Mathematics

Senior 3 Part I

MELVIN CHIA

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Introduction

Why this book?

Disclaimer

Acknowledgements

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Chapter 22

Exponents and Logarithms

22.1 Exponents

Definition and Properties of Exponents

Back in Senior 1, we have learnt the following definitions of exponents:

Positive exponent	$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$
Zero exponent	$a^0 = 1$
Negative exponent	$a^{-n} = \dfrac{1}{a^n}$ $(a \neq 0, n \in \mathbb{Z}^+)$
Fractional exponent	$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$ $(a \geq 0, n > 1, m, n \in \mathbb{Z}^+)$

The exponent of rational numbers have the following properties:

1. $a^m \times a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^n = a^n b^n$
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ $(b \neq 0)$

Practice 1

Without using the calculator, find the value of the following expressions (Question 1 to 2):

1. $2^{-2} + 2{-}5 - (-2)^{-3}$

   Sol.

$$\begin{aligned} 2^{-2} + 2^{-5} - (-2)^{-3} &= \frac{1}{4} + \frac{1}{32} - \left(-\frac{1}{8}\right) \\ &= \frac{8}{32} + \frac{1}{32} + \frac{4}{32} \\ &= \frac{13}{32} \end{aligned}$$

2. $\left(3\frac{6}{25}\right)^{-\frac{1}{2}}$

   Sol.

$$\begin{aligned} \left(3\frac{6}{25}\right)^{-\frac{1}{2}} &= \left(\frac{131}{25}\right)^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{\frac{81}{25}}} \\ &= \frac{1}{\frac{9}{5}} \\ &= \frac{5}{9} \end{aligned}$$

3. Simplify $a^{-}4 \div a^{-5} \times (b^{-3})^{-4}$

   Sol.

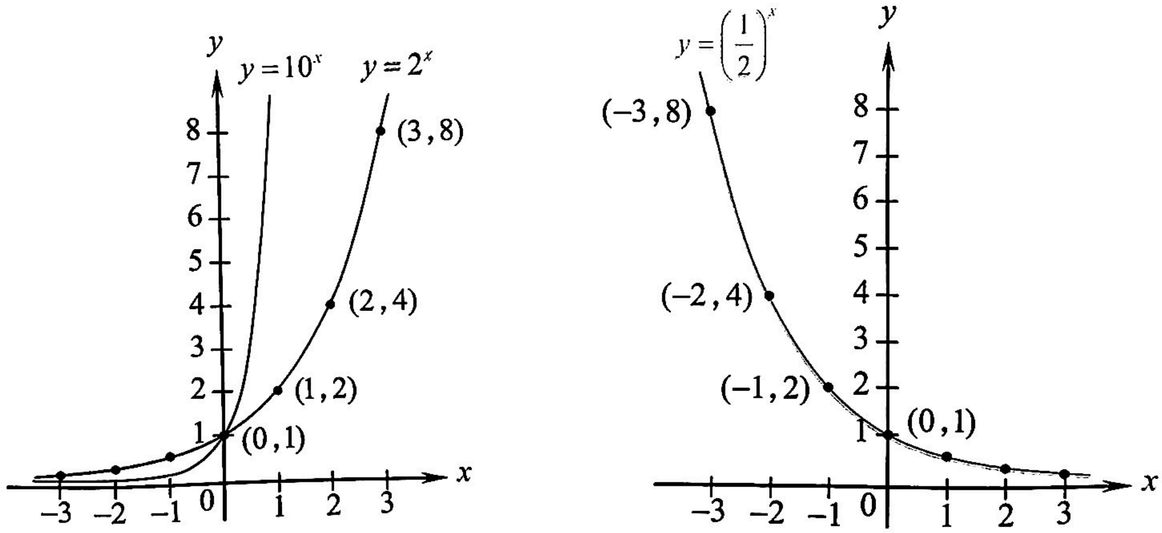
$$\begin{aligned} a^{-4} \div a^{-5} \times (b^{-3})^{-4} &= a^{-4} \times a^{5} \times b^{12} \\ &= a^{-4+5} \times b^{12} \\ &= ab^{12} \end{aligned}$$

Exponential Functions and Graphs

Let a is a constant that is bigger than zero and not equal to 1, then the function being expressed in the form of $y = a^x$ is called an *exponential function*. The domain of an exponential function is $\mathbb{R}$.

Consider the following: a cell divides into two cells, and then each of the two cells divides into two cells again, and so on. If we let x be the number of divisions, the number of cells after the divisions be y, then the functional relationship between x and y is $y = 2^x$, which is an exponential function.

In order to look into the graph and its properties of an exponential function $y = a^x$, we sketch the graph of some exponential functions, the graph of $y = 2^x$, $y = 10^x$, and $y = \left(\frac{1}{2}\right)^x$ are shown in the diagram below.



From the diagram above, we can see that:

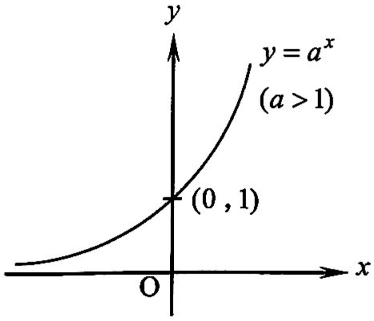
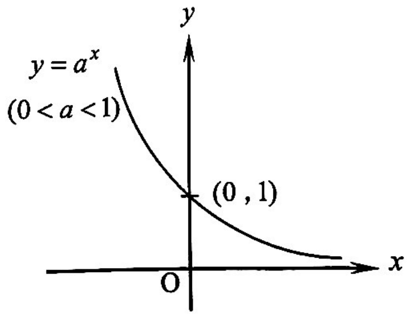
(1) The graph of the function $y = 2^x$, $y = 10^x$, and $y = \left(\frac{1}{2}\right)^x$ are only at the top of the x-axis. Actually, when $a > 0$, $a^x > 0$. Therefore, the value of the exponential function $y = a^x$ is always positive.

(2) When $x = 0$, $y = 1$. Hence, the graph of exponential functions $y = a^x$ always passes through the point $(0, 1)$.

(3) For the function $y = 2^x$, when $x > 0$ and $y = 10^x$, when $x < 0$, $y < 1$; when $x > 0$, $y > 1$. When the value of x increases, the value of y increases, that is, the function is an increasing function in the interval $(-\infty, +\infty)$.

(4) For the function $y = \left(\frac{1}{2}\right)^x$, when $x > 0$, $y > 1$; when $x < 0$, $y < 1$. When the value of x increases, the value of y decreases, that is, the function is a decreasing function in the interval $(-\infty, +\infty)$.

When we are discussing about the graph and its properties of an exponential function $y = a^x$, the following two cases are considered:

	$a > 1$	$0 < a < 1$
Graph		
Properties	$y > 0$	
	When $x = 0$, $y = 1$	
	When $x > 0$, $y > 1$; When $x < 0$, $0 < y < 1$	When $x > 0$, $0 < y < 1$; When $x < 0$, $y > 1$
	It is an increasing function in the interval $(-\infty, +\infty)$	It is a decreasing function in the interval $(-\infty, +\infty)$

Practice 2

1. Without using the calculator, compare the value of the following expressions:

(a) $\pi^{2.1}$ and $\pi^{3.5}$

Sol.

$\because \pi > 1$, $y = \pi^x$ is an increasing function in the interval $(-\infty, +\infty)$

$\because 2.1 < 3.5$,

$\therefore \pi^{2.1} < \pi^{3.5}$

(b) $0.5^{-2.3}$ and $0.5^{-3.8}$

Sol.

$\because 0.5 < 1$, $y = 0.5^x$ is an increasing function in the interval $(-\infty, +\infty)$

$\because -2.3 > -3.8$,

$\therefore 0.5^{-2.3} < 0.5^{-3.8}$

2. Given the exponential functions $f(x) = 3^{x^2-3x+5}$ and $g(x) = 3^{x+10}$. Find the value of x such that $f(x) = g(x)$.

