6 \cdot 3!$$
$$= 6 \cdot 6$$
$$= 36$$

(f) $\frac{_{12}P_{12} - _{11}P_{11}}{_{10}P_{10}}$

Sol.

$$\frac{_{12}P_{12} - _{11}P_{11}}{_{10}P_{10}} = \frac{12! - 11!}{10!}$$
$$= \frac{12 \cdot 11! - 11!}{10!}$$
$$= \frac{11 \cdot 11!}{10!}$$
$$= \frac{11 \cdot 11 \cdot 10!}{10!}$$
$$= 11 \cdot 11$$
$$= 121$$

4. Simplify the following:

(a) $\frac{(n+1)!}{(n-1)!}$

Sol.

$$\frac{(n+1)!}{(n-1)!} = \frac{(n+1)n(n-1)!}{(n-1)!}$$
$$= n(n+1)$$

(b) $\frac{(20-r)!}{(18-r)!}$

Sol.

$$\frac{(20-r)!}{(18-r)!} = \frac{(20-r)(19-r)(18-r)!}{(18-r)!}$$
$$= (20-r)(19-r)$$

5. Find the value of n or r of the following expressions:

(a) $\frac{(n+1)!}{n!} = 42$

Sol.

$$\frac{(n+1)!}{n!} = 42$$
$$\frac{(n+1)n!}{n!} = 42$$
$$n + 1 = 42$$
$$n = 41$$

(b) $127(_{2n}P_3) = _{2n}P_4$

Sol.

$$127(_{2n}P_3) = _{2n}P_4$$
$$\frac{_{2n}P_4}{_{2n}P_3} = 127$$
$$\frac{(2n)!}{(2n-4)!} \cdot \frac{(2n-3)!}{(2n)!} = 127$$
$$\frac{(2n-3)!}{(2n-4)!} = 127$$
$$\frac{(2n-3)(2n-4)!}{(2n-4)!} = 127$$
$$2n - 3 = 127$$
$$2n = 130$$
$$n = 65$$

(c) $18(_{n-1}P_2) = _nP_4$

Sol.

$$18(_{n-1}P_2) = {_nP_4}$$

$$\frac{_nP_4}{_{n-1}P_2} = 18$$

$$\frac{n!}{(n-4)!} \cdot \frac{(n-3)!}{(n-1)!} = 18$$

$$\frac{n(n-1)!}{(n-4)!} \cdot \frac{(n-3)!}{(n-1)!} = 18$$

$$\frac{n(n-3)!}{(n-4)!} = 18$$

$$\frac{n(n-3)(n-4)!}{(n-4)!} = 18$$

$$n(n-3) = 18$$

$$n^2 - 3n - 18 = 0$$

$$(n-6)(n+3) = 0$$

$$n = 6 \text{ or } n-3$$

$$\because n \in \mathbb{N}, \; n \geq 4$$

$$\therefore n = 6$$

(d) $_{2n+1}P_4 = 132(_nP_3)$

Sol.

$$_{2n+1}P_4 = 132(_nP_3)$$

$$\frac{_{2n+1}P_4}{_nP_3} = 132$$

$$\frac{(2n+1)!}{(2n-3)!} \cdot \frac{(n-3)!}{n!} = 132$$

$$\frac{(2n+1)(2n)(2n-1)(2n-2)(2n-3)!}{(2n-3)!} \cdot \frac{(n-3)!}{n(n-1)(n-2)(n-3)!} = 132$$

$$\frac{(2n+1)(2n)(2n-1)(2n-2)}{n(n-1)(n-2)} = 132$$

$$\frac{4n(n-1)(2n+1)(2n-1)}{n(n-1)(n-2)} = 132$$

$$\frac{(2n+1)(2n-1)}{n-2} = 33$$

$$4n^2 - 1 = 33n - 66$$

$$4n^2 - 33n + 65 = 0$$

$$(n-5)(4n-13) = 0$$

$$n = 5 \text{ or } n = \frac{13}{4}$$

$$\because n \in \mathbb{N}, \; n \geq 3$$

$$\therefore n = 5$$

(e) $4(_{10}P_{r-1}) = {_{10}P_r}$

Sol.

$$4(_{10}P_{r-1}) = {_{10}P_r}$$

$$\frac{_{10}P_r}{_{10}P_{r-1}} = 4$$

$$\frac{10!}{(10-r)!} \cdot \frac{(10-r+1)!}{10!} = 4$$

$$\frac{(10-r+1)!}{(10-r)!} = 4$$

$$\frac{(10-r+1)(10-r)!}{(10-r)!} = 4$$

$$10 - r + 1 = 4$$

$$11 - r = 4$$

$$r = 7$$

(f) $6(_9P_{r-2}) = {_9P_r}$

Sol.

$$6(_9P_{r-2}) = {_9P_r}$$

$$\frac{_9P_r}{_9P_{r-2}} = 6$$

$$\frac{9!}{(9-r)!} \cdot \frac{(9-r+2)!}{9!} = 6$$

$$\frac{(9-r+2)!}{(9-r)!} = 6$$

$$\frac{(9-r+2)(9-r+1)(9-r)!}{(9-r)!} = 6$$

$$(9-r+2)(9-r+1) = 6$$

$$(11-r)(10-r) = 6$$

$$r^2 - 21r + 110 = 6$$

$$r^2 - 21r + 104 = 0$$

$$(r-13)(r-8) = 0$$

$$r = 13 \text{ or } r = 8$$

$$\because r \in \mathbb{N}, \; r \leq 11$$

$$\therefore r = 8$$

19.2.4 Practice 4

1. How many 3 digit numbers with no repeated digits can be formed using the digits 1, 2, 3, 4, 5?

   Sol.

   There are $_5P_3 = 5 \cdot 4 \cdot 3 = 60$ ways to form a 3 digit number with these 5 digits.

2. How many 3 digit numbers with no repeated digits can be formed using the digits 0, 1, 2, 3, 4?

   Sol.

Since the hundred's digit cannot be 0, there are 4 digits to choose from. Therefore, there are $4 \cdot {}_4P_2 = 4 \cdot 4 \cdot 3 = 48$ ways to form a 3 digit number with these 5 digits.

19.2.5 Practice 5

1. There are 50 seats and 50 students in a class. How many ways can the students be seated in the class?

   Sol.

   There are ${}_{50}P_{50} = 50!$ ways to seat the students.

2. Person A and B has two choose two adjacent seats in a row of 5 chairs. How many ways can they be seated?

   Sol.

   There are 4 ways to seat one person in one chair and the other person in the chair at the right of the first person. There are 2! ways to arrange person A and B in these two chairs, so there are $4 \cdot 2 = 8$ ways to seat the two persons.

3. 4 boys and 2 girls are standing in a row to take a photo. Assume that the two girls has to stand next to each other, how many ways can they be arranged?

   Sol.

   Treat these two girls as one, there are 5! ways to arrange the 5 persons. The two girls can switch places, so there are 2! ways to arrange them. Therefore, there are $5! \cdot 2! = 240$ ways to arrange the 6 persons.

19.2.6 Exercise 19.2b

1. Assume that there is no repeated digits, how many 5 digit numbers can be formed using the digits 1, 2, 3, 4, 5?

   Sol.

   There are ${}_5P_5 = 5! = 120$ ways to form a 5 digit number with these 5 digits.

2. How many ways are there to arrange the flags of 10 ASEAN members in a row?

   Sol.

   There are ${}_{10}P_{10} = 10! = 3628800$ ways to arrange the flags of the 10 ASEAN members in a row.

3. 7 novel stories are to be compiled into a book. The sortest story must be placed at the beginning of the book, while the longest story must be placed at the end. How many ways can the stories be arranged?

Sol.

Since the first book and the last book are fixed, there are ${}_5P_5 = 5! = 120$ ways to arrange the 5 books in the middle.

4. Ten students are to be arranged in a row. Two of the tallest students must be placed at the beginning of the row. How many ways can the students be arranged?

   Sol.

   Since the first two students are fixed, while these two students can switch places, there are $2! \cdot 8! = 80640$ ways to arrange the 10 students.

5. There are nine programmes in a literature festival. If one of the programmes is to be placed at the middle or at the end, how many ways can the programmes be arranged?

   Sol.

   There are 2! ways to arrange the one of the programme, either at the middle or at the end. There are ${}_8P_8 = 8! = 40320$ ways to arrange the other 8 programmes at the rest of the positions. Therefore, there are $2! \cdot 8! = 80640$ ways to arrange the 9 programmes.

6. How many permutations of the letters in the word *EQUATION* are there? if the letter E and N are to be placed at the beginning and at the end respectively, how many ways can the letters be arranged?

   Sol.

   There are ${}_8P_8 = 8! = 40320$ ways to arrange these 8 letters. If the first letter and the last letter are fixed, there are ${}_6P_6 = 6! = 720$ ways to arrange the other 6 letters.

7. There are 4 mobile phones that are to be registered a mobile phone number. Chosen 7 phone numbers for pairing, how many ways can the mobile phones be paired with the phone numbers?

   Sol.

   Since each of these 4 phones can be paired with any of the 7 phone numbers, here are ${}_7P_4 = 840$ ways to pair the 4 mobile phones with the 7 phone numbers.

8. There are 4 passengers sitting inside a 6 seats SUV. How many ways can the passengers be seated in the SUV?

   Sol.

Since these 4 passengers can choose any of the 6 seats in the SUV, here are $_6P_4 = 360$ ways to seat the 4 passengers in the SUV.

9. A ping pong coach wants to choose 3 players from a total of 5 players to be the first single, second single and third single respectively. If an elite player has to be chosen as the first or the second single, how many ways can the players be chosen?

   Sol.

   The elite player can choose either the first single or the second single. There are $_4P_2$ ways to choose the other two positions among the 4 players excluding the elite player. Therefore, there are $2! \cdot {}_4P_2 = 24$ ways to choose the players.

10. 8 chairs are to be arranged in two rows of 4 chairs each, in order to provide a place for 8 people to sit. If 3 out of the 8 people are to be seated in the first row, how many ways can the people be arranged?

    Sol.

    The 3 people who need to be seated at the first row can choose any of the 4 chairs in the first row. There are $_4P_3$ ways to arrange the 3 people in the first row. The other 5 people can choose any of the other 5 chairs. There are $_5P_5$ ways to arrange these 5 people. Therefore, there are $_4P_3 \cdot {}_5P_5 = 24 \cdot 120 = 2880$ ways to arrange the 8 people.

11. How many permutations of the 5 different letters in the word *TRIANGLE* are there? If the beginning and the end of the word are consonants, how many ways can the letters be arranged?

    Sol.

    There are $_8P_5 = 6720$ ways to arrange the 5 letters among the 8 letters in the word *TRIANGLE*. If the beginning and the end of the word are consonants, there are $_3P_2$ ways to arrange the 2 of the 3 consonants at the beginning and the end of the word. There are $_6P_3$ ways to arrange the 3 of the other 6 letters in the middle of the word. Therefore, there are $_3P_2 \cdot {}_6P_3 = 6 \cdot 240 = 2400$ ways to arrange the letters.

12. All the letters in the word *FANCIES* are to be arranged. If the vowels are to be arranged at even positions, how many ways can the letters be arranged?

    Sol.

    There are $_3P_3 = 3!$ ways to arrange the 3 vowels in the word, and there are $_4P_4 = 4!$ ways to arrange the 4 consonants in the word. Therefore, there are $3! \cdot 4! = 6 \cdot 24 = 144$ ways to arrange the letters.

13. Rearranging all the letters in the word *NUMERICAL*, how many ways can the letters be arranged? if 4 vowels are to be put together, how many ways can the letters be arranged?

    Sol.

    There are $_9P_9 = 10! = 362880$ ways to arrange the 10 letters in the word. If all vowels are to be put together, we can treat the vowels as one letter, and there are $_6P_6 = 6!$ ways to arrange the 6 letters. There are $_4P_4 = 4!$ ways to arrange the four vowels. Therefore, there are $6! \cdot 4! = 720 \cdot 24 = 17280$ ways to arrange the letters.

14. Examinations of 7 subjects are to be arranged in a row of 7 days, with one subject at a day. If two of which cannot be arranged to be held on two consecutive days, how many ways can the examinations of these 7 subjects be arranged?

    Sol.

    If two of the examinations are to be held on two consecutive days, we treat these two examinations as one subject. There are $_6P_6 = 6!$ ways to arrange the 6 subjects. Since these two subjects can switch places, there are $2!$ ways to arrange the two subjects. Therefore, there are $6! \cdot 2!$ ways to arrange the 7 subjects such that two of which are arranged to be held on two consecutive days.

    With no criteria on the arrangement of the examinations, there are $_7P_7 = 7!$ ways to arrange the 7 subjects.

    Therefore, there are $_7P_7 - {}_6P_6 \cdot 2! = 5040 - 1440 = 3600$ ways to arrange the 7 subjects such that two of which are not arranged to be held on two consecutive days.

15. From 5 numbers 1, 2, 3, 4, 5, how many ways can the following numbers with no repeated digits be formed:

    (a) 5 digits odd numbers

    Sol.

    Since the number must be odd, the last digit must be 1, 3, or 5. There are 3 ways to choose the last digit. There are $_4P_4 = 4!$ ways to arrange the other 4 digits. Therefore, there are $3 \cdot 4! = 3 \cdot 24 = 72$ ways to form the 5 digits odd numbers

    (b) 5 digits even numbers

    Sol.

Since the number must be even, the last digit must be 2 or 4. There are 2 ways to choose the last digit. There are $_4P_4 = 4!$ ways to arrange the other 4 digits. Therefore, there are $2 \cdot 4! = 2 \cdot 24 = 48$ ways to form the 5 digits even numbers.

16. If the digits are not repeated, from the numbers 0 to 5, how many ways can the 6 digits odd numbers be formed?

Sol.

Since the number must be odd, the last digit must be 1, 3, or 5. There are 3 ways to choose the last digit. Since the first digit must not be 0, there are 4 ways to choose the first digit. There are $_4P_4 = 4!$ ways to arrange the other 4 digits. Therefore, there are $3 \cdot 4! \cdot 4 = 3 \cdot 24 \cdot 4 = 288$ ways to form the 6 digits odd numbers.

17. From the 8 numbers 0, 1, 2, 3, 4, 5, 6, 7, how many 5 digits numbers can be formed such that their digits are not repeated and can be divided by 25?

Sol.

Since the number must be divisible by 25 and cannot have repeated digits, the last two digits must be either 25, 50, or 75. Since the first digit cannot be 0, if the first two digits are 50, there are 6 ways to choose the last three digits. If the first two digits are 25 or 75, there are 5 ways to choose the last three digits. Therefore, there are $6 + 5 \cdot 2 = 16$ ways to choose the first digit and the last two digits. For the other two digits, there are $_5P_2$ ways to choose them. Therefore, there are $16 \cdot {}_5P_2 = 16 \cdot 20 = 320$ ways to form the 5 digits numbers.

18. From the 6 numbers 0, 1, 2, 3, 4, 5, how many 4 digits numbers can be formed such that their digits are not repeated and can be divided by 4?

Sol.

Since the number must be divisible by 4, the last two digits divisible by 4. Among the numbers from 0 to 5, 04, 12, 20, 24, 32, 40, 52 are the only numbers with no repeated digits that are divisible by 4. Since the first digit cannot be 0, if the last two digit are either 04, 20, or 40, there are 4 ways to choose the first digit. If the last two digits are either 12, 24, 32 or 52, there are 3 ways to choose the first digit. Therefore, there are $3 \cdot 4 + 4 \cdot 3 = 24$ ways to choose the first digit and the last two digits. For the other digits, there are 3 ways to

choose them. Therefore, there are $24 \cdot 3 = 72$ ways to form the 4 digits numbers.

19.3 Circular Permutations

In the permutation we have discussed in the previous section, all the elements are arranged in a row. This kind of permutation is called *linear permutation*. Its identity is that it has a beginning and an end. The permutations we are going to discuss in this section are arranged on a closed curve line. This kind of permutation is called *circular permutation*. The identity of circular permutation is that it has no beginning and no end.

For this, we will use an example to explain the concept of circular permutation.

Four people are to be seated in a circle. How many ways can they be seated?

Sol.

Let these four people be *A*, *B*, *C* and *D* respectively.

If these people are arranged in a row, there are 4! ways to arrange them.

Let's take a look at the arrangements below:

$$
\begin{array}{cccc}
A & B & C & D \\
B & C & D & A \\
C & D & A & B \\
D & A & B & C
\end{array}
$$

You'll notice that the arrangements above are all the same if being arranged on a circle, and the only difference is the position of the first person.
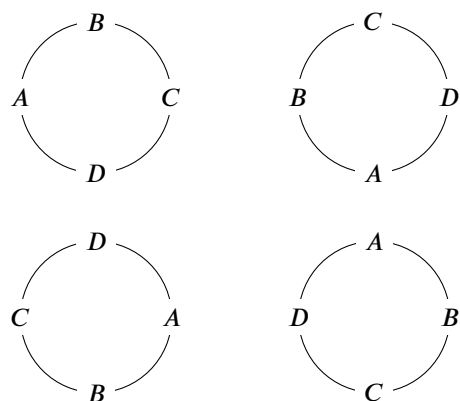
From the above example, each circular permutation is corresponded to 4 linear permutations. Given that the permutation of 4 people arranging in a row has 4! ways, so there are $\frac{4!}{4} = 3! = 6$ ways to arrange these four people in a circle.

If we generalize the above example, we can get the following formula:

1. The formula of circular permutation of n elements:

$$\frac{_nP_n}{n} = \frac{n!}{n} = (n-1)!$$

2. The formula of circular permutation of r elements from n elements ($r \le n$):

$$\frac{_nP_r}{r} = \frac{n!}{r(n-r)!}$$

19.3.1 Practice 6

1. Choose 5 people from 6 males and 5 females to be seated in a circle. How many ways can they be seated?

   Sol.

   There are a total of $6 + 5 = 11$ people, so they are a total of $\frac{_{11}P_5}{5} = 11088$ ways to be seated.

2. 6 males and 4 females are to be seated around a circular table. If females can't seat beside each other, how many ways can they be seated?

   Sol.

   First, arrange the 6 males into a circle, there are $(6-1)! = 5!$ ways to do it. Next, arrange the the 4 females to sit in between the 6 males, there are $_6P_4$ ways to do it. So there are a total of $5! \cdot {_6P_4} = 120 \cdot 360 = 43200$ ways to arrange the seat.

19.3.2 Exercise 19.3

1. 8 people are to be seated in a circle. How many ways can they be seated?

   Sol.

   There are a total of $(8-1)! = 7! = 5040$ ways to arrange them.

2. 10 children are to be arranged in a circle. How many ways can they be arraged? If a child must be seated in the primary position, how many ways can they be arranged?

   Sol.

   There are a total of $(10-1)! = 9! = 362880$ ways to arrange them. If a child must be seated in the primary position, there are a total of $9! = 362880$ ways to arrange them.

3. 6 people are to be formed into a circle. If two people must seat together, how many ways can they be arranged?

   Sol.

   We can treat those two people as one person, so there are a total of $(5-1)! = 4!$ ways to arrange them. Since these two people can switch their position, so there are a total of $2!$ ways to arrange them. Therefore, there are a total of $4! \cdot 2! = 24 \cdot 2 = 48$ ways to arrange them.

4. 4 males and 3 females are to be seated around a circular table. If none of the females can seat together, how many ways can they be seated?

   Sol.

   First, arrange the 4 males into a circle, there are $(4-1)! = 3!$ ways to do it. Next, put 3 females in between those 4 males, there are a total of $_4P_3$ ways to do it. So there are a total of $3! \cdot {_4P_3} = 6 \cdot 24 = 144$ ways to arrange the seat.

5. 4 pairs of couples and one child are to be seated around a circular table. If the couples must sit together, how many ways can they be seated?

   Sol.

   We can treat the couples as one person, so there are a total of $(3-1)! = 2!$ ways to arrange them. Next, in between those couples, there are $4!$ ways to arrange them. Next, in between each couples, since they can switch their position, so there are a total of $2!$ ways to arrange them. Therefore, there are a total of $2! \cdot 4! \cdot 2! = 8 \cdot 24 \cdot 2 = 384$ ways to arrange them.

6. A family of 7 people are sitting together around a circular table for a dinner. If the grandfather, grandmother, father and mother must sit together, of which the grandfather and the grandmother, the father and the mother must sit together, how many ways can they be seated?

   Sol.

   Since grandfather, grandmother, father and mother must sit together, we can treat them as one person, so there are a total of $(4-1)! = 3!$ ways to arrange them. In between those 4 people who must be seated together, since the grandfather and the grandmother, the father and the mother can switch their position, so there are a total of $2!$ ways to arrange them. Next, in between the grandfather and grandmother, father and mother, since they can switch their position, so there are a total of $2 \cdot 2!$ ways to arrange them. Therefore, there are a total of $3! \cdot 2! \cdot 2 \cdot 2! = 48$ ways to arrange them.

7. If a linear permutation of n people is 6 times of a circular permutation of n people, find these two permutations.

Sol.

$$n! = 6(n-1)!$$
$$\frac{n!}{(n-1)!} = 6$$
$$\frac{n(n-1)!}{(n-1)!} = 6$$
$$n = 6$$

So there are a total of $6! = 720$ linear permutations and $(6-1)! = 5! = 120$ circular permutations.

19.4 Full Permutations of Inexactly Distinct Elements

In all the previous questions of permutations, the given elements are all distinct. However, in some cases, there are some elements that are the same, this kind of permutation are considered as *permutations with repetition*. Let's discusse the folowing example:

For a full permutation of three elements a, a, and b, how many ways can they be arranged?

Let's treat the identical elemnents a as two different elements a_1 and a_2, there will be 3! ways to arrange these three different element a_1, a_2, and b, as listed below:

$$\begin{array}{ccc ccc} a_1 & a_2 & b & a_2 & a_1 & b \\ a_1 & b & a_2 & a_2 & b & a_1 \\ b & a_1 & a_2 & b & a_2 & a_1 \end{array}$$

In the three rows above, for each row, the position of b is fixed, a_1 and a_2 has $2! = 2$ ways to be arranged.

If we change a_1 and a_2 above back to a, then the two different arrangements of a_1 and a_2 will be counted as one arrangement of a. Hence, there are only $\frac{3!}{2!} = 3$ ways for full permutation of three elements a, a, and b.

Generalize the above example, given n elements, where there are n_1 elements a_1, n_2 elements a_2, $\cdots$, n_k elements a_k, where $n_1 + n_2 + \cdots + n_k = n$, then the number of full permutations of these n elements is:

$$\frac{n!}{n_1! n_2! \cdots n_k!}$$

19.4.1 Practice 7

1. Giving 9 children 2 pens, 3 ball-point pens, 4 pencils, how many ways can they be given if each child must be given one pen?

Sol.

There are a total of $\frac{9!}{2!3!4!} = 1260$ ways to give the children the pens.

2. Find the full permutation of the letters in the word *EXPRESSION* with 9 letters.

Sol. There are 2 *E*s, 2 *S*s, 1 *P*, 1 *O*, 1 *N*, 1 *R* and 1 *X*. Hence, there are a total of $\frac{10!}{2!2!1!1!1!1!1!} = 907200$ ways to arrange the letters in the word *EXPRESSION*.

19.4.2 Exercise 19.4

1. How many 8 digit numbers can be formed using 8 digits 1, 1, 2, 2, 3, 3, 4, 5?

Sol.

There are 2 1s, 2 2s, 2 3s, 1 4 and 1 5 in these 8 digits. Hence, there are a total of $\frac{8!}{2!2!2!1!1!} = 5040$ ways to arrange the 8 digits.

2. Arranging all the letters in the word *MALAYSIA*, how many ways can they be arranged such that the three "A"s are not totally together?

Sol.

There are 3 *A*s, 1 *L*, 1 *M*, 1 *S*, 1 *I* and 1 *Y* in the word *MALAYSIA*. Hence, there are a total of $\frac{8!}{3!1!1!1!1!1!} = 6720$ ways to arrange the letters in the word *MALAYSIA*. If all the *A*s are together, treat all the *A* as one element, there are $6! = 720$ ways to arrange them. Hence, there are a total of $6720 - 720 = 6000$ ways to arrange the letters in the word *MALAYSIA* such that the three "A"s are not totally together.

3. Arranging all the letters in the word *MATHEMATICAL*, how many ways can they be arranged such that the three "A"s are not totally together?

Sol.

