Mathematics

Senior 3 Part I

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Introduction

Why this book?

Disclaimer

Acknowledgements

Contents

Revision Exercise 22

1. Determine whether the following mappings from set $A = \{1, 2, 3, 4\}$ to set $B = \{a, b, c, d\}$ are functions or not.

   (a) $1 \to a, 2 \to c, 4 \to b$

   Sol.

   Since $3 \in A$ does not have an image in B, this is not a function.

   (b) $1 \to a, 2 \to d, 3 \to b, 4 \to a$

   Sol.

   Since each element in A has an image in B, this is a function.

   (c) $1 \to c, 2 \to c, 3 \to b, 4 \to b$

   Sol.

   Since each element in A has an image in B, this is a function.

   (d) $1 \to a, 2 \to c, 2 \to b, 4 \to d$

   Sol.

   Since $2 \in A$ has two images b and c in B, $3 \in A$ has two images in B, this is not a function.

   (e) $1 \to c, 2 \to b, 3 \to d, 4 \to c, 4 \to a$

   Sol.

   Since $4 \in A$ has two images c and a in B, this is not a function.

2. Given the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 3x - 2, & x < -3 \\ 2x^2 + 4, & -3 \leq x < 2 \\ -2x + 9, & x \geq 2 \end{cases}$, find

   (a) $f(-4)$

   Sol.

   $$\begin{aligned} f(-4) &= 3(-4) - 2 \\ &= -14 \end{aligned}$$

   (b) $f(0)$

   Sol.

   $$\begin{aligned} f(0) &= 2(0)^2 + 4 \\ &= 4 \end{aligned}$$

   (c) $f(2)$

   Sol.

   $$\begin{aligned} f(2) &= -2(2) + 9 \\ &= 5 \end{aligned}$$

   (d) $f(3)$

   Sol.

   $$\begin{aligned} f(3) &= -2(3) + 9 \\ &= 3 \end{aligned}$$

3. Find the domain and range of the following functions:

   (a) $f : 1 \to 3, 2 \to 5, 4 \to 8$

   Sol.

   $D_f = \{1, 2, 4\}$, $R_f = \{3, 5, 8\}$

   (b) $g : 2 \to 4, 4 \to 5, 5 \to 7, 6 \to 9$

   Sol.

   $D_g = \{2, 4, 5, 6\}$, $R_g = \{4, 5, 7, 9\}$

   (c) $h : 1 \to 3, 2 \to 5, 3 \to 6, 4 \to 8$

   Sol.

   $D_h = \{1, 2, 3, 4\}$, $R_h = \{3, 5, 6, 8\}$

4. The table below shows a function f:

x	-3	-2	-1	0	1
f(x)	-22	-3	4	5	6

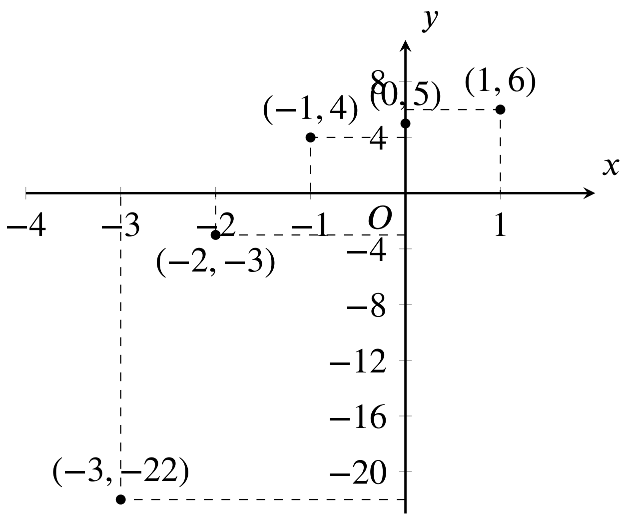
   (a) Find the domain and range of the function;

   Sol.

   $D_f = \{-3, -2, -1, 0, 1\}$, $R_f = \{-22, -3, 4, 5, 6\}$

   (b) Express the function using graph.

   Sol.



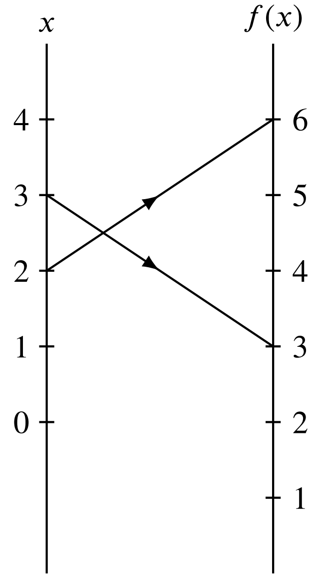
   (c) Determine if the inverse function of f exists.

   Sol.

   Since each element in the codomain of f is mapped to exactly one element in the domain of f, the function f is a one-to-one function. Since each element in the codomain of f has preimage in the domain of f, the function f is an onto function.

   Hence, the function f is a one to one onto function. According to the definition of inverse function, the inverse function of f exists.

