1. (a) Find x, correct to two decimal places, such that $2^x \cdot 3^x = 18$.

Solution:

$$\begin{aligned} 2^x \cdot 3^x &= 18 \\ (2 \cdot 3)^x &= 18 \\ 6^x &= 18 \\ \log 6^x &= \log 18 \\ x \log 6 &= \log 18 \\ x &= \frac{\log 18}{\log 6} \\ &\approx 1.63 \end{aligned}$$

(b) Given that the curve $y = kx^n$ passes through the points $(2, 2)$ and $\left(5, 31\frac{1}{4}\right)$, find the values of k and n.

Solution:

When $x = 2$, $y = 2$, $2 = k \cdot 2^n \cdots (1)$.
When $x = 5$, $y = 31\frac{1}{4}$, $31\frac{1}{4} = k \cdot 5^n \cdots (2)$.
Dividing (2) by (1),

$$\begin{aligned} \frac{31\frac{1}{4}}{2} &= \frac{k \cdot 5^n}{k \cdot 2^n} \\ \frac{125}{8} &= \frac{5^n}{2^n} \\ \left(\frac{5}{2}\right)^n &= \frac{125}{8} \\ &= \left(\frac{5}{2}\right)^3 \\ n &= 3 \end{aligned}$$

Substituting $n = 3$ into (1),

$$\begin{aligned} 2 &= k \cdot 2^3 \\ k &= \frac{1}{4} \end{aligned}$$

2. Solve the equations $\log_2 y^2 = 3 + \log_2(y + 6)$.

Solution:

$$\begin{aligned} \log_2 y^2 &= 3 + \log_2(y + 6) \\ \log_2 y^2 &= \log_2 8(y + 6) \\ y^2 &= 8(y + 6) \\ y^2 - 8y - 48 &= 0 \\ (y - 12)(y + 4) &= 0 \\ y &= 12 \text{ or } -4 \end{aligned}$$

3. (a) Given that $q^p = 25$, express $\log_5 q$ in terms of p. **Solution:**

$$\begin{aligned} q^p &= 25 \\ q &= 25^{\frac{1}{p}} \\ \log_5 q &= \log_5 25^{\frac{1}{p}} \\ &= \frac{1}{p} \log_5 25 \\ &= \frac{1}{p} \cdot 2 \\ &= \frac{2}{p} \end{aligned}$$

(b) Given that $2 \lg x^2 y = 3 + \lg x - \lg y$, where x and y are both positive, express, in its simplest form, y in terms of x.

Solution:

$$\begin{aligned} 2 \lg x^2 y &= 3 + \lg x - \lg y \\ \lg x^4 y^2 &= \lg \frac{1000x}{y} \\ x^4 y^2 &= \frac{1000x}{y} \\ x^4 y^3 &= 1000x \\ y^3 &= \frac{1000x}{x^4} \\ &= \frac{1000}{x^3} \\ y &= \sqrt[3]{\frac{1000}{x^3}} \\ &= \frac{10}{x} \end{aligned}$$

4. (a) Solve the equation $\lg y + \lg(2y - 1) = 1$.

Solution:

$$\begin{aligned} \lg y + \lg(2y - 1) &= 1 \\ \lg y(2y - 1) &= 1 \\ y(2y - 1) &= 10 \\ 2y^2 - y &= 10 \\ 2y^2 - y - 10 &= 0 \\ (2y - 5)(y + 2) &= 0 \\ y &= \frac{5}{2} \ (y > 0) \end{aligned}$$

(b) Given that $5\log_p 6 - \log_p 96 = 4$, find the value of p.

Solution:

$$\begin{aligned}
5\log_p 6 - \log_p 96 &= 4\\
\log_p 6^5 - \log_p 96 &= 4\\
\log_p \frac{6^5}{96} &= 4\\
\frac{7776}{96} &= p^4\\
p^4 &= 81\\
p &= 3
\end{aligned}$$

5. (a) Denoting $\log_3 a$ by p, express in terms of p

i. $\log_3 a^3$,

Solution:

$$\begin{aligned}
\log_3 a^3 &= 3\log_3 a\\
&= 3p
\end{aligned}$$

ii. $\log_3 \left(\frac{1}{a}\right)$,

Solution:

$$\begin{aligned}
\log_3 \left(\frac{1}{a}\right) &= \log_3 a^{-1}\\
&= -\log_3 a\\
&= -p
\end{aligned}$$

iii. $\log_9 a$.

Solution:

$$\begin{aligned}
\log_9 a &= \frac{1}{2}\log_3 a\\
&= \frac{1}{2}p
\end{aligned}$$

(b) Given that $y = ax^n - 20$ and that $y = 12$ when $x = 2$, and $y = 140$ when $x = 4$, find n and a

Solution:

When $x = 2$, $y = 12$,

$$\begin{aligned}
12 &= a\cdot 2^n - 20\\
a\cdot 2^n &= 32 \ \cdots (1)
\end{aligned}$$

When $x = 4$, $y = 140$,

$$\begin{aligned}
140 &= a\cdot 4^n - 20\\
a\cdot 4^n &= 160 \ \cdots (2)
\end{aligned}$$

Dividing (2) by (1),

$$\begin{aligned}
\frac{a\cdot 4^n}{a\cdot 2^n} &= \frac{160}{32}\\
2^n &= 5\\
n &= \log_2 5\\
&\approx 2.32
\end{aligned}$$