There are 3 *A*s, 2 *M*s, 2 *T*s, 1 *C*, 1 *E*, 1 *H*, 1 *I*, 1 *L*, and 1 *U* in the word *MATHEMATICAL* with 12 letters. There are a total of $\frac{12!}{3!2!2!1!1!1!1!1!1!} = 19958400$ ways to arrange the letters in the word. If all the *A*s are together, treat all the *A* as one element, there are $\frac{10!}{2!2!1!1!1!1!1!} = 907200$ ways to arrange them. Hence, there are a total of $19958400 - 907200 = 19051200$ ways to arrange the letters in the word *MATHEMATICAL* such that the three "A"s are not totally together.

4. The diagram below shows a city with 5 north-west roads and 5 south-east roads, where each road shares the same length.
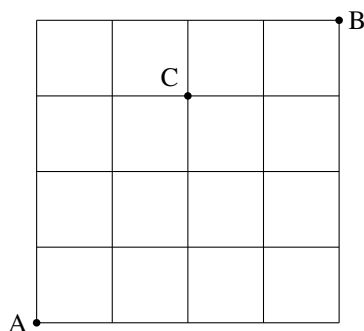
(a) How many shortest paths are there from *A* through *C* to *B*?

Sol.

From *A* to *C*, we must walk 3 steps to the north and 2 steps to the east. There are a total of $\frac{5!}{3!2!} = 10$ ways go from *A* to *C*.

From *C* to *B*, we must walk 1 step to the north and 2 steps to the east. There are a total of $\frac{3!}{1!2!} = 3$ ways go from *C* to *B*.

Hence, there are a total of $10 \cdot 3 = 30$ shortest paths from *A* through *C* to *B*.

(b) How many shortest paths are there from A to B without passing through C?

Sol.

From *A* to *B*, we must walk 4 steps to the north and 4 steps to the east. There are a total of $\frac{8!}{4!4!} = 70$ ways go from *A* to *B*.

Hence, there are a total of $70 - 30 = 40$ shortest paths from *A* to *B* without passing through *C*.

5. Arranging all the letters in the word *GEOMETRIC*, find the number of permutations such that the following conditions are satisfied:

(a) No limitaion on the arrangement of the letters.

Sol.

There are 2 *E*s, 1 *G*, 1 *I*, 1 *M*, 1 *O*, 1 *R*, 1 *C*, and 1 *T* in the word *GEOMETRIC* with 9 letters. There are a total of $\frac{9!}{2!1!1!1!1!1!1!1!} = 181400$ ways to arrange the letters in the word.

(b) All the vowels must be together.

Sol.

Treat all vowels as one element, there are 6! = 720 ways to arrange them. since all vowels can

switch position with each other, there are $\frac{4!}{2!1!1!} = 12$ ways to arrange them. Hence, there are a total of $720 \cdot 12 = 8640$ ways to arrange the letters in the word *GEOMETRIC* such that all the vowels are together.

(c) None of the vowels are adjacent to each other.

Sol.

Arrange all the consonants together. There are 5! = 120 ways to arrange them. Next, put 4 vowels at the beginning of the word, the end of the word, or in between consonants. There are a total of $\frac{{}_6P_4}{2!1!1!} = 180$ ways to do so. Hence, there are a total of $120 \cdot 180 = 21600$ ways to arrange the letters in the word *GEOMETRIC* such that none of the vowels are adjacent to each other.

19.5 Permutations with Repetition

If the elements can be chosen more than once, arranging *r* elements from *n* elements is called *permutation with repetition* of *r* elements from *n* elements.

Given *n* distinct elements, each element can be chosen more than once, then there are *n* ways to choose the first element, *n* ways to choose the second element, $\cdots$, *n* ways to choose the *r*th element. According to the multiplication rule, its permutations are:

$$\underbrace{n \times n \times \cdots \times n}_{r} = n^r$$

19.5.1 Practice 8

1. How many ways are there to shoot 3 balls into 6 baskets?

Sol.

There are 6 baskets, each basket can be shoot more than once, so there are $6^3 = 216$ ways to shoot 3 balls into 6 baskets.

2. If the all the digits can be used more than once, how many 3 digit numbers can be formed using the digits 0, 1, 2, 3, 4?

Sol.

Since the first digit cannot be 0, there are 4 ways to choose it. For the other two digits, there are 5 ways to choose each of them. Hence, there are a total of $4 \cdot 5 \cdot 5 = 100$ 3 digit numbers can be formed using the digits 0, 1, 2, 3, 4.

19.5.2 Exercise 19.5

1. The office documents of a company are coded with a combination of two of the first four letters of the alphabet A, B, C, D. If each letter can be used more than once, how many different codes can be formed?

   Sol.

   There are two letters in each code, each letter can be chosen more than once, so there are $4^2 = 16$ different codes can be formed.

2. There are eight paths from A to B. How many ways are there to go from A to B and then back to A?

   Sol.

   There are 8 paths from A to B, and 8 paths from B to A. Hence, there are a total of $8 \cdot 8 = 64$ ways to go from A to B and then back to A.

3. A four digit number is to be formed with 4 numbers 3, 4, 5, 7, how many ways can it be formed such that its thousand digit and hundred digit are the same?

   Sol.

   Since the thousand digit and hundred digit are the same, there are 4 ways to choose the thousand and hundred digit combined. For the other two digits, there are 4 ways to choose each of them. Hence, there are a total of $4 \cdot 4 \cdot 4 = 64$ ways to form a four digit number such that its thousand digit and hundred digit are the same.

4. A basketball match has two possible result, either team A wins or team B wins. How many possible results are there if there are 10 matches?

   Sol.

   There are 2 possible results, either team A wins or team B wins. Hence, there are a total of $2^{10} = 1024$ possible results if there are 10 matches.

5. A restaurant has introduced 7 sets of special dishes for lunch. There are 3 people come for lunch, each person orders one dish. How many different combinations of dishes can be ordered?

   Sol.

   There are 7 sets of special dishes, each person can order any of the 7 dishes. Hence, there are a total of $7^3 = 343$ different combinations of dishes can be ordered.

6. A building is assigned one security guard every night. If the security company has three assignable guards, how many different ways of assignments are there for a week?

   Sol.

   There are 7 nights in a week, each night can be assigned to any of the 3 guards. Hence, there are a total of $3^7 = 2187$ different ways of assignments are there for a week.

7. A football match has three possible results, either team A wins, team B wins or the match is a draw. How many possible results are there if there are 8 matches?

   There are 3 possible results, either team A wins, team B wins or the match is a draw. Hence, there are a total of $3^8 = 6561$ possible results if there are 8 matches.

8. A school is going to hold a trilingual speech competition. The rule stated that each class can only send one representative for each language. If there are 40 students in a class, how many different combinations of students can be chosen for the competition?

   Sol.

   There are 3 languages, each class can only send one representative for each of the 3 languages. Hence, there are a total of $40^3 = 64000$ different combinations of students can be chosen for the competition.

9. How many 6 digits odd number can be formed using the digitis 0, 1, 2, $\cdots$, 5 if each digit can be used more than once?

   Sol.

   Since the number is odd, the last digit must be odd. Hence, there are 3 ways to choose it. Since the first digit cannot be 0, there are 5 ways to choose it. For the other four digits, there are 6 ways to choose each of them. Hence, there are a total of $3 \cdot 5 \cdot 6^4 = 19440$ 6 digits odd number can be formed using the digitis 0, 1, 2, $\cdots$, 5 if each digit can be used more than once.

10. The phone number format of a country is going from 7 digits to 8 digits. but the leading digit cannot be 0 or 1. How many new phone numbers will be formed after the change?

    Sol.

    When there are only 7 digits, there are 8 ways to choose the first digit (2 - 10), 10 ways to choose the rest of the digits (0 - 10). Hence, there are $8 \cdot 10^6$ phone numbers. When there are 8 digits, there are 8 ways to choose the first digit, 10 ways to choose the rest of the digits.

Hence, there are $8 \cdot 10^7$ phone numbers. Hence, there are:

$$8 \cdot 10^7 - 8 \cdot 10^6 = 80,000,000 - 8,000,000$$
$$= 72,000,000$$

new phone numbers.

19.6 Combinations and Combination Formula

Grouping r elements from n elements ($r \leq n$) without considering the order of the elements is called *combination* of r elements from n elements. The number of combinations of r elements from n elements is denoted by ${}^nC_r$, ${}_nC_r$, or $\binom{n}{r}$.

Take for example: taking two elements from three distinct element a, b, c to form a group. Below are the combinations of any two elements from a, b, c:

Permutations	ab	ba	ac	ca	bc	cb
Combinations	ab		ac		bc	

It can be considered as two steps:

First, combine any two elements from the three distinct elements to form a group. There are ${}_3C_2$ possible combinations.

Second, make full permutations for each combination. There are 2! possible permutations for each combination.

According to the multiplication rule,

$$\begin{aligned} {}_3P_2 &= {}_3C_2 \cdot 2! \\ \therefore {}_3C_2 &= \frac{{}_3P_2}{2!} \\ &= 3 \end{aligned}$$

Generalizing the above example, there are two steps to find the combinations of r elements from n elements:

First, combine any r elements from the n elements to form a group. There are ${}_nC_r$ possible combinations.

Second, make full permutations for each combination. There are $r!$ possible permutations for each combination.

According to the multiplication rule,

$$\begin{aligned} {}_nP_r &= {}_nC_r \cdot r! \\ \therefore {}_nC_r &= \frac{{}_nP_r}{r!} \\ &= \frac{n!}{(n-r)!r!}, \ r \leq n \end{aligned}$$

According to the definition ${}_nC_r = \frac{n!}{(n-r)!r!}$,

$$\begin{aligned} {}_nC_{n-r} &= \frac{n!}{(n-(n-r))!(n-r)!} \\ &= \frac{n!}{r!(n-r)!} \\ &= {}_nC_r \end{aligned}$$

That is,

$$ {}_nC_r = {}_nC_{n-r} $$

Note that:

1. If $r = n$, then ${}_nC_n = \frac{n!}{0!n!} = 1$.

2. If $r = 0$, then ${}_nC_0 = \frac{n!}{n!0!} = 1$.

19.6.1 Practice 9

1. There are six main cities in a country, each city has roads connecting to the other five cities. How many roads are there connecting the six cities?

   Sol.

   a road is formed between 2 cities, there are ${}_6C_2 = 15$ roads connecting the six cities.

2. 5 people are to be distributed in 4 cars, each car must have at least one person. How many ways are there to distribute the 5 people into 4 cars?

   Sol.

   There are one car with two people, so there are ${}_5C_2 = 10$ ways of distribution. For these 3 people and one group of 2 people, there are ${}_4P_4 = 4! = 24$ ways of distribution. Hence, there are $10 \cdot 24 = 240$ ways of distributing the 5 people into 4 cars.

19.6.2 Practice 10

There are 4 different books. With the following criteria, how many ways are there to distribute the books?

(a) Distribute evenly to two people.

   Sol.

   First, distribute the book to one person. There are ${}_4C_2 = 6$ ways of distribution. Then, distribute the remaining books to the other person. There are ${}_2C_2 = 1$ way of distribution. Hence, there are $N_1 = 6 \cdot 1 = 6$ ways of distributing the books.

(b) Seperate evenly into two piles.

   Sol.

Destructing the solution of (a). First, distribute the book into two piles. There are N_2 ways of distribution. Then, distribute each pile to the two people. There are $2!$ ways of distribution. There are $N_1 = N_2 \cdot 2!$ ways of distributing the books. Hence, there are $N_2 = \frac{6}{2!} = 3$ ways of distributing the books into two piles.

19.6.3 Exercise 19.6

1. Find the value of n and r of the following expessions:

(a) $_{16}C_{r+3} = {_{16}C_{7-r}}$

Sol.

$$_{16}C_{r+3} = {_{16}C_{7-r}}$$
$$r + 3 = 7 - r$$
$$2r = 4$$
$$r = 2$$

(b) $_{30}C_r = {_{30}C_{r+2}}$

Sol.

$$_{30}C_r = {_{30}C_{r+2}}$$
$$30 - r = r + 2$$
$$2r = 28$$
$$r = 14$$

(c) $_nC_8 = {_nC_7}$

Sol.

$$_nC_8 = {_nC_7}$$
$$_nC_r = {_nC_{n-r}}$$
$$r = 8$$
$$n - r = 7$$
$$n - 8 = 7$$
$$n = 15$$

2. Assume that $3(_nC_4) = 5(_{n-1}C_5)$, find the value of $_nC_9$.

Sol.

$$3(_nC_4) = 5(_{n-1}C_5)$$
$$\frac{_nC_4}{_{n-1}C_5} = \frac{5}{3}$$
$$\frac{n!}{4!(n-4)!} \cdot \frac{5!(n-6)!}{(n-1)!} = \frac{5}{3}$$
$$\frac{n(n-1)!}{4!(n-4)!} \cdot \frac{5 \cdot 4!(n-6)!}{(n-1)!} = \frac{5}{3}$$
$$\frac{5n(n-6)!}{(n-4)!} = \frac{5}{3}$$
$$\frac{5n(n-6)!}{(n-4)(n-5)(n-6)!} = \frac{5}{3}$$
$$\frac{n}{(n-4)(n-5)} = \frac{1}{3}$$
$$3n = (n-4)(n-5)$$
$$n^2 - 9n + 20 - 3n = 0$$
$$n^2 - 12n + 20 = 0$$
$$(n-10)(n-2) = 0$$
$$n = 10 \text{ or } n = 2$$

$\because n \in \mathbb{N}, \ n > 6$

$\therefore n = 10$

$$_nC_9 = {_{10}C_9}$$
$$= \frac{10!}{9!(10-9)!}$$
$$= \frac{10!}{9!1!}$$
$$= \frac{10 \cdot 9!}{9!}$$
$$= 10$$

3. There are 17 teams participating in a football competition. If each team plays against every other team, how many matches are there?

Sol.

Choosing two teams from 17 teams. There are $_{17}C_2 = 136$ ways of matching.

4. How many diagonals can be drawn in a convex nonagon?

Sol.

There are 9 vertices in a nonagon. Since the diagonals do not include the 9 sides of the nonagon, there are $_9C_2 - 9 = 36 - 9 = 27$ diagonals in a nonagon.

5. There are 6 students on duty in a class, 1 student is in charge of cleaning the whiteboard, 1 is in charge of cleaning the rubbish bin, 2 students are in charge of sweeping the floor and 2 students are in charge of ar-

ranging the desks. How many ways are there to distribute the 6 students into the 4 jobs?

Sol.

To choose 1 from 6 students for cleaning the whiteboard, there are $_6C_1 = 6$ ways. To choose 1 from 5 students for cleaning the rubbish bin, there are $_5C_1 = 5$ ways. To choose 2 from 4 students for sweeping the floor, there are $_4C_2 = 6$ ways. The two students left will be in charge of arranging the desks. Therefore, there are $_6C_1 \cdot _5C_1 \cdot _4C_2 = 6 \cdot 5 \cdot 6 = 180$ ways to distribute the 6 students into the 4 jobs.

6. 4 people are to be choosen from 5 couples, where each couple cannot be chosen together. How many ways are there to choose the 4 people?

Sol.

Choosing 4 people from 10 people or 5 couples with no restriction, there are $_{10}C_4$ ways. Choosing 4 people with at least one couple, there are $_5C_1 \cdot _8C_2$ ways. Choosing 4 people with two couples, there are $_5C_2$ ways. Therefore, using inclusion-exclusion rule, there are $_{10}C_4 - _5C_1 \cdot _8C_2 + _5C_2 = 210 - 140 + 10 = 80$ ways to choose 4 people from 5 couples where each couple cannot be chosen together.

7. Three signal flags are to be chosen from 6 flags, one of which is red, two of which are yellow and the rest are blue. How many ways are there to choose the 3 flags?

Sol.

Choosing 1 flag from each color, there are $_1C_1 \cdot _2C_1 \cdot _3C_1 = 6$ ways. Choosing 1 red and 2 yellow, there are $1 \cdot _2C_2 = 1$ ways. Choosing 1 red and 2 blue, there are $1 \cdot _3C_2 = 3$ ways. Choosing 2 yellow and 1 blue, there are $_2C_2 \cdot 1 = 2$ ways. Choosing 1 yellow and 2 blue, there are $_2C_1 \cdot _3C_2 = 6$ ways. Choosing 3 blue, there are $_3C_3 = 1$ way. Therefore, there are $6 + 1 + 3 + 2 + 6 + 1 = 19$ ways to choose 3 flags from 6 flags.

8. A delegation with 6 members is to be formed from 9 students who major in Mathematics and 4 students who major in Education. With the following criteria, how many ways are there to form the delegation?

(a) There are exactly two students who major in Education.

Sol.

Choose 2 from 4 students who major in Education, there are $_4C_2$ ways. Choose the other 4

students from 9 students who major in Mathematics, there are $_9C_4$ ways. Therefore, there are $_4C_2 \cdot _9C_4 = 6 \cdot 126 = 756$ ways to form the delegation.

(b) There are at least two students who major in Education.

Sol.

According to the critera, the combination of less than two students who major in Education is not allowed. There are a total of $_{13}C_6$ ways to form the delegation with no restriction, and there are $_9C_6$ ways to form the delegation with no Education major, $_4C_1 \cdot _9C_5$ ways to form the delegation with one Education major. Therefore, there are $_{13}C_6 - _9C_6 - _4C_1 \cdot _9C_5 = 1716 - 84 - 504 = 1118$ ways to form the delegation with at least two Education majors.

9. Separate 14 students evenly into 2 groups, how many ways are there to do so? How many ways are there to seperate these 14 students evenly into two classrooms?

Sol.

Choose 7 students from 14 students to form a group, there are $_{14}C_7$ ways. Since the order of the groups does not matter, there are $\frac{_{14}C_7}{2!} = \frac{3432}{2} = 1716$ ways of seperating 14 students evenly into 2 groups.

If the 14 students are to be seperated into two classrooms, since the these two groups of stuents can switch classrooms, there are $1716 \cdot 2! = 3432$ ways to do so.

19.7   Revision Exercise 19

1. Evaluate the following:

(a) $\frac{7!}{3!4!} + \frac{7!}{2!5!}$

Sol.

$$\begin{aligned}
\frac{7!}{3!4!} + \frac{7!}{2!5!} &= \frac{7 \cdot 6 \cdot 5!}{6 \cdot 24} + \frac{7 \cdot 6 \cdot 5!}{2 \cdot 120} \\
&= \frac{7 \cdot 5!}{24} + \frac{7 \cdot 5!}{40} \\
&= \frac{7 \cdot 120}{24} + \frac{7 \cdot 120}{40} \\
&= 7 \cdot 5 + 7 \cdot 3 \\
&= 35 + 21 \\
&= 56
\end{aligned}$$

(b) $\frac{_{11}P_5 + _{11}P_4}{_{12}P_5 + _{12}P_4}$

Sol.

$$\frac{_{11}P_5 + _{11}P_4}{_{12}P_5 + _{12}P_4} = \frac{\frac{11!}{(11-5)!} + \frac{11!}{(11-4)!}}{\frac{12!}{(12-5)!} + \frac{12!}{(12-4)!}}$$

$$= \frac{\frac{11!}{6!} + \frac{11!}{7!}}{\frac{12!}{7!} + \frac{12!}{8!}}$$

$$= \frac{\frac{11!(6!+7!)}{6!7!}}{\frac{12!(7!+8!)}{7!8!}}$$

$$= \frac{11!(6! + 7!)}{6!7!} \cdot \frac{7!8!}{12!(7! + 8!)}$$

$$= \frac{11!(6! + 7!)}{6!} \cdot \frac{8 \cdot 7 \cdot 6!}{12!(7! + 8!)}$$

$$= \frac{56 \cdot 11!(6! + 7!)}{12 \cdot 11!(7! + 8!)}$$

$$= \frac{14(6! + 7!)}{3(7! + 8!)}$$

$$= \frac{14(6! + 7 \cdot 6!)}{3(7! + 8 \cdot 7!)}$$

$$= \frac{14 \cdot 6! \cdot 8}{3 \cdot 7! \cdot 9}$$

$$= \frac{14 \cdot 6! \cdot 8}{3 \cdot 7 \cdot 6! \cdot 9}$$

$$= \frac{14 \cdot 8}{3 \cdot 7 \cdot 9}$$

$$= \frac{2 \cdot 8}{3 \cdot 9}$$

$$= \frac{16}{27}$$

2. If $\frac{(n+6)!}{(n+4)!} = 18(n + 1)$, find the value of n.

Sol.

$$\frac{(n + 6)!}{(n + 4)!} = 18(n + 1)$$

$$\frac{(n + 6)(n + 5)(n + 4)!}{(n + 4)!} = 18(n + 1)$$

$$(n + 6)(n + 5) = 18(n + 1)$$

$$n^2 + 11n + 30 = 18n + 18$$

$$n^2 - 7n + 12 = 0$$

$$(n - 3)(n - 4) = 0$$

$$n = 3 \text{ or } n = 4$$

3. Assume that $_{2n}C_3 : {_n}C_2 = 44 : 3$, find the value of n.

Sol.

$$_{2n}C_3 : {_n}C_2 = 44 : 3$$

$$\frac{_{2n}C_3}{_n C_2} = \frac{44}{3}$$

$$\frac{(2n!)}{3!(2n - 3)!} \cdot \frac{2!(n - 2)!}{n!} = \frac{44}{3}$$

$$\frac{(2n!)}{6(2n - 3)!} \cdot \frac{2(n - 2)!}{n!} = \frac{44}{3}$$

$$\frac{(2n!)}{3(2n - 3)!} \cdot \frac{(n - 2)!}{n!} = \frac{44}{3}$$

$$\frac{(2n!)}{(2n - 3)!} \cdot \frac{(n - 2)!}{n!} = 44$$

$$\frac{(2n!)}{(2n - 3)!} \cdot \frac{(n - 2)!}{n(n - 1)(n - 2)!} = 44$$

$$\frac{2n(2n - 1)(2n - 2)(2n - 3)!}{(2n - 3)!} \cdot \frac{1}{n(n - 1)} = 44$$

$$\frac{4n(2n - 1)(n - 1)}{n(n - 1)} = 44$$

$$4(2n - 1) = 44$$

$$2n - 1 = 11$$

$$2n = 12$$

$$n = 6$$

4. All the letters from the word *TRIANGLE* are to be arranged. How many ways of arrangements are there such that the vowels are all seperated from each other?

Sol.

First, arrange the consonants. There are $_5P_5 = 5!$ ways to do so. Then, put the vowels in the beginning of the word, the end of the word, or in between the consonants. There are $_6P_3$ ways to do so. Thus, there are $_6P_3 \cdot 5! = 120 \cdot 120 = 14400$ ways to arrange the letters.

5. One vowel and one consonant are to be chosen from the word *VOLUME* to form a word. How many ways are there to do so?

Sol.

There are 3 consonants and 3 vowels. Pick one from each of them, there are 9 ways. Since the consonant and vowel can be chosen in any order, there are $9 \cdot 2 = 18$ ways.

6. A basketball coach wants to choose two vanguards from four shooters, one center from two tall players, and two defenders from four ball handlers. How many ways are there to form a team?

Sol.

There are $_4C_2 = 6$ ways to choose two vanguards, $_2C_1 = 2$ ways to choose one center, and $_4C_2 = 6$ ways to choose two defenders. Thus, there are $6 \cdot 2 \cdot 6 = 72$ ways to form a team.

7. A committee must be formed by one lawyer, two engineers and two doctors chosen from 3 lawyers, 6 engineers and 7 doctors. How many ways are there to form the committee?

   Sol.

   There are $_3C_1 = 3$ ways to choose one lawyer, $_6C_2 = 15$ ways to choose two engineers, and $_7C_2 = 21$ ways to choose two doctors. Thus, there are $3 \cdot 15 \cdot 21 = 945$ ways to form the committee.

8. There is one of each note of the following value: $1, $5, $10, $20, $50, $100. How many distinct values of currency can be made from these notes?

   Sol.

   Pick one note, there are $_6C_1 = 6$ ways to do so.

   Pick two notes, there are $_6C_2 = 15$ ways to do so.

   $\vdots$

   Pick six notes, there are $_6C_6 = 1$ way to do so.

   Thus, there are $6 + 15 + 20 + 15 + 6 + 1 = 63$ distinct values of currency that can be made from these notes.

9. There are 4 different history books, 5 different geography books and 3 different literature books on the shelf. If they are to be arranged in a row, how many ways are there to arrange them such that books of the same subject are arranged together?

   Sol.

   There are $_4P_4 = 4!$ ways to arrange the history books.

   There are $_5P_5 = 5!$ ways to arrange the geography books.

   There are $_3P_3 = 3!$ ways to arrange the literature books.

   Since the books of each subject can be arranged in any order, there are $3! \cdot 4! \cdot 5! \cdot 3! = 6 \cdot 24 \cdot 120 \cdot 6 = 103680$ ways to arrange the books.

10. 6 characters from 10 different characters A, B, C, D, E, and 0, 1, 2, 3, 4 are to be chosen to form a password. How many password can be formed such that there are no repeated characters and the password does not start with 0?

