高一第十章

三角方程式

(作答题)

1. 解方程式 $\sin 4x + \cos 2x = 0$。

解:

$$
\begin{aligned}
\sin 4x + \cos 2x &= 0 \\
2\sin 2x \cos 2x + \cos 2x &= 0 \\
\cos 2x(2\sin 2x + 1) &= 0 \\
\cos 2x = 0 \quad \text{or} \quad \sin 2x &= -\frac{1}{2} \\
2x = k\pi + \frac{\pi}{2} \quad \text{or} \quad 2x &= k\pi + (-1)^{k+1}\frac{\pi}{6} \\
x = k\pi + \frac{\pi}{4} \quad \text{or} \quad x &= k\pi + (-1)^{k+1}\frac{\pi}{12} \quad \text{where } k \in \mathbb{Z}
\end{aligned}
$$

■

2. 解 $\dfrac{\cos x - \sin x}{\cos x + \sin x} = \cos^2 x - \sin^2 x$。

解:

$$
\begin{aligned}
\frac{\cos x - \sin x}{\cos x + \sin x} &= \cos^2 x - \sin^2 x \\
\frac{(\cos x - \sin x)^2}{(\cos^2 x - \sin^2 x)} &= \cos^2 x - \sin^2 x \\
\frac{\cos^2 x - 2\sin x \cos x + \sin^2 x}{\cos 2x} &= \cos 2x \\
1 - \sin 2x &= \cos^2 2x \\
1 - \sin 2x &= 1 - \sin^2 2x \\
\sin^2 2x - \sin 2x &= 0 \\
\sin 2x(\sin 2x - 1) &= 0 \\
\sin 2x = 0 \quad \text{or} \quad \sin 2x &= 1 \\
2x = k\pi \quad \text{or} \quad 2x &= 2k\pi + \frac{\pi}{2} \\
x = k\pi \quad \text{or} \quad x &= k\pi + \frac{\pi}{4} \qquad \text{where } k \in \mathbb{Z}
\end{aligned}
$$

■

3. 解方程式 $\tan\left(\frac{\pi}{4}-x\right)+\cot\left(\frac{\pi}{4}-x\right)=4$。

解:

$$\begin{aligned}
\tan\left(\frac{\pi}{4}-x\right)+\cot\left(\frac{\pi}{4}-x\right)&=4\\
\tan\left(\frac{\pi}{4}-x\right)+\frac{1}{\tan\left(\frac{\pi}{4}-x\right)}&=4\\
\frac{\tan^2\left(\frac{\pi}{4}-x\right)+1}{\tan\left(\frac{\pi}{4}-x\right)}&=4\\
\frac{\sec^2\left(\frac{\pi}{4}-x\right)}{\tan\left(\frac{\pi}{4}-x\right)}&=4\\
\frac{1}{\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right)}&=4\\
\frac{1}{2\sin\left(\frac{\pi}{2}-2x\right)}&=2\\
\sin\left(\frac{\pi}{2}-2x\right)&=\frac{1}{2}\\
\cos 2x&=\frac{1}{2}\\
2x&=2k\pi\pm\frac{\pi}{3}\\
x&=k\pi\pm\frac{\pi}{6}\qquad\text{where } k\in\mathbb{Z}
\end{aligned}$$

■

4. 试证 $4\cos x-3\sin x\leq 5$。若 $4\cos x-3\sin x=5$, 求 x 之一般值。

解:

$$\begin{aligned}
y&=4\cos x-3\sin x\\
\frac{dy}{dx}&=-4\sin x-3\cos x=0\\
-4\sin x&=3\cos x\\
\tan x&=-\frac{3}{4}\\
x&\approx-36.87^\circ\\
\frac{d^2y}{dx^2}&=-4\cos x+3\sin x
\end{aligned}$$

When $x=-36.87^\circ$,

$$\begin{aligned}
\frac{d^2y}{dx^2}&=-4\cos x+3\sin x\\
&=-4\cos(-36.87^\circ)+3\sin(-36.87^\circ)\\
&\approx-4\times 0.8+3\times-0.6\\
&\approx-5
\end{aligned}$$

$\therefore y$ is maximum when $x=-36.87^\circ$ and $y=5 \implies 4\cos x-3\sin x\leq 5$. ■

Let $t=\tan\frac{x}{2}$, then $\cos x=\frac{1-t^2}{1+t^2}$ and $\sin x=\frac{2t}{1+t^2}$.

$$\begin{aligned}
4\cos x-3\sin x&=5\\
4\left(\frac{1-t^2}{1+t^2}\right)-3\left(\frac{2t}{1+t^2}\right)&=5\\
4-4t^2-6t&=5+5t^2\\
9t^2+6t+1&=0\\
(3t+1)^2&=0\\
t&=-\frac{1}{3}\\
\tan\frac{x}{2}&=-\frac{1}{3}\\
\frac{x}{2}&=180^\circ k-18.43^\circ\\
x&=360^\circ k-36.87^\circ \quad \text{where } k\in\mathbb{Z}
\end{aligned}$$

■

5. 求满足方程式 $4\sin x-2\cos x=3$ 的所有自 0° 至 360° 的角。

解:

Let $t=\tan\frac{x}{2}$, then $\cos x=\frac{1-t^2}{1+t^2}$ and $\sin x=\frac{2t}{1+t^2}$.

$$\begin{aligned}
4\sin x-2\cos x&=3\\
4\left(\frac{2t}{1+t^2}\right)-2\left(\frac{1-t^2}{1+t^2}\right)&=3\\
8t-2+2t^2&=3+3t^2\\
t^2-8t+5&=0\\
t&=\frac{8\pm\sqrt{64-4\times 5}}{2}\\
&=4\pm\sqrt{11}\\
\tan\frac{x}{2}&=4\pm\sqrt{11}\\
\frac{x}{2}&=180^\circ k+82.22^\circ \quad \text{or} \quad 180^\circ k-34.35^\circ\\
x&=360^\circ k+164.44^\circ \quad \text{or} \quad 360^\circ k-68.7^\circ \quad \text{where } k\in\mathbb{Z}
\end{aligned}$$

■

6. 解方程式 $3\sin 2x=2\tan x$, 其中 $0\le x\le 2\pi$。

解:

$$\begin{aligned}
3\sin 2x&=2\tan x\\
3\sin x\cos x&=\frac{\sin x}{\cos x}\\
3\sin x\cos^2 x-\sin x&=0\\
\sin x(3\cos^2 x-1)&=0\\
\sin x&=0 \quad \text{or} \quad 3\cos^2 x-1=0
\end{aligned}$$

$$x = k\pi \quad \text{or} \quad \cos x = \pm\frac{1}{\sqrt{3}}$$
$$x = k\pi \quad \text{or} \quad x = 2k\pi \pm 0.955$$

When $k = 0$, $x = 0$ or $x = \pm 0.955$.

When $k = 1$, $x = \pi$ or $x = \pi \pm 0.955$.

When $k = 2$, $x = 2\pi$ or $x = 2\pi \pm 0.955$.

Since $x \leq 2\pi$, the solutions are $0, \pi, 2\pi, 0.955, \pi \pm 0.955, 2\pi - 0.955$. ■

7. 求满足方程式 $2\sin 3x + \cos 2x = 1$ 在 $0^\circ \leq x \leq 360^\circ$ 范围内 x 的角度。

解:

$$\begin{aligned}
2\sin 3x + \cos 2x &= 1 \\
2(\sin x \cos 2x + \cos x \sin 2x) + 1 - 2\sin^2 x &= 1 \\
2(\sin x(2\cos^2 x - 1) + \cos x(2\sin x \cos x)) - 2\sin^2 x &= 0 \\
2(2\cos^2 x \sin x - \sin x + 2\sin x \cos^2 x) - 2\sin^2 x &= 0 \\
2(4\sin x \cos^2 x - \sin x) - 2\sin^2 x &= 0 \\
2(4\sin x(1 - \sin^2 x) - \sin x) - 2\sin^2 x &= 0 \\
2(4\sin x - 4\sin^3 x - \sin x) - 2\sin^2 x &= 0 \\
2(3\sin x - 4\sin^3 x) - 2\sin^2 x &= 0 \\
6\sin x - 8\sin^3 x - 2\sin^2 x &= 0 \\
\sin x(4\sin^2 x + \sin x - 3) &= 0 \\
\sin x(\sin x + 1)(4\sin x - 3) &= 0 \\
\sin x = 0 \quad \text{or} \quad \sin x = -1 \quad \text{or} \quad \sin x &= \frac{3}{4} \\
x = 180^\circ k \quad \text{or} \quad x = 360^\circ k - 90^\circ \quad &\text{or} \quad x = 180^\circ k + (-1)^k \cdot 48.59^\circ
\end{aligned}$$

When $k = 0$, $x = 0$ or $x = -90$ or $x = 48.59$.