Substituting $n = \log_2 5$ into (1),

$$\begin{aligned}
a\cdot 2^{\log_2 5} &= 32\\
a\cdot 5 &= 32\\
a &= \frac{32}{5}\\
&= 6.4
\end{aligned}$$

6. (a) Without using tables or calculators evaluate

i. $2\log_2 12 + 3\log_2 5 - \log_2 15 - \log_2 150$,

Solution:

$$\begin{aligned}
&2\log_2 12 + 3\log_2 5 - \log_2 15 - \log_2 150\\
&= \log_2 12^2 + \log_2 5^3 - \log_2 15 - \log_2 150\\
&= \log_2 \frac{12^2\cdot 5^3}{15\cdot 150}\\
&= \log_2 \frac{144\cdot 125}{2250}\\
&= \log_2 \frac{18000}{2250}\\
&= \log_2 8\\
&= 3
\end{aligned}$$

ii. $\log_8 32$.

Solution:

$$\begin{aligned}
\log_8 32 &= \frac{\log_2 32}{\log_2 8}\\
&= \frac{5}{3}
\end{aligned}$$

(b) Show that $5^n + 5^{n+1} + 5^{n+2}$ is divisible by 31 for all positive integer values of n.

Solution:

$$\begin{aligned}
5^n + 5^{n+1} + 5^{n+2} &= 5^n(1+5+25)\\
&= 31\cdot 5^n
\end{aligned}$$

$\because \forall n \in \mathbb{Z}_+,\ 5^n \in \mathbb{Z}_+$,
$\therefore 31 | (31\cdot 5^n)$.
$\therefore \forall n \in \mathbb{Z}_+,\ 31 | (5^n + 5^{n+1} + 5^{n+2})$. ■

7. (a) Solve the equation $\lg 25 + \lg x - \lg(x-1) = 2$.

Solution:

$$\begin{aligned}
\lg 25 + \lg x - \lg(x-1) &= 2 \\
\lg \frac{25x}{x-1} &= 2 \\
\frac{25x}{x-1} &= 100 \\
25x &= 100x - 100 \\
75x &= 100 \\
x &= \frac{4}{3}
\end{aligned}$$

(b) Solve the equation $5^y = 10$.

Solution:

$$\begin{aligned}
5^y &= 10 \\
y &= \log_5 10 \\
&= \frac{\log 10}{\log 5} \\
&\approx 1.43
\end{aligned}$$

(c) Given that $\lg z = k$ find, in terms of k, an expression for $\log_z 10z$.

Solution:

$$\begin{aligned}
\log_z 10z &= \log_z 10 + \log_z z \\
&= \log_z 10 + 1 \\
&= \frac{\log 10}{\log z} + 1 \\
&= \frac{1}{k} + 1
\end{aligned}$$

8. (a) Solve the equation $\sqrt{4x-9} = 2\sqrt{x} - 1$.

Solution:

$$\begin{aligned}
\sqrt{4x-9} &= 2\sqrt{x} - 1 \\
4x - 9 &= 4x - 4\sqrt{x} + 1 \\
4\sqrt{x} &= 10 \\
\sqrt{x} &= \frac{5}{2} \\
x &= \frac{25}{4}
\end{aligned}$$

Upon checking, $x = \frac{25}{4}$ is a valid solution.

(b) Solve the equation $7^{x^2} - 49^{6-2x} = 0$.

Solution:

$$\begin{aligned}
7^{x^2} - 49^{6-2x} &= 0 \\
7^{x^2} &= 49^{6-2x} \\
7^{x^2} &= 7^{2(6-2x)} \\
x^2 &= 2(6-2x) \\
x^2 + 4x - 12 &= 0 \\
(x+6)(x-2) &= 0 \\
x &= -6 \text{ or } 2
\end{aligned}$$

(c) Evaluate $\log_3 7 \cdot \log_7 2 \cdot \log_2 3$.

Solution:

$$\begin{aligned}
\log_3 7 \cdot \log_7 2 \cdot \log_2 3 &= \frac{\log_7 7}{\log_7 3} \cdot \frac{\log_7 2}{\log_7 7} \cdot \frac{\log_7 3}{\log_7 2} \\
&= 1
\end{aligned}$$

9. (a) Solve the equation $\frac{6}{\sqrt{x-1}} - 2\sqrt{x-1} = 1$,

Solution:

$$\begin{aligned}
\frac{6}{\sqrt{x-1}} - 2\sqrt{x-1} &= 1 \\
6 - 2(x-1) &= \sqrt{x-1} \\
8 - 2x &= \sqrt{x-1} \\
4x^2 - 32x + 64 &= x - 1 \\
4x^2 - 33x + 65 &= 0 \\
(4x-13)(x-5) &= 0 \\
x &= \frac{13}{4} \text{ or } x = 5
\end{aligned}$$

Upon checking, $x = \frac{13}{4}$ is the only valid solution.

(b) If $5^x 25^{2y} = 1$ and $3^{5x} 9^y = \frac{1}{9}$ calculate the value of x and of y.

