Mathematics

Senior 3 Part I

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Introduction

Why this book?

Disclaimer

Acknowledgements

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Chapter 22

Differentiation

22.1 Gradient of Tangent Line on a Curve

As shown in the diagram below, sketch the graph of curve C on the Cartesian plane. and pick two points P and Q on the curve, the straight line that passes through P and Q is the secant line of the curve. When point Q moves closer to point P along the curve, the secant line PQ rotates along point P and approaches a limit straight line PT. The line PT is called the *tangent line* of the curve C at point P.

We know the fact that a line can be determined by a point on the line and its gradient. Therefore, in order to find the gradient of the tangent line of the curve C at point P, we have to first find the gradient of the tangent.

To find the gradient of the tangent line of curve $y = f(x)$ at point $P\left(x_0, f(x_0)\right)$, we can pick a point $\left(x_0 + \Delta x, f(x_0 + \Delta x)\right)$ near to point P on the curve, as shown in the diagram above. Denote the variable PR on the horizontal axis as Δx, and the corresponding variable RQ on the vertical axis as Δy, we have $\Delta y = f(x_0 + \Delta x) - f(x_0)$.

Hence, the gradient of the secant line PQ is $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$.

When $\Delta x \to 0$, if the limit of the expression above exist, it is the gradient of the tangent line of the curve $y = f(x)$ at point P, denoted as m, that is

$$\boxed{m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}}$$

22.1.1 Practice 1

1. Find the gradient of the tangent line of the curve $y = 2 - x^2$ at $x = 1$.

   Sol.

   Let $f(x) = 2 - x^2$.

   When $x = 1$, $f(1) = 2 - 1^2 = 1$.

   Take nearby point $Q(1 + \Delta x, f(1 + \Delta x))$ of $x = 1$ on the curve, we have

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(1+\Delta x) - f(1)}{\Delta x}\\ &= \frac{2 - (1+\Delta x)^2 - 1}{\Delta x}\\ &= \frac{2 - 1 - 2\Delta x - (\Delta x)^2 - 1}{\Delta x}\\ &= \frac{-2\Delta x - (\Delta x)^2}{\Delta x}\\ &= -2 - \Delta x\end{aligned}$$

   $\therefore$ The gradient of the tangent line of the curve at $x = 1$ is $m = \lim_{\Delta x \to 0}(-2 - \Delta x) = -2$.

2. Find the gradient of the tangent line of the curve $y = x^2 + 3x$ at $x = 2$.

   Sol.

   Let $f(x) = x^2 + 3x$. When $x = 2$, $f(2) = 2^2 + 3 \times 2 = 10$.

   Take nearby point $Q(2 + \Delta x, f(2 + \Delta x))$ of $x = 2$ on the curve, we have

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(2+\Delta x) - f(2)}{\Delta x}\\ &= \frac{(2+\Delta x)^2 + 3(2+\Delta x) - 10}{\Delta x}\\ &= \frac{4 + 4\Delta x + (\Delta x)^2 + 6 + 3\Delta x - 10}{\Delta x}\\ &= \frac{(\Delta x)^2 + 7\Delta x}{\Delta x}\\ &= \Delta x + 7\end{aligned}$$

   $\therefore$ The gradient of the tangent line of the curve at $x = 2$ is $m = \lim_{\Delta x \to 0}(\Delta x + 7) = 7$.

22.1.2 Exercise 25.1

1. Given the curve $y = x^2 + 1$. Assume that P is the point on the curve at $x = 2$, Q is a nearby point,

   (a) Find the variable Δx and the corresponding variable Δy when the x coordinates of point Q is 2.5, 2.25, 2.1, 2.05, 2.01, and 2.001 respectively. Hence, complete the table below.

x-coords of Q	y-coords of Q	Δx	Δy	Gradient of secant PQ $\dfrac{\Delta y}{\Delta x}$
$x = 2.5$				
$x = 2.25$				
$x = 2.1$				
$x = 2.05$				
$x = 2.01$				
$x = 2.001$				

   Sol.

   When $x = 2$, $y = 2^2 + 1 = 5$.

x-coords of Q	y-coords of Q	Δx	Δy	Gradient of secant PQ $\dfrac{\Delta y}{\Delta x}$
$x = 2.5$	$y = 7.25$	0.5	2.25	4.5
$x = 2.25$	$y = 6.0625$	0.25	1.0625	4.25
$x = 2.1$	$y = 5.41$	0.1	0.41	4.1
$x = 2.05$	$y = 5.2025$	0.05	0.2025	4.05
$x = 2.01$	$y = 5.0401$	0.01	0.0401	4.01
$x = 2.001$	$y = 5.004001$	0.001	0.004001	4.001

   (b) Inspect the gradient of secant PQ as point Q approaches point P. Hence, find the gradient of the tangent line of the curve at point P.

   Sol.

   As point Q approaches point P, the gradient of secant PQ approaches 4. Hence, the gradient of the tangent line of the curve at point P is $m = 4$.

2. Find the gradient of the tangent line of the curve $y = \frac{1}{3}x^2$ at $x = 2$.

Sol.

Let $f(x) = \frac{1}{3}x^2$.

When $x = 2$, $f(2) = \frac{1}{3} \times 2^2 = \frac{4}{3}$.

Take nearby point $Q(2 + \Delta x, f(2 + \Delta x))$ of $x = 2$ on the curve, we have

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{f(2+\Delta x) - f(2)}{\Delta x} \\
&= \frac{\frac{1}{3}(2+\Delta x)^2 - \frac{4}{3}}{\Delta x} \\
&= \frac{\frac{4}{3} + \frac{4}{3}\Delta x + \frac{1}{3}(\Delta x)^2 - \frac{4}{3}}{\Delta x} \\
&= \frac{\frac{4}{3}\Delta x + \frac{1}{3}(\Delta x)^2}{\Delta x} \\
&= \frac{4}{3} + \frac{1}{3}\Delta x
\end{aligned}$$

$\therefore$ The gradient of the tangent line of the curve at $x = 2$ is $m = \lim_{\Delta x \to 0} \left(\frac{4}{3} + \frac{1}{3}\Delta x \right) = \frac{4}{3}$.

3. Find the gradient of the tangent line of the curve $y = \frac{10}{x}$ at point $Q(2, 5)$.

Sol.

Let $f(x) = \frac{10}{x}$.

When $x = 2$, $f(2) = 5$.

Take nearby point $P(2 + \Delta x, f(2 + \Delta x))$ of $x = 2$ on the curve, we have

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{f(2+\Delta x) - f(2)}{\Delta x} \\
&= \frac{\frac{10}{2+\Delta x} - 5}{\Delta x} \\
&= \frac{\frac{10 - 5(2+\Delta x)}{(2+\Delta x)}}{\Delta x} \\
&= \frac{-5\Delta x}{(2+\Delta x)\Delta x} \\
&= \frac{-5}{(2+\Delta x)}
\end{aligned}$$

$\therefore$ The gradient of the tangent line of the curve at $x = 2$ is $m = \lim_{\Delta x \to 0} \frac{-5}{(2+\Delta x)} = -\frac{5}{2}$

4. Find the gradient of the tangent line of the curve $y = \dfrac{4}{x} - 5$ at $x = 1$.

Sol.

Let $f(x) = \dfrac{4}{x} - 5$.

When $x = 1$, $f(1) = \dfrac{4}{1} - 5 = -1$.

Take nearby point $Q(1 + \Delta x, f(1 + \Delta x))$ of $x = 1$ on the curve, we have

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{f(1 + \Delta x) - f(1)}{\Delta x} \\
&= \frac{\dfrac{4}{1 + \Delta x} - 5 - (-1)}{\Delta x} \\
&= \frac{\dfrac{4 - 4(1 + \Delta x)}{(1 + \Delta x)}}{\Delta x} \\
&= \frac{-4\Delta x}{(1 + \Delta x)\Delta x} \\
&= \frac{-4}{(1 + \Delta x)}
\end{aligned}$$

$\therefore$ The gradient of the tangent line of the curve at $x = 1$ is $m = \displaystyle\lim_{\Delta x \to 0} \frac{-4}{(1 + \Delta x)} = -4$

5. Find the gradient of the tangent line of the curve $y = \dfrac{1}{2}x^3 + 1$ at point $P(-2, -3)$.

Sol.

Let $f(x) = \dfrac{1}{2}x^3 + 1$.

When $x = -2$, $f(-2) = \dfrac{1}{2}(-2)^3 + 1 = -3$.

Take nearby point $Q(-2 + \Delta x, f(-2 + \Delta x))$ of $x = -2$ on the curve, we have

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{f(-2 + \Delta x) - f(-2)}{\Delta x} \\
&= \frac{\frac{1}{2}(-2 + \Delta x)^3 + 1 - (-3)}{\Delta x} \\
&= \frac{(-2 + \Delta x)^3 + 8}{2\Delta x} \\
&= \frac{(\Delta x)^3 - 6(\Delta x)^2 + 12\Delta x}{2\Delta x} \\
&= \frac{(\Delta x)^2 - 6\Delta x + 12}{2}
\end{aligned}$$

$\therefore$ The gradient of the tangent line of the curve at $x = -2$ is $m = \displaystyle\lim_{\Delta x \to 0} \frac{(\Delta x)^2 - 6\Delta x + 12}{2} = 6$

22.2 Gradient of Tangent Line and Derivative

In the last section, we have learnt that, if $P\left(x_0, f(x_0)\right)$ and $Q\left(x_0+\Delta x, f(x_0+\Delta x)\right)$ are two points on the curve $y = f(x)$, then the gradient of the secant line PQ is $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$. As point Q approaches point P, that is, as $\Delta x \to 0$, the gradient of the secant line PQ approaches the tangent line PT.

Hence, the gradient of the tangent line PT is

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

Definition of Derivative

Let the function $y = f(x)$ be defined at $x = x_0$ and its nearby points, when there exists a variable Δx of x at x_0, there exist a corresponding variable $\Delta y = f(x_0+\Delta x) - f(x_0)$ of the function y. When $\Delta x \to 0$, the limit of $\dfrac{\Delta y}{\Delta x}$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

exists, $y = f(x)$ is said to be *differentiable* at $x = x_0$, and the limit is called the *derivative* of $y = f(x)$ at $x = x_0$, denoted as $f'(x_0)$, that is

$$\boxed{f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}}$$

If the limit above does not exist, $y = f(x)$ is said to be *non-differentiable* at $x = x_0$.

If function $y = f(x)$ is differentiable at every point of an interval (a, b). Each defined value x_0 in the interval (a, b) corresponds to a derivative value $f'(x_0)$, thus forming a new function in the interval (a, b). This new function is called the *derivative function* of $f(x)$, denoted as $f'(x)$ or y'.

From the definition of derivative, we can get the derivative

$$y' = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$f'(x)$ is often being denoted as $\dfrac{dy}{dx}$ as well, and is called *differentiation of* $y = f(x)$ *with respect to* x.

