

Praktis 2

Differentiation

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2.1 Limit and its Relation to Differentiation

1. Find the value of each of the following.

(a) $\lim_{x \rightarrow 1} (x - 1)$

Sol.

$$\begin{aligned}\lim_{x \rightarrow 1} (x - 1) &= 1 - 1 \\ &= 0 \quad \square\end{aligned}$$

(b) $\lim_{x \rightarrow 1} \frac{x^2 - 2}{x}$

Sol.

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^2 - 2}{x} &= \frac{1^2 - 2}{1} \\ &= \frac{-1}{1} \\ &= -1 \quad \square\end{aligned}$$

(c) $\lim_{x \rightarrow 0} \frac{2x - 5}{x + 3}$

Sol.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{2x - 5}{x + 3} &= \frac{2(0) - 5}{0 + 3} \\ &= \frac{-5}{3} \\ &= -\frac{5}{3} \quad \square\end{aligned}$$

(d) $\lim_{x \rightarrow a} (x - a)$

Sol.

$$\begin{aligned}\lim_{x \rightarrow a} (x - a) &= a - a \\ &= 0 \quad \square\end{aligned}$$

2. Calculate the value for each of the following.

(a) $\lim_{x \rightarrow 0} \frac{2x^2 - 5x}{x}$

Sol.

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{2x^2 - 5x}{x} &= \lim_{x \rightarrow 0} \frac{x(2x - 5)}{x} \\ &= \lim_{x \rightarrow 0} (2x - 5) \\ &= 2(0) - 5 \\ &= -5 \quad \square\end{aligned}$$

(b) $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$

Sol.

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} &= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} (x + 2) \\ &= 2 + 2 \\ &= 4 \quad \square\end{aligned}$$

(c) $\lim_{x \rightarrow 5} \frac{x^2 + 4x - 45}{x - 5}$

Sol.

$$\begin{aligned}\lim_{x \rightarrow 5} \frac{x^2 + 4x - 45}{x - 5} &= \lim_{x \rightarrow 5} \frac{(x - 5)(x + 9)}{x - 5} \\ &= \lim_{x \rightarrow 5} (x + 9) \\ &= 5 + 9 \\ &= 14 \quad \square\end{aligned}$$

(d) $\lim_{x \rightarrow 1} \frac{\log_{10} x^2}{\log_{10} x}$

Sol.

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{\log_{10} x^2}{\log_{10} x} &= \lim_{x \rightarrow 1} \frac{\log_{10} x^2}{\log_{10} x} \\ &= \lim_{x \rightarrow 1} \frac{2 \log_{10} x}{\log_{10} x} \\ &= \lim_{x \rightarrow 1} 2 \\ &= 2 \quad \square\end{aligned}$$

3. Find the value for each of the following.

(a) $\lim_{x \rightarrow 9} \frac{x - 9}{\sqrt{x} - 3}$

Sol.

$$\begin{aligned}\lim_{x \rightarrow 9} \frac{x - 9}{\sqrt{x} - 3} &= \lim_{x \rightarrow 9} \frac{(x - 9)'}{(\sqrt{x} - 3)'} \\ &= \lim_{x \rightarrow 9} \frac{1}{\frac{1}{2\sqrt{x}}} \\ &= \lim_{x \rightarrow 9} (2\sqrt{x}) \\ &= 2\sqrt{9} \\ &= 2(3) \\ &= 6 \quad \square\end{aligned}$$

(b) $\lim_{x \rightarrow -1} \frac{x + 1}{\sqrt{x + 5} - 2}$

Sol.

$$\begin{aligned}\lim_{x \rightarrow -1} \frac{x + 1}{\sqrt{x + 5} - 2} &= \lim_{x \rightarrow -1} \frac{(x + 1)'}{(\sqrt{x + 5} - 2)'} \\ &= \lim_{x \rightarrow -1} \frac{1}{\frac{1}{2\sqrt{x + 5}}} \\ &= \lim_{x \rightarrow -1} (2\sqrt{x + 5}) \\ &= 2\sqrt{-1 + 5} \\ &= 2\sqrt{4} \\ &= 2(2) \\ &= 4 \quad \square\end{aligned}$$

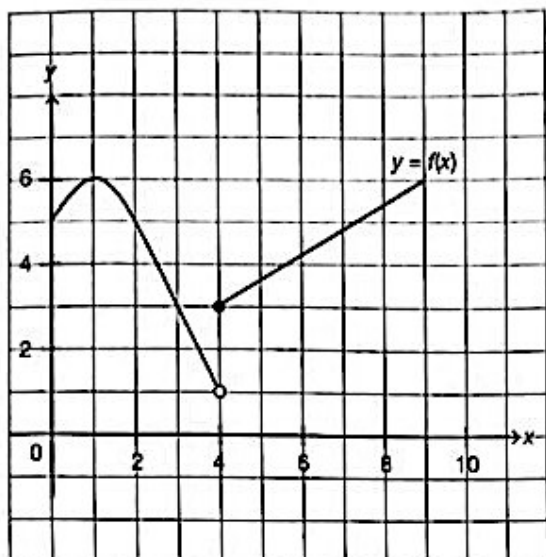
(c) $\lim_{x \rightarrow 9} \frac{\sqrt{x+7} - 4}{x-9}$
Sol.

$$\begin{aligned} \lim_{x \rightarrow 9} \frac{\sqrt{x+7} - 4}{x-9} &= \lim_{x \rightarrow 9} \frac{(\sqrt{x+7} - 4)'}{(x-9)'} \\ &= \lim_{x \rightarrow 9} \frac{1}{2\sqrt{x+7}} \\ &= \frac{1}{2\sqrt{9+7}} \\ &= \frac{1}{8} \end{aligned}$$

(d) $\lim_{x \rightarrow 2} \frac{\sqrt{6-x} - 2}{3 - \sqrt{11-x}}$
Sol.

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{\sqrt{6-x} - 2}{3 - \sqrt{11-x}} &= \lim_{x \rightarrow 2} \frac{(\sqrt{6-x} - 2)'}{(3 - \sqrt{11-x})'} \\ &= \lim_{x \rightarrow 2} \frac{1}{\frac{2\sqrt{6-x}}{-2\sqrt{11-x}}} \\ &= \lim_{x \rightarrow 2} \frac{-2\sqrt{11-x}}{2\sqrt{6-x}} \\ &= \lim_{x \rightarrow 2} \frac{-\sqrt{11-x}}{\sqrt{6-x}} \\ &= -\frac{\sqrt{11-2}}{\sqrt{6-2}} \\ &= -\frac{3}{2} \quad \square \end{aligned}$$

4. The following diagram shows part of a graph $y = f(x)$.



Based on this graph, find

(a) $f(4)$
Sol.

$$f(4) = 3$$

(b) $\lim_{x \rightarrow 4} f(x)$ and explain your answer.
Sol.

$$\begin{aligned} \lim_{x \rightarrow 4^-} f(x) &\neq 4 \\ \lim_{x \rightarrow 4^+} f(x) &= 4 \end{aligned}$$

Since the left limit and right limit are different, $f(4)$ does not exist.