Solution:

$$\begin{aligned}
5^x 25^{2y} &= 1 \\
5^x 5^{4y} &= 1 \\
5^{x+4y} &= 1 \\
x + 4y &= 0 \quad \cdots \ (1)
\end{aligned}$$

$$
\begin{aligned}
3^{5x}9^{y} &= 3^{-2}\\
3^{5x}3^{2y} &= 3^{-2}\\
3^{5x+2y} &= 3^{-2}\\
5x+2y &= -2 \ \cdots\ (2)
\end{aligned}
$$

Multiplying (2) by 2,

$$10x+4y = -4 \ \cdots\ (3)$$

Subtracting (1) from (3),

$$
\begin{aligned}
9x &= -4\\
x &= -\frac{4}{9}
\end{aligned}
$$

Substituting $x = -\frac{4}{9}$ into (1),

$$
\begin{aligned}
-\frac{4}{9}+4y &= 0\\
4y &= \frac{4}{9}\\
y &= \frac{1}{9}
\end{aligned}
$$

Therefore, $x = -\frac{4}{9}$ and $y = \frac{1}{9}$.

10. (a) Given that $\log_p 7 + \log_p k = 0$, find k.

Solution:

$$
\begin{aligned}
\log_p 7 + \log_p k &= 0\\
\log_p 7k &= 0\\
7k &= p^0\\
7k &= 1\\
k &= \frac{1}{7}
\end{aligned}
$$

(b) Given that $4\log_q 3 + 2\log_q 2 - \log_q 144 = 2$ find q.

Solution:

$$
\begin{aligned}
4\log_q 3 + 2\log_q 2 - \log_q 144 &= 2\\
\log_q 3^4 + \log_q 2^2 - \log_q 144 &= 2\\
\log_q \frac{3^4\cdot 2^2}{144} &= 2\\
\frac{3^4\cdot 2^2}{144} &= q^2\\
q^2 &= \frac{9}{4}\\
q &= \frac{3}{2}\ (q>0)
\end{aligned}
$$

(c) Given that $\log_3 2 = 0.631$ and tha $\log_3 5 = 1.465$, evaluate $\log_3 1.2$, without using tables or a calculator. **Solution:**

$$
\begin{aligned}
\log_3 1.2 &= \log_3 \frac{12}{10}\\
&= \log_3 12 - \log_3 10\\
&= \log_3 2^2 + \log_3 3 - \log_3 2 - \log_3 5\\
&= \log_3 2 + 1 - \log_3 5\\
&= 0.631 + 1 - 1.465\\
&= 0.166
\end{aligned}
$$

11. (a) Solve the equation $\log_5 x = 16\log_x 5$.

Solution:

$$
\begin{aligned}
\log_5 x &= 16\log_x 5\\
\log_5 x &= \frac{16}{\log_5 x}\\
\log_5^2 x &= 16\\
\log_5 x &= \pm 4\\
x &= 5^4 \text{ or } 5^{-4}\\
&= 625 \text{ or } \frac{1}{625}
\end{aligned}
$$

(b) Find the values of y which satisfy the equation $(8^y)^y \cdot \frac{1}{32^y} = 4$

Solution:

$$
\begin{aligned}
(8^y)^y \cdot \frac{1}{32^y} &= 4\\
8^{y^2} \cdot \frac{1}{2^{5y}} &= 4\\
8^{y^2} \cdot 2^{-5y} &= 4\\
2^{3y^2} \cdot 2^{-5y} &= 4\\
2^{3y^2-5y} &= 4\\
3y^2-5y &= 2\\
3y^2-5y-2 &= 0\\
(3y+1)(y-2) &= 0\\
y &= -\frac{1}{3} \text{ or } 2
\end{aligned}
$$

(c) Express $\frac{4+\sqrt{2}}{2-\sqrt{2}}$ in the form $p+\sqrt{q}$, where E and q are integers.

Solution:

$$\begin{aligned}\frac{4+\sqrt{2}}{2-\sqrt{2}} &= \frac{(4+\sqrt{2})(2+\sqrt{2})}{(2-\sqrt{2})(2+\sqrt{2})}\\ &= \frac{8+6\sqrt{2}+2}{4-2}\\ &= \frac{10+6\sqrt{2}}{2}\\ &= 5+3\sqrt{2}\\ &= 5+\sqrt{18}\end{aligned}$$

12. Given that $y = x^{-\frac{1}{3}}$ use the calculus to determine an approximate value for $\frac{1}{\sqrt[3]{0.9}}$.

Solution:

$$\begin{aligned}y &= x^{-\frac{1}{3}}\\ \frac{dy}{dx} &= -\frac{1}{3}x^{-\frac{4}{3}}\\ \frac{\Delta y}{\Delta x} &\approx \frac{dy}{dx}\\ \Delta y &\approx \frac{dy}{dx}\Delta x\end{aligned}$$

When $x = 1$, $y = 1$, $\frac{dy}{dx}$, $\Delta x = -0.1$,

$$\begin{aligned}\Delta y &\approx \frac{dy}{dx}\Delta x\\ &= -\frac{1}{3}\cdot 1\cdot(-0.1)\\ &= \frac{1}{30}\end{aligned}$$

$$\therefore\ \frac{1}{\sqrt[3]{0.9}} \approx 1+\frac{1}{30} = \frac{31}{30} \approx 1.033.$$

13. (a) Solve the equation $2\lg 3 + \lg 2x - \lg(3x+1) = 0$

Solution:

$$\begin{aligned}2\lg 3 + \lg 2x - \lg(3x+1) &= 0\\ \lg 3^2 + \lg 2x - \lg(3x+1) &= 0\\ \lg\frac{9\cdot 2x}{3x+1} &= 0\\ \frac{9\cdot 2x}{3x+1} &= 1\\ 18x &= 3x+1\\ 15x &= 1\\ x &= \frac{1}{15}\end{aligned}$$

(b) Given that $\frac{5^{3x}}{25^y} = 3125$ and $2^x 4^{(y-1)} = 32$, finc the value of x and of y.