    Sol.

Since the first character cannot be 0, there are 9 ways to choose the first character. Since there are 2 categories of characters: alphabets and digits, there are $\frac{_{10}P_5}{2}$ ways to choose the remaining 5 characters. Thus, there are $9 \cdot \frac{_{10}P_5}{2} = 9 \cdot 151020 = 136080$ passwords that can be formed.

11. How many 7 digit even numbers can be formed from full permutations of the digits 0, 1, 2, 3, 4, 5, 6? How many multiple of 10 are there in these numbers?

    Sol.

    Since the first digit cannot be 0, there are 6 ways to choose the first digit, 3 even and 3 odd. Since the number must be even, the last digit must be 2, 4, 6, or 0. If the first digit is even, there are 3 ways to choose the last digit. If the first digit is odd, there are 4 ways to choose the last digit. Thus, there are $3 \cdot 3 + 3 \cdot 4 = 21$ ways to choose the first and the last digit. For the other 5 digits, there are $_5P_5 = 5!$ ways to choose them. Thus, there are $21 \cdot 5! = 21 \cdot 120 = 2520$ 7 digit even numbers that can be formed.

12. How many permutations of all the characters in the word "fei li wu shi, fei li wu ting" are there?

    Sol.

    There are 2 "fei"s, 2 "li"s, 2 "wu"s, 1 "shi", and 1 "ting". Therefore, there are $\frac{8!}{2!2!2!1!1!} = 5040$ permutations of all the characters in the word "fei li wu shi, fei li wu ting".

13. How many ways are there to arrange all the letters in the word *ARRANGEMENT*?

    Sol.

    There are 2 "A"s, 2 "R"s, 2 "N"s, 1 "G", 2 "E"s, 1 "M", and 1 "T". Therefore, there are $\frac{11!}{2!2!2!2!1!1!1!} = 2494800$ ways to arrange all the letters in the word *ARRANGEMENT*.

14. Arranging 6 cups with different color and 4 canned juices with different flavour into a circle, how many ways are there to do so such that all the canned juices are not next to each other?

    Sol.

    First, arrange the 6 cups into a circle. There are $(6 - 1)! = 5!$ ways of doing so.

    Then, arrange the 4 canned juices in between the cups. There are $_6P_4$ ways of doing so.

Thus, there are $5! \cdot {}_6P_4 = 120 \cdot 360 = 43200$ ways to arrange the 6 cups and 4 canned juices into a circle such that all the canned juices are not next to each other.

15. In a box of Chinese chess, there are 1 general, 2 advisors, 2 elephants, 2 horses, 2 chariots, 2 cannons and 5 soldiers of red color. Now we arrange these 16 pieces into a circle:

   (a) All the pieces of the same type are arranged together.

   Sol.

   There are 7 types of pieces. Thus, there are $(7-1)! = 6! = 720$ ways to arrange the 7 types of pieces into a circle.

   (b) All the pieces of the same type are symmetrically arranged on one diameter.

   Sol.

   Since there are only one general, we arrange the general and one soldier to form a symmetrical pair. There are $\frac{2!}{2}$ ways of arranging them. Then, we arrange the other 4 soldiers to form two symmetrical pairs with 2 soldiers each. Next, we arrange the 2 advisors, 2 horses, 2 chariots, 2 cannons and 2 elephants to form symmetrical pairs respectively. Then, insert these 7 symmetrical pairs besides the general and soldiers. Since the two pairs of soldiers are the same, there are $\frac{7!}{2!}$ ways of arranging these 7 pairs into the circle. Thus, there are $\frac{2!}{2} \cdot \frac{7!}{2!} = 2520$ ways to arrange the pieces into a circle.

16. If there is no repeated digits, how many odd numbers in between 4000 and 9000 can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9?

   Sol.

   Since the numbers are in between 4000 and 9000, the first digit must be 4, 5, 6, 7, or 8. There are 5 ways to choose the first digit. Since the numbers must be odd, the last digit must be 1, 3, 5, 7, or 9. If the first digit is 4, 6, or 8, there are 5 ways to choose the last digit. If the first digit is 5 or 7, there are 4 ways to choose the last digit. Thus, there are $3 \cdot 5 + 2 \cdot 4 = 23$ ways to choose the first and the last digit. For the other 2 digits, there are ${}_8P_2 = 56$ ways to choose them. Thus, there are $23 \cdot 56 = 1288$ odd numbers in between 4000 and 9000 that can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

17. How many 4 digit numbers with exactly 2 repeated digits are there if the digits are chosen from 1, 2, 3, 4, 5 while the digits can be chosen more than once?

   Sol.

   There are 3 distinct digits in the numbers. First, choose the digit that occurs twice. There are 5 ways to choose the digits. Then, choose the other 2 digits. There are ${}^3P_2$ ways to choose them. Since the digits can switch places, there are $\frac{4!}{2!1!1!} = 12$ ways to arrange the 4 digits. Thus, there are $5 \cdot {}_3P_2 \cdot 12 = 360$ 4 digit numbers with exactly 2 repeated digits.

18. From the class committee of 10 members, at least 2 members, at most 8 members are to be chosen as representatives to attend a forum. How many ways are there to choose the representatives?

   Sol.

   If two members are chosen, there are ${}_{10}C_2 = 45$ ways to choose the representatives.

   If three members are chosen, there are ${}_{10}C_3 = 120$ ways to choose the representatives.

   $\vdots$

   If eight members are chosen, there are ${}_{10}C_8 = 45$ ways to choose the representatives.

   Thus, there are ${}_{10}C_2 + {}_{10}C_3 + {}_{10}C_4 + {}_{10}C_5 + {}_{10}C_6 + {}_{10}C_7 + {}_{10}C_8 = 45 + 120 + 210 + 252 + 210 + 120 + 45 = 1002$ representatives.

19. A 4 members team of at least 2 engineers and 1 technician is to be formed from 5 engineers and 4 technicians. How many ways are there to form the team?

   Sol.

   According to the question, team with no engineers or only one engineer is not a valid team. Also, a team with no technicians is not a valid team as well.

   With no criteria given, there are ${}_9C_4 = 126$ ways to form the team.

   If the team has no engineers, there are ${}_4C_4$ ways to form the team.

   If the team has only one engineer, there are ${}_5C_1 \cdot {}_4C_3$ ways to form the team.

   If the team has no technicians, there are ${}_5C_4$ ways to form the team.

   Therefore, there are ${}_9C_4 - {}_4C_4 - {}_5C_1 \cdot {}_4C_3 - {}_5C_4 = 126 - 1 - 20 - 5 = 100$ ways to form the team.

20. 9 different books are to be distributed to 3 people A, B and C. With the following conditions, how many ways are there to distribute the books?

(a) Each people get 3 books.

Sol.

First, choose three books to give to A. There are $_9C_3$ ways to do so.

Then, choose three books from the remaining 6 books to give to B. There are $_6C_3$ ways to do so.

Finally, choose three books from the remaining 3 books to give to C. There are $_3C_3$ ways to do so.

Thus, there are $_9C_3 \cdot {_6C_3} \cdot {_3C_3} = 1680$ ways to distribute the books.

(b) A gets 2 books, B gets 3 books, C gets 4 books.

Sol.

First, choose two books to give to A. There are $_9C_2$ ways to do so.

Then, choose three books from the remaining 7 books to give to B. There are $_7C_3$ ways to do so.

Finally, choose four books from the remaining 4 books to give to C. There are $_4C_4$ ways to do so.

Thus, there are $_9C_2 \cdot {_7C_3} \cdot {_4C_4} = 1260$ ways to distribute the books.

(c) One people get 2 books, one people get 3 books and one people get 4 books.

Sol.

First, make a pile of two books. There are $_9C_2$ ways to do so.

Then, make a pile of three books from the remaining 7 books. There are $_7C_3$ ways to do so.

Finally, make a pile of four books from the remaining 4 books. There are $_4C_4$ ways to do so.

Since the piles can be distributed to any of the three people, there are 3! ways to distribute the piles.

Thus, there are $_9C_2 \cdot {_7C_3} \cdot {_4C_4} \cdot 3! = 7560$ ways to distribute the books.

(d) Seperated into 3 groups, each group has 3 books.

Sol.

First, choose three books to give to the first group. There are $_9C_3$ ways to do so.

Then, choose three books from the remaining 6 books to give to the second group. There are $_6C_3$ ways to do so.

Finally, choose three books from the remaining 3 books to give to the third group. There are $_3C_3$ ways to do so.

Since the groups can be seperated in any order, there are 3! ways to do so.

Thus, there are $\frac{_9C_3 \cdot {_6C_3} \cdot {_3C_3}}{3!} = \frac{1680}{3!} = 280$ ways to distribute the books.

Chapter 20

Bionomial Theorem

20.1 Bionomial Theorem when n is a Natural Number

Back in junior high school, we have learnt

$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Now, let's discuss the expansion of $(a + b)^4$.

$$(a + b)^4 = (a + b)(a + b)(a + b)(a + b)$$

Each term in the expasion of $(a + b)^4$ is the product of one letter taken from each of the four brackets, that is to say, the expansion of should contain the following terms: a^4, a^3b, a^2b^2, ab^3, b^4.

Using the knowledge of permutation and combination, we can get the coefficient of each term in the expansion:

In the 4 brackets, if no b is chosen, then there are $_4P_0$ ways to do so, therefore the coefficient of a^4 is $_4P_0$.

In the 4 brackets, if 1 b is chosen, then there are $_4P_1$ ways to do so, therefore the coefficient of a^3b is $_4P_1$.

In the 4 brackets, if 2 b is chosen, then there are $_4P_2$ ways to do so, therefore the coefficient of a^2b^2 is $_4P_2$.

In the 4 brackets, if 3 b is chosen, then there are $_4P_3$ ways to do so, therefore the coefficient of ab^3 is $_4P_3$.

In the 4 brackets, if all 4 b is chosen, then there are $_4P_4$ ways to do so, therefore the coefficient of b^4 is $_4P_4$.

Therefore, $(a + b)^4 = {}_4C_0a^4 + {}_4C_1a^3b + {}_4C_2a^2b^2 + {}_4C_3ab^3 + {}_4C_4b^4$.

Generalize the above expansion, we have the following formula:

$$(a + b)^n = {}_nC_0a^n + {}_nC_1a^{n-1}b + {}_nC_1a^{n-2}b^2 \cdots + $$
$$ {}_nC_{n-r}a^{n-r}b^r + \cdots + {}_nC_nb^n$$

where $n \in \mathbb{N}$.

This formula is called the *Bionomial Theorem*, the polynomial in the right hand side is called the *Bionomial Expansion* of $(a + b)^n$, of which $_nC_0$, $_nC_1$, $\cdots$, $_nC_n$ are called the *Bionomial Coefficients*.

By looking at the formula above, we can know that,

1. The sum of the indices of a and b in each term is equal to the binomial expression, the index of a decreases by 1 from n to 0, while the index of b increases by 1 from 0 to n.

2. The bionomial expansion has $n+1$ terms, that is to say, it has one term more than the exponent of the binomial expression.

3. Since $_nC_r = {}_nC_{n-r}$, therefore $_nC_0 = {}_nC_n$, $_nC_1 = {}_nC_{n-1}$, $\cdots$, $_nC_2 = {}_nC_{n-2}$, $\cdots$.

Bionomial expression can also be calculated by the following table (Pascal Triangle):

$(a + b)^0$				1		
$(a + b)^1$				1 1		
$(a + b)^2$			1	2	1	
$(a + b)^3$		1	3	3	1	
$(a + b)^4$	1	4	6	4	1	
$(a + b)^5$	1 5	10	10	5	1	

$\vdots$

In the table above, for each row, except the beginning and the end being 1, all numbers except 1 are the sum of the two numbers above it, aka $_nC_{n+1}r = {}_nC_{r-1} + {}_nC_r$.

In bionomial theorem, let $a = 1$, $b = x$, then we get the following formula:

$$(1 + x)^n = 1 + {}_nC_1x + {}_nC_2x^2 + \cdots + {}_nC_rx^r + \cdots + x^n$$

20.1.1 Practice 1

Expand the following expression:

1. $(1 + x)^7$

Sol.

$$(1 + x)^7 = 1 + {}_7C_1 x + {}_7C_2 x^2 + {}_7C_3 x^3 + {}_7C_4 x^4$$
$$+ {}_7C_5 x^5 + {}_7C_6 x^6 + {}_7C_7 x^7$$
$$= 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5$$
$$+ 7x^6 + x^7$$

2. $(2 + 3x)^5$

Sol.

$$(2 + 3x)^5 = {}_5C_0 2^5 + {}_5C_1 2^4(3x) + {}_5C_2 2^3(3x)^2$$
$$+ {}_5C_3 2^2(3x)^3 + {}_5C_4 2(3x)^4 + {}_5C_5(3x)^5$$
$$= 32 + 240x + 720x^2 + 1080x^3 + 810x^4$$
$$+ 243x^5$$

20.1.2 Exercise 20.1

Expand the following expression (1 to 9):

1. $(m + n)^7$

Sol.

$$(m + n)^7 = {}_7C_0 m^7 + {}_7C_1 m^6 n + {}_7C_2 m^5 n^2 + {}_7C_3 m^4 n^3$$
$$+ {}_7C_4 m^3 n^4 + {}_7C_5 m^2 n^5 + {}_7C_6 mn^6 + {}_7C_7 n^7$$
$$= m^7 + 7m^6 n + 21m^5 n^2 + 35m^4 n^3 + 35m^3 n^4$$
$$+ 21m^2 n^5 + 7mn^6 + n^7$$

2. $(3 + 2x)^4$

Sol.

$$(3 + 2x)^4 = {}_4C_0 3^4 + {}_4C_1 3^3(2x) + {}_4C_2 3^2(2x)^2$$
$$+ {}_4C_3 3(2x)^3 + {}_4C_4(2x)^4$$
$$= 81 + 216x + 216x^2 + 96x^3 + 16x^4$$

3. $(x - 3)^5$

Sol.

$$(x - 3)^5 = {}_5C_0 x^5 + {}_5C_1 x^4(-3) + {}_5C_2 x^3(-3)^2$$
$$+ {}_5C_3 x^2(-3)^3 + {}_5C_4 x(-3)^4 + {}_5C_5(-3)^5$$
$$= x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$$

4. $(x + y^2)^6$

Sol.

$$(x + y^2)^6 = {}_6C_0 x^6 + {}_6C_1 x^5 y^2 + {}_6C_2 x^4(y^2)^2$$
$$+ {}_6C_3 x^3(y^2)^3 + {}_6C_4 x^2(y^2)^4 + {}_6C_5 x(y^2)^5$$
$$+ {}_6C_6(y^2)^6$$
$$= x^6 + 6x^5 y^2 + 15x^4 y^4 + 20x^3 y^6 + 15x^2 y^8$$
$$+ 6xy^{10} + y^{12}$$

5. $\left(2 + \frac{1}{x}\right)^5$

Sol.

$$\left(2 + \frac{1}{x}\right)^5 = {}_5C_0 2^5 + {}_5C_1 2^4\left(\frac{1}{x}\right) + {}_5C_2 2^3\left(\frac{1}{x}\right)^2$$
$$+ {}_5C_3 2^2\left(\frac{1}{x}\right)^3 + {}_5C_4 2\left(\frac{1}{x}\right)^4 + {}_5C_5\left(\frac{1}{x}\right)^5$$
$$= 32 + \frac{80}{x} + \frac{80}{x^2} + \frac{40}{x^3} + \frac{10}{x^4} + \frac{1}{x^5}$$

6. $\left(\frac{x}{3} + \frac{2}{x}\right)^4$

Sol.

$$\left(\frac{x}{3} + \frac{2}{x}\right)^4 = {}_4C_0\left(\frac{x}{3}\right)^4 + {}_4C_1\left(\frac{x}{3}\right)^3\left(\frac{2}{x}\right)$$
$$+ {}_4C_2\left(\frac{x}{3}\right)^2\left(\frac{2}{x}\right)^2 + {}_4C_3\left(\frac{x}{3}\right)\left(\frac{2}{x}\right)^3$$
$$+ {}_4C_4\left(\frac{2}{x}\right)^4$$
$$= \frac{x^4}{81} + \frac{8x^2}{27} + \frac{8}{3} + \frac{32}{3x^2} + \frac{16}{x^4}$$

7. $\left(x - \sqrt[3]{x^2}\right)^3$

Sol.

$$\left(x - \sqrt[3]{x^2}\right)^3 = {}_3C_0 x^3 + {}_3C_1 x^2\left(-\sqrt[3]{x^2}\right)$$
$$+ {}_3C_2 x\left(-\sqrt[3]{x^2}\right)^2 + {}_3C_3\left(-\sqrt[3]{x^2}\right)^3$$
$$= x^3 - 3x^2\sqrt[3]{x^2} + 3x^2\sqrt[3]{x} - x^2$$

8. $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^6$

Sol.

$$\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^6 = {}_6C_0\left(\sqrt{x}\right)^6 + {}_6C_1\left(\sqrt{x}\right)^5\left(-\frac{1}{\sqrt{x}}\right)$$

$$+ {}_6C_2\left(\sqrt{x}\right)^4\left(-\frac{1}{\sqrt{x}}\right)^2$$

$$+ {}_6C_3\left(\sqrt{x}\right)^3\left(-\frac{1}{\sqrt{x}}\right)^3$$

$$+ {}_6C_4\left(\sqrt{x}\right)^2\left(-\frac{1}{\sqrt{x}}\right)^4$$

$$+ {}_6C_5\left(\sqrt{x}\right)\left(-\frac{1}{\sqrt{x}}\right)^5$$

$$+ {}_6C_6\left(-\frac{1}{\sqrt{x}}\right)^6$$

$$= x^3 - 6x^2 + 15x - 20 + \frac{15}{x} - \frac{6}{x^2} + \frac{1}{x^3}$$

9. $(1 + x + x^2)^3$

Sol.

$$(1 + x + x^2)^3 = {}_3C_0(1 + x)^3 + {}_3C_1(1 + x)^2 x^2$$

$$+ {}_3C_2(1 + x)(x^2)^2 + {}_3C_3(x^2)^3$$

$$= x^3 + 3x^2 + 3x + 1 + 3(x^4 + 2x^3 + x^2)$$

$$+ 3(x^6 + 2x^5 + x^4) + x^6$$

$$= x^3 + 3x^2 + 3x + 1 + 3x^4 + 6x^3 + 3x^2$$

$$+ 3x^4 + 3x^5 + x^6$$

$$= 1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$$

10. Calculate $\left(1 + \sqrt{x}\right)^5 + \left(1 - \sqrt{x}\right)^5$.

Sol.

$$\left(1 + \sqrt{x}\right)^5 = 1 + {}_5C_1\sqrt{x} + {}_5C_2\left(\sqrt{x}\right)^2 + {}_5C_3\left(\sqrt{x}\right)^3$$

$$+ {}_5C_4\left(\sqrt{x}\right)^4 + {}_5C_5\left(\sqrt{x}\right)^5$$

$$\left(1 - \sqrt{x}\right)^5 = 1 - {}_5C_1\sqrt{x} + {}_5C_2\left(\sqrt{x}\right)^2 - {}_5C_3\left(\sqrt{x}\right)^3$$

$$+ {}_5C_4\left(\sqrt{x}\right)^4 - {}_5C_5\left(\sqrt{x}\right)^5$$

$$\left(1 + \sqrt{x}\right)^5 + \left(1 - \sqrt{x}\right)^5$$

$$= 2 + 2{}_5C_2\left(\sqrt{x}\right)^2 + 2{}_5C_4\left(\sqrt{x}\right)^4$$

$$= 2 + 20x + 10x^2$$

20.2 General Form of Bionomial Expansion

In the bionomial expansion,

$$(a + b)^n = {}_nC_0 a^n + {}_nC_1 a^{n-1}b + \cdots + {}_nC_{n-r}ab^{n-r} + \cdots + {}_nC_n b^n$$

The $(r + 1)$th term is

$$T_{r+1} = {}_nC_r a^{n-r}b^r$$

This is the general form of bionomial expansion.

20.2.1 Practice 2

Find the fourth term of $(x^3 + 2x)^7$ after expanding it in descending power of x.

Sol.

$$\because r + 1 = 4$$

$$\therefore r = 3$$

$$T_3 = {}_7C_3(x^3)^4(2x)^3$$

$$= 35 \cdot x^{12} \cdot 8x^3$$

$$= 280x^{15}$$

20.2.2 Exercise 20.2

1. Find the coefficient of the fourth term of $(x + 1)^9$ after expanding it in descending power of x.

   Sol.

   Using general formula $T_{r+1} = {}_nC_r a^{n-r}b^r$,

   $$\because r + 1 = 4$$

   $$\therefore r = 3$$

   $$T_3 = {}_9C_3 x^6$$

   $$= 84x^6$$

2. Find the third term of $(3x + 2)^5$ after expanding it in descending power of x.

   Sol.

   Using general formula $T_{r+1} = {}_nC_r a^{n-r}b^r$,

   $$\because r + 1 = 3$$

   $$\therefore r = 2$$

   $$T_2 = {}_5C_2(3x)^3(2)^2$$

   $$= 10 \cdot 27x^3 \cdot 4$$

   $$= 1080x^3$$

3. Find the fourth term of $\left(1 + \frac{x^2}{2}\right)^{10}$ after expanding it in ascending order of x.

Sol.

Using general formula $T_{r+1} = {}_nC_r a^{n-r} b^r$,

$$\because r + 1 = 4$$
$$\therefore r = 3$$
$$T_3 = {}_{10}C_3 \left(\frac{x^2}{2}\right)^3$$
$$= 120 \cdot \frac{x^6}{8}$$
$$= 15x^6$$

4. Find the coefficient of the middle term of $\left(\sqrt{x} - 2\sqrt[3]{x}\right)^8$ after expanding it in ascending order.

Sol.

There are 9 terms in the expansion. The middle term is the fifth term.

Using general formula $T_{r+1} = {}_nC_r a^{n-r} b^r$,

$$\because r + 1 = 5$$
$$\therefore r = 4$$
$$T_4 = {}_8C_4 \left(\sqrt{x}\right)^4 \left(2\sqrt[3]{x}\right)^4$$
$$= 70 \cdot x^2 \cdot x\sqrt[3]{x}$$
$$= 70x^7 x^3 \sqrt[3]{x}$$

5. Find the coefficient of x^2 in $(2 - 3x)^7$.

Sol.

From the general formula, $T_{r+1} = {}_7C_r 2^{7-r}(3x)^r$

$$= {}_7C_r 2^{7-r} 3^r x^r$$

For x^2, $r = 2$

$$T_{r+1} = {}_7C_2 2^{7-2} 3^2 x^2$$
$$= 21 \cdot 32 \cdot 9 \cdot x^2$$
$$= 6048x^2$$

Therefore, the coefficient of x^2 is 6048.

6. Find the constant term of $\left(x + \frac{1}{x}\right)^{10}$.

Sol.

From the general formula, $T_{r+1} = {}_{10}C_r x^{10-r} \left(\frac{1}{x}\right)^r$

$$= {}_{10}C_r x^{10-r} \left(x^{-1}\right)^r$$
$$= {}_{10}C_r x^{10-2r}$$

For the constant term, $10 - 2r = 0$

$$2r = 10$$
$$r = 5$$
$$T_5 = {}_{10}C_5 x^0$$
$$= 252$$

Therefore, the constant term is 252.

7. Find the coefficient of $\frac{1}{x^5}$ in the expansion of $\left(x - \frac{1}{x}\right)^9$.

Sol.

From the general formula, $T_{r+1} = {}_9C_r x^{9-r} \left(-\frac{1}{x}\right)^r$

$$= {}_9C_r(-1)^r \left(\frac{1}{x}\right)^{r-9} \left(\frac{1}{x}\right)^r$$
$$= {}_9C_r(-1)^r \left(\frac{1}{x}\right)^{2r-9}$$

For $\frac{1}{x^5}$, $2r - 9 = 5$

$$2r = 14$$
$$r = 7$$
$$T_7 = {}_9C_7(-1)^5 \left(\frac{1}{x}\right)^5$$
$$= -36 \cdot \frac{1}{x^5}$$

Therefore, the coefficient of $\frac{1}{x^5}$ is -36.

8. Find the coefficient of x^4 in the expansion of $\left(2x + \frac{1}{\sqrt[3]{x}}\right)^8$.

Sol.

