Coordinate Transformation

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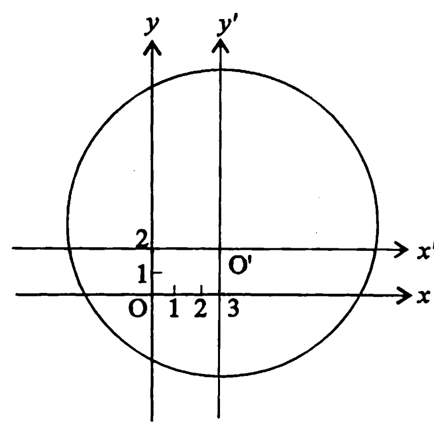
Chapter 4

Coordinate Transformation

4.1 Translation of Axes

The coordinates of points and the equation of curves are for certain axes. For example, in the circle shown below, its center is at $(3,2)$ in the coordinate system xOy, and its equation is $(x-3)^2+(y-2)^2=5$. If we were to use the coordinate system $x'Oy'(O'x'//Ox, O'y'//Oy)$, they would become $(3,2)$ and $x'^2+y'^2=5$ respectively.

That is to say, for the same point or the same curve, the coordinates of the point and the equation of the curve would be different if different coordinate systems were chosen. As we can see in the example on the right, changing the coordinate system to a more suitable one can simplify the equation of the curve, hence making it easier to study the properties of the curve.

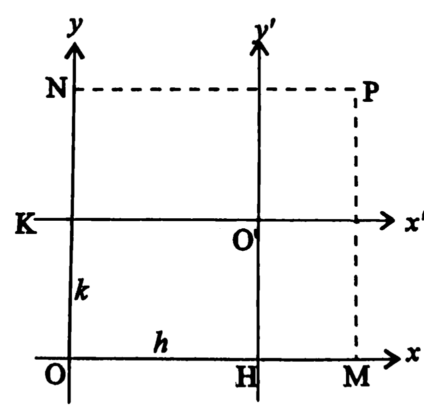


Without changing the direction and the unit of length of the axes while changing only the position of the origin, this kind of transformation is called **translation of axes**.

Here we study the relationship between the coordinates of a point in two different coordinate systems when performing a translation.

Let the coordinates of O' at the original coordinate system xOy be (h,k), translate the axes with O' as the origin, and create a new coordinate system $x'O'y'$.

As shown in the figure on the right, the coordinates of an arbitrary point P in the original coordinate system are (x,y), and its coordinates in the new coordinate system are (x',y'), the perpendicular foot from P to the x-axis and the x'-axis are M and N respectively. We can see that,



$$
\begin{aligned}
x &= OH + HM \\
&= h + x' \\
y &= OK + KN \\
&= k + y'
\end{aligned}
$$

Hence the coordinates of P in the original and the new coordinate systems has the following relationship,

$$\boxed{x = x' + h,\ y = y' + k}$$

The formula above is called the **translation formula**.

Example 1 Translate the axes by moving the origin to $O'(3, -4)$ (as shown in the diagram below), find the new coordinates of the following points:

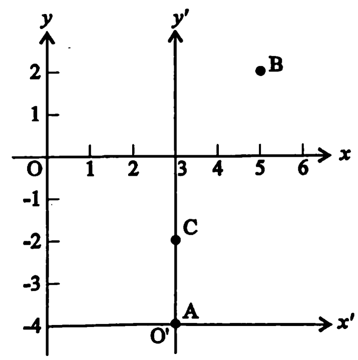
$O(0, 0)$, $A(3, -4)$, $B(5, 2)$, $C(3, -2)$.

Sol. Substituting the original coordinates of the given points into

$$x = x' + 3$$
$$y = y' - 4$$

We get the new coordinates of them:

$$O'(-3, 4),\ A'(0, 0),\ B'(2, 6),\ C'(0, 2)$$



4.2 Simplifying Quadratic Equation of Two Variables by Translation of Axes

Now, we study how choose the coordinate system to simplify the equation. First, take a look at the following example.

Example 2 Translate the axes and simplify the equation $x^2 - y^2 + 8x - 14y - 133 = 0$.

Sol. 1 Substitute $x = x' + h$ and $y = y' + k$ into the equation, we get

$$(x' + h)^2 - (y' + k)^2 + 8(x' + h) - 14(y' + k) - 133 = 0$$

That is $\quad x'^2 - y'^2 + (2h + 8)x' - (2k + 14)y' + h^2 - k^2 + 8h - 14k - 133 = 0 \ \cdots\cdots \ (1)$

Let $2h + 8 = 0$ and $2k + 14 = 0$, we get $h = -4$ and $k = -7$.

