Solution Book of Mathematic

Ssnior 2 Part I

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Contents

11.1 Revision Exercise 14

Calculate the following (Question 1 to 4):

1. $5\begin{pmatrix}-3 & -1\\ 3 & 4\end{pmatrix}+4\begin{pmatrix}6 & 2\\ 1 & -1\end{pmatrix}$

Sol.

$$\begin{aligned}&5\begin{pmatrix}-3 & -1\\ 3 & 4\end{pmatrix}+4\begin{pmatrix}6 & 2\\ 1 & -1\end{pmatrix}\\ &=\begin{pmatrix}-15 & -5\\ 15 & 20\end{pmatrix}+\begin{pmatrix}24 & 8\\ 4 & -4\end{pmatrix}\\ &=\begin{pmatrix}9 & 3\\ 19 & 16\end{pmatrix}\end{aligned}$$

2. $-4\begin{pmatrix}3 & 0\\ -1 & 5\end{pmatrix}-3\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}$

Sol.

$$\begin{aligned}&-4\begin{pmatrix}3 & 0\\ -1 & 5\end{pmatrix}-3\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}\\ &=\begin{pmatrix}-12 & 0\\ 4 & -20\end{pmatrix}-\begin{pmatrix}3 & 0\\ -3 & -3\end{pmatrix}\\ &=\begin{pmatrix}-15 & 0\\ 7 & 17\end{pmatrix}\end{aligned}$$

3. $\begin{pmatrix}2 & 6 & -1\\ 8 & -4 & 3\\ 5 & 7 & -2\end{pmatrix}+\begin{pmatrix}4 & -2 & 1\\ -2 & 1 & 0\\ -3 & 5 & -4\end{pmatrix}$

Sol.

$$\begin{aligned}&\begin{pmatrix}2 & 6 & -1\\ 8 & -4 & 3\\ 5 & 7 & -2\end{pmatrix}+\begin{pmatrix}4 & -2 & 1\\ -2 & 1 & 0\\ -3 & 5 & -4\end{pmatrix}\\ &=\begin{pmatrix}6 & 4 & 0\\ 6 & -3 & 3\\ 2 & 12 & -6\end{pmatrix}\end{aligned}$$

4. $2\begin{pmatrix}1 & -3 & 5\\ 7 & 2 & 0\\ 2 & 4 & -4\end{pmatrix}-\begin{pmatrix}-2 & -6 & 2\\ -5 & 1 & 1\\ 0 & 0 & -2\end{pmatrix}$

Sol.

$$\begin{aligned}&2\begin{pmatrix}1 & -3 & 5\\ 7 & 2 & 0\\ 2 & 4 & -4\end{pmatrix}-\begin{pmatrix}-2 & -6 & 2\\ -5 & 1 & 1\\ 0 & 0 & -2\end{pmatrix}\\ &=\begin{pmatrix}2 & -6 & 10\\ 14 & 4 & 0\\ 4 & 8 & -8\end{pmatrix}-\begin{pmatrix}-2 & -6 & 2\\ -5 & 1 & 1\\ 0 & 0 & -2\end{pmatrix}\\ &=\begin{pmatrix}4 & 0 & 8\\ 19 & 3 & -1\\ 4 & 8 & -6\end{pmatrix}\end{aligned}$$

5. Given that $\begin{pmatrix}2\\ -3\end{pmatrix}+3\begin{pmatrix}5\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$, find the value of x and y.

