Mathematics

Senior 3 Part I

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Introduction

Why this book?

Disclaimer

Acknowledgements

Contents

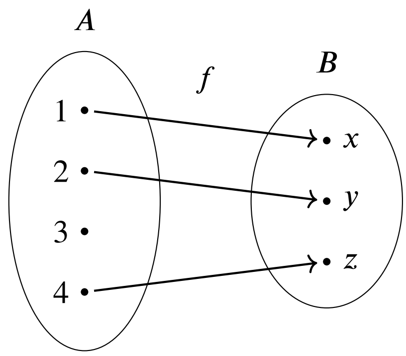
Chapter 22

Function

22.1 Definition of a Function

Mapping, Preimage and Image

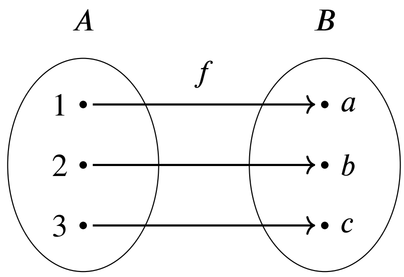
For two non-empty sets A and B, If an element a inside set A has a corresponding element b inside set B, denoted as $a \to b$, then we say that a is mapped to b or a and b are paired. The mapping between two sets is normally denoted as f, g, h, etc. The mapping shown in the diagram below can be denoted as $f : 1 \to x,\ 2 \to y,\ 4 \to z$.



Let $f : A \to B$ is a mapping, a is an element in A. If a is mapped to b under the mapping f, then b is said to be the image of a under the mapping f, denoted as $b = f(a)$; a is said to be the preimage of b under the mapping f. In the diagram above, under the mapping f, the image of 1, 2, and 4 are x, y, and z respectively, while the preimage of x, y, and z are 1, 2, and 4 respectively.

> Let A and B be two non-empty sets, f is a mapping from A to B such that for all elements in A, there is a unique corresponding element in B, then f is a function or a mapping from A to B, denoted as $f : A \to B$.

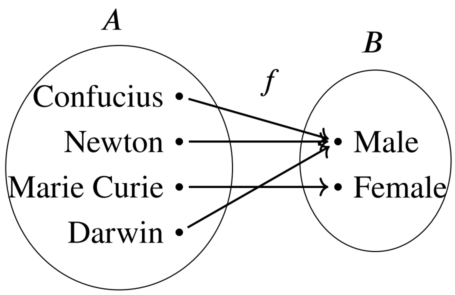
The mapping shown in the diagram below is a function.



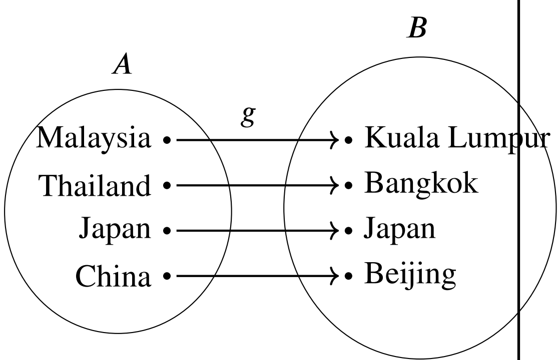
Practice 1

1. For the following mappings, list the image of each element in A and the preimage of each element in B, and determine whether the mapping is a function or not:

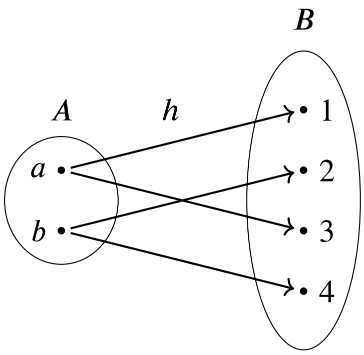
(a)



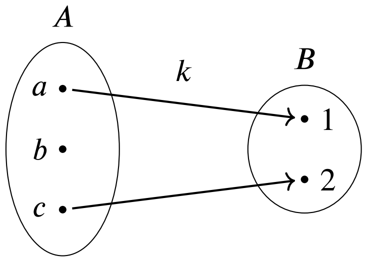
(b)



(c)



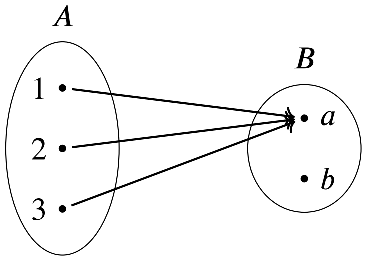
(d)



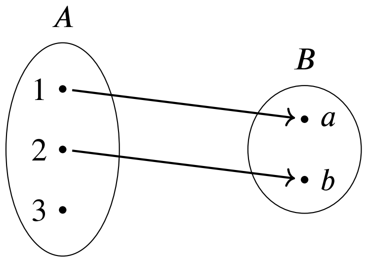
2. Given a mapping $g : x \to x + 3$, $x \in \{-2, -1, 0, 1, 2, 3\}$, find the image of each x.

3. Determine whether the following mappings are functions.

(a)



(b)



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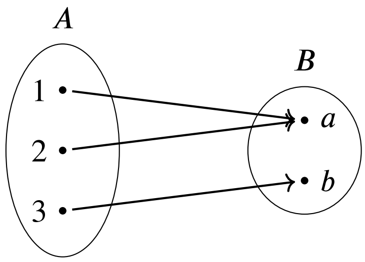
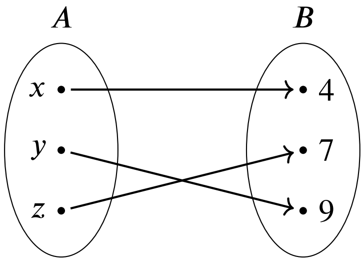
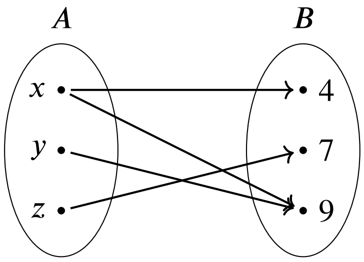


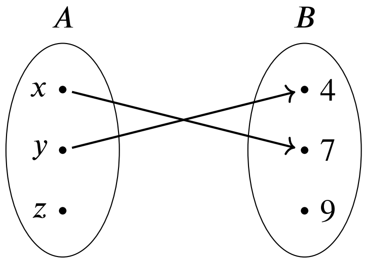
(d)



(e)



(f)



The function $f : A \to B$ can be written as $y = f(x)$, x is the element of A and y is the element of B. When x changes, y changes as well. x is called independent variable, while y is called dependent variable.

Keep in mind that $f(x)$ is NOT the product of f and x.

Representation of Functions

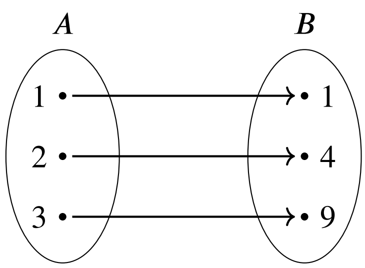
Generally speaking, there are a few ways to represent a function:

1. **Narrative Form**: express the function of two sets in words. For example, Let $A = \{1, 2, 3\}$ and $B = \{1, 4, 9\}$, f is a function from A to B, its definition is that for any element x in A, its corresponding element is x^2 in B.