From the definition of derivative, we can conclude the following steps to find the derivative of a function $y = f(x)$:

1. Find the variable $\Delta y = f(x+\Delta x) - f(x)$ of the function.
2. Find the ratio of the variable Δy to Δx, that is, $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$.
3. Find the limit of the ratio above, that is, $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$.

The method above to find the derivative using the definition of derivative is called the **first principle** of differentiation.

Note that derivative function is also called the derivative. Unless otherwise stated, finding the derivative means finding the derivative function.

Also note that the derivative function $f'(x)$ of the function $y = f(x)$ is different from the derivative $f'(x_0)$ of the function $y = f(x)$ at $x = x_0$. $f'(x)$ is a function, while $f'(x_0)$ is a value, but they are related to each other. $f'(x_0)$ is the function value of the derivative function $f'(x)$ at $x = x_0$, and is called the *derivative value* at $x = x_0$.

22.2.1 Practice 2

1. Find the derivative of the function $y = 2x^2$, and find the derivative value at $x = 1$.

   Sol.

$$\begin{aligned}\Delta y &= 2(x+\Delta x)^2 - 2x^2 \\ &= 2x^2 + 4x\Delta x + 2\Delta x^2 - 2x^2 \\ &= 4x\Delta x + 2\Delta x^2\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{4x\Delta x + 2\Delta x^2}{\Delta x} \\ &= 4x + 2\Delta x\end{aligned}$$

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0} 4x + 2\Delta x \\ \frac{dy}{dx} &= 4x\end{aligned}$$

   $\therefore$ The derivative value at $x = 1$ is $4(1) = 4$.

2. Let function $y = 4 - 3x + x^2$. Find:

   (a) $\dfrac{\Delta y}{\Delta x}$

   Sol.

$$\begin{aligned}\Delta y &= 4 - 3(x+\Delta x) + (x+\Delta x)^2 - (4 - 3x + x^2) \\ &= 4 - 3x - 3\Delta x + x^2 + 2x\Delta x + \Delta x^2 - 4 + 3x - x^2 \\ &= -3\Delta x + 2x\Delta x + \Delta x^2\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{-3\Delta x + 2x\Delta x + \Delta x^2}{\Delta x} \\ &= \Delta x + 2x - 3\end{aligned}$$

(b) $\dfrac{dy}{dx}$

Sol.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \Delta x + 2x - 3$$
$$\frac{dy}{dx} = 2x - 3$$

(c) The derivative value of the function at $x = 2$.

Sol.

$$\begin{aligned} \frac{dy}{dx} &= 2x - 3 \\ &= 2(2) - 3 \\ &= 1 \end{aligned}$$

22.2.2 Exercise 25.2

Find teh derivative of the following functions using the definition of derivative. Hence, find the derivative value at $x = 2$ (Question 1 to 12):

1. $f(x) = 3x + 2$

Sol.

$$\begin{aligned} \Delta y &= 3(x + \Delta x) + 2 - (3x + 2) \\ &= 3x + 3\Delta x + 2 - 3x - 2 \\ &= 3\Delta x \end{aligned}$$

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{3\Delta x}{\Delta x} \\ &= 3 \end{aligned}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} 3$$
$$f'(x) = 3$$

$\therefore$ The derivative value at $x = 2$ is 3.

2. $f(x) = x^2 + 1$

Sol.

$$\begin{aligned} \Delta y &= (x + \Delta x)^2 + 1 - (x^2 + 1) \\ &= x^2 + 2x\Delta x + \Delta x^2 - x^2 - 1 \\ &= 2x\Delta x + \Delta x^2 \end{aligned}$$

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{2x\Delta x + \Delta x^2}{\Delta x} \\ &= 2x + \Delta x \end{aligned}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x)$$
$$f'(x) = 2x$$

$\therefore$ The derivative value at $x = 2$ is $2(2) = 4$.

3. $f(x) = 4x^2 - 3$

Sol.

$$\begin{aligned}\Delta y &= 4(x+\Delta x)^2 - 3 - (4x^2 - 3)\\ &= 4x^2 + 8x\Delta x + 4\Delta x^2 - 3 - 4x^2 + 3\\ &= 8x\Delta x + 4\Delta x^2\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{8x\Delta x + 4\Delta x^2}{\Delta x}\\ &= 8x + 4\Delta x\end{aligned}$$

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0}(8x + 4\Delta x)\\ f'(x) &= 8x\end{aligned}$$

$\therefore$ The derivative value at $x = 2$ is $8(2) = 16$.

4. $f(x) = 2 - x^2$

Sol.

$$\begin{aligned}\Delta y &= 2 - (x+\Delta x)^2 - (2 - x^2)\\ &= 2 - x^2 - 2x\Delta x - \Delta x^2 - 2 + x^2\\ &= -2x\Delta x - \Delta x^2\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{-2x\Delta x - \Delta x^2}{\Delta x}\\ &= -2x - \Delta x\end{aligned}$$

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0}(-2x - \Delta x)\\ f'(x) &= -2x\end{aligned}$$

$\therefore$ The derivative value at $x = 2$ is $-2(2) = -4$.

5. $y = x^2 - 3x$

Sol.

$$\begin{aligned}\Delta y &= (x+\Delta x)^2 - 3(x+\Delta x) - (x^2 - 3x)\\ &= x^2 + 2x\Delta x + \Delta x^2 - 3x - 3\Delta x - x^2\\ &\quad + 3x\\ &= 2x\Delta x + \Delta x^2 - 3\Delta x\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{2x\Delta x + \Delta x^2 - 3\Delta x}{\Delta x}\\ &= 2x + \Delta x - 3\end{aligned}$$

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0}(2x + \Delta x - 3)\\ \frac{dy}{dx} &= 2x - 3\end{aligned}$$

$\therefore$ The derivative value at $x = 2$ is $2(2) - 3 = 1$.

6. $y = x^2 - x - 3$

Sol.

$$\begin{aligned}\Delta y &= (x+\Delta x)^2 - (x+\Delta x) - 3 - (x^2 - x - 3)\\ &= x^2 + 2x\Delta x + \Delta x^2 - x - \Delta x - 3 - x^2 + x\\ &\quad + 3\\ &= 2x\Delta x + \Delta x^2 - \Delta x\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{2x\Delta x + \Delta x^2 - \Delta x}{\Delta x}\\ &= 2x + \Delta x - 1\end{aligned}$$

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0}(2x + \Delta x - 1)\\ \frac{dy}{dx} &= 2x - 1\end{aligned}$$

$\therefore$ The derivative value at $x = 2$ is $2(2) - 1 = 3$.

7. $y = 2x^2 + 3x - 1$

Sol.

$$\begin{aligned}\Delta y &= 2(x+\Delta x)^2 + 3(x+\Delta x) - 1 \\ &\quad - (2x^2 + 3x - 1) \\ &= 2(x^2 + 2x\Delta x + \Delta x^2) + 3x + 3\Delta x \\ &\quad - 1 - 2x^2 \\ &\quad - 3x + 1 \\ &= 4x\Delta x + 2\Delta x^2 + 3\Delta x\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{4x\Delta x + 2\Delta x^2 + 3\Delta x}{\Delta x} \\ &= 4x + 2\Delta x + 3\end{aligned}$$

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0}(4x + 2\Delta x + 3) \\ \frac{dy}{dx} &= 4x + 3\end{aligned}$$

$\therefore$ The derivative value at $x = 2$ is $4(2) + 3 = 11$.

8. $y = x^3 + 1$

Sol.

$$\begin{aligned}\Delta y &= (x+\Delta x)^3 + 1 - (x^3 + 1) \\ &= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + 1 \\ &\quad - x^3 - 1 \\ &= 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x} \\ &= 3x^2 + 3x\Delta x + \Delta x^2\end{aligned}$$

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0}\left(3x^2 + 3x\Delta x + \Delta x^2\right) \\ \frac{dy}{dx} &= 3x^2\end{aligned}$$

$\therefore$ The derivative value at $x = 2$ is $3(2)^2 = 12$.

9. $y = x^3 - 2x$

Sol.

$$\begin{aligned}\Delta y &= (x+\Delta x)^3 - 2(x+\Delta x) - (x^3 - 2x) \\ &= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 \\ &\quad - 2x - 2\Delta x - x^3 + 2x \\ &= 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - 2\Delta x\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - 2\Delta x}{\Delta x} \\ &= 3x^2 + 3x\Delta x + \Delta x^2 - 2\end{aligned}$$

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} &= \lim_{\Delta x\to 0}\left(3x^2 + 3x\Delta x + \Delta x^2 - 2\right) \\ \frac{dy}{dx} &= 3x^2 - 2\end{aligned}$$

$\therefore$ The derivative value at $x = 2$ is $3(2)^2 - 2 = 10$.

10. $y = \sqrt{x+2}$

Sol.

$$\Delta y = \sqrt{x+\Delta x+2} - \sqrt{x+2}$$

$$\frac{\Delta y}{\Delta x} = \frac{\sqrt{x+\Delta x+2} - \sqrt{x+2}}{\Delta x}$$

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+2} - \sqrt{x+2}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+2} - \sqrt{x+2}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x+2} + \sqrt{x+2}}{\sqrt{x+\Delta x+2} + \sqrt{x+2}} \\
&= \lim_{\Delta x \to 0} \frac{x+\Delta x+2-x+-2}{\Delta x\left(\sqrt{x+\Delta x+2} + \sqrt{x+2}\right)} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x+2} + \sqrt{x+2}\right)} \\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x+2} + \sqrt{x+2}} \\
&= \frac{1}{\sqrt{x+2} + \sqrt{x+2}} \\
\frac{dy}{dx} &= \frac{1}{2\sqrt{x+2}}
\end{aligned}$$

$\therefore$ The derivative value at $x = 2$ is $\dfrac{1}{2\sqrt{2+2}} = \dfrac{1}{4}$.

11. $y = \dfrac{2}{x^2}$

Sol.

$$\begin{aligned}
\Delta y &= \frac{2}{(x+\Delta x)^2} - \frac{2}{x^2} \\
&= \frac{2}{x^2 + 2x\Delta x + \Delta x^2} - \frac{2}{x^2} \\
&= \frac{2x^2 - 2x^2 - 4x\Delta x - 2\Delta x^2}{x^2\left(x^2 + 2x\Delta x + \Delta x^2\right)} \\
&= \frac{-4x\Delta x - 2\Delta x^2}{x^2\left(x^2 + 2x\Delta x + \Delta x^2\right)} \\
&= \frac{-4x\Delta x - 2\Delta x^2}{x^4 + 2x^3\Delta x + x^2\Delta x^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{-4x\Delta x - 2\Delta x^2}{x^4 + 2x^3\Delta x + x^2\Delta x^2} \cdot \frac{1}{\Delta x} \\
&= \frac{-4x - 2\Delta x}{x^4 + 2x^3\Delta x + x^2\Delta x^2}
\end{aligned}$$

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{-4x - 2\Delta x}{x^4 + 2x^3\Delta x + x^2\Delta x^2} \\
\frac{dy}{dx} &= \frac{-4x}{x^4} \\
&= -\frac{4}{x^3}
\end{aligned}$$

$\therefore$ The derivative value at $x = 2$ is $-\dfrac{4}{2^3} = -\dfrac{1}{2}$.