From the general formula, $T_{r+1} = {}_8C_r(2x)^{8-r}\left(\dfrac{1}{\sqrt[3]{x}}\right)^r$

$$= {}_8C_r2^{8-r}x^{8-r}\left(x^{-\frac{1}{3}}\right)^r$$

$$= {}_8C_r2^{8-r}x^{8-\frac{4}{3}r}$$

For x^4, $8 - \dfrac{4}{3}r = 4$

$$24 - 4r = 12$$

$$4r = 12$$

$$r = 3$$

$$T_3 = {}_8C_32^{8-3}x^4$$

$$= 56 \cdot 32 \cdot x^4$$

$$= 1792x^4$$

Therefore, the coefficient of x^4 is 1792.

20.3   Revision Exercise 20

1. Find the expansion of $(1 - 2x)^5$.

Sol.

$(1 - 2x)^5 = 1 + {}_5C_1(-2x) + {}_5C_2(2x)^2 + {}_5C_3(-2x)^3$
$$+ {}_5C_4(2x)^4 + {}_5C_5(-2x)^5$$
$$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$$

2. Expend $\left(2\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^6$.

Sol.

$$\left(2\sqrt{x} - \frac{1}{\sqrt{x}}\right)^6 = {}_6C_0(2\sqrt{x})^6 + {}_6C_1(2\sqrt{x})^5\left(-\frac{1}{\sqrt{x}}\right)$$

$$+ {}_6C_2(2\sqrt{x})^4\left(-\frac{1}{\sqrt{x}}\right)^2$$

$$+ {}_6C_3(2\sqrt{x})^3\left(-\frac{1}{\sqrt{x}}\right)^3$$

$$+ {}_6C_4(2\sqrt{x})^2\left(-\frac{1}{\sqrt{x}}\right)^4$$

$$+ {}_6C_5(2\sqrt{x})\left(-\frac{1}{\sqrt{x}}\right)^5$$

$$+ {}_6C_6\left(-\frac{1}{\sqrt{x}}\right)^6$$

$$= 64x^3 - 192x^2 + 240x - 160$$

$$+ \frac{60}{x} - \frac{12}{x^2} + \frac{1}{x^3}$$

3. Find the eighth term of $\left(\dfrac{3x^2}{2} - \dfrac{1}{3x}\right)^{11}$ after expanding it in descending power of x.

Sol.

Using general formula $T_{r+1} = {}_nC_ra^{n-r}b^r$,

$$\because r + 1 = 8$$

$$\therefore r = 7$$

$$T_7 = {}_{11}C_7\left(\frac{3x^2}{2}\right)^4\left(-\frac{1}{3x}\right)^7$$

$$= -330 \cdot \frac{81x^8}{16} \cdot \frac{1}{2187x^7}$$

$$= -55 \cdot \frac{x}{8} \cdot \frac{1}{9}$$

$$= -55 \cdot \frac{x}{72}$$

4. Find the middle term of $\left(x + \dfrac{1}{2\sqrt{x}}\right)^8$ after expanding it in descending power of x.

Sol.

There are 9 terms in the expansion. The middle term

is the 5th term. Therefore,

Using general formula $T_{r+1} = {_nC_r}a^{n-r}b^r$,

$$\because r + 1 = 5$$

$$\therefore r = 4$$

$$T_4 = {_8C_4}x^4 \left(\frac{1}{2\sqrt{x}}\right)^4$$

$$= 70 \cdot x^4 \cdot \frac{1}{16x^2}$$

$$= \frac{35x^2}{8}$$

5. Find the coefficient of x^{-12} in the expansion of the bionomial expression $\left(x^3 - \frac{1}{x}\right)^{24}$.

Sol.

From general formula, $T_{r+1} = {_{24}C_r}(x^3)^{24-r}\left(-\frac{1}{x}\right)^r$

$$= {_{24}C_r}x^{72-3r}(-1)^r x^{-r}$$

$$= (-1)^r {_{24}C_r}x^{72-4r}$$

For x^{-12}, $72 - 4r = -12$,

$$4r = 84$$

$$r = 21$$

$$T_{21} = (-1)^{21} {_{24}C_{21}}x^{12}$$

$$= -2024x^{12}$$

Therefore, the coefficient of x^{-12} is -2024.

6. If the coefficient of x^4 in the expansion of of the bionomial expression $(1 + ax)^5$ is 80, find the value of a.

Sol.

From general formula, $T_{r+1} = {_5C_r}(ax)^r$

$$= {_5C_r}a^r x^r$$

For x^4, $r = 4$

$$T_4 = {_5C_4}a^4 x^4$$

$$80 = 5a^4$$

$$a^4 = 16$$

$$a = \pm 2$$

7. Given that the coefficient of the second, third, and fourth term if the expansion of $(1+x)^n$ after expanding it in ascending power of x form an arithmetic progression, find the value of n.

Sol.

Using general formula $T_{r+1} = {_nC_r}a^{n-r}b^r$,

$$T_1 = {_nC_1}x$$

$$T_2 = {_nC_2}x^2$$

$$T_3 = {_nC_3}x^3$$

$\because {_nC_1}, {_nC_2}, {_nC_3}$ form an arithmetic progression,

$$\therefore \frac{{_nC_3} + {_nC_1}}{2} = {_nC_2}$$

$$\frac{{_nC_3} + {_nC_1}}{{_nC_2}} = 2$$

$$\frac{{_nC_3} + {_nC_1}}{{_nC_2}} = 2$$

$$\frac{\frac{n!}{3!(n-3)!} + \frac{n!}{1!(n-1)!}}{\frac{n!}{2!(n-2)!}} = 2$$

$$\left(\frac{n!}{6(n-3)!} + \frac{n!}{(n-1)!}\right) \cdot \frac{2(n-2)!}{n!} = 2$$

$$\frac{(n-2)![(n-1)! + 6(n-3)!]}{6(n-3)!(n-1)!} = 1$$

$$\frac{(n-2)(n-3)![(n-1)! + 6(n-3)!]}{(n-3)!(n-1)!} = 6$$

$$\frac{(n-2)[(n-1)! + 6(n-3)!]}{(n-1)(n-2)(n-3)!} = 6$$

$$\frac{(n-1)! + 6(n-3)!}{(n-1)(n-3)!} = 6$$

$$\frac{(n-1)(n-2)(n-3)! + 6(n-3)!}{(n-1)(n-3)!} = 6$$

$$\frac{(n-3)![(n-1)(n-2) + 6]}{(n-1)(n-3)!} = 6$$

$$\frac{(n-1)(n-2) + 6}{(n-1)} = 6$$

$$(n-1)(n-2) + 6 = 6(n-1)$$

$$n^2 - 3n + 2 + 6 = 6n - 6$$

$$n^2 - 3n + 8 - 6n + 6 = 0$$

$$n^2 - 9n + 14 = 0$$

$$(n-7)(n-2) = 0$$

$$n = 7 \text{ or } n = 2$$

$\because n \in \mathbb{N}$ and $n \geq 3$,

$$\therefore n = 7$$

8. Find the fourth term of $\left(px + \frac{q}{x}\right)^n$ after expanding it in descending power of x. If this is a constant term, find the value of n.

Sol.

Using general formula $T_{r+1} = {_n}C_r a^{n-r} b^r$,

$$\because r + 1 = 4$$

$$\therefore r = 3$$

$$T_3 = {_n}C_3 (px)^{n-3} \left(\frac{q}{x}\right)^3$$

$$= {_n}C_3 p^{n-3} x^{n-3} q^3 x^{-3}$$

$$= {_n}C_3 p^{n-3} q^3 x^{n-6}$$

$\because T_3$ is a constant term, $n - 6 = 0$

$$\therefore n = 6$$

Chapter 21

Probability

In our daily life, a lot of stuff will yield certain results in certain conditions or situations. For example, by throwing a stone into the sky, the stone will fall down to the ground; the pure water will boil at 100°C. However, in some cases, there may be more than one possible result in a certain situation. For example, when we throw a coin into the air, it may land on the head or the tail, and the result is unpredictable. Nonetheless, if we do the experiment many times under the same conditions, we'll find certain patterns in the result after some analysis.

In order to find the pattern of a coin landing on the head, there are a lot of people who've conducted thousands of coin-tossing experiments, and here are the results:

Experimenter	Tosses (n)	Heads (m)	Freq. $\left(\frac{m}{n}\right)$
De Morgan	2048	1061	0.5181
Buffon	4048	2048	0.5059
Feller	10000	4979	0.4979
Pearson	12000	6019	0.5016
Pearson	24000	12012	0.5005

From the results, we can see that when the number of tosses is large enough, the frequency $\frac{m}{n}$ of the coin landing on the head (m) will always be close to 0.5.

From that, we can see two obvious facts about this experiment:

1. Contingency: The result cannot be predicted in advance.

2. Inevitability: The results of the same experiment being conducted numerous times show a statistical regularity.

Probability theory is a branch of mathematics that studies statistical regularity in a mathematical way. In this chapter, we'll study the basic concepts of probability theory.

21.1 Sample Space and Events

Every possible results of a trial is called a *sample point* of the trial, and the set of all possible results is called the *sample space* of the trial, typically denoted by S. Take coin-tossing as an example, there are two possible resuls: head and tail. If we denote head by H and tail by T, then the sample space of the coin-tossing experiment is $S = \{H, T\}$.

Although there are only two sample points in the coin-tossing experiment, there may be infinite sample points in some trials. For example, choose a number between 0 and 1, there will be an infinite amount of sample points, e.g. 0.1, 0.12, 0.145, etc.

21.1.1 Practice 1

1. Write down the sample space of throwing two coins once.

   Sol.

   Let H and T denote head and tail, respectively.
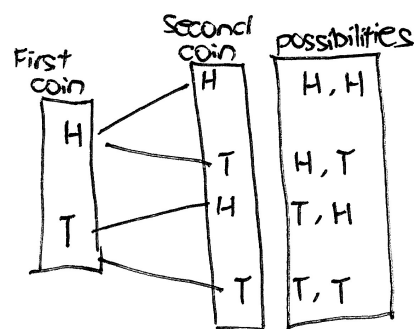
   $$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

2. Write down the sample space of rolling a die once.

   Sol.

   $$S = \{1, 2, 3, 4, 5, 6\}$$

3. Select any number from 0, 1, $\cdots$, 9, and write down its sample space.

   Sol.

   $$S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

4. Write down the sample space of throwing a coin three times.

Sol.

$$S = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),$$
$$(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$$

Within a trial, the set of a few sample points, that is, a subset of the sample space S, is called an *event* of the trial, and is usually denoted by capital letters A, B, C, etc. The sample space S in itself is also an event that will surely happen, and is called the *sure event*. Empty set $\emptyset$ is also an event that will never happen, and is called the *impossible event* or *null event*. When an event only contains one sample point, that is, there is only one element of S in the event, it is called a *simple event*.

For example, in the trial of throwing a dice for the dice points, the sure event $S = \{1,2,3,4,5,6\}$, showing a dice points of 7 is an impossible event, and its denoted as $\emptyset$. The event of showing any of the dice points from 1 to 6 is a simple event.

Take another example, draw a card from a deck of 52 poker cards, there are 52 possible results. Hence, the sample space of this trial is a set of 52 elements, and any event is a subset of the sample space. Below are some example of events in this sample space:

1. The card drawn is a number 11 (impossible event)

2. The card drawn is a red heart 3 (simple event)

3. The card drawn is a number 9 (event with 4 elements)

4. The card drawn is a black spade (event with 13 elements)

5. The card drawn is not a club 5 (event with 51 elements)

Since the events are expressed as sets, listed below are some set operations used to describe relationships between events:

Let A and B be two events, then:

1. $A \cup B$ means that at least one of the events A and B will happen.

2. $A \cap B$ means that both events A and B will happen.

3. A' means that the event A will not happen.

21.1.2 Practice 2

Express the following events in set notation (1 to 4):

1. Throwing a dice, event $A =$ "showing a prime number dice points".
   Sol.

   $$A = \{2,3,5\}$$

2. Throwing three dices, event $B =$ "total dice points less than 6".
   Sol.

   $$B = \{(1,1,1),(1,1,2),(1,1,3),(1,2,1),$$
   $$(1,2,2),(1,3,1),(2,1,1),(2,1,2),$$
   $$(2,2,1),(3,1,1)\}$$

3. Tossing two coins once, event $K =$ "showing exactly one head", event $L =$ "showing at least one head", event $M =$ "showing at most one head".
   Sol.

   Let H and T denote head and tail, respectively.

   $$K = \{(H,T),(T,H)\}$$
   $$L = \{(H,T),(T,H),(H,H)\}$$
   $$M = \{(H,T),(T,H),(T,T)\}$$

4. Tossing a coin three times:

   (a) $D =$ "get at least two heads".
   Sol.

   $$D = \{(T,H,H),(H,T,H),(H,H,T),(H,H,H)\}$$

   (b) $E =$ "the number of heads is lesser than the number of tails".
   Sol.

   $$E = \{(H,T,T),(T,H,T),(T,T,H),(T,T,T)\}$$

21.1.3 Exercise 12.1

1. Choose any two letters from the letters K, O, T, A, find:

   (a) The sample space S.
   Sol.

   $$S = \{(K,O),(K,T),(K,A),(O,T),(O,A),(T,A)\}$$

   (b) Event $A =$ "one letter is a vovel, the other is a consonant".

Sol.

$A = \{(K, O), (K, A), (O, T), (T, A)\}$

(c) Event B = "both letters are vovels".'
Sol.

$B = \{(O, A)\}$

(d) Event C = "at least one letter is a vovel".
Sol.

$C = \{(K, O), (K, A), (O, T), (O, A), (T, A)\}$

2. Throwing two dices, find the event where the sum of the dice points is a multiple of 3.
Sol.

$S = \{(1, 2), (2, 1), (3, 3), (4, 2), (2, 4), (5, 1), (1, 5),$
$\quad (6, 3), (3, 6), (4, 5), (5, 4), (6, 6)\}$

3. Throwing three dices, find the event where the sum of the dice points is 15.
Sol.

$S = \{(6, 6, 3), (6, 3, 6), (3, 6, 6), (6, 5, 4), (6, 4, 5),$
$\quad (5, 6, 4), (5, 4, 6), (4, 6, 5), (4, 5, 6), (5, 5, 5)\}$

4. Choose any two letters from the letters A, B, C, D, E to form a row, find:

   (a) The sample space S.
   Sol.

   $S = \{(A, B), (A, C), (A, D), (A, E), (B, A), (B, C),$
   $\quad (B, D), (B, E), (C, A), (C, B), (C, D), (C, E),$
   $\quad (D, A), (D, B), (D, C), (D, E), (E, A), (E, B),$
   $\quad (E, C), (E, D)\}$

   (b) Event M = "there are exactly one vowel".
   Sol.

   $M = \{(A, B), (A, C), (A, D), (B, A), (B, E),$
   $\quad (C, A), (C, E), (D, A), (D, E), (E, B),$
   $\quad (E, C), (E, D)\}$

5. Throwing three coins, find:

   (a) The sample space S.
   Sol.

   $S = \{(H, H, H), (H, H, T), (H, T, H), (H, T, T),$
   $\quad (T, H, H), (T, H, T), (T, T, H), (T, T, T)\}$

   (b) Event A = "all coins show heads".
   Sol.

   $A = \{(H, H, H)\}$

   (c) Event B = "two coins show heads, one coin shows tail".
   Sol.

   $B = \{(H, H, T), (H, T, H), (T, H, H)\}$

   (d) Event C = "at least two coin shows heads".
   Sol.

   $C = \{(H, H, H), (H, H, T), (H, T, H), (T, H, H)\}$

6. Throwing two dices, express the following events in set notation:

   (a) A = "the dice points of two dices are equal".
   Sol.

   $A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

   (b) B = "The dice points of one dice is twice the dice points of another".
   Sol.

   $B = \{(1, 2), (2, 1), (3, 6), (6, 3)\}$

   (c) C = "The sum of the dice points is a multiple of 5 or 6".
   Sol.

   $C = \{(1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 3)$
   $\quad (5, 1), (1, 5), (6, 6), (6, 4), (4, 6), (5, 5)\}$

21.2   Definition of Probability

In this section, we'll cover two definitions of probability, the classical and the statistical definition.

Statistical Definition of Probability

Doing a trial numerous times in the same conditions, the frequency of an event will show a certain regularity. Let's discuss the following example of dice-tossing:

Tossing a dice multiple time and recording the numbers of time of getting 1 dice point and the total number of tosses, we get the following results:

Tosses (n)	No. of 1 dice points (m)	Frequency $\left(\frac{m}{n}\right)$
1000	174	0.1740
2000	350	0.1750
3000	499	0.1663
4000	673	0.1683
5000	837	0.1674
6000	9999	0.1665

From the table, we can see that as the number of tosses increases, the frequency of getting 1 dice point keep approaching a constant value of $\frac{1}{6} = 0.1667$.

When performing a large amount of repeated trials, the frequency of an event A $\left(\frac{m}{n}\right)$ always approaches a constant value. This constant value is called the *probability* of event A, and denoted as $P(A)$. This is the statistical definition of probability.

Since the occurrence of an event will never exceed the total number of trials, its frequency will always be a number between 0 and 1, that is, $0 \leq \frac{m}{n} \leq 1$. Hence, according to the statistical definition of probability, for any event A, its probability $0 \leq P(A) \leq 1$.

A sure event S will always happen in every trial, so $P(S) = 1$. For an impossible event $\emptyset$, no matter how many trials we do, its occurrence will always be 0, so $P(\emptyset) = 0$.

Classical Definition of Probability

Assume that a trial satisfies the following conditions:

1. The outcome of the trial is finite.

2. The probability of each outcome is equal.

This kind of trial model is called a *classical probabilistic model*.

Let S be the sample space of the trial that contains n equally probable, A is an event that contains m outcomes, then the probability of event A is:

$$P(A) = \frac{m}{n} = \frac{n(A)}{n(S)}$$

This is the classical definition of probability.

21.2.1 Practice 3

1. Randomly draw two cards from a deck of 52 poker cards, find the probability of two "K" cards.

   Sol.

   S = "Randomly draw two cards from a deck of 52 poker cards", $n(S) = {}_{52}C_2 = 1326$.

   Let event A = "Two "K" cards", $n(A) = {}_4C_2 = 6$.

   $P(A) = \frac{n(A)}{n(S)} = \frac{6}{1326} = \frac{1}{221}$.

2. Between all students in a class, there are 13 students with type A blood, 10 students with type B blood, 2 students with type AB blood, and 15 students with type O blood. If we randomly choose 4 students, find the probability of the following events:

   Sol.

   S = "Randomly choose 4 students", $n(S) = {}_{40}C_4 = 91390$.

   (a) 2 type A blood, 2 type B blood

   Sol.

   Let event A = "2 type A blood, 2 type B blood", $n(A) = {}_{13}C_2 \cdot {}_{10}C_2 = 3510$.

   $P(A) = \frac{n(A)}{n(S)} = \frac{3510}{91390} = \frac{27}{703}$.

   (b) 2 type A blood, 1 type AB blood, 1 type O blood

   Sol.

   Let event B = "2 type A blood, 1 type AB blood, 1 type O blood", $n(A) = {}_{13}C_2 \cdot {}_2C_1 \cdot {}_{15}C_1 = 2340$.

   $P(B) = \frac{n(B)}{n(S)} = \frac{2340}{91390} = \frac{18}{703}$.

   (c) 1 for each type of blood.

   Sol.

   Let event C = "1 for each type of blood", $n(C) = {}_{13}C_1 \cdot {}_{10}C_1 \cdot {}_2C_1 \cdot {}_{15}C_1 = 3900$.

   $P(C) = \frac{n(C)}{n(S)} = \frac{3900}{91390} = \frac{39}{703}$.

3. Arranging all the letters in the word *GERMANY*, find the probability of the following events:

   Sol.

   S = "Arranging all the letters in the word *GERMANY*", $n(S) = 7!$.

   (a) 5 adjacent consonants

   Sol.

   Let event A = "5 adjacent consonants".

   5 consonants can be treated as one letter, there are 3! ways to arrange 3 letters. All the consonants

can switch their positions, so there are 5! ways to arrange 5 consonants.

$n(A) = 3! \cdot 5!.$

$P(A) = \frac{n(A)}{n(S)} = \frac{3! \cdot 5!}{7!} = \frac{1}{7}.$

(b) 5 non fully adjacent consonants

Sol.

Let event B = "5 non fully adjacent consonants".

$n(B) = n(S) - n(A) = 7! - 3! \cdot 5!.$

$P(B) = \frac{n(B)}{n(S)} = \frac{7! - 3! \cdot 5!}{7!} = \frac{6}{7}.$

21.2.2   Exercise 21.2

1. A bag contains 9 balls, of which 2 are white, 3 are red, and 4 are yellow. Randomly drawing one ball, find the probability of the following events:

   Sol.

   S = "Randomly drawing one ball", $n(S) = {}_9C_1.$

   (a) The ball drawn is red

   Sol.

   Let event A = "The ball drawn is red", $n(A) = {}_3C_1.$

   $P(A) = \frac{n(A)}{n(S)} = \frac{{}_3C_1}{{}_9C_1} = \frac{1}{3}.$

   (b) The ball drawn is not red

   Sol.

   Let event B = "The ball drawn is not red", $n(B) = n(S) - n(A) = {}_9C_1 - {}_3C_1.$

   $P(B) = \frac{n(B)}{n(S)} = \frac{{}_9C_1 - {}_3C_1}{9} = \frac{2}{3}.$

   (c) The ball drawn is yellow

   Sol.

   Let event C = "The ball drawn is yellow", $n(C) = {}_4C_1.$

   $P(C) = \frac{n(C)}{n(S)} = \frac{{}_4C_1}{{}_9C_1} = \frac{4}{9}.$

2. A box contains 3 throat lozenges and 5 bubble gums. Randomly drawing two of them, find the probability of getting two bubble gums.

   Sol.

   S = "Randomly drawing two candies, $n(S) = {}_8C_2.$

   Let event A = "Getting two bubble gums", $n(A) = {}_5C_2.$

   $P(A) = \frac{n(A)}{n(S)} = \frac{{}_5C_2}{{}_8C_2} = \frac{5}{14}.$

3. Randomly drawing 3 cards from a deck of 52 poker cards, find the probability of getting 3 cards of spades.

   Sol.

S = "Randomly drawing 3 cards from a deck of 52 poker cards", $n(S) = {}_{52}C_3.$

Let event A = "Getting 3 cards of spades", $n(A) = {}_{13}C_3.$

$P(A) = \frac{n(A)}{n(S)} = \frac{{}_{13}C_3}{{}_{52}C_3} = \frac{11}{850}.$

4. There are 4 novels and 8 essay collections in a box. Randomly drawing 3 books from the box, find the probability of the following events:

   Sol.

   S = "Randomly drawing 3 books from the box", $n(S) = {}_{12}C_3.$

   (a) All three books are novels

   Sol.

   Let event A = "All three books are novels", $n(A) = {}_4C_3.$

   $P(A) = \frac{n(A)}{n(S)} = \frac{{}_4C_3}{{}_{12}C_3} = \frac{1}{55}.$

   (b) Two books of novels, one book of essay collection

   Sol.

   Let event B = "Two books of novels, one book of essay collection", $n(B) = {}_4C_2 \cdot {}_8C_1.$

   $P(B) = \frac{n(B)}{n(S)} = \frac{{}_4C_2 \cdot {}_8C_1}{{}_{12}C_3} = \frac{12}{55}.$

   (c) All three books are essay collections

   Sol.

   Let event C = "All three books are essay collections", $n(C) = {}_8C_3.$

   $P(C) = \frac{n(C)}{n(S)} = \frac{{}_8C_3}{{}_{12}C_3} = \frac{14}{55}.$

5. A committee is to be formed by selecting 4 people from 6 males and 5 females. Find the probability of the following events:

   Sol.

   Let event S = "Selecting 4 people from 11 people", $n(S) = {}_{11}C_4.$

   (a) All 4 people are male

   Sol.

   Let event A = "All 4 people are male", $n(A) = {}_6C_4.$

   $P(A) = \frac{n(A)}{n(S)} = \frac{{}_6C_4}{{}_{11}C_4} = \frac{1}{22}.$

   (b) All 4 people are female

   Sol.

   Let event B = "All 4 people are female", $n(B) = {}_5C_4.$

   $P(B) = \frac{n(B)}{n(S)} = \frac{{}_5C_4}{{}_{11}C_4} = \frac{1}{66}.$

(c) 2 people from each gender

Sol.

Let event C = "2 people from each gender", $n(C) = {}_6C_2 \cdot {}_5C_2$.

$P(C) = \frac{n(C)}{n(S)} = \frac{{}_6C_2 \cdot {}_5C_2}{{}_{11}C_4} = \frac{5}{11}$.

6. Of 100 products, 95 are quality products, and 5 are defective products. Randomly drawing 2 products, find the probability of the following events:

Sol.

S = "Randomly drawing 2 products", $n(S) = {}_{100}C_2$.

