Solution Book of Mathematic

Ssnior 2 Part I

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Contents

11.0.1 Exercise 14.4

Calculate the following products (Question 1 to 8):

1. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Sol.

$$\begin{aligned}
&\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 1(1) + 2(2) + 3(3) \end{bmatrix} \\
&= \begin{bmatrix} 14 \end{bmatrix}
\end{aligned}$$

2. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

Sol.

$$\begin{aligned}
&\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}
\end{aligned}$$

3. $\begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Sol.

$$\begin{aligned}
&\begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2(1) + (-3)(0) & 2(0) + (-3)(1) \\ 1(1) + 5(0) & 1(0) + 5(1) \end{bmatrix} \\
&= \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}
\end{aligned}$$

4. $\begin{bmatrix} -6 & -4 & 2 \\ 7 & 8 & -5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

Sol.

$$\begin{aligned}
&\begin{bmatrix} -6 & -4 & 2 \\ 7 & 8 & -5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -6(1) + (-4)(2) + 2(3) \\ 7(1) + 8(2) + (-5)(3) \end{bmatrix} \\
&= \begin{bmatrix} -8 \\ 8 \end{bmatrix}
\end{aligned}$$

5. $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 4 & 2 \end{bmatrix}$

Sol.

$$\begin{aligned}
&\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & 1 \\ 4 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2(2) + 3(3) + 4(4) & 2(0) + 3(1) + 4(2) \\ 0(2) + 1(3) + 2(4) & 0(0) + 1(1) + 2(2) \end{bmatrix} \\
&= \begin{bmatrix} 27 & 11 \\ 11 & 5 \end{bmatrix}
\end{aligned}$$

6. $\begin{bmatrix} 6 & 4 & 2 \\ 5 & -2 & 0 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$

Sol.

$$\begin{aligned}
&\begin{bmatrix} 6 & 4 & 2 \\ 5 & -2 & 0 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 6(5) + 4(2) + 2(3) \\ 5(5) + (-2)(2) + 0(3) \\ 0(5) + 3(2) + 1(3) \end{bmatrix} \\
&= \begin{bmatrix} 44 \\ 21 \\ 9 \end{bmatrix}
\end{aligned}$$

7. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol.

$$\begin{aligned}
&\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0(0)+1(1)+0(0) & 0(1)+1(0)+0(0) & 0(0)+1(0)+0(1) \\ 1(0)+0(1)+0(0) & 1(1)+0(0)+0(0) & 1(0)+0(0)+0(1) \\ 0(0)+0(1)+1(0) & 0(1)+0(0)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

8. $\begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

Sol.

$$\begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1+(-2)+1 & 2+(-4)+2 & 3+(-6)+3 \\ (-3)+4+(-1) & (-6)+8+(-2) & (-9)+12+(-3) \\ (-2)+2+0 & (-4)+4+0 & (-6)+6+0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

11.1 Determinants

The determinant of an n-order matrix $A = (a_{ij})_{n\times n}$ is denoted as $\det(A)$. When $n \leq 2$, the deternimant can also be denoted as $|A|$. The determinant is a value.

When $n = 1$, the determinant is the value of the only element in the matrix.

Determinant of a 2x2 matrix

For a 2x2 matrix, the determinant is defined as:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

11.1.1 Practice 4

Find the value of the following determinants.

1. $\begin{vmatrix} 2 & -3 \\ 5 & 7 \end{vmatrix}$

   Sol.

$$\begin{vmatrix} 2 & -3 \\ 5 & 7 \end{vmatrix}$$
$$= 2(7) - (-3)(5)$$
$$= 14 + 15$$
$$= 29$$

2. $\begin{vmatrix} -6 & -7 \\ -8 & -9 \end{vmatrix}$

   Sol.

$$\begin{vmatrix} -6 & -7 \\ -8 & -9 \end{vmatrix}$$
$$= (-6)(-9) - (-7)(-8)$$
$$= 54 - 56$$
$$= -2$$

3. $\begin{vmatrix} 12 & -20 \\ -21 & 35 \end{vmatrix}$

   Sol.

$$\begin{vmatrix} 12 & -20 \\ -21 & 35 \end{vmatrix}$$
$$= 12(35) - (-20)(-21)$$
$$= 420 - 420$$
$$= 0$$

Determinant of a 3x3 matrix

For a 3x3 matrix, the determinant is defined as:

$$\begin{aligned}\det(A) &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 \\ &\quad - c_3a_2b_1\end{aligned}$$

A 3x3 matrix can be expanded using the Sarrus method. The Sarrus method is defined as:

$$\begin{array}{ccc:cc} + & + & + & & \\ a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \\ - & - & - & & \end{array}$$

Note that the Sarrus method is only applicable to 3x3 matrices.

