Solution Book of Mathematic

Ssnior 2 Part I

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Contents

Chapter 12

Sequence and Series

12.1 Sequence and Series

12.1.1 Practice 1

1. Find the first 5 terms of the sequence $a_n = \frac{2^n}{n+1}$.

   sol. $a_1 = \frac{2}{2} = 1, a_2 = \frac{4}{3}, a_3 = \frac{8}{4}, a_4 = \frac{16}{5}, a_5 = \frac{32}{6}$

2. Write the general term of the sequence $1, 8, 27, 64, \cdots$

   sol. $a_n = n^3$

12.1.2 Practice 2

1. Express the series $\sum_{n=1}^{10} n^2 + 1$ in the form of numbers.

   sol. $$\begin{aligned}&\sum_{n=1}^{10} n^2 + 1\\ &= (1^2+1)+(2^2+1)+(3^2+1)+(4^2+1)\\ &+(5^2+1)+(6^2+1)+(7^2+1)\\ &+(8^2+1)+(9^2+1)+(10^2+1)\\ &= 2+5+10+17+26+37+50+65\\ &+82+101\end{aligned}$$

2. Write the first term, last term and the number of terms of the series $\sum_{n=1}^{10} (3^n - 2^n)$.

   sol. $$\begin{aligned}First\ term &= (3^1 - 2^1) = 1\\ Last\ term &= (3^{10} - 2^{10}) = 59049\\ Number\ of\ terms &= 10\end{aligned}$$

3. Express the series $2\times5+3\times7+4\times9+\cdots+15\times31$ in the form of $\sum$.

   sol.
   $$\begin{aligned}a_1 &= 2\times5 = 10\\ a_2 &= 3\times7 = 21\\ a_3 &= 4\times9 = 36\\ a_4 &= 5\times11 = 55\\ &\vdots\\ a_{15} &= 15\times31 = 465\\ \therefore & 2\times5+3\times7+4\times9+\cdots+15\times31\\ &= \sum_{n=1}^{15} a_n\end{aligned}$$

12.1.3 Exercise 12.1

1. Find the general term of the following sequences.

   (a) $5, 8, 11, 14, \ldots$
   sol. $a_n = 3n + 2$

   (b) $2, 4, 8, 16, \ldots$
   sol. $a_n = 2^n$

   (c) $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \cdots$
   sol. $a_n = \frac{n+1}{n}$

   (d) $\frac{2}{5}, \frac{4}{7}, \frac{6}{9}, \frac{8}{11}, \cdots$
   sol. $a_n = \frac{2n}{2n+1}$

2. Find the first 5 terms of the following sequences.

   (a) $a_n = 2n + 3$
   sol. $a_1 = 2\times1+3 = 5, a_2 = 2\times2+3 = 7, a_3 = 2\times3+3 = 9, a_4 = 2\times4+3 = 11, a_5 = 2\times5+3 = 13$

   (b) $a_n = n(n-2)$
   sol. $a_1 = 1\times(-1) = -1, a_2 = 2\times0 = 0, a_3 = 3\times1 = 3, a_4 = 4\times2 = 8, a_5 = 5\times3 = 15$

   (c) $a_n = \frac{n}{2n+1}$
   sol. $a_1 = \frac{1}{2\times1+1} = \frac{1}{3}, a_2 = \frac{2}{2\times2+1} = \frac{2}{5}, a_3 = \frac{3}{2\times3+1} = \frac{3}{7}, a_4 = \frac{4}{2\times4+1} = \frac{4}{9}, a_5 = \frac{5}{2\times5+1} = \frac{5}{11}$

   (d) $a_n = (-3)^n$
   sol. $a_1 = (-3)^1 = -3, a_2 = (-3)^2 = 9, a_3 = (-3)^3 = -27, a_4 = (-3)^4 = 81, a_5 = (-3)^5 = -243$

3. Express the following series in the form of numbers.

   (a) $\sum_{n=1}^{5} n(n+3)$

sol. $\sum_{n=1}^{5} n(n+3)$
$= (1\times 4) + (2\times 5) + (3\times 6) + (4\times 7)$
$+ (5\times 8)$
$= 4 + 10 + 18 + 28 + 40$

(b) $\sum_{n=2}^{6} \frac{1}{3^n}$

sol. $\sum_{n=2}^{6} \frac{1}{3^n}$
$= \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6}$
$= \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729}$

(c) $\sum_{n=1}^{6} \frac{1}{n(2n+1)}$

sol. $\sum_{n=1}^{6} \frac{1}{n(2n+1)}$
$= \frac{1}{1(2\times 1+1)} + \frac{1}{2(2\times 2+1)}$
$+ \frac{1}{3(2\times 3+1)} + \frac{1}{4(2\times 4+1)}$
$+ \frac{1}{5(2\times 5+1)} + \frac{1}{6(2\times 6+1)}$
$= \frac{1}{3} + \frac{1}{10} + \frac{1}{21} + \frac{1}{36} + \frac{1}{55} + \frac{1}{78}$

(d) $\sum_{n=2}^{5} \frac{1}{n^2+2}$

sol. $\sum_{n=2}^{5} \frac{1}{n^2+2}$
$= \frac{1}{4+2} + \frac{1}{9+2} + \frac{1}{16+2} + \frac{1}{25+2}$
$= \frac{1}{6} + \frac{1}{11} + \frac{1}{18} + \frac{1}{27}$

4. Find the first term, last term and the number of terms of the following series.

(a) $\sum_{n=3}^{10} 2^2$
sol. $a_3 = 2^2 = 4, a_{10} = 2^2 = 4, n = 10 - 3 + 1 = 8$

(b) $\sum_{n=1}^{8} \frac{n+2}{n}$

sol. $a_1 = \frac{1+2}{1} = \frac{3}{1} = 3, a_8 = \frac{8+2}{8} = \frac{10}{8} = \frac{5}{4}, n = 8 - 1 + 1 = 8$

(c) $\sum_{n=1}^{10} 3n^2 - n$

sol. $a_1 = 3\times 1^2 - 1 = 2, a_{10} = 3\times 10^2 - 10 = 290, n = 10 - 1 + 1 = 10$

(d) $\sum_{n=9}^{14} n^2(n-7)$

sol. $a_9 = 9^2(9-7) = 9^2\times 2 = 162, a_{14} = 14^2(14-7) = 14^2\times 7 = 2744, n = 14-9+1 = 6$

5. Express the following series in the form of $\sum$.

(a) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{30}$

sol.

$$a_1 = 1$$
$$a_2 = \frac{1}{2}$$
$$a_3 = \frac{1}{3}$$
$$\vdots$$
$$a_{30} = \frac{1}{30}$$
$$\therefore 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{30} = \sum_{n=1}^{30} \frac{1}{n}$$

(b) $1^3 + 2^3 + 3^3 + \cdots + 50^3$

sol.

$$a_1 = 1^3$$
$$a_2 = 2^3$$
$$a_3 = 3^3$$
$$\vdots$$
$$a_{50} = 50^3$$
$$\therefore 1^3 + 2^3 + 3^3 + \cdots + 50^3 = \sum_{n=1}^{50} n^3$$

(c) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16}$

sol.

$$\begin{aligned}
a_1 &= (-\frac{1}{2})^{1-1} \\
a_2 &= (-\frac{1}{2})^{2-1} \\
a_3 &= (-\frac{1}{2})^{3-1} \\
a_4 &= (-\frac{1}{2})^{4-1} \\
a_5 &= (-\frac{1}{2})^{5-1} \\
\therefore 1 &- \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} \\
&= \sum_{n=1}^{5} (-\frac{1}{2})^{n-1}
\end{aligned}$$

(d) $2 \times 4 + 4 \times 7 + 6 \times 10 + 8 \times 13 + 10 \times 16$

sol.

$$\begin{aligned}
a_1 &= 2 \times 1 \times (3 \times 1 + 1) \\
a_2 &= 2 \times 2 \times (3 \times 2 + 1) \\
a_3 &= 2 \times 3 \times (3 \times 3 + 1) \\
a_4 &= 2 \times 4 \times (3 \times 4 + 1) \\
a_5 &= 2 \times 5 \times (3 \times 5 + 1) \\
\therefore & 2 \times 4 + 4 \times 7 + 6 \times 10 + 8 \times 13 \\
&+ 10 \times 16 = \sum_{n=1}^{5} 2n(3n+1)
\end{aligned}$$

12.2 Arithmetic Progression

General term of an Arithmetic Progression (AP) is given by

$$a_n = a_1 + (n-1)d$$

where a_1 is the first term, d is the common difference and n is the number of terms.

12.2.1 Practice 3

1. Find the number of terms of the AP $-4 - 2\frac{3}{4} - 1\frac{1}{2} - \frac{1}{4} + \cdots + 16$.

$$\begin{aligned}
a_1 &= -4 \\
a_n &= 16 \\
d &= -2\frac{3}{4} - (-4) \\
&= -2\frac{3}{4} + 4 \\
&= \frac{5}{4} \\
16 &= -4 + (n-1)\frac{5}{4} \\
20 &= \frac{5}{4}(n-1) \\
80 &= 5(n-1) \\
n - 1 &= 16 \\
n &= 17
\end{aligned}$$

2. Given that $a_2 = 4$ and $a_6 = -8$, find the 10th term of the AP.

sol.

$$\begin{aligned}
a_2 &= 4 \\
a + (2-1)d &= 4 \\
a_6 &= -8 \\
a + (6-1)d &= -8
\end{aligned}$$

$$\begin{cases} a + d = 4 & (1) \\ a + 5d = \text{-}8 & (2) \end{cases}$$

$$\begin{aligned}
(2) - (1) : 4d &= -12 \\
d &= -3 \\
a + (-3) &= 4 \\
a &= 7 \\
\therefore a_{10} &= 7 + (10-1)(-3) \\
&= 7 - 27 \\
&= -20
\end{aligned}$$

3. How many multiples of 7 are there between 50 and 500?

sol.

$$\begin{aligned} a_1 &= 56 \\ a_n &= 497 \\ d &= 7 \\ 497 = 56 + (n-1)7 \\ 441 = 7(n-1) \\ n - 1 &= 63 \\ n &= 64 \end{aligned}$$

4. Find 5 numbers between 30 and 54 such that these numbers form an AP.

   sol.

$$\begin{aligned} &a_1 = 30 \\ &a_7 = 54 \\ &54 = 30 + (7-1)d \\ &24 = 6d \\ &d = 4 \end{aligned}$$

$$\therefore These\ 5\ numbers\ are\ 34,\ 38,\ 42,\ 46,\ and\ 50.$$

Arithmetic mean

If A is in between x and y, and x, A, y are in AP, then

$$A = \frac{x+y}{2}$$

12.2.2 Practice 4

1. If 9, x, 17 are in AP, find x.

   sol.

$$\begin{aligned} x &= \frac{9+17}{2} \\ &= \frac{26}{2} \\ &= 13 \end{aligned}$$

2. Find the arithmetic mean of 26 and -11.

   sol.

$$\begin{aligned} A &= \frac{26-11}{2} \\ &= \frac{15}{2} \end{aligned}$$

3. Find x and y when 3, x, 12, y, 21 are in AP.

   sol.

$$\begin{aligned} x &= \frac{3+12}{2} \\ &= \frac{15}{2} \\ y &= \frac{12+21}{2} \\ &= \frac{33}{2} \end{aligned}$$

Summation of Arithmetic Progression

The summation formula for AP is given by

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

or

$$S_n = \frac{n}{2}(a_1 + a_n)$$

12.2.3 Practice 5

1. Find the sum of the first 16 terms of the AP $22 + 18 + 14 + 10 + \cdots$

   sol.

$$\begin{aligned} a_1 &= 22 \\ n &= 16 \\ d &= -4 \\ S_n &= \frac{16}{2}(2 \times 22 + (-4)(16-1)) \\ &= \frac{16}{2}(44 + (-4)(15)) \\ &= \frac{16}{2}(44 - 60) \\ &= \frac{16}{2}(-16) \\ &= -128 \end{aligned}$$

2. If the sum of AP $23+19+15+\cdots$ is 72, find the number of terms.

sol.

$$\begin{aligned}
a_1 &= 23\\
S_n &= 72\\
d &= -4\\
72 &= \frac{n}{2}(2\times 23 + (-4)(n-1))\\
72 &= \frac{n}{2}(46 + (-4)(n-1))\\
144 &= n(46 + (-4)(n-1))\\
144 &= n(46 - 4n + 4)\\
144 &= n(50 - 4n)\\
144 &= 50n - 4n^2\\
72 &= 25n - 2n^2\\
2n^2 - 25n + 72 &= 0\\
(n-8)(2n-9) &= 0\\
n &= 8
\end{aligned}$$

3. Given that $S_n = 2n + 3n^2$, find the first term and the common difference of the AP.

sol.

$$\begin{aligned}
S_n &= 2n + 3n^2\\
2n + 3n^2 &= \frac{n}{2}(2a + (n-1)d)\\
4n + 6n^2 &= n(2a + (n-1)d)\\
4n + 6n^2 &= 2na + (n-1)nd\\
4n + 6n^2 &= 2na + n^2d - nd\\
4n + 6n^2 &= (2a-d)n + dn^2
\end{aligned}$$

$$\begin{aligned}
&\textit{Comparing both sides,}\\
2a - d &= 4\\
a &= 6\\
d &= 2
\end{aligned}$$

12.2.4 Exercise 12.2

1. Find the 10th terms of the AP $5, 13, 21, \cdots$

sol.

$$\begin{aligned}
a_1 &= 5\\
n &= 10\\
d &= 8\\
a_{10} &= 5 + (10-1)\times 8\\
&= 5 + 72\\
&= 77
\end{aligned}$$

2. Find the 8th term of the AP $5, 4\frac{1}{4}, 3\frac{1}{2}, 2\frac{3}{4}, \cdots$

sol.

$$\begin{aligned}
a_1 &= 5\\
n &= 8\\
d &= -\frac{3}{4}\\
a_8 &= 5 + (8-1)\times -\frac{3}{4}\\
&= 5 - \frac{3}{4}\times 7\\
&= 5 - \frac{21}{4}\\
&= -\frac{1}{4}
\end{aligned}$$

3. Find the number of terms of the following AP.

(a) $4, 9, \ldots, 64$

sol.

$$\begin{aligned}
a_1 &= 4\\
a_n &= 64\\
d &= 5\\
64 &= 4 + (n-1)\times 5\\
60 &= 5(n-1)\\
12 &= n-1\\
n &= 13
\end{aligned}$$

(b) $4\frac{1}{3}, 3\frac{2}{3}, 3, \ldots, -10\frac{1}{3}$

sol.

$$
\begin{aligned}
a_1 &= 4\frac{1}{3} \\
a_n &= -10\frac{1}{3} \\
d &= -\frac{2}{3} \\
-10\frac{1}{3} &= 4\frac{1}{3} + (n-1) \times -\frac{2}{3} \\
-\frac{31}{3} &= \frac{13}{3} - \frac{1}{3}(n-1) \\
-31 &= 13 - 2n + 2 \\
-46 &= 2n \\
n &= 23
\end{aligned}
$$

4. The 6th term of an AP is 43, and its 10th term is 75. Find the first term and common difference of this AP.

   sol.

$$
\begin{aligned}
a_6 &= 43 \\
a_{10} &= 75 \\
43 &= a + (6-1)d \\
75 &= a + (10-1)d \\
32 &= 4d \\
d &= 8 \\
43 &= a + 5 \times 8 \\
43 &= a + 40 \\
3 &= a \\
a &= 3 \\
&\therefore a_1 = 3,\ d = 8
\end{aligned}
$$

5. The 7th term of an AP is -10, and the 12th term -25, find the 15th term of this AP.

   sol.

$$
\begin{aligned}
a_7 &= -10 \\
a_{12} &= -25 \\
-10 &= a + (7-1)d \\
-25 &= a + (12-1)d \\
-15 &= 5d \\
d &= -3 \\
-10 &= a + 6 \times -3 \\
-10 &= a - 18 \\
a &= 8 \\
a_{15} &= 8 + (15-1) \times -3 \\
&= 8 - 42 \\
&= -34
\end{aligned}
$$

