Mathematics

Senior 3 Part I

MELVIN CHIA

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Introduction

Why this book?

Disclaimer

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Chapter 22

Limits

22.1 Concept of Limits

Limit is a fundamental concept of calculus. It is the theoretical basis for studies on the changes and trends of functions. We will first introduce an example related to the idea of limits.

Cyclotomic Method by Liu Wei

The circle is not a shape with straight edges, so where does its area formula $A = \pi r^2$ come from?

Let the side length of a regular n -gon inscribed in a circle be a_n, length from the center to the side be r_n, as shown in the diagram below, the area of the regular n-gon is $A_n = n \cdot \frac{1}{2} a_n \cdot r_n$, while its circumference is $P_n = na_n$, then $A_n = \frac{1}{2} P_n \cdot r_n$.

When the value of n becomes larger and larger, the area A_n of the n-gon is indefinitely close to the area A of the circle, denoted as $A = \lim_{n\to\infty} A_n$, the limit of A_n is said to be A when n approaches infinity.

When $n \to \infty$, the circumference P_n and the length between center and side r_n of the inscribed regular n-gon, approaches the circumference P and the radius r of the circle respectively. That is to say, $\lim_{n\to\infty} P_n = P$, $\lim_{n\to\infty} r_n = r$.

$$\begin{aligned} \therefore A &= \lim_{n\to\infty} A_n \\ &= \lim_{n\to\infty} \frac{1}{2} P_n \cdot r_n \\ &= \frac{1}{2} Pr \end{aligned}$$

from $P = 2\pi r$. we get $A = \pi r^2$.

The concept above uses the idea of "replacing curves with straight lines", which treats the area of a circle as the limit of the area of a regular n-gon when n approaches infinity. This way of calculating the area of a circle using the limit is invented by Liu Hui, a mathematician back in the 3rd century, and is called the "Cyclotomic Method". Quoted from his own words, "The smaller the circle is divided, the lesser the error is; divide and divide, until the circle is unable to be divided, then the error will be negligible and it will be the same as the circle."

22.2 Limits of Functions

If the value of a variable x approaches a certain constant a, we say that x tends to a.

When x approaches x_0 (but not equal to x_0), if the value of $f(x)$ approaches a certain constant A, we say that as x approaches x_0, the limit of $f(x)$ is A, denoted as $\lim_{x \to x_0} f(x) = A$.

In the definition of the limit $\lim_{x \to x_0} f(x) = A$, when approaches x_0 from the left ($x < x_0$) and right ($x > x_0$), the limit of $f(x)$ approaches the same constant A.

When x approaches x_0 from the left ($x < x_0$), denoted as $x \to x_0^-$, if the value of $f(x)$ approaches a certain constant A, then A is the left limit of $f(x)$ when x approaches x_0, denoted as $\lim_{x \to x_0^-} f(x) = A$.

Similarly, when x approaches x_0 from the right ($x > x_0$), denoted as $x \to x_0^+$, if the value of $f(x)$ approaches a certain constant A, then A is the right limit of $f(x)$ when x approaches x_0, denoted as $\lim_{x \to x_0^+} f(x) = A$.

If x approaches x_0 from the left (or right), but the value of $f(x)$ does not approach a certain constant, then the left (or right) limit does not exist.

From the definition of the limit, left limit and right limit, we can conclude the following theorem:

When $x \to x_0$, if the left and right limit of the function $f(x)$ exist and are equal, then the limit of $f(x)$ exists, and $\lim_{x \to x_0} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$.

Contrarily, if the left limit or right limit of the function $f(x)$ does not exist, or the left and right limit are not equal, then the limit of $f(x)$ does not exist.

Note that the limit $\lim_{x \to x_0} f(x)$ and the function value $f(x_0)$ are two different concepts. When $\lim_{x \to x_0} f(x)$ exists, it does not mean that $f(x_0)$ exists. Even if $f(x_0)$ exists, it does not guarantee to be equal to $\lim_{x \to x_0} f(x)$.

