Solution Book of Mathematic

Senior 2 Part I

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Contents

Chapter 12

Sequence and Series

12.1 Sequence and Series

12.1.1 Practice 1

1. Find the first 5 terms of the sequence $a_n = \frac{2^n}{n+1}$.

   sol. $a_1 = \frac{2}{2} = 1, a_2 = \frac{4}{3}, a_3 = \frac{8}{4}, a_4 = \frac{16}{5}, a_5 = \frac{32}{6}$

2. Write the general term of the sequence $1, 8, 27, 64, \cdots$

   sol. $a_n = n^3$

12.1.2 Practice 2

1. Express the series $\sum_{n=1}^{10} n^2 + 1$ in the form of numbers.

   sol. $\displaystyle\sum_{n=1}^{10} n^2 + 1$
   $$= (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1)$$
   $$+ (5^2 + 1) + (6^2 + 1) + (7^2 + 1)$$
   $$+ (8^2 + 1) + (9^2 + 1) + (10^2 + 1)$$
   $$= 2 + 5 + 10 + 17 + 26 + 37 + 50 + 65$$
   $$+ 82 + 101$$

2. Write the first term, last term and the number of terms of the series $\sum_{n=1}^{10} (3^n - 2^n)$.

   sol. $First\ term = (3^1 - 2^1) = 1$
   $Last\ term = (3^{10} - 2^{10}) = 59049$
   $Number\ of\ terms = 10$

3. Express the series $2 \cdot 5 + 3 \cdot 7 + 4 \cdot 9 + \cdots + 15 \cdot 31$ in the form of $\sum$.

   sol.
   $$a_1 = 2 \cdot 5 = 10$$
   $$a_2 = 3 \cdot 7 = 21$$
   $$a_3 = 4 \cdot 9 = 36$$
   $$a_4 = 5 \cdot 11 = 55$$
   $$\vdots$$
   $$a_{15} = 15 \cdot 31 = 465$$
   $$\therefore 2 \cdot 5 + 3 \cdot 7 + 4 \cdot 9 + \cdots + 15 \cdot 31$$
   $$= \sum_{n=1}^{15} a_n$$

12.1.3 Exercise 12.1

1. Find the general term of the following sequences.

   (a) $5, 8, 11, 14, \ldots$
   sol. $a_n = 3n + 2$

   (b) $2, 4, 8, 16, \ldots$
   sol. $a_n = 2^n$

   (c) $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \cdots$
   sol. $a_n = \frac{n+1}{n}$

   (d) $\frac{2}{5}, \frac{4}{7}, \frac{6}{9}, \frac{8}{11}, \cdots$
   sol. $a_n = \frac{2n}{2n+1}$

2. Find the first 5 terms of the following sequences.

   (a) $a_n = 2n + 3$
   sol. $a_1 = 2 \cdot 1 + 3 = 5, a_2 = 2 \cdot 2 + 3 = 7, a_3 = 2 \cdot 3 + 3 = 9, a_4 = 2 \cdot 4 + 3 = 11, a_5 = 2 \cdot 5 + 3 = 13$

   (b) $a_n = n(n - 2)$
   sol. $a_1 = 1 \cdot (-1) = -1, a_2 = 2 \cdot 0 = 0, a_3 = 3 \cdot 1 = 3, a_4 = 4 \cdot 2 = 8, a_5 = 5 \cdot 3 = 15$

   (c) $a_n = \frac{n}{2n+1}$
   sol. $a_1 = \frac{1}{2 \cdot 1+1} = \frac{1}{3}, a_2 = \frac{2}{2 \cdot 2+1} = \frac{2}{5}, a_3 = \frac{3}{2 \cdot 3+1} = \frac{3}{7}, a_4 = \frac{4}{2 \cdot 4+1} = \frac{4}{9}, a_5 = \frac{5}{2 \cdot 5+1} = \frac{5}{11}$

   (d) $a_n = (-3)^n$
   sol. $a_1 = (-3)^1 = -3, a_2 = (-3)^2 = 9, a_3 = (-3)^3 = -27, a_4 = (-3)^4 = 81, a_5 = (-3)^5 = -243$

3. Express the following series in the form of numbers.

   (a) $\sum_{n=1}^{5} n(n + 3)$

sol. $\sum_{n=1}^{5} n(n+3)$
$= (1 \cdot 4) + (2 \cdot 5) + (3 \cdot 6) + (4 \cdot 7)$
$+ (5 \cdot 8)$
$= 4 + 10 + 18 + 28 + 40$

(b) $\sum_{n=2}^{6} \frac{1}{3^n}$

sol. $\sum_{n=2}^{6} \frac{1}{3^n}$
$= \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6}$
$= \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729}$

(c) $\sum_{n=1}^{6} \frac{1}{n(2n+1)}$

sol. $\sum_{n=1}^{6} \frac{1}{n(2n+1)}$
$= \frac{1}{1(2 \cdot 1 + 1)} + \frac{1}{2(2 \cdot 2 + 1)}$
$+ \frac{1}{3(2 \cdot 3 + 1)} + \frac{1}{4(2 \cdot 4 + 1)}$
$+ \frac{1}{5(2 \cdot 5 + 1)} + \frac{1}{6(2 \cdot 6 + 1)}$
$= \frac{1}{3} + \frac{1}{10} + \frac{1}{21} + \frac{1}{36} + \frac{1}{55} + \frac{1}{78}$