6. How many multiples of 7 are there between 100 and 200?

   sol.

$$
\begin{aligned}
a &= 105 \\
d &= 7 \\
a_n &= 196 \\
196 &= 105 + (n-1) \times 7 \\
91 &= 7(n-1) \\
13 &= n - 1 \\
n &= 14
\end{aligned}
$$

7. Find the arithmetic mean o fthe following number pairs.

   (a) $(8, 20)$

   sol.

$$\frac{8+20}{2} = 14$$

   (b) $(-9, 17)$

   sol.

$$\frac{-9+17}{2} = 4$$

8. Find 5 numbers between 22 and 58 such that these 7 numbers are in AP.

   sol.

$$
\begin{aligned}
a_1 &= 22 \\
a_7 &= 58 \\
58 &= 22 + (7-1)d \\
36 &= 6d \\
d &= 6 \\
&\therefore These\ 5\ numbers\ are\ 22, 28, 34, 40, 46
\end{aligned}
$$

9. Find the sum of first 20 terms of AP $12 + 15 + 18 + \cdots$

   sol.

$$
\begin{aligned}
a_1 &= 12 \\
n &= 20 \\
d &= 3 \\
S_{20} &= \frac{20}{2}(2 \times 12 + (20-1) \times 3) \\
&= 10(24 + 57) \\
&= 10(81) \\
&= 810
\end{aligned}
$$

10. Find the sum of first 12 terms of the AP $18 + 10 + 2 - 6 - \dots$

sol.

$$\begin{aligned} a_1 &= 18 \\ n &= 12 \\ d &= -8 \\ S_{12} &= \frac{12}{2}(2 \times 18 + (12-1) \times -8) \\ &= 6(36 - 88) \\ &= 6(-52) \\ &= -312 \end{aligned}$$

11. Find the sum of first 14 terms of the AP $\frac{1}{6} + \frac{4}{3} + \frac{5}{2} + \cdots$

sol.

$$\begin{aligned} a_1 &= \frac{1}{6} \\ n &= 14 \\ d &= \frac{7}{6} \\ S_{14} &= \frac{14}{2}(2 \times \frac{1}{6} + (14-1) \times \frac{7}{6}) \\ &= 7(\frac{1}{3} + \frac{91}{6}) \\ &= 7 \times \frac{93}{6} \\ &= 7 \times \frac{31}{2} \\ &= \frac{217}{2} \end{aligned}$$

12. Find the sum of all the multiples of 13 in between 200 and 800.

sol.

$$\begin{aligned} a_1 &= 208 \\ a_n &= 793 \\ d &= 13 \\ 793 &= 208 + (n-1) \times 13 \\ 585 &= 13(n-1) \\ 45 &= n - 1 \\ n &= 46 \end{aligned}$$

$$\begin{aligned} S_{46} &= \frac{46}{2}(2 \times 208 + (46-1) \times 13) \\ &= 23(416 + 585) \\ &= 23(1001) \\ &= 23023 \end{aligned}$$

13. If the sum of first n terms of the AP $-3, -7, -11, \cdots$ is -903, find the value of n.

sol.

$$\begin{aligned} a_1 &= -3 \\ d &= -4 \\ -903 &= \frac{n}{2}(2 \times (-3) - 4(n-1)) \\ -1806 &= -2n - 4n^2 \\ 4n^2 + 2n - 1806 &= 0 \\ 2n^2 + n - 903 &= 0 \\ (n-21)(2n+43) &= 0 \\ n &= 21, -43(invalid) \\ \therefore n &= 21 \end{aligned}$$

14. Given that the first 3 terms of an AP are $x,\ 3x-4,\ 2x+7$, find:

(a) The value of x

sol.

$$\begin{aligned} 3x - 4 &= \frac{x + 2x + 7}{2} \\ 6x - 8 &= 3x + 7 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

(b) The common difference

sol.

$$\begin{aligned} a_1 &= x = 5 \\ a_2 &= 3x - 4 = 3 \times 5 - 4 = 11 \\ d &= 11 - 5 \\ &= 6 \end{aligned}$$

(c) The sum of first 10 terms.

sol.

$$\begin{aligned} a_1 &= x = 5 \\ n &= 10 \\ d &= 6 \\ S_{10} &= \frac{10}{2}(2 \times 5 + (10-1) \times 6) \\ &= 5(10 + 54) \\ &= 5(64) \\ &= 320 \end{aligned}$$

15. Let the sum of the first n terms of an AP to be $S_n = \frac{n(n+1)}{4}$, find:

(a) The first term

sol.

$$\begin{aligned}\frac{n(n+1)}{4} &= \frac{n}{2}(2a+(n-1)d)\\ n(n+1) &= 2n(2a+dn-d)\\ n^2+n &= 4na+2dn^2-2nd\\ n^2+n &= 2dn^2+(4a-2d)n\end{aligned}$$

Comparing both sides,

$$\begin{aligned}2d &= 1\\ d &= \frac{1}{2}\\ 4a-2d &= 1\\ 4a-1 &= 1\\ 4a &= 2\\ a &= \frac{1}{2}\end{aligned}$$

(b) The common difference

sol.

$$d = \frac{1}{2}$$

gg

(c) The 6th terms

sol.

$$\begin{aligned}a_1 &= \frac{1}{2}\\ n &= 6\\ d &= \frac{1}{2}\\ a_6 &= \frac{1}{2}+(6-1)\times\frac{1}{2}\\ &= \frac{1}{2}+\frac{5}{2}\\ &= 3\end{aligned}$$

(d) The sum from 6th term to 10th term

sol.

$$\begin{aligned}a &= \frac{1}{2}\\ d &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}S_{10} &= \frac{10}{2}(2\times\frac{1}{2}+(10-1)\times\frac{1}{2})\\ &= \frac{10}{2}(1+\frac{9}{2})\\ &= 5\times\frac{11}{2}\\ &= \frac{55}{2}\end{aligned}$$

$$\begin{aligned}S_5 &= \frac{5}{2}(2\times\frac{1}{2}+(5-1)\times\frac{1}{2})\\ &= \frac{5}{2}(1+2)\\ &= \frac{15}{2}\end{aligned}$$

$$\begin{aligned}S_{10}-S_6 &= \frac{55}{2}-\frac{15}{2}\\ &= \frac{40}{2}\\ &= 20\end{aligned}$$

16. Given three numbers in an AP, the sum of these three numbers is 30, and the sum of square of these numbers is 318, find these three numbers.

sol.

$$
\begin{aligned}
a_1 + a_2 + a_3 &= 30 \\
a_1^2 + a_2^2 + a_3^2 &= 318 \\
a_2 - a_1 &= a_3 - a_2 \\
a_1 - 2a_2 + a_3 &= 0 \\
3a_2 &= 30 \\
a_2 &= 10 \\
a_1 - 20 + a_3 &= 0 \\
a_1 + a_3 &= 20 \\
a_3 &= 20 - a_1 \\
a_1^2 + 100 + (20 - a_1)^2 &= 318 \\
a_1^2 + 100 + 400 + a_1^2 - 40a_1 &= 318 \\
2a_1^2 - 40a_1 + 182 &= 0 \\
a_1^2 - 20a_1 + 91 &= 0 \\
(a_1 - 7)(a_1 - 13) &= 0 \\
a_1 &= 7 or a_1 = 13
\end{aligned}
$$

$$\therefore These\ three\ numbers\ are\ 7,\ 10,\ and\ 13$$

17. Find the sum of all the numbers between 100 and 200 that are both the multiples of 2 and 3.

sol.

$$
\begin{aligned}
a_1 &= 102 \\
d &= 6 \\
a_n &= 198 \\
198 &= 102 + (n-1) \times 6 \\
96 &= 6(n-1) \\
6n - 6 &= 96 \\
6n &= 102 \\
n &= 17
\end{aligned}
$$

$$
\begin{aligned}
S_{17} &= \frac{17}{2}(2 \times 102 + (17-1) \times 6) \\
&= \frac{17}{2}(204 + 96) \\
&= \frac{17}{2}(300) \\
&= 150 \times 17 \\
&= 2550
\end{aligned}
$$

18. Given an AP $-100 - 96 - 92 - \cdots$:

(a) Find the term where the number become positive.

sol.

$$
\begin{aligned}
a_1 &= -100 \\
d &= 4 \\
a_n &= -100 + (n-1) \times 4 > 0 \\
-100 + 4n - 4 &> 0 \\
4n &> 104 \\
n &> 26
\end{aligned}
$$

$$\therefore n = 27$$

(b) Find the term where the sum of this AP becomes positive.

sol.

$$
\begin{aligned}
S_n = \frac{n}{2}(2(-100) + (n-1) \times (4)) &> 0 \\
\frac{n}{2}(-200 + 4n - 4) &> 0 \\
\frac{n}{2}(-204 + 4n) &> 0 \\
n(2n - 102) &> 0 \\
n(n - 51) &> 0 \\
n &> 51
\end{aligned}
$$

$$\therefore n = 52$$

19. Find the first negative term of the AP $20, 19\frac{1}{5}, 18\frac{2}{5}, \cdots$

sol.

$$
\begin{aligned}
a_1 &= 20 \\
d &= -\frac{4}{5} \\
a_n = 20 + (n-1) \times (-\frac{4}{5}) &< 0 \\
100 - 4n + 4 &< 0 \\
4n &> 104 \\
n &> 26
\end{aligned}
$$

$$\therefore n = 27$$

20. Given an AP $10 + 9\frac{1}{5} + 8\frac{2}{5} + \cdots$, what is the first negative term? When will the sum of the terms become negative, and what's the value of it?

sol.

$$\begin{aligned} a_n &= 10 + (n-1) \times (-\frac{4}{5}) < 0 \\ 10 - \frac{4}{5}(n-1) &< 0 \\ 50 - 4n + 4 &< 0 \\ -4n &< -54 \\ n &> 13\frac{1}{2} \\ \\ \therefore n &= 14 \end{aligned}$$

$$\begin{aligned} S_n = \frac{n}{2}(2 \times 10 + (n-1) \times (-\frac{4}{5})) &< 0 \\ \frac{n}{2}(20 - \frac{4}{5}(n-1)) &< 0 \\ 20n - \frac{4}{5}(n^2 - n) &< 0 \\ 100n - 4n^2 + 4n &< 0 \\ 25n - n^2 + n &< 0 \\ 26n - n^2 &< 0 \\ n(n-26) &> 0 \\ n &> 26 \\ \\ \therefore n &= 27 \end{aligned}$$

$$\begin{aligned} S_{27} &= \frac{27}{2}(2 \times 10 + (27-1) \times (-\frac{4}{5})) \\ &= \frac{27}{2}(20 - \frac{4}{5}(27-1)) \\ &= \frac{27}{2}(20 - \frac{4}{5}(26)) \\ &= \frac{27}{2} \times (-\frac{4}{5}) \\ &= -\frac{54}{5} \end{aligned}$$

$\therefore The\ first\ negative\ term\ is\ the\ 14th\ term$

$\therefore The\ first\ term\ where\ the\ sum\ of\ the\ terms\ becomes\ negative\ is\ the\ 27th\ term$

$\therefore The\ value\ of\ the\ sum\ of\ the\ terms\ when\ it\ becomes\ negative\ is\ -\frac{54}{5}$

21. Given a polygon which all their internal angles are in AP. The common difference of this AP is 6°, the largest angle is 135°. How many sides does this polygon have?

sol.

$$\begin{aligned} a_1 &= 135 \\ d &= -6 \\ \frac{n}{2}(2 \times 135 + (n-1) \times (-6)) &= 180(n-2) \\ n(270 - 6(n-1)) &= 360(n-2) \\ n(276 - 6n) &= 360n - 720 \\ 276n - 6n^2 &= 360n - 720 \\ 46n - n^2 &= 60n - 120 \\ n^2 + 14n - 120 &= 0 \\ (n+20)(n-6) &= 0 \\ n &= -20\ (invalid) \\ n &= 6 \\ \therefore The\ number\ of\ sides\ is\ 6 \end{aligned}$$

22. Given an AP which its 5th term is 3 and the sum of its first 10 terms is $26\frac{1}{4}$. Whcih term in this AP is 0?

sol.

$$\begin{aligned} a_5 = a + (5-1)d &= 3 \\ a + 4d &= 3 \\ S_{10} = \frac{10}{2}(2a + (10-1)d) &= 26\frac{1}{4} \\ 5(2a + 9d) &= 26\frac{1}{4} \\ 20(2a + 9d) &= 105 \\ 4(2a + 9d) &= 21 \\ 8a + 36d &= 21 \\ 8a + 32d &= 24 \\ 4d &= -3 \\ d &= -\frac{3}{4} \\ a &= 3 + \frac{3}{4} \times 4 \\ &= 6 \\ a_n = 6 + (n-1) \times (-\frac{3}{4}) &= 0 \\ 6 - \frac{3}{4}(n-1) &= 0 \\ 24 - 3n + 3 &= 0 \\ 3n &= 27 \\ n &= 9 \end{aligned}$$

23. Given that the sum of the first 6 terms of an AP is 96, and the sum of the first 20 terms is 3 times the sum of the first 10 terms of this AP. Find the first term and the 10th term of it.

sol.

$$
\begin{aligned}
S_6 = \frac{6}{2}(2a + (6-1)d) &= 96 \\
3(2a + 5d) &= 96 \\
2a + 5d &= 32 \\
S_{20} &= 3S_{10} \\
\frac{20}{2}(2a + (20-1)d) &= 3 \times \frac{10}{2}(2a \\
&\quad + (10-1)d) \\
10(2a + 19d) &= 15(2a + 9d) \\
2(2a + 19d) &= 3(2a + 9d) \\
4a + 38d &= 6a + 27d \\
2a - 11d &= 0 \\
16d &= 32 \\
d &= 2 \\
a &= \frac{11 \times 2}{2} \\
&= 11 \\
a_{10} &= 11 + (10-1) \times 2 \\
&= 29
\end{aligned}
$$

24. Given that $5^2 \times 5^4 \times 5^6 \times \cdots \times 5^{2n} = (0.04)^{-28}$, find the value of n.

sol.

$$
\begin{aligned}
(0.04)^{-28} &= \frac{1}{25}^{-28} \\
&= (5^(-2))^{-28} \\
&= 5^{56} \\
\because n^a \times n^b &= n^{a+b} \\
2 + 4 + 6 + \cdots + 2n &= 56 \\
S_n = \frac{n}{2}(2 \times 2 + (n-1) \times 2) &= 56 \\
n(4 + 2(n-1)) &= 112 \\
n(2 + 2n) &= 112 \\
2n^2 + 2n &= 112 \\
n^2 + n - 56 &= 0 \\
(n+8)(n-7) &= 0 \\
n &= -8 \ (invalid) \\
n &= 7
\end{aligned}
$$

25. Given that the 9th term of an AP is double the 5th term of it. Find the ratio of the sum of first 9 terms and the sum of first 5 terms of the AP.

sol.

$$
\begin{aligned}
a_9 &= 2a_5 \\
a + (9-1)d &= 2(a + (5-1)d) \\
a + 8d &= 2a + 8d \\
a &= 0 \\
S_9 : S_5 &= \frac{9}{2}(2a + a_9) : \frac{5}{2}(2a + a_5) \\
&= \frac{9}{2}(2a + 2a_5) : \frac{5}{2}(2a + a_5) \\
&= 9(a + a_5) : \frac{5}{2}(2a + a_5) \\
\frac{S_9}{S_5} &= \frac{9(a + a_5)}{\frac{5}{2}(2a + a_5)} \\
&= \frac{18(a + a_5)}{5(2a + a_5)} \\
&= \frac{18 \times a_5}{5 \times a_5} \\
&= \frac{18}{5} \\
\therefore S_9 : S_5 &= 18 : 5
\end{aligned}
$$

12.3 Geometric Progression

The general formula of a geometric progression (GP) is given by

$$a_n = a_1 \times r^{n-1}$$

where a_1 is the first term, r is the common ratio, and n is the number of terms.

