Solution Book of Mathematic

Senior 2 Part I

MELVIN CHIA

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Chapter 12

Sequence and Series

12.1 Sequence and Series

12.1.1 Practice 1

1. Find the first 5 terms of the sequence $a_n = \frac{2^n}{n+1}$.

   Sol. $a_1 = \frac{2}{2} = 1, a_2 = \frac{4}{3}, a_3 = \frac{8}{4}, a_4 = \frac{16}{5}, a_5 = \frac{32}{6}$

2. Write the general term of the sequence 1, 8, 27, 64, …

   Sol. $a_n = n^3$

12.1.2 Practice 2

1. Express the series $\sum_{n=1}^{10} n^2 + 1$ in the form of numbers.

   Sol. $$\sum_{n=1}^{10} n^2 + 1$$
   $$= (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1)$$
   $$+ (5^2 + 1) + (6^2 + 1) + (7^2 + 1)$$
   $$+ (8^2 + 1) + (9^2 + 1) + (10^2 + 1)$$
   $$= 2 + 5 + 10 + 17 + 26 + 37 + 50 + 65$$
   $$+ 82 + 101$$

2. Write the first term, last term and the number of terms of the series $\sum_{n=1}^{10} (3^n - 2^n)$.

   Sol. $First\ term = (3^1 - 2^1) = 1$
   $Last\ term = (3^{10} - 2^{10}) = 59049$
   $Number\ of\ terms = 10$

3. Express the series $2 \times 5 + 3 \times 7 + 4 \times 9 + \ldots + 15 \times 31$ in the form of $\sum$.

   Sol.
   $$a_1 = 2 \times 5 = 10$$
   $$a_2 = 3 \times 7 = 21$$
   $$a_3 = 4 \times 9 = 36$$
   $$a_4 = 5 \times 11 = 55$$
   $$\vdots$$
   $$a_{15} = 15 \times 31 = 465$$
   $$\therefore 2 \times 5 + 3 \times 7 + 4 \times 9 + \ldots + 15 \times 31$$
   $$= \sum_{n=1}^{15} a_n$$

12.1.3 Exercise 12.1

1. Find the general term of the following sequences.

   (a) 5, 8, 11, 14, …
   Sol. $a_n = 3n + 2$

   (b) 2, 4, 8, 16, …
   Sol. $a_n = 2^n$

   (c) $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$
   Sol. $a_n = \frac{n+1}{n}$

   (d) $\frac{2}{5}, \frac{4}{7}, \frac{6}{9}, \frac{8}{11}, \ldots$
   Sol. $a_n = \frac{2n}{2n+1}$

2. Find the first 5 terms of the following sequences.

   (a) $a_n = 2n + 3$
   Sol. $a_1 = 2\times1+3 = 5, a_2 = 2\times2+3 = 7, a_3 = 2\times3+3 = 9, a_4 = 2\times4+3 = 11, a_5 = 2\times5+3 = 13$

   (b) $a_n = n(n - 2)$
   Sol. $a_1 = 1 \times (-1) = -1, a_2 = 2 \times 0 = 0, a_3 = 3 \times 1 = 3, a_4 = 4 \times 2 = 8, a_5 = 5 \times 3 = 15$

   (c) $a_n = \frac{n}{2n+1}$
   Sol. $a_1 = \frac{1}{2\times1+1} = \frac{1}{3}, a_2 = \frac{2}{2\times2+1} = \frac{2}{5}, a_3 = \frac{3}{2\times3+1} = \frac{3}{7}, a_4 = \frac{4}{2\times4+1} = \frac{4}{9}, a_5 = \frac{5}{2\times5+1} = \frac{5}{11}$

   (d) $a_n = (-3)^n$
   Sol. $a_1 = (-3)^1 = -3, a_2 = (-3)^2 = 9, a_3 = (-3)^3 = -27, a_4 = (-3)^4 = 81, a_5 = (-3)^5 = -243$

3. Express the following series in the form of numbers.

   (a) $\sum_{n=1}^{5} n(n + 3)$

Sol. $\sum_{n=1}^{5} n(n+3)$
$= (1\times 4)+(2\times 5)+(3\times 6)+(4\times 7)$
$+ (5\times 8)$
$= 4+10+18+28+40$

(b) $\sum_{n=2}^{6} \frac{1}{3^n}$

Sol. $\sum_{n=2}^{6} \frac{1}{3^n}$
$= \frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\frac{1}{3^5}+\frac{1}{3^6}$
$= \frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\frac{1}{729}$

(c) $\sum_{n=1}^{6} \frac{1}{n(2n+1)}$

Sol. $\sum_{n=1}^{6} \frac{1}{n(2n+1)}$
$= \frac{1}{1(2\times 1+1)}+\frac{1}{2(2\times 2+1)}$
$+ \frac{1}{3(2\times 3+1)}+\frac{1}{4(2\times 4+1)}$
$+ \frac{1}{5(2\times 5+1)}+\frac{1}{6(2\times 6+1)}$
$= \frac{1}{3}+\frac{1}{10}+\frac{1}{21}+\frac{1}{36}+\frac{1}{55}+\frac{1}{78}$

(d) $\sum_{n=2}^{5} \frac{1}{n^2+2}$

Sol. $\sum_{n=2}^{5} \frac{1}{n^2+2}$
$= \frac{1}{4+2}+\frac{1}{9+2}+\frac{1}{16+2}+\frac{1}{25+2}$
$= \frac{1}{6}+\frac{1}{11}+\frac{1}{18}+\frac{1}{27}$

4. Find the first term, last term and the number of terms of the following series.

(a) $\sum_{n=3}^{10} 2^2$
Sol. $a_3 = 2^2 = 4, a_{10} = 2^2 = 4, n = 10-3+1 = 8$

(b) $\sum_{n=1}^{8} \frac{n+2}{n}$

Sol. $a_1 = \frac{1+2}{1} = \frac{3}{1} = 3, a_8 = \frac{8+2}{8} = \frac{10}{8} = \frac{5}{4}, n = 8-1+1 = 8$

(c) $\sum_{n=1}^{10} 3n^2 - n$

Sol. $a_1 = 3\times 1^2 - 1 = 2, a_{10} = 3\times 10^2 - 10 = 290, n = 10-1+1 = 10$

(d) $\sum_{n=9}^{14} n^2(n-7)$

Sol. $a_9 = 9^2(9-7) = 9^2\times 2 = 162, a_{14} = 14^2(14-7) = 14^2\times 7 = 2744, n = 14-9+1 = 6$

5. Express the following series in the form of $\sum$.

(a) $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{30}$

Sol.

$$a_1 = 1$$
$$a_2 = \frac{1}{2}$$
$$a_3 = \frac{1}{3}$$
$$\vdots$$
$$a_{30} = \frac{1}{30}$$
$$\therefore 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{30} = \sum_{n=1}^{30}\frac{1}{n}$$

(b) $1^3+2^3+3^3+\ldots+50^3$

Sol.

$$a_1 = 1^3$$
$$a_2 = 2^3$$
$$a_3 = 3^3$$
$$\vdots$$
$$a_{50} = 50^3$$
$$\therefore 1^3+2^3+3^3+\ldots+50^3 = \sum_{n=1}^{50} n^3$$

(c) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}$

Sol.

$$\begin{aligned} a_1 &= (-\frac{1}{2})^{1-1} \\ a_2 &= (-\frac{1}{2})^{2-1} \\ a_3 &= (-\frac{1}{2})^{3-1} \\ a_4 &= (-\frac{1}{2})^{4-1} \\ a_5 &= (-\frac{1}{2})^{5-1} \\ \therefore 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} & \\ = \sum_{n=1}^{5} (-\frac{1}{2})^{n-1} & \end{aligned}$$

(d) $2 \times 4 + 4 \times 7 + 6 \times 10 + 8 \times 13 + 10 \times 16$

Sol.

$$\begin{aligned} a_1 &= 2 \times 1 \times (3 \times 1 + 1) \\ a_2 &= 2 \times 2 \times (3 \times 2 + 1) \\ a_3 &= 2 \times 3 \times (3 \times 3 + 1) \\ a_4 &= 2 \times 4 \times (3 \times 4 + 1) \\ a_5 &= 2 \times 5 \times (3 \times 5 + 1) \\ \therefore 2 \times 4 + 4 \times 7 + 6 \times 10 + 8 \times 13 & \\ + 10 \times 16 = \sum_{n=1}^{5} 2n(3n+1) & \end{aligned}$$

12.2 Arithmetic Progression

General term of an Arithmetic Progression (AP) is given by

$$a_n = a_1 + (n-1)d$$

where a_1 is the first term, d is the common difference and n is the number of terms.

12.2.1 Practice 3

1. Find the number of terms of the AP $-4 - 2\frac{3}{4} - 1\frac{1}{2} - \frac{1}{4} + \ldots + 16$.

$$\begin{aligned} a_1 &= -4 \\ a_n &= 16 \\ d &= -2\frac{3}{4} - (-4) \\ &= -2\frac{3}{4} + 4 \\ &= \frac{5}{4} \\ 16 &= -4 + (n-1)\frac{5}{4} \\ 20 &= \frac{5}{4}(n-1) \\ 80 &= 5(n-1) \\ n - 1 &= 16 \\ n &= 17 \end{aligned}$$

2. Given that $a_2 = 4$ and $a_6 = -8$, find the 10th term of the AP.

Sol.

$$\begin{aligned} a_2 &= 4 \\ a + (2-1)d &= 4 \\ a_6 &= -8 \\ a + (6-1)d &= -8 \end{aligned}$$

$$\begin{cases} a + d = 4 & (1) \\ a + 5d = -8 & (2) \end{cases}$$

$$\begin{aligned} (2) - (1) : 4d &= -12 \\ d &= -3 \\ a + (-3) &= 4 \\ a &= 7 \\ \therefore a_{10} &= 7 + (10-1)(-3) \\ &= 7 - 27 \\ &= -20 \end{aligned}$$

3. How many multiples of 7 are there between 50 and 500?

Sol.

$$\begin{aligned}
a_1 &= 56\\
a_n &= 497\\
d &= 7\\
497 &= 56 + (n-1)7\\
441 &= 7(n-1)\\
n-1 &= 63\\
n &= 64
\end{aligned}$$

4. Find 5 numbers between 30 and 54 such that these numbers form an AP.

   Sol.

$$\begin{aligned}
&a_1 = 30\\
&a_7 = 54\\
&54 = 30 + (7-1)d\\
&24 = 6d\\
&d = 4
\end{aligned}$$

$$\therefore These\ 5\ numbers\ are\ 34,\ 38,\ 42,\ 46,\ and\ 50.$$

Arithmetic mean

If A is in between x and y, and x, A, y are in AP, then

$$A = \frac{x+y}{2}$$

12.2.2 Practice 4

1. If 9, x, 17 are in AP, find x.

   Sol.

$$\begin{aligned}
x &= \frac{9+17}{2}\\
&= \frac{26}{2}\\
&= 13
\end{aligned}$$

2. Find the arithmetic mean of 26 and -11.

   Sol.

$$\begin{aligned}
A &= \frac{26-11}{2}\\
&= \frac{15}{2}
\end{aligned}$$

3. Find x and y when 3, x, 12, y, 21 are in AP.

   Sol.

$$\begin{aligned}
x &= \frac{3+12}{2}\\
&= \frac{15}{2}\\
y &= \frac{12+21}{2}\\
&= \frac{33}{2}
\end{aligned}$$

Summation of Arithmetic Progression

The summation formula for AP is given by

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

or

$$S_n = \frac{n}{2}(a_1 + a_n)$$

12.2.3 Practice 5

1. Find the sum of the first 16 terms of the AP $22 + 18 + 14 + 10 + \ldots$

   Sol.

$$\begin{aligned}
a_1 &= 22\\
n &= 16\\
d &= -4\\
S_n &= \frac{16}{2}(2 \times 22 + (-4)(16-1))\\
&= \frac{16}{2}(44 + (-4)(15))\\
&= \frac{16}{2}(44 - 60)\\
&= \frac{16}{2}(-16)\\
&= -128
\end{aligned}$$

2. If the sum of AP $23 + 19 + 15 + \ldots$ is 72, find the number of terms.

Sol.

$$\begin{aligned}
a_1 &= 23\\
S_n &= 72\\
d &= -4\\
72 &= \frac{n}{2}(2 \times 23 + (-4)(n-1))\\
72 &= \frac{n}{2}(46 + (-4)(n-1))\\
144 &= n(46 + (-4)(n-1))\\
144 &= n(46 - 4n + 4)\\
144 &= n(50 - 4n)\\
144 &= 50n - 4n^2\\
72 &= 25n - 2n^2\\
2n^2 - 25n + 72 &= 0\\
(n-8)(2n-9) &= 0\\
n &= 8
\end{aligned}$$