(d) $\sum_{n=2}^{5} \frac{1}{n^2+2}$

sol. $\sum_{n=2}^{5} \frac{1}{n^2+2}$
$= \frac{1}{4+2} + \frac{1}{9+2} + \frac{1}{16+2} + \frac{1}{25+2}$
$= \frac{1}{6} + \frac{1}{11} + \frac{1}{18} + \frac{1}{27}$

4. Find the first term, last term and the number of terms of the following series.

(a) $\sum_{n=3}^{10} 2^2$
sol. $a_3 = 2^2 = 4, a_{10} = 2^2 = 4, n = 10 - 3 + 1 = 8$

(b) $\sum_{n=1}^{8} \frac{n+2}{n}$

sol. $a_1 = \frac{1+2}{1} = \frac{3}{1} = 3, a_8 = \frac{8+2}{8} = \frac{10}{8} = \frac{5}{4}, n = 8 - 1 + 1 = 8$

(c) $\sum_{n=1}^{10} 3n^2 - n$

sol. $a_1 = 3 \cdot 1^2 - 1 = 2, a_{10} = 3 \cdot 10^2 - 10 = 290, n = 10 - 1 + 1 = 10$

(d) $\sum_{n=9}^{14} n^2(n-7)$

sol. $a_9 = 9^2(9-7) = 9^2 \cdot 2 = 162, a_{14} = 14^2(14-7) = 14^2 \cdot 7 = 2744, n = 14 - 9 + 1 = 6$

5. Express the following series in the form of $\sum$.

(a) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{30}$

Sol.

$$a_1 = 1$$
$$a_2 = \frac{1}{2}$$
$$a_3 = \frac{1}{3}$$
$$\vdots$$
$$a_{30} = \frac{1}{30}$$
$$\therefore 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{30} = \sum_{n=1}^{30} \frac{1}{n}$$

(b) $1^3 + 2^3 + 3^3 + \cdots + 50^3$

Sol.

$$a_1 = 1^3$$
$$a_2 = 2^3$$
$$a_3 = 3^3$$
$$\vdots$$
$$a_{50} = 50^3$$
$$\therefore 1^3 + 2^3 + 3^3 + \cdots + 50^3 = \sum_{n=1}^{50} n^3$$

(c) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16}$

Sol.

$$a_1 = (-\frac{1}{2})^{1-1}$$
$$a_2 = (-\frac{1}{2})^{2-1}$$
$$a_3 = (-\frac{1}{2})^{3-1}$$
$$a_4 = (-\frac{1}{2})^{4-1}$$
$$a_5 = (-\frac{1}{2})^{5-1}$$
$$\therefore 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} = \sum_{n=1}^{5} (-\frac{1}{2})^{n-1}$$

(d) $2 \cdot 4 + 4 \cdot 7 + 6 \cdot 10 + 8 \cdot 13 + 10 \cdot 16$

Sol.

$$a_1 = 2 \cdot 1 \cdot (3 \cdot 1 + 1)$$
$$a_2 = 2 \cdot 2 \cdot (3 \cdot 2 + 1)$$
$$a_3 = 2 \cdot 3 \cdot (3 \cdot 3 + 1)$$
$$a_4 = 2 \cdot 4 \cdot (3 \cdot 4 + 1)$$
$$a_5 = 2 \cdot 5 \cdot (3 \cdot 5 + 1)$$
$$\therefore 2 \cdot 4 + 4 \cdot 7 + 6 \cdot 10 + 8 \cdot 13 + 10 \cdot 16 = \sum_{n=1}^{5} 2n(3n+1)$$

12.2 Arithmetic Progression

General term of an Arithmetic Progression (AP) is given by

$$a_n = a_1 + (n-1)d$$

where a_1 is the first term, d is the common difference and n is the number of terms.

12.2.1 Practice 3

1. Find the number of terms of the AP $-4 - 2\frac{3}{4} - 1\frac{1}{2} - \frac{1}{4} + \cdots + 16$.

$$a_1 = -4$$
$$a_n = 16$$
$$d = -2\frac{3}{4} - (-4) = -2\frac{3}{4} + 4 = \frac{5}{4}$$
$$16 = -4 + (n-1)\frac{5}{4}$$
$$20 = \frac{5}{4}(n-1)$$
$$80 = 5(n-1)$$
$$n - 1 = 16$$
$$n = 17$$

2. Given that $a_2 = 4$ and $a_6 = -8$, find the 10th term of the AP.

Sol.

$$a_2 = 4$$
$$a + (2-1)d = 4$$
$$a_6 = -8$$
$$a + (6-1)d = -8$$

$$\begin{cases} a + d = 4 & (1) \\ a + 5d = \text{-8} & (2) \end{cases}$$

$$(2) - (1) : 4d = -12$$
$$d = -3$$
$$a + (-3) = 4$$
$$a = 7$$
$$\therefore a_{10} = 7 + (10-1)(-3) = 7 - 27 = -20$$

3. How many multiples of 7 are there between 50 and 500?

Sol.

$$\begin{aligned} a_1 &= 56 \\ a_n &= 497 \\ d &= 7 \\ 497 = 56 + (n-1)7 \\ 441 = 7(n-1) \\ n-1 &= 63 \\ n &= 64 \end{aligned}$$