2. **Arrow Method**: draw an arrow to connect the preimage and image of a function such that the preimage is corresponding to the image. To express the example above, we express it as $f : 1 \to 1,\ 2 \to 4,\ 3 \to 9$.

3. **Analytical Method**: express the function in the form of mathematical expression to represent the relationship between the independent variable and the dependent variable. For example, $f(x) = x^2, x \in A$.

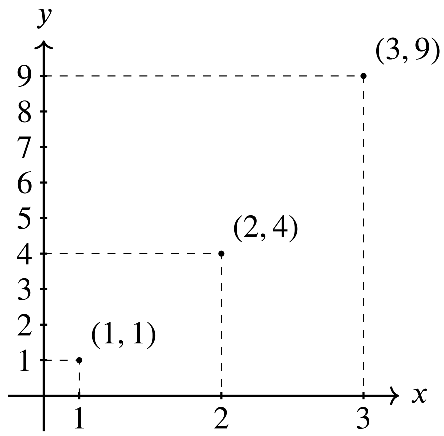
4. **Venn Diagram**: draw arrows between the venn diagram of two sets to represent the function, as shown below:



5. **Table Method**: express the function in the form of table, showing the relationship of the chosen value between independent variable x and the value of its corresponding dependent variable y, as shown below:

x	1	2	3
y	1	4	9

6. **Graphical Method**: draw a graph to represent the function of the two variables, as shown below:



Practice 2

Express the following functions using analytical method, venn diagram, table method and graphical method.

(a) f mapping each integers from -3 to 3 to its squares plus 4.

(b) g mapping each natural numbers from 1 to 4 to its cubes.

Exercise 22.1

1. Express the mapping from set A to set B, and determine which of the following mappings are functions.

	Set A	Set B	Mapping
(a)	{0, 3, 9, 12}	{0, 1, 2, 3}	Divide by 3
(b)	{-2, -1, 0, 1, 2}	{0, 1, 4, 9, 16}	Power of 4
(c)	{-2, -1, 0, 1, 2}	{0, 1, 4}	Square
(d)	{30°, 45°, 60°}	$\left\{\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right\}$	Sine
(e)	{-1, 0, 1, 2}	{-1, 0, 1}	Cube

2. Let function $f(x) = 3x^2 + 1$.

(a) Find the image of the following elements:

   i. -3

   ii. -2

   iii. 0

   iv. 2

   v. 5

(b) Find the preimage of the following elements:

   i. 13

   ii. 28

   iii. 1

   iv. 0

   v. 4

3. Let function $g(x) = 5x - 2$. Find:

   (a) $g(-2)$

   (b) $g(-1)$

   (c) $g(0)$

4. Let function $f(x) = \begin{cases} 2x, & x \leq -1 \\ x - 1, & -1 \leq x < 3 \\ 4x + 2, & x \geq 3 \end{cases}$,

   find

   (a) $f(-5)$

   (b) $f(-2)$

   (c) $f(0)$

   (d) $f(2)$

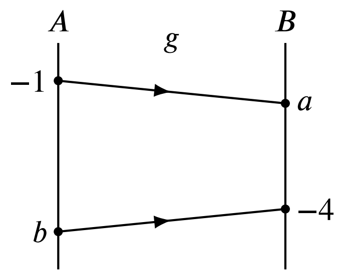
   (e) $f(10)$

5. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$. Find the image of -1, 0, 1, and 2 under f.

6. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$. Find the preimage of 0, 1, and 4 under f.

   In $\mathbb{R}$, which element does not have a preimage?

7. In the diagram below, given that function $g : A \to B$ is defined as $g : x \to 2x - 8$. Find the value of a and b.

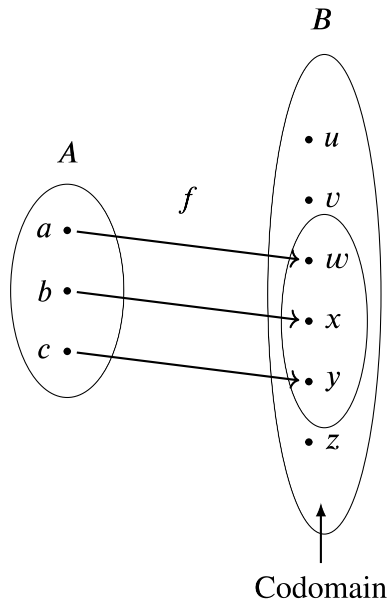


8. Using narrative form, arrow method, venn diagram, table method and graphical method, express the function $f(x) = 2x$, $x \in \{-2, -1, 0, 1, 2\}$.

22.2 Domain and Range

Let f is a function from set A to set B, then set A is called the domain of f, denoted by D_f; set B is called the codomain of f; the set of the images of all elements of A under f is called the range of f, denoted by R_f.

If the domain A and range B of function $f : A \to B$ are both subsets of real number set $\mathbb{R}$, then this function is called real valued function / real function. This book primarily discusses about real valued functions. When the domain of a real function is not mentioned and only the mapping rule is given, its domain is assumed to be the set of all real numbers that yield defined values $f(x)$. After the domain and the mapping rule are determined, the range of a function will then be determined.



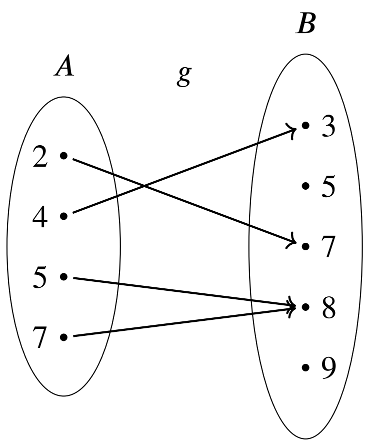
Interval Notation

Let a and b be two real number, $a < b$.

Intervals	Set Notations
(a, b)	$\{x \| x \in \mathbb{R}, a < x < b\}$
$[a, b)$	$\{x \| x \in \mathbb{R}, a \leq x < b\}$
$(a, b]$	$\{x \| x \in \mathbb{R}, a < x \leq b\}$
$[a, b]$	$\{x \| x \in \mathbb{R}, a \leq x \leq b\}$
(a, ∞)	$\{x \| x \in \mathbb{R}, x > a\}$
$[a, \infty)$	$\{x \| x \in \mathbb{R}, x \leq a\}$
$(-\infty, a)$	$\{x \| x \in \mathbb{R}, x < a\}$
$(-\infty, a]$	$\{x \| x \in \mathbb{R}, x \leq a\}$

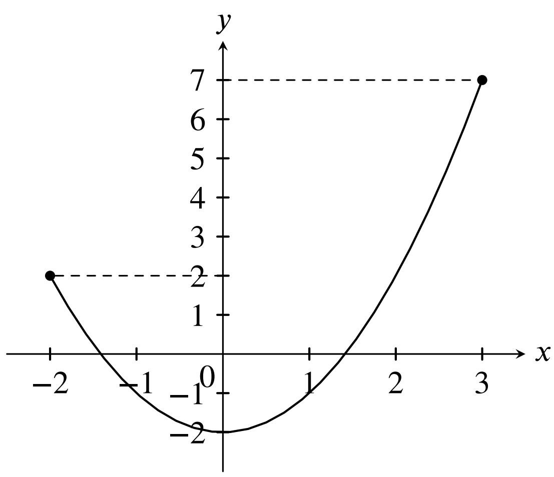
Practice 3

1. Let $A = \{2, 4, 5, 7\}$ and $B = \{3, 5, 7, 8, 9\}$, the definition of function g is given by the diagram below. Find the domain, codomain and range of function g.