3. Given that $S_n = 2n + 3n^2$, find the first term and the common difference of the AP.

Sol.

$$\begin{aligned}
S_n &= 2n + 3n^2\\
2n + 3n^2 &= \frac{n}{2}(2a + (n-1)d)\\
4n + 6n^2 &= n(2a + (n-1)d)\\
4n + 6n^2 &= 2na + (n-1)nd\\
4n + 6n^2 &= 2na + n^2 d - nd\\
4n + 6n^2 &= (2a - d)n + dn^2
\end{aligned}$$

$$\begin{aligned}
&\textit{Comparing both sides,}\\
2a - d &= 4\\
a &= 6\\
d &= 2
\end{aligned}$$

12.2.4 Exercise 12.2

1. Find the 10th terms of the AP $5, 13, 21, \ldots$

Sol.

$$\begin{aligned}
a_1 &= 5\\
n &= 10\\
d &= 8\\
a_{10} &= 5 + (10-1) \times 8\\
&= 5 + 72\\
&= 77
\end{aligned}$$

2. Find the 8th term of the AP $5, 4\frac{1}{4}, 3\frac{1}{2}, 2\frac{3}{4}, \ldots$

Sol.

$$\begin{aligned}
a_1 &= 5\\
n &= 8\\
d &= -\frac{3}{4}\\
a_8 &= 5 + (8-1) \times -\frac{3}{4}\\
&= 5 - \frac{3}{4} \times 7\\
&= 5 - \frac{21}{4}\\
&= -\frac{1}{4}
\end{aligned}$$

3. Find the number of terms of the following AP.

(a) $4, 9, \ldots, 64$

Sol.

$$\begin{aligned}
a_1 &= 4\\
a_n &= 64\\
d &= 5\\
64 &= 4 + (n-1) \times 5\\
60 &= 5(n-1)\\
12 &= n - 1\\
n &= 13
\end{aligned}$$

(b) $4\frac{1}{3}, 3\frac{2}{3}, 3, \ldots, -10\frac{1}{3}$

Sol.

$$
\begin{aligned}
a_1 &= 4\frac{1}{3} \\
a_n &= -10\frac{1}{3} \\
d &= -\frac{2}{3} \\
-10\frac{1}{3} &= 4\frac{1}{3} + (n-1) \times -\frac{2}{3} \\
-\frac{31}{3} &= \frac{13}{3} - \frac{1}{3}(n-1) \\
-31 &= 13 - 2n + 2 \\
-46 &= 2n \\
n &= 23
\end{aligned}
$$

4. The 6th term of an AP is 43, and its 10th term is 75. Find the first term and common difference of this AP.

   Sol.

$$
\begin{aligned}
a_6 &= 43 \\
a_{10} &= 75 \\
43 &= a + (6-1)d \\
75 &= a + (10-1)d \\
32 &= 4d \\
d &= 8 \\
43 &= a + 5 \times 8 \\
43 &= a + 40 \\
3 &= a \\
a &= 3 \\
\therefore a_1 &= 3, \; d = 8
\end{aligned}
$$

5. The 7th term of an AP is -10, and the 12th term -25, find the 15th term of this AP.

   Sol.

$$
\begin{aligned}
a_7 &= -10 \\
a_{12} &= -25 \\
-10 &= a + (7-1)d \\
-25 &= a + (12-1)d \\
-15 &= 5d \\
d &= -3 \\
-10 &= a + 6 \times -3 \\
-10 &= a - 18 \\
a &= 8 \\
a_{15} &= 8 + (15-1) \times -3 \\
&= 8 - 42 \\
&= -34
\end{aligned}
$$

6. How many multiples of 7 are there between 100 and 200?

   Sol.

$$
\begin{aligned}
a &= 105 \\
d &= 7 \\
a_n &= 196 \\
196 &= 105 + (n-1) \times 7 \\
91 &= 7(n-1) \\
13 &= n - 1 \\
n &= 14
\end{aligned}
$$

7. Find the arithmetic mean o fthe following number pairs.

   (a) $(8, 20)$

   Sol.

$$\frac{8+20}{2} = 14$$

   (b) $(-9, 17)$

   Sol.

$$\frac{-9+17}{2} = 4$$

8. Find 5 numbers between 22 and 58 such that these 7 numbers are in AP.

Sol.

$$\begin{aligned}
a_1 &= 22\\
a_7 &= 58\\
58 &= 22 + (7-1)d\\
36 &= 6d\\
d &= 6\\
&\therefore These\ 5\ numbers\ are\ 22, 28, 34, 40, 46
\end{aligned}$$

9. Find the sum of first 20 terms of AP $12 + 15 + 18 + \ldots$

Sol.

$$\begin{aligned}
a_1 &= 12\\
n &= 20\\
d &= 3\\
S_{20} &= \frac{20}{2}(2 \times 12 + (20-1) \times 3)\\
&= 10(24 + 57)\\
&= 10(81)\\
&= 810
\end{aligned}$$

10. Find the sum of first 12 terms of the AP $18 + 10 + 2 - 6 - \ldots$

Sol.

$$\begin{aligned}
a_1 &= 18\\
n &= 12\\
d &= -8\\
S_{12} &= \frac{12}{2}(2 \times 18 + (12-1) \times -8)\\
&= 6(36 - 88)\\
&= 6(-52)\\
&= -312
\end{aligned}$$

11. Find the sum of first 14 terms of the AP $\frac{1}{6} + \frac{4}{3} + \frac{5}{2} + \ldots$

Sol.

$$\begin{aligned}
a_1 &= \frac{1}{6}\\
n &= 14\\
d &= \frac{7}{6}\\
S_{14} &= \frac{14}{2}(2 \times \frac{1}{6} + (14-1) \times \frac{7}{6})\\
&= 7(\frac{1}{3} + \frac{91}{6})\\
&= 7 \times \frac{93}{6}\\
&= 7 \times \frac{31}{2}\\
&= \frac{217}{2}
\end{aligned}$$

12. Find the sum of all the multiples of 13 in between 200 and 800.

Sol.

$$\begin{aligned}
a_1 &= 208\\
a_n &= 793\\
d &= 13\\
793 &= 208 + (n-1) \times 13\\
585 &= 13(n-1)\\
45 &= n - 1\\
n &= 46
\end{aligned}$$

$$\begin{aligned}
S_{46} &= \frac{46}{2}(2 \times 208 + (46-1) \times 13)\\
&= 23(416 + 585)\\
&= 23(1001)\\
&= 23023
\end{aligned}$$

13. If the sum of first n terms of the AP $-3, -7, -11, \ldots$ is -903, find the value of n.

Sol.

$$\begin{aligned} a_1 &= -3 \\ d &= -4 \\ -903 &= \frac{n}{2}(2 \times (-3) - 4(n-1)) \\ -1806 &= -2n - 4n^2 \\ 4n^2 + 2n - 1806 &= 0 \\ 2n^2 + n - 903 &= 0 \\ (n-21)(2n+43) &= 0 \\ n &= 21, -43(invalid) \\ \therefore n &= 21 \end{aligned}$$

14. Given that the first 3 terms of an AP are $x,\ 3x-4,\ 2x+7$, find:

    (a) The value of x

    Sol.

    $$\begin{aligned} 3x - 4 &= \frac{x + 2x + 7}{2} \\ 6x - 8 &= 3x + 7 \\ 3x &= 15 \\ x &= 5 \end{aligned}$$

    (b) The common difference

    Sol.

    $$\begin{aligned} a_1 &= x = 5 \\ a_2 &= 3x - 4 = 3 \times 5 - 4 = 11 \\ d &= 11 - 5 \\ &= 6 \end{aligned}$$

    (c) The sum of first 10 terms.

    Sol.

    $$\begin{aligned} a_1 &= x = 5 \\ n &= 10 \\ d &= 6 \\ S_{10} &= \frac{10}{2}(2 \times 5 + (10-1) \times 6) \\ &= 5(10 + 54) \\ &= 5(64) \\ &= 320 \end{aligned}$$

15. Let the sum of the first n terms of an AP to be $S_n = \frac{n(n+1)}{4}$, find:

    (a) The first term

    Sol.

    $$\begin{aligned} \frac{n(n+1)}{4} &= \frac{n}{2}(2a + (n-1)d) \\ n(n+1) &= 2n(2a + dn - d) \\ n^2 + n &= 4na + 2dn^2 - 2nd \\ n^2 + n &= 2dn^2 + (4a - 2d)n \end{aligned}$$

    $$\begin{aligned} &Comparing\ both\ sides, \\ 2d &= 1 \\ d &= \frac{1}{2} \\ 4a - 2d &= 1 \\ 4a - 1 &= 1 \\ 4a &= 2 \\ a &= \frac{1}{2} \end{aligned}$$

    (b) The common difference

    Sol.

    $$d = \frac{1}{2}$$

    gg

    (c) The 6th terms

    Sol.

    $$\begin{aligned} a_1 &= \frac{1}{2} \\ n &= 6 \\ d &= \frac{1}{2} \\ a_6 &= \frac{1}{2} + (6-1) \times \frac{1}{2} \\ &= \frac{1}{2} + \frac{5}{2} \\ &= 3 \end{aligned}$$

    (d) The sum from 6th term to 10th term

Sol.

$$a = \frac{1}{2}$$
$$d = \frac{1}{2}$$

$$\begin{aligned} S_{10} &= \frac{10}{2}(2 \times \frac{1}{2} + (10-1) \times \frac{1}{2}) \\ &= \frac{10}{2}(1 + \frac{9}{2}) \\ &= 5 \times \frac{11}{2} \\ &= \frac{55}{2} \end{aligned}$$

$$\begin{aligned} S_5 &= \frac{5}{2}(2 \times \frac{1}{2} + (5-1) \times \frac{1}{2}) \\ &= \frac{5}{2}(1+2) \\ &= \frac{15}{2} \end{aligned}$$

$$\begin{aligned} S_{10} - S_6 &= \frac{55}{2} - \frac{15}{2} \\ &= \frac{40}{2} \\ &= 20 \end{aligned}$$

16. Given three numbers in an AP, the sum of these three numbers is 30, and the sum of square of these numbers is 318, find these three numbers.

Sol.

$$\begin{aligned} a_1 + a_2 + a_3 &= 30 \\ a_1^2 + a_2^2 + a_3^2 &= 318 \\ a_2 - a_1 &= a_3 - a_2 \\ a_1 - 2a_2 + a_3 &= 0 \\ 3a_2 &= 30 \\ a_2 &= 10 \\ a_1 - 20 + a_3 &= 0 \\ a_1 + a_3 &= 20 \\ a_3 &= 20 - a_1 \\ a_1^2 + 100 + (20 - a_1)^2 &= 318 \\ a_1^2 + 100 + 400 + a_1^2 - 40a_1 &= 318 \\ 2a_1^2 - 40a_1 + 182 &= 0 \\ a_1^2 - 20a_1 + 91 &= 0 \\ (a_1 - 7)(a_1 - 13) &= 0 \\ a_1 &= 7 or a_1 = 13 \end{aligned}$$

$$\therefore These\ three\ numbers\ are\ 7,\ 10,\ and\ 13$$

17. Find the sum of all the numbers between 100 and 200 that are both the multiples of 2 and 3.

Sol.

$$\begin{aligned} a_1 &= 102 \\ d &= 6 \\ a_n &= 198 \\ 198 &= 102 + (n-1) \times 6 \\ 96 &= 6(n-1) \\ 6n - 6 &= 96 \\ 6n &= 102 \\ n &= 17 \end{aligned}$$

$$\begin{aligned} S_{17} &= \frac{17}{2}(2 \times 102 + (17-1) \times 6) \\ &= \frac{17}{2}(204 + 96) \\ &= \frac{17}{2}(300) \\ &= 150 \times 17 \\ &= 2550 \end{aligned}$$

18. Given an AP $-100 - 96 - 92 - \ldots$:

(a) Find the term where the number become positive.

Sol.

$$\begin{aligned} a_1 &= -100 \\ d &= 4 \\ a_n = -100 + (n-1) \times 4 &> 0 \\ -100 + 4n - 4 &> 0 \\ 4n &> 104 \\ n &> 26 \end{aligned}$$

$$\therefore n = 27$$

(b) Find the term where the sum of this AP becomes positive.