12. $y = \dfrac{1}{1-x}$

Sol.

$$\begin{aligned}
\Delta y &= \frac{1}{1-(x+\Delta x)} - \frac{1}{1-x} \\
&= \frac{1}{1-x-\Delta x} - \frac{1}{1-x} \\
&= \frac{1-x-1+x+\Delta x}{(1-x-\Delta x)(1-x)} \\
&= \frac{\Delta x}{(1-x-\Delta x)(1-x)}
\end{aligned}$$

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{\Delta x}{(1-x-\Delta x)(1-x)} \cdot \frac{1}{\Delta x} \\
&= \frac{1}{(1-x-\Delta x)(1-x)}
\end{aligned}$$

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{1}{(1-x-\Delta x)(1-x)} \\
\frac{dy}{dx} &= \frac{1}{(1-x)(1-x)} \\
&= \frac{1}{(1-x)^2}
\end{aligned}$$

$\therefore$ The derivative value at $x = 2$ is $\dfrac{1}{(1-2)^2} = 1$.

13. Given the function $y = \dfrac{3}{x-2}$, $x \neq 2$, find the following:

(a) $\dfrac{\Delta y}{\Delta x}$

Sol.

$$\begin{aligned}
\Delta y &= \frac{3}{(x+\Delta x)-2} - \frac{3}{x-2} \\
&= \frac{3}{x+\Delta x-2} - \frac{3}{x-2} \\
&= \frac{3(x-2) - 3(x+\Delta x-2)}{(x+\Delta x-2)(x-2)} \\
&= \frac{3x-6-3x-3\Delta x+6}{(x+\Delta x-2)(x-2)} \\
&= \frac{-3\Delta x}{(x+\Delta x-2)(x-2)}
\end{aligned}$$

$$\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{-3\Delta x}{(x+\Delta x-2)(x-2)} \cdot \frac{1}{\Delta x} \\
&= -\frac{3}{(x+\Delta x-2)(x-2)}
\end{aligned}$$

(b) $\dfrac{dy}{dx}$

Sol.

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} -\frac{3}{(x+\Delta x-2)(x-2)} \\
&= -\frac{3}{(x-2)^2}
\end{aligned}$$

(c) the derivative value at $x = 3$

Sol.

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{3}{(x-2)^2} \\
&= -\frac{3}{(3-2)^2} \\
&= -3
\end{aligned}$$

14. Given the function $f(x) = \dfrac{2}{\sqrt{x}}$, $x \neq 0$. Find (a) $f'(x)$; (b) $f'(1)$.

Sol.

$$
\begin{aligned}
\Delta y &= \frac{2}{\sqrt{x+\Delta x}} - \frac{2}{\sqrt{x}} \\
&= \frac{2\sqrt{x} - 2\sqrt{x+\Delta x}}{\sqrt{x(x+\Delta x)}} \\
&= \frac{2\left(\sqrt{x} - \sqrt{x+\Delta x}\right)}{\sqrt{x^2 + x\Delta x}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\Delta y}{\Delta x} &= \frac{2\left(\sqrt{x} - \sqrt{x+\Delta x}\right)}{\Delta x\sqrt{x^2 + x\Delta x}} \\
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{2\left(\sqrt{x} - \sqrt{x+\Delta x}\right)}{\Delta x\sqrt{x^2 + x\Delta x}} \\
&= \lim_{\Delta x \to 0} \frac{2\left(\sqrt{x} - \sqrt{x+\Delta x}\right)}{\Delta x\sqrt{x^2 + x\Delta x}} \cdot \frac{\sqrt{x} + \sqrt{x+\Delta x}}{\sqrt{x} + \sqrt{x+\Delta x}} \\
&= \lim_{\Delta x \to 0} \frac{2\left(x - x - \Delta x\right)}{\Delta x\left(\sqrt{x^2 + x\Delta x}\right)\left(\sqrt{x} + \sqrt{x+\Delta x}\right)} \\
&= \lim_{\Delta x \to 0} \frac{-2\Delta x}{\Delta x\left(\sqrt{x^2 + x\Delta x}\right)\left(\sqrt{x} + \sqrt{x+\Delta x}\right)} \\
&= \lim_{\Delta x \to 0} \frac{-2}{\left(\sqrt{x^2 + x\Delta x}\right)\left(\sqrt{x} + \sqrt{x+\Delta x}\right)}
\end{aligned}
$$

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{-2}{\left(\sqrt{x^2 + x\Delta x}\right)\left(\sqrt{x} + \sqrt{x+\Delta x}\right)} \\
&= \frac{-2}{x\left(\sqrt{x} + \sqrt{x}\right)} \\
&= \frac{-2}{2x\sqrt{x}} \\
&= -\frac{1}{\sqrt{x^3}}
\end{aligned}
$$

$$
\begin{aligned}
f'(1) &= -\frac{1}{\sqrt{1^3}} \\
&= -1
\end{aligned}
$$

15. Given the curve $y = 3 - 4x - 5x^2$.

(a) Find $\dfrac{dy}{dx}$ using the definition of derivative.

Sol.

$$\begin{aligned}\Delta y &= 3 - 4(x + \Delta x) - 5(x + \Delta x)^2 - (3 - 4x - 5x^2)\\ &= 3 - 4x - 4\Delta x - 5x^2 - 10x\Delta x - 5\Delta x^2 - 3 + 4x + 5x^2\\ &= -4\Delta x - 10x\Delta x - 5\Delta x^2\end{aligned}$$

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{-4\Delta x - 10x\Delta x - 5\Delta x^2}{\Delta x}\\ &= -4 - 10x - 5\Delta x\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x}\\ &= \lim_{\Delta x\to 0} -4 - 10x - 5\Delta x\\ &= -4 - 10x\end{aligned}$$

(b) Find the gradient of the tangent to the curve at the point where $x = -2$.

Sol.

$$\begin{aligned}\frac{dy}{dx} &= -4 - 10x\\ &= -4 - 10(-2)\\ &= 16\end{aligned}$$

(c) If the gradient of the tangent to the curve at point P is 6, find the coordinates of point P.

Sol.

$$\begin{aligned}\frac{dy}{dx} &= 6\\ -4 - 10x &= 6\\ -10x &= 10\\ x &= -1\\ y &= 3 - 4x - 5x^2\\ &= 3 - 4(-1) - 5(-1)^2\\ &= 3 + 4 - 5\\ &= 2\end{aligned}$$

$$P = (-1, 2)$$

22.3 Law of Differentiation

Since the method of finding the derivative using the definition of derivative is very complicated, we need to derive some formulas from the definition of the derivative, then we can use these formulas to find the derivatives.

Derivative of Power Function

Let $y = f(x) = c$, where c is a constant.

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(x+\Delta x) - f(x)}{\Delta x}\\ &= \frac{c-c}{\Delta x}\\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(c) &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\\ &= \lim_{\Delta x \to 0} 0\\ &= 0\end{aligned}$$

$$\boxed{\frac{d}{dx}(c) = 0}$$

Let $y = f(x) = x^n$, where n is a positive integer.

$$\begin{aligned}\Delta y = f(x+\Delta x) - f(x) &= f(x-\Delta x) - f(x)\\ &= (x+\Delta x)^n - x^n\\ &= \left[x^n + {}_nC_1 x^{n-1}\Delta x + {}_nC_2 x^{n-2}(\Delta x)^2 + \cdots + {}_nC_n(\Delta x)^n\right] - x^n\\ &= {}_nC_1 x^{n-1}\Delta x + {}_nC_2 x^{n-2}(\Delta x)^2 + \cdots + {}_nC_n(\Delta x)^n\\ \frac{\Delta y}{\Delta x} &= {}_nC_1 x^{n-1} + {}_nC_2 x^{n-2}\Delta x + \cdots + {}_nC_n(\Delta x)^{n-1}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}(x^n) &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\\ &= \lim_{\Delta x \to 0} \left[{}_nC_1 x^{n-1} + {}_nC_2 x^{n-2}\Delta x + \cdots + {}_nC_n(\Delta x)^{n-1}\right]\\ &= {}_nC_1 x^{n-1}\end{aligned}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The derivation above only considered the case where n is a positive integer. In fact, the formula above is also valid when n is a negative integer or a rational number.

Derivative of Product of Function and Constant

Let $y = cu$, where c is a constant and u is a differentiable function of x.

$$\begin{aligned}
y + \Delta y &= c(u + \Delta u) = cu + c\Delta u \\
\Delta y &= c\Delta u \\
\frac{\Delta y}{\Delta x} &= c\frac{\Delta u}{\Delta x}
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dx}(cu) &= \lim_{\Delta x \to 0}\left(c\frac{\Delta u}{\Delta x}\right) \\
&= c\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} \\
&= c\frac{du}{dx}
\end{aligned}$$

$$\frac{d}{dx}(cu) = c\frac{du}{dx} \quad \text{(where } c \text{ is a constant, } u \text{ is a differentiable function of } x\text{)}$$

Derivative of Sum and Difference of Functions

Let $y = u \pm v$, where u and v are differentiable functions of x.

$$
\begin{aligned}
y + \Delta y &= (u + \Delta u) \pm (v + \Delta v) \\
\Delta y &= \Delta u \pm \Delta v \\
\frac{\Delta y}{\Delta x} &= \frac{\Delta u}{\Delta x} \pm \frac{\Delta v}{\Delta x}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}(u \pm v) &= \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} \pm \frac{\Delta v}{\Delta x} \right) \\
&= \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \pm \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \\
&= \frac{du}{dx} \pm \frac{dv}{dx}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}(u + v) &= \frac{du}{dx} + \frac{dv}{dx} \\
\frac{d}{dx}(u - v) &= \frac{du}{dx} - \frac{dv}{dx}
\end{aligned}
\qquad \text{(where } u \text{ and } v \text{ are differentiable functions of } x\text{)}
$$

The above formulae can be extended to the case where there are more than two functions being added or subtracted. That is,

$$
\frac{d}{dx}(u_1 \pm u_2 \pm \cdots \pm u_n) = \frac{du_1}{dx} \pm \frac{du_2}{dx} \pm \cdots \pm \frac{du_n}{dx}
$$

Product Rule

Let $y = uv$, where u and v are differentiable functions of x.

$$
\begin{aligned}
y + \Delta y &= (u + \Delta u)(v + \Delta v) \\
&= uv + u\Delta v + v\Delta u + \Delta u \Delta v \\
\Delta y &= u\Delta v + v\Delta u + \Delta u \Delta v \\
\frac{\Delta y}{\Delta x} &= u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}
\end{aligned}
$$

Given that $\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx}$ and $\lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = \frac{dv}{dx}$,

$$\begin{aligned}
\text{Hence, } \lim_{\Delta x \to 0} \Delta v &= \lim_{\Delta x \to 0} \left(\frac{\Delta v}{\Delta x} \cdot \Delta x \right) \\
&= \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta x \\
&= v'(x) \cdot 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\therefore \lim_{\Delta x \to 0} (uv) &= \lim_{\Delta x \to 0} \left(u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta v \frac{\Delta v}{\Delta x} \right) \\
&= u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \right) \left(\lim_{\Delta x \to 0} \Delta v \right) \\
&= u\frac{dv}{dx} + v\frac{du}{dx} + 0 \\
&= u\frac{dv}{dx} + v\frac{du}{dx}
\end{aligned}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{(where } u \text{ and } v \text{ are differentiable functions of } x\text{)}$$

This has proven that if $\frac{dv}{dx}$ exists at $x = x_0$, then

$$\begin{aligned}
&\lim_{\Delta x \to 0} v(x_0 + \Delta x) - v(x_0) \\
&= \lim_{\Delta x \to 0} \Delta v \\
&= 0
\end{aligned}$$

From that, we can conclude that

$$\lim_{\Delta x \to 0} v(x_0 + \Delta x) = v(x_0)$$

That is, $y = v(x)$ is continuous at $x = x_0$.
Note that $\frac{d}{dx}(uv) \neq \frac{d}{dx}(u) \cdot \frac{d}{dx}(v)$.

