Calculus

Melvin Chia

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41. $\lim_{x\to 1} \dfrac{1-\sqrt[4]{x}}{1-\sqrt[3]{x}}$

42. $\lim_{x\to 0} \dfrac{\sqrt[n]{1+x}-1}{x} \quad (n \in \mathbb{W})$

43. $\lim_{x\to 1} \dfrac{2-\sqrt{x+3}}{x^2-1}$

44. $\lim_{x\to 16} \dfrac{\sqrt[4]{x}-2}{\sqrt{x}-4}$

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46. $\lim_{x\to -1} \dfrac{x^2+2}{x^2+x+3}$

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48. $\lim_{x\to 1} \dfrac{x^5-x^4}{x^3-x}$

49. $\lim_{x\to a} \dfrac{x^2+ax-2a^2}{x^2-a^2}, a \neq 0$

50. $\lim_{x\to a} \dfrac{\sqrt{3x-a}-\sqrt{x+a}}{x-a}$

51. Given that $f(x) = x^2 - 3x$, find $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

52. $\lim_{x\to 2} \sqrt{2x^2+1}$

53. $\lim_{x\to 7} \dfrac{x^2\sqrt{x+2}}{x^2+14}$

54. $\lim_{x\to 0} \dfrac{\sqrt{3x+4}-2}{x}$

55. $\lim_{x\to 0} \dfrac{1}{x^2}$

56. $\lim_{x\to 1} \dfrac{1}{x-1}$

57. $\lim_{x\to 1} \dfrac{4x-3}{x^2-5x+4}$

58. $\lim_{x\to \infty} \dfrac{3x^3-4x^2+2}{7x^3+5x^2-3}$

59. $\lim_{n\to \infty} x^2$

60. $\lim_{x\to \infty} \dfrac{3x^2-2x-1}{2x^3-x^2+5}$

61. $\lim_{n\to \infty} \dfrac{1}{n^2}(1+2+3+\cdots+n)$

62. $\lim_{n\to \infty} \left[\dfrac{1+2+3+\cdots+n}{n+2} - \dfrac{n}{2}\right]$

63. $\lim_{n\to \infty} \left[\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \cdots + \dfrac{1}{n(n+1)}\right]$

64. $\lim_{x\to \infty} \dfrac{5x^3+4x^2-6x+2}{8x^3-7x^2+4x-1}$

65. $\lim_{x\to \infty} \dfrac{x^4-2x^3+x^2+3}{x^5-x^4+1}$

66. $\lim_{x\to \infty} \dfrac{x^3-8x^2+4x-1}{x^2-6x+3}$

67. $\lim_{x\to \infty} (\sqrt{x^4+1}-x^2)$

68. Find $\lim_{n\to\infty} a_n$ of the following:

   (a) $a_n = 1 - \dfrac{1}{n^2}$

   (b) $a_n = \dfrac{4n-3}{8+6n}$

   (c) $\lim_{n\to\infty} \sum_{k=1}^{n} \dfrac{k^2}{n^3}$

   (d) $a_n = \dfrac{n+2n^2+3n^3}{4n^3-7}$

   (e) $a_n = \dfrac{1+2+3+\cdots+n}{n^2}$

69. Let $a_n = \left(\dfrac{(-1)^{n+1}}{\sqrt[3]{n^2+1}}\right)\dfrac{1}{n}$, find $\lim_{n\to\infty} a_n$.

70. $\lim_{n\to \infty} \sqrt{4+\left(\dfrac{1}{n}\right)^2}$

71. $\lim_{n\to \infty} \left(\dfrac{n!}{n^n}\right)$

72. $\lim_{n\to \infty} \dfrac{4^n}{n!}$

73. Given $|\dfrac{f(x)-f(c)}{x-c}| \leq M, \ \forall x \neq c$

74. If $|f(x)| \leq B$ for all x, show that $\lim_{x\to 0} x^2 f(x) = 0$

75. $\lim_{x\to \infty} \left(\dfrac{1}{\sqrt{n^2+1}} + \cdots + \dfrac{1}{\sqrt{n^2+2}} + \cdots + \dfrac{1}{\sqrt{n^2+n}}\right)$