Substitute them into (1), we get $\qquad x'^2 - y'^2 = 100$

Sol. 2 Completing the square of the equation $x'^2 - y'^2 + 8x - 14y - 133 = 0$,

We get
$$x^2 + 8x - (y^2 + 14y) = 133$$
$$(x + 4)^2 - 16 - (y + 7)^2 + 49 = 133$$

Let $\quad x' = x + 4,\ y' = y + 7$, we get $\qquad x'^2 - y'^2 = 100$

Example 2 above shows that, for a quadratic equation of two variables that does not contain xy term,

$$ax^2 + by^2 + 2gx + 2fy + c = 0 \quad (a\ b \text{ are not zero at the same time})$$

We can translate the axes to simplify the equation.

Exercise 4a

1. Translate the axes by moving the origin to $O'(4,5)$. Find the new coordinates of the following points, and sketch the new coordinate system and each point:

   $A(3,-6)$, $B(7,0)$, $C(-4,5)$, $D(0,-8)$

2. After the translation of the axes, the new coordinates of the point $(-1,2)$ is $(4,-3)$, find the original coordinates of the new point of origin.

3. After the translation of the axes, the new coordinate of the points $(3,-3)$ and $(-2,2)$ are $(2,-1)$ and (a,b) respectively. Find a and b.

4. After the translation of the axes, the new coordinates of the points $(-4,2)$ and $(6,-2)$ are $(6,\alpha)$ and $(\beta,4)$ respectively. Find α and β.

5. Translate the axes by moving the origin to $O'(2,-3)$. Find the equation of $x^2+y^2-4x+6y-3=0$ in the new coordinate system, and sketch the coordinates system and the graph.

6. Using the translation of axes, simplify the following equations:

   (a) $x^2+y^2-6x+12y-4=0$

   (b) $8x^2-9y^2-16x+36y-77=0$

   (c) $x^2+y^2-2x+6y-6=0$

   (d) $9x^2+4y^2-18x+16y-11=0$

7. After the translation of the axes and moving the origin to $O'(1,-2)$, the new coordinates of the points A, B, C, and D are $(-1,1)$, $(0,-2)$, $(3,2)$, and $(2,0)$ respectively, find their original coordinates, and sketch the original coordinate system and each point.

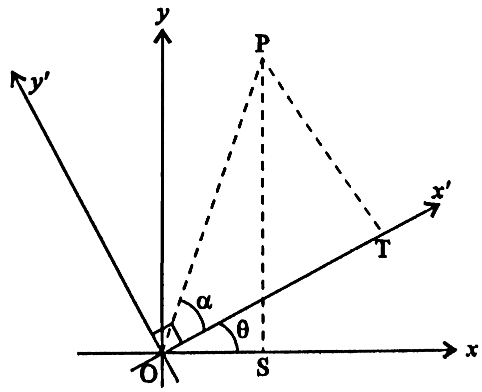
8. After the transformation of the axes, the coordinates of the point A changes from $(2,-1)$ to $(-2,1)$. Find the coordinates the point of origin in the new coordinate system.

4.3 Rotation of Axes

Without changing the point of origin and the length unit of the coordinate system, only rotate the coordinate system to the same direction by the same degree around to the point of origin, we call this transformation **rotation of axes**.

Let's derive the formula of the transformation of the coordinates under the rotation of axes.

Let the angle of rotation of the axes be θ, and pick an arbitrary point P on the plane, its coordinates in the coordinate system xOy and $x'O'y'$ are (x, y) and (x', y') respectively, as shown in the figure on the right.



Draw PS, PT perpendicular to the x-axis and the x'-axis respectively. Connect OP, let $\angle POT = \alpha$, then

$$
\begin{aligned}
x' &= |OP| \cos a \\
y' &= |OP| \sin a
\end{aligned}
$$

Hence
$$
\begin{aligned}
x &= |OP| \cos(\alpha + \theta) \\
&= |OP|(\cos\theta \sin\alpha - \sin\theta \cos\alpha) \\
&= x' \cos\theta - y' \sin\theta \\
y &= |OP| \sin(\alpha + \theta) \\
&= |OP|(\sin\theta \sin\alpha + \cos\theta \cos\alpha) \\
&= x' \sin\theta + y' \cos\theta
\end{aligned}
$$

That is,

$$
\boxed{
\begin{aligned}
x &= x' \cos\theta - y' \sin\theta \\
y &= x' \sin\theta + y' \cos\theta
\end{aligned}
}
$$

The formula above is called the **rotation formula**.

Example 3 Rotating the axes by $\frac{\pi}{6}$, find the coordinate of the point $P\left(-1, \sqrt{3}\right)$ in the new coordinate system.

Sol. Substitute $\theta = \frac{\pi}{6}$, $x = -1$, $y = \sqrt{3}$ into the rotation formula, we get

$$
\begin{aligned}
-1 &= x' \cos\frac{\pi}{6} - y' \sin\frac{\pi}{6} \\
-2 &= \sqrt{3}x' - y' \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (1)
\end{aligned}
$$

And
$$
\begin{aligned}
\sqrt{3} &= x' \sin\frac{\pi}{6} + y' \cos\frac{\pi}{6} \\
2\sqrt{3} &= x' + \sqrt{3}y' \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (2)
\end{aligned}
$$

$(1) \times \sqrt{3}$, we get
$$-2\sqrt{3} = 3x' - \sqrt{3}y' \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (3)$$

$(2) + (3)$, we get
$$
\begin{aligned}
x' &= 0 \\
y' &= 2
\end{aligned}
$$

$\therefore$ The coordinate of the point P in the new coordinate system is $(0, 2)$.