Sol.

$$
\begin{aligned}
f(x) &= g(x) \\
3^{x^2-3x+5} &= 3^{x+10} \\
x^2 - 3x + 5 &= x + 10 \\
x^2 - 4x - 5 &= 0 \\
(x-5)(x+1) &= 0 \\
x &= 5 \text{ or } -1
\end{aligned}
$$

Exercise 23.1

Without using the calculator, find the value of the following expressions (Question 1 to 10):

1. $\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^0 - \left(-\frac{1}{3}\right)^{-2}$

   Sol.

$$\begin{aligned}\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^0 - \left(-\frac{1}{3}\right)^{-2} &= \frac{1}{9} + 1 - 9 \\ &= \frac{10}{9} - 9 \\ &= \frac{10 - 81}{9} \\ &= -\frac{71}{9}\end{aligned}$$

2. $\left(\frac{3^{-5} \cdot 3^2}{3^{-3}}\right)^{-2}$

   Sol.

$$\begin{aligned}\left(\frac{3^{-5} \cdot 3^2}{3^{-3}}\right)^{-2} &= \left(\frac{3^{-3}}{3^{-3}}\right)^{-2} \\ &= 1^{-2} \\ &= 1\end{aligned}$$

3. $6^{-8} \div 6^{-5} + 3^{-3}$

   Sol.

$$\begin{aligned}6^{-8} \div 6^{-5} + 3^{-3} &= 6^{-8} \times 6^5 + 3^{-3} \\ &= 6^{-3} + 3^{-3} \\ &= \frac{1}{6^3} + \frac{1}{3^3} \\ &= \frac{1}{216} + \frac{1}{27} \\ &= \frac{1+8}{216} \\ &= \frac{9}{216} \\ &= \frac{1}{24}\end{aligned}$$

4. $12^{\frac{1}{3}} \times 6^{\frac{1}{3}} \div 27^{\frac{1}{6}} \div 3^{\frac{1}{6}}$

Sol.

$$\begin{aligned}
12^{\frac{1}{3}} \times 6^{\frac{1}{3}} \div 27^{\frac{1}{6}} \div 3^{\frac{1}{6}} &= 72^{\frac{1}{3}} \div 81^{\frac{1}{6}} \\
&= 72^{\frac{1}{3}} \div (9^2)^{\frac{1}{6}} \\
&= 72^{\frac{1}{3}} \div 9^{\frac{1}{3}} \\
&= 8^{\frac{1}{3}} \\
&= 2
\end{aligned}$$

5. $(0.2)^{-2} \times (0.125)^{\frac{2}{3}}$

Sol.

$$\begin{aligned}
(0.2)^{-2} \times (0.125)^{\frac{2}{3}} &= \left(\frac{1}{5}\right)^{-2} \times \left(\frac{1}{8}\right)^{\frac{2}{3}} \\
&= 5^2 \times \frac{1}{64^{\frac{1}{3}}} \\
&= 25 \times \frac{1}{4} \\
&= \frac{25}{4}
\end{aligned}$$

6. $(0.3)^{-\frac{1}{3}} \times (0.0081)^{\frac{1}{3}} + (0.064)^{\frac{1}{3}}$

Sol.

$$\begin{aligned}
(0.3)^{-\frac{1}{3}} \times (0.0081)^{\frac{1}{3}} + (0.064)^{\frac{1}{3}} &= \left(\frac{3}{10}\right)^{-\frac{1}{3}} \cdot \left(\frac{81}{10000}\right)^{\frac{1}{3}} + \left(\frac{64}{1000}\right)^{\frac{1}{3}} \\
&= \left(\frac{3}{10}\right)^{-\frac{1}{3}} \cdot \left(\frac{3}{10} \cdot \frac{27}{1000}\right)^{\frac{1}{3}} + \frac{4}{10} \\
&= \left(\frac{3}{10}\right)^{-\frac{1}{3}} \cdot \left(\frac{3}{10}\right)^{\frac{1}{3}} \cdot \frac{3}{10} + \frac{4}{10} \\
&= \frac{3}{10} + \frac{4}{10} \\
&= \frac{7}{10}
\end{aligned}$$

7. $\left(\frac{81}{16}\right)^{-0.25} \times \left(\frac{8}{27}\right)^{-\frac{2}{3}} \times (0.25)^{-2.5}$

Sol.

$$\begin{aligned}\left(\frac{81}{16}\right)^{-0.25} \times \left(\frac{8}{27}\right)^{-\frac{2}{3}} \times (0.25)^{-2.5} &= \left(\frac{16}{81}\right)^{\frac{1}{4}} \times \frac{27}{8}^{\frac{2}{3}} \times 4^{\frac{5}{2}} \\ &= \frac{2}{3} \times \left(\frac{3}{2}\right)^{2} \times 2^{5} \\ &= \frac{2}{3} \times \frac{9}{4} \times 32 \\ &= 48\end{aligned}$$

8. $\left(\frac{1}{2}\right)^{-2} + 125^{\frac{2}{3}} + 343^{\frac{1}{3}} - \left(\frac{1}{27}\right)^{-\frac{1}{3}}$

Sol.

$$\begin{aligned}\left(\frac{1}{2}\right)^{-2} + 125^{\frac{2}{3}} + 343^{\frac{1}{3}} - \left(\frac{1}{27}\right)^{-\frac{1}{3}} &= 2^2 + 5^2 + 7 - 3^3 \\ &= 4 + 25 + 7 - 3 \\ &= 33\end{aligned}$$

9. $\left(2\frac{1}{4}\right)^{-\frac{3}{2}} + \left(1\frac{11}{25}\right)^{-\frac{1}{2}} - \left(2\frac{2}{3}\right)^{0}$

Sol.

$$\begin{aligned}\left(2\frac{1}{4}\right)^{-\frac{3}{2}} + \left(1\frac{11}{25}\right)^{-1} - \left(2\frac{2}{3}\right)^{0} &= \left(\frac{9}{4}\right)^{-\frac{3}{2}} + \left(\frac{36}{25}\right)^{-\frac{1}{2}} - 1 \\ &= \left(\frac{2}{3}\right)^{3} + \frac{5}{6} - 1 \\ &= \frac{8}{27} + \frac{5}{6} - 1 \\ &= \frac{48}{162} + \frac{135}{162} - \frac{162}{162} \\ &= \frac{21}{162} \\ &= \frac{7}{54}\end{aligned}$$

10. $\dfrac{5\sqrt{4}\sqrt{8}\left(\sqrt[3]{\sqrt[5]{4}}\right)^2}{\sqrt[3]{\sqrt{2}}}$

Sol.

$$
\begin{aligned}
\frac{\sqrt[5]{4}\sqrt{8}\left(\sqrt[3]{\sqrt[5]{4}}\right)^2}{\sqrt[3]{\sqrt{2}}} &= \frac{\left(\sqrt[5]{2}\right)^2\left(\sqrt{2}\right)^3\left(\sqrt[15]{2}\right)^4}{\sqrt[6]{2}} \\
&= \frac{2^{\frac{2}{5}}\cdot 2^{\frac{3}{2}}\cdot 2^{\frac{4}{15}}}{2^{\frac{1}{6}}} \\
&= 2^{\frac{2}{5}+\frac{3}{2}+\frac{4}{15}-\frac{1}{6}} \\
&= 2^{\frac{12+45+8-5}{30}} \\
&= 2^{\frac{60}{30}} \\
&= 2^2 \\
&= 4
\end{aligned}
$$

Simplify the following expressions (Question 11 to 24):

11. $a^{\frac{1}{2}}\cdot a^{\frac{1}{4}}\cdot a^{-\frac{1}{8}}\cdot a^{\frac{1}{6}}$

Sol.

$$
\begin{aligned}
a^{\frac{1}{2}}\cdot a^{\frac{1}{4}}\cdot a^{-\frac{1}{8}}\cdot a^{\frac{1}{6}} &= a^{\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{3}{8}} \\
&= a^{\frac{12+6-3+9}{24}} \\
&= a^{\frac{24}{24}} \\
&= a
\end{aligned}
$$

12. $\left(9a^2b^{-2}c^{-4}\right)^{-1}$

Sol.

$$
\begin{aligned}
\left(9a^2b^{-2}c^{-4}\right)^{-1} &= \frac{1}{9a^2b^{-2}c^{-4}} \\
&= \frac{b^2c^4}{9a^2}
\end{aligned}
$$

13. $\left(x^4y^{-5}\right)\left(x^{-2}y^2\right)^2$

Sol.

$$\begin{aligned}\left(x^4y^{-5}\right)\left(x^{-2}y^2\right)^2 &= x^4y^{-5}x^{-4}y^4\\ &= x^{4-4}y^{-5+4}\\ &= y^{-1}\\ &= \frac{1}{y}\end{aligned}$$

14. $3a^{-2}b^{-3} \div \left(-3^{-1}a^2b^{-3}\right)$

Sol.

$$\begin{aligned}3a^{-2}b^{-3} \div \left(-3^{-1}a^2b^{-3}\right) &= \frac{3a^{-2}b^{-3}}{-\frac{1}{3}a^2b^{-3}}\\ &= -\frac{9}{a^4}\end{aligned}$$

