Mathematical Induction

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Chapter 1

Mathematical Induction

1.1 Mathematical Induction

We inspect the following example:

$$\begin{aligned}
&1^3 = 1 = 1^2 \\
&1^3 + 2^3 = 9 = (1+2)^2 \\
&1^3 + 2^3 + 3^3 = 36 = (1+2+3)^2 \\
&1^3 + 2^3 + 3^3 + 4^3 = 100 = (1+2+3+4)^2 \\
&1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225 = (1+2+3+4+5)^2 \\
&\quad \vdots
\end{aligned}$$

From the above example, we can conclude that

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + n^3 &= (1+2+3+\cdots+n)^2 \\
&= \left[\frac{n(n+1)}{2}\right]^2
\end{aligned}$$

Reasoning in the way of obtaining a general formula from a few examples is called **induction**. We can use induction to help us derive a general formula from a few examples. However, the general formula obtained from only a few examples may not be correct. For example:

$$a^n = (n^2 - 5n + 5)^2$$

can easily be proven

$$\begin{aligned}
a_1 &= (1^2 - 5 \times 1 + 5)^2 = 1 \\
a_2 &= (2^2 - 5 \times 2 + 5)^2 = 1 \\
a_3 &= (3^2 - 5 \times 3 + 5)^2 = 1 \\
a_4 &= (4^2 - 5 \times 4 + 5)^2 = 1
\end{aligned}$$

If we make a conclusion based on the above examples: for all natural number n,

$$a^n = (n^2 - 5n + 5)^2 = 1$$

is true, then we are wrong. In fact,

$$a_5 = (5^2 - 5 \times 5 + 5)^2 = 25 \neq 1$$

That is to say, the general formula of propositions related to natural numbers obtained from induction is not necessarily true. In order to prove its truth, we usually adopt the following method.

First, we prove that the proposition is true for the first value n_1 (for example $n_1 = 1$). Next, we assume that the proposition is true for $n = k$ ($k \in \mathbb{N}, k \geq n_1$), and then prove that the proposition is true for $n = k + 1$. In this way, we can prove that the proposition is true for all natural numbers n after n_1. This method is called **mathematical induction**.

For example, we use mathematical induction to prove that the following equation is true for all natural numbers:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

(1) When $n = 1$, LHS $= 1^3 = 1$, RHS $= \left[\frac{1(1+1)}{2}\right]^2 = 1$, so the equation is true for $n = 1$.

(2) Assume that the equation is true for $n = k$, that is,

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$$

Hence, when $n = k + 1$,

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 &= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= \left[\frac{(k+1)(k+2)}{2}\right]^2 \end{aligned}$$

Therefore, the equation is true when $n = k + 1$.

From (1), when $n = 1$, the equation is true. From (2), when $n = 1 + 1 = 2$, the equation is also true. Since the equation is true when $n = 2$, from (2), when $n = 2 + 1 = 3$, the equation is also true. Recursively, we can prove that the equation is true for $n = 4, 5, 6, \cdots$. Therefore, from (1) and (2), we can conclude that the equation is true for all $n \in \mathbb{N}$.

From the example above, we can see that the process of mathematical induction is as follows:

(1) Prove that the proposition is true for the first value n_1 (for example $n_1 = 1$ or 2).

(2) Assume that the proposition is true for $n = k$ ($k \in \mathbb{N}, k \geq n_1$), and then prove that the proposition is true for $n = k + 1$.

After the above two steps, we can conclude that the proposition is true for all natural numbers n after n_1.

It is worth noting that the two steps above are indispensable. From the calculation of the value of each term of

$$a_n = (n^2 - 5n + 5)^2$$

earlier, we can see that completing step (1) but not completing step (2) will result in a wrong conclusion. It's because we can't prove recursively that the proposition is true for $n = 2, 3, 4, 5, \cdots$. Similarly, if we complete step (2) but not step (1), we will also get a wrong conclusion.

For example, assume that when $n = k$, the equation $2 + 6 + 10 + \cdots + 2(2n - 1) = 2n^2 + 2$ is true, that is,

$$2 + 6 + 10 + \cdots + 2(2k - 1) = 2k^2 + 2$$

So, when $n = k + 1$,

$$\begin{aligned} 2 + 6 + 10 + \cdots + 2(2k - 1) + 2[2(k + 1) - 1] &= 2k^2 + 2 + 4(k + 1) - 2 \\ &= 2k^2 + 2 + 4k + 2 \\ &= 2(k + 1)^2 + 2 \end{aligned}$$

That is, if the equation is true for $n = k$, then it is also true for $n = k + 1$. However, if we make a conclusion that the equation is true for all $n \in \mathbb{N}$ based on this, we will be wrong. In fact,

When $n = 1$, LHS $= 2$, RHS $= 2 \times 1^2 + 2 = 4$,

$$\text{LHS} \neq \text{RHS}.$$

This indicates that step (2) is meaningless if step (1) is not completed.

Example 1 Prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ using mathematical induction.