4. Find 5 numbers between 30 and 54 such that these numbers form an AP.

   Sol.

$$\begin{aligned} &a_1 = 30 \\ &a_7 = 54 \\ &54 = 30 + (7-1)d \\ &24 = 6d \\ &d = 4 \end{aligned}$$

$$\therefore These\ 5\ numbers\ are\ 34,\ 38,\ 42,\ 46,\ and\ 50.$$

Arithmetic mean

If A is in between x and y, and x, A, y are in AP, then

$$A = \frac{x+y}{2}$$

12.2.2 Practice 4

1. If 9, x, 17 are in AP, find x.

   Sol.

$$\begin{aligned} x &= \frac{9+17}{2} \\ &= \frac{26}{2} \\ &= 13 \end{aligned}$$

2. Find the arithmetic mean of 26 and -11.

   Sol.

$$\begin{aligned} A &= \frac{26-11}{2} \\ &= \frac{15}{2} \end{aligned}$$

3. Find x and y when 3, x, 12, y, 21 are in AP.

   Sol.

$$\begin{aligned} x &= \frac{3+12}{2} \\ &= \frac{15}{2} \\ y &= \frac{12+21}{2} \\ &= \frac{33}{2} \end{aligned}$$

Summation of Arithmetic Progression

The summation formula for AP is given by

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

or

$$S_n = \frac{n}{2}(a_1 + a_n)$$

12.2.3 Practice 5

1. Find the sum of the first 16 terms of the AP $22 + 18 + 14 + 10 + \cdots$

   Sol.

$$\begin{aligned} a_1 &= 22 \\ n &= 16 \\ d &= -4 \\ S_n &= \frac{16}{2}(2 \cdot 22 + (-4)(16-1)) \\ &= \frac{16}{2}(44 + (-4)(15)) \\ &= \frac{16}{2}(44 - 60) \\ &= \frac{16}{2}(-16) \\ &= -128 \end{aligned}$$

2. If the sum of AP $23+19+15+\cdots$ is 72, find the number of terms.

Sol.

$$\begin{aligned}
a_1 &= 23 \\
S_n &= 72 \\
d &= -4 \\
72 &= \frac{n}{2}(2 \cdot 23 + (-4)(n-1)) \\
72 &= \frac{n}{2}(46 + (-4)(n-1)) \\
144 &= n(46 + (-4)(n-1)) \\
144 &= n(46 - 4n + 4) \\
144 &= n(50 - 4n) \\
144 &= 50n - 4n^2 \\
72 &= 25n - 2n^2
\end{aligned}$$

$$\begin{aligned}
2n^2 - 25n + 72 &= 0 \\
(n-8)(2n-9) &= 0 \\
n &= 8
\end{aligned}$$

3. Given that $S_n = 2n + 3n^2$, find the first term and the common difference of the AP.

Sol.

$$\begin{aligned}
S_n &= 2n + 3n^2 \\
2n + 3n^2 &= \frac{n}{2}(2a + (n-1)d) \\
4n + 6n^2 &= n(2a + (n-1)d) \\
4n + 6n^2 &= 2na + (n-1)nd \\
4n + 6n^2 &= 2na + n^2 d - nd \\
4n + 6n^2 &= (2a - d)n + dn^2
\end{aligned}$$

$$\begin{aligned}
&\textit{Comparing both sides,} \\
2a - d &= 4 \\
a &= 6 \\
d &= 2
\end{aligned}$$

12.2.4 Exercise 12.2

1. Find the 10th terms of the AP $5, 13, 21, \cdots$

Sol.

$$\begin{aligned}
a_1 &= 5 \\
n &= 10 \\
d &= 8 \\
a_{10} &= 5 + (10-1) \cdot 8 \\
&= 5 + 72 \\
&= 77
\end{aligned}$$

2. Find the 8th term of the AP $5, 4\frac{1}{4}, 3\frac{1}{2}, 2\frac{3}{4}, \cdots$

Sol.

$$\begin{aligned}
a_1 &= 5 \\
n &= 8 \\
d &= -\frac{3}{4} \\
a_8 &= 5 + (8-1) \cdot -\frac{3}{4} \\
&= 5 - \frac{3}{4} \cdot 7 \\
&= 5 - \frac{21}{4} \\
&= -\frac{1}{4}
\end{aligned}$$

3. Find the number of terms of the following AP.

(a) $4, 9, \ldots, 64$

Sol.

$$\begin{aligned}
a_1 &= 4 \\
a_n &= 64 \\
d &= 5 \\
64 &= 4 + (n-1) \cdot 5 \\
60 &= 5(n-1) \\
12 &= n - 1 \\
n &= 13
\end{aligned}$$

(b) $4\frac{1}{3}, 3\frac{2}{3}, 3, \ldots, -10\frac{1}{3}$

Sol.

$$
\begin{aligned}
a_1 &= 4\frac{1}{3} \\
a_n &= -10\frac{1}{3} \\
d &= -\frac{2}{3} \\
-10\frac{1}{3} &= 4\frac{1}{3} + (n-1) \cdot -\frac{2}{3} \\
-\frac{31}{3} &= \frac{13}{3} - \frac{1}{3}(n-1) \\
-31 &= 13 - 2n + 2 \\
-46 &= 2n \\
n &= 23
\end{aligned}
$$

4. The 6th term of an AP is 43, and its 10th term is 75. Find the first term and common difference of this AP.

   Sol.