5. As shown in the diagram below, let a function $f : x \to ax + b$. Find the value of $f(4)$ and $f^{-1}(5)$.



Sol.

$$
\begin{aligned}
f(3) &= 3a + b = 3 \\
f(2) &= 2a + b = 6 \\
f(3) - f(2) &= a = -3 \\
3(-3) + b &= 3 \\
b &= 12
\end{aligned}
$$

$$\therefore f(x) = -3x + 12$$

$$f(4) = -3(4) + 12 = 0$$

$$
\begin{aligned}
&\text{Let } y = f^{-1}(x) \\
f(y) &= x \\
-3y + 12 &= x \\
y &= -\frac{x-12}{3}
\end{aligned}
$$

$$
\begin{aligned}
\therefore f^{-1}(x) &= -\frac{x-12}{3} \\
f^{-1}(5) &= -\frac{5-12}{3} \\
&= \frac{7}{3}
\end{aligned}
$$

6. Given the function $f : x \to x^2 - x + 1$, $-1 \le x \le 3$, find its range.

Sol.

$$
\begin{aligned}
f(x) &= x^2 - x + 1 \\
&= \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}
\end{aligned}
$$

$$\text{Vertex} : \left(\frac{1}{2}, \frac{3}{4}\right)$$

$$\because a > 0, y_{\min} = \frac{3}{4}$$

$$f(-1) = (-1)^2 - (-1) + 1 = 3$$

$$f(3) = 3^2 - 3 + 1 = 7$$

$$\therefore R_f = \left\{ y \middle| y \in \mathbb{R}, \frac{3}{4} \le y \le 7 \right\}$$

7. Let function $f : x \to 2x^2 - 4x + 3$.

   (a) If $D_f = \mathbb{R}$, find the range of f;

   Sol.

$$
\begin{aligned}
f(x) &= 2x^2 - 4x + 3 \\
&= 2(x^2 - 2) + 3 \\
&= 2(x-1)^2 + 1
\end{aligned}
$$

$$\text{Vertex} : (1,1)$$

$$\because a > 0, y_{\min} = 1$$

$$\therefore R_f = \{y | y \in \mathbb{R}, y \geq 1\}$$

   (b) If $D_f = \{x | x \geq 3\}$, find the range of f.

   Sol.

$$f(3) = 2(3)^2 - 4(3) + 3 = 9$$

$$\therefore R_f = \{y | y \in \mathbb{R}, y \geq 9\}$$

8. Find the domain and range of the following functions:

   (a) $f(x) = \dfrac{1}{x}$

   Sol.

   $\because f(x)$ is defined when $x \neq 0$,

   $\therefore D_f = \mathbb{R} \setminus \{0\}$.

   $\because f(x) = \dfrac{1}{x} \neq 0$,

   $\therefore R_f = \mathbb{R} \setminus \{0\}$.

   (b) $f(x) = \sqrt{2x - 5}$

   Sol.

   $\because f(x)$ is defined when $2x - 5 \geq 0$,

   $\therefore D_f = \left\{x | x \geq \dfrac{5}{2}\right\}$.

   $\because f(x) = \sqrt{2x-5} \geq 0$,

   $\therefore R_f = \{y | y \geq 0\}$.

   (c) $f(x) = x^2 + 4x + 7$

   Sol.

   $\because f(x)$ is defined for all $x \in \mathbb{R}$,

   $\therefore D_f = \mathbb{R}$.

$$
\begin{aligned}
f(x) &= x^2 + 4x + 7 \\
&= (x+2)^2 + 3
\end{aligned}
$$

$$\text{Vertex} : (-2,3)$$

$$\because a > 0, y_{\min} = 3$$

$$\therefore R_f = \{y | y \in \mathbb{R}, y \geq 3\}$$

   (d) $f(x) = \dfrac{1}{x^2 + 4}$

   Sol. $\because x^2 + 4 \geq 4$ for all $x \in \mathbb{R}$,

   $\therefore D_f = \mathbb{R}$.

   $\because f(x) = x^2 + 4 \geq 4$ for all $x \in \mathbb{R}$,

   $\therefore f(x) \geq \dfrac{1}{4}$ for all $x \in \mathbb{R}$,

   $\therefore R_f = \left\{y | y \in \mathbb{R}, y \geq \dfrac{1}{4}\right\}$.

9. Find the domain of the following functions:

(a) $f(x) = \dfrac{2x}{x-3}$

Sol.

$\because f(x)$ is defined when $x - 3 \neq 0$,

$\therefore D_f = \mathbb{R} \setminus \{3\}$.

(b) $f(x) = \sqrt{4 - x^2}$

Sol.

$\because f(x)$ is defined when $4 - x^2 \geq 0$,

$\therefore D_f = \{x | x \in \mathbb{R}, -2 \leq x \leq 2\}$.

(c) $f(x) = \dfrac{x-2}{2x^2 - 5x + 2}$

Sol.

$\because f(x)$ is defined when $2x^2 - 5x + 2 \neq 0$,

$\therefore D_f = \left\{x | x \in \mathbb{R}, x \neq \dfrac{1}{2}, x \neq 2\right\}$.

(d) $f(x) = \dfrac{x-3}{\sqrt{x^2 - 9}}$

Sol.

