Exercise 11d

Find the following indefinite integrals:

1. $\int \sin^2 x\,dx$

   Sol.

   $$\begin{aligned}\int \sin^2 x\,dx &= \int \frac{1-\cos 2x}{2}\,dx \\ &= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx \\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C\end{aligned}$$

2. $\dfrac{1}{\sec^2 4x}\,dx$

   Sol.

   $$\begin{aligned}\int \frac{1}{\sec^2 4x}\,dx &= \int \cos^2 4x\,dx \\ &= \int \frac{1+\cos 8x}{2}\,dx \\ &= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 8x\,dx \\ &= \frac{1}{2}x + \frac{1}{16}\sin 8x + C\end{aligned}$$

3. $\int \cos^4 x \sin^3 x\,dx$

   Sol.

   $$\begin{aligned}\int \cos^4 x \sin^3 x\,dx &= \int \cos^4 x \sin^2 x \sin x\,dx \\ &= \int \cos^4 x(1-\cos^2 x)\sin x\,dx \qquad (\text{Let } u=\cos x,\ du=-\sin x\,dx) \\ &= -\int u^4(1-u^2)du \\ &= -\int (u^4-u^6)du \\ &= -\frac{1}{5}u^5 + \frac{1}{7}u^7 + C \\ &= -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C\end{aligned}$$

4. $\int \cos^2(3x-1)\,dx$

   Sol.

   $$\begin{aligned}\int \cos^2(3x-1)\,dx &= \int \frac{1+\cos(6x-2)}{2}\,dx \\ &= \frac{1}{2}\int dx + \frac{1}{2}\int \cos(6x-2)\,dx \\ &= \frac{1}{2}x + \frac{1}{12}\sin(6x-2) + C\end{aligned}$$

5. $\int \sin^3 \cos^2 x \, dx$

Sol.

$$\begin{aligned}
\int \sin^3 \cos^2 x \, dx &= \int \sin^2 \cos^2 x \sin x \, dx \\
&= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx \qquad (\text{Let } u = \cos x,\ du = -\sin x \, dx) \\
&= -\int (1 - u^2) u^2 du \\
&= -\int (u^2 - u^4) du \\
&= -\frac{1}{3} u^3 + \frac{1}{5} u^5 + C \\
&= -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C
\end{aligned}$$

6. $\int \sin^2 4x \cos 4x \, dx$

Sol.

$$\begin{aligned}
\int \sin^2 4x \cos 4x \, dx &= \int (1 - \cos^2 4x) \cos 4x \, dx \\
&= \int (\cos 4x - \cos^3 4x) \, dx \\
&= \int \cos 4x \, dx - \int \cos^3 4x \, dx \\
&= \frac{1}{4} \sin 4x - \int \cos^2 4x \cos 4x \, dx + C' \\
&= \frac{1}{4} \sin 4x - \int (1 - \sin^2 4x) \cos 4x \, dx + C' \qquad (\text{Let } u = \sin 4x,\ du = 4 \cos 4x \, dx) \\
&= \frac{1}{4} \sin 4x - \frac{1}{4} \int (1 - u^2) du + C' \\
&= \frac{1}{4} \sin 4x - \frac{1}{4} \left(u - \frac{1}{3} u^3 \right) + C' \\
&= \frac{1}{4} \sin 4x - \frac{1}{4} \sin 4x + \frac{1}{12} \sin^3 4x + C' \\
&= \frac{1}{12} \sin^3 4x + C
\end{aligned}$$

7. $\int \sin^3 \frac{x}{2} \, dx$

Sol.

$$\begin{aligned}
\int \sin^3 \frac{x}{2} \, dx &= \int \sin^2 \frac{x}{2} \sin \frac{x}{2} \, dx \\
&= \int (1 - \cos^2 \frac{x}{2}) \sin \frac{x}{2} \, dx \qquad (\text{Let } u = \cos \frac{x}{2},\ du = -\frac{1}{2} \sin \frac{x}{2} \, dx) \\
&= -2 \int (1 - u^2) du \\
&= 2 \int (u^2 - 1) du \\
&= 2 \left(\frac{1}{3} u^3 - u \right) + C \\
&= \frac{2}{3} \cos^3 \frac{x}{2} - 2 \cos \frac{x}{2} + C
\end{aligned}$$

8. $\int \cos^3 x\, dx$

Sol.

$$\begin{aligned}\int \cos^3 x\, dx &= \int \cos^2 x \cos x\, dx \\ &= \int (1-\sin^2 x)\cos x\, dx \qquad (\text{Let } u=\sin x,\ du=\cos x\, dx) \\ &= \int (1-u^2)du \\ &= u - \frac{1}{3}u^3 + C \\ &= \sin x - \frac{1}{3}\sin^3 x + C\end{aligned}$$

9. $\int 2x\sin^3 x^2\, dx$

Sol.

$$\begin{aligned}\int 2x\sin^3 x^2\, dx &= \int 2x\sin^2 x^2 \sin x^2\, dx \\ &= \int 2x(1-\cos^2 x^2)\sin x^2\, dx \qquad (\text{Let } u=\cos x^2,\ du=-2x\sin x^2\, dx) \\ &= -\int (1-u^2)du \\ &= \int (u^2-1)du \\ &= \frac{1}{3}u^3 - u + C \\ &= \frac{1}{3}\cos^3 x^2 - \cos x^2 + C\end{aligned}$$

10. $\int \cos^4 x\, dx$

Sol.

$$\begin{aligned}\int \cos^4 x\, dx &= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)\, dx \\ &= \frac{1}{4}\int dx + \frac{1}{2}\int \cos 2x\, dx + \frac{1}{4}\int \cos^2 2x\, dx \\ &= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}\int (1+\cos 4x)\, dx \\ &= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C\end{aligned}$$

11. $\int (2-\sin x)^2\, dx$

Sol.

$$\begin{aligned}\int (2-\sin x)^2\, dx &= \int (4-4\sin x+\sin^2 x)\, dx \\ &= \int 4\, dx - \int 4\sin x\, dx + \int \sin^2 x\, dx \\ &= 4x + 4\cos x + \frac{1}{2}\int (1-\cos 2x)\, dx \\ &= \frac{9}{2}x + 4\cos x - \frac{1}{4}\sin 2x + C\end{aligned}$$

12. $\displaystyle\int \sin^2\frac{x}{2}\cos^2\frac{x}{2}\,dx$

Sol.

$$\begin{aligned}\int \sin^2\frac{x}{2}\cos^2\frac{x}{2}\,dx &= \int\left(\sin\frac{x}{2}\cos\frac{x}{2}\right)^2\,dx\\ &= \frac{1}{4}\int \sin^2 x\,dx\\ &= \frac{1}{8}\int(1-\cos 2x)\,dx\\ &= \frac{1}{8}x-\frac{1}{16}\sin 2x+C\end{aligned}$$

13. $\displaystyle\int \sin^4 ax\,dx$

Sol.

$$\begin{aligned}\int \sin^4 ax\,dx &= \int\left(\frac{1-\cos 2ax}{2}\right)^2\,dx\\ &= \frac{1}{4}\int(1-2\cos 2ax+\cos^2 2ax)\,dx\\ &= \frac{1}{4}\int dx-\frac{1}{2}\int\cos 2ax\,dx+\frac{1}{4}\int\cos^2 2ax\,dx\\ &= \frac{1}{4}x-\frac{1}{4a}\sin 2ax+\frac{1}{8}\int(1+\cos 4ax)\,dx\\ &= \frac{3}{8}x-\frac{1}{4a}\sin 2ax+\frac{1}{32a}\sin 4ax+C\end{aligned}$$