2. Let $A = \{-2, -1, 0, 1, 2\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = x^2 - 1$. Find the domain and range of f.

3. The curve in the diagram below represents the function $y = f(x)$, $-2 \leq x \leq 3$. Find the domain and range of f.



4. Find the domain and range of the following functions:

   (a) $f(x) = -4x + 5$

   (b) $g(x) = x^2 - 1$

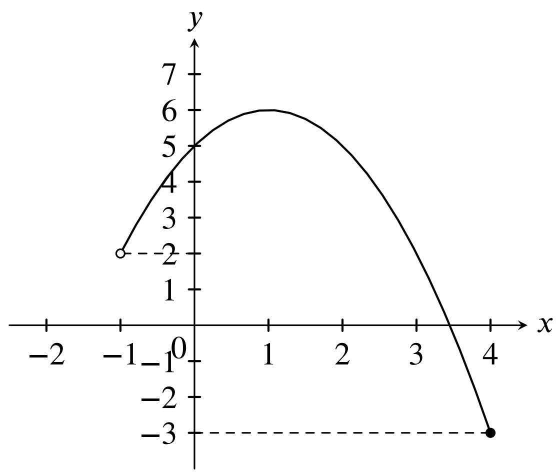
   (c) $h(x) = \dfrac{1}{4x + 7}$

   (d) $k(x) = \sqrt{6 - x}$

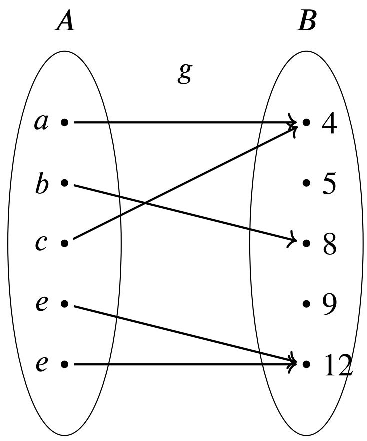
Exercise 22.2

1. Let $X = \{a, b, c, d\}$ and $Y = \{-1, 2, 9, 11\}$, function $f : X \to Y$ is defined by $f(a) = 2$, $f(b) = -1$, $f(c) = 2$, $f(d) = 9$. Find the domain and range of the f.

2. The curve in the diagram below represents the function $y = f(x)$, $-1 < x \leq 4$. Find the domain and range of f.



3. Let $A = \{a, b, c, d, e\}$ and $B = \{4, 5, 8, 9, 12\}$, the definition of function $g : A \to B$ is given by the digram below. Find the domain, codomain and range of function g.



4. Let $A = \{-1, 0, 1, 2\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = 3x^2 - 2$, find the domain and range of f.

5. Let $A = \{-1, 0, 2, 5, 11\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = x^2 - x - 2$, find the domain and range of f.

6. Find the domain and range of the following functions:

   (a) $f(x) = x^3$

   (b) $g(x) = \sqrt{1 - x^2}$

   (c) $h(x) = \dfrac{1}{2x + 3}$

   (d) $k(x) = x^2 - 2x + 4$

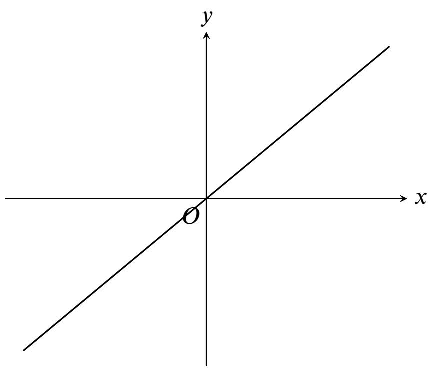
22.3 Graphs of Functions and Their Transformations

Graphs of Simple Functions

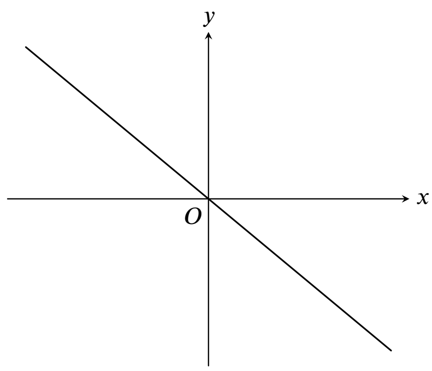
On a Cartesian plane, the graphs formed by all the point (x, y) that satisfied the equation $y = f(x)$ are called graphs of function f. Below are some examples of graphs of simple functions.

Note that any line that is parallel to the y-axis intersects the graph of a function at most once.