Sol.

$$\begin{aligned}\begin{pmatrix}2\\ -3\end{pmatrix}+3\begin{pmatrix}5\\ y\end{pmatrix}&=\begin{pmatrix}x\\ y\end{pmatrix}\\ \begin{pmatrix}2\\ -3\end{pmatrix}+\begin{pmatrix}15\\ 3y\end{pmatrix}&=\begin{pmatrix}x\\ y\end{pmatrix}\\ \begin{pmatrix}17\\ -3+3y\end{pmatrix}&=\begin{pmatrix}x\\ y\end{pmatrix}\\ x&=17\\ y&=-3+3y\\ 2y&=3\\ y&=\frac{3}{2}\\ \therefore x=17,\ y&=\frac{3}{2}\end{aligned}$$

6. Let $P=\begin{pmatrix}3 & -2 & 1\\ -1 & 2 & -3\\ 4 & 0 & -2\end{pmatrix}$, $Q=\begin{pmatrix}1 & -5 & -4\\ -2 & 0 & 6\\ 3 & 2 & 3\end{pmatrix}$ and $R=\begin{pmatrix}4 & 0 & 5\\ 1 & -2 & -7\\ 0 & -2 & 1\end{pmatrix}$. Find the following:

(a) $2Q+R'$

Sol.

$$\begin{aligned}2Q+R'&=2\begin{pmatrix}1 & -5 & -4\\ -2 & 0 & 6\\ 3 & 2 & 3\end{pmatrix}+\begin{pmatrix}4 & 0 & 5\\ 1 & -2 & -7\\ 0 & -2 & 1\end{pmatrix}'\\ &=\begin{pmatrix}2 & -10 & -8\\ -4 & 0 & 12\\ 6 & 4 & 6\end{pmatrix}+\begin{pmatrix}4 & 1 & 0\\ 0 & -2 & -2\\ 5 & -7 & 1\end{pmatrix}\\ &=\begin{pmatrix}6 & -9 & -8\\ -4 & -2 & 10\\ 11 & -3 & 7\end{pmatrix}\end{aligned}$$

(b) $(P - R) + 2Q'$

Sol.

$$
\begin{aligned}
&(P-R)+2Q' \\
&= \begin{pmatrix} 3 & -2 & 1 \\ -1 & 2 & -3 \\ 4 & 0 & -2 \end{pmatrix} - \begin{pmatrix} 4 & 0 & 5 \\ 1 & -2 & -7 \\ 0 & -2 & 1 \end{pmatrix} \\
&\quad + 2\begin{pmatrix} 1 & -5 & -4 \\ -2 & 0 & 6 \\ 3 & 2 & 3 \end{pmatrix}' \\
&= \begin{pmatrix} -1 & -2 & -4 \\ -2 & 4 & 4 \\ 4 & 2 & -3 \end{pmatrix} + \begin{pmatrix} 2 & -4 & 6 \\ -10 & 0 & 4 \\ -8 & 12 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 6 & 2 \\ -12 & 4 & 8 \\ -4 & 14 & 3 \end{pmatrix}
\end{aligned}
$$

(c) $[2(Q - P)]'$

Sol.

$$
\begin{aligned}
&[2(Q-P)]' \\
&= \left\{ 2\left[\begin{pmatrix} 1 & -5 & -4 \\ -2 & 0 & 6 \\ 3 & 2 & 3 \end{pmatrix} - \begin{pmatrix} 3 & -2 & 1 \\ -1 & 2 & -3 \\ 4 & 0 & -2 \end{pmatrix} \right] \right\}' \\
&= \left[2\begin{pmatrix} -2 & -3 & -5 \\ -1 & -2 & 9 \\ -1 & 2 & 5 \end{pmatrix} \right]' \\
&= \begin{pmatrix} -4 & -6 & -10 \\ -2 & -4 & 18 \\ -2 & 4 & 10 \end{pmatrix}' \\
&= \begin{pmatrix} -4 & -2 & -2 \\ -6 & -4 & 4 \\ -10 & 18 & 10 \end{pmatrix}
\end{aligned}
$$

(d) $(R' - Q)'$

Sol.