Sol.

$$\begin{aligned} S_n = \frac{n}{2}(2(-100) + (n-1) \times (4)) &> 0 \\ \frac{n}{2}(-200 + 4n - 4) &> 0 \\ \frac{n}{2}(-204 + 4n) &> 0 \\ n(2n - 102) &> 0 \\ n(n - 51) &> 0 \\ n &> 51 \end{aligned}$$

$$\therefore n = 52$$

19. Find the first negative term of the AP $20, 19\frac{1}{5}, 18\frac{2}{5}, \ldots$

Sol.

$$\begin{aligned} a_1 &= 20 \\ d &= -\frac{4}{5} \\ a_n = 20 + (n-1) \times (-\frac{4}{5}) &< 0 \\ 100 - 4n + 4 &< 0 \\ 4n &> 104 \\ n &> 26 \end{aligned}$$

$$\therefore n = 27$$

20. Given an AP $10 + 9\frac{1}{5} + 8\frac{2}{5} + \ldots$, what is the first negative term? When will the sum of the terms become negative, and what's the value of it?

Sol.

$$\begin{aligned} a_n = 10 + (n-1) \times (-\frac{4}{5}) &< 0 \\ 10 - \frac{4}{5}(n-1) &< 0 \\ 50 - 4n + 4 &< 0 \\ -4n &< -54 \\ n &> 13\frac{1}{2} \end{aligned}$$

$$\therefore n = 14$$

$$\begin{aligned} S_n = \frac{n}{2}(2 \times 10 + (n-1) \times (-\frac{4}{5})) &< 0 \\ \frac{n}{2}(20 - \frac{4}{5}(n-1)) &< 0 \\ 20n - \frac{4}{5}(n^2 - n) &< 0 \\ 100n - 4n^2 + 4n &< 0 \\ 25n - n^2 + n &< 0 \\ 26n - n^2 &< 0 \\ n(n - 26) &> 0 \\ n &> 26 \end{aligned}$$

$$\therefore n = 27$$

$$\begin{aligned} S_{27} &= \frac{27}{2}(2 \times 10 + (27-1) \times (-\frac{4}{5})) \\ &= \frac{27}{2}(20 - \frac{4}{5}(27-1)) \\ &= \frac{27}{2}(20 - \frac{4}{5}(26)) \\ &= \frac{27}{2} \times (-\frac{4}{5}) \\ &= -\frac{54}{5} \end{aligned}$$

$\therefore The\ first\ negative\ term\ is\ the\ 14th\ term$

$\therefore The\ first\ term\ where\ the\ sum\ of\ the\ terms\ becomes\ negative\ is\ the\ 27th\ term$

$\therefore The\ value\ of\ the\ sum\ of\ the\ terms\ when\ it\ becomes\ negative\ is\ -\frac{54}{5}$

21. Given a polygon which all their internal angles are in AP. The common difference of this AP is 6°, the largest angle is 135°. How many sides does this polygon have?

Sol.

$$\begin{aligned}
a_1 &= 135 \\
d &= -6 \\
\frac{n}{2}(2\times 135 + (n-1)\times(-6)) &= 180(n-2) \\
n(270 - 6(n-1)) &= 360(n-2) \\
n(276 - 6n) &= 360n - 720 \\
276n - 6n^2 &= 360n - 720 \\
46n - n^2 &= 60n - 120 \\
n^2 + 14n - 120 &= 0 \\
(n+20)(n-6) &= 0 \\
n &= -20\ (invalid) \\
n &= 6 \\
\therefore The\ number\ of\ sides\ is\ 6
\end{aligned}$$

22. Given an AP which its 5th term is 3 and the sum of its first 10 terms is $26\frac{1}{4}$. Whcih term in this AP is 0?

Sol.

$$\begin{aligned}
a_5 = a + (5-1)d &= 3 \\
a + 4d &= 3 \\
S_{10} = \frac{10}{2}(2a + (10-1)d) &= 26\frac{1}{4} \\
5(2a+9d) &= 26\frac{1}{4} \\
20(2a+9d) &= 105 \\
4(2a+9d) &= 21 \\
8a + 36d &= 21 \\
8a + 32d &= 24 \\
4d &= -3 \\
d &= -\frac{3}{4} \\
a &= 3 + \frac{3}{4}\times 4 \\
&= 6 \\
a_n = 6 + (n-1)\times(-\frac{3}{4}) &= 0 \\
6 - \frac{3}{4}(n-1) &= 0 \\
24 - 3n + 3 &= 0 \\
3n &= 27 \\
n &= 9
\end{aligned}$$

23. Given that the sum of the first 6 terms of an AP is 96, and the sum of the first 20 terms is 3 times the sum of the first 10 terms of this AP. Find the first term and the 10th term of it.

Sol.

$$\begin{aligned}
S_6 = \frac{6}{2}(2a + (6-1)d) &= 96 \\
3(2a+5d) &= 96 \\
2a + 5d &= 32 \\
S_{20} &= 3S_{10} \\
\frac{20}{2}(2a + (20-1)d) &= 3\times\frac{10}{2}(2a \\
&\quad + (10-1)d) \\
10(2a+19d) &= 15(2a+9d) \\
2(2a+19d) &= 3(2a+9d) \\
4a + 38d &= 6a + 27d \\
2a - 11d &= 0 \\
16d &= 32 \\
d &= 2 \\
a &= \frac{11\times 2}{2} \\
&= 11 \\
a_{10} &= 11 + (10-1)\times 2 \\
&= 29
\end{aligned}$$

24. Given that $5^2\times 5^4\times 5^6\times\ldots\times 5^{2n} = (0.04)^{-28}$, find the value of n.

Sol.

$$\begin{aligned}
(0.04)^{-28} &= \frac{1}{25}^{-28} \\
&= (5^{-2})^{-28} \\
&= 5^{56} \\
\because n^a \times n^b &= n^{a+b} \\
2 + 4 + 6 + \ldots + 2n &= 56 \\
S_n = \frac{n}{2}(2\times 2 + (n-1)\times 2) &= 56 \\
n(4 + 2(n-1)) &= 112 \\
n(2+2n) &= 112 \\
2n^2 + 2n &= 112 \\
n^2 + n - 56 &= 0 \\
(n+8)(n-7) &= 0 \\
n &= -8\ (invalid) \\
n &= 7
\end{aligned}$$

25. Given that the 9th term of an AP is double the 5th term of it. Find the ratio of the sum of first 9 terms and the sum of first 5 terms of the AP.

Sol.

$$\begin{aligned}
a_9 &= 2a_5 \\
a + (9-1)d &= 2(a + (5-1)d) \\
a + 8d &= 2a + 8d \\
a &= 0 \\
S_9 : S_5 &= \frac{9}{2}(2a + a_9) : \frac{5}{2}(2a + a_5) \\
&= \frac{9}{2}(2a + 2a_5) : \frac{5}{2}(2a + a_5) \\
&= 9(a + a_5) : \frac{5}{2}(2a + a_5) \\
\frac{S_9}{S_5} &= \frac{9(a + a_5)}{\frac{5}{2}(2a + a_5)} \\
&= \frac{18(a + a_5)}{5(2a + a_5)} \\
&= \frac{18 \times a_5}{5 \times a_5} \\
&= \frac{18}{5} \\
\therefore S_9 : S_5 &= 18 : 5
\end{aligned}$$

12.3 Geometric Progression

The general formula of a geometric progression (GP) is given by

$$a_n = a_1 \times r^{n-1}$$

where a_1 is the first term, r is the common ratio, and n is the number of terms.

12.3.1 Practice 6

1. Find the 6th term of the GP $12, -18, 27, \ldots$

Sol.

$$\begin{aligned}
a_1 &= 12 \\
r &= \frac{-18}{12} \\
&= -\frac{3}{2} \\
a_6 &= 12 \times (-\frac{3}{2})^{6-1} \\
&= 12 \times (-\frac{3}{2})^5 \\
&= 12 \times (-\frac{243}{32}) \\
&= -\frac{729}{8}
\end{aligned}$$

2. Find the number of terms of GP $\frac{1}{64} - \frac{1}{32} + \frac{1}{16} - \frac{1}{8} + \ldots - 512$

Sol.

$$\begin{aligned}
a_1 &= \frac{1}{64} \\
r &= \frac{-\frac{1}{32}}{\frac{1}{64}} \\
&= -2 \\
-512 &= \frac{1}{64}(-2)^{n-1} \\
(-2)^9 &= \frac{1}{2^6}(-2)^{n-1} \\
(-2)^{15} &= (-2)^{n-1} \\
n - 1 &= 15 \\
n &= 16
\end{aligned}$$

3. The 5th term of a GP is 3, and its 9th term is $\frac{1}{27}$, find the first term and the common ratio of this GP.

Sol.

$$\begin{aligned}
a_5 &= ar^4 = 3 \\
a_9 &= ar^8 = \frac{1}{27} \\
r^4 &= \frac{1}{27} \times \frac{1}{3} \\
&= \frac{1}{81} \\
r &= \frac{1}{3} \\
a_1 &= 3 \times 81 \\
&= 243
\end{aligned}$$

4. Find 5 numbers between $\frac{1}{2}$ and $frac1128$ such that these 7 numbers are in GP.

Sol.

$$
\begin{aligned}
a_1 &= \frac{1}{2} \\
n &= 7 \\
\frac{1}{128} &= \frac{1}{2}r^{7-1} \\
r^6 &= \frac{1}{64} \\
r &= \frac{1}{2}
\end{aligned}
$$

$$\therefore These\ 5\ numbers\ are\ \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$$

Geometric Mean

The geometric mean G of two numbers x and y is given by

$$
\begin{aligned}
\frac{G}{x} &= \frac{G}{y} \\
G^2 &= xy \\
G &= \mp\sqrt[2]{xy}
\end{aligned}
$$

12.3.2 Practice 7

Find the geometric mean of $\frac{27}{8}$ and $\frac{2}{3}$.

Sol.

$$
\begin{aligned}
G &= \pm\sqrt[2]{\frac{27}{8} \times \frac{2}{3}} \\
&= \pm\sqrt[2]{\frac{9}{4}} \\
&= \pm\frac{3}{2}
\end{aligned}
$$

Summation of Geometric Progression

The sum of n terms of a GP is given by

$$S_n = \frac{a_1(1-r^n)}{1-r}\ (r \neq 1)$$

12.3.3 Practice 8

1. Find the sum of the first 8 terms of GP $3 + 6 + 12 + \ldots$

Sol.

$$
\begin{aligned}
a_1 &= 3 \\
r &= \frac{6}{3} \\
&= 2 \\
n &= 8 \\
S_n &= \frac{3(1-2^8)}{1-2} \\
&= \frac{3(1-256)}{1-2} \\
&= 3 \times 255 \\
&= 765
\end{aligned}
$$

2. Find the sum of the GP $1 + \sqrt{3} + 3 + \ldots + 81$

Sol.

$$
\begin{aligned}
a_1 &= 1 \\
r &= \sqrt{3} \\
81 &= 1 \times (\sqrt{3})^{n-1} \\
3^4 &= (\sqrt{3})^{n-1} \\
(\sqrt{3})^8 &= (\sqrt{3})^{n-1} \\
n-1 &= 8 \\
n &= 9 \\
S_n &= \frac{1(1-(\sqrt{3})^9)}{1-\sqrt{3}} \\
&= \frac{1-81\sqrt{3}}{1-\sqrt{3}} \\
&= \frac{(1-81\sqrt{3})(1+\sqrt{3})}{-2} \\
&= \frac{1-81\sqrt{3}+\sqrt{3}-243}{-2} \\
&= \frac{-242-80\sqrt{(3)}}{-2} \\
&= 121 + 40\sqrt{3}
\end{aligned}
$$

3. Given that the sum of the first n terms of GP $4\frac{4}{5}, 1\frac{3}{5}, \frac{8}{15}, \ldots$ is $7\frac{145}{729}$, find n.