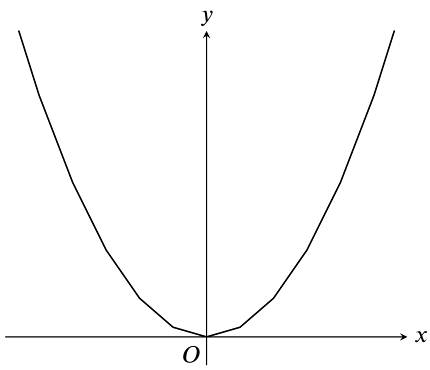
(a) $y = x$



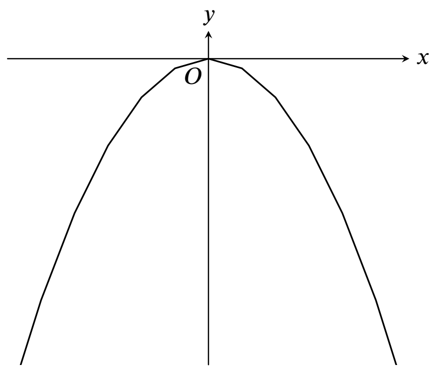
(b) $y = -x$



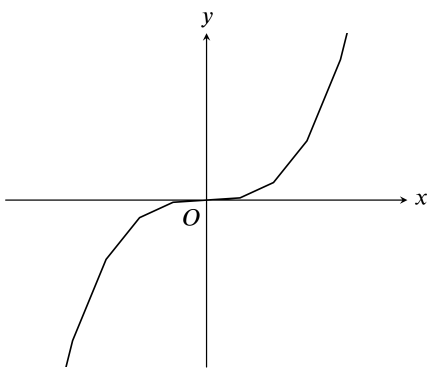
(c) $y = x^2$



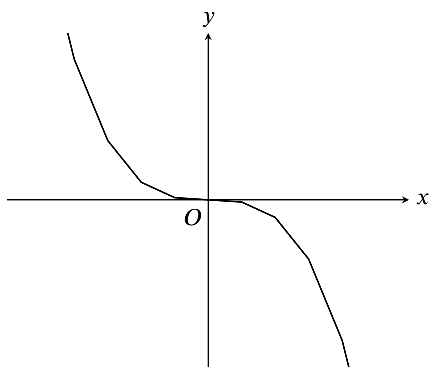
(d) $y = x^2$



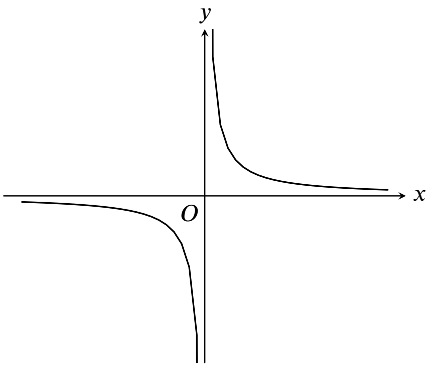
(e) $y = x^3$



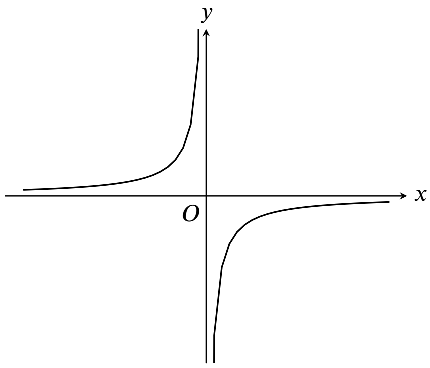
(f) $y = -x^3$



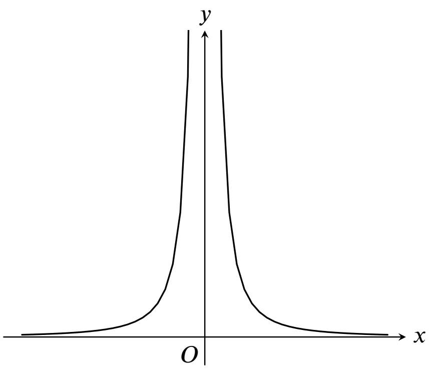
(g) $y = \frac{1}{x}$



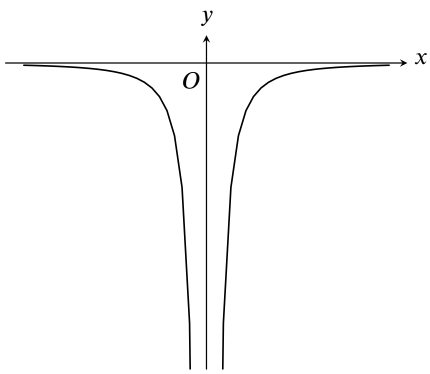
(h) $y = -\frac{1}{x}$



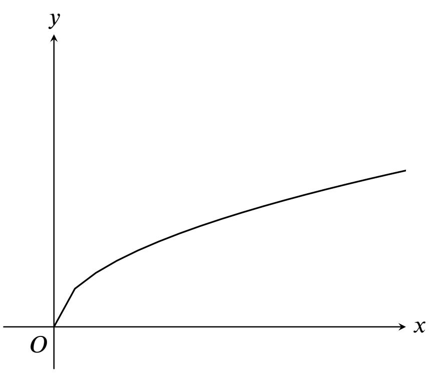
(i) $y = \frac{1}{x^2}$



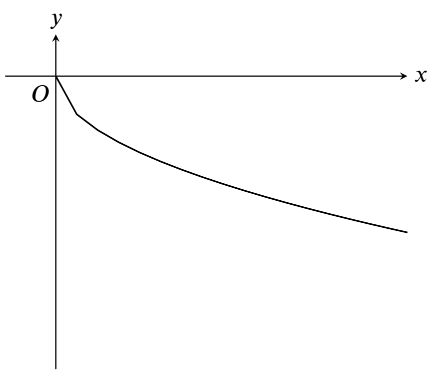
(j) $y = -\frac{1}{x^2}$



(k) $y = \sqrt{x}$



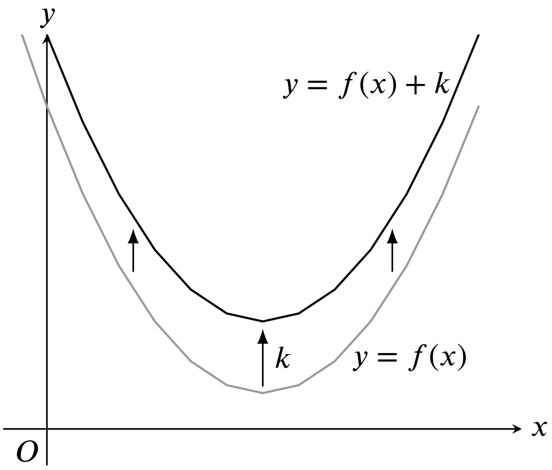
(l) $y = -\sqrt{x}$

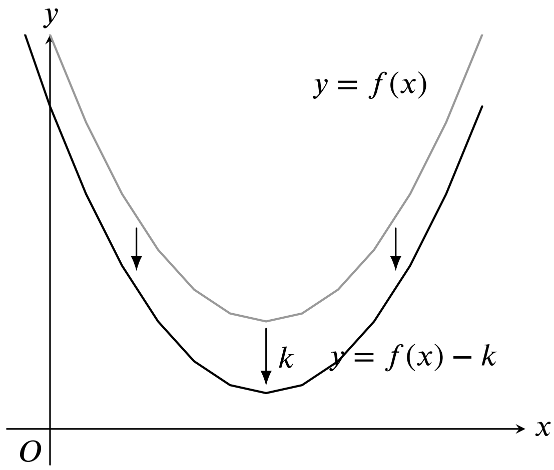


Transformations of Graphs

- If $k > 0$, translate the graph of $y = f(x)$ vertically upwards by k units, the graph of $y = f(x) + k$ is obtained.

- If $k > 0$, translate the graph of $y = f(x)$ vertically downwards by k units, the graph of $y = f(x) - k$ is obtained.





- If $h > 0$, translate the graph of $y = f(x)$ horizontally to the right by h units, the graph of $y = f(x+h)$ is obtained.
- If $h > 0$, translate the graph of $y = f(x)$ horizontally to the left by h units, the graph of $y = f(x-h)$ is obtained.

- If $k > 0$, reflect the graph of $y = f(x)$ about the x-axis, the graph of $y = -f(x)$ is obtained.
- If $k > 0$, reflect the graph of $y = f(x)$ about the y-axis, the graph of $y = f(-x)$ is obtained.

If $a > 0$, zooming (when $a > 1$) or shrinking (when $0 < a < 1$) the graph of $y = f(x)$ by a factor of a in the y-direction, the graph of $y = af(x)$ is obtained.

If $a > 0$, shrinking (when $a > 1$) or zooming (when $0 < a < 1$) the graph of $y = f(x)$ by a factor of $\frac{1}{a}$ in the x-direction, the graph of $y = f(ax)$ is obtained.

Practice 4

Find the line of symmetry and vertex of the following parabola, and sketch its graph. (Question 1 to 2):

1. $y = 2x^2 + 8x + 11$

2. $y = -3x^2 + 18x - 7$

Sketch the graph of the following functions. (Question 3 to 4):

3. $y = \dfrac{4}{(x+2)^2}$

4. $y = \sqrt{x-1} + 3$

Exercise 22.3

Find the line of symmetry and vertex of the following parabola, and sketch its graph.

1. $y = 2x^2 + 4x + 5$

2. $y = -3x^2 + 12x - 4$

3. $y = 4x^2 - 20x + 19$

4. $y = -3x^2 - 6x - 4$

Sketch the graph of the following functions.