$$
\begin{aligned}
(R'-Q)' &= \left[\begin{pmatrix} 4 & 0 & 5 \\ 1 & -2 & -7 \\ 0 & -2 & 1 \end{pmatrix}' - \begin{pmatrix} 1 & -5 & -4 \\ -2 & 0 & 6 \\ 3 & 2 & 3 \end{pmatrix} \right]' \\
&= \begin{pmatrix} 4 & 0 & 5 \\ 1 & -2 & -7 \\ 0 & -2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -5 & -4 \\ -2 & 0 & 6 \\ 3 & 2 & 3 \end{pmatrix}' \\
&= \begin{pmatrix} 4 & 0 & 5 \\ 1 & -2 & -7 \\ 0 & -2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 3 \\ -5 & 0 & 2 \\ -4 & 6 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 2 & 2 \\ 6 & -2 & -9 \\ 4 & -8 & -2 \end{pmatrix}
\end{aligned}
$$

7. Let $M = \begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix}$ and $N = \begin{pmatrix} 3 & 6 \\ 7 & -1 \\ -4 & 2 \end{pmatrix}$. Find the matrix X in the following equations:

(a) $2N - 3M = 2M - X$

Sol.

$$
\begin{aligned}
2N - 3M &= 2M - X \\
X &= 5M - 2N \\
&= 5\begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix} - 2\begin{pmatrix} 3 & 6 \\ 7 & -1 \\ -4 & 2 \end{pmatrix} \\
&= \begin{pmatrix} -5 & 0 \\ 20 & -15 \\ 10 & 20 \end{pmatrix} - \begin{pmatrix} 6 & 12 \\ 14 & -2 \\ -8 & 4 \end{pmatrix} \\
&= \begin{pmatrix} -11 & -12 \\ 6 & -13 \\ 18 & 16 \end{pmatrix}
\end{aligned}
$$

(b) $2(M - 2N) + X = M + N$

Sol.

$$\begin{aligned}
2(M-2N)+X &= M+N \\
X &= M+N-2(M-2N) \\
&= M+N-2M+4N \\
&= -M+5N \\
&= -\begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix} + 5\begin{pmatrix} 3 & 6 \\ 7 & -1 \\ -4 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ -4 & 3 \\ -2 & -4 \end{pmatrix} + \begin{pmatrix} 15 & 30 \\ 35 & -5 \\ -20 & 10 \end{pmatrix} \\
&= \begin{pmatrix} 16 & 30 \\ 31 & -2 \\ -22 & 6 \end{pmatrix}
\end{aligned}$$

(c) $(M+2N)' = X$

Sol.

$$\begin{aligned}
(M+2N)' &= X \\
X &= (M+2N)' \\
&= \left[\begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix} + 2\begin{pmatrix} 3 & 6 \\ 7 & -1 \\ -4 & 2 \end{pmatrix}\right]' \\
&= \left[\begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 6 & 12 \\ 14 & -2 \\ -8 & 4 \end{pmatrix}\right] \\
&= \begin{pmatrix} 5 & 12 \\ 18 & -5 \\ -6 & 8 \end{pmatrix}' \\
&= \begin{pmatrix} 5 & 18 & -6 \\ 12 & -5 & 8 \end{pmatrix}
\end{aligned}$$

(d) $3N' - M' = 2X$

Sol.

$$\begin{aligned}
3N' - M' &= 2X \\
2X &= (3N-M)' \\
&= \left[3\begin{pmatrix} 3 & 6 \\ 7 & -1 \\ -4 & 2 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix}\right]' \\
&= \left[\begin{pmatrix} 9 & 18 \\ 21 & -3 \\ -12 & 6 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 4 & -3 \\ 2 & 4 \end{pmatrix}\right]' \\
&= \begin{pmatrix} 10 & 18 \\ 17 & 0 \\ -14 & 2 \end{pmatrix}' \\
&= \begin{pmatrix} 10 & 17 & -14 \\ 18 & 0 & 2 \end{pmatrix} \\
X &= \begin{pmatrix} 5 & \frac{17}{2} & -7 \\ 9 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Of the following matrices, determine if AB and BA are defined. If any of them is defined, find the value of them (Question 8 to 11):

8. $A = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

Sol.