When $k = 1$, $x = 180$ or $x = 270$ or $x = 131.41$.

When $k = 2$, $x = 360$ or $x = 630$ or $x = 408.59$.

Since $0 \leq x \leq 360$, the solutions are $0^0, 180^0, 270^0, 360^0, 48.59^0, 131.41^0$. ■

8. 求满足方程式 $5\cos 2x + 8\sin x = 3$ 的所有自 0° 至 360° 的 x 值。

解:

$$\begin{aligned}
5\cos 2x + 8\sin x &= 3 \\
5(1 - 2\sin^2 x) + 8\sin x &= 3 \\
5 - 10\sin^2 x + 8\sin x &= 3 \\
10\sin^2 x - 8\sin x - 2 &= 0 \\
5\sin^2 x - 4\sin x - 1 &= 0
\end{aligned}$$

$$(5\sin x+1)(\sin x-1)=0$$
$$\sin x=-\frac{1}{5}\quad\text{or}\quad\sin x=1$$
$$x=180^\circ k+(-1)^{k+1}\cdot 11.54^\circ\quad\text{or}\quad x=180^\circ k+90^\circ$$

When $k=0$, $x=-11.54$ or $x=90$.

When $k=1$, $x=191.54$ or $x=270$.

When $k=2$, $x=348.46$ or $x=450$.

Since $0\le x\le 360$, the solutions are $90^\circ, 270^\circ, 191.54^\circ, 348.46^\circ$. ■

9. 解方程式 $20\cos x-15\sin x=9$, 式中 $0^\circ\le x\le 360^\circ$。

解:

Let $t=\tan\frac{x}{2}$, then $\cos x=\frac{1-t^2}{1+t^2}$ and $\sin x=\frac{2t}{1+t^2}$.

$$20\cos x-15\sin x=9$$
$$20\left(\frac{1-t^2}{1+t^2}\right)-15\left(\frac{2t}{1+t^2}\right)=9$$
$$20-20t^2-30t=9+9t^2$$
$$29t^2+30t-11=0$$
$$t=\frac{-30\pm\sqrt{30^2-4\times 29\times -11}}{2\times 29}$$
$$=\frac{-15\pm 4\sqrt{34}}{29}$$
$$\tan\frac{x}{2}=\frac{-15\pm 4\sqrt{34}}{29}$$
$$\frac{x}{2}=180^\circ k+\arctan\left(\frac{-15\pm 4\sqrt{34}}{29}\right)$$
$$x=360^\circ k+32.03^\circ\quad\text{or}\quad x=360^\circ k-105.8^\circ\quad\text{where } k\in\mathbb{Z}$$

When $k=0$, $x=32.03$ or $x=-105.8$.

When $k=1$, $x=392.03$ or $x=254.2$.

Since $0\le x\le 360$, the solutions are $32.03^\circ, 254.2^\circ$. ■

10. 试不用计算机或对数表, 求三角方程式 $\cos\theta+\sqrt{3}\sin\theta=\sqrt{2}$ 之一般解。

解:

$$\cos\theta+\sqrt{3}\sin\theta=R\cos(\theta-\alpha)=R\cos\theta\cos\alpha+R\sin\theta\sin\alpha$$

$$\begin{cases} R\cos\alpha &=1\ \cdots\ (1)\\ R\sin\alpha &=\sqrt{3}\ \cdots\ (2)\end{cases}$$

$$(1)^2+(2)^2\Rightarrow R^2(\cos^2\alpha+\sin^2\alpha)=1+3$$

$$R^2 = 4$$
$$R = \pm 2$$
$$\frac{(2)}{(1)} \Rightarrow \tan\alpha = \sqrt{3}$$
$$\alpha = \frac{\pi}{3}$$

$$\begin{aligned}
\cos\theta + \sqrt{3}\sin\theta &= \sqrt{2} \\
\pm 2\cos\left(\theta - \frac{\pi}{3}\right) &= \sqrt{2} \\
\cos\left(\theta - \frac{\pi}{3}\right) &= \pm\frac{\sqrt{2}}{2} \\
\theta - \frac{\pi}{3} &= 2k\pi \pm \frac{\pi}{4} \\
\theta &= 2k\pi + \frac{\pi}{3} \pm \frac{\pi}{4} \\
\theta &= 2k\pi + \frac{7\pi}{12} \quad \text{or} \quad \theta = 2k\pi + \frac{\pi}{12} \quad \text{where } k \in \mathbb{Z}
\end{aligned}$$
■

11. (a) 解方程式 $3\cos x - \sin x = 1$, 式中 $0^\circ \le x \le 360^\circ$。

解:

Let $t = \tan\frac{x}{2}$, then $\cos x = \frac{1-t^2}{1+t^2}$ and $\sin x = \frac{2t}{1+t^2}$.

$$\begin{aligned}
3\cos x - \sin x &= 1 \\
3\left(\frac{1-t^2}{1+t^2}\right) - \left(\frac{2t}{1+t^2}\right) &= 1 \\
3 - 3t^2 - 2t &= 1 + t^2 \\
4t^2 + 2t - 2 &= 0 \\
2t^2 + t - 1 &= 0 \\
(2t-1)(t+1) &= 0 \\
t &= \frac{1}{2} \quad \text{or} \quad t = -1 \\
\tan\frac{x}{2} &= \frac{1}{2} \quad \text{or} \quad \tan\frac{x}{2} = -1 \\
\frac{x}{2} &= 180^\circ k + 26.57^\circ \quad \text{or} \quad \frac{x}{2} = 180^\circ k - 45^\circ \\
x &= 360^\circ k + 53.14^\circ \quad \text{or} \quad x = 360^\circ k - 90^\circ
\end{aligned}$$

When $k = 0$, $x = 53.14$ or $x = -90$.

When $k = 1$, $x = 413.14$ or $x = 270$.

Since $0 \le x \le 360$, the solutions are $53.14^\circ, 270^\circ$. ■

(b) 求下列方程式的一般解:

i. $\sin 2\theta + \cos^2\theta = 1$;

解:

$$\begin{aligned}
\sin 2\theta + \cos^2\theta &= 1 \\
2\sin\theta\cos\theta + 1 - \sin^2\theta &= 1 \\
2\sin\theta\cos\theta - \sin^2\theta &= 0 \\
\sin\theta(2\cos\theta - \sin\theta) &= 0 \\
\sin\theta = 0 \text{ or } 2\cos\theta - \sin\theta &= 0 \\
\sin\theta = 0 \text{ or } \tan\theta &= 2 \\
\theta = k\pi \text{ or } \theta = k\pi + 1.107 \quad \text{where } k \in \mathbb{Z}
\end{aligned} \qquad \blacksquare$$

ii. $\cos 3\theta + 2\cos\theta = 0$。

解:

$$\begin{aligned}
\cos 3\theta + 2\cos\theta &= 0 \\
\cos 2\theta\cos\theta - \sin 2\theta\sin\theta + 2\cos\theta &= 0 \\
(1 - 2\sin^2\theta)\cos\theta - 2\sin^2\theta\cos\theta + 2\cos\theta &= 0 \\
3\cos\theta - 4\sin^2\theta\cos\theta &= 0 \\
\cos\theta(3 - 4\sin^2\theta) &= 0 \\
\cos\theta = 0 \text{ or } 3 - 4(1 - \cos^2\theta) &= 0 \\
\cos\theta = 0 \text{ or } \cos^2\theta &= \frac{1}{4} \\
\cos\theta = 0 \text{ or } 2\cos^2\theta &= \frac{1}{2} \\
\cos\theta = 0 \text{ or } 1 - 2\cos^2\theta &= -\frac{1}{2} \\
\cos\theta = 0 \text{ or } \cos 2\theta &= -\frac{1}{2} \\
\theta = k\pi + \frac{\pi}{2} \text{ or } 2\theta = 2k\pi \pm \frac{2\pi}{3} \\
\theta = k\pi + \frac{\pi}{2} \text{ or } \theta = k\pi \pm \frac{\pi}{3} \quad \text{where } k \in \mathbb{Z}
\end{aligned} \qquad \blacksquare$$