(c) $\lim_{x \rightarrow 1} f(x)$
Sol.

$$\lim_{x \rightarrow 1} f(x) = 6$$

5. Find $\frac{dy}{dx}$ by using the first principle.

(a) $y = 3x + 5$

Sol.

$$y = 3x + 5 \quad (1)$$

$$y + \delta y = 3(x + \delta x) + 5$$

$$y + \delta y = 3x + 3\delta x + 5 \quad (2)$$

(2) - (1) :

$$\delta y = 3\delta x$$

$$\frac{\delta y}{\delta x} = 3$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) \\ &= \lim_{\delta x \rightarrow 0} 3 \\ &= 3 \quad \square \end{aligned}$$

(b) $y = x^2 - 7$

Sol.

$$y = x^2 - 7 \quad (1)$$

$$y + \delta y = (x + \delta x)^2 - 7$$

$$y + \delta y = x^2 + 2x\delta x + (\delta x)^2 - 7 \quad (2)$$

(2) - (1) :

$$\delta y = 2x\delta x + (\delta x)^2$$

$$\frac{\delta y}{\delta x} = 2x + 2\delta x$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) \\ &= \lim_{\delta x \rightarrow 0} (2x + 2\delta x) \\ &= 2x \quad \square \end{aligned}$$

(c) $y = x^2 + 2x + 1$

Sol.

$$y = x^2 + 2x + 1 \quad (1)$$

$$y + \delta y = (x + \delta x)^2 + 2(x + \delta x) + 1$$

$$y + \delta y = x^2 + 2x\delta x + (\delta x)^2 + 2x + 2\delta x + 1 \quad (2)$$

$$(2) - (1) :$$

$$\delta y = 2x\delta x + (\delta x)^2 + 2\delta x$$

$$\frac{\delta y}{\delta x} = 2x + \delta x + 2$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) \\ &= \lim_{\delta x \rightarrow 0} (2x + \delta x + 2) \\ &= 2x + 2 \quad \square \end{aligned}$$

(d) $y = -x^3 + 9$

Sol.

$$y = -x^3 + 9 \quad (1)$$

$$y + \delta y = -(x + \delta x)^3 + 9$$

$$y + \delta y = -x^3 - 3x^2\delta x - 3x(\delta x)^2 - \delta x^3 + 9 \quad (2)$$

$$(2) - (1) :$$

$$\delta y = -3x^2\delta x - 3x(\delta x)^2 - \delta x^3$$

$$\frac{\delta y}{\delta x} = -3x^2 - 3x\delta x - (\delta x)^2$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) \\ &= \lim_{\delta x \rightarrow 0} [-3x^2 - 3x\delta x - (\delta x)^2] \\ &= -3x^2 \quad \square \end{aligned}$$

(e) $y = 2 - \frac{3}{x}$

Sol.

$$y = 2 - 3x^{-1} \quad (1)$$

$$y + \delta y = 2 - 3(x + \delta x)^{-1} \quad (2)$$

$$(2) - (1) :$$

$$\delta y = -3(x + \delta x)^{-1} + 3x^{-1}$$

$$= -\frac{3}{x + \delta x} + \frac{3}{x}$$

$$= \frac{-3x + 3x + 3\delta x}{x(x + \delta x)}$$

$$= \frac{3\delta x}{x^2 + x\delta x}$$

$$\frac{\delta y}{\delta x} = \frac{3}{x^2 + x\delta x}$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) \\ &= \lim_{\delta x \rightarrow 0} \left(\frac{3}{x^2 + x\delta x} \right) \\ &= \frac{3}{x^2} \quad \square \end{aligned}$$

6. Given a curve $y = x^2 - ax + b$

- (a) By using the first principle, find the gradient function to the curve.

Sol.

$$y = x^2 - ax + b \quad (1)$$

$$y + \delta y = (x + \delta x)^2 - a(x + \delta x) + b$$

$$y + \delta y = x^2 + 2x\delta x + (\delta x)^2 - ax - a\delta x + b \quad (2)$$

$$(2) - (1) :$$

$$\delta y = 2x\delta x + (\delta x)^2 - a\delta x$$

$$\frac{\delta y}{\delta x} = 2x + \delta x - a$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) \\ &= \lim_{\delta x \rightarrow 0} (2x + \delta x - a) \\ &= 2x - a \quad \square \end{aligned}$$

- (b) Given that the value of gradient of the curve at $(2, -3)$ is 2, find the value of a and b .

Sol.

$$\frac{dy}{dx} = 2x - a$$

$$2 = 2(2) - a$$

$$\therefore a = 2 \quad \square$$

$$y = x^2 - 2x + b$$

$$-3 = (2)^2 - 2(2) + b$$

$$-3 = 4 - 4 + b$$

$$\therefore b = -3 \quad \square$$

2.2 The First Derivative

7. Find the first derivative for each of the following functions.

(a) $y = 6x^2$

Sol.

$$\frac{dy}{dx} = 12x \quad \square$$

(b) $y = -x^4$

Sol.

$$\frac{dy}{dx} = -4x^3 \quad \square$$

(c) $y = \sqrt[3]{x^4}$

Sol.

$$y = x^{\frac{4}{3}}$$

$$\frac{dy}{dx} = \frac{4}{3}x^{\frac{4}{3}-1}$$

$$= \frac{4}{3}\sqrt[3]{x} \quad \square$$

(d) $y = -\frac{2}{x^2}$

Sol.

$$\begin{aligned} y &= -2x^{-2} \\ \frac{dy}{dx} &= -2(-2x^{-3}) \\ &= 4x^{-3} \\ &= \frac{4}{x^3} \quad \square \end{aligned}$$

8. Find each of the following.

(a) $\frac{d}{dx}(2x^2 + 3x - 9)$

Sol.

$$\frac{d}{dx}(2x^2 + 3x - 9) = 4x + 3 \quad \square$$

(b) $\frac{d}{dx}\left(x^2 + \frac{2}{x}\right)$

Sol.

$$\begin{aligned} \frac{d}{dx}\left(x^2 + \frac{2}{x}\right) &= \frac{d}{dx}(x^2 + 2x^{-1}) \\ &= 2x - 2x^{-2} \\ &= 2x - \frac{2}{x^2} \quad \square \end{aligned}$$

(c) $\frac{d}{dx}\left(5x^3 + 2x^2 + 4x - 7 - \frac{1}{x} + \frac{3}{x^2}\right)$

Sol.