12.3.1 Practice 6

1. Find the 6th term of the GP $12, -18, 27, \cdots$

sol.

$$
\begin{aligned}
a_1 &= 12 \\
r &= \frac{-18}{12} \\
&= -\frac{3}{2} \\
a_6 &= 12 \times (-\frac{3}{2})^{6-1} \\
&= 12 \times (-\frac{3}{2})^5 \\
&= 12 \times (-\frac{243}{32}) \\
&= -\frac{729}{8}
\end{aligned}
$$

2. Find the number of terms of GP $\frac{1}{64} - \frac{1}{32} + \frac{1}{16} - \frac{1}{8} + \cdots - 512$

   sol.

$$
\begin{aligned}
a_1 &= \frac{1}{64} \\
r &= \frac{-\frac{1}{32}}{\frac{1}{64}} \\
&= -2 \\
-512 &= \frac{1}{64}(-2)^{n-1} \\
(-2)^9 &= \frac{1}{2^6}(-2)^{n-1} \\
(-2)^{15} &= (-2)^{n-1} \\
n - 1 &= 15 \\
n &= 16
\end{aligned}
$$

3. The 5th term of a GP is 3, and its 9th term is $\frac{1}{27}$, find the first term and the common ratio of this GP.

   sol.

$$
\begin{aligned}
a_5 &= ar^4 = 3 \\
a_9 &= ar^8 = \frac{1}{27} \\
r^4 &= \frac{1}{27} \times \frac{1}{3} \\
&= \frac{1}{81} \\
r &= \frac{1}{3} \\
a_1 &= 3 \times 81 \\
&= 243
\end{aligned}
$$

4. Find 5 numbers between $\frac{1}{2}$ and $frac1128$ such that these 7 numbers are in GP.

   sol.

$$
\begin{aligned}
a_1 &= \frac{1}{2} \\
n &= 7 \\
\frac{1}{128} &= \frac{1}{2}r^{7-1} \\
r^6 &= \frac{1}{64} \\
r &= \frac{1}{2}
\end{aligned}
$$

$$
\therefore These\ 5\ numbers\ are\ \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}
$$

Geometric Mean

The geometric mean G of two numbers x and y is given by

$$
\begin{aligned}
\frac{G}{x} &= \frac{G}{y} \\
G^2 &= xy \\
G &= \mp\sqrt[2]{xy}
\end{aligned}
$$

12.3.2 Practice 7

Find the geometric mean of $\frac{27}{8}$ and $\frac{2}{3}$.

sol.

$$
\begin{aligned}
G &= \pm\sqrt[2]{\frac{27}{8} \times \frac{2}{3}} \\
&= \pm\sqrt[2]{\frac{9}{4}} \\
&= \pm\frac{3}{2}
\end{aligned}
$$

Summation of Geometric Progression

The sum of n terms of a GP is given by

$$
S_n = \frac{a_1(1 - r^n)}{1 - r}\ (r \neq 1)
$$

12.3.3 Practice 8

1. Find the sum of the first 8 terms of GP $3 + 6 + 12 + \cdots$

sol.

$$
\begin{aligned}
a_1 &= 3 \\
r &= \frac{6}{3} \\
&= 2 \\
n &= 8 \\
S_n &= \frac{3(1-2^8)}{1-2} \\
&= \frac{3(1-256)}{1-2} \\
&= 3 \times 255 \\
&= 765
\end{aligned}
$$

2. Find the sum of the GP $1 + \sqrt{3} + 3 + \cdots + 81$

sol.

$$
\begin{aligned}
a_1 &= 1 \\
r &= \sqrt{3} \\
81 &= 1 \times (\sqrt{3})^{n-1} \\
3^4 &= (\sqrt{3})^{n-1} \\
(\sqrt{3})^8 &= (\sqrt{3})^{n-1} \\
n - 1 &= 8 \\
n &= 9 \\
S_n &= \frac{1(1-(\sqrt{3})^9)}{1-\sqrt{3}} \\
&= \frac{1-81\sqrt{3}}{1-\sqrt{3}} \\
&= \frac{(1-81\sqrt{3})(1+\sqrt{3})}{-2} \\
&= \frac{1-81\sqrt{3}+\sqrt{3}-243}{-2} \\
&= \frac{-242-80\sqrt{(3)}}{-2} \\
&= 121 + 40\sqrt{3}
\end{aligned}
$$

3. Given that the sum of the first n terms of GP $4\frac{4}{5}, 1\frac{3}{5}, \frac{8}{15}, \cdots$ is $7\frac{145}{729}$, find n.

sol.

$$
\begin{aligned}
a_1 &= \frac{24}{5} \\
r &= \frac{8}{5} \times \frac{5}{24} \\
&= \frac{1}{3} \\
S_n &= \frac{24}{5} \times \frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}} \\
\frac{5248}{729} &= \frac{24}{5} \times \frac{1-(\frac{1}{3})^n}{\frac{2}{3}} \\
\frac{5248}{729} \times \frac{5}{24} \times \frac{2}{3} &= 1 - (\frac{1}{3})^n \\
\frac{6560}{6561} &= 1 - (\frac{1}{3})^n \\
-\frac{1}{6561} &= -(\frac{1}{3})^n \\
(\frac{1}{3})^8 &= (\frac{1}{3})^n \\
n &= 8
\end{aligned}
$$

Summation of Infinite Geometric Progression

The sum of infinite GP is given by

$$S_\infty = \frac{a_1}{1-r} \ (-1 < r < 1)$$

12.3.4 Practice 9

1. Find the sum of the following infinite GP.

   (a) $16 + 8 + 4 + \cdots$

   sol.

$$
\begin{aligned}
a_1 &= 16 \\
r &= \frac{8}{16} \\
&= \frac{1}{2} \\
S_\infty &= \frac{16}{1-\frac{1}{2}} \\
&= \frac{16}{\frac{1}{2}} \\
&= 32
\end{aligned}
$$

   (b) $18 - 12 + 8 + \cdots$

sol.

$$\begin{aligned} a_1 &= 18 \\ r &= \frac{8}{-12} \\ &= -\frac{2}{3} \\ S_\infty &= \frac{18}{1+\frac{2}{3}} \\ &= \frac{18}{\frac{5}{3}} \\ &= \frac{54}{5} \end{aligned}$$

(c) $1 + \frac{3}{4} + \frac{9}{16} + \cdots$

sol.

$$\begin{aligned} a_1 &= 1 \\ r &= \frac{9}{16} \times \frac{16}{9} \\ &= \frac{3}{4} \\ S_\infty &= \frac{1}{1-\frac{3}{4}} \\ &= \frac{1}{\frac{1}{4}} \\ &= 4 \end{aligned}$$

(d) $\sqrt{2} + 1 + \frac{1}{\sqrt{2}} + \cdots$

sol.

$$\begin{aligned} a_1 &= \sqrt{2} \\ r &= \frac{1}{\sqrt{2}} \\ S_\infty &= \frac{\sqrt{2}}{1-\frac{1}{\sqrt{2}}} \\ &= \frac{\sqrt{2}}{\frac{\sqrt{2}-1}{\sqrt{2}}} \\ &= \frac{2}{\sqrt{2}-1} \\ &= 2(\sqrt{2}+1) \end{aligned}$$

2. Convert the following recurring decimals to fraction using the summation of inifinite geometric series.

(a) $0.\overline{3}$

sol.

$$\begin{aligned} a_1 &= 0.3 \\ r &= 0.1 \\ S_\infty &= \frac{0.3}{1-0.1} \\ &= \frac{0.3}{0.9} \\ &= \frac{1}{3} \end{aligned}$$

$$\therefore 0.\overline{3} = \frac{1}{3}$$

(b) $0.5\overline{3}$

sol.

$$\begin{aligned} a_1 &= 0.03 \\ r &= 0.01 \\ S_\infty &= \frac{0.03}{1-0.01} \\ &= \frac{0.03}{0.99} \\ &= \frac{3}{99} \end{aligned}$$

$$\begin{aligned} \therefore 0.5\overline{3} &= \frac{5}{10} + \frac{3}{99} \\ &= \frac{53}{99} \end{aligned}$$

12.3.5 Exercise 12.3

1. Find the 10th term of the GP $2, 4, 8, \cdots$

sol.

$$\begin{aligned} a_1 &= 2 \\ r &= \frac{4}{2} \\ &= 2 \\ a_{10} &= 2 \times 2^{10-1} \\ &= 2 \times 512 \\ &= 1024 \end{aligned}$$

2. Find the 8th term of the GP $243, -162, 108, \cdots$

sol.

$$\begin{aligned} a_1 &= 243 \\ r &= \frac{-162}{243} \\ &= -\frac{2}{3} \\ a_8 &= 243 \times (-\frac{2}{3})^{8-1} \\ &= 243 \times (-\frac{128}{2187}) \\ &= -\frac{128}{9} \end{aligned}$$

3. Find the number of terms of the following GP.

(a) $8, 4, 2, 1, \ldots, \frac{1}{64}$

sol.

$$\begin{aligned} a_1 &= 8 \\ r &= \frac{4}{8} \\ &= \frac{1}{2} \\ \frac{1}{64} &= 8 \times (\frac{1}{2})^{n-1} \\ \frac{1}{512} &= (\frac{1}{2})^{n-1} \\ \frac{1}{2^9} &= (\frac{1}{2})^{n-1} \\ n-1 &= 9 \\ n &= 10 \end{aligned}$$

(b) $6, -18, 54, \ldots, -13122$

sol.

$$\begin{aligned} a_1 &= 6 \\ r &= \frac{-18}{6} \\ &= -3 \\ -13122 &= 6 \times (-3)^{n-1} \\ -2187 &= (-3)^{n-1} \\ (-3)^7 &= (-3)^{n-1} \\ n-1 &= 7 \\ n &= 8 \end{aligned}$$

(c) $54, 36, 24, \ldots, 3\frac{13}{81}$

sol.

$$\begin{aligned} a_1 &= 54 \\ r &= \frac{36}{54} \\ &= \frac{2}{3} \\ \frac{256}{81} &= 54 \times (\frac{2}{3})^{n-1} \\ \frac{256}{81} \times \frac{1}{54} &= (\frac{2}{3})^{n-1} \\ \frac{128}{2187} &= (\frac{2}{3})^{n-1} \\ (\frac{2}{3})^7 &= (\frac{2}{3})^{n-1} \\ n-1 &= 7 \\ n &= 8 \end{aligned}$$

4. Given that the 2nd term of a GP is 12, and its 4th term is 108, find the first term and the common ratio of it.

sol.

$$\begin{aligned} a_2 &= ar = 12 \\ a_4 &= ar^3 = 109 \\ r^2 &= 9 \\ r &= \pm 3 \\ a_1 &= \pm 4 \\ \therefore a_1 &= 4, r = 3 \text{ or } a_1 = -4, r = -3 \end{aligned}$$

5. Given that the 3rd term of an GP is $1\frac{1}{3}$, and its 8th term is $-10\frac{1}{8}$. Find the 5th term of this AP.

sol.

$$
\begin{aligned}
a_3 &= ar^2 = \frac{4}{3} \\
a_8 &= ar^7 = -\frac{81}{8} \\
r^5 &= -\frac{81}{8} \times \frac{3}{4} \\
&= -\frac{243}{32} \\
&= (-\frac{3}{2})^5 \\
r &= -\frac{3}{2} \\
a &= \frac{4}{3} \times \frac{4}{9} \\
&= \frac{16}{27} \\
a_5 &= \frac{16}{27} \times (\frac{3}{2})^4 \\
&= \frac{16}{27} \times \frac{81}{16} \\
&= 3
\end{aligned}
$$

6. Find the geometric mean of 2 and 18.

sol.

$$
\begin{aligned}
G &= \pm\sqrt[2]{2 \times 18} \\
&= \pm\sqrt[2]{36} \\
&= \pm 6
\end{aligned}
$$

7. Given that x+12, x+4 and x-2 are in GP, find the value of x and the common ratio of this GP.

sol.

$$
\begin{aligned}
x + 4 &= \pm\sqrt{(x+12)(x-2)} \\
x^2 + 8x + 16 &= x^2 + 10x - 24 \\
2x &= 40 \\
x &= 20 \\
a_1 &= 20 + 12 = 32 \\
a_2 &= 20 + 4 = 24 \\
r &= \frac{24}{32} \\
&= \frac{3}{4}
\end{aligned}
$$

8. Find 3 numbers between 14 and 224 such that these 5 numbersare in GP.

sol.

$$
\begin{aligned}
a_1 &= 14 \\
a_5 &= 224 \\
244 &= 14 \times r^4 \\
16 &= r^4 \\
(\pm 2)^4 &= r^4 \\
r &= \pm 2
\end{aligned}
$$

$$
\begin{aligned}
&\therefore These\ 3\ numbers\ are\ 28,\ 56,\ 112 \\
&\quad or\ -28, 56,\ -112
\end{aligned}
$$

9. Calculate the sum of the first 6 terms of the GP $2+6+18+\cdots$

sol.

$$
\begin{aligned}
a_1 &= 2 \\
r &= \frac{6}{2} \\
&= 3 \\
S_6 &= \frac{2(1-3^6)}{1-3} \\
&= \frac{2(1-729)}{-2} \\
&= 728
\end{aligned}
$$

10. Calculate the sum of the first 8 terms of the GP $32 - 16 + 8 - \cdots$

sol.

$$
\begin{aligned}
a_1 &= 32 \\
r &= \frac{-16}{32} \\
&= -\frac{1}{2} \\
S_8 &= \frac{32(1-(\frac{1}{2})^8)}{1+\frac{1}{2}} \\
&= \frac{32(1-\frac{1}{256})}{\frac{3}{2}} \\
&= 32 \times \frac{255}{256} \times \frac{2}{3} \\
&= \frac{85}{4}
\end{aligned}
$$

11. Find the sum of the GP $14 - 28 + 56 - \cdots + 3584$

sol.

$$\begin{aligned}
a_1 &= 14 \\
r &= \frac{-28}{14} = -2 \\
3584 &= 14 \times (-2)^{n-1} \\
256 &= (-2)^{n-1} \\
(-2)^8 &= (-2)^{n-1} \\
n-1 &= 8 \\
n &= 9 \\
S_9 &= \frac{14(1-(-2)^9)}{1-(-2)} \\
&= \frac{14(1+512)}{3} \\
&= \frac{14 \times 513}{3} \\
&= 2394
\end{aligned}$$

12. If the first term of a GP is 7, its common ratio is 3, and the sum of its terms is 847, find the number of terms and the last term of this GP.

sol.

$$\begin{aligned}
a_1 &= 7 \\
r &= 3 \\
S_n &= \frac{7(1-3^n)}{1-3} = 847 \\
7(1-3^n) &= -1694 \\
1-3^n &= -242 \\
3^n &= 243 \\
3^n &= 3^5 \\
n &= 5 \\
a_5 &= 7 \times 3^4 = 567
\end{aligned}$$

13. Find the sum of the following infinite GP.

(a) $24 + 18 + 13\frac{1}{2} + \cdots$

sol.

$$\begin{aligned}
a_1 &= 24 \\
r &= \frac{18}{24} = \frac{3}{4} \\
S_\infty &= \frac{24}{1-\frac{3}{4}} \\
&= \frac{24}{\frac{1}{4}} \\
&= 96
\end{aligned}$$

(b) $27 - 9 + 3 - 1 + \cdots$

sol.

$$\begin{aligned}
a_1 &= 27 \\
r &= \frac{-9}{27} = -\frac{1}{3} \\
S_\infty &= \frac{27}{1+\frac{1}{3}} \\
&= \frac{27}{\frac{4}{3}} \\
&= \frac{81}{4}
\end{aligned}$$

(c) $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \cdots$

sol.

$$\begin{aligned}
a_1 &= 2 \\
r &= \frac{-\frac{1}{2}}{2} = -\frac{1}{4} \\
S_\infty &= \frac{2}{1+\frac{1}{4}} \\
&= \frac{2}{\frac{5}{4}} \\
&= \frac{8}{5}
\end{aligned}$$