Sol.

$$\begin{aligned}
a_1 &= \frac{24}{5}\\
r &= \frac{8}{5} \times \frac{5}{24}\\
&= \frac{1}{3}\\
S_n &= \frac{24}{5} \times \frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}}\\
\frac{5248}{729} &= \frac{24}{5} \times \frac{1-(\frac{1}{3})^n}{\frac{2}{3}}\\
\frac{5248}{729} \times \frac{5}{24} \times \frac{2}{3} &= 1-(\frac{1}{3})^n\\
\frac{6560}{6561} &= 1-(\frac{1}{3})^n\\
-\frac{1}{6561} &= -(\frac{1}{3})^n\\
(\frac{1}{3})^8 &= (\frac{1}{3})^n\\
n &= 8
\end{aligned}$$

Summation of Infinite Geometric Progression

The sum of infinite GP is given by

$$S_\infty = \frac{a_1}{1-r} \; (-1 < r < 1)$$

12.3.4 Practice 9

1. Find the sum of the following infinite GP.

   (a) $16 + 8 + 4 + \ldots$

   Sol.

$$\begin{aligned}
a_1 &= 16\\
r &= \frac{8}{16}\\
&= \frac{1}{2}\\
S_\infty &= \frac{16}{1-\frac{1}{2}}\\
&= \frac{16}{\frac{1}{2}}\\
&= 32
\end{aligned}$$

   (b) $18 - 12 + 8 + \ldots$

   Sol.

$$\begin{aligned}
a_1 &= 18\\
r &= \frac{8}{-12}\\
&= -\frac{2}{3}\\
S_\infty &= \frac{18}{1+\frac{2}{3}}\\
&= \frac{18}{\frac{5}{3}}\\
&= \frac{54}{5}
\end{aligned}$$

   (c) $1 + \frac{3}{4} + \frac{9}{16} + \ldots$

   Sol.

$$\begin{aligned}
a_1 &= 1\\
r &= \frac{9}{16} \times \frac{16}{9}\\
&= \frac{3}{4}\\
S_\infty &= \frac{1}{1-\frac{3}{4}}\\
&= \frac{1}{\frac{1}{4}}\\
&= 4
\end{aligned}$$

   (d) $\sqrt{2} + 1 + \frac{1}{\sqrt{2}} + \ldots$

   Sol.

$$\begin{aligned}
a_1 &= \sqrt{2}\\
r &= \frac{1}{\sqrt{2}}\\
S_\infty &= \frac{\sqrt{2}}{1-\frac{1}{\sqrt{2}}}\\
&= \frac{\sqrt{2}}{\frac{\sqrt{2}-1}{\sqrt{2}}}\\
&= \frac{2}{\sqrt{2}-1}\\
&= 2(\sqrt{2}+1)
\end{aligned}$$

2. Convert the following recurring decimals to fraction using the summation of inifinite geometric series.

(a) $0.\overline{3}$

Sol.

$$\begin{aligned} a_1 &= 0.3 \\ r &= 0.1 \\ S_\infty &= \frac{0.3}{1-0.1} \\ &= \frac{0.3}{0.9} \\ &= \frac{1}{3} \end{aligned}$$

$$\therefore 0.\overline{3} = \frac{1}{3}$$

(b) $0.5\overline{3}$

Sol.

$$\begin{aligned} a_1 &= 0.03 \\ r &= 0.01 \\ S_\infty &= \frac{0.03}{1-0.01} \\ &= \frac{0.03}{0.99} \\ &= \frac{3}{99} \end{aligned}$$

$$\begin{aligned} \therefore 0.5\overline{3} &= \frac{5}{10} + \frac{3}{99} \\ &= \frac{53}{99} \end{aligned}$$

12.3.5 Exercise 12.3

1. Find the 10th term of the GP $2, 4, 8, \ldots$

Sol.

$$\begin{aligned} a_1 &= 2 \\ r &= \frac{4}{2} \\ &= 2 \\ a_{10} &= 2 \times 2^{10-1} \\ &= 2 \times 512 \\ &= 1024 \end{aligned}$$

2. Find the 8th term of the GP $243, -162, 108, \ldots$

Sol.

$$\begin{aligned} a_1 &= 243 \\ r &= \frac{-162}{243} \\ &= -\frac{2}{3} \\ a_8 &= 243 \times (-\frac{2}{3})^{8-1} \\ &= 243 \times (-\frac{128}{2187}) \\ &= -\frac{128}{9} \end{aligned}$$

3. Find the number of terms of the following GP.

(a) $8, 4, 2, 1, \ldots, \frac{1}{64}$

Sol.

$$\begin{aligned} a_1 &= 8 \\ r &= \frac{4}{8} \\ &= \frac{1}{2} \\ \frac{1}{64} &= 8 \times (\frac{1}{2})^{n-1} \\ \frac{1}{512} &= (\frac{1}{2})^{n-1} \\ \frac{1}{2^9} &= (\frac{1}{2})^{n-1} \\ n-1 &= 9 \\ n &= 10 \end{aligned}$$

(b) $6, -18, 54, \ldots, -13122$

Sol.

$$\begin{aligned} a_1 &= 6 \\ r &= \frac{-18}{6} \\ &= -3 \\ -13122 &= 6 \times (-3)^{n-1} \\ -2187 &= (-3)^{n-1} \\ (-3)^7 &= (-3)^{n-1} \\ n-1 &= 7 \\ n &= 8 \end{aligned}$$

(c) $54, 36, 24, \ldots, 3\frac{13}{81}$

Sol.

$$\begin{aligned}
a_1 &= 54 \\
r &= \frac{36}{54} \\
&= \frac{2}{3} \\
\frac{256}{81} &= 54 \times (\frac{2}{3})^{n-1} \\
\frac{256}{81} \times \frac{1}{54} &= (\frac{2}{3})^{n-1} \\
\frac{128}{2187} &= (\frac{2}{3})^{n-1} \\
(\frac{2}{3})^7 &= (\frac{2}{3})^{n-1} \\
n - 1 &= 7 \\
n &= 8
\end{aligned}$$

4. Given that the 2nd term of a GP is 12, and its 4th term is 108, find the first term and the common ratio of it.

Sol.

$$\begin{aligned}
&a_2 = ar = 12 \\
&a_4 = ar^3 = 109 \\
&r^2 = 9 \\
&r = \pm 3 \\
&a_1 = \pm 4 \\
&\therefore a_1 = 4, r = 3 \text{ or } a_1 = -4, r = -3
\end{aligned}$$

5. Given that the 3rd term of an GP is $1\frac{1}{3}$, and its 8th term is $-10\frac{1}{8}$. Find the 5th term of this AP.

Sol.

$$\begin{aligned}
a_3 &= ar^2 = \frac{4}{3} \\
a_8 &= ar^7 = -\frac{81}{8} \\
r^5 &= -\frac{81}{8} \times \frac{3}{4} \\
&= -\frac{243}{32} \\
&= (-\frac{3}{2})^5 \\
r &= -\frac{3}{2} \\
a &= \frac{4}{3} \times \frac{4}{9} \\
&= \frac{16}{27} \\
a_5 &= \frac{16}{27} \times (\frac{3}{2})^4 \\
&= \frac{16}{27} \times \frac{81}{16} \\
&= 3
\end{aligned}$$

6. Find the geometric mean of 2 and 18.

Sol.

$$\begin{aligned}
G &= \pm \sqrt[2]{2 \times 18} \\
&= \pm \sqrt[2]{36} \\
&= \pm 6
\end{aligned}$$

7. Given that x+12, x+4 and x-2 are in GP, find the value of x and the common ratio of this GP.

Sol.

$$\begin{aligned}
x + 4 &= \pm\sqrt{(x + 12)(x - 2)} \\
x^2 + 8x + 16 &= x^2 + 10x - 24 \\
2x &= 40 \\
x &= 20 \\
a_1 &= 20 + 12 = 32 \\
a_2 &= 20 + 4 = 24 \\
r &= \frac{24}{32} \\
&= \frac{3}{4}
\end{aligned}$$

8. Find 3 numbers between 14 and 224 such that these 5 numbersare in GP.

Sol.

$$\begin{aligned} a_1 &= 14 \\ a_5 &= 224 \\ 244 &= 14 \times r^4 \\ 16 &= r^4 \\ (\pm 2)^4 &= r^4 \\ r &= \pm 2 \end{aligned}$$

$$\begin{aligned} \therefore These\ 3\ numbers\ are\ 28,\ 56,\ 112 \\ or\ -28, 56, -112 \end{aligned}$$

9. Calculate the sum of the first 6 terms of the GP $2 + 6 + 18 + \ldots$

Sol.

$$\begin{aligned} a_1 &= 2 \\ r &= \frac{6}{2} \\ &= 3 \\ S_6 &= \frac{2(1-3^6)}{1-3} \\ &= \frac{2(1-729)}{-2} \\ &= 728 \end{aligned}$$

10. Calculate the sum of the first 8 terms of the GP $32 - 16 + 8 - \ldots$

Sol.

$$\begin{aligned} a_1 &= 32 \\ r &= \frac{-16}{32} \\ &= -\frac{1}{2} \\ S_8 &= \frac{32(1-(\frac{1}{2})^8)}{1+\frac{1}{2}} \\ &= \frac{32(1-\frac{1}{256})}{\frac{3}{2}} \\ &= 32 \times \frac{255}{256} \times \frac{2}{3} \\ &= \frac{85}{4} \end{aligned}$$

11. Find the sum of the GP $14 - 28 + 56 - \ldots + 3584$

Sol.

$$\begin{aligned} a_1 &= 14 \\ r &= \frac{-28}{14} = -2 \\ 3584 &= 14 \times (-2)^{n-1} \\ 256 &= (-2)^{n-1} \\ (-2)^8 &= (-2)^{n-1} \\ n-1 &= 8 \\ n &= 9 \\ S_9 &= \frac{14(1-(-2)^9)}{1-(-2)} \\ &= \frac{14(1+512)}{3} \\ &= \frac{14 \times 513}{3} \\ &= 2394 \end{aligned}$$

12. If the first term of a GP is 7, its common ratio is 3, and the sum of its terms is 847, find the number of terms and the last term of this GP.

Sol.

$$\begin{aligned} a_1 &= 7 \\ r &= 3 \\ S_n &= \frac{7(1-3^n)}{1-3} = 847 \\ 7(1-3^n) &= -1694 \\ 1-3^n &= -242 \\ 3^n &= 243 \\ 3^n &= 3^5 \\ n &= 5 \\ a_5 &= 7 \times 3^4 = 567 \end{aligned}$$

13. Find the sum of the following infinite GP.

(a) $24 + 18 + 13\frac{1}{2} + \ldots$

Sol.

$$\begin{aligned} a_1 &= 24 \\ r &= \frac{18}{24} = \frac{3}{4} \\ S_\infty &= \frac{24}{1-\frac{3}{4}} \\ &= \frac{24}{\frac{1}{4}} \\ &= 96 \end{aligned}$$

(b) $27 - 9 + 3 - 1 + \dots$

Sol.

$$\begin{aligned} a_1 &= 27 \\ r &= \frac{-9}{27} = -\frac{1}{3} \\ S_\infty &= \frac{27}{1+\frac{1}{3}} \\ &= \frac{27}{\frac{4}{3}} \\ &= \frac{81}{4} \end{aligned}$$

(c) $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \dots$

Sol.

$$\begin{aligned} a_1 &= 2 \\ r &= \frac{-\frac{1}{2}}{2} = -\frac{1}{4} \\ S_\infty &= \frac{2}{1+\frac{1}{4}} \\ &= \frac{2}{\frac{5}{4}} \\ &= \frac{8}{5} \end{aligned}$$

14. Given an infinite GP which has a sum of 24 and first term of 30, find the common difference.

Sol.

$$\begin{aligned} a_1 &= 30 \\ S_\infty &= 24 \\ 24 &= \frac{30}{1-r} \\ 24(1-r) &= 30 \\ 24 - 24r &= 30 \\ -24r &= 6 \\ r &= -\frac{1}{4} \end{aligned}$$

15. Convert the following recurrring decimals into fractions.

(a) $0.\overline{45}$

Sol.

$$\begin{aligned} a_1 &= 0.45 \\ r &= 0.01 \\ S_\infty &= \frac{0.45}{1-0.01} \\ &= \frac{0.45}{0.99} \\ &= \frac{45}{99} \\ &= \frac{5}{11} \end{aligned}$$

$$\therefore 0.\overline{45} = \frac{5}{11}$$

(b) $0.\overline{037}$

Sol.

$$\begin{aligned} a_1 &= 0.037 \\ r &= 0.001 \\ S_\infty &= \frac{0.037}{1-0.001} \\ &= \frac{0.037}{0.999} \\ &= \frac{37}{999} \\ &= \frac{1}{27} \end{aligned}$$

$$\therefore 0.\overline{037} = \frac{1}{27}$$

(c) $0.2\overline{18}$

Sol.

$$
\begin{aligned}
a_1 &= 0.018 \\
r &= 0.01 \\
S_\infty &= \frac{0.018}{1-0.01} \\
&= \frac{0.018}{0.99} \\
&= \frac{18}{990} \\
&= \frac{1}{55}
\end{aligned}
$$

$$
\begin{aligned}
\therefore 0.2\overline{18} &= \frac{1}{5} + \frac{1}{55} \\
&= \frac{12}{55}
\end{aligned}
$$

(d) $1.\overline{3}$

Sol.