$$\begin{aligned} \frac{d}{dx}\left(5x^3 + 2x^2 + 4x - 7 - \frac{1}{x} + \frac{3}{x^2}\right) &= \frac{d}{dx}(5x^3 + 2x^2 + 4x - 7 - x^{-1} + 3x^{-2}) \\ &= 15x^2 + 4x + 4 + x^{-2} - 6x^{-3} \\ &= 15x^2 + 4x + 4 + \frac{1}{x^2} - \frac{6}{x^3} \quad \square \end{aligned}$$

9. Differentiate each of the following functions with respect to x.

(a) $f(x) = x\left(\frac{1}{2}x^4 - x^2 - 5x\right)$

Sol.

$$\begin{aligned} f(x) &= \frac{1}{2}x^5 - x^3 - 5x^2 \\ \frac{d}{dx} &= \frac{5}{2}x^4 - 3x^2 - 10x \quad \square \end{aligned}$$

(b) $f(x) = (x^2 - 5)(x + 3)$

Sol.

$$\begin{aligned} f(x) &= x^3 + 3x^2 - 5x - 15 \\ \frac{d}{dx} &= 3x^2 + 6x - 5 \quad \square \end{aligned}$$

(c) $f(x) = \frac{(x^3 - x + 4)}{x}$

Sol.

$$\begin{aligned} f(x) &= \frac{x^2}{x} - 1 + \frac{4}{x} \\ &= x^2 - 1 + 4x^{-1} \\ \frac{d}{dx} &= 2x - 4x^{-2} \\ &= 2x - \frac{4}{x^2} \quad \square \end{aligned}$$

(d) $f(x) = \frac{(x^2 - x - 2)}{(x - 2)}$

Sol.

$$\begin{aligned} f(x) &= \frac{(x - 2)(x + 1)}{x - 2} \\ &= x + 1 \\ \frac{d}{dx} &= 1 \quad \square \end{aligned}$$

10. Find $f'(x)$ for each of the following functions.

(a) $f(x) = (3x - 5)^4$

Sol.

$$\begin{aligned} f'(x) &= 4(3x - 5)^3 \cdot \frac{d}{dx}(3x - 5) \\ &= 4(3x - 5)^3 \cdot 3 \\ &= 12(3x - 5)^2 \quad \square \end{aligned}$$

(b) $f(x) = 5(x^3 + 4x)^3$

Sol.

$$\begin{aligned} f'(x) &= 5(x^3 + 4x)^3 \cdot \frac{d}{dx}(x^3 + 4x) \\ &= 5(x^3 + 4x)^3 \cdot (3x^2 + 4) \\ &= 15(3x^2 + 4)(x^3 + 4x)^2 \quad \square \end{aligned}$$

(c) $f(x) = \frac{2}{(5x^2 - 3x)^{10}}$

Sol.

$$\begin{aligned} f(x) &= \frac{-20 \cdot \frac{d}{dx}(5x^2 - 3x)}{(5x^2 - 3x)^{11}} \\ &= \frac{-20(10x - 3)}{(5x^2 - 3x)^{11}} \end{aligned}$$

11. Find the first derivative for each of the following functions by using the product rule.

(a) $y = 6x^2(x + 5x^2)^3$

Sol.

$$\begin{aligned}y &= 6x^2[x(1+5x)]^3 \\&= 6x^2(x^3)(1+5x)^3 \\&= 6x^5(1+5x)^3\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= 6x^5 \frac{d}{dx}(1+5x)^3 + (1+5x)^3 \frac{d}{dx}6x^5 \\&= 6x^5 \cdot 5 \cdot 3(1+5x)^2 + 30x^4(1+5x)^3 \\&= 90x^5 \cdot (1+5x)^2 + 30x^4(1+5x)^3 \\&= 30x^4(1+5x)^2(1+5x+3x) \\&= 30x^4(5x+1)^2(8x+1)\end{aligned}$$

(b) $y = x(7x+3)^5$

Sol.

$$\begin{aligned}\frac{dy}{dx} &= x \frac{d}{dx}(7x+3)^5 + (7x+3)^5 \frac{d}{dx}x \\&= x \cdot 5(7x+3)^4 \cdot 7 + (7x+3)^5 \cdot 1 \\&= 35x(7x+3)^4 + (7x+3)^5 \\&= (7x+3)^4(35x+7x+3) \\&= (7x+3)^4(42x+3) \quad \square\end{aligned}$$

(c) $y = (4x^2-3x)(1-2x^2)^{10}$

Sol.

$$\begin{aligned}\frac{dy}{dx} &= (4x^2-3x) \frac{d}{dx}(1-2x^2)^{10} + (1-2x^2)^{10} \frac{d}{dx}(4x^2-3x) \\&= (4x^2-3x) \cdot 10(1-2x^2)^9 \cdot (-4x) + (1-2x^2)^{10}(8x-3) \\&= (1-2x^2)^9 [(-40x)(4x^2-3x) + (1-2x^2)(8x-3)] \\&= (1-2x^2)^9 [-160x^3 + 120x^2 + 8x - 3 - 16x^3 + 6x^2] \\&= (1-2x^2)^9 [-176x^3 + 126x^2 + 8x - 3] \quad \square\end{aligned}$$

12. Find $\frac{dy}{dx}$ for each of the following functions by using the quotient rule.

(a) $y = \frac{x-2}{2x+1}$

Sol.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2x+1) \frac{d}{dx}(x-2) - (x-2) \frac{d}{dx}(2x+1)}{(2x+1)^2} \\&= \frac{2x+1 - 2(x-2)}{(2x+1)^2} \\&= \frac{5}{(2x+1)^2} \quad \square\end{aligned}$$

(b) $y = \frac{x^2+3x-4}{x-1}$

Sol.

$$\begin{aligned}y &= \frac{(x+4)(x-1)}{x-1} \\&= x+4\end{aligned}$$

$$\frac{dy}{dx} = 1 \quad \square$$

(c) $y = \frac{x^3}{(2x-1)^2}$

Sol.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2x-1)^2 \frac{d}{dx}(x^3) - (x^3) \frac{d}{dx}(2x-1)^2}{(2x-1)^4} \\&= \frac{(2x-1)^2 \cdot 3x^2 - x^3 \cdot 4(2x-1)}{(2x-1)^4} \\&= \frac{(2x-1)[3x^2(2x-1) - 4x^3]}{(2x-1)^4} \\&= \frac{6x^3 - 3x^2 - 4x^3}{(2x-1)^3} \\&= \frac{2x^3 - 3x^2}{(2x-1)^3} \\&= \frac{x^2(2x-3)}{(2x-1)^3} \quad \square\end{aligned}$$

13. Find the gradient function to the curve $y = \sqrt{x}(4x+1)$. Hence, find the value of the gradient of the curve at $x = 4$.