14. Given an infinite GP which has a sum of 24 and first term of 30, find the common difference.

sol.

$$\begin{aligned}
a_1 &= 30 \\
S_\infty &= 24 \\
24 &= \frac{30}{1-r} \\
24(1-r) &= 30 \\
24 - 24r &= 30 \\
-24r &= 6 \\
r &= -\frac{1}{4}
\end{aligned}$$

15. Convert the following recurrring decimals into fractions.

(a) $0.\overline{45}$

sol.

$$\begin{aligned} a_1 &= 0.45 \\ r &= 0.01 \\ S_\infty &= \frac{0.45}{1-0.01} \\ &= \frac{0.45}{0.99} \\ &= \frac{45}{99} \\ &= \frac{5}{11} \end{aligned}$$

$$\therefore 0.\overline{45} = \frac{5}{11}$$

(b) $0.\overline{037}$

sol.

$$\begin{aligned} a_1 &= 0.037 \\ r &= 0.001 \\ S_\infty &= \frac{0.037}{1-0.001} \\ &= \frac{0.037}{0.999} \\ &= \frac{37}{999} \\ &= \frac{1}{27} \end{aligned}$$

$$\therefore 0.\overline{037} = \frac{1}{27}$$

(c) $0.2\overline{18}$

sol.

$$\begin{aligned} a_1 &= 0.018 \\ r &= 0.01 \\ S_\infty &= \frac{0.018}{1-0.01} \\ &= \frac{0.018}{0.99} \\ &= \frac{18}{990} \\ &= \frac{1}{55} \end{aligned}$$

$$\begin{aligned} \therefore 0.2\overline{18} &= \frac{1}{5} + \frac{1}{55} \\ &= \frac{12}{55} \end{aligned}$$

(d) $1.\overline{3}$

sol.

$$\begin{aligned} a_1 &= 0.3 \\ r &= 0.1 \\ S_\infty &= \frac{0.3}{1-0.1} \\ &= \frac{0.3}{0.9} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \therefore 1.\overline{3} &= 1 + \frac{1}{3} \\ &= \frac{4}{3} \end{aligned}$$

16. Three integers are in GP, summing up to 42 while accumulating up to 512, find these three integers.

sol.

$$\begin{aligned} a_1 + a_2 + a_3 &= 42 \\ a_1 a_2 a_3 &= 512 \\ a_2 &= \pm\sqrt{a_1 a_3} \\ a_1 a_3 &= a_2^2 \\ a_2^3 &= 512 \\ a_2 &= \sqrt[3]{512} \\ &= 8 \\ a_1 a_3 &= 64 \\ a_3 &= \frac{64}{a_1} \\ a_1 + 8 + \frac{64}{a_1} &= 42 \\ a_1 + \frac{64}{a_1} &= 34 \\ a_1^2 + 64 &= 34a_1 \\ a_1^2 - 34a_1 + 64 &= 0 \\ (a_1 - 32)(a_1 - 2) &= 0 \\ a_1 &= 32 \ or \ a_1 = 2 \end{aligned}$$

$$\therefore These\ three\ integers\ are\ 2,\ 8,\ 32$$

17. The sum of first 6 term of a GP is 9 times the sum of first 3 terms. Find the common ratio.

sol.

$$\begin{aligned}
S_6 &= 9S_3 \\
\frac{a(1-r^6)}{1-r} &= 9\times\frac{a(1-r^3)}{1-r} \\
a(1-r^6) &= 9a(1-r^3) \\
1-r^6 &= 9(1-r^3) \\
&= 9-9r^3 \\
r^6-9r^3+8 &= 0 \\
(r^3-8)(r^3-1) &= 0 \\
r^3 &= 8\ or\ r^3=1 \\
r &= 1\ (invalid) \\
r &= 2
\end{aligned}$$

18. Given a GP, its first term is 16, last term is $\frac{1}{2}$ and its sum is $31\frac{1}{2}$, find its common ratio and number of terms.

sol.

$$\begin{aligned}
a_1 &= 16 \\
\frac{1}{2} &= 16r^{n-1} \\
\frac{1}{32} &= r^{n-1} \\
&= r^n\times\frac{1}{r} \\
r^n &= \frac{r}{32} \\
\frac{63}{2} &= \frac{16(1-r^n)}{1-r} \\
63(1-r) &= 32(1-r^n) \\
63-63r &= 32-32r^n \\
-31 &= 32r^n-63r \\
-31 &= r-63r \\
-31 &= -62r \\
r &= \frac{1}{2} \\
(\frac{1}{2})^{n-1} &= \frac{1}{32} \\
&= (\frac{1}{2})^5 \\
n-1 &= 5 \\
n &= 6
\end{aligned}$$

19. Given a GP, its 3rd term is 6 less than its 2nd term, ant its 2nd term is 9 less than its 1st term. Find the 4th term and the sum of the first 4 terms.

sol.

$$\begin{aligned}
Let\ x &= a_2 \\
a_3 &= x-6 \\
a_1 &= x+9 \\
x &= \pm\sqrt{(x-6)(x+9)} \\
x^2 &= x^2+3x-54 \\
3x-54 &= 0 \\
x &= 18 \\
a_2 &= 18 \\
a_1 &= 27 \\
r &= \frac{12}{18} \\
&= \frac{2}{3} \\
a_4 &= 27\times(\frac{2}{3})^3 \\
&= 8 \\
S_4 &= \frac{27(1-(\frac{16}{3})^4)}{1-\frac{2}{3}} \\
&= \frac{27(1-\frac{8}{81})}{\frac{1}{3}} \\
&= 81\times\frac{65}{81} \\
&= 65
\end{aligned}$$

20. GIven an infinite GP, its common ratio is positive and the sum of it is 9. The sum of the first two terms is 5, find the 4th term.

sol.

$$\begin{aligned}
S_\infty &= \frac{a}{1-r} = 9 \\
a &= 9(1-r) \\
&= 9 - 9r \\
S_2 &= \frac{a(1-r^2)}{1-r} = 5 \\
a - ar^2 &= 5 - 5r \\
9 - 9r - (9-9r)r^2 &= 5 - 5r \\
9 - 9r - 9r^2 + 9r^3 &= 5 - 5r \\
4 - 4r - 9r^2 + 9r^3 &= 0 \\
4(1-r) - 9r^2(1-r) &= 0 \\
(4 - 9r^2)(1-r) &= 0 \\
(9r^2 - 4)(r-1) &= 0 \\
(3r^2+2)(3r^2-2)(r-1) &= 0 \\
r &= 1\ (invalid) \\
r &= -\frac{2}{3}(invalid) \\
r &= \frac{2}{3} \\
a &= 9(1-\frac{2}{3}) \\
&= 3 \\
a_4 &= 3(\frac{2}{3})^3 \\
&= 3 \times \frac{8}{27} \\
&= \frac{8}{9}
\end{aligned}$$

21. If $x+1, x-2, \frac{1}{2}x$ are the first three terms of an infinite GP, find:

   (a) The value of x

   sol.

$$\begin{aligned}
x - 2 &= \pm\sqrt{(x+1)(\frac{1}{2}x)} \\
x^2 - 4x + 4 &= \frac{1}{2}x(x+1) \\
2x^2 - 8x + 8 &= x^2 + x \\
x^2 - 9x + 8 &= 0 \\
(x-8)(x-1) &= 0 \\
x &= 8 \ or\ x = 1
\end{aligned}$$

   (b) The common ratio

   sol.

$$\begin{aligned}
When\ x &= 8, \\
r &= \frac{8-2}{8+1} \\
&= \frac{6}{9} \\
&= \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
When\ x &= 1, \\
r &= \frac{1-2}{1+1} \\
&= -\frac{1}{2}
\end{aligned}$$

   (c) The sum of the GP

   sol.

$$\begin{aligned}
When\ x &= 8, \\
S_\infty &= \frac{a}{1-r} \\
&= \frac{9}{1-\frac{2}{3}} \\
&= 9 \times 3 \\
&= 27
\end{aligned}$$

$$\begin{aligned}
When\ x &= 1, \\
S_\infty &= \frac{a}{1-r} \\
&= \frac{2}{1+\frac{1}{2}} \\
&= 2 \times \frac{2}{3} \\
&= \frac{4}{3}
\end{aligned}$$

12.4 Simple Summation of Special Series

Sum formula of natural number:

$$\sum_{i=1}^{n} k = \frac{n(n+1)}{2}$$

Sum formula of square of natural number:

$$\sum_{i=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum formula of cube of natural number:

$$\sum_{i=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

12.4.1 Practice 10

1. Find the sum of the following series.

(a) $\sum_{k=1}^{8} 3k$

sol.

$$\begin{aligned}\sum_{k=1}^{8} 3k = 3\sum_{k=1}^{8} k &= 3 \times \frac{8(8+1)}{2}\\ &= 3 \times \frac{8 \times 9}{2}\\ &= 3 \times \frac{72}{2}\\ &= 3 \times 36\\ &= 108\end{aligned}$$

(b) $\sum_{k=1}^{12} k^2$

sol.

$$\begin{aligned}\sum_{k=1}^{12} k^2 &= \frac{12(12+1)(2 \times 12+1)}{6}\\ &= \frac{12 \times 13 \times 25}{6}\\ &= 650\end{aligned}$$

(c) $\sum_{k=3}^{10}(2k-3)$

sol.

$$\begin{aligned}&\sum_{k=3}^{10}(2k-3)\\ &= 2\sum_{k=3}^{10} k - \sum_{k=3}^{10} 3\\ &= 2\left[\sum_{k=1}^{10} k - \sum_{k=1}^{2} k\right] - (30-6)\\ &= 2\left[\frac{10(10+1)}{2} - \frac{2(2+1)}{2}\right] - 8\\ &= 2(55-3) - 24\\ &= 2 \times 52 - 24\\ &= 104 - 24\\ &= 80\end{aligned}$$

(d) $\sum_{k=7}^{13} 3k^2$

sol.

$$\begin{aligned}&\sum_{k=7}^{13} 3k^2\\ &= 3\left[\sum_{k=1}^{13} k^2 - \sum_{k=1}^{6} k^2\right]\\ &= 3 \times \left[\frac{13(13+1)(2 \times 13+1)}{6}\right.\\ &\quad \left. - \frac{6(6+1)(2 \times 6+1)}{6}\right]\\ &= 3 \times \left[\frac{13 \times 14 \times 27}{6} - \frac{6 \times 7 \times 13}{6}\right]\\ &= 3 \times \left[\frac{4914}{6} - \frac{546}{6}\right]\\ &= 3 \times \frac{4368}{6}\\ &= 3 \times 728\\ &= 2184\end{aligned}$$

2. Given that the nth term of a series is n (n+3), find the sum of the first 20 terms of the series.

sol.

$$\begin{aligned}&\sum_{k=1}^{20} k(k+3)\\ &= \sum_{k=1}^{20} k^2 + 3k\\ &= \sum_{k=1}^{20} k^2 + 3\sum_{k=1}^{20} k\\ &= \frac{20(20+1)(2 \times 20+1)}{6} + 3 \times \frac{20(20+1)}{2}\\ &= \frac{20 \times 21 \times 41}{6} + 3 \times \frac{20 \times 21}{2}\\ &= 2870 + 630\\ &= 3500\end{aligned}$$

3. Find the sum of series $1{\times}3+2{\times}4+3{\times}5+\cdots+n(n+2)$.

sol.

$$\begin{aligned}
&\sum_{k=1}^{n} k(k+2) \\
&= \sum_{k=1}^{n} k^2 + 2k \\
&= \sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k \\
&= \frac{n(n+1)(2n+1)}{6} + 2 \times \frac{n(n+1)}{2} \\
&= \frac{n(n+1)(2n+1)}{6} + n(n+1) \\
&= \frac{n(n+1)(2n+1) + 6n(n+1)}{6} \\
&= \frac{n(n+1)(2n+7)}{6}
\end{aligned}$$

12.4.2 Exercise 12.4

1. Find the sum of the following series.

(a) $\sum k = 1^8 5k^2$

sol.

$$\begin{aligned}
\sum_{k=1}^{8} 5k^2 = 5\sum_{k=1}^{8} k^2 &= 5 \times \frac{8(8+1)(2\times 8+1)}{6} \\
&= 5 \times \frac{8 \times 9 \times 17}{6} \\
&= 5 \times \frac{1368}{6} \\
&= 5 \times 204 \\
&= 1020
\end{aligned}$$

(b) $\sum_{k=1}^{9} k^3$

sol.

$$\begin{aligned}
\sum_{k=1}^{9} k^3 &= \left[\frac{9(9+1)}{2}\right]^2 \\
&= 45^2 \\
&= 2025
\end{aligned}$$

(c) $\sum_{n=1}^{10}(3n-5)$

sol.

$$\begin{aligned}
\sum_{n=1}^{10}(3n-5) = 3\sum_{n=1}^{10} n - 5\sum_{n=1}^{10} 1 &= 3 \times \frac{10(10+1)}{2} - 5 \times 10 \\
&= 3 \times \frac{10 \times 11}{2} - 5 \times 10 \\
&= 3 \times 55 - 50 \\
&= 3 \times 5 - 50 \\
&= 165 - 50 \\
&= 115
\end{aligned}$$

(d) $\sum_{k=3}^{6} 2k^3$

sol.

$$\begin{aligned}
\sum_{k=3}^{6} 2k^3 = 2\sum_{k=3}^{6} k^3 &= 2\left(\sum_{k=1}^{6} k^3 - \sum_{k=1}^{2} k^3\right) \\
&= 2\left\{\left[\frac{6(6+1)}{2}\right]^2 \right. \\
&\quad \left. - \left[\frac{2(2+1)}{2}\right]^2\right\} \\
&= 2(21^2 - 3^2) \\
&= 2(441 - 9) \\
&= 2 \times 432 \\
&= 864
\end{aligned}$$

(e) $\sum_{k=6}^{10}(2k^2+3)$

sol.

$$\sum_{k=6}^{10}(2k^2+3)$$
$$= 2\sum_{k=6}^{10}k^2 + 3\sum_{k=6}^{10}1$$
$$= 2\left(\sum_{k=1}^{10}k^2 - \sum_{k=1}^{5}k^2\right)$$
$$+ 3\times(10-5)$$
$$= 2\times\left[\frac{10\times11\times21}{6} - \frac{5\times6\times11}{6}\right]$$
$$+ 3\times5$$
$$= 2\times\left[\frac{2310}{6} - \frac{330}{6}\right] + 3\times5$$
$$= 2\times\frac{1980}{6} + 3\times5$$
$$= 2\times330 + 3\times5$$
$$= 660 + 15$$
$$= 675$$

(f) $\sum_{n=11}^{15}(n^2+2n)$

sol.

$$\sum_{n=11}^{15}(n^2+2n)$$
$$= \sum_{n=11}^{15}n^2 + 2\sum_{n=11}^{15}n$$
$$= \left[\sum_{n=1}^{15}n^2 - \sum_{n=1}^{10}n^2\right]$$
$$+ 2\left[\sum_{n=1}^{15}n - \sum_{n=1}^{10}n\right]$$
$$= \left[\frac{15\times16\times31}{6} - \frac{10\times11\times21}{6}\right]$$
$$+ 2\left[\frac{15\times16}{2} - \frac{10\times11}{2}\right]$$
$$= 985$$

(g) $\sum_{n=2}^{6} n(n^2-n+1)$

sol.

$$\sum_{n=2}^{6} n(n^2-n+1)$$
$$= \sum_{n=2}^{6}n^3 - \sum_{n=2}^{6}n^2 + \sum_{n=2}^{6}n$$
$$= \left[\sum_{n=1}^{6}n^3 - \sum_{n=1}^{1}n^3\right] - \left[\sum_{n=1}^{6}n^2 - \sum_{n=1}^{1}n^2\right]$$
$$+ \left[\sum_{n=1}^{6}n - \sum_{n=1}^{1}n\right]$$
$$= \left[\left(\frac{6\times7}{2}\right)^2 - \left(\frac{1\times2}{2}\right)^2\right]$$
$$- \left(\frac{6\times7\times13}{6} - \frac{1\times2\times3}{6}\right)$$
$$+ \left(\frac{6\times7}{2} - \frac{1\times2}{2}\right)$$
$$= 21^2 - 1^2 - (7\times13-1) + (3\times7-1)$$
$$= 440 - 90 + 20$$
$$= 370$$