76. Given $a > b > 0$, find $\lim_{n\to\infty} (a^n+b^n)^{\frac{1}{n}}$

77. $\lim_{n\to \infty} \dfrac{|-1+h|-|-1|}{h}$

78. $\lim_{n\to \infty} \dfrac{3^n+2^n}{3^n}$

79. $\lim_{n\to \infty} \dfrac{4^n+1}{4^n}$

80. $\lim_{n\to \infty} \dfrac{1-3^n}{3^n}$

81. Find $\lim_{n\to\infty} a_n$ of the following:

   (a) $a_n = 3^n + 3^{-n}$

   (b) $a_n = (\sqrt{3})^n 2^{-n}$

   (c) $a_n = \dfrac{5^{1-n}}{6^{1-n}}$

   (d) $a_n = (2\sin 45^\circ \cos 45^\circ)$

   (e) $a_n = \dfrac{(-1)^n}{2n^2} + 1$

(f) $a_n = \frac{1}{n}\left[n + (-1)^n\right]$

82. $\lim\limits_{x\to\infty} \frac{1 - x^{2n}}{1 + x^{2n}} \quad (x \in \mathbb{R})$

83. Find $\lim\limits_{n\to\infty} a_n$ of the following:

(a) $a_n = \frac{1}{n}(\cos 60°)^n$

(b) $a_n = \left(\frac{1}{2}\right)^n \cos nx$

(c) $a_n = 2^n - 2^{-n}$

(d) $a_n = (\sin^2 30° + \cos^2 30°)^n$

84. $\lim\limits_{n\to\infty} (\sqrt{n^2 + n} - n)$

85. $\lim\limits_{n\to\infty} (\sqrt{n^2 - 2n} - n)$

86. $\lim\limits_{n\to\infty} (\sqrt{n+1} - \sqrt{n})$

87. $\lim\limits_{x\to\infty} (\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x})$

88. $\lim\limits_{t\to 2} \frac{\sqrt{1 + \sqrt{2 + t}} - \sqrt{3}}{t - 2}$

89. $\lim\limits_{x\to 1} \left(\frac{1}{1 - x} - \frac{3}{1 - x^3}\right)$

90. $\lim\limits_{x\to 1} \frac{12x^{11} - 12x}{4x(x^2 - 1)}$

91. $\lim\limits_{x\to 5} \left[\left(\frac{1}{x} - \frac{1}{5}\right)\left(\frac{1}{x - 5}\right)\right]$

92. $\lim\limits_{x\to 2} \left(\frac{1}{x^2 - x - 2} - \frac{1}{2x^2 - 5x + 2}\right)$

93. $\lim\limits_{x\to 0} \frac{1}{x}\left[\frac{1}{(x + 2)^2} - \frac{1}{4}\right]$

94. $\lim\limits_{x\to 1} \left(\frac{x^3 - 1}{x^2 - 1} - \frac{x - \frac{1}{x}}{x - 1}\right)$

95. $\lim\limits_{x\to 1} \left(\frac{1}{x - 1} - \frac{2}{x^2 - 1}\right)$

96. $\lim\limits_{x\to -\infty} \frac{\sqrt{x^2 + x + 1}}{x + 3}$

97. $\lim\limits_{x\to -2} \left(\frac{x}{x - 1} + \frac{x^2 + 3x + 2}{x^2 - 4}\right)$

98. $\lim\limits_{x\to 6} (\frac{2x - 17}{x^2 - 7x + 6} + \frac{x - 5}{x - 6})$

99. $\lim\limits_{x\to -\infty} \frac{\sqrt[146]{x}}{\sqrt[7]{10^7} + \sqrt[3]{10^3} + \sqrt[7]{x + 10^7}}$

100. $\lim\limits_{x\to -\infty} \frac{\sqrt{\frac{9x^2 + 1}{x + 4}}}{x + 4}$

101. $\lim\limits_{x\to -\infty} \frac{x\sqrt{-x}}{\sqrt{1 - 4x^3}}$

102. $\lim\limits_{n\to\infty} \frac{\sqrt[3]{\frac{x^3 + x + 4}{4}}}{1 + x}$

103. $\lim\limits_{n\to 4} \frac{\sqrt{x + 5} - 3}{x - 4}$

104. $\lim\limits_{x\to -8} \frac{\sqrt{1 - x} - 3}{2 + \sqrt[3]{x}}$

105. $\lim\limits_{x\to 0} \frac{\sqrt{x + 5} - \sqrt{5}}{2x}$

106. $\lim\limits_{x\to 0} \frac{\sqrt{9 + x} - \sqrt{9 - x}}{x}$

107. $\lim\limits_{x\to 4} \frac{\sqrt{x + 5} - 3}{x - 4}$

108. $\lim\limits_{x\to 0} \frac{\sqrt{x + 4} - 2}{x}$

109. $\lim\limits_{t\to 0} \frac{9 - t}{3 - \sqrt{t}}$

110. $\lim\limits_{x\to 2} \frac{\sqrt{x + 7} - 3}{\sqrt{x + 2} - 2}$

111. $\lim\limits_{x\to a} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}}{x - a}$

