Praktis 3
Integration

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Praktis Formatif

3.1 Integration as the Inverse of Differentiation

1. (a) Given $\frac{d}{dx}(2x^3 + 5x^2 - 7x) = 6x^2 + 10x - 7$, find $\int 6x^2 + 10x - 7\,dx$.

Sol.

$$\int 6x^2 + 10x - 7\,dx = 2x^3 + 5x^2 - 7x \quad \square$$

(b) Given $\frac{d}{dx}(5x^4 + 3x^2 + x) = 20x^3 + 6x + 1$, find $\int 20x^3 + 6x + 1\,dx$.

Sol.

$$\int 20x^3 + 6x + 1\,dx = 5x^4 + 3x^2 + x \quad \square$$

2. (a) Given $\frac{d}{dx}(4x - 5x^2 + 2x^3) = 4 - 10x + 6x^2$, find $\int 2 - 5x + 3x^2\,dx$.

Sol.

$$\begin{aligned}\int 2 - 5x + 3x^2\,dx &= \frac{2}{2}\int 2 - 5x + 3x^2\,dx \\ &= \frac{1}{2}\int 4 - 10x + 6x^2\,dx \\ &= \frac{1}{2}(4x - 5x^2 + 2x^3) \\ &= 2x - \frac{5}{2}x^2 + x^3 \quad \square\end{aligned}$$

(b) Given $\frac{d}{dx}\left(2x - \frac{3}{x^4}\right) = 2 + \frac{12}{x^5}$, find $\int 6 + \frac{36}{x^5}\,dx$

Sol.

$$\begin{aligned}\int 6 + \frac{36}{x^5}\,dx &= 6\int 1 + \frac{6}{x^5}\,dx \\ &= 3\int 2 + \frac{12}{x^5}\,dx \\ &3\left(2x - \frac{3}{x^4}\right) \\ &= 6x - \frac{9}{x^4} \quad \square\end{aligned}$$

(c) Given $f(x) = \frac{d}{dx}[g(x)]$, find $\int 2f(x)\,dx$.

Sol.

$$\begin{aligned}\int 2f(x)\,dx &= 2\int f(x)\,dx \\ &= 2g(x) \quad \square\end{aligned}$$

(d) Differentiate $\frac{2x^2}{3x-1}$ with respect to x and hence, find $\int \frac{6x(3x-2)}{(3x-1)^2}\,dx$.

Sol.

$$\begin{aligned}\frac{d}{dx}\left(\frac{2x^2}{3x-1}\right) &= \frac{4x(3x-1) - 3(2x^2)}{(3x-1)^2} \\ &= \frac{12x^2 - 4x - 6x^2}{(3x-1)^2} \\ &= \frac{6x^2 - 4x}{(3x-1)^2} \\ &= \frac{2x(3x-2)}{(3x-1)^2}\end{aligned}$$

$$\begin{aligned}\int \frac{6x(3x-2)}{(3x-1)^2}\,dx &= 3\int \frac{2x(3x-2)}{(3x-1)^2}\,dx \\ &= 3\left(\frac{2x^2}{3x-1}\right) \\ &= \frac{6x^2}{3x-1} \quad \square\end{aligned}$$

3. The daily production of bread of a bakery shop is given b y the function $R(x) = -50(x^2 - 12x)$, where x represents the number of bakers who work in the shop with condition x is not more than 6.

(a) Find the rate of daily production of bread in terms of x.

Sol.

$$R'(x) = -100x + 600 \quad \square$$

(b) If the rate of daily production of bread becomes $300 - 50x$ on a particular day, calculate the revenue of the bakery shop if all the loaves of bread baked by three bakers on that day are sold out at a price of RM5.50 for each loaf.