$$
\begin{aligned}
a_1 &= 0.3 \\
r &= 0.1 \\
S_\infty &= \frac{0.3}{1-0.1} \\
&= \frac{0.3}{0.9} \\
&= \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
\therefore 1.\overline{3} &= 1 + \frac{1}{3} \\
&= \frac{4}{3}
\end{aligned}
$$

16. Three integers are in GP, summing up to 42 while accumulating up to 512, find these three integers.

Sol.

$$
\begin{aligned}
a_1 + a_2 + a_3 &= 42 \\
a_1 a_2 a_3 &= 512 \\
a_2 &= \pm\sqrt{a_1 a_3} \\
a_1 a_3 &= a_2^2 \\
a_2^3 &= 512 \\
a_2 &= \sqrt[3]{512} \\
&= 8 \\
a_1 a_3 &= 64 \\
a_3 &= \frac{64}{a_1} \\
a_1 + 8 + \frac{64}{a_1} &= 42 \\
a_1 + \frac{64}{a_1} &= 34 \\
a_1^2 + 64 &= 34a_1 \\
a_1^2 - 34a_1 + 64 &= 0 \\
(a_1 - 32)(a_1 - 2) &= 0 \\
a_1 &= 32 \; or \; a_1 = 2
\end{aligned}
$$

$$
\therefore These\ three\ integers\ are\ 2,\ 8,\ 32
$$

17. The sum of first 6 term of a GP is 9 times the sum of first 3 terms. Find the common ratio.

Sol.

$$
\begin{aligned}
S_6 &= 9S_3 \\
\frac{a(1-r^6)}{1-r} &= 9 \times \frac{a(1-r^3)}{1-r} \\
a(1-r^6) &= 9a(1-r^3) \\
1 - r^6 &= 9(1-r^3) \\
&= 9 - 9r^3 \\
r^6 - 9r^3 + 8 &= 0 \\
(r^3 - 8)(r^3 - 1) &= 0 \\
r^3 &= 8 \; or \; r^3 = 1 \\
r &= 1 \; (invalid) \\
r &= 2
\end{aligned}
$$

18. Given a GP, its first term is 16, last term is $\frac{1}{2}$ and its sum is $31\frac{1}{2}$, find its common ratio and number of terms.

Sol.

$$
\begin{aligned}
a_1 &= 16 \\
\frac{1}{2} &= 16r^{n-1} \\
\frac{1}{32} &= r^{n-1} \\
&= r^n \times \frac{1}{r} \\
r^n &= \frac{r}{32} \\
\frac{63}{2} &= \frac{16(1-r^n)}{1-r} \\
63(1-r) &= 32(1-r^n) \\
63-63r &= 32-32r^n \\
-31 &= 32r^n - 63r \\
-31 &= r-63r \\
-31 &= -62r \\
r &= \frac{1}{2} \\
(\frac{1}{2})^{n-1} &= \frac{1}{32} \\
&= (\frac{1}{2})^5 \\
n-1 &= 5 \\
n &= 6
\end{aligned}
$$

19. Given a GP, its 3rd term is 6 less than its 2nd term, ant its 2nd term is 9 less than its 1st term. Find the 4th term and the sum of the first 4 terms.

Sol.

$$
\begin{aligned}
Let\ x &= a_2 \\
a_3 &= x-6 \\
a_1 &= x+9 \\
x &= \pm\sqrt{(x-6)(x+9)} \\
x^2 &= x^2+3x-54 \\
3x-54 &= 0 \\
x &= 18 \\
a_2 &= 18 \\
a_1 &= 27 \\
r &= \frac{12}{18} \\
&= \frac{2}{3} \\
a_4 &= 27 \times (\frac{2}{3})^3 \\
&= 8 \\
S_4 &= \frac{27(1-(\frac{16}{3})^4)}{1-\frac{2}{3}} \\
&= \frac{27(1-\frac{8}{81})}{\frac{1}{3}} \\
&= 81 \times \frac{65}{81} \\
&= 65
\end{aligned}
$$

20. GIven an infinite GP, its common ratio is positive and the sum of it is 9. The sum of the first two terms is 5, find the 4th term.

Sol.

$$
\begin{aligned}
S_\infty &= \frac{a}{1-r} = 9 \\
a &= 9(1-r) \\
&= 9-9r \\
S_2 &= \frac{a(1-r^2)}{1-r} = 5 \\
a - ar^2 &= 5-5r \\
9-9r-(9-9r)r^2 &= 5-5r \\
9-9r-9r^2+9r^3 &= 5-5r \\
4-4r-9r^2+9r^3 &= 0 \\
4(1-r)-9r^2(1-r) &= 0 \\
(4-9r^2)(1-r) &= 0 \\
(9r^2-4)(r-1) &= 0 \\
(3r^2+2)(3r^2-2)(r-1) &= 0 \\
r &= 1\ (invalid) \\
r &= -\frac{2}{3}(invalid) \\
r &= \frac{2}{3} \\
a &= 9(1-\frac{2}{3}) \\
&= 3 \\
a_4 &= 3(\frac{2}{3})^3 \\
&= 3 \times \frac{8}{27} \\
&= \frac{8}{9}
\end{aligned}
$$

21. If $x+1, x-2, \frac{1}{2}x$ are the first three terms of an infinite GP, find:

(a) The value of x

Sol.

$$
\begin{aligned}
x-2 &= \pm\sqrt{(x+1)(\frac{1}{2}x)} \\
x^2-4x+4 &= \frac{1}{2}x(x+1) \\
2x^2-8x+8 &= x^2+x \\
x^2-9x+8 &= 0 \\
(x-8)(x-1) &= 0 \\
x &= 8\ or\ x = 1
\end{aligned}
$$

(b) The common ratio

Sol.

$$
\begin{aligned}
&When\ x = 8, \\
r &= \frac{8-2}{8+1} \\
&= \frac{6}{9} \\
&= \frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
&When\ x = 1, \\
r &= \frac{1-2}{1+1} \\
&= -\frac{1}{2}
\end{aligned}
$$

(c) The sum of the GP

Sol.

$$
\begin{aligned}
&When\ x = 8, \\
S_\infty &= \frac{a}{1-r} \\
&= \frac{9}{1-\frac{2}{3}} \\
&= 9 \times 3 \\
&= 27
\end{aligned}
$$

$$
\begin{aligned}
&When\ x = 1, \\
S_\infty &= \frac{a}{1-r} \\
&= \frac{2}{1+\frac{1}{2}} \\
&= 2 \times \frac{2}{3} \\
&= \frac{4}{3}
\end{aligned}
$$

12.4 Simple Summation of Special Series

Sum formula of natural number:

$$\sum_{i=1}^{n} k = \frac{n(n+1)}{2}$$

Sum formula of square of natural number:

$$\sum_{i=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum formula of cube of natural number:

$$\sum_{i=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

12.4.1 Practice 10

1. Find the sum of the following series.

   (a) $\sum_{k=1}^{8} 3k$

   Sol.

   $$\begin{aligned}\sum_{k=1}^{8} 3k &= 3\sum_{k=1}^{8} k\\ &= 3 \times \frac{8(8+1)}{2}\\ &= 3 \times \frac{8 \times 9}{2}\\ &= 3 \times \frac{72}{2}\\ &= 3 \times 36\\ &= 108\end{aligned}$$

   (b) $\sum_{k=1}^{12} k^2$

   Sol.

   $$\begin{aligned}\sum_{k=1}^{12} k^2 &= \frac{12(12+1)(2 \times 12+1)}{6}\\ &= \frac{12 \times 13 \times 25}{6}\\ &= 650\end{aligned}$$

   (c) $\sum_{k=3}^{10}(2k-3)$

   Sol.

   $$\begin{aligned}&\sum_{k=3}^{10}(2k-3)\\ &= 2\sum_{k=3}^{10} k - \sum_{k=3}^{10} 3\\ &= 2\left[\sum_{k=1}^{10} k - \sum_{k=1}^{2} k\right] - (30-6)\\ &= 2\left[\frac{10(10+1)}{2} - \frac{2(2+1)}{2}\right] - 8\\ &= 2(55-3) - 24\\ &= 2 \times 52 - 24\\ &= 104 - 24\\ &= 80\end{aligned}$$

   (d) $\sum_{k=7}^{13} 3k^2$

   Sol.

   $$\begin{aligned}&\sum_{k=7}^{13} 3k^2\\ &= 3\left[\sum_{k=1}^{13} k^2 - \sum_{k=1}^{6} k^2\right]\\ &= 3 \times \left[\frac{13(13+1)(2 \times 13+1)}{6}\right.\\ &\qquad \left. - \frac{6(6+1)(2 \times 6+1)}{6}\right]\\ &= 3 \times \left[\frac{13 \times 14 \times 27}{6} - \frac{6 \times 7 \times 13}{6}\right]\\ &= 3 \times \left[\frac{4914}{6} - \frac{546}{6}\right]\\ &= 3 \times \frac{4368}{6}\\ &= 3 \times 728\\ &= 2184\end{aligned}$$

2. Given that the nth term of a series is n(n+3), find the sum of the first 20 terms of the series.

Sol.

$$\sum_{k=1}^{20} k(k+3)$$
$$= \sum_{k=1}^{20} k^2 + 3k$$
$$= \sum_{k=1}^{20} k^2 + 3\sum_{k=1}^{20} k$$
$$= \frac{20(20+1)(2\times 20+1)}{6} + 3\times\frac{20(20+1)}{2}$$
$$= \frac{20\times 21\times 41}{6} + 3\times\frac{20\times 21}{2}$$
$$= 2870 + 630$$
$$= 3500$$

3. Find the sum of series $1\times3+2\times4+3\times5+\cdots+n(n+2)$.

Sol.

$$\sum_{k=1}^{n} k(k+2)$$
$$= \sum_{k=1}^{n} k^2 + 2k$$
$$= \sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k$$
$$= \frac{n(n+1)(2n+1)}{6} + 2\times\frac{n(n+1)}{2}$$
$$= \frac{n(n+1)(2n+1)}{6} + n(n+1)$$
$$= \frac{n(n+1)(2n+1)+6n(n+1)}{6}$$
$$= \frac{n(n+1)(2n+7)}{6}$$

12.4.2 Exercise 12.4

1. Find the sum of the following series.

(a) $\sum k = 1^8 5k^2$

Sol.

$$\sum_{k=1}^{8} 5k^2 = 5\sum_{k=1}^{8} k^2$$
$$= 5\times\frac{8(8+1)(2\times 8+1)}{6}$$
$$= 5\times\frac{8\times 9\times 17}{6}$$
$$= 5\times\frac{1368}{6}$$
$$= 5\times 204$$
$$= 1020$$

(b) $\sum_{k=1}^{9} k^3$

Sol.

$$\sum_{k=1}^{9} k^3 = \left[\frac{9(9+1)}{2}\right]^2$$
$$= 45^2$$
$$= 2025$$

(c) $\sum_{n=1}^{10}(3n-5)$

Sol.

$$\sum_{n=1}^{10}(3n-5) = 3\sum_{n=1}^{10} n - 5\sum_{n=1}^{10} 1$$
$$= 3\times\frac{10(10+1)}{2} - 5\times 10$$
$$= 3\times\frac{10\times 11}{2} - 5\times 10$$
$$= 3\times 55 - 50$$
$$= 3\times 5 - 50$$
$$= 165 - 50$$
$$= 115$$

(d) $\sum_{k=3}^{6} 2k^3$

Sol.

$$\begin{aligned}\sum_{k=3}^{6} 2k^3 &= 2\sum_{k=3}^{6} k^3\\ &= 2\left(\sum_{k=1}^{6} k^3 - \sum_{k=1}^{2} k^3\right)\\ &= 2\left\{\left[\frac{6(6+1)}{2}\right]^2\right.\\ &\quad \left. - \left[\frac{2(2+1)}{2}\right]^2\right\}\\ &= 2(21^2 - 3^2)\\ &= 2(441-9)\\ &= 2\times 432\\ &= 864\end{aligned}$$

(e) $\sum_{k=6}^{10}(2k^2+3)$

Sol.

$$\begin{aligned}&\sum_{k=6}^{10}(2k^2+3)\\ &= 2\sum_{k=6}^{10} k^2 + 3\sum_{k=6}^{10} 1\\ &= 2\left(\sum_{k=1}^{10} k^2 - \sum_{k=1}^{5} k^2\right)\\ &\quad + 3\times(10-5)\\ &= 2\times\left[\frac{10\times 11\times 21}{6} - \frac{5\times 6\times 11}{6}\right]\\ &\quad + 3\times 5\\ &= 2\times\left[\frac{2310}{6} - \frac{330}{6}\right] + 3\times 5\\ &= 2\times\frac{1980}{6} + 3\times 5\\ &= 2\times 330 + 3\times 5\\ &= 660 + 15\\ &= 675\end{aligned}$$