112. $\lim\limits_{h\to 0} \frac{\sqrt[3]{a + h} - \sqrt[3]{a - h}}{h} \quad (a \neq 0)$

113. $\lim\limits_{t\to 0} \frac{\sqrt{1 + t} - 1}{t}$

114. $\lim\limits_{h\to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}$

115. $\lim\limits_{x\to 0} \frac{1}{x}(\sqrt{1 + x + x^2} - 1)$

116. $\lim\limits_{x\to 2} \frac{\sqrt{1 + \sqrt{2 + x}} - \sqrt{3}}{x - 2}$

117. $\lim\limits_{x\to 0} \frac{\sqrt{1 + x^2} - \left(1 + \frac{x}{2}\right)}{x^2}$

118. $\lim\limits_{x\to 2} \left(\frac{1}{x^2 - 4} - \frac{\sqrt{x - 1}}{x^2 - 4}\right)$

119. $\lim\limits_{x\to\infty} \frac{\sqrt[3]{8 + x + x^3} - \sqrt[3]{8 + x}}{\sqrt[3]{8 + x} - \sqrt[3]{8 + x^3}}$

120. $\lim\limits_{x\to\infty} (\sqrt{x^2 + 1} - x)$

121. $\lim\limits_{x\to 0} \frac{\sqrt{2 + x} - \sqrt{3x - 2}}{\sqrt{4x + 1} - \sqrt{5x - 1}}$

122. $\lim\limits_{x\to\infty} (\sqrt{x^2 + 1} - \sqrt{x^2 - 4x})$

123. $\lim\limits_{x\to\infty} (\sqrt{x^2 + x - 1} - \sqrt{x^2 - x})$

124. $\lim_{x\to\infty}(\sqrt{x^2+x}-x)$

125. $\lim_{x\to27}\dfrac{\sqrt{1+\sqrt[3]{x}}-2}{x-27}$

126. $\lim_{x\to\infty}(\sqrt{x^2+9}-x)$

127. $\lim_{x\to\infty}\sqrt{x}(\sqrt{x+a}-\sqrt{x+b})$

128. $\lim_{x\to\infty}(\sqrt{x^2+x}-\sqrt{x^2+10})$

129. Find the value of each of the following.

130. $\lim_{x\to1}(x-1)$

131. $\lim_{x\to1}\dfrac{x^2-2}{x}$

132. $\lim_{x\to0}\dfrac{2x-5}{x+3}$

133. $\lim_{x\to a}(x-a)$

134. $\lim_{x\to0}\dfrac{2x^2-5x}{x}$

135. $\lim_{x\to2}\dfrac{x^2-4}{x-2}$

136. $\lim_{x\to5}\dfrac{x^2+4x-45}{x-5}$

137. $\lim_{x\to1}\dfrac{\log_{10}x^2}{\log_{10}x}$

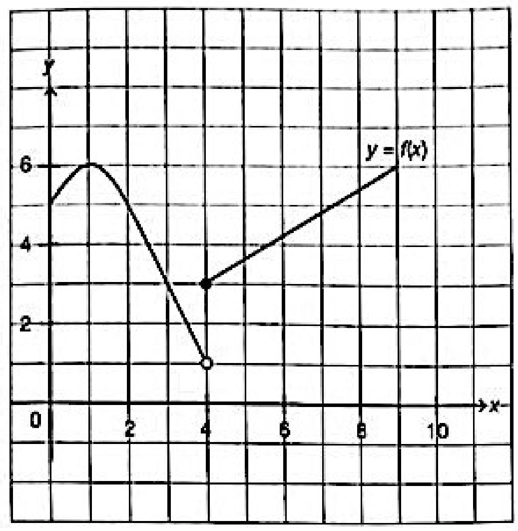
138. $\lim_{x\to9}\dfrac{x-9}{\sqrt{x}-3}$

139. $\lim_{x\to-1}\dfrac{x+1}{\sqrt{x+5}-2}$

140. $\lim_{x\to9}\dfrac{\sqrt{x+7}-4}{x-9}$

141. $\lim_{x\to2}\dfrac{\sqrt{6-x}-2}{3-\sqrt{11-x}}$

142. The following diagram shows part of a graph $y=f(x)$.



Based on this graph, find

(a) $f(4)$

(b) $\lim_{x\to4}f(x)$ and explain your answer. Since the left limit and right limit are different, $f(4)$ does not exist.