2. Fiven that the nth term of a series is $3n^2+n$, find the sum of the first 10 terms of the series.

sol.

$$\sum_{n=1}^{10}3n^2+n = 3\sum_{n=1}^{10}n^2 + \sum_{n=1}^{10}n$$
$$= 3\left(\frac{10\times11\times21}{6}\right) + \left(\frac{10\times11}{2}\right)$$
$$= 3\times\frac{2310}{6} + \frac{110}{2}$$
$$= 3\times385 + 55$$
$$= 1210$$

3. Find the sum of first nth term of series $1\times3+2\times7+3\times11+\cdots$

sol.

$$\begin{aligned}
&\sum_{n=1}^{n} n \times (4n-1)\\
&= 4\sum_{n=1}^{n} n^2 - \sum_{n=1}^{n} n\\
&= 4\left(\frac{n(n+1)(2n+1)}{6}\right) - \left(\frac{n(n+1)}{2}\right)\\
&= \frac{4n(n+1)(2n+1) - 3n(n+1)}{6}\\
&= \frac{n(n+1)(8n+1)}{6}
\end{aligned}$$

4. Find the sum fo the series $1^2 + 3^2 + 5^2 + \cdots + 15^2$

sol.

$$\begin{aligned}
\sum_{n=1}^{8}(2n-1)^2 &= \sum_{n=1}^{8}(4n^2-4n+1)\\
&= 4\sum_{n=1}^{8} n^2 - 4\sum_{n=1}^{8} n + \sum_{n=1}^{8} 1\\
&= 4\left(\frac{8\times 9\times 17}{6}\right) - 4\left(\frac{8\times 9}{2}\right) + 8\\
&= 4\times 204 - 4\times 36 + 8\\
&= 816 - 144 + 8\\
&= 680
\end{aligned}$$

12.5 Revision Exerise 12

1. Express the following series in form of $\sum$.

(a) $\frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \cdots + \frac{49}{50}$

sol.

$$\begin{aligned}
a_1 &= \frac{2\times 1 - 1}{2\times 1}\\
a_2 &= \frac{2\times 2 - 1}{2\times 2}\\
a_3 &= \frac{2\times 3 - 1}{2\times 3}\\
&\vdots\\
a_{25} &= \frac{2\times 25 - 1}{2\times 25}\\
\therefore \frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \cdots + \frac{49}{50} &= \sum_{n=1}^{25}\frac{2n-1}{2n}
\end{aligned}$$

(b) $6 - 7 + 8 - 9 + \cdots$

sol.

$$\begin{aligned}
a_1 &= (-1)^6 \times 6\\
a_2 &= (-1)^7 \times 7\\
a_3 &= (-1)^8 \times 8\\
&\vdots\\
a_n &= (-1)^n n \therefore \quad 6 - 7 + 8 - 9 + \cdots = \sum_{n=1}^{\infty}(-1)^n n
\end{aligned}$$

(c) $2\times 5 + 3\times 7 + 4\times 9 + \cdots + 15\times 31$

sol.

$$\begin{aligned}
a_1 &= (1+1)(2\times 1 + 3)\\
a_2 &= (2+1)(2\times 2 + 3)\\
a_3 &= (3+1)(2\times 3 + 3)\\
&\vdots\\
a_{14} &= (14+1)(2\times 14 + 3)\\
&\therefore 2\times 5 + 3\times 7 + 4\times 9 + \cdots + 15\times 31\\
&= \sum_{n=1}^{14}(n+1)(2n+3)
\end{aligned}$$

2. Given a general formula $a_n = \frac{3^n}{2n-3}$, state the first 5 terms of the sequence.

sol.

$$\begin{aligned}
a_1 &= \frac{3^1}{2\times 1 - 3} = -3\\
a_2 &= \frac{3^2}{2\times 2 - 3} = 9\\
a_3 &= \frac{3^3}{2\times 3 - 3} = 9\\
a_4 &= \frac{3^4}{2\times 4 - 3} = \frac{81}{5}\\
a_5 &= \frac{3^5}{2\times 5 - 3} = \frac{243}{7}
\end{aligned}$$

3. Express the series $\sum_{k=1}^{10}(2k^2-3)$

sol.

$$\begin{aligned}
&\sum_{k=1}^{10}(2k^2-3)\\
&=(2\times 1^2-3)+(2\times 2^2-3)+(2\times 3^2-3)\\
&\quad+(2\times 4^2-3)+(2\times 5^2-3)+(2\times 6^2-3)\\
&\quad+(2\times 7^2-3)+(2\times 8^2-3)+(2\times 9^2-3)\\
&\quad+(2\times 10^2-3)\\
&=-1+5+15+29+47+69+95+125\\
&\quad+159+197
\end{aligned}$$

4. State the first term, last term and the number of terms of theh series $\sum_{k=3}^{7}(3^k-2^k-k)$

   sol.

$$\begin{aligned}
a_3 &= 3^3-2^3-3=27-8-3=16\\
a_7 &= 3^7-2^7-7=2187-128-7=2052\\
n &= 5
\end{aligned}$$

5. Find the number of terms of the AP $-4-2\frac{3}{4}-112-\frac{1}{4}+\cdots+16$

   sol.

$$\begin{aligned}
a &= -4\\
d &= \frac{5}{4}\\
16 &= -4+(n-1)\frac{5}{4}\\
20 &= \frac{5}{4}(n-1)\\
5n-5 &= 80\\
5n &= 85\\
n &= 17
\end{aligned}$$

6. If x+1, 2x+1, x-3 are the first 3 terms of AP, find:

   (a) The value of x

   sol.

$$\begin{aligned}
2x+1 &= \frac{x+1+x-3}{2}\\
4x+2 &= 2x-2\\
2x &= -4\\
x &= -2
\end{aligned}$$

   (b) Sum from the 10th term to the 20th term

   sol.

$$\begin{aligned}
a_1 &= -1\\
a_2 &= -3\\
r &= -2\\
S &= S_{20}-S_9\\
&= \frac{20}{2}(-2+(20-1)(-2))\\
&\quad-\frac{9}{2}(-2+(9-1)(-2))\\
&= 10\times(-40)-9\times(-9)\\
&= -400+81\\
&= -319
\end{aligned}$$

7. Find 4 numbers between 28 and -12 such that these 6 numbers form an AP.

   sol.

$$\begin{aligned}
a_1 &= 28\\
a_n &= -12\\
n &= 6\\
-12 &= 28+5d\\
5d &= 40\\
d &= 8
\end{aligned}$$

$$\therefore These\ 4\ numbers\ are\ -4,4,12,20$$

8. Find the sum of the following AP.

   (a) $7+11+15+\cdots$ up to the 10th term

   sol.

$$\begin{aligned}
a_1 &= 7\\
d &= 4\\
n &= 10\\
S_{10} &= \frac{10}{2}(2\times 7+(10-1)4)\\
&= 5(14+36)\\
&= 250
\end{aligned}$$

   (b) $20+18\frac{1}{2}+17+\cdots$ up to the 16tm term

sol.

$$
\begin{aligned}
a_1 &= 20 \\
d &= -\frac{3}{2} \\
n &= 16 \\
S_{16} &= \frac{16}{2}(2\times 20 + (16-1)(-\frac{3}{2})) \\
&= 8(40 - \frac{45}{2}) \\
&= 8 \times \frac{35}{2} \\
&= 140
\end{aligned}
$$

(c) $2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} + \cdots + 13\sqrt{2}$

sol.

$$
\begin{aligned}
a_1 &= 2\sqrt{2} \\
d &= \sqrt{2} \\
n &= 12 \\
S_{12} &= \frac{12}{2}(2\times 2\sqrt{2} + (12-1)\sqrt{2}) \\
&= 6(4\sqrt{2} + 11\sqrt{2}) \\
&= 6 \times 15\sqrt{2} \\
&= 90\sqrt{2}
\end{aligned}
$$

9. Given an AP which the sum of the first n terms $S_n = n(1+2n)$, find:

(a) First term

sol.

$$
\begin{aligned}
\frac{n}{2}(2a + (n-1)d) &= n(1+2n) \\
n(2a + (n-1d)) &= 2n(1+2n) \\
2an + dn^2 - dn &= 2n - 4n^2 \\
(2a-d)n + dn^2 &= 2n - 4n^2
\end{aligned}
$$

$$
\begin{aligned}
&Comparing\ both\ sides, \\
&a = 3 \\
&d = 4
\end{aligned}
$$

(b) Common Difference

sol.

$$
\begin{aligned}
&According\ to\ the\ sol.\ of\ (a), \\
&d = 4
\end{aligned}
$$

(c) Sum of the first 20 terms.

sol.

$$
\begin{aligned}
&According\ to\ the\ sol.\ of\ (a), \\
a &= 3 \\
d &= 4 \\
n &= 20 \\
S_{20} &= \frac{20}{2}(2\times 3 + (20-1)4) \\
&= 10(6+76) \\
&= 10 \times 82 \\
&= 820
\end{aligned}
$$

10. Given an AP $33 + 27 + 21 + \cdots$

(a) If the first sum of the first n terms is 105, find the value of n.

sol.

$$
\begin{aligned}
a_1 &= 33 \\
d &= -6 \\
105 &= \frac{n}{2}(2\times 33 + (n-1)\times(-6)) \\
210 &= n(66 - (n-1)6) \\
35 &= 11n - n^2 + n \\
n^2 - 12n + 35 &= 0 \\
(n-7)(n-5) &= 0 \\
n &= 7\ or\ n = 5
\end{aligned}
$$

(b) If the sum of the first n terms is negative value, find the minimum value of n.

sol.

$$
\begin{aligned}
a_1 &= 33 \\
d &= -6 \\
\frac{n}{2}(2\times 33 + (n-1)\times(-6)) &< 0 \\
n(66 - 6n + 6) &< 0 \\
12n - n^2 &< 0 \\
n(12-n) &< 0 \\
n &> 12
\end{aligned}
$$

$$\therefore The\ minimum\ value\ of\ n\ is\ 13$$

11. Find the sum of the numbers between 150 and 300 that are multiple of both 5 and 3.

sol.

$$\begin{aligned} a_1 &= 165 \\ a_n &= 285 \\ d &= 15 \\ 285 &= 165 + (n-1) \times 15 \\ 8 &= n - 1 \\ n &= 9 \end{aligned}$$

$$\begin{aligned} S_9 &= \frac{9}{2}(2 \times 165 + (9-1) \times 15) \\ &= \frac{9}{2} \times 450 \\ &= 2025 \end{aligned}$$

12. Find the sum of all the numbers between 100 and 200 that can be divided by 2 or 3.

sol.

$$\begin{aligned} a_1 &= 102 \\ a_n &= 198 \end{aligned}$$

When $d = 2$,

$$\begin{aligned} 198 &= 102 + (n-1) \times 2 \\ 48 &= n - 1 \\ n &= 49 \\ S_{49} &= \frac{49}{2}(2 \times 102 + (49-1) \times 2) \\ &= \frac{49}{2} \times (204 + 96) \\ &= 7350 \end{aligned}$$

When $d = 3$,

$$\begin{aligned} 198 &= 102 + (n-1) \times 3 \\ 32 &= n - 1 \\ n &= 33 \\ S_{33} &= \frac{33}{2}(2 \times 102 + (33-1) \times 3) \\ &= \frac{33}{2} \times (204 + 96) \\ &= 4950 \end{aligned}$$

When $d = 6$,

$$\begin{aligned} 198 &= 102 + (n-1) \times 6 \\ 16 &= n - 1 \\ n &= 17 \\ S_{17} &= \frac{17}{2}(2 \times 102 + (17-1) \times 6) \\ &= \frac{17}{2} \times (204 + 96) \\ &= 2550 \end{aligned}$$

$$\begin{aligned} \therefore S &= 7350 + 4950 - 2550 \\ &= 9750 \end{aligned}$$

13. Find the sum of the numbers between 50 and 100 that cannot be divided by 5.

sol.

When $d = 1$,

$$\begin{aligned}
a_1 &= 51\\
a_n &= 99\\
99 &= 51 + (n-1)\times 1\\
48 &= n-1\\
n &= 49\\
S_{49} &= \frac{49}{2}(2\times 51 + (49-1)\times 1)\\
&= \frac{49}{2}\times(102+48)\\
&= 3675
\end{aligned}$$

When $d = 5$,

$$\begin{aligned}
a_1 &= 55\\
a_n &= 95\\
95 &= 55 + (n-1)\times 5\\
8 &= n-1\\
n &= 9\\
S_9 &= \frac{9}{2}(2\times 55 + (9-1)\times 5)\\
&= \frac{9}{2}\times(110+40)\\
&= 675
\end{aligned}$$

$$\begin{aligned}
\therefore S &= 3675 - 675\\
&= 3000
\end{aligned}$$

14. Which term is the first negative term of the AP $20 + 16\frac{1}{4} + 12\frac{1}{2} + \cdots$?

sol.

$$\begin{aligned}
a_1 &= 20\\
d &= -\frac{15}{4}\\
a_n = 20 - (n-1)\times\frac{15}{4} &< 0\\
80 - 15(n-1) &< 0\\
16 - 3n + 3 &< 0\\
3n &> 19\\
n &> 6\frac{1}{3}
\end{aligned}$$

$$\therefore The\ first\ negative\ term\ is\ 7$$

15. Three numbers are in AP, thier sum is 15 while the sum of the square of these numbers is 83. Find this three numbers.

sol.

$$\begin{aligned}
a_1 + a_2 + a_3 &= 15\\
a_1^2 + a_2^2 + a_3^2 &= 83\\
a_2 - a_1 &= a_3 - a_2\\
a_1 + a_3 &= 2a_2\\
3a_2 &= 15\\
a_2 &= 5\\
a_3 &= 10 - a_1\\
a_1^2 + a_3^2 &= 83 - 25\\
&= 58\\
a_1^2 + (10-a_1)^2 &= 58\\
a_1^2 + 100 - 20a_1 + a_1^2 &= 58\\
2a_1^2 - 20a_1 + 100 &= 58\\
2a_1^2 - 20a_1 + 42 &= 0\\
a_1^2 - 10a_1 + 21 &= 0\\
(a_1 - 7)(a_1 - 3) &= 0\\
a_1 &= 7\ or\ a_1 = 3
\end{aligned}$$

$$\therefore The\ three\ numbers\ are\ 7, 5, 3$$

16. Find the sum of the series $18^2 - 17^2 + 16^2 - 15^2 + 14^2 - 13^2 + \cdots + 2^2 - 1^2$

sol.

$$\begin{aligned}
&18^2 - 17^2 + 16^2 - 15^2 + \cdots + 2^2 - 1^2\\
&= (18^2 - 17^2) + (16^2 - 15^2) + \cdots + (2^2 - 1^2)\\
&= ((2\times 9)^2 - (2\times 9 - 1)^2) + ((2\times 8)^2 - (2\times 8 - 1)^2)\\
&\quad + \cdots + ((2\times 1)^2 - (2\times 1 - 1)^2)\\
&= \sum_{n=1}^{9}\left[(2n)^2 - (2n-1)^2\right]\\
&= \sum_{n=1}^{9}(4n-1)\\
&= 4\sum_{n=1}^{9} n - \sum_{n=1}^{9} 1\\
&= 4\times\frac{9\times 10}{2} - 9\\
&= 180 - 9\\
&= 171
\end{aligned}$$

17. State the general formula of the series $20, -10, 5, -2\frac{1}{2}, \cdots$

sol.

$$a_1 = 20$$
$$r = -\frac{1}{2}$$
$$a_n = 20(-\frac{1}{2})^{n-1}$$

18. Given three integers x-3, x+1, 4x-2 that are in GP. If the sum of this GP is S, common ratio is r, find the value of S+r.