Sol.

$$\begin{aligned}\int 300 - 50x\,dx &= \frac{1}{2}\int (600 - 100x)\,dx \\ &= \frac{1}{2}(-50x^2 + 600x) \\ &= -25x^2 + 300x \\ R(3) &= -25(3)^2 + 300(3) \\ &= -225 + 900 \\ &= 675\end{aligned}$$

$$\begin{aligned}\text{Revenue} &= 675 \times 5.50 \\ &= \text{RM}3712.50 \quad \square\end{aligned}$$

4. Given $f(x) = x^4 - 2x^3$ and $f'(x) = 4x^3 - 6x^2$. Express $f'(x)\int f'(x)\,dx$ in factored form.

Sol.

$$\begin{aligned}f'(x)\int f'(x)\,dx &= (4x^3 - 6x^2)(x^4 - 2x^3) \\ &= 2x^5(2x-3)(x-2) \quad \square\end{aligned}$$

5. Given $y = \dfrac{2x-6}{x}$.

   (a) Find $\dfrac{dy}{dx}$.

   Sol.

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x-2x-6}{x^2}\\ &= -\frac{6}{x^2} \quad \square\end{aligned}$$

   (b) Solve $4 + \displaystyle\int \left(\frac{dy}{dx}\right)\,dx = 0$.

   Sol.

$$\begin{aligned}4 + \int \left(\frac{dy}{dx}\right)\,dx &= 0\\ 4 + \int \left(-\frac{6}{x^2}\right)\,dx &= 0\\ 4 + \frac{2x-6}{x} &= 0\\ 4x + 2x - 6 &= 0\\ 6x &= 6\\ x &= 1 \quad \square\end{aligned}$$

6. Given $f'(x) = g(x)$. Find $\dfrac{3f(x)}{\displaystyle\int g(x)dx}$.

   Sol.

$$\begin{aligned}f'(x) &= g(x)\\ f(x) &= \int g(x)dx\\ \frac{3f(x)}{\displaystyle\int g(x)dx} &= \frac{3f(x)}{f(x)}\\ &= 3 \quad \square\end{aligned}$$

7. The population of town A is given by a function $P(t) = \frac{5}{6}(2.72^{1.2t}) - t^2 + 1495$ and the population continues to increase at the rate of $2.72^{1.2t} - 2t$ people per year where t is the number of years. Given that the population of town b increases at twice the rate of the population of town A based on the same model, find, to the nearest integer,

   (a) the rate of increase of the population of town B at $t = 5$ years.

   Sol.

$$\begin{aligned}P'_B(5) &= 2[2.72^{1.2(5)} - 2(5)]\\ &= 2[404.96 - 10]\\ &= 2(394.96)\\ &= 789.92\\ &= 790 \text{ people per year} \quad \square\end{aligned}$$

   (b) the population of town B after 5 years.

   Sol.

$$\begin{aligned}P_B(5) &= 2\left[\frac{5}{6}(2.72^{1.2\cdot 5}) - (5)^2 + 1495\right]\\ &= \frac{5}{3}(2.72^6) - 50 + 2990\\ &= 3614.93\\ &= 3615 \text{ people} \quad \square\end{aligned}$$

3.2 Indefinite Integral

8. By using the indefinite integral formula, find the integral of each of the following constants or algebraic functions.

   (a) $\displaystyle\int 3\,dx$

   Sol.

$$\int 3\,dx = 3x + C \quad \square$$

   (b) $\displaystyle\int 24x\,dx$

   Sol.

$$\int 24x\,dx = 12x^2 + C \quad \square$$

   (c) $\displaystyle\int 6x^2\,dx$

   Sol.

$$\int 6x^2\,dx = 2x^3 + C \quad \square$$

   (d) $\displaystyle\int 3x^2 + 4x\,dx$

   Sol.

$$\int 3x^2 + 4x\,dx = x^3 + 2x^2 + C \quad \square$$

   (e) $\displaystyle\int \frac{2}{x^4}\,dx$

   Sol.

$$\int \frac{2}{x^4}\,dx = -\frac{2}{x^3} + C \quad \square$$

   (f) $\displaystyle\int x^2(x-3)\,dx$

   Sol.