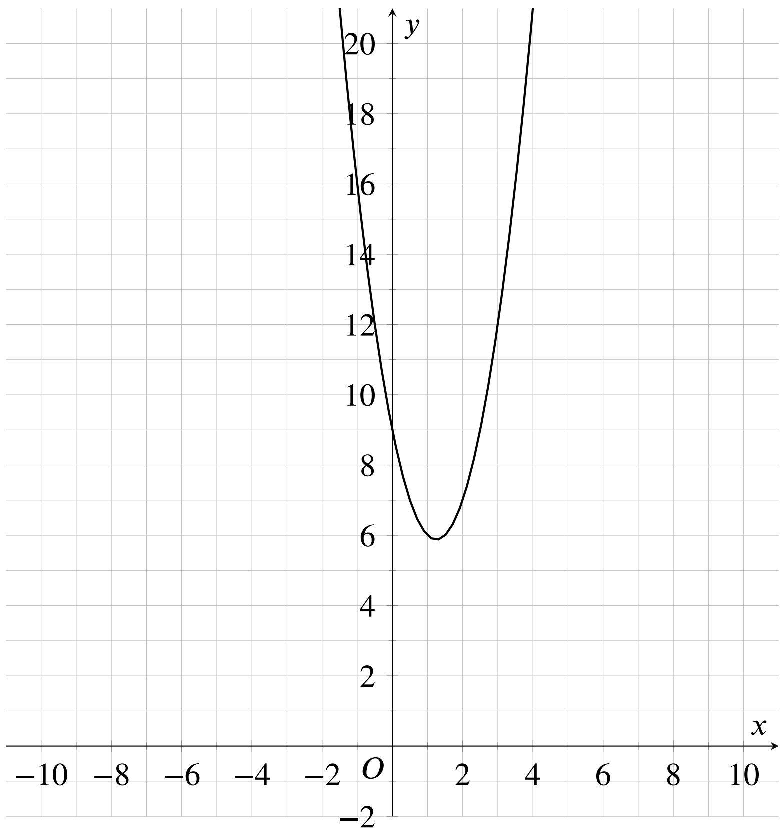
$\because f(x)$ is defined when $x^2 - 9 > 0$,

$\therefore D_f = \{x | x \in \mathbb{R}, x < -3 \text{ or } x > 3\}$.

10. Sketch the graph for the following functions:

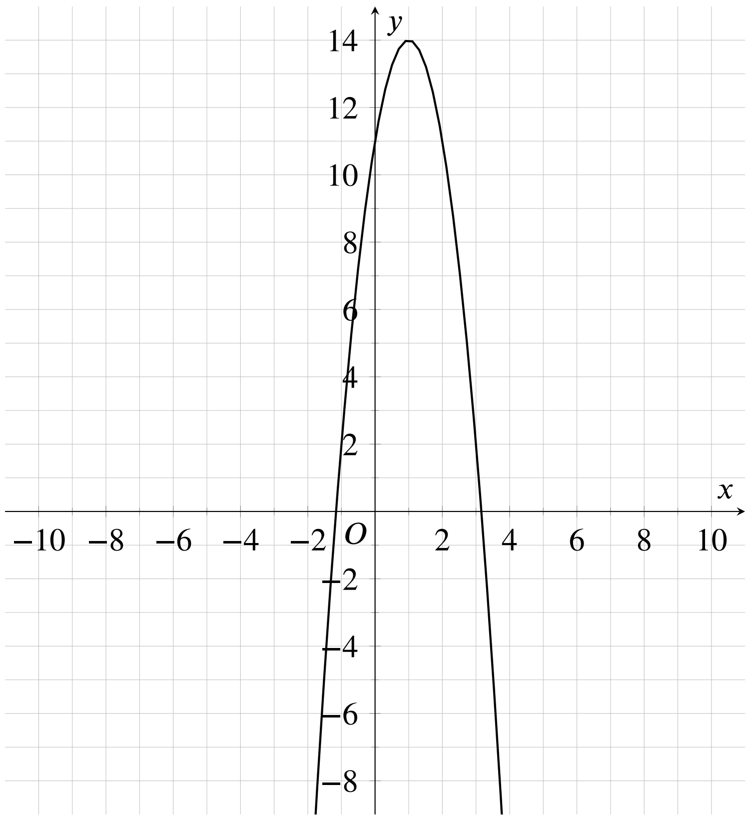
(a) $f(x) = 2x^2 - 5x + 9$

Sol.



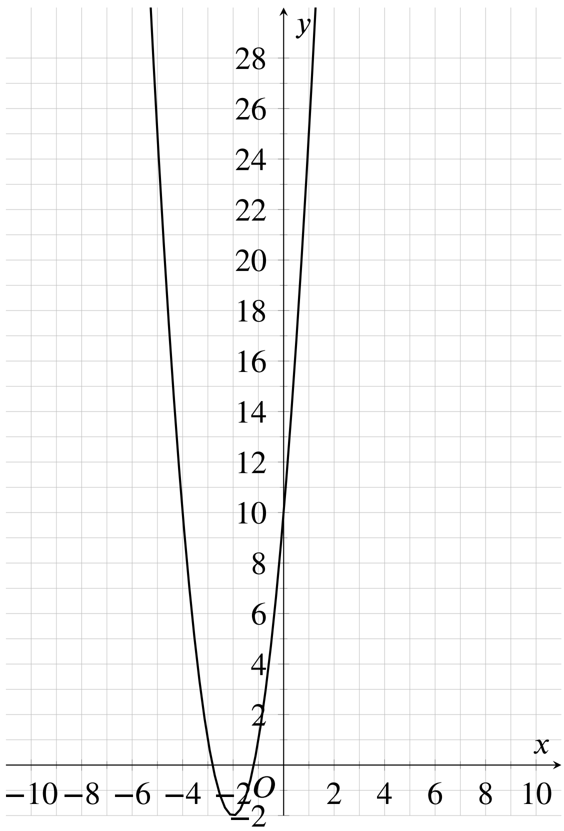
(b) $f(x) = -3x^2 + 6x + 11$

Sol.