(a) All 2 products are quality products

Sol.

Let event A = "All 2 products are quality products", $n(A) = {}_{95}C_2$.

$P(A) = \frac{n(A)}{n(S)} = \frac{{}_{95}C_2}{{}_{100}C_2} = \frac{893}{990}$.

(b) All 2 products are defective products

Sol.

Let event B = "All 2 products are defective products", $n(B) = {}_5C_2$.

$P(B) = \frac{n(B)}{n(S)} = \frac{{}_5C_2}{{}_{100}C_2} = \frac{1}{495}$.

(c) 1 quality products, 1 defective product

Sol.

Let event C = "1 quality products, 1 defective product", $n(C) = {}_{95}C_1 \cdot {}_5C_1$.

$P(C) = \frac{n(C)}{n(S)} = \frac{{}_{95}C_1 \cdot {}_5C_1}{{}_{100}C_2} = \frac{19}{198}$.

7. Randomly choose 3 letters from the word *TRIANGLE*, find the probability of the following events:

Sol.

S = "Randomly choose 3 letters from the word *TRIANGLE*", $n(S) = {}_8C_3$.

(a) More vowels than consonants

Sol.

Let event A = "More vowels than consonants".

Choose three vowels from the word *TRIANGLE*, there are ${}_3C_3$ ways to do so.

Choose two vowels and one consonant from the word *TRIANGLE*, there are ${}_3C_2 \cdot {}_5C_1$ ways to do so.

$n(A) = {}_3C_3 + {}_3C_2 \cdot {}_5C_1$.

$P(A) = \frac{n(A)}{n(S)} = \frac{{}_3C_3 + {}_3C_2 \cdot {}_5C_1}{{}_8C_3} = \frac{2}{7}$.

(b) More consonants than vowels

Sol.

Let event B = "More consonants than vowels".

Choose three consonants from the word *TRIANGLE*, there are ${}_5C_3$ ways to do so.

Choose two consonants and one vowel from the word *TRIANGLE*, there are ${}_5C_2 \cdot {}_3C_1$ ways to do so.

$n(B) = {}_5C_3 + {}_5C_2 \cdot {}_3C_1$.

$P(B) = \frac{n(B)}{n(S)} = \frac{{}_5C_3 + {}_5C_2 \cdot {}_3C_1}{{}_8C_3} = \frac{5}{7}$.

8. Tossing three dices at the same time once, find the probability of the sum of dice points larger than 15.

Sol.

S = "Tossing three dices at the same time once", $n(S) = 6^3 = 216$.

Let event A = "The sum of dice points larger than 15".

$A = \{(6,6,6), (6,6,5), (6,5,6), (6,6,5), (6,6,4), (6,4,6), (4,6,6), (6,5,5), (5,6,5), (5,5,6)\}$.

$n(A) = 10$.

$P(A) = \frac{n(A)}{n(S)} = \frac{10}{216} = \frac{5}{108}$.

9. Randomly shuffling each digits in 2233344455, find the probability of two 2s being adjacent to each other.

Sol.

S = "Randomly shuffling each digits in 2233344455", $n(S) = \frac{10!}{2! \cdot 3! \cdot 3! \cdot 2!} = 25200$.

Let event A = "Two 2s being adjacent to each other".

Treat two 2s as a single 2, there are $\frac{9!}{1! \cdot 3! \cdot 3! \cdot 2!} = 5040$ ways to do so.

$n(A) = 5040$.

$P(A) = \frac{n(A)}{n(S)} = \frac{5040}{25200} = \frac{1}{5}$.

10. Randomly shuffling 5 cards that are assigned to 1, 2, 3, 4, 5 respectively and arranging them into a 5 digit number, find the probability of the number being an even number.

Sol.

S = "Randomly shuffling 5 cards that are assigned to 1, 2, 3, 4, 5 respectively"

$n(S) = 5!$.

Let event A = "The number being an even number".

Since the number is even, the last digit must be 2 or 4. There are 2 ways to arrange it. For the other 4 digits, there are 4! ways to do arrange them.

$n(A) = 2 \cdot 4!$.

$P(A) = \frac{n(A)}{n(S)} = \frac{2 \cdot 4!}{5!} = \frac{2}{5}$.

21.3 Addition Rule

Mutually Exclusive Events and Inclusive Events

When two events A and B can happen at the same time, event A and B are said to be *inclusive*. For example, when tossing a dice once, the events "getting an even number" and "getting a multiple of 3" can happen at the same time. Therefore, these two events are inclusive events. Now let's find the probability of getting an even number or a multiple of 3 when tossing a dice once.

Let the sample space of the trial be $S = \{1, 2, 3, 4, 5, 6\}$, $n(S) = 6$.

Let event A = "getting an even number" = $\{2, 4, 6\}$, $n(A) = 3$, therefore $P(A) = \frac{3}{6}$.

Let event B = "getting a multiple of 3" = $\{3, 6\}$, $n(B) = 2$, therefore $P(B) = \frac{2}{6}$.

$A \cup B$ = "getting an even number or a multiple of 3" = $\{2, 3, 4, 6\}$, $n(A \cup B) = 4$, therefore $P(A \cup B) = \frac{4}{6} = \frac{2}{3}$.

$A \cap B$ = "getting an even number that is also a multiple of 3" = $\{6\}$, $n(A \cap B) = 1$, therefore $P(A \cap B) = \frac{1}{6}$.

Generally speaking, according to the formula of cardinality of the union of two sets,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Dividing both side by $n(S)$, we get:

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}$$

That is,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

The relationship is the *addition rule* of probability. In the example above,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{3}{6} + \frac{2}{6} - \frac{1}{6}$$
$$= \frac{2}{3}$$

When two events A and B cannot happen at the same time, that is, $A \cap B = \emptyset$, A and B is said to be *mutually exclusive*. For example, there are a red ball and a white ball in a bag. When randomly drawing one ball, the ball will be either red or white, but not both. This is a mutually exclusive event.

Since A and B are mutually exclusive, $P(A \cap B) = 0$.

$$P(A \cup B) = P(A) + P(B)$$

The relationship above is the additional rule of mutually exclusive events.

From that, we know that if event A and event B are mutually exclusive, their probability is the sum of their individual probabilities.

21.3.1 Practice 4

1. A bag contains 5 cards each for the color red, blue and green. Randomly drawing one card, find the probability of getting a red card or a yellow card.

   Sol.

   Let event A = "getting a red card", $P(A) = \frac{5}{15}$.

   Let event B = "getting a yellow card", $P(B) = \frac{5}{15}$.

   $A \cup B$ = "getting a red card or a yellow card".

   Since A and B are mutually exclusive,

   $$\therefore P(A \cup B) = P(A) + P(B)$$
   $$= \frac{5}{15} + \frac{5}{15}$$
   $$= \frac{10}{15}$$
   $$= \frac{2}{3}$$

2. Drawing a card from a deck of 52 poker cards, find the probability of getting a heart or number that is a multiple of 5.

   Sol.

   Let event A = "getting a heart", $P(A) = \frac{13}{52}$.

   Let event B = "getting a number that is a multiple of 5", $P(B) = \frac{8}{52}$.

   $A \cup B$ = "getting a heart or a number that is a multiple of 5".

   $A \cap B$ = "getting a heart that is also a number that is a multiple of 5", $P(A \cap B) = \frac{2}{52}$.

   $$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
   $$= \frac{13}{52} + \frac{8}{52} - \frac{2}{52}$$
   $$= \frac{19}{52}$$

3. Of a class of 45 students, 20 of them has visited Bali or Jakarta, of which 14 of them has visited Bali and 10 of them has visited Jakarta. Randomly choosing one student, find the probability of the student has visited both cities.

Sol.

Let event $A =$ "visited Bali", $P(A) = \frac{14}{45}$.

Let event $B =$ "visited Jakarta", $P(B) = \frac{10}{45}$.

$A \cup B =$ "visited Bali or Jakarta", $P(A \cup B) = \frac{20}{45}$.

$A \cap B =$ "visited Bali and Jakarta".

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$\therefore P(A \cap B) = P(A \cup B) - P(A) - P(B)$$
$$= \frac{20}{45} - \frac{14}{45} - \frac{10}{45}$$
$$= \frac{4}{45}$$

Complementary Events

During a trial, if one of two mutually exclusive events A and B must happen, then these two events are said to be complementary events. The complementary event of an event A is denoted by A'. Take coin-tossing as an example, if event A is "getting a head", then its complementary event A' is "getting a tail". Since either event A or A' must happen, therefore $A \cup A' = S$. Applying additional rule for mutually exclusive events,

$$P(A) + P(A') = P(A \cup A')$$
$$= P(S)$$
$$= 1$$
$$\therefore P(A) = 1 - P(A')$$

The relationship above can be used to calculat the probability of complementary events.

21.3.2 Practice 5

1. There are 22 boys and 23 girls in a class. Randomly picking 2 students from them, find the probability of having at least one boy in these 2 people.

Sol.

$S =$ "picking 2 students from the class".

$n(S) = {}_{45}C_2 = 990$.

Let event $A =$ "Having at least one boy in these 2 people",

then $A' =$ "All people are girls".

$n(A') = {}_{23}C_2 = 253$. $P(A') = \frac{23}{90}$.

$$\therefore P(A) = 1 - P(A')$$
$$= 1 - \frac{23}{90}$$
$$= \frac{67}{90}$$

2. In a lucky draw, there are 60 boxes, of which 5 of them contain a prize. Randomly picking 2 boxes, find the probability of getting at least one prize.

Sol.

$S =$ "picking 2 boxes from the lucky draw".

$n(S) = {}_{60}C_2 = 1770$.

Let event $A =$ "Getting at least one prize",

then $A' =$ "Getting no prize".

$n(A') = {}_{55}C_2 = 1485$. $P(A') = \frac{99}{118}$.

$$\therefore P(A) = 1 - P(A')$$
$$= 1 - \frac{99}{118}$$
$$= \frac{19}{118}$$

21.3.3 Exercise 21.3

1. There are 5 red balls, 6 yellow balls, and 8 black balls in a box. Randomly drawing one ball, find the probability of getting a red ball or a yellow ball.

Sol.

Let event $A =$ "getting a red ball", $P(A) = \frac{5}{19}$.

Let event $B =$ "getting a yellow ball", $P(B) = \frac{6}{19}$.

$A \cup B =$ "getting a red ball or a yellow ball".

Since A and B are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B)$$
$$= \frac{5}{19} + \frac{6}{19}$$
$$= \frac{11}{19}$$

2. There are 18 different reference books on the shelf, of which 6 of them are Chinese and 5 of them are Maths, and the rest of them are Ecomoics. Randomly picking two books from the shelf, find the probability of getting at least one Math book or one Chinese book.

Sol.

S = "picking two books from the shelf".

$n(S) = {}_{18}C_2 = 153$.

Let event A = "getting one Math book or one Chinese book",

then A' = "all two books are Economics books".

$n(A') = {}_7C_2 = 21$. $P(A') = \frac{21}{153}$.

$$\begin{aligned} \therefore P(A) &= 1 - P(A') \\ &= 1 - \frac{21}{153} \\ &= \frac{44}{153} \end{aligned}$$

3. There are 15 shirts, 10 vests and 5 T-shirts on the rack. Randomly picking two clothes from the rack, find the probability of getting at least one T-shirt.

    Sol.

    S = "picking two clothes from the rack".

    $n(S) = {}_{30}C_2 = 435$.

    Let event A = "getting at least one T-shirt",

    then A' = "getting no T-shirt".

    $n(A') = {}_{25}C_2 = 300$. $P(A') = \frac{20}{29}$.

    $$\begin{aligned} \therefore P(A) &= 1 - P(A') \\ &= 1 - \frac{20}{29} \\ &= \frac{9}{29} \end{aligned}$$

4. There are 50 prizes in a lucky draw, of which 1 of them is cash prize worth \$800, 2 of them are cash prize worth \$500, 5 of them are cash prize worth \$100, and the rest of them are bookshop vouchers worth \$10. One person make two draws, find the probability of getting at least one cash prize.

    Sol.

    S = "making two draws".

    $n(S) = {}_{50}C_2 = 1225$.

    Let event A = "getting at least one cash prize",

    then A' = "getting no cash prize".

    $n(A') = {}_{42}C_2 = 861$. $P(A') = \frac{123}{175}$.

    $$\begin{aligned} \therefore P(A) &= 1 - P(A') \\ &= 1 - \frac{123}{175} \\ &= \frac{52}{175} \end{aligned}$$

5. There are 50 people doing a meeting in a classroom, of which 35 of them are students, 12 of them are parents, and 3 of them are teachers. Randomly picking one spokesperson, find the probability of the following events:

    (a) The spokesperson is either a teacher or a student.

    Sol.

    Let event A = "the spokesperson is a teacher", $P(A) = \frac{3}{50}$.

    Let event B = "the spokesperson is a student", $P(B) = \frac{35}{50}$.

    $A \cup B$ = "the spokesperson is either a teacher or a student".

    Since A and B are mutually exclusive events,

    $$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{3}{50} + \frac{35}{50} \\ &= \frac{19}{25} \end{aligned}$$

    (b) The spokesperson is either a teacher or a parent.

    Sol.

    Let event A = "the spokesperson is a teacher", $P(A) = \frac{3}{50}$.

    Let event B = "the spokesperson is a parent", $P(B) = \frac{12}{50}$.

    $A \cup B$ = "the spokesperson is either a teacher or a parent".

    Since A and B are mutually exclusive events,

    $$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{3}{50} + \frac{12}{50} \\ &= \frac{3}{10} \end{aligned}$$

    (c) The spokesperson is either a student or a parent.

    Sol.

    Let event A = "the spokesperson is a student", $P(A) = \frac{35}{50}$.

    Let event B = "the spokesperson is a parent", $P(B) = \frac{12}{50}$.

    $A \cup B$ = "the spokesperson is either a student or a parent".

Since A and B are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B)$$
$$= \frac{35}{50} + \frac{12}{50}$$
$$= \frac{47}{50}$$

6. There are 100 lottery tickets, of which 3 of them are winning tickets. One person has bought 10 tickets, find the probability of the following events:

   (a) All tickets are not winning tickets.

   Sol.

   $S = $ "bought 10 tickets". $n(S) = {}_{100}C_{10}$.

   Let event $A = $ "all tickets are not winning tickets". $n(A) = {}_{97}C_{10}$.

   $P(A) = \frac{n(A)}{n(S)} = \frac{{}_{97}C_{10}}{{}_{100}C_{10}} = \frac{178}{245}$.

   (b) At least one ticket is a winning ticket.

   Sol.

   Let event $B = $ "at least one ticket is a winning ticket",

   then $B' = $ "all tickets are not winning tickets".

   $$P(B) = 1 - P(B')$$
   $$= 1 - \frac{178}{245}$$
   $$= \frac{67}{245}$$

7. There are 11 out of 45 students in a class who have donated their blood before. Randomly picking three students, find the probability of the following events:

   (a) All three students have donated their blood before.

   Sol.

   $S = $ "picking three students". $n(S) = {}_{45}C_3$.

   Let event $A = $ "all three students have donated their blood before".

   $n(A) = {}_{11}C_3$.

   $P(A) = \frac{n(A)}{n(S)} = \frac{{}_{11}C_3}{{}_{45}C_3} = \frac{1}{86}$.

   (b) All three students have never donated their blood before.

   Sol.

   Let $B = $ "all three students have never donated their blood before".

   $n(B) = {}_{34}C_3$.

   $P(B) = \frac{n(B)}{n(S)} = \frac{{}_{34}C_3}{{}_{45}C_3} = \frac{272}{645}$.

   (c) At least one student has donated his/her blood before.

   Sol.

   Let $C = $ "at least one student has donated his/her blood before",

   then $C' = $ "all three students have never donated their blood before".

   $$P(C) = 1 - P(C')$$
   $$= 1 - \frac{272}{645}$$
   $$= \frac{373}{645}$$

8. Tossing two dices at the same time, find the probability of the sum of the dice points being at least 9.

   Sol.

   $S = $ "tossing two dices". $n(S) = 6^2 = 36$.

   Let event $A = $ "the sum of the dice points is at least 9".

   $A = \{(3,6), (4,5), (5,4), (6,3), (5,5), (6,4), (4,6), (6,5), (5,6), (6,6)\}$.

   $n(A) = 10$.

   $P(A) = \frac{n(A)}{n(S)} = \frac{10}{36} = \frac{5}{18}$.

9. Continuously tossing a coin 5 times, find the probability of getting at least one head.

   Sol.

   $S = $ "tossing a coin 5 times". $n(S) = 2^5 = 32$.

   Let event $A = $ "getting at least one head",

   then $A' = $ "getting no head".

   $A' = \{(T,T,T,T,T)\}$.

   $n(A') = 1$.

   $P(A) = 1 - P(A') = 1 - \frac{1}{32} = \frac{31}{32}$.

10. There are 7 red balls and 10 white balls in a box. Randomly drawing 3 balls, find the probability of the following events:

    Sol.

    $S = $ "drawing 3 balls". $n(S) = {}_{17}C_3$.

    (a) Getting at least one red ball.

    Sol.

    Let event $A = $ "getting at least one red ball",

then $A' = $ "getting all white ball". $n(A') = {}_{10}C_3$.
$P(A') = \frac{n(A')}{n(S)} = \frac{{}_{10}C_3}{{}_{17}C_3} = \frac{3}{17}$.

$$P(A) = 1 - P(A')$$
$$= 1 - \frac{3}{17}$$
$$= \frac{14}{17}$$

(b) Getting at least one white ball.

Sol.

Let event $B = $ "getting at least one white ball",

then $B' = $ "getting all red ball". $n(B') = {}_7C_3$.
$P(B') = \frac{n(B')}{n(S)} = \frac{{}_7C_3}{{}_{17}C_3} = \frac{7}{136}$.

$$P(B) = 1 - P(B')$$
$$= 1 - \frac{7}{136}$$
$$= \frac{129}{136}$$

(c) Getting at least two red balls.

Sol.

Let event $C = $ "getting at least two red balls",

then $C' = $ "getting no red ball or only one red ball".

Let event $D = $ "getting no red ball",

then $D' = $ "getting at least one red ball" $= A$.

$$P(D) = 1 - P(A)$$
$$= 1 - \frac{14}{17}$$
$$= \frac{3}{17}$$

Let event $E = $ "getting only one red ball", $n(E) = {}_7C_1 \cdot {}_{10}C_2$.
$P(E) = \frac{n(E)}{n(S)} = \frac{{}_7C_1 \cdot {}_{10}C_2}{{}_{17}C_3} = \frac{63}{136}$.
Since event E and event D are mutually exclusive,

$$P(C') = P(D) + P(E)$$
$$= \frac{3}{17} + \frac{63}{136}$$
$$= \frac{87}{136}$$
$$\therefore P(C) = 1 - P(C')$$
$$= 1 - \frac{87}{136}$$
$$= \frac{49}{136}$$

11. Tossing 3 dices once, find the probabily of hte following events:

Sol.

$S = $ "tossing 3 dices". $n(S) = 6^3 = 216$.

(a) Exactly one dice shows 6.

Sol.

Let event $A = $ "exactly one dice shows 6".

There are ${}_3C_1 = 3$ outcomes for one dice to shows 6. The other two dices show 1, 2, 3, 4, or 5, there are 5^2 outcomes. The total number of outcomes is $35^2 \cdot = 75$.
$P(A) = \frac{n(A)}{n(S)} = \frac{75}{216} = \frac{25}{72}$.

(b) Exactly one dice or all three dices show 1.

Sol.

Let event $B = $ "exactly one dice shows 1".

There are ${}_3C_1 = 3$ outcomes for one dice to shows 1. The other two dices show 2, 3, 4, 5, or 6, there are 5^2 outcomes. The total number of outcomes is $35^2 = 75$.
$P(B) = \frac{n(B)}{n(S)} = \frac{75}{216} = \frac{25}{72}$.
Let event $C = $ "all three dices show 1". $n(C) = 1$.
$P(C) = \frac{n(C)}{n(S)} = \frac{1}{216}$.

Since event B and event C are mutually exclusive,

$$P(B \cup C) = P(B) + P(C)$$
$$= \frac{25}{72} + \frac{1}{216}$$
$$= \frac{19}{54}$$

21.4   Multiplication Rule

Independent Events

For two events A and B, if the occurrence of event A does not affect the occurrence of event B, then the two events are said to be independent events. For example, tossing a dice twice, the result of the first toss does not affect the result of the second toss, vice versa. Therefore, the two events are independent events. The probability of the two events occurring is the product of their individual probabilities:

$$P(A \cap B) = P(A) \cdot P(B)$$

21.4.1 Practice 6

1. There are 10 oranges and 12 apples in two baskets respectively, of which 2 oranges and 4 apples are rotten. Randomly picking one fruit from each basket, find the probability of getting two rotten fruits.

   Sol.

   Let event A = "getting a rotten orange", $P(A) = \frac{2}{10} = \frac{1}{5}$.

   Let event B = "getting a rotten apple", $P(B) = \frac{4}{12} = \frac{1}{3}$.

   Since event A and event B are independent events,

   $$P(A \cap B) = P(A) \cdot P(B)$$
   $$= \frac{1}{5} \cdot \frac{1}{3}$$
   $$= \frac{1}{15}$$

2. Person A and person B are participating in a archery competition. Person A and person B have a probability of $\frac{5}{8}$ and $\frac{7}{15}$ respectively to hit the bullseye. Find the probability the following events:

   (a) Both person A and person B hit the bullseye.

   Sol.

   Let event A = "person A and person B both hit the bullseye".

   $$P(A) = \frac{5}{8} \cdot \frac{7}{15}$$
   $$= \frac{7}{24}$$

   (b) Both of them does not hit the bullseye.

   Sol.

   Let B be the event of person A not hitting the bullseye,

   then B' is the event of person A hitting the bullseye.

   Let C be the event of person B not hitting the bullseye,

   then C' is the event of person B hitting the bullseye.

   $$P(B \cap C) = P(B) \cdot P(C)$$
   $$= 1 - P(B') \cdot 1 - P(C')$$
   $$= 1 - \frac{5}{8} \cdot 1 - \frac{7}{15}$$
   $$= \frac{3}{8} \cdot \frac{8}{15}$$
   $$= \frac{1}{5}$$

   (c) At least one of them hits the bullseye.

   Sol.

   Let event D = "at least one of them hits the bullseye",

   then D' is the event of both of them not hitting the bullseye.

   $$P(D) = 1 - P(D')$$
   $$= 1 - \frac{1}{5}$$
   $$= \frac{4}{5}$$

21.4.2 Exercise 21.4a

1. Tossing a dice three times, find the probability of getting 2 for the first and the second toss, and an odd number for the third toss.

   Sol.

   Let event A = "getting 2 for the first and the second toss", $P(A) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

   Let event B = "getting an odd number for the third toss", $P(B) = \frac{3}{6} = \frac{1}{2}$.

   Since event A and event B are independent events,

   $$P(A \cap B) = P(A) \cdot P(B)$$
   $$= \frac{1}{36} \cdot \frac{1}{2}$$
   $$= \frac{1}{72}$$

2. Tossing a coin 5 times, find the probability of getting exactly 4 heads continuously.

   Sol.

   S = "tossing a coin 5 times", $n(S) = 2^5 = 32$.

   Let event A = "getting exactly 4 heads continuously".

   The four heads can be treat as one coin. and the other coin must be tail, there are $_2C_1$ ways to get the four heads. $n(A) = {_2C_1} = 2$.

   $$P(A) = \frac{n(A)}{n(S)}$$
   $$= \frac{2}{32}$$
   $$= \frac{1}{16}$$

3. The forecast accuracy of a weather station is 80%. Find probability of getting 4 accurate forecasts our of 5 forecasts. (Round to the nearest 4 decimal places)

Sol.

Each forecast is independent, so the probability of getting 4 accurate forecasts is

$$P(A) = \left(\frac{8}{10}\right)^4$$
$$= \frac{4096}{10000}$$
$$= 0.4096$$

4. There are three processes for processing a component, the failure rate of each process are 1.2%, 1.8%, and 0.8% respectively. If the chance of failure of the component does not depend on other processes, find the success rate of the component.

   Sol.

   Since the chance of failure of the component does not depend on other processes, each process is independent.