5. $y = (x+2)^3 - 5$

6. $y = \sqrt{x-5}$

7. $y = \dfrac{1}{(x+2)^2}$

8. $y = -\dfrac{1}{2(x-1)^2}$

9. $y = 3\sqrt{x+1} - 4$

10. $y = \dfrac{4}{2x+3}$

11. $y = \begin{cases} 4x+9, & x \leq 0 \\ 9-2x, & x > 0 \end{cases}$

12. $y = \begin{cases} x, & x < -1 \\ \sqrt{x+1}, & x \geq -1 \end{cases}$

13. Sketch the graph for the function $f(x) = x^2-6x+12, -2 \leq x \leq 8$, and find its domain and range.

14. Sketch the graph for the function $g(x) = -x^2 - 4x - 7, -2 \leq x \leq 5$, and find its domain and range.

15. Sketch the graph for the function $f(x) = -x^2+2x+10$, and find its domain and range.

16. Sketch the graph of the function $y = \sqrt{x}$, and transform it according the following steps. Sketch the graph of each function after each step on the same diagram, and write down the corresponding function.

    Step 1: Translate 4 units to the left;

    Step 2: Scale up by a factor of 2 in the x-direction;

    Step 3: Reflect about the y-axis;

    Step 4: Translate 3 units downwards.

    Step 5: Scale down by half in the y-direction.

22.4 Composite Functions

Let A, B, and C be three non-empty sets, $f : A \to B$ and $g : B \to C$ be two functions, an element x in set A is mapped to an element $f(x)$ in set B by function f, and $f(x)$ is mapped to an element $g\big(f(x)\big)$ in set C by function g. In other words, x in set A is mapped to an element $g\big(f(x)\big)$ in C after two mappings. That is:

$$x \xrightarrow{f} f(x) \xrightarrow{g} g\big(f(x)\big)$$

The combination of these two mappings are a function from set A to set C, this function is called the *composite function* of f and g, denoted by $g \circ f$. When defining the composite function $g \circ f$, the range of f must be a subset of the domain of g, that is, $R_f \subseteq D_g$.

Note that $D_{g \circ f} = D_f$, $R_{g \circ f} \subseteq R_g$.

$\forall n \in \mathbb{N}$, $f^{n+1} = f \circ f^n$.

Generally speaking, $g \circ f \neq f \circ g$.

If $f \circ (g \circ h)$ is defined, then $(f \circ g) \circ h$ is also defined, and $f \circ (g \circ h) = (f \circ g) \circ h$. Therefore, we can write $f \circ g \circ h$ without ambiguity.

Practice 5

1. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x+3$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 5 - x$. Find $(g \circ f)(x)$ and $(f \circ g)(x)$.
2. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 2x + 3$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 3x - 4$. Find
   (a) $g \circ f$ and $f \circ g$;
   (b) $g\big(f(2)\big)$, $f\big(g(2)\big)$, $(g \circ f)(2)$, and $(f \circ g)(2)$.
3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4 - x^2$ and $g : \{x | x \leq 4\} \to \mathbb{R}$, $g(x) = \sqrt{4 - x}$. Prove the existence of $f \circ g$ and $g \circ f$ respectively.

22.5 One to One Function, Onto Function and One to One Onto Function

One to One Function

Let $f : A \to B$ be a function, if there is at most one preimage in set A for each element in set B, then f is called a *one to one function*.

As shown in the diagram above, each element in the codomain B of the function $f : A \to B$ has at most one preimage in the domain A of the function, thus f is a one to one function; while the element b_2

in the codomain B of the function $g : A \to B$ has two preimages a_2 and a_3, thus g is not a one to one function.

A function $y = f(x)$ is a one to one function, if and only if any line parallel to the x-axis intersects the graph of the function at most once.

Onto Function

If each element in the codomain B of the function $f : A \to B$ has at least one preimage under the function f, then f is said to be an *onto function*.

As shown in the diagram above, each element in the codomain B of the function $f : A \to B$ has at least one preimage under the function f, therefore f is an onto function; while the element b_3 in the codomain B of the function $g : A \to B$ has no preimage under the function g, therefore g is not an onto function.

One to One Onto Function

If a function is both a one to one function and an onto function, then it is a *one to one onto function*, as shown in the diagram above.

Practice 7

Determine whether the following functions are one to one functions or onto functions.

Exercise 22.5

1. Let $A = \{1, 2, 3\}$, $f : A \to A$ is defined by $f : 1 \to 1, 2 \to 3, 3 \to 2$. Determine if f is a one to one function or an onto function.
2. Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$, $f : A \to B$ is defined by $f : a \to y, b \to x, c \to z, d \to y$. Determine if f is a one to one function or an onto function.
3. Let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 2x + 1$. Determine if g is a one to one function or an onto function.
4. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x^3 - 3$. Determine if f is a one to one function or an onto function.
5. Let the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = \dfrac{1}{x}$. Determine if f is a one to one function or an onto function.
6. Let the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = \sqrt{x}$. Determine if f is a one to one function or an onto function.

7. Determine whether the following functions are one to one, onto or one to one onto functions.

   (a) $A = \{a, b, c\}$, $B = \{x, y, z\}$, $f : A \to B$, $f : a \to x, b \to x, c \to y$

   (b) $A = \{a, b, c\}$, $B = \{x, y, z\}$, $g : A \to B$, $g : a \to x, b \to y, c \to z$

   (c) $A = \{a, b, c\}$, $B = \{x, y\}$, $h : A \to B$, $h : a \to x, b \to y, c \to y$

   (d) $A = \{a, b, c\}$, $B = \{x, y\}$, $k : A \to B$, $k : a \to x, a \to y, c \to y$

   (e) $A = \{a, b, c\}$, $B = \{x, y\}$, $f : A \to B$, $f : a \to x, a \to y, b \to x, c \to y$

   (f) $A = \{a, b, c, d\}$, $B = \{u, v, x, y, z\}$, $g : A \to B$, $g : a \to u, b \to v, c \to x, d \to y$

8. Determine whether the following functions mapping A to B are one to one functions or onto functions.

22.6 Inverse Functions

If $f : A \to B$ is a one to one onto function, then there exist a function $g : B \to A$, such that if $y = f(x)$, then $g(y) = x$. The function g is called the *inverse function* of f, and is denoted by f^{-1}.

from the diagram above, we can conclude the following:

$$x \xrightarrow{f} y = f(x) \xrightarrow{f^{-1}} f^{-1}\big(f(x)\big) = f^{-1}(y)$$

or

$$y \xrightarrow{f^{-1}} x = f^{-1}(y) \xrightarrow{f} f\big(f^{-1}(y)\big) = f(x)$$

If both $(g\circ f)^{-}1$ and $f^{-1}\circ g^{-1}$ exist, then $(g\circ f)^{-1} = f^{-1}\circ g^{-1}$.

Practice 8

Exercise 22.5

1. Find the inverse function of the following functions:
   (a) $f : x \to 7x - 3$
   (b) $g : x \to \frac{1}{2}x + 9$
   (c) $h : x \to \frac{x+1}{x-8}$, $x \neq 8$
   (d) $k : x \to \frac{x-1}{2x}$, $x \neq 0$
2. Given the function $f : x \to 2x + 1$ and $g : x \to \frac{1}{x-4}$, $x \neq 4$. Find:
   (a) f^{-1}
   (b) g^{-1}
   (c) $f^{-1}\circ g^{-1}$
   (d) $g^{-1}\circ f^{-1}$
   (e) $(f\circ g)^{-1}$
   (f) $(g\circ f)^{-1}$

Graph of Inverse Functions

If f is a one to one function, then the graph of f^{-1} is the reflection of the graph of f about the line $y = x$.

Practice 9

Given the function $g : \mathbb{R}^+ \cup 0 \to \mathbb{R}^+ \cup 0$, $g : x \to x^2$. On the same set of axes, draw the graph of the function g and its inverse function g^{-1}.