(c) $\lim_{x\to1}f(x)$

Chapter 2

Differentiation

2.1 The First Derivative

1. Find the first derivative for each of the following functions.

   (a) $y = 6x^2$

   (b) $y = -x^4$

   (c) $y = \sqrt[3]{x^4}$

   (d) $y = -\dfrac{2}{x^2}$

2. Find each of the following.

   (a) $\dfrac{d}{dx}\left(2x^2 + 3x - 9\right)$

   (b) $\dfrac{d}{dx}\left(x^2 + \dfrac{2}{x}\right)$

   (c) $\dfrac{d}{dx}\left(5x^3 + 2x^2 + 4x - 7 - \dfrac{1}{x} + \dfrac{3}{x^2}\right)$

3. Differentiate each of the following functions with respect to x.

   (a) $f(x) = x\left(\dfrac{1}{2}x^4 - x^2 - 5x\right)$

   (b) $f(x) = (x^2 - 5)(x + 3)$

   (c) $f(x) = \dfrac{(x^3 - x + 4)}{x}$

   (d) $f(x) = \dfrac{(x^2 - x - 2)}{(x - 2)}$

4. Find $f'(x)$ for each of the following functions.

   (a) $f(x) = (3x - 5)^4$

   (b) $f(x) = 5(x^3 + 4x)^3$

   (c) $f(x) = \dfrac{2}{(5x^2 - 3x)^{10}}$

5. Find the first derivative for each of the following functions by using the product rule.

   (a) $y = 6x^2(x + 5x^2)^3$

   (b) $y = x(7x + 3)^5$

   (c) $y = (4x^2 - 3x)(1 - 2x^2)^{10}$

6. Find $\dfrac{dy}{dx}$ for each of the following functions by using the quotient rule.

   (a) $y = \dfrac{x - 2}{2x + 1}$

   (b) $y = \dfrac{x^2 + 3x - 4}{x - 1}$

   (c) $y = \dfrac{x^3}{(2x - 1)^2}$

7. Find the gradient function to the curve $y = \sqrt{x}(4x + 1)$. Hence, find the value of the gradient of the curve at $x = 4$.

8. Given $x^2y = 5$, find $\dfrac{dy}{dx}$ when $x = 2$.

9. Given $y = 5x^m$ and $\dfrac{dy}{dx} = x^n$, find the value of m and n.

10. Given $f(x) = ax^3 - bx^2 + 9x + 5$ where $a, b > 0$. Show that $f'(x)$ is always positive for all the values of x when $b^2 < 27a$.

11. Given $\dfrac{d}{dx}(ax^m + bx^n) = 12x^s + 9x^t$ where $a, b > 0$.

    (a) Find $\dfrac{s}{t}$ in terms of a and b.

    (b) Find the values of a and b if $3s = 5t$ and $\dfrac{m}{n} = \dfrac{3}{2}$.

    (c) Hence, or otherwise, find the values of m, n, s, and t.

12. Given $\delta y = 4x\delta x + 2(\delta x)^2 + 3\delta x$. Find $\dfrac{dy}{dx}$ when $x = 2$.

13. Given $\dfrac{d}{dx}\left(\dfrac{x^3}{3 - x^3}\right) = \dfrac{kx^m}{(3 - x^3)^n}$, determine the values of k, m, and n.

14. Given $y = 5$ and $\dfrac{dy}{dx} = kx^m$. Based on the formula for the first derivative, state the value of k and m.

15. Given the equation of a curve $y = 2x^2 + 7x - 1$. Find the coordinates of a point on the curve that has a gradient of 5. Hence, find the value of constant p such that $y = 5x + p$ is the tangent to the curve.

16. Show that the gradient of the curve $y = 3x^3 - 18x^2 + 42x - 29$ is never negative for all the values of x.

2.2 The Second Derivative

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following.