15. $\sqrt[3]{\dfrac{a^2b^{-1}}{a^{\frac{1}{2}}b^5}}$

Sol.

$$\begin{aligned}\sqrt[3]{\frac{a^2b^{-1}}{a^{\frac{1}{2}}b^5}} &= \sqrt[3]{a^2b^{-1}a^{-\frac{1}{2}}b^{-5}}\\ &= \left(a^{\frac{3}{2}}b^{-6}\right)^{\frac{1}{3}}\\ &= a^{\frac{1}{2}}b^{-2}\\ &= \frac{\sqrt{a}}{b^2}\end{aligned}$$

16. $5a^{-2}b^{-3} \div \left(5^{-1}a^2b^{-3}\right) \times 5^{-2}ab^4c$

Sol.

$$\begin{aligned}5a^{-2}b^{-3} \div \left(5^{-1}a^2b^{-3}\right) \times 5^{-2}ab^4c &= 5^{1-(-1)-2}a^{-2-2+1}b^{-3-(-3)+4}c\\ &= a^{-3}b^4c\\ &= \frac{b^4c}{a^3}\end{aligned}$$

17. $\dfrac{a^{-2}-b^{-2}}{a^{-2}+b^{-2}}$

Sol.

$$\begin{aligned}\frac{a^{-2}-b^{-2}}{a^{-2}+b^{-2}} &= \frac{a^{-2}-b^{-2}}{a^{-2}+b^{-2}}\cdot\frac{a^2b^2}{a^2b^2}\\ &= \frac{a^{-2}a^2b^2-b^{-2}a^2b^2}{a^{-2}a^2b^2+b^{-2}a^2b^2}\\ &= \frac{b^2-a^2}{b^2+a^2}\end{aligned}$$

18. $\left(a^{-1}+b^{-1}\right)(a+b)^{-1}$

Sol.

$$\begin{aligned}\left(a^{-1}+b^{-1}\right)(a+b)^{-1} &= \frac{a^{-1}+b^{-1}}{a+b}\\ &= \frac{a^{-1}+b^{-1}}{a+b}\cdot\frac{ab}{ab}\\ &= \frac{b+a}{ab\,(a+b)}\\ &= \frac{1}{ab}\end{aligned}$$

19. $\left(x+x^{-1}\right)\left(x-x^{-1}\right)$

Sol.

$$\begin{aligned}\left(x+x^{-1}\right)\left(x-x^{-1}\right) &= x^2-1+1-x^{-2}\\ &= x^2-x^{-2}\\ &= x^2-\frac{1}{x^2}\\ &= \frac{x^4-1}{x^2}\end{aligned}$$

20. $\left(-2x^{\frac{1}{4}}y^{-\frac{1}{3}}\right)\left(3x^{-\frac{1}{2}}y^{\frac{2}{3}}\right)\left(-4x^{\frac{1}{4}}y^{\frac{2}{3}}\right)$

Sol.

$$\begin{aligned}\left(-2x^{\frac{1}{4}}y^{-\frac{1}{3}}\right)\left(3x^{-\frac{1}{2}}y^{\frac{2}{3}}\right)\left(-4x^{\frac{1}{4}}y^{\frac{2}{3}}\right) &= -2\times 3\times(-4)x^{\frac{1}{4}-\frac{1}{2}+\frac{1}{4}}y^{-\frac{1}{3}+\frac{2}{3}+\frac{2}{3}}\\ &= 24x^0y^1\\ &= 24y\end{aligned}$$

21. $2x^{-\frac{1}{3}}\left(\frac{1}{2}x^{\frac{1}{3}}-2x^{-\frac{2}{3}}\right)$

Sol.

$$\begin{aligned}2x^{-\frac{1}{3}}\left(\frac{1}{2}x^{\frac{1}{3}}-2x^{-\frac{2}{3}}\right)&=x^{-\frac{1}{3}}x^{\frac{1}{3}}-x^{-\frac{1}{3}}4x^{-\frac{2}{3}}\\&=x^{0}-4x^{-1}\\&=1-\frac{4}{x}\\&=\frac{x-4}{x}\end{aligned}$$

22. $\left(\sqrt[3]{x^3}\cdot\sqrt{y}\right)^2\cdot\left(\sqrt{y}\cdot\sqrt{x^3}\right)^3$

Sol.

$$\begin{aligned}\left(\sqrt[3]{x^3}\cdot\sqrt{y}\right)^2\cdot\left(\sqrt{y}\cdot\sqrt{x^3}\right)^3&=\left(xy^{\frac{1}{2}}\right)^2\cdot\left(y^{\frac{1}{2}}x^{\frac{3}{2}}\right)^3\\&=x^2y\cdot y^{\frac{3}{2}}x^{\frac{9}{2}}\\&=x^{\frac{13}{2}}y^{\frac{5}{2}}\\&=x^6y^2\sqrt{xy}\end{aligned}$$

23. $\dfrac{3\times 2^n-4\times 2^{n-2}}{2^n-2^{n-1}}$

Sol.

$$\begin{aligned}\frac{3\times 2^n-4\times 2^{n-2}}{2^n-2^{n-1}}&=\frac{3\times 2^n-4\times 2^n\times 2^{-2}}{2^n-2^n\times 2^{-1}}\\&=\frac{3\times 2^n-2^n}{2^n\left(1-\frac{1}{2}\right)}\\&=\frac{2\times 2^n}{\frac{1}{2}\times 2^n}\\&=4\end{aligned}$$

24. $\left(3^{n+6}-5\times 3^{n+1}\right)\div\left(7\times 3^{n+2}\right)$

Sol.

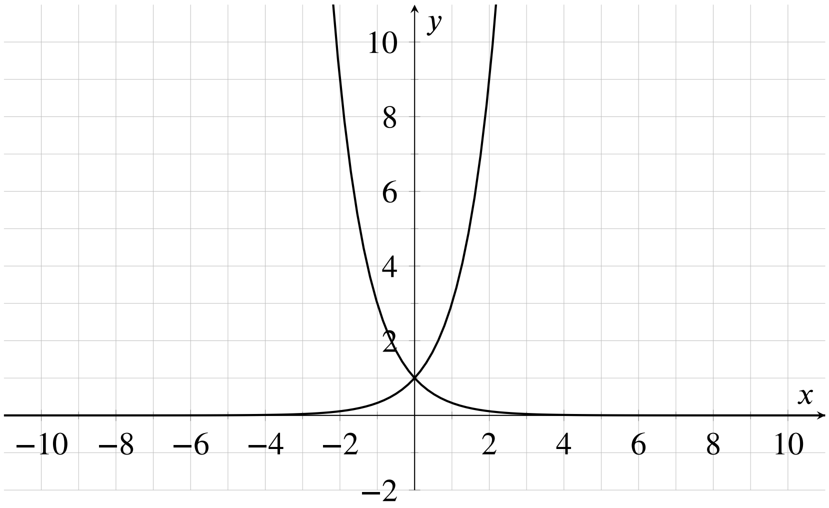
$$\begin{aligned}\left(3^{n+6}-5\times 3^{n+1}\right)\div\left(7\times 3^{n+2}\right)&=\frac{3^n\times 3^6-5\times 3^n\times 3}{7\times 3^n\times 9}\\&=\frac{3^n\left(3^6-15\right)}{63\times 3^n}\\&=\frac{3^6-15}{63}\\&=\frac{34}{3}\end{aligned}$$

25. Sketch the graph of the following functions on the same diagram:

(a) $y = 3^x$

(b) $y = \left(\frac{1}{3}\right)^x$

Sol.



26. Without using the calculator, compare the value of the following expressions:

(a) $2.5^{7.1}$ and $2.5^{8.5}$

Sol.

$\because 2.5 > 1,\ y = 2.5^x$ is an increasing function in the interval $(-\infty, \infty)$

$\because 7.1 < 8.5$,

$\therefore 2.5^{7.1} < 2.5^{8.5}$

(b) $0.35^{6.5}$ and $0.35^{5.6}$

Sol.

$\because 0.35 < 1,\ y = 0.35^x$ is a decreasing function in the interval $(-\infty, \infty)$

$\because 6.5 > 5.6$,

$\therefore 0.35^{6.5} < 0.35^{5.6}$

(c) $1.03^{-2.1}$ and $1.03^{-3.2}$

Sol.

$\because 1.03 > 1,\ y = 1.03^x$ is an increasing function in the interval $(-\infty, \infty)$

$\because -2.1 > -3.2$,

$\therefore 1.03^{-2.1} < 1.03^{-3.2}$

(d) $\left(\sqrt{2}\right)^{\pi}$ and $\left(\sqrt{2}\right)^{\pi-3.5}$

Sol.