(f) $\sum_{n=11}^{15}(n^2+2n)$

Sol.

$$\begin{aligned}&\sum_{n=11}^{15}(n^2+2n)\\ &= \sum_{n=11}^{15} n^2 + 2\sum_{n=11}^{15} n\\ &= \left[\sum_{n=1}^{15} n^2 - \sum_{n=1}^{10} n^2\right]\\ &\quad + 2\left[\sum_{n=1}^{1} 5n - \sum_{n=1}^{10} n\right]\\ &= \left[\frac{15\times 16\times 31}{6} - \frac{10\times 11\times 21}{6}\right]\\ &\quad + 2\left[\frac{15\times 16}{2} - \frac{10\times 11}{2}\right]\\ &= 985\end{aligned}$$

(g) $\sum_{n=2}^{6} n(n^2-n+1)$

Sol.

$$\begin{aligned}&\sum_{n=2}^{6} n(n^2-n+1)\\ &= \sum_{n=2}^{6} n^3 - \sum_{n=2}^{6} n^2 + \sum_{n=2}^{6} n\\ &= \left[\sum_{n=1}^{6} n^3 - \sum_{n=1}^{1} n^3\right] - \left[\sum_{n=1}^{6} n^2 - \sum_{n=1}^{1} n^2\right]\\ &\quad + \left[\sum_{n=1}^{6} n - \sum_{n=1}^{1} n\right]\\ &= \left[\left(\frac{6\times 7}{2}\right)^2 - \left(\frac{1\times 2}{2}\right)^2\right]\\ &\quad - \left(\frac{6\times 7\times 13}{6} - \frac{1\times 2\times 3}{6}\right)\\ &\quad + \left(\frac{6\times 7}{2} - \frac{1\times 2}{2}\right)\\ &= 21^2 - 1^2 - (7\times 13 - 1) + (3\times 7 - 1)\\ &= 440 - 90 + 20\\ &= 370\end{aligned}$$

2. Fiven that the nth term of a series is $3n^2 + n$, find the sum of the first 10 terms of the series.

Sol.

$$\begin{aligned}\sum_{n=1}^{10} 3n^2+n &= 3\sum_{n=1}^{10} n^2+\sum_{n=1}^{10} n\\ &= 3\left(\frac{10\times 11\times 21}{6}\right)+\left(\frac{10\times 11}{2}\right)\\ &= 3\times\frac{2310}{6}+\frac{110}{2}\\ &= 3\times 385+55\\ &= 1210\end{aligned}$$

3. Find the sum of first nth term of series $1\times 3+2\times 7+3\times 11+\ldots$

Sol.

$$\begin{aligned}&\sum_{n=1}^{n} n\times(4n-1)\\ &= 4\sum_{n=1}^{n} n^2-\sum_{n=1}^{n} n\\ &= 4\left(\frac{n(n+1)(2n+1)}{6}\right)-\left(\frac{n(n+1)}{2}\right)\\ &= \frac{4n(n+1)(2n+1)-3n(n+1)}{6}\\ &= \frac{n(n+1)(8n+1)}{6}\end{aligned}$$

4. Find the sum fo the series $1^2+3^2+5^2+\ldots+15^2$

Sol.

$$\begin{aligned}\sum_{n=1}^{8}(2n-1)^2 &= \sum_{n=1}^{8}(4n^2-4n+1)\\ &= 4\sum_{n=1}^{8} n^2-4\sum_{n=1}^{8} n+\sum_{n=1}^{8} 1\\ &= 4\left(\frac{8\times 9\times 17}{6}\right)-4\left(\frac{8\times 9}{2}\right)+8\\ &= 4\times 204-4\times 36+8\\ &= 816-144+8\\ &= 680\end{aligned}$$

12.5 Revision Exerise 12

1. Express the following series in form of $\sum$.

(a) $\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\ldots+\frac{49}{50}$

Sol.

$$\begin{aligned}a_1 &= \frac{2\times 1-1}{2\times 1}\\ a_2 &= \frac{2\times 2-1}{2\times 2}\\ a_3 &= \frac{2\times 3-1}{2\times 3}\\ &\vdots\\ a_{25} &= \frac{2\times 25-1}{2\times 25}\\ &\therefore \frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\ldots+\frac{49}{50}=\sum_{n=1}^{25}\frac{2n-1}{2n}\end{aligned}$$

(b) $6-7+8-9+\ldots$

Sol.

$$\begin{aligned}a_1 &= (-1)^6\times 6\\ a_2 &= (-1)^7\times 7\\ a_3 &= (-1)^8\times 8\\ &\vdots\\ a_n &= (-1)^n n \therefore \quad 6-7+8-9+\ldots=\sum_{n=1}^{\infty}(-1)^n n\end{aligned}$$

(c) $2\times 5+3\times 7+4\times 9+\ldots+15\times 31$

Sol.

$$\begin{aligned}a_1 &= (1+1)(2\times 1+3)\\ a_2 &= (2+1)(2\times 2+3)\\ a_3 &= (3+1)(2\times 3+3)\\ &\vdots\\ a_{14} &= (14+1)(2\times 14+3)\\ &\therefore 2\times 5+3\times 7+4\times 9+\ldots+15\times 31\\ &= \sum_{n=1}^{14}(n+1)(2n+3)\end{aligned}$$

2. Given a general formula $a_n=\frac{3^n}{2n-3}$, state the first 5 terms of the sequence.

Sol.

$$\begin{aligned}
a_1 &= \frac{3^1}{2\times 1-3} = -3\\
a_2 &= \frac{3^2}{2\times 2-3} = 9\\
a_3 &= \frac{3^3}{2\times 3-3} = 9\\
a_4 &= \frac{3^4}{2\times 4-3} = \frac{81}{5}\\
a_5 &= \frac{3^5}{2\times 5-3} = \frac{243}{7}
\end{aligned}$$

3. Express the series $\sum_{k=1}^{1} 0(2k^2-3)$

   Sol.

$$\begin{aligned}
&\sum_{k=1}^{10}(2k^2-3)\\
&= (2\times 1^2-3)+(2\times 2^2-3)+(2\times 3^2-3)\\
&\quad +(2\times 4^2-3)+(2\times 5^2-3)+(2\times 6^2-3)\\
&\quad +(2\times 7^2-3)+(2\times 8^2-3)+(2\times 9^2-3)\\
&\quad +(2\times 10^2-3)\\
&= -1+5+15+29+47+69+95+125\\
&\quad +159+197
\end{aligned}$$

4. State the first term, last term and the number of terms of theh series $\sum_{k=3}^{7}(3^k-2^k-k)$

   Sol.

$$\begin{aligned}
a_3 &= 3^3-2^3-3 = 27-8-3 = 16\\
a_7 &= 3^7-2^7-7 = 2187-128-7 = 2052\\
n &= 5
\end{aligned}$$

5. Find the number of terms of the AP $-4-2\frac{3}{4}-112-\frac{1}{4}+\ldots+16$

   Sol.

$$\begin{aligned}
a &= -4\\
d &= \frac{5}{4}\\
16 &= -4+(n-1)\frac{5}{4}\\
20 &= \frac{5}{4}(n-1)\\
5n-5 &= 80\\
5n &= 85\\
n &= 17
\end{aligned}$$

6. If x+1, 2x+1, x-3 are the first 3 terms of AP, find:

   (a) The value of x

   Sol.

$$\begin{aligned}
2x+1 &= \frac{x+1+x-3}{2}\\
4x+2 &= 2x-2\\
2x &= -4\\
x &= -2
\end{aligned}$$

   (b) Sum from the 10th term to the 20th term

   Sol.

$$\begin{aligned}
a_1 &= -1\\
a_2 &= -3\\
r &= -2\\
S &= S_{20}-S_9\\
&= \frac{20}{2}(-2+(20-1)(-2))\\
&\quad -\frac{9}{2}(-2+(9-1)(-2))\\
&= 10\times(-40)-9\times(-9)\\
&= -400+81\\
&= -319
\end{aligned}$$

7. Find 4 numbers between 28 and -12 such that these 6 numbers form an AP.

Sol.

$$
\begin{aligned}
a_1 &= 28 \\
a_n &= -12 \\
n &= 6 \\
-12 &= 28 + 5d \\
5d &= 40 \\
d &= 8
\end{aligned}
$$

$$\therefore These\ 4\ numbers\ are\ -4, 4, 12, 20$$

8. Find the sum of the following AP.

   (a) $7 + 11 + 15 + \dots$ up to the 10th term

   Sol.

$$
\begin{aligned}
a_1 &= 7 \\
d &= 4 \\
n &= 10 \\
S_10 &= \frac{10}{2}(2 \times 7 + (10 - 1)4) \\
&= 5(14 + 36) \\
&= 250
\end{aligned}
$$

   (b) $20 + 18\frac{1}{2} + 17 + \dots$ up to the 16tm term

   Sol.

$$
\begin{aligned}
a_1 &= 20 \\
d &= -\frac{3}{2} \\
n &= 16 \\
S_{16} &= \frac{16}{2}(2 \times 20 + (16 - 1)(-\frac{3}{2})) \\
&= 8(40 - \frac{45}{2}) \\
&= 8 \times \frac{35}{2} \\
&= 140
\end{aligned}
$$

   (c) $2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} + \dots + 13\sqrt{2}$

   Sol.

$$
\begin{aligned}
a_1 &= 2\sqrt{2} \\
d &= \sqrt{2} \\
n &= 12 \\
S_{12} &= \frac{12}{2}(2 \times 2\sqrt{2} + (12 - 1)\sqrt{2}) \\
&= 6(4\sqrt{2} + 11\sqrt{2}) \\
&= 6 \times 15\sqrt{2} \\
&= 90\sqrt{2}
\end{aligned}
$$

9. Given an AP which the sum of the first n terms $S_n = n(1 + 2n)$, find:

   (a) First term

   Sol.

$$
\begin{aligned}
\frac{n}{2}(2a + (n - 1)d) &= n(1 + 2n) \\
n(2a + (n - 1d)) &= 2n(1 + 2n) \\
2an + dn^2 - dn &= 2n - 4n^2 \\
(2a - d)n + dn^2 &= 2n - 4n^2
\end{aligned}
$$

$$
\begin{aligned}
&Comparing\ both\ sides, \\
&a = 3 \\
&d = 4
\end{aligned}
$$

   (b) Common Difference

   Sol.

$$
\begin{aligned}
&According\ to\ the\ sol.\ of\ (a), \\
&d = 4
\end{aligned}
$$

   (c) Sum of the first 20 terms.

   Sol.

$$
\begin{aligned}
&According\ to\ the\ sol.\ of\ (a), \\
&a = 3 \\
&d = 4 \\
&n = 20 \\
S_{20} &= \frac{20}{2}(2 \times 3 + (20 - 1)4) \\
&= 10(6 + 76) \\
&= 10 \times 82 \\
&= 820
\end{aligned}
$$

10. Given an AP $33 + 27 + 21 + \dots$

(a) If the first sum of the first n terms is 105, find the value of n.

Sol.

$$\begin{aligned} a_1 &= 33 \\ d &= -6 \\ 105 &= \frac{n}{2}(2 \times 33 + (n-1) \times (-6)) \\ 210 &= n(66 - (n-1)6) \\ 35 &= 11n - n^2 + n \\ n^2 - 12n + 35 &= 0 \\ (n-7)(n-5) &= 0 \\ n &= 7 \ or \ n = 5 \end{aligned}$$

(b) If the sum of the first n terms is negative value, find the minimum value of n.