Solution:

$$\begin{aligned}\frac{5^{3x}}{25^y} &= 3125\\ 5^{3x} &= 5^{5+2y}\\ 3x &= 5+2y\\ 3x-2y &= 5\ \cdots\ (1)\end{aligned}$$

$$\begin{aligned}2^x 4^{(y-1)} &= 32\\ 2^x 2^{2(y-1)} &= 2^5\\ 2^{x+2y-2} &= 2^5\\ x+2y-2 &= 5\\ x+2y &= 7\ \cdots\ (2)\end{aligned}$$

Adding (1) and (2),

$$\begin{aligned}4x &= 12\\ x &= 3\end{aligned}$$

Substituting $x = 3$ into (1),

$$\begin{aligned}9-2y &= 5\\ y &= 2\end{aligned}$$

(c) Solve the equation $3x - \sqrt{9x^2-20} = 4$.

Solution:

$$\begin{aligned}3x-\sqrt{9x^2-20} &= 4\\ \sqrt{9x^2-20} &= 3x-4\\ 9x^2-20 &= 9x^2-24x+16\\ 24x &= 36\\ x &= \frac{3}{2}\end{aligned}$$

14. (a) Solve the equation $2\lg 15 + \lg(5-x) - \lg 4x = 2$.

Solution:

$$\begin{aligned}
2\lg 15 + \lg(5-x) - \lg 4x &= 2 \\
\lg \frac{15^2(5-x)}{4x} &= 2 \\
\frac{15^2(5-x)}{4x} &= 100 \\
225(5-x) &= 400x \\
5 - x &= \frac{16x}{9} \\
45 - 9x &= 16x \\
25x &= 45 \\
x &= \frac{9}{5}
\end{aligned}$$

(b) Solve the simultaneous equations $\frac{125^x}{25^y} = 625$, $2 \times 4^x = 32^y$.

Solution:

$$\begin{aligned}
\frac{125^x}{25^y} &= 625 \\
\frac{5^{3x}}{5^{2y}} &= 5^4 \\
5^{3x-2y} &= 5^4 \\
3x - 2y &= 4 \ \cdots \ (1)
\end{aligned}$$

$$\begin{aligned}
2 \cdot 4^x &= 32^y \\
2 \cdot 2^{2x} &= 2^{5y} \\
2x + 1 &= 5y \\
2x - 5y &= -1 \ \cdots \ (2)
\end{aligned}$$

Multiplying (1) by 2,

$$6x - 4y = 8 \ \cdots \ (3)$$

Multiplying (2) by 3,

$$6x - 15y = -3 \ \cdots \ (4)$$

Subtracting (4) from (3),

$$\begin{aligned}
11y &= 11 \\
y &= 1
\end{aligned}$$

Substituting $y = 1$ into (1),

$$\begin{aligned}
3x - 2 &= 4 \\
x &= 2
\end{aligned}$$

(c) Without using tables or a calculator, find the value of $\frac{3}{\sqrt{2}-1} - \frac{6}{\sqrt{2}}$

Solution:

$$\begin{aligned}
\frac{3}{\sqrt{2}-1} - \frac{6}{\sqrt{2}} &= 3(\sqrt{2}-1) - 3\sqrt{2} \\
&= 3
\end{aligned}$$

15. (a) Given that $\log_a N = \frac{1}{2}(\log_a 24 - \log_a 0.375 - 6\log_a 3\ )$, find the value of N.

Find also the value of $\log_a N$ when $a = \frac{2}{3}$.

Solution:

$$\begin{aligned}
\log_a N &= \frac{1}{2}(\log_a 24 - \log_a 0.375 - 6\log_a 3) \\
&= \frac{1}{2}(\log_a 24 - \log_a \frac{3}{8} - \log_a 3^6) \\
&= \frac{1}{2}\log_a \left(\frac{2^6}{3^6}\right) \\
&= \log_a \frac{2^3}{3^3} \\
&= \log_a \frac{8}{27} \\
N &= \frac{8}{27}
\end{aligned}$$

$$\begin{aligned}
\log_{\frac{2}{3}} \frac{8}{27} &= \log_{\frac{2}{3}} \left(\frac{2}{3}\right)^3 \\
&= 3
\end{aligned}$$

(b) Find the value of x which satisfies the equation $\sqrt{3x-5} - \sqrt{x+2} = \sqrt{x-6}$

Solution:

$$\begin{aligned}
\sqrt{3x-5} - \sqrt{x+2} &= \sqrt{x-6} \\
4x - 3 - 2\sqrt{(3x-5)(x+2)} &= x - 6 \\
2\sqrt{3x^2 + x - 10} &= 3x + 3 \\
4(3x^2 + x - 10) &= 9x^2 + 18x + 9 \\
12x^2 + 4x - 40 &= 9x^2 + 18x + 9 \\
3x^2 - 14x - 49 &= 0 \\
(3x+7)(x-7) &= 0 \\
x &= -\frac{7}{3} \text{ or } 7
\end{aligned}$$

Upon checking, $x = 7$ is the only valid solution.