sol.

$$x + 1 = \pm\sqrt{(x-3)(4x-2)}$$
$$x^2 + 2x + 1 = 4x^2 - 14x + 6$$
$$3x^2 - 16^x + 5 = 0$$
$$(3x-1)(x-5) = 0$$
$$x = 5 \text{ or } x = \frac{1}{3}$$

$$a_1 = x - 3 = 5 - 3 = 2$$
$$a_2 = x + 1 = 5 + 1 = 6$$
$$a_3 = 4x - 2 = 4(5) - 2 = 18$$

$$\begin{aligned} S &= a_1 + a_2 + a_3 \\ &= 2 + 6 + 18 \\ &= 26 \end{aligned}$$

$$r = \frac{a_3}{a_2} = \frac{18}{6} = 3$$

$$\begin{aligned} \therefore S + r &= 26 + 3 \\ &= 29 \end{aligned}$$

19. Find the geometric mean of $\frac{1}{3}$ and $\frac{1}{5}$

sol.

$$\begin{aligned} G &= \pm\sqrt{\frac{1}{3} \times \frac{1}{5}} \\ &= \pm\sqrt{\frac{1}{15}} \\ &= \pm\frac{1}{\sqrt{15}} \\ &= \pm\frac{\sqrt{15}}{15} \end{aligned}$$

20. Find 5 numbers between $-\frac{1}{4}$ and $-\frac{1}{256}$ such that these 7 numbers form a GP.

sol.

$$a_1 = -\frac{1}{4}$$
$$n = 7$$
$$-\frac{1}{256} = -\frac{1}{4}r^6$$
$$\frac{1}{64} = r^6$$
$$\left(\pm\frac{1}{2}\right)^6 = r^6$$
$$r = \pm\frac{1}{2}$$

When $r = \frac{1}{2}$,
These 5 numbers are
$\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$

When $r = -\frac{1}{2}$,
These 5 numbers are
$\frac{1}{8}, -\frac{1}{16}, \frac{1}{32}, -\frac{1}{64}, \frac{1}{128}$

21. Find the sum of the series $\sum_{n=5}^{15} n^2(3n+1)$

sol.

$$\begin{aligned}
\sum_{n=5}^{15} n^2(3n+1) &= \sum_{n=5}^{15} n^3 + \sum_{n=5}^{15} 3n^2 \\
&= 3\sum_{n=5}^{15} n^3 + \sum_{n=5}^{15} n^2 \\
&= 3\left[\sum_{n=1}^{15} n^3 - \sum_{n=1}^{4} n^3\right] \\
&\quad + \left[\sum_{n=1}^{15} n^2 - \sum_{n=1}^{4} n^2\right] \\
&= 3\left[\left(\frac{15\times 16}{2}\right)^2 - \left(\frac{4\times 5}{2}\right)^2\right] \\
&\quad + \left[\frac{15\times 16\times 31}{6} - \frac{4\times 5\times 9}{6}\right] \\
&= 3\left[(15\times 8)^2 - (2\times 5)^2\right] \\
&\quad + 1240 - 30 \\
&= 3(14400 - 100) + 1210 \\
&= 42900 + 1210 \\
&= 44110
\end{aligned}$$

22. Find the sum of the series $5^2 + 7^2 + 9^2 + \cdots + 25^2$

sol.

$$\begin{aligned}
&\sum_{n=1}^{11} (2n+3)^2 \\
&= \sum_{n=1}^{11} 4n^2 + 12n + 9 \\
&= 4\sum_{n=1}^{11} n^2 + 12\sum_{n=1}^{11} n + 11 \\
&= 4\left[\frac{11\times 12\times 23}{6}\right] + 12\left[\frac{11\times 12}{2}\right] + 99 \\
&= 2024 + 792 + 99 \\
&= 2915
\end{aligned}$$

23. Find the sum of the series $2\times 3 + 3\times 12 + 4\times 27 + \cdots + (n+1)\times 3n^2$

sol.

$$\begin{aligned}
&\sum_{n=1}^{n} (n+1)3n^2 \\
&= \sum_{n=1}^{n} 3n^3 + \sum_{n=1}^{n} 3n^2 \\
&= 3\left[\sum_{n=1}^{n} n^3 + \sum_{n=1}^{n} n^2\right] \\
&= 3\left[\left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6}\right] \\
&= 3\left[\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}\right] \\
&= 3\left[\frac{3n^2(n+1)^2 + 2n(n+1)(2n+1)}{12}\right] \\
&= \frac{n(n+1)\left[3n^2 + 3n + 4n + 2\right]}{4} \\
&= \frac{n(n+1)\left[3n^2 + 7n + 2\right]}{4} \\
&= \frac{n(n+1)(n+2)(3n+1)}{4}
\end{aligned}$$

Chapter 13

System of Equations

13.1 System of Equations with Two Variables

13.1.1 Practice 1

Solve the following system of equations.

1.

$$\begin{cases} 2x - 3y &= 11 \\ xy &= -5 \end{cases}$$

sol.

$$\begin{cases} 2x - 3y = 11 & (1) \\ xy = -5 & (2) \end{cases}$$

$$\begin{aligned} (2) \Rightarrow y &= -\frac{5}{x} \qquad (3) \\ \text{Sub (3) into (1)} \Rightarrow 2x - \frac{15}{x} &= 11 \\ 2x^2 - 15 &= 11x \\ 2x^2 - 11x - 15 &= 0 \\ (2x-5)(x-3) &= 0 \\ x = 3 \; or \; x &= \frac{5}{2} \\ \text{Sub x = 3 into (2)} \Rightarrow y &= -\frac{5}{3} \\ \text{Sub x} = \frac{5}{2} \text{ into (2)} \Rightarrow y &= -\frac{5}{\frac{5}{2}} \\ \Rightarrow y &= -\frac{5}{5} \\ \Rightarrow y &= -1 \end{aligned}$$

$$\therefore \begin{cases} x = 3 \\ y = -\frac{5}{3} \end{cases} or \begin{cases} x = \frac{5}{2} \\ y = -1 \end{cases}$$

2.

$$\begin{cases} 3x + y &= 5 \\ x^2 - 2xy &= 8 \end{cases}$$

sol.

$$\begin{cases} 3x + y = 5 & (1) \\ x^2 - 2xy = 8 & (2) \end{cases}$$

$$\begin{aligned} 3(1) \Rightarrow y &= 5 - 3x \qquad (3) \\ \text{Sub (3) into (2)} \Rightarrow x^2 - 2x(5-3x) &= 8 \\ x^2 - 10x + 6x^2 &= 8 \\ 7x^2 - 10x + 8 &= 0 \\ (7x+4)(x-2) &= 0 \\ x = -\frac{4}{7} \; or \; x &= 2 \\ \text{Sub } x = -\frac{4}{7} \text{ into (1)} \Rightarrow y &= 5 - 3\left(-\frac{4}{7}\right) \\ \Rightarrow y &= \frac{47}{7} \\ \text{Sub } x = 2 \text{ into (1)} \Rightarrow y &= -1 \end{aligned}$$

$$\therefore \begin{cases} x = -\frac{4}{7} \\ y = \frac{47}{7} \end{cases} or \begin{cases} x = 2 \\ y = -1 \end{cases}$$

13.1.2 Exercise 13.1

Solve the following system of equations.

1.

$$\begin{cases} x - y &= 1 \\ xy &= 6 \end{cases}$$

sol.

$$\begin{cases} x - y = 1 & (1) \\ xy = 6 & (2) \end{cases}$$

$$(1) \Rightarrow y = x - 1 \quad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow x(x-1) = 6$$

$$x^2 - x = 6$$

$$x^2 - x - 6 = 0$$

$$(x+2)(x-3) = 0$$

$$x = -2 \text{ } or \text{ } x = 3$$

$$\text{Sub } x = -2 \text{ into (1)} \Rightarrow y = -2 - 1$$

$$\Rightarrow y = -3$$

$$\text{Sub } x = 3 \text{ into (1)} \Rightarrow y = 3 - 1$$

$$\Rightarrow y = 2$$

$$\therefore \begin{cases} x = -2 \\ y = -3 \end{cases} or \begin{cases} x = 3 \\ y = 2 \end{cases}$$

2.

$$\begin{cases} 3x - y = 4 \\ xy = 4 \end{cases}$$

sol.

$$\begin{cases} 3x - y = 4 & (1) \\ xy = 4 & (2) \end{cases}$$

$$(1) \Rightarrow y = 3x - 4 \quad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow x(3x-4) = 4$$

$$3x^2 - 4x = 4$$

$$3x^2 - 4x - 4 = 0$$

$$(3x+2)(x-2) = 0$$

$$x = -\frac{2}{3} \text{ } or \text{ } x = 2$$

$$\text{Sub } x = -\frac{2}{3} \text{ into (1)} \Rightarrow y = 3\left(-\frac{2}{3}\right) - 4$$

$$\Rightarrow y = -6$$

$$\text{Sub } x = 2 \text{ into (1)} \Rightarrow y = 3(2) - 4$$

$$\Rightarrow y = 2$$

$$\therefore \begin{cases} x = -\frac{2}{3} \\ y = -6 \end{cases} or \begin{cases} x = 2 \\ y = 2 \end{cases}$$

3.

$$\begin{cases} 3x + 4y = -39 \\ xy = 30 \end{cases}$$

sol.

$$\begin{cases} 3x + 4y = -39 & (1) \\ xy = 30 & (2) \end{cases}$$

$$(2) \Rightarrow y = \frac{30}{x} \quad (3)$$

$$\text{Sub (3) into (1)} \Rightarrow 3x + 4\frac{30}{x} = -39$$

$$3x^2 + 120 = -39x$$

$$3x^2 + 39x + 120 = 0$$

$$x^2 + 13x + 40 = 0$$

$$(x+5)(x+8) = 0$$

$$x = -5 \text{ } or \text{ } x = -8$$

$$\text{Sub } x = -5 \text{ into (1)} \Rightarrow y = \frac{30}{-5} - 39$$

$$\Rightarrow y = -6$$

$$\text{Sub } x = -8 \text{ into (1)} \Rightarrow y = \frac{30}{-8} - 39$$

$$\Rightarrow y = -\frac{15}{4}$$

$$\therefore \begin{cases} x = -5 \\ y = -6 \end{cases} or \begin{cases} x = -8 \\ y = -\frac{15}{4} \end{cases}$$

4.

$$\begin{cases} y = 2x + 3 \\ y = x^2 - 2x + 1 \end{cases}$$

sol.

$$\begin{cases} y = 2x + 3 & (1) \\ y = x^2 & (2) \end{cases}$$

$$(1) = (2) \Rightarrow 2x + 3 = x^2$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow (x+1)(x-3) = 0$$

$$\Rightarrow x = -1 \text{ } or \text{ } x = 3$$

$$\text{Sub } x = -1 \text{ into (1)} \Rightarrow y = 2(-1) + 3$$

$$\Rightarrow y = 1$$

$$\text{Sub } x = 3 \text{ into (1)} \Rightarrow y = 2(3) + 3$$

$$\Rightarrow y = 9$$

$$\therefore \begin{cases} x = -1 \\ y = 1 \end{cases} or \begin{cases} x = 3 \\ y = 9 \end{cases}$$

5.

$$\begin{cases} x - y = 1 \\ x^2 + y^2 = 25 \end{cases}$$

sol.

$$\begin{cases} x - y = 1 & (1) \\ x^2 + y^2 = 25 & (2) \end{cases}$$

$$\begin{aligned}
(1) &\Rightarrow x = y + 1 && (3) \\
\text{Sub (3) into (2)} &\Rightarrow (y+1)^2 + y^2 = 25 \\
&\Rightarrow y^2 + 2y + 1 + y^2 = 25 \\
&\Rightarrow 2y^2 + 2y = 24 \\
&\Rightarrow y^2 + y = 12 \\
&\Rightarrow y^2 + y - 12 = 0 \\
&\Rightarrow (y+4)(y-3) = 0 \\
&\Rightarrow y = -4 \text{ or } y = 3 \\
\text{Sub } y = -4 \text{ into (1)} &\Rightarrow x = -4 + 1 \\
&\Rightarrow x = -3 \\
\text{Sub } y = 3 \text{ into (1)} &\Rightarrow x = 3 + 1 \\
&\Rightarrow x = 4
\end{aligned}$$

$$\therefore \begin{cases} x = -3 \\ y = -4 \end{cases} \text{ or } \begin{cases} x = 4 \\ y = 3 \end{cases}$$

6.

$$\begin{cases} 5x - y = 3 \\ y^2 - 6x^2 = 25 \end{cases}$$

sol.

$$\begin{cases} 5x - y = 3 & (1) \\ y^2 - 6x^2 = 25 & (2) \end{cases}$$

$$\begin{aligned}
(1) &\Rightarrow y = 5x - 3 && (3) \\
\text{Sub (3) into (2)} &\Rightarrow (5x-3)^2 - 6x^2 = 25 \\
&\Rightarrow 25x^2 - 30x + 9 \\
&\quad - 6x^2 = 25 \\
&\Rightarrow 19x^2 - 30x + 16 = 0 \\
&\Rightarrow (19x+8)(x-2) = 0 \\
&\Rightarrow x = -\frac{8}{19} \text{ or } x = 2 \\
\text{Sub } x = -\frac{8}{19} \text{ into (1)} &\Rightarrow y = 5(-\frac{8}{19}) - 3 \\
&\Rightarrow y = -\frac{97}{19} \\
\text{Sub } x = 2 \text{ into (1)} &\Rightarrow y = 7
\end{aligned}$$

$$\therefore \begin{cases} x = -\frac{8}{19} \\ y = -\frac{97}{19} \end{cases} \text{ or } \begin{cases} x = 2 \\ y = 7 \end{cases}$$

7.

$$\begin{cases} x + y = 3 \\ (x+2)(y+3) = 12 \end{cases}$$

sol.

$$\begin{cases} x + y = 3 & (1) \\ (x+2)(y+3) = 12 & (2) \end{cases}$$

$$\begin{aligned}
(1) &\Rightarrow x = 3 - y && (3) \\
\text{Sub (3) into (2)} &\Rightarrow (3 - y + 2)(y+3) = 12 \\
&\Rightarrow (5-y)(y+3) = 12 \\
&\Rightarrow 5y + 15 - y^2 - 3y = 12 \\
&\Rightarrow 2y - y^2 = -3 \\
&\Rightarrow y^2 - 2y - 3 = 0 \\
&\Rightarrow (y+1)(y-3) = 0 \\
&\Rightarrow y = -1 \text{ or } y = 3 \\
\text{Sub } y = -1 \text{ into (1)} &\Rightarrow x = 4 \\
\text{Sub } y = 3 \text{ into (1)} &\Rightarrow x = 0
\end{aligned}$$

$$\therefore \begin{cases} x = 4 \\ y = -1 \end{cases} \text{ or } \begin{cases} x = 0 \\ y = 3 \end{cases}$$

8.

$$\begin{cases} 5x - 6y = -1 \\ 25x^2 + 36y^2 = 61 \end{cases}$$

sol.

$$\begin{cases} 5x - 6y = -1 & (1) \\ 25x^2 + 36y^2 = 61 & (2) \end{cases}$$

$$(1) \Rightarrow y = \frac{5x+1}{6} \quad (3)$$

$$\begin{aligned}
\text{Sub (3) into (2)} &\Rightarrow 25x^2 + 36\left(\frac{5x+1}{6}\right)^2 \\
&= 61 \\
&\Rightarrow 25x^2 + 36\left(\frac{5x+1}{6}\right)^2 \\
&+ 36 = 61 \\
&\Rightarrow 25x^2 + 25x^2 + 10x \\
&+ 1 = 61 \\
&\Rightarrow 50x^2 + 10x = 60 \\
&\Rightarrow 5x^2 + x - 6 = 0 \\
&\Rightarrow (5x+6)(x-1) = 0 \\
&\Rightarrow x = -\frac{6}{5} \text{ or } x = 1 \\
\text{Sub } x = -\frac{6}{5} \text{ into (1)} &\Rightarrow y = \frac{5(-\frac{6}{5}) + 1}{6} \\
&\Rightarrow y = -\frac{5}{6} \\
\text{Sub } x = 1 \text{ into (1)} &\Rightarrow y = \frac{5(1) + 1}{6} \\
&\Rightarrow y = \frac{6}{6} \\
&\Rightarrow y = 1
\end{aligned}$$

$$\therefore \begin{cases} x = -\frac{6}{5} \\ y = -\frac{5}{6} \end{cases} or \begin{cases} x = 1 \\ y = 1 \end{cases}$$