   Let event A = "the component is successfully processed by the first process",

   then A' = "the component is failed to be processed by the first process".

   $$P(A) = 100\% - P(A')$$
   $$= 100\% - 1.2\%$$
   $$= 98.8$$

   Let event B = "the component is successfully processed by the second process",

   then B' = "the component is failed to be processed by the second process".

   $$P(B) = 100\% - P(B')$$
   $$= 100\% - 1.8\%$$
   $$= 98.2$$

   Let event C = "the component is successfully processed by the third process",

   then C' = "the component is failed to be processed by the third process".

   $$P(C) = 100\% - P(C')$$
   $$= 100\% - 0.8\%$$
   $$= 99.2$$

The success rate of the component in each process is independent, so the success rate of the component is

$$P(D) = P(A) \cdot P(B) \cdot P(C)$$
$$= 98.8\% \cdot 98.2\% \cdot 99.2\%$$
$$= 96.25\%$$

5. Randomly drawing 1 card from a deck of 52 cards, and draw another card without putting the first card back into the deck. Find the probability of getting a spade for the second card.

   Sol.

   Let event A = "getting a spade for the first card", $P(A) = \frac{13}{52} = \frac{1}{4}$.

   Let event B = "getting a spade for the second card".

   If the first card is a spade, $P(B|A) = \frac{12}{51}$.

   If the first card is not a spade, $P(B|A') = \frac{13}{51}$.

   Note: $P(B|A)$ and $P(B|A')$ are conditional probability, means the probability of getting a spade for the second card given the first card is a spade or not a spade.

   Since these two event B are mutually exclusive, and are independent to event A, the probability of getting a spade for the second card is

   $$P(B) = P(A) \cdot P(B|A) + P(A') \cdot P(B|A')$$
   $$= \frac{1}{4} \cdot \frac{12}{51} + \frac{3}{4} \cdot \frac{13}{51}$$
   $$= \frac{1}{4}$$

6. 3 people are decrypting a message on their own. The probability of each of them decrypting the message correctly is $\frac{1}{5}$, $\frac{1}{4}$ and $\frac{1}{3}$ respectively. Find the probability the message getting decrypted.

   Sol.

   Let event A = "the message getting decrypted by at least one person",

then $A' =$ "the message getting decrypted by no one".

$$P(A') = \left(1 - \frac{1}{5}\right) \cdot \left(1 - \frac{1}{4}\right) \cdot \left(1 - \frac{1}{3}\right)$$
$$= \frac{4}{5} \cdot \frac{3}{4} \cdot \frac{2}{3}$$
$$= \frac{24}{60}$$
$$= \frac{2}{5}$$
$$\therefore P(A) = 1 - P(A')$$
$$= 1 - \frac{2}{5}$$
$$= \frac{3}{5}$$

7. 3 people are participating in an archery competition. person A make three hits out of five, person B make two hits out of three, person C make one hit out of two. Now each person gets one shot, find the probability of the following events:

Sol.

Let event $A =$ "person A make a hit", $P(A) = \frac{3}{5}$.

Let event $B =$ "person B make a hit", $P(B) = \frac{2}{3}$.

Let event $C =$ "person C make a hit", $P(C) = \frac{1}{2}$.

(a) All three of them make a hit.

Sol.

Since these three events are independent, the probability of all three of them make a hit is

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$
$$= \frac{3}{5} \cdot \frac{2}{3} \cdot \frac{1}{2}$$
$$= \frac{6}{30}$$
$$= \frac{1}{5}$$

(b) Only one of them make a hit.

Sol.

If person A make a hit, then person B and C make no hit.
$$P(A \cap B' \cap C') = P(A) \cdot P(B') \cdot P(C') = \frac{3}{5} \cdot \frac{1}{3} \cdot \frac{1}{2}$$
If person B make a hit, then person A and C make no hit.
$$P(A' \cap B \cap C') = P(A') \cdot P(B) \cdot P(C') = \frac{2}{5} \cdot \frac{2}{3} \cdot \frac{1}{2}$$
If person C make a hit, then person A and B make no hit.
$$P(A' \cap B' \cap C) = P(A') \cdot P(B') \cdot P(C) = \frac{2}{5} \cdot \frac{1}{3} \cdot \frac{1}{2}$$

Since these three events are mutually exclusive, the probability of only one of them make a hit is

$$P(A \cap B' \cap C') + P(A' \cap B \cap C') + P(A' \cap B' \cap C)$$
$$= \frac{3}{5} \cdot \frac{1}{3} \cdot \frac{1}{2} + \frac{2}{5} \cdot \frac{2}{3} \cdot \frac{1}{2} + \frac{2}{5} \cdot \frac{1}{3} \cdot \frac{1}{2}$$
$$= \frac{3}{30} + \frac{4}{30} + \frac{2}{30}$$
$$= \frac{3}{10}$$

(c) At least one of them make a hit.

Sol.

Since these three events are mutually exclusive, the probability of at least one of them make a hit is

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap$$
$$B) - P(A \cap C) - P(B \cap C) +$$
$$P(A \cap B \cap C)$$
$$= \frac{3}{5} + \frac{2}{3} + \frac{1}{2} - \frac{6}{15} - \frac{3}{10} - \frac{2}{6} + \frac{1}{5}$$
$$= \frac{14}{15}$$

8. There are 5 black balls, 4 yellow balls, and 3 white balls in a box. Randomly draw one balls and put it back into the box. Repeat the process 3 times. Find the probability of the following events:

Sol.

(a) Only get one yellow ball

Sol.

Let event $A =$ "draw a yellow ball", $P(A) = \frac{4}{12}$, then $A' =$ "draw a non-yellow ball", $P(A') = \frac{8}{12}$.

Since these two events are independent, the probability of only get one yellow ball in 3 draws is

$$P(A \cap A' \cap A') = P(A) \cdot P(A') \cdot P(A')$$
$$= \frac{4}{12} \cdot \frac{8}{12} \cdot \frac{8}{12}$$
$$= \frac{4}{9}$$

(b) All three balls are yellow balls

Sol.

Let event $B =$ "draw a yellow ball", $P(B) = \frac{4}{12}$,

Since these three events are independent, the probability of all three balls are yellow balls in

3 draws is

$$P(B \cap B \cap B) = P(B) \cdot P(B) \cdot P(B)$$
$$= \frac{4}{12} \cdot \frac{4}{12} \cdot \frac{4}{12}$$
$$= \frac{1}{27}$$

(c) Get one black ball, one yellow ball, and one white ball

Sol.

Let event C = "draw a black ball in the first draw", $P(C) = \frac{5}{12}$,

Let event D = "draw a yellow ball in the seocnd ", $P(D) = \frac{4}{12}$,

Let event E = "draw a white ball", $P(E) = \frac{3}{12}$.

Since these three events are independent, the probability of get one black ball, one yellow ball, and one white ball in order is

$$P(C \cap D \cap E) = P(C) \cdot P(D) \cdot P(E)$$
$$= \frac{5}{12} \cdot \frac{4}{12} \cdot \frac{3}{12}$$
$$= \frac{5}{144}$$

Since these three balls can be drawn in any order, there are $3! = 6$ ways to get one black ball, one yellow ball, and one white ball.

$$P = \frac{5}{144} \cdot 6$$
$$= \frac{5}{24}$$

21.4.3 Practice 7

There are 80% of the families living in a town have signed up for fibre optic cable. Now choose 10 random families from the town for a survey, find the probability of the following events (round to the nearest 4 decimal places):
Sol.

The probability of a family has signed up for fibre optic cable is 0.8, while the probability of a family has not signed up for fibre optic cable is 0.2.

1. Exactly half of the families have signed up for fibre optic cable.

   Sol.

   $$\therefore P = {}_nC_{10}5 \cdot 0.8^5 \cdot 0.2^5$$
   $$\approx 0.0264$$

2. All of the families have signed up for fibre optic cable.

   Sol.

   $$\therefore P = {}_nC_{10}10 \cdot 0.8^{10} \cdot 0.2^0$$
   $$\approx 0.1074$$

3. At least one family has not signed up for fibre optic cable.

   Sol. Let event A = "at least one family has not signed up for fibre optic cable",

   then A' = "all families have signed up for fibre optic cable",

   $$P(A) = 1 - P(A')$$
   $$= 1 - 0.1074$$
   $$\approx 0.8926$$

21.4.4 Exercise 21.4b

1. The survival rate of a plant is 0.6. Now 10 plants are planted, find the probability exactly 5 of them survive.

   Sol.

   The probability of a plant survives is 0.6, while the probability of a plant dies is 0.4.

   $$\therefore P = {}_nC_{10}5 \cdot 0.6^5 \cdot 0.4^5$$
   $$\approx 0.2007$$

2. The hit rate of a person shooting a basket ball is 0.4. Find the probability of him making 10 hit out of 25 shots.

   Sol.

   The probability of a person making a hit is 0.4, while the probability of a person missing a shot is 0.6.

   $$\therefore P = {}_nC_{25}10 \cdot 0.4^{10} \cdot 0.6^{15}$$
   $$\approx 0.1612$$

3. According to statistical data, there are 85% of the population in a city has Hepatitis B Vaccination. Now randomly choose 8 people from the city for a health check, find the probability of at most two of them have not been vaccinated.

   Sol.

   The probability of a person has been vaccinated is 0.85,

while the probability of a person has not been vaccinated is 0.15.

If all 8 people have been vaccinated, then the probability is

$$P = {}_nC_8 8 \cdot 0.85^8 \cdot 0.15^0$$
$$\approx 0.2725$$

If 7 people have been vaccinated and 1 person has not been vaccinated, then the probability is

$$P = {}_nC_8 7 \cdot 0.85^7 \cdot 0.15^1$$
$$\approx 0.3847$$

If 6 people have been vaccinated and 2 people have not been vaccinated, then the probability is

$$P = {}_nC_8 6 \cdot 0.85^6 \cdot 0.15^2$$
$$\approx 0.2376$$

Since these three events are mutually exclusive, the probability of at most two of them have not been vaccinated is

$$\therefore P = 0.2725 + 0.3847 + 0.2376$$
$$\approx 0.8948$$

4. The probability of a medicine successfully curing a cold is 0.96. Now 10 cold patients are taking the medicine, find the probability of at least 8 of them are cured.

   Sol.

   The probability of a medicine successfully curing a cold is 0.96, while the probability of a medicine failing to cure a cold is 0.04.

   If 10 patients are cured, then the probability is

   $$P = {}_nC_{10} 10 \cdot 0.96^{10} \cdot 0.04^0$$
   $$\approx 0.66483$$

   If 9 patients are cured and 1 patient is not cured, then the probability is

   $$P = {}_nC_{10} 9 \cdot 0.96^9 \cdot 0.04^1$$
   $$\approx 0.27701$$

If 8 patients are cured and 2 patients are not cured, then the probability is

$$P = {}_nC_{10} 8 \cdot 0.96^8 \cdot 0.04^2$$
$$\approx 0.05194$$

Since these three events are mutually exclusive, the probability of at least 8 of them are cured is

$$\therefore P = 0.66483 + 0.27701 + 0.05194$$
$$\approx 0.9938$$

5. A factory produces a component, the probability of the component being defective is 0.04. Now 20 components are produced, find the probability of the following events:

   Sol.

   The probability of a component being defective is 0.04, while the probability of a component being non-defective is 0.96.

   (a) Exactly 1 of them is defective.

   Sol.

   $$P = {}_nC_{20} 1 \cdot 0.04^1 \cdot 0.96^{19}$$
   $$\approx 0.3683$$

   (b) Exactly 2 of them are defective.

   Sol.

   $$P = {}_nC_{20} 2 \cdot 0.04^2 \cdot 0.96^{18}$$
   $$\approx 0.1458$$

   (c) At most one of them is defective.

   Sol.

   The probability of no component being defective is

   $$P = {}_nC_{20} 0 \cdot 0.04^0 \cdot 0.96^{20}$$
   $$\approx 0.4420$$

   The probability of exactly one component being defective is 0.3683.

   Since these two events are mutually exclusive, the probability of at most one of them is defec-

tive is

$$\therefore P = 0.4420 + 0.3683$$
$$\approx 0.8103$$

6. Tossing a coin 5 times, find the probability of the following events:

   Sol.

   The probability of getting a head and getting a tail are both $\frac{1}{2}$.

   (a) Exactly 3 heads.

   Sol.

   $$P = {}_nC_5 3 \cdot \frac{1}{2}^3 \cdot \frac{1}{2}^2$$
   $$\frac{5}{16}$$

   (b) At least 3 heads.

   Sol.

   The probability of getting exactly 3 heads is $\frac{5}{16}$.
   The probability of getting exactly 4 heads is

   $$P = {}_nC_5 4 \cdot \frac{1}{2}^4 \cdot \frac{1}{2}^1$$
   $$= \frac{5}{32}$$

   The probability of getting exactly 5 heads is

   $$P = {}_nC_5 5 \cdot \frac{1}{2}^5 \cdot \frac{1}{2}^0$$
   $$= \frac{1}{32}$$

   Since these three events are not mutually exclusive, the probability of at least 3 heads is

   $$\therefore P = \frac{5}{16} + \frac{5}{32} + \frac{1}{32}$$
   $$= \frac{1}{2}$$

   (c) The number of heads is odd.

   The probability of getting exactly 1 head is

   $$P = {}_nC_5 1 \cdot \frac{1}{2}^1 \cdot \frac{1}{2}^4$$
   $$= \frac{5}{32}$$

   The probability of getting exactly 3 heads is $\frac{5}{16}$.
   The probability of getting exactly 5 heads is $\frac{1}{32}$.

Since these three events are mutually exclusive, the probability of the number of heads is odd is

$$\therefore P = \frac{5}{32} + \frac{5}{16} + \frac{1}{32}$$
$$= \frac{1}{2}$$

7. A person comes across four traffic lights on his way to work. Given that the duration of red, yellow, and green lights are $90s$, $5s$ and $25s$ respectively for each light. Find the probability of the following events:

   Sol.

   The probability of getting a red light is $\frac{90}{120} = \frac{3}{4}$.

   The probability of getting a yellow light is $\frac{5}{120} = \frac{1}{24}$.

   The probability of getting a green light is $\frac{25}{120} = \frac{5}{24}$.

   (a) Get a red light for the every light.

   Sol.

   $$P = {}_4C_4 \cdot \frac{3}{4}^4 \cdot \frac{1}{24}^0 \cdot \frac{5}{24}^0$$
   $$= \frac{81}{256}$$

   (b) Only get a red light for the first two lights.

   Sol.

   The probability of getting a red light for the first light is $\frac{3}{4}$.

   The probability of getting a red light for the second light is $\frac{3}{4}$.

   The probability of not getting a red light for the third light is $\frac{1}{4}$.

   The probability of not getting a red light for the fourth light is $\frac{1}{4}$.

   Since these four events are independent, the probability of only getting a red light for the first two lights is

   $$\therefore P = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4}$$
   $$= \frac{9}{256}$$

   (c) Get exactly two red lights.

   Sol.

   $$P = {}_4C_2 \cdot \frac{3}{4}^2 \cdot \frac{1}{4}^2$$
   $$= \frac{27}{128}$$

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21.5 Mathematical Expectation

Consider the following scenario: during a commercial activity, the probability of a person getting a profit of $300 is 0.6, and the probability of a person getting a loss of $100 is 0.4. He did the activity 10 times. According to the probability, this guy was expected to get a profit of $300 for 6 times, and a loss of $100 for 4 times. Therefore, the expected profit of these 10 times of commercial activities is $300 \cdot 6 + (-100) \cdot 4$.

$$
\begin{aligned}
\text{Average Profit} &= \frac{300 \cdot 6 + (-100) \cdot 4}{10} \\
&= \frac{1800 - 400}{10} \\
&= \frac{1400}{10} \\
&= 140
\end{aligned}
$$

This average value is called the *mathematical expectated value*, or *expected value* for short, of this people doing the commercial activity.

The expect value does not mean that the actual profit of the person will be $140 for every activity he did. It is just the average value of the profit he will get if he did the activity a large number of times.

Generalize the above example, let the probability of someone getting a profit of x_1, x_2, $\cdots$, x_k be p_1, p_2, $\cdots$, p_k respectively, of which $p_1 + p_2 + \cdots + p_k = 1$. Then the expected value of the profit is

$$
E = x_1 p_1 + x_2 p_2 + \cdots + x_k p_k
$$

21.5.1 Practice 8

1. One person pays $2 to play a a game. The chance of winning the game is 0.4, and he will get $3 if he wins. Find the expected value of profit of this person in this game.

   Sol.

   The chance of winning the game is 0.4, and the profit of winning the game is $1;

   The chance of losing the game is 0.6, and the loss of losing the game is $2.

   Therefore, the expected value of profit of this person in this game is

   $$
   \begin{aligned}
   E &= 0.4 \cdot 1 + 0.6 \cdot (-2) \\
   &= 0.4 - 1.2 \\
   &= -\$0.80
   \end{aligned}
   $$

2. During an investment activity, the probability of a person getting a profit of $2,500 is 0.55, and the probability of a person getting a loss of $1,200 is 0.45. Find the expected value of profit of this person in this investment activity.

   Sol.

   The expected value of profit of this person in this investment activity is

   $$
   \begin{aligned}
   E &= 2500 \cdot 0.55 + (-1200) \cdot 0.45 \\
   &= 1375 - 540 \\
   &= \$835
   \end{aligned}
   $$

21.5.2 Exercise 21.5

1. During a commercial activity, the probability of a person getting a profit of $10,000 is $\frac{3}{5}$, and the probability of a person getting a loss of $6,000 is $\frac{2}{5}$. Find the expected value of profit of this person in this commercial activity.

   Sol.

   The expected value of profit of this person in this commercial activity is

   $$
   \begin{aligned}
   E &= 10000 \cdot \frac{3}{5} + (-6000) \cdot \frac{2}{5} \\
   &= 6000 - 2400 \\
   &= \$3600
   \end{aligned}
   $$

2. A company produces light bulbs. The profit of producing a quality light bulb is $2, and the loss of producing a defective light bulb is $0.50. Assume that the chance of producing a defective light bulb is 0.02. Find the expected value of profit of this company in producing each light bulb.

   Sol.

   The chance of producing a quality light bulb is $1 - 0.02 = 0.98$, while the chance of producing a defective light bulb is 0.02.

   The expected value of profit of this company in producing each light bulb is

   $$
   \begin{aligned}
   E &= 2 \cdot 0.98 + (-0.50) \cdot 0.02 \\
   &= 1.96 - 0.01 \\
   &= \$1.95
   \end{aligned}
   $$

3. A company has insured accident insurance worth $60

for its employees, and the employees will get $1,200$ if they are involved in an accident. Given that the probability of an an accident happening is 0.005, find the expected value of profit of the insurance company in this insurance.

Sol.

The chance of an accident not happening is $1 - 0.005 = 0.995$, while the chance of an accident happening is 0.005.

The expected value of profit of the insurance company in this insurance is

$$E = 60 \cdot 0.995 + (-1200) \cdot 0.005$$
$$= 59.70 + 6.00$$
$$= \$53.70$$
$$\approx \$54$$

4. An airline provides an $8 aviation assurance plan to its passengers. If flight is delayed for more than an hour, the passengers will get $250 compensation. Given that the percentage of a flight being delayed for more than an hours is 2%, find the expected value of profit of the airline.

Sol.

The chance of a flight not being delayed for more than an hour is $1 - 0.02 = 0.98$, while the chance of a flight being delayed for more than an hour is 0.02.

The expected value of profit of the airline is

$$E = 8 \cdot 0.98 + (-250) \cdot 0.02$$
$$= 7.84 - 5$$
$$= \$2.84$$
$$\approx \$3$$

5. The price of a lottery ticket is $2. The probability of winning the lottery is as follows: $\frac{1}{10000}$ for $5,000$, $\frac{1}{1000}$ for 500. Find the expected value of someone who buys a lottery ticket.

Sol.

The chance of winning the lottery is $\frac{1}{10000} + \frac{1}{1000} = \frac{10100}{100000} = 0.101$, while the chance of losing the lottery is $1 - 0.101 = 0.899$.

The expected value of someone who buys a lottery

ticket is

$$E = (-2) \cdot 0.899 + 5000 \cdot \frac{1}{10000} + 500 \cdot \frac{1}{1000}$$
$$= -1.798 + 0.5 + 0.5$$
$$= \$ - 0.798$$
$$\approx -\$1$$

6. A person pays $1 to play a game. The probability of him getting $3,000$, $2,000$, and $1,000$ are all $\frac{1}{10000}$. Find the expected value of this person in this game, and determine of this game is worth playing.

Sol.

The chance of him getting nothing is $1 - \frac{1}{10000} - \frac{1}{10000} - \frac{1}{10000} = 0.9997$, while the chance of him getting $3,000$, $2,000$, and $1,000$ are all $\frac{1}{10000}$.

The expected value of this person in this game is

$$E = (-1) \cdot 0.9997 + 3000 \cdot \frac{1}{10000} + 2000 \cdot \frac{1}{10000}$$
$$+ 1000 \cdot \frac{1}{10000}$$
$$= -0.9997 + 0.3 + 0.2 + 0.1$$
$$= -\$0.3997$$
$$\approx -\$0.40$$

7. A high school has released 8000 $1 lottery tickets during a fun fair, 5 of which has a prize of $500, 8 of which has a prize of $300, 10 of which has a prize of $100, and 50 of which has a prize of $10. Find the expected value of someone who buys a lottery ticket.

The chance of winning the lottery is $\frac{5}{8000} + \frac{8}{8000} + \frac{10}{8000} + \frac{50}{8000} = \frac{73}{8000}$, while the chance of losing the lottery is $1 - \frac{73}{8000} = \frac{7927}{8000}$.

The expected value of someone who buys a lottery ticket is

$$E = (-1) \cdot \frac{7927}{8000} + 500 \cdot \frac{5}{8000} + 300 \cdot \frac{8}{8000} + 100$$
$$\cdot \frac{10}{8000} + 10 \cdot \frac{50}{8000}$$
$$= -\$0.1909$$
$$\approx -\$0.20$$

21.6   Normal Distribution

In the probability models that we have discussed in the previous sections, the number of results of a trial is limited, that is, the limited sets in the sample space. However, there are

many situations in which the results are real numbers within a certain range.

For example, measuring the height of senior 2 boy students, the results (in cm) is a real number bigger than 0. As the number of students getting measured become larger and larger, the frequency polygon of the height of the students will become a bell curve as shown in the figure below. This bell curve is called the *normal curve*.
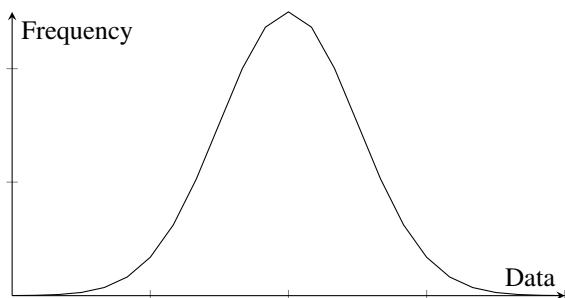

The normal distribution is the most common probability model in real life. A normal distribution consists of two parameters, the mean value μ and the standard deviation σ, thier corresponding functional expression of normal curve is:

$$y = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Assume that the observation X of a trial is normal distribution with mean value μ and standard deviation σ, then its denoted as:

$$X \sim N(\mu, \sigma^2)$$

The probability $P(a < X < b)$ of the observation X being in the interval (a, b) is the area formed by the x-axis, line $x = a$, line $x = b$, and the normal curve, as shown in the figure below.
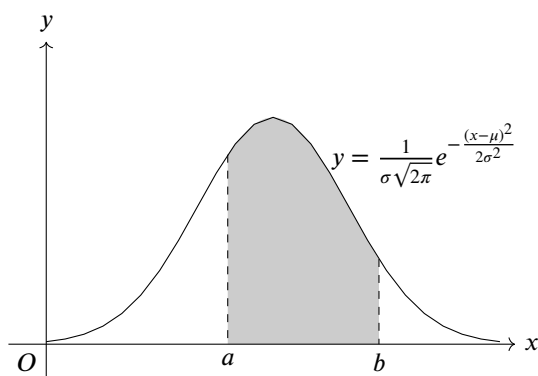
A normal distribution with mean value of 0 and standard deviation of 1 is called the standard normal distribution, its normal curve

$$y = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

is symmetrical about y-axis, as shown in the figure below.
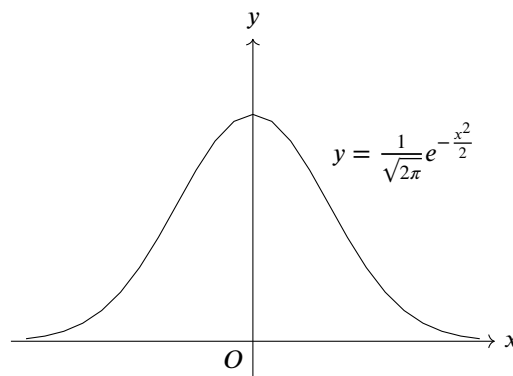
Assume that X is a normal distribution with mean value μ and standard deviation σ, then

$$Z = \frac{X - \mu}{\sigma}$$

is a standard normal distribution. That is,

If $x \sim N(\mu, \sigma^2)$, then $Z = \frac{x - \mu}{\sigma}$, $Z \sim N(0, 1)$

Hence, General problems of normal distribution can be solved by converting into standard normal distribution problems.

