Mathematics

Senior 3 Part I

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Introduction

Why this book?

Disclaimer

Acknowledgements

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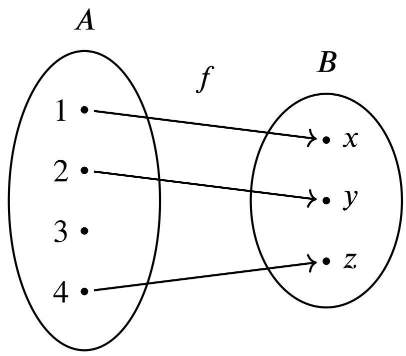
Chapter 22

Function

22.1 Definition of a Function

Mapping, Preimage and Image

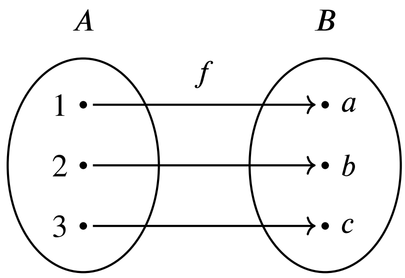
For two non-empty sets A and B, If an element a inside set A has a corresponding element b inside set B, denoted as $a \to b$, then we say that a is mapped to b or a and b are paired. The mapping between two sets is normally denoted as f, g, h, etc. The mapping shown in the diagram below can be denoted as $f : 1 \to x,\ 2 \to y,\ 4 \to z$.



Let $f : A \to B$ is a mapping, a is an element in A. If a is mapped to b under the mapping f, then b is said to be the image of a under the mapping f, denoted as $b = f(a)$; a is said to be the preimage of b under the mapping f. In the diagram above, under the mapping f, the image of 1, 2, and 4 are x, y, and z respectively, while the preimage of x, y, and z are 1, 2, and 4 respectively.

Let A and B be two non-empty sets, f is a mapping from A to B such that for all elements in A, there is a unique corresponding element in B, then f is a function or a mapping from A to B, denoted as $f : A \to B$.

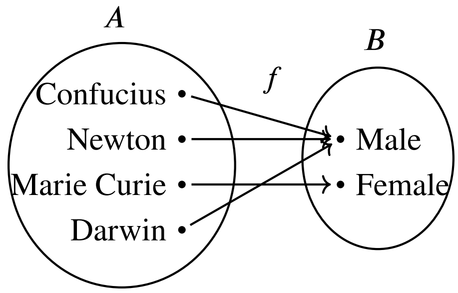
The mapping shown in the diagram below is a function.



Practice 1

1. For the following mappings, list the image of each element in A and the preimage of each element in B, and determine whether the mapping is a function or not:

(a)



Sol.

The image of each element in A:

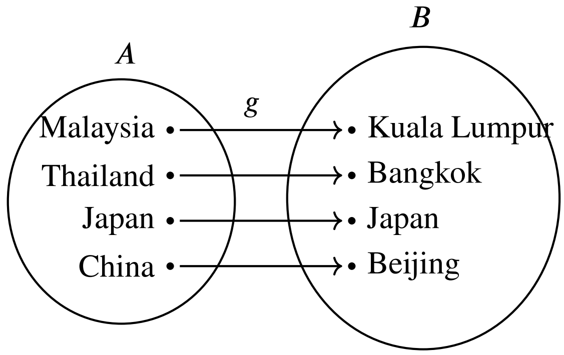
- Confucius → Male
- Newton → Male
- Marie Curie → Female
- Darwin → Male

The preimage of each element in B:

- Male → {Confucius, Newton, Darwin}
- Female → Marie Curie

Since each element in A has a unique corresponding element in B, the mapping f is a function.

(b)



Sol.

The image of each element in A:

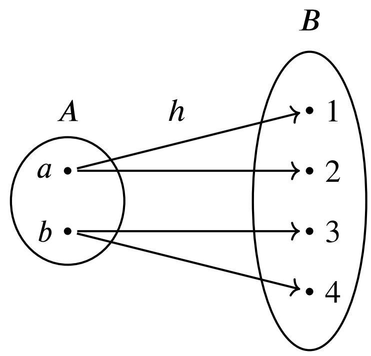
- Malaysia → Kuala Lumpur
- Thailand → Bangkok
- Japan → Japan
- China → Beijing

The preimage of each element in B:

- Kuala Lumpur → Malaysia
- Bangkok → Thailand
- Japan → Japan
- Beijing → China

Since each element in A has a unique corresponding element in B, the mapping g is a function.

(c)



Sol.

The image of each element in A:

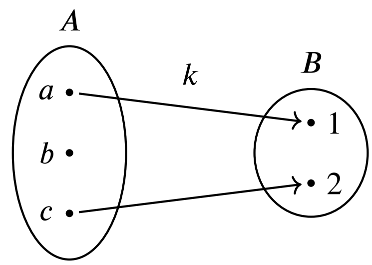
- $a \to \{1, 2\}$
- $b \to \{3, 4\}$

The preimage of each element in B:

- $1 \to a$
- $2 \to a$
- $3 \to b$
- $4 \to b$

Since $a \in A$ has two corresponding elements 1 and 2 in B, $b \in A$ has two corresponding elements 3 and 4 in B, the mapping h is not a function.

(d)



Sol.

The image of each element in A:

- $a \to \{1\}$
- $b \to \emptyset$
- $c \to \{2\}$

The preimage of each element in B:

- $1 \to a$
- $2 \to c$

Since $b \in A$ has no corresponding element in B, the mapping k is not a function.

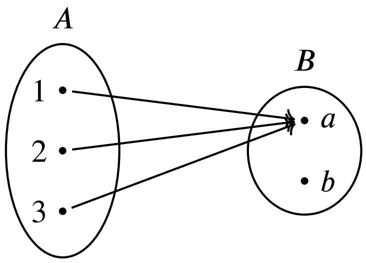
2. Given a mapping $g : x \to x + 3$, $x \in \{$-2, -1, 0, 1, 2, 3$\}$, find the image of each x.

Sol.

$$g(-2) = -2 + 3 = 1$$
$$g(-1) = -1 + 3 = 2$$
$$g(0) = 0 + 3 = 3$$
$$g(1) = 1 + 3 = 4$$
$$g(2) = 2 + 3 = 5$$
$$g(3) = 3 + 3 = 6$$

3. Determine whether the following mappings are functions.

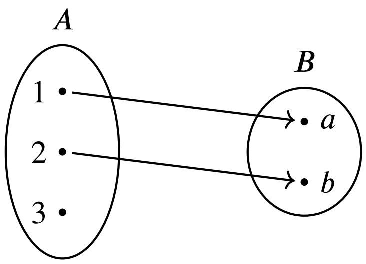
(a)



Sol.

Since each element in A has a corresponding element in B, the mapping is a function.

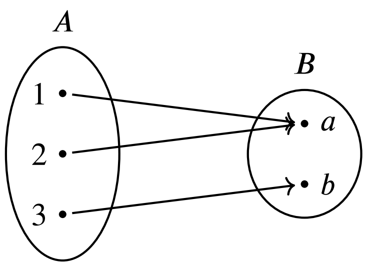
(b)



Sol.

Since $3 \in A$ has no corresponding element in B, the mapping is not a function.

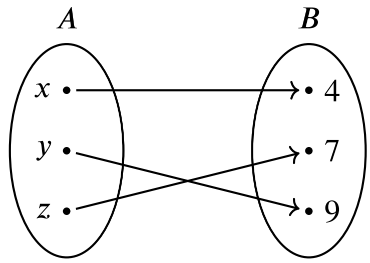
(c)



Sol.

Since each element in A has a corresponding element in B, the mapping is a function.

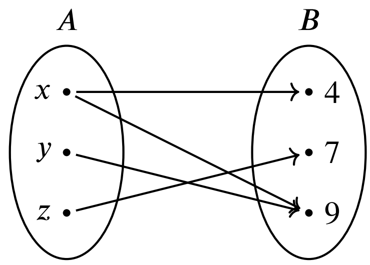
(d)



Sol.

Since each element in A has a corresponding element in B, the mapping is a function.

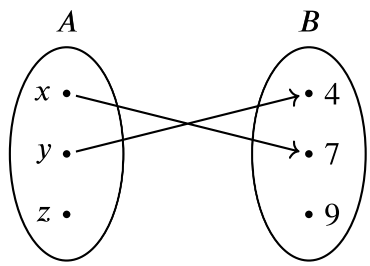
(e)



Sol.

Since $x \in A$ has two corresponding elements 4 and 9 in B, the mapping is not a function.

(f)



Sol.

Since $z \in A$ has no corresponding element in B, the mapping is not a function.

The function $f : A \to B$ can be written as $y = f(x)$, x is the element of A and y is the element of B. When x changes, y changes as well. x is called independent variable, while y is called dependent variable. Keep in mind that $f(x)$ is NOT the product of f and x.

Representation of Functions

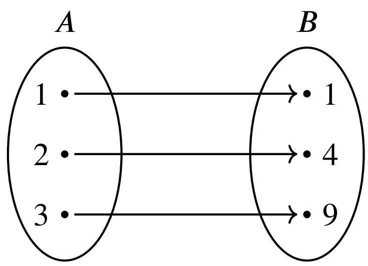
Generally speaking, there are a few ways to represent a function:

1. **Narrative Form**: express the function of two sets in words. For example, Let $A = \{1, 2, 3\}$ and $B = \{1, 4, 9\}$, f is a function from A to B, its definition is that for any element x in A, its corresponding element is x^2 in B.

2. **Arrow Method**: draw an arrow to connect the preimage and image of a function such that the preimage is corresponding to the image. To express the example above, we express it as $f\ :\ 1 \to 1,\ 2 \to 4,\ 3 \to 9$.

3. **Analytical Method**: express the function in the form of mathematical expression to represent the relationship between the independent variable and the dependent variable. For example, $f(x) = x^2, x \in A$.

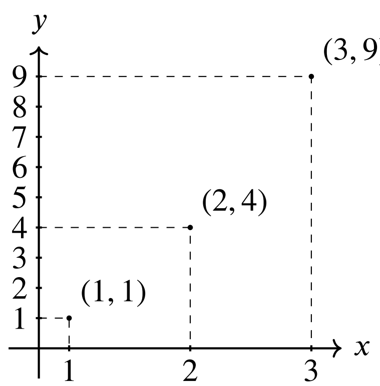
4. **Venn Diagram**: draw arrows between the venn diagram of two sets to represent the function, as shown below:



5. **Table Method**: express the function in the form of table, showing the relationship of the chosen value between independent variable x and the value of its corresponding dependent variable y, as shown below:

x	1	2	3
y	1	4	9

6. **Graphical Method**: draw a graph to represent the function of the two variables, as shown below:



Practice 2

Express the following functions using analytical method, venn diagram, table method and graphical method.

(a) f mapping each integers from -3 to 3 to its squares plus 4.

Sol.

Analytical method: $f(x) = x^2 + 4$, $-3 \le x \le 3$, $x \in \mathbb{Z}$.

Venn diagram:

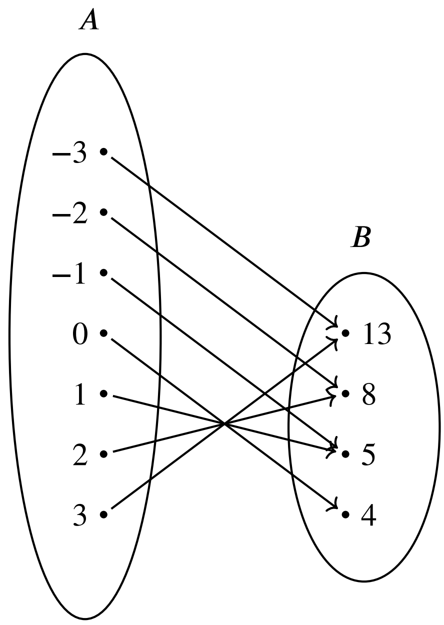
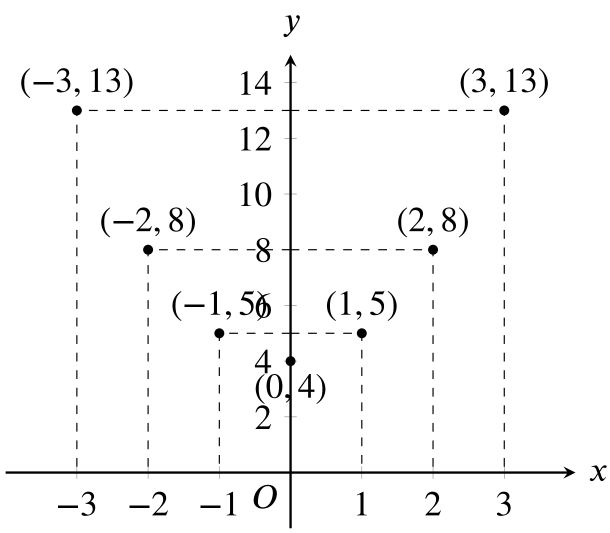


Table method:

x	-3	-2	-1	0	1	2	3
y	13	8	5	4	5	8	13

Graphical method:



(b) g mapping each natural numbers from 1 to 4 to its cubes.

Sol.

Analytical method: $g(x) = x^3$, $1 \leq x \leq 4$, $x \in \mathbb{N}$.

Venn diagram:

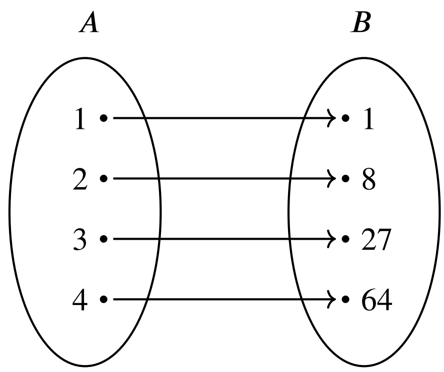
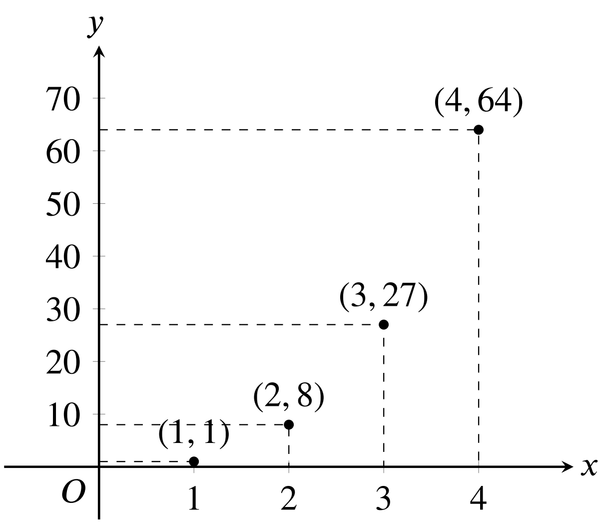


Table method:

x	1	2	3	4
y	1	8	27	64

Graphical method:



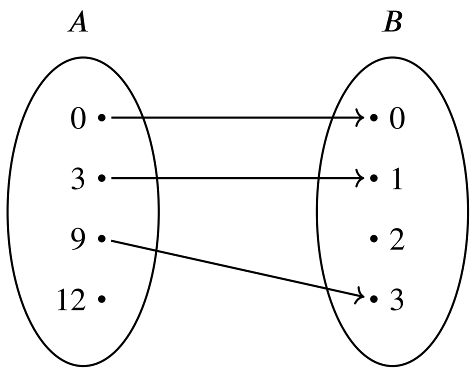
Exercise 22.1

1. Express the mapping from set A to set B using venn diagram, and determine which of the following mappings are functions.

	Set A	Set B	Mapping
(a)	{0, 3, 9, 12}	{0, 1, 2, 3}	Divide by 3
(b)	{-2, -1, 0, 1, 2}	{0, 1, 4, 9, 16}	Power of 4
(c)	{-2, -1, 0, 1, 2}	{0, 1, 4}	Square
(d)	{30°, 45°, 60°}	$\left\{\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right\}$	Sine
(e)	{-1, 0, 1, 2}	{-1, 0, 1}	Cube

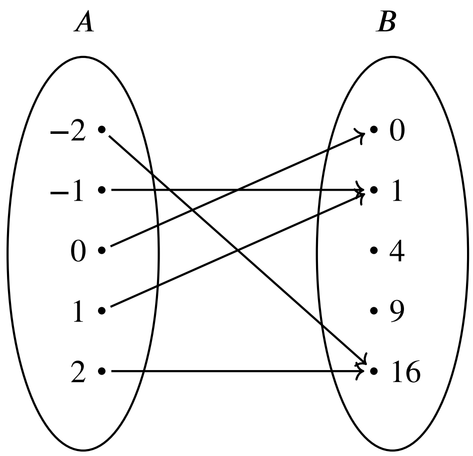
Sol.

(a)



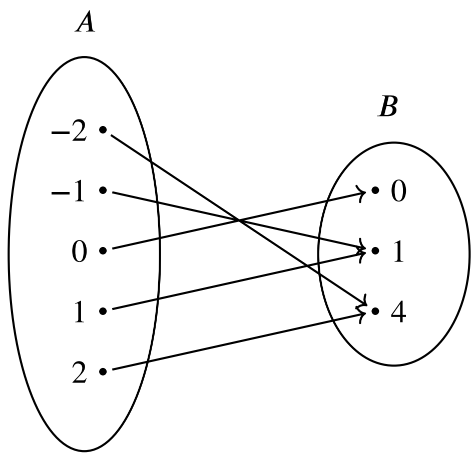
Since $12 \in A$ has no image in B, this mapping is not a function.

(b)



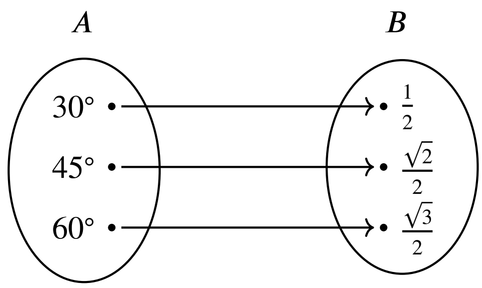
Since each element in A has an image in B, this mapping is a function.

(c)



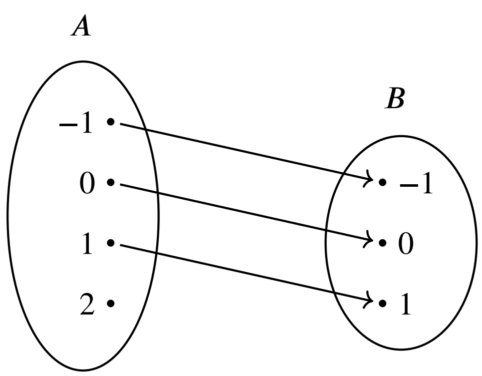
Since each element in A has an image in B, this mapping is a function.

(d)



Since each element in A has an image in B, this mapping is a function.

(e)



Since $2 \in A$ does not have an image in B, this mapping is not a function.

2. Let function $f(x) = 3x^2 + 1$.

   (a) Find the image of the following elements:

   i. -3
   Sol.

   $$f(-3) = 3(-3)^2 + 1$$
   $$= 28$$

   ii. -2
   Sol.

   $$f(-2) = 3(-2)^2 + 1$$
   $$= 13$$

   iii. 0
   Sol.

   $$f(0) = 3(0)^2 + 1$$
   $$= 1$$

   iv. 2
   Sol.

   $$f(2) = 3(2)^2 + 1$$
   $$= 13$$

   v. 5
   Sol.

   $$f(5) = 3(5)^2 + 1$$
   $$= 76$$

   (b) Find the preimage of the following elements:

   i. 13
   Sol.

   $$13 = 3x^2 + 1$$
   $$12 = 3x^2$$
   $$x = \pm 2$$

   ii. 28
   Sol.

   $$28 = 3x^2 + 1$$
   $$27 = 3x^2$$
   $$x = \pm 3$$

   iii. 1
   Sol.

   $$1 = 3x^2 + 1$$
   $$0 = 3x^2$$
   $$x = 0$$

   iv. 0
   Sol.

   $$0 = 3x^2 + 1$$
   $$-\frac{1}{3} = x^2$$
   x is not a real no.

   v. 4
   Sol.

   $$4 = 3x^2 + 1$$
   $$1 = x^2$$
   $$x = \pm 1$$

3. Let function $g(x) = 5x - 2$. Find:

(a) $g(-2)$

Sol.

$$\begin{aligned} g(-2) &= 5(-2) - 2 \\ &= -12 \end{aligned}$$

(b) $g(-1)$

Sol.

$$\begin{aligned} g(-1) &= 5(-1) - 2 \\ &= -7 \end{aligned}$$

(c) $g(0)$

Sol.

$$\begin{aligned} g(0) &= 5(0) - 2 \\ &= -2 \end{aligned}$$

4. Let function $f(x) = \begin{cases} 2x, & x \leq -1 \\ x - 1, & -1 \leq x < 3 \\ 4x + 2, & x \geq 3 \end{cases}$, find

(a) $f(-5)$

Sol.

$$\begin{aligned} f(-5) &= 2(-5) \\ &= -10 \end{aligned}$$

(b) $f(-2)$

Sol.

$$\begin{aligned} f(-2) &= 2(-2) \\ &= -4 \end{aligned}$$

(c) $f(0)$

Sol.

$$\begin{aligned} f(0) &= 0 - 1 \\ &= -1 \end{aligned}$$

(d) $f(2)$

Sol.

$$\begin{aligned} f(2) &= 2 - 1 \\ &= 1 \end{aligned}$$

(e) $f(10)$

Sol.

$$\begin{aligned} f(10) &= 4(10) + 2 \\ &= 42 \end{aligned}$$

5. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$. Find the image of -1, 0, 1, and 2 under f.

Sol.

$$\begin{aligned} f(-1) &= (-1)^4 = 1 \\ f(0) &= (0)^4 = 0 \\ f(1) &= (1)^4 = 1 \\ f(2) &= (2)^4 = 16 \end{aligned}$$

6. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Find the preimage of 0, 1, and 4 under f.

   In $\mathbb{R}$, which element does not have a preimage?

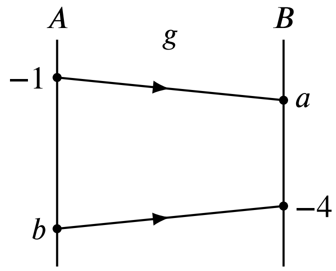
   Sol.

   $$\begin{aligned} 0 &= x^4 \\ x &= 0 \end{aligned} \qquad \begin{aligned} 1 &= x^4 \\ x &= \pm 1 \end{aligned} \qquad \begin{aligned} 4 &= x^4 \\ x &= \pm 2 \end{aligned}$$

   $\because \forall x \in \mathbb{R}, f(x) \geq 0$

   $\therefore x \in \mathbb{R}^-$ does not have a preimage.

7. In the diagram below, given that function $g : A \to B$ is defined as $g : x \to 2x - 8$. Find the value of a and b.



   Sol.

   $$\begin{aligned} a &= 2(-1) - 8 \\ &= -10 \end{aligned} \qquad\qquad \begin{aligned} -4 &= 2b - 8 \\ 2b &= 4 \\ b &= 2 \end{aligned}$$

8. Using narrative form, arrow method, venn diagram, table method and graphical method, express the function $f(x) = 2x$, $x \in \{-2, -1, 0, 1, 2\}$.

Sol.

Narrative form:

Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{-4, -2, 0, 2, 4\}$, f is a function from A to B, its definition is that for any element x in A, its corresponding element is $2x$ in B.

Arrow method:

$f: -2 \to -4, -1 \to -2, 0 \to 0, 1 \to 2, 2 \to 4$

Venn diagram:

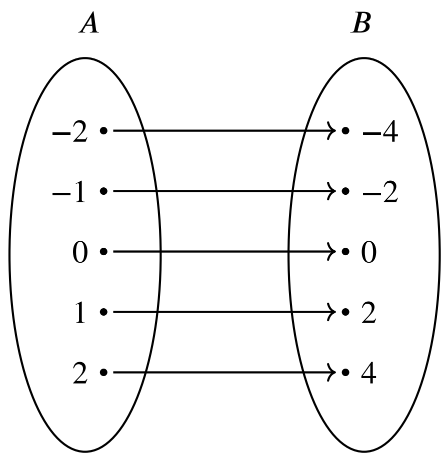
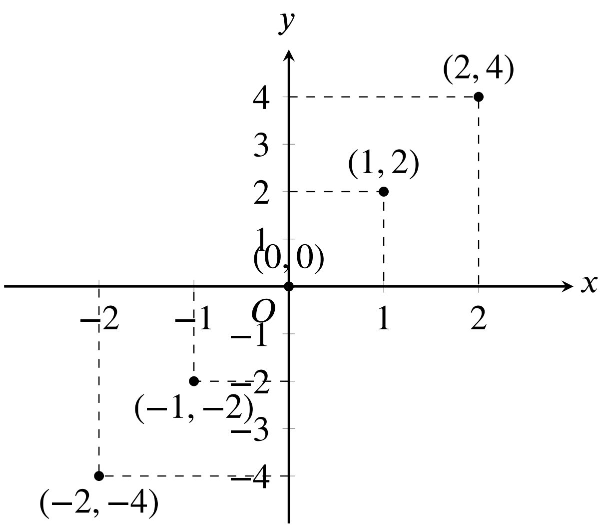


Table method:

x	-2	-1	0	1	2
$f(x)$	-4	-2	0	2	4

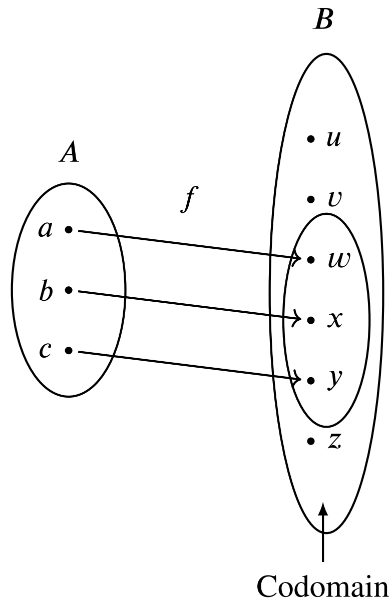
Graphical method:



22.2 Domain and Range

Let f is a function from set A to set B, then set A is called the domain of f, denoted by D_f; set B is called the codomain of f; the set of the images of all elements of A under f is called the range of f, denoted by R_f.

If the domain A and range B of function $f : A \to B$ are both subsets of real number set $\mathbb{R}$, then this function is called real valued function / real function. This book primarily discusses about real valued functions. When the domain of a real function is not mentioned and only the mapping rule is given, its domain is assumed to be the set of all real numbers that yield defined values $f(x)$. After the domain and the mapping rule are determined, the range of a function will then be determined.



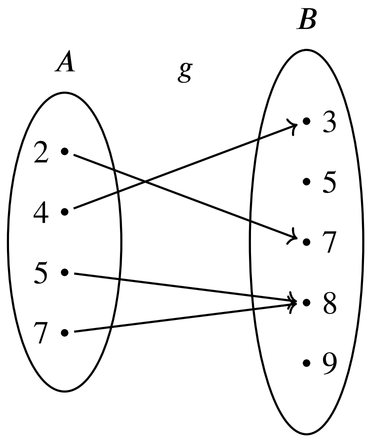
Interval Notation

Let a and b be two real number, $a < b$.

Intervals	Set Notations
(a, b)	$\{x \mid x \in \mathbb{R}, a < x < b\}$
$[a, b)$	$\{x \mid x \in \mathbb{R}, a \leq x < b\}$
$(a, b]$	$\{x \mid x \in \mathbb{R}, a < x \leq b\}$
$[a, b]$	$\{x \mid x \in \mathbb{R}, a \leq x \leq b\}$
(a, ∞)	$\{x \mid x \in \mathbb{R}, x > a\}$
$[a, \infty)$	$\{x \mid x \in \mathbb{R}, x \leq a\}$
$(-\infty, a)$	$\{x \mid x \in \mathbb{R}, x < a\}$
$(-\infty, a]$	$\{x \mid x \in \mathbb{R}, x \leq a\}$

Practice 3

1. Let $A = \{2, 4, 5, 7\}$ and $B = \{3, 5, 7, 8, 9\}$, the definition of function g is given by the diagram below. Find the domain, codomain and range of function g.



Sol.

$D_g = \{2, 4, 5, 7\}$, $R_g = \{3, 5, 8\}$

Codomain: $\{3, 5, 7, 8, 9\}$

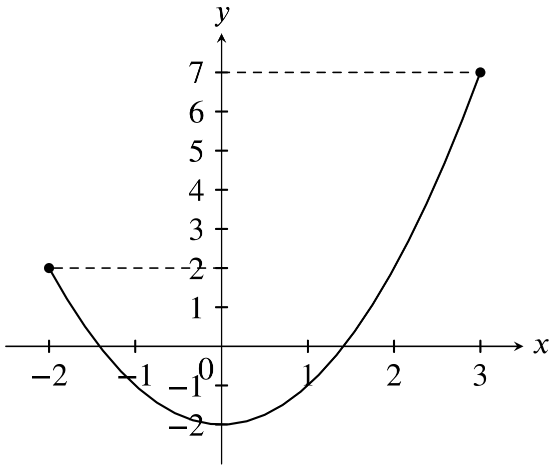
2. Let $A = \{-2, -1, 0, 1, 2\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = x^2 - 1$. Find the domain and range of f.

Sol.

$$\begin{aligned} f(-2) &= (-2)^2 - 1 = 3 \\ f(-1) &= (-1)^2 - 1 = 0 \\ f(0) &= (0)^2 - 1 = -1 \\ f(1) &= (1)^2 - 1 = 0 \\ f(2) &= (2)^2 - 1 = 3 \end{aligned}$$

$D_f = \{-2, -1, 0, 1, 2\}$, $R_f = \{-1, 0, 3\}$

3. The curve in the diagram below represents the function $y = f(x)$, $-2 \le x \le 3$. Find the domain and range of f.



Sol.

$D_f = \{x | x \in \mathbb{R}, -2 \le x \le 3\}$, $R_f = \{y | y \in \mathbb{R}, -2 \le y \le 7\}$

4. Find the domain and range of the following functions:

(a) $f(x) = -4x + 5$

Sol.

$D_f = \mathbb{R}$, $R_f = \mathbb{R}$

(b) $g(x) = x^2 - 1$

Sol.

$D_g = \mathbb{R}$, $R_g = [-1, \infty)$

(c) $h(x) = \dfrac{1}{4x+7}$

Sol.

$\because \dfrac{1}{4x+7}$ is defined only when $x \neq -\dfrac{7}{4}$

$\therefore D_h = \left\{ x | x \in \mathbb{R}, x \neq -\dfrac{7}{4} \right\}$

$\because \dfrac{1}{4x+7} \neq 0$

$\therefore R_h = \{y | y \in \mathbb{R}, y \neq 0\}$

(d) $k(x) = \sqrt{6-x}$

Sol.

$\because \sqrt{6-x}$ is defined only when $6 - x \ge 0$

$\therefore D_k = (-\infty, 6]$

$\because \sqrt{6-x} \ge 0$

$\therefore R_k = [0, \infty)$

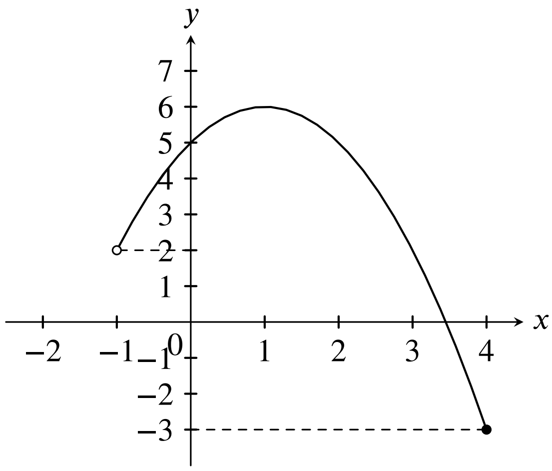
Exercise 22.2

1. Let $X = \{a, b, c, d\}$ and $Y = \{-1, 2, 9, 11\}$, function $f : X \to Y$ is defined by $f(a) = 2$, $f(b) = -1$, $f(c) = 2$, $f(d) = 9$. Find the domain and range of the f.

   Sol.

   $D_f = \{a, b, c, d\}$, $R_f = \{-1, 2, 9\}$.

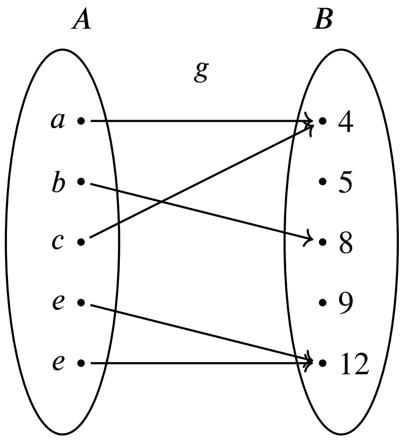
2. The curve in the diagram below represents the function $y = f(x)$, $-1 < x \leq 4$. Find the domain and range of f.



   Sol.

   $D_f = (-1, 4]$, $R_f = [-3, 6]$.

3. Let $A = \{a, b, c, d, e\}$ and $B = \{4, 5, 8, 9, 12\}$, the definition of function $g : A \to B$ is given by the digram below. Find the domain, codomain and range of function g.



   Sol.

   $D_g = \{a, b, c, d, e\}$, $R_g = \{4, 8, 12\}$

   Codomain $= \{4, 5, 8, 9, 12\}$

4. Let $A = \{-1, 0, 1, 2\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = 3x^2 - 2$, find the domain and range of f.

Sol.

$$f(-1) = 3(-1)^2 - 2 = 1 \qquad f(1) = 3(1)^2 - 2 = 1$$

$$f(0) = 3(0)^2 - 2 = -2 \qquad f(2) = 3(2)^2 - 2 = 10$$

$D_f = \{-1, 0, 1, 2\}$, $R_f = \{-2, 1, 10\}$

5. Let $A = \{-1, 0, 2, 5, 11\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = x^2 - x - 2$, find the domain and range of f.

Sol.

$$f(-1) = (-1)^2 - (-1) - 2 = 0 \qquad f(5) = 5^2 - 5 - 2 = 18$$

$$f(0) = 0^2 - 0 - 2 = -2 \qquad f(11) = 11^2 - 11 - 2 = 118$$

$$f(2) = 2^2 - 2 - 2 = 0$$

$D_f = \{-1, 0, 2, 5, 11\}$, $R_f = \{-2, 0, 18, 118\}$.

6. Find the domain and range of the following functions:

(a) $f(x) = x^3$

Sol.

$D_f = \mathbb{R}$, $R_f = \mathbb{R}$.

(b) $g(x) = \sqrt{1 - x^2}$

Sol.

$\because \sqrt{1 - x^2}$ is defined only when

$1 - x^2 \geq 0$

$\therefore D_g = [-1, 1]$

$\because \sqrt{1 - x^2} \geq 0$, the maximum value of

$g(x)$ is 1 when $x = 0$

$\therefore R_g = [0, 1]$

(c) $h(x) = \dfrac{1}{2x + 3}$

Sol.

$\because \dfrac{1}{2x + 3}$ is defined only when $2x + 3 \neq 0$

$\therefore D_h = \left\{ x | x \in \mathbb{R}, x \neq -\dfrac{3}{2} \right\}$

$\because \dfrac{1}{2x + 3} \neq 0$

$\therefore R_h = \{ y | y \in \mathbb{R}, y \neq 0 \}$

(d) $k(x) = x^2 - 2x + 4$

Sol.