22.3.1 Practice 3

Find the derivative of the following functions:

1. $y = 5x$

   Sol.

   $$y' = 5$$

2. $y = \sqrt{x}$

   Sol.

   $$y' = \frac{1}{2\sqrt{x}}$$

3. $y = 6x^3 + 3x^2 - 5$

   Sol.

   $$y' = 18x^2 + 6x$$

4. $y = (2x-3)\left(x^2+2\right)$

   Sol.

   $$\begin{aligned} y' &= (x^2+2)(2x-3)' + (2x-3)(x^2+2)' \\ &= (x^2+2)(2) + (2x-3)(2x) \\ &= 2x^2 + 4 + 4x^2 - 6x \\ &= 6x^2 - 6x + 4 \end{aligned}$$

5. $y = \frac{2}{x} + \frac{1}{x^2}$

   Sol.

   $$\begin{aligned} y' &= \left(\frac{2x+1}{x^2}\right)' \\ &= \frac{(2x+1)'x^2 - (2x+1)(x^2)'}{x^4} \\ &= \frac{2x^2 - (2x+1)(2x)}{x^4} \\ &= \frac{2x^2 - 4x^2 - 2x}{x^4} \\ &= \frac{-2x^2 - 2x}{x^4} \\ &= \frac{-2x(x+1)}{x^4} \\ &= -\frac{2(x+1)}{x^3} \end{aligned}$$

6. $y = 2x^3 - 3x + \frac{7}{\sqrt{x}}$

   Sol.

   $$\begin{aligned} y' &= 6x^2 - 3 - \frac{7}{2\sqrt{x^3}} \\ &= 6x^2 - 3 - \frac{7}{2}x^{-\frac{3}{2}} \end{aligned}$$

22.3.2 Exercise 25.3a

Find the derivative of the following functions (Question 1 to 18):

1. $y = 2x^3 + 2$

   Sol.

   $$y' = 6x^2$$

2. $y = \frac{1}{2}x^2 - \frac{1}{3}x + 2$

   Sol.

   $$y' = x - \frac{1}{3}$$

3. $y = x^5 - \frac{1}{4}x^4 + 3x^2 - 4$

   Sol.

   $$y' = 5x^4 - x^3 + 6x$$

4. $y = 2x^2 + 4x^3 - 7x^4$

   Sol.

   $$y' = 4x + 12x^2 - 28x^3$$

5. $y = x^2 - 3x + \frac{2}{x} - \frac{4}{x^2}$

   Sol.

   $$y' = 2x - 3 + \frac{2}{x^2} - \frac{8}{x^3}$$

6. $y = 2x^3 - 4x - \frac{3}{x^2} + \frac{4}{x^3}$

   Sol.

   $$y' = 6x^2 - 4 + \frac{6}{x^3} - \frac{12}{x^4}$$

7. $y = \sqrt{x} + \frac{2}{\sqrt{x}} + 3$

   Sol.

   $$\begin{aligned} y' &= \left(x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} + 3\right)' \\ &= \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} \end{aligned}$$

8. $y = 3\sqrt{x} - 5\sqrt[3]{x^2}$

   Sol.

   $$\begin{aligned} y' &= \left(3x^{\frac{1}{2}} - 5x^{\frac{2}{3}}\right)' \\ &= \frac{3}{2}x^{-\frac{1}{2}} - \frac{10}{3}x^{-\frac{1}{3}} \end{aligned}$$

9. $y = (2x-5)^2$

   Sol.

   $$\begin{aligned} y' &= (4x^2 - 20x + 25)' \\ &= 8x - 20 \end{aligned}$$

10. $y = \left(x + \frac{1}{x}\right)^2$

    Sol.

    $$\begin{aligned} y' &= \left(x^2 + 2 + \frac{1}{x^2}\right)' \\ &= \left(x^2 + 2 + x^{-2}\right)' \\ &= 2x - 2x^{-3} \\ &= 2x - \frac{2}{x^3} \end{aligned}$$

11. $y = \dfrac{x^3 + 4x^2 - x + 2}{x}$

Sol.

$$\begin{aligned} y' &= \left(x^2 + 4x - 1 + 2x^{-1}\right)' \\ &= 2x + 4 - \frac{2}{x^2} \end{aligned}$$

12. $y = \dfrac{2x^4 + 3x^2 - 6}{x^2}$

Sol.

$$\begin{aligned} y' &= \left(2x^2 + 3 - 6x^{-2}\right)' \\ &= 4x + 12x^{-3} \\ &= 4x + \frac{12}{x^3} \end{aligned}$$

13. $y = \dfrac{\sqrt[3]{x} - 2}{\sqrt{x}}$

Sol.

$$\begin{aligned} y' &= \left(x^{-\frac{1}{6}} - 2x^{-\frac{1}{2}}\right)' \\ &= -\frac{1}{6}x^{-\frac{7}{6}} + x^{-\frac{3}{2}} \end{aligned}$$

14. $y = (x + 2)\left(x^2 + 3x - 8\right)$

Sol.

$$\begin{aligned} y' &= \left(x^3 + 3x^2 - 8x + 2x^2 + 6x - 16\right)' \\ &= \left(x^3 + 5x^2 - 2x - 16\right)' \\ &= 3x^2 + 10x - 2 \end{aligned}$$

15. $y = (2 + 3x)\left(1 + x - x^2\right)$

Sol.

$$\begin{aligned} y' &= (1 + x - x^2)(2 + 3x)' + (2 + 3x) \\ &\quad (1 + x - x^2)' \\ &= (1 + x - x^2)(3) + (2 + 3x)(1 - 2x) \\ &= 3 + 3x - 3x^2 + 2 - x - 6x^2 \\ &= 5 + 2x - 9x^2 \end{aligned}$$

16. $y = (x + 2)(x - 2)\left(x^2 + 4\right)$

Sol.

$$\begin{aligned} y' &= \left[(x^2 - 4)(x^2 + 4)\right]' \\ &= \left(x^4 - 16\right)' \\ &= 4x^3 \end{aligned}$$

17. $y = (x - 1)^2(3x + 5)$

Sol.

$$\begin{aligned} y' &= \left[(x^2 - 2x + 1)(3x + 5)\right]' \\ &= (x^2 - 2x + 1)(3x + 5)' \\ &\quad + (3x + 5)(x^2 - 2x + 1)' \\ &= (x^2 - 2x + 1)(3) + (3x + 5)(2x - 2) \\ &= 3x^2 - 6x + 3 + 6x^2 + 4x - 10 \\ &= 9x^2 - 2x - 7 \end{aligned}$$

18. $y = \left(x^2 + 1\right)(3x - 1)\left(1 - x^2\right)$

Sol.

$$\begin{aligned} y' &= \left[(1 - x^4)(3x - 1)\right]' \\ &= (1 - x^4)'(3x - 1) + (3x - 1)'(1 - x^4) \\ &= (-4x^3)(3x - 1) + (3)(1 - x^4) \\ &= -12x^4 + 4x^3 + 3 - 3x^4 \\ &= -15x^4 + 4x^3 + 3 \end{aligned}$$

19. If the gradient of the curve $y = x^3 + 6x^2 + 45x + 12$ at point A is 36, find the coordinates of A.

Sol.

$$
\begin{aligned}
y' &= \left(x^3 + 6x^2 + 45x + 12\right)' \\
36 &= 3x^2 + 12x + 45 \\
12 &= x^2 + 4x + 15 \\
0 &= x^2 + 4x + 3 \\
0 &= (x+3)(x+1)
\end{aligned}
$$

$$x = -3 \text{ or } x = -1$$

$$
\begin{aligned}
y(-3) &= -27 + 54 - 135 + 12 \\
&= -96 \\
y(-1) &= -1 + 6 - 45 + 12 \\
&= -28
\end{aligned}
$$

Therefore, the coordinates of A are $(-3, -96)$ and $(-1, -28)$.

20. Given the functions $f(x) = f(x) = x^2 + 3x + 4$ and $g(x) = x^3 + x^2 + 7$. If $f'(a) = g'(a)$, find the value of a.

Sol.

$$
\begin{aligned}
f'(x) &= 2x + 3 \\
g'(x) &= 3x^2 + 2x
\end{aligned}
$$

$$
\begin{aligned}
f'(a) &= g'(a) \\
2a + 3 &= 3a^2 + 2a \\
3a^2 - 3 &= 0 \\
a^2 &= 1 \\
a &= \pm 1
\end{aligned}
$$

Quotient Rule

Let $y = \dfrac{u}{v}$, where u and v are differentiable functions of x.

$$\begin{aligned}
y + \Delta y &= \frac{u + \Delta u}{v + \Delta v} \\
\Delta y &= \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} \\
&= \frac{v\Delta u - u\Delta v}{v(v + \Delta v)} \\
\frac{\Delta y}{\Delta x} &= \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}
\end{aligned}$$

When $\dfrac{\Delta x}{\Delta x}$ exists, $\lim\limits_{\Delta x \to 0} \Delta v = 0$.

$$\begin{aligned}
\therefore \frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)} \\
&= \frac{v \lim\limits_{\Delta x \to 0} \dfrac{\Delta u}{\Delta x} - u \lim\limits_{\Delta x \to 0} \dfrac{\Delta v}{\Delta x}}{\lim\limits_{\Delta x \to 0} v^2 + v \lim\limits_{\Delta x \to 0} \Delta v} \\
&= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}
\end{aligned}$$

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \quad (v \neq 0,\ u \text{ and } v \text{ are differentiable functions of } x)}$$

22.3.3 Practice 4

Find the derivative of the following functions:

1. $y = \dfrac{2x}{x+2}$

   Sol.

$$\begin{aligned} y' &= \frac{(x+2)(2x)' - 2x(2x+2)'}{(x+2)^2} \\ &= \frac{(x+2)(2) - 2x(2)}{(x+2)^2} \\ &= \frac{2x+4-4x}{(x+2)^2} \\ &= \frac{-2x+4}{(x+2)^2} \end{aligned}$$

2. $y = \dfrac{3x-2}{x+2}$

   Sol.