11.1.2 Practice 5

Calculate the value of the following determinants.

1. $\begin{vmatrix} 1 & 5 & 1 \\ 1 & 6 & 3 \\ 9 & 8 & 9 \end{vmatrix}$

   Sol.

$$\begin{array}{ccc:cc} + & + & + & & \\ 1 & 5 & 1 & 1 & 5 \\ 1 & 6 & 3 & 1 & 6 \\ 9 & 8 & 9 & 9 & 8 \\ - & - & - & & \end{array}$$

$$\begin{vmatrix} 1 & 5 & 1 \\ 1 & 6 & 3 \\ 9 & 8 & 9 \end{vmatrix} = 54 + 135 + 8 - 54 - 24 - 45$$

$$= 74$$

2. $\begin{vmatrix} 3 & 1 & -2 \\ 0 & -1 & 1 \\ 4 & 2 & 5 \end{vmatrix}$

Sol.

$$\begin{array}{ccc:cc} + & + & + & & \\ 3 & 1 & -2 & 3 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ 4 & 2 & 5 & 4 & 2 \\ - & - & - & & \end{array}$$

$$\begin{vmatrix} 3 & 1 & -2 \\ 0 & -1 & 1 \\ 4 & 2 & 5 \end{vmatrix} = -15 + 4 - 0 - 8 - 6 - 0$$

$$= -25$$

Minor and Cofactor

The minor of an element in a matrix is the determinant of the matrix obtained by deleting the row and column containing the element. Take $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ as an example. The minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, the minor of c_2 is $\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$, and so on.

The cofactor of an element in a matrix is the minor of the element multiplied by $(-1)^{i+j}$, where i and j are the row and column indices of the element. The cofactor of a_1 is $(-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, the cofactor of c_2 is $(-1)^{3+2} \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$, and so on.

Let A_1, B_1, C_1 are the cofactors of a_1, b_1, c_1 respectively. Then

$$A_1 = (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

$$B_1 = (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

$$C_1 = (-1)^{1+3} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Thus,

$$|A| = a_1 A_1 + a_2 B_1 + a_3 C_1$$

That is, the value of the determinant is the elements of the first row multiplied by the cofactors of the elements of the first row.

The sign of the cofactor is determined by the sum of the row and column indices of the element. If the sum is even, the cofactor is positive; if the sum is odd, the cofactor is negative.

Generally, a 3x3 determinant has the following theorem:

Theorem 1. *The determinant of a 3x3 matrix is the sum of the elements of any row or column multiplied by the cofactors of the elements of that row or column.*

That is, we can use the cofactor expansion to calculate the determinant of a 3x3 matrix.

$$\begin{aligned} |A| &= a_1 A_1 + b_1 B_1 + c_1 C_1 \\ &= a_2 B_2 + b_2 B_2 + c_2 C_2 \\ &= a_3 C_3 + b_3 C_3 + c_3 C_3 \\ &= a_1 A_1 + a_2 A_2 + a_3 A_3 \\ &= b_1 B_1 + b_2 B_2 + b_3 B_3 \\ &= c_1 C_1 + c_2 C_2 + c_3 C_3 \end{aligned}$$

The determinant of any order matrix can also be calculated by the cofactor expansion.

Theorem 2. *The product of the elements of any row or column and the cofactor of corresponding elements of another row or column of a determinant is 0.*

For example, the product of the elements of the second row and the corresponding element of the cofactor of first row of the determinant is 0. That is,

$$\begin{aligned} & a_2 B_1 + b_2 B_1 + c_2 C_1 \\ &= a_2 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_2 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_2 b_2 c_3 + a_2 b_3 c_2 - a_2 b_2 c_3 + a_3 b_2 c_2 + a_2 b_3 c_2 - a_3 b_2 c_2 \\ &= 0 \end{aligned}$$