$x^2 - 2x + 4 = (x - 1)^2 + 3$

$\because \min_{k(x)} = 3$

$\therefore D_k = \mathbb{R}$, $R_k = [3, \infty)$

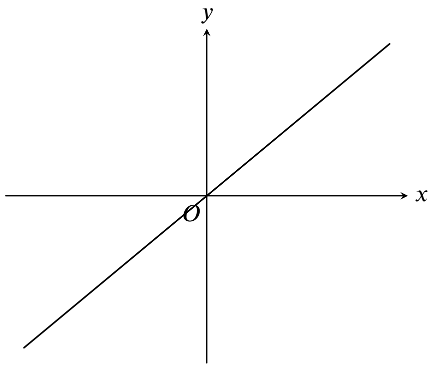
22.3 Graphs of Functions and Their Transformations

Graphs of Simple Functions

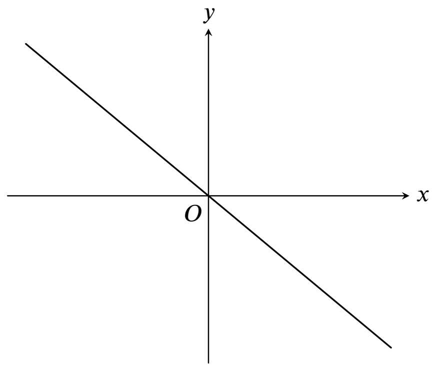
On a Cartesian plane, the graphs formed by all the point (x, y) that satisfied the equation $y = f(x)$ are called graphs of function f. Below are some examples of graphs of simple functions.

Note that any line that is parallel to the y-axis intersects the graph of a function at most once.

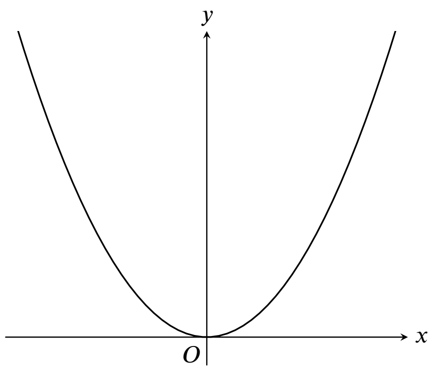
(a) $y = x$



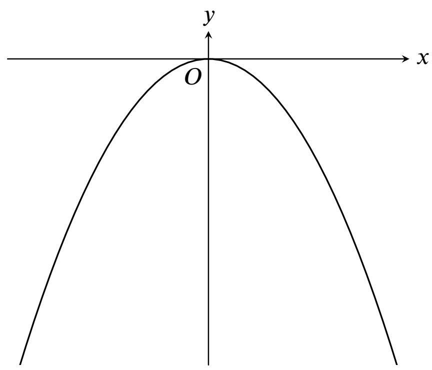
(b) $y = -x$



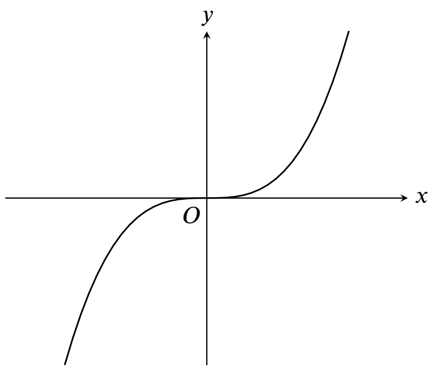
(c) $y = x^2$



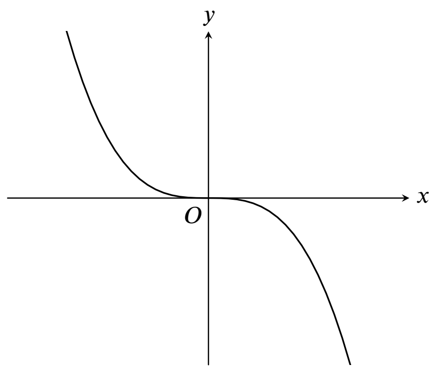
(d) $y = x^2$



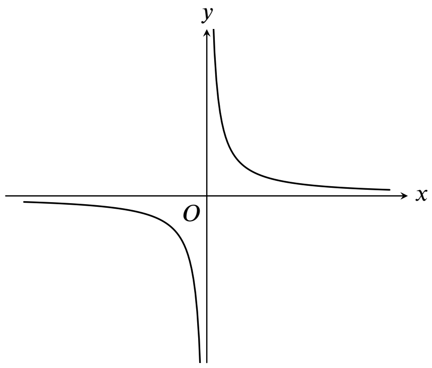
(e) $y = x^3$



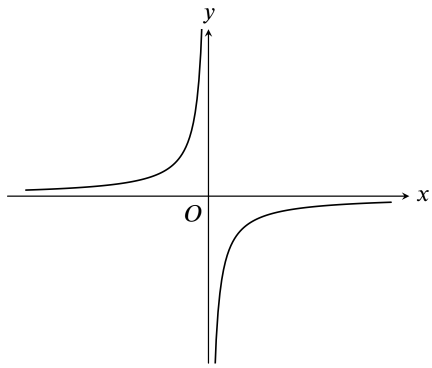
(f) $y = -x^3$



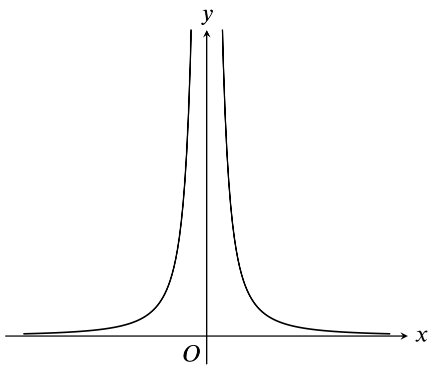
(g) $y = \frac{1}{x}$



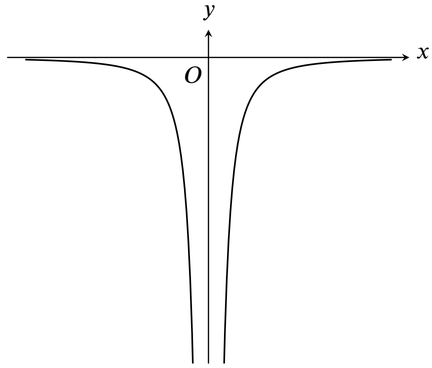
(h) $y = -\frac{1}{x}$



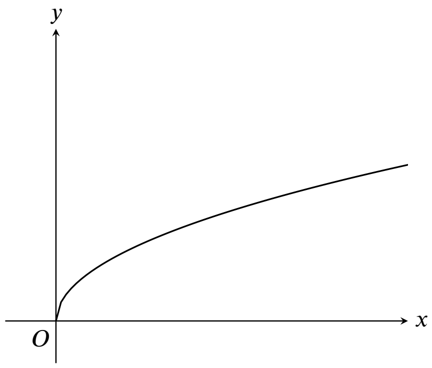
(i) $y = \frac{1}{x^2}$



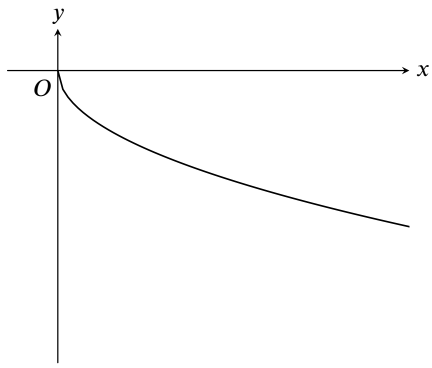
(j) $y = -\frac{1}{x^2}$



(k) $y = \sqrt{x}$

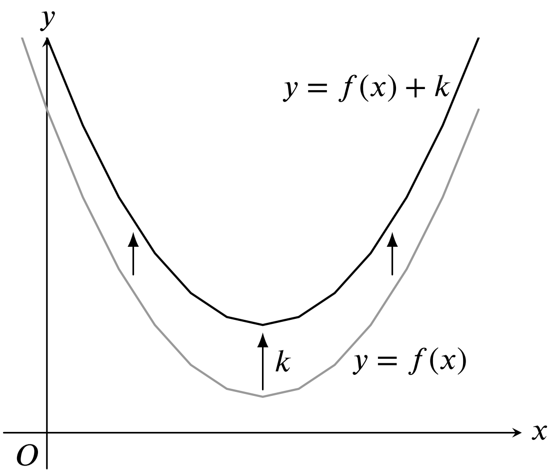


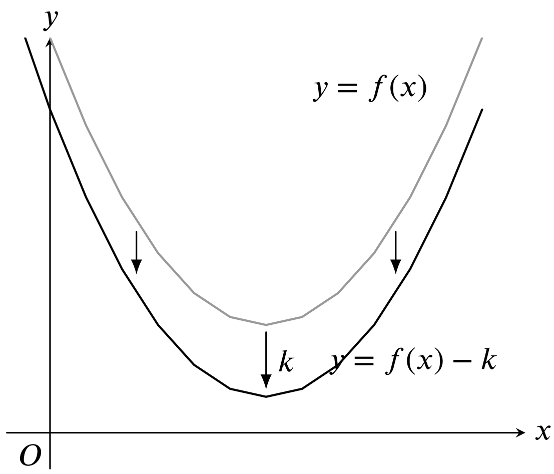
(l) $y = -\sqrt{x}$



Transformations of Graphs

- If $k > 0$, translate the graph of $y = f(x)$ vertically upwards by k units, the graph of $y = f(x) + k$ is obtained.
- If $k > 0$, translate the graph of $y = f(x)$ vertically downwards by k units, the graph of $y = f(x) - k$ is obtained.





- If $h > 0$, translate the graph of $y = f(x)$ horizontally to the right by h units, the graph of $y = f(x+h)$ is obtained.
- If $h > 0$, translate the graph of $y = f(x)$ horizontally to the left by h units, the graph of $y = f(x-h)$ is obtained.

- If $k > 0$, reflect the graph of $y = f(x)$ about the x-axis, the graph of $y = -f(x)$ is obtained.
- If $k > 0$, reflect the graph of $y = f(x)$ about the y-axis, the graph of $y = f(-x)$ is obtained.

If $a > 0$, zooming (when $a > 1$) or shrinking (when $0 < a < 1$) the graph of $y = f(x)$ by a factor of

a in the y-direction, the graph of $y = af(x)$ is obtained.

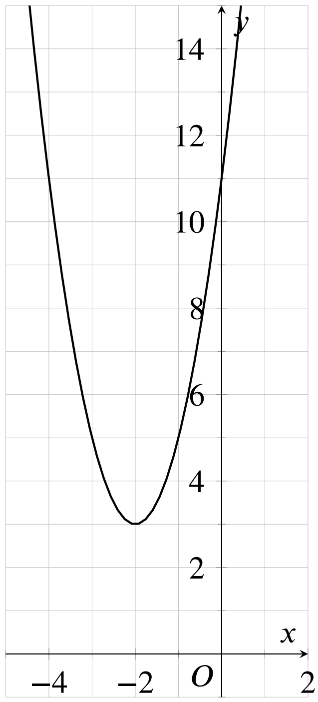
If $a > 0$, shrinking (when $a > 1$) or zooming (when $0 < a < 1$) the graph of $y = f(x)$ by a factor of $\frac{1}{a}$ in the x-direction, the graph of $y = f(ax)$ is obtained.

Practice 4

Find the line of symmetry and vertex of the following parabola, and sketch its graph. (Question 1 to 2):

1. $y = 2x^2 + 8x + 11$

Sol.

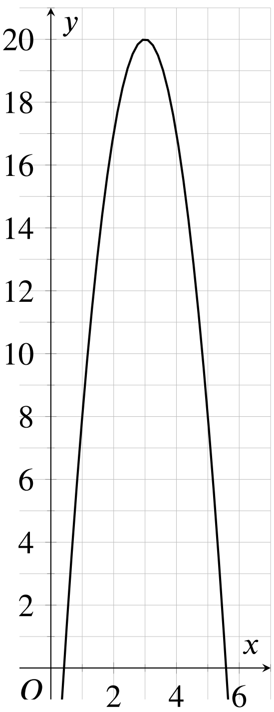


$$\begin{aligned} y &= 2x^2 + 8x + 11 \\ &= 2(x^2 + 4x) + 11 \\ &= 2[(x^2 + 4x + 4) - 4] + 11 \\ &= 2[(x+2)^2 - 4] + 11 \\ &= 2(x+2)^2 - 8 + 11 \\ &= 2(x+2)^2 + 3 \end{aligned}$$

Vertex: $(-2, 3)$
Line of symmetry: $x = -2$

2. $y = -3x^2 + 18x - 7$

Sol.



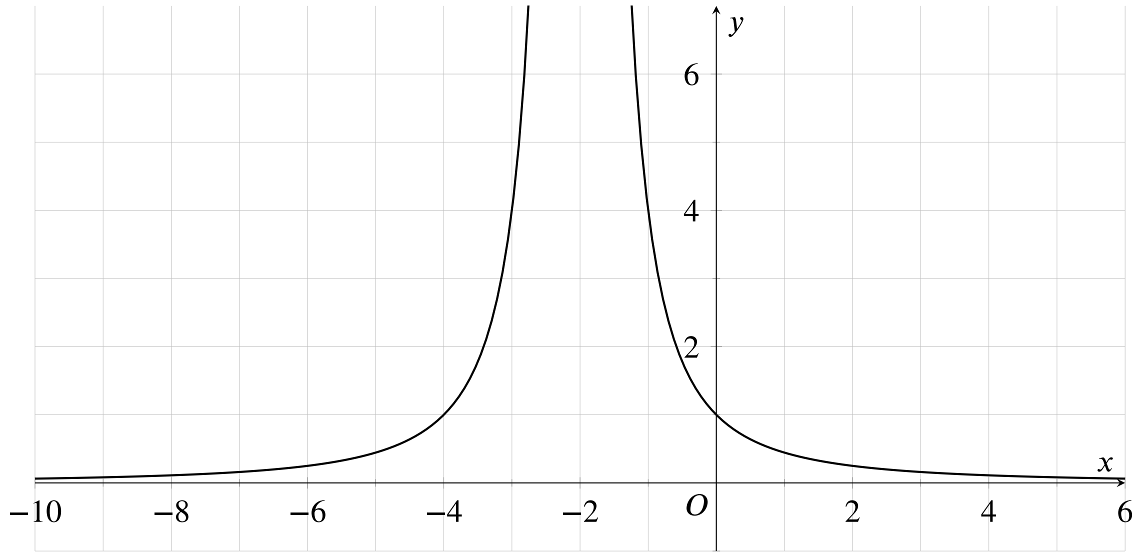
$$\begin{aligned} y &= -3x^2 + 18x - 7 \\ &= -3(x^2 - 6x) - 7 \\ &= -3[(x^2 - 6x + 9 - 9)] - 7 \\ &= -3[(x-3)^2 - 9] - 7 \\ &= -3(x-3)^2 + 27 - 7 \\ &= -3(x-3)^2 + 20 \end{aligned}$$

Vertex: $(3, 20)$
Line of symmetry: $x = 3$

Sketch the graph of the following functions. (Question 3 to 4):

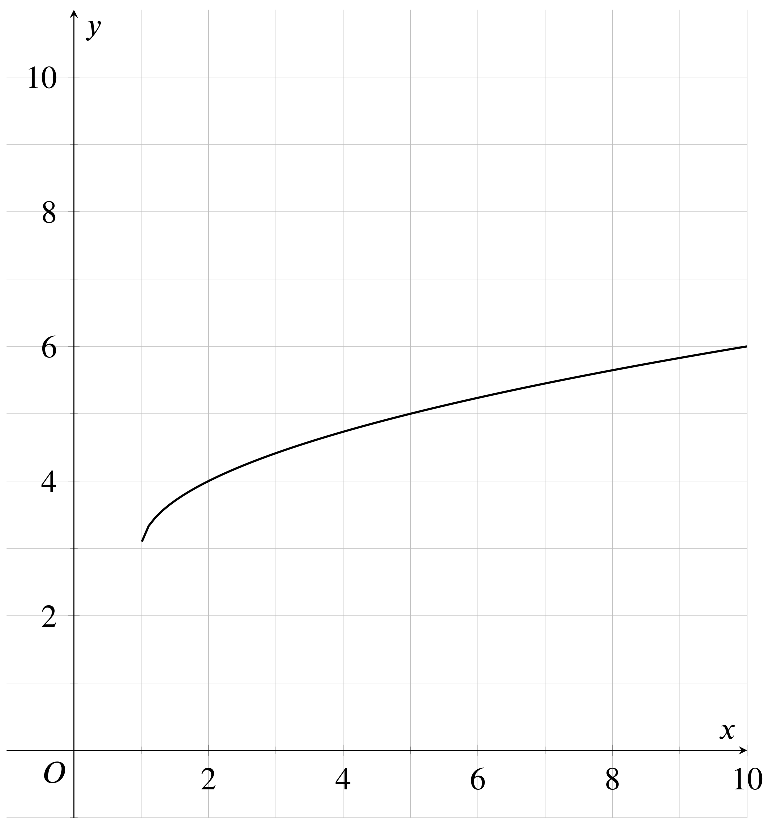
3. $y = \dfrac{4}{(x+2)^2}$

Sol.



4. $y = \sqrt{x-1} + 3$

Sol.

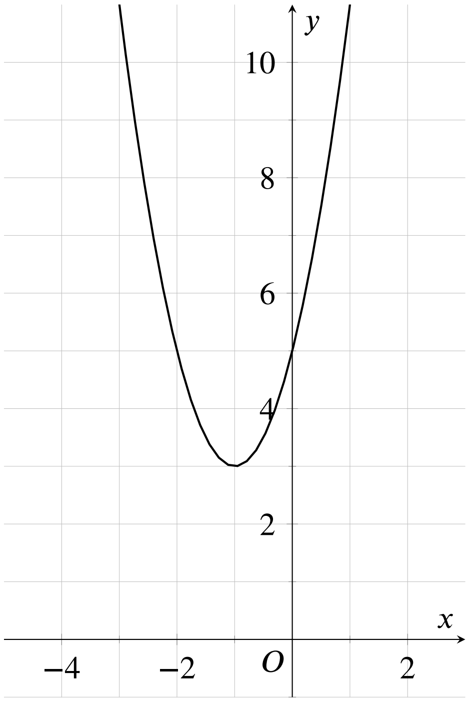


Exercise 22.3

Find the line of symmetry and vertex of the following parabola, and sketch its graph.

1. $y = 2x^2 + 4x + 5$

   Sol.

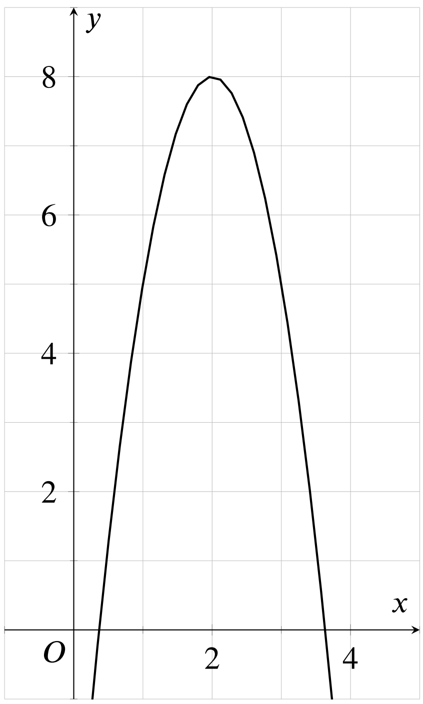


$$\begin{aligned} y &= 2x^2 + 4x + 5 \\ &= 2(x^2 + 2x) + 5 \\ &= 2[(x^2 + 2x + 1) - 1] + 5 \\ &= 2(x + 1)^2 - 2 + 5 \\ &= 2(x + 1)^2 + 3 \end{aligned}$$

   Vertex: $(-1, 3)$
   Line of symmetry: $x = -1$

2. $y = -3x^2 + 12x - 4$

   Sol.

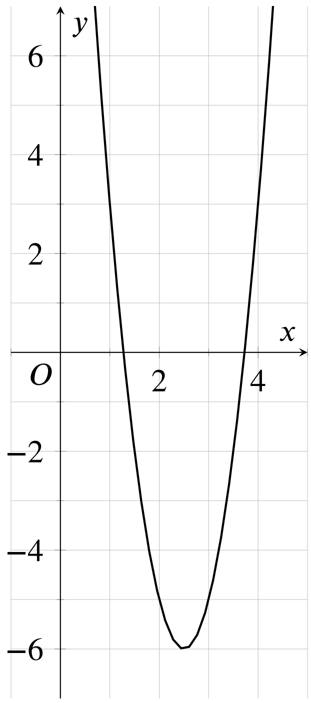


$$\begin{aligned} y &= -3x^2 + 12x - 4 \\ &= -3(x^2 - 4x) - 4 \\ &= -3[(x^2 - 4x + 4) - 4] - 4 \\ &= -3(x - 2)^2 + 12 - 4 \\ &= -3(x - 2)^2 + 8 \end{aligned}$$

   Vertex: $(2, 8)$
   Line of symmetry: $x = 2$

3. $y = 4x^2 - 20x + 19$

Sol.

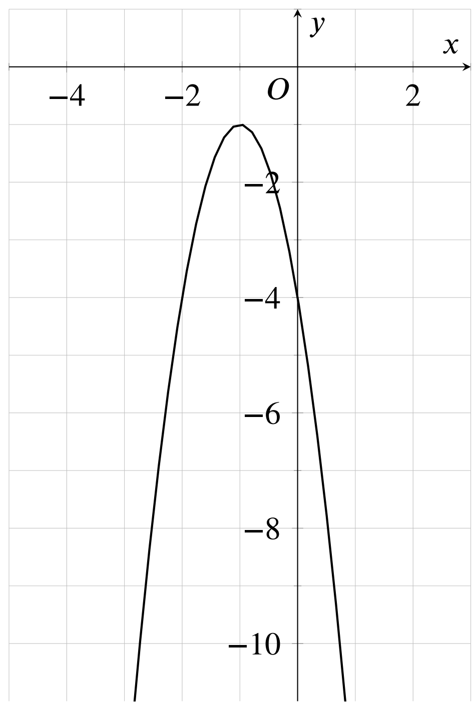


$$
\begin{aligned}
y &= 4x^2 - 20x + 19 \\
&= 4(x^2 - 5x) + 19 \\
&= 4\left[\left(x^2 - 5x + \frac{25}{4}\right) - \frac{25}{4}\right] + 19 \\
&= 4\left(x - \frac{5}{2}\right)^2 - 25 + 19 \\
&= 4\left(x - \frac{5}{2}\right)^2 - 6
\end{aligned}
$$

Vertex: $(3, -5)$
Line of symmetry: $x = \dfrac{5}{2}$

4. $y = -3x^2 - 6x - 4$

Sol.



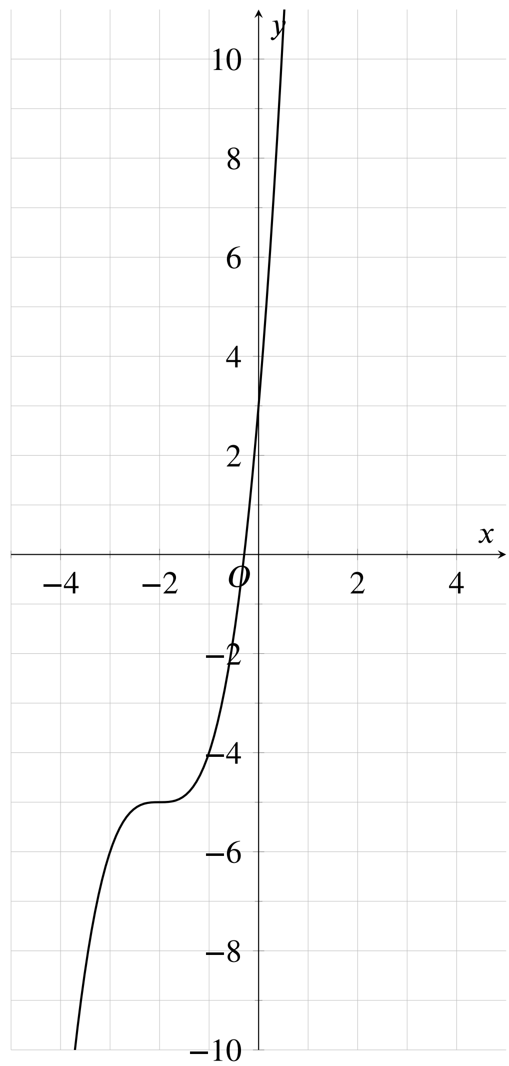
$$
\begin{aligned}
y &= -3x^2 - 6x - 4 \\
&= -3(x^2 + 2x) - 4 \\
&= -3[(x^2 + 2x + 1) - 1] - 4 \\
&= -3(x + 1)^2 + 3 - 4 \\
&= -3(x + 1)^2 - 1
\end{aligned}
$$

Vertex: $(-1, -1)$
Line of symmetry: $x = -1$

Sketch the graph of the following functions.

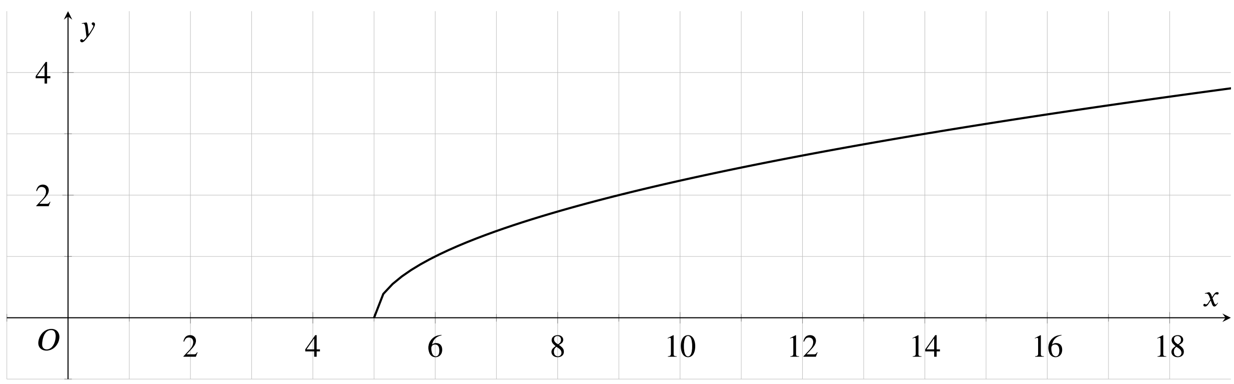
5. $y = (x+2)^3 - 5$

Sol.



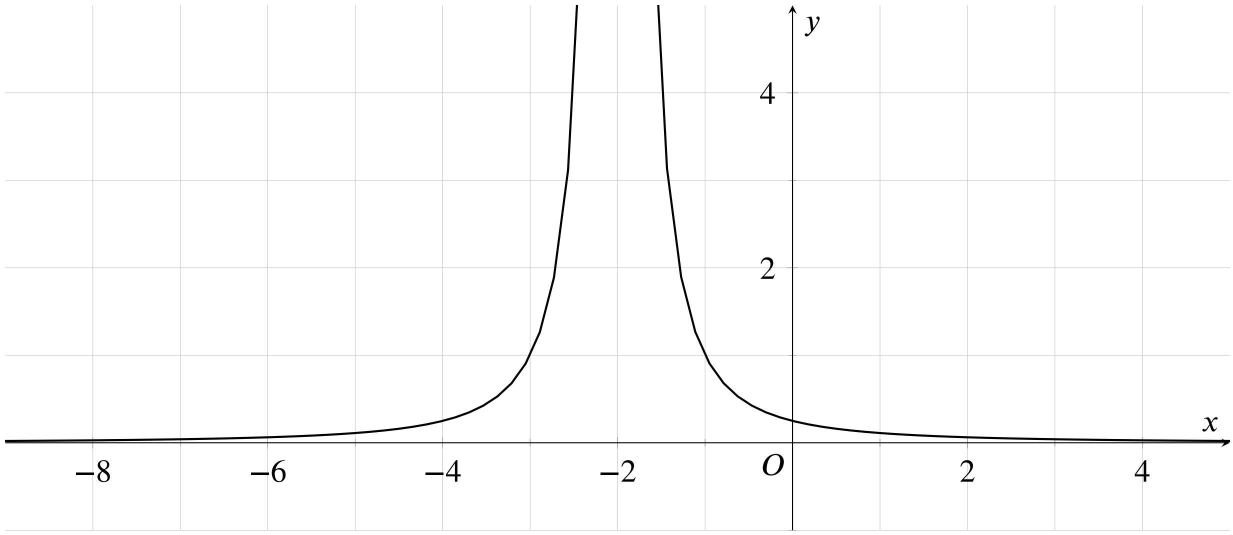
6. $y = \sqrt{x-5}$

Sol.



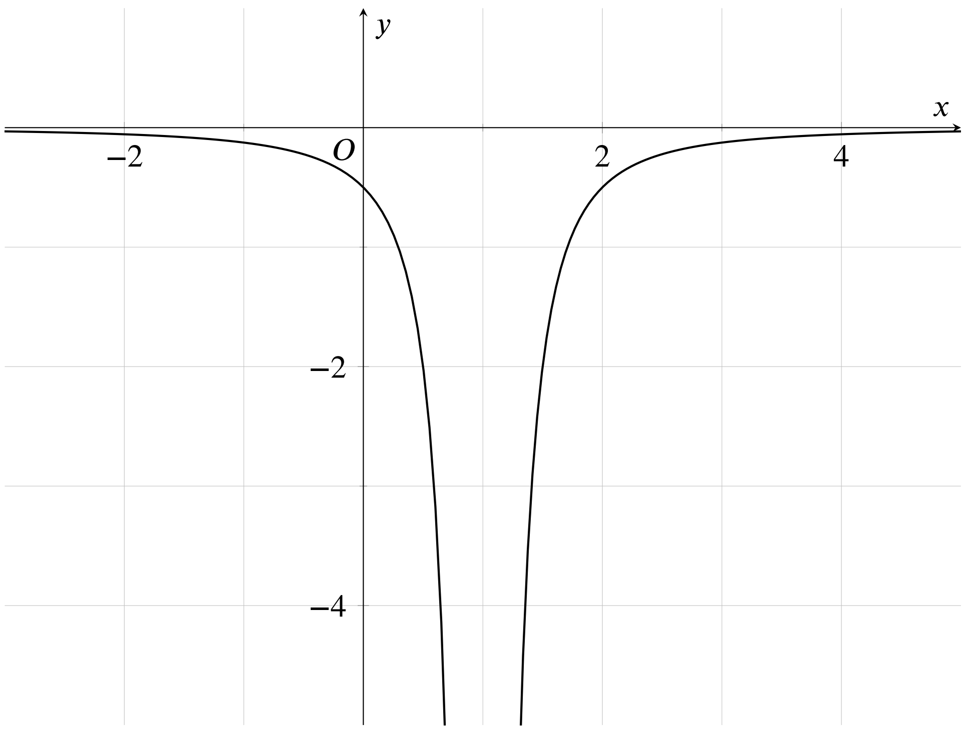
7. $y = \dfrac{1}{(x+2)^2}$

Sol.



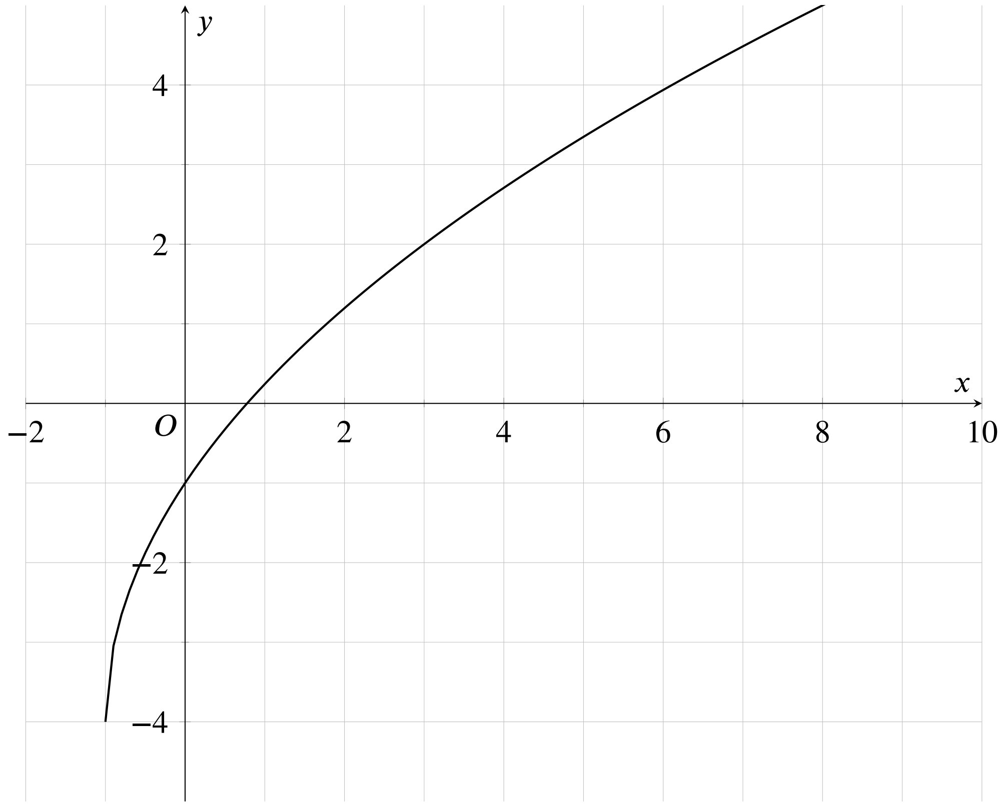
8. $y = -\dfrac{1}{2(x-1)^2}$

Sol.



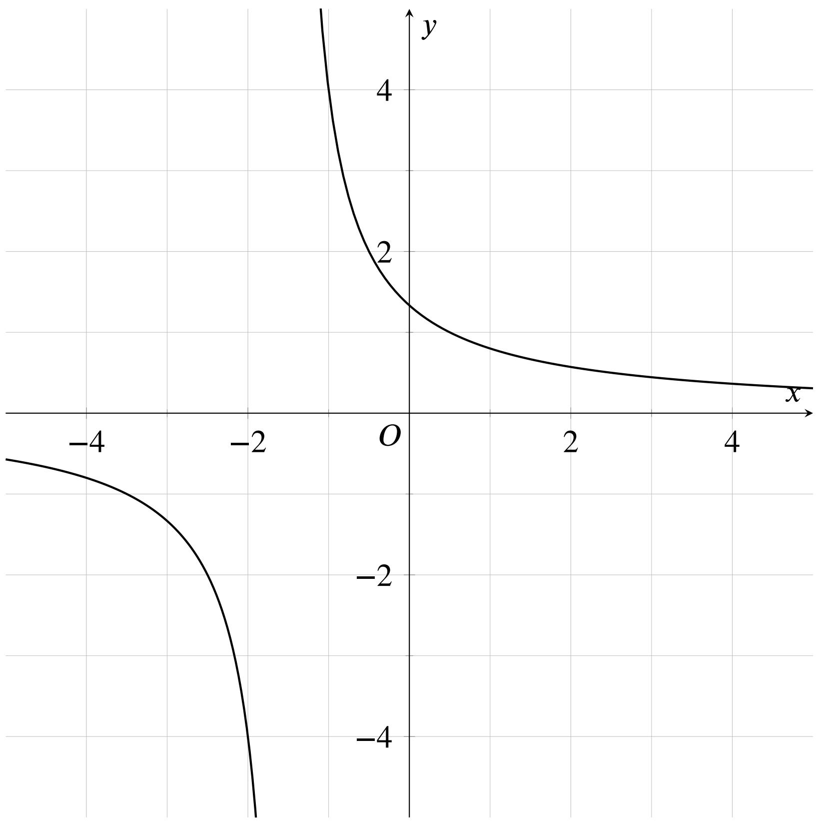
9. $y = 3\sqrt{x+1} - 4$

Sol.