$\because \sqrt{2} > 1,\ y = \left(\sqrt{2}\right)^x$ is an increasing function in the interval $(-\infty, \infty)$

$\because \pi > \pi - 3.5$,

$\therefore \left(\sqrt{2}\right)^{\pi} > \left(\sqrt{2}\right)^{\pi-3.5}$

(e) $0.01^{-\frac{1}{3}}$ and $0.01^{-\frac{1}{2}}$

Sol.

$\because 0.01 < 1,\ y = 0.01^x$ is a decreasing function in the interval $(-\infty, \infty)$

$\because -\dfrac{1}{3} > -\dfrac{1}{2},$

$\therefore 0.01^{-\frac{1}{3}} < 0.01^{-\frac{1}{2}}$

(f) $2.7^{\sqrt{20}}$ and $2.7^{\sqrt[3]{35}}$

Sol.

$\because 2.7 > 1,\ y = 2.7^x$ is an increasing function in the interval $(-\infty, \infty)$

$\because \sqrt{20} > \sqrt[3]{35},$

$\therefore 2.7^{\sqrt{20}} > 2.7^{\sqrt[3]{35}}$

27. Given that $f_1 : x \to 2^{3x}$ and $f_2 : x \to 2^{x^2+2}$. Find the value of x such that $f_1(x) = f_2(x)$.

Sol.

$$\begin{aligned} 2^{3x} &= 2^{x^2+2} \\ x^2 - 3x + 2 &= 0 \\ (x-1)(x-2) &= 0 \\ x &= 1 \text{ or } x = 2 \end{aligned}$$

28. Given the function $f(x) = (0.4)^{x^2-x+1}$ and $g(x) = (0.4)^{6x+19}$. Find the value of x such that $f(x) = g(x)$.

Sol.

$$\begin{aligned} (0.4)^{x^2-x+1} &= (0.4)^{6x+19} \\ x^2 - x + 1 &= 6x + 19 \\ x^2 - 7x + 18 &= 0 \\ (x-9)(x+2) &= 0 \\ x &= 9 \text{ or } x = -2 \end{aligned}$$

22.2 Logarithms

Definition of Logarithms

If $a_n = x$, where $a > 0$ and $a \neq 1$, then we define $\log_a x = n$, and we say that n is the logarithm of x to the base a. In $\log_a x$, a is called the base, x is called the antilogarithm.

On the other hand, if $\log_a x = n$, then $a_n = x$. This is the inversible relationship between exponents and logarithms. That is,

$$\boxed{\log_a x = n \iff a^n = x \qquad a > 0,\ a \neq 1,\ x > 0}$$

Logarithms with base 10 are called common logarithms, and are usually written as $\log a$.

Another common logarithm is the natural logarithm, which has base e ($e \approx 2.71828182846$), and is usually written as $\ln x$.

Practice 3

Find the value of x in the following equations:

1. $\log x = 3$

   Sol.

   $$\begin{aligned} \log x &= 3 \\ 10^3 &= x \\ x &= 1000 \end{aligned}$$

2. $\log_x 27 = \dfrac{3}{2}$

   Sol.

   $$\begin{aligned} \log_x 27 &= \frac{3}{2} \\ x^{\frac{3}{2}} &= 27 \\ x &= \sqrt[3]{27^2} \\ &= 9 \end{aligned}$$

3. $2\log_x(3\sqrt{3}) = 1$

   Sol.

   $$\begin{aligned} 2\log_x(3\sqrt{3}) &= 1 \\ \log_x(3\sqrt{3}) &= \frac{1}{2} \\ x^{\frac{1}{2}} &= 3\sqrt{3} \\ x &= 27 \end{aligned}$$

4. $\log_2(16\sqrt{2}) = x$

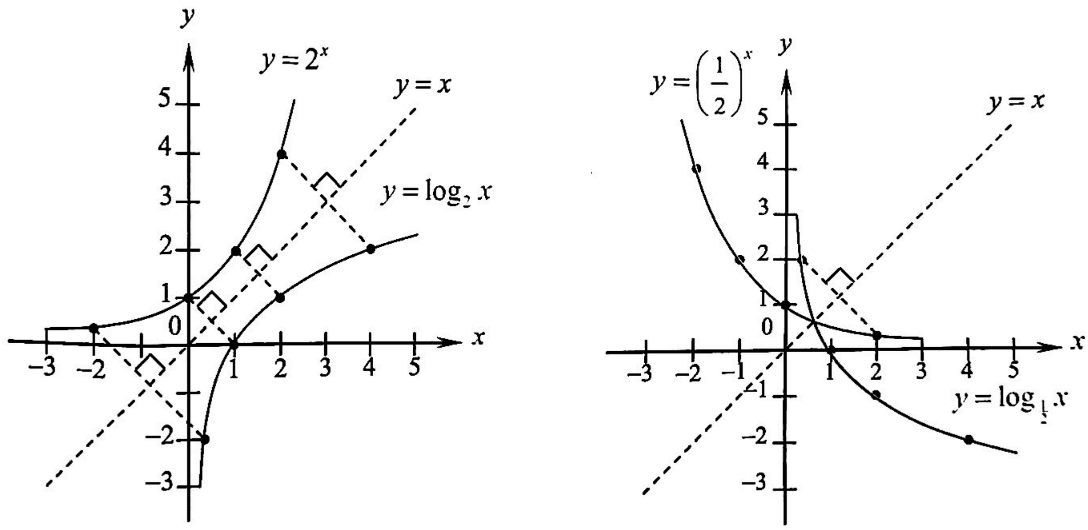
   Sol.

   $$\begin{aligned} \log_2(16\sqrt{2}) &= x \\ x &= \log_2(16\sqrt{2}) \\ &= \log_2(16) + \log_2(\sqrt{2}) \\ &= 4 + \frac{1}{2} \\ &= \frac{9}{2} \end{aligned}$$

Logarithmic Functions and Graphs

From the definition of logarithms, we can see that if $y = a^x$, then $x = \log_a y$. From the concept of inverse functions, we know that $y = \log_a x$ is the inverse function of $y = a^x$. Function $y = \log_a x$ is called the logarithmic function, where $a > 0$ and $a \neq 1$. Since the domain of $y = a^x$ is $\mathbb{R}$, and its range is $\mathbb{R}^+$, so the domain of $y = \log_a x$ is $\mathbb{R}^+$, and its range is $\mathbb{R}$.

Since the logarithmic function $y = \log_a x$ is the inverse function of the exponential function $y = a^x$, so the graph of $y = \log_a x$ is the reflection of the graph of $y = a^x$ about the line $y = x$. If we draw a curve of $y = a^x$, then reflect it about the line $y = x$, we can get the graph of $y = \log_a x$. For example, in the diagram below, the curves that are the reflection of the graphs of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ about the line $y = x$ are the graphs of $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ respectively.



From the diagram above, we can see that:

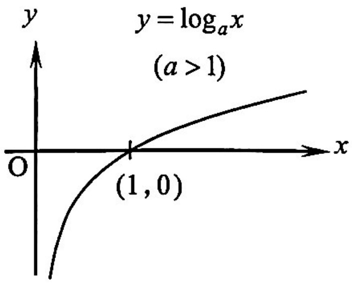
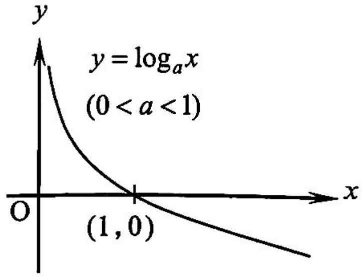
(1) Since the domain of $y = \log_a x$ is $x > 0$, so the graph of the function $y = \log_2 x$ and $y = \log_{\frac{1}{2}} x$ are only at the right side of the y-axis.

(2) When $x = 1$, $y = 0$. Hence, the graph of logarithmic functions $y = a^x$ always passes through the point $(1, 0)$.

(3) For the function $y = \log_2 x$, when $x > 1$, $y > 0$; when $0 < x < 1$, $y < 0$. When the value of x increases, the value of y increases, that is, the function is an increasing function in the interval $(0, +\infty)$.

(4) For the function $y = \log_{\frac{1}{2}} x$, when $x > 1$, $y < 0$; when $0 < x < 1$, $y > 0$. When the value of x increases, the value of y decreases, that is, the function is a decreasing function in the interval $(0, +\infty)$.