9.

$$\begin{cases} x + 4y = 5 \\ 2x^2 + 21xy + 27y^2 = 0 \end{cases}$$

sol.

$$\begin{cases} x + 4y = 5 & (1) \\ 2x^2 + 21xy + 27y^2 = 0 & (2) \end{cases}$$

$$(1) \Rightarrow x = 5 - 4y \quad (3)$$

$$\begin{aligned}
\text{Sub (3) into (2)} &\Rightarrow 2(5-4y)^2 + 21(5-4y)y \\
&+ 27y^2 = 0 \\
&\Rightarrow 2\left(25 - 40y + 16y^2\right) \\
&+ 105y - 84y^2 + 27y^2 = 0 \\
&\Rightarrow 50 - 80y + 32y^2 + 105y \\
&- 57y^2 = 0 \\
&\Rightarrow 25y^2 - 25y - 50 = 0 \\
&\Rightarrow y^2 - y - 2 \\
&\Rightarrow (y+1)(y-2) = 0 \\
&\Rightarrow y = -1 \text{ or } y = 2 \\
\text{Sub } y = -1 \text{ into (1)} &\Rightarrow x = 5 - 4(-1) = 9 \\
\text{Sub } y = 2 \text{ into (1)} &\Rightarrow x = 5 - 4(2) = -3
\end{aligned}$$

$$\therefore \begin{cases} x = 9 \\ y = -1 \end{cases} or \begin{cases} x = -3 \\ y = 2 \end{cases}$$

10.

$$\begin{cases} \frac{x}{3} - \frac{y}{10} = \frac{5}{6} \\ x(y-2) = 2y + 3 \end{cases}$$

sol.

$$\begin{cases} \frac{x}{3} - \frac{y}{10} = \frac{5}{6} & (1) \\ x(y-2) = 2y + 3 & (2) \end{cases}$$

$$
\begin{aligned}
(1) &\Rightarrow 10x - 3y = 25 && (3)\\
(2) &\Rightarrow x = \frac{2y+3}{y-2} && (4)\\
\text{Sub (4) int 3} &\Rightarrow 10\left(\frac{2y+3}{y-2}\right) - 3y = 25\\
&\Rightarrow 10(2y+3) - 3y(y-2)\\
&\quad = 25(y-2)\\
&\Rightarrow 20y + 30 - 3y^2 + 6y\\
&\quad = 25y - 50\\
&\Rightarrow 3y^2 - y - 80 = 0\\
&\Rightarrow (y+5)(3y-16) = 0\\
&\Rightarrow y = -5 \text{ or } y = \frac{16}{3}\\
\text{Sub } y = -5 \text{ into (1)} &\Rightarrow 10x - 3(-5) = 25\\
&\Rightarrow 10x + 15 = 25\\
&\Rightarrow 10x = 10\\
&\Rightarrow x = 1\\
\text{Sub } y = \frac{16}{3} \text{ into (1)} &\Rightarrow 10x - 3\left(\frac{16}{3}\right) = 25\\
&\Rightarrow 10x = 41\\
&\Rightarrow x = \frac{41}{10}
\end{aligned}
$$

$$
\therefore \begin{cases} x = 1 \\ y = -5 \end{cases} \text{ or } \begin{cases} x = \frac{41}{10} \\ y = \frac{16}{3} \end{cases}
$$

13.2 System of Equations with Three Variables

13.2.1 Practice 2

Solve the system of equation

$$
\begin{cases} x + 2y - z = -5 \\ 2x - y + z = 6 \\ x - y - 3z = -3 \end{cases}
$$

sol.

$$
\begin{cases} x + 2y - z = -5 & (1) \\ 2x - y + z = 6 & (2) \\ x - y - 3z = -3 & (3) \end{cases}
$$

$$
\begin{aligned}
(1) \times 3 &\Rightarrow 3x + 6y - 3z = -15 && (4)\\
(2) \times 3 &\Rightarrow 6x - 3y + 3z = 18 && (5)\\
(3) + (5) &\Rightarrow 7x - 4y = 15 && (6)\\
(4) + (5) &\Rightarrow 9x + 3y = 3 && (7)\\
(6) \times 3 &\Rightarrow 21x - 12y = 45 && (8)\\
(7) \times 4 &\Rightarrow 36x + 12y = 12 && (9)\\
(8) + (9) &\Rightarrow 57x = 57 && (10)\\
&\Rightarrow x = 1\\
\text{Sub } x = 1 \text{ into (7)} &\Rightarrow -4y = 8\\
&\Rightarrow y = -2\\
\text{Sub } y = -2 \text{ and } x = 1 \text{ into (1)} &\Rightarrow -z = -2\\
&\Rightarrow z = 2\\
\therefore x = 1, y = -2, z = 2
\end{aligned}
$$

13.2.2 Exercise 13.2

Solve the following system of equations.

1.

$$
\begin{cases} x + y - z = 1 \\ 2x - 3y + z = 0 \\ 2x + y + 2z = 5 \end{cases}
$$

sol.

$$
\begin{cases} x + y - z = 1 & (1) \\ 2x - 3y + z = 0 & (2) \\ 2x + y + 2z = 5 & (3) \end{cases}
$$

$$
\begin{aligned}
(1) \times 2 &\Rightarrow 2x + 2y - 2z = 2 && (4)\\
(4) - (3) &\Rightarrow y - 4z = -3 && (5)\\
(3) - (2) &\Rightarrow 4y + z = 5 && (6)\\
(5) \times 4 &\Rightarrow 4y - 16z = -12 && (7)\\
(6) - (7) &\Rightarrow 17z = 17\\
&\Rightarrow z = 1\\
\text{Sub } z = 1 \text{ into (5)} &\Rightarrow y = 1\\
\text{Sub } y = 1 \text{ and } z = 1 \text{ into (1)} &\Rightarrow z = 1
\end{aligned}
$$

$$
\therefore x = 1, y = 1, z = 1
$$

2.

$$
\begin{cases} x - 2y = 5 \\ 2x + y - 3z = 8 \\ x + 4y - z = 0 \end{cases}
$$

sol.

$$\begin{cases} x - 2y = 5 & (1) \\ 2x + y - 3z = 8 & (2) \\ x + 4y - z = 0 & (3) \end{cases}$$

$$\begin{aligned}
(3) \times 3 &\Rightarrow 3x + 12y - 3z = 0 \quad (4) \\
(4) - (2) &\Rightarrow x + 11y = -8 \quad (5) \\
(5) - (1) &\Rightarrow 13y = -13 \\
&\Rightarrow y = -1 \\
\text{Sub } y = -1 \text{ into } (1) &\Rightarrow x + 2 = 5 \\
&\Rightarrow x = 3 \\
\text{Sub } x = 3 & \\
\text{and } y = -1 \text{ into } (2) &\Rightarrow -3z = 3 \\
&\Rightarrow z = -1
\end{aligned}$$

$\therefore x = 3, y = -1, z = -1$

3.

$$\begin{cases} x + y = z - 5 \\ y + z = x - 3 \\ z + x = y + 1 \end{cases}$$

sol.

$$\begin{cases} x + y = z - 5 & (1) \\ y + z = x - 3 & (2) \\ z + x = y + 1 & (3) \end{cases}$$

$$\begin{aligned}
(1) &\Rightarrow x + y - z = -5 \quad (4) \\
(2) &\Rightarrow -x + y + z = -3 \quad (5) \\
(3) &\Rightarrow x - y + z = 1 \quad (6) \\
(4) + (5) &\Rightarrow 2y = -8 \\
&\Rightarrow y = -4 \\
(5) + (6) &\Rightarrow 2z = -2 \\
&\Rightarrow z = -1 \\
\text{Sub } y = -4 & \\
\text{and } z = -1 \text{ into } (2) &\Rightarrow x - 3 = -5 \\
&\Rightarrow x = -2
\end{aligned}$$

$\therefore x = -2, y = -4, z = -1$

4.

$$\begin{cases} x + 4y + 2z = 4 \\ 2x - 2y + z = 4 \\ x - 2y + 3z = 3 \end{cases}$$

sol.

$$\begin{cases} x + 4y + 2z = 4 & (1) \\ 2x - 2y + z = 4 & (2) \\ x - 2y + 3z = 3 & (3) \end{cases}$$

$$\begin{aligned}
(1) \times 2 &\Rightarrow 2x + 8y + 4z = 8 \quad (4) \\
(3) \times 2 &\Rightarrow 2x - 4y + 6z = 6 \quad (5) \\
(4) - (2) &\Rightarrow 10y + 3z = 4 \quad (6) \\
(5) - (4) &\Rightarrow -12y + 2z = -2 \quad (7) \\
(6) \times 2 &\Rightarrow 20y + 6z = 8 \quad (8) \\
(7) \times 3 &\Rightarrow -36y + 6z = -6 \quad (9) \\
(8) - (9) &\Rightarrow 56y = 14 \\
&\Rightarrow y = \frac{1}{4} \\
\text{Sub } y = \frac{1}{4} \text{ into } (6) &\Rightarrow 6z = 3 \\
&\Rightarrow z = \frac{1}{2} \\
\text{Sub } y = \frac{1}{4} \text{ and } z = \frac{1}{2} \text{ into } (1) &\Rightarrow x + 1 + 1 = 4 \\
&\Rightarrow x = 2
\end{aligned}$$

$\therefore x = 2, y = \frac{1}{4}, z = \frac{1}{2}$

5.

$$\begin{cases} x - y - z = 0 \\ 3x + 2y = 13 \\ y - 3z = -1 \end{cases}$$

sol.

$$\begin{cases} x - y - z = 0 & (1) \\ 3x + 2y = 13 & (2) \\ y - 3z = -1 & (3) \end{cases}$$

$$\begin{aligned}
(3) &\Rightarrow y = 3z - 1 && (4)\\
\text{Sub (4) into (1)} &\Rightarrow x - (3z-1) - z = 0\\
&\Rightarrow x - 4z = -1 && (5)\\
\text{Sub (4) into (2)} &\Rightarrow 3x + 2(3z-1) = 13\\
&\Rightarrow 3x + 6z = 15 && (6)\\
(5)\times 3 &\Rightarrow 3x - 12z = -3 && (7)\\
(6)-(7) &\Rightarrow 18z = 18\\
&\Rightarrow z = 1\\
\text{Sub } z=1 \text{ into (4)} &\Rightarrow y = 2\\
\text{Sub } z=1 \text{ into (5)} &\Rightarrow x - 4 = -1\\
&\Rightarrow x = 3
\end{aligned}$$

$\therefore x = 3, y = 2, z = 1$

6.

$$\begin{cases} 2x + 2y - z = -1\\ x + 3y + z = -8\\ 3x - 2y + 3z = 9 \end{cases}$$

sol.

$$\begin{cases} 2x + 2y - z = -1 & (1)\\ x + 3y + z = -8 & (2)\\ 3x - 2y + 3z = 9 & (3) \end{cases}$$

$$\begin{aligned}
(1)\times 3 &\Rightarrow 6x + 6y - 3z = -3 && (4)\\
(2)\times 3 &\Rightarrow 3x + 9y + 3z = -24 && (5)\\
(3)+(4) &\Rightarrow 9x + 4y = 6 && (6)\\
(4)+(5) &\Rightarrow 9x + 15y = -27 && (7)\\
(7)-(6) &\Rightarrow 11y = -33\\
&\Rightarrow y = -3\\
\text{Sub } y=-3 \text{ into (6)} &\Rightarrow 9x = 18\\
&\Rightarrow x = 2\\
\text{Sub } x = 2 &\\
\text{and } y=-3 \text{ into (2)} &\Rightarrow -7 + z = -8\\
&\Rightarrow z = -1
\end{aligned}$$

$\therefore x = 2, y = -3, z = -1$

7.

$$\begin{cases} \frac{3}{x} + \frac{1}{y} + \frac{4}{z} = 0\\ \frac{1}{x} + \frac{4}{y} - \frac{2}{z} = 4\\ \frac{2}{x} - \frac{3}{y} - \frac{1}{z} = -11 \end{cases}$$

sol.

$$\begin{cases} \dfrac{3}{x} + \dfrac{1}{y} + \dfrac{4}{z} = 0 & (1)\\ \dfrac{1}{x} + \dfrac{4}{y} - \dfrac{2}{z} = 4 & (2)\\ \dfrac{2}{x} - \dfrac{3}{y} - \dfrac{1}{z} = -11 & (3) \end{cases}$$

$$\begin{aligned}
\text{Let } u = \frac{1}{x}, v &= \frac{1}{y}, w = \frac{1}{z}\\
(1) &\Rightarrow 3u + v + 4w = 0 && (4)\\
(2) &\Rightarrow u + 4v - 2w = 4 && (5)\\
(3) &\Rightarrow 2u - 3v - w = -11 && (6)\\
(5)\times 2 &\Rightarrow 2u + 8v - 4w = 8 && (7)\\
(6)\times 4 &\Rightarrow 8u - 12v - 4w = -44 && (8)\\
(4)+(7) &\Rightarrow 5u + 9v = 8 && (9)\\
(4)+(8) &\Rightarrow 11u - 11v = -44\\
&\Rightarrow u - v = -4 && (10)\\
(10)\times 5 &\Rightarrow 5u - 5v = -20 && (11)\\
(9)-(11) &\Rightarrow 14v = 28 && (12)\\
&\Rightarrow v = 2\\
\text{Sub } v = 2 \text{ into (10)} &\Rightarrow u = -2\\
\text{Sub } u = -2 &\\
\text{and } v = 2 \text{ into (4)} &\Rightarrow -4 + 4w = 0\\
&\Rightarrow w = 1
\end{aligned}$$

$\therefore u = -2, v = 2, w = 1$

$\therefore x = -\dfrac{1}{2}, y = \dfrac{1}{2}, z = 1$

13.3 Revision Exercise 13

Solve the following system of equations.