12. 求满足方程式 $\sin 5x + \sin 3x = \sin 8x$ 在 $0 \leq x \leq \pi$ 的所有 x 之值。

解:

$$\begin{aligned}
\sin 5x + \sin 3x &= \sin 8x \\
2\sin 4x\cos x &= 2\sin 4x\cos 4x \\
\sin 4x(\cos x - \cos 4x) &= 0 \\
\sin 4x\left(2\sin\frac{5x}{2}\sin\frac{3x}{2}\right) &= 0 \\
\sin 4x\sin\frac{5x}{2}\sin\frac{3x}{2} &= 0
\end{aligned}$$

$$\sin 4x = 0 \text{ or } \sin\frac{5x}{2} = 0 \text{ or } \sin\frac{3x}{2} = 0$$
$$4x = k\pi \text{ or } \frac{5x}{2} = k\pi \text{ or } \frac{3x}{2} = k\pi$$
$$x = \frac{k\pi}{4} \text{ or } x = \frac{2k\pi}{5} \text{ or } x = \frac{2k\pi}{3}$$

When $k = 0$, $x = 0$ or $x = 0$ or $x = 0$.

When $k = 1$, $x = \frac{\pi}{4}$ or $x = \frac{2\pi}{5}$ or $x = \frac{2\pi}{3}$.

When $k = 2$, $x = \frac{\pi}{2}$ or $x = \frac{4\pi}{5}$ or $x = \frac{4\pi}{3}$.

When $k = 3$, $x = \frac{3\pi}{4}$ or $x = \frac{6\pi}{5}$ or $x = \frac{6\pi}{3}$.

When $k = 4$, $x = \pi$ or $x = \frac{8\pi}{5}$ or $x = 2\pi$.

Since $0 \le x \le \pi$, the solutions are $0, \frac{\pi}{4}, \frac{2\pi}{5}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{4\pi}{5}, \pi$. ■

13. 解方程式 $3\sin 2\theta - 4\cos 2\theta = 2$, 式中 $0^\circ \le \theta \le 180^\circ$。

解:

$$\begin{aligned}
3\sin 2\theta - 4\cos 2\theta &= 2\\
6\sin\theta\cos\theta - 4(\cos^2\theta - \sin^2\theta) &= 2\\
6\sin\theta\cos\theta - 4\cos^2\theta + 4\sin^2\theta &= 2\\
3\sin\theta\cos\theta - 2\cos^2\theta + 2\sin^2\theta &= 1\\
3\sin\theta\cos\theta - 2\cos^2\theta + 2\sin^2\theta &= \sin^2\theta + \cos^2\theta\\
3\sin\theta\cos\theta - 3\cos^2\theta + \sin^2\theta &= 0\\
\tan^2\theta + 3\tan\theta - 3 &= 0
\end{aligned}$$
$$\tan\theta = \frac{-3 \pm \sqrt{21}}{2}$$
$$\theta = 180^\circ k + \arctan\left(\frac{-3 \pm \sqrt{21}}{2}\right)$$
$$\theta = 180^\circ k + 38.35^\circ \quad \text{or} \quad \theta = 180^\circ k - 75.22^\circ$$

When $k = 0$, $\theta = 38.35$ or $\theta = -75.22$.

When $k = 1$, $\theta = 218.35$ or $\theta = 104.78$.

Since $0 \le \theta \le 180$, the solutions are $38.35^\circ, 104.78^\circ$. ■

14. 已知 α 是锐角且 $\cos\alpha = x-1$, 证明 $\cos 2\alpha - 3\cos\alpha\sin^2\alpha = 3x^3-7x^2+2x+1$。然后解方程式 $\cos 2\alpha - 3\cos\alpha\sin^2\alpha+1=0$。

解:

$$\begin{aligned}
\cos 2\alpha - 3\cos\alpha\sin^2\alpha &= 2\cos^2\alpha - 1 - 3\cos\alpha(1-\cos^2\alpha) \\
&= 2\cos^2\alpha - 1 - 3\cos\alpha + 3\cos^3\alpha \\
&= 2(x-1)^2 - 1 - 3(x-1) + 3(x-1)^3 \\
&= 2(x^2-2x+1) - 1 - 3x + 3 + 3(x^3-3x^2+3x-1) \\
&= 2x^2-4x+2-1-3x+3+3x^3-9x^2+9x-3 \\
&= 3x^3-7x^2+2x+1 \qquad \blacksquare
\end{aligned}$$

$$\begin{aligned}
\cos 2\alpha - 3\cos\alpha\sin^2\alpha + 1 &= 0 \\
3x^3-7x^2+2x+1+1 &= 0 \\
3x^3-7x^2+2x+2 &= 0 \\
(x-1)(3x^2-4x-2) &= 0 \\
x-1=0 \text{ or } 3x^2-4x-2 &= 0 \\
\cos\alpha = 1 \text{ or } \cos\alpha+1 &= \frac{2\pm\sqrt{10}}{3} \\
\alpha = k\pi + \frac{\pi}{2} &\text{ or } \alpha = 180^\circ k \pm 43.88^\circ
\end{aligned}$$

When $n=0$, $\alpha = 90^\circ$ or $\alpha = 43.88^\circ$ or $\alpha = -43.88^\circ$.

$\because \alpha$ is acute, $\therefore \alpha = 43.88^\circ$. $\blacksquare$

15. (a) 若 $5\cos\theta - 12\sin\theta = R\cos(\theta+\alpha)$, 式中 R 为常数, α 为锐角, 试求 R 和 α 之值如果 $5\cos\theta-12\sin\theta = k$, 试证 $-13 \le k \le 13$。

由此, 试解方程式 $5\cos\theta - 12\sin\theta = 4$, 式中 $0^\circ \le \theta \le 360^\circ$。

解:

$$5\cos\theta - 12\sin\theta = R\cos(\theta+\alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

$$\begin{cases} R\cos\alpha &= 5 \ \cdots \ (1) \\ R\sin\alpha &= 12 \ \cdots \ (2) \end{cases}$$

$$\begin{aligned}
(1)^2+(2)^2 \Rightarrow R^2(\cos^2\alpha+\sin^2\alpha) &= 5^2+12^2 \\
R &= 13 \qquad \blacksquare \\
\frac{(2)}{(1)} \Rightarrow \tan\alpha &= \frac{12}{5} \\
\alpha &= 67.38^\circ \qquad \blacksquare
\end{aligned}$$

$$
\begin{aligned}
5\cos\theta - 12\sin\theta &= k \\
13\cos(\theta + 67.38^\circ) &= k \\
-1 \le \cos(\theta + 67.38^\circ) &\le 1 \\
-13 \le 13\cos(\theta + 67.38^\circ) &\le 13 \\
-13 \le k &\le 13
\end{aligned}
$$

$$
\begin{aligned}
5\cos\theta - 12\sin\theta &= 4 \\
13\cos(\theta + 67.38^\circ) &= 4 \\
\cos(\theta + 67.38^\circ) &= \frac{4}{13} \\
\theta + 67.38^\circ &= 360^\circ k \pm 72.08^\circ \\
\theta &= 360^\circ k - 67.38^\circ \pm 72.08^\circ \\
\theta &= 360^\circ k + 4.7^\circ \quad \text{or} \quad \theta = 360^\circ k - 139.46^\circ
\end{aligned}
$$

When $k = 0$, $\theta = 4.7$ or $\theta = -139.46$.