Exercise 22.6

1. Find the inverse function of the following functions:

   (a) $f : x \to 2x - 7$

   (b) $g : x \to \dfrac{1}{x-2}$, $x \neq 2$

   (c) $h : x \to \dfrac{2x-5}{x-2}$, $x \neq 2$

   (d) $k : x \to \dfrac{3x}{x-4}$, $x \neq 4$

2. Given that $f : x \to \dfrac{160}{ax+b}$, $f(5) = 8$ and $f(9) = 10$. Find

   (a) the values of a and b;

   (b) $f^{-1}(16)$.

3. Given that $f : x \to \dfrac{a}{x+b}$, $f(3) = -1$, and $f(-9) = 3$. Find

   (a) the values of a and b;

   (b) the value of x such that $f(x) = f^{-1}(x)$.

4. Given the function $g : x \to \dfrac{6}{x} - 3$, $x \neq 0$. Find

   (a) g^{-1};

   (b) the value of x such that $g^{-1}(x) = x - 2$.

5. Given the function $f : x \to ax + b$ and $f^2 : x \to 4x + 12$. If $a > 0$, find

   (a) the values of a and b;

   (b) $f^{-1}(3)$.

6. Given the function $f : x \to 3x - 2$ and $g : x \to \dfrac{x}{x+4}$, $x \neq -4$. Find

   (a) f^{-1}

   (b) g^{-1}

   (c) $f^{-1} \circ g^{-1}$

   (d) $g^{-1} \circ f^{-1}$

   (e) $(f \circ g)^{-1}$

   (f) $(g \circ f)^{-1}$

7. Given the function $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{0\}$, $f : x \to \dfrac{1}{x-2}$.

   (a) Find f^{-1}.

   (b) On the same set of axes, draw the graph of f and f^{-1}.

8. Given the function $f : x \to 2\sqrt{x+4}$, $x \geq -4$,

   (a) Find f^{-1}.

   (b) On the same set of axes, draw the graph of f and f^{-1}.

Revision Exercise 22

1. Determine whether the following mappings from set $A = \{1, 2, 3, 4\}$ to set $B = \{a, b, c, d\}$ are functions or not.

   (a) $1 \to a, 2 \to c, 4 \to b$

   (b) $1 \to a, 2 \to d, 3 \to b, 4 \to a$

   (c) $1 \to c, 2 \to c, 3 \to b, 4 \to b$

   (d) $1 \to a, 2 \to c, 2 \to b, 4 \to d$

   (e) $1 \to c, 2 \to b, 3 \to d, 4 \to c, 4 \to a$

2. Given the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 3x-2, & x < -3 \\ 2x^2+4, & -3 \le x < 2 \\ -2x+9, & x \ge 2 \end{cases}$, find

   (a) $f(-4)$

   (b) $f(0)$

   (c) $f(2)$

   (d) $f(3)$

3. Find the domain and range of the following functions:

   (a) $f : 1 \to 3, 2 \to 5, 4 \to 8$

   (b) $g : 2 \to 4, 4 \to 5, 5 \to 7, 6 \to 9$

   (c) $h : 1 \to 3, 2 \to 5, 3 \to 6, 4 \to 8$

4. The table below shows a function f:

x	-3	-2	-1	0	1
f(x)	-22	-3	4	5	6

   (a) Find the domain and range of the function;

   (b) Sketch the graph of the function.

   (c) Determine if the inverse function of f exists.

5. As shown in the diagram below, let a function $f : x \to ax + b$. Find the value of $f(4)$ and $f^{-1}(5)$.

6. Given the function $f : x \to x^2 - x + 1$, $-1 \le x \le 3$, find its range.

7. Let function $f : x \to 2x^2 - 4x + 3$.

   (a) If $D_f = \mathbb{R}$, find the range of f;

   (b) If $D_f = \{x | x \ge 3\}$, find the range of f.

8. Find the domain and range of the following functions:

   (a) $f(x) = \dfrac{1}{x}$

   (b) $f(x) = \sqrt{2x-5}$

   (c) $f(x) = x^2 + 4x + 7$

   (d) $f(x) = \dfrac{1}{x^2+4}$

9. Find the domain of the following functions:

   (a) $f(x) = \dfrac{2x}{x-3}$

   (b) $f(x) = \sqrt{4-x^2}$

   (c) $f(x) = \dfrac{x-2}{2x^2-5x+2}$

   (d) $f(x) = \dfrac{x-3}{\sqrt{x^2-9}}$

10. Sketch the graph for the following functions:

   (a) $f(x) = 2x^2 - 5x + 9$

   (b) $f(x) = -3x^2 + 6x + 11$

   (c) $f(x) = 3x^2 + 12x + 10$

   (d) $f(x) = -5x^2 + 6x + 11$

   (e) $f(x) = 2x^3 - 7$

   (f) $f(x) = \sqrt{3x-9}$

   (g) $f(x) = \dfrac{4}{2x+11}$

   (h) $f(x) = \dfrac{2x+7}{x-1}$

   (i) $f(x) = 2\sqrt{x+5} - 4$

   (j) $f(x) = \dfrac{1}{(2x-3)^2}$

   (k) $f(x) = \begin{cases} 2x+1, & x < 0 \\ x^2, & x \ge 0 \end{cases}$

(l) $f(x) = \begin{cases} 1-x^2, & x \le 1 \\ x^2+2x-3, & x > 1 \end{cases}$

11. Given the function $f : x \to 2x^2$ and $g : x \to 3x-4$. Find the value of m such that $(f \circ g)(m) = (g \circ f)(m)$.

12. Given the function $f : x \to x^2+2x-3$ and $g : x \to 3x-4$. If $(f \circ g)(k) = (g \circ f)(k)$, find the value of k.

13. Given that $f(x) = 3x+1$, $x \neq 0$. If $(f \circ g)(x) = 6x^2 - 9x + 4$, find $g(x)$.

14. Given that $f(x) = \frac{x+1}{x}$, $x \neq 0$. IF $(f \circ g)(x) = x$, find $g(x)$.

15. A function f is defined by $f : x \to x-3$. Find another function g such that $g \circ f : x \to 4x^2 - 20x + 25$.

16. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} -2, & x \le -3 \\ |x| - 2x, & -3 < x < 3 \\ 2x-1, & x \ge 3 \end{cases}$. Find $(f \circ f \circ f)(-1000)$.

17. Let function $f : A \to \mathbb{R}$ be defined by $f : x \to 2x^2$. Determine if f is one to one function when A is the following sets.
    (a) $A = \{x | 0 \le x < 6\}$
    (b) $A = \{x | x < 0\}$
    (c) $A = \{x | -2 \le x < 2\}$
    (d) $A = \{x | x > 3\}$

18. Determine whether the following functions are one to one functions or onto functions.
    (a) $f : \mathbb{R}^+ \to \mathbb{R}$, $f : x \to |x| - 2$
    (b) $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{1\}$, $f : x \to \frac{x}{x-2}$
    (c) $f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$, $f : x \to |x|$

19. Let $A = \mathbb{R} \setminus \left\{-\frac{1}{2}\right\}$ and $A = \mathbb{R} \setminus \left\{\frac{1}{2}\right\}$, function $f : A \to B$ is defined by $f(x) = \frac{x-3}{2x+1}$. Find
    (a) $f^{-1}(-2)$
    (b) $f^{-1}(0)$
    (c) $f^{-1}(3)$

20. Let function $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = x^2+2x+1$. Find $f^{-1}(4)$ and $f^{-1}(9)$.

21. A function f is defined by $f : x \to \frac{x}{2} + 1$. If $g \circ f^{-1} : x \to 4x^2 - 8x + 7$, find the function g.