$$\begin{aligned} y' &= \frac{(x+2)(3x)' - (3x-2)(x+2)'}{(x+2)^2} \\ &= \frac{(x+2)(3) - (3x-2)(1)}{(x+2)^2} \\ &= \frac{3x+6-3x+2}{(x+2)^2} \\ &= \frac{8}{(x+2)^2} \end{aligned}$$

3. $y = \dfrac{x}{x^2-5}$

   Sol.

$$\begin{aligned} y' &= \frac{(x^2-5)(x)' - x(x^2-5)'}{(x^2-5)^2} \\ &= \frac{(x^2-5)(1) - x(2x)}{(x^2-5)^2} \\ &= \frac{x^2-5-2x^2}{(x^2-5)^2} \\ &= -\frac{x^2+5}{(x^2-5)^2} \end{aligned}$$

4. $y = \dfrac{x^2-1}{3x-2}$

   Sol.

$$\begin{aligned} y' &= \frac{(3x-2)(x^2-1)' - (x^2-1)(3x-2)'}{(3x-2)^2} \\ &= \frac{(3x-2)(2x) - (x^2-1)(3)}{(3x-2)^2} \\ &= \frac{6x^2-4x-3x^2+3}{(3x-2)^2} \\ &= \frac{3x^2-4x+3}{(3x-2)^2} \end{aligned}$$

22.3.4 Exercise 25.3b

Find the derivative of the following functions:

1. $y = \dfrac{x-2}{x+2}$

   Sol.

$$\begin{aligned} y' &= \frac{(x+2)(x-2)' - (x-2)(x+2)'}{(x+2)^2} \\ &= \frac{(x+2)(1) - (x-2)(1)}{(x+2)^2} \\ &= \frac{x+2-x+2}{(x+2)^2} \\ &= \frac{4}{(x+2)^2} \end{aligned}$$

2. $y = \dfrac{x-a}{2x+a}$here a is a constant

   Sol.

$$\begin{aligned} y' &= \frac{(2x+a)(x-a)' - (x-a)(2x+a)'}{(2x+a)^2} \\ &= \frac{(2x+a)(1) - (x-a)(2)}{(2x+a)^2} \\ &= \frac{2x+a-2x+2a}{(2x+a)^2} \\ &= \frac{3a}{(2x+a)^2} \end{aligned}$$

3. $y = \dfrac{2x^3}{x+2}$

Sol.

$$\begin{aligned} y' &= \frac{(x+2)(2x^3)' - 2x^3(x+2)'}{(x+2)^2} \\ &= \frac{(x+2)(6x^2) - 2x^3(1)}{(x+2)^2} \\ &= \frac{6x^3 + 12x^2 - 2x^3}{(x+2)^2} \\ &= \frac{4x^3 + 12x^2}{(x+2)^2} \end{aligned}$$

4. $y = \dfrac{2x+3}{x^2+1}$

Sol.

$$\begin{aligned} y' &= \frac{(x^2+1)(2x+3)' - (2x+3)(x^2+1)'}{(x^2+1)^2} \\ &= \frac{(x^2+1)(2) - (2x+3)(2x)}{(x^2+1)^2} \\ &= \frac{2x^2 + 2 - 4x^2 - 6x}{(x^2+1)^2} \\ &= \frac{-2x^2 - 6x + 2}{(x^2+1)^2} \end{aligned}$$

5. $y = \dfrac{3x}{x^2 - 4x}$

Sol.

$$\begin{aligned} y' &= \frac{(x^2-4x)(3x)' - 3x(x^2-4x)'}{(x^2-4x)^2} \\ &= \frac{(x^2-4x)(3) - 3x(2x-4)}{(x^2-4x)^2} \\ &= \frac{3x^2 - 12x - 6x^2 + 12x}{(x^2-4x)^2} \\ &= \frac{-3x^2}{(x^2-4x)^2} \\ &= -\frac{3}{(x-4)^2} \end{aligned}$$

6. $y = \dfrac{3x^2 + x - 1}{2x - 1}$

Sol.

$$\begin{aligned} y' &= \frac{(2x-1)(6x+1) - (3x^2+x-1)(2)}{(2x-1)^2} \\ &= \frac{12x^2 - 4x - 1 - 6x^2 - 2x + 2}{(2x-1)^2} \\ &= \frac{6x^2 - 6x + 1}{(2x-1)^2} \end{aligned}$$

7. $y = \dfrac{2x^4}{(x+3)^2}$

Sol.

$$\begin{aligned} y' &= \frac{(x+3)^2(2x^4)' - 2x^4\left[(x+3)^2\right]'}{(x+3)^4} \\ &= \frac{(x+3)^2(8x^3) - 2x^4(2)(x+3)}{(x+3)^4} \\ &= \frac{(x+3)\left[(x+3)(8x^3) - 4x^4\right]}{(x+3)^4} \\ &= \frac{8x^4 + 24x^3 - 4x^4}{(x+3)^3} \\ &= \frac{4x^4 + 24x^3}{(x+3)^3} \end{aligned}$$

8. $y = \dfrac{2}{1+x} + \dfrac{2}{1-x}$

Sol.

$$\begin{aligned} y' &= \frac{2(1 - x + 1 + x)}{(1-x)(1+x)} \\ &= \frac{4}{1-x^2} \\ &= \frac{(1-x^2)(4)' - 4(1-x^2)'}{(1-x^2)^2} \\ &= \frac{-4(-2x)}{(1-x^2)^2} \\ &= \frac{8x}{(1-x^2)^2} \end{aligned}$$

9. $y = \dfrac{4-x}{3-2x+x^2}$

Sol.

$$
\begin{aligned}
y' &= \frac{(3-2x+x^2)(4-x)' - (4-x)(3-2x+x^2)'}{(3-2x+x^2)^2} \\
&= \frac{(3-2x+x^2)(-1) - (4-x)(-2+2x)}{(3-2x+x^2)^2} \\
&= \frac{-3+2x-x^2+8-2x+2x^2}{(3-10x+x^2)^2} \\
&= \frac{x^2-8x+5}{(3-2x+x^2)^2}
\end{aligned}
$$

10. $y = \dfrac{1+x-x^2}{1-x+x^2}$

Sol.

$$
\begin{aligned}
y' &= \frac{(1-x+x^2)(1+x-x^2)' - (1+x-x^2)(1-x+x^2)'}{(1-x+x^2)^2} \\
&= \frac{(1-x+x^2)(1-2x) - (1+x-x^2)(-1+2x)}{(1-x+x^2)^2} \\
&= \frac{(1-2x)(1-x+x^2+1+x-x^2)}{(1-x+x^2)^2} \\
&= \frac{(1-2x)(2)}{(1-x+x^2)^2} \\
&= \frac{-4x+2}{(1-x+x^2)^2}
\end{aligned}
$$

22.4 Chain Rule - Differentiation of Composite Functions

If $y = f(u)$ and $u = g(x)$ are both differentiable functions of x, then $y = f\left(g(x)\right)$ is also a differentiable function of x. In a composite function $y = f\left(g(x)\right)$, when x changes by Δx, u changes by Δu and y also changes by Δy, where $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$.

If the derivative of the functions $y = f(u)$ and $u = g(x)$ are $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$ respectively, then when $\Delta x \to 0$, $\Delta u \to 0$, and $\lim\limits_{\Delta x \to 0} \dfrac{\Delta u}{\Delta x} = \dfrac{du}{dx}$, $\lim\limits_{\Delta u \to 0} \dfrac{\Delta y}{\Delta u} = \dfrac{dy}{du}$.

$$\begin{aligned}\text{Hence, } \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \right) \\ &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\ &= \frac{dy}{du} \cdot \frac{du}{dx}\end{aligned}$$

Therefore, the derivative of the composite function $y = f\left(g(x)\right)$ is

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'\left(g(x)\right) \cdot g'(x)}$$

Further extending the chain rule, if $y = f(v)$, $v = g(u)$, and $u = h(x)$, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}}$$

The rule above is called the *chain rule*.

22.4.1 Practice 5

Find the derivative of the following functions:

1. $y = \left(2x^3 - 4\right)^3$

Sol.

$$\begin{aligned} y' &= 3\left(2x^3 - 4\right)^2 \cdot \left(2x^3 - 4\right)' \\ &= 3\left(2x^3 - 4\right)^2 \cdot 6x^2 \\ &= 18x^2\left(2x^3 - 4\right)^2 \\ &= 72x^2\left(x^3 - 2\right)^2 \end{aligned}$$

2. $y = \dfrac{6}{\sqrt{2x - 3}}$

Sol.

$$\begin{aligned} y' &= 6(2x - 3)^{-\frac{1}{2}} \\ &= 6 \cdot \left(-\frac{1}{2}\right)(2x - 3)^{-\frac{3}{2}} \cdot 2 \\ &= -6(2x - 3)^{-\frac{3}{2}} \\ &= -\frac{6}{\sqrt{(2x - 3)^3}} \end{aligned}$$

3. $y = (x + 1)(x - 3)^3$

Sol.

$$\begin{aligned} y' &= (x + 1)'(x - 3)^3 + (x + 1)\left[(x - 3)^3\right]' \\ &= (x - 3)^3 + (x + 1) \cdot 3(x - 3)^2 \cdot (x - 3)' \\ &= (x - 3)^3 + 3(x - 3)^2(x + 1) \\ &= (x - 3)^2\left[(x - 3) + 3(x + 1)\right] \\ &= (x - 3)^2(x - 3 + 3x + 3) \\ &= 4x(x - 3)^2 \end{aligned}$$

4. $y = \left(x^2 - 2\right)\sqrt{1 + x}$

Sol.

$$\begin{aligned} y' &= \left[(x^2 - 2)(1 + x)^{\frac{1}{2}}\right]' \\ &= (x^2 - 2)'(1 + x)^{\frac{1}{2}} + (x^2 - 2)\left[(1 + x)^{\frac{1}{2}}\right]' \\ &= 2x(1 + x)^{\frac{1}{2}} + (x^2 - 2) \cdot \frac{1}{2}(1 + x)^{-\frac{1}{2}} \\ &= 2x(1 + x)^{\frac{1}{2}} + \frac{x^2 - 2}{2(1 + x)^{\frac{1}{2}}} \\ &= \frac{4x(1 + x) + (x^2 - 2)}{2(1 + x)^{\frac{1}{2}}} \\ &= \frac{4x^2 + 4x + x^2 - 2}{2\sqrt{1 + x}} \\ &= \frac{5x^2 + 4x - 2}{2\sqrt{x + 1}} \end{aligned}$$

5. $y = \dfrac{x}{(2x+3)^3}$

Sol.

$$\begin{aligned}
y' &= \frac{(2x+3)^3 \cdot x' - x\left[(2x+3)^3\right]'}{(2x+3)^3} \\
&= \frac{(2x+3)^3 - x \cdot 3(2x+3)^2 \cdot (2x+3)'}{(2x+3)^6} \\
&= \frac{(2x+3)^3 - 6x(2x+3)^2}{(2x+3)^6} \\
&= \frac{(2x+3)^2(2x+3-6x)}{(2x+3)^6} \\
&= \frac{-4x+3}{(2x+3)^4}
\end{aligned}$$