When $k = 1$, $\theta = 364.7$ or $\theta = 220.54$.

Since $0 \le \theta \le 360$, the solutions are $4.7^\circ, 220.54^\circ$. ■

(b) 因式分解 $8\cos^3 x + 6\cos^2 x - 3\cos x - 1$。

由此试求满足方程式 $8\cos^3 x + 6\cos^2 x - 3\cos x - 1 = 0$ 的所有 x 之值, 且 $0^\circ \le \theta \le 360^\circ$。

解:

Let $u = \cos x$.

$$
\begin{aligned}
8\cos^3 x + 6\cos^2 x - 3\cos x - 1 &= 8u^3 + 6u^2 - 3u - 1 \\
&= (2u-1)(4u^2 + 5u + 1) \\
&= (2u-1)(u+1)(4u+1) \\
&= (2\cos x - 1)(\cos x + 1)(4\cos x + 1) \quad ■
\end{aligned}
$$

$$
\begin{aligned}
8\cos^3 x + 6\cos^2 x - 3\cos x - 1 &= 0 \\
(2\cos x - 1)(\cos x + 1)(4\cos x + 1) &= 0 \\
2\cos x - 1 = 0 \text{ or } \cos x + 1 &= 0 \text{ or } 4\cos x + 1 = 0 \\
\cos x = \frac{1}{2} \text{ or } \cos x = -1 \text{ or } \cos x &= -\frac{1}{4} \\
x = 360^\circ k \pm 60^\circ \text{ or } x = 360^\circ k + 180^\circ \text{ or } x &= 360^\circ k \pm 104.48^\circ
\end{aligned}
$$

When $k = 0$, $x = 60$ or $x = -60$ or $x = 180$ or $x = 104.48$ or $x = -104.48$.

When $k = 1$, $x = 420$ or $x = 300$ or $x = 540$ or $x = 464.48$ or $x = 255.52$.

Since $0 \le x \le 360$, the solutions are $60^\circ, 180^\circ, 300^\circ, 104.48^\circ, 255.52^\circ$. ■

16. 试在 $0 \le x \le 2\pi$ 范围内解方程式: $\sin x + \sin 3x + \sin 5x = 0$。

解:

$$
\begin{aligned}
\sin x + \sin 3x + \sin 5x &= 0 \\
\sin x + 2\sin 4x \cos x &= 0 \\
\sin x + 4\sin 2x \cos 2x \cos x &= 0 \\
\sin x + 4\sin 2x(1-2\sin^2 x)\cos x &= 0 \\
\sin x + 4\sin 2x \cos x - 8\sin^2 x \sin 2x \cos x &= 0 \\
\sin x + 4(2\sin x\cos x)\cos x - 8\sin^2 x(2\sin x \cos x)\cos x &= 0 \\
\sin x + 8\sin x\cos^2 x - 16\sin^3 x\cos^2 x &= 0 \\
\sin x + 8\sin x(1-\sin^2 x) - 16\sin^3 x(1-\sin^2 x) &= 0 \\
\sin x + 8\sin x - 8\sin^3 x - 16\sin^3 x + 16\sin^5 x &= 0 \\
16\sin^5 x - 24\sin^3 x + 9\sin x &= 0 \\
\sin x(16\sin^4 x - 24\sin^2 x + 9) &= 0 \\
\sin x(4\sin^2 x - 3)^2 &= 0 \\
\sin x = 0 \text{ or } 4\sin^2 x - 3 &= 0 \\
x = k\pi \text{ or } \sin x &= \pm\frac{\sqrt{3}}{2} \\
x = k\pi \text{ or } x &= k\pi \pm (-1)^k\left(\frac{\pi}{3}\right) \\
x = k\pi \text{ or } x &= k\pi \pm \frac{\pi}{3}
\end{aligned}
$$

When $k=0$, $x=0$ or $x=\frac{\pi}{3}$ or $x=-\frac{\pi}{3}$.

When $k=1$, $x=\pi$ or $x=\frac{4\pi}{3}$ or $x=\frac{2\pi}{3}$.

When $k=2$, $x=2\pi$ or $x=\frac{5\pi}{3}$ or $x=\frac{7\pi}{3}$.

Since $0 \le x \le 2\pi$, the solutions are $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$. ■

17. 如果 $3\sin x + 2\cos x \equiv R\sin(x+\alpha)$, 式中 R 是常数, α 是锐角, 求 R 及 α 的值。据此或其他方法, 求:

解:

$$
\begin{aligned}
3\sin x + 2\cos x &= R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha \\
3\sin x + 2\cos x &= R\sin x\cos\alpha + R\cos x\sin\alpha
\end{aligned}
$$

$$
\begin{cases}
R\cos\alpha &= 3 \ \cdots\ (1) \\
R\sin\alpha &= 2 \ \cdots\ (2)
\end{cases}
$$

$$
\begin{aligned}
(1)^2 + (2)^2 \Rightarrow R^2(\cos^2\alpha + \sin^2\alpha) &= 3^2 + 2^2 \\
R &= \sqrt{13}
\end{aligned}
$$

■

$$\frac{(2)}{(1)} \Rightarrow \tan\alpha = \frac{2}{3}$$
$$\alpha = 33.69^\circ$$ ■

(a) $\dfrac{1}{(3\sin x + 2\cos x)^2}$ 的最小值;

解:

$$\frac{1}{(3\sin x + 2\cos x)^2} = \frac{1}{13\sin^2(x + 33.69^\circ)}$$

Since $\sin^2(x + 33.69^\circ) \le 1$, the minimum value is $\dfrac{1}{13}$. ■

(b) 方程式 $3\sin x + 2\cos x = 3$ 在 $0^\circ \le x \le 180^\circ$ 的解。

解:

$$\begin{aligned}
3\sin x + 2\cos x &= 3 \\
\sqrt{13}\sin(x + 33.69^\circ) &= 3 \\
\sin(x + 33.69^\circ) &= \frac{3}{\sqrt{13}} \\
x + 33.69^\circ &= 180^\circ k + (-1)^k \cdot 56.31^\circ \\
x &= 180^\circ k + (-1)^k \cdot 56.31^\circ - 33.69^\circ
\end{aligned}$$

When $k = 0$, $x = 22.62$.

When $k = 1$, $x = 90$.

Since $0 \le x \le 180$, the solutions are $22.62^\circ, 90^\circ$. ■

18. 解方程式 $3\cos x + 4\sin x = 2$, 式中 $0^\circ \le x \le 360^\circ$。

解:

$$3\cos x + 4\sin x = R\sin(x + \alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$$

$$\begin{cases} R\sin\alpha &= 3 \ \cdots\ (1) \\ R\cos\alpha &= 4 \ \cdots\ (2) \end{cases}$$

$$\begin{aligned}
(1)^2 + (2)^2 &\Rightarrow R^2(\sin^2\alpha + \cos^2\alpha) = 3^2 + 4^2 \\
R &= 5 \\
\frac{(1)}{(2)} &\Rightarrow \tan\alpha = \frac{3}{4} \\
\alpha &= 36.87^\circ
\end{aligned}$$

$$\begin{aligned}
3\cos x + 4\sin x &= 2 \\
5\sin(x + 36.87^\circ) &= 2 \\
\sin(x + 36.87^\circ) &= \frac{2}{5}
\end{aligned}$$

$$
\begin{aligned}
x + 36.87^\circ &= 180^\circ k + (-1)^k \cdot 23.58^\circ \\
x &= 180^\circ k + (-1)^k \cdot 23.58^\circ - 36.87^\circ
\end{aligned}
$$