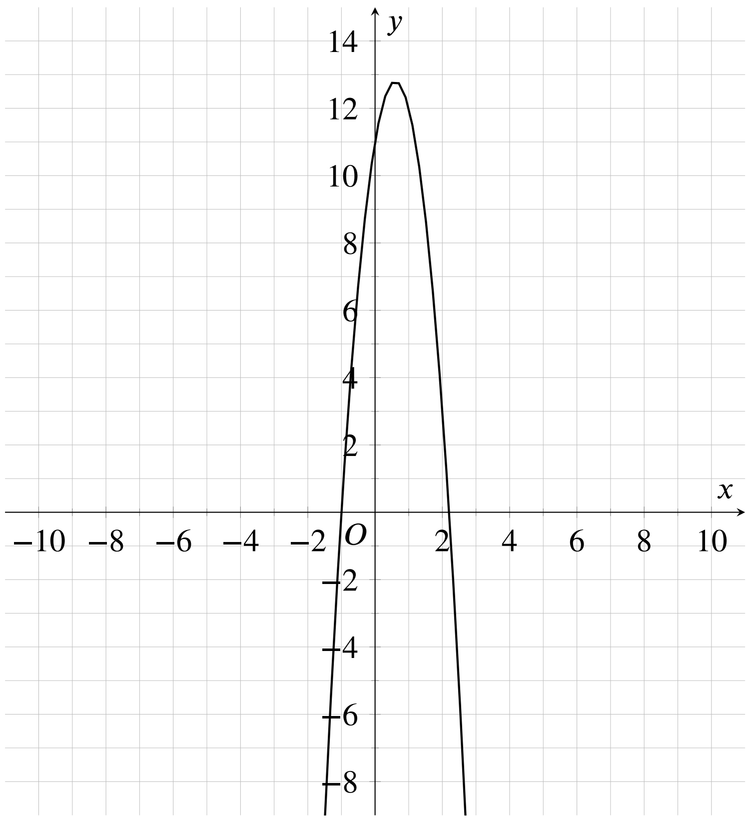
(c) $f(x) = 3x^2 + 12x + 10$

Sol.



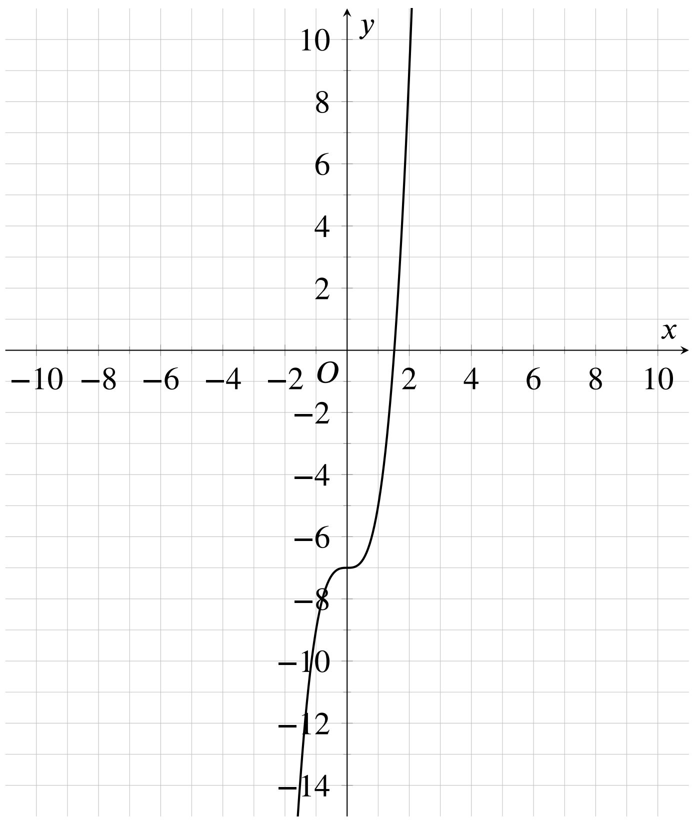
(d) $f(x) = -5x^2 + 6x + 11$

Sol.