In order to calculate the probability of events related to standard normal distribution, we have attached a standard normal distribution table in appendix A. Assume that Z is a standard normal distribution, z is a positive number lesser than or equal to 3.49, we can use the table to calculate the probability of $Q(z) = P(Z \leq z)$.

For example, to find $P(Z \leq 1.23)$, first find the row of $z = 1.2$. The value in the column of this row corresponding to 3 (2nd decimal place) is 0.1093, this is the probability of $P(Z \leq 1.23)$.

z	0	1	2	3	$\cdots$
1.2	0.1151	0.1131	0.1112	**0.1093**	

Any probability of events related to standard normal distribution can be solved using symmetry and properties of probability.

Since the probability of sure event is 1, the area of the region under the curve of standard normal distribution above the x-axis is 1.

Assume that X is a normal distribution, x is any real number, $P(X = x) = 0$. Therefore, $P(X < x) = P(X \leq x)$, $P(X > x) = P(X \geq x)$.

Given that Z is a standard normal distribution, when $a > 0$, $P(Z < a) > 0.5$, $P(Z > a) < 0.5$; when $a < 0$, $P(Z < a) < 0.5$, $P(Z > a) > 0.5$.

21.6.1 Practice 9

There are 2500 students who have attended geography exam during the Senior UEC exam. their marks can be assumed to be a normal distribution with mean value of 60 and standard deviation of 11.

Sol.

Let X be the marks of students, then $Z = \frac{X-60}{11}$ is a standard normal distribution.

(a) Find the number of students whose marks are lower than 40.

Sol.

$$P(X < 40) = P\left(\frac{X - 60}{11} < \frac{40 - 60}{11}\right)$$
$$= P(Z < -1.82)$$
$$= P(Z > 1.82)$$
$$\approx 0.0344$$

Therefore, the number of students whose marks are lower than 40 is $2500 \cdot 0.0344 = 86$.

(b) If grade $A1$ is 78 marks or above, find the percentage of students who obtained grade $A1$.

Sol.

$$P(X \geq 78) = P\left(\frac{X - 60}{11} \geq \frac{78 - 60}{11}\right)$$
$$= P(Z \geq 1.64)$$
$$= 0.0505$$

Therefore, the percentage of students who obtained grade $A1$ is $0.0505 \cdot 100\% = 5.05\%$.

(c) If the passing rate is 90%, find the minimum marks required to pass the exam. (Round to the nearest integer)

Sol.

Let the minimum marks required to pass the exam be

x, then

$$P(X \geq x) = 0.9$$
$$P\left(Z \geq \frac{x - 60}{11}\right) = 0.9$$
$$P(Z < \frac{x - 60}{11}) = 0.1$$
$$P(Z > -\frac{x - 60}{11}) = 0.1$$
$$-\frac{x - 60}{11} \approx 1.28$$
$$x \approx 45.92$$
$$\approx 46$$

21.6.2 Exercise 21.6

1. If the observation of a trial Z is a standard normal distribution, find the probability of the following events:

   (a) $P(Z > 0.91)$

   Sol.

   $$P(Z > 0.91) \approx 0.1814$$

   (b) $P(Z \leq -2.01)$

   Sol.

   $$P(Z \leq -2.01) = P(Z \geq 2.01)$$
   $$\approx 0.0222$$

   (c) $P(Z \geq -0.5)$

   Sol.

   $$P(Z \geq -0.5) = P(Z \leq 0.5)$$
   $$= 1 - P(Z \geq 0.5)$$
   $$\approx 1 - 0.3085$$
   $$\approx 0.6915$$

   (d) $P(0.24 < Z \leq 1.79)$

   Sol.

   $$P(0.24 < Z \leq 1.79) = 1 - P(Z \leq 0.24) - P(Z > 1.79)$$
   $$= 1 - [1 - P(Z > 0.24)] - P(Z > 1.79)$$
   $$= P(Z > 0.24) - P(Z > 1.79)$$
   $$\approx 0.4052 - 0.0367$$
   $$\approx 0.3685$$

(e) $P(-2.21 < Z < -0.97)$

Sol.

$$P(-2.21 < Z < -0.97) = 1 - P(Z \le -2.21)-$$
$$P(Z \ge -0.97)$$
$$= 1 - P(Z \ge 2.21)-$$
$$P(Z \le 0.97)$$
$$= 1 - P(Z \ge 2.21)-$$
$$[1 - P(Z > 0.97)]$$
$$= -P(Z \ge 2.21)+$$
$$P(Z > 0.97)$$
$$\approx -0.0136 + 0.1660$$
$$\approx 0.1524$$

(f) $P(-2.39 < Z \le 0.56)$

Sol.

$$P(-2.39 < Z \le 0.56) = 1 - P(Z \le -2.39)-$$
$$P(Z > 0.56)$$
$$= 1 - P(Z \ge 2.39)-$$
$$P(Z > 0.56)$$
$$\approx 1 - 0.0084 - 0.2877$$
$$\approx 0.7039$$

2. If the observation of a trial Z is a standard normal distribution, find the value of a of the following events:

(a) $P(Z > a) = 0.0505$

Sol.

$$P(Z > a) = 0.0505$$
$$a \approx 1.64$$

(b) $P(Z < a) = 0.8980$

Sol. $\because 0.8980 > 0.5, \therefore a > 0$

$$P(Z < a) = 0.8980$$
$$P(Z \ge a) = 1 - 0.8980$$
$$= 0.1020$$
$$a \approx 1.27$$

(c) $P(Z < a) = 0.3632$

Sol. $\because 0.3632 < 0.5, \therefore a < 0$

$$P(Z < a) = 0.3632$$
$$P(Z > -a) = 0.3632$$
$$-a \approx 0.35$$
$$a \approx -0.35$$

(d) $P(Z > a) = 0.8599$

Sol. $\because 0.8599 > 0.5, \therefore a < 0$

$$P(Z > a) = 0.8599$$
$$P(Z \le a) = 1 - 0.8599$$
$$= 0.1401$$
$$P(Z \ge -a) = 0.1401$$
$$-a \approx 1.08$$
$$a \approx -1.08$$

(e) $P(|Z| > a) = 0.0142$

Sol.

$$P(|Z| > a) = 0.0142$$
$$2P(Z > a) = 0.0142$$
$$P(Z > a) = 0.0071$$
$$a \approx 2.45$$

(f) $P(|Z| < a) = 0.7888$

Sol.

$$P(|Z| < a) = 0.7888$$
$$P(|Z| \ge a) = 1 - 0.7888$$
$$= 0.2112$$
$$2P(Z \ge a) = 0.2112$$
$$'P(Z \ge a) = 0.1056$$
$$a \approx 1.25$$

3. A factory produces canned coffee. The capacity of each can can be assumed to be a normal distribution with mean value of $244.5ml$ and standard deviation of $5.4ml$.

Sol.

Let X be the capacity of each can of coffee, then $Z = \frac{X-244.5}{5.4}$ is a standard normal distribution.

(a) Randomly pick a can of coffee, find the probability that the capacity of the can is less than $235ml$.

Sol.

$$P(X < 235) = P\left(Z < \frac{235 - 244.5}{5.4}\right)$$
$$= P(Z < -1.76)$$
$$= P(Z \geq 1.76)$$
$$\approx 0.0392$$

(b) Randomly pick a can of coffee, find the probability that the capacity of the can is in between 240*ml* and 250*ml*.

Sol.

$$P(240 < X \leq 250) = P\left(\frac{240 - 244.5}{5.4} < Z \leq \frac{250 - 244.5}{5.4}\right)$$
$$= P(-0.83 < Z \leq 1.02)$$
$$= 1 - P(Z \leq -0.83) - P(Z > 1.02)$$
$$= 1 - P(Z \geq 0.83) - P(Z > 1.02)$$
$$\approx 1 - 0.2033 - 0.1539$$
$$\approx 0.6428$$

(c) The factory has sold $18,000$ can of coffee in a week. How many cans have a capacity greater than 260*ml*?

Sol.

$$P(X > 260) = P\left(Z > \frac{260 - 244.5}{5.4}\right)$$
$$= P(Z > 2.87)$$
$$\approx 0.0021$$

Therefore, there are approximately $18,000 \cdot 0.0021 = 38$ cans of coffee with a capacity greater than 260*ml*.

(d) Someone has bought 7 cans of coffee, find the probability that 4 cans of coffee have less than 242*ml* of capacity each.

Sol.

The probability of a can of coffee having less than

242*ml* of capacity is

$$P(X < 242) = P\left(Z < \frac{242 - 244.5}{5.4}\right)$$
$$= P(Z < -0.46)$$
$$= P(Z \geq 0.46)$$
$$\approx 0.3228$$

While the probability of a can of coffee having more than 242*ml* of capacity is $1 - 0.3228 = 0.6772$.

Therefore, the probability of 4 cans of coffee having less than 242*ml* of capacity each is

$$P = {}_7C_4 (0.3228)^4 (0.6772)^3$$
$$\approx 0.1180$$

4. A machine produces wrapped cookies, the weight of each cookie can be assumed to be a normal distribution with mean value of $501.25g$ and standard deviation of $5.32g$.

Sol.

Let X be the weight of each cookie, then $Z = \frac{X - 501.25}{5.32}$ is a standard normal distribution.

(a) Find the percentage of wrapped cookies that have a weight less than 490*g*.

Sol.

$$P(X < 490) = P\left(Z < \frac{490 - 501.25}{5.32}\right)$$
$$= P(Z < -2.11)$$
$$= P(Z \geq 2.11)$$
$$\approx 0.0174$$

Therefore, the percentage of cookies that have a weight less than 490*g* is $0.0174 \cdot 100\% = 1.74\%$.

(b) Of the $10,000$ cookies produced by the machine, how many wrapped cookies have a weight greater than 510*g*?

Sol.

$$P(X > 510) = P\left(Z > \frac{510 - 501.25}{5.32}\right)$$
$$= P(Z > 1.64)$$
$$\approx 0.0505$$

Therefore, there are approximately $10,000 \cdot 0.0505 = 505$ wrapped cookies that have a

weight greater than 510g.

(c) If there are 2.5% of the wrapped cookies that have a weight less than a grams, find the value of a.

Sol.

$$P(X < a) = 0.0250$$
$$P\left(Z < \frac{a - 501.25}{5.32}\right) = 0.0250$$
$$P(Z > -\frac{a - 501.25}{5.32}) = 0.0250$$
$$-\frac{a - 501.25}{5.32} \approx 1.96$$
$$a \approx 490.82$$

(d) If there are 99.02% of the wrapped cookies that have a weight between $(501.25 - c)$ grams and $(501.25 + c)$ grams, find the value of c.

Sol.

$$P\left(501.25 - c < X < 501.25 + c\right) = 0.9902$$
$$P\left(\frac{501.25 - c - 501.25}{5.32} < Z < \frac{501.25 + c - 501.25}{5.32}\right) = 0.9902$$
$$P\left(-\frac{c}{5.32} < Z < \frac{c}{5.32}\right) = 0.9902$$
$$1 - P\left(Z \le -\frac{c}{5.32}\right) - P\left(Z \ge \frac{c}{5.32}\right) = 0.9902$$
$$1 - P\left(Z \ge \frac{c}{5.32}\right) - P\left(Z \ge \frac{c}{5.32}\right) = 0.9902$$
$$1 - 2P\left(Z \ge \frac{c}{5.32}\right) = 0.9902$$
$$2P\left(Z \ge \frac{c}{5.32}\right) = 0.0098$$
$$P\left(Z \ge \frac{c}{5.32}\right) = 0.0049$$
$$\frac{c}{5.32} \approx 2.58$$
$$c \approx 13.73$$

5. The life span of brand A television can be assumed to be a normal distribution with mean value of 7.28 years and standard deviation of 2.23 years. The warranty of the television is 3 years. If the television is broken within the warranty period, the company will replace it with a new one for free. Find the percentage of television that will be replaced.

Sol.

Let X be the life span of the television, then $Z =$

$\frac{X - 7.28}{2.23}$ is a standard normal distribution.

$$P(X \le 3) = P\left(Z \le \frac{3 - 7.28}{2.23}\right)$$
$$= P(Z \le -1.92)$$
$$= P(Z \ge 1.92)$$
$$\approx 0.0274$$

Therefore, the percentage of television that will be replaced is $0.0274 \cdot 100\% = 2.74\%$.

6. The number of letters received by a company in a working day. Given that the probability of receiving more than 150 letters in a working day is 0.1210, while the probability of receiving more than 50 letters in a working day is 0.9495. Find the mean value and standard deviation of the number of letters received by the company in a working day.

Sol.

Let X be the number of letters received by the company in a working day, then $Z = \frac{X - \mu}{\sigma}$ is a standard normal distribution.

$$P(X > 150) = 0.1210$$
$$P\left(Z > \frac{150 - \mu}{\sigma}\right) = 0.1210$$
$$\frac{150 - \mu}{\sigma} \approx 1.17$$
$$150 - \mu \approx 1.17\sigma \qquad (1)$$

$$P(X > 50) = 0.9495$$
$$P\left(Z > \frac{50 - \mu}{\sigma}\right) = 0.9495$$
$$P\left(Z \le \frac{50 - \mu}{\sigma}\right) = 0.0505$$
$$P\left(Z \ge -\frac{50 - \mu}{\sigma}\right) = 0.0505$$
$$-\frac{50 - \mu}{\sigma} \approx 1.64$$
$$50 - \mu \approx -1.64\sigma \qquad (2)$$

$$(1) - (2) : \ 2.81\sigma \approx 100$$
$$\sigma \approx 35.59$$
$$\mu = 150 - 1.17\sigma$$
$$\approx 108.36$$

$$\therefore \mu \approx 108.36, \sigma \approx 35.59$$

7. There are $2,256$ students sitting on the entrance exam for a high school. THe full marks of the exam is 400 marks, and there are $1,200$ students who get accepted into the high school. The marks of the students can be assumed to be a normal distribution with mean value of 189 marks and standard deviation of 53.

Sol.

Let X be the marks of the students, then $Z = \frac{X-189}{53}$ is a standard normal distribution.

(a) Find the number of students who get less than 160 marks.

Sol.

$$P(X < 160) = P\left(Z < \frac{160 - 189}{53}\right)$$
$$= P(Z < -0.55)$$
$$= P(Z > 0.55)$$
$$\approx 0.2912$$

Therefore, there are $0.2912 \cdot 2256 \approx 657$ students who get less than 160 marks.

(b) Find the lowest marks that a student can get to be accepted into the high school. (Round to the nearest integer)

There are $1,200$ students who get accepted into the high school, which is $\frac{1,200}{2,256} \approx 0.5319$ of the total students.

Let x be the lowest marks that a student can get to be accepted into the high school,

$$P(X > x) \approx 0.5319$$
$$P\left(Z > \frac{x - 189}{53}\right) \approx 0.5319$$
$$P\left(Z \leq \frac{x - 189}{53}\right) \approx 0.4681$$
$$P\left(Z \geq -\frac{x - 189}{53}\right) \approx 0.4681$$
$$-\frac{x - 189}{53} \approx 0.08$$
$$x \approx 184.76$$
$$\approx 185$$

(c) If the students who get more than 300 marks are eligible to get a scholarship, find the number of students who are eligible to get a scholarship, and their percentage over the total number of accepted students.

Sol.

$$P(X > 300) = P\left((Z > \frac{300 - 189}{53}\right)$$
$$= P(Z > 2.09)$$
$$\approx 0.0183$$

Therefore, there are $0.0183 \cdot 2256 \approx 41$ students who are eligible to get a scholarship, and their percentage is $\frac{41}{1200} \approx 3.42\%$.

(d) The school administration requires the students who get less than 200 marks to take a remedial class. Find the number of students who have to take a remedial class.

Sol.

$$P(X < 200) = P\left(Z < \frac{200 - 189}{53}\right)$$
$$= P(Z < 0.21)$$
$$= 1 - P(Z \geq 0.21)$$
$$\approx 1 - 0.4168$$
$$\approx 0.5832$$

There are $0.5832 \cdot 2256 \approx 1,316$ students who get less than 200 marks.

Since there are $2256 - 1200 = 1056$ students who failed to get accepted into the high school, there are $1,316 - 1056 = 260$ students who have to take a remedial class.

8. A group of data is normally distributed with mean value of μ and standard deviation of σ. Find in this group of data:

Sol.

μ is the mean value of the data, and σ is the standard deviation of the data, then $Z = \frac{X-\mu}{\sigma}$ is a standard normal distribution.

(a) The percentage of data that is in between $\mu - \sigma$ and $\mu + \sigma$.

Sol.

$P(\mu - \sigma \le X \le \mu + \sigma)$

$= P\left(\dfrac{\mu - \sigma - \mu}{\sigma} \le Z \le \dfrac{\mu + \sigma - \mu}{\sigma}\right)$

$= P(-1 \le Z \le 1)$

$= 1 - P(Z < -1) - P(Z > 1)$

$= 1 - P(Z > 1) - P(Z > 1)$

$= 1 - 2P(Z > 1)$

$= 1 - 2P(Z > 1)$

$\approx 1 - 2 \cdot 0.1587$

$\approx 1 - 0.3174$

≈ 0.6826

Therefore, the percentage of data that is in between $\mu - \sigma$ and $\mu + \sigma$ is $0.6826 \cdot 100\% \approx 68.26\%$.

(b) The percentage of data that is in between $\mu - 2\sigma$ and $\mu + 2\sigma$.

Sol.

$P(\mu - 2\sigma \le X \le \mu + 2\sigma)$

$= P\left(\dfrac{\mu - 2\sigma - \mu}{\sigma} \le Z \le \dfrac{\mu + 2\sigma - \mu}{\sigma}\right)$

$= P(-2 \le Z \le 2)$

$= 1 - P(Z < -2) - P(Z > 2)$

$= 1 - P(Z > 2) - P(Z > 2)$

$= 1 - 2P(Z > 2)$

$= 1 - 2P(Z > 2)$

$\approx 1 - 2 \cdot 0.0228$

$\approx 1 - 0.0456$

≈ 0.9544

Therefore, the percentage of data that is in between $\mu - 2\sigma$ and $\mu + 2\sigma$ is $0.9544 \cdot 100\% \approx 95.44\%$.

(c) The percentage of data that is in between $\mu - 3\sigma$ and $\mu + 3\sigma$.

Sol.

$P(\mu - 3\sigma \le X \le \mu + 3\sigma)$

$= P\left(\dfrac{\mu - 3\sigma - \mu}{\sigma} \le Z \le \dfrac{\mu + 3\sigma - \mu}{\sigma}\right)$

$= P(-3 \le Z \le 3)$

$= 1 - P(Z < -3) - P(Z > 3)$

$= 1 - P(Z > 3) - P(Z > 3)$

$= 1 - 2P(Z > 3)$

$= 1 - 2P(Z > 3)$

$\approx 1 - 2 \cdot 0.0013$

$\approx 1 - 0.0026$

≈ 0.9974

Therefore, the percentage of data that is in between $\mu - 3\sigma$ and $\mu + 3\sigma$ is $0.9974 \cdot 100\% \approx 99.74\%$.

21.7 Revision Exercise 21

1. Tossing two dices at the same time, find the event where the sum of the two dices is 8.

   Sol.

   Let event $A =$ "the sum of the two dices is 8"

   $A = \{(4,4), (2,6), (3,5), (5,3), (6,2)\}$

2. There are 30 boys and 12 girls in a class. Randomly pick a representative, find the probability of a boy getting picked.

   Sol.

   $S =$ "randomly pick a representative", $n(S) = 30 + 12 = 42$.

   Let event $A =$ "pick a boy", $n(A) = 30$.

   $P(A) = \dfrac{n(A)}{n(S)}$

   $ = \dfrac{30}{42}$

   $ = \dfrac{5}{7}$

3. Randomly pick a card from a deck of 52 poker card, find the probability of the following events:

   Sol.

   $S =$ "draw a card", $n(S) = 52$.

   (a) Getting a spade.

Sol.

Let event A = "getting a spade", $n(A) = 13$.

$$P(A) = \frac{n(A)}{n(S)}$$
$$= \frac{13}{52}$$
$$= \frac{1}{4}$$

(b) Not getting a spade.

Sol.

Let event A = "not getting a spade",

then A' = "getting a spade".

$$P(A) = 1 - P(A')$$
$$= 1 - \frac{1}{4}$$
$$= \frac{3}{4}$$

4. Randomly pick a card from a deck of 52 poker card, find the probability of the following events:

Sol.

S = "draw a card", $n(S) = 52$.

(a) Getting a club.

Sol.

Let event A = "getting a club", $n(A) = 13$.

$$P(A) = \frac{n(A)}{n(S)}$$
$$= \frac{13}{52}$$
$$= \frac{1}{4}$$

(b) Getting an ace.

Sol.

Let event B = "getting an ace", $n(A) = 4$.

$$P(B) = \frac{n(B)}{n(S)}$$
$$= \frac{4}{52}$$
$$= \frac{1}{13}$$

(c) Getting a club ace.

Sol.

Since the events are inclusive, the probability of

getting a club ace is

$$P(A \cap B) = P(A) \times P(B)$$
$$= \frac{1}{4} \times \frac{1}{13}$$
$$= \frac{1}{52}$$

(d) Getting a club or an ace.

Sol.

Since the events are mutually exclusive, the probability of getting a club or an ace is

$$P(A \cup B) = P(A) + P(B)$$
$$= \frac{1}{4} + \frac{1}{13}$$
$$= \frac{4}{13}$$

5. Randomly write down a two digit number, find the probability of the following events:

Sol.

S = "write down a two digit number".

Since the number must be two digit, the first digit can be any digit from 1 to 9. There are 9 possible first digits.

The second digit can be any digit from 0 to 9. There are 10 possible second digits.

Therefore, the total number of possible two digit numbers is $9 \cdot 10 = 90$, which means, $n(S) = 90$.

(a) Larger than 20.

Sol.

Let event A = "the two digit number is larger than 20",

Since the number must be larger than 20, the first digit must be larger or equal to 2. There are 8 possible first digits,

If the first digit is 2, the second digit can be any digit from 1 to 9. There are 9 possible second digits.

If the first digit is larger than 2, the second digit can be any digit from 0 to 9. There are 10 possible second digits.

Therefore, the total number of possible two digit numbers is $1 \cdot 9 + 7 \cdot 10 = 79$, which means,

$n(A) = 79$.

$$P(A) = \frac{n(A)}{n(S)}$$
$$= \frac{79}{90}$$

(b) An even number.

Sol.

Let event $B =$ "the two digit number is an even number",

The first digit can be any digit from 1 to 9. There are 9 possible first digits.

The second digit can be any even digit from 0 to 9. There are 5 possible second digits.

Therefore, the total number of possible two digit numbers is $9 \cdot 5 = 45$, which means, $n(B) = 45$.

$$P(B) = \frac{n(B)}{n(S)}$$
$$= \frac{45}{90}$$
$$= \frac{1}{2}$$

(c) An odd number.

Sol.

Let event $C =$ "the two digit number is an odd number",

The first digit can be any digit from 1 to 9. There are 9 possible first digits.

The second digit can be any odd digit from 0 to 9. There are 5 possible second digits.

Therefore, the total number of possible two digit numbers is $9 \cdot 5 = 45$, which means, $n(C) = 45$.

$$P(C) = \frac{n(C)}{n(S)}$$
$$= \frac{45}{90}$$
$$= \frac{1}{2}$$

6. Let a phone number consist of 7 digits formed by 0, 1, 2, ⋯, 9. Someone only remembers the first three digits and forgots the last four digits of the phone number of his colleague. Find the probability him calling the right person with just one dial.

Sol.

Let event $A =$ "entering a right digit", $P(A) = \frac{1}{10}$.

Since the events are independent, the probability of him calling the right person is

$$P(A \cap A \cap A \cap A) = P(A)^4$$
$$= \left(\frac{1}{10}\right)^4$$
$$= \frac{1}{10000}$$

7. Person A and B toss two dices each at the same time. Person B gets 10 points, what is the probability of person A winning?

Sol.

Person A wins if the sum of the two dices is larger than 10.

$S =$ "throwing two dices each at the same time", $n(S) = 6 \cdot 6 = 36$.

Let event $A =$ "two dices of person A sum up to a number larger than 10",

$A = \{(6, 6), (5, 6), (6, 5)\}$.

$n(A) = 3$.

$$P(A) = \frac{n(A)}{n(S)}$$
$$= \frac{3}{36}$$
$$= \frac{1}{12}$$

8. In a archery competition, the probability of person A and person B winning are 0.35 and 0.45 respectively. Find the probability of both of them losing.