When $k = 0$, $x = -13.29$

When $k = 1$, $x = 119.55$

When $k = 2$, $x = 346.71$

Since $0 \leq x \leq 360$, the solutions are $119.55^\circ, 346.71^\circ$. ■

19. 求方程式 $\cos x + \cos 7x = \cos 4x$ 的一般解, 答案以弧度表示。

解:

$$
\begin{aligned}
\cos x + \cos 7x &= \cos 4x \\
2\cos 4x \cos 3x &= \cos 4x \\
\cos 4x(2\cos 3x - 1) &= 0 \\
\cos 4x = 0 \text{ or } 2\cos 3x - 1 &= 0 \\
4x = 2k\pi \pm \frac{\pi}{2} \text{ or } 3x &= 2k\pi \pm \frac{\pi}{3} \\
x &= \frac{k\pi}{2} \pm \frac{\pi}{8} \text{ or } x = \frac{2k\pi}{3} \pm \frac{\pi}{9} \quad \text{where } k \in \mathbb{Z}
\end{aligned}
$$
■

20. 求方程式 $2\cos^2\theta + \sqrt{3}\sin\theta + 1 = 0$ 的一般解。

解:

$$
\begin{aligned}
2\cos^2\theta + \sqrt{3}\sin\theta + 1 &= 0 \\
2(1 - \sin^2\theta) + \sqrt{3}\sin\theta + 1 &= 0 \\
2 - 2\sin^2\theta + \sqrt{3}\sin\theta + 1 &= 0 \\
2\sin^2\theta - \sqrt{3}\sin\theta - 3 &= 0 \\
(\sin\theta - \sqrt{3})(2\sin\theta + \sqrt{3}) &= 0 \\
\sin\theta = \sqrt{3} \text{ or } \sin\theta &= -\frac{\sqrt{3}}{2} \\
\theta = k\pi + (-1)^k\frac{\pi}{3} &\text{ or } \theta = k\pi + (-1)^{k+1}\frac{\pi}{3}
\end{aligned}
$$

$\because \theta = k\pi + (-1)^k\frac{\pi}{3}$ is included in $\theta = k\pi + (-1)^{k+1}\frac{\pi}{3}$,

$\therefore$ the general solution is $\theta = k\pi + (-1)^{k+1}\frac{\pi}{3}$. ■

21. 试证 $\cos\theta + 2\cos 2\theta + \cos 3\theta = 4\cos 2\theta\cos^2\dfrac{1}{2}\theta$。

据此, 求满足方程式 $\cos\theta + 2\cos 2\theta + \cos 3\theta = 0$ 的所有 θ 的值, 且 $0 \leq \theta \leq 2\pi$。

解:

$$\begin{aligned}
L.H.S. = \cos\theta + 2\cos 2\theta + \cos 3\theta &= \cos\theta + 2(2\cos^2\theta - 1) + \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \\
&= \cos\theta + 4\cos^2\theta - 2 + (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta \\
&= 4\cos^2\theta - 2 + 2\cos^3\theta - 2(1-\cos^2\theta)\cos\theta \\
&= 4\cos^2\theta - 2 + 2\cos^3\theta - 2\cos\theta + 2\cos^3\theta \\
&= 4\cos^2\theta - 2 + 4\cos^3\theta - 2\cos\theta \\
R.H.S. = 4\cos 2\theta\cos^2\frac{1}{2}\theta &= 4(2\cos^2\theta - 1)\left(\frac{1+\cos\theta}{2}\right) \\
&= 2(2\cos^2\theta - 1)(1+\cos\theta) \\
&= 2(2\cos^2\theta - 1 + 2\cos^3\theta - \cos\theta) \\
&= 4\cos^2\theta - 2 + 4\cos^3\theta - 2\cos\theta
\end{aligned}$$

Since $L.H.S. = R.H.S.$, the equation is true.

$$\begin{aligned}
\cos\theta + 2\cos 2\theta + \cos 3\theta &= 0 \\
4\cos 2\theta\cos^2\frac{1}{2}\theta &= 0 \\
\cos 2\theta\cos^2\frac{1}{2}\theta &= 0 \\
\cos 2\theta = 0 \text{ or } \cos\frac{1}{2}\theta &= 0 \\
2\theta = 2k\pi \pm \frac{\pi}{2} \text{ or } \frac{1}{2}\theta &= 2k\pi \pm \frac{\pi}{2} \\
\theta = k\pi \pm \frac{\pi}{4} \text{ or } \theta &= 4k\pi \pm \pi
\end{aligned}$$

When $k=0$, $\theta = \dfrac{\pi}{4}$ or $\theta = -\dfrac{\pi}{4}$ or $\theta = \pi$ or $\theta = -\pi$.

When $k=1$, $\theta = \dfrac{5\pi}{4}$ or $\theta = \dfrac{3\pi}{4}$ or $\theta = 3\pi$ or $\theta = 5\pi$.

When $k=2$, $\theta = \dfrac{9\pi}{4}$ or $\theta = \dfrac{7\pi}{4}$ or $\theta = 7\pi$ or $\theta = 9\pi$.

Since $0 \leq \theta \leq 2\pi$, the solutions are $\dfrac{\pi}{4}, \dfrac{3\pi}{4}, \pi, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$. ■

22. 求 $\dfrac{1-\tan x}{1+\tan x} = \cos 2x$ 的一般解。

解:

$$\begin{aligned}
\frac{1-\tan x}{1+\tan x} &= \cos 2x \\
\frac{1-\dfrac{\sin x}{\cos x}}{1+\dfrac{\sin x}{\cos x}} &= \cos 2x \\
\frac{\cos x - \sin x}{\cos x + \sin x} &= \cos 2x
\end{aligned}$$

$$
\begin{aligned}
\frac{\cos x - \sin x}{\cos x + \sin x} &= \frac{1 - 2\sin^2 x}{1 + 2\sin x \cos x} \\
\frac{(\cos x - \sin x)^2}{\cos^2 x - \sin^2 x} &= \cos 2x \\
\frac{\cos^2 x - 2\cos x \sin x + \sin^2 x}{\cos 2x} &= \cos 2x \\
1 - \sin 2x &= \cos^2 2x \\
1 - \sin 2x &= 1 - \sin^2 2x \\
\sin^2 2x - \sin 2x &= 0 \\
\sin 2x(\sin 2x - 1) &= 0 \\
\sin 2x = 0 \text{ or } \sin 2x &= 1 \\
2x = k\pi \text{ or } 2x &= \frac{\pi}{2} + 2k\pi \\
x = k\pi \text{ or } x &= \frac{\pi}{4} + k\pi \quad \text{where } k \in \mathbb{Z}
\end{aligned}
$$

■

23. 将 $y = \cos x - \sqrt{3}\sin x$ 表达成 $\mathrm{R}\cos(x + \alpha)$ 的形式, 其中 $\mathrm{R} > 0$ 及 $0^\circ < \alpha < 90^\circ$。据此或用其他方法, 求

解:

$$\cos x - \sqrt{3}\sin x = R\cos(x + \alpha) = R\cos x \cos\alpha - R\sin x \sin\alpha$$

$$\begin{cases} R\cos\alpha & = 1 \ \cdots \ (1) \\ R\sin\alpha & = \sqrt{3} \ \cdots \ (2) \end{cases}$$

$$
\begin{aligned}
(1)^2 + (2)^2 \Rightarrow R^2(\cos^2\alpha + \sin^2\alpha) &= 1^2 + 3 \\
R &= 2 \\
\frac{(2)}{(1)} \Rightarrow \tan\alpha &= \sqrt{3} \\
\alpha &= 60^\circ \\
y &= 2\cos(x + 60^\circ)
\end{aligned}
$$

■

(a) y 的极大值与极小值; **解**:

y is maximum when $\cos(x + 60^\circ) = 1$, hence the maximum value is 2.