16. Without using tables or a calculator, solve the following equations.

 (i) $\lg x - \lg\left(\frac{10}{x^2}\right) = 2$

 Solution:

$$\begin{aligned} \lg x - \lg\left(\frac{10}{x^2}\right) &= 2 \\ \lg \frac{x^3}{10} &= 2 \\ \frac{x^3}{10} &= 10^2 \\ x^3 &= 1000 \\ x &= 10 \end{aligned}$$

 (ii) $3^{y^2+3} = 9^{2y}$

 Solution:

$$\begin{aligned} 3^{y^2+3} &= 9^{2y} \\ 3^{y^2+3} &= 3^{4y} \\ y^2+3 &= 4y \\ y^2-4y+3 &= 0 \\ (y-1)(y-3) &= 0 \\ y &= 1 \text{ or } 3 \end{aligned}$$

 (iii) $\log_z 16 = 8$.

 Solution:

$$\begin{aligned} \log_z 16 &= 8 \\ \frac{\log_2 16}{\log_2 z} &= 8 \\ \frac{4}{\log_2 z} &= 8 \\ \log_2 z &= \frac{1}{2} \\ z &= 2^{\frac{1}{2}} \\ &= \sqrt{2} \end{aligned}$$

17. Solve the equations

 (i) $2^{x-1} = 10$,

 Solution:

$$\begin{aligned} 2^{x-1} &= 10 \\ x-1 &= \log_2 10 \\ &\approx 3.32 \\ x &\approx 4.32 \end{aligned}$$

 (ii) $\log_y 8 = \frac{1}{3}$

 Solution:

$$\begin{aligned} \log_y 8 &= \frac{1}{3} \\ y^{\frac{1}{3}} &= 8 \\ y &= 8^3 \\ &= 512 \end{aligned}$$

 (iii) $z + \sqrt{32-z} = 2$.

 Solution:

$$\begin{aligned} z + \sqrt{32-z} &= 2 \\ \sqrt{32-z} &= 2-z \\ 32-z &= 4-4z+z^2 \\ z^2-3z-28 &= 0 \\ (z-7)(z+4) &= 0 \\ z &= 7 \text{ or } -4 \end{aligned}$$

18. Solve the equations

 (i) $3^{x+1} = 7$,

 Solution:

$$\begin{aligned} 3^{x+1} &= 7 \\ x+1 &= \log_3 7 \\ &\approx 1.771 \\ x &\approx 0.771 \end{aligned}$$

 (ii) $y = \sqrt{y+9} + 3$,

 Solution:

$$\begin{aligned} y &= \sqrt{y+9} + 3 \\ y-3 &= \sqrt{y+9} \\ y^2-6y+9 &= y+9 \\ y(y-7) &= 0 \\ y &= 0 \text{ or } 7 \end{aligned}$$

 (iii) $2\lg z = \lg(3z+4)$.

 Solution:

$$\begin{aligned} 2\lg z &= \lg(3z+4) \\ \lg z^2 &= \lg(3z+4) \\ z^2-3z &= 4 \\ (z-4)(z+1) &= 0 \\ z &= 4 \text{ or } -1 \end{aligned}$$

 Upon checking, $z = 4$ is the only valid solution.

19. (a) Solve the equations

i. $2 \times 4^{x+1} = 16^{2x}$,

Solution:

$$\begin{aligned} 2 \cdot 4^{x+1} &= 16^{2x} \\ 2 \cdot 2^{2x+2} &= 2^{8x} \\ 2x + 3 &= 8x \\ 3 &= 6x \\ x &= \frac{1}{2} \end{aligned}$$

ii. $\log_2 y^2 = 4 + \log_2(y+5)$.

Solution:

$$\begin{aligned} \log_2 y^2 &= 4 + \log_2(y+5) \\ \log_2 y^2 &= \log_2 16(y+5) \\ y^2 &= 16(y+5) \\ y^2 - 16y &= 80 \\ (y-20)(y+4) &= 0 \\ y &= 20 \text{ or } -4 \end{aligned}$$

(b) Given that $y = ax^n + 3$, that $y = 4.4$ when $x = 10$ and $y = 12.8$ when $x = 100$, find the value of n and of a.

Solution:

When $x = 10$, $y = 4.4$,

$$\begin{aligned} 4.4 &= a \cdot 10^n + 3 \\ a &= \frac{1.4}{10^n} \end{aligned}$$

When $x = 100$, $y = 12.8$,

$$\begin{aligned} 12.8 &= a \cdot 100^n + 3 \\ a &= \frac{9.8}{100^n} \end{aligned}$$

$$\begin{aligned} \frac{1.4}{10^n} &= \frac{9.8}{100^n} \\ 9.8 \cdot 10^n &= 1.4 \cdot 10^{2n} \\ 10^n &= 7 \\ n &= \log_{10} 7 \\ &\approx 0.8451 \end{aligned}$$

Substituting $n = \log_{10} 7$ into $a = \dfrac{1.4}{10^n}$,

$$\begin{aligned} a &= \frac{1.4}{10^{\log_{10} 7}} \\ &= \frac{1.4}{7} \\ &= 0.2 \end{aligned}$$

20. (a) By using the substitution $y = e^x$, find the value of x such that $e^{2x} = e^x + 12$.