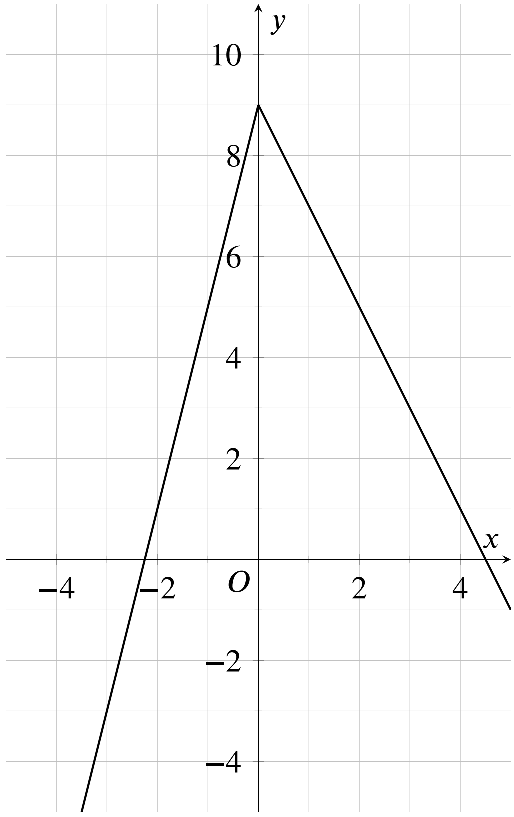
10. $y = \dfrac{4}{2x+3}$

Sol.



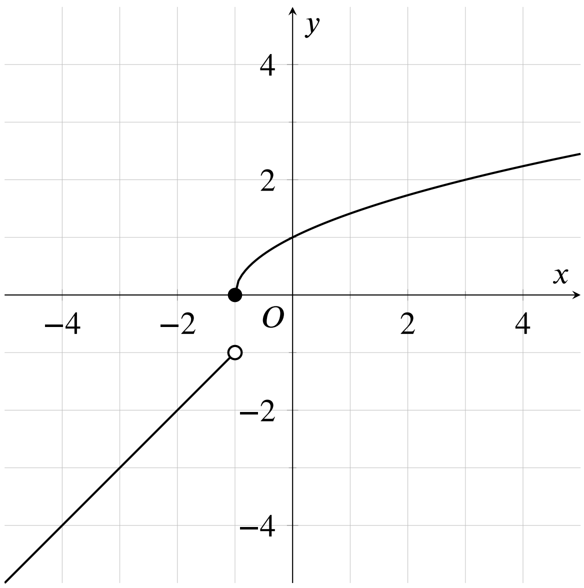
11. $y = \begin{cases} 4x + 9, & x \le 0 \\ 9 - 2x, & x > 0 \end{cases}$

Sol.



12. $y = \begin{cases} x, & x < -1 \\ \sqrt{x+1}, & x \ge -1 \end{cases}$

Sol.

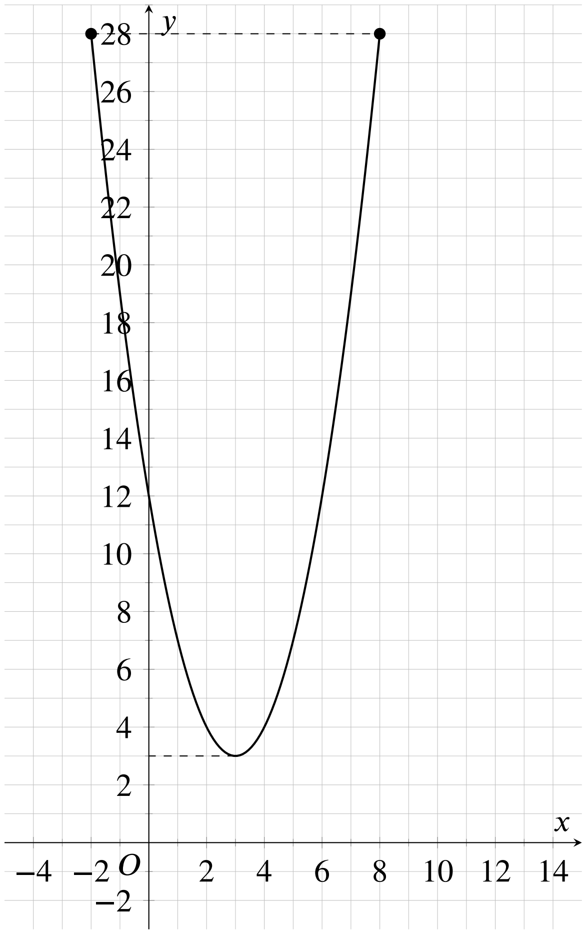


13. Sketch the graph for the function $f(x) = x^2 - 6x + 12$, $-2 \leq x \leq 8$, and find its domain and range.

Sol.

$$
\begin{aligned}
f(x) &= x^2 - 6x + 12 \\
&= x^2 - 6x + 9 - 9 + 12 \\
&= (x-3)^2 + 3
\end{aligned}
$$

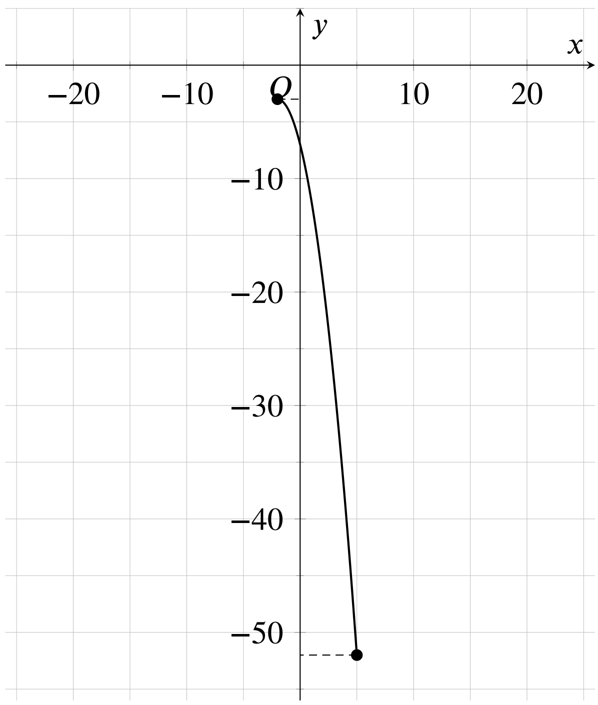
$D_f = [-2, 8]$, $R_f = [3, 28]$



14. Sketch the graph for the function $g(x) = -x^2 - 4x - 7$, $-2 \leq x \leq 5$, and find its domain and range.

$$
\begin{aligned}
f(x) &= -x^2 - 4x - 7 \\
&= -[(x^2 + 4x + 4) - 4] - 7 \\
&= -[(x+2)^2 - 4] - 3 \\
&= -(x+2)^2 + 4 - 3 \\
&= -(x+2)^2 + 1
\end{aligned}
$$

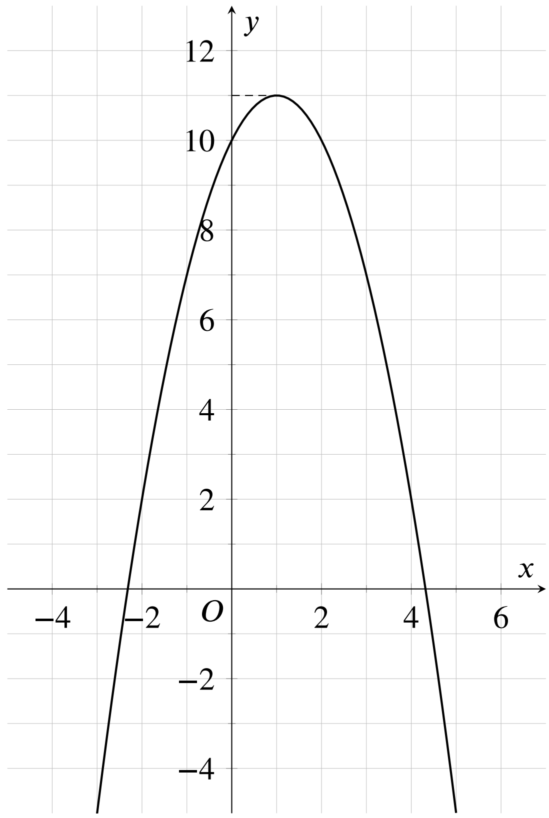
$D_f = [-2, 5]$, $R_f = [-52, 3]$



15. Sketch the graph for the function $f(x) = -x^2 + 2x + 10$, and find its domain and range.

$$\begin{aligned} f(x) &= -x^2 + 2x + 10 \\ &= -[(x^2 - 2x + 1) - 1] + 10 \\ &= -[(x-1)^2 - 1] + 10 \\ &= -(x-1)^2 + 1 + 10 \\ &= -(x-1)^2 + 11 \end{aligned}$$

$\boldsymbol{D}_f = \mathbb{R}$, $\boldsymbol{R}_f = [11, \infty)$



16. Sketch the graph of the function $y = \sqrt{x}$, and transform it according the following steps. Sketch the graph of each function after each step on the same diagram, and write down the corresponding function.

Step 1: Translate 4 units to the left;

Step 2: Scale up by a factor of 2 in the x-direction;

Step 3: Reflect about the y-axis;

Step 4: Translate 3 units downwards.

Step 5: Scale down by half in the y-direction.

Sol.

Step 0: $y = \sqrt{x}$

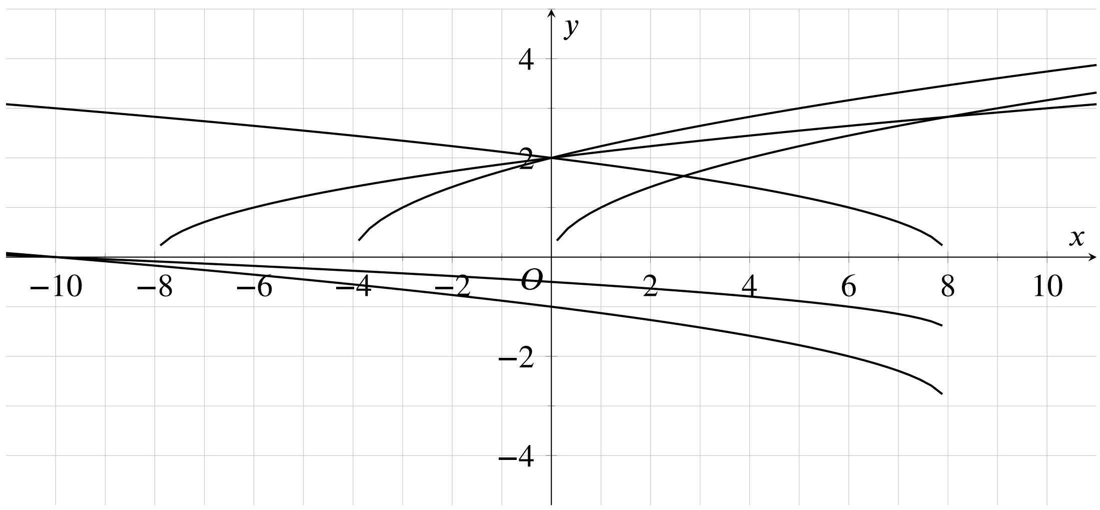
Step 1: $y = \sqrt{x+4}$

Step 2: $y = \sqrt{\dfrac{x}{2} + 4}$

Step 3: $y = \sqrt{-\dfrac{x}{2} + 4}$

Step 4: $y = \sqrt{-\dfrac{x}{2} + 4} - 3$

Step 5: $y = \dfrac{1}{2}\left(\sqrt{-\dfrac{x}{2} + 4} - 3\right)$


y
4
2
x
−10
−8
−6
−4
−2
O
2
4
6
8
10
−2
−4

22.4 Composite Functions

Let A, B, and C be three non-empty sets, $f : A \to B$ and $g : B \to C$ be two functions, an element x in set A is mapped to an element $f(x)$ in set B by function f, and $f(x)$ is mapped to an element $g\big(f(x)\big)$ in set C by function g. In other words, x in set A is mapped to an element $g\big(f(x)\big)$ in C after two mappings. That is:

$$x \xrightarrow{f} f(x) \xrightarrow{g} g\big(f(x)\big)$$

The combination of these two mappings are a function from set A to set C, this function is called the *composite function* of f and g, denoted by $g \circ f$. When defining the composite function $g \circ f$, the range of f must be a subset of the domain of g, that is, $R_f \subseteq D_g$.

Note that $D_{g \circ f} = D_f$, $R_{g \circ f} \subseteq R_g$.

$\forall n \in \mathbb{N}$, $f^{n+1} = f \circ f^n$.

Generally speaking, $g \circ f \neq f \circ g$.

If $f \circ (g \circ h)$ is defined, then $(f \circ g) \circ h$ is also defined, and $f \circ (g \circ h) = (f \circ g) \circ h$. Therefore, we can write $f \circ g \circ h$ without ambiguity.

Practice 5

1. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 3$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 5 - x$. Find $(g \circ f)(x)$ and $(f \circ g)(x)$.

   Sol.

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(2x + 3) \\ &= 5 - (2x + 3) \\ &= -2x + 2 \end{aligned}$$

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(5 - x) \\ &= 2(5 - x) + 3 \\ &= -2x + 13 \end{aligned}$$

2. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 2x + 3$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 3x - 4$. Find

   (a) $g \circ f$ and $f \circ g$;

   Sol.

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(x^2 - 2x + 3) \\ &= 3(x^2 - 2x + 3) - 4 \\ &= 3x^2 - 6x + 5 \end{aligned}$$

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(3x - 4) \\ &= (3x - 4)^2 - 2(3x - 4) + 3 \\ &= 9x^2 - 30x + 27 \end{aligned}$$

(b) $g\big(f(2)\big)$, $f\big(g(2)\big)$, $(g \circ f)(2)$, and $(f \circ g)(2)$.

Sol.

$$\begin{aligned} g\big(f(2)\big) &= g(2^2 - 2(2) + 3) \\ &= g(3) \\ &= 3(3) - 4 \\ &= 5 \end{aligned}$$

$$\begin{aligned} f\big(g(2)\big) &= f(3(2) - 4) \\ &= f(2) \\ &= 2^2 - 2(2) + 3 \\ &= 3 \end{aligned}$$

$$\begin{aligned} (g \circ f)(2) &= g(f(2)) \\ &= g(3) \\ &= 3(3) - 4 \\ &= 5 \end{aligned}$$

$$\begin{aligned} (f \circ g)(2) &= f(g(2)) \\ &= f(3) \\ &= 3^2 - 2(3) + 3 \\ &= 3 \end{aligned}$$

3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4 - x^2$ and $g : \{x | x \leq 4\} \to \mathbb{R}$, $g(x) = \sqrt{4 - x}$. Prove the existence of $f \circ g$ and $g \circ f$ respectively.

Sol.

$$\begin{aligned} &\boldsymbol{R_g} = \mathbb{R},\ \boldsymbol{D_f} = \mathbb{R} \\ &\because \boldsymbol{R_g} \subset \boldsymbol{D_f} \\ &\therefore \exists\ f \circ g \end{aligned}$$

$$\begin{aligned} &\because \forall\, x \in \mathbb{R},\ x^2 \geq 0,\ \boldsymbol{R_f} = (\infty, 4] \\ &\boldsymbol{R_f} = (\infty, 4],\ \boldsymbol{D_g} = (\infty, 4] \\ &\because \boldsymbol{R_f} \subset \boldsymbol{D_g} \\ &\therefore \exists\ g \circ f \end{aligned}$$

Practice 6

1. Given that $f : x \to 2x + 1$ and $f \circ g : x \to 2x - 1$. Find the function g.

   Sol.

$$
\begin{aligned}
f \circ g &= f(g(x)) \\
2x - 1 &= 2g(x) + 1 \\
2g(x) &= 2x - 2 \\
g(x) &= x - 1
\end{aligned}
$$

2. The function f is defined as $f : x \to 5 - x$. Find another function g such that $g \circ f : x \to x^2 - 10x + 8$.

   Sol.

$$
\begin{aligned}
(g \circ f)(x) &= x^2 - 10x + 8 \\
g(5 - x) &= x^2 - 10x + 8
\end{aligned}
$$

   Let $y = 5 - x$

$$
\begin{aligned}
x &= 5 - y \\
g(y) &= (5 - y)^2 - 10(5 - y) + 8 \\
&= 25 - 10y + y^2 - 50 + 10y + 8 \\
&= y^2 - 17
\end{aligned}
$$

   $\therefore g(x) = x^2 - 17$

3. The function f is defined as $f : x \to 2x - 3$. Find another function g such that $f \circ g : x \to 4x^2 - 14x + 9$.

   Sol.

$$
\begin{aligned}
(f \circ g)(x) &= 4x^2 - 14x + 9 \\
2(g(x)) - 3 &= 4x^2 - 14x + 9 \\
2g(x) &= 4x^2 - 14x + 12 \\
g(x) &= 2x^2 - 7x + 6
\end{aligned}
$$

Exercise 22.4

1. Given the functions f and g, find $f \circ g$ and $g \circ f$:

   (a) $f : x \to 3x$, $g : x \to x - 1$

   Sol.

   $$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x-1) \\ &= 3(x-1) \\ &= 3x - 3\end{aligned}$$

   $$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x) \\ &= 3x - 1\end{aligned}$$

   (b) $f : x \to 5x$, $g : x \to \frac{1}{5}x$

   Sol.

   $$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{1}{5}x\right) \\ &= 5\left(\frac{1}{5}x\right) \\ &= x\end{aligned}$$

   $$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(5x) \\ &= \frac{1}{5}(5x) \\ &= x\end{aligned}$$

   (c) $f : x \to x^2$, $g : x \to x + 2$

   Sol.

   $$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4\end{aligned}$$

   $$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^2) \\ &= x^2 + 2\end{aligned}$$

   (d) $f : x \to 2x + 1$, $g : x \to x^2 - 2$

   Sol.

   $$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x^2 - 2) \\ &= 2(x^2 - 2) + 1 \\ &= 2x^2 - 3\end{aligned}$$

   $$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x+1) \\ &= (2x+1)^2 - 2 \\ &= 4x^2 + 4x + 1 - 2 \\ &= 4x^2 + 4x - 1\end{aligned}$$

(e) $f : x \to 3x^2 - 5x + 2, g : x \to 4 - x$

Sol.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(4 - x) \\ &= 3(4 - x)^2 - 5(4 - x) + 2 \\ &= 3(16 - 8x + x^2) - 20 + 5x + 2 \\ &= 48 - 24x + 3x^2 - 20 + 5x + 2 \\ &= 3x^2 - 19x + 30\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(3x^2 - 5x + 2) \\ &= 4 - (3x^2 - 5x + 2) \\ &= -3x^2 + 5x + 2\end{aligned}$$

2. Let the function $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 3$ and $g(x) = x + 5$ respectively. Find the following:

(a) $g \circ f$ and $f \circ g$

Sol.

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^2 - 3) \\ &= x^2 - 3 + 5 \\ &= x^2 + 2\end{aligned}$$

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x + 5) \\ &= (x + 5)^2 - 3 \\ &= x^2 + 10x + 25 - 3 \\ &= x^2 + 10x + 22\end{aligned}$$

3. Let the function $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x - 2$ and $g(x) = x + 1$ respectively. Find the following:

(a) $f \circ g$

Sol.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x + 1) \\ &= 3(x + 1) - 2 \\ &= 3x + 3 - 2 \\ &= 3x + 1\end{aligned}$$

(b) $(f \circ g)(3)$, $(f \circ g)(-1)$, and $(f \circ g)\left(\frac{1}{4}\right)$

Sol.

i. $(f \circ g)(3)$

$$\begin{aligned}(f \circ g)(3) &= 3(3) + 1\\ &= 9 + 1\\ &= 10\end{aligned}$$

ii. $(f \circ g)(-1)$

$$\begin{aligned}(f \circ g)(-1) &= 3(-1) + 1\\ &= -3 + 1\\ &= -2\end{aligned}$$

iii. $(f \circ g)\left(\frac{1}{4}\right)$

$$\begin{aligned}(f \circ g)\left(\frac{1}{4}\right) &= 3\left(\frac{1}{4}\right) + 1\\ &= \frac{7}{4}\end{aligned}$$

(c) $g \circ f$

Sol.

$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(3x - 2)\\ &= (3x - 2) + 1\\ &= 3x - 2 + 1\\ &= 3x - 1\end{aligned}$$

(d) $(g \circ f)(2)$, $(g \circ f)(-2)$, and $(g \circ f)(0)$

Sol.

i. $(g \circ f)(2)$

$$\begin{aligned}(g \circ f)(2) &= 3(2) - 1\\ &= 6 - 1\\ &= 5\end{aligned}$$

ii. $(g \circ f)(-2)$

$$\begin{aligned}(g \circ f)(-2) &= 3(-2) - 1\\ &= -6 - 1\\ &= -7\end{aligned}$$

iii. $(g \circ f)(0)$

$$\begin{aligned}(g \circ f)(0) &= 3(0) - 1\\ &= 0 - 1\\ &= -1\end{aligned}$$

(e) f^2

Sol.

$$\begin{aligned}f^2(x) &= f(f(x))\\ &= f(3x - 2)\\ &= 3(3x - 2) - 2\\ &= 9x - 8\end{aligned}$$

(f) g^2

Sol.

$$\begin{aligned}g^2(x) &= g(g(x))\\ &= g(x + 1)\\ &= (x + 1) + 1\\ &= x + 2\end{aligned}$$

4. Let the function $g : \{x|x \in \mathbb{R}, x \neq 1\} \to \mathbb{R}$ be defined as $g : x \to \dfrac{x+1}{x-1}$. Find $g^2(x)$ and $g^4(x)$. Hence, find $g^{33}(x)$ and $g^{50}(x)$.

Sol.

$$
\begin{aligned}
g^2(x) &= g(g(x)) \\
&= g\left(\frac{x+1}{x-1}\right) \\
&= \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} \\
&= \frac{\frac{x+1+x-1}{x-1}}{\frac{x+1-x+1}{x-1}} \\
&= \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
g^4(x) &= g^2(g^2(x)) \\
&= g^2(x) \\
&= x
\end{aligned}
$$

We can see that for $g^n(x)$, where n is a multiple of 2, $g^n(x) = x$.

$$
\begin{aligned}
g^{33}(x) &= g^{32}(g(x)) \\
&= g^{32}\left(\frac{x+1}{x-1}\right) \\
&= \frac{x+1}{x-1}
\end{aligned}
$$

$$g^{50}(x) = x$$

5. Let $f : x \to 3x-2$ and $g : x \to kx+2$ be functions from $\mathbb{R}$ to $\mathbb{R}$. If $g \circ f = f \circ g$, find the value of k.

Sol.

$$
\begin{aligned}
(g \circ f)(x) &= (f \circ g)(x) \\
g(3x-2) &= f(kx+2) \\
k(3x-2)+2 &= 3(kx+2)-2 \\
-2k &= 2 \\
k &= -1
\end{aligned}
$$

6. Given the function $f : x \to x+2$, find the function g such that $f \circ g : x \to \dfrac{3-2x}{x+5}$.

Sol.

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
\frac{3-2x}{x+5} &= g(x)+2 \\
g(x) &= \frac{3-2x}{x+5}-2 \\
&= \frac{3-2x-2x-10}{x+5} \\
&= \frac{-4x-7}{x+5}
\end{aligned}
$$

7. Given the function $f : x \to 3x - 1$. Find the function g such that $f \circ g : x \to 6x^2 - 3x + 2$.

Sol.

$$\begin{aligned} f(g(x)) &= 6x^2 - 3x + 2 \\ 3g(x) - 1 &= 6x^2 - 3x + 2 \\ 3g(x) &= 6x^2 - 3x + 3 \\ g(x) &= 2x^2 - x + 1 \end{aligned}$$

8. Given the function $g(x) = x - 2$. Find the function h such that $(h \circ g)(x) = 2x^2 - 8x + 5$.

Sol.

$$\begin{aligned} (h \circ g)(x) &= 2x^2 - 8x + 5 \\ h(x-2) &= 2x^2 - 8x + 5 \end{aligned}$$

Let $y = x - 2$

$$x = y + 2$$

$$\begin{aligned} h(y) &= 2(y+2)^2 - 8(y+2) + 5 \\ &= 2y^2 + 8y + 8 - 8y - 11 \\ &= 2y^2 - 3 \end{aligned}$$

$\therefore h(x) = 2x^2 - 3$

9. Given the functions g and $f \circ g$. Find the function f:

(a) $g : x \to 2x - 4$ and $f \circ g : x \to 4x^2 - 16x + 1$

Sol.

$$\begin{aligned} f \circ g(x) &= 4x^2 - 16x + 1 \\ f(2x-4) &= 4x^2 - 16x + 1 \end{aligned}$$

Let $y = 2x - 4$

$$x = \frac{y+4}{2}$$

$$\begin{aligned} f(y) &= 4\left(\frac{y+4}{2}\right)^2 - 16\left(\frac{y+4}{2}\right) + 1 \\ &= y^2 + 8y + 16 - 8y - 31 \\ &= y^2 - 15 \end{aligned}$$

$\therefore f(x) = x^2 - 15$

(b) $g : x \to x - 3$ and $f \circ g : x \to x^2 - 5x + 3$

Sol.

$$\begin{aligned} f \circ g(x) &= x^2 - 5x + 3 \\ f(x-3) &= x^2 - 5x + 3 \end{aligned}$$

Let $y = x - 3$

$$x = y + 3$$

$$\begin{aligned} f(y) &= (y+3)^2 - 5(y+3) + 3 \\ &= y^2 + 6y + 9 - 5y - 15 + 3 \\ &= y^2 + y - 3 \end{aligned}$$

$\therefore f(x) = x^2 + x - 3$

10. Given the function $f : x \to x - 3$ and the function $g \circ f : x \to x^2 - 6x + 7$. Find the function $f \circ g$.

Sol.

$$
\begin{aligned}
g \circ f &= x^2 - 6x + 7 \\
g(f(x)) &= x^2 - 6x + 7 \\
g(x-3) &= x^2 - 6x + 7
\end{aligned}
$$

Let $y = x - 3$

$$x = y + 3$$

$$
\begin{aligned}
g(y) &= (y+3)^2 - 6(y+3) + 7 \\
&= y^2 + 6y + 9 - 6y - 18 + 7 \\
&= y^2 - 2
\end{aligned}
$$

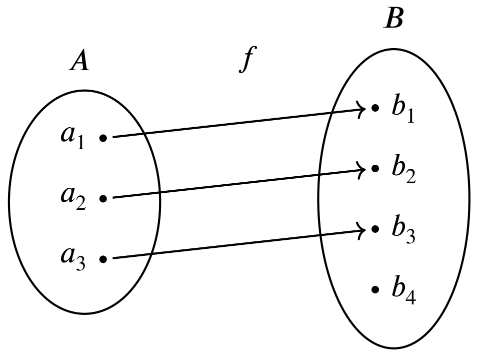
$\therefore g(x) = x^2 - 2$

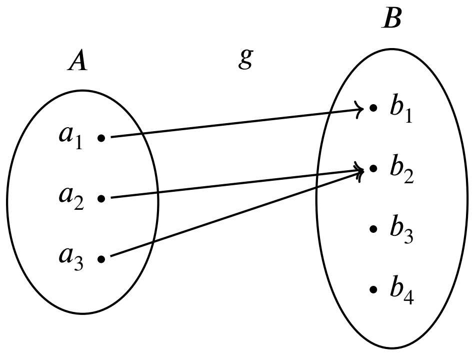
$$
\begin{aligned}
f \circ g &= f(x^2 - 2) \\
&= (x^2 - 2) - 3 \\
&= x^2 - 5
\end{aligned}
$$

22.5 One to One Function, Onto Function and One-one Onto Function

One to One Function

Let $f : A \to B$ be a function, if there is at most one preimage in set A for each element in set B, then f is called a *one to one function*.



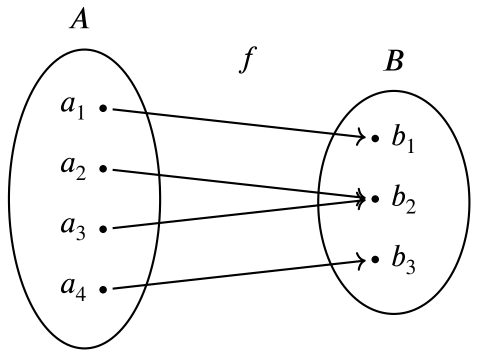


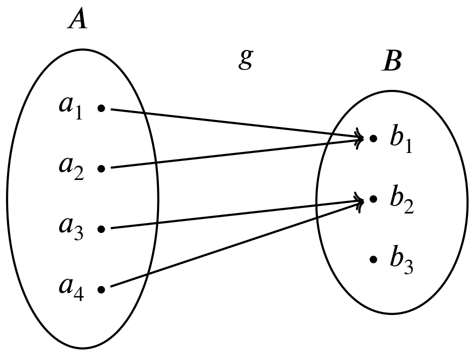
As shown in the diagram above, each element in the codomain B of the function $f : A \to B$ has at most one preimage in the domain A of the function, thus f is a one to one function; while the element b_2 in the codomain B of the function $g : A \to B$ has two preimages a_2 and a_3, thus g is not a one to one function.

A function $y = f(x)$ is a one to one function, if and only if any line parallel to the x-axis intersects the graph of the function at most once.

Onto Function

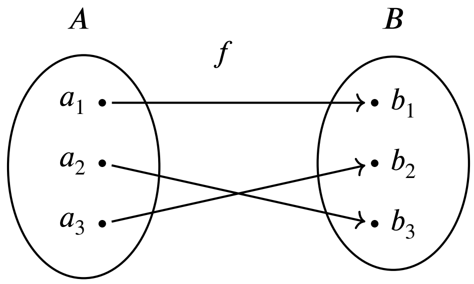
If each element in the codomain B of the function $f : A \to B$ has at least one preimage under the function f, then f is said to be an *onto function*.





As shown in the diagram above, each element in the codomain B of the function $f : A \to B$ has at least one preimage under the function f, therefore f is an onto function; while the element b_3 in the codomain B of the function $g : A \to B$ has no preimage under the function g, therefore g is not an onto function.

One-one onto function

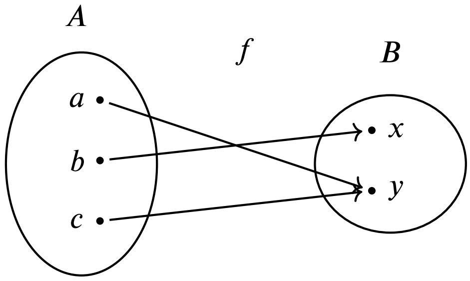


If a function is both a one to one function and an onto function, then it is a *one-one onto function*, as shown in the diagram above.

Practice 7

Determine whether the following functions are one to one functions or onto functions.

(a)

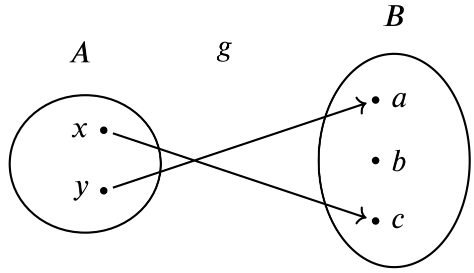


Sol.

Since $y \in B$ has two preimages a and c, f is not a one to one function.

Since all elements in B have preimages, f is an onto function.

(b)

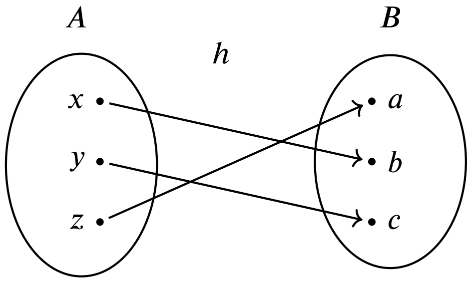


Sol.

Since each element in B has at most one preimage, g is a one to one function.

Since $b \in B$ has no preimage, g is not an onto function.

(c)

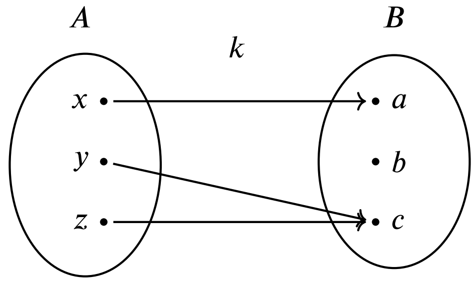


Sol.

Since each element in B has exactly one preimage, h is a one to one function. Since all elements in B have preimages, h is an onto function.

Hence, h is a one-one onto function.

(d)



Sol.

Since $c \in B$ has two preimages y and z, k is not a one to one function.

Since $b \in B$ has no preimage, k is not an onto function.

Exercise 22.5

1. Let $A = \{1, 2, 3\}$, $f : A \to A$ is defined by $f : 1 \to 1, 2 \to 3, 3 \to 2$. Determine if f is a one to one function or an onto function.

   Sol.

   Since each element in the codomain is mapped to exactly one element in the domain, f is a one to one function.

   Since each element in the codomain is mapped to at least one element in the domain, f is an onto function.

   Hence, f is a one-one onto function.

2. Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$, $f : A \to B$ is defined by $f : a \to y, b \to x, c \to z, d \to y$. Determine if f is a one to one function or an onto function.

   Sol.

   Since $y \in B$ is mapped to two elements a and d in A, f is not a one to one function.

   Since each element in the codomain is mapped to at least one element in the domain, f is an onto function.

3. Let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 2x + 1$. Determine if g is a one to one function or an onto function.

   Sol.

   Since each element in the codomain is mapped to exactly one element in the domain, g is a one to one function.

   Since each element in the codomain is mapped to at least one element in the domain, g is an onto function.

   Hence, g is a one-one onto function.

4. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x^3 - 3$. Determine if f is a one to one function or an onto function.

   Sol.

   Since each element in the codomain is mapped to exactly one element in the domain, f is a one to one function.

   Since each element in the codomain is mapped to at least one element in the domain, f is an onto function.

   Hence, f is a one-one onto function.

5. Let the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = \frac{1}{x}$. Determine if f is a one to one function or an onto function.

   Sol.

   Since each element in the codomain is mapped to exactly one element in the domain, f is a one to one function.

   Since each element in the codomain is mapped to at least one element in the domain, f is an onto function.

   Hence, f is a one-one onto function.

6. Let the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = \sqrt{x}$. Determine if f is a one to one function or an onto function.

   Sol.

   Since each element in the codomain is mapped to exactly one element in the domain, f is a one to one function.

   Since each element in the codomain is mapped to at least one element in the domain, f is an onto function.

   Hence, f is a one-one onto function.

7. Determine whether the following functions are one to one, onto or one to one onto functions.

   (a) $A = \{a, b, c\}$, $B = \{x, y, z\}$, $f : A \to B$, $f : a \to x, b \to x, c \to y$

   Sol.

   Since each element in the domain is mapped to exactly one element in the codomain, f is a function.

   Since $x \in B$ is mapped to two elements a and b in A, f is not a one to one function.

   Since $z \in B$ has no preimage in A, f is not an onto function.

   (b) $A = \{a, b, c\}$, $B = \{x, y, z\}$, $g : A \to B$, $g : a \to x, b \to y, c \to z$

   Sol.

   Since each element in the domain is mapped to exactly one element in the codomain, g is a function.

   Since each element in the codomain is mapped to exactly one element in the domain, g is a one to one function.

   Since each element in the codomain has preimage in A, g is an onto function.