22. Given the functioon $f : x \to 3x^2 + 5x + 9$, $x \le a$. Find the maximum value of a such that the inverse function of f exists.

23. Let the function f and g be defined as $f : x \to 5x+3$ and $g : x \to 2x-7$ respectively. Find
    (a) $f \circ g$
    (b) f^{-1}
    (c) g^{-1}

24. Given the function $f : x \to 2x+3$ and $g : x \to 3 - x2x + 5$, $x \neq -\frac{5}{2}$. Find
    (a) $f \circ g$
    (b) f^{-1}

(c) g^{-1}

Show that $g^{-1} \circ f^{-1} = (f \circ g)^{-1}$.

25. Given the function $f : x \to \sqrt{x}$, $x \neq 0$ and $g : x \to x^3$. Find

(a) $g \circ f$

(b) f^{-1}

(c) g^{-1}

(d) $(g \circ f)^{-1}$

(e) $g^{-1} \circ f^{-1}$

26. Given the function $f : x \to 2\sqrt{x-4} + 3$, $x \geq 4$.

(a) Find the range of f.

(b) Find the inverse function f^{-1} of f.

(c) On the same diagram, sketch the graphs of f and f^{-1}.

Chapter 23

Exponents and Logarithms

23.1 Exponents

Definition and Properties of Exponents

Back in Junior 1, we have learnt the following definitions of exponents:

Positive exponent	$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$
Zero exponent	$a^0 = 1$
Negative exponent	$a^{-n} = \dfrac{1}{a^n}$ $(a \neq 0, n \in \mathbb{Z}^+)$
Fractional exponent	$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ $(a \geq 0, n > 1, m, n \in \mathbb{Z}^+)$

The exponent of rational numbers have the following properties:

1. $a^m \times a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^n = a^n b^n$
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ $(b \neq 0)$

Practice 1

Without using the calculator, find the value of the following expressions (Question 1 to 2):

1. $2^{-2} + 2 - 5 - (-2)^{-3}$
2. $\left(3\frac{6}{25}\right)^{-\frac{1}{2}}$
3. Simplify $a^{-}4 \div a^{-5} \times (b^{-3})^{-4}$

Exponential Functions and Graphs

Let a is a constant that is bigger than zero and not equal to 1, then the function being expressed in the form of $y = a^x$ is called an *exponential function*. The domain of an exponential function is $\mathbb{R}$.

Consider the following: a cell divides into two cells, and then each of the two cells divides into two cells again, and so on. If we let x be the number of divisions, the number of cells after the divisions be y, then the functional relationship between x and y is $y = 2^x$, which is an exponential function.

In order to look into the graph and its properties of an exponential function $y = a^x$, we sketch the graph of some exponential functions, the graph of $y = 2^x$, $y = 10^x$, and $y = \left(\frac{1}{2}\right)^x$ are shown in the diagram below.

From the diagram above, we can see that:

(1) The graph of the function $y = 2^x$, $y = 10^x$, and $y = \left(\frac{1}{2}\right)^x$ are only at the top of the x-axis. Actually, when $a > 0$, $a^x > 0$. Therefore, the value of the exponential function $y = a^x$ is always positive.

(2) When $x = 0$, $y = 1$. Hence, the graph of exponential functions $y = a^x$ always passes through the point $(0, 1)$.

(3) For the function $y = 2^x$, when $x > 0$ and $y = 10^x$, when $x < 0$, $y < 1$; when $x > 0$, $y > 1$. When the value of x increases, the value of y increases, that is, the function is an increasing function in the interval $(-\infty, +\infty)$.

(4) For the function $y = \left(\frac{1}{2}\right)^x$, when $x > 0$, $y > 1$; when $x < 0$, $y < 1$. When the value of x increases, the value of y decreases, that is, the function is a decreasing function in the interval $(-\infty, +\infty)$.

When we are discussing about the graph and its properties of an exponential function $y = a^x$, the following two cases are considered:

Practice 2

1. Without using the calculator, compare the value of the following expressions:

   (a) $\pi^{2.1}$ and $\pi^{3.5}$

   (b) $0.5^{-2.3}$ and $0.5^{-3.8}$

2. Given the exponential functions $f(x) = 3^{x^2-3x+5}$ and $g(x) = 3^{x+10}$. Find the value of x such that $f(x) = g(x)$.

Exercise 23.1

Without using the calculator, find the value of the following expressions (Question 1 to 10):

1. $\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^0 - \left(-\frac{1}{3}\right)^{-2}$
2. $\left(\dfrac{3^{-5} \cdot 3^2}{3^{-3}}\right)^{-2}$
3. $6^{-8} \div 6^{-5} + 3^{-3}$
4. $12^{\frac{1}{3}} \times 6^{\frac{1}{3}} \div 27^{\frac{1}{6}} \div 3^{\frac{1}{6}}$
5. $(0.2)^{-2} \times (0.125)^{\frac{2}{3}}$
6. $(0.3)^{-\frac{1}{3}} \times (0.0081)^{\frac{1}{3}} + (0.064)^{\frac{1}{3}}$
7. $\left(\frac{81}{16}\right)^{-0.25} \times \left(\frac{8}{27}\right)^{-\frac{2}{3}} \times (0.25)^{-2.5}$
8. $\left(\frac{1}{2}\right)^{-2} + 125^{\frac{2}{3}} + 343^{\frac{1}{3}} - \left(\frac{1}{27}\right)^{-\frac{1}{3}}$
9. $\left(2\frac{1}{4}\right)^{-\frac{3}{2}} + \left(1\frac{11}{25}\right)^{-1} - \left(2\frac{2}{3}\right)^0$
10. $\dfrac{5\sqrt{4}\sqrt{8}\left(\sqrt[3]{\sqrt[5]{4}}\right)^2}{\sqrt[3]{\sqrt{2}}}$

Simplify the follwoing expressions (Question 11 to 24):

11. $a^{\frac{1}{2}} \cdot a^{\frac{1}{4}} \cdot a^{-\frac{1}{8}} \cdot a^{\frac{1}{6}}$
12. $\left(9a^2b^{-2}c^4\right)^{-1}$
13. $\left(x^4y^{-5}\right)\left(x^{-2}y^2\right)^2$
14. $3a^{-2}b^{-3} \div \left(-3^{-1}a^2b^{-3}\right)$
15. $\sqrt[3]{\dfrac{a^2b^{-1}}{a^{\frac{1}{2}}b^5}}$
16. $5a^{-2}b^{-3} \div \left(5^{-1}a^2b^{-3}\right) \times 5^{-2}ab^4c$
17. $\dfrac{a^{-2} - b^{-2}}{a^{-2} + b^{-2}}$
18. $\left(a^{-1} + b^{-1}\right)(a+b)^{-1}$
19. $\left(x + x^{-1}\right)\left(x - x^{-1}\right)$
20. $\left(-2x^{\frac{1}{4}}y^{-\frac{1}{3}}\right)\left(3x^{-\frac{1}{2}}y^{\frac{2}{3}}\right)\left(-4x^{\frac{1}{4}}y^{\frac{2}{3}}\right)$
21. $2x^{-\frac{1}{3}}\left(\frac{1}{2}x^{\frac{1}{3}} - 2x^{-\frac{2}{3}}\right)$
22. $\left(\sqrt{x^3} \cdot \sqrt{y}\right)^2 \cdot \left(\sqrt{y} \cdot \sqrt{x^3}\right)^3$
23. $\dfrac{3 \times 2^n - 4 \times 2^{n-2}}{2^n - 2^{n-1}}$
24. $\left(3^{n+6} - 5 \times 3^{n+1}\right) \div \left(7 \times 3^{n+2}\right)$
25. Sketch the graph of the following functions on the same diagram:

    (a) $y = 3^x$

    (b) $y = \left(\frac{1}{3}\right)^x$
26. Without using the calculator, compare the value of the following expressions:

(a) $2.5^{7.1}$ and $2.5^{8.5}$

(b) $0.35^{6.5}$ and $0.35^{5.6}$

(c) $1.03^{-2.1}$ and $1.03^{-3.2}$

(d) $\left(\sqrt{2}\right)^{\pi}$ and $\left(\sqrt{2}\right)^{\pi-3.5}$

(e) $0.01^{-\frac{1}{3}}$ and $0.01^{-\frac{1}{2}}$

(f) $2.7^{\sqrt{20}}$ and $2.7^{\sqrt[3]{35}}$

27. Given that $f_1 : x \to 2^{3x}$ and $f_2 : x \to 2^{x^2+2}$. Find the value of x such that $f_1(x) = f_2(x)$.

28. Given the function $f(x) = (0.4)^{x^2-x+1}$ and $g(x) = (0.4)^{6x+19}$. Find the value of x such that $f(x) = g(x)$.

23.2 Logarithms

Definition of Logarithms

If $a_n = x$, where $a > 0$ and $a \neq 1$, then we define $\log_a x = n$, and we say that n is the logarithm of x to the base a. In $\log_a x$, a is called the base, x is called the antilogarithm.

On the other hand, if $\log_a x = n$, then $a_n = x$. This is the inversible relationship between exponents and logarithms. That is,

$$\log_a x = n \iff a^n = x \qquad a > 0,\ a \neq 1,\ x > 0$$

Logarithms with base 10 are called common logarithms, and are usually written as $\log a$.

Another common logarithm is the natural logarithm, which has base e ($e \approx 2.71828182846$), and is usually written as $\ln x$.

Practice 3

Find the value of x in the following equations:

1. $\log x = 3$
2. $\log_x 27 = \dfrac{3}{2}$
3. $2\log_x(3\sqrt{3}) = 1$
4. $\log_2(16\sqrt{2}) = x$

Logarithmic Functions and Graphs

From the definition of logarithms, we can see that if $y = a^x$, then $x = \log_a y$. From the concept of inverse functions, we know that $y = \log_a x$ is the inverse function of $y = a^x$. Function $y = \log_a x$ is called the logarithmic function, where $a > 0$ and $a \neq 1$. Since the domain of $y = a^x$ is $\mathbb{R}$, and its range is $\mathbb{R}^+$, so the domain of $y = \log_a x$ is $\mathbb{R}^+$, and its range is $\mathbb{R}$.

Since the logarithmic function $y = \log_a x$ is the inverse function of the exponential function $y = a^x$, so the graph of $y = \log_a x$ is the reflection of the graph of $y = a^x$ about the line $y = x$. If we draw a curve of $y = a^x$, then reflect it about the line $y = x$, we can get the graph of $y = \log_a x$. For example, in the diagram below, the curves that are the reflection of the graphs of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ about the line $y = x$ are the graphs of $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ respectively.

From the diagram above, we can see that:

(1) Since the domain of $y = \log_a x$ is $x > 0$, so the graph of the function $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ are only at the right side of the y-axis.

(2) When $x = 1$, $y = 0$. Hence, the graph of logarithmic functions $y = a^x$ always passes through the point $(1, 0)$.

(3) For the function $y = \log_2 x$, when $x > 1$, $y > 0$; when $0 < x < 1$, $y < 0$. When the value of x increases, the value of y increases, that is, the function is an increasing function in the interval $(0, +\infty)$.

(4) For the function $y = \log_{\frac{1}{2}} x$, when $x > 1$, $y < 0$; when $0 < x < 1$, $y > 0$. When the value of x increases, the value of y decreases, that is, the function is a decreasing function in the interval $(0, +\infty)$.

When we are discussing about the graph and properties of a logarithmic function $y = \log_a x$, the following two cases are considered:

Practice 4

1. Without using the calculator, Compare the value of the following expressions:

2. (a) $\log 6$ and $\log 9$

   (b) $\log_{0.5} 4.2$ and $\log_{0.5} 3.9$

   (c) $\log_2 1.8$ abd $\log_4 5.8$.

3. Find the domain of the following functions:

   (a) $y = \log_a(x + 2)$

   (b) $y = \log_2(x^2 - 9)$

   (c) $y = \log_7 \dfrac{2}{3 - 2x}$

   (d) $y - \sqrt{\log_5(2 - x)}$

Exercise 23.2

1. Find the value of x for the following expression:

   (a) $\log_2 x = 4$

   (b) $\log_{125} x = \dfrac{1}{3}$

   (c) $\log_{16}(2\sqrt{2}) = x$

   (d) $\log_{\frac{1}{3}} 81 = x$

   (e) $\log_x 81 = 4$

   (f) $\log_x 49 = -2$

2. Sketch the graph of the following functions on the same set of axes:

   (a) $y = \log_5 x$

   (b) $y = \log_{\frac{1}{5}} x$

3. Without using the calculator, compare the value of the following expressions:

   (a) $\log_3 5$ and $\log_3 6$

   (b) $\log 1.51.4$ and $\log_{1.5} 1.6$

   (c) $\log_{\sqrt{3}} 4.8$ and $\log_{\sqrt{3}} 5.8$

   (d) $\log_{2.3} \pi$ and $\log_{2.3}(\pi - 3)$

   (e) $\log_{0.4} \sqrt{2}$ and $\log_{0.4} \sqrt{3}$

   (f) $\log_{\frac{1}{2}} 3$ and $\log_{\frac{1}{2}} \dfrac{1}{4}$

4. Find the domain of the following functions:

   (a) $y = \log_2(3 - 2x)$

   (b) $y = \log(x^2 + 1)$

   (c) $y = \log_5(9 - 16x^2)$

   (d) $y = \log_9 \dfrac{1}{x - 2}$

   (e) $y = \log_8 \sqrt{2x^2 - x - 3}$

   (f) $y = \dfrac{1}{\log_3(7x - 5)}$

23.3 Arithmetic Properties of Logarithms and Base Changing Formula

23.4 Exponential Equations

23.5 Logarithmic Equations

23.6 Compound Interest and Annuity

Chapter 24

Limits

24.1 Concept of Limits

24.2 Limits of Functions

24.3 Arithmetic Properties of Limits of Functions

Chapter 25

Differentiation

25.1 Gradient of Tangent Line on a Curve

25.2 Gradient of Tangent Line and Derivative

25.3 Law of Differentiation

25.4 Chain Rule - Differentiation of Composite Functions

25.5 Higher Order Derivatives

25.6 Implicit Differentiation

25.7 Two Basic Limits

25.8 Derivatives of Trigonometric Functions

25.9 Derivatives of Exponential Functions

25.10 Derivatives of Logarithmic Functions