Solution (1) When $n = 1$, LHS $= 1$, RHS $= 1^2 = 1$, so the equation is true for $n = 1$.

(2) Assume that the equation is true for $n = k$, that is,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

Hence, when $n = k + 1$,

$$\begin{aligned} &1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] \\ &= k^2 + [2(k + 1) - 1] \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

Therefore, the equation is true when $n = k + 1$.

From (1) and (2), we can conclude that the equation is true for all $n \in \mathbb{N}$.

Example 3 Prove that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n + 1)(2n + 1)}{6}$ using mathematical induction.

Solution (1) When $n = 1$, LHS $= 1^2 = 1$, RHS $= \dfrac{1(1 + 1)(2 \times 1 + 1)}{6} = 1$, so the equation is true for $n = 1$.

(2) Assume that the equation is true for $n = k$, that is,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Hence, when $n = k + 1$,

$$\begin{aligned}
&1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}
\end{aligned}$$

Therefore, the equation is true when $n = k + 1$.

From (1) and (2), we can conclude that the equation is true for all $n \in \mathbb{N}$.

Exercise 1a

Use mathematical induction to prove the following statements (1 - 7).

1. $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$
2. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{1}{3}n(n+1)(n+2)$
3. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(4n^2-1)}{3}$
4. $1 \cdot 4 + 2 \cdot 7 + 3 \cdot 10 + \cdots + n(3n+1) = n(n+1)^2$
5. $2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + (n+1) \cdot 2^n = n \cdot 2^{n+1}$
6. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$

1.2 Application of Mathematical Induction

Mathematical induction can be used to prove proposition that contains any natural number. In the previous section, by the introduction of mathematical induction, we have shown the application of it in proving equations through **Example 1** and **Example 2**.

Example 3 By using mathematical induction, prove that

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n} \quad (n>1,\ n\in\mathbb{N})$$

Solution (1) When $n=2$, LHS $=\left(1-\dfrac{1}{4}\right)=\dfrac{3}{4}$

RHS $=\dfrac{2+1}{2\times 2}=\dfrac{3}{4}$

so the equation is true for $n=2$.

(2) Assume that the equation is true for $n=k$, that is,

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$$

Hence, when $n=k+1$,

$$\begin{aligned}
&\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{k^2}\right)\left[1-\frac{1}{(k+1)^{\frac{1}{2}}}\right]\\
&=\frac{k+1}{2k}\left[1-\frac{1}{(k+1)^2}\right]\\
&=\frac{k+1}{2k}\cdot\frac{k^2+2k}{(k+1)^2}\\
&=\frac{k+2}{2(k+1)}\\
&=\frac{(k+1)+1}{2(k+1)}
\end{aligned}$$

Therefore, the equation is true when $n=k+1$.

From (1) and (2), we can conclude that the equation is true for all $n\in\mathbb{N}$.

Example 4 Prove that $x^{2n} - y^{2n}$ ($n \in \mathbb{N}$) is divisible by $x + y$

Solution (1) When $n = 1$, $x^2 - y^2 = (x+y)(x-y)$ can be divided by $x + y$, so the statement is true for $n = 1$.

(2) Assume that the statement is true for $n = k$, that is, $x^{2k} - y^{2k}$ is divisible by $x + y$.

Hence, when $n = k + 1$,

$$
\begin{aligned}
x^{2(k+1)} - y^{2(k+1)} &= x^2 \cdot x^{2k} - y^2 \cdot y^{2k} \\
&= x^2 \cdot x^{2k} - x^2 \cdot y^{2k} + x^2 \cdot y^{2k} - y^2 \cdot y^{2k} \\
&= x^2(x^{2k} - y^{2k}) + y^{2k}(x^2 - y^2)
\end{aligned}
$$

Since $x^{2k} - y^{2k}$ and $x^2 - y^2$ are both divisible by $x + y$,

The above sum $x^2(x^{2k} - y^{2k}) + y^{2k}(x^2 - y^2)$ is also divisible by $x + y$.

Therefore, when $n = k + 1$, $x^{2(k+1)} - y^{2(k+1)}$ is divisible by $x + y$.

From (1) and (2), we can conclude that the statement is true for all $n \in \mathbb{N}$.

Example 5 $p > -1 \implies \forall\, n \in \mathbb{N}, (1+p)^n \geq 1 + np$

Solution (1) When $n = 1$, LHS $= (1+p)^1 = 1 + p$

RHS $= 1 + 1 \times p = 1 + p$

so the inequality is true for $n = 1$.

(2) Assume that the inequality is true for $n = k$, that is,

$$(1+p)^k \geq 1 + kp$$

Hence, when $n = k + 1$,

$$
\begin{aligned}
(1+p)^{k+1} &= (1+p)^k(1+p) \\
&\geq (1+kp)(1+p) \\
&= 1 + p + kp + kp^2 \\
&= 1 + (k+1)p + kp^2 \\
&\geq 1 + (k+1)p
\end{aligned}
$$

Therefore, when $n = k + 1$, the inequality is true.