22.2.1 Practice 1

Complete the following table, state the changes of the value of the function $f(x) = 2x + 1$ when $x \to 1$, and find $\lim_{x \to 1} f(x)$.

x	0.9	0.99	0.999	0.9999	0.99999
$f(x) = 2x + 1$					

x	1.1	1.01	1.001	1.0001	1.00001
$f(x) = 2x + 1$					

If x approaches positive infinity, the value of function $f(x)$ approaches a certain constant A, then A is the limit of $f(x)$ when $x \to \infty$, denoted as $\lim_{x\to\infty} f(x) = A$.

Similarly, if x approaches negative infinity, the value of function $f(x)$ approaches a certain constant B, then B is the limit of $f(x)$ when $x \to -\infty$, denoted as $\lim_{x\to-\infty} f(x) = B$.

If x approaches positive infinity (or negative infinity), but the value of $f(x)$ does not approach a certain constant, then the limit of $f(x)$ does not exist when $x \to \infty$ (or $x \to -\infty$).

22.2.2 Practice 2

Complete the following table, state the changes of the value of the function $f(x) = \frac{1}{x}$ when $x \to \infty$, and find $\lim_{x\to\infty} f(x)$.

x	-1	-10	-100	-1000	-10000	$\cdots$
$f(x) = \frac{1}{x}$						

22.2.3 Exercise 24.2

1. Complete the following table. Hence, find the left limit, right limit, and limit of the function $f(x) = 3x - 1$ and $x = 1$.

x	0.9	0.99	0.999	0.9999	0.99999
$f(x) = 3x - 1$					

x	1.1	1.01	1.001	1.0001	1.00001
$f(x) = 3x - 1$					

2. Complete the following tables, state the changes of the value of the function $f(x)$ when $x \to x_0$, and find $\lim_{x\to x_0} f(x)$.

(a) $f(x) = x^3 + 1$, $x_0 = 0$

x	0.1	0.01	0.001	0.0001	0.00001
$f(x) = x^3 + 1$					

x	-0.1	-0.01	-0.001	-0.0001	-0.00001
$f(x) = x^3 + 1$					

(b) $f(x) = \frac{x^2 - 4}{x + 2}$, $x_0 = -2$

x	-2.1	-2.01	-2.001	-2.0001	-2.00001
$f(x) = \frac{x^2 - 4}{x + 2}$					

x	-1.9	-1.99	-1.999	-1.9999	-1.99999
$f(x)$					

(c) $f(x) = \begin{cases} 4-x, & x<1 \\ x^2+2, & x \geq 1 \end{cases}$, $x_0 = 1$

x	0.9	0.99	0.999	0.9999	0.99999
$f(x)$					

x	1.1	1.01	1.001	1.0001	1.00001
$f(x)$					

3. Complete the following table, state the changes of the function $f(x) = \dfrac{2x+1}{x+2}$ when $x \to x_0$, and find $\lim\limits_{x\to\infty} f(x)$.

x	1	10	100	1000	10000	100000
$f(x)$						

4. Complete the following table, state the changes of the function $f(x) = 2^x$ when $x \to -\infty$, and find $\lim\limits_{x\to-\infty} f(x)$.

x	-1	-2	-3	-4	-5	-10
$f(x)$						

22.3 Arithmetic Rules of Limits of Functions

If $\lim\limits_{x\to x_0} f(x)$ and $\lim\limits_{x\to x_0} g(x)$ exist, then

1. $\lim\limits_{x\to x_0}[f(x) \pm g(x)] = \lim\limits_{x\to x_0} f(x) \pm \lim\limits_{x\to x_0} g(x)$
2. $\lim\limits_{x\to x_0}[f(x) \cdot g(x)] = \lim\limits_{x\to x_0} f(x) \cdot \lim\limits_{x\to x_0} g(x)$
3. $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}$, provided that $\lim\limits_{x\to x_0} g(x) \neq 0$
4. $\lim\limits_{x\to x_0} k \cdot f(x) = k \cdot \lim\limits_{x\to x_0} f(x)$, where k is a constant
5. $\lim\limits_{x\to x_0}[f(x)]^n = \left[\lim\limits_{x\to x_0} f(x)\right]^n$, where n is a positive integer
6. $\lim\limits_{x\to x_0} k = k$, where k is a constant