   (a) $y = 4x^3 + 7x^{-1}$

   (b) $y = (2x^3 - 3)^5$

   (c) $y = \frac{4}{3}\pi x^3$

   (d) $y = \frac{3}{(x^2+1)^2}$

2. Given a curve $y = 4x^3 - 2x^2 + 5$. Find the first and the second derivatives for the curve y when $x = 2$.

3. Given $y = \frac{1}{x}$. Prove that $y + \frac{d^2y}{dx^2} = y^3(x^2 + 2)$.

4. Prove that for all values, of x,

$$\frac{d^2}{dx^2}\left(\frac{x^4}{12} - x^3 + \frac{9}{2}x^2 + 6x - 3\right)$$

   is never negative.

5. Given $h(x) = 3x^3 + mx^2 + x - 1$. Find the value of m if $h''(1) = 10$.

6. Given $f(x) = \frac{1}{2}x^4 + px^3 + \frac{3}{2}x^2 - 16x$. Determine the range of values for p such that the equation $f''(x) = 0$ has at least one real solution.

7. In the diagram in the answer space, sketch a graph $y = f(x)$ that satisfies the following conditions:

   (a) The points $A(x_a, y_a)$, $B(x_b, y_b)$, and $C(x_c, y_c)$ lies on the curve y.

   (b) $f'(x_a) = f'(x_b) = f'(x_c) = 0$.

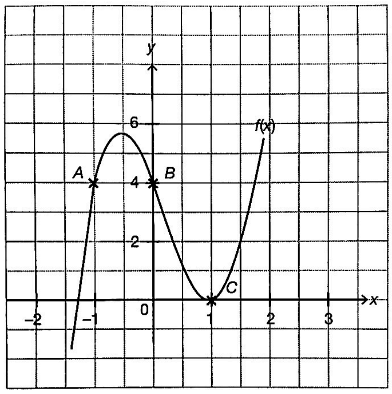
   (c) $f''(x_c) < f''(x_b) < f''(x_a)$.

   (d) $f''(x_b) = 0$.

2.3 Application of Differentiation

2.3.1 Tangent and Normal Lines

1. The following diagram shows the graph of part of the curve $f(x) = 3x^3 - 2x^2 - 5x + 4$. The points $A(-1, 4)$, $B(0, 4)$, and $C(1, 0)$ lie on the curve.



   (a) Find the gradient function of the tangent to the curve $f(x)$.

   (b) i. Find the values of gradient of the tangents to the curve at points A, B, and C.

   ii. Hence, elaborate the situations of the tangents at points A, B, and C based on the values of the gradient obtained in (i).

2. Find the gradient of the tangent for each of the following curves at the given point P.

   (a) $y = 4x - \frac{8}{x}$; $P(4, 14)$

   (b) $y = \frac{4 - 3x^2}{3 - 2x}$; $P(2, 8)$

3. (a) Find the value of gradient of the tangent to the curve $y = 2x^3 - 3x^2$ when $x = 1$.

   (b) Find the coordinates of points to the curve $y = \frac{x^3}{3} + x^2 - 1$ such that the gradient to the curve at the points is 8.

   (c) Given the curve $y = ax^2 + bx + 3$ has the gradient 5 when $x = 2$ and the gradient 0 when $x = -3$. Determine the values of a and b.

4. Find the equations of tangent and normal to the curve $y = 8 - 2x - x^2$ at each of the following points.

   (a) $A(1, 5)$

(b) $C(-1, 9)$

5. (a) Find the equation of normal to the curve $y = 3x^2 + 8x - 7$ at point $(-2, 6)$.

   (b) Given the tangent to the curve $y = ax^2 + bx$ at the point $P(4, 8)$ is perpendicular to the straight line that passes through the point $A(4, 1)$ and the point $B(12, 0)$. Find the values of a and b.

6. A curve $y_1 = x^2 - x - 5$ intersects another curve $y_2 = x^2 - \frac{31}{5}x + \frac{53}{5}$ at point A.

   (a) Determine the gradient functions for both curves at the point of intersection A.

   (b) Show that the tangents of both curves at point A are normal to each other.

7. Given the equation of a normal for a curve $y = x^2 + 2x - 5$ at point $A(2, 3)$ is given bt $y = ax + b$. Find the values of a and b.