$$\begin{aligned}\int x^2(x-3)\,dx &= \int x^3 - 3x^2\,dx\\ &= \frac{1}{4}x^4 - x^3 + C \quad \square\end{aligned}$$

   (g) $\displaystyle\int (x+2)(2x^4-1)\,dx$

   Sol.

$$\begin{aligned}&\int (x+2)(2x^4-1)\,dx\\ &= \int 2x^5 - x + 4x^4 - 2\\ &= \frac{1}{3}x^6 + \frac{4}{5}x^5 - \frac{1}{2}x^2 - 2x + C \quad \square\end{aligned}$$

(h) $\displaystyle\int \frac{x^2+3x+2}{x+2}\,dx$

Sol.

$$\begin{aligned}\int \frac{x^2+3x+2}{x+2}\,dx &= \int \frac{(x+2)(x+1)}{x+2}\,dx\\ &= \int x+1\,dx\\ &= \frac{1}{2}x^2+x+C \quad \square\end{aligned}$$

9. Find the indefinite integral for each of the following by using

(a) the substitution method.

(b) the indefinite integral formula.

i. $\displaystyle\int \frac{2}{(x+2)^5}\,dx$

Sol.

(a) Let $v=(x+2)$.

$$\begin{aligned}\int \frac{2}{(x+2)^5}\,dx &= \int \frac{2}{v^5}\,dv\\ &= \int 2v^{-5}\,dv\\ &= -\frac{1}{2}v^{-4}+C\\ &= -\frac{1}{2v^4}+C\\ &= -\frac{1}{2(x+2)^4}+C \quad \square\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{2}{(x+2)^5}\,dx &= \int 2(x+2)^{-5}\,dx\\ &= 2\int (x+2)^{-5}\,dx\\ &= 2\left[\frac{(x+2)^{-4}}{-4}\right]+C\\ &= -\frac{1}{2(x+2)^4}+C \quad \square\end{aligned}$$

ii. $\displaystyle\int \frac{3}{5}(3x+2)^8\,dx$

Sol.

(a) Let $v=3x+2$, $\dfrac{dv}{dx}=3$.

$$\begin{aligned}\int \frac{3}{5}(3x+2)^8\,dx &= \int \frac{3}{5}v^8\,dv\\ &= \int \frac{3}{5}v^8\frac{dv}{3}\\ &= \int \frac{1}{5}v^8\,dv\\ &= \frac{1}{45}v^9+C\\ &= \frac{(3x+2)^9}{45}+C \quad \square\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{3}{5}(3x+2)^8\,dx &= \frac{3}{5}\int (3x+2)^8\,dx\\ &= \frac{3}{5}\left[\frac{(3x+2)^9}{27}\right]+C\\ &= \frac{(3x+2)^9}{45}+C \quad \square\end{aligned}$$

10. Determine the equation of a curve based on the following information.

(a) The gradient function of the curve is $\dfrac{dy}{dx}=3x^2+x-2$ and it passes through the point $p(2,15)$.

Sol.

$$\begin{aligned}\frac{dy}{dx} &= 3x^2+x-2\\ y &= \int 3x^2+x-2\,dx\\ &= x^3+\frac{x^2}{2}-2x+C\end{aligned}$$

When $x=2$, $y=15$,

$$\begin{aligned}15 &= 2^3+\frac{2^2}{2}-2(2)+C\\ 15 &= 8+2-4+C\\ 15 &= 6+C\\ C &= 9\end{aligned}$$

Hence, the equation of the curve is $y=x^3+\dfrac{x^2}{2}-2x+9$. $\square$

(b) The gradient function of the curve is $f'(x)=2x+9$ and $f(3)=21$.

