Solution Book of Mathematic

Ssnior 2 Part I

MELVIN CHIA

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Contents

Appendix A

Cheat Sheet

A.12 Sequence and Series

1. Series:
   (a) Finite series: $a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k$
   (b) Infinite series: $a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$
2. General term:
   (a) Arithmetic sequence: $a_n = a_1 + (n-1)d$
   (b) Geometric sequence: $a_n = a_1 r^{n-1}$
3. Summation formula:
   (a) Arithmetic sequence:
      i. $S_n = \frac{1}{2}n(a_1 + a_n)$
      ii. $S_n = \frac{n}{2}[2a_1 + (n-1)d]$
   (b) Geometric sequence:
      i. When $r \neq 1$: $S_n = \frac{a_1(1-r^n)}{1-r}$
      ii. When $r = 1$: $S_n = na_1$
4. Mean:
   (a) Arithmetic mean: $A = x + y2$
   (b) Geometric mean: $G = \pm\sqrt{xy}$
5. Summation of infinite geometric series:

$$S = \frac{a_1}{1-r} \quad (1 < r < 1)$$

6. Simple summation formulas:
   (a) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
   (b) $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
   (c) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

A.14 Matrices and Determinants

1. General form of matrix: $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$
2. Square matrix: $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$
3. Zero matrix: $A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$
4. Identity matrix: $I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$
5. Transpose of a matrix: $A' = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$
6. Addition of matrices: $A+B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$
7. Subtraction of matrices: $A-B = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{pmatrix}$
8. Properties of addition and subtraction of matrices:
   (a) $A + B = B + A$
   (b) $(A + B) + C = A + (B + C)$
   (c) $A + O = A$
   (d) $(A \pm B)' = A' \pm B'$
9. Scalar product of a matrix: $kA = \begin{pmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{pmatrix}$
10. Multiplication of matrices: $(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$
11. Properties of multiplication of matrices:
    (a) $A(BC) = (AB)C$
    (b) $A(B + C) = AB + AC$
    (c) $(B + C)A = BA + CA$
    (d) $k(AB) = A(kB)$
    (e) $(AB)' = B'A'$
12. Determinants:
    (a) 2×2 determinant: $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

(b) 3×3 determinant:

$$\begin{aligned}
\det(A) &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\
&= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\
&= a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - a_3 b_2 c_1 \\
&\quad - b_3 c_2 a_1 - c_3 a_2 b_1
\end{aligned}$$

13. The Sarrus method:

$$\begin{array}{ccc:cc}
+ & + & + & & \\
a_1 & b_1 & c_1 & a_1 & b_1 \\
a_2 & b_2 & c_2 & a_2 & b_2 \\
a_3 & b_3 & c_3 & a_3 & b_3 \\
- & - & - & &
\end{array}$$

14. Minor of an element in a matrix: the determinant of the matrix obtained by removing the row and column containing the element

15. Cofactor of an element in a matrix: the minor of the element multiplied by $(-1)^{i+j}$

16. Theorems of 3×3 determinants:

Theorem 1. *The determinant of a 3x3 matrix is the sum of the elements of any row or column multiplied by the cofactors of the elements of that row or column.*

$$\begin{aligned}
|A| &= a_1 A_1 + b_1 B_1 + c_1 C_1 \\
&= a_2 B_2 + b_2 B_2 + c_2 C_2 \\
&= a_3 C_3 + b_3 C_3 + c_3 C_3 \\
&= a_1 A_1 + a_2 A_2 + a_3 A_3 \\
&= b_1 B_1 + b_2 B_2 + b_3 B_3 \\
&= c_1 C_1 + c_2 C_2 + c_3 C_3
\end{aligned}$$

Theorem 2. *The product of the elements of any row or column and the cofactor of corresponding elements of another row or column of a determinant is 0.*

$$\begin{aligned}
&a_2 B_1 + b_2 B_1 + c_2 C_1 \\
&= a_2 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_2 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\
&= a_2 b_2 c_3 + a_2 b_3 c_2 - a_2 b_2 c_3 + a_3 b_2 c_2 + a_2 b_3 c_2 \\
&\quad - a_3 b_2 c_2 \\
&= 0
\end{aligned}$$

17. Identities of determinants:

Theorem 1. *The value of a determinant is the same as the value of its transpose, aka* $|A| = |A'|$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

Theorem 2. *Switching any two rows or columns of a determinants results in the opposite value.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

Theorem 3. *If two rows or cols of a determinant are identical, the value of the determinant is zero.*

$$\begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix} = 0$$

Theorem 4. *If all elements of a row (or column) of a determinant are multiplied by some scalar number k, the value of the new determinant is k times of the given determinant.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Theorem 5. *In a determinant each element in any row (or column) consists of the sum of two terms, then the determinant can be expressed as sum of two determinants of same order.*

$$\begin{vmatrix} a_1 + d_1 & b_1 & c_1 \\ a_2 + d_2 & b_2 & c_2 \\ a_3 + d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Theorem 6. *If a determinant is obtained by adding a row or column multiplied by a some scalar number k to a different row or column, then the value of the new determinant is the same as the original determinant.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Theorem 7. *The determinant of product of two matrices of equal size is equal to the product of determinants of each matrix, aka* $|AB| = |A||B|$.

18. Inverse of a matrix:
    (a) $AB = BA = I$
    (b) $B = A^{-1}$, $A = B^{-1}$

19. Formulas of inverse matrix:
    (a) Inverse of a 2×2 matrix: $A^{-1} = \frac{1}{ad-bc} \begin{vmatrix} d & -b \\ -c & a \end{vmatrix}$
    (b) Inverse of a 3×3 matrix: $A^{-1} = \frac{1}{|A|} \operatorname{adj} A$

20. Adjoint of a matrix:

$$\operatorname{adj} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$$

21. Gauss elimination:

(a) Interchange two rows:
$R_i \leftrightarrow R_j$: interchange row i and row j.

(b) Multiply a row by a nonzero constant:
$R_i \rightarrow kR_i$: multiply row i by k, where k is a nonzero constant.

(c) Add a multiple of one row to another row:
$R_i \rightarrow R_i + kR_j$: add k times row j to row i.

22. Inverse a matrix with Gauss elimination:

$$\left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

23. Cramer's Rule:

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}$$

$$\Delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$\Delta_x = \begin{pmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{pmatrix}$$

$$\Delta_y = \begin{pmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{pmatrix}$$

$$\Delta_z = \begin{pmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{pmatrix}$$

$$x = \frac{\Delta_x}{\Delta},\ y = \frac{\Delta_y}{\Delta},\ z = \frac{\Delta_z}{\Delta}$$

A.15 Inequalities

1. Inequalities signs:

   (a) $<$: less than

   (b) $>$: greater than

   (c) $\leq$: less than or equal to

   (d) $\geq$: greater than or equal to

2. Compare numbers by their difference:

   (a) If $a - b > 0$, then $a > b$

   (b) If $a - b < 0$, then $a < b$

3. Identities of inequalities:

   Theorem 1. *If $a > b$, $b > c$, then $a > c$*

   Theorem 2. *If $a > b$ then $a + c > b + c$*

   Theorem 3. *If $a > b$, $c > d$, then $a + c > b + d$*

   Theorem 4. *If $a > b$, then:*

   (a) When $c > 0$, $ac > bc$

   (b) When $c = 0$, $ac = bc$

   (c) When $c < 0$, $ac < bc$