$\because$ The number of columns of A is not equal to the number of rows of B
$\therefore$ AB is not defined

$\because$ The number of columns of B is equal to the number of rows of A
$\therefore$ BA is defined

$$\therefore BA = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 11 \\ -2 \\ 0 \end{pmatrix}$$

9. $A = \begin{pmatrix} -5 & 4 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 \\ -3 & 0 \\ 1 & -4 \end{pmatrix}$

Sol.

$\because$ The number of columns of A is equal to the number of rows of B

$\therefore AB$ is defined

$$\therefore AB = \begin{pmatrix} -5 & 4 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -3 & 0 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -17 \\ -5 & -11 \end{pmatrix}$$

$\because$ The number of columns of B is equal to the number of rows of A

$\therefore BA$ is defined

$$\therefore BA = \begin{pmatrix} -2 & 1 \\ -3 & 0 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} -5 & 4 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -6 & -3 \\ 13 & -12 & -9 \\ -9 & -4 & -9 \end{pmatrix}$$

10. $A = \begin{pmatrix} 3 & 2 & -1 \\ 5 & 4 & 0 \\ -2 & 6 & 1 \end{pmatrix}, B = \begin{pmatrix} 9 & 0 \\ -7 & 1 \\ 2 & 3 \end{pmatrix}$

Sol.

$\because$ The number of columns of A is equal to the number of rows of B

$\therefore AB$ is defined

$$\therefore AB = \begin{pmatrix} 3 & 2 & -1 \\ 5 & 4 & 0 \\ -2 & 6 & 1 \end{pmatrix} \begin{pmatrix} 9 & 0 \\ -7 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 11 & -1 \\ 17 & 4 \\ -58 & 9 \end{pmatrix}$$

$\because$ The number of columns of B is not equal to the number of rows of A

$\therefore BA$ is not defined

11. $A = \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 5 & 2 \end{pmatrix}$

Sol.

$\because$ The number of columns of A is not equal to the number of rows of B

$\therefore AB$ is not defined

$\because$ The number of columns of B is equal to the number of rows of A

$\therefore BA$ is defined

$$\therefore BA = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 10 & 4 \\ 18 & 2 \end{pmatrix}$$

12. Given that $A = \begin{pmatrix} a & 3a \\ 2b & b \end{pmatrix}$, $B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $AB = \begin{pmatrix} 45 \\ 48 \end{pmatrix}$, find the value of a and b.

Sol.

$$AB = \begin{pmatrix} a & 3a \\ 2b & b \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 45 \\ 48 \end{pmatrix}$$

$$\begin{pmatrix} 3a + 6a \\ 6b + 2b \end{pmatrix} = \begin{pmatrix} 45 \\ 48 \end{pmatrix}$$

$$9a = 45$$

$$8b = 48$$

$$a = 5$$

$$b = 6$$

13. Given that $A = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$, $B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $A + B = AB$, find the value of a, b and c.

Sol.

$$A + B = AB$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

$$\begin{pmatrix} 3 + a & b \\ 0 & 4 + c \end{pmatrix} = \begin{pmatrix} 3a & 3b \\ 0 & 4c \end{pmatrix}$$

$$3 + a = 3a$$

$$2a = 3$$

$$b = 3b$$

$$2b = 0$$

$$4 + c = 4c$$

$$3c = 4$$

$$a = \frac{3}{2},\ b = 0,\ c = \frac{4}{3}$$

Find the value of the following determinants (Question 14 to 22):

14. $\begin{vmatrix} 20 & 15 \\ 8 & 6 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 20 & 15 \\ 8 & 6 \end{vmatrix} = 20 \cdot 6 - 15 \cdot 8$$

$$= 120 - 120$$

$$= 0$$

15. $\begin{vmatrix} 6 & -7 \\ 15 & -2 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 6 & -7 \\ 15 & -2 \end{vmatrix} = 6 \cdot -2 - (-7) \cdot 15$$
$$= -12 + 105$$
$$= 93$$