2.3.2 Turning Points, Concavity, and Stationary Points

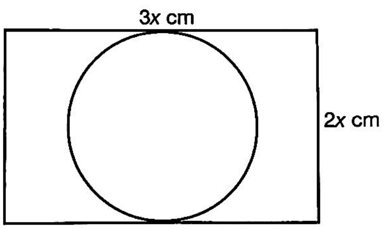
1. Find the coordinates of the turning points for each of the following curves. Hence, determine the nature of the turning points.

   (a) $y = 5x^2 - 2x + 1$

   (b) $y = \dfrac{x^2}{x+1}$

   (c) $y = 7 - x^3$

2. The following diagram shows the plan of a cuboid in which its centre in the shape of a cylinder is taken out. The cuboid measures $3x cm \times 2x cm \times (45 - 5x) cm$.



   Find the value of x that makes the volume of the cylinder taken out a maximum.

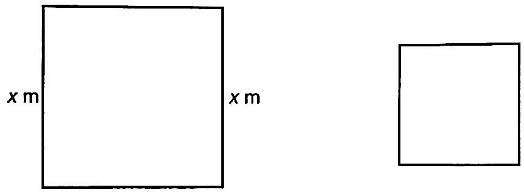
3. Given $A = bh$ where $b^2 + h^2 = 40$ and $b, h > 0$. Find the values of b and h so that A becomes a stationary point and show that the value of A is maximum.

4. A piece of wire with a length of $120 cm$ is divided into two parts where is each is bent to form an equilateral triangle with an edge of $x cm$ and a square with an edge of $y cm$ respectively. Express y in terms of x. Hence, show that the total area of both shapes, $A cm^2$ is given by

$$A = \frac{9(40 - x)^2 + 4\sqrt{3}x^2}{16}$$

   Calculate the value of x so that A has a stationary value. Determine whether this value of x makes A a maximum of a minimum.

5. Chan wants to build two separate pens by using a fence of $100m$. Both pens are square in shape.

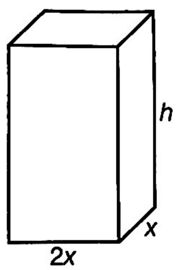


   If the edge of the larger pen is xm,

   (a) find the length of the side of the smaller pen in terms of x.

   (b) find the value of x such that the total area of both pens is minimum.

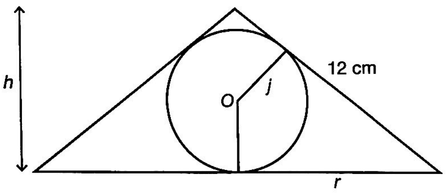
6. A factory needs to produce containers of the same size for a type of breakfast cereals as shown in the diagram below. Each container must have a volume of $2666\frac{2}{3} cm^3$. The base of the containers is rectangular in shape with its length twice the width. In order to reduce the production cost, the total surface area of each container must be minimum.



   (a) Find the dimensions of the containers produced.

   (b) Hence, find the total cost of production for 20000 units of containers if the cost of production for a containers is RM0.002 per cm^2.

7. An opened right circular cone with the base radius $r\,cm$ and slant height $12cm$ is used to cover a ball with radius $j\,cm$ such that the ball is inscribed in the cone as shown in the cross-section diagram below.

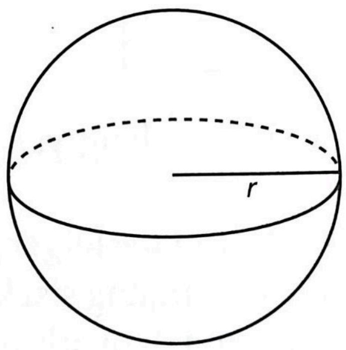


Show that the volume of the cone is given by $V = \dfrac{\pi}{3}(144h - h^3)$. Hence, determine the dimensions of the right circular cone such that its volume is maximum and find the radius of the ball, j, which corresponded to the maximum volume of the cone.

2.3.3 Rates of Change

1. The total surface area, Acm^2, of a metal solid which consists of a cone and a cylinder with a common radius, $r\,cm$ is given by $A = 2\pi\left(\dfrac{18}{r} + \dfrac{r^2}{3}\right)$. When it is heated, its total surface area changes at the rate of $2.1\pi\,cm^2s^{-1}$. Find the rate of change of the radius, in cms^{-1}, at the instant $r = 6cm$.