Sol.

$$\begin{aligned}f'(x) &= 2x+9\\ f(x) &= \int 2x+9\,dx\\ &= x^2+9x+C\\ f(3) &= 3^2+9(3)+C\\ 21 &= 9+27+C\\ C &= -15\end{aligned}$$

Hence, the equation of the curve is $f(x)=x^2+9x-15$. $\square$

(c) The gradient function of the curve is given by $g(t)=\dfrac{5t^2-6t+1}{t^3(t-1)}$ and it passes through the point $(1,3)$.

Sol.

$$\begin{aligned}
g(t) &= \frac{5t^2-6t+1}{t^3(t-1)} \\
&= \frac{(5t-1)(t-1)}{t^3(t-1)} \\
&= \frac{5t-1}{t^3} \\
&= \frac{5}{t^2} - \frac{1}{t^3} \\
&= 5t^{-2} - t^{-3} \\
f(t) &= \int 5t^{-2} - t^{-3}\, dt \\
&= -\frac{5}{t} + \frac{1}{2t^2} + C
\end{aligned}$$

When $t = 1$, $f(1) = 3$,

$$\begin{aligned}
3 &= -5 + \frac{1}{2} + C \\
3 &= -\frac{9}{2} + C \\
C &= \frac{15}{2}
\end{aligned}$$

Hence, the equation of the curve is $f(t) = -\frac{5}{t} + \frac{1}{2t^2} + \frac{15}{2}$. ▯

11. Tommy moves in his roller skates at the rate of change in displacement, $\frac{ds}{dt} = t^2 + 9$ metres per second, where t is the time in seconds. At $t = 3$ seconds, Tommy is 4 metres away from his starting place. Find the displacement, s metres, when $t = 10$ seconds.

Sol.

$$\begin{aligned}
\frac{ds}{dt} &= t^2 + 9 \\
s &= \int t^2 + 9\, dt \\
&= \frac{t^3}{3} + 9t + C
\end{aligned}$$

When $t = 3$, $s = 4$,

$$\begin{aligned}
4 &= \frac{3^3}{3} + 9(3) + C \\
4 &= 9 + 27 + C \\
4 &= 54 + C \\
C &= -32 \\
s &= \frac{t^3}{3} + 9t - 32
\end{aligned}$$

When $t = 10$,

$$\begin{aligned}
s &= \frac{10^3}{3} + 9(10) - 32 \\
&= 333 + 90 - 32 \\
&= 391\frac{1}{3}m
\end{aligned}$$ ▯

12. Given the gradient function of a curve is $\frac{dy}{dx} = kx^2 + 2x$ where k is a constant. The curve passes through point $A(1, 6)$ and point $B(-2, 0)$. Determine the equation of the curve.

3.3 Definite Integral

13. Calculate each of the following.

   (a) $\int_2^1 \left(\sqrt{x} + \frac{1}{x}\right)$

   (b) $\int_0^3 \left(\frac{x^4 + 3x}{x}\right) dx$

   (c) $\int_{-2}^{-1} \left(\frac{(4-x)(3-x)}{x^5}\right) dx$

14. Given $\int_a^b f(x)\, dx = 5$, $\int_b^c f(x)\, dx = 8$ and $\int_b^a g(x)\, dx = 2$. Find each of the following.

   [answer can be in terms of a and/or b.]

   (a) $\int_a^b 3f(x)\, dx$

   (b) $\int_a^c f(x)\, dx$

   (c) $\int_a^b [f(x) + g(x)]\, dx$

   (d) $\int_c^a f(x)\, dx$

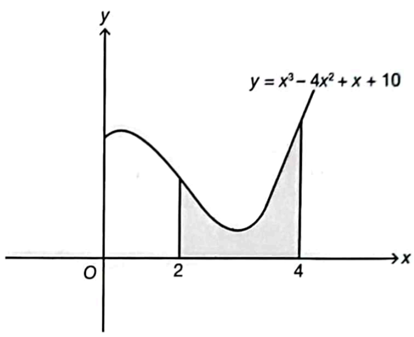
   (e) $\int_a^b [g(x) + 3]\, dx$

   (f) $\int_a^a f(x)\, dx$

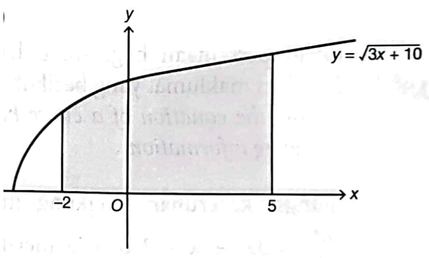
   (g) The value of k such that $\int_b^a [f(x) + kx]\, dx = 25$ if $a = 1$ and $b = 4$.