6. $y = \dfrac{(3x+4)^2}{(x-5)^3}$

Sol.

$$\begin{aligned}
y' &= \frac{(x-5)^3\left[(3x+4)^2\right]' - (3x+4)^2\left[(x-5)^3\right]'}{(x-5)^6} \\
&= \frac{(x-5)^3 \cdot 2(3x+4) \cdot (3x+4)' - (3x+4)^2 \cdot 3(x-5)^2 \cdot (x-5)'}{(x-5)^6} \\
&= \frac{6(x-5)^3(3x+4) - 3(3x+4)^2(x-5)^2}{(x-5)^6} \\
&= \frac{(x-5)^2\left[6(x-5)(3x+4) - 3(3x+4)^2\right]}{(x-5)^6} \\
&= \frac{6(3x^2-11x-20) - 3(9x^2-24x+16)}{(x-5)^4} \\
&= \frac{18x^2-66x-120-27x^2-72x-48}{(x-5)^4} \\
&= \frac{-9x^2-138x-168}{(x-5)^4} \\
&= -\frac{3(3x^2+46x+56)}{(x-5)^4} \\
&= -\frac{3(3x+4)(x+14)}{(x-5)^4}
\end{aligned}$$

22.4.2 Exercise 25.4

Find the derivative of the following functions:

1. $y = (2x-3)^8$

   Sol.

$$
\begin{aligned}
y' &= 8(2x-3)^7 \cdot (2x-3)' \\
&= 16(2x-3)^7
\end{aligned}
$$

2. $y = \left(3x^2 - 6x + 5\right)^4$

   Sol.

$$
\begin{aligned}
y' &= 4(3x^2-6x+5)^3 \cdot (3x^2-6x+5)' \\
&= 4(6x-6)(3x^2-6x+5)^3 \\
&= 24(x-1)(3x^2-6x+5)^3
\end{aligned}
$$

3. $y = \sqrt{2x^4 - 4x^2 + 5}$

   Sol.

$$
\begin{aligned}
y' &= \left(2x^4-4x^2+5\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\left(2x^4-4x^2+5\right)^{-\frac{1}{2}} \cdot \left(2x^4-4x^2+5\right)' \\
&= \frac{1}{2}\left(2x^4-4x^2+5\right)^{-\frac{1}{2}} \cdot (8x^3-8x) \\
&= (4x^3-x)\left(2x^4-4x^2+5\right)^{-\frac{1}{2}} \\
&= \frac{4x^3-x}{\sqrt{2x^4-4x^2+5}}
\end{aligned}
$$

4. $y = \sqrt[3]{3x^2 - 3x + 2}$

   Sol.

$$
\begin{aligned}
y' &= \left[\left(3x^2-3x+2\right)^{\frac{1}{3}}\right]' \\
&= \frac{1}{3}(3x^2-3x+2)^{-\frac{2}{3}} \cdot (3x^2-3x+2)' \\
&= \frac{1}{3}(3x^2-3x+2)^{-\frac{2}{3}} \cdot (6x-3) \\
&= (3x^2-3x+2)^{-\frac{2}{3}}(2x-1) \\
&= \frac{2x-1}{\sqrt[3]{((2x^4-4x^2+5)^2}}
\end{aligned}
$$

5. $y = \dfrac{3}{6x^2 - 4}$

Sol.

$$
\begin{aligned}
y' &= \frac{(6x^2-4)(3)' - 3(6x^2-4)'}{(6x^2-4)^2} \\
&= \frac{-3(12x)}{(6x^2-4)^2} \\
&= \frac{-36x}{(6x^2-4)^2} \\
&= -\frac{36x}{4(3x^2-2)^2} \\
&= \frac{9x}{(3x^2-2)^2}
\end{aligned}
$$

6. $y = \dfrac{9}{\sqrt{6x^3 - 9x}}$

Sol.

$$
\begin{aligned}
y' &= \frac{(9')\left(\sqrt{6x^3-9x}\right) - 9\left(\sqrt{6x^3-9x}\right)'}{6x^3-9x} \\
&= \frac{-9\left[\frac{1}{2}\left(6x^3-9x\right)^{-\frac{1}{2}}(6x^3-9x)'\right]}{6x^3-9x} \\
&= \frac{-9\left[\frac{1}{2}\left(6x^3-9x\right)^{-\frac{1}{2}}(18x^2-9)\right]}{6x^3-9x} \\
&= \frac{-9(18x^2-9)}{2(6x^3-9x)\left(6x^3-9x\right)^{\frac{1}{2}}} \\
&= \frac{-18(2x^2-1)}{2\left(6x^3-9x\right)^{\frac{3}{2}}} \\
&= -\frac{18(2x^2-1)}{2\sqrt{\left(6x^3-9x\right)^3}}
\end{aligned}
$$

7. $y = \dfrac{3x}{(x+5)^2}$

Sol.

$$\begin{aligned}
y' &= \frac{(3x)'(x+5)^2 - 3x\left[(x+5)^2\right]'}{(x+5)^4} \\
&= \frac{3(x+5)^2 - 3x \cdot 2(x+5) \cdot (x+5)'}{(x+5)^4} \\
&= \frac{3(x+5)^2 - 6x(x+5)}{(x+5)^4} \\
&= \frac{(x+5)\left[3(x+5) - 6x\right]}{(x+5)^4} \\
&= \frac{3x + 15 - 6x}{(x+5)^3} \\
&= \frac{-3x + 15}{(x+5)^3} \\
&= -\frac{3(x-5)}{(x+5)^3}
\end{aligned}$$

8. $y = \dfrac{3x-1}{\sqrt{x-1}}$

9. $y = \sqrt{x}(x-3)^5$

10. $y = x^2(x-3)(x+2)^2$

11. $y = (2x+1)^2(x+1)^3$

12. $y = \left(\dfrac{1+x}{1-x}\right)^2$

13. $y = x^2\sqrt{1+x^2}$

14. $y = \sqrt{\dfrac{1+x^2}{1-x^2}}$

22.5 Higher Order Derivatives

If the derivative $f'(x)$ of a function $y = f(x)$ is differentiable at $x = x_0$, then the derivative of f' at $x = x_0$ is called the *second derivative* of $y = f(x)$ at $x = x_0$, and is denoted by $f''(x_0)$, that is,

$$\lim_{\Delta x \to 0} \frac{f'(x_0 + \Delta x) - f'(x_0)}{\Delta x} = f''(x_0)$$

The derivative of the derivative $f'(x)$ of a function $y = f(x)$ is called the *second derivative* of $y = f(x)$, and is denoted by $(f')'(x)$ or $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2} = \dfrac{d^2\big(f(x)\big)}{dx^2}$; while $f'(x)$ is called the *first derivative* of $y = f(x)$. Using the same way, we can define the third derivative $f'''(x)$, the fourth derivative $f^{(4)}(x)$, up to the nth derivative $f^{(n)}(x)$.

The way of finding higher order derivatives is similar to the way of finding the first derivative, as we just need differentiate the function again and again until we get the desired order of derivative.

22.5.1 Practice 6

1. Find the first to sixth derivatives of $y = 3x^5 + 2x^4 - 5x^2 - 8x + 9$

Find the third derivative of the following functions (Question 2 to 5):

2. $y = \dfrac{3}{x^4}$
3. $y = 3x^3 + 6x^2 - 5x$
4. $y = \sqrt{2x - 3}$
5. $y = \dfrac{1}{(2x + 5)^3}$

22.5.2 Exercise 25.5

1. Given the function $y = ax^6 + 2bx^4 - 3cx^3 + 4x^2 - 5$ where a, b and c are constants, find the first to seventh derivatives of y.

Find the second derivative of the following functions (Question 2 to 5):

2. $y = 2x - \dfrac{x^3}{2}$
3. $y = \left(2 + x^2\right)^3$
4. $y = x + \dfrac{1}{x} + \dfrac{1}{x^2}$
5. $y = x^2 - \dfrac{1}{x^2}$
6. $y = \sqrt{a^2 - x^2}$ where a is a constant
7. $y = \dfrac{2}{\sqrt{x - 2}}$
8. $y = \sqrt{3x^2 + 2}$
9. $y = \dfrac{x}{\sqrt{1 - x^2}}$
10. Given the function $y = 2x^3 - 3x^2 - 6x + 10$, find the value of y', y'' and y''' at $x = 1$.
11. Given the function $f(x) = \dfrac{5}{(2x + 1)^2}$, find the value of $f''(2)$.
12. Find the second derivative y'', of the function $y = 2x^3 + 3x^2 - 72x + 15$, and find the value of x at $y'' = 0$.

22.6 Implicit Differentiation

Consider the function $x^2 + y^2 = x^3$ and $y^2 - 2xy - 3 = 0$. For these kind of functions, y is not expressed in terms of x explicitly, hence we say that y is an *implicit function* of x. In some cases, the equations above are hard if not impossible to be rewritten in the form of $y = f(x)$. To find the derivative of implicit functions, we can differentiate both sides of the equation with respect to x and use the chain rule at the same time to get an equation in terms of x, y and $\dfrac{dy}{dx}$, then we can solve for $\dfrac{dy}{dx}$.

22.6.1 Practice 7

Find $\dfrac{dy}{dx}$ for the following implicit functions:

1. $x^2 + y^2 - 6x + 8y - 9 = 0$
2. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 5$ where a and b are constants
3. $2x^2 - 4xy + 2y^2 = 5$
4. $x^5 + 2x^2y^3 + y^4 = 1$

22.6.2 Exercise 25.6

Find $\frac{dy}{dx}$ for the following implicit functions:

1. $x^2 + y^3 = 3$
2. $\sqrt{x} + \sqrt{y} = \sqrt{a}$ where a is a constant
3. $x^3 + y^3 = 2xy + 3$
4. $y^2(x + 1) = 3x^2$
5. $3x^2 - 6xy + 3y^2 = 25$
6. $xy^3 = 2x^2 - 2y^2$
7. $\frac{1}{x} + \frac{1}{y} = 2x$
8. $\frac{3x^2}{y} + \frac{3y^2}{x} = 1$
9. $xy - x + xy^3 = 10$
10. $4x^3 + 2xy^2 - xy = 0$

22.7 Two Basic Limits

A. $\lim_{x\to 0} \frac{\sin x}{x}$

Consider the value of $\frac{\sin x}{x}$ when x approaches 0. When $x = 0$, the value of $\sin x$ and x are both 0, hence $\frac{\sin x}{x} = \frac{0}{0}$, which is undefined. However, the value of $\lim_{x\to 0} \frac{\sin x}{x}$ is defined, as shown in the diagram below:

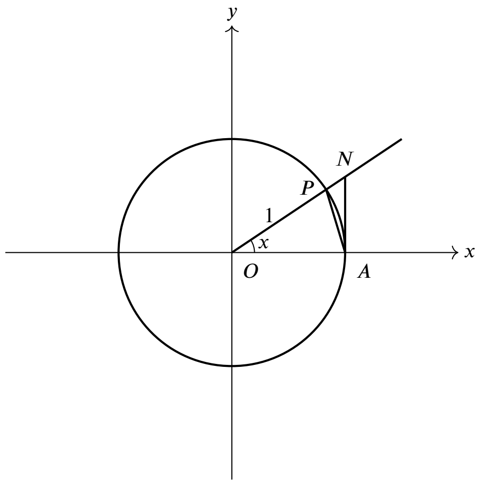
x (in rad)	$\sin x$	$\frac{\sin x}{x}$
1	0.84147098	0.84147098
0.500	0.479425539	0.958851077
0.250	0.247403959	0.989615837
0.100	0.099833417	0.99833417
0.010	0.0099998333	0.99998333
0.001	0.00099999983	0.99999983
$\vdots$	$\vdots$	$\vdots$

Note that the trigonometric independent variable x is in radian.