From (1) and (2), we can conclude that the inequality is true for all $n \in \mathbb{N}$.

Example 6 A plane has n lines such that no two of them are parallel and no three of them are concurrent. Prove that the amount of point of intersection is $V_n = \frac{1}{2}n(n-1)$, $n \geq 2$.

Solution (1) When $n = 2$, there is only one point of intersection, that is, $V_2 = 1$.
Also, when $n = 2$, $\frac{1}{2} \times 2 \times (2-1) = 1$.
Therefore, the statement is true for $n = 2$.

(2) Assume that the statement is true for $n = k (k \geq 2)$, that is, the amount of point of intersection that satisfies the given k lines on the plane is $V_k = \frac{1}{2}k(k-1)$.
Now consider the case when adding one more line to the plane, that is, the case when there are $k+1$ lines. The new line will intersect with the other k lines, that is, there will be k new points of intersection. Therefore, the amount of point of intersection on the plane is

$$\begin{aligned} V_{k+1} = V_k + k &= \frac{1}{2}k(k-1) + k \\ &= \frac{1}{2}k\left[(k-1) + 2\right] \\ &= \frac{1}{2}(k+1)[(k+1) - 1] \end{aligned}$$

Therefore, when $n = k+1$, the statement is true.

From (1) and (2), we can conclude that the statement is true for all $n \geq 2$ and $n \in \mathbb{N}$.

Example 7 Prove that $3^{4n+2} + 5^{2n+1}$ is divisible by 14 for all $n \geq 0$, $n \in \mathbb{Z}$.

Solution (1) When $n = 0$, $3^2 + 5^1 = 14$ can be divided by 14,
that is, the statement is true for $n = 0$.

(2) Assume that the statement is true for $n = k$, that is, $3^{4k+2} + 5^{2k+1}$ is divisible by 14.
Hence, when $n = k+1$,

$$\begin{aligned} 3^{4(k+1)+2} + 5^{2(k+1)+1} &= 3^{4k+2+4} + 5^{2k+1+2} \\ &= 81 \cdot 3^{4k+2} + 25 \cdot 5^{2k+1} \\ &= 81(3^{4k+2} + 5^{2k+1}) - 56 \cdot 5^{2k+1} \end{aligned}$$

Since $3^{4k+2} + 5^{2k+1}$ and $56 \cdot 5^{2k+1}$ are both divisible by 14,
$3^{4(k+1)+2} + 5^{2(k+1)+1}$ is divisible by 14,
that is to say, when $n = k+1$, the statement is true.

From (1) and (2), we can conclude that the statement is true for all $n \geq 0$, $n \in \mathbb{Z}$.

Example 8 Prove that $\sum 2^{n-1} = 2^n - 1$ for all $n \in \mathbb{N}$ using the method of mathematical induction.

Solution (1) When $n = 1$, LHS $= 2^{1-1} = 1$

RHS $= 2^1 - 1 = 1$

$\therefore$ the formula is true for $n = 1$.

(2) Assume that the formula is true for $n = k$, that is, $2_0 + 2_1 + \cdots + 2_{k-1} = 2^k - 1$.

Hence, when $n = k + 1$, $2^0 + 2^1 + \cdots + 2^{k-1} + 2^k = 2^k - 1 + 2^k$

$$= 2 \cdot 2^k - 1$$
$$= 2^{k+1} - 1$$

Therefore, when $n = k + 1$, the formula is true.

From (1) and (2), we can conclude that the formula is true for all $n \in \mathbb{N}$.

Exercise 1b

Prove the following statements using the method of mathematical induction:

1. $-1 + 3 - 5 + \cdots + (-1)^n(2n - 1) = (-1)^n \cdot n$
2. $\sum(5n - 1) = \dfrac{n(5n + 3)}{2}, n \in \mathbb{N}$
3. $\sum 3^{n-1} = \dfrac{3^n - 1}{2}, n \in \mathbb{N}$
4. $2^n > n^2$, $n > 4$ and $n \in \mathbb{N}$
5. $2^n + 2 > n^2$, $n \in \mathbb{N}$
6. The sum of the interior angles of a polygon with n sides is $(n - 2)\pi$, $n \geq 3$.
7. $(a^n - b^n)$ is divisible by $(a - b)$.
8. $x^{n+2} + (x + 1)^{2n+1}$ is divisible by $x^2 + x + 1$, $n \geq 0$ and $n \in \mathbb{Z}$.
9. $x^n + 5n$ ($n \in \mathbb{N}$) is divisible by 6.
10. The sum of the cube of three consecutive integers is divisible by 9.
11. For all natural number n, $9^n - 8n - 1$ is a multiple of 64, $n \geq 2$.
12. Determine the general formula for the following, and prove it using the method of mathematical induction.

$$\begin{aligned}
1 &= 1 \\
3 + 5 &= 8 \\
7 + 9 + 11 &= 27 \\
13 + 15 + 17 + 19 &= 64 \\
21 + 23 + 25 + 27 + 29 &= 125
\end{aligned}$$