   Hence, g is a one-one onto function.

(c) $A = \{a, b, c\}$, $B = \{x, y\}$, $h : A \to B$, $h : a \to x, b \to y, c \to y$

Sol.

Since each element in the domain is mapped to exactly one element in the codomain, h is a function.

Since $y \in B$ is mapped to two elements b and c in A, h is not a one to one function.

Since each element in the codomain has preimage in A, h is an onto function.

(d) $A = \{a, b, c\}$, $B = \{x, y\}$, $k : A \to B$, $k : a \to x, a \to y, c \to y$

Sol.

Since $b \in A$ has no image in B, k is not a function.

(e) $A = \{a, b, c\}$, $B = \{x, y\}$, $f : A \to B$, $f : a \to x, a \to y, b \to x, c \to y$

Sol.

Since $a \in A$ has two images x and y in B, f is not a function.

(f) $A = \{a, b, c, d\}$, $B = \{u, v, x, y, z\}$, $g : A \to B$, $g : a \to u, b \to v, c \to x, d \to y$

Sol.

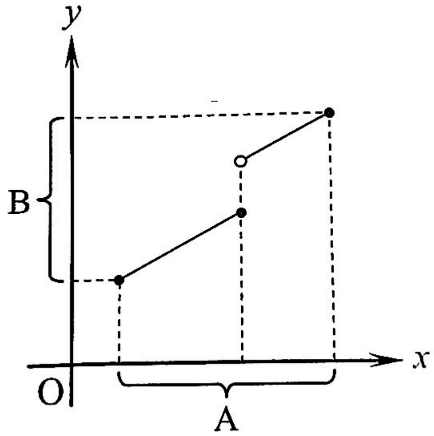
Since each element in the domain is mapped to exactly one element in the codomain, g is a function.

Since each element in the codomain is mapped to at most one element in the domain, g is a one to one function.

Since $z \in B$ has no preimage in A, g is not an onto function.

8. Determine whether the following functions mapping A to B are one to one functions or onto functions.

(a)

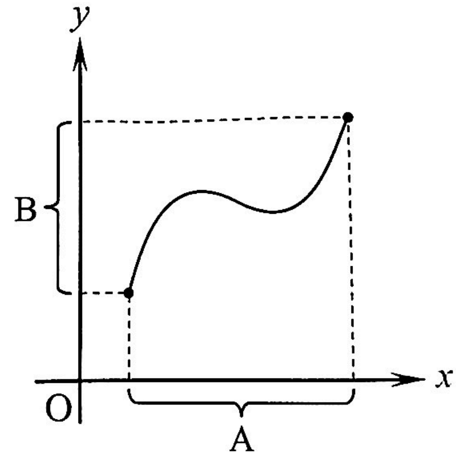


Sol.

Since each element in the codomain B is mapped to at most one element in the domain A, f is a one to one function.

Since not each element in the codomain B has preimage in the domain A, f (no preimage at the white dot), f is not an onto function.

(b)



Sol.

Since some elements in the codomain B are mapped to more than one element in the domain A, f is not a one to one function.

Since each element in the codomain B has preimage in the domain A, f is an onto function.

22.6 Inverse Functions

If $f : A \to B$ is a one-one onto function, then there exist a function $g : B \to A$, such that if $y = f(x)$, then $g(y) = x$. The function g is called the *inverse function* of f, and is denoted by f^{-1}.

from the diagram above, we can conclude the following:

$$x \xrightarrow{f} y = f(x) \xrightarrow{f^{-1}} f^{-1}\big(f(x)\big) = f^{-1}(y)$$

or

$$y \xrightarrow{f^{-1}} x = f^{-1}(y) \xrightarrow{f} f\big(f^{-1}(y)\big) = f(x)$$

If both $(g \circ f)^{-}1$ and $f^{-1} o g^{-1}$ exist, then $(g \circ f)^{-1} = f^{-1} o g^{-1}$.

Practice 8

1. Find the inverse function of the following functions:

(a) $f : x \to 7x - 3$

Sol.

$$\text{Let } y = f^{-1}(x)$$
$$f(y) = x$$
$$7y - 3 = x$$
$$7y = x + 3$$
$$y = \frac{x+3}{7}$$

$$\therefore f^{-1}(x) = \frac{x+3}{7}$$

(b) $g : x \to \frac{1}{2}x + 9$

Sol.

$$\text{Let } y = g^{-1}(x)$$
$$g(y) = x$$
$$\frac{1}{2}y + 9 = x$$
$$\frac{1}{2}y = x - 9$$
$$y = 2x - 18$$

$$\therefore g^{-1}(x) = 2x - 18$$

(c) $h : x \to \dfrac{x+1}{x-8},\ x \neq 8$

Sol.

$$\begin{aligned}
\text{Let } y &= h^{-1}(x)\\
h(y) &= x\\
\frac{y+1}{y-8} &= x\\
y+1 &= x(y-8)\\
y+1 &= xy-8x\\
y-xy &= -8x-1\\
y(1-x) &= -8x-1\\
y &= \frac{-8x-1}{1-x}\\
&= \frac{8x+1}{x-1} \quad (x \neq 1)
\end{aligned}$$

$$\therefore h^{-1}(x) = \frac{8x+1}{x-1} \quad (x \neq 1)$$

(d) $k : x \to \dfrac{x-1}{2x},\ x \neq 0$

Sol.

$$\begin{aligned}
\text{Let } y &= k^{-1}(x)\\
k(y) &= x\\
\frac{y-1}{2y} &= x\\
y-1 &= 2xy\\
y-2xy &= 1\\
y(1-2x) &= 1\\
y &= \frac{1}{1-2x} \quad \left(x \neq \frac{1}{2}\right)
\end{aligned}$$

$$\therefore k^{-1}(x) = \frac{1}{1-2x} \quad \left(x \neq \frac{1}{2}\right)$$

2. Given the function $f : x \to 2x+1$ and $g : x \to \dfrac{1}{x-4},\ x \neq 4$. Find:

(a) f^{-1}

Sol.

$$\begin{aligned}
\text{Let } y &= f^{-1}(x)\\
f(y) &= x\\
2y+1 &= x\\
2y &= x-1\\
y &= \frac{x-1}{2}
\end{aligned}$$

$$\therefore f^{-1}(x) = \frac{x-1}{2}$$

(b) g^{-1}

Sol.

$$\begin{aligned}
\text{Let } y &= g^{-1}(x)\\
g(y) &= x\\
\frac{1}{y-4} &= x\\
1 &= x(y-4)\\
1 &= xy-4x\\
xy &= 4x+1\\
y &= \frac{4x+1}{x} \quad (x \neq 0)
\end{aligned}$$

$$\therefore g^{-1}(x) = \frac{4x+1}{x} \quad (x \neq 0)$$

(c) $f^{-1} \circ g^{-1}$

Sol.

$$\begin{aligned}
f^{-1} \circ g^{-1} &= f^{-1}(g^{-1}(x)) \\
&= f^{-1}\left(\frac{4x+1}{x}\right) \\
&= \frac{\frac{4x+1}{x} - 1}{2} \\
&= \frac{\frac{4x+1-x}{x}}{2} \\
&= \frac{\frac{3x+1}{x}}{2} \\
&= \frac{3x+1}{2x} \quad (x \neq 0)
\end{aligned}$$

(d) $g^{-1} \circ f^{-1}$

Sol.

$$\begin{aligned}
g^{-1} \circ f^{-1} &= g^{-1}(f^{-1}(x)) \\
&= g^{-1}\left(\frac{x-1}{2}\right) \\
&= \frac{4\left(\frac{x-1}{2}\right) + 1}{\frac{x-1}{2}} \\
&= \frac{2x-2+1}{\frac{x-1}{2}} \\
&= \frac{2x-1}{\frac{x-1}{2}} \\
&= \frac{4x-2}{x-1} \quad (x \neq 1)
\end{aligned}$$

(e) $(f \circ g)^{-1}$

Sol.

$$\begin{aligned}
f \circ g &= f(g(x)) \\
&= f\left(\frac{1}{x-4}\right) \\
&= 2\left(\frac{1}{x-4}\right) + 1 \\
&= \frac{2+x-4}{x-4} \\
&= \frac{x-2}{x-4}
\end{aligned}$$

$$\text{Let } y = (f \circ g)^{-1}(x)$$

$$\begin{aligned}
(f \circ g)(y) &= x \\
\frac{y-2}{y-4} &= x \\
y - 2 &= x(y-4) \\
y - xy &= -4x + 2 \\
y(1-x) &= -4x + 2 \\
y &= \frac{-4x+2}{1-x} \\
&= \frac{4x-2}{x-1} \quad (x \neq 1)
\end{aligned}$$

$$\therefore (f \circ g)^{-1}(x) = \frac{4x-2}{x-1} \quad (x \neq 1)$$

(f) $(g \circ f)^{-1}$

Sol.

$$\begin{aligned}
g \circ f &= g(f(x)) \\
&= g(2x+1) \\
&= \frac{1}{(2x+1)-4} \\
&= \frac{1}{2x-3}
\end{aligned}$$

$$\text{Let } y = (g \circ f)^{-1}(x)$$

$$\begin{aligned}
(g \circ f)(y) &= x \\
\frac{1}{2y-3} &= x \\
1 &= x(2y-3) \\
1 &= 2xy - 3x \\
2xy &= 3x + 1 \\
y &= \frac{3x+1}{2x} \quad (x \neq 0)
\end{aligned}$$

$$\therefore (g \circ f)^{-1}(x) = \frac{3x+1}{2x} \quad (x \neq 0)$$

Graph of Inverse Functions

If f is a one to one function, then the graph of f^{-1} is the reflection of the graph of f about the line $y = x$.

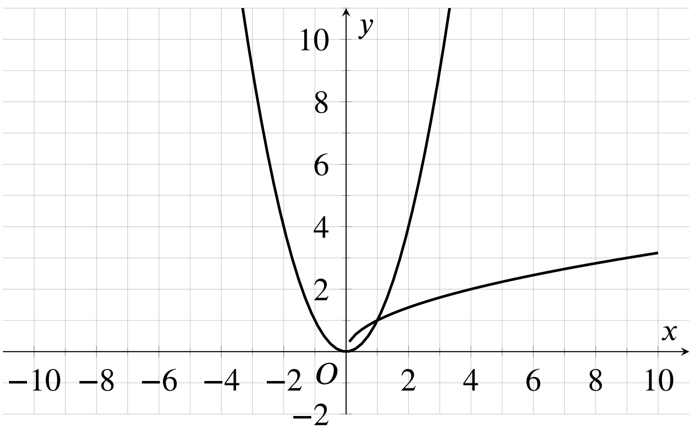
Practice 9

Given the function $g : \mathbb{R}^+ \cup 0 \to \mathbb{R}^+ \cup 0$, $g : x \to x^2$. On the same set of axes, draw the graph of the function g and its inverse function g^{-1}.

Sol.

$$\begin{aligned} \text{Let } y &= g^{-1}(x) \\ g(y) &= x \\ y^2 &= x \\ y &= \pm\sqrt{x} \end{aligned}$$

$$\because R_g = \mathbb{R}^+ \cup 0$$
$$\therefore g^{-1}(x) = \sqrt{x}$$



Exercise 22.6

1. Find the inverse function of the following functions:

(a) $f : x \to 2x - 7$

Sol.

$$\begin{aligned} \text{Let } y &= f^{-1}(x) \\ f(y) &= x \\ 2y - 7 &= x \\ y &= \frac{x+7}{2} \end{aligned}$$

$$\therefore f^{-1}(x) = \frac{x+7}{2}$$

(b) $g : x \to \frac{1}{x-2}$, $x \neq 2$

Sol.

$$\begin{aligned} \text{Let } y &= g^{-1}(x) \\ g(y) &= x \\ \frac{1}{y-2} &= x \\ x(y-2) &= 1 \\ xy - 2x &= 1 \\ xy &= 2x + 1 \\ y &= \frac{2x+1}{x} \quad (x \neq 0) \end{aligned}$$

$$\therefore g^{-1}(x) = \frac{2x+1}{x} \quad (x \neq 0)$$

(c) $h : x \to \dfrac{2x-5}{x-2},\ x \neq 2$

Sol.

$$\begin{aligned}
\text{Let } y &= h^{-1}(x) \\
h(y) &= x \\
\frac{2y-5}{y-2} &= x \\
2y-5 &= x(y-2) \\
2y-5 &= xy-2x \\
2y-xy &= -2x+5 \\
y(2-x) &= -2x+5 \\
y &= \frac{-2x+5}{2-x} \\
&= \frac{2x-5}{x-2} \quad (x \neq 2)
\end{aligned}$$

$$\therefore h^{-1}(x) = \frac{2x-5}{x-2} \quad (x \neq 2)$$

(d) $k : x \to \dfrac{3x}{x-4},\ x \neq 4$

Sol.

$$\begin{aligned}
\text{Let } y &= k^{-1}(x) \\
k(y) &= x \\
\frac{3y}{y-4} &= x \\
3y &= x(y-4) \\
3y &= xy-4x \\
3y-xy &= -4x \\
y(3-x) &= -4x \\
y &= \frac{-4x}{3-x} \\
&= \frac{4x}{x-3} \quad (x \neq 3)
\end{aligned}$$

$$\therefore k^{-1}(x) = \frac{4x}{x-3} \quad (x \neq 3)$$

2. Given that $f : x \to \dfrac{160}{ax+b}$, $f(5) = 8$ and $f(9) = 10$. Find

(a) the values of a and b;

Sol.

$$\begin{aligned}
f(5) &= 8 \\
8 &= \frac{160}{5a+b} \\
8(5a+b) &= 160 \\
40a+8b &= 160 \\
5a+b &= 20 \quad \cdots (1)
\end{aligned}$$

$$\begin{aligned}
f(9) &= 10 \\
10 &= \frac{160}{9a+b} \\
10(9a+b) &= 160 \\
90a+10b &= 160 \\
9a+b &= 16 \quad \cdots (2)
\end{aligned}$$

$$(1)-(2) \implies 4a = -4$$
$$a = -1$$

$$\begin{aligned}
5(-1)+b &= 20 \\
-5+b &= 20 \\
b &= 25
\end{aligned}$$

$$\therefore a = -1,\ b = 25$$

(b) $f^{-1}(16)$.

Sol.

$$
\begin{aligned}
f(x) &= \frac{160}{-x+25} \\
\text{Let } y &= f^{-1}(x) \\
f(y) &= x \\
\frac{160}{-y+25} &= x \\
160 &= x(-y+25) \\
&= -xy + 25x \\
xy &= 25x - 160 \\
y &= \frac{25x-160}{x} \quad (x \neq 0)
\end{aligned}
$$

$$\therefore f^{-1}(x) = \frac{25x-160}{x} \quad (x \neq 0)$$

$$
\begin{aligned}
f^{-1}(16) &= \frac{25(16)-160}{16} \\
&= 15
\end{aligned}
$$

3. Given that $f : x \to \dfrac{a}{x+b}$, $f(3) = -1$, and $f(-9) = 3$. Find

(a) the values of a and b;

Sol.

$$
\begin{aligned}
f(3) &= -1 \\
-1 &= \frac{a}{3+b} \\
-1(3+b) &= a \\
-3 - b &= a \\
-b &= a + 3 \\
b &= -a - 3 \quad \cdots (1)
\end{aligned}
$$

$$
\begin{aligned}
f(-9) &= 3 \\
3 &= \frac{a}{-9+b} \\
-27 + 3b &= a \\
3b &= a + 27 \\
b &= \frac{a+27}{3} \quad \cdots (2)
\end{aligned}
$$

$$
\begin{aligned}
(1) - (2) \implies -\frac{a+27}{3} &= -a - 3 \\
-a - 27 &= -3a - 9 \\
2a &= -18 \\
a &= -9
\end{aligned}
$$

$$
\begin{aligned}
b &= -(-9) - 3 \\
&= 6
\end{aligned}
$$

$$\therefore a = -9,\ b = 6$$

(b) the value of x such that $f(x) = f^{-1}(x)$.

Sol.

$$f(x) = \frac{-9}{x+6}$$

Let $y = f^{-1}(x)$

$$f(y) = x$$

$$\frac{-9}{y+6} = x$$

$$-9 = x(y+6)$$

$$xy = -6x - 9$$

$$y = \frac{-6x-9}{x} \quad (x \neq 0)$$

$$\therefore f(x) = \frac{-6x-9}{x} \quad (x \neq 0)$$

$$f(x) = f^{-1}(x)$$

$$\frac{-9}{x+6} = \frac{-6x-9}{x}$$

$$(-6x-9)(x+6) = -9x$$

$$6x^2 + 45x + 54 = 9x$$

$$6x^2 + 36x + 54 = 0$$

$$x^2 + 6x + 9 = 0$$

$$(x+3)^2 = 0$$

$$x = -3$$

4. Given the function $g : x \to \dfrac{6}{x} - 3,\ x \neq 0$. Find

(a) g^{-1};

Sol.

Let $y = g^{-1}(x)$

$$g(y) = x$$

$$\frac{6}{y} - 3 = x$$

$$\frac{6}{y} = x + 3$$

$$6 = y(x+3)$$

$$y = \frac{6}{x+3} \quad (x \neq -3)$$

$$\therefore g^{-1}(x) = \frac{6}{x+3} \quad (x \neq -3)$$

(b) the value of x such that $g^{-1}(x) = x - 2$.

Sol.

$$\begin{aligned} g^{-1}(x) &= x - 2 \\ \frac{6}{x+3} &= x - 2 \\ 6 &= (x-2)(x+3) \\ 6 &= x^2 + x - 6 \\ x^2 + x - 12 &= 0 \\ (x+4)(x-3) &= 0 \\ x &= -4 \text{ or } x = 3 \end{aligned}$$

5. Given the function $f : x \to ax + b$ and $f^2 : x \to 4x + 12$. If $a > 0$, find

(a) the values of a and b;

Sol.

$$\begin{aligned} f(x) &= ax + b \\ f^2(x) &= f(f(x)) \\ 4x + 12 &= a(ax + b) + b \\ 4x + 12 &= a^2x + ab + b \end{aligned}$$

Comparing both sides,

$$\begin{aligned} a^2 &= 4 \\ a &= 2 \ (a > 0) \end{aligned}$$

$$\begin{aligned} 2b + b &= 12 \\ 3b &= 12 \\ b &= 4 \end{aligned}$$

$$\therefore a = 2,\ b = 4$$

(b) $f^{-1}(3)$.

Sol.

$$\begin{aligned} f(x) &= 2x + 4 \\ \text{Let } y &= f^{-1}(x) \\ f(y) &= x \\ 2y + 4 &= x \\ 2y &= x - 4 \\ y &= \frac{x-4}{2} \end{aligned}$$

$$\begin{aligned} \therefore f^{-1}(x) &= \frac{x-4}{2} \\ f^{-1}(3) &= \frac{3-4}{2} \\ &= -\frac{1}{2} \end{aligned}$$

6. Given the function $f : x \to 3x - 2$ and $g : x \to \dfrac{x}{x+4}, x \neq -4$. Find

(a) f^{-1}

Sol.

$$\begin{aligned} \text{Let } y &= f^{-1}(x) \\ f(y) &= x \\ 3y - 2 &= x \\ 3y &= x + 2 \\ y &= \frac{x+2}{3} \end{aligned}$$

$$\therefore f^{-1}(x) = \frac{x+2}{3}$$

(b) g^{-1}

Sol.

$$\begin{aligned} \text{Let } y &= g^{-1}(x) \\ g(y) &= x \\ \frac{y}{y+4} &= x \\ y &= x(y+4) \\ y &= xy + 4x \\ y - xy &= 4x \\ y(1-x) &= 4x \\ y &= \frac{4x}{1-x} \quad (x \neq 1) \end{aligned}$$

$$\therefore g^{-1}(x) = \frac{4x}{1-x} \quad (x \neq 1)$$

(c) $f^{-1} \circ g^{-1}$

Sol.

$$\begin{aligned} f^{-1} \circ g^{-1} &= f^{-1}(g^{-1}(x)) \\ &= f^{-1}\left(\frac{4x}{1-x}\right) \\ &= \frac{\frac{4x}{1-x} + 2}{3} \\ &= \frac{4x + 2 - 2x}{3(1-x)} \\ &= \frac{2x+2}{3-3x} \quad (x \neq 1) \end{aligned}$$

(d) $g^{-1} \circ f^{-1}$

Sol.

$$\begin{aligned} g^{-1} \circ f^{-1} &= g^{-1}(f^{-1}(x)) \\ &= g^{-1}\left(\frac{x+2}{3}\right) \\ &= \frac{4\left(\frac{x+2}{3}\right)}{1 - \left(\frac{x+2}{3}\right)} \\ &= \frac{\frac{4x+8}{3}}{\frac{3-x-2}{3}} \\ &= \frac{4x+8}{1-x} \quad (x \neq 1) \end{aligned}$$

(e) $(f \circ g)^{-1}$

Sol.

$$\begin{aligned} f \circ g &= f(g(x)) \\ &= f\left(\frac{x}{x+4}\right) \\ &= 3\left(\frac{x}{x+4}\right) - 2 \\ &= \frac{3x}{x+4} - 2 \\ &= \frac{3x - 2x - 8}{x+4} \\ &= \frac{x-8}{x+4} \quad (x \neq -4) \end{aligned}$$

$$\begin{aligned} \text{Let } y &= (f \circ g)^{-1}(x) \\ f(g(y)) &= x \\ \frac{y-8}{y+4} &= x \\ y - 8 &= x(y+4) \\ y &= xy + 4x + 8 \\ y - xy &= 4x + 8 \\ y(1-x) &= 4x + 8 \\ y &= \frac{4x+8}{1-x} \quad (x \neq 1) \end{aligned}$$

$$\therefore (f \circ g)^{-1}(x) = \frac{4x+8}{1-x} \quad (x \neq 1)$$

(f) $(g \circ f)^{-1}$

Sol.

$$\begin{aligned} g \circ f &= g(f(x)) \\ &= g(3x-2) \\ &= \frac{3x-2}{(3x-2)+4} \\ &= \frac{3x-2}{3x+2} \quad \left(x \neq \frac{2}{3}\right) \end{aligned}$$

$$\begin{aligned} \text{Let } y &= (g \circ f)^{-1}(x) \\ g(f(y)) &= x \\ \frac{3y-2}{3y+2} &= x \\ 3y - 2 &= x(3y+2) \\ 3y - 2 &= 3xy + 2x \\ 3y - 3xy &= 2x + 2 \\ y(3-3x) &= 2x + 2 \\ y &= \frac{2x+2}{3-3x} \quad (x \neq 1) \end{aligned}$$

$$\therefore (g \circ f)^{-1}(x) = \frac{2x+2}{3-3x} \quad (x \neq 1)$$

7. Given the function $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{0\}$, $f : x \to \dfrac{1}{x-2}$.

(a) Find f^{-1}.

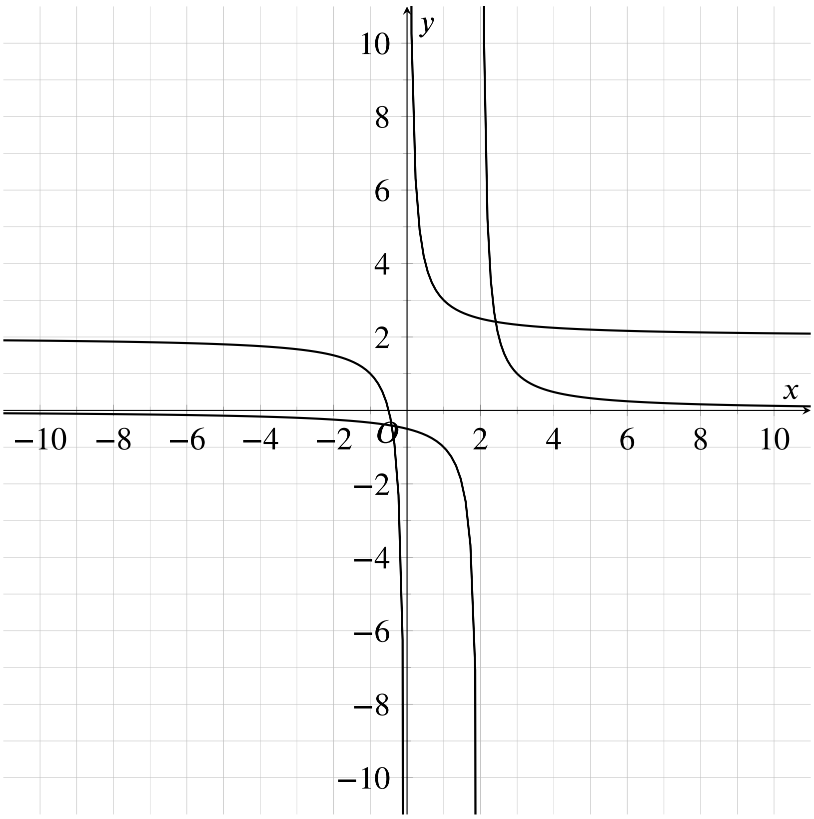
Sol.

$$\begin{aligned}
\text{Let } y &= f^{-1}(x) \\
f(y) &= x \\
\frac{1}{y-2} &= x \\
1 &= x(y-2) \\
1 &= xy - 2x \\
xy &= 2x + 1 \\
y &= \frac{2x+1}{x} \quad (x \neq 0)
\end{aligned}$$

$$\therefore f^{-1}(x) = \frac{2x+1}{x} \quad (x \neq 0)$$

(b) On the same set of axes, draw the graph of f and f^{-1}.

Sol.



8. Given the function $f : x \to 2\sqrt{x+4}$, $x \geq -4$,

(a) Find f^{-1}.

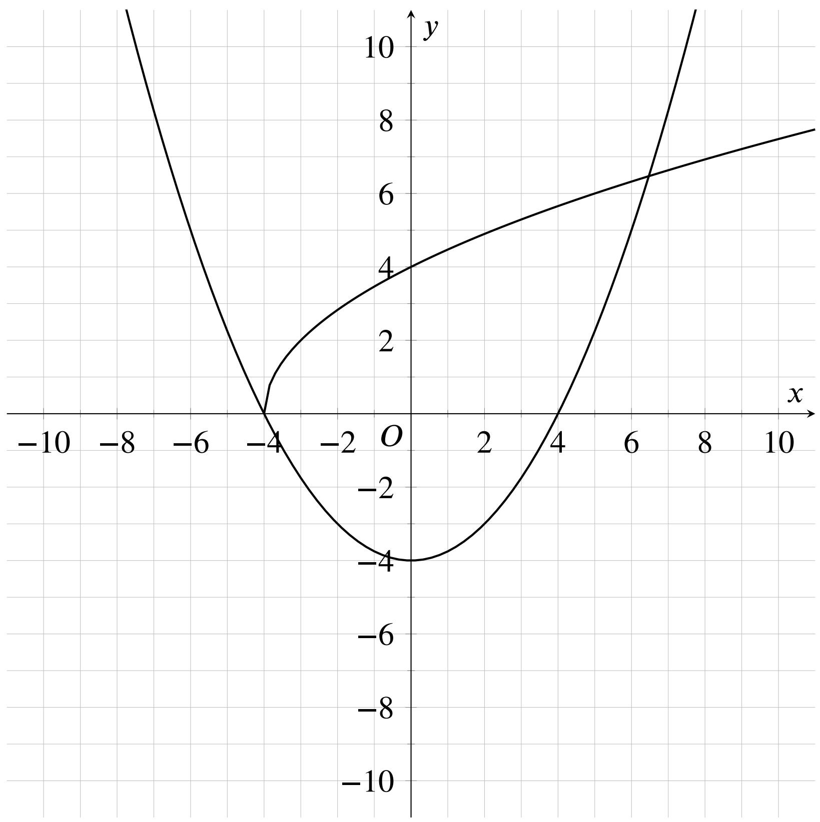
Sol.

$$\begin{aligned}
\text{Let } y &= f^{-1}(x) \\
f(y) &= x \\
2\sqrt{y+4} &= x \\
\sqrt{y+4} &= \frac{x}{2} \\
y+4 &= \frac{x^2}{4} \\
y &= \frac{x^2}{4} - 4
\end{aligned}$$

$$\begin{aligned}
\therefore f^{-1}(x) &= \frac{x^2}{4} - 4 \\
&= \frac{x^2 - 16}{4}
\end{aligned}$$

(b) On the same set of axes, draw the graph of f and f^{-1}.

Sol.



Revision Exercise 22

1. Determine whether the following mappings from set $A = \{1, 2, 3, 4\}$ to set $B = \{a, b, c, d\}$ are functions or not.

   (a) $1 \to a, 2 \to c, 4 \to b$

   Sol.

   Since $3 \in A$ does not have an image in B, this is not a function.

   (b) $1 \to a, 2 \to d, 3 \to b, 4 \to a$

   Sol.

   Since each element in A has an image in B, this is a function.

   (c) $1 \to c, 2 \to c, 3 \to b, 4 \to b$

   Sol.

   Since each element in A has an image in B, this is a function.

   (d) $1 \to a, 2 \to c, 2 \to b, 4 \to d$

   Sol.

   Since $2 \in A$ has two images b and c in B, $3 \in A$ has two images in B, this is not a function.

   (e) $1 \to c, 2 \to b, 3 \to d, 4 \to c, 4 \to a$

   Sol.

   Since $4 \in A$ has two images c and a in B, this is not a function.

2. Given the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 3x - 2, & x < -3 \\ 2x^2 + 4, & -3 \le x < 2 \\ -2x + 9, & x \ge 2 \end{cases}$, find

   (a) $f(-4)$

   Sol.

   $$\begin{aligned} f(-4) &= 3(-4) - 2 \\ &= -14 \end{aligned}$$

   (b) $f(0)$

   Sol.

   $$\begin{aligned} f(0) &= 2(0)^2 + 4 \\ &= 4 \end{aligned}$$

   (c) $f(2)$

   Sol.

   $$\begin{aligned} f(2) &= -2(2) + 9 \\ &= 5 \end{aligned}$$

   (d) $f(3)$

   Sol.

   $$\begin{aligned} f(3) &= -2(3) + 9 \\ &= 3 \end{aligned}$$

3. Find the domain and range of the following functions:

   (a) $f : 1 \to 3, 2 \to 5, 4 \to 8$

   Sol.

   $D_f = \{1, 2, 4\}$, $R_f = \{3, 5, 8\}$

   (b) $g : 2 \to 4, 4 \to 5, 5 \to 7, 6 \to 9$

   Sol.

   $D_g = \{2, 4, 5, 6\}$, $R_g = \{4, 5, 7, 9\}$

   (c) $h : 1 \to 3, 2 \to 5, 3 \to 6, 4 \to 8$

   Sol.

   $D_h = \{1, 2, 3, 4\}$, $R_h = \{3, 5, 6, 8\}$

4. The table below shows a function f:

x	-3	-2	-1	0	1
f(x)	-22	-3	4	5	6

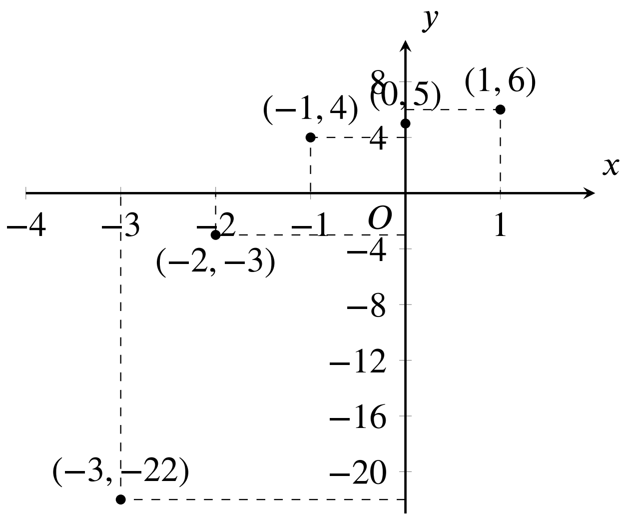
   (a) Find the domain and range of the function;

   Sol.

   $D_f = \{-3, -2, -1, 0, 1\}$, $R_f = \{-22, -3, 4, 5, 6\}$

   (b) Express the function using graph.

   Sol.



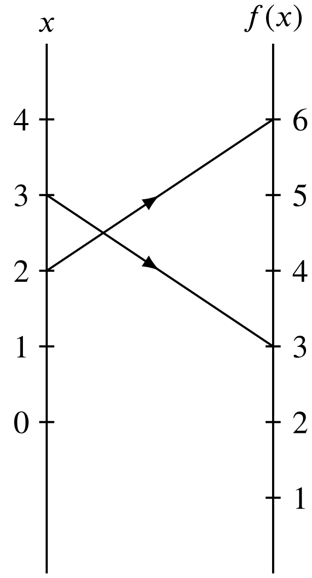
   (c) Determine if the inverse function of f exists.

   Sol.

   Since each element in the codomain of f is mapped to exactly one element in the domain of f, the function f is a one-to-one function. Since each element in the codomain of f has preimage in the domain of f, the function f is an onto function.

   Hence, the function f is a one-one onto function. According to the definition of inverse function, the inverse function of f exists.

5. As shown in the diagram below, let a function $f : x \to ax + b$. Find the value of $f(4)$ and $f^{-1}(5)$.



Sol.

$$\begin{aligned} f(3) &= 3a + b = 3 \\ f(2) &= 2a + b = 6 \\ f(3) - f(2) &= a = -3 \\ 3(-3) + b &= 3 \\ b &= 12 \end{aligned}$$

$$\therefore f(x) = -3x + 12$$

$$f(4) = -3(4) + 12 = 0$$

$$\begin{aligned} &\text{Let } y = f^{-1}(x) \\ f(y) &= x \\ -3y + 12 &= x \\ y &= -\frac{x-12}{3} \end{aligned}$$

$$\begin{aligned} \therefore f^{-1}(x) &= -\frac{x-12}{3} \\ f^{-1}(5) &= -\frac{5-12}{3} \\ &= \frac{7}{3} \end{aligned}$$

6. Given the function $f : x \to x^2 - x + 1$, $-1 \le x \le 3$, find its range.

Sol.

$$\begin{aligned} f(x) &= x^2 - x + 1 \\ &= \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \end{aligned}$$

Vertex : $\left(\frac{1}{2}, \frac{3}{4}\right)$

$\because a > 0, y_{\min} = \frac{3}{4}$

$$f(-1) = (-1)^2 - (-1) + 1 = 3$$

$$f(3) = 3^2 - 3 + 1 = 7$$

$$\therefore R_f = \left\{ y \mid y \in \mathbb{R}, \frac{3}{4} \le y \le 7 \right\}$$

7. Let function $f : x \to 2x^2 - 4x + 3$.

(a) If $D_f = \mathbb{R}$, find the range of f;

Sol.

$$\begin{aligned} f(x) &= 2x^2 - 4x + 3 \\ &= 2(x^2 - 2) + 3 \\ &= 2(x-1)^2 + 1 \end{aligned}$$

$$\text{Vertex} : (1, 1)$$

$$\because a > 0, y_{\min} = 1$$

$$\therefore R_f = \{y | y \in \mathbb{R}, y \geq 1\}$$

(b) If $D_f = \{x | x \geq 3\}$, find the range of f.

Sol.

$$f(3) = 2(3)^2 - 4(3) + 3 = 9$$

$$\therefore R_f = \{y | y \in \mathbb{R}, y \geq 9\}$$

8. Find the domain and range of the following functions:

(a) $f(x) = \dfrac{1}{x}$

Sol.

$\because f(x)$ is defined when $x \neq 0$,

$\therefore D_f = \mathbb{R} \setminus \{0\}$.