By looking at the table above, as x approaches 0, the value of $\frac{\sin x}{x}$ approaches 1. Hence, we can speculate that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

We can also prove the speculation above.



In the unit circle above, the central angle x from the initial arm OA, its terminal side intersects with the circle at point P, AN is the tangent line that passes through point A, AN and OP intersects at point N.

From the diagram above, we can see that $\overset{\frown}{PA} = x$

$$AN = \tan x$$

and the area of $\Delta OAP <$ the area of sector $OAP <$ the area of ΔOAN.

$$\therefore \frac{1}{2}\sin x < \frac{1}{2}(1)^2 x < \frac{1}{2}(1)\tan x$$

$$\sin x < x < \frac{\sin x}{\cos x}$$

When x is a positive angle that is smaller than 90°, $\sin x > 0$. Dividing the inequality above by $\sin x$, we get

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$\cos x < \frac{\sin x}{x} < 1$$

When $x < 0$, $\frac{\sin x}{x} = \frac{\sin(-x)}{-x}$, $\cos(-x) = \cos x$.

$\therefore \lim_{x \to 0} \frac{\sin x}{x}$ is in between $\cos x$ and $1 \because \lim_{x \to 0} \cos x = 1 \therefore \lim_{x \to 0} \frac{\sin x}{x} = 1$

The statement above uses the Squeeze Theorem. From the proof above, we can also conclude that

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{1}{\frac{\sin x}{x}} = 1$$

22.7.1 Practice 8

Find the limit of the following:

1. $\lim_{x\to 0}\frac{\sin 4x}{5x}$
2. $\lim_{x\to 0}\frac{7x}{\sin 5x}$
3. $\lim_{1\to 0}\frac{\sin 5x}{\sin 4x}$
4. $\lim_{x\to 0}\frac{1-\cos 2x}{3x^2}$
5. $\lim_{s\to 0}\frac{\tan 9x}{\sin 5x}$
6. $\lim_{x\to 0}\frac{\tan 2x\sin 3x}{x^2}$

22.7.2 Exercise 25.7a

Find the limit of the following:

1. $\lim_{x\to 0}\frac{x\cos x}{\sin 2x}$
2. $\lim_{x\to 0}(x\cot x)$
3. $\lim_{x\to 0}\frac{2x}{\sin 7x}$
4. $\lim_{x\to 0}\frac{\sin\frac{5}{2}x}{2x}$
5. $\lim_{x\to 0}\frac{\sin(2x^2)}{x}$
6. $\lim_{x\to 0}\frac{\sin(2x^2)}{x}$
7. $\lim_{x\to 0}\frac{x^2}{\sin^2 bx}$
8. $\lim_{x\to 0}\frac{\sin^2 3x}{2x^2}$
9. $\lim_{x\to 0}\frac{4x}{\tan 8x}$
10. $\lim_{x\to 0}\frac{\sin 5x}{\tan 3x}$
11. $\lim_{x\to 0}\frac{1-\cos 4x}{x\sin 5x}$
12. $\lim_{x\to 0}\frac{5x^2}{2\tan^2 2x}$
13. $\lim_{x\to 0}\frac{\cos 3x-1}{3x^2}$

B. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

Consider the limit of $\left(1+\frac{1}{x}\right)^x$ as x approaches infinity. As $x\to\infty$, the changes of the function $f(x) = \left(1+\frac{1}{x}\right)^x$ is shown in the table below:

x	$f(x) = \left(1 + \frac{1}{x}\right)^x$
1	$\left(1 + \frac{1}{1}\right)^1 = 2$
10	$\left(1 + \frac{1}{10}\right)^{10} = 2.59374$
100	$\left(1 + \frac{1}{100}\right)^{100} = 2.70481$
1000	$\left(1 + \frac{1}{1000}\right)^{1000} = 2.71692$
10000	$\left(1 + \frac{1}{10000}\right)^{10000} = 2.71815$
100000	$\left(1 + \frac{1}{100000}\right)^{100000} = 2.71827$
$\vdots$	$\vdots$

By looking at the table above, we can see that as x approaches infinity, the value of $\left(1 + \frac{1}{x}\right)^x$ approaches a constant value that is denoted as e. e is an irrational number and is approximately equal to 2.71828182846. It is called the **natural base**, and is one of the most important numbers in mathematics and appears in many applications.

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$$

If $y = \frac{1}{x}$, then $x = \frac{1}{y}$. As $x \to \infty$, $y \to 0$.
Therefore, $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{y\to 0} (1 + y)^{\frac{1}{y}} = e$.

22.7.3 Practice 9

Find the limit of the following:

1. $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{x+2}$
2. $\lim_{x\to\infty} (1 - 3x)^{\frac{2}{x}}$
3. $\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^{\frac{x}{2}}$
4. $\lim_{x\to\infty} \left(\frac{2 - 3x}{2}\right)^2$

22.7.4 Exercise 25.7b

Find the limit of the following:

1. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{\frac{x}{2}}$

2. $\lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}\right)^{2x}$

3. $\lim\limits_{x\to\infty}\left(\dfrac{x+5}{x}\right)^{2x}$

4. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{2x}\right)^{-3x}$

5. $\lim\limits_{x\to 0}\left(\dfrac{3-x}{3}\right)^{-\frac{3}{x}}$

6. $\lim\limits_{x\to\infty}\left(1+\dfrac{5}{x}\right)^{2-x}$

7. $\lim\limits_{x\to\infty}\left(1+\dfrac{3}{2x}\right)^{3+x}$

8. $\lim\limits_{x\to 0}(1-4x)^{-\frac{3}{x}}$

9. $\lim\limits_{x\to 0}(1+2x)^{1-\frac{2}{x}}$

10. $\lim\limits_{x\to\infty}\left(\dfrac{x+2}{x+1}\right)^{x}$

22.8 Derivatives of Trigonometric Functions

Here we will derive the derivatives of the sine, cosine, and tangent functions.

Let $y = \sin x$.

$$\begin{aligned}
y+\Delta y &= \sin(x+\Delta x)\\
\Delta y &= \sin(x+\Delta x)-\sin x\\
&= \sin x\cos\Delta x+\cos x\sin\Delta x-\sin x\\
&= \sin x(\cos\Delta x-1)+\cos x\sin\Delta x\\
\frac{\Delta y}{\Delta x} &= \sin x\frac{\cos\Delta x-1}{\Delta x}+\cos x\frac{\sin\Delta x}{\Delta x}
\end{aligned}$$

$$\begin{aligned}
\because \lim_{\Delta x\to 0}\frac{\cos\Delta x-1}{\Delta x} &= \lim_{\Delta x\to 0}\frac{-2\sin^2\dfrac{\Delta x}{2}}{\Delta x}\\
&= -\frac{1}{2}\lim_{\Delta x\to 0}\left(\frac{\sin\dfrac{\Delta x}{2}}{\dfrac{\Delta x}{2}}\right)\cdot\Delta x\\
&= 0\\
\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x} &= 1
\end{aligned}$$

$$\begin{aligned}
\therefore \frac{d}{dx}(\sin x) &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\\
&= \sin x\cdot 0+\cos x\cdot 1\\
&= \cos x
\end{aligned}$$

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

$$\because \ \cos x = \sin\left(\frac{\pi}{2} - x\right)$$

$$\begin{aligned}
\therefore \ \frac{d}{dx}(\cos x) &= \frac{d}{dx}\left(\sin\left(\frac{\pi}{2} - x\right)\right) \\
&= \cos\left(\frac{\pi}{2} - x\right)(-1) \\
&= -sinx
\end{aligned}$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

$$\because \ \tan x = \frac{\sin x}{\cos x}$$

$$\begin{aligned}
\therefore \ \frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

Below are the derivatives of the remaining trigonometric functions.

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

22.8.1 Practice 10

Find the derivatives of the following functions:

1. $y = \cos\left(5x^2\right)$
2. $y = \sin^2(2x - 3)$
3. $y = \tan^4 3x$

22.8.2 Exercise 25.8

Find the derivatives of the following functions:

1. $y = \sin^2 x$
2. $y = \sin 3x - \cos 3x$
3. $y = \tan 4x - \cot 5x$
4. $y = \sec 3x + \csc 5x$
5. $y = 2x \sec x$
6. $y = \dfrac{2x}{\sin x}$
7. $y = \cos 3x°$
8. $y = \cos^4(1 - 2x)$
9. $y = \tan^3\left(2x^2\right)$
10. $y = \cos^2 x - \sin^2 x$
11. $y = 4 \sin x \cos x$
12. $y = \sin 3x \tan 6x$
13. $y = \sqrt{\cos 2x}$
14. $y = \sin^2 \sqrt{1 + x^2}$
15. $y = x \tan^2(3x - 2)$
16. $y = \dfrac{3 \sin 2x}{\cos x}$
17. $y = \dfrac{2}{3 \tan^3 x}$
18. $y = \dfrac{1 + \cos x}{1 - \cos x}$
19. $y = \dfrac{2 \tan x}{1 - \tan^2 x}$
20. $y = \dfrac{\sin\left(2x - \frac{\pi}{4}\right)}{\sin\left(2x + \frac{\pi}{4}\right)}$
21. Given the function $y = \dfrac{x}{2 + \cos x}$. Find the derivative value when $x = 0$, $x = \dfrac{\pi}{2}$, and $x = \pi$.
22. If the function $y = \sec x + \tan x$, prove that $\dfrac{dy}{dx} = y \sec x$.