Example 4 Rotate the axes by $\frac{\pi}{3}$, find the equation of the curve $2x^2 - \sqrt{3}xy + y^2 = 10$ in the new coordinate system.

Sol. Substitute $\theta = \frac{\pi}{3}$ into the rotation formula, we get

$$\begin{aligned} x &= x' \cos\frac{\pi}{3} - y' \sin\frac{\pi}{3} \\ &= \frac{1}{2}x' - \frac{\sqrt{3}}{2}y' \\ y &= x' \sin\frac{\pi}{3} + y' \cos\frac{\pi}{3} \\ &= \frac{\sqrt{3}}{2}x' + \frac{1}{2}y' \end{aligned}$$

Substitute into the original equation, we get the equation of the curve in the new coordinate system

$$2\left(\frac{1}{2}x' - \frac{\sqrt{3}}{2}y'\right)^2 - \sqrt{3}\left(\frac{1}{2}x' - \frac{\sqrt{3}}{2}y'\right)\left(\frac{\sqrt{3}}{2}x' + \frac{1}{2}y'\right) + \left(\frac{\sqrt{3}}{2}x' + \frac{1}{2}y'\right)^2 = 10$$

Simplifying the equation above, we get $\quad \dfrac{x'^2}{20} + \dfrac{y'^2}{4} = 1.$

Exercise 4b

1. Let the angle of rotation $\theta = -\frac{\pi}{4}$, find the original coordinates of two points $A(-3, 2)$ and $B(1, 0)$ in the new coordinate system.
2. Let the angle of rotation $\theta = \frac{\pi}{6}$, find the new coordinates of two points $C(2, -1)$ and $D(0, 1)$ in the original coordinate system.
3. Transform the following equations by rotating the axes by the given angle:
   (a) $x + y = 0, \qquad \theta = \frac{\pi}{4}$
   (b) $x - y = 0, \qquad \theta = \frac{\pi}{2}$
   (c) $x^2 + y^2 = 9, \qquad \theta = -\frac{\pi}{3}$
   (d) $x^2 - 2\sqrt{3}xy + 3y^2 = 8, \qquad \theta = -\frac{\pi}{3}$
4. If the origin of the coordinate system does not change, by what angle should the axes be rotated so that the point $A(-2, 3)$ is transformed into the point that lies on the horizontal axis?

Revision Exercise 4

1. (a) Translate the axes by moving the origin to which point for the points to have the following changes in coordinates?

$$A(1,0) \to A(4,3),\ B(2,4) \to B(2,-3)$$

   (b) After the translation of axes that moves the origin to the point $O'(3,-2)$, the new coordinates of the points A, B, C, and D are $(0,2)$, $(-3,0)$, $(-1,3)$, and $(1,1)$ respectively. Find their original coordinates, and sketch the new coordinate system and each point.

2. Translate the axes by moving the origin to $O'(-4,2)$, find the new coordinates of the following points, and sketch the new coordinate system and each points:

$$A(-8,3),\ B(2,3),\ C(-4,2),\ O(0,0)$$

3. Translate the axes by moving the origin to O'. Find the new equation of the following curves.
   (a) $y = 3$, $\quad O'(-2,1)$
   (b) $3x - 4y = 6$, $\quad O'(3,0)$
   (c) $x^2 + y^2 - 3x - 2y = 0$, $\quad O'(2,1)$
   (d) $x^2 - 6x - y + 11 = 0$, $\quad O'(-3,2)$

4. Simplify the following equations by translating the axes:
   (a) $x^2 + y^2 + 4x + 8y = 5$
   (b) $x^2 + 2y^2 - 4x + 8y - 5 = 0$
   (c) $4x^2 - 9y^2 + 16y - 54y - 29 = 0$
   (d) $x^2 - 4x - y + 5 = 0$

5. By what angle should the axes be rotated so that the point $A(4,3)$ is transformed into the point at $(3,4)$?

6. By what angle should the axes be rotated such that the horizontal coordinate and the vertical coordinate of the point $A(1, 2+\sqrt{3})$ are the same?

7. After rotating the axes by $\frac{\pi}{3}$, the coordinate of the point M becomes $(1,2)$, find the coordinates of the point M in the original coordinate system.

8. After rotating the axes by $-\frac{\pi}{2}$, find the equation of the curve $x^2 + \frac{y^2}{2} = 1$ in the new coordinate system.

9. After rotating the axes by $\frac{\pi}{2}$, find the equation of the curve $y = 2px$ in the new coordinate system.

10. Prove that no matter how the axes are rotated, the equation $x^2 + y^2 = r^2$ remains unchanged.