y is minimum when $\cos(x + 60^\circ) = -1$, hence the minimum value is -2. ■

(b) 当 $y = 1$ 时，x 的一般解。

解:

$$
\begin{aligned}
2\cos(x + 60^\circ) &= 1 \\
\cos(x + 60^\circ) &= \frac{1}{2} \\
x + 60^\circ &= 360^\circ k \pm 60^\circ \\
x &= 360^\circ k - 60^\circ \pm 60^\circ \\
x &= 360^\circ k \text{ or } x = 360^\circ k - 120^\circ \quad \text{where } k \in \mathbb{Z}
\end{aligned}
$$

■

24. 证明 $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$。

据此, 或用其它方法, 解方程式 $\dfrac{1+\tan x}{1-\tan x} = 1+\sin 2x$, 式中 $2 < x < 12$, 且 x 为弧度。

解:

$$\begin{aligned}
\frac{2\tan x}{1+\tan^2 x} &= \frac{2\left(\dfrac{\sin x}{\cos x}\right)}{\sec^2 x} \\
&= \frac{2\sin x}{\cos x} \times \cos^2 x \\
&= 2\sin x\cos x \\
&= \sin 2x \qquad \blacksquare
\end{aligned}$$

$$\begin{aligned}
\frac{1+\tan x}{1-\tan x} &= 1+\sin 2x \\
\frac{1+\tan x}{1-\tan x} &= 1+\frac{2\tan x}{1+\tan^2 x} \\
\frac{1+\tan x}{1-\tan x} &= \frac{1+\tan^2 x+2\tan x}{1+\tan^2 x} \\
\frac{(1+\tan x)^2}{1-\tan^2 x} &= \frac{(1+\tan x)^2}{1+\tan^2 x} \\
(1+\tan x)^2\left(\frac{1}{1-\tan^2 x}-\frac{1}{1+\tan^2 x}\right) &= 0 \\
(1+\tan x)^2\left(\frac{1+\tan^2 x-1+\tan^2 x}{1-\tan^4 x}\right) &= 0 \\
(1+\tan x)^2\left(\frac{2\tan^2 x}{1-\tan^4 x}\right) &= 0 \\
1+\tan x = 0 \text{ or } \tan x &= 0 \\
\tan x = -1 \text{ or } \tan x &= 0 \\
x = k\pi - \frac{\pi}{4} \text{ or } x &= k\pi
\end{aligned}$$

When $k=0$, $x = -\dfrac{\pi}{4}$ or $x = 0$.

When $k=1$, $x = \dfrac{3\pi}{4}$ or $x = \pi$.

When $k=2$, $x = \dfrac{7\pi}{4}$ or $x = 2\pi$.

When $k=3$, $x = \dfrac{11\pi}{4}$ or $x = 3\pi$.

When $k=4$, $x = \dfrac{15\pi}{4}$ or $x = 4\pi$.

Since $2 < x < 12$, the solutions are $\dfrac{3\pi}{4}, \pi, \dfrac{7\pi}{4}, 2\pi, \dfrac{11\pi}{4}, 3\pi, \dfrac{15\pi}{4}$. $\blacksquare$

25. 试证明 $\operatorname{cosec}\theta - \cot\theta = \tan\frac{1}{2}\theta$。

解:

$$\begin{aligned}\operatorname{cosec}\theta - \cot\theta &= \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta}\\ &= \frac{1-\cos\theta}{\sin\theta}\\ &= \tan\frac{1}{2}\theta\end{aligned} \qquad \blacksquare$$

据此或用其他方法,

(a) 解方程式 $\operatorname{cosec}3\theta - \cot 3\theta = \sqrt{3}$, 式中 $0 \le \theta \le \pi$。

解:

$$\begin{aligned}\operatorname{cosec}3\theta - \cot 3\theta &= \sqrt{3}\\ \tan\frac{3\theta}{2} &= \sqrt{3}\\ \frac{3\theta}{2} &= \frac{\pi}{3} + k\pi\\ \theta &= \frac{2\pi}{9} + \frac{2k\pi}{3}\end{aligned}$$

When $k = 0$, $\theta = \dfrac{2\pi}{9}$.
When $k = 1$, $\theta = \dfrac{8\pi}{9}$.
Since $0 \le \theta \le \pi$, the solutions are $\dfrac{2\pi}{9}, \dfrac{8\pi}{9}$. $\blacksquare$

(b) 求 $\tan\frac{3}{8}\pi$ 的值, 答案以根式表示。

解:

$$\begin{aligned}\tan\frac{3}{8}\pi &= \tan\left(\frac{1}{2}\cdot\frac{3}{4}\pi\right)\\ &= \operatorname{cosec}\frac{3}{4}\pi - \cot\frac{3}{4}\pi\\ &= \sqrt{2} - 1\end{aligned} \qquad \blacksquare$$

26. 求 $\sin^2 x - \cos^2 x = 1 + \frac{1}{2}\sin 2x$ 的一般解。

解:

$$\begin{aligned}\sin^2 x - \cos^2 x &= 1 + \frac{1}{2}\sin 2x\\ -\cos 2x &= 1 + \frac{1}{2}\sin 2x\\ -2\cos 2x &= 2 + \sin 2x\end{aligned}$$

Let $u = 2x$.

$$-2\cos u = 2 + \sin u$$
$$\sin u + 2\cos u = -2$$

Let $\tan\frac{u}{2} = t$, then $\cos u = \frac{1-t^2}{1+t^2}$ and $\sin u = \frac{2t}{1+t^2}$.

$$\begin{aligned}
\frac{2t}{1+t^2} + 2\left(\frac{1-t^2}{1+t^2}\right) &= -2 \\
2t &= -4 \\
t &= -2 \\
\tan\frac{u}{2} &= -2 \\
\tan x &= -2 \\
x &= k\pi - \arctan 2 \quad \text{where } k \in \mathbb{Z}
\end{aligned}$$

27. (a) 将 $12\cos\theta - 5\sin\theta$ 表达成 $R\cos(\theta+\alpha)$ 的形成, 式中 $R > 0$ 及 $0^\circ < \alpha < 90^\circ$。

解:

$$12\cos\theta - 5\sin\theta = R\cos(\theta+\alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

$$\begin{cases} R\cos\alpha & = 12 \ \cdots \ (1) \\ R\sin\alpha & = 5 \ \cdots \ (2) \end{cases}$$

$$\begin{aligned}
(1)^2 + (2)^2 \Rightarrow R^2(\cos^2\alpha + \sin^2\alpha) &= 12^2 + 5^2 \\
R &= 13 \\
\frac{(2)}{(1)} \Rightarrow \tan\alpha &= \frac{5}{12} \\
\alpha &= 22.62^\circ \\
12\cos\theta - 5\sin\theta &= 13\cos(\theta + 22.62^\circ)
\end{aligned}$$

■

(b) 函数 $g(x)$ 定义成 $g(x) = 27\cos^2 x - 10\sin x\cos x + 3\sin^2 x$。试将 $g(x)$ 表达成 $a\cos 2x + b\sin 2x + c$ 的形式, 式中 a, b 及 c 都是待定的常数。

解:

$$\begin{aligned}
g(x) &= 27\cos^2 x - 10\sin x\cos x + 3\sin^2 x \\
&= 27\left(\frac{1+\cos 2x}{2}\right) - 5(2\sin x\cos x) + 3\left(\frac{1-\cos 2x}{2}\right) \\
&= \frac{27 + 27\cos 2x + 3 - 3\cos 2x}{2} - 5\sin 2x \\
&= \frac{30 + 24\cos 2x}{2} - 5\sin 2x \\
&= 12\cos 2x - 5\sin 2x + 15
\end{aligned}$$

■

(c) 根据 (a) 和 (b) 的结果, 或用其他方法, 求

i $g(x)$ 的极大值与极小值;

解:

$$\begin{aligned} g(x) &= 12\cos 2x - 5\sin 2x + 15 \\ &= 13\cos(2x + 22.62^\circ) + 15 \end{aligned}$$

$g(x)$ is maximum when $\cos(2x - 22.62^\circ) = 1$, hence the maximum value is 28.