Sol.

$$\begin{aligned} a_1 &= 33 \\ d &= -6 \\ \frac{n}{2}(2 \times 33 + (n-1) \times (-6)) &< 0 \\ n(66 - 6n + 6) &< 0 \\ 12n - n^2 &< 0 \\ n(12 - n) &< 0 \\ n &> 12 \end{aligned}$$

$$\therefore The\ minimum\ value\ of\ n\ is\ 13$$

11. Find the sum of the numbers between 150 and 300 that are multiple of both 5 and 3.

Sol.

$$\begin{aligned} a_1 &= 165 \\ a_n &= 285 \\ d &= 15 \\ 285 &= 165 + (n-1) \times 15 \\ 8 &= n - 1 \\ n &= 9 \end{aligned}$$

$$\begin{aligned} S_9 &= \frac{9}{2}(2 \times 165 + (9-1) \times 15) \\ &= \frac{9}{2} \times 450 \\ &= 2025 \end{aligned}$$

12. Find the sum of all the numbers between 100 and 200 that can be divided by 2 or 3.

Sol.

$$a_1 = 102$$
$$a_n = 198$$

When $d = 2$,

$$198 = 102 + (n-1) \times 2$$
$$48 = n - 1$$
$$n = 49$$
$$S_49 = \frac{49}{2}(2 \times 102 + (49-1) \times 2)$$
$$= \frac{49}{2} \times (204 + 96)$$
$$= 7350$$

When $d = 3$,

$$198 = 102 + (n-1) \times 3$$
$$32 = n - 1$$
$$n = 33$$
$$S_33 = \frac{33}{2}(2 \times 102 + (33-1) \times 3)$$
$$= \frac{33}{2} \times (204 + 96)$$
$$= 4950$$

When $d = 6$,

$$198 = 102 + (n-1) \times 6$$
$$16 = n - 1$$
$$n = 17$$
$$S_17 = \frac{17}{2}(2 \times 102 + (17-1) \times 6)$$
$$= \frac{17}{2} \times (204 + 96)$$
$$= 2550$$

$$\therefore S = 7350 + 4950 - 2550$$
$$= 9750$$

13. Find the sum of the numbers between 50 and 100 that cannot be divided by 5.

Sol.

When $d = 1$,

$$a_1 = 51$$
$$a_n = 99$$
$$99 = 51 + (n-1) \times 1$$
$$48 = n - 1$$
$$n = 49$$
$$S_{49} = \frac{49}{2}(2 \times 51 + (49-1) \times 1)$$
$$= \frac{49}{2} \times (102 + 48)$$
$$= 3675$$

When $d = 5$,

$$a_1 = 55$$
$$a_n = 95$$
$$95 = 55 + (n-1) \times 5$$
$$8 = n - 1$$
$$n = 9$$
$$S_9 = \frac{9}{2}(2 \times 55 + (9-1) \times 5)$$
$$= \frac{9}{2} \times (110 + 40)$$
$$= 675$$

$$\therefore S = 3675 - 675$$
$$= 3000$$

14. Which term is the first negative term of the AP $20 + 16\frac{1}{4} + 12\frac{1}{2} + \ldots$?

Sol.

$$a_1 = 20$$
$$d = -\frac{15}{4}$$
$$a_n = 20 - (n-1) \times \frac{15}{4} < 0$$
$$80 - 15(n-1) < 0$$
$$16 - 3n + 3 < 0$$
$$3n > 19$$
$$n > 6\frac{1}{3}$$

$$\therefore The\ first\ negative\ term\ is\ 7$$

15. Three numbers are in AP, thier sum is 15 while the

sum of the square of these numbers is 83. Find this three numbers.

Sol.

$$\begin{aligned}
a_1 + a_2 + a_3 &= 15 \\
a_1^2 + a_2^2 + a_3^2 &= 83 \\
a_2 - a_1 &= a_3 - a_2 \\
a_1 + a_3 &= 2a_2 \\
3a_2 &= 15 \\
a_2 &= 5 \\
a_3 &= 10 - a_1 \\
a_1^2 + a_3^2 &= 83 - 25 \\
&= 58 \\
a_1^2 + (10 - a_1)^2 &= 58 \\
a_1^2 + 100 - 20a_1 + a_1^2 &= 58 \\
2a_1^2 - 20a_1 + 100 &= 58 \\
2a_1^2 - 20a_1 + 42 &= 0 \\
a_1^2 - 10a_1 + 21 &= 0 \\
(a_1 - 7)(a_1 - 3) &= 0 \\
a_1 &= 7 \text{ or } a_1 = 3
\end{aligned}$$

$$\therefore \textit{The three numbers are } 7, 5, 3$$

16. Find the sum of the series $18^2 - 17^2 + 16^2 - 15^2 + 14^2 - 13^2 + \cdots + 2^2 - 1^2$

Sol.

$$\begin{aligned}
& 18^2 - 17^2 + 16^2 - 15^2 + \cdots + 2^2 - 1^2 \\
&= (18^2 - 17^2) + (16^2 - 15^2) + \cdots + (2^2 - 1^2) \\
&= ((2\times 9)^2 - (2\times 9 - 1)^2) + ((2\times 8)^2 - (2\times 8 - 1)^2) \\
&\quad + \cdots + ((2\times 1)^2 - (2\times 1 - 1)^2) \\
&= \sum_{n=1}^{9} \left[(2n)^2 - (2n-1)^2\right] \\
&= \sum_{n=1}^{9} (4n - 1) \\
&= 4\sum_{n=1}^{9} n - \sum_{n=1}^{9} 1 \\
&= 4 \times \frac{9\times 10}{2} - 9 \\
&= 180 - 9 \\
&= 171
\end{aligned}$$

17. State the general formula of the series $20, -10, 5, -2\frac{1}{2}, \cdots$

Sol.

$$\begin{aligned}
a_1 &= 20 \\
r &= -\frac{1}{2} \\
a_n &= 20(-\frac{1}{2})^{n-1}
\end{aligned}$$

18. Given three integers x-3, x+1, 4x-2 that are in GP. If the sum of this GP is S, common ratio is r, find the value of S+r.

Sol.

$$\begin{aligned}
x + 1 &= \pm\sqrt{(x-3)(4x-2)} \\
x^2 + 2x + 1 &= 4x^2 - 14x + 6 \\
3x^2 - 16^x + 5 &= 0 \\
(3x - 1)(x - 5) &= 0 \\
x &= 5 \text{ or } x = \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
a_1 &= x - 3 = 5 - 3 = 2 \\
a_2 &= x + 1 = 5 + 1 = 6 \\
a_3 &= 4x - 2 = 4(5) - 2 = 18
\end{aligned}$$

$$\begin{aligned}
S &= a_1 + a_2 + a_3 \\
&= 2 + 6 + 18 \\
&= 26
\end{aligned}$$

$$r = \frac{a_3}{a_2} = \frac{18}{6} = 3$$

$$\begin{aligned}
\therefore S + r &= 26 + 3 \\
&= 29
\end{aligned}$$

19. Find the geometric mean of $\frac{1}{3}$ and $\frac{1}{5}$

Sol.

$$G = \pm\sqrt{\frac{1}{3}\times\frac{1}{5}}$$
$$= \pm\sqrt{\frac{1}{15}}$$
$$= \pm\frac{1}{\sqrt{15}}$$
$$= \pm\frac{\sqrt{15}}{15}$$

20. Find 5 numbers between $-\frac{1}{4}$ and $-\frac{1}{256}$ such that these 7 numbers form a GP.

Sol.

$$a_1 = -\frac{1}{4}$$
$$n = 7$$
$$-\frac{1}{256} = -\frac{1}{4}r^6$$
$$\frac{1}{64} = r^6$$
$$\left(\pm\frac{1}{2}\right)^6 = r^6$$
$$r = \pm\frac{1}{2}$$

When $r = \frac{1}{2}$,
These 5 numbers are
$\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$

When $r = -\frac{1}{2}$,
These 5 numbers are
$\frac{1}{8}, -\frac{1}{16}, \frac{1}{32}, -\frac{1}{64}, \frac{1}{128}$

21. Find the sum of the series $\sum_{n=5}^{1} 5n^2(3n+1)$

Sol.

$$\sum_{n=5}^{15} n^2(3n+1) = \sum_{n=5}^{15} n^3 + \sum_{n=5}^{15} 3n^2$$
$$= 3\sum_{n=5}^{15} n^3 + \sum_{n=5}^{15} n^2$$
$$= 3\left[\sum_{n=1}^{15} n^3 - \sum_{n=1}^{4} n^3\right]$$
$$+ \left[\sum_{n=1}^{15} n^2 - \sum_{n=1}^{4} n^2\right]$$
$$= 3\left[\left(\frac{15\times 16}{2}\right)^2 - \left(\frac{4\times 5}{2}\right)^2\right]$$
$$+ \left[\frac{15\times 16\times 31}{6} - \frac{4\times 5\times 9}{6}\right]$$
$$= 3\left[(15\times 8)^2 - (2\times 5)^2\right]$$
$$+ 1240 - 30$$
$$= 3(14400 - 100) + 1210$$
$$= 42900 + 1210$$
$$= 44110$$

22. Find the sum of the series $5^2 + 7^2 + 9^2 + \cdots + 25^2$

Sol.

$$\sum_{n=1}^{11}(2n+3)^2$$
$$= \sum_{n=1}^{11} 4n^2 + 12n + 9$$
$$= 4\sum_{n=1}^{11} n^2 + 12\sum_{n=1}^{11} n + 11$$
$$= 4\left[\frac{11\times 12\times 23}{6}\right] + 12\left[\frac{11\times 12}{2}\right] + 99$$
$$= 2024 + 792 + 99$$
$$= 2915$$

23. Find the sum of the series $2\times 3 + 3\times 12 + 4\times 27 + \cdots + (n+1)\times 3n^2$

Sol.

$$\sum_{n=1}^{n}(n+1)3n^2$$
$$= \sum_{n=1}^{n} 3n^3 + \sum_{n=1}^{n} 3n^2$$
$$= 3\left[\sum_{n=1}^{n} n^3 + \sum_{n=1}^{n} n^2\right]$$
$$= 3\left[\left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6}\right]$$
$$= 3\left[\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}\right]$$
$$= 3\left[\frac{3n^2(n+1)^2 + 2n(n+1)(2n+1)}{12}\right]$$
$$= \frac{n(n+1)\left[3n^2+3n+4n+2\right]}{4}$$
$$= \frac{n(n+1)\left[3n^2+7n+2\right]}{4}$$
$$= \frac{n(n+1)(n+2)(3n+1)}{4}$$

Chapter 13

System of Equations

13.1 System of Equations with Two Variables

13.1.1 Practice 1

Solve the following system of equations.

1.

$$\begin{cases} 2x - 3y &= 11 \\ xy &= -5 \end{cases}$$

Sol.

$$\begin{cases} 2x - 3y = 11 & (1) \\ xy = -5 & (2) \end{cases}$$

$$(2) \Rightarrow y = -\frac{5}{x} \quad (3)$$

$$\begin{aligned} \text{Sub (3) into (1)} \Rightarrow 2x - \frac{15}{x} &= 11 \\ 2x^2 - 15 &= 11x \\ 2x^2 - 11x - 15 &= 0 \\ (2x-5)(x-3) &= 0 \\ x = 3 \text{ or } x &= \frac{5}{2} \end{aligned}$$

$$\begin{aligned} \text{Sub x} = 3 \text{ into (2)} &\Rightarrow y = -\frac{5}{3} \\ \text{Sub x} = \frac{5}{2} \text{ into (2)} &\Rightarrow y = -\frac{5}{\frac{5}{2}} \\ &\Rightarrow y = -\frac{5}{5} \\ &\Rightarrow y = -1 \end{aligned}$$

$$\therefore \begin{cases} x = 3 \\ y = -\frac{5}{3} \end{cases} or \begin{cases} x = \frac{5}{2} \\ y = -1 \end{cases}$$

2.

$$\begin{cases} 3x + y &= 5 \\ x^2 - 2xy &= 8 \end{cases}$$

Sol.

$$\begin{cases} 3x + y = 5 & (1) \\ x^2 - 2xy = 8 & (2) \end{cases}$$

$$3(1) \Rightarrow y = 5 - 3x \quad (3)$$

$$\begin{aligned} \text{Sub (3) into (2)} \Rightarrow x^2 - 2x(5-3x) &= 8 \\ x^2 - 10x + 6x^2 &= 8 \\ 7x^2 - 10x + 8 &= 0 \\ (7x+4)(x-2) &= 0 \\ x = -\frac{4}{7} &\text{ or } x = 2 \end{aligned}$$

$$\begin{aligned} \text{Sub } x = -\frac{4}{7} \text{ into (1)} &\Rightarrow y = 5 - 3\left(-\frac{4}{7}\right) \\ &\Rightarrow y = \frac{47}{7} \\ \text{Sub } x = 2 \text{ into (1)} &\Rightarrow y = -1 \end{aligned}$$

$$\therefore \begin{cases} x = -\frac{4}{7} \\ y = \frac{47}{7} \end{cases} or \begin{cases} x = 2 \\ y = -1 \end{cases}$$

13.1.2 Exercise 13.1

Solve the following system of equations.