$$
\begin{aligned}
a_6 &= 43 \\
a_{10} &= 75 \\
43 &= a + (6-1)d \\
75 &= a + (10-1)d \\
32 &= 4d \\
d &= 8 \\
43 &= a + 5 \cdot 8 \\
43 &= a + 40 \\
3 &= a \\
a &= 3 \\
\therefore\ a_1 &= 3,\ d = 8
\end{aligned}
$$

5. The 7th term of an AP is -10, and the 12th term -25, find the 15th term of this AP.

   Sol.

$$
\begin{aligned}
a_7 &= -10 \\
a_{12} &= -25 \\
-10 &= a + (7-1)d \\
-25 &= a + (12-1)d \\
-15 &= 5d \\
d &= -3 \\
-10 &= a + 6 \cdot -3 \\
-10 &= a - 18 \\
a &= 8 \\
a_{15} &= 8 + (15-1) \cdot -3 \\
&= 8 - 42 \\
&= -34
\end{aligned}
$$

6. How many multiples of 7 are there between 100 and 200?

   Sol.

$$
\begin{aligned}
a &= 105 \\
d &= 7 \\
a_n &= 196 \\
196 &= 105 + (n-1) \cdot 7 \\
91 &= 7(n-1) \\
13 &= n - 1 \\
n &= 14
\end{aligned}
$$

7. Find the arithmetic mean o fthe following number pairs.

   (a) $(8, 20)$

   Sol.

$$\frac{8+20}{2} = 14$$

   (b) $(-9, 17)$

   Sol.

$$\frac{-9+17}{2} = 4$$

8. Find 5 numbers between 22 and 58 such that these 7 numbers are in AP.

   Sol.

$$
\begin{aligned}
a_1 &= 22 \\
a_7 &= 58 \\
58 &= 22 + (7-1)d \\
36 &= 6d \\
d &= 6 \\
\therefore\ & These\ 5\ numbers\ are\ 22, 28, 34, 40, 46
\end{aligned}
$$

9. Find the sum of first 20 terms of AP $12 + 15 + 18 + \cdots$

   Sol.

$$
\begin{aligned}
a_1 &= 12 \\
n &= 20 \\
d &= 3 \\
S_{20} &= \frac{20}{2}(2 \cdot 12 + (20-1) \cdot 3) \\
&= 10(24 + 57) \\
&= 10(81) \\
&= 810
\end{aligned}
$$

10. Find the sum of first 12 terms of the AP $18 + 10 + 2 - 6 - \dots$

Sol.

$$\begin{aligned}
a_1 &= 18 \\
n &= 12 \\
d &= -8 \\
S_{12} &= \frac{12}{2}(2 \cdot 18 + (12-1) \cdot -8) \\
&= 6(36 - 88) \\
&= 6(-52) \\
&= -312
\end{aligned}$$

11. Find the sum of first 14 terms of the AP $\frac{1}{6} + \frac{4}{3} + \frac{5}{2} + \cdots$

Sol.

$$\begin{aligned}
a_1 &= \frac{1}{6} \\
n &= 14 \\
d &= \frac{7}{6} \\
S_{14} &= \frac{14}{2}(2 \cdot \frac{1}{6} + (14-1) \cdot \frac{7}{6}) \\
&= 7(\frac{1}{3} + \frac{91}{6}) \\
&= 7 \cdot \frac{93}{6} \\
&= 7 \cdot \frac{31}{2} \\
&= \frac{217}{2}
\end{aligned}$$

12. Find the sum of all the multiples of 13 in between 200 and 800.

Sol.

$$\begin{aligned}
a_1 &= 208 \\
a_n &= 793 \\
d &= 13 \\
793 &= 208 + (n-1) \cdot 13 \\
585 &= 13(n-1) \\
45 &= n - 1 \\
n &= 46
\end{aligned}$$

$$\begin{aligned}
S_{46} &= \frac{46}{2}(2 \cdot 208 + (46-1) \cdot 13) \\
&= 23(416 + 585) \\
&= 23(1001) \\
&= 23023
\end{aligned}$$

13. If the sum of first n terms of the AP $-3, -7, -11, \cdots$ is -903, find the value of n.

Sol.

$$\begin{aligned}
a_1 &= -3 \\
d &= -4 \\
-903 &= \frac{n}{2}(2 \cdot (-3) - 4(n-1)) \\
-1806 &= -2n - 4n^2 \\
4n^2 + 2n - 1806 &= 0 \\
2n^2 + n - 903 &= 0 \\
(n-21)(2n+43) &= 0 \\
n &= 21, -43(invalid) \\
\therefore n &= 21
\end{aligned}$$

14. Given that the first 3 terms of an AP are $x,\ 3x-4,\ 2x+7$, find:

    (a) The value of x

    Sol.

$$\begin{aligned}
3x - 4 &= \frac{x + 2x + 7}{2} \\
6x - 8 &= 3x + 7 \\
3x &= 15 \\
x &= 5
\end{aligned}$$

    (b) The common difference

    Sol.

$$\begin{aligned}
a_1 &= x = 5 \\
a_2 &= 3x - 4 = 3 \cdot 5 - 4 = 11 \\
d &= 11 - 5 \\
&= 6
\end{aligned}$$

    (c) The sum of first 10 terms.