1.

$$\begin{cases} 3x + 4y = 24\\ xy = 12 \end{cases}$$

sol.

$$\begin{cases} 3x + 4y = 24 & (1)\\ xy = 12 & (2) \end{cases}$$

$$(2) \Rightarrow y = \frac{12}{x} \quad (3)$$

$$\text{Sub (3) into (1)} \Rightarrow 3x + 4(\frac{12}{x}) = 24$$

$$\Rightarrow 3x^2 + 48 = 24x$$

$$\Rightarrow x^2 - 8x + 16 = 0$$

$$\Rightarrow (x-4)^2 = 0$$

$$\Rightarrow x = 4, x = -4$$

$$\text{Sub } x = 4 \text{ into (3)} \Rightarrow y = \frac{12}{4} = 3$$

$$\text{Sub } x = -4 \text{ into (3)} \Rightarrow y = \frac{12}{-4} = -3$$

$$\therefore \begin{cases} x = 4 \\ y = 3 \end{cases} \text{ or } \begin{cases} x = -4 \\ y = -3 \end{cases}$$

2.

$$\begin{cases} x + 2y = 5 \\ 5x^2 + 4y^2 + 12x = 29 \end{cases}$$

sol.

$$\begin{cases} x + 2y = 5 & (1) \\ 5x^2 + 4y^2 + 12x = 29 & (2) \end{cases}$$

$$(1) \Rightarrow x = 5 - 2y \quad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow 5(5-2y)^2 + 4y^2$$

$$+ 12(5-2y) = 29$$

$$\Rightarrow 5(25 - 20y + 4y^2)$$

$$+ 4y^2 + 60 - 24y = 29$$

$$\Rightarrow 125 - 100y + 20y^2$$

$$+ 4y^2 + 60 - 24y = 29$$

$$\Rightarrow 24y^2 + 124y + 156 = 0$$

$$\Rightarrow 6y^2 + 31y + 39 = 0$$

$$\Rightarrow (y-3)(6y-13) = 0$$

$$\Rightarrow y = 3, y = \frac{13}{6}$$

$$\text{Sub } y = 3 \text{ into (1)} \Rightarrow x = 5 - 2(3) = -1$$

$$\text{Sub } y = \frac{13}{6} \text{ into (1)} \Rightarrow x = 5 - 2(\frac{13}{6}) = \frac{2}{3}$$

$$\therefore \begin{cases} x = -1 \\ y = 3 \end{cases} \text{ or } \begin{cases} x = \frac{2}{3} \\ y = \frac{13}{6} \end{cases}$$

3.

$$\begin{cases} 2x + y = 7 \\ x^2 - xy + y^2 = 7 \end{cases}$$

sol.

$$\begin{cases} 2x + y = 7 & (1) \\ x^2 - xy + y^2 = 7 & (2) \end{cases}$$

$$(1) \Rightarrow y = 7 - 2x \quad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow x^2 - x(7-2x)$$

$$+ (7-2x)^2 = 7$$

$$\Rightarrow x^2 - 7x + 2x^2 - 28x$$

$$+ 49 + 4x^2 = 7$$

$$\Rightarrow 7x^2 - 35x + 42 = 0$$

$$\Rightarrow x^2 - 5x + 6 = 0$$

$$\Rightarrow (x-2)(x-3) = 0$$

$$\Rightarrow x = 2, x = 3$$

$$\text{Sub } x = 2 \text{ into (3)} \Rightarrow y = 7 - 2(2) = 3$$

$$\text{Sub } x = 3 \text{ into (3)} \Rightarrow y = 7 - 2(3) = 1$$

$$\therefore \begin{cases} x = 2 \\ y = 3 \end{cases} \text{ or } \begin{cases} x = 3 \\ y = 1 \end{cases}$$

4.

$$\begin{cases} 2x + 3y = 7 \\ x^2 + xy + y^2 = 7 \end{cases}$$

sol.

$$\begin{cases} 2x + 3y = 7 & (1) \\ x^2 + xy + y^2 = 7 & (2) \end{cases}$$

$$(1) \Rightarrow y = \frac{7-2x}{3} \qquad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow x^2 + x\left(\frac{7-2x}{3}\right) + \left(\frac{7-2x}{3}\right)^2 = 7$$

$$\Rightarrow x^2 + \frac{7x-2x^2}{3} + \frac{49-28x+4x^2}{9} = 7$$

$$\Rightarrow 9x^2 + 21x - 6x^2 + 49 - 28x + 4x^2 = 63$$

$$\Rightarrow 7x^2 - 7x - 14 = 0$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x+1)(x-2) = 0$$

$$\Rightarrow x = -1, x = 2$$

$$\text{Sub } x = -1 \text{ into (3)} \Rightarrow y = \frac{7-2(-1)}{3} = 3$$

$$\text{Sub } x = 2 \text{ into (3)} \Rightarrow y = \frac{7-2(2)}{3} = 1$$

$$\therefore \begin{cases} x = -1 \\ y = 3 \end{cases} \text{ or } \begin{cases} x = 2 \\ y = 1 \end{cases}$$

5.

$$\begin{cases} 4x - 3y + 2 = 0 \\ 2y + 5z - 19 = 0 \\ 5x - 7z + 16 = 0 \end{cases}$$

sol.

$$\begin{cases} 4x - 3y + 2 = 0 & (1) \\ 2y + 5z - 19 = 0 & (2) \\ 5x - 7z + 16 = 0 & (3) \end{cases}$$

$$(1) \times 2 \Rightarrow 8x - 6y + 4 = 0 \qquad (4)$$

$$(2) \times 3 \Rightarrow 6y + 15z - 57 = 0 \qquad (5)$$

$$(4) + (5) \Rightarrow 8x + 15z - 53 = 0 \qquad (6)$$

$$(3) \times 8 \Rightarrow 40x - 56z + 128 = 0 \qquad (7)$$

$$(6) \times 5 \Rightarrow 40x + 75z - 265 = 0 \qquad (8)$$

$$(7) - (8) \Rightarrow -131z + 393 = 0 \qquad (9)$$

$$\Rightarrow 131z = 393$$

$$\Rightarrow z = 3$$

$$\text{Sub } z = 3 \text{ into (8)} \Rightarrow 40x + 75(3) - 265 = 0$$

$$\Rightarrow 40x + 225 - 265 = 0$$

$$\Rightarrow 40x - 40 = 0$$

$$\Rightarrow x = 1$$

$$\text{Sub } z = 3 \text{ into (2)} \Rightarrow 6y - 12 = 0$$

$$\Rightarrow y = 2$$

$$\therefore x = 1, y = 2, z = 3$$

6.

$$\begin{cases} x + y + z = 9 \\ 3x + y - 2z = 1 \\ x - 2y + z = 0 \end{cases}$$

sol.

$$\begin{cases} x + y + z = 9 & (1) \\ 3x + y - 2z = 1 & (2) \\ x - 2y + z = 0 & (3) \end{cases}$$

$$(1) \Rightarrow x + z = 9 - y \qquad (4)$$

$$\text{Sub (4) into (3)} \Rightarrow 9 - y - 2y = 0$$

$$\Rightarrow 3y = 9$$

$$\Rightarrow y = 3$$

$$\text{Sub } y = 3 \text{ into (2)} \Rightarrow 3x - 2z = -2 \qquad (5)$$

$$\text{Sub } y = 3 \text{ into (3)} \Rightarrow x + z = 6 \qquad (6)$$

$$(6) \times 2 \Rightarrow 2x + 2z = 12 \qquad (7)$$

$$(5) + (7) \Rightarrow 5x = 10$$

$$\Rightarrow x = 2$$

$$\text{Sub } x = 2 \text{ into (6)} \Rightarrow z = 4$$

$$\therefore x = 2, y = 3, z = 4$$

7.

$$\begin{cases} 2x-3y-z=4 \\ 4x+y+2z=3 \\ x-4y-3z=2 \end{cases}$$

sol.

$$\begin{cases} 2x-3y-z=4 & (1) \\ 4x+y+2z=3 & (2) \\ x-4y-3z=2 & (3) \end{cases}$$

$$\begin{aligned}
(1)\times 2 &\Rightarrow 4x-6y-2z=8 && (4) \\
(3)\times 4 &\Rightarrow 4x-16y-12z=8 && (5) \\
(2)-(4) &\Rightarrow 7y+4z=-5 && (6) \\
(4)-(5) &\Rightarrow 10y+10z=0 \\
&\Rightarrow y+z=0 \\
&\Rightarrow y=-z \\
\text{Sub } y=-z \text{ into } (6) &\Rightarrow 7(-z)+4z=-5 \\
&\Rightarrow 3z=5 \\
&\Rightarrow z=\frac{5}{3} \\
y=-z &\Rightarrow y=-\frac{5}{3} \\
\text{Sub } y=-\frac{5}{3} \\
\text{and } z=\frac{5}{3} \text{ into } (1) &\Rightarrow 2x-3(-\frac{5}{3})-\frac{5}{3}=4 \\
&\Rightarrow 2x-\frac{5}{3}=-1 \\
&\Rightarrow 2x=\frac{2}{3} \\
&\Rightarrow x=\frac{1}{3}
\end{aligned}$$

$$\therefore x=\frac{1}{3},\ y=-\frac{5}{3},\ z=\frac{5}{3}$$

8.

$$\begin{cases} \frac{3}{x+1}-\frac{1}{y+2}+\frac{1}{z-1}=2 \\ \frac{2}{x+1}-\frac{3}{y+2}-\frac{1}{z-1}=7 \\ \frac{1}{x+1}+\frac{1}{y+2}-\frac{4}{z-1}=8 \end{cases}$$

sol.

$$\begin{cases} \frac{3}{x+1}-\frac{1}{y+2}+\frac{1}{z-1}=2 & (1) \\ \frac{2}{x+1}-\frac{3}{y+2}-\frac{1}{z-1}=7 & (2) \\ \frac{1}{x+1}+\frac{1}{y+2}-\frac{4}{z-1}=8 & (3) \end{cases}$$

Let $u=\frac{1}{x+1},\ v=\frac{1}{y+2},\ w=\frac{1}{z-1}$

$$\begin{aligned}
(1) &\Rightarrow 3u-v+w=2 && (4) \\
(2) &\Rightarrow 2u-3v-w=7 && (5) \\
(3) &\Rightarrow u+v-4w=8 && (6) \\
(4)\times 3 &\Rightarrow 9u-3v+3w=6 && (7) \\
(6)\times 3 &\Rightarrow 3u+3v-12w=24 && (8) \\
(5)+(8) &\Rightarrow 5u-13w=31 && (9) \\
(7)+(8) &\Rightarrow 12u-9w=30 \\
&\Rightarrow 4u-3w=10 && (10) \\
(9)\times 4 &\Rightarrow 20u-52w=124 && (11) \\
(10)\times 5 &\Rightarrow 20u-15w=50 && (12) \\
(12)-(11) &\Rightarrow 37w=-74 && (13) \\
&\Rightarrow w=-2 \\
\text{Sub } w=-2 \text{ into } (10) &\Rightarrow 4u=4 \\
&\Rightarrow u=1 \\
\text{Sub } u=1 \\
\text{and } w=-2 \text{ into } (6) &\Rightarrow 9+v=8 \\
&\Rightarrow v=-1
\end{aligned}$$

$$\therefore u=1,\ v=-1,\ w=-2$$

$$\therefore x=0,\ y=-3,\ z=\frac{1}{2}$$

Chapter 14

Marix and Determinant

14.1 Matrix

Definition of Matrix

A matrix is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. It is generally denoted as:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where m is the number of rows and n is the number of columns.

Each number in the matrix is called *an entry of the matrix*, the number in the i^{th} row and j^{th} column is denoted as a_{ij}. Thus, a matrix can also be denoted as $A = (a_{ij})$, or $A = (a_{ij})_{mn}$ where m is the number of rows and n is the number of columns.

A matrix with m rows and n columns is called an $m \times n$ matrix, where $m \times n$ is called the *order of the matrix*. For example, the following matrix is a **3×4 matrix**:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

When $m = n$, the matrix is called a *square matrix*. For example, the following matrix is a **third-order square matrix**:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

When $m = 1$, the matrix is called a *row matrix*. For example, the following matrix is a **row matrix**:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

When $n = 1$, the matrix is called a *column matrix*. For example, the following matrix is a **column matrix**:

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

Equal Matrices

Two matrices A and B are equal if they have the same order and the same entries. That is, $A = B$ if and only if $A_{ij} = B_{ij}$ for all i and j.

Zero Matrix

The matrix with all entries equal to zero is called the *zero matrix* and is denoted as O. Zero matrix can be in any order. For exmaple, the matrix below is a **2×2 zero matrix** or a **second-order square zero matrix**:

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Identity Matrix

The matrix with all entries equal to zero except the entries on the main diagonal, which are equal to one, is called the *identity matrix* and is denoted as I. Identity matrix can be in any order. The form of an identity matrix is:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Transpose Matrix

The transpose of a matrix A is denoted as A', A^t or A^T and is obtained by interchanging the rows and columns of A. For example, given the matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

The transpose of A is:

$$A' = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

Thus, we know that the transpose matrix of $m \times n$ matrix is a $n \times m$ matrix.

14.1.1 Exercise 14.1

1. State the order of the following matrices.

   (a) $A = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

   sol. A is a matrix with order $\mathbf{3 \times 1}$.

   (b) $B = \begin{pmatrix} 2 & 1 & 3 & 0 \\ 4 & 2 & 1 & 5 \end{pmatrix}$

   sol. B is a matrix with order $\mathbf{2 \times 4}$.

   (c) $C = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 3 & 0 \\ 2 & 1 & -1 \end{pmatrix}$

   sol. C is a matrix with order $\mathbf{3 \times 3}$.

2. Given $A = \begin{pmatrix} 1 & 5 & -2 & 4 \\ 2 & -4 & 3 & 1 \\ 0 & 6 & 4 & 7 \end{pmatrix}$, what is a_{23} and a_{34}?

   sol. $a_{23} = 3$ and $a_{34} = 7$.

3. If $\begin{pmatrix} 2 & 0 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & x \end{pmatrix}$, what is x?

   sol. $x = -4$.

14.2 Matrix Addition and Substraction

Given two matrices A and B of the same order, the sum of A and B is defined as the matrix $A + B$ whose (i, j)-th entry is the sum of the (i, j)-th entries of A and B. That is:

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

The difference of A and B is defined as the matrix $A - B$ whose (i, j)-th entry is the difference of the (i, j)-th entries of A and B. That is:

$$A - B = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdots & a_{2n} - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \cdots & a_{mn} - b_{mn} \end{pmatrix}$$

Note that the order of A and B must be the same. For example, the following metrices cannot be added or subtracted:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

The addition of matrices has the following properties:

- Commutative: $A + B = B + A$.
- Associative: $(A + B) + C = A + (B + C)$.
- Identity: $A \pm O = A$.
- Inverse: $A + (-A) = O$.
- Transpose: $(A \pm B)' = A' \pm B'$.

where A, B, C are matrices of the same order and O is the zero matrix of the same order as A.

Given a matrix A, if $A = A'$, then A is called a *symmetric matrix*. If $A = -A'$, then A is called an *antisymmetric matrix*.

For any given matrix A, $A + A'$ is symmetric, and $A - A'$ is antisymmetric.

14.2.1 Exercise 14.2

Let $P = \begin{pmatrix} -5 & 4 & 2 \\ 6 & -4 & 3 \\ -2 & 1 & 6 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & 1 \\ 0 & 0 & 4 \end{pmatrix}$. Evaluate the following:

1. $(P + Q)'$

   sol.

$$P + Q = \begin{pmatrix} -4 & 2 & 2 \\ 9 & -2 & 4 \\ -2 & 1 & 10 \end{pmatrix}$$

$$\therefore (P + Q)' = \begin{pmatrix} -4 & 9 & -2 \\ 2 & -2 & 1 \\ 2 & 4 & 10 \end{pmatrix}$$

2. $Q' - P'$

   sol.

$$Q - P = \begin{pmatrix} 6 & -6 & 2 \\ -3 & 6 & -2 \\ 2 & -1 & -2 \end{pmatrix}$$

$$\therefore Q' - P' = (Q - P)' = \begin{pmatrix} 6 & -3 & 2 \\ -6 & 6 & -1 \\ 2 & -2 & -2 \end{pmatrix}$$

3. $(P' - Q)'$

   sol.

$$P' = \begin{pmatrix} -5 & 6 & -2 \\ 4 & -4 & 1 \\ 2 & 3 & 6 \end{pmatrix}$$

$$P' - Q = \begin{pmatrix} -6 & 8 & -2 \\ 1 & -6 & 0 \\ 2 & 3 & 2 \end{pmatrix}$$

$$\therefore (P' - Q)' = \begin{pmatrix} -6 & 1 & 2 \\ 8 & -6 & 3 \\ -2 & 0 & 2 \end{pmatrix}$$

4. $P' - (I - Q)'$

sol.

$$\begin{aligned} P' - (I - Q)' &= P' - I' + Q' \\ &= (P + Q)' - I' \\ &= (P + Q - I)' \end{aligned}$$

$$P + Q - I = \begin{pmatrix} -4 & 2 & 2 \\ 9 & -2 & 4 \\ -2 & 1 & 10 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 2 & 2 \\ 9 & -3 & 4 \\ -2 & 1 & 9 \end{pmatrix}$$

$$\begin{aligned} \therefore P' - (I - Q)' &= (P + Q - I)' \\ &= \begin{pmatrix} -5 & 9 & -2 \\ 2 & -3 & 1 \\ 2 & 4 & 9 \end{pmatrix} \end{aligned}$$