16. $\begin{vmatrix} -4 & -10 \\ 12 & 7 \end{vmatrix}$

Sol.

$$\begin{vmatrix} -4 & -10 \\ 12 & 7 \end{vmatrix} = -4 \cdot 7 - (-10) \cdot 12$$
$$= -28 + 120$$
$$= 92$$

17. $\begin{vmatrix} 3 & 3 & -6 \\ 2 & -2 & 0 \\ 3 & 0 & -3 \end{vmatrix}$

$$\begin{array}{ccc:cc} + & + & + & & \\ 3 & 3 & -6 & 3 & 3 \\ 2 & -2 & 0 & 2 & -2 \\ 3 & 0 & -3 & 3 & 0 \\ - & - & - & & \end{array}$$

Sol.

$$\begin{vmatrix} 3 & 3 & -6 \\ 2 & -2 & 0 \\ 3 & 0 & -3 \end{vmatrix} = 18 + 0 + 0 - 36 - 0 + 18$$
$$= 0$$

18. $\begin{vmatrix} 5 & 7 & 1 \\ -3 & 6 & 9 \\ 4 & 7 & 3 \end{vmatrix}$

$$\begin{array}{ccc:cc} + & + & + & & \\ 5 & 7 & 1 & 5 & 7 \\ -3 & 6 & 9 & -3 & 6 \\ 4 & 7 & 3 & 4 & 7 \\ - & - & - & & \end{array}$$

Sol.

$$\begin{vmatrix} 5 & 7 & 1 \\ -3 & 6 & 9 \\ 4 & 7 & 3 \end{vmatrix} = 90 + 252 - 21 - 24 - 315 + 63$$
$$= 45$$

19. $\begin{vmatrix} -2 & 7 & -4 \\ 3 & -5 & 2 \\ -1 & 0 & -3 \end{vmatrix}$

$$\begin{array}{ccc:cc} + & + & + & & \\ -2 & 7 & -4 & -2 & 7 \\ 3 & -5 & 2 & 3 & -5 \\ -1 & 0 & -3 & -1 & 0 \\ - & - & - & & \end{array}$$

Sol.

$$\begin{vmatrix} -2 & 7 & -4 \\ 3 & -5 & 2 \\ -1 & 0 & -3 \end{vmatrix} = -30 - 14 - 0 + 20 + 0 + 63$$
$$= -39$$

20. $\begin{vmatrix} 1 & 0 & -1 \\ 3 & -2 & 5 \\ -1 & 1 & 3 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 1 & 0 & -1 \\ 3 & -2 & 5 \\ -1 & 1 & 3 \end{vmatrix} = \begin{vmatrix} -2 & 5 \\ 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ -2 & 5 \end{vmatrix}$$
$$= -11 - 3 + 2 = -12$$

21. $\begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 1 \\ -2 & -6 & 5 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 2 & 6 & 4 \\ 1 & 3 & 1 \\ -2 & -6 & 5 \end{vmatrix} = 3\begin{vmatrix} 2 & 2 & 4 \\ 1 & 1 & 1 \\ -2 & -2 & 5 \end{vmatrix}$$
$$= 0 \quad \text{(col 1 and 2 are the same)}$$

22. $\begin{vmatrix} 10 & 8 & -2 \\ 15 & 16 & -3 \\ -5 & -4 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 10 & 8 & -2 \\ 15 & 16 & -3 \\ -5 & -4 & 1 \end{vmatrix} = -5\begin{vmatrix} 2 & 8 & 2 \\ 3 & 16 & 3 \\ -1 & -4 & -1 \end{vmatrix}$$
$$= 0 \quad \text{(col 1 and 3 are the same)}$$