The rules above also applied to $x \to \infty$ and $x \to -\infty$. Obviously, $\lim\limits_{x\to x_0} x = x$, where k is a constant.

Using the arithmetic rules above, we can calculate the limit of relatively complicated functions using given limit of simpler functions. For example, from the rules above, we can get

$$\begin{aligned} &\lim_{x\to x_0}\left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right) \\ &= a_n x_0^n + a_{n-1}x_0^{n-1} + \cdots + a_1 x_0 + a_0 \end{aligned}$$

22.3.1 Practice 3

Find the limit of the followings:

1. $\lim\limits_{x \to -1} \dfrac{\sqrt{x+2}-3}{x+4}$
2. $\lim\limits_{x \to 1} \dfrac{x^2+2x-3}{x^2-3x+2}$
3. $\lim\limits_{x \to \infty} \dfrac{x^2+1}{2x^2+x+1}$

22.3.2 Practice 4

State whether the left limit, right limit, and limit of the function $f(x) = \dfrac{1}{x}$ exist when $x \to 0$.

22.3.3 Exercise 24.3

Find the limit of the followings (Question 1 to 31):

1. $\lim\limits_{x \to -2} \left(x^2+x-3\right)$
2. $\lim\limits_{x \to 4} x^2(x-1)$
3. $\lim\limits_{x \to -2} x\left(9-x^2\right)$
4. $\lim\limits_{x \to -1} (x+3)(x-1)$
5. $\lim\limits_{x \to \frac{1}{2}} (2x-1)\left(x^2+3x+4\right)$
6. $\lim\limits_{x \to -2} \left(4x^3+2x^2+3x+1\right)$
7. $\lim\limits_{x \to 2} \dfrac{x^2+2}{x-5}$
8. $\lim\limits_{x \to -1} \dfrac{(x+2)(x-3)}{x-1}$
9. $\lim\limits_{x \to 0} \dfrac{2x^2+3x-4}{x-4}$
10. $\lim\limits_{x \to -3} \dfrac{(x+5)(x+3)}{x+3}$
11. $\lim\limits_{x \to 2} \dfrac{x^2-5x+6}{x-2}$
12. $\lim\limits_{x \to -2} \dfrac{x^3+8}{x+2}$
13. $\lim\limits_{x \to 1} \dfrac{x^2-2x+1}{x^2-1}$
14. $\lim\limits_{x \to 0} \dfrac{2x^5+3x}{x^2+2x}$
15. $\lim\limits_{x \to 0} \dfrac{4x^3+x^2}{x^2-3x}$
16. $\lim\limits_{x \to 1} \dfrac{x^3+3x^2-2x-2}{2x^2+x-3}$
17. $\lim\limits_{x \to 2} \sqrt{2x^2+1}$
18. $\lim\limits_{x \to 2} \dfrac{\sqrt{x+2}-3}{x^2-9}$
19. $\lim\limits_{x \to 4} \dfrac{\sqrt{x}-2}{x-4}$
20. $\lim\limits_{x \to 0} \dfrac{\sqrt{3x+4}-2}{x}$
21. $\lim\limits_{x \to 1} \dfrac{2-\sqrt{x+3}}{x^2-1}$
22. $\lim\limits_{x \to 2} \dfrac{\sqrt{3x-2}-\sqrt{x+2}}{x-2}$
23. $\lim\limits_{x \to 1} \left(\dfrac{x+3}{x^2-1}-\dfrac{x+1}{x^2-x}\right)$
24. $\lim\limits_{x \to \infty} \dfrac{2x^3-1}{x^3-x+2}$