    Sol.

$$\begin{aligned}
a_1 &= x = 5 \\
n &= 10 \\
d &= 6 \\
S_{10} &= \frac{10}{2}(2 \cdot 5 + (10-1) \cdot 6) \\
&= 5(10 + 54) \\
&= 5(64) \\
&= 320
\end{aligned}$$

15. Let the sum of the first n terms of an AP to be $S_n = \frac{n(n+1)}{4}$, find:

    (a) The first term

Sol.

$$\begin{aligned}\frac{n(n+1)}{4} &= \frac{n}{2}(2a+(n-1)d)\\ n(n+1) &= 2n(2a+dn-d)\\ n^2+n &= 4na+2dn^2-2nd\\ n^2+n &= 2dn^2+(4a-2d)n\end{aligned}$$

$$\begin{aligned}&Comparing\ both\ sides,\\ 2d &= 1\\ d &= \frac{1}{2}\\ 4a-2d &= 1\\ 4a-1 &= 1\\ 4a &= 2\\ a &= \frac{1}{2}\end{aligned}$$

(b) The common difference

Sol.

$$d = \frac{1}{2}$$

gg

(c) The 6th terms

Sol.

$$\begin{aligned}a_1 &= \frac{1}{2}\\ n &= 6\\ d &= \frac{1}{2}\\ a_6 &= \frac{1}{2}+(6-1)\cdot\frac{1}{2}\\ &= \frac{1}{2}+\frac{5}{2}\\ &= 3\end{aligned}$$

(d) The sum from 6th term to 10th term

Sol.

$$\begin{aligned}a &= \frac{1}{2}\\ d &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}S_{10} &= \frac{10}{2}(2\cdot\frac{1}{2}+(10-1)\cdot\frac{1}{2})\\ &= \frac{10}{2}(1+\frac{9}{2})\\ &= 5\cdot\frac{11}{2}\\ &= \frac{55}{2}\end{aligned}$$

$$\begin{aligned}S_5 &= \frac{5}{2}(2\cdot\frac{1}{2}+(5-1)\cdot\frac{1}{2})\\ &= \frac{5}{2}(1+2)\\ &= \frac{15}{2}\end{aligned}$$

$$\begin{aligned}S_{10}-S_6 &= \frac{55}{2}-\frac{15}{2}\\ &= \frac{40}{2}\\ &= 20\end{aligned}$$

16. Given three numbers in an AP, the sum of these three numbers is 30, and the sum of square of these numbers is 318, find these three numbers.

Sol.

$$\begin{aligned}
a_1 + a_2 + a_3 &= 30 \\
a_1^2 + a_2^2 + a_3^2 &= 318 \\
a_2 - a_1 &= a_3 - a_2 \\
a_1 - 2a_2 + a_3 &= 0 \\
3a_2 &= 30 \\
a_2 &= 10 \\
a_1 - 20 + a_3 &= 0 \\
a_1 + a_3 &= 20 \\
a_3 &= 20 - a_1 \\
a_1^2 + 100 + (20 - a_1)^2 &= 318 \\
a_1^2 + 100 + 400 + a_1^2 - 40a_1 &= 318 \\
2a_1^2 - 40a_1 + 182 &= 0 \\
a_1^2 - 20a_1 + 91 &= 0 \\
(a_1 - 7)(a_1 - 13) &= 0 \\
a_1 &= 7 or a_1 = 13
\end{aligned}$$

$$\therefore These\ three\ numbers\ are\ 7,\ 10,\ and\ 13$$

17. Find the sum of all the numbers between 100 and 200 that are both the multiples of 2 and 3.

Sol.

$$\begin{aligned}
a_1 &= 102 \\
d &= 6 \\
a_n &= 198 \\
198 &= 102 + (n-1) \cdot 6 \\
96 &= 6(n-1) \\
6n - 6 &= 96 \\
6n &= 102 \\
n &= 17
\end{aligned}$$

$$\begin{aligned}
S_{17} &= \frac{17}{2}(2 \cdot 102 + (17-1) \cdot 6) \\
&= \frac{17}{2}(204 + 96) \\
&= \frac{17}{2}(300) \\
&= 150 \cdot 17 \\
&= 2550
\end{aligned}$$

18. Given an AP $-100 - 96 - 92 - \cdots$:

(a) Find the term where the number become positive.

Sol.

$$\begin{aligned}
a_1 &= -100 \\
d &= 4 \\
a_n &= -100 + (n-1) \cdot 4 > 0 \\
-100 + 4n - 4 &> 0 \\
4n &> 104 \\
n &> 26
\end{aligned}$$

$$\therefore n = 27$$

(b) Find the term where the sum of this AP becomes positive.

Sol.

$$\begin{aligned}
S_n &= \frac{n}{2}(2(-100) + (n-1) \cdot (4)) > 0 \\
&\frac{n}{2}(-200 + 4n - 4) > 0 \\
&\frac{n}{2}(-204 + 4n) > 0 \\
&n(2n - 102) > 0 \\
&n(n - 51) > 0 \\
&n > 51
\end{aligned}$$

$$\therefore n = 52$$

19. Find the first negative term of the AP $20, 19\frac{1}{5}, 18\frac{2}{5}, \cdots$

Sol.

$$\begin{aligned}
a_1 &= 20 \\
d &= -\frac{4}{5} \\
a_n &= 20 + (n-1) \cdot (-\frac{4}{5}) < 0 \\
100 - 4n + 4 &< 0 \\
4n &> 104 \\
n &> 26
\end{aligned}$$

$$\therefore n = 27$$

20. Given an AP $10 + 9\frac{1}{5} + 8\frac{2}{5} + \cdots$, what is the first negative term? When will the sum of the terms become negative, and what's the value of it?