Using the identities of determinant, prove the following equations (Question 23 to 24):

23. $\begin{vmatrix} bc & 1 & bc(b+c) \\ ca & 1 & ca(c+a) \\ ab & 1 & ab(a+b) \end{vmatrix} = 0$

Proof.

$$\begin{vmatrix} bc & 1 & bc(b+c) \\ ca & 1 & ca(c+a) \\ ab & 1 & ab(a+b) \end{vmatrix}$$

$$= a^2b^2c^2 \begin{vmatrix} 1 & \frac{1}{bc} & b+c \\ 1 & \frac{1}{ca} & c+a \\ 1 & \frac{1}{ab} & a+b \end{vmatrix}$$

$$= a^2b^2c^2 \begin{vmatrix} 1 & \frac{1}{bc} & -a \\ 1 & \frac{1}{ca} & -b \\ 1 & \frac{1}{ab} & -c \end{vmatrix} \qquad C_3 \to C_3 + (a+b+c)C_1$$

$$= a^2b^2c^2 \begin{vmatrix} 1 & a & -a \\ 1 & b & -b \\ 1 & c & -c \end{vmatrix} \qquad C_2 \to C_2 + abcC_1$$

$$= -a^2b^2c^2 \begin{vmatrix} 1 & a & a \\ 1 & b & b \\ 1 & c & c \end{vmatrix}$$

$$= 0 \qquad C_2 = C_3$$

24. $\begin{vmatrix} a & 1 & a^2(b+c) \\ b & 1 & b^2(c+a) \\ c & 1 & c^2(a+b) \end{vmatrix} = 0$

Proof.

$$\begin{vmatrix} a & 1 & a^2(b+c) \\ b & 1 & b^2(c+a) \\ c & 1 & c^2(a+b) \end{vmatrix}$$

$$= \begin{vmatrix} a & 1 & a^2(b+c) \\ b-a & 0 & b^2(c+a) - a^2(b+c) \\ c-a & 0 & c^2(a+b) - a^2(b+c) \end{vmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$= \begin{vmatrix} b-a & b^2(c+a) - a^2(b+c) \\ c-a & c^2(a+b) - a^2(b+c) \end{vmatrix}$$

$$= (b-a)[c^2(a+b) - a^2(b+c)]$$
$$- (c-a)[b^2(c+a) - a^2(b+c)]$$
$$= c^2(b-a)(b+a) - a^2(b+c)(c-a)$$
$$- b^2(c-a)(c+a) + a^2(b+c)(c-a)$$
$$= c^2(b^2-a^2) - b^2(c^2-a^2)$$
$$= b^2c^2 - a^2c^2 - b^2c^2 + a^2c^2$$
$$= 0$$

Find the value of x in the following expressions (Question 25 to 26):

25. $\begin{vmatrix} 2x & 3 & x+5 \\ -3 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 5x - 1$

Sol.

$$\begin{vmatrix} 2x & 3 & x+5 \\ -3 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 5x - 1$$
$$x+5 \begin{vmatrix} -3 & 2 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 2x & 3 \\ 2 & 1 \end{vmatrix} = 5x - 1$$
$$-7(x+5) - (2x-6) = 5x - 1$$
$$-7x - 35 - 2x + 6 = 5x - 1$$
$$-14x = 28$$
$$x = -2$$

26. $\begin{vmatrix} x+3 & 1 & 0 \\ x & 3 & 0 \\ 1 & 0 & -x-2 \end{vmatrix} = x + 6$

Sol.

$$\begin{vmatrix} x+3 & 1 & 0 \\ x & 3 & 0 \\ 1 & 0 & -x-2 \end{vmatrix} = x + 6$$
$$-x-2 \begin{vmatrix} x+3 & 1 \\ x & 3 \end{vmatrix} = x + 6$$
$$-(x+2)(3x+9-x) = x + 6$$
$$(x+2)(2x+9) = -x - 6$$
$$2x^2 + 13x + 18 = -x - 6$$
$$2x^2 + 14x + 24 = 0$$
$$x^2 + 7x + 12 = 0$$
$$(x+4)(x+3) = 0$$
$$x = -4 \text{ or } x = -3$$

27. Given an identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $(2I + J)^{-1} = rI + sJ$, find the value of r and s.