$\because f(x) = \dfrac{1}{x} \neq 0$,

$\therefore R_f = \mathbb{R} \setminus \{0\}$.

(b) $f(x) = \sqrt{2x - 5}$

Sol.

$\because f(x)$ is defined when $2x - 5 \geq 0$,

$\therefore D_f = \left\{x | x \geq \dfrac{5}{2}\right\}$.

$\because f(x) = \sqrt{2x - 5} \geq 0$,

$\therefore R_f = \{y | y \geq 0\}$.

(c) $f(x) = x^2 + 4x + 7$

Sol.

$\because f(x)$ is defined for all $x \in \mathbb{R}$,

$\therefore D_f = \mathbb{R}$.

$$\begin{aligned} f(x) &= x^2 + 4x + 7 \\ &= (x+2)^2 + 3 \end{aligned}$$

$$\text{Vertex} : (-2, 3)$$

$$\because a > 0, y_{\min} = 3$$

$$\therefore R_f = \{y | y \in \mathbb{R}, y \geq 3\}$$

(d) $f(x) = \dfrac{1}{x^2 + 4}$

Sol. $\because x^2 + 4 \geq 4$ for all $x \in \mathbb{R}$,

$\therefore D_f = \mathbb{R}$.

$\because f(x) = x^2 + 4 \geq 4$ for all $x \in \mathbb{R}$,

$\therefore f(x) \geq \dfrac{1}{4}$ for all $x \in \mathbb{R}$,

$\therefore R_f = \left\{y | y \in \mathbb{R}, y \geq \dfrac{1}{4}\right\}$.

9. Find the domain of the following functions:

(a) $f(x) = \dfrac{2x}{x-3}$

Sol.

$\because f(x)$ is defined when $x - 3 \neq 0$,

$\therefore D_f = \mathbb{R} \setminus \{3\}$.

(b) $f(x) = \sqrt{4 - x^2}$

Sol.

$\because f(x)$ is defined when $4 - x^2 \geq 0$,

$\therefore D_f = \{x | x \in \mathbb{R}, -2 \leq x \leq 2\}$.

(c) $f(x) = \dfrac{x-2}{2x^2 - 5x + 2}$

Sol.

$\because f(x)$ is defined when $2x^2 - 5x + 2 \neq 0$,

$\therefore D_f = \left\{x | x \in \mathbb{R}, x \neq \dfrac{1}{2}, x \neq 2\right\}$.

(d) $f(x) = \dfrac{x-3}{\sqrt{x^2 - 9}}$

Sol.

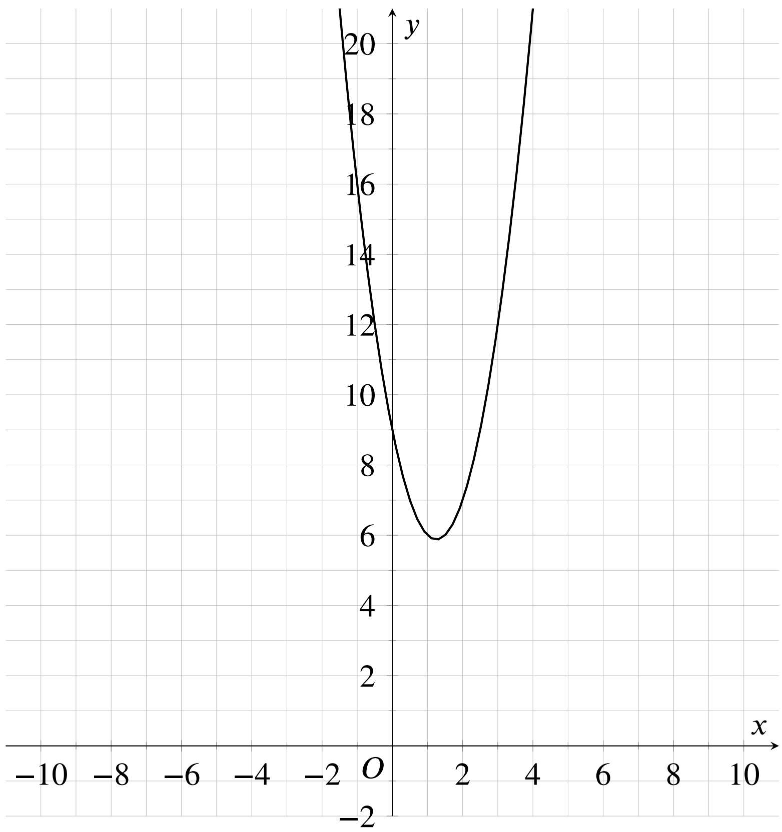
$\because f(x)$ is defined when $x^2 - 9 > 0$,

$\therefore D_f = \{x | x \in \mathbb{R}, x < -3 \text{ or } x > 3\}$.

10. Sketch the graph for the following functions:

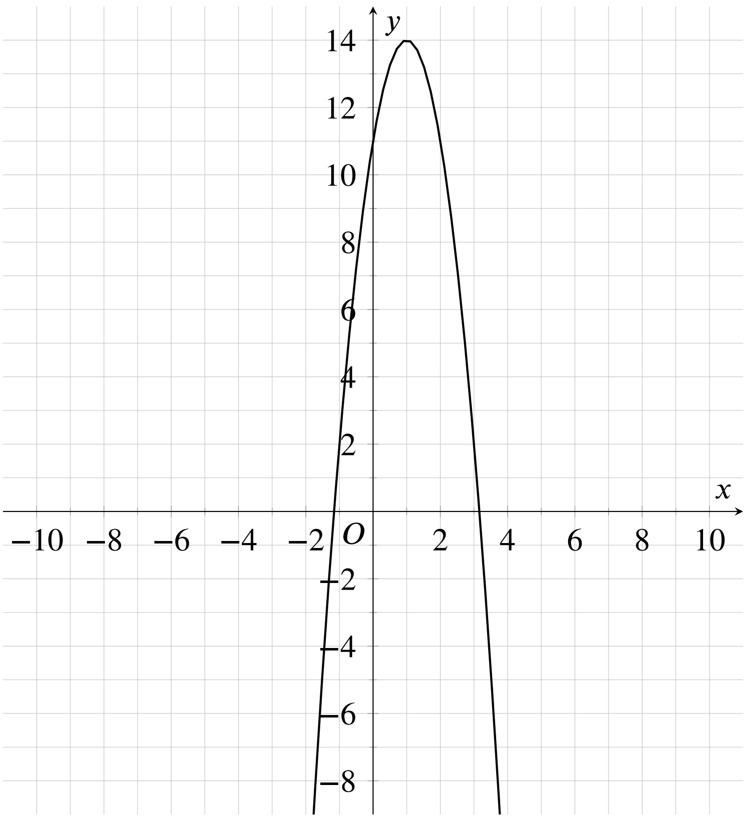
(a) $f(x) = 2x^2 - 5x + 9$

Sol.



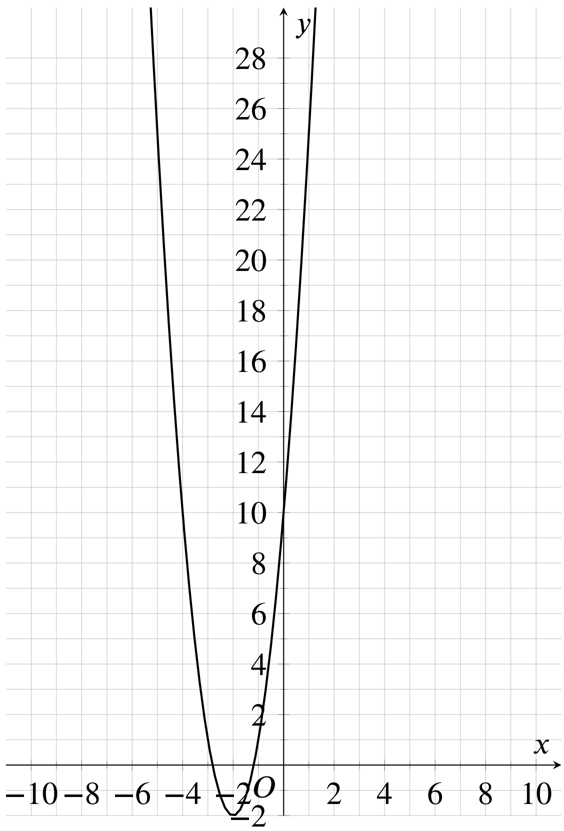
(b) $f(x) = -3x^2 + 6x + 11$

Sol.



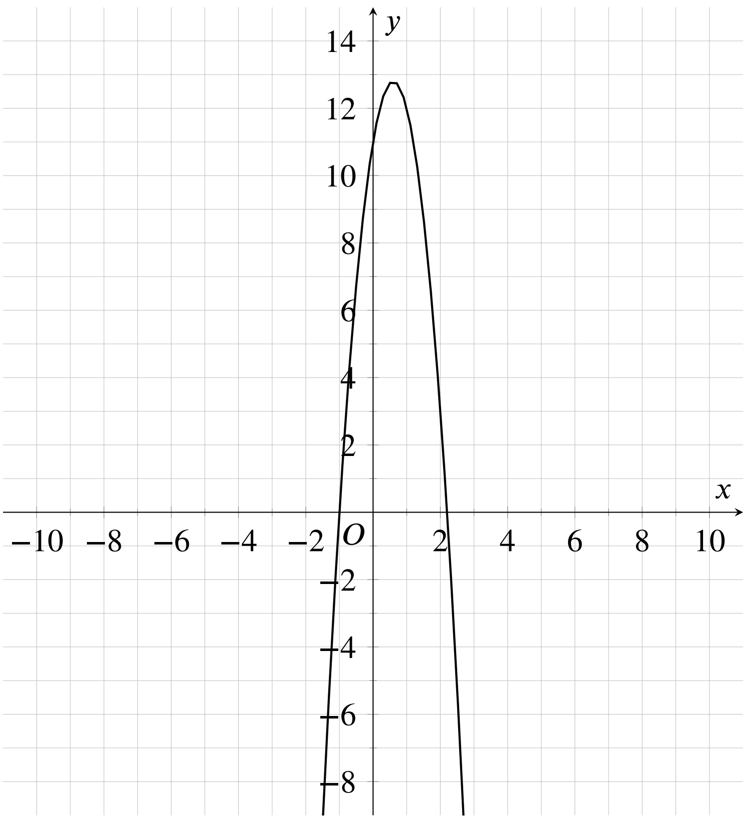
(c) $f(x) = 3x^2 + 12x + 10$

Sol.



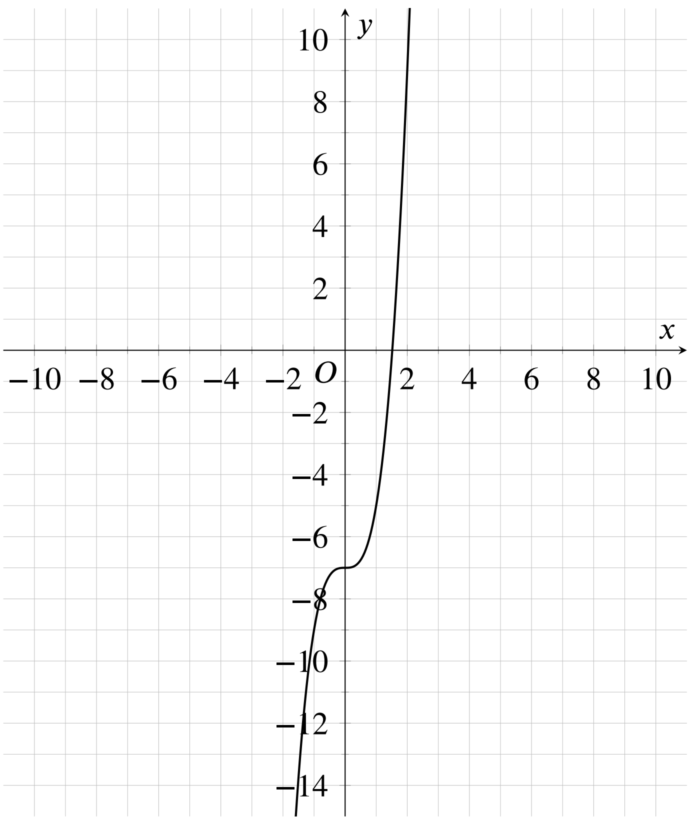
(d) $f(x) = -5x^2 + 6x + 11$

Sol.



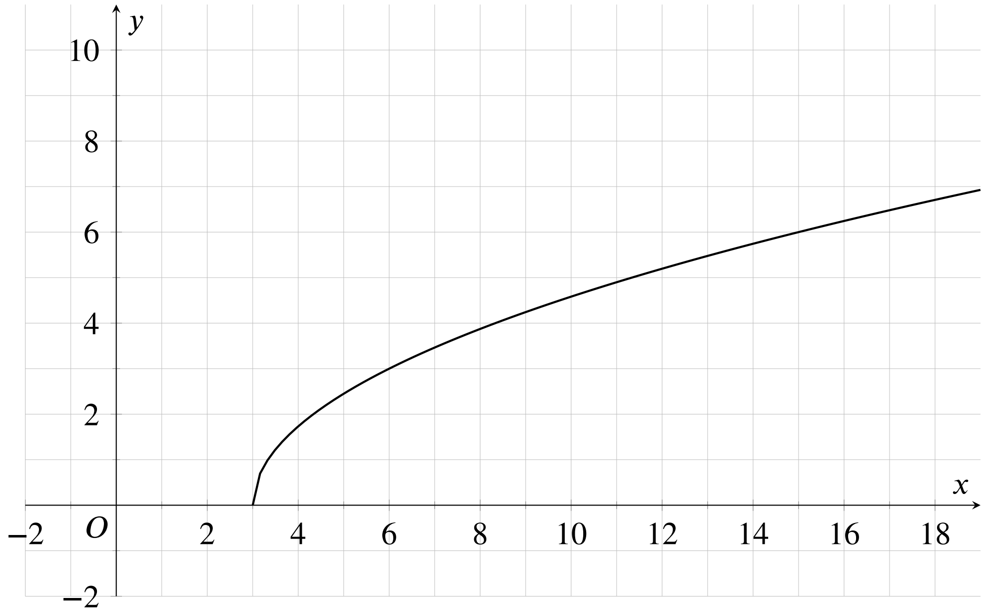
(e) $f(x) = 2x^3 - 7$

Sol.



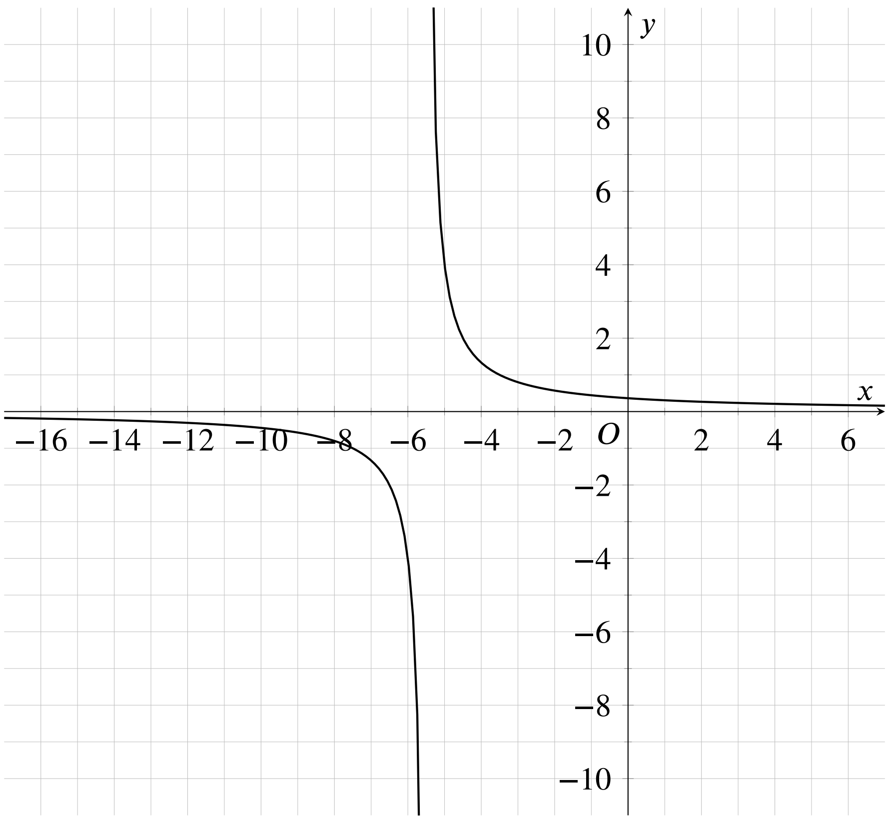
(f) $f(x) = \sqrt{3x - 9}$

Sol.



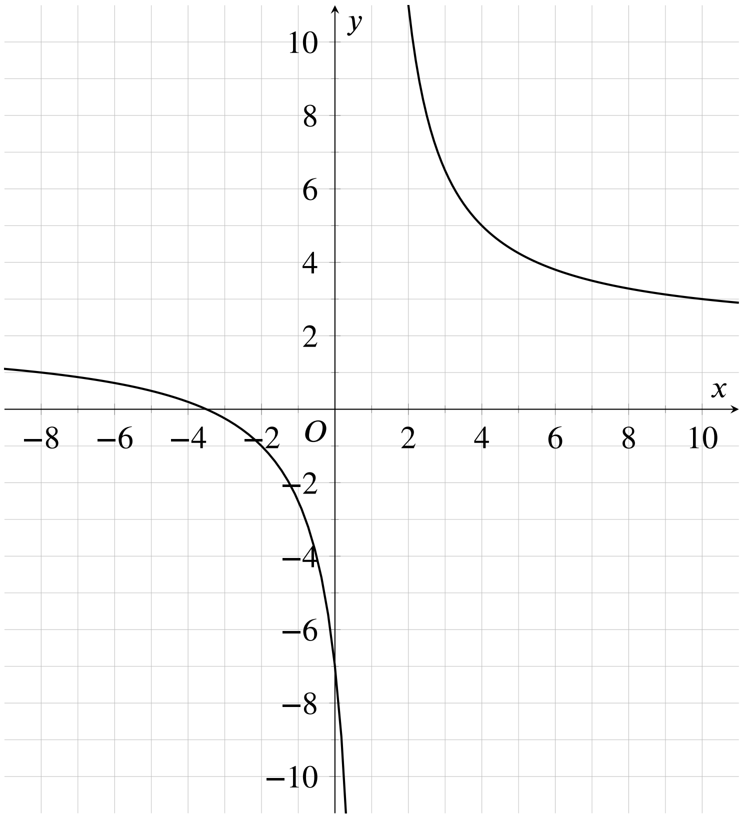
(g) $f(x) = \dfrac{4}{2x + 11}$

Sol.



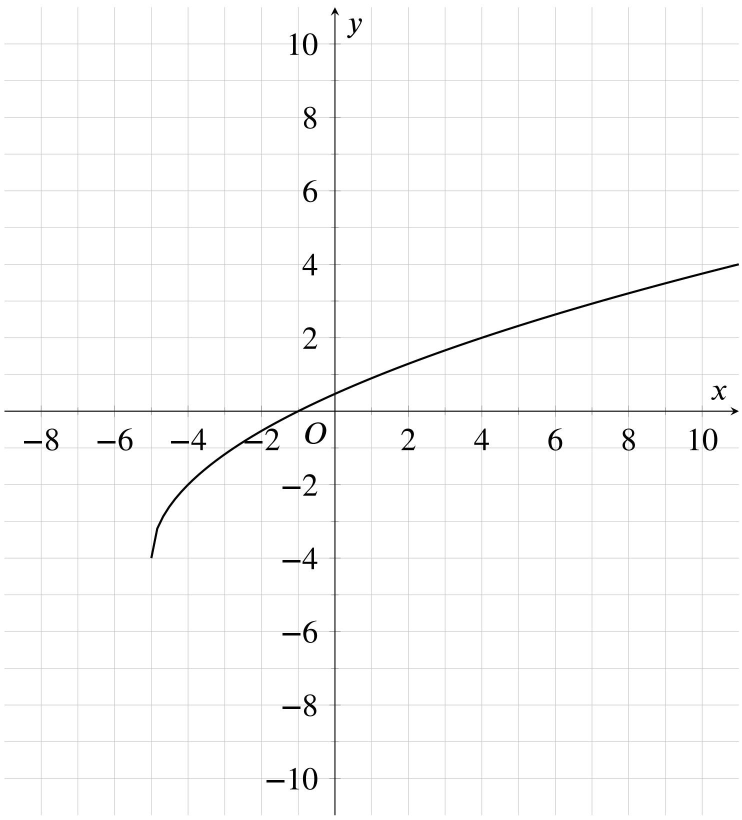
(h) $f(x) = \dfrac{2x+7}{x-1}$

Sol.



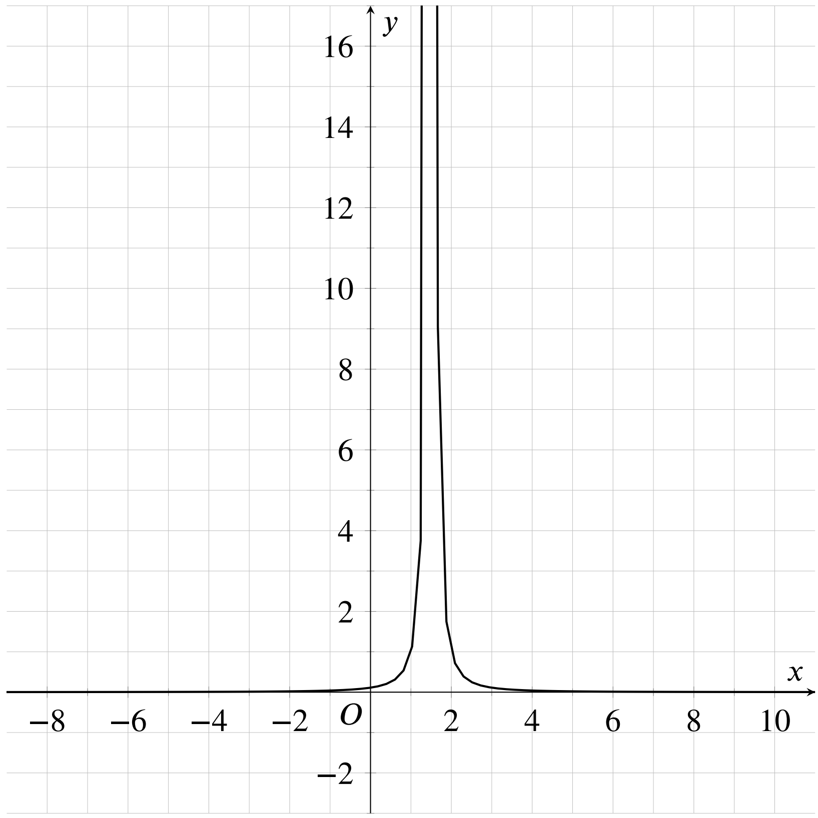
(i) $f(x) = 2\sqrt{x+5} - 4$

Sol.



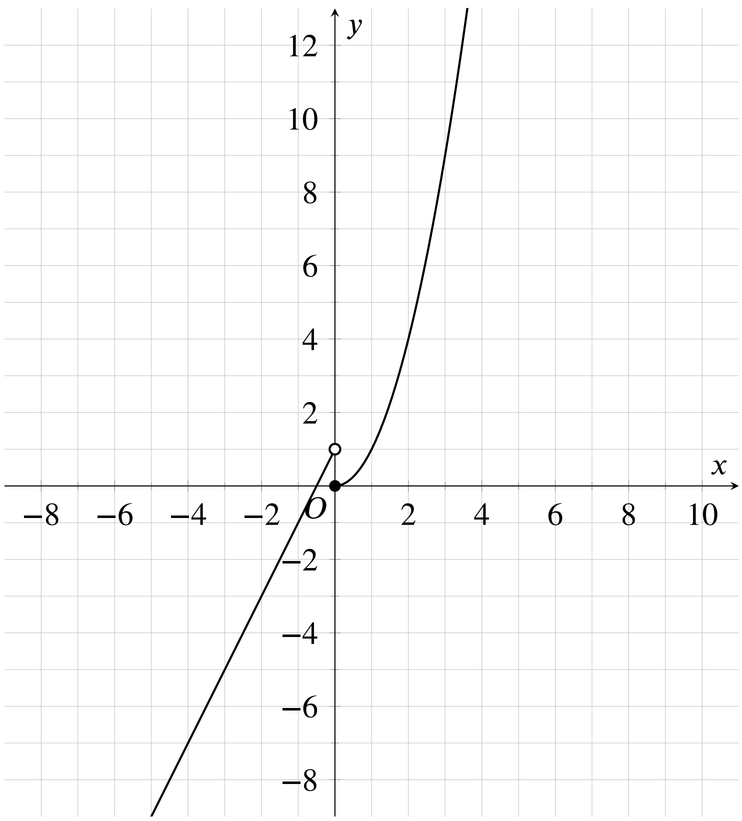
(j) $f(x) = \dfrac{1}{(2x-3)^2}$

Sol.



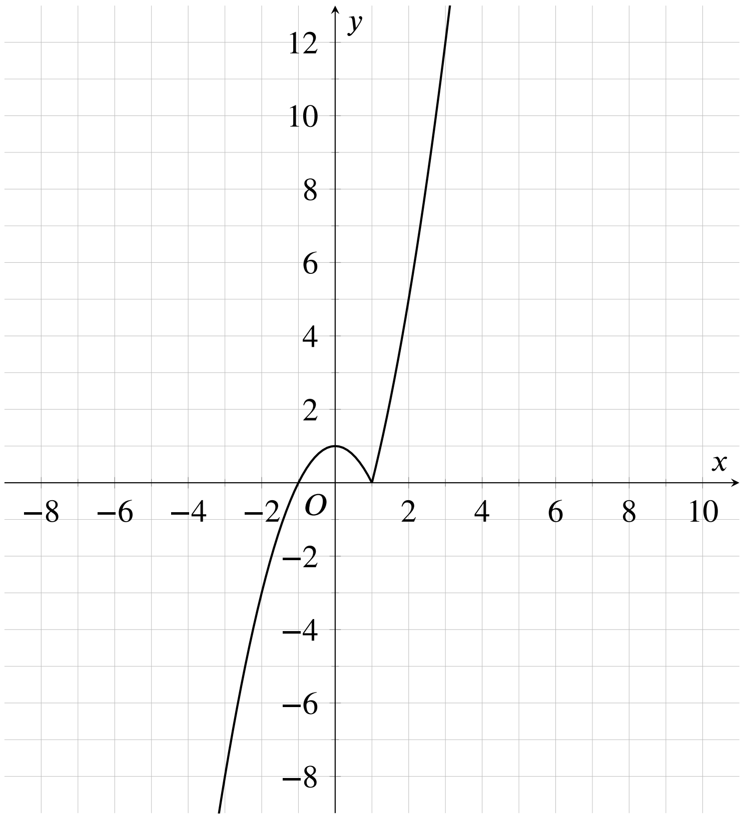
(k) $f(x) = \begin{cases} 2x+1, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

Sol.



(l) $f(x) = \begin{cases} 1 - x^2, & x \le 1 \\ x^2 + 2x - 3, & x > 1 \end{cases}$

Sol.



11. Given the function $f : x \to 2x^2$ and $g : x \to 3x - 4$. Find the value of m such that $(f \circ g)(m) = (g \circ f)(m)$.

Sol.

$$\begin{aligned}
(f \circ g)(m) &= (g \circ f)(m) \\
f(g(m)) &= g(f(m)) \\
f(3m-4) &= g(2m^2) \\
2(3m-4)^2 &= 3(2m^2) - 4 \\
18m^2 - 48m + 32 &= 6m^2 - 4 \\
12m^2 - 48m + 36 &= 0 \\
3m^2 - 12m + 9 &= 0 \\
(3m-3)(m-3) &= 0 \\
m &= 3 \text{ or } m = 1
\end{aligned}$$

12. Given the function $f : x \to x^2 + 2x - 3$ and $g : x \to 3x - 4$. If $(f \circ g)(k) = (g \circ f)(k)$, find the value of k.

Sol.

$$\begin{aligned}
(f \circ g)(k) &= (g \circ f)(k) \\
f(g(k)) &= g(f(k)) \\
f(3k-4) &= g(k^2 + 2k - 3) \\
(3k-4)^2 + 2(3k-4) - 3 &= 3(k^2 + 2k - 3) - 4 \\
9k^2 - 24k + 16 + 6k - 8 - 3 &= 3k^2 + 6k - 9 - 4 \\
9k^2 - 18k + 5 &= 3k^2 + 6k - 13 \\
6k^2 - 24k + 18 &= 0 \\
k^2 - 4k + 3 &= 0 \\
(k-3)(k-1) &= 0 \\
k &= 3 \text{ or } k = 1
\end{aligned}$$

13. Given that $f(x) = 3x + 1$, $x \neq 0$. If $(f \circ g)(x) = 6x^2 - 9x + 4$, find $g(x)$.

Sol.

$$\begin{aligned}
(f \circ g)(x) &= 6x^2 - 9x + 4 \\
f(g(x)) &= 6x^2 - 9x + 4 \\
3g(x) + 1 &= 6x^2 - 9x + 4 \\
3g(x) &= 6x^2 - 9x + 3 \\
g(x) &= 2x^2 - 3x + 1
\end{aligned}$$

14. Given that $f(x) = \dfrac{x+1}{x}$, $x \neq 0$. If $(f \circ g)(x) = x$, find $g(x)$.

Sol.

$$\begin{aligned}
(f \circ g)(x) &= x \\
f(g(x)) &= x \\
\frac{g(x)+1}{g(x)} &= x \\
g(x) + 1 &= xg(x) \\
g(x) - xg(x) &= -1 \\
g(x)(1-x) &= -1 \\
g(x) &= \frac{-1}{1-x} \\
&= \frac{1}{x-1} \quad (x \neq 1)
\end{aligned}$$

15. A function f is defined by $f : x \to x - 3$. Find another function g such that $g \circ f : x \to 4x^2 - 20x + 25$.

Sol.

$$\begin{aligned}(g \circ f)(x) &= 4x^2 - 20x + 25 \\ g(f(x)) &= 4x^2 - 20x + 25 \\ g(x-3) &= 4x^2 - 20x + 25\end{aligned}$$

$$\begin{aligned}\text{Let } y &= x - 3 \\ x &= y + 3\end{aligned}$$

$$\begin{aligned}g(y) &= 4(y+3)^2 - 20(y+3) + 25 \\ &= 4(y^2 + 6y + 9) - 20y - 60 + 25 \\ &= 4y^2 + 24y + 36 - 20y - 35 \\ &= 4y^2 + 4y + 1 \\ &= (2y+1)^2\end{aligned}$$

$$\therefore g(x) = (2x+1)^2$$

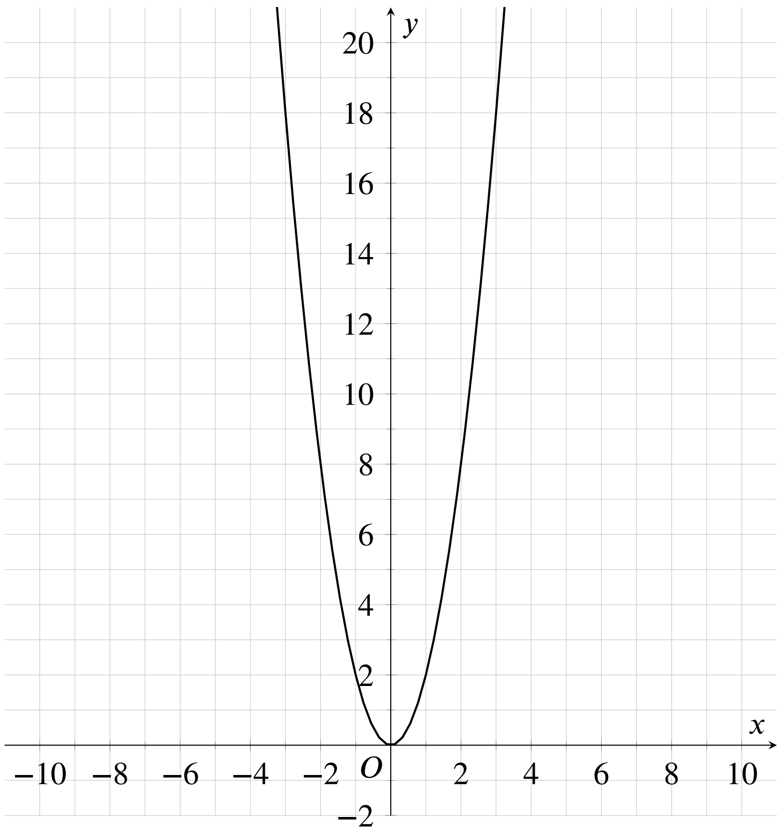
16. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} -2, & x \le -3 \\ |x| - 2x, & -3 < x < 3 \\ 2x - 1, & x \ge 3 \end{cases}$. Find $(f \circ f \circ f)(-1000)$.

Sol.

$$\begin{aligned}(f \circ f \circ f)(-1000) &= f(f(f(-1000))) \\ &= f(f(-2)) \\ &= f(|-2| - 2(-2)) \\ &= f(2+4) \\ &= f(6) \\ &= 2(6) - 1 \\ &= 11\end{aligned}$$

17. Let function $f : A \to \mathbb{R}$ be defined by $f : x \to 2x^2$. Determine if f is one to one function when A is the following sets.

Sol.



(a) $A = \{x|0 \le x < 6\}$

Sol.

Since any real number x has at most one preimage in A, f is a one to one function.

(b) $A = \{x|x < 0\}$

Sol.

Since $f(x) = 2x^2 >= 0$ for all $x \in \mathbb{R}$, all elements in A have no image in R, f is neither a one to one function nor a function.

(c) $A = \{x| - 2 \le x < 2\}$

Sol.

Since any real number x has at most one preimage in A, f is a one to one function.

(d) $A = \{x|x > 3\}$

Sol.

Since any real number x has at most one preimage in A, f is a one to one function.

18. Determine whether the following functions are one to one functions or onto functions.

    (a) $f : \mathbb{R}^+ \to \mathbb{R}, f : x \to |x| - 2$

    Sol.

    Since any real number x in the codomain has at most one preimage in the domain, f is a one to one function. (Since the domain is limited to positive real numbers)

    Since $x < -2$ has no preimage in the domain, f is not an onto function.

    (b) $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{1\}, f : x \to \dfrac{x}{x-2}$

    Sol.

    Since any real number x in the codomain has at most one preimage in the domain, f is a one to one function.

    Since any real number x except $x = 1$ has at least one preimage in the domain, f is an onto function.

    Hence, f is a one-one onto function.

    (c) $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}, f : x \to |x|$

    Sol.

    Since any real number x in the codomain has two preimages in the domain, f is a one to one function. For example, $f(1) = 1$ and $f(-1) = 1$.

    Since any real number x in the codomain has at least one preimage in the domain, f is an onto function.

19. Let $A = \mathbb{R} \setminus \left\{-\frac{1}{2}\right\}$ and $B = \mathbb{R} \setminus \left\{\frac{1}{2}\right\}$, function $f : A \to B$ is defined by $f(x) = \dfrac{x-3}{2x+1}$. Find

    Sol.

$$
\begin{aligned}
\text{Let } y &= f^{-1}(x) \\
f(y) &= x \\
\frac{y-3}{2y+1} &= x \\
y - 3 &= 2xy + x \\
y - 2xy &= x + 3 \\
y(1-2x) &= x + 3 \\
f^{-1}(x) = y &= \frac{x+3}{1-2x}
\end{aligned}
$$

    (a) $f^{-1}(-2)$

    Sol.

    $$f^{-1}(-2) = \frac{-2+3}{1-2(-2)} = \frac{1}{5}$$

    (b) $f^{-1}(0)$

    Sol.

    $$f^{-1}(0) = \frac{0+3}{1-2(0)} = 3$$

    (c) $f^{-1}(3)$

    Sol.

    $$f^{-1}(3) = \frac{3+3}{1-2(3)} = -\frac{6}{5}$$

20. Let function $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = x^2 + 2x + 1$. Find $f^{-1}(4)$ and $f^{-1}(9)$.