Sol.

$$\begin{aligned}
a_n = 10 + (n-1)\cdot(-\frac{4}{5}) &< 0 \\
10 - \frac{4}{5}(n-1) &< 0 \\
50 - 4n + 4 &< 0 \\
-4n &< -54 \\
n &> 13\frac{1}{2} \\
\\
\therefore n &= 14
\end{aligned}$$

$$\begin{aligned}
S_n = \frac{n}{2}(2\cdot 10 + (n-1)\cdot(-\frac{4}{5})) &< 0 \\
\frac{n}{2}(20 - \frac{4}{5}(n-1)) &< 0 \\
20n - \frac{4}{5}(n^2 - n) &< 0 \\
100n - 4n^2 + 4n &< 0 \\
25n - n^2 + n &< 0 \\
26n - n^2 &< 0 \\
n(n-26) &> 0 \\
n &> 26 \\
\\
\therefore n &= 27
\end{aligned}$$

$$\begin{aligned}
S_{27} &= \frac{27}{2}(2\cdot 10 + (27-1)\cdot(-\frac{4}{5})) \\
&= \frac{27}{2}(20 - \frac{4}{5}(27-1)) \\
&= \frac{27}{2}(20 - \frac{4}{5}(26)) \\
&= \frac{27}{2}\cdot(-\frac{4}{5}) \\
&= -\frac{54}{5}
\end{aligned}$$

$\therefore The\ first\ negative\ term\ is\ the\ 14th\ term$

$\therefore The\ first\ term\ where\ the\ sum\ of\ the\ terms\ becomes\ negative\ is\ the\ 27th\ term$

$\therefore The\ value\ of\ the\ sum\ of\ the\ terms\ when\ it\ becomes\ negative\ is\ -\frac{54}{5}$

21. Given a polygon which all their internal angles are in AP. The common difference of this AP is 6°, the largest angle is 135°. How many sides does this polygon have?

Sol.

$$\begin{aligned}
a_1 &= 135 \\
d &= -6 \\
\frac{n}{2}(2\cdot 135 + (n-1)\cdot(-6)) &= 180(n-2) \\
n(270 - 6(n-1)) &= 360(n-2) \\
n(276 - 6n) &= 360n - 720 \\
276n - 6n^2 &= 360n - 720 \\
46n - n^2 &= 60n - 120 \\
n^2 + 14n - 120 &= 0 \\
(n+20)(n-6) &= 0 \\
n &= -20\ (invalid) \\
n &= 6 \\
\therefore The\ number\ of\ sides\ is\ 6
\end{aligned}$$

22. Given an AP which its 5th term is 3 and the sum of its first 10 terms is $26\frac{1}{4}$. Whcih term in this AP is 0?

Sol.

$$\begin{aligned}
a_5 = a + (5-1)d &= 3 \\
a + 4d &= 3 \\
S_{10} = \frac{10}{2}(2a + (10-1)d) &= 26\frac{1}{4} \\
5(2a + 9d) &= 26\frac{1}{4} \\
20(2a + 9d) &= 105 \\
4(2a + 9d) &= 21 \\
8a + 36d &= 21 \\
8a + 32d &= 24 \\
4d &= -3 \\
d &= -\frac{3}{4} \\
a &= 3 + \frac{3}{4}\cdot 4 \\
&= 6 \\
a_n = 6 + (n-1)\cdot(-\frac{3}{4}) &= 0 \\
6 - \frac{3}{4}(n-1) &= 0 \\
24 - 3n + 3 &= 0 \\
3n &= 27 \\
n &= 9
\end{aligned}$$

23. Given that the sum of the first 6 terms of an AP is 96, and the sum of the first 20 terms is 3 times the sum of the first 10 terms of this AP. Find the first term and the 10th term of it.

Sol.

$$\begin{aligned}
S_6 = \frac{6}{2}(2a + (6-1)d) &= 96 \\
3(2a+5d) &= 96 \\
2a+5d &= 32 \\
S_{20} &= 3S_{10} \\
\frac{20}{2}(2a + (20-1)d) &= 3 \cdot \frac{10}{2}(2a \\
&\quad + (10-1)d) \\
10(2a+19d) &= 15(2a+9d) \\
2(2a+19d) &= 3(2a+9d) \\
4a+38d &= 6a+27d \\
2a-11d &= 0 \\
16d &= 32 \\
d &= 2 \\
a &= \frac{11 \cdot 2}{2} \\
&= 11 \\
a_{10} &= 11 + (10-1) \cdot 2 \\
&= 29
\end{aligned}$$

24. Given that $5^2 \cdot 5^4 \cdot 5^6 \cdot \cdots \cdot 5^{2n} = (0.04)^{-28}$, find the value of n.

Sol.

$$\begin{aligned}
(0.04)^{-28} &= \frac{1}{25}^{-28} \\
&= (5^(-2))^{-28} \\
&= 5^{56} \\
\because n^a \cdot n^b &= n^{a+b} \\
2+4+6+\cdots+2n &= 56 \\
S_n = \frac{n}{2}(2 \cdot 2 + (n-1) \cdot 2) &= 56 \\
n(4+2(n-1)) &= 112 \\
n(2+2n) &= 112 \\
2n^2+2n &= 112 \\
n^2+n-56 &= 0 \\
(n+8)(n-7) &= 0 \\
n &= -8\ (invalid) \\
n &= 7
\end{aligned}$$

25. Given that the 9th term of an AP is double the 5th term of it. Find the ratio of the sum of first 9 terms and the sum of first 5 terms of the AP.