15. Find the area of the shaded region for each of the following diagrams.

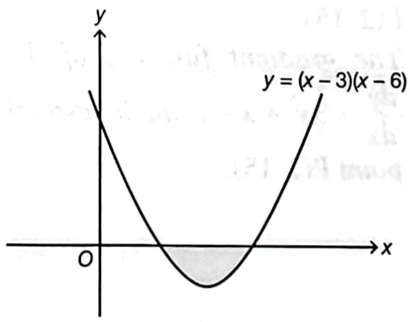
   (a)



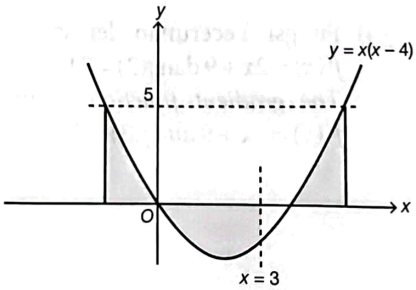
(b)



(c)

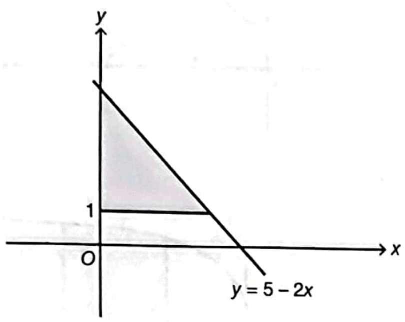


(d)

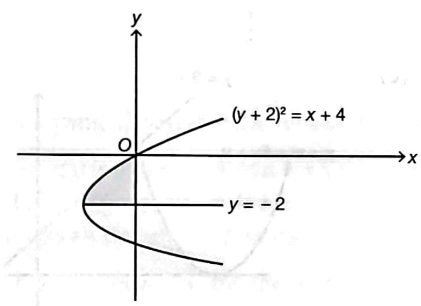


16. Determine the area bounded by the curve, the horizontal line(s) and the y-axis.

(a)



(b)



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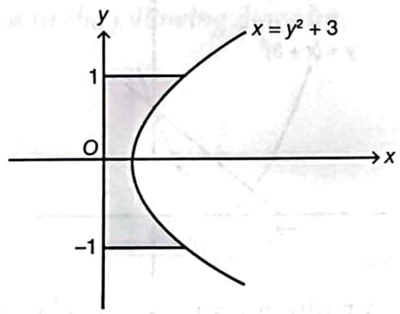

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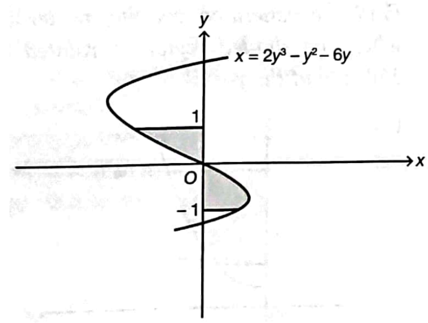
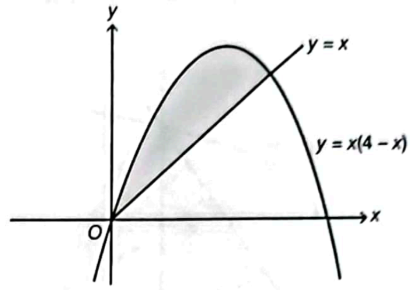
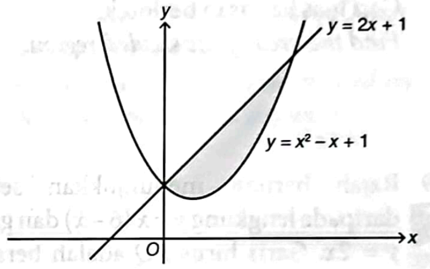