25. $\lim_{x\to\infty} \frac{3x^3-2x+1}{2x^3+x^2-5}$

26. $\lim_{x\to\infty} \frac{2x+7}{x^3+2x^2-4}$

27. $\lim_{x\to\infty} \frac{x^4+2x^3-x^2+1}{x^5+x^3-2}$

28. $\lim_{x\to\infty} \left(\frac{x^2+3x}{2_x+1} - \frac{x}{2}\right)$

29. $\lim_{x\to\infty} 2^x$

30. $\lim_{x\to\infty} \frac{x^2}{x+1}$

31. $\lim_{x\to\infty} \frac{x^3-3x+1}{2x^2+x+1}$

32. $f(x) = \begin{cases} -1, & x < 0 \\ 2x, & x \geq 0 \end{cases}$, find $\lim_{x\to 0} f(x)$

33. $f(x) = \begin{cases} -x+1, & x < 1 \\ 3, & x = 1 \\ 2x-2, & x > 1 \end{cases}$, find $\lim_{x\to 1} f(x)$

Determine if the limit of the following functions exists at $x = x_0$. If it exists, find their limit.

34. $f(x) = \begin{cases} x+1, & x < 0 \\ 0, & x = 0 \\ x-1, & x > 0 \end{cases}$, $x_0 = 0$

35. $f(x) = \begin{cases} -x+1, & x \leq 2 \\ x-3, & x > 2 \end{cases}$, $x_0 = 0, x_0 = 2$

36. $f(x) = \frac{1}{x+3}$, $x_0 = -3$

22.4 Revision Exercise 24

1. Complete the following table, state the changes of the function $f(x) = x^3 - 1$ at $x = 1$. Hence, find the left limit, right limit and limit of the function at $x = 1$.

x	0.9	0.99	0.999	0.9999	1	1.0001	1.001	1.01	1.1
$f(x)$									

Find the limit of the followings (Question 2 to 13):

2. $\lim_{x\to 1} \left(x^2+x-2\right)$

3. $\lim_{x\to 2} \left(x^2-1\right)\sqrt{x+2}$

4. $\lim_{x\to -2} \frac{2x^2+3x-2}{3x^2+4x-4}$

5. $\lim_{x\to 2} \frac{3-\sqrt{x+7}}{x^2-1}$

6. $\lim_{x\to 0} \frac{2-\sqrt{3x+4}}{x^2+x}$

7. $\lim_{x\to -1} \frac{\sqrt{2x+5}-\sqrt{3}}{x+1}$

8. $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x^2-x-6}$

9. $\lim_{x\to\infty} \frac{4x+3}{x^2+2x-1}$

10. $\lim_{x\to\infty} \frac{2x^4-x^3+3x+1}{3x^4+4x^2-x-2}$

11. $\lim_{x\to\infty} \frac{x^3-2x^2+3}{2x^2-1}$

12. $\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{2}{x^2-1}\right)$

13. $\lim_{x\to\infty} \left(\frac{x^3}{2x^2+1} - \frac{x^2}{2x+3}\right)$

Determine if the limit of the following functions exists at $x = x_0$. If it exists, find their limit. (Question 14 to 17)

14. $f(x) = \begin{cases} 1-3x, & x < 0 \\ 2x+1, & x \geq 0 \end{cases}$, $x_0 = 0$

15. $f(x) = \begin{cases} \sqrt{x+3}, & x < -2 \\ x+1, & x > -2 \end{cases}$, $x_0 = -2$

16. $f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ \dfrac{1}{2}, & x = 1 \end{cases}$, $x_0 = 1$

17. $f(x) = \begin{cases} 2x+1, & x \leq 1 \\ x^2+1, & 1 < x \leq 2 \\ 3x-1, & x > 2 \end{cases}$, $x_0 = 1$, $x_0 = 2$