Sol.

Let event $A =$ "person A lost",

then $A' =$ "person A won", $P(A') = 1 - P(A) = 0.65$.

Let event $B =$ "person B lost",

then $B' =$ "person B won", $P(B') = 1 - P(B) = 0.55$.

Since the events are independent, the probability of both of them losing is

$$P(A \cap B) = P(A) \cdot P(B)$$
$$= 0.65 \cdot 0.55$$
$$= 0.3575$$

9. There are 200 staff members in a company, a quarter of which are foreigners. There are 115 male staff

members and 85 female staff members in the company. Given that 20 female staff members are foreigners. Now randomly pick a staff member, find the probability of the staff member picked is a male and a native.

Sol.

Let event A = "the staff member picked is a male and a native".

There are 200 staff members in the company, a quarter ($200 \cdot 25\% = 50$) of which are foreigners, while the others (150) are natives.

20 female staff members are foreigners, while the others ($85 - 20 = 65$) are natives.

20 foreigners are females, while the others ($50 - 20 = 30$) are males.

30 males are foreigners, while the others ($115 - 30 = 85$) are natives.

Therefore, $n(A) = 85$.

$$P(A) = \frac{n(A)}{n(S)}$$
$$= \frac{85}{200}$$
$$= \frac{17}{40}$$

10. Drawing three cards from a deck of 52 poker card, find the probability of drawing at least one face card.

Sol.

$n(S) = {}_{52}C_3 = 22100$.

Let event A = "draw at least one face card",

then A' = "draw no face card", $n(A') = {}_{40}C_3 = 9880$.

$$P(A) = 1 - P(A')$$
$$= 1 - \frac{n(A')}{n(S)}$$
$$= 1 - \frac{9880}{22100}$$
$$= 1 - \frac{38}{85}$$
$$= \frac{47}{85}$$

11. The hit rate of a person shooting a basketball is 0.8. If he shoots three times, find the probability of him scoring exactly two times.

Sol.

$$P = {}_3C_2 \cdot 0.8^2 \cdot 0.2^1$$
$$= \frac{3}{10} \cdot 0.64 \cdot 0.2$$
$$= \frac{3}{10} \cdot 0.128$$
$$= 0.384$$

12. The accuracy of forecast of a weather station is 89%. Find the probability of five accurate forecasts in a week.

Sol.

The probability of an accurate forecast is 0.89, while the probability of an inaccurate forecast is 0.11.

$$P = {}_7C_5 \cdot 0.89^5 \cdot 0.11^2$$
$$= 0.1419$$

13. Given that the probability of a 18-year-old teenager being drawn for national service is 0.2. Given that a community has 4 18-year-old teenagers, find the probability of at least one of them being drawn for national service.

Sol.

The chance of a 18-year-old teenager being drawn for national service is 0.2, while the chance of a 18-year-old teenager not being drawn for national service is 0.8.

Let event A = "at least one of them being drawn for national service",

then A' = "none of them being drawn for national service",

$$P(A') = {}_4C_0 \cdot 0.8^4 \cdot 0.2^0$$
$$= 0.4096$$
$$P(A) = 1 - P(A')$$
$$= 1 - 0.4096$$
$$= 0.5904$$

14. Tossing a dice, getting a number 6 can get $30, while getting other numbers can get $3. Find the expected value of the game.

Sol.

The expected value of the game is

$$E = \frac{1}{6} \cdot 30 + \frac{5}{6} \cdot 3$$
$$= \$7.50$$

15. There are 4 50 cent coins and and 6 20 cent coins in a bag. A person randomly pick two coins from the bag. Find the expected value he gets.

    Sol.

    The expected value of picking a coin from the bag is

    $$E = \frac{4}{10} \cdot 0.50 + \frac{6}{10} \cdot 0.20$$
    $$= \$0.32$$

    The expected value of picking two coins from the bag is

    $$E = 2 \cdot 0.32$$
    $$= \$0.64$$

16. A food stall perpares 250 packets of nasi lemak every day. The cost of each packet is $1.50, and the selling price is $5.00, unsold packets are thrown away. According to statistical data, the stall can sell 57% of the nasi lemaks. Find the expected value of the profit of the stall.

    Sol.

    The profit of selling a packet of nasi lemak is $3.50, while the loss of a packet of nasi lemak being thrown away is $1.50.

    The expected value of the profit of each packet of nasi lemak is

    $$E = \frac{57}{100} \cdot 3.50 + \frac{43}{100} \cdot (-1.50)$$
    $$= \$1.35$$

    The expected value of the profit of the stall is

    $$E = 250 \cdot 1.35$$
    $$= \$337.50$$

17. In a lucky draw, there are 15 envelopes with cash prizes inside, one of which has $100, two of which has $50, three of which has $10, four of which has $5, and five of which has $1. One person draw one envelope from the lucky draw, find the expected value. If the person pays $15 to draw one envelope, determine whether it is worth it to pay for the lucky draw.

    Sol.

    The expected value of the lucky draw is

    $$E = -15 + \frac{1}{15} \cdot 100 + \frac{2}{15} \cdot 50 + \frac{3}{15} \cdot 10 + \frac{4}{15} \cdot 5 + \frac{5}{15} \cdot 1$$
    $$= \$17$$

    Since the expected value is positive, it is worth it to pay for the lucky draw, and he will get a profit of $2.

18. The winning rate of a guessing game in a charity fair are as follows: winning the probabilities of $2,000, $500, $200 are all $\frac{1}{5000}$, while the probability of winning $150 is $\frac{1}{3000}$. If the fee to play the game once is $1, find the expected value of the prize. Is it worth it for the player to play the game?

    Sol.

    The expected value of the game is

    $$E = -1 + \frac{1}{5000} \cdot 2000 + \frac{1}{5000} \cdot 500 + \frac{1}{5000} \cdot 200 + \frac{1}{3000} \cdot 150$$
    $$= \$0.59$$

    Since the expected value is lesser than the fee, it is not worth it for the player to play the game, and he will lose $0.41.

19. The weight distribution of 2,524 girls in a school can be assumed to be a normal distribution with mean weight of 53.79kg and standard deviation of 7.24kg.

    Sol.

    Let X be the weight of the girls, then $Z = \frac{X-53.79}{7.24}$ is a standard normal distribution.

    (a) Randomly pick a girl, find the probability of her weight lesser than 40kg.

        Sol.

        $$P(X < 40) = P(Z < \frac{40 - 53.79}{7.24})$$
        $$= P(Z < -1.90)$$
        $$= P(Z > 1.90)$$
        $$\approx 0.0287$$

    (b) Find the number of girls whose weight is greater than 65kg.

Sol.

$$P(X > 65) = P(Z > \frac{65 - 53.79}{7.24})$$

$$= P(Z > 1.55)$$

$$\approx 0.0606$$

Therefore, there are approximately $2524 \cdot 0.0606 \approx 153$ girls whose weight is greater than $65kg$.

(c) Find the percentage of girls whose weight is between $45kg$ and $55kg$.

Sol.

$$P(45 < X < 55)$$

$$= P(\frac{45 - 53.79}{7.24} < Z < \frac{55 - 53.79}{7.24})$$

$$= P(-1.21 < Z < 0.17)$$

$$= 1 - P(Z \leq -1.21) - P(Z \geq 0.17)$$

$$= 1 - P(Z \geq 1.21) - P(Z \geq 0.17)$$

$$\approx 1 - 0.1131 - 0.4325$$

$$\approx 0.4544$$

Therefore, there are approximately $0.4544 \cdot 100\% = 45.44\%$ of the girls whose weight is between $45kg$ and $55kg$.

(d) Assume that there are 10.03% of the girls whose weight is greater than $c$kg, find c.

Sol.

$$P(X > c) = 0.1003$$

$$P(Z > \frac{c - 53.79}{7.24}) = 0.1003$$

$$\frac{c - 53.79}{7.24} = 1.28$$

$$c = 53.79 + 1.28 \cdot 7.24$$

$$= 63.06$$

(e) Randomly pick 10 girls, find the probability of at least two of them having weight lesser than $55kg$.

Sol.

The probability of a girl having weight lesser than

$55kg$ is

$$P(X < 55) = P(Z < \frac{55 - 53.79}{7.24})$$

$$= P(Z < 0.17)$$

$$= 1 - P(Z \geq 0.17)$$

$$= 1 - 0.4325$$

$$= 0.5675$$

And the probability of a girl having weight greater than or equal to $55kg$ is $1 - 0.5675 = 0.4325$.

Let event A = "at least two of them having weight lesser than $55kg$",

then A' = "none of them or only one of them having weight lesser than $55kg$".

The probability of none of them lesser than $55kg$ is

$$P = {}_{10}C_0 \cdot (0.5675)^0 \cdot (0.4325)^{10}$$

$$= 0.0002$$

The probability of only one of them lesser than $55kg$ is

$$P = {}_{10}C_1 \cdot (0.5675)^1 \cdot (0.4325)^9$$

$$= 0.0030$$

Since these events are mutually exclusive, the probability of A' is

$$P(A') = 0.0002 + 0.0030$$

$$= 0.0032$$

$$P(A) = 1 - P(A')$$

$$= 1 - 0.0032$$

$$= 0.9968$$

20. The duration of phone calls of customer service of a company can be assumed to be a normal distribution. Given that of all the phone calls, 1.02% of them are more than 30 minutes long, 25.14% of them are less than 20 minutes long. Find the mean value and standard deviation of the duration of phone calls.

Sol.

Let X be the duration of phone calls, then $Z = \frac{X - \mu}{\sigma}$ is

a standard normal distribution.

$$P(X > 30) = 0.0102$$

$$P\left(Z > \frac{30 - \mu}{\sigma}\right) = 0.0102$$

$$\frac{30 - \mu}{\sigma} \approx 2.32$$

$$30 - \mu \approx 2.32\sigma \qquad (1)$$

$$P(X < 20) = 0.2514$$

$$P\left(Z < \frac{20 - \mu}{\sigma}\right) = 0.2514$$

$$P\left(Z > -\frac{20 - \mu}{\sigma}\right) = 0.2514$$

$$-\frac{20 - \mu}{\sigma} \approx 0.67$$

$$20 - \mu \approx -0.67\sigma \qquad (2)$$

$$(1) - (2): \ 2.99\sigma \approx 10$$

$$\sigma \approx 3.34$$

$$\mu = 20 + 0.67 \cdot 3.34$$

$$\approx 22.24$$

$$\therefore \mu \approx 22.24, \sigma \approx 3.34$$

Appendix A

Standard Normal Distribution Table

Listed in the table below are $Q(z) = P(Z \leq z)$, in which Z is a standard normal distribution $N(0, 1)$.

z	0	1	2	3	4	5	6	7	8	9
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
3.1	0.0010	0.0009	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0007	0.0007
3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002

Appendix B

Cheat Sheet

B.18   Statistics

1. Frequency Distribution Table

Range	Frequency
$1.27 - 1.29$	1
$1.30 - 1.32$	4
$1.33 - 1.35$	7
$1.36 - 1.38$	22
$1.39 - 1.41$	24
$1.42 - 1.44$	24
$1.45 - 1.47$	10
$1.48 - 1.50$	6
$1.51 - 1.53$	1
$1.54 - 1.56$	1

2. Histogram and Frequency Polygon

Histogram and Frequency Polygon
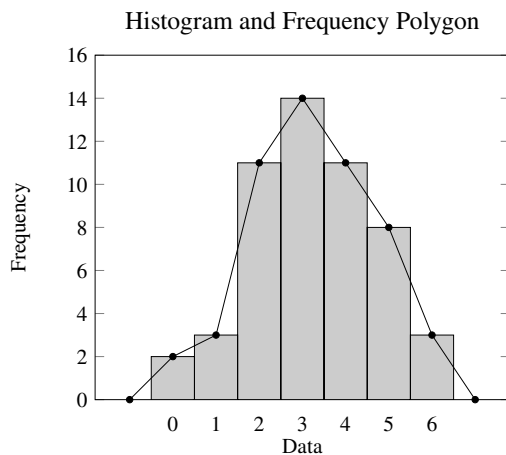
3. Cumulative Frequency Distribution Table

Range	Freq.	Lower Than	Cumm. Freq.
50 - 55	3	55	3
55 - 60	8	60	11
60 - 65	25	65	36
65 - 70	38	70	74
70 - 75	46	75	120
75 - 80	19	80	139
80 - 85	12	85	151
85 - 90	4	90	155

4. Cumulative Frequency Polygon

Cumulative Frequency Polygon
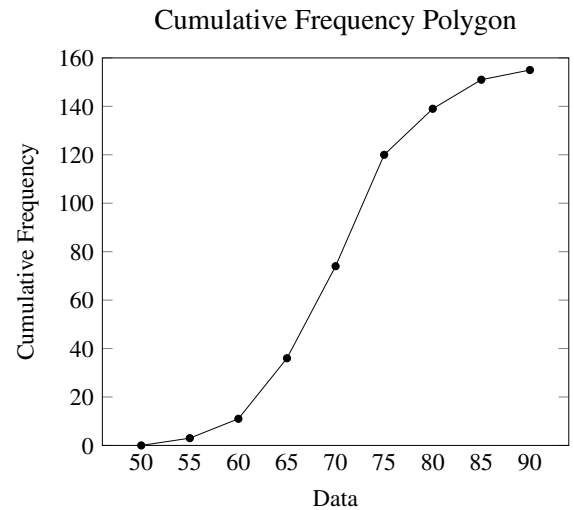
5. Central Tendency

   (a) Mean

      i. Arithmetic Mean

      $$\bar{x} = \frac{\sum x_i}{n}$$

      ii. Mean of Grouped Data

      $$\bar{x} = \frac{\sum x_i f_i}{\sum f_i}$$

      iii. Weighted Mean

      $$\bar{x} = \frac{\sum x_i w_i}{\sum w_i}$$

   (b) Median

      i. When n is odd, the median is the number in $\frac{n+1}{2}$ position.

      ii. When n is even, the median is the mean of the number in $\frac{n}{2}$ and $\frac{n}{2} + 1$ position.

      iii. Median of Grouped Data

      $$M = L_m + \left(\frac{\frac{n}{2} - F_m}{f_m} \right) C_m$$

   (c) Mode

i. The mode is the number that occurs most frequently.

ii. There can be more than one mode in a set of data.

iii. If all the values in a dataset occur with the same frequency, then there is no mode for the data.

iv. Mode of Grouped Data

$$mode = L + \left(\frac{d_1}{d_1 + d_2} \right) C$$

6. Measure of Dispersion

 (a) Range

   i. Range
   The difference between the largest and the smallest value in the dataset

   ii. Range for Grouped Data
   The difference between the upper limit of the highest class and the lower limit of the lowest class

 (b) Quartiles

   i. First Quartile (Q_1)
   The median of the lower half of the data

   ii. Second Quartile (Q_2)
   The median of the data

   iii. Third Quartile (Q_3)
   The median of the upper half of the data

 (c) Quartiles for grouped data

   i. First Quartile

$$Q_1 = L_1 + \left(\frac{\frac{n}{4} - F_1}{f_1} \right) C_1$$

   ii. Third Quartile

$$Q_3 = L_3 + \left(\frac{\frac{3n}{4} - F_2}{f_2} \right) C_2$$

 (d) Interquartile Range (IQR)

$$Interquartile\ range = Q_3 - Q_1$$

 (e) Quartile Deviation

$$QD = \frac{Q_3 - Q_1}{2}$$

 (f) Mean Deviation

   i. Mean Deviation

$$M.D. = \frac{\sum |x_i - \bar{x}|}{n}$$

   ii. Mean Deviation for grouped data

$$M.D. = \frac{\sum |x_i - \bar{x}| f_i}{\sum f_i}$$

iii. For ranged data, we take the midpoints of the classes as the representative value x_i.

 (g) Variance

   i. Variance

$$\sigma^2 = \frac{\sum x_i^2}{n} - \bar{x}^2$$

   ii. Variance for Grouped Data

$$\sigma^2 = \frac{\sum x_i^2 f_i}{\sum f_i} - \bar{x}^2$$

   iii. For ranged data, we take the midpoints of the classes as the representative value x_i.

 (h) Standard Deviation

   i. Standard Deviation

$$\sigma = \sqrt{\frac{\sum x_i^2}{n} - \bar{x}^2}$$

   ii. Standard Deviation for Grouped Data

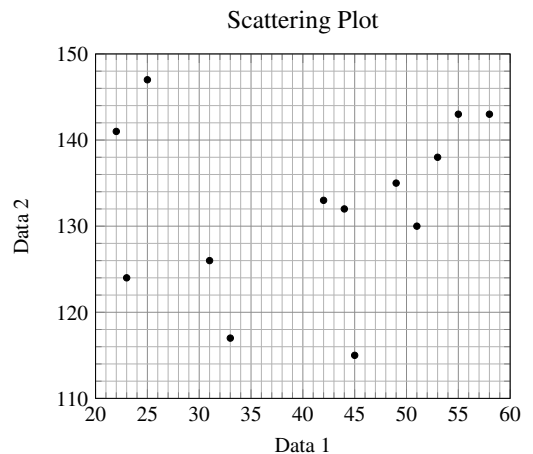$$\sigma = \sqrt{\frac{\sum x_i^2 f_i}{\sum f_i} - \bar{x}^2}$$

   iii. For ranged data, we take the midpoints of the classes as the representative value x_i.

7. Coefficient of Variation

$$v = \frac{\sigma}{\bar{x}} \cdot 100\%$$

8. Correlation and Correlation Coefficient

 (a) Scatter Plot

Scattering Plot

 (b) Linear Correlation

   i. If every single point in the scatter plot is on the line of best fit, then it's a perfect positive correlation. If the slope of the line of best fit is positive, then it's a positive correlation. If the slope of the line of best fit is negative, then it's a negative correlation.

ii. If the points in the scatter plot are scattered around the line of best fit with non-zero slope, then the closer the points are to the line of best fit, the stronger the correlation is.

iii. If the points in the scatter plot are scattered evenly around the whole plot with no obvious pattern, then there is no correlation between the two variables, aka zero correlation.

(c) Correlation Coefficient

$$r = \frac{\frac{\sum x_i y_i}{n} - \bar{x}\bar{y}}{\sqrt{\left(\frac{\sum x_i^2}{n} - \bar{x}^2\right)\left(\frac{\sum y_i^2}{n} - \bar{y}^2\right)}}$$

i. $|r| = 1$: perfect correlation
ii. $0 < |r| < 0.3$: weak correlation
iii. $0.3 \leq |r| < 0.7$: moderate correlation
iv. $0.7 \leq |r| \leq 1$: strong correlation

9. Statistical Index

(a) Statistical Index

$$I = \frac{Q_1}{Q_0} \cdot 100$$

(b) Price Index

$$\text{P.R.} = \frac{P_1}{P_0} \cdot 100$$

(c) Composite Index

$$\text{Composite Index} = \frac{\sum w_i x_i}{\sum w_i}$$

B.19 Permutation and Combination

1. Addition Rule

The task can be done in whatever way using whatever method
$$m_1 + m_2 + \cdots + m_n$$

2. Multiplication Rule

The task must be done in multiple steps, and each step can be done in whatever way

$$m_1 \times m_2 \times \cdots \times m_n$$

3. Permutation

(a) Factorial

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

(b) Normal Permutation

i. Permutation of n objects taken r at a time

$$_nP_r = \frac{n!}{(n-r)!}$$

ii. Permutation of n objects taken n at a time

$$_nP_n = n!$$

(c) Circular Permutation

i. Circular permutation of n elements:

$$\frac{_nP_n}{n} = (n-1)!$$

ii. Circular permutation of r elements from n elements ($r \leq n$):

$$\frac{_nP_r}{r} = \frac{n!}{r(n-r)!}$$

(d) Full Permutation of Inexactly Distinct Elements

$$\frac{n!}{n_1! n_2! \cdots n_k!}$$

(e) Permutation with Repetition

$$\underbrace{n \times n \times \cdots \times n}_{r} = n^r$$

4. Combination

(a) Combination Formula

$$_nC_r = \frac{n!}{(n-r)!r!}, \ r \leq n$$

(b) Properties of Combination

i. $_nC_{n-r} = {_nC_r}$
ii. If $r = n$, then $_nC_n = \frac{n!}{0!n!} = 1$.
iii. If $r = 0$, then $_nC_0 = \frac{n!}{n!0!} = 1$.

B.20 Bionomial Theorem

1. Formula

(a) Formula

$$(a+b)^n = {_nC_0}a^n + {_nC_1}a^{n-1}b + {_nC_1}a^{n-2}b^2 \cdots + {_nC_{n-r}}a^{n-r}b^r + \cdots + {_nC_n}b^n$$

(b) Formula for $a = 1, b = x$

$$(1+x)^n = 1 + {_nC_1}x + {_nC_2}x^2 + \cdots + {_nC_r}x^r + \cdots + x^n$$

2. Pascal Triangle

$$
\begin{array}{c}
(a+b)^0 \qquad\qquad 1 \\
(a+b)^1 \qquad\quad 1 \quad 1 \\
(a+b)^2 \qquad\; 1 \quad 2 \quad 1 \\
(a+b)^3 \qquad 1 \quad 3 \quad 3 \quad 1 \\
(a+b)^4 \quad\; 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
(a+b)^5 \quad 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
\vdots
\end{array}
$$

3. General Form of Bionomial Theorem

$$
T_{r+1} = {}_nC_r a^{n-r} b^r
$$

B.21 Probability

1. Sample Space

   The set of all possible results of a trial

2. Event

   A subset of the sample space

   (a) Simple Event: an event that contains only one outcome

   (b) Sure Event: the sample space itself

   (c) Impossible Event / Null Event: an empty set

3. Notation

   (a) $A \cup B$: at least one of the events A and B will happen.

   (b) $A \cap B$: both events A and B will happen.

   (c) A': the event A will not happen.

4. Definition of Probability

$$
P(A) = \frac{n(A)}{n(S)}
$$

5. Addition Rule

   (a) Inclusive Event

   Two events A and B can happen at the same time

$$
P(A \cup B) = P(A) + P(B) - P(A \cap B)
$$

   (b) Exclusive Event

   Two events A and B cannot happen at the same time
$$
P(A \cup B) = P(A) + P(B)
$$

   (c) Complementary Event

   Two events A and A' are complement to each other if they are both mutually exclusive and one of them will happen

$$
P(A') = 1 - P(A)
$$

6. Multiplication Rule

   (a) Independent Events

   Two events A and B are independent if the occurrence of one event does not affect the occurrence of the other event

$$
P(A \cap B) = P(A) \cdot P(B)
$$

7. Mathematical Expectation

$$
E = x_1 p_1 + x_2 p_2 + \cdots + x_k p_k
$$

8. Normal Distribution

   (a) Normal Curve

   i. Function

   A. Expression

$$
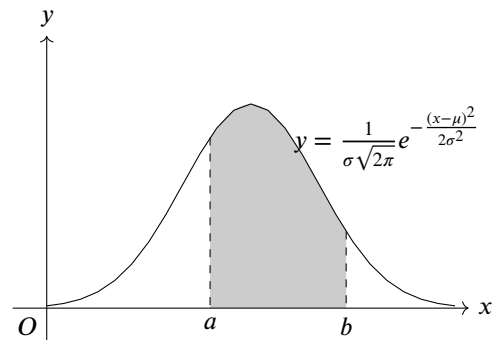f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}
$$

   B. Parameters
   - μ: mean
   - σ: standard deviation

   C. Notation

$$
X \sim N(\mu, \sigma^2)
$$

   ii. Diagram
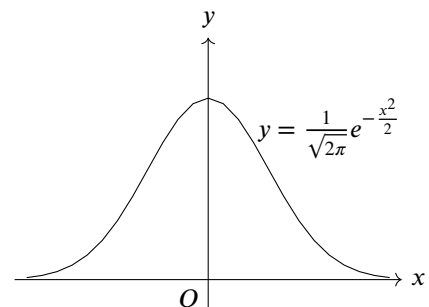

   (b) Standard Normal Distribution

   i. Definition
   A normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$

   ii. Function

$$
y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}
$$

   iii. Graph

iv. Convert to Standard Normal Distribution

If $x \sim N(\mu, \sigma^2)$, then $Z = \frac{x-\mu}{\sigma}$, $Z \sim N(0,1)$

(c) Properties

i. $P(Z \leq z) = P(Z < z)$

ii. $P(Z \geq z) = P(Z > z)$

iii. $P(Z = z) = 0$

iv. $P(Z < z) = P(Z > -z) = 1 - P(Z \geq z)$

v. $P(Z > z) = P(Z < -z) = 1 - P(Z \leq z)$

vi. $P(|Z| < z) = 2P(Z < z)$

vii. $P(|Z| > z) = 2P(Z > z)$