Sol.

$$\begin{aligned}y &= \sqrt{x}(4x+1) \\&= 4x\sqrt{x} + \sqrt{x} \\&= 4x \cdot x^{\frac{1}{2}} + x^{\frac{1}{2}} \\&= 4x^{\frac{3}{2}} + x^{\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= 4 \left(\frac{3}{2} x^{\frac{1}{2}} \right) + \frac{1}{2} x^{-\frac{1}{2}} \\&= 6\sqrt{x} + \frac{1}{2\sqrt{x}} \quad \square\end{aligned}$$

When $x = 4$,

$$\begin{aligned}\frac{dy}{dx} &= 6\sqrt{4} + \frac{1}{2\sqrt{4}} \\&= 12 + \frac{1}{4} \\&= \frac{49}{4} \quad \square\end{aligned}$$

14. Given $x^2y = 5$, find $\frac{dy}{dx}$ when $x = 2$.

Sol.

$$\begin{aligned}x^2y &= 5 \\y &= \frac{5}{x^2} \\&= 5x^{-2}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= 5(-2x^{-3}) \\&= -10x^{-3}\end{aligned}$$

When $x = 2$,

$$\begin{aligned}\frac{dy}{dx} &= -10(2)^{-3} \\&= -10\left(\frac{1}{8}\right) \\&= -\frac{5}{4} \quad \square\end{aligned}$$

15. Given $y = 5x^m$ and $\frac{dy}{dx} = x^n$, find the value of m and n .

Sol.

$$\begin{aligned}\frac{dy}{dx} &= 5m \cdot x^{m-1} \\x^n &= 5mx^{m-1}\end{aligned}$$

Comparing both sides,

$$\begin{aligned}5m &= 1 \\m &= \frac{1}{5} \quad \square \\m - 1 &= n \\\frac{1}{5} - 1 &= n \\n &= -\frac{4}{5} \quad \square\end{aligned}$$

16. Given $f(x) = ax^3 - bx^2 + 9x + 5$ where $a, b > 0$. Show that $f'(x)$ is always positive for all the values of x when $b^2 < 27a$.

Sol.

$$\begin{aligned}f'(x) &= 3ax^2 - 2bx + 9 \\&= 3a\left(x^2 - \frac{2b}{3a}x\right) + 9 \\&= 3a\left[\left(x^2 - \frac{2b}{3a}x + \frac{b^2}{9a^2}\right) - \frac{b^2}{9a^2}\right] + 9 \\&= 3a\left(x - \frac{b}{3a}\right)^2 - \frac{b^2}{3a} + 9\end{aligned}$$

$f'(x)$ is always positive when $-\frac{b^2}{3a} + 9 > 0$.

$$\begin{aligned}-\frac{b^2}{3a} + 9 &> 0 \\\frac{b^2}{3a} &< 9 \\b^2 &< 27a \quad (\text{shown}) \quad \square\end{aligned}$$

17. Given $\frac{d}{dx}(ax^m + bx^n) = 12x^s + 9x^t$ where $a, b > 0$.

- (a) Find $\frac{s}{t}$ in terms of a and b .

Sol.

$$\begin{aligned}\frac{d}{dx}(ax^m + bx^n) &= max^{m-1} + nbx^{n-1} \\12x^s + 9x^t &= max^{m-1} + nbx^{n-1}\end{aligned}$$

Comparing both sides,

$$\begin{aligned}ma &= 12 \\m &= \frac{12}{a} \\nb &= 9 \\n &= \frac{9}{b} \\s &= m - 1 \\&= \frac{12}{a} - 1 \\&= \frac{12 - a}{a} \\t &= n - 1 \\&= \frac{9}{b} - 1 \\&= \frac{9 - b}{b} \\\frac{s}{t} &= \frac{12 - a}{a} \cdot \frac{b}{9 - b} \\&= \frac{b(12 - a)}{a(9 - b)} \quad \square\end{aligned}$$

- (b) Find the values of a and b if $3s = 5t$ and $\frac{m}{n} = \frac{3}{2}$.

Sol.

$$\begin{aligned}3s &= 5t \\3\left(\frac{12 - a}{a}\right) &= 5\left(\frac{9 - b}{b}\right) \\\frac{36 - 3a}{a} &= \frac{45 - 5b}{b} \\36b - 3ab &= 45a - 5ab \\45a - 36b - 2ab &= 0\end{aligned} \quad (1)$$

$$\begin{aligned}\frac{m}{n} &= \frac{3}{2} \\\frac{12}{a} \cdot \frac{b}{9} &= \frac{3}{2} \\\frac{12b}{9a} &= \frac{3}{2} \\\frac{4b}{3a} &= \frac{3}{2} \\8b &= 9a \\a &= \frac{8b}{9}\end{aligned} \quad (2)$$

Substituting (2) in (1),

$$\begin{aligned}
 45 \left(\frac{8b}{9} \right) - 36b - 2 \left(\frac{8b}{9} \right) b &= 0 \\
 40b - 36b - \frac{16b^2}{9} &= 0 \\
 4b - \frac{16b^2}{9} &= 0 \\
 16b^2 - 36b &= 0 \\
 b(4b - 9) &= 0 \\
 b = 0 \quad \text{or} \quad b = \frac{9}{4} \\
 \because b > 0, b = 0 \text{ is not possible} \\
 \therefore b = \frac{9}{4} \quad \square
 \end{aligned}$$

Substituting $b = \frac{9}{4}$ in (2),

$$\begin{aligned}
 a &= \frac{8b}{9} \\
 &= \frac{8 \cdot \frac{9}{4}}{9} \\
 &= \frac{18}{9} \\
 &= 2 \quad \square
 \end{aligned}$$

(c) Hence, or otherwise, find the values of m , n , s , and t .

Sol.

Substituting $a = 2$ in $m = \frac{12}{a}$,

$$\begin{aligned}
 m &= \frac{12}{a} \\
 &= \frac{12}{2} \\
 &= 6 \quad \square
 \end{aligned}$$

Substituting $b = \frac{9}{4}$ in $n = \frac{9}{b}$,

$$\begin{aligned}
 n &= \frac{9}{b} \\
 &= \frac{9}{\frac{9}{4}} \\
 &= 4 \quad \square
 \end{aligned}$$

Substituting $a = 2$ in $s = m - 1$,

$$\begin{aligned}
 s &= m - 1 \\
 &= 6 - 1 \\
 &= 5 \quad \square
 \end{aligned}$$

Substituting $b = \frac{9}{4}$ in $t = n - 1$,

$$\begin{aligned}
 t &= n - 1 \\
 &= 4 - 1 \\
 &= 3 \quad \square
 \end{aligned}$$

2.3 The Second Derivative

18. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following.

(a) $y = 4x^3 + 7x^{-1}$

Sol.

$$\begin{aligned}
 \frac{dy}{dx} &= 12x^2 - 7x^{-2} \quad \square \\
 \frac{d^2y}{dx^2} &= 24x + 14x^{-3} \quad \square
 \end{aligned}$$

(b) $y = (2x^3 - 3)^5$

Sol.

$$\begin{aligned}
 \frac{dy}{dx} &= 5(2x^3 - 3)^4(6x^2) \\
 &= 30x^2(2x^3 - 3)^4 \quad \square \\
 \frac{d^2y}{dx^2} &= 60x(2x^3 - 3)^4 + 720x^4(2x^3 - 3)^3 \\
 &= 60x(2x^3 - 3)^3(2x^3 - 3 + 12x^3) \\
 &= 60x(2x^3 - 3)^3(14x^3 - 3) \quad \square
 \end{aligned}$$

(c) $y = \frac{4}{3}\pi x^3$

Sol.