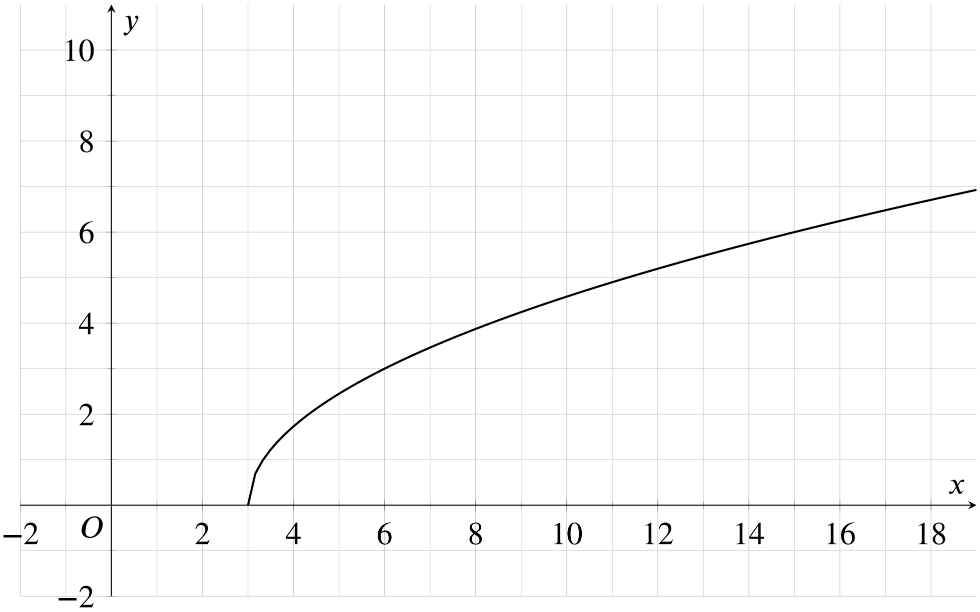
(e) $f(x) = 2x^3 - 7$

Sol.



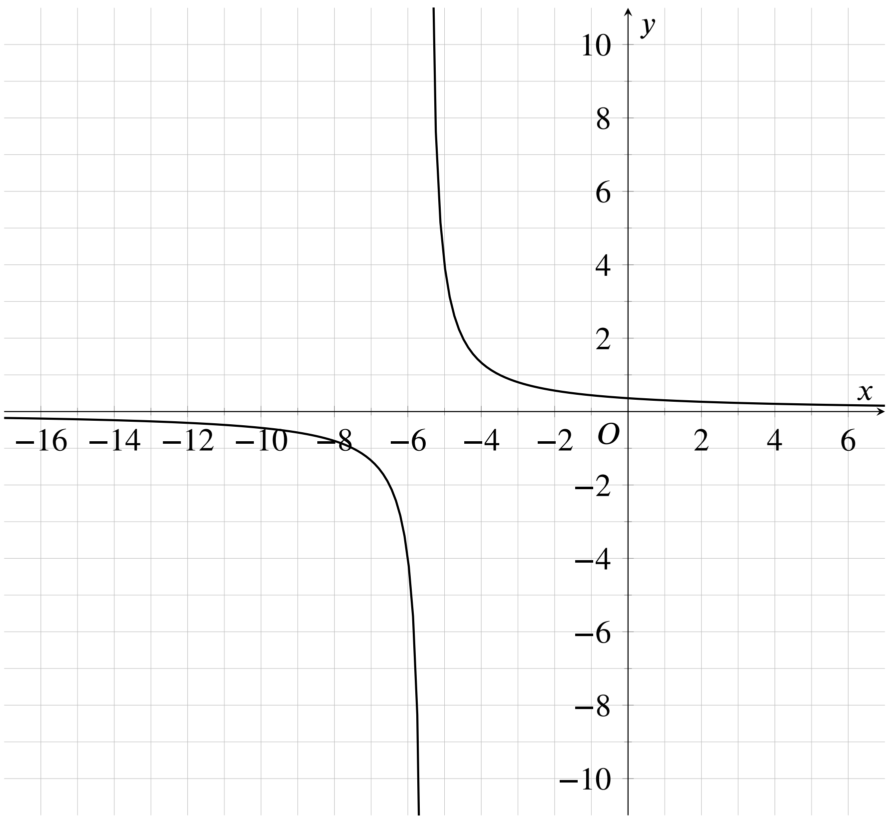
(f) $f(x) = \sqrt{3x - 9}$

Sol.



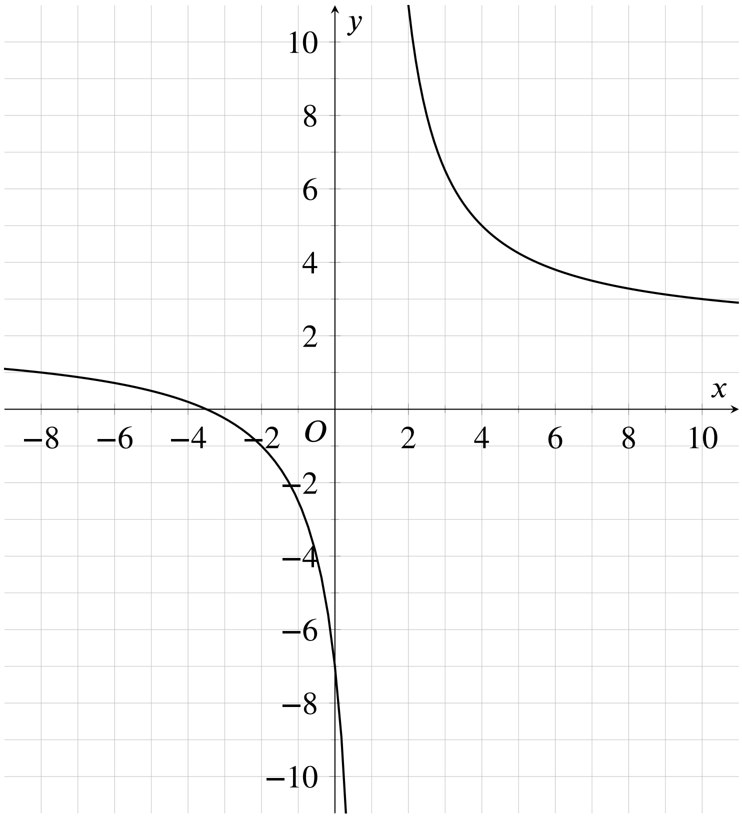
(g) $f(x) = \dfrac{4}{2x + 11}$

Sol.