Find the value of a in the following matrices if they are non-inversible (Question 28 to 31):

28. $\begin{pmatrix} 3 & a \\ -2 & 6 \end{pmatrix}$

Sol.

$$\begin{aligned}\begin{vmatrix} 3 & a \\ -2 & 6 \end{vmatrix} &= 0 \\ 18 + 2a &= 0 \\ 2a &= -18 \\ a &= -9\end{aligned}$$

29. $\begin{pmatrix} 5a+2 & 4 \\ 6 & a \end{pmatrix}$

Sol.

$$\begin{aligned}\begin{vmatrix} 5a+2 & 4 \\ 6 & a \end{vmatrix} &= 0 \\ 5a^2 + 2a - 24 &= 0 \\ (x-2)(5x+12) &= 0 \\ x = 2 \text{ or } x &= \frac{12}{5}\end{aligned}$$

30. $\begin{pmatrix} -7 & a & 3 \\ 2 & -3 & 1 \\ 0 & -a & 4 \end{pmatrix}$

Sol.

$$\begin{aligned}\begin{vmatrix} -7 & a & 3 \\ 2 & -3 & 1 \\ 0 & -a & 4 \end{vmatrix} &= 0 \\ -7\begin{vmatrix} -3 & 1 \\ -a & 4 \end{vmatrix} - 2\begin{vmatrix} a & 3 \\ -a & 4 \end{vmatrix} &= 0 \\ -7(-12 + a) - 2(4a + 3a) &= 0 \\ 84 - 7a - 14a &= 0 \\ 21a &= 84 \\ a &= 4\end{aligned}$$

31. $\begin{pmatrix} a & -1 & 0 \\ 4 & 0 & -2 \\ a+4 & a & -8 \end{pmatrix}$

Sol.

$$\begin{aligned}\begin{vmatrix} a & -1 & 0 \\ 4 & 0 & -2 \\ a+4 & a & -8 \end{vmatrix} &= 0 \\ a\begin{vmatrix} 0 & -2 \\ a & -8 \end{vmatrix} + \begin{vmatrix} 4 & -2 \\ a+4 & -8 \end{vmatrix} & \\ 2a^2 - 32 + 2a + 8 &= 0 \\ 2a^2 + 2a - 24 &= 0 \\ a^2 + a - 12 &= 0 \\ (a+4)(a-3) &= 0 \\ a = -4 \text{ or } a &= 3\end{aligned}$$

Find the inverse of the following matrices (Question 32 to 37):

32. $\begin{pmatrix} 2 & 5 \\ 2 & 3 \end{pmatrix}$

33. $\begin{pmatrix} -2 & -1 \\ 4 & 6 \end{pmatrix}$

34. $\begin{pmatrix} 1 & 0 & 3 \\ 3 & 1 & 9 \\ -2 & 2 & -4 \end{pmatrix}$

35. $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & -2 \\ -2 & 5 & 1 \end{pmatrix}$

36. $\begin{pmatrix} 4 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & -4 & 4 \end{pmatrix}$

37. $\begin{pmatrix} 3 & 2 & 3 \\ -1 & -3 & \\ 1 & 0 & 3 \end{pmatrix}$

Solve the following system of equations using the method of Gauss elimination (Question 38 to 41):

38. $\begin{cases} 2x - y + 4z = 5 \\ 2x + 3y - 4z = -7 \\ x + y + z = 2 \end{cases}$

39. $\begin{cases} x - 2y - 3z = -4 \\ 3x + y - 4z = -5 \\ 2x + 4y - z = -5 \end{cases}$