23. If the function $y = \sqrt{\dfrac{1-\cos x}{1+\cos x}}$, find $\dfrac{dy}{dx}$.

24. If the function $r = \sin 3t - 2\cos t$, find $\dfrac{d^2r}{dt^2}$ when $t = \dfrac{\pi}{3}$.

22.9 Derivatives of Exponential Functions

Derivative of Natural Logarithmic Functions

Let $y = \ln x$ where $x > 0$.

$$
\begin{aligned}
y + \Delta y &= \ln(x + \Delta x) \\
\Delta y &= \ln(x + \Delta x) - \ln x \\
&= \ln\left(\frac{x + \Delta x}{x}\right) \\
&= \ln\left(1 + \frac{\Delta x}{x}\right) \\
\frac{\Delta y}{\Delta x} &= \frac{1}{\Delta x}\ln\left(1 + \frac{\Delta x}{x}\right) \\
&= \frac{1}{\Delta x}\cdot\frac{x}{\Delta x}\cdot\ln\left(1 + \frac{\Delta x}{x}\right) \\
&= \frac{1}{x}\ln\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} \\
\frac{dy}{dx} &= \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} \\
&= \frac{1}{x}\lim_{\Delta x \to 0}\ln\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}} \\
&= \frac{1}{x}\ln\left(\lim_{\Delta x \to 0}\left(1 + \frac{\Delta x}{x}\right)^{\frac{x}{\Delta x}}\right) \\
&= \frac{1}{x}\ln e \\
&= \frac{1}{x}
\end{aligned}
$$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

If $x < 0$, then $-x > 0$.

$$
\begin{aligned}
\frac{d}{dx}(\ln -x) &= \frac{1}{-x}\frac{d}{dx}(-x) \\
&= \frac{1}{x}
\end{aligned}
$$

$$\boxed{\frac{d}{dx}(\ln|x|) = \frac{1}{x}}$$

Derivative of Other Logarithmic Functions

$$\begin{aligned}
\text{Let } y &= \log_a x \\
&= \frac{\ln x}{\ln a} \\
&= \frac{1}{\ln a}\ln x \\
\frac{dy}{dx} &= \frac{1}{\ln a}\cdot\frac{d}{dx}(\ln x) \\
&= \frac{1}{\ln a}\cdot\left(\frac{1}{x}\right) \\
&= \frac{1}{x\ln a}
\end{aligned}$$

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}}$$

Generally, when solving for the derivative of a logarithmic function, we can first convert it to a natural logarithmic function and then solve for the derivative.

22.9.1 Practice 11

Find the derivative of the following functions:

1. $y = \ln\left(x^2 - 2x + 1\right)$
2. $y = \log_2 \sin 3x$
3. $y = \ln(\sec x)$
4. $y = x\ln x$
5. $y = \ln\frac{1+x}{1-x}$
6. $y = \log_2\sqrt{1+x^2}$
7. $y = \ln\dfrac{1+x}{1-x}$
8. $y = \log_2\sqrt{1+x^2}$

22.9.2 Exercise 25.9

Find the derivative of the following functions:

1. $y = \ln(5x - 3)$
2. $y = \ln\left(ax^3\right)$
3. $y = \ln \dfrac{3}{x^2}$
4. $y = \ln\left(2x^2 - 3x + 4\right)$
5. $y = \log_5(3x)$
6. $y = \log_2\left(x^2 - 4x + 3\right)$
7. $y = \log_a\left(2ax^2 - 4ax\right)$
8. $y = \log_a(\ln x)$
9. $y = 3x^2 \ln(5x)$
10. $y = \ln\left(\cos^2 x\right)$
11. $y = \ln(4x + 3)^2$
12. $y = \log_5\left(2x^2 - 3\right)$
13. $y = \log_8 \sqrt{x^2 - 2}$
14. $y = \log_b(\sin 5x)$
15. $y = \log(5x) + \ln(\tan x)$
16. $y = \ln^2(\sec x)$
17. $y = \log_3 \dfrac{2}{x^2 - 1}$
18. $y = \dfrac{1 + \ln x}{1 - \ln x}$
19. $y = \ln \dfrac{2 + x^2}{2 - x^2}$
20. $y = \ln \dfrac{2 \sin x}{\sec x}$

22.10 Derivatives of Logarithmic Functions

Derivative of exponential functions $y = e^x$

If $y = e^x$, then $\ln y = e^x = x$.

Differentiate both sides with respect to x, we get
$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y \\ &= e^x \end{aligned}$$

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

Derivative of other exponential functions

Let $y = a^x$ where $a > 0$.

Take the natural logarithm of both sides, we get
$$\begin{aligned} \ln y &= \ln a^x \\ &= x \ln a \end{aligned}$$

Differentiate both sides with respect to x, we get
$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= \ln a \\ \frac{dy}{dx} &= y \ln a \\ &= a^x \ln a \end{aligned}$$

Generally, when solving for the derivative of an exponential function, we can first convert it to a natural exponential function and then solve for the derivative.

22.10.1 Practice 12

Find the derivative of the following functions:

1. $y = e^{-3x}$
2. $y = -e^{-\frac{1}{3}x}$
3. $y = 2a^{5x}$
4. $y = a^{2x} - e^{-x} + x^a$
5. $y = \left(e^x + \frac{1}{e^x}\right)^2$
6. $y = \frac{e^{3x} - e^{-3x}}{e^x}$

22.10.2 Exercise 25.10

Find the derivative of the following functions:

1. $y = e^{-5x}$
2. $y = 3e^{-\frac{1}{3}x}$
3. $y = e^{x^2+3x-1}$
4. $y = e^{5x} \cdot e^{-6x} \cdot 4e^2$
5. $y = 5^{2x}$
6. $y = (3b)^{7x}$
7. $y = 4^{3-2x^2}$
8. $y = 4^{2x} - 4^{-2x}$
9. $y = 3e^{\tan x}$
10. $y = e^x \ln x$
11. $y = 3^{2x} \cos 5x$
12. $y = \dfrac{e^x + 1}{e^x - 1}$
13. $y = \dfrac{e^{2x} - e^{-2x}}{2e^x}$
14. $y = \dfrac{3e^{5x} - 2e^{-2x}}{6e^x}$

22.11 Revision Exercise 25

1. Find the gradient of the tangent to the curve $y = 2x^2 + 1$ at the point where $x = 2$.

2. Find the gradient of the curve $y = 3x^2 - 1$ at the point $A(-1, 2)$.

3. Find the gradient of the curve $y = 2x - x^3$ at the point $B(-1, -1)$.

4. Find the derivative of the following functions using the definition of the derivative, and find the value of the derivative at the point where $x = 1$:

   (a) $f(x) = x^2 + 2x$

   (b) $g(x) = x^3$

   (c) $h(x) = \dfrac{5}{x}$

   (d) $k(x) = \sqrt{x+3}$

5. Find the derivative of the following functions:

   (a) $y = 2x^4 - 3x^3 + 5x - 8$

   (b) $y = 2x + \dfrac{2}{x} - \dfrac{3}{x^2}$

   (c) $y = \sqrt[3]{x} - \dfrac{1}{\sqrt{3x}}$

   (d) $y = \left(x^3 - 4x\right)\left(x^2 + 3x - 1\right)$

   (e) $y = (x-1)^5\sqrt{x+2}$

   (f) $y = (2x+5)\left(x^2 - 2\right)\left(x^3 - 1\right)$

   (g) $y = \dfrac{2x^3 - 3x^2 + 4}{x^2}$

   (h) $y = \dfrac{x^2 + 4}{x + 1}$

   (i) $y = \dfrac{x + 2}{x^2 + 5x + 6}$

   (j) $y = \dfrac{x^2}{\left(x^2 - 1\right)^3}$

6. Find the derivative of the following functions:

   (a) $y = \left(x^3 - 1\right)^4$

   (b) $y = (5x + 3)^6$

   (c) $y = \left(x^3 - 3x\right)^5$

   (d) $y = \sqrt{x^2 - 2x}$

   (e) $y = \dfrac{1}{\sqrt[3]{2x^2 - 1}}$

(f) $y = \dfrac{2x-1}{\sqrt{1-2x}}$

7. Find the second derivative of the following functions:

   (a) $y = x^2(3x-4)$

   (b) $y = 2x^5 - 6x^4 - 3x + 5$

   (c) $y = \dfrac{3}{x^5}$

   (d) $y = \sqrt{2x+1}$

8. If the function $y = \dfrac{x^3}{(x-1)^2}$, find y' and y''.

9. Given the function $y = 2x^3 + 3x^2 - 72x + 21$, find the value of x when $\dfrac{dy}{dx} = 0$.

10. Given the function $y = (2-3x^2)^4$, find the value of $\dfrac{d^2y}{dx^2}$ when $x = 1$.

11. If the function $f(x) = \dfrac{x^3}{3} - \dfrac{5x^2}{2} + x + 1$, find

    (a) $f'(1)$ and $f''(2)$

    (b) the value of x when $f''(x) = 0$

12. If the function $f(x) = \sqrt{x\sqrt{x\sqrt{x}}}$, find $f'(1)$, $f''(1)$, $f'''(1)$ and $f^{(4)}(1)$.

13. Find the derivative $\dfrac{dy}{dx}$ of the following implicit functions:

    (a) $x^2 + 2y = 2x + 3$

    (b) $x^2 + 3x = y^2 - 5y$

    (c) $3x^2 + 7xy - 9y^2 = 2$

    (d) $x^5y + xy^5 = 3xy$

14. Find the gradient of the tangent to the curve $x^2 + xy + y^2 = 4$ at the point $A(2, -2)$.

15. Find the limit of the following:

    (a) $\lim\limits_{x\to 0} \dfrac{\sin 4x}{x}$

    (b) $\lim\limits_{x\to 0} \dfrac{\tan 2x}{\tan 5x}$

    (c) $\lim\limits_{x\to 0} \dfrac{x^2}{\tan^2 3x}$

    (d) $\lim\limits_{x\to\infty} \left(1 + \dfrac{1}{x}\right)^{4x+1}$

    (e) $\lim\limits_{x\to 0} (1-3x)^{\frac{2}{x}}$

(f) $\lim_{x\to\infty}\left(\dfrac{x+3}{x+1}\right)^{x}$

16. Find the derivative of the following functions:

(a) $y = \tan^2 3x$

(b) $y = \cos^4 2x$

(c) $y = \sec^3 2x$

(d) $y = \sec^2(3x+5)$

(e) $y = (1+\sin x)^3$

(f) $y = \sin(\cos x)$

(g) $y = \sin 2x \cos 2x$

(h) $y = \dfrac{1}{\sin x + \cos x}$

(i) $y = \dfrac{\cos 5x}{\sin 3x}$

(j) $y = \dfrac{1+\cos x}{\sin x}$

17. Find the derivative of the following functions:

(a) $y = 5^{3x-2}$

(b) $y = 3e^{2x^2}$

(c) $y = a^{3x} + e^{-3x}$

(d) $y = \dfrac{e^{3x} - e^{2x} + e^{5x}}{e^{2x}}$

(e) $y = x^a - 2a^x$

(f) $y = e^{2x}\csc 2x$

18. If the function $y = \dfrac{\sin 2x}{e^x}$, prove that $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 0$.

19. Find the derivative of the following functions:

(a) $y = \ln\left(x^2+5\right)$

(b) $y = \ln\left(3x^2+6x\right)$

(c) $y = \ln(e^x+2)$

(d) $y = \ln\left(\sin^2 4x\right)$

(e) $y = \log\left(x^3+3x-4\right)$

(f) $y = \log_5(3x+7)$

(g) $y = \log_2\dfrac{x}{x+3}$

(h) $y = \dfrac{1+\log x}{1+\ln x}$

20. If the function $y = \ln(x+1)$, find the value of $\dfrac{d^2y}{dx^2}$ when $x = 1$.