When we are discussing about the graph and properties of a logarithmic function $y = \log_a x$, the following two cases are considered:

<table>
<tr><td></td><td>$a > 1$</td><td>$0 < a < 1$</td></tr>
<tr><td>Graph</td><td>
</td><td>
</td></tr>
<tr><td rowspan="4">Properties</td><td colspan="2">x > 0</td></tr>
<tr><td colspan="2">When $x = 1$, $y = 0$</td></tr>
<tr><td>When $x > 1$, $y > 0$;
When $0 < y < 1$, $y < 0$</td><td>When $x > 1$, $y < 0$;
When $0 < x < 1$, $y > 0$</td></tr>
<tr><td>It is an increasing function in the interval $(0, +\infty)$</td><td>It is a decreasing function in the interval $(0, +\infty)$</td></tr>
</table>

Practice 4

1. Without using the calculator, Compare the value of the following expressions:

   (a) $\log 6$ and $\log 9$

   Sol.

   $\because 10 > 1$, $y = \log x$ is an increasing function in the interval $(0, +\infty)$,

   $\because 6 < 9$,

   $\therefore \log 6 < \log 9$

   (b) $\log_{0.5} 4.2$ and $\log_{0.5} 3.9$

   Sol.

   $\because 0 < 0.5 < 1$, $y = \log x$ is a decreasing function in the interval $(-\infty, 0)$,

   $\because 4.2 > 3.9$,

   $\therefore \log_{0.5} 4.2 < \log_{0.5} 3.9$

   (c) $\log_2 1.8$ abd $\log_4 5.8$.

   Sol.

   $\because \log_2 1.8 < 1$, $\log_4 5.8 > 1$,

   $\therefore \log_2 1.8 < \log_4 5.8$

2. Find the domain of the following functions:

   (a) $y = \log_a(x+2)$

   Sol.

   $$x + 2 > 0 \implies x > -2$$
   $$\therefore \text{Domain} = (-2, +\infty)$$

   (b) $y = \log_2(x^2 - 9)$

   Sol.

   $$x^2 - 9 > 0 \implies x < -3 \text{ or } x > 3$$
   $$\therefore \text{Domain} = (-\infty, -3) \cup (3, +\infty)$$

   (c) $y = \log_7 \dfrac{2}{3-2x}$

   Sol.

   $$3 - 2x > 0 \implies x < \frac{3}{2}$$
   $$\therefore \text{Domain} = \left(-\infty, \frac{3}{2}\right)$$

   (d) $y - \sqrt{\log_5(2-x)}$

   Sol.

   $$2 - x > 0 \implies x < 2$$
   $$\therefore \text{Domain} = (-\infty, 2)$$

Exercise 23.2

1. Find the value of x for the following expression:

(a) $\log_2 x = 4$

Sol.

$$\begin{aligned} \log_2 x &= 4 \\ x &= 2^4 \\ &= 16 \end{aligned}$$

(b) $\log_{125} x = \frac{1}{3}$

Sol.

$$\begin{aligned} \log_{125} x &= \frac{1}{3} \\ x &= 125^{\frac{1}{3}} \\ &= 5 \end{aligned}$$

(c) $\log_{16}(2\sqrt{2}) = x$

Sol.

$$\begin{aligned} \log_{16}(2\sqrt{2}) &= x \\ x &= \log_{16}(2\sqrt{2}) \\ &= \log_{16} 2 + \log_{16} \sqrt{2} \\ &= \log_{16} 2 + \frac{1}{2} \log_{16} 2 \\ &= \frac{3}{2} \log_{16} 2 \\ &= \frac{3}{2} \log_{16} 16^{\frac{1}{4}} \\ &= \frac{3}{2} \cdot \frac{1}{4} \\ &= \frac{3}{8} \end{aligned}$$

(d) $\log_{\frac{1}{3}} 81 = x$

Sol.

$$\begin{aligned} \log_{\frac{1}{3}} 81 &= x \\ x &= \log_{\frac{1}{3}} 81 \\ &= \log_{\frac{1}{3}} \left(\frac{1}{3}\right)^{-4} \\ &= -4 \end{aligned}$$

(e) $\log_x 81 = 4$

Sol.

$$\begin{aligned} \log_x 81 &= 4 \\ x^4 &= 81 \\ x &= \sqrt[4]{81} \\ &= 3 \end{aligned}$$

(f) $\log_x 49 = -2$

Sol.

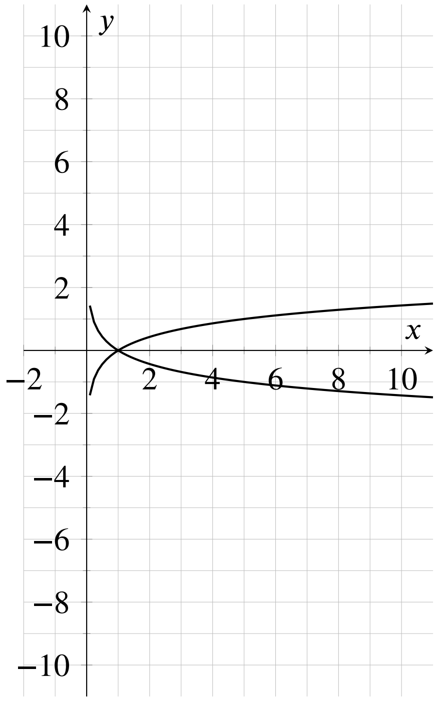
$$\begin{aligned} \log_x 49 &= -2 \\ x^{-2} &= 49 \\ x &= \frac{1}{\sqrt{49}} \\ &= \frac{1}{7} \end{aligned}$$

2. Sketch the graph of the following functions on the same set of axes:

(a) $y = \log_5 x$

(b) $y = \log_{\frac{1}{5}} x$

Sol.



3. Without using the calculator, compare the value of the following expressions:

(a) $\log_3 5$ and $\log_3 6$

$\because 3 > 1,\ y = \log x$ is an increasing function in the interval $(0, +\infty)$,

$\because 5 < 6$,

$\therefore\ \log 5 < \log 6$

(b) $\log_{1.5} 1.4$ and $\log_{1.5} 1.6$

Sol.

$\because 1.5 > 1,\ y = \log x$ is an increasing function in the interval $(0, +\infty)$,

$\because 1.4 < 1.6$,

$\therefore\ \log_{1.5} 1.4 < \log_{1.5} 1.6$

(c) $\log_{\sqrt{3}} 4.8$ and $\log_{\sqrt{3}} 5.8$

Sol.

$\because \sqrt{3} > 1,\ y = \log x$ is an increasing function in the interval $(0, +\infty)$,

$\because 4.8 < 5.8$,

$\therefore\ \log_{\sqrt{3}} 4.8 < \log_{\sqrt{3}} 5.8$

(d) $\log_{2.3} \pi$ and $\log_{2.3}(\pi - 3)$

Sol.

$\because 2.3 > 1,\ y = \log x$ is an increasing function in the interval $(0, +\infty)$,

$\because \pi > \pi - 3$,

$\therefore\ \log_{2.3} \pi > \log_{2.3} (\pi - 3)$

(e) $\log_{0.4}\sqrt{2}$ and $\log_{0.4}\sqrt{3}$

Sol.

$\because 0.4 < 1,\ y = \log x$ is a decreasing function in the interval $(-\infty, 0)$,

$\because \sqrt{2} < \sqrt{3}$,

$\therefore \log_{0.4}\sqrt{2} > \log_{0.4}\sqrt{3}$

(f) $\log_{\frac{1}{2}} 3$ and $\log_{\frac{1}{3}} \frac{1}{4}$

Sol.

$\because \frac{1}{2} < 1,\ y = \log x$ is a decreasing function in the interval $(-\infty, 0)$,

$\because \log_{\frac{1}{2}} 3 > 1,\ \log_{\frac{1}{3}} \frac{1}{4} < 1$,

$\therefore \log_{\frac{1}{2}} 3 < \log_{\frac{1}{3}} \frac{1}{4}$

4. Find the domain of the following functions:

(a) $y = \log_2(3 - 2x)$

Sol.

$3 - 2x > 0 \implies x < \frac{3}{2}$

$\therefore$ Domain $= \left(-\infty, \frac{3}{2}\right)$

(b) $y = \log(x^2 + 1)$

Sol.

$x^2 + 1 > 0$ for all $x \in \mathbb{R}$,

$\therefore$ Domain $= \mathbb{R}$

(c) $y = \log_5(9 - 16x^2)$

Sol.

$9 - 16x^2 > 0 \implies -\frac{3}{4} < x < \frac{3}{4}$

$\therefore$ Domain $= \left(-\frac{3}{4}, \frac{3}{4}\right)$

(d) $y = \log_9 \frac{1}{x - 2}$

Sol.