40. $\begin{cases} x - 2y - z = 3 \\ 4x - y + 2z = 1 \\ x + 3y = 5 \end{cases}$

41. $\begin{cases} 2x - y - z = 0 \\ 4x - 3y + 2z = 1 \\ 3x - 2y - 4z = -1 \end{cases}$

Solve the following system of equations using the Cramer's rule (Question 42 to 45):

42. $\begin{cases} x - 3y - 2z = 1 \\ 7x + 4y - 5z = 0 \\ 3x + 9y + z = -1 \end{cases}$

Sol.

$$\Delta = \begin{vmatrix} 1 & -3 & -2 \\ 7 & 4 & -5 \\ 3 & 9 & 1 \end{vmatrix} = 13$$

$$\Delta_x = \begin{vmatrix} 1 & -3 & -2 \\ 0 & 4 & -5 \\ -1 & 9 & 1 \end{vmatrix} = 26$$

$$\Delta_y = \begin{vmatrix} 1 & 1 & -2 \\ 7 & 0 & -5 \\ 3 & -1 & 1 \end{vmatrix} = -13$$

$$\Delta_z = \begin{vmatrix} 1 & -3 & 1 \\ 7 & 4 & 0 \\ 3 & 9 & -1 \end{vmatrix} = 26$$

$$\therefore x = \frac{26}{13} = 2,\ y = \frac{-13}{13} = -1,\ z = \frac{26}{13} = 2$$

43. $\begin{cases} x - 2y + 3z = 6 \\ 2x + 3y - 4z = 20 \\ 3x - 2y - 5z = 6 \end{cases}$

Sol.

$$\Delta = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -4 \\ 3 & -2 & -5 \end{vmatrix} = -58$$

$$\Delta_x = \begin{vmatrix} 6 & -2 & 3 \\ 20 & 3 & -4 \\ 6 & -2 & -5 \end{vmatrix} = -464$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 20 & -4 \\ 3 & 6 & -5 \end{vmatrix} = -232$$

$$\Delta_z = \begin{vmatrix} 1 & -2 & 6 \\ 2 & 3 & 20 \\ 3 & -2 & 6 \end{vmatrix} = -116$$

$$\therefore x = \frac{-464}{-58} = 8,\ y = \frac{-232}{-58} = 4,\ z = \frac{-116}{-58} = 2$$

44. $\begin{cases} 2x - 2y - 4z + 3 = 0 \\ 2x + 3y + 4z - 2 = 0 \\ 7x + 3y - 2z - 2 = 0 \end{cases}$

Sol.

$$\begin{cases} 2x - 2y - 4z = -3 \\ 2x + 3y + 4z = 2 \\ 7x + 3y - 2z = 2 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & -2 & -4 \\ 2 & 3 & 4 \\ 7 & 3 & -2 \end{vmatrix} = -40$$

$$\Delta_x = \begin{vmatrix} -3 & -2 & -4 \\ 2 & 3 & 4 \\ 2 & 3 & -2 \end{vmatrix} = 30$$

$$\Delta_y = \begin{vmatrix} 2 & -3 & -4 \\ 2 & 2 & 4 \\ 7 & 2 & -2 \end{vmatrix} = -80$$

$$\Delta_z = \begin{vmatrix} 2 & -2 & -3 \\ 2 & 3 & 2 \\ 7 & 3 & 2 \end{vmatrix} = 25$$

$$\therefore x = \frac{30}{-40} = -\frac{3}{4},\ y = \frac{-80}{-40} = 2,\ z = \frac{25}{-40} = -\frac{5}{8}$$

45. $\begin{cases} \frac{2}{x} - \frac{5}{y} + \frac{4}{z} = -3 \\ \frac{4}{x} + \frac{1}{y} - \frac{2}{z} = 7 \\ \frac{7}{x} - \frac{3}{z} = 4 \end{cases}$