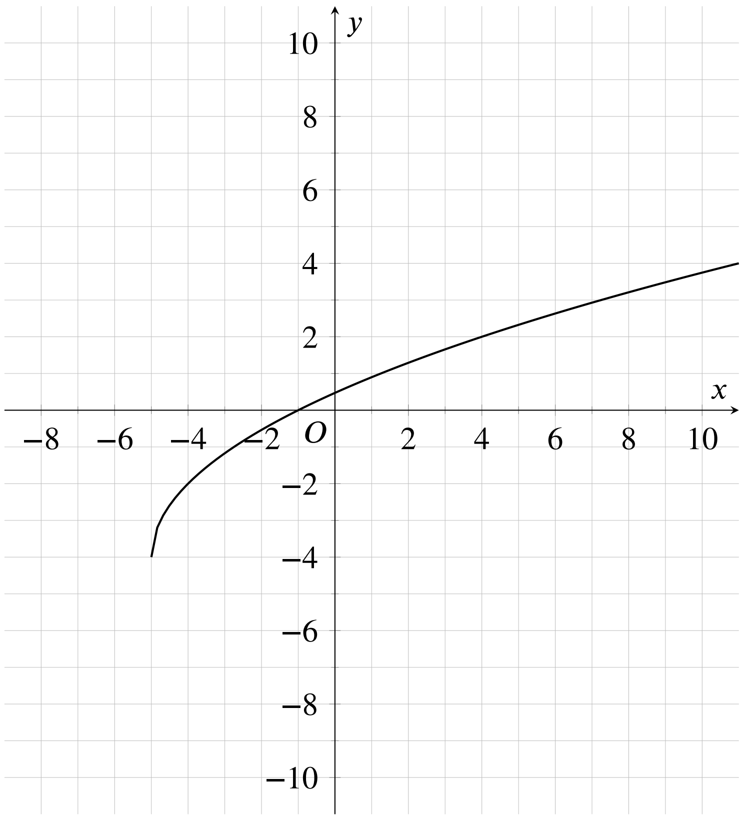
(h) $f(x) = \dfrac{2x+7}{x-1}$

Sol.



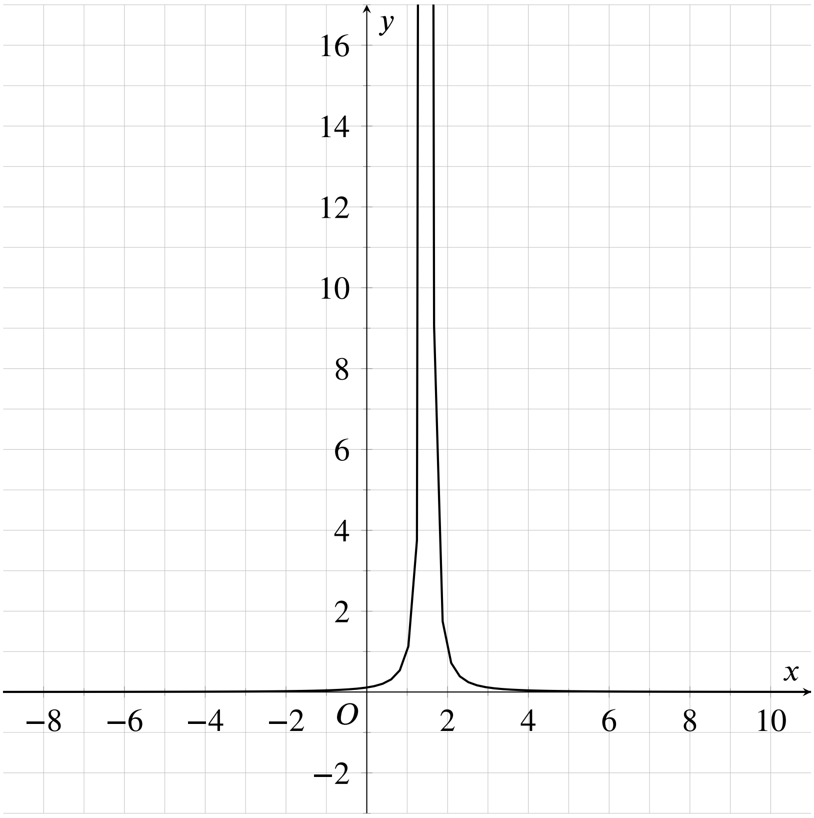
(i) $f(x) = 2\sqrt{x+5} - 4$

Sol.