$g(x)$ is minimum when $\cos(2x - 22.62^\circ) = -1$, hence the minimum value is 2. ■

ii 方程式 $g(x) = 2$ 的一般解。

解:

$$\begin{aligned} 13\cos(2x + 22.62^\circ) + 15 &= 2 \\ \cos(2x + 22.62^\circ) &= -1 \\ 2x + 22.62^\circ &= 360^\circ k + 180^\circ \\ 2x &= 360^\circ k + 157.38^\circ \\ x &= 180^\circ k + 78.69^\circ \quad \text{where } k \in \mathbb{Z} \end{aligned}$$ ■

28. 求方程式 $5\sin^2 x + \sin 2x - 3\cos^2 x = 2$ 的一般解。

解:

$$\begin{aligned} 5\sin^2 x + \sin 2x - 3\cos^2 x &= 2 \\ 5\sin^2 x + 2\sin x\cos x - 3\cos^2 x &= 2\sin^2 x + 2\cos^2 x \\ 3\sin^2 x + 2\sin x\cos x - 5\cos^2 x &= 0 \\ 3\tan^2 x + 2\tan x - 5 &= 0 \\ (3\tan x + 5)(\tan x - 1) &= 0 \end{aligned}$$

$$\tan x = \frac{-5}{3} \text{ or } \tan x = 1$$

$$x = k\pi - \arctan\frac{5}{3} \text{ or } x = k\pi + \frac{\pi}{4} \quad \text{where } k \in \mathbb{Z}$$ ■

29. 求三角方程式 $\sin 4\theta = \cos 5\theta$ 的一般解。

解:

$$\begin{aligned} \sin 4\theta &= \cos 5\theta \\ \cos 5\theta &= \cos\left(\frac{\pi}{2} - 4\theta\right) \\ 5\theta &= 2k\pi \pm \left(\frac{\pi}{2} - 4\theta\right) \\ 5\theta &= 2k\pi + \frac{\pi}{2} - 4\theta \text{ or } 5\theta = 2k\pi - \frac{\pi}{2} + 4\theta \\ 9\theta &= 2k\pi + \frac{\pi}{2} \text{ or } \theta = 2k\pi - \frac{\pi}{2} \\ \theta &= \frac{2k\pi}{9} + \frac{\pi}{18} \text{ or } \theta = 2k\pi - \frac{\pi}{2} \\ \theta &= \frac{\pi}{18}(4k+1) \text{ or } \theta = \frac{\pi}{2}(4k-1) \quad \text{where } k \in \mathbb{Z} \end{aligned}$$ ■

30. 解方程式 $\sin^2\theta+\sin^2 2\theta=\sin^2 3\theta$。

解:

$$\begin{aligned}
\sin^2\theta+\sin^2 2\theta &= \sin^2 3\theta\\
\sin^2\theta+4\sin^2\theta\cos^2\theta &= (\sin 2\theta\cos\theta+\cos 2\theta\sin\theta)^2\\
\sin^2\theta+4\sin^2\theta\cos^2\theta &= [(2\sin\theta\cos\theta)\cos\theta+(1-2\sin^2\theta)\sin\theta]^2\\
\sin^2\theta+4\sin^2\theta\cos^2\theta &= [2\sin\theta(1-\sin^2\theta)+\sin\theta-2\sin^3\theta]^2\\
\sin^2\theta+4\sin^2\theta\cos^2\theta &= (3\sin\theta-4\sin^3\theta)^2\\
\sin^2\theta+4\sin^2\theta\cos^2\theta &= 9\sin^2\theta-24\sin^4\theta+16\sin^6\theta\\
\sin^2\theta+4\sin^2\theta\cos^2\theta-9\sin^2\theta+24\sin^4\theta-16\sin^6\theta &= 0\\
\sin^2\theta(-8+4\cos^2\theta+24\sin^2\theta-16\sin^4\theta) &= 0\\
\sin^2\theta(2-\cos^2\theta-6\sin^2\theta+4\sin^4\theta) &= 0\\
\sin^2\theta(2-\cos^2\theta-\sin^2\theta-5\sin^2\theta+4\sin^4\theta) &= 0\\
\sin^2\theta(1-5\sin^2\theta+4\sin^4\theta) &= 0\\
\sin^2\theta(\sin^2\theta-1)(4\sin^2\theta-1) &= 0\\
\sin^2\theta(-\cos^2\theta)(4\sin^2\theta-1) &= 0\\
\sin\theta=0 \text{ or } \cos\theta=0 \text{ or } \sin\theta &= \pm\frac{1}{2}\\
\theta=k\pi \text{ or } \theta=\frac{\pi}{2}+k\pi \text{ or } \theta &= k\pi\pm\frac{\pi}{6}\\
\theta=\frac{k\pi}{2} \text{ or } \theta &= \frac{k\pi}{3}+\frac{\pi}{6}\\
\theta=\frac{k\pi}{2} \text{ or } \theta &= \frac{1}{6}(2k+1)\pi \quad \text{where } k\in\mathbb{Z} \qquad \blacksquare
\end{aligned}$$

31. 已知 $4\cos\theta+3\sin\theta=R\cos(\theta-\alpha)$, 式中 R >0 且 α 为锐角, 试求 R 与 α 的值。据之解方程式 $4\cos\theta+3\sin\theta=3, 0^\circ<\theta<360^\circ$。

解:

$$4\cos\theta+3\sin\theta=R\cos(\theta-\alpha)=R\cos\theta\cos\alpha+R\sin\theta\sin\alpha$$

$$\begin{cases} R\cos\alpha &= 4 \ \cdots\ (1)\\ R\sin\alpha &= 3 \ \cdots\ (2)\end{cases}$$

$$\begin{aligned}
(1)^2+(2)^2 \Rightarrow R^2(\cos^2\alpha+\sin^2\alpha) &= 16+9\\
R &= 5 \qquad \blacksquare\\
\frac{(2)}{(1)} \Rightarrow \tan\alpha &= \frac{3}{4}\\
\alpha &= 36.87^\circ \qquad \blacksquare
\end{aligned}$$

$$
\begin{aligned}
4\cos\theta + 3\sin\theta &= 3 \\
5\cos(\theta - 36.87^\circ) &= 3 \\
\cos(\theta - 36.87^\circ) &= \frac{3}{5} \\
\theta - 36.87^\circ &= 360^\circ k \pm 53.13^\circ \\
\theta &= 360^\circ k + 36.87^\circ + 53.13^\circ \text{ or } \theta = 360^\circ k + 36.87^\circ - 53.13^\circ \\
\theta &= 360^\circ k + 90^\circ \text{ or } \theta = 360^\circ k - 16.26^\circ
\end{aligned}
$$

When $k = 0$, $\theta = 90^\circ$ or $\theta = -16.26^\circ$.

When $k = 1$, $\theta = 450^\circ$ or $\theta = 343.74^\circ$.

