Exercise 5e

1. Find the equation of the rectangular hyperbola that satisfies the following conditions:

   (a) Vertex coordinates are $(-2, 0)$ and $(2, 0)$.

   Sol.

   Since the vertices are on the x-axis, the equation of the hyperbola is of the form

   $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

   Since $a = b$, the equation becomes

   $$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{a^2} &= 1 \\ \frac{x^2 - y^2}{a^2} &= 1 \\ x^2 - y^2 &= a^2 \end{aligned}$$

   Substituting the coordinates of the vertices, we get

   $$\begin{aligned} (-2)^2 - 0^2 &= a^2 \\ 4 &= a^2 \end{aligned}$$

   Therefore, the equation of the hyperbola is

   $$x^2 - y^2 = 4 \qquad \square$$

   (b) Focus coordinates are $(0, 0)$ and $(0, 4)$.

   Sol.

   The center of the hyperbola is at the midpoint of the foci, which is $(0, 2)$.

   Translate the foci to the origin and form a new coordinates system such that

   $$x' = x - 0 \qquad \text{and} \qquad y' = y - 2$$

   The foci are now at $(0, -2)$ and $(0, 2)$.

   Let the rectangular hyperbola be of the form

   $$x'^2 - y'^2 = a^2$$

   $$\begin{aligned} ae &= 2 \\ a^2 &= a^2e^2 - a^2 \\ &= 4 - a^2 \\ 2a^2 &= 4 \\ a^2 &= 2 \end{aligned}$$

   Therefore, the equation of the hyperbola is

   $$(y - 2)^2 - x^2 = 2 \implies y^2 - 4y + 4 - x^2 = 2 \implies x^2 - y^2 + 4y - 2 = 0 \qquad \square$$

2. Given that the equations of the asymptotes of a rectangular hyperbola are $x - y - 1 = 0$ and $x + y - 3 = 0$, and it passes through the point $(4, 1 + \sqrt{3})$, find its equation.

Sol.

$$\begin{aligned}
x - y - 1 = 0 &\text{ or } x + y - 3 = 0 \\
x = y + 1 &\text{ or } x = -y + 3 \\
x - 2 = y - 1 &\text{ or } x - 2 = -y + 1 \\
x - 2 &= \pm(y - 1) \\
(x-2)^2 &= (y-1)^2 \\
(x-2)^2 - (y-1)^2 &= k
\end{aligned}$$

Substituting the point $(4, 1 + \sqrt{3})$ into the equation, we get

$$\begin{aligned}
(4-2)^2 - (1 + \sqrt{3} - 1)^2 &= k \\
2^2 - (\sqrt{3})^2 &= k \\
4 - 3 &= k \\
k &= 1
\end{aligned}$$

Therefore, the equation of the hyperbola is

$$\begin{aligned}
(x-2)^2 - (y-1)^2 &= 1 \\
x^2 - 4x + 4 - y^2 + 2y - 1 &= 1 \\
x^2 - y^2 - 4x + 2y + 2 &= 0 \qquad \square
\end{aligned}$$

3. Find a point P(a, b) on the right branch of the rectangular hyperbola $x^2 - y^2 = 1$ such that the distance from point P to the line $x - y = 0$ is $\sqrt{2}$.

Sol.

Substituting $P(a, b)$ into the equation of the line, we get

$$\begin{aligned}
a^2 - b^2 = 1 &\implies a^2 = 1 + b^2 \\
a > 0 \implies a = \sqrt{1 + b^2} > \sqrt{b^2} &= |b| \implies a > b \\
\frac{|a - b|}{\sqrt{2}} &= \sqrt{2} \\
a - b &= 2 \\
a &= b + 2 \\
(b+2)^2 - b^2 &= 1 \\
b^2 + 4b + 4 - b^2 &= 1 \\
4b &= -3 \\
b &= -\frac{3}{4} \\
a &= -\frac{3}{4} + 2 = \frac{5}{4}
\end{aligned}$$

Therefore, the point P is $\left(\frac{5}{4}, -\frac{3}{4}\right)$. $\square$

4. Prove that the product of the distances from any point on a rectangular hyperbola to its two foci is equal to the square of the distance from that point to the center of the hyperbola.

Proof.

Let the rectangular hyperbola be of the form

$$x^2 - y^2 = a^2$$

The center of the hyperbola is at the origin, and the foci are at $(\pm ae, 0)$.

$$\begin{aligned} a^2 &= a^2e^2 - a^2 \\ 2a^2 &= e^2a^2 \\ e^2 &= 2 \\ e &= \sqrt{2} \end{aligned}$$

Therefore, the foci are at $(\pm\sqrt{2}a, 0)$.

The distance from P to the foci is

$$\begin{aligned} d_1d_2 &= \sqrt{(x-\sqrt{2}a)^2+y^2}\sqrt{(x+\sqrt{2}a)^2+y^2} \\ &= \sqrt{x^2-2\sqrt{2}ax+2a^2+y^2}\sqrt{x^2+2\sqrt{2}ax+2a^2+y^2} \\ &= \sqrt{(x^2+y^2+2a^2)^2-(2\sqrt{2}ax)^2} \\ &= \sqrt{x^4+y^4+4a^4+2x^2y^2+4a^2x^2+4a^2y^2-8a^2x^2} \\ &= \sqrt{(x^2+y^2)^2+4a^4+4a^2y^2-4a^2x^2} \\ &= \sqrt{(x^2+y^2)^2+4a^4-4a^2(x^2-y^2)} \\ &= \sqrt{(x^2+y^2)^2+4a^4-4a^2a^2} \\ &= \sqrt{(x^2+y^2)^2} \\ &= x^2+y^2 \end{aligned}$$

The distance from P to the center of the hyperbola is

$$d_3^2 = (\sqrt{x^2+y^2})^2 = x^2+y^2$$

$\because d_1d_2 = d_3^2$,

$\therefore$ the product of the distances from any point on a rectangular hyperbola to its two foci is equal to the square of the distance from that point to the center of the hyperbola. $\square$