Solution:

$$\begin{aligned} e^{2x} &= e^x + 12 \\ y^2 &= y + 12 \\ y^2 - y - 12 &= 0 \\ (y-4)(y+3) &= 0 \\ y &= 4 \text{ or } -3 \end{aligned}$$

When $y = 4$, $x = \ln 4 \approx 1.39$.
When $y = -3$, $x = \ln(-3)$, which is not a real number.

(b) Given that $y = ax^b + 2$, and that $y = 7$ when $x = 3$ and $y = 52$ when $x = 9$, find the value of a and of b.

Solution: When $x = 3$, $y = 7$,

$$\begin{aligned} 7 &= a \cdot 3^b + 2 \\ a &= \frac{5}{3^b} \end{aligned}$$

When $x = 9$, $y = 52$,

$$\begin{aligned} 52 &= a \cdot 9^b + 2 \\ a &= \frac{50}{9^b} \end{aligned}$$

$$\begin{aligned} \frac{5}{3^b} &= \frac{50}{9^b} \\ 9^b \cdot 5 &= 3^b \cdot 50 \\ 3^{2b} \cdot 5 &= 3^b \cdot 50 \\ 3^{2b-b} &= 10 \\ 3^b &= 10 \\ b &= \log_3 10 \\ &\approx 2.1 \end{aligned}$$

Substituting $b = \log_3 10$ into $a = \dfrac{5}{3^b}$,

$$\begin{aligned} a &= \frac{5}{3^{\log_3 10}} \\ &= \frac{5}{10} \\ &= \frac{1}{2} \end{aligned}$$

(c) Given that $\log_b\left(x^3y\right) = p$ and $\log_b\left(\frac{y}{x^2}\right) = q$, express $\log_b(xy)$ in terms of p and q.

Solution:

$$\begin{aligned}\log_b(x^3y) &= p\\ \log_b(y) &= p - 3\log_b x\end{aligned}$$

$$\begin{aligned}\log_b\left(\frac{y}{x^2}\right) &= q\\ \log_b(y) - 2\log_b x &= q\\ \log_b(y) &= q + 2\log_b x\\ p - 3\log_b x &= q + 2\log_b x\\ p &= 5\log_b x + q\\ \log_b x &= \frac{p-q}{5}\end{aligned}$$

$$\begin{aligned}\log_b(xy) &= \log_b x + \log_b y\\ &= \frac{p-q}{5} + p - 3\cdot\frac{p-q}{5}\\ &= \frac{-2p+2q}{5} + p\\ &= \frac{3p+2q}{5}\end{aligned}$$

21. (a) Sketch the graph of $y = \ln x$ for $x > 0$. Express $xe^x = 7.39$ in the form $\ln x = ax + b$ and state the value of a and of b. Insert on your sketch the additional graph required to illustrate how a graphical solution of the equation $xe^x = 7.39$ may be obtained.

Solution:

Lazy to draw the graph. :P

(b) Given that $\log_3 x = r$ and $\log_9 y = s$ express xy^2 and $\frac{x^2}{y}$ as powers of 3 . Hence, given that $xy^2 = 81$ and $\frac{x^2}{y} = \frac{1}{3}$, determine the value of r and of s.

Solution:

$$\begin{aligned}\log_3 x &= r\\ x &= 3^r\end{aligned}$$

$$\begin{aligned}\log_9 y &= s\\ y &= 9^s\\ &= 3^{2s}\end{aligned}$$

$$\begin{aligned}xy^2 &= 3^r \cdot 3^{4s}\\ &= 3^{r+4s}\end{aligned}$$

$$\begin{aligned}\frac{x^2}{y} &= \frac{3^{2r}}{3^{2s}}\\ &= 3^{2r-2s}\end{aligned}$$

$$\begin{aligned}xy^2 &= 81\\ 3^{r+4s} &= 81\\ r + 4s &= 4\\ r &= 4 - 4s\end{aligned}$$

$$\begin{aligned}\frac{x^2}{y} &= \frac{1}{3}\\ 3^{2r-2s} &= \frac{1}{3}\\ 2r - 2s &= -1\\ 2(4-4s) - 2s &= -1\\ 8 - 8s - 2s &= -1\\ 10s &= 9\\ s &= \frac{9}{10}\end{aligned}$$

Substituting $s = \frac{9}{10}$ into $r = 4 - 4s$,

$$\begin{aligned}r &= 4 - 4\cdot\frac{9}{10}\\ &= 4 - \frac{18}{5}\\ &= \frac{2}{5}\end{aligned}$$

22. (a) Solve the equation $5^{x+1} = 6$.

Solution:

$$\begin{aligned}5^{x+1} &= 6\\ x + 1 &= \log_5 6\\ &\approx 1.113\\ x &\approx 0.113\end{aligned}$$

(b) Solve the equation $\log_2 x + \log_2(6x+1) = 1$.