Since $0^\circ < \theta < 360^\circ$, the solutions are $90^\circ, 343.74^\circ$. ■

32. 已知函数 $f(x) = \cos 2x + 4\sin^2 x - \cos x - 2$。

(a) 解方程式 $f(x) = 0$。

解:

$$
\begin{aligned}
\cos 2x + 4\sin^2 x - \cos x - 2 &= 0 \\
2\cos^2 x - 1 + 4(1 - \cos^2 x) - \cos x - 2 &= 0 \\
2\cos^2 x - 1 + 4 - 4\cos^2 x - \cos x - 2 &= 0 \\
-2\cos^2 x - \cos x + 1 &= 0 \\
2\cos^2 x + \cos x - 1 &= 0 \\
(2\cos x - 1)(\cos x + 1) &= 0 \\
\cos x &= \frac{1}{2} \text{ or } \cos x = -1 \\
x &= 2k\pi \pm \frac{\pi}{3} \text{ or } x = 2k\pi + \pi \\
x &= \frac{\pi}{3} + \frac{2k\pi}{3} \\
x &= \frac{\pi}{3}(2k+1) \quad \text{where } k \in \mathbb{Z}
\end{aligned}
$$ ■

(b) 在 $0 \le x \le 2\pi$ 的条件下, 解不等式 $f(x) > 0$。

解:

$$
\begin{aligned}
\cos 2x + 4\sin^2 x - \cos x - 2 &> 0 \\
2\cos^2 x - 1 + 4(1 - \cos^2 x) - \cos x - 2 &> 0 \\
2\cos^2 x - 1 + 4 - 4\cos^2 x - \cos x - 2 &> 0 \\
-2\cos^2 x - \cos x + 1 &> 0 \\
2\cos^2 x + \cos x - 1 &< 0 \\
(2\cos x - 1)(\cos x + 1) &< 0 \\
-1 < \cos x &< \frac{1}{2}
\end{aligned}
$$ ■

33. 如果 $\sin^4\theta+\sin^2\theta\cos^2\theta+\cos^4\theta\leq\frac{3}{4}$, 式中 $0\leq\theta\leq2\pi$, 求 θ 的值。

解:

$$\begin{aligned}
\sin^4\theta+\sin^2\theta\cos^2\theta+\cos^4\theta &\leq \frac{3}{4}\\
\sin^2\theta(\sin^2\theta+\cos^2\theta)+\cos^4\theta &\leq \frac{3}{4}\\
\sin^2\theta+\cos^4\theta &\leq \frac{3}{4}\\
1-\cos^2\theta+\cos^4\theta &\leq \frac{3}{4}\\
\cos^4\theta-\cos^2\theta+\frac{1}{4} &\leq 0\\
\left(\cos^2\theta-\frac{1}{2}\right)^2 &\leq 0\\
\cos^2\theta &= \frac{1}{2}\\
\theta &= k\pi\pm\frac{\pi}{4}
\end{aligned}$$

When $k=0$, $\theta=\frac{\pi}{4}$ or $\theta=-\frac{\pi}{4}$.
When $k=1$, $\theta=\frac{5\pi}{4}$ or $\theta=\frac{3\pi}{4}$.
When $k=2$, $\theta=\frac{9\pi}{4}$ or $\theta=\frac{7\pi}{4}$.
Since $0\leq\theta\leq2\pi$, the solutions are $\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$. ■

34. 解方程式 $\cos^3 x\sin x-\sin^3 x\cos x=\frac{\sqrt{3}}{8}$, 其中 x 以弧度为单位且 $0<x<\frac{\pi}{2}$。

解:

$$\begin{aligned}
\cos^3 x\sin x-\sin^3 x\cos x &= \frac{\sqrt{3}}{8}\\
\sin x\cos x(\cos^2 x-\sin^2 x) &= \frac{\sqrt{3}}{8}\\
2\sin x\cos x\cos 2x &= \frac{\sqrt{3}}{4}\\
\sin 2x\cos 2x &= \frac{\sqrt{3}}{4}\\
2\sin 2x\cos 2x &= \frac{\sqrt{3}}{2}\\
\sin 4x &= \frac{\sqrt{3}}{2}\\
4x &= k\pi+(-1)^k\frac{\pi}{3}\\
x &= \frac{k\pi}{4}+(-1)^k\frac{\pi}{12}
\end{aligned}$$

When $k=0$, $x=\frac{\pi}{12}$.
When $k=1$, $x=\frac{\pi}{6}$.
Since $0<x<\frac{\pi}{2}$, the solutions are $\frac{\pi}{12},\frac{\pi}{6}$. ■

35. 证明 $\dfrac{1-\cos\alpha}{1+\cos\alpha}=\tan^2\dfrac{\alpha}{2}$。

据此, 或用其他方法, 证明若 $\alpha\in(\pi,2\pi)$, 则 $\sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}+\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}}=-\dfrac{2}{\sin\alpha}$。

解:

$$\begin{aligned}\frac{1-\cos\alpha}{1+\cos\alpha}&=\frac{1-\cos\alpha}{2}\div\frac{1+\cos\alpha}{2}\\&=\left(\pm\sqrt{\frac{1-\cos\alpha}{2}}\right)^2\div\left(\pm\sqrt{\frac{1+\cos\alpha}{2}}\right)^2\\&=\frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}\\&=\tan^2\frac{\alpha}{2}\end{aligned}$$

$$\begin{aligned}\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}+\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}&=\tan\frac{\alpha}{2}+\cot\frac{\alpha}{2}\\&=\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}+\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\\&=\frac{\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}}{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\\&=\frac{1}{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\\&=\frac{2}{\sin\alpha}\end{aligned}$$

$\because \alpha\in(\pi,2\pi)$, $\therefore \sin\alpha<0$.

$\therefore \sqrt{\dfrac{1-\cos\alpha}{1+\cos\alpha}}+\sqrt{\dfrac{1+\cos\alpha}{1-\cos\alpha}}=-\dfrac{2}{\sin\alpha}$. ■

36. 证明 $\cos\theta+2\cos2\theta+\cos3\theta=4\cos2\theta\cos^2\dfrac{\theta}{2}$。

据此, 或用其它方法, 解方程式 $\cos\theta+2\cos2\theta+\cos3\theta=0$, 其中 $0\le\theta\le\pi$。

解:

Same as question 21.

The solutions are $\dfrac{\pi}{4},\dfrac{3\pi}{4},\pi$. ■

37. 求方程式 $\cos^2 x + 3\cos^2 2x = \cos^2 3x$ 的一般解。

解:

$$\begin{aligned}
\cos^2 x + 3\cos^2 2x &= \cos^2 3x \\
\cos^2 x + 3(2\cos^2 x - 1)^2 &= (4\cos^3 x - 3\cos x)^2 \\
\cos^2 x + 3(4\cos^4 x - 4\cos^2 x + 1) &= 16\cos^6 x - 24\cos^4 x + 9\cos^2 x \\
\cos^2 x + 12\cos^4 x - 12\cos^2 x + 3 &= 16\cos^6 x - 24\cos^4 x + 9\cos^2 x \\
16\cos^6 x - 36\cos^4 x + 20\cos^2 x - 3 &= 0
\end{aligned}$$

Let $u = \cos^2 x$.

$$\begin{aligned}
16u^3 - 36u^2 + 20u - 3 &= 0 \\
(2u-3)(2u-1)(4u-1) &= 0 \\
u &= \frac{3}{2} \text{ or } u = \frac{1}{2} \text{ or } u = \frac{1}{4} \\
\cos^2 x &= \frac{3}{2} \text{ or } \cos^2 x = \frac{1}{2} \text{ or } \cos^2 x = \frac{1}{4} \\
x &= 2k\pi \pm \frac{\pi}{3} \text{ or } x = 2k\pi \pm \frac{\pi}{4} \quad \text{where } k \in \mathbb{Z}
\end{aligned}$$

■

38. 证明 $\sin^4 x + \cos^4 x = 1 - \frac{1}{2}\sin^2 2x$。

据此, 求三角方程式 $2\sin^4 x + 2\cos^4 x = \sin 2x$ 的一般解。

解:

$$\begin{aligned}
\sin^4 x + \cos^4 x &= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x \\
&= 1 - \frac{1}{2}(2\sin x\cos x)^2 \\
&= 1 - \frac{1}{2}\sin^2 2x
\end{aligned}$$

■

$$\begin{aligned}
2\sin^4 x + 2\cos^4 x &= \sin 2x \\
2 - \sin^2 2x &= \sin 2x \\
\sin^2 2x + \sin 2x - 2 &= 0 \\
(\sin 2x + 2)(\sin 2x - 1) &= 0 \\
\sin 2x &= -2 \text{ or } \sin 2x = 1 \\
2x &= 2k\pi + \frac{\pi}{2} \\
x &= k\pi + \frac{\pi}{4} \quad \text{where } k \in \mathbb{Z}
\end{aligned}$$

■