$x - 2 > 0 \implies x > 2$

$\frac{1}{x - 2} > 0 \implies x > 2$

$\therefore$ Domain $= (2, +\infty)$

(e) $y = \log_8 \sqrt{2x^2 - x - 3}$

Sol.

$2x^2 - x - 3 > 0 \implies x < -1$ or $x > \frac{3}{2}$

$\therefore$ Domain $= (-\infty, -1) \cup \left(\frac{3}{2}, +\infty\right)$

(f) $y = \frac{1}{\log_3(7x - 5)}$

Sol.

$$\begin{cases} 7x - 5 > 0 \\ \log_3(7x - 5) > 0 \end{cases} \implies x > \frac{5}{7},\ x \neq \frac{6}{7}$$

$\therefore$ Domain $= \left(\frac{5}{7}, +\infty\right) \setminus \left\{\frac{6}{7}\right\}$

22.3 Arithmetic Properties of Logarithms and Base Changing Formula

Identties and Arithmetic Properties of Logarithms

Logarithms have the following identities:

$$\log_a 1 = 0$$
$$\log_a a = 1$$

Logarithms have the following arithmetic properties:

$$\begin{aligned}&\log_a(xy) = \log_a x + \log_a y \quad (x > 0, y > 0)\\&\log_a \frac{x}{y} = \log_a x - \log_a y \quad (x > 0, y > 0)\\&\log_a x^n = n\log_a x \quad (x > 0)\\&a^{\log_a x} = x\end{aligned}$$

Base Changing Formula

The base of a logarithm can be changed from one to another. Let $\log_a x = n$, then $a^n = x$. Change both sides of the equation to logarithm with base b, we have

$$\begin{aligned}\log_b a^n &= \log_b x\\ n\log_b a &= \log_b x\\ \because a \neq 1,\ \therefore\ \log_b a &\neq 0\\ n &= \frac{\log_b x}{\log_b a}\\ \therefore\ \log_a x &= \frac{\log_b x}{\log_b a}\end{aligned}$$

The expression above is called the *base changing formula*.
When $x = b$, we have $\log_a b = \dfrac{1}{\log_b a}$.
In this book, the value of logarithms are rounded to 4 decimal places.

Practice 5

Exercise 23.2

Without using the calculator, compare the value of the following expressions (Question 1 to 4):

1. $5^{2\log_5 4}$
2. $4^{3\log_2 \sqrt{2}}$
3. $2\log 5 - \log \frac{1}{3} + \frac{1}{2}\log \frac{16}{9}$
4. $\frac{\log_4 27}{\log_2 3}$
5. Given that $\log_2 4 = a$ and $\log_2 5 = b$. Express the following expressions in terms of a and b:

   (a) $\log_2 90$

   (b) $\log_3 270$

   (c) $\log_9 1.8$
6. Given that $\log_{16} y = \frac{1}{2} + \log_4 x$. Express x in terms of y.

Exercise 23.3

Exercise 23.2

Simplify the following expressions (Question 1 to 6):

1. $\log_2 4^x$
2. $\log_2 a^{\log_a 2}$
3. $3^{\log_3 x - \log_3 y}$
4. $\log_3 (9^x \times 27^y)$
5. $2^{-\log_8 x}$
6. $3\log_4 2^x$

Without using the calculator, evaluate the following expressions (Question 7 to 22):

7. $\log_7 \sqrt[3]{49}$
8. $49^{\log_7 3}$
9. $2^{2\log_2 7} + \left(\frac{1}{2}\right)^{-\log_2 7}$
10. $\log_3 5 - \log_3 15$
11. $\frac{\log \sqrt{3}}{\log \frac{1}{9}}$
12. $\log_5 \frac{1}{5} + \log_5 \sqrt[3]{5} - \log_5 25$
13. $\log_3 \sqrt[3]{27\sqrt[4]{81}}$
14. $\log (0.1)^4 - \log \sqrt[3]{0.001}$
15. $\frac{\log 4 + \log 3}{1 + \log 0.4 + \frac{1}{2}\log 9}$
16. $\log_{36} 6 - \log_6 36 - \log_6 \frac{1}{6}$
17. $\log_2 \frac{1}{25} \log_3 \frac{1}{8} \cdot \log_5 \frac{1}{9}$
18. $\log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8$
19. $\log_4 8 - \log_{\frac{1}{9}} 3 - \log_{\sqrt{2}} 4$
20. $\left(\log_2 3 + \log_2 \sqrt{3}\right) \log_{\sqrt{3}} 2$
21. $\frac{\log_5 \sqrt{2} \cdot \log_7 9}{\log_7 \sqrt[3]{4} \cdot \log_5 \frac{1}{3}}$
22. $\frac{1}{2}\log \frac{81}{17} + 2\log \frac{5}{3} - \log \frac{17}{4} + \frac{3}{2}\log 17$
23. Given that $\log_2 3 = a$ and $\log_2 5 = b$. Express a and b in terms of $\log_4 15$.

24. Given that $\log_3 5 = m$ and $\log_5 6 = n$. Express m and n in terms of $\log_2 554$.

25. Given that $\log_2 3 = a$ and $\log_3 7 = b$. Express a and b in terms of $\log_4 214$.

26. Given that $\log_3 6 = x$. Epress x in terms of $\log_9 12$.

27. Given that $\log_3 y - \log_9 \sqrt[3]{x} = 1 + \log_2 7x$. Exress x in terms of y.

28. Given that $\log_2 5(2x-1) = \log_5(x-3) + \log_2 55$, prove that $5x^2 - 32x + 46 = 0$.

29. If $a > 0$, $b > 0$, and $a^2 + b^2 = 7ab$, prove that $2\log_3(a+b) = 2 + \log_3 a + \log_3 b$.

30. If $x > 0$, $y > 0$, and $x^2 + 9y^2 = 10xy$, prove that $2\log(x+3y) - 4\log 2 = \log x + \log y$.

22.4 Exponential Equations

All the equations with terms that contain the variable in the exponent are called exponential equations. For example, $9^x = 3^{x-1}$, $3^x = 5$, and $2^{x-1} + 2^x - 2 = 0$ are all exponential equations.

Practice 6

Solve the following exponential equations:

1. $3^{2x} = -\frac{1}{9}$
2. $2^{x^2+4x} = \frac{1}{8}$
3. $6^x = 5^{x-1}$
4. $4^{x-1} + 2^{x-1} = 20$

Exercise 23.4

Solve the following exponential equations:

1. $8^{x-3} = \frac{1}{256}$
2. $3^{2x+1} = 243$
3. $10^{x^2-4} = 1$
4. $3^{x^2+3} = 27^{x+7}$
5. $4^{x^2} = 2^{5x+7}$
6. $5^{2x^2+x} = 25^{1+x-2x^2}$
7. $\left(\frac{9}{16}\right)^x = \left(\frac{4}{3}\right)^{x-6}$
8. $5^{2^x+1} = 5^{4^x+1}$
9. $2^{2x+3} \cdot 4^{x+6} = (8^x)^x$
10. $\frac{5^{x^2}}{5} = 7^{(x+1)(x-1)}$
11. $3^{x+1} = 4^{x-1}$
12. $7^{5-3x} = 5^{x+2}$
13. $13^{2x+5} = 14^{x+7}$
14. $2^{x^2-1} = 3^{x+1}$
15. $\left(\frac{1}{3}\right)^x - \left(\frac{1}{3}\right)^{-x} = \frac{80}{9}$
16. $3^{x+1} + 9^x - 18 = 0$
17. $25^x - 23 \cdot 5^x - 50 = 0$
18. $3^{x-1} + 3^{3-x} - 10 = 0$
19. $3^{2x} - 3^{x+1} + 2 = 0$
20. $2^{x+2} + 3\left(2^{1-x}\right) - 14 = 0$
21. $2^{2x-1} - 3 \cdot 2^{x-1} + 1 = 0$
22. $3^x - 5^{x+2} = 3^{x+1} - 5^{x+3}$

22.5 Logarithmic Equations

All the equations with logarithmic terms which contains variable in the base or in the argument are called logarithmic equations. For example, $\log(x-1) = 3$, $\log_x 9 = 2$, and $2\log_3 x + \log_9 x = 1$ are all logarithmic equations. The results acquired when solving logarithmic equations need to be checked.