Solution:

$$\begin{aligned}
\log_2 x + \log_2(6x+1) &= 1 \\
\log_2 x(6x+1) &= 1 \\
x(6x+1) &= 2 \\
6x^2 + x &= 2 \\
6x^2 + x - 2 &= 0 \\
(3x+2)(2x-1) &= 0 \\
x &= -\frac{2}{3} \text{ or } \frac{1}{2}
\end{aligned}$$

Upon checking, $x = \frac{1}{2}$ is the only valid solution.

(c) Given that $\lg x = a$ and $\lg y = b$, express $\lg \sqrt{\frac{1000x^3}{y}}$ in terms of a and b.

Solution:

$$\begin{aligned}
\lg \sqrt{\frac{1000x^3}{y}} &= \frac{1}{2} \lg \frac{1000x^3}{y} \\
&= \frac{1}{2}(\lg 1000 + \lg x^3 - \lg y) \\
&= \frac{1}{2}(3 + 3a - b) \\
&= \frac{3}{2} + \frac{3a}{2} - \frac{b}{2}
\end{aligned}$$

(d) Sketch the graph of $y = e^{2x}$, for $-1 \leq x \leq 2$, and state the coordinates of the point where the graph crosses the y-axis.

Solution:

Lazy to draw the graph. :P

(e) Sketch the graph of $y = \ln 3x$, for $0 \leq x \leq 2$, and state the coordinates of the point where the graph crosses the x - axis.

Solution:

Lazy to draw the graph. :P

23. (a) Draw the graph of $y = e^x$ for $0 \leq x \leq 1$, taking intervals of 0.25 .

By drawing a straight line on your diagram, obtain an approximate solution to the equation $e^x = 5 - 5x$

Solution:

Lazy to draw the graph. :P

(b) Solve the equation $\lg\left(x^2 + 12x - 3\right) = 1 + 2\lg x$.

(c) By means of the substitution $y = 2^x$, find the value of x such that $2^{x+2} - 3 = 7 \times 2^{x-1}$

(d) Solve the equations

(i) $\lg\left(x^2 - 2x + 8\right) = 2\lg x$,

Solution:

$$\begin{aligned}
\lg\left(x^2 - 2x + 8\right) &= 2\lg x \\
\lg\left(x^2 - 2x + 8\right) &= \lg x^2 \\
x^2 - 2x + 8 &= x^2 \\
-2x + 8 &= 0 \\
x &= 4
\end{aligned}$$

(ii) $3^y = 7$

Solution:

$$\begin{aligned}
3^y &= 7 \\
y &= \log_3 7 \\
&\approx 1.77
\end{aligned}$$

(iii) $\lg 5z - \lg(3 - 2z) = 1$

Solution:

$$\begin{aligned}
\lg 5z - \lg(3 - 2z) &= 1 \\
\lg \frac{5z}{3 - 2z} &= 1 \\
\frac{5z}{3 - 2z} &= 10 \\
5z &= 30 - 20z \\
25z &= 30 \\
z &= \frac{6}{5}
\end{aligned}$$

(e) i. Solve the equation $2^x = 5$.

Solution:

$$\begin{aligned}
2^x &= 5 \\
x &= \log_2 5 \\
&\approx 2.32
\end{aligned}$$

ii. Solve the equation $\lg x + \lg(3x+1) = 1$.

Solution:

$$\begin{aligned}
\lg x + \lg(3x+1) &= 1 \\
\lg x(3x+1) &= 1 \\
x(3x+1) &= 10 \\
3x^2 + x &= 10 \\
3x^2 + x - 10 &= 0 \\
(3x-5)(x+2) &= 0 \\
x &= \frac{5}{3} \text{ or } -2
\end{aligned}$$

Upon checking, $x = \frac{5}{3}$ is the only valid solution.

iii. By using the substitution $y = e^x$, find the value if x such that $8e^{-x} - e^x = 2$.

Solution:

$$\begin{aligned}
8e^{-x} - e^x &= 2 \\
\frac{8}{y} - y &= 2 \\
8 - y^2 &= 2y \\
y^2 + 2y - 8 &= 0 \\
(y+4)(y-2) &= 0 \\
y &= -4 \text{ or } 2
\end{aligned}$$

When $y = 2$, $x = \ln 2$.
When $y = -4$, $x = \ln(-4)$, which is not a real number.

iv. Given that $y = ax^b$, that $y = 2$ when $x = 3$ and that $y = \frac{2}{9}$ when $x = 9$, find the value of a and of b.

Solution: When $x = 3$, $y = 2$,

$$\begin{aligned}
2 &= a \cdot 3^b \\
a &= \frac{2}{3^b}
\end{aligned}$$

When $x = 9$, $y = \frac{2}{9}$,

$$\begin{aligned}
\frac{2}{9} &= a \cdot 9^b \\
a &= \frac{2}{9^{b+1}}
\end{aligned}$$

$$\begin{aligned}
\frac{2}{3^b} &= \frac{2}{9^{b+1}} \\
9^{b+1} &= 3^b \\
3^{2b+2} &= 3^b \\
2b + 2 &= b \\
b &= -2 \\
a &= \frac{2}{3^{-2}} \\
&= \frac{2}{\frac{1}{9}} \\
&= 18
\end{aligned}$$

24. (a) The population of a village at the beginning of the year 1800 was 240 . The population increased so that, after a period of n years, the new population was $240(1.06)^n$, Find

(i) the population at the beginning of 1820,

Solution:

$$\begin{aligned}
240(1.06)^n &= 240(1.06)^{20} \\
&\approx 770
\end{aligned}$$

(ii) the year in which the population first reached 2500.