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{4}{3}\pi(3x^2) \\
 &= 4\pi x^2 \quad \square \\
 \frac{d^2y}{dx^2} &= \frac{4}{3}\pi(6x) \\
 &= 8\pi x \quad \square
 \end{aligned}$$

(d) $y = \frac{3}{(x^2 + 1)^2}$

Sol.

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{0 - 3(2)(x^2 + 1)(2x)}{(x^2 + 1)^4} \\
 &= \frac{-12x(x^2 + 1)}{(x^2 + 1)^4} \\
 &= \frac{-12x}{(x^2 + 1)^3} \quad \square \\
 \frac{d^2y}{dx^2} &= \frac{-12(x^2 + 1)^3 + 12x(3)(x^2 + 1)^2(2x)}{(x^2 + 1)^6} \\
 &= \frac{-12(x^2 + 1)^3 + 72x^2(x^2 + 1)^2}{(x^2 + 1)^6} \\
 &= \frac{-12(x^2 + 1)^2(x^2 + 1 - 6x^2)}{(x^2 + 1)^6} \\
 &= \frac{-12(1 - 5x^2)}{(x^2 + 1)^4} \quad \square
 \end{aligned}$$

19. Given a curve $y = 4x^3 - 2x^2 + 5$. Find the first and the second derivatives for the curve y when $x = 2$.

Sol.

$$\begin{aligned}\frac{dy}{dx} &= 12x^2 - 4x \\ \frac{d^2y}{dx^2} &= 24x - 4\end{aligned}$$

Substituting $x = 2$ in the above,

$$\begin{aligned}\frac{dy}{dx} &= 12(2)^2 - 4(2) \\ &= 48 - 8 \\ &= 40 \quad \square \\ \frac{d^2y}{dx^2} &= 24(2) - 4 \\ &= 48 - 4 \\ &= 44 \quad \square\end{aligned}$$

20. Given $y = \frac{1}{x}$. Prove that $y + \frac{d^2y}{dx^2} = y^3(x^2 + 2)$.

Proof.

$$\begin{aligned}y &= x^{-1} \\ \frac{dy}{dx} &= -x^{-2} \\ \frac{d^2y}{dx^2} &= 2x^{-3} \\ &= \frac{2}{x^3} \\ L.H.S. &= y + \frac{d^2y}{dx^2} = \frac{1}{x} + \frac{2}{x^3} \\ &= \frac{x^2 + 2}{x^3} \\ R.H.S. &= y^3(x^2 + 2) = \frac{1}{x^3}(x^2 + 2) \\ &= \frac{x^2 + 2}{x^3}\end{aligned}$$

$$\therefore L.H.S. = R.H.S.$$

$$\therefore y + \frac{d^2y}{dx^2} = y^3(x^2 + 2) \quad \square$$

21. Prove that for all values, of x ,

$$\frac{d^2}{dx^2} \left(\frac{x^4}{12} - x^3 + \frac{9}{2}x^2 + 6x - 3 \right) \text{ is never negative.}$$

Proof.

$$\begin{aligned}\frac{d}{dx} &= \frac{1}{3}x^3 - 3x^2 + 9x + 6 \\ \frac{d^2}{dx^2} &= x^2 - 6x + 9 \\ &= (x - 3)^2\end{aligned}$$

$$\forall x \in \mathbb{R},$$

$$\therefore (x - 3)^2 \geq 0$$

$$\therefore \frac{d^2}{dx^2} \left(\frac{x^4}{12} - x^3 + \frac{9}{2}x^2 + 6x - 3 \right) \geq 0 \quad \square$$

22. Given $h(x) = 3x^3 + mx^2 + x - 1$. Find the value of m if $h''(1) = 10$.

Sol.

$$\begin{aligned}h'(x) &= 9x^2 + 2mx + 1 \\ h''(x) &= 18x + 2m \\ h''(1) &= 10 \\ 18 + 2m &= 10 \\ 2m &= -8 \\ m &= -4 \quad \square\end{aligned}$$

23. Given $f(x) = \frac{1}{2}x^4 + px^3 + \frac{3}{2}x^2 - 16x$. Determine the range of values for p such that the equation $f''(x) = 0$ has at least one real solution.

Sol.

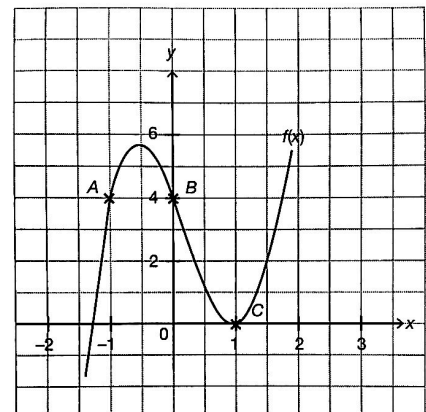
$$\begin{aligned}f'(x) &= 2x^3 + 3px^2 + 3x - 16 \\ f''(x) &= 6x^2 + 6px + 3 \\ f''(x) &= 0 \\ 6x^2 + 6px + 3 &= 0 \\ 2x^2 + 2px + 1 &= 0\end{aligned}$$

When $f''(x) = 0$ has at least one real solution,

$$\begin{aligned}4p^2 - 8 &\geq 0 \\ p^2 &\geq 2 \\ p &\leq -\sqrt{2} \text{ or } p \geq \sqrt{2} \quad \square\end{aligned}$$

2.4 Application of Differentiation

24. The following diagram shows the graph of part of the curve $f(x) = 3x^3 - 2x^2 - 5x + 4$. The points $A(-1, 4)$, $B(0, 4)$, and $C(1, 0)$ lie on the curve.



- (a) Find the gradient function of the tangent to the curve $f(x)$.

Sol.

$$f'(x) = 9x^2 - 4x - 5 \quad \square$$

- (b) i. Find the values of gradient of the tangents to the curve at points A , B , and C .

Sol.

$$\begin{aligned}
 m_A &= f'(-1) \\
 &= 9(-1)^2 - 4(-1) - 5 \\
 &= 9 + 4 - 5 \\
 &= 8 \quad \square \\
 m_B &= f'(0) \\
 &= 9(0)^2 - 4(0) - 5 \\
 &= -5 \quad \square \\
 m_C &= f'(1) \\
 &= 9(1)^2 - 4(1) - 5 \\
 &= 9 - 4 - 5 \\
 &= 0 \quad \square
 \end{aligned}$$

- ii. Hence, elaborate the situations of the tangents at points A , B , and C based on the values of the gradient obtained in (i).