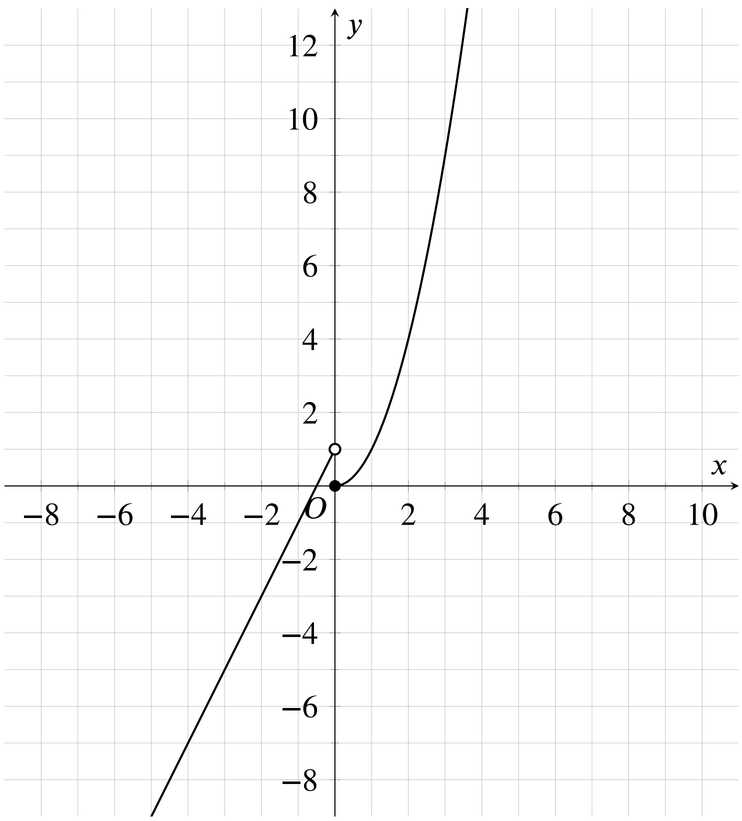
(j) $f(x) = \dfrac{1}{(2x-3)^2}$

Sol.



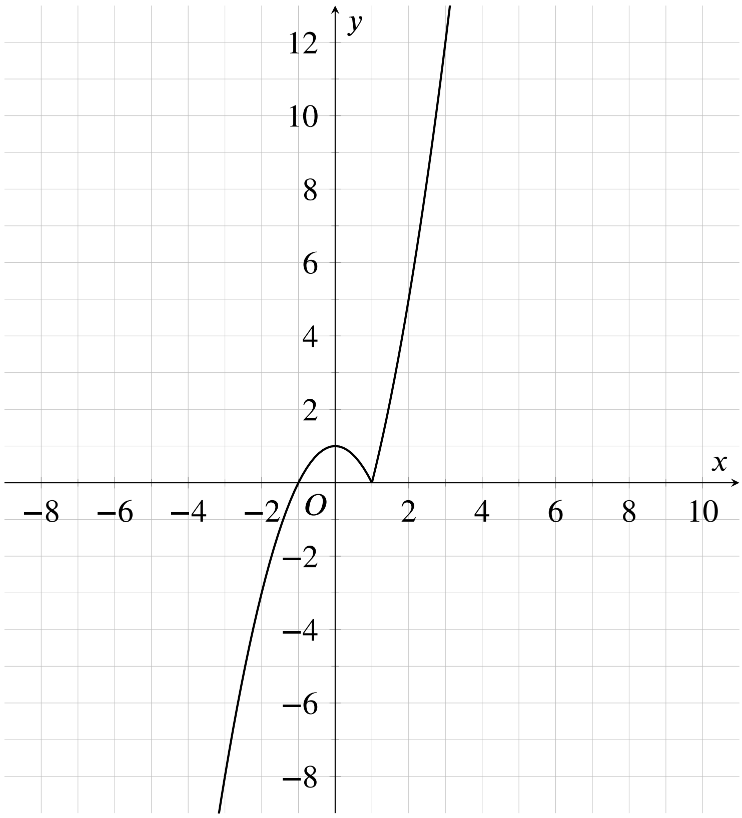
(k) $f(x) = \begin{cases} 2x+1, & x < 0 \\ x^2, & x \geq 0 \end{cases}$

Sol.



(l) $f(x) = \begin{cases} 1 - x^2, & x \le 1 \\ x^2 + 2x - 3, & x > 1 \end{cases}$

Sol.



11. Given the function $f : x \to 2x^2$ and $g : x \to 3x - 4$. Find the value of m such that $(f \circ g)(m) = (g \circ f)(m)$.

Sol.

$$\begin{aligned}
(f \circ g)(m) &= (g \circ f)(m) \\
f(g(m)) &= g(f(m)) \\
f(3m-4) &= g(2m^2) \\
2(3m-4)^2 &= 3(2m^2) - 4 \\
18m^2 - 48m + 32 &= 6m^2 - 4 \\
12m^2 - 48m + 36 &= 0 \\
3m^2 - 12m + 9 &= 0 \\
(3m-3)(m-3) &= 0 \\
m &= 3 \text{ or } m = 1
\end{aligned}$$

12. Given the function $f : x \to x^2 + 2x - 3$ and $g : x \to 3x - 4$. If $(f \circ g)(k) = (g \circ f)(k)$, find the value of k.