Sol.

$$\begin{aligned}
\text{Let } y &= f^{-1}(x) \\
f(y) &= x \\
y^2 + 2y + 1 &= x \\
(y+1)^2 &= x \\
y + 1 &= \sqrt{x} \\
y &= \sqrt{x} - 1
\end{aligned}$$

$$\therefore f^{-1}(x) = \sqrt{x} - 1$$

$$\begin{aligned}
f^{-1}(4) &= \sqrt{4} - 1 = 1 \\
f^{-1}(9) &= \sqrt{9} - 1 = 2
\end{aligned}$$

21. A function f is defined by $f : x \to \dfrac{x}{2} + 1$. If $g \circ f^{-1} : x \to 4x^2 - 8x + 7$, find the function g.

Sol.

$$\begin{aligned}
\text{Let } y &= f^{-1}(x) \\
f(y) &= x \\
\frac{y}{2} + 1 &= x \\
y &= 2x - 2 \\
f^{-1}(x) &= 2x - 2
\end{aligned}$$

$$\begin{aligned}
g \circ f^{-1} &= g(2x - 2) \\
g(2x - 2) &= 4x^2 - 8x + 7 \\
\text{Let } z &= 2x - 2 \\
x &= \frac{z+2}{2} \\
g(z) &= 4\left(\frac{z+2}{2}\right)^2 - 8\left(\frac{z+2}{2}\right) + 7 \\
&= 4\left(\frac{z^2 + 4z + 4}{4}\right) - 4z - 8 + 7 \\
&= z^2 + 4z + 4 - 4z - 1 \\
&= z^2 + 3 \\
\therefore g(x) &= x^2 + 3
\end{aligned}$$

22. Given the functioon $f : x \to 3\left(x + \frac{5}{6}\right)^2 + \frac{25}{12}, x \leq a$. Find the maximum value of a such that the inverse function of f exists.

Sol.

The inverse function of f exists if f is a one-one onto function. For f to be a one-one onto function, $f(x) \geq 0$.

Since $3x^2 + 5x + 9$ is a quadratic function that opens upwards, any y value greater than its vertex will corresponds to two x values. Therefore, f is not a one-one function above its vertex.

$$\begin{aligned}
3x^2 + 5x + 9 &= 3\left(x^2 + \frac{5}{3}x\right) + 9 \\
&= 3\left(x^2 + \frac{5}{3}x + \frac{25}{36} - \frac{25}{36}\right) + 9 \\
&= 3\left(x + \frac{5}{6}\right)^2 - \frac{25}{12} + 9 \\
&= 3\left(x + \frac{5}{6}\right)^2 + \frac{83}{12}
\end{aligned}$$

$$\text{Vertex } = \left(-\frac{5}{6}, \frac{83}{12}\right)$$

Therefore, f is a one-one onto function if $x \leq -\frac{5}{6}$, i.e. $a = -\frac{5}{6}$.

23. Let the function f and g be defined as $f : x \to 5x + 3$ and $g : x \to 2x - 7$ respectively. Find

(a) $f \circ g$

Sol.

$$\begin{aligned}
f \circ g &= f(2x - 7) \\
&= 5(2x - 7) + 3 \\
&= 10x - 35 + 3 \\
&= 10x - 32
\end{aligned}$$

(b) $(f \circ g)^{-1}$

Sol.

$$\begin{aligned}
\text{Let } y &= f \circ g^{-1}(x) \\
f(g(y)) &= x \\
10y - 32 &= x \\
y &= \frac{x + 32}{10}
\end{aligned}$$

$$\therefore f \circ g^{-1}(x) = \frac{x + 32}{10}$$

(c) $g^{-1} \circ f^{-1}$

Sol.

$$\begin{aligned} \text{Let } y &= g^{-1}(x) \\ g(y) &= x \\ 2y - 7 &= x \\ y &= \frac{x+7}{2} \\ \therefore g^{-1}(x) &= \frac{x+7}{2} \end{aligned}$$

$$\begin{aligned} \text{Let } z &= f^{-1}(x) \\ f(z) &= x \\ 5z + 3 &= x \\ z &= \frac{x-3}{5} \\ \therefore f^{-1}(x) &= \frac{x-3}{5} \end{aligned}$$

$$\begin{aligned} g^{-1} \circ f^{-1}(x) &= g^{-1}\left(\frac{x-3}{5}\right) \\ &= \frac{\frac{x-3}{5} + 7}{2} \\ &= \frac{x - 3 + 35}{10} \\ &= \frac{x+32}{10} \end{aligned}$$

24. Given the function $f : x \to 2x + 3$ and $g : x \to 3 - x2x + 5, x \neq -\frac{5}{2}$. Find

(a) $f \circ g$

Sol.

$$\begin{aligned} f \circ g &= f\left(\frac{3-x}{2x+5}\right) \\ &= 2\left(\frac{3-x}{2x+5}\right) + 3 \\ &= \frac{6-2x}{2x+5} + \frac{3(2x+5)}{2x+5} \\ &= \frac{6 - 2x + 6x + 15}{2x+5} \\ &= \frac{4x+21}{2x+5} \quad \left(x \neq -\frac{5}{2}\right) \end{aligned}$$

(b) f^{-1}

Sol.

$$\begin{aligned} \text{Let } y &= f^{-1}(x) \\ f(y) &= x \\ 2y + 3 &= x \\ y &= \frac{x-3}{2} \\ \therefore f^{-1}(x) &= \frac{x-3}{2} \end{aligned}$$

(c) g^{-1}

Sol.

$$\begin{aligned} \text{Let } y &= g^{-1}(x) \\ g(y) &= x \\ \frac{3-y}{2y+5} &= x \\ 3 - y &= 2xy + 5x \\ y(2x+1) &= 3 - 5x \\ \therefore g^{-1}(x) = y &= \frac{3-5x}{2x+1} \quad \left(x \neq -\frac{1}{2}\right) \end{aligned}$$

Show that $g^{-1} \circ f^{-1} = (f \circ g)^{-1}$.

Sol.

$$
\begin{aligned}
g^{-1} \circ f^{-1} &= g^{-1}\left(\frac{x-3}{2}\right) \\
&= \frac{3-5\left(\frac{x-3}{2}\right)}{2\left(\frac{x-3}{2}\right)+1} \\
&= \frac{\frac{6-5x+15}{2}}{x-3+1} \\
&= \frac{21-5x}{2x-4}
\end{aligned}
$$

$$
\begin{aligned}
\text{Let } y &= (f \circ g)^{-1}(x) \\
(f \circ g)(y) &= x \\
\frac{4y+21}{2y+5} &= x \\
4y+21 &= 2xy+5x \\
y(2x-4) &= 21-5x \\
y &= \frac{21-5x}{2x-4} \quad (x \neq 2)
\end{aligned}
$$

$$\therefore (f \circ g)^{-1}(x) = \frac{21-5x}{2x-4}$$

$\therefore g^{-1} \circ f^{-1} = (f \circ g)^{-1}$ (shown)

25. Given the function $f : x \to \sqrt{x}$, $x \neq 0$ and $g : x \to x^3$. Find

(a) $g \circ f$

Sol.

$$
\begin{aligned}
g \circ f &= g\left(\sqrt{x}\right) \\
&= \sqrt{x^3}
\end{aligned}
$$

(b) f^{-1}

Sol.

$$
\begin{aligned}
\text{Let } y &= f^{-1}(x) \\
f(y) &= x \\
\sqrt{y} &= x \\
y &= x^2
\end{aligned}
$$

$$\therefore f^{-1}(x) = x^2$$

(c) g^{-1}

Sol.

$$
\begin{aligned}
\text{Let } y &= g^{-1}(x) \\
g(y) &= x \\
y^3 &= x \\
y &= \sqrt[3]{x}
\end{aligned}
$$

$$\therefore g^{-1}(x) = y = \sqrt[3]{x}$$

(d) $(g \circ f)^{-1}$

Sol.

$$
\begin{aligned}
\text{Let } y &= (g \circ f)^{-1}(x) \\
(g \circ f)(y) &= x \\
\sqrt{y^3} &= x \\
y^3 &= x^2
\end{aligned}
$$

$$\therefore (g \circ f)^{-1}(x) = y = \sqrt[3]{x^2}$$

(e) $g^{-1} \circ f^{-1}$

Sol.

$$
\begin{aligned}
g^{-1} \circ f^{-1} &= g^{-1}\left(x^2\right) \\
&= \sqrt[3]{x^2}
\end{aligned}
$$

26. Given the function $f : x \to 2\sqrt{x-4}+3$, $x \geq 4$.

(a) Find the range of f.

Sol.

$\because 2\sqrt{x-4} \geq 0$ for all $x \geq 4$

$\therefore 2\sqrt{x-4}+3 \geq 3$ for all $x \geq 4$

$\therefore R_f = \{y | y \in \mathbb{R}, y \geq 3\}$

(b) Find the inverse function f^{-1} of the function f.

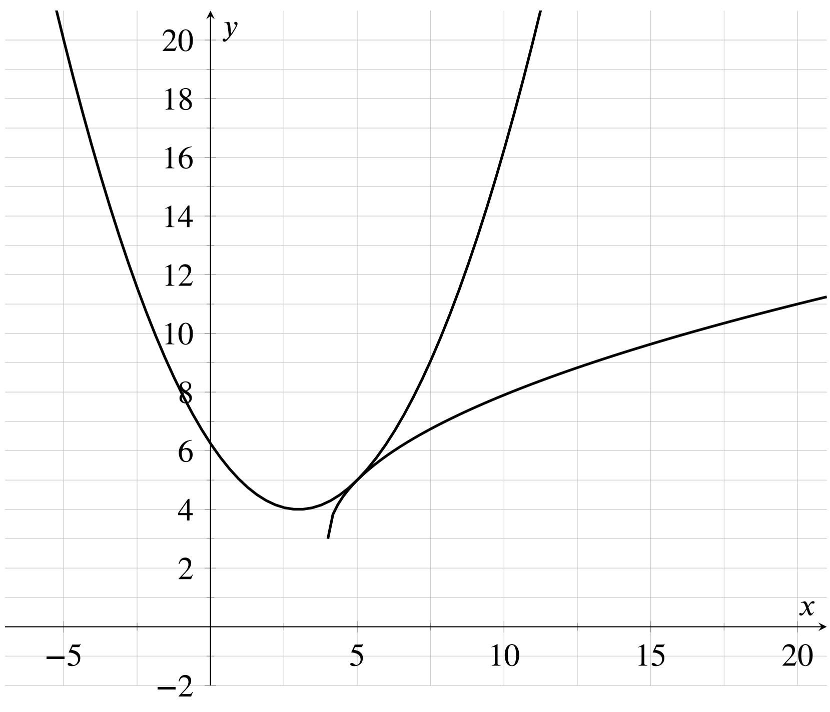
Sol.

$$\begin{aligned}
\text{Let } y &= f^{-1}(x) \\
f(y) &= x \\
2\sqrt{y-4}+3 &= x \\
2\sqrt{y-4} &= x-3 \\
\sqrt{y-4} &= \frac{x-3}{2} \\
y-4 &= \frac{x^2-6x+9}{4} \\
y &= \frac{x^2-6x+25}{4}
\end{aligned}$$

$$\therefore f^{-1}(x) = \frac{x^2-6x+25}{4}$$

(c) On the same diagram, sketch the graphs of f and f^{-1}.

Sol.



Chapter 23

Exponents and Logarithms

23.1 Exponents

Definition and Properties of Exponents

Back in Senior 1, we have learnt the following definitions of exponents:

Positive exponent	$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$
Zero exponent	$a^0 = 1$
Negative exponent	$a^{-n} = \dfrac{1}{a^n}\ (a \neq 0, n \in \mathbb{Z}^+)$
Fractional exponent	$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}\ (a \geq 0, n > 1, m, n \in \mathbb{Z}^+)$

The exponent of rational numbers have the following properties:

1. $a^m \times a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^n = a^n b^n$
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}\ (b \neq 0)$

Practice 1

Without using the calculator, find the value of the following expressions (Question 1 to 2):

1. $2^{-2} + 2{-}5 - (-2)^{-3}$
2. $\left(3\dfrac{6}{25}\right)^{-\frac{1}{2}}$
3. Simplify $a^{-}4 \div a^{-5} \times (b^{-3})^{-4}$

Exponential Functions and Graphs

Let a is a constant that is bigger than zero and not equal to 1, then the function being expressed in the form of $y = a^x$ is called an *exponential function*. The domain of an exponential function is $\mathbb{R}$.

Consider the following: a cell divides into two cells, and then each of the two cells divides into two cells again, and so on. If we let x be the number of divisions, the number of cells after the divisions be y, then the functional relationship between x and y is $y = 2^x$, which is an exponential function.

In order to look into the graph and its properties of an exponential function $y = a^x$, we sketch the graph of some exponential functions, the graph of $y = 2^x$, $y = 10^x$, and $y = \left(\frac{1}{2}\right)^x$ are shown in the diagram below.

From the diagram above, we can see that:

(1) The graph of the function $y = 2^x$, $y = 10^x$, and $y = \left(\frac{1}{2}\right)^x$ are only at the top of the x-axis. Actually, when $a > 0$, $a^x > 0$. Therefore, the value of the exponential function $y = a^x$ is always positive.

(2) When $x = 0$, $y = 1$. Hence, the graph of exponential functions $y = a^x$ always passes through the point $(0, 1)$.

(3) For the function $y = 2^x$, when $x > 0$ and $y = 10^x$, when $x < 0$, $y < 1$; when $x > 0$, $y > 1$. When the value of x increases, the value of y increases, that is, the function is an increasing function in the interval $(-\infty, +\infty)$.

(4) For the function $y = \left(\frac{1}{2}\right)^x$, when $x > 0$, $y > 1$; when $x < 0$, $y < 1$. When the value of x increases, the value of y decreases, that is, the function is a decreasing function in the interval $(-\infty, +\infty)$.

When we are discussing about the graph and its properties of an exponential function $y = a^x$, the following two cases are considered:

Practice 2

1. Without using the calculator, compare the value of the following expressions:

   (a) $\pi^{2.1}$ and $\pi^{3.5}$

   (b) $0.5^{-2.3}$ and $0.5^{-3.8}$

2. Given the exponential functions $f(x) = 3^{x^2-3x+5}$ and $g(x) = 3^{x+10}$. Find the value of x such that $f(x) = g(x)$.

Exercise 23.1

Without using the calculator, find the value of the following expressions (Question 1 to 10):

1. $\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^0 - \left(-\frac{1}{3}\right)^{-2}$

2. $\left(\dfrac{3^{-5} \cdot 3^2}{3^{-3}}\right)^{-2}$

3. $6^{-8} \div 6^{-5} + 3^{-3}$

4. $12^{\frac{1}{3}} \times 6^{\frac{1}{3}} \div 27^{\frac{1}{6}} \div 3^{\frac{1}{6}}$

5. $(0.2)^{-2} \times (0.125)^{\frac{2}{3}}$

6. $(0.3)^{-\frac{1}{3}} \times (0.0081)^{\frac{1}{3}} + (0.064)^{\frac{1}{3}}$

7. $\left(\dfrac{81}{16}\right)^{-0.25} \times \left(\dfrac{8}{27}\right)^{-\frac{2}{3}} \times (0.25)^{-2.5}$

8. $\left(\frac{1}{2}\right)^{-2} + 125^{\frac{2}{3}} + 343^{\frac{1}{3}} - \left(\dfrac{1}{27}\right)^{-\frac{1}{3}}$

9. $\left(2\dfrac{1}{4}\right)^{-\frac{3}{2}} + \left(1\dfrac{11}{25}\right)^{-1} - \left(2\dfrac{2}{3}\right)^0$

10. $\dfrac{5\sqrt{4}\sqrt{8}\left(\sqrt[3]{\sqrt[5]{4}}\right)^2}{\sqrt[3]{\sqrt{2}}}$

Simplify the follwoing expressions (Question 11 to 24):

11. $a^{\frac{1}{2}} \cdot a^{\frac{1}{4}} \cdot a^{-\frac{1}{8}} \cdot a^{\frac{1}{6}}$

12. $\left(9a^2b^{-2}c^4\right)^{-1}$

13. $\left(x^4y^{-5}\right)\left(x^{-2}y^2\right)^2$

14. $3a^{-2}b^{-3} \div \left(-3^{-1}a^2b^{-3}\right)$

15. $\sqrt[3]{\dfrac{a^2b^{-1}}{a^{\frac{1}{2}}b^5}}$

16. $5a^{-2}b^{-3} \div \left(5^{-1}a^2b^{-3}\right) \times 5^{-2}ab^4c$

17. $\dfrac{a^{-2} - b^{-2}}{a^{-2} + b^{-2}}$

18. $\left(a^{-1} + b^{-1}\right)(a + b)^{-1}$

19. $\left(x + x^{-1}\right)\left(x - x^{-1}\right)$

20. $\left(-2x^{\frac{1}{4}}y^{-\frac{1}{3}}\right)\left(3x^{-\frac{1}{2}}y^{\frac{2}{3}}\right)\left(-4x^{\frac{1}{4}}y^{\frac{2}{3}}\right)$

21. $2x^{-\frac{1}{3}}\left(\dfrac{1}{2}x^{\frac{1}{3}} - 2x^{-\frac{2}{3}}\right)$

22. $\left(\sqrt{x^3} \cdot \sqrt{y}\right)^2 \cdot \left(\sqrt{y} \cdot \sqrt{x^3}\right)^3$

23. $\dfrac{3 \times 2^n - 4 \times 2^{n-2}}{2^n - 2^{n-1}}$

24. $\left(3^{n+6} - 5 \times 3^{n+1}\right) \div \left(7 \times 3^{n+2}\right)$

25. Sketch the graph of the following functions on the same diagram:

    (a) $y = 3^x$

    (b) $y = \left(\dfrac{1}{3}\right)^x$

26. Without using the calculator, compare the value of the following expressions:

    (a) $2.5^{7.1}$ and $2.5^{8.5}$

    (b) $0.35^{6.5}$ and $0.35^{5.6}$

    (c) $1.03^{-2.1}$ and $1.03^{-3.2}$

    (d) $\left(\sqrt{2}\right)^{\pi}$ and $\left(\sqrt{2}\right)^{\pi-3.5}$

    (e) $0.01^{-\frac{1}{3}}$ and $0.01^{-\frac{1}{2}}$

(f) $2.7^{\sqrt{20}}$ and $2.7^{\sqrt[3]{35}}$

27. Given that $f_1 : x \to 2^{3x}$ and $f_2 : x \to 2^{x^2+2}$. Find the value of x such that $f_1(x) = f_2(x)$.

28. Given the function $f(x) = (0.4)^{x^2-x+1}$ and $g(x) = (0.4)^{6x+19}$. Find the value of x such that $f(x) = g(x)$.

23.2 Logarithms

Definition of Logarithms

If $a_n = x$, where $a > 0$ and $a \neq 1$, then we define $\log_a x = n$, and we say that n is the logarithm of x to the base a. In $\log_a x$, a is called the base, x is called the antilogarithm.

On the other hand, if $\log_a x = n$, then $a_n = x$. This is the inversible relationship between exponents and logarithms. That is,

$$\boxed{\log_a x = n \iff a^n = x \qquad a > 0,\ a \neq 1,\ x > 0}$$

Logarithms with base 10 are called common logarithms, and are usually written as $\log a$.

Another common logarithm is the natural logarithm, which has base e ($e \approx 2.71828182846$), and is usually written as $\ln x$.

Practice 3

Find the value of x in the following equations:

1. $\log x = 3$
2. $\log_x 27 = \dfrac{3}{2}$
3. $2\log_x(3\sqrt{3}) = 1$
4. $\log_2(16\sqrt{2}) = x$

Logarithmic Functions and Graphs

From the definition of logarithms, we can see that if $y = a^x$, then $x = \log_a y$. From the concept of inverse functions, we know that $y = \log_a x$ is the inverse function of $y = a^x$. Function $y = \log_a x$ is called the logarithmic function, where $a > 0$ and $a \neq 1$. Since the domain of $y = a^x$ is $\mathbb{R}$, and its range is $\mathbb{R}^+$, so the domain of $y = \log_a x$ is $\mathbb{R}^+$, and its range is $\mathbb{R}$.

Since the logarithmic function $y = \log_a x$ is the inverse function of the exponential function $y = a^x$, so the graph of $y = \log_a x$ is the reflection of the graph of $y = a^x$ about the line $y = x$. If we draw a curve of $y = a^x$, then reflect it about the line $y = x$, we can get the graph of $y = \log_a x$. For example, in the diagram below, the curves that are the reflection of the graphs of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ about the line $y = x$ are the graphs of $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ respectively.

From the diagram above, we can see that:

(1) Since the domain of $y = \log_a x$ is $x > 0$, so the graph of the function $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ are only at the right side of the y-axis.

(2) When $x = 1$, $y = 0$. Hence, the graph of logarithmic functions $y = a^x$ always passes through the point $(1, 0)$.

(3) For the function $y = \log_2 x$, when $x > 1$, $y > 0$; when $0 < x < 1$, $y < 0$. When the value of x increases, the value of y increases, that is, the function is an increasing function in the interval $(0, +\infty)$.

(4) For the function $y = \log_{\frac{1}{2}} x$, when $x > 1$, $y < 0$; when $0 < x < 1$, $y > 0$. When the value of x increases, the value of y decreases, that is, the function is a decreasing function in the interval $(0, +\infty)$.

When we are discussing about the graph and properties of a logarithmic function $y = \log_a x$, the following two cases are considered:

Practice 4

1. Without using the calculator, Compare the value of the following expressions:

2. (a) $\log 6$ and $\log 9$
   (b) $\log_{0.5} 4.2$ and $\log_{0.5} 3.9$
   (c) $\log_2 1.8$ abd $\log_4 5.8$.

3. Find the domain of the following functions:
   (a) $y = \log_a(x+2)$
   (b) $y = \log_2(x^2 - 9)$
   (c) $y = \log_7 \dfrac{2}{3-2x}$
   (d) $y - \sqrt{\log_5(2-x)}$

Exercise 23.2

1. Find the value of x for the following expression:
   (a) $\log_2 x = 4$
   (b) $\log_{125} x = \dfrac{1}{3}$
   (c) $\log_{16}(2\sqrt{2}) = x$
   (d) $\log_{\frac{1}{3}} 81 = x$
   (e) $\log_x 81 = 4$
   (f) $\log_x 49 = -2$

2. Sketch the graph of the following functions on the same set of axes:
   (a) $y = \log_5 x$
   (b) $y = \log_{\frac{1}{5}} x$

3. Without using the calculator, compare the value of the following expressions:
   (a) $\log_3 5$ and $\log_3 6$
   (b) $\log 1.51.4$ and $\log_{1.5} 1.6$
   (c) $\log_{\sqrt{3}} 4.8$ and $\log_{\sqrt{3}} 5.8$
   (d) $\log_{2.3} \pi$ and $\log_{2.3}(\pi - 3)$
   (e) $\log_{0.4} \sqrt{2}$ and $\log_{0.4} \sqrt{3}$
   (f) $\log_{\frac{1}{2}} 3$ and $\log_{\frac{1}{2}} \dfrac{1}{4}$

4. Find the domain of the following functions:
   (a) $y = \log_2(3 - 2x)$
   (b) $y = \log(x^2 + 1)$
   (c) $y = \log_5(9 - 16x^2)$
   (d) $y = \log_9 \dfrac{1}{x-2}$
   (e) $y = \log_8 \sqrt{2x^2 - x - 3}$
   (f) $y = \dfrac{1}{\log_3(7x-5)}$

23.3 Arithmetic Properties of Logarithms and Base Changing Formula

Identties and Arithmetic Properties of Logarithms

Logarithms have the following identities:

$$\log_a 1 = 0$$
$$\log_a a = 1$$

Logarithms have the following arithmetic properties:

$$\log_a(xy) = \log_a x + \log_a y \quad (x > 0, y > 0)$$
$$\log_a \frac{x}{y} = \log_a x - \log_a y \quad (x > 0, y > 0)$$
$$\log_a x^n = n \log_a x \quad (x > 0)$$
$$a^{\log_a x} = x$$

Base Changing Formula

The base of a logarithm can be changed from one to another. Let $\log_a x = n$, then $a^n = x$. Change both sides of the equation to logarithm with base b, we have

$$\begin{aligned}
\log_b a^n &= \log_b x \\
n \log_b a &= \log_b x \\
\because a \neq 1, \ \therefore \ \log_b a &\neq 0 \\
n &= \frac{\log_b x}{\log_b a} \\
\therefore \ \log_a x &= \frac{\log_b x}{\log_b a}
\end{aligned}$$

The expression above is called the *base changing formula*.
When $x = b$, we have $\log_a b = \dfrac{1}{\log_b a}$.
In this book, the value of logarithms are rounded to 4 decimal places.

Practice 5

Exercise 23.2

Without using the calculator, compare the value of the following expressions (Question 1 to 4):

1. $5^{2 \log_5 4}$

2. $4^{3 \log_2 \sqrt{2}}$

3. $2\log 5 - \log\frac{1}{3} + \frac{1}{2}\log\frac{16}{9}$

4. $\dfrac{\log_4 27}{\log_2 3}$

5. Given that $\log_2 4 = a$ and $\log_2 5 = b$. Express the following expressions in terms of a and b:

   (a) $\log_2 90$

   (b) $\log_3 270$

   (c) $\log_9 1.8$

6. Given that $\log_{16} y = \frac{1}{2} + \log_4 x$. Express x in terms of y.

Exercise 23.3

Exercise 23.2

Simplify the following expressions (Question 1 to 6):

1. $\log_2 4^x$

2. $\log_2 a^{\log_a 2}$

3. $3^{\log_3 x - \log_3 y}$

4. $\log_3 (9^x \times 27^y)$

5. $2^{-\log_8 x}$

6. $3\log_4 2^x$

Without using the calculator, evaluate the following expressions (Question 7 to 22):

7. $\log_7 \sqrt[3]{49}$

8. $49^{\log_7 3}$

9. $2^{2\log_2 7} + \left(\frac{1}{2}\right)^{-\log_2 7}$

10. $\log_3 5 - \log_3 15$

11. $\dfrac{\log\sqrt{3}}{\log\frac{1}{9}}$

12. $\log_5 \frac{1}{5} + \log_5 \sqrt[3]{5} - \log_5 25$

13. $\log_3 \sqrt[3]{27\sqrt[4]{81}}$

14. $\log (0.1)^4 - \log\sqrt[3]{0.001}$

15. $\dfrac{\log 4 + \log 3}{1 + \log 0.4 + \frac{1}{2}\log 9}$

16. $\log_{36} 6 - \log_6 36 - \log_6 \frac{1}{6}$

17. $\log_2 \frac{1}{25} \log_3 \frac{1}{8} \cdot \log_5 \frac{1}{9}$

18. $\log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8$

19. $\log_4 8 - \log_{\frac{1}{9}} 3 - \log_{\sqrt{2}} 4$

20. $\left(\log_2 3 + \log_2 \sqrt{3}\right)\log_{\sqrt{3}} 2$

21. $\dfrac{\log_5 \sqrt{2} \cdot \log_7 9}{\log_7 \sqrt[3]{4} \cdot \log_5 \frac{1}{3}}$

22. $\frac{1}{2}\log\frac{81}{17} + 2\log\frac{5}{3} - \log\frac{17}{4} + \frac{3}{2}\log 17$

23. Given that $\log_2 3 = a$ and $\log_2 5 = b$. Express a and b in terms of $\log_4 15$.

24. Given that $\log_3 5 = m$ and $\log_5 6 = n$. Express m and n in terms of $\log_2 554$.

25. Given that $\log_2 3 = a$ and $\log_3 7 = b$. Express a and b in terms of $\log_4 214$.

26. Given that $\log_3 6 = x$. Epress x in terms of $\log_9 12$.

27. Given that $\log_3 y - \log_9 \sqrt[3]{x} = 1 + \log_2 7x$. Exress x in terms of y.

28. Given that $\log_2 5(2x-1) = \log_5(x-3) + \log_2 55$, prove that $5x^2 - 32x + 46 = 0$.

29. If $a > 0$, $b > 0$, and $a^2 + b^2 = 7ab$, prove that $2\log_3(a+b) = 2 + \log_3 a + \log_3 b$.

30. If $x > 0$, $y > 0$, and $x^2 + 9y^2 = 10xy$, prove that $2\log(x+3y) - 4\log 2 = \log x + \log y$.

23.4 Exponential Equations

All the equations with terms that contain the variable in the exponent are called exponential equations. For example, $9^x = 3^{x-1}$, $3^x = 5$, and $2^{x-1} + 2^x - 2 = 0$ are all exponential equations.

Practice 6

Solve the following exponential equations:

1. $3^{2x} = -\dfrac{1}{9}$
2. $2^{x^2+4x} = \dfrac{1}{8}$
3. $6^x = 5^{x-1}$
4. $4^{x-1} + 2^{x-1} = 20$

Exercise 23.4

Solve the following exponential equations:

1. $8^{x-3} = \dfrac{1}{256}$
2. $3^{2x+1} = 243$
3. $10^{x^2-4} = 1$
4. $3^{x^2+3} = 27^{x+7}$
5. $4^{x^2} = 2^{5x+7}$
6. $5^{2x^2+x} = 25^{1+x-2x^2}$
7. $\left(\dfrac{9}{16}\right)^x = \left(\dfrac{4}{3}\right)^{x-6}$
8. $5^{2^x+1} = 5^{4^x+1}$
9. $2^{2x+3} \cdot 4^{x+6} = (8^x)^x$
10. $\dfrac{5^{x^2}}{5} = 7^{(x+1)(x-1)}$
11. $3^{x+1} = 4^{x-1}$
12. $7^{5-3x} = 5^{x+2}$
13. $13^{2x+5} = 14^{x+7}$
14. $2^{x^2-1} = 3^{x+1}$
15. $\left(\dfrac{1}{3}\right)^x - \left(\dfrac{1}{3}\right)^{-x} = \dfrac{80}{9}$
16. $3^{x+1} + 9^x - 18 = 0$
17. $25^x - 23 \cdot 5^x - 50 = 0$
18. $3^{x-1} + 3^{3-x} - 10 = 0$
19. $3^{2x} - 3^{x+1} + 2 = 0$
20. $2^{x+2} + 3\left(2^{1-x}\right) - 14 = 0$
21. $2^{2x-1} - 3 \cdot 2^{x-1} + 1 = 0$
22. $3^x - 5^{x+2} = 3^{x+1} - 5^{x+3}$

23.5 Logarithmic Equations

All the equations with logarithmic terms which contains variable in the base or in the argument are called logarithmic equations. For example, $\log(x-1) = 3$, $\log_x 9 = 2$, and $2\log_3 x + \log_9 x = 1$ are all logarithmic equations. The results acquired when solving logarithmic equations need to be checked.

Practice 7

Solve the following logarithmic equations:

1. $\log_3 x = 5$
2. $\log_5(x-2) = 0$
3. $\log(x^2+2x-3) - \log(x+3) = 0$
4. $\log_3(3x+1) + 1 = \log_3(2x-1) + \log_3 5$
5. $\log_x 3 + \log_x 81 = 5$
6. $3\log_2^2 x + 5\log_2 x - 2 = 0$
7. $\log_2 x - \log_x 8 = 2$
8. $x^{\log x} = 100x$

Exercise 23.5

Solve the following logarithmic equations:

1. $\log_{\sqrt{3}} x = -2$
2. $\log_2 x^4 = 4$
3. $\log \frac{x-2}{x+2} = \log \frac{1}{x-1}$
4. $2\log x + \log 7 = \log 14$
5. $\log x + \log(x-3) = 1$
6. $\log(x+6) - \log(x-3) = 1$
7. $\log_6 x + \log_6(x^2-7) = 1$
8. $\log_{1,2}(15x^2-2x-12) = 0$
9. $\log_8(x^2-3x-2) = \frac{1}{3}$
10. $\log_2(x^2-x-2) - \log_2(x+1) = 0$
11. $\log_3(2x-3) + \log_3(3x+2) = \log_3(2x-1)$
12. $\frac{1}{2}(\log x - \log 5) = \log 2 - \frac{1}{2}\log(9-x)$
13. $\log(x+6) - \frac{1}{2}\log(2x-3) = 2 - \log 25$
14. $\log_2 x = \log_8 x + 1$
15. $3^{\log x} = 2^{\log 3}$
16. $4x^{\log_2 x} = x^3$
17. $2\left(\log_3 x\right)^2 + \log_3 x - 1 = 0$
18. $\log_4^2 x - 5\log_4 x + 6 = 0$
19. $6\log^2 x + \log x^3 - 3 = 0$
20. $(\log x)^2 = 2\log x$
21. $\log_x 25 - \log_{25} x = 0$
22. $2\log_4 x - 3\log_x 4 + 5 = 0$
23. $2\log_x 10 - \log x + 1 = 0$
24. $\log_5\left[\log_2\left(\log_x 5\right)\right] = 0$

23.6 Compound Interest and Annuity

Simple interest and compound interest are two different methods of calculating interest. Simple interest is calculated on the principal amount of a loan only. Compound interest is calculated on the principal amount and also on the accumulated interest of previous periods, and can thus be regarded as interest on interest.

For example, a fund amounted to RMp is deposited into a bank account with a yearly interest rate of $r\%$.

$$\text{Principal amount} = \text{RM}p$$

When $t = 1$,

$$\begin{aligned}\text{Interest earned} &= p \times r\% = \frac{pr}{100}\\ \text{Accumulated amount} &= p + \frac{pr}{100} = p\left(1 + \frac{r}{100}\right)\end{aligned}$$

When $t = 2$,

$$\begin{aligned}\text{Interest earned} &= \left(p + \frac{pr}{100}\right) \times r\% = \frac{pr}{100}\left(1 + \frac{r}{100}\right)\\ \text{Accumulated amount} &= p\left(1 + \frac{r}{100}\right) + \frac{pr}{100}\left(1 + \frac{r}{100}\right)\\ &= p\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right)\\ &= p\left(1 + \frac{r}{100}\right)^2\end{aligned}$$

When $t = 3$,

$$\begin{aligned}\text{Interest earned} &= \left(p\left(1 + \frac{r}{100}\right)^2\right) \times r\% = \frac{pr}{100}\left(1 + \frac{r}{100}\right)^2\\ \text{Accumulated amount} &= p\left(1 + \frac{r}{100}\right)^2 + \frac{pr}{100}\left(1 + \frac{r}{100}\right)^2\\ &= p\left(1 + \frac{r}{100}\right)^2\left(1 + \frac{r}{100}\right)\\ &= p\left(1 + \frac{r}{100}\right)^3\end{aligned}$$

In general, the accumulated amount after t years is given by

$$A = p\left(1 + \frac{r}{100}\right)^t$$

where p is called the *present value* of A.

If the interest is compounded m times per year, then the accumulated amount is given by

$$A = p\left(1 + \frac{r}{100m}\right)^{mt}$$

Annuity and Present Value of Annuity

An annuity is a series of equal payments made at equal intervals of time according to some kind of contract, standing order or the amount received. For example, all sorts of instalment, insurance premiums, house rent, car loan, etc. are annuities. In this book, we will only consider annuities with equal payments made or received at equal intervals of time.