Practice 7

Solve the following logarithmic equations:

1. $\log_3 x = 5$
2. $\log_5(x-2) = 0$
3. $\log(x^2+2x-3) - \log(x+3) = 0$
4. $\log_3(3x+1) + 1 = \log_3(2x-1) + \log_3 5$
5. $\log_x 3 + \log_x 81 = 5$
6. $3\log_2^2 x + 5\log_2 x - 2 = 0$
7. $\log_2 x - \log_x 8 = 2$
8. $x^{\log x} = 100x$

Exercise 23.5

Solve the following logarithmic equations:

1. $\log_{\sqrt{3}} x = -2$
2. $\log_2 x^4 = 4$
3. $\log \frac{x-2}{x+2} = \log \frac{1}{x-1}$
4. $2\log x + \log 7 = \log 14$
5. $\log x + \log(x-3) = 1$
6. $\log(x+6) - \log(x-3) = 1$
7. $\log_6 x + \log_6(x^2-7) = 1$
8. $\log_{1,2}(15x^2-2x-12) = 0$
9. $\log_8(x^2-3x-2) = \frac{1}{3}$
10. $\log_2(x^2-x-2) - \log_2(x+1) = 0$
11. $\log_3(2x-3) + \log_3(3x+2) = \log_3(2x-1)$
12. $\frac{1}{2}(\log x - \log 5) = \log 2 - \frac{1}{2}\log(9-x)$
13. $\log(x+6) - \frac{1}{2}\log(2x-3) = 2 - \log 25$
14. $\log_2 x = \log_8 x + 1$
15. $3^{\log x} = 2^{\log 3}$
16. $4x^{\log_2 x} = x^3$
17. $2\left(\log_3 x\right)^2 + \log_3 x - 1 = 0$
18. $\log_4^2 x - 5\log_4 x + 6 = 0$
19. $6\log^2 x + \log x^3 - 3 = 0$
20. $(\log x)^2 = 2\log x$
21. $\log_x 25 - \log_{25} x = 0$
22. $2\log_4 x - 3\log_x 4 + 5 = 0$
23. $2\log_x 10 - \log x + 1 = 0$
24. $\log_5\left[\log_2\left(\log_x 5\right)\right] = 0$

22.6 Compound Interest and Annuity

Simple interest and compound interest are two different methods of calculating interest. Simple interest is calculated on the principal amount of a loan only. Compound interest is calculated on the principal amount and also on the accumulated interest of previous periods, and can thus be regarded as interest on interest.

For example, a fund amounted to RMp is deposited into a bank account with a yearly interest rate of $r\%$.

$$\text{Principal amount} = \text{RM}p$$

When $t = 1$,

$$\begin{aligned}\text{Interest earned} &= p \times r\% = \frac{pr}{100}\\ \text{Accumulated amount} &= p + \frac{pr}{100} = p\left(1 + \frac{r}{100}\right)\end{aligned}$$

When $t = 2$,

$$\begin{aligned}\text{Interest earned} &= \left(p + \frac{pr}{100}\right) \times r\% = \frac{pr}{100}\left(1 + \frac{r}{100}\right)\\ \text{Accumulated amount} &= p\left(1 + \frac{r}{100}\right) + \frac{pr}{100}\left(1 + \frac{r}{100}\right)\\ &= p\left(1 + \frac{r}{100}\right)\left(1 + \frac{r}{100}\right)\\ &= p\left(1 + \frac{r}{100}\right)^2\end{aligned}$$

When $t = 3$,

$$\begin{aligned}\text{Interest earned} &= \left(p\left(1 + \frac{r}{100}\right)^2\right) \times r\% = \frac{pr}{100}\left(1 + \frac{r}{100}\right)^2\\ \text{Accumulated amount} &= p\left(1 + \frac{r}{100}\right)^2 + \frac{pr}{100}\left(1 + \frac{r}{100}\right)^2\\ &= p\left(1 + \frac{r}{100}\right)^2\left(1 + \frac{r}{100}\right)\\ &= p\left(1 + \frac{r}{100}\right)^3\end{aligned}$$

In general, the accumulated amount after t years is given by

$$\boxed{A = p\left(1 + \frac{r}{100}\right)^t}$$

where p is called the *present value* of A.

If the interest is compounded m times per year, then the accumulated amount is given by

$$A = p\left(1 + \frac{r}{100m}\right)^{mt}$$

Annuity and Present Value of Annuity

An annuity is a series of equal payments made at equal intervals of time according to some kind of contract, standing order or the amount received. For example, all sorts of instalment, insurance premiums, house rent, car loan, etc. are annuities. In this book, we will only consider annuities with equal payments made or received at equal intervals of time.

Note that the annuity is not limited to once a year.

We can compare which payment plan is better by comparing the present values of the annuities. From the formula $A = p(1 + r\%)^t$, we can know that the present value $p = \dfrac{A}{(1 + r\%)^t}$. If the yearly interest rate is $r\%$, the annuity is RMA, the payment is made once per year, then the present value of the amount paid after a year is $A(1+r\%)^{-1}$, the present value of the amount paid after two years is $A(1+r\%)^{-2}$, and so on. The present value of the amount paid after n years is $A(1+r\%)^{-n}$. Hence, the sum of the present values of the amount paid after n years is

$$\begin{aligned}
&\frac{A}{1 + r\%} + \frac{A}{(1 + r\%)^2} + \cdots + \frac{A}{(1 + r\%)^n} \\
&= A\left[\frac{1}{1 + r\%} + \frac{1}{(1 + r\%)^2} + \cdots + \frac{1}{(1 + r\%)^n}\right] \\
&= A\left[\frac{1 - \dfrac{1}{(1 + r\%)^n}}{1 - \dfrac{1}{1 + r\%}}\right] \\
&= \frac{A}{r\%}\left(1 - \frac{1}{(1 + r\%)^n}\right)
\end{aligned}$$

Annuity that is paid indefinitely is called *perpetuity*, $n \to \infty$, $\dfrac{1}{(1 + r\%)^n} \to 0$. From that, we can know that the present value of perpetuity is $\dfrac{A}{r\%}$.

Practice 8

1. Given that the principal amount is RM75,000, interest rate is 4.5%. Using composite interest method, find the accumulated amount after 10 years.

2. A person has deposited RM40,000 into a bank account. The bank pays 8% interest per annum compounded half yearly. Using the compound interest method, find the amount in the account after 3 years.

3. Given that the interest rate is 6%, the inter-

est is compounded half yearly. Using the compound interest method, the accumulated amount after 5 years is RM4031.75, find the principal amount.

4. Given that the interest rate is 4%, the annuity is RM3,500, the payment is made once per year. The payment has since been made for 15 years continuously. Find the present value. Hence, find the present value of the perpetuity.

Exercise 23.6

1. Given that the principal amount is RM90,000, the interest rate is 5%. Compounding the interest once per year, find the accumulated amount after 10 years.

2. A person has deposited a fund into a bank account. The bank pays 8% interest per annum compounded yearly. The amount in the account after 3 years has increased by RM779.14. Find the amount of the fund deposited.

3. RM80,000 was deposited into a financial institution. The interest rate is 8% per annum compounded once per three months. Find the amount in the account after 5 years.

4. Prove that the accumulated amount after being compounded with an interest of 5 for 15 years will exceed twice the principal amount.

5. Given that the principal amount is RM15,000, the interest rate is 6% being compouned once per year. How long does it take for the accumulated amount to be more than RM30,000?

6. Given that the principal amount is RM120,000, the interest rate is 5.5% being compounded half yearly. How long does it take for the accumulated amount to be more than RM200,000?

7. A person deposited RM2,500 into his bank account at the beginning of the yaar, the interest rate is 4.5% compounded once per year. Find the amount in the account after 15 years.

8. If the present value is RM15,443.46, the interest rate is 5%, find the annuity if the payment is made for 10 years.

9. Given that the annuity is RM5,000, the interest rate is 5%, the payment is made once per year for 25 years. Find the present value. Hence, find the present value of the perpetuity.

10. Given that the annuity is RM2,500, the interest rate is 4.5%, the payment is made once per year. How many years does it take for the present value to exceed RM30,000?

11. If a bank has introduced an annuity scheme, the investors can receive RM1,000 per year for life after paying RM20,000. If the annuity plan is considered approximately to be a perpetuity, find the interest rate.