Sol.

The gradient of the tangent at point A is positive, hence the tangent is rising.

The gradient of the tangent at point B is negative, hence the tangent is falling.

The gradient of the tangent at point C is zero, hence the tangent is horizontal.

25. Find the gradient of the tangent for each of the following curves at the given point P .

(a) $y = 4x - \frac{8}{x}$; $P(4, 14)$

Sol.

$$\begin{aligned}
 y' &= 4 - \frac{8}{x^2} \\
 &= 4 + \frac{8}{x^2} \\
 &= 4 + \frac{8}{(4)^2} \\
 &= 4 + \frac{1}{2} \\
 &= 4.5 \quad \square
 \end{aligned}$$

(b) $y = \frac{4 - 3x^2}{3 - 2x}$; $P(2, 8)$

Sol.

$$\begin{aligned}
 y' &= \frac{(3 - 2x)(-6x) - (4 - 3x^2)(-2)}{(3 - 2x)^2} \\
 &= \frac{-18x + 12x^2 + 8 - 6x^2}{(3 - 2x)^2} \\
 &= \frac{6x^2 - 18x + 8}{(3 - 2x)^2} \\
 &= \frac{2(3x^2 - 9x + 4)}{(3 - 2x)^2} \\
 &= \frac{2[3(2)^2 - 9(2) + 4]}{(3 - 2(2))^2} \\
 &= -4 \quad \square
 \end{aligned}$$

26. (a) Find the value of gradient of the tangent to the curve $y = 2x^3 - 3x^2$ when $x = 1$.

Sol.

$$\begin{aligned}
 \frac{dy}{dx} &= 6x^2 - 6x \\
 &= 6(1) - 6(1) \\
 &= 0 \quad \square
 \end{aligned}$$

- (b) Find the coordinates of points to the curve $y = \frac{x^3}{3} + x^2 - 1$ such that the gradient to the curve at the points is 8.

Sol.

$$\begin{aligned}
 \frac{dy}{dx} &= x^2 + 2x \\
 8 &= x^2 + 2x \\
 x^2 + 2x - 8 &= 0 \\
 (x + 4)(x - 2) &= 0 \\
 x &= -4 \text{ or } x = 2
 \end{aligned}$$

When $x = -4$,

$$\begin{aligned}
 y &= \frac{(-4)^3}{3} + (-4)^2 - 1 \\
 &= -\frac{64}{3} + 16 - 1 \\
 &= -\frac{19}{3}
 \end{aligned}$$

When $x = 2$,

$$\begin{aligned}
 y &= \frac{2^3}{3} + 2^2 - 1 \\
 &= \frac{8}{3} + 4 - 1 \\
 &= \frac{17}{3}
 \end{aligned}$$

Therefore, the coordinates of the points are $(-4, -\frac{19}{3})$ and $(2, \frac{17}{3})$. \square

- (c) Given the curve $y = ax^2 + bx + 3$ has the gradient 5 when $x = 2$ and the gradient 0 when $x = -3$. Determine the values of a and b .

Sol.

$$\begin{aligned}\frac{dy}{dx} &= 2ax + b \\ 5 &= 2a(2) + b \\ 4a + b &= 5 \quad (1) \\ 0 &= 2a(-3) + b \\ -6a + b &= 0 \quad (2) \\ (1) - (2) : 10a &= 5 \\ a &= \frac{1}{2} \quad \square \\ \text{Substituting } a = \frac{1}{2} \text{ into (1),} \\ 4\left(\frac{1}{2}\right) + b &= 5 \\ b &= 3 \quad \square\end{aligned}$$

27. Find the equations of tangent and normal to the curve $y = 8 - 2x - x^2$ at each of the following points.

Sol.

$$\frac{dy}{dx} = -2 - 2x$$

- (a) $A(1, 5)$

Sol.

At point $A(1, 5)$, the gradient of the tangent is $-2 - 2(1) = -4$.

Hence, the equation of the tangent is

$$\begin{aligned}y - 5 &= -4(x - 1) \\ y - 5 &= -4x + 4 \\ y &= -4x + 9 \quad \square\end{aligned}$$

At point $A(1, 5)$, the gradient of the normal is $\frac{1}{4}$.

Hence, the equation of the normal is

$$\begin{aligned}y - 5 &= \frac{1}{4}(x - 1) \\ 4y - 20 &= x - 1 \\ x - 4y + 19 &= 0 \quad \square\end{aligned}$$

- (b) $C(-1, 9)$

Sol.

At point $C(-1, 9)$, the gradient of the tangent is $-2 - 2(-1) = 0$.

Hence, the equation of the tangent is

$$\begin{aligned}y - 9 &= 0(x + 1) \\ y - 9 &= 0 \\ y &= 9 \quad \square\end{aligned}$$

At point $C(-1, 9)$, the gradient of the normal is *undefined*.

Hence, the equation of the normal is

$$\begin{aligned}x + 1 &= 0 \\ x &= -1 \quad \square\end{aligned}$$

28. (a) Find the equation of normal to the curve $y = 3x^2 + 8x - 7$ at point $(-2, 6)$.

Sol.

$$\frac{dy}{dx} = 6x + 8$$

The gradient of the tangent is $6(-2) + 8 = -4$.

The gradient of the normal is $\frac{1}{4}$.

Hence, the equation of the normal is

$$\begin{aligned}y - 6 &= \frac{1}{4}(x + 2) \\ 4y - 24 &= x + 2 \\ x - 4y + 26 &= 0 \quad \square\end{aligned}$$

- (b) Given the tangent to the curve $y = ax^2 + bx$ at the point $P(4, 8)$ is perpendicular to the straight line that passes through the point $A(4, 1)$ and the point $B(12, 0)$. Find the values of a and b .

Sol.

$$m_{AB} = -\frac{1}{8}$$

Since the tangent at point $P(4, 8)$ is perpendicular to the straight line AB ,

The gradient of the tangent at point $P(4, 8)$ is 8.

$$\frac{dy}{dx} = 2ax + b$$

The gradient of the tangent at point $P(4, 8)$ is $2a(4) + b = 8a + b$.

$$8a + b = 8 \quad \dots \quad (1)$$

At point $P(4, 8)$

$$\begin{aligned}a(4)^2 + b(4) &= 8 \\ 16a + 4b &= 8 \\ 4a + b &= 2 \quad \dots \quad (2)\end{aligned}$$

$$(1) - (2) : 4a = 6$$

$$a = \frac{3}{2} \quad \square$$

Substituting $a = \frac{3}{2}$ into (2),

$$\begin{aligned}b &= 2 - 4\left(\frac{3}{2}\right) \\ &= 2 - 6 \\ &= -4 \quad \square\end{aligned}$$

29. Find the coordinates of the turning points for each of the following curves. Hence, determine the nature of the turning points.