Note that the annuity is not limited to once a year.

We can compare which payment plan is better by comparing the present values of the annuities. From the formula $A = p(1 + r\%)^t$, we can know that the present value $p = \frac{A}{(1 + r\%)^t}$. If the yearly interest rate is $r\%$, the annuity is RMA, the payment is made once per year, then the present value of the amount paid after a year is $A(1 + r\%)^{-1}$, the present value of the amount paid after two years is $A(1 + r\%)^{-2}$, and so on. The present value of the amount paid after n years is $A(1 + r\%)^{-n}$. Hence, the sum of the present values of the amount paid after n years is

$$\begin{aligned}
&\frac{A}{1 + r\%} + \frac{A}{(1 + r\%)^2} + \cdots + \frac{A}{(1 + r\%)^n} \\
&= A\left[\frac{1}{1 + r\%} + \frac{1}{(1 + r\%)^2} + \cdots + \frac{1}{(1 + r\%)^n}\right] \\
&= A\left[\frac{1 - \frac{1}{(1 + r\%)^n}}{1 - \frac{1}{1 + r\%}}\right] \\
&= \frac{A}{r\%}\left(1 - \frac{1}{(1 + r\%)^n}\right)
\end{aligned}$$

Annuity that is paid indefinitely is called *perpetuity*, $n \to \infty$, $\frac{1}{(1 + r\%)^n} \to 0$. From that, we can know that the present value of perpetuity is $\frac{A}{r\%}$.

Practice 8

1. Given that the principal amount is RM75,000, interest rate is 4.5%. Using composite interest method, find the accumulated amount after 10 years.

2. A person has deposited RM40,000 into a bank account. The bank pays 8% interest per annum compounded half yearly. Using the compound interest method, find the amount in the account after 3 years.

3. Given that the interest rate is 6%, the inter-

est is compounded half yearly. Using the compound interest method, the accumulated amount after 5 years is RM4031.75, find the principal amount.

4. Given that the interest rate is 4%, the annuity is RM3,500, the payment is made once per year. The payment has since been made for 15 years continuously. Find the present value. Hence, find the present value of the perpetuity.

Exercise 23.6

1. Given that the principal amount is RM90,000, the interest rate is 5%. Compounding the interest once per year, find the accumulated amount after 10 years.

2. A person has deposited a fund into a bank account. The bank pays 8% interest per annum compounded yearly. The amount in the account after 3 years has increased by RM779.14. Find the amount of the fund deposited.

3. RM80,000 was deposited into a financial institution. The interest rate is 8% per annum compounded once per three months. Find the amount in the account after 5 years.

4. Prove that the accumulated amount after being compounded with an interest of 5 for 15 years will exceed twice the principal amount.

5. Given that the principal amount is RM15,000, the interest rate is 6% being compouned once per year. How long does it take for the accumulated amount to be more than RM30,000?

6. Given that the principal amount is RM120,000, the interest rate is 5.5% being compounded half yearly. How long does it take for the accumulated amount to be more than RM200,000?

7. A person deposited RM2,500 into his bank account at the beginning of the yaar, the interest rate is 4.5% compounded once per year. Find the amount in the account after 15 years.

8. If the present value is RM15,443.46, the interest rate is 5%, find the annuity if the payment is made for 10 years.

9. Given that the annuity is RM5,000, the interest rate is 5%, the payment is made once per year for 25 years. Find the present value. Hence, find the present value of the perpetuity.

10. Given that the annuity is RM2,500, the interest rate is 4.5%, the payment is made once per year. How many years does it take for the present value to exceed RM30,000?

11. If a bank has introduced an annuity scheme, the investors can receive RM1,000 per year for life after paying RM20,000. If the annuity plan is considered approximately to be a perpetuity, find the interest rate.

Revision Exercise 23

1. Without using a calculator, find the value of the following:

   (a) $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^6 + \left(-\frac{1}{2}\right)^{-2}$

   (b) $5^{\frac{1}{2}} + 5^{-\frac{1}{2}} - \left(\frac{1}{5}\right)^{\frac{1}{2}} + \left(\frac{1}{5}\right)^{-\frac{1}{2}}$

   (c) $\left(\frac{1}{3}\right)^2 \times \left(-\frac{1}{3}\right)^2 \times \left(\frac{1}{2}\right)^{-2}$

   (d) $\sqrt{2\sqrt[3]{3}} \div \sqrt[3]{\frac{\sqrt{8}}{3}}$

2. Simplify the following expressions:

   (a) $\left(\frac{b}{2a^2}\right)^3 \div \left(\frac{2b^2}{3a}\right)^0 \times \left(-\frac{b}{a}\right)^{-3}$

   (b) $\dfrac{3^{n+2} - 2 \times 3^n}{5(3^{n+1})}$

   (c) $\dfrac{\left(x^{-1} + y^{-1}\right)\left(x^{-1} - y^{-1}\right)}{x^{-2}y^{-2}}$

   (d) $\left(x^{\frac{1}{4}} - y^{-\frac{1}{4}}\right)\left(x^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)\left(x^{\frac{1}{4}} + y^{-\frac{1}{4}}\right)$

   (e) $\dfrac{\left(\sqrt[4]{p^3}\right)^{\frac{1}{6}} \sqrt[9]{p^{-3}}}{\left(\sqrt{p^{-7}}\right)^{\frac{1}{6}}}$

   (f) $\dfrac{\left(a^2 + a^{-2} + 2\right)^2}{\left(a^2 + 1\right)^4}$

3. Without using a calculator, compare the value of the following:

   (a) 2.3^{-2} and 2.3^{-1}

   (b) $0.15^{-\frac{1}{2}}$ and $0.15^{-\frac{1}{3}}$

   (c) $\left(\frac{1}{3}\right)^{\frac{2}{5}}$ and $3^{-\frac{5}{3}}$

   (d) $\left(\frac{3}{5}\right)^2$ and $\left(\frac{5}{3}\right)^{3.1}$

4. Without using a calculator, compare the value of the following:

   (a) $\log_{3.2} 3$ and $\log_{3.2} 2$

   (b) $\log_{0.5} 5.3$ and $\log_{0.5} 3.5$

   (c) $\log_3 2$ and $\log_2 3$

   (d) $\log_2 2.3$ and $\log_4 4.8$

5. Find the domain of the following functions:

   (a) $y = \log_{0.5}\left(16 - x^2\right)$

   (b) $y = \log_2\left(2x^2 - 5x - 12\right)$

   (c) $y = \sqrt{3 - 3^x}$

   (d) $y = \log_2(x-3)^2$

   (e) $y = \log_5\left(x^2 - 2x\right)$

   (f) $y = \log_3 \dfrac{2}{3-x}$

   (g) $y = \dfrac{1}{\log(x+1) - 1}$

   (h) $y = \dfrac{\log_3(2-x)}{\log_3(2+x)}$

   (i) $y = \sqrt{\log_3(x-2)}$

   (j) $y = \dfrac{2}{\sqrt{1 - \log x}}$

6. Simplify the following expressions:

   (a) $\log_3 27^x$

   (b) $\log_x b^{a \log_b x}$

   (c) $\log_5 (25^x \cdot 5^y)$

   (d) $3^{2\log_3 x - \log_3 y}$

   (e) $5^{-2\log_{25} x}$

7. If $\log_2 5 = p$, express $\log_2 100$ in terms of p.

8. If $\log_3 12 = a$, express the following in terms of a:

   (a) $\log_3 24$

   (b) $\log_9 36$

9. Without using a calculator, find the value of the following:

   (a) $\log_c \frac{1}{5} + \log_c 5$

   (b) $\log_2\left(2\sqrt{2}\right) - 2\log_2\sqrt{2}$

   (c) $\log_8 \dfrac{2}{7} - \log_8(-2)^2 - \log_8 \dfrac{1}{7}$

   (d) $\log \dfrac{5}{32} - 2\log\dfrac{5}{6} + \log\dfrac{40}{9}$

   (e) $\left(\log_2 3\right)\left(\log_3 4\right)$

(f) $\dfrac{\log_{16} 5}{\log_{32} 5}$

(g) $\log_3 5 \cdot \log_5 7 \cdot \log_7 27$

(h) $\log_2 \dfrac{1}{9} \cdot \log_3 \dfrac{1}{25} \cdot \log_5 \sqrt{8}$

(i) $\dfrac{1}{3} \log_2 8 + \log_3 27 - \dfrac{1}{4} \log_4 16$

(j) $\log^2 2 + \log 2 \cdot \log 5 + \log 5$

(k) $2 \log_3 15 + 3 \log_3 12 - \log_3 25 - 6 \log_3 2$

(l) $\dfrac{\log \sqrt{27} + \log \sqrt{8} - \log \sqrt{125}}{\log 6 - \log 5}$

(m) $\log_8 \left(\log_2 \sqrt{8 + 4\sqrt{3}} + \log_2 \sqrt{8 - 4\sqrt{3}} \right)$

10. If $\log_3 2 = a$, $\log_2 5 = b$, prove that $\log_5 3 = \dfrac{1}{a(b+1)}$.

11. If $\log_2 3 = p$, $\log_3 7 = q$, prove that $\log_2 114 = \dfrac{pq+1}{p(q+1)}$.

12. Given that $2 \log_5(x+y) = 1 + \log_5 x + \log_5 y$. Prove that $x^2 + y^2 = 3xy$.

13. Given that $x = 5^k$ and $y = 5^n$. Express the following in terms of k and n:

(a) $\log_5 \dfrac{xy^3}{125}$

(b) $\log_{25} \left(5\sqrt{xy}\right)$

14. Given that $2 + \log_4 y = 2 \log_1 6x$. Express x in terms of y.

15. Solve the following exponential equations:

(a) $3^{3x-2} = 243$

(b) $4^{1-x} = \left(\dfrac{1}{8}\right)^2$

(c) $2^{x^2} = (2^x)^2$

(d) $3^{5^x} = 3$

(e) $5^{8^x} = 625$

(f) $3^{x+1} = 6^x$

(g) $7^x - 7^{x-1} = 6$

(h) $3^{x+1} = 10\,(3^x) - 3$

(i) $2^{2x+1} = 3\,(2^x) - 1$

(j) $5^{2x+1} = 26\,(5^x) - 5$

(k) $2^{2x+3} - 2^x = 1 - 2^{x+3}$

(l) $2^{2x+8} - 32\,(2^x) + 1 = 0$

16. If $\log_2 x + \log_4 x = \dfrac{9}{2}$, find the value of x.

17. Solve the follwoing logarithmic equations:

(a) $2 \log x - 3 \log 4 = 2$

(b) $2 \log x = \log 32 + \log 2$

(c) $\log x + \log (x+3) = \log (x+8)$

(d) $\left(\log_2 x\right)^2 = \log_2 x + 6$

(e) $\log_3 x + 6 \log_x 3 = 5$

(f) $4^{\log x} = 2^{\log x + 1}$

(g) $\log_{x+1} \left(x^2 - 5x - 13\right) = 2$

(h) $\log_r \sqrt{2x^2 - 5x + 6} = 1$

(i) $\log_2(x+1) + \log_2(x+3) = 3 + \log_2 x$

(j) $\log_2 \left[\log_3 \left(\log_5 x\right)\right] = 0$

(k) $3 \log_s x - 2 \log_2 x + 2 = 0$

(l) $\log_4(x+4) + 1 = \log_2(x+1)$

(m) $2 \log_2 x \cdot \log_8 x = \log_2 x + \log_8 x$

18. A person has deposited a fund into a bank account that pays 5.5% interest compounded annually. If the balance in the account has increased by RM1,432.95 after 4 years, how much was deposited initially?

19. Given that there is a principal of RM75,000 at an interest rate of 3.5% per annum compounded once per three months. Find the accumulated value of the principal after 8 years.

20. Given that there is a principal of RM150,000 at an interest rate of 5.25% per annum compounded once per annum. How many years does it take to accumulate at least RM300,000?

21. If the present value is RM24,924.44, the interest rate is 5% per annum compounded once per annum, find the annuity payment if the payment is made for 20 years.

22. Given that the annuity payment is RM8,000, the interest rate is 4.5% per annum compounded once per annum, and the payment is made for 15 years. Find the present value. Hence, find the present value of the perpetuity.

23. Given that the annuity amount is RM4,500, the interest rate is 4.5% per annum compounded once per annum. How many years does it take for the present value to exceed RM50,000?

24. The price of a branded laptop is RM2,500, the payment can be paid in full or by instalment. If the payment is made by instalment, the monthly payment is RM110 for 2 years. If the interest rate is 4% per annum compounded once per month, which payment method is more economical considering the present value of the payment?

Chapter 24

Limits

24.1 Concept of Limits

Limit is a fundamental concept of calculus. It is the theoretical basis for studies on the changes and trends of functions. We will first introduce an example related to the idea of limits.

Cyclotomic Method by Liu Wei

The circle is not a shape with straight edges, so where does its area formula $A = \pi r^2$ come from?

Let the side length of a regular n -gon inscribed in a circle be a_n, length from the center to the side be r_n, as shown in the diagram below, the area of the regular n-gon is $A_n = n \cdot \frac{1}{2} a_n \cdot r_n$, while its circumference is $P_n = na_n$, then $A_n = \frac{1}{2} P_n \cdot r_n$.

When the value of n becomes larger and larger, the area A_n of the n-gon is indefinitely close to the area A of the circle, denoted as $A = \lim\limits_{n\to\infty} A_n$, the limit of A_n is said to be A when n approaches infinity.

When $n \to \infty$, the circumference P_n and the length between center and side r_n of the inscribed regular n-gon, approaches the circumference P and the radius r of the circle respectively. That is to say, $\lim\limits_{n\to\infty} P_n = P$, $\lim\limits_{n\to\infty} r_n = r$.

$$\begin{aligned} \therefore A &= \lim_{n\to\infty} A_n \\ &= \lim_{n\to\infty} \frac{1}{2} P_n \cdot r_n \\ &= \frac{1}{2} Pr \end{aligned}$$

from $P = 2\pi r$. we get $A = \pi r^2$.

The concept above uses the idea of "replacing curves with straight lines", which treats the area of a circle as the limit of the area of a regular n-gon when n approaches infinity. This way of calculating the area of a circle using the limit is invented by Liu Hui, a mathematician back in the 3rd century, and is called the "Cyclotomic Method". Quoted from his own words, "The smaller the circle is divided, the lesser the error is; divide and divide, until the circle is unable to be divided, then the error will be negligible and it will be the same as the circle."

24.2 Limits of Functions

If the value of a variable x approaches a certain constant a, we say that x tends to a.

When x approaches x_0 (but not equal to x_0), if the value of $f(x)$ approaches a certain constant A, we say that as x approaches x_0, the limit of $f(x)$ is A, denoted as $\lim\limits_{x \to x_0} f(x) = A$.

In the definition of the limit $\lim\limits_{x \to x_0} f(x) = A$, when approaches x_0 from the left ($x < x_0$) and right ($x > x_0$), the limit of $f(x)$ approaches the same constant A.

When x approaches x_0 from the left ($x < x_0$), denoted as $x \to x_0^-$, if the value of $f(x)$ approaches a certain constant A, then A is the left limit of $f(x)$ when x approaches x_0, denoted as $\lim\limits_{x \to x_0^-} f(x) = A$.

Similarly, when x approaches x_0 from the right ($x > x_0$), denoted as $x \to x_0^+$, if the value of $f(x)$ approaches a certain constant A, then A is the right limit of $f(x)$ when x approaches x_0, denoted as $\lim\limits_{x \to x_0^+} f(x) = A$.

If x approaches x_0 from the left (or right), but the value of $f(x)$ does not approach a certain constant, then the left (or right) limit does not exist.

From the definition of the limit, left limit and right limit, we can conclude the following theorem:

> When $x \to x_0$, if the left and right limit of the function $f(x)$ exist and are equal, then the limit of $f(x)$ exists, and $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x)$.
>
> Contrarily, if the left limit or right limit of the function $f(x)$ does not exist, or the left and right limit are not equal, then the limit of $f(x)$ does not exist.

Note that the limit $\lim\limits_{x \to x_0} f(x)$ and the function value $f(x_0)$ are two different concepts. When $\lim\limits_{x \to x_0} f(x)$ exists, it does not mean that $f(x_0)$ exists. Even if $f(x_0)$ exists, it does not guarantee to be equal to $\lim\limits_{x \to x_0} f(x)$.

Practice 1

Complete the following table, state the changes of the value of the function $f(x) = 2x + 1$ when $x \to 1$, and find $\lim\limits_{x \to 1} f(x)$.

x	0.9	0.99	0.999	0.9999	0.99999
$f(x) = 2x + 1$					

x	1.1	1.01	1.001	1.0001	1.00001
$f(x) = 2x + 1$					

If x approaches positive infinity, the value of function $f(x)$ approaches a certain constant A, then A is the limit of $f(x)$ when $x \to \infty$, denoted as $\lim_{x\to\infty} f(x) = A$.

Similarly, if x approaches negative infinity, the value of function $f(x)$ approaches a certain constant B, then B is the limit of $f(x)$ when $x \to -\infty$, denoted as $\lim_{x\to-\infty} f(x) = B$.

If x approaches positive infinity (or negative infinity), but the value of $f(x)$ does not approach a certain constant, then the limit of $f(x)$ does not exist when $x \to \infty$ (or $x \to -\infty$).

Practice 2

Complete the following table, state the changes of the value of the function $f(x) = \frac{1}{x}$ when $x \to \infty$, and find $\lim_{x\to\infty} f(x)$.

x	-1	-10	-100	-1000	-10000	$\cdots$
$f(x) = \frac{1}{x}$						

Exercise 24.2

1. Complete the following table. Hence, find the left limit, right limit, and limit of the function $f(x) = 3x - 1$ and $x = 1$.

x	0.9	0.99	0.999	0.9999	0.99999
$f(x) = 3x - 1$					

x	1.1	1.01	1.001	1.0001	1.00001
$f(x) = 3x - 1$					

2. Complete the following tables, state the changes of the value of the function $f(x)$ when $x \to x_0$, and find $\lim_{x\to x_0} f(x)$.

(a) $f(x) = x^3 + 1$, $x_0 = 0$

x	0.1	0.01	0.001	0.0001	0.00001
$f(x) = x^3 + 1$					

x	-0.1	-0.01	-0.001	-0.0001	-0.00001
$f(x) = x^3 + 1$					

(b) $f(x) = \frac{x^2 - 4}{x + 2}$, $x_0 = -2$

x	-2.1	-2.01	-2.001	-2.0001	-2.00001
$f(x) = \frac{x^2 - 4}{x + 2}$					

x	-1.9	-1.99	-1.999	-1.9999	-1.99999
$f(x)$					

(c) $f(x) = \begin{cases} 4 - x, & x < 1 \\ x^2 + 2, & x \geq 1 \end{cases}$, $x_0 = 1$

x	0.9	0.99	0.999	0.9999	0.99999
$f(x)$					

x	1.1	1.01	1.001	1.0001	1.00001
$f(x)$					

3. Complete the following table, state the changes of the function $f(x) = \dfrac{2x+1}{x+2}$ when $x \to x_0$, and find $\lim\limits_{x\to\infty} f(x)$.

x	1	10	100	1000	10000	100000
$f(x)$						

4. Complete the following table, state the changes of the function $f(x) = 2^x$ when $x \to -\infty$, and find $\lim\limits_{x\to-\infty} f(x)$.

x	-1	-2	-3	-4	-5	-10
$f(x)$						

24.3 Arithmetic Rules of Limits of Functions

If $\lim\limits_{x\to x_0} f(x)$ and $\lim\limits_{x\to x_0} g(x)$ exist, then

1. $\lim\limits_{x\to x_0} [f(x) \pm g(x)] = \lim\limits_{x\to x_0} f(x) \pm \lim\limits_{x\to x_0} g(x)$
2. $\lim\limits_{x\to x_0} [f(x) \cdot g(x)] = \lim\limits_{x\to x_0} f(x) \cdot \lim\limits_{x\to x_0} g(x)$
3. $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}$, provided that $\lim\limits_{x\to x_0} g(x) \neq 0$
4. $\lim\limits_{x\to x_0} k \cdot f(x) = k \cdot \lim\limits_{x\to x_0} f(x)$, where k is a constant
5. $\lim\limits_{x\to x_0} [f(x)]^n = \left[\lim\limits_{x\to x_0} f(x)\right]^n$, where n is a positive integer
6. $\lim\limits_{x\to x_0} k = k$, where k is a constant

The rules above also applied to $x \to \infty$ and $x \to -\infty$. Obviously, $\lim\limits_{x\to x_0} x = x$, where k is a constant.

Using the arithmetic rules above, we can calculate the limit of relatively complicated functions using given limit of simpler functions. For example, from the rules above, we can get

$$\begin{aligned}
&\lim_{x\to x_0} \left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right) \\
&= a_n x_0^n + a_{n-1}x_0^{n-1} + \cdots + a_1 x_0 + a_0
\end{aligned}$$

Practice 3

Find the limit of the followings:

1. $\lim\limits_{x\to -1} \dfrac{\sqrt{x+2}-3}{x+4}$
2. $\lim\limits_{x\to 1} \dfrac{x^2+2x-3}{x^2-3x+2}$
3. $\lim\limits_{x\to \infty} \dfrac{x^2+1}{2x^2+x+1}$

Practice 4

State whether the left limit, right limit, and limit of the function $f(x) = \dfrac{1}{x}$ exist when $x \to 0$.

Exercise 24.3

Find the limit of the followings (Question 1 to 31):

1. $\lim\limits_{x\to -2} \left(x^2+x-3\right)$
2. $\lim\limits_{x\to 4} x^2(x-1)$
3. $\lim\limits_{x\to -2} x\left(9-x^2\right)$
4. $\lim\limits_{x\to -1} (x+3)(x-1)$
5. $\lim\limits_{x\to \frac{1}{2}} (2x-1)\left(x^2+3x+4\right)$
6. $\lim\limits_{x\to -2} \left(4x^3+2x^2+3x+1\right)$
7. $\lim\limits_{x\to 2} \dfrac{x^2+2}{x-5}$
8. $\lim\limits_{x\to -1} \dfrac{(x+2)(x-3)}{x-1}$
9. $\lim\limits_{x\to 0} \dfrac{2x^2+3x-4}{x-4}$
10. $\lim\limits_{x\to -3} \dfrac{(x+5)(x+3)}{x+3}$
11. $\lim\limits_{x\to 2} \dfrac{x^2-5x+6}{x-2}$
12. $\lim\limits_{x\to -2} \dfrac{x^3+8}{x+2}$
13. $\lim\limits_{x\to 1} \dfrac{x^2-2x+1}{x^2-1}$
14. $\lim\limits_{x\to 0} \dfrac{2x^5+3x}{x^2+2x}$
15. $\lim\limits_{x\to 0} \dfrac{4x^3+x^2}{x^2-3x}$
16. $\lim\limits_{x\to 1} \dfrac{x^3+3x^2-2x-2}{2x^2+x-3}$
17. $\lim\limits_{x\to 2} \sqrt{2x^2+1}$
18. $\lim\limits_{x\to 2} \dfrac{\sqrt{x+2}-3}{x^2-9}$
19. $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$
20. $\lim\limits_{x\to 0} \dfrac{\sqrt{3x+4}-2}{x}$
21. $\lim\limits_{x\to 1} \dfrac{2-\sqrt{x+3}}{x^2-1}$
22. $\lim\limits_{x\to 2} \dfrac{\sqrt{3x-2}-\sqrt{x+2}}{x-2}$
23. $\lim\limits_{x\to 1} \left(\dfrac{x+3}{x^2-1}-\dfrac{x+1}{x^2-x}\right)$
24. $\lim\limits_{x\to \infty} \dfrac{2x^3-1}{x^3-x+2}$

25. $\lim\limits_{x\to\infty} \dfrac{3x^3 - 2x + 1}{2x^3 + x^2 - 5}$

26. $\lim\limits_{x\to\infty} \dfrac{2x + 7}{x^3 + 2x^2 - 4}$

27. $\lim\limits_{x\to\infty} \dfrac{x^4 + 2x^3 - x^2 + 1}{x^5 + x^3 - 2}$

28. $\lim\limits_{x\to\infty} \left(\dfrac{x^2 + 3x}{2_x + 1} - \dfrac{x}{2} \right)$

29. $\lim\limits_{x\to\infty} 2^x$

30. $\lim\limits_{x\to\infty} \dfrac{x^2}{x + 1}$

31. $\lim\limits_{x\to\infty} \dfrac{x^3 - 3x + 1}{2x^2 + x + 1}$

32. $f(x) = \begin{cases} -1, & x < 0 \\ 2x, & x \geq 0 \end{cases}$, find $\lim\limits_{x\to 0} f(x)$

33. $f(x) = \begin{cases} -x + 1, & x < 1 \\ 3, & x = 1 \\ 2x - 2, & x > 1 \end{cases}$, find $\lim\limits_{x\to 1} f(x)$

Determine if the limit of the following functions exists at $x = x_0$. If it exists, find their limit.

34. $f(x) = \begin{cases} x + 1, & x < 0 \\ 0, & x = 0 \\ x - 1, & x > 0 \end{cases}$, $x_0 = 0$

35. $f(x) = \begin{cases} -x + 1, & x \leq 2 \\ x - 3, & x > 2 \end{cases}$, $x_0 = 0, x_0 = 2$

36. $f(x) = \dfrac{1}{x + 3}$, $x_0 = -3$

Revision Exercise 24

1. Complete the follwoing table, state the changes of the fucntion $f(x) = x^3 - 1$ at $x = 1$. Hence, find the left limit, right limit and limit of the function at $x = 1$.

x	0.9	0.99	0.999	0.9999	1	1.0001	1.001	1.01	1.1
$f(x)$									

Find the limit of the followings (Question 2 to 13):

2. $\lim\limits_{x\to 1} (x^2 + x - 2)$

3. $\lim\limits_{x\to 2} (x^2 - 1)\sqrt{x + 2}$

4. $\lim\limits_{x\to -2} \dfrac{2x^2 + 3x - 2}{3x^2 + 4x - 4}$

5. $\lim\limits_{x\to 2} \dfrac{3 - \sqrt{x + 7}}{x^2 - 1}$

6. $\lim\limits_{x\to 0} \dfrac{2 - \sqrt{3x + 4}}{x^2 + x}$

7. $\lim\limits_{x\to -1} \dfrac{\sqrt{2x + 5} - \sqrt{3}}{x + 1}$

8. $\lim\limits_{x\to 3} \dfrac{\sqrt{x + 1} - 2}{x^2 - x - 6}$

9. $\lim\limits_{x\to\infty} \dfrac{4x + 3}{x^2 + 2x - 1}$

10. $\lim\limits_{x\to\infty} \dfrac{2x^4 - x^3 + 3x + 1}{3x^4 + 4x^2 - x - 2}$

11. $\lim\limits_{x\to\infty} \dfrac{x^3 - 2x^2 + 3}{2x^2 - 1}$

12. $\lim\limits_{x\to 1} \left(\dfrac{1}{x - 1} - \dfrac{2}{x^2 - 1} \right)$

13. $\lim\limits_{x\to\infty} \left(\dfrac{x^3}{2x^2 + 1} - \dfrac{x^2}{2x + 3} \right)$

Determine if the limit of the following functions exists at $x = x_0$. If it exists, find their limit. (Question 14 to 17)

14. $f(x) = \begin{cases} 1 - 3x, & x < 0 \\ 2x + 1, & x \geq 0 \end{cases}$, $x_0 = 0$

15. $f(x) = \begin{cases} \sqrt{x+3}, & x < -2 \\ x+1, & x > -2 \end{cases}$, $x_0 = -2$

16. $f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ \dfrac{1}{2}, & x = 1 \end{cases}$, $x_0 = 1$

17. $f(x) = \begin{cases} 2x+1, & x \leq 1 \\ x^2+1, & 1 < x \leq 2 \\ 3x-1, & x > 2 \end{cases}$, $x_0 = 1$, $x_0 = 2$

Chapter 25

Differentiation

25.1 Gradient of Tangent Line on a Curve

As shown in the diagram below, sketch the graph of curve C on the Cartesian plane. and pick two points P and Q on the curve, the straight line that passes through P and Q is the secant line of the curve. When point Q moves closer to point P along the curve, the secant line PQ rotates along point P and approaches a limit straight line PT. The line PT is called the *tangent line* of the curve C at point P.

We know the fact that a line can be determined by a point on the line and its gradient. Therefore, in order to find the gradient of the tangent line of the curve C at point P, we have to first find the gradient of the tangent.

To find the gradient of the tangent line of curve $y = f(x)$ at point $P\left(x_0, f(x_0)\right)$, we can pick a point $\left(x_0 + \Delta x, f(x_0 + \Delta x)\right)$ near to point P on the curve, as shown in the diagram above. Denote the variable PR on the horizontal axis as Δx, and the corresponding variable RQ on the vertical axis as Δy, we have $\Delta y = f(x_0 + \Delta x) - f(x_0)$.

Hence, the gradient of the secant line PQ is $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$.

When $\Delta x \to 0$, if the limit of the expression above exist, it is the gradient of the tangent line of the curve $y = f(x)$ at point P, denoted as m, that is

$$\boxed{m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}}$$

Pratice 1

1. Find the gradient of the tangent line of the curve $y = 2 - x^2$ at $x = 1$.

2. Find the gradient of the tangent line of the curve $y = x^2 + 3x$ at $x = 2$.

Exercise 25.1

1. Given the curve $y = x^2 + 1$. Assume that P is the point on the curve at $x = 2$, Q is a nearby point,

   (a) Find the variable Δx and the corresponding variable Δy when the x coordinates of point Q is 2.5, 2.25, 2.1, 2.05, 2.01, and 2.001 respectively. Hence, complete the table below.

x-coords of point Q	y-coords of point Q	Δx	Δy	Gradient of secant PQ $\frac{\Delta y}{\Delta x}$
$x = 2.5$				
$x = 2.25$				
$x = 2.1$				
$x = 2.05$				
$x = 2.01$				
$x = 2.001$				

   (b) Inspect the gradient of secant PQ as point Q approaches point P. Hence, find the gradient of the tangent line of the curve at point P.

2. Find the gradient of the tangent line of the curve $y = \frac{1}{3}x^2$ at $x = 2$.

3. Find the gradient of the tangent line of the curve $y = \frac{10}{x}$ at point $Q(2, 5)$.

4. Find the gradient of the tangent line of the curve $y = \frac{4}{x} - 5$ at $x = 1$.

5. Find the gradient of the tangent line of the curve $y = \frac{1}{2}x^3 + 1$ at point $P(-2, -3)$.

25.2 Gradient of Tangent Line and Derivative

In the last section, we have learnt that, if $P\left(x_0, f(x_0)\right)$ and $Q\left(x_0 + \Delta x, f(x_0 + \Delta x)\right)$ are two points on the curve $y = f(x)$, then the gradient of the secant line PQ is $\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$. As point Q approaches point P, that is, as $\Delta x \to 0$, the gradient of the secant line PQ approaches the tangent line PT.

Hence, the gradient of the tangent line PT is

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Definition of Derivative

Let the function $y = f(x)$ be defined at $x = x_0$ and its nearby points, when there exists a variable Δx of x at x_0, there exist a corresponding variable $\Delta y = f(x_0 + \Delta x) - f(x_0)$ of the function y. When $\Delta x \to 0$, the limit of $\frac{\Delta y}{\Delta x}$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exists, $y = f(x)$ is said to be *differentiable* at $x = x_0$, and the limit is called the *derivative* of $y = f(x)$ at $x = x_0$, denoted as $f'(x_0)$, that is

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

If the limit above does not exist, $y = f(x)$ is said to be *non-differentiable* at $x = x_0$.

If function $y = f(x)$ is differentiable at every point of an interval (a, b). Each defined value x_0 in the interval (a, b) corresponds to a derivative value $f'(x_0)$, thus forming a new function in the interval (a, b). This new function is called the *derivative function* of $f(x)$, denoted as $f'(x)$ or y'.

From the definition of derivative, we can get the derivative

$$y' = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$f'(x)$ is often being denoted as $\frac{dy}{dx}$ as well, and is called *differentiation of* $y = f(x)$ *with respect to* x.

From the definition of derivative, we can conclude the following steps to find the derivative of a function $y = f(x)$:

1. Find the variable $\Delta y = f(x + \Delta x) - f(x)$ of the function.
2. Find the ratio of the variable Δy to Δx, that is, $\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$.
3. Find the limit of the ratio above, that is, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$.

The method above to find the derivative using the definition of derivative is called the **first principle** of differentiation.

Note that derivative function is also called the derivative. Unless otherwise stated, finding the derivative means finding the derivative function.

Also note that the derivative function $f'(x)$ of the function $y = f(x)$ is different from the derivative $f'(x_0)$ of the function $y = f(x)$ at $x = x_0$. $f'(x)$ is a function, while $f'(x_0)$ is a value, but they are related to each other. $f'(x_0)$ is the function value of the derivative function $f'(x)$ at $x = x_0$, and is called the *derivative value* at $x = x_0$.

25.3 Law of Differentiation

Since the method of finding the derivative using the definition of derivative is very complicated, we need to derive some formulas from the definition of the derivative, then we can use these formulas to find the derivatives.

Derivative of Power Function

Let $y = f(x) = c$, where c is a constant.

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \frac{c-c}{\Delta x} \\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(c) &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} 0 \\ &= 0\end{aligned}$$

$$\boxed{\frac{d}{dx}(c) = 0}$$

Let $y = f(x) = x^n$, where n is a positive integer.

$$\begin{aligned}\Delta y = f(x+\Delta x) - f(x) &= f(x-\Delta x) - f(x) \\ &= (x+\Delta x)^n - x^n \\ &= \left[x^n + {}_nC_1 x^{n-1}\Delta x + {}_nC_2 x^{n-2}(\Delta x)^2 + \cdots + {}_nC_n(\Delta x)^n\right] - x^n \\ &= {}_nC_1 x^{n-1}\Delta x + {}_nC_2 x^{n-2}(\Delta x)^2 + \cdots + {}_nC_n(\Delta x)^n \\ \frac{\Delta y}{\Delta x} &= {}_nC_1 x^{n-1} + {}_nC_2 x^{n-2}\Delta x + \cdots + {}_nC_n(\Delta x)^{n-1}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(x^n) &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left[{}_nC_1 x^{n-1} + {}_nC_2 x^{n-2}\Delta x + \cdots + {}_nC_n(\Delta x)^{n-1}\right] \\ &= {}_nC_1 x^{n-1}\end{aligned}$$

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}}$$

The derivation above only considered the case where n is a positive integer. In fact, the formula above is also valid when n is a negative integer or a rational number.