Revision Exercise 23

1. Without using a calculator, find the value of the following:

(a) $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^6 + \left(-\frac{1}{2}\right)^{-2}$

(b) $5^{\frac{1}{2}} + 5^{-\frac{1}{2}} - \left(\dfrac{1}{5}\right)^{\frac{1}{2}} + \left(\dfrac{1}{5}\right)^{-\frac{1}{2}}$

(c) $\left(\frac{1}{3}\right)^2 \times \left(-\frac{1}{3}\right)^2 \times \left(\frac{1}{2}\right)^{-2}$

(d) $\sqrt{2\sqrt[3]{3}} \div \sqrt[3]{\dfrac{\sqrt{8}}{3}}$

2. Simplify the following expressions:

(a) $\left(\dfrac{b}{2a^2}\right)^3 \div \left(\dfrac{2b^2}{3a}\right)^0 \times \left(-\dfrac{b}{a}\right)^{-3}$

(b) $\dfrac{3^{n+2} - 2 \times 3^n}{5(3^{n+1})}$

(c) $\dfrac{\left(x^{-1} + y^{-1}\right)\left(x^{-1} - y^{-1}\right)}{x^{-2}y^{-2}}$

(d) $\left(x^{\frac{1}{4}} - y^{-\frac{1}{4}}\right)\left(x^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)\left(x^{\frac{1}{4}} + y^{-\frac{1}{4}}\right)$

(e) $\dfrac{\left(\sqrt[4]{p^3}\right)^{\frac{1}{6}} \sqrt[9]{p^{-3}}}{\left(\sqrt{p^{-7}}\right)^{\frac{1}{6}}}$

(f) $\dfrac{\left(a^2 + a^{-2} + 2\right)^2}{\left(a^2 + 1\right)^4}$

3. Without using a calculator, compare the value of the following:

(a) 2.3^{-2} and 2.3^{-1}

(b) $0.15^{-\frac{1}{2}}$ and $0.15^{-\frac{1}{3}}$

(c) $\left(\dfrac{1}{3}\right)^{\frac{2}{5}}$ and $3^{-\frac{5}{3}}$

(d) $\left(\dfrac{3}{5}\right)^2$ and $\left(\dfrac{5}{3}\right)^{3.1}$

4. Without using a calculator, compare the value of the following:

(a) $\log_{3.2} 3$ and $\log_{3.2} 2$

(b) $\log_{0.5} 5.3$ and $\log_{0.5} 3.5$

(c) $\log_3 2$ and $\log_2 3$

(d) $\log_2 2.3$ and $\log_4 4.8$

5. Find the domain of the following functions:

(a) $y = \log_{0.5}\left(16 - x^2\right)$

(b) $y = \log_2\left(2x^2 - 5x - 12\right)$

(c) $y = \sqrt{3 - 3^x}$

(d) $y = \log_2(x - 3)^2$

(e) $y = \log_5\left(x^2 - 2x\right)$

(f) $y = \log_3 \dfrac{2}{3 - x}$

(g) $y = \dfrac{1}{\log(x + 1) - 1}$

(h) $y = \dfrac{\log_3(2 - x)}{\log_3(2 + x)}$

(i) $y = \sqrt{\log_3(x - 2)}$

(j) $y = \dfrac{2}{\sqrt{1 - \log x}}$

6. Simplify the following expressions:

(a) $\log_3 27^x$

(b) $\log_x b^{a \log_b x}$

(c) $\log_5\left(25^x \cdot 5^y\right)$

(d) $3^{2\log_3 x - \log_3 y}$

(e) $5^{-2\log_{25} x}$

7. If $\log_2 5 = p$, express $\log_2 100$ in terms of p.

8. If $\log_3 12 = a$, express the following in terms of a:

(a) $\log_3 24$

(b) $\log_9 36$

9. Without using a calculator, find the value of the following:

(a) $\log_c \frac{1}{5} + \log_c 5$

(b) $\log_2\left(2\sqrt{2}\right) - 2\log_2 \sqrt{2}$

(c) $\log_8 \dfrac{2}{7} - \log_8(-2)^2 - \log_8 \dfrac{1}{7}$

(d) $\log \dfrac{5}{32} - 2\log \dfrac{5}{6} + \log \dfrac{40}{9}$

(e) $\left(\log_2 3\right)\left(\log_3 4\right)$

(f) $\dfrac{\log_{16} 5}{\log_{32} 5}$

(g) $\log_3 5 \cdot \log_5 7 \cdot \log_7 27$

(h) $\log_2 \dfrac{1}{9} \cdot \log_3 \dfrac{1}{25} \cdot \log_5 \sqrt{8}$

(i) $\dfrac{1}{3} \log_2 8 + \log_3 27 - \dfrac{1}{4} \log_4 16$

(j) $\log^2 2 + \log 2 \cdot \log 5 + \log 5$

(k) $2\log_3 15 + 3\log_3 12 - \log_3 25 - 6\log_3 2$

(l) $\dfrac{\log\sqrt{27} + \log\sqrt{8} - \log\sqrt{125}}{\log 6 - \log 5}$

(m) $\log_8 \left(\log_2 \sqrt{8 + 4\sqrt{3}} + \log_2 \sqrt{8 - 4\sqrt{3}} \right)$

10. If $\log_3 2 = a$, $\log_2 5 = b$, prove that $\log_5 3 = \dfrac{1}{a(b+1)}$.

11. If $\log_2 3 = p$, $\log_3 7 = q$, prove that $\log_2 114 = \dfrac{pq+1}{p(q+1)}$.

12. Given that $2\log_5(x+y) = 1 + \log_5 x + \log_5 y$. Prove that $x^2 + y^2 = 3xy$.

13. Given that $x = 5^k$ and $y = 5^n$. Express the following in terms of k and n:

(a) $\log_5 \dfrac{xy^3}{125}$

(b) $\log_{25} \left(5\sqrt{xy}\right)$

14. Given that $2 + \log_4 y = 2\log_1 6x$. Express x in terms of y.

15. Solve the following exponential equations:

(a) $3^{3x-2} = 243$

(b) $4^{1-x} = \left(\dfrac{1}{8}\right)^2$

(c) $2^{x^2} = (2^x)^2$

(d) $3^{5^x} = 3$

(e) $5^{8^x} = 625$

(f) $3^{x+1} = 6^x$

(g) $7^x - 7^{x-1} = 6$

(h) $3^{x+1} = 10\,(3^x) - 3$

(i) $2^{2x+1} = 3\,(2^x) - 1$

(j) $5^{2x+1} = 26\,(5^x) - 5$

(k) $2^{2x+3} - 2^x = 1 - 2^{x+3}$

(l) $2^{2x+8} - 32\,(2^x) + 1 = 0$

16. If $\log_2 x + \log_4 x = \dfrac{9}{2}$, find the value of x.

17. Solve the follwoing logarithmic equations:

(a) $2\log x - 3\log 4 = 2$

(b) $2\log x = \log 32 + \log 2$

(c) $\log x + \log(x+3) = \log(x+8)$

(d) $\left(\log_2 x\right)^2 = \log_2 x + 6$

(e) $\log_3 x + 6\log_x 3 = 5$

(f) $4^{\log x} = 2^{\log x + 1}$

(g) $\log_{x+1}\left(x^2 - 5x - 13\right) = 2$

(h) $\log_r \sqrt{2x^2 - 5x + 6} = 1$

(i) $\log_2(x+1) + \log_2(x+3) = 3 + \log_2 x$

(j) $\log_2\left[\log_3\left(\log_5 x\right)\right] = 0$

(k) $3\log_s x - 2\log_2 x + 2 = 0$

(l) $\log_4(x+4) + 1 = \log_2(x+1)$

(m) $2\log_2 x \cdot \log_8 x = \log_2 x + \log_8 x$

18. A person has deposited a fund into a bank account that pays 5.5% interest compounded annually. If the balance in the account has increased by RM1,432.95 after 4 years, how much was deposited initially?

19. Given that there is a principal of RM75,000 at an interest rate of 3.5% per annum compounded once per three months. Find the accumulated value of the principal after 8 years.

20. Given that there is a principal of RM150,000 at an interest rate of 5.25% per annum compounded once per annum. How many years does it take to accumulate at least RM300,000?

21. If the present value is RM24,924.44, the interest rate is 5% per annum compounded once per annum, find the annuity payment if the payment is made for 20 years.

22. Given that the annuity payment is RM8,000, the interest rate is 4.5% per annum compounded once per annum, and the payment is made for 15 years. Find the present value. Hence, find the present value of the perpetuity.

23. Given that the annuity amount is RM4,500, the interest rate is 4.5% per annum compounded once per annum. How many years does it take for the present value to exceed RM50,000?

24. The price of a branded laptop is RM2,500, the payment can be paid in full or by instalment. If the payment is made by instalment, the monthly payment is RM110 for 2 years. If the interest rate is 4% per annum compounded once per month, which payment method is more economical considering the present value of the payment?