(a) $y = 5x^2 - 2x + 1$

Sol.

$$\begin{aligned}\frac{dy}{dx} &= 10x - 2 \\ 10x - 2 &= 0 \\ 10x &= 2 \\ x &= \frac{1}{5} \quad \square\end{aligned}$$

When $x = \frac{1}{5}$,

$$\begin{aligned}y &= 5\left(\frac{1}{5}\right)^2 - 2\left(\frac{1}{5}\right) + 1 \\ &= \frac{1}{5} - \frac{2}{5} + 1 \\ &= \frac{4}{5}\end{aligned}$$

Hence, the coordinates of the turning point are $\left(\frac{1}{5}, \frac{4}{5}\right)$. \square

$$\frac{d^2y}{dx^2} = 10 > 0$$

Hence, $\left(\frac{1}{5}, \frac{4}{5}\right)$ is a *maximum point*. \square

(b) $y = \frac{x^2}{x+1}$

Sol.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x+1)(2x) - x^2}{(x+1)^2} \\ &= \frac{2x^2 + 2x - x^2}{(x+1)^2} \\ &= \frac{x(x+2)}{(x+1)^2} \\ \frac{x(x+2)}{(x+1)^2} &= 0 \\ x(x+2) &= 0 \\ x &= 0 \text{ or } x = -2\end{aligned}$$

When $x = 0$,

$$\begin{aligned}y &= \frac{0^2}{0+1} \\ &= 0\end{aligned}$$

When $x = -2$,

$$\begin{aligned}y &= \frac{(-2)^2}{(-2)+1} \\ &= \frac{4}{-1} \\ &= -4\end{aligned}$$

Hence, the coordinates of the turning points are $(0, 0)$ and $(-2, -4)$. \square

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^2 + 2x}{(x+1)^2} \\ \frac{d^2y}{dx^2} &= \frac{(2x+2)(x+1)^2 - 2(x+1)(x^2+2x)}{(x+1)^4} \\ &= \frac{(x+1)[(2x+2)(x+1) - 2(x^2+2x)]}{(x+1)^4} \\ &= \frac{2x^2 + 4x + 2 - 2x^2 - 4x}{(x+1)^3} \\ &= \frac{2}{(x+1)^3}\end{aligned}$$

When $x = 0$,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{2}{(0+1)^3} \\ &= \frac{2}{1^3} \\ &= 2 > 0\end{aligned}$$

Hence, $(0, 0)$ is a *maximum point*. \square

When $x = -2$,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{2}{(-2+1)^3} \\ &= \frac{2}{(-1)^3} \\ &= -2 < 0\end{aligned}$$

Hence, $(-2, -4)$ is a *minimum point*. \square

(c) $y = 7 - x^3$

Sol.

$$\begin{aligned}\frac{dy}{dx} &= -3x^2 \\ -3x^2 &= 0 \\ x &= 0\end{aligned}$$

When $x = 0$,

$$\begin{aligned}y &= 7 - 0^3 \\ &= 7\end{aligned}$$

Hence, the coord. of the turning point is $(0, 7)$. \square

$$\frac{d^2y}{dx^2} = -6x$$

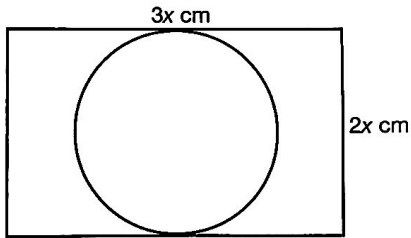
When $x = 0$,

$$\begin{aligned}\frac{d^2y}{dx^2} &= -6(0) \\ &= 0\end{aligned}$$

Hence, $(0, 7)$ is a *inflection point*. \square

30. Solve the following problems related to stationary points.

- (a) The following diagram shows the plan of a cuboid in which its centre in the shape of a cylinder is taken out. The cuboid measures $3x\text{cm} \times 2x\text{cm} \times (45 - 5x)\text{cm}$.



Find the value of x that makes the volume of the cylinder taken out a maximum.

Sol.

$$\begin{aligned} r &= x \\ V &= \pi r^2 h \\ &= \pi x^2 (45 - 5x) \\ &= 45\pi x^2 - 5\pi x^3 \end{aligned}$$

V is maximum when

$$\begin{aligned} \frac{dV}{dx} &= 0 \\ 90\pi x - 15\pi x^2 &= 0 \\ 6x - x^2 &= 0 \\ x(x - 6) &= 0 \\ x &= 0 \text{ or } x = 6 \\ x &\neq 0, x = 6 \\ x = 6, \frac{d^2V}{dx^2} &= 90\pi - 30\pi(6) \\ &= 90\pi - 180\pi \\ &= -90\pi < 0 \end{aligned}$$

Hence, $x = 6$ is the value of x that makes the volume of the cylinder taken out a maximum. \square

- (b) Given $A = bh$ where $b^2 + h^2 = 40$ and $b, h > 0$. Find the values of b and h so that A becomes a stationary point and show that the value of A is maximum.

Sol.

$$\begin{aligned} b^2 + h^2 &= 40 \\ b^2 &= 40 - h^2 \\ b &= \sqrt{40 - h^2} \\ &= (40 - h^2)^{\frac{1}{2}} \\ A &= bh \\ &= (40 - h^2)^{\frac{1}{2}} h \end{aligned}$$

A is stationary when

$$\begin{aligned} \frac{dA}{dh} &= 0 \\ -\frac{2h^2}{2\sqrt{40 - h^2}} + \sqrt{40 - h^2} &= 0 \\ -\frac{2h^2}{2\sqrt{40 - h^2}} + \sqrt{40 - h^2} &= 0 \\ \frac{-h^2 + 40 - h^2}{\sqrt{40 - h^2}} &= 0 \\ -2h^2 + 40 &= 0 \\ h^2 &= 20 \\ h &= \sqrt{20} \quad (h > 0) \quad \square \\ b &= \sqrt{40 - (\sqrt{20})^2} \\ &= \sqrt{20} \quad \square \end{aligned}$$

$$A = \sqrt{20} \cdot \sqrt{20} = 20$$

h	4	$\sqrt{20}$	5
$\frac{dA}{dh}$	1.63	0	-2.58
Tangent Sketch	/	—	\
Graph Sketch	/ — \		

Hence, $A = 20$ is maximum when $h = \sqrt{20}$. \square

- (c) A piece of wire with a length of 120cm is divided into two parts where each is bent to form an equilateral triangle with an edge of $x\text{cm}$ and a square with an edge of $y\text{cm}$ respectively. Express y in terms of x . Hence, show that the total area of both shapes, $A\text{cm}^2$ is given by

$$A = \frac{9(40 - x)^2 + 4\sqrt{3}x^2}{16}$$

Calculate the value of x so that A has a stationary value. Determine whether this value of x makes A a maximum of a minimum.