Solution:

$$\begin{aligned}
240(1.06)^n &= 2500 \\
(1.06)^n &= \frac{2500}{240} \\
&= \frac{25}{24} \\
n &= \log_{1.06} \frac{250}{24} \\
&= \frac{\log \frac{250}{24}}{\log 1.06} \\
&\approx 40
\end{aligned}$$

$\therefore$ The year is 1840.

(b) Find the value of q for which $\lg q = 1 + \lg 2 - 2\lg 5$.

Solution:

$$\begin{aligned}\lg q &= 1 + \lg 2 - 2\lg 5\\ &= \lg(\frac{2 \cdot 10}{25})\\ &= \lg \frac{4}{5}\\ q &= \frac{4}{5}\end{aligned}$$

(c) By using the substitution $u = 2^x$, solve the equation $4^x - 9(2^x) + 8 = 0$.

Solution:

$$\begin{aligned}4^x - 9 \cdot 2^x + 8 &= 0\\ u^2 - 9u + 8 &= 0\\ (u-8)(u-1) &= 0\\ u &= 8 \text{ or } 1\\ 2^x &= 8 \text{ or } 1\\ x &= 3 \text{ or } 0\end{aligned}$$

(d) Sketch the curve $y = e^{2x-1}$ and calculate, correct to two decimal places, the gradient of the curve at the point where it meets the y-axis.

Solution:

Lazy to draw the graph. :P

25. (a) The curve $y = ab^x$ passes through the points $(1, 96), (2, 1152)$ and $(3, p)$. Find the exact values of a, b and p.

Solution: When $x = 1$, $y = 96$,

$$\begin{aligned}96 &= ab^1\\ a &= \frac{96}{b}\end{aligned}$$

When $x = 2$, $y = 1152$,

$$\begin{aligned}1152 &= ab^2\\ a &= \frac{1152}{b^2}\end{aligned}$$

$$\begin{aligned}\frac{96}{b} &= \frac{1152}{b^2}\\ 96b &= 1152\\ b &= 12\end{aligned}$$

Substituting $b = 12$ into $a = \dfrac{96}{b}$,

$$\begin{aligned}a &= \frac{96}{12}\\ &= 8\end{aligned}$$

When $x = 3$, $y = p$,

$$\begin{aligned}p &= 8 \cdot 12^3\\ &= 13824\end{aligned}$$

(b) Solve the equation $\lg(4x+5) = 1+\lg(x-1)$.

Solution:

$$\begin{aligned}\lg(4x+5) &= 1 + \lg(x-1)\\ \lg(4x+5) &= \lg 10(x-1)\\ 4x+5 &= 10(x-1)\\ 4x+5 &= 10x - 10\\ 6x &= 15\\ x &= \frac{5}{2}\end{aligned}$$

(c) Find the coordinates of the stationary point of the curve $y = xe^{-x}$.
Draw the curve $y = xe^{-x}$ for $-1 \le x \le 2$ and use your graph to estimale the solution of the equation $x + e^x = 0$.

Solution:

Lazy to draw the graph. :P

26. (a) Solve the equation

(i) $3\lg(x-1) = \lg 8$,

Solution:

$$\begin{aligned}3\lg(x-1) &= \lg 8\\ \lg(x-1)^3 &= \lg 8\\ (x-1)^3 &= 8\\ x - 1 &= 2\\ x &= 3\end{aligned}$$

(ii) $\lg(20y) - \lg(y-8) = 2$.

Solution:

$$\begin{aligned}\lg(20y) - \lg(y-8) &= 2\\ \lg\frac{20y}{y-8} &= 2\\ \frac{20y}{y-8} &= 100\\ 20y &= 100y - 800\\ 80y &= 800\\ y &= 10\end{aligned}$$

(b) By using the substitution $y = e^{2x}$, solve the equation

$$e^{2x} + 4e^{-2x} = 4$$

Solution:

$$\begin{aligned}
e^{2x} + 4e^{-2x} &= 4 \\
y + \frac{4}{y} &= 4 \\
y^2 + 4 &= 4y \\
y^2 - 4y + 4 &= 0 \\
(y-2)^2 &= 0 \\
y &= 2 \\
e^{2x} &= 2 \\
2x &= \ln 2 \\
x &= \frac{\ln 2}{2}
\end{aligned}$$

27. Solve

(a) $3^x = 2$,

Solution:

$$\begin{aligned}
3^x &= 2 \\
x &= \log_3 2 \\
&\approx 0.631
\end{aligned}$$

(b) $\log_3(4x) + \log_3(x-1) = 1$.

Solution:

$$\begin{aligned}
\log_3(4x) + \log_3(x-1) &= 1 \\
\log_3[4x(x-1)] &= 1 \\
4x(x-1) &= 3 \\
4x^2 - 4x &= 3 \\
4x^2 - 4x - 3 &= 0 \\
(2x-3)(2x+1) &= 0 \\
x &= -\frac{1}{2} \text{ or } \frac{3}{2}
\end{aligned}$$

Upon checking, $x = \dfrac{3}{2}$ is the only valid solution.