17. Find the area of the shaded region for each of the following.

(a)



(b)



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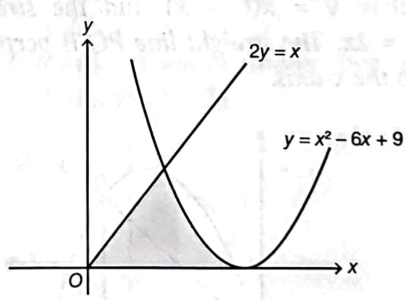
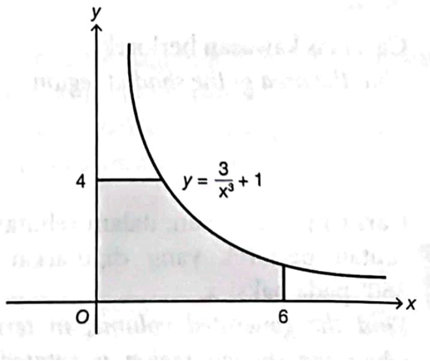
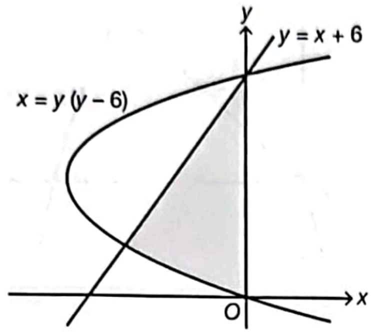


(d)

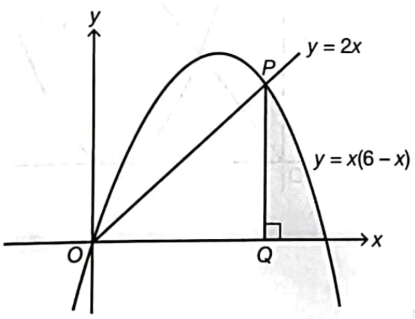


18. The following diagram shows a part of the curve $x = y(y-6)$ and the straight line $y = x + 6$.



Find the area of the shaded region.

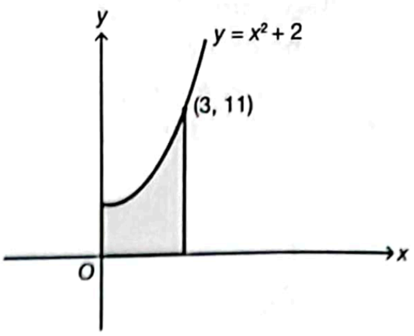
19. The following diagram shows a part of the curve $y = x(6-x)$ and a straight line $y = 2x$. The straight line PQ is perpendicular to the x-axis.



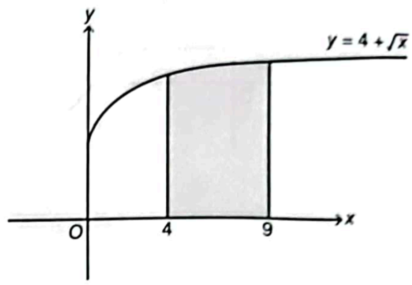
Find the area of the shaded region.