Derivative of Product of Function and Constant

Let $y = cu$, where c is a constant and u is a differentiable function of x.

$$\begin{aligned} y + \Delta y &= c(u + \Delta u) = cu + c\Delta u \\ \Delta y &= c\Delta u \\ \frac{\Delta y}{\Delta x} &= c\frac{\Delta u}{\Delta x} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}(cu) &= \lim_{\Delta x \to 0}\left(c\frac{\Delta u}{\Delta x}\right) \\ &= c\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} \\ &= c\frac{du}{dx} \end{aligned}$$

$$\boxed{\frac{d}{dx}(cu) = c\frac{du}{dx} \quad \text{(where } c \text{ is a constant, } u \text{ is a differentiable function of } x\text{)}}$$

Derivative of Sum and Difference of Functions

Let $y = u \pm v$, where u and v are differentiable functions of x.

$$\begin{aligned} y + \Delta y &= (u + \Delta u) \pm (v + \Delta v) \\ \Delta y &= \Delta u \pm \Delta v \\ \frac{\Delta y}{\Delta x} &= \frac{\Delta u}{\Delta x} \pm \frac{\Delta v}{\Delta x} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}(u \pm v) &= \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x} \pm \frac{\Delta v}{\Delta x}\right) \\ &= \lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} \pm \lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x} \\ &= \frac{du}{dx} \pm \frac{dv}{dx} \end{aligned}$$

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$
$$\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$$
(where u and v are differentiable functions of x)

The above formulae can be extended to the case where there are more than two functions being added or subtracted. That is,

$$\frac{d}{dx}(u_1 \pm u_2 \pm \cdots \pm u_n) = \frac{du_1}{dx} \pm \frac{du_2}{dx} \pm \cdots \pm \frac{du_n}{dx}$$

Product Rule

Let $y = uv$, where u and v are differentiable functions of x.

$$\begin{aligned} y + \Delta y &= (u + \Delta u)(v + \Delta v) \\ &= uv + u\Delta v + v\Delta u + \Delta u \Delta v \\ \Delta y &= u\Delta v + v\Delta u + \Delta u \Delta v \\ \frac{\Delta y}{\Delta x} &= u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x} \end{aligned}$$

Given that $\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx}$ and $\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \frac{dv}{dx}$,

$$\begin{aligned} \text{Hence, } \lim_{\Delta x \to 0} \Delta v &= \lim_{\Delta x \to 0} \left(\frac{\Delta v}{\Delta x} \cdot \Delta x \right) \\ &= \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta x \\ &= v'(x) \cdot 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \therefore \lim_{\Delta x \to 0} (uv) &= \lim_{\Delta x \to 0} \left(u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta v\frac{\Delta v}{\Delta x} \right) \\ &= u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \right) \left(\lim_{\Delta x \to 0} \Delta v \right) \\ &= u\frac{dv}{dx} + v\frac{du}{dx} + 0 \\ &= u\frac{dv}{dx} + v\frac{du}{dx} \end{aligned}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{(where } u \text{ and } v \text{ are differentiable functions of } x\text{)}$$

This has proven that if $\frac{dv}{dx}$ exists at $x = x_0$, then

$$\begin{aligned}
&\lim_{\Delta x \to 0} v(x_0 + \Delta x) - v(x_0) \\
&= \lim_{\Delta x \to 0} \Delta v \\
&= 0
\end{aligned}$$

From that, we can conclude that

$$\lim_{\Delta x \to 0} v(x_0 + \Delta x) = v(x_0)$$

That is, $y = v(x)$ is continuous at $x = x_0$.
Note that $\frac{d}{dx}(uv) \neq \frac{d}{dx}(u) \cdot \frac{d}{dx}(v)$.

Practice 3

Find the derivative of the following functions:

1. $y = 5x$
2. $y = \sqrt{x}$
3. $y = 6x^3 + 3x^2 - 5$
4. $y = (2x - 3)\left(x^2 + 2\right)$
5. $y = \frac{2}{x} + \frac{1}{x^2}$
6. $y = 2x^3 - 3x + \frac{7}{\sqrt{x}}$

Exercise 25.3a

Find the derivative of the following functions (Question 1 to 18):

1. $y = 2x^3 + 2$
2. $y = \frac{1}{2}x^2 - \frac{1}{3}x + 2$
3. $y = x^5 - \frac{1}{4}x^4 + 3x^2 - 4$
4. $y = 2x^2 + 4x^3 - 7x^4$
5. $y = x^2 - 3x + \frac{2}{x} - \frac{4}{x^2}$
6. $y = 2x^3 - 4x - \frac{3}{x^2} + \frac{4}{x^3}$
7. $y = \sqrt{x} + \frac{2}{\sqrt{x}} + 3$
8. $y = 3\sqrt{x} - 5\sqrt[3]{x^2}$
9. $y = (2x - 5)^2$
10. $y = \left(x + \frac{1}{x}\right)^2$
11. $y = \frac{x^3 + 4x^2 - x + 2}{x}$
12. $y = \frac{2x^4 + 3x^2 - 6}{x^2}$

13. $y = \dfrac{\sqrt[3]{x} - 2}{\sqrt{x}}$

14. $y = (x+2)\left(x^2 + 3x - 8\right)$

15. $y = (2+3x)\left(1 + x - x^2\right)$

16. $y = (x+2)(x-2)\left(x^2+4\right)$

17. $y = (x-1)^2(3x+5)$

18. $y = \left(x^2+1\right)(3x-1)\left(1-x^2\right)$

19. If the gradient of the curve $y = x^3 + 6x^2 + 45x + 12$ at point A is 36, find the coordinates of A.

20. Given the functions $f(x) = f(x) = x^2 + 3x + 4$ and $g(x) = x^3 + x^2 + 7$. If $f'(a) = g'(a)$, find the value of a.

Quotient Rule

Let $y = \dfrac{u}{v}$, where u and v are differentiable functions of x.

$$\begin{aligned}
y + \Delta y &= \frac{u + \Delta u}{v + \Delta v} \\
\Delta y &= \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} \\
&= \frac{v\Delta u - u\Delta v}{v(v + \Delta v)} \\
\frac{\Delta y}{\Delta x} &= \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}
\end{aligned}$$

When $\dfrac{\Delta x}{\Delta x}$ exists, $\lim\limits_{\Delta x \to 0} \Delta v = 0$.

$$\begin{aligned}
\therefore \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)} \\
&= \frac{v \lim\limits_{\Delta x \to 0} \dfrac{\Delta u}{\Delta x} - u \lim\limits_{\Delta x \to 0} \dfrac{\Delta v}{\Delta x}}{\lim\limits_{\Delta x \to 0} v^2 + v \lim\limits_{\Delta x \to 0} \Delta v} \\
&= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}
\end{aligned}$$

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \quad (v \neq 0,\ u \text{ and } v \text{ are differentiable functions of } x)}$$

Practice 4

Find the derivative of the following functions:

1. $y = \dfrac{2x}{x+2}$
2. $y = \dfrac{3x-2}{x+2}$
3. $y = \dfrac{x}{x^2-5}$
4. $y = \dfrac{x^2-1}{3x-2}$

Exercise 25.3b

Find the derivative of the following functions:

1. $y = \dfrac{x-2}{x+2}$
2. $y = \dfrac{x-a}{2x+a}$ where a is a constant
3. $y = \dfrac{2x^3}{x+2}$
4. $y = \dfrac{2x+3}{x^2+1}$
5. $y = \dfrac{3x}{x^2-4x}$
6. $y = \dfrac{3x^2+x-1}{2x-1}$
7. $y = \dfrac{2x^4}{(x+3)^2}$
8. $y = \dfrac{2}{1+x} + \dfrac{2}{1-x}$
9. $y = \dfrac{4-x}{3-2x+x^2}$
10. $y = \dfrac{1+x-x^2}{1-x+x^2}$

25.4 Chain Rule - Differentiation of Composite Functions

If $y = f(u)$ and $u = g(x)$ are both differentiable functions of x, then $y = f\big(g(x)\big)$ is also a differentiable function of x. In a composite function $y = f\big(g(x)\big)$, when x changes by Δx, u changes by Δu and y also changes by Δy, where $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$.

If the derivative of the functions $y = f(u)$ and $u = g(x)$ are $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$ respectively, then when $\Delta x \to 0$, $\Delta u \to 0$, and $\lim\limits_{\Delta x \to 0} \dfrac{\Delta u}{\Delta x} = \dfrac{du}{dx}$, $\lim\limits_{\Delta u \to 0} \dfrac{\Delta y}{\Delta u} = \dfrac{dy}{du}$.

$$\begin{aligned}
\text{Hence, } \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \right) \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \frac{dy}{du} \cdot \frac{du}{dx}
\end{aligned}$$

Therefore, the derivative of the composite function $y = f\big(g(x)\big)$ is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'\big(g(x)\big) \cdot g'(x)$$

Further extending the chain rule, if $y = f(v)$, $v = g(u)$, and $u = h(x)$, then

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

The rule above is called the *chain rule*.

Practice 5

Find the derivative of the following functions:

1. $y = \left(2x^3 - 4\right)^3$
2. $y = \dfrac{6}{\sqrt{2x - 3}}$
3. $y = (x + 1)(x - 3)^3$
4. $y = \left(x^2 - 2\right) \sqrt{1 + x}$
5. $y = \dfrac{x}{(2x + 3)^3}$
6. $y = \dfrac{(3x + 4)^2}{(x - 5)^3}$

Exercise 25.4

Find the derivative of the following functions:

1. $y = (2x - 3)^8$
2. $y = \left(3x^2 - 6x + 5\right)^4$
3. $y = \sqrt{2x^4 - 4x^2 + 5}$
4. $y = \sqrt[3]{3x^2 - 3x + 2}$
5. $y = \dfrac{3}{6x^2 - 4}$
6. $y = \dfrac{9}{\sqrt{6x^3 - 9x}}$
7. $y = \dfrac{3x}{(x + 5)^2}$
8. $y = \dfrac{3x - 1}{\sqrt{x - 1}}$
9. $y = \sqrt{x}\,(x - 3)^5$
10. $y = x^2(x - 3)(x + 2)^2$
11. $y = (2x + 1)^2(x + 1)^3$
12. $y = \left(\dfrac{1 + x}{1 - x}\right)^2$
13. $y = x^2\sqrt{1 + x^2}$
14. $y = \sqrt{\dfrac{1 + x^2}{1 - x^2}}$

25.5 Higher Order Derivatives

If the derivative $f'(x)$ of a function $y = f(x)$ is differentiable at $x = x_0$, then the derivative of f' at $x = x_0$ is called the *second derivative* of $y = f(x)$ at $x = x_0$, and is denoted by $f''(x_0)$, that is,

$$\lim_{\Delta x \to 0} \frac{f'(x_0 + \Delta x) - f'(x_0)}{\Delta x} = f''(x_0)$$

The derivative of the derivative $f'(x)$ of a function $y = f(x)$ is called the *second derivative* of $y = f(x)$, and is denoted by $(f')'(x)$ or $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = \frac{d^2\big(f(x)\big)}{dx^2}$; while $f'(x)$ is called the *first derivative* of $y = f(x)$. Using the same way, we can define the third derivative $f'''(x)$, the fourth derivative $f^{(4)}(x)$, up to the nth derivative $f^{(n)}(x)$.

The way of finding higher order derivatives is similar to the way of finding the first derivative, as we just need differentiate the function again and again until we get the desired order of derivative.

Practice 6

1. Find the first to sixth derivatives of $y = 3x^5 + 2x^4 - 5x^2 - 8x + 9$

Find the third derivative of the following functions (Question 2 to 5):

2. $y = \frac{3}{x^4}$
3. $y = 3x^3 + 6x^2 - 5x$
4. $y = \sqrt{2x - 3}$
5. $y = \frac{1}{(2x + 5)^3}$

Exercise 25.5

1. Given the function $y = ax^6 + 2bx^4 - 3cx^3 + 4x^2 - 5$ where a, b and c are constants, find the first to seventh derivatives of y.

Find the second derivative of the following functions (Question 2 to 5):

2. $y = 2x - \frac{x^3}{2}$
3. $y = \left(2 + x^2\right)^3$
4. $y = x + \frac{1}{x} + \frac{1}{x^2}$
5. $y = x^2 - \frac{1}{x^2}$
6. $y = \sqrt{a^2 - x^2}$ where a is a constant
7. $y = \frac{2}{\sqrt{x - 2}}$
8. $y = \sqrt{3x^2 + 2}$
9. $y = \frac{x}{\sqrt{1 - x^2}}$
10. Given the function $y = 2x^3 - 3x^2 - 6x + 10$, find the value of y', y'' and y''' at $x = 1$.
11. Given the function $f(x) = \frac{5}{(2x + 1)^2}$, find the value of $f''(2)$.
12. Find the second derivative y'', of the function $y = 2x^3 + 3x^2 - 72x + 15$, and find the value of x at $y'' = 0$.

25.6 Implicit Differentiation

Consider the function $x^2 + y^2 = x^3$ and $y^2 - 2xy - 3 = 0$. For these kind of functions, y is not expressed in terms of x explicitly, hence we say that y is an *implicit function* of x. In some cases, the equations above are hard if not impossible to be rewritten in the form of $y = f(x)$. To find the derivative of implicit functions, we can differnetiate both sides of the equation with respect to x and use the chain rule at the same time to get an equation in terms of x, y and $\frac{dy}{dx}$, then we can solve for $\frac{dy}{dx}$.

Practice 7

Find $\frac{dy}{dx}$ for the following implicit functions:

1. $x^2 + y^2 - 6x + 8y - 9 = 0$
2. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 5$ where a and b are constants
3. $2x^2 - 4xy + 2y^2 = 5$
4. $x^5 + 2x^2y^3 + y^4 = 1$

Exercise 25.6

Find $\frac{dy}{dx}$ for the following implicit functions:

1. $x^2 + y^3 = 3$
2. $\sqrt{x} + \sqrt{y} = \sqrt{a}$ where a is a constant
3. $x^3 + y^3 = 2xy + 3$
4. $y^2(x + 1) = 3x^2$
5. $3x^2 - 6xy + 3y^2 = 25$
6. $xy^3 = 2x^2 - 2y^2$
7. $\frac{1}{x} + \frac{1}{y} = 2x$
8. $\frac{3x^2}{y} + \frac{3y^2}{x} = 1$
9. $xy - x + xy^3 = 10$
10. $4x^3 + 2xy^2 - xy = 0$

25.7 Two Basic Limits

A. $\lim\limits_{x \to 0} \frac{\sin x}{x}$

Consider the value of $\frac{\sin x}{x}$ when x approaches 0. When $x = 0$, the value of $\sin x$ and x are both 0, hence $\frac{\sin x}{x} = \frac{0}{0}$, which is undefined. However, the value of $\lim\limits_{x \to 0} \frac{\sin x}{x}$ is defined, as shown in the diagram below:

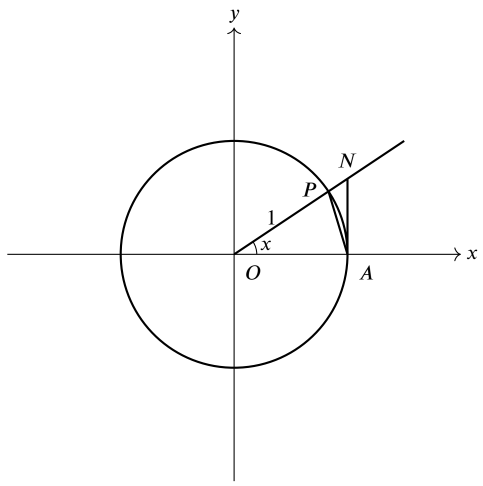
x (in rad)	$\sin x$	$\dfrac{\sin x}{x}$
1	0.84147098	0.84147098
0.500	0.479425539	0.958851077
0.250	0.247403959	0.989615837
0.100	0.099833417	0.99833417
0.010	0.0099998333	0.99998333
0.001	0.00099999983	0.99999983
$\vdots$	$\vdots$	$\vdots$

Note that the trigonometric independent variable x is in radian.

By looking at the table above, as x approaches 0, the value of $\dfrac{\sin x}{x}$ approaches 1. Hence, we can speculate that

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x} = 1}$$

We can also prove the speculation above.



In the unit circle above, the central angle x from the initial arm OA, its terminal side intersects with the circle at point P, AN is the tangent line that passes through point A, AN and OP intersects at point N.

From the diagram above, we can see that $\overset{\frown}{PA} = x$

$$AN = \tan x$$

and the area of $\Delta OAP <$ the area of sector $OAP <$ the area of ΔOAN.

$$\therefore \frac{1}{2}\sin x < \frac{1}{2}(1)^2 x < \frac{1}{2}(1)\tan x$$
$$\sin x < x < \frac{\sin x}{\cos x}$$

When x is a positive angle that is smaller than 90°, $\sin x > 0$. Dividing the inequality above by $\sin x$, we get

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$
$$\cos x < \frac{\sin x}{x} < 1$$

When $x < 0$, $\dfrac{\sin x}{x} = \dfrac{\sin(-x)}{-x}$, $\cos(-x) = \cos x$.

$$\therefore \lim_{x\to 0}\frac{\sin x}{x} \text{ is in between } \cos x \text{ and } 1 \because \lim_{x\to 0}\cos x = 1 \therefore \lim_{x\to 0}\frac{\sin x}{x} = 1$$

The statement above uses the Squeeze Theorem. From the proof above, we can also conclude that

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{1}{\frac{\sin x}{x}} = 1$$

Practice 8

Find the limit of the following:

1. $\lim\limits_{x\to 0}\dfrac{\sin 4x}{5x}$
2. $\lim\limits_{x\to 0}\dfrac{7x}{\sin 5x}$
3. $\lim\limits_{1\to 0}\dfrac{\sin 5x}{\sin 4x}$
4. $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{3x^2}$
5. $\lim\limits_{s\to 0}\dfrac{\tan 9x}{\sin 5x}$
6. $\lim\limits_{x\to 0}\dfrac{\tan 2x\sin 3x}{x^2}$

Exercise 25.7a

Find the limit of the following:

1. $\lim\limits_{x\to 0}\dfrac{x\cos x}{\sin 2x}$
2. $\lim\limits_{x\to 0}(x\cot x)$
3. $\lim\limits_{x\to 0}\dfrac{2x}{\sin 7x}$
4. $\lim\limits_{x\to 0}\dfrac{\sin\frac{5}{2}x}{2x}$
5. $\lim\limits_{x\to 0}\dfrac{\sin(2x^2)}{x}$
6. $\lim\limits_{x\to 0}\dfrac{\sin(2x^2)}{x}$
7. $\lim\limits_{x\to 0}\dfrac{x^2}{\sin^2 bx}$
8. $\lim\limits_{x\to 0}\frac{\sin^2 3x}{2x^2}$

9. $\lim\limits_{x\to 0} \frac{4x}{\tan 8x}$

10. $\lim\limits_{x\to 0} \frac{\sin 5x}{\tan 3x}$

11. $\lim\limits_{x\to 0} \frac{1-\cos 4x}{x\sin 5x}$

12. $\lim\limits_{x\to 0} \frac{5x^2}{2\tan^2 2x}$

13. $\lim\limits_{x\to 0} \frac{\cos 3x-1}{3x^2}$

B. $\lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

Consider the limit of $\left(1+\frac{1}{x}\right)^x$ as x approaches infinity. As $x \to \infty$, the changes of the function $f(x) = \left(1+\frac{1}{x}\right)^x$ is shown in the table below:

x	$f(x) = \left(1+\frac{1}{x}\right)^x$
1	$\left(1+\frac{1}{1}\right)^1 = 2$
10	$\left(1+\frac{1}{10}\right)^{10} = 2.59374$
100	$\left(1+\frac{1}{100}\right)^{100} = 2.70481$
1000	$\left(1+\frac{1}{1000}\right)^{1000} = 2.71692$
10000	$\left(1+\frac{1}{10000}\right)^{10000} = 2.71815$
100000	$\left(1+\frac{1}{100000}\right)^{100000} = 2.71827$
$\vdots$	$\vdots$

By looking at the table above, we can see that as x approaches infinity, the value of $\left(1+\frac{1}{x}\right)^x$ approaches a costant value that is denoted as e. e is an irrational number and is approximately equal to 2.71828182846. It is called the **natural base**, and is one of the most important numbers in mathematics and appears in many applications.

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$$

If $y = \frac{1}{x}$, then $x = \frac{1}{y}$. As $x \to \infty$, $y \to 0$.

Therefore, $\lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim\limits_{y\to 0} (1+y)^{\frac{1}{y}} = e$.

Practice 9

Find the limit of the following:

1. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+2}$
2. $\lim_{x\to\infty}(1-3x)^{\frac{2}{x}}$
3. $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{\frac{x}{2}}$
4. $\lim_{x\to\infty}\left(\frac{2-3x}{2}\right)^{2}$

Exercise 25.7b

Find the limit of the following:

1. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{\frac{x}{2}}$
2. $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{2x}$
3. $\lim_{x\to\infty}\left(\frac{x+5}{x}\right)^{2x}$
4. $\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^{-3x}$
5. $\lim_{x\to 0}\left(\frac{3-x}{3}\right)^{-\frac{3}{x}}$
6. $\lim_{x\to\infty}\left(1+\frac{5}{x}\right)^{2-x}$
7. $\lim_{x\to\infty}\left(1+\frac{3}{2x}\right)^{3+x}$
8. $\lim_{x\to 0}(1-4x)^{-\frac{3}{x}}$
9. $\lim_{x\to 0}(1+2x)^{1-\frac{2}{x}}$
10. $\lim_{x\to\infty}\left(\frac{x+2}{x+1}\right)^{x}$

25.8 Derivatives of Trigonometric Functions

Here we will derive the derivatives of the sine, cosine, and tangent functions.

Let $y = \sin x$.

$$\begin{aligned}
y + \Delta y &= \sin(x + \Delta x) \\
\Delta y &= \sin(x + \Delta x) - \sin x \\
&= \sin x \cos \Delta x + \cos x \sin \Delta x - \sin x \\
&= \sin x(\cos \Delta x - 1) + \cos x \sin \Delta x \\
\frac{\Delta y}{\Delta x} &= \sin x \frac{\cos \Delta x - 1}{\Delta x} + \cos x \frac{\sin \Delta x}{\Delta x}
\end{aligned}$$

$$
\begin{aligned}
\because \lim_{\Delta x \to 0} \frac{\cos \Delta x - 1}{\Delta x} &= \lim_{\Delta x \to 0} \frac{-2\sin^2 \frac{\Delta x}{2}}{\Delta x} \\
&= -\frac{1}{2} \lim_{\Delta x \to 0} \left(\frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \right) \cdot \Delta x \\
&= 0 \\
\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x} &= 1
\end{aligned}
$$

$$
\begin{aligned}
\therefore \frac{d}{dx}(\sin x) &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \sin x \cdot 0 + \cos x \cdot 1 \\
&= \cos x
\end{aligned}
$$

$$
\boxed{\frac{d}{dx}(\sin x) = \cos x}
$$

$$\because \ \cos x = \sin\left(\frac{\pi}{2} - x\right)$$

$$\begin{aligned}
\therefore \ \frac{d}{dx}(\cos x) &= \frac{d}{dx}\left(\sin\left(\frac{\pi}{2} - x\right)\right) \\
&= \cos\left(\frac{\pi}{2} - x\right)(-1) \\
&= -sinx
\end{aligned}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

$$\because \ \tan x = \frac{\sin x}{\cos x}$$

$$\begin{aligned}
\therefore \ \frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

Below are the derivatives of the remaining trigonometric functions.

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Practice 10

Find the derivatives of the following functions:

1. $y = \cos\left(5x^2\right)$
2. $y = \sin^2(2x - 3)$
3. $y = \tan^4 3x$

Exercise 25.8

Find the derivatives of the following functions:

1. $y = \sin^2 x$
2. $y = \sin 3x - \cos 3x$
3. $y = \tan 4x - \cot 5x$
4. $y = \sec 3x + \csc 5x$
5. $y = 2x \sec x$
6. $y = \dfrac{2x}{\sin x}$
7. $y = \cos 3x°$
8. $y = \cos^4(1 - 2x)$
9. $y = \tan^3\left(2x^2\right)$
10. $y = \cos^2 x - \sin^2 x$
11. $y = 4 \sin x \cos x$
12. $y = \sin 3x \tan 6x$
13. $y = \sqrt{\cos 2x}$
14. $y = \sin^2 \sqrt{1 + x^2}$
15. $y = x \tan^2(3x - 2)$
16. $y = \dfrac{3 \sin 2x}{\cos x}$
17. $y = \dfrac{2}{3 \tan^3 x}$
18. $y = \dfrac{1 + \cos x}{1 - \cos x}$
19. $y = \dfrac{2 \tan x}{1 - \tan^2 x}$
20. $y = \dfrac{\sin\left(2x - \frac{\pi}{4}\right)}{\sin\left(2x + \frac{\pi}{4}\right)}$
21. Given the function $y = \dfrac{x}{2 + \cos x}$. Find the derivative value when $x = 0$, $x = \dfrac{\pi}{2}$, and $x = \pi$.
22. If the function $y = \sec x + \tan x$, prove that $\dfrac{dy}{dx} = y \sec x$.

23. If the function $y = \sqrt{\dfrac{1-\cos x}{1+\cos x}}$, find $\dfrac{dy}{dx}$.

24. If the function $r = \sin 3t - 2\cos t$, find $\dfrac{d^2r}{dt^2}$ when $t = \dfrac{\pi}{3}$.

25.9 Derivatives of Exponential Functions

Derivative of Natural Logarithmic Functions

Let $y = \ln x$ where $x > 0$.

$$\begin{aligned}
y + \Delta y &= \ln(x + \Delta x) \\
\Delta y &= \ln(x + \Delta x) - \ln x \\
&= \ln\left(\frac{x + \Delta x}{x}\right) \\
&= \ln\left(1 + \frac{\Delta x}{x}\right) \\
\frac{\Delta y}{\Delta x} &= \frac{1}{\Delta x}\ln\left(1 + \frac{\Delta x}{x}\right) \\
&= \frac{1}{\Delta x} \cdot \frac{x}{\Delta x} \cdot \ln\left(1 + \frac{\Delta x}{x}\right) \\
&= \frac{1}{x}\ln\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} \\
\frac{dy}{dx} &= \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} \\
&= \frac{1}{x}\lim_{\Delta x \to 0}\ln\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} \\
&= \frac{1}{x}\ln\left(\lim_{\Delta x \to 0}\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right) \\
&= \frac{1}{x}\ln e \\
&= \frac{1}{x}
\end{aligned}$$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

If $x < 0$, then $-x > 0$.

$$\begin{aligned}
\frac{d}{dx}(\ln -x) &= \frac{1}{-x}\frac{d}{dx}(-x) \\
&= \frac{1}{x}
\end{aligned}$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

Derivative of Other Logarithmic Functions

$$\begin{aligned}
\text{Let } y &= \log_a x \\
&= \frac{\ln x}{\ln a} \\
&= \frac{1}{\ln a} \ln x \\
\frac{dy}{dx} &= \frac{1}{\ln a} \cdot \frac{d}{dx}(\ln x) \\
&= \frac{1}{\ln a} \cdot \left(\frac{1}{x}\right) \\
&= \frac{1}{x \ln a}
\end{aligned}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Generally, when solving for the derivative of a logarithmic function, we can first convert it to a natural logarithmic function and then solve for the derivative.

Practice 11

Find the derivative of the following functions:

1. $y = \ln\left(x^2 - 2x + 1\right)$
2. $y = \log_2 \sin 3x$
3. $y = \ln(\sec x)$
4. $y = x \ln x$
5. $y = \ln \frac{1+x}{1-x}$
6. $y = \log_2 \sqrt{1 + x^2}$
7. $y = \ln \dfrac{1 + x}{1 - x}$
8. $y = \log_2 \sqrt{1 + x^2}$

Exercise 25.9

Find the derivative of the following functions:

1. $y = \ln(5x - 3)$
2. $y = \ln\left(ax^3\right)$
3. $y = \ln\dfrac{3}{x^2}$
4. $y = \ln\left(2x^2 - 3x + 4\right)$
5. $y = \log_5(3x)$
6. $y = \log_2\left(x^2 - 4x + 3\right)$
7. $y = \log_a\left(2ax^2 - 4ax\right)$
8. $y = \log_a(\ln x)$
9. $y = 3x^2\ln(5x)$
10. $y = \ln\left(\cos^2 x\right)$
11. $y = \ln(4x + 3)^2$
12. $y = \log_5\left(2x^2 - 3\right)$
13. $y = \log_8\sqrt{x^2 - 2}$
14. $y = \log_b(\sin 5x)$
15. $y = \log(5x) + \ln(\tan x)$
16. $y = \ln^2(\sec x)$
17. $y = \log_3\dfrac{2}{x^2 - 1}$
18. $y = \dfrac{1 + \ln x}{1 - \ln x}$
19. $y = \ln\dfrac{2 + x^2}{2 - x^2}$
20. $y = \ln\dfrac{2\sin x}{\sec x}$

25.10 Derivatives of Logarithmic Functions

Derivative of exponential functions $y = e^x$

If $y = e^x$, then $\ln y = e^x = x$.

Differentiate both sides with respect to x, we get
$$\begin{aligned}\frac{1}{y}\cdot\frac{dy}{dx} &= 1\\ \frac{dy}{dx} &= y\\ &= e^x\end{aligned}$$

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

Derivative of other exponential functions

Let $y = a^x$ where $a > 0$.

Take the natural logarithm of both sides, we get
$$\begin{aligned} \ln y &= \ln a^x \\ &= x \ln a \end{aligned}$$

Differentiate both sides with respect to x, we get
$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= \ln a \\ \frac{dy}{dx} &= y \ln a \\ &= a^x \ln a \end{aligned}$$

Generally, when solving for the derivative of an exponential function, we can first convert it to a natural exponential function and then solve for the derivative.

Practice 12

Find the derivative of the following functions:

1. $y = e^{-3x}$
2. $y = -e^{-\frac{1}{3}x}$
3. $y = 2a^{5x}$
4. $y = a^{2x} - e^{-x} + x^a$
5. $y = \left(e^x + \frac{1}{e^x}\right)^2$
6. $y = \frac{e^{3x} - e^{-3x}}{e^x}$

Exercise 25.10

Find the derivative of the following functions:

1. $y = e^{-5x}$
2. $y = 3e^{-\frac{1}{3}x}$
3. $y = e^{x^2+3x-1}$
4. $y = e^{5x} \cdot e^{-6x} \cdot 4e^2$
5. $y = 5^{2x}$
6. $y = (3b)^{7x}$
7. $y = 4^{3-2x^2}$
8. $y = 4^{2x} - 4^{-2x}$
9. $y = 3e^{\tan x}$
10. $y = e^x \ln x$
11. $y = 3^{2x} \cos 5x$
12. $y = \dfrac{e^x + 1}{e^x - 1}$
13. $y = \dfrac{e^{2x} - e^{-2x}}{2e^x}$
14. $y = \dfrac{3e^{5x} - 2e^{-2x}}{6e^x}$

Revision Exercise 25

1. Find the gradient of the tangent to the curve $y = 2x^2 + 1$ at the point where $x = 2$.

2. Find the gradient of the curve $y = 3x^2 - 1$ at the point $A(-1, 2)$.

3. Find the gradient of the curve $y = 2x - x^3$ at the point $B(-1, -1)$.

4. Find the derivative of the follwoing functions using the definition of the derivative, and find the value of the derivative at the point where $x = 1$:

   (a) $f(x) = x^2 + 2x$

   (b) $g(x) = x^3$

   (c) $h(x) = \dfrac{5}{x}$

   (d) $k(x) = \sqrt{x+3}$

5. Find the derivative of the following functions:

   (a) $y = 2x^4 - 3x^3 + 5x - 8$

   (b) $y = 2x + \dfrac{2}{x} - \dfrac{3}{x^2}$

   (c) $y = \sqrt[3]{x} - \dfrac{1}{\sqrt{3x}}$

   (d) $y = \left(x^3 - 4x\right)\left(x^2 + 3x - 1\right)$

   (e) $y = (x-1)^5\sqrt{x+2}$

   (f) $y = (2x+5)\left(x^2 - 2\right)\left(x^3 - 1\right)$

   (g) $y = \dfrac{2x^3 - 3x^2 + 4}{x^2}$

   (h) $y = \dfrac{x^2 + 4}{x + 1}$

   (i) $y = \dfrac{x + 2}{x^2 + 5x + 6}$

   (j) $y = \dfrac{x^2}{\left(x^2 - 1\right)^3}$

6. Find the derivative of the following functions:

   (a) $y = \left(x^3 - 1\right)^4$

   (b) $y = (5x + 3)^6$

   (c) $y = \left(x^3 - 3x\right)^5$

   (d) $y = \sqrt{x^2 - 2x}$

   (e) $y = \dfrac{1}{\sqrt[3]{2x^2 - 1}}$

(f) $y = \dfrac{2x-1}{\sqrt{1-2x}}$

7. Find the second derivative of the following functions:

   (a) $y = x^2(3x-4)$

   (b) $y = 2x^5 - 6x^4 - 3x + 5$

   (c) $y = \dfrac{3}{x^5}$

   (d) $y = \sqrt{2x+1}$

8. If the function $y = \dfrac{x^3}{(x-1)^2}$, find y' and y''.

9. Given the function $y = 2x^3 + 3x^2 - 72x + 21$, find the value of x when $\dfrac{dy}{dx} = 0$.

10. Given the function $y = (2-3x^2)^4$, find the value of $\dfrac{d^2y}{dx^2}$ when $x = 1$.

11. If the function $f(x) = \dfrac{x^3}{3} - \dfrac{5x^2}{2} + x + 1$, find

    (a) $f'(1)$ and $f''(2)$

    (b) the value of x when $f''(x) = 0$

12. If the function $f(x) = \sqrt{x\sqrt{x\sqrt{x}}}$, find $f'(1)$, $f''(1)$, $f'''(1)$ and $f^{(4)}(1)$.

13. Find the derivative $\dfrac{dy}{dx}$ of the following implicit functions:

    (a) $x^2 + 2y = 2x + 3$

    (b) $x^2 + 3x = y^2 - 5y$

    (c) $3x^2 + 7xy - 9y^2 = 2$

    (d) $x^5y + xy^5 = 3xy$

14. Find the gradient of the tangent to the curve $x^2 + xy + y^2 = 4$ at the point $A(2, -2)$.

15. Find the limit of the following:

    (a) $\lim\limits_{x\to 0} \dfrac{\sin 4x}{x}$

    (b) $\lim\limits_{x\to 0} \dfrac{\tan 2x}{\tan 5x}$

    (c) $\lim\limits_{x\to 0} \dfrac{x^2}{\tan^2 3x}$

    (d) $\lim\limits_{x\to \infty} \left(1 + \dfrac{1}{x}\right)^{4x+1}$

    (e) $\lim\limits_{x\to 0} (1-3x)^{\frac{2}{x}}$

(f) $\lim_{x\to\infty}\left(\dfrac{x+3}{x+1}\right)^x$

16. Find the derivative of the following functions:

    (a) $y = \tan^2 3x$

    (b) $y = \cos^4 2x$

    (c) $y = \sec^3 2x$

    (d) $y = \sec^2(3x+5)$

    (e) $y = (1+\sin x)^3$

    (f) $y = \sin(\cos x)$

    (g) $y = \sin 2x \cos 2x$

    (h) $y = \dfrac{1}{\sin x + \cos x}$

    (i) $y = \dfrac{\cos 5x}{\sin 3x}$

    (j) $y = \dfrac{1+\cos x}{\sin x}$

17. Find the derivative of the following functions:

    (a) $y = 5^{3x-2}$

    (b) $y = 3e^{2x^2}$

    (c) $y = a^{3x} + e^{-3x}$

    (d) $y = \dfrac{e^{3x} - e^{2x} + e^{5x}}{e^{2x}}$

    (e) $y = x^a - 2a^x$

    (f) $y = e^{2x}\csc 2x$

18. If the function $y = \dfrac{\sin 2x}{e^x}$, prove that $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0$.

19. Find the derivative of the following functions:

    (a) $y = \ln\left(x^2+5\right)$

    (b) $y = \ln\left(3x^2+6x\right)$

    (c) $y = \ln(e^x+2)$

    (d) $y = \ln\left(\sin^2 4x\right)$

    (e) $y = \log\left(x^3+3x-4\right)$

    (f) $y = \log_5(3x+7)$

    (g) $y = \log_2 \dfrac{x}{x+3}$

    (h) $y = \dfrac{1+\log x}{1+\ln x}$

20. If the function $y = \ln(x+1)$, find the value of $\dfrac{d^2y}{dx^2}$ when $x = 1$.