Sol.

$$\begin{aligned} C_{\text{triangle}} &= 3x \\ C_{\text{square}} &= 4y \\ 3x + 4y &= 120 \\ y &= \frac{120 - 3x}{4} \end{aligned}$$

$$\begin{aligned}
A_{\text{triangle}} &= \frac{1}{2}bh \\
&= \frac{1}{2}x\sqrt{x^2 - \left(\frac{x}{2}\right)^2} \\
&= \frac{1}{2}x\sqrt{x^2 - \frac{x^2}{4}} \\
&= \frac{1}{2}x\sqrt{\frac{3x^2}{4}} \\
&= \frac{1}{2}x\left(\frac{1}{2}x\sqrt{3}\right) \\
&= \frac{\sqrt{3}x^2}{4}
\end{aligned}$$

$$\begin{aligned}
A_{\text{square}} &= y^2 \\
&= \left(\frac{120-3x}{4}\right)^2 \\
&= \frac{(120-3x)^2}{16} \\
&= \frac{[3(40-x)]^2}{16} \\
&= \frac{9(40-x)^2}{16} \\
A &= A_{\text{square}} + A_{\text{triangle}} \\
&= \frac{9(40-x)^2}{16} + \frac{\sqrt{3}x^2}{4} \\
&= \frac{9(40-x)^2 + 4\sqrt{3}x^2}{16} \quad (\text{shown}) \quad \square
\end{aligned}$$

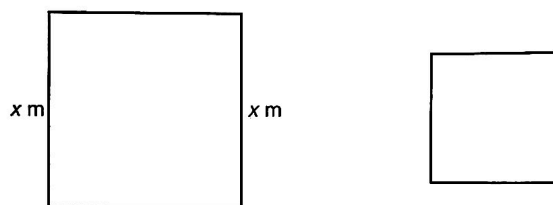
A is stationary when $\frac{dA}{dx} = 0$.

$$\begin{aligned}
\frac{dA}{dx} &= \frac{d}{dx} \left[\frac{9(40-x)^2 + 4\sqrt{3}x^2}{16} \right] \\
&= \frac{d}{dx} \left[\frac{9(40-x)^2}{16} + \frac{\sqrt{3}x^2}{4} \right] \\
&= \frac{-18(40-x)}{16} + \frac{2\sqrt{3}x}{4} \\
&= -\frac{9(40-x)}{8} + \frac{4\sqrt{3}x}{8} \\
&= -\frac{9(40-x) - 4\sqrt{3}x}{8} \\
-\frac{9(40-x) - 4\sqrt{3}x}{8} &= 0 \\
360 - 9x - 4\sqrt{3}x &= 0 \\
9x + 4\sqrt{3}x &= 360 \\
(9 + 4\sqrt{3})x &= 360 \\
x &= \frac{360}{9 + 4\sqrt{3}} \\
&\approx 22.6\text{cm} \quad \square
\end{aligned}$$

$$\frac{d^2A}{dx^2} = \frac{9 + 4\sqrt{3}}{8} > 0$$

Hence, A is a minimum when $x = 22.6\text{cm}$. \square

31. Chan wants to build two separate pens by using a fence of 100m . Both pens are square in shape.

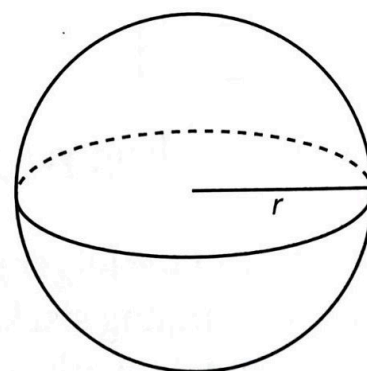


If the edge of the larger pen is $x\text{m}$,

- find the length of the side of the smaller pen in terms of x .
- find the value of x such that the total area of both pens is minimum.

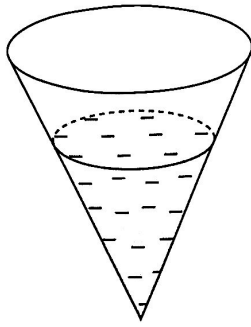
32. Solve the following problems related to the rates of change.

- The total surface area, $A\text{cm}^2$, of a metal solid which consists of a cone and a cylinder with a common radius, $r\text{cm}$ is given by $A = 2\pi\left(\frac{18}{r} + \frac{r^2}{3}\right)$. When it is heated, its total surface area changes at the rate of $2.1\pi\text{cm}^2\text{s}^{-1}$. Find the rate of change of the radius, in cm s^{-1} , at the instant $r = 6\text{cm}$.
- A spherical balloon experiences a constant rate of increase of $6\text{cm}^2\text{s}^{-1}$.



At the instant when the radius is 5cm , find

- the rate of increase, in cm s^{-1} , of the radius.
 - the rate of increase of volume, in cm^3s^{-1} , of the sphere.
- (c) The following diagram shows a container in the shape of a cone. Given its height is equal to its base radius. Water is poured into the container at the rate of $80\text{cm}^3\text{s}^{-1}$. The volume of the water in the container is $\frac{1}{3}\pi x^3\text{cm}^3$, when the depth of the water is $x\text{cm}$.



Calculate, at the instant when the depth of the water is 10cm ,

- i. the rate of increase of the depth, in cms^{-1} , of the water.
- ii. the rate of increase of the horizontal surface area, in cm^2s^{-1} , of the water.

33. Solve the following problems related to the small changes and approximations.

(a) Given that $y = 2x^3 - 5x^2 + x - 1$, find the value of $\frac{dy}{dx}$ when $x = 1$. Hence, find the small changes in y when x increases from 1 to 1.02.

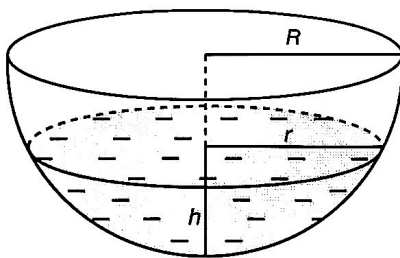
(b) Given the equation of a curve is $y = \frac{9}{(2x - 5)^2}$, find, in terms of p , where p is a small value, the approximate change in

- i. y when x increases from 3 to $3 + p$.
- ii. x when y decreases from 1 to $1 - p$.

(c) Given $y = x^4$, by using the calculus method, find the approximate value of

- i. 2.03^4 .
- ii. 1.99^4 .

34. A hemispherical bowl of radius $R\text{cm}$ is filled with water to a depth of $h\text{cm}$.



The volume of the water in the bowl is given by $V = \frac{\pi}{3}(3Rh^2 - h^3)$.

- (a) Show that the radius of the water surface, r , is given by $r = \sqrt{2Rh - h^2}$.
- (b) Water is poured into the bown at a constand rate of $300\text{cm}^3\text{s}^{-1}$. Find, in terms of R , the rate of increase of the surface area, in $\text{cm}^2\text{min}^{-1}$, of the water when $2h = R$.