2. A spherical balloon experiences a constant rate of increase of $6cm^2s^{-1}$.

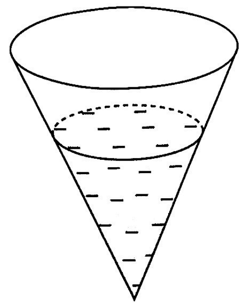


At the instant when the radius is $5cm$, find

(a) the rate of increase, in cms^{-1}, of the radius.

(b) the rate of increase if volume, in cm^3s^{-1}, of the sphere.

3. The following diagram shows a container in the shape of a cone. Given its height is equal to its base radius. Water is poured into the container at the rate of $80cm^3s^{-1}$. The volume of the water in the container is $\dfrac{1}{3}\pi x^3 cm^3$, when the depth of the water is $x\,cm$.

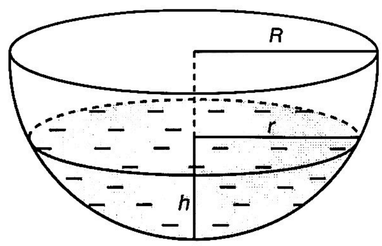


Calculate, at the instant when the depth of the water is $10cm$,

(a) the rate of increase of the depth, in cms^{-1}, of the water.

(b) the rate of increase of the horizontal surface area, in cm^2s^{-1}, of the water.

4. A hemispherical bowl of radius Rcm is filled with water to a depth of hcm.

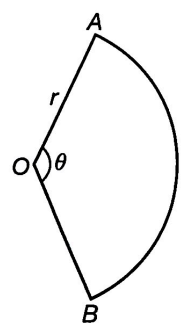


The volume of the water in the bowl is given by $V = \dfrac{\pi}{3}(3Rh^2 - h^3)$.

(a) Show that the radius of the water surface, r, is given by $r = \sqrt{2Rh - h^2}$.

(b) Water is poured into the bowl at a constant rate of $300cm^3min^{-1}$. Find, in terms of R, the rate of increase of the surface area, in cm^2min^{-1}, of the water when $2h = R$.

5. A wire of length of $26cm$ is bent to form a sector with centre O and radius $r\,cm$ as in the diagram below.



(a) Express θ in terms of r.

(b) If the radius increases at the rate of $0.1 cms^{-1}$, find, at the instant $r = 2cm$,

i. the rate of change, in $rads^{-1}$, of θ.

ii. the rate of change of the area, in $cm^2 s^{-1}$, of the sector.

2.3.4 Small Changes and Approximations

1. Given that $y = 2x^3 - 5x^2 + x - 1$, find the value of $\frac{dy}{dx}$ when $x = 1$. Hence, find the small changes in y when x increases from 1 to 1.02.

2. Given the equation of a curve is $y = \frac{9}{(2x-5)^2}$, find, in terms of p, where p is a small value, the approximate change in

   (a) y when x increases from 3 to $3 + p$.

   (b) x when y decreases from 1 to $1 - p$.

3. Given $y = x^4$, by using the calculus method, find the approximate value of

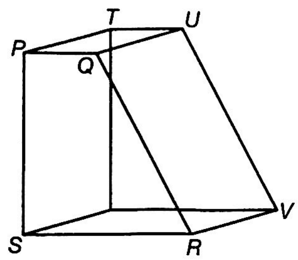
   (a) 2.03^4.

   (b) 1.99^4.

4. Given the equation of a curve $y = 2x^3 + x$. By using the differentiation method, find in terms of p, the approximate percentage increase in y when x increases from 2 by $p\%$, where p is a small value.

5. By using the calculus method, show the steps to determine the value of $9.02^{-\frac{1}{2}}$.

6. Diagram below shows a metal solid with a uniform cross-section in the shape of a right trapezium $PQRS$.



Given that PQ is $x cm$, $PQ : PS : SR = 2 : 5 : 3$ and the area of its cross-section is $A cm^2$.

   (a) Express A in terms of x.

   (b) i. When the metal is heated, x increases at the rate of $0.02 cms^{-1}$. Find the rate of change of the area, in $cm^2 s^{-1}$, of the cross-section when $x = 4cm$.

   ii. Given the thickness of the metal is $\frac{2}{5} x cm$, find the approximate change of the volume, in cm^3, of the metal when x changes from $4cm$ to $4.05cm$.