Sol.

$$\begin{aligned}
a_9 &= 2a_5 \\
a + (9-1)d &= 2(a + (5-1)d) \\
a+8d &= 2a+8d \\
a &= 0 \\
S_9 : S_5 &= \frac{9}{2}(2a+a_9) : \frac{5}{2}(2a+a_5) \\
&= \frac{9}{2}(2a+2a_5) : \frac{5}{2}(2a+a_5) \\
&= 9(a+a_5) : \frac{5}{2}(2a+a_5) \\
\frac{S_9}{S_5} &= \frac{9(a+a_5)}{\frac{5}{2}(2a+a_5)} \\
&= \frac{18(a+a_5)}{5(2a+a_5)} \\
&= \frac{18 \cdot a_5}{5 \cdot a_5} \\
&= \frac{18}{5} \\
\therefore S_9 : S_5 &= 18 : 5
\end{aligned}$$

12.3 Geometric Progression

The general formula of a geometric progression (GP) is given by

$$a_n = a_1 \cdot r^{n-1}$$

where a_1 is the first term, r is the common ratio, and n is the number of terms.

12.3.1 Practice 6

1. Find the 6th term of the GP $12, -18, 27, \cdots$

Sol.

$$
\begin{aligned}
a_1 &= 12 \\
r &= \frac{-18}{12} \\
&= -\frac{3}{2} \\
a_6 &= 12 \cdot (-\frac{3}{2})^{6-1} \\
&= 12 \cdot (-\frac{3}{2})^{5} \\
&= 12 \cdot (-\frac{243}{32}) \\
&= -\frac{729}{8}
\end{aligned}
$$

2. Find the number of terms of GP $\frac{1}{64} - \frac{1}{32} + \frac{1}{16} - \frac{1}{8} + \cdots - 512$

   Sol.

$$
\begin{aligned}
a_1 &= \frac{1}{64} \\
r &= \frac{-\frac{1}{32}}{\frac{1}{64}} \\
&= -2 \\
-512 &= \frac{1}{64}(-2)^{n-1} \\
(-2)^9 &= \frac{1}{2^6}(-2)^{n-1} \\
(-2)^{15} &= (-2)^{n-1} \\
n-1 &= 15 \\
n &= 16
\end{aligned}
$$

3. The 5th term of a GP is 3, and its 9th term is $\frac{1}{27}$, find the first term and the common ratio of this GP.

   Sol.

$$
\begin{aligned}
a_5 &= ar^4 = 3 \\
a_9 &= ar^8 = \frac{1}{27} \\
r^4 &= \frac{1}{27} \cdot \frac{1}{3} \\
&= \frac{1}{81} \\
r &= \frac{1}{3} \\
a_1 &= 3 \cdot 81 \\
&= 243
\end{aligned}
$$

4. Find 5 numbers between $\frac{1}{2}$ and $frac1128$ such that these 7 numbers are in GP.

   Sol.

$$
\begin{aligned}
a_1 &= \frac{1}{2} \\
n &= 7 \\
\frac{1}{128} &= \frac{1}{2}r^{7-1} \\
r^6 &= \frac{1}{64} \\
r &= \frac{1}{2}
\end{aligned}
$$

$$
\therefore These\ 5\ numbers\ are\ \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}
$$

Geometric Mean

The geometric mean G of two numbers x and y is given by

$$
\begin{aligned}
\frac{G}{x} &= \frac{G}{y} \\
G^2 &= xy \\
G &= \mp\sqrt[2]{xy}
\end{aligned}
$$

12.3.2 Practice 7

Find the geometric mean of $\frac{27}{8}$ and $\frac{2}{3}$.
Sol.

$$
\begin{aligned}
G &= \pm\sqrt[2]{\frac{27}{8} \cdot \frac{2}{3}} \\
&= \pm\sqrt[2]{\frac{9}{4}} \\
&= \pm\frac{3}{2}
\end{aligned}
$$

Summation of Geometric Progression

The sum of n terms of a GP is given by

$$
S_n = \frac{a_1(1-r^n)}{1-r} \ (r \neq 1)
$$

12.3.3 Practice 8

1. Find the sum of the first 8 terms of GP $3 + 6 + 12 + \cdots$

Sol.

$$\begin{aligned} a_1 &= 3 \\ r &= \frac{6}{3} \\ &= 2 \\ n &= 8 \\ S_n &= \frac{3(1-2^8)}{1-2} \\ &= \frac{3(1-256)}{1-2} \\ &= 3 \cdot 255 \\ &= 765 \end{aligned}$$

2. Find the sum of the GP $1 + \sqrt{3} + 3 + \cdots + 81$

Sol.