20. Find the generated volume, in terms of π, when the shaded region is rotated through $360°$ about the x-axis.

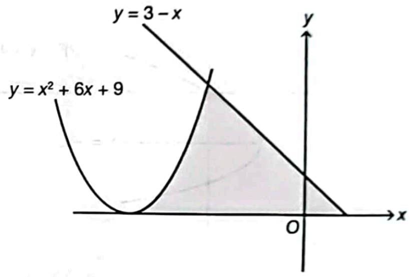
(a)



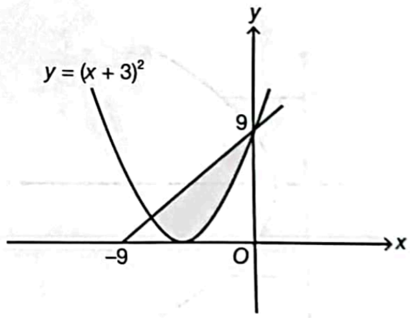
(b)



(c)

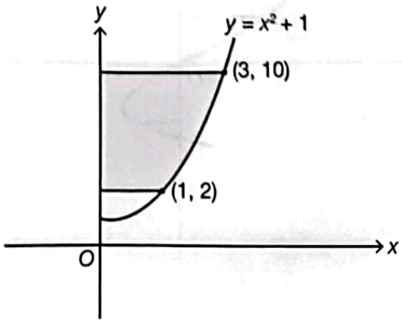


(d)

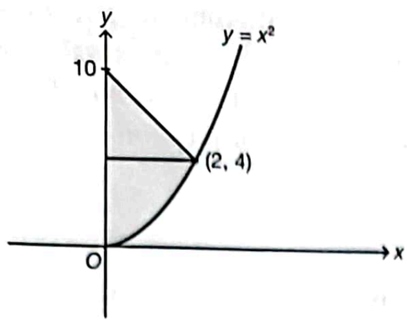


21. Find the generated volume, in terms of π, when the shaded region is rotated through $360°$ about the y-axis.

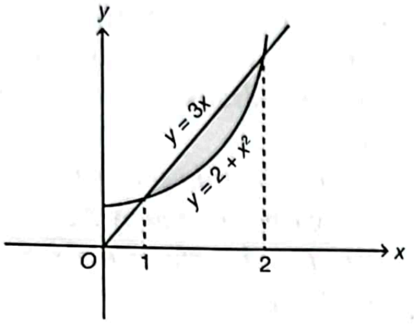
(a)



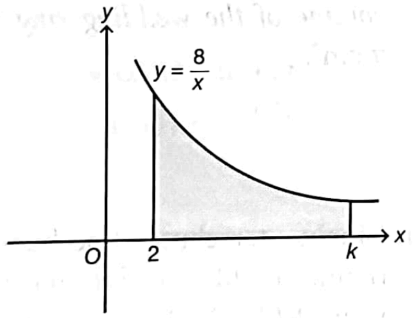
(b)



(c)

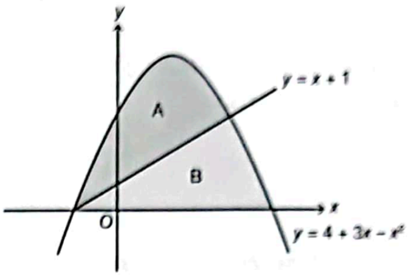


22. The region bounded by the curve $y = \dfrac{8}{x}$, the x-axis, and the straight line $x = 2$ and $x = k$ is rotated through $360°$ about the x-axis as shown in the following diagram.



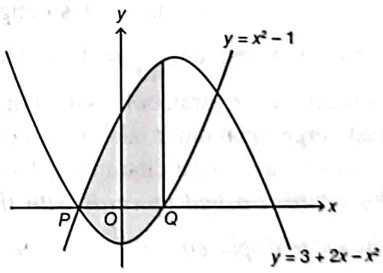
Express the volume generated by the region in terms of k. If the value of k becomes extremely large, deduce the nearest value of volume.

23. The following diagram shows a part of the curve $y = 4 + 3x - x^2$ and the straight line $y = x + 1$.



Find the ration of the area of the shaded region A to the area of the shaded region B.

24. The following diagram shows two curves $y = x^2 - 1$ and $y = 3 + 2x - x^2$.



Find the coordinate of the points P and Q. Hence, calculate the area of the shaded region.