1.

$$\begin{cases} x - y & = 1 \\ xy & = 6 \end{cases}$$

Sol.

$$\begin{cases} x - y = 1 & (1) \\ xy = 6 & (2) \end{cases}$$

$$\begin{aligned} (1) \Rightarrow y &= x - 1 && (3) \\ \text{Sub (3) into (2)} \Rightarrow x(x-1) &= 6 \\ x^2 - x &= 6 \\ x^2 - x - 6 &= 0 \\ (x+2)(x-3) &= 0 \\ x = -2 \ &or\ x = 3 \\ \text{Sub } x = -2 \text{ into (1)} \Rightarrow y &= -2 - 1 \\ \Rightarrow y &= -3 \\ \text{Sub } x = 3 \text{ into (1)} \Rightarrow y &= 3 - 1 \\ \Rightarrow y &= 2 \end{aligned}$$

$$\therefore \begin{cases} x = -2 \\ y = -3 \end{cases} or \begin{cases} x = 3 \\ y = 2 \end{cases}$$

2.

$$\begin{cases} 3x - y = 4 \\ xy = 4 \end{cases}$$

Sol.

$$\begin{cases} 3x - y = 4 & (1) \\ xy = 4 & (2) \end{cases}$$

$$\begin{aligned} (1) \Rightarrow y &= 3x - 4 && (3) \\ \text{Sub (3) into (2)} \Rightarrow x(3x-4) &= 4 \\ 3x^2 - 4x &= 4 \\ 3x^2 - 4x - 4 &= 0 \\ (3x+2)(x-2) &= 0 \\ x = -\frac{2}{3} \ &or\ x = 2 \\ \text{Sub } x = -\frac{2}{3} \text{ into (1)} \Rightarrow y &= 3\left(-\frac{2}{3}\right) - 4 \\ \Rightarrow y &= -6 \\ \text{Sub } x = 2 \text{ into (1)} \Rightarrow y &= 3(2) - 4 \\ \Rightarrow y &= 2 \end{aligned}$$

$$\therefore \begin{cases} x = -\frac{2}{3} \\ y = -6 \end{cases} or \begin{cases} x = 2 \\ y = 2 \end{cases}$$

3.

$$\begin{cases} 3x + 4y = -39 \\ xy = 30 \end{cases}$$

Sol.

$$\begin{cases} 3x + 4y = -39 & (1) \\ xy = 30 & (2) \end{cases}$$

$$\begin{aligned} (2) \Rightarrow y &= \frac{30}{x} && (3) \\ \text{Sub (3) into (1)} \Rightarrow 3x + 4\frac{30}{x} &= -39 \\ 3x^2 + 120 &= -39x \\ 3x^2 + 39x + 120 &= 0 \\ x^2 + 13x + 40 &= 0 \\ (x+5)(x+8) &= 0 \\ x = -5 \ &or\ x = -8 \\ \text{Sub } x = -5 \text{ into (1)} \Rightarrow y &= \frac{30}{-5} - 39 \\ \Rightarrow y &= -6 \\ \text{Sub } x = -8 \text{ into (1)} \Rightarrow y &= \frac{30}{-8} - 39 \\ \Rightarrow y &= -\frac{15}{4} \end{aligned}$$

$$\therefore \begin{cases} x = -5 \\ y = -6 \end{cases} or \begin{cases} x = -8 \\ y = -\frac{15}{4} \end{cases}$$

4.

$$\begin{cases} y = 2x + 3 \\ y = x^2 - 2x + 1 \end{cases}$$

Sol.

$$\begin{cases} y = 2x + 3 & (1) \\ y = x^2 & (2) \end{cases}$$

$$\begin{aligned} (1) = (2) &\Rightarrow 2x + 3 = x^2 \\ &\Rightarrow x^2 - 2x - 3 = 0 \\ &\Rightarrow (x+1)(x-3) = 0 \\ &\Rightarrow x = -1 \text{ or } x = 3 \\ \text{Sub } x = -1 \text{ into } (1) &\Rightarrow y = 2(-1) + 3 \\ &\Rightarrow y = 1 \\ \text{Sub } x = 3 \text{ into } (1) &\Rightarrow y = 2(3) + 3 \\ &\Rightarrow y = 9 \end{aligned}$$

$$\therefore \begin{cases} x = -1 \\ y = 1 \end{cases} \text{ or } \begin{cases} x = 3 \\ y = 9 \end{cases}$$

5.

$$\begin{cases} x - y = 1 \\ x^2 + y^2 = 25 \end{cases}$$

Sol.

$$\begin{cases} x - y = 1 & (1) \\ x^2 + y^2 = 25 & (2) \end{cases}$$

$$\begin{aligned} (1) &\Rightarrow x = y + 1 \qquad (3) \\ \text{Sub (3) into (2)} &\Rightarrow (y+1)^2 + y^2 = 25 \\ &\Rightarrow y^2 + 2y + 1 + y^2 = 25 \\ &\Rightarrow 2y^2 + 2y = 24 \\ &\Rightarrow y^2 + y = 12 \\ &\Rightarrow y^2 + y - 12 = 0 \\ &\Rightarrow (y+4)(y-3) = 0 \\ &\Rightarrow y = -4 \text{ or } y = 3 \\ \text{Sub } y = -4 \text{ into } (1) &\Rightarrow x = -4 + 1 \\ &\Rightarrow x = -3 \\ \text{Sub } y = 3 \text{ into } (1) &\Rightarrow x = 3 + 1 \\ &\Rightarrow x = 4 \end{aligned}$$

$$\therefore \begin{cases} x = -3 \\ y = -4 \end{cases} \text{ or } \begin{cases} x = 4 \\ y = 3 \end{cases}$$

6.

$$\begin{cases} 5x - y = 3 \\ y^2 - 6x^2 = 25 \end{cases}$$

Sol.

$$\begin{cases} 5x - y = 3 & (1) \\ y^2 - 6x^2 = 25 & (2) \end{cases}$$

$$\begin{aligned} (1) &\Rightarrow y = 5x - 3 \qquad (3) \\ \text{Sub (3) into (2)} &\Rightarrow (5x-3)^2 - 6x^2 = 25 \\ &\Rightarrow 25x^2 - 30x + 9 \\ &\quad - 6x^2 = 25 \\ &\Rightarrow 19x^2 - 30x + 16 = 0 \\ &\Rightarrow (19x+8)(x-2) = 0 \\ &\Rightarrow x = -\frac{8}{19} \text{ or } x = 2 \\ \text{Sub } x = -\frac{8}{19} \text{ into } (1) &\Rightarrow y = 5(-\frac{8}{19}) - 3 \\ &\Rightarrow y = -\frac{97}{19} \\ \text{Sub } x = 2 \text{ into } (1) &\Rightarrow y = 7 \end{aligned}$$

$$\therefore \begin{cases} x = -\frac{8}{19} \\ y = -\frac{97}{19} \end{cases} \text{ or } \begin{cases} x = 2 \\ y = 7 \end{cases}$$

7.

$$\begin{cases} x + y = 3 \\ (x+2)(y+3) = 12 \end{cases}$$

Sol.

$$\begin{cases} x + y = 3 & (1) \\ (x+2)(y+3) = 12 & (2) \end{cases}$$

$$\begin{aligned}
(1) &\Rightarrow x = 3 - y & (3) \\
\text{Sub (3) into (2)} &\Rightarrow (3 - y + 2)(y+3) = 12 \\
&\Rightarrow (5-y)(y+3) = 12 \\
&\Rightarrow 5y + 15 - y^2 - 3y = 12 \\
&\Rightarrow 2y - y^2 = -3 \\
&\Rightarrow y^2 - 2y - 3 = 0 \\
&\Rightarrow (y+1)(y-3) = 0 \\
&\Rightarrow y = -1 \text{ or } y = 3 \\
\text{Sub } y = -1 \text{ into (1)} &\Rightarrow x = 4 \\
\text{Sub } y = 3 \text{ into (1)} &\Rightarrow x = 0
\end{aligned}$$

$$\therefore \begin{cases} x = 4 \\ y = -1 \end{cases} or \begin{cases} x = 0 \\ y = 3 \end{cases}$$

8.

$$\begin{cases} 5x - 6y = -1 \\ 25x^2 + 36y^2 = 61 \end{cases}$$

Sol.

$$\begin{cases} 5x - 6y = -1 & (1) \\ 25x^2 + 36y^2 = 61 & (2) \end{cases}$$

$$\begin{aligned}
(1) &\Rightarrow y = \frac{5x+1}{6} & (3) \\
\text{Sub (3) into (2)} &\Rightarrow 25x^2 + 36\left(\frac{5x+1}{6}\right)^2 \\
&= 61 \\
&\Rightarrow 25x^2 + 36\left(\frac{5x+1}{6}\right)^2 \\
&+ 36 = 61 \\
&\Rightarrow 25x^2 + 25x^2 + 10x \\
&+ 1 = 61 \\
&\Rightarrow 50x^2 + 10x = 60 \\
&\Rightarrow 5x^2 + x - 6 = 0 \\
&\Rightarrow (5x+6)(x-1) = 0 \\
&\Rightarrow x = -\frac{6}{5} \text{ or } x = 1 \\
\text{Sub } x = -\frac{6}{5} \text{ into (1)} &\Rightarrow y = \frac{5(-\frac{6}{5}) + 1}{6} \\
&\Rightarrow y = -\frac{5}{6} \\
\text{Sub } x = 1 \text{ into (1)} &\Rightarrow y = \frac{5(1) + 1}{6} \\
&\Rightarrow y = \frac{6}{6} \\
&\Rightarrow y = 1
\end{aligned}$$

$$\therefore \begin{cases} x = -\frac{6}{5} \\ y = -\frac{5}{6} \end{cases} or \begin{cases} x = 1 \\ y = 1 \end{cases}$$

9.

$$\begin{cases} x + 4y = 5 \\ 2x^2 + 21xy + 27y^2 = 0 \end{cases}$$

Sol.

$$\begin{cases} x + 4y = 5 & (1) \\ 2x^2 + 21xy + 27y^2 = 0 & (2) \end{cases}$$

$$(1) \Rightarrow x = 5 - 4y \quad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow 2\,(5-4y)^2 + 21\,(5-4y)\,y + 27y^2 = 0$$

$$\Rightarrow 2\left(25 - 40y + 16y^2\right) + 105y - 84y^2 + 27y^2 = 0$$

$$\Rightarrow 50 - 80y + 32y^2 + 105y - 57y^2 = 0$$

$$\Rightarrow 25y^2 - 25y - 50 = 0$$

$$\Rightarrow y^2 - y - 2$$

$$\Rightarrow (y+1)(y-2) = 0$$

$$\Rightarrow y = -1 \text{ or } y = 2$$

$$\text{Sub } y = -1 \text{ into (1)} \Rightarrow x = 5 - 4(-1) = 9$$

$$\text{Sub } y = 2 \text{ into (1)} \Rightarrow x = 5 - 4(2) = -3$$

$$\therefore \begin{cases} x = 9 \\ y = -1 \end{cases} \text{ or } \begin{cases} x = -3 \\ y = 2 \end{cases}$$

10.

$$\begin{cases} \frac{x}{3} - \frac{y}{10} = \frac{5}{6} \\ x(y-2) = 2y + 3 \end{cases}$$

Sol.

$$\begin{cases} \dfrac{x}{3} - \dfrac{y}{10} = \dfrac{5}{6} & (1) \\ x(y-2) = 2y + 3 & (2) \end{cases}$$

$$(1) \Rightarrow 10x - 3y = 25 \quad (3)$$

$$(2) \Rightarrow x = \frac{2y+3}{y-2} \quad (4)$$

$$\text{Sub (4) int 3} \Rightarrow 10\left(\frac{2y+3}{y-2}\right) - 3y = 25$$

$$\Rightarrow 10(2y+3) - 3y(y-2) = 25(y-2)$$

$$\Rightarrow 20y + 30 - 3y^2 + 6y = 25y - 50$$

$$\Rightarrow 3y^2 - y - 80 = 0$$

$$\Rightarrow (y+5)(3y-16) = 0$$

$$\Rightarrow y = -5 \text{ or } y = \frac{16}{3}$$

$$\text{Sub } y = -5 \text{ into (1)} \Rightarrow 10x - 3(-5) = 25$$

$$\Rightarrow 10x + 15 = 25$$

$$\Rightarrow 10x = 10$$

$$\Rightarrow x = 1$$

$$\text{Sub } y = \frac{16}{3} \text{ into (1)} \Rightarrow 10x - 3\left(\frac{16}{3}\right) = 25$$

$$\Rightarrow 10x = 41$$

$$\Rightarrow x = \frac{41}{10}$$

$$\therefore \begin{cases} x = 1 \\ y = -5 \end{cases} \text{ or } \begin{cases} x = \frac{41}{10} \\ y = \frac{16}{3} \end{cases}$$

13.2 System of Equations with Three Variables