Sol.

$$\begin{aligned}
(f \circ g)(k) &= (g \circ f)(k) \\
f(g(k)) &= g(f(k)) \\
f(3k-4) &= g(k^2 + 2k - 3) \\
(3k-4)^2 + 2(3k-4) - 3 &= 3(k^2 + 2k - 3) - 4 \\
9k^2 - 24k + 16 + 6k - 8 - 3 &= 3k^2 + 6k - 9 - 4 \\
9k^2 - 18k + 5 &= 3k^2 + 6k - 13 \\
6k^2 - 24k + 18 &= 0 \\
k^2 - 4k + 3 &= 0 \\
(k-3)(k-1) &= 0 \\
k &= 3 \text{ or } k = 1
\end{aligned}$$

13. Given that $f(x) = 3x + 1$, $x \neq 0$. If $(f \circ g)(x) = 6x^2 - 9x + 4$, find $g(x)$.

Sol.

$$\begin{aligned}
(f \circ g)(x) &= 6x^2 - 9x + 4 \\
f(g(x)) &= 6x^2 - 9x + 4 \\
3g(x) + 1 &= 6x^2 - 9x + 4 \\
3g(x) &= 6x^2 - 9x + 3 \\
g(x) &= 2x^2 - 3x + 1
\end{aligned}$$

14. Given that $f(x) = \dfrac{x+1}{x}$, $x \neq 0$. If $(f \circ g)(x) = x$, find $g(x)$.

Sol.

$$\begin{aligned}
(f \circ g)(x) &= x \\
f(g(x)) &= x \\
\frac{g(x)+1}{g(x)} &= x \\
g(x) + 1 &= xg(x) \\
g(x) - xg(x) &= -1 \\
g(x)(1-x) &= -1 \\
g(x) &= \frac{-1}{1-x} \\
&= \frac{1}{x-1} \quad (x \neq 1)
\end{aligned}$$

15. A function f is defined by $f : x \to x - 3$. Find another function g such that $g \circ f : x \to 4x^2 - 20x + 25$.

Sol.

$$
\begin{aligned}
(g \circ f)(x) &= 4x^2 - 20x + 25 \\
g(f(x)) &= 4x^2 - 20x + 25 \\
g(x-3) &= 4x^2 - 20x + 25
\end{aligned}
$$

$$
\begin{aligned}
\text{Let } y &= x - 3 \\
x &= y + 3
\end{aligned}
$$

$$
\begin{aligned}
g(y) &= 4(y+3)^2 - 20(y+3) + 25 \\
&= 4(y^2 + 6y + 9) - 20y - 60 + 25 \\
&= 4y^2 + 24y + 36 - 20y - 35 \\
&= 4y^2 + 4y + 1 \\
&= (2y+1)^2
\end{aligned}
$$

$$\therefore g(x) = (2x+1)^2$$

16. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} -2, & x \le -3 \\ |x| - 2x, & -3 < x < 3 \\ 2x - 1, & x \ge 3 \end{cases}$. Find $(f \circ f \circ f)(-1000)$.

Sol.

$$
\begin{aligned}
(f \circ f \circ f)(-1000) &= f(f(f(-1000))) \\
&= f(f(-2)) \\
&= f(|-2| - 2(-2)) \\
&= f(2+4) \\
&= f(6) \\
&= 2(6) - 1 \\
&= 11
\end{aligned}
$$

17. Let function $f : A \to \mathbb{R}$ be defined by $f : x \to 2x^2$. Determine if f is one to one function when A is the following sets.

    (a) $A = \{x|0 \leq x < 6\}$

    (b) $A = \{x|x < 0\}$

    (c) $A = \{x| -2 \leq x < 2\}$

    (d) $A = \{x|x > 3\}$

18. Determine whether the following functions are one to one functions or onto functions.

    (a) $f : \mathbb{R}^+ \to \mathbb{R}, f : x \to |x| - 2$

    (b) $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{1\}, f : x \to \dfrac{x}{x-2}$

    (c) $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}, f : x \to |x|$

19. Let $A = \mathbb{R} \setminus \left\{-\frac{1}{2}\right\}$ and $A = \mathbb{R} \setminus \left\{\frac{1}{2}\right\}$, function $f : A \to B$ is defined by $f(x) = \dfrac{x-3}{2x+1}$. Find

    (a) $f^{-1}(-2)$

    (b) $f^{-1}(0)$

    (c) $f^{-1}(3)$

20. Let function $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = x^2 + 2x + 1$. Find $f^{-1}(4)$ and $f^{-1}(9)$.

21. A function f is defined by $f : x \to \dfrac{x}{2} + 1$. If $g \circ f^{-1} : x \to 4x^2 - 8x + 7$, find the function g.

22. Given the functioon $f : x \to 3x^2 + 5x + 9$, $x \leq a$. Find the maximum value of a such that the inverse function of f exists.

23. Let the function f and g be defined as $f : x \to 5x + 3$ and $g : x \to 2x - 7$ respectively. Find

    (a) $f \circ g$

    (b) f^{-1}

    (c) g^{-1}

24. Given the function $f : x \to 2x + 3$ and $g : x \to 3 - x2x + 5$, $x \neq -\dfrac{5}{2}$. Find

    (a) $f \circ g$

    (b) f^{-1}

    (c) g^{-1}

    Show that $g^{-1} \circ f^{-1} = (f \circ g)^{-1}$.

25. Given the function $f : x \to \sqrt{x}$, $x \neq 0$ and $g : x \to x^3$. Find

    (a) $g \circ f$

(b) f^{-1}

(c) g^{-1}

(d) $(g \circ f)^{-1}$

(e) $g^{-1} \circ f^{-1}$

26. Given the function $f : x \to 2\sqrt{x-4} + 3$, $x \geq 4$.

(a) Find the range of f.

(b) Find the inverse function f^{-1} of f.

(c) On the same diagram, sketch the graphs of f and f^{-1}.