$$\begin{aligned} a_1 &= 1 \\ r &= \sqrt{3} \\ 81 &= 1 \cdot (\sqrt{3})^{n-1} \\ 3^4 &= (\sqrt{3})^{n-1} \\ (\sqrt{3})^8 &= (\sqrt{3})^{n-1} \\ n-1 &= 8 \\ n &= 9 \\ S_n &= \frac{1(1-(\sqrt{3})^9)}{1-\sqrt{3}} \\ &= \frac{1-81\sqrt{3}}{1-\sqrt{3}} \\ &= \frac{(1-81\sqrt{3})(1+\sqrt{3})}{-2} \\ &= \frac{1-81\sqrt{3}+\sqrt{3}-243}{-2} \\ &= \frac{-242-80\sqrt{(3)}}{-2} \\ &= 121+40\sqrt{3} \end{aligned}$$

3. Given that the sum of the first n terms of GP $4\frac{4}{5}, 1\frac{3}{5}, \frac{8}{15}, \cdots$ is $7\frac{145}{729}$, find n.

Sol.

$$\begin{aligned} a_1 &= \frac{24}{5} \\ r &= \frac{8}{5} \cdot \frac{5}{24} \\ &= \frac{1}{3} \\ S_n &= \frac{24}{5} \cdot \frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}} \\ \frac{5248}{729} &= \frac{24}{5} \cdot \frac{1-(\frac{1}{3})^n}{\frac{2}{3}} \\ \frac{5248}{729} \cdot \frac{5}{24} \cdot \frac{2}{3} &= 1-(\frac{1}{3})^n \\ \frac{6560}{6561} &= 1-(\frac{1}{3})^n \\ -\frac{1}{6561} &= -(\frac{1}{3})^n \\ (\frac{1}{3})^8 &= (\frac{1}{3})^n \\ n &= 8 \end{aligned}$$

Summation of Infinite Geometric Progression

The sum of infinite GP is given by

$$S_\infty = \frac{a_1}{1-r} \ (-1 < r < 1)$$

12.3.4 Practice 9

1. Find the sum of the following infinite GP.

(a) $16 + 8 + 4 + \cdots$

Sol.

$$\begin{aligned} a_1 &= 16 \\ r &= \frac{8}{16} \\ &= \frac{1}{2} \\ S_\infty &= \frac{16}{1-\frac{1}{2}} \\ &= \frac{16}{\frac{1}{2}} \\ &= 32 \end{aligned}$$

(b) $18 - 12 + 8 + \cdots$

Sol.

$$
\begin{aligned}
a_1 &= 18 \\
r &= \frac{8}{-12} \\
&= -\frac{2}{3} \\
S_\infty &= \frac{18}{1+\frac{2}{3}} \\
&= \frac{18}{\frac{5}{3}} \\
&= \frac{54}{5}
\end{aligned}
$$

(c) $1 + \frac{3}{4} + \frac{9}{16} + \cdots$

Sol.

$$
\begin{aligned}
a_1 &= 1 \\
r &= \frac{9}{16} \cdot \frac{16}{9} \\
&= \frac{3}{4} \\
S_\infty &= \frac{1}{1-\frac{3}{4}} \\
&= \frac{1}{\frac{1}{4}} \\
&= 4
\end{aligned}
$$

(d) $\sqrt{2} + 1 + \frac{1}{\sqrt{2}} + \cdots$

Sol.

$$
\begin{aligned}
a_1 &= \sqrt{2} \\
r &= \frac{1}{\sqrt{2}} \\
S_\infty &= \frac{\sqrt{2}}{1-\frac{1}{\sqrt{2}}} \\
&= \frac{\sqrt{2}}{\frac{\sqrt{2}-1}{\sqrt{2}}} \\
&= \frac{2}{\sqrt{2}-1} \\
&= 2(\sqrt{2}+1)
\end{aligned}
$$

2. Convert the following recurring decimals to fraction using the summation of inifinite geometric series.

(a) $0.\overline{3}$

Sol.

$$
\begin{aligned}
a_1 &= 0.3 \\
r &= 0.1 \\
S_\infty &= \frac{0.3}{1-0.1} \\
&= \frac{0.3}{0.9} \\
&= \frac{1}{3}
\end{aligned}
$$

$$\therefore 0.\overline{3} = \frac{1}{3}$$

(b) $0.5\overline{3}$

Sol.

$$
\begin{aligned}
a_1 &= 0.03 \\
r &= 0.01 \\
S_\infty &= \frac{0.03}{1-0.01} \\
&= \frac{0.03}{0.99} \\
&= \frac{3}{99}
\end{aligned}
$$

$$
\begin{aligned}
\therefore 0.5\overline{3} &= \frac{5}{10} + \frac{3}{99} \\
&= \frac{53}{99}
\end{aligned}
$$

12.3.5 Exercise 12.3

1. Find the 10th term of the GP $2, 4, 8, \cdots$

Sol.

$$
\begin{aligned}
a_1 &= 2 \\
r &= \frac{4}{2} \\
&= 2 \\
a_{10} &= 2 \cdot 2^{10-1} \\
&= 2 \cdot 512 \\
&= 1024
\end{aligned}
$$

2. Find the 8th term of the GP $243, -162, 108, \cdots$

Sol.

$$\begin{aligned} a_1 &= 243 \\ r &= \frac{-162}{243} \\ &= -\frac{2}{3} \\ a_8 &= 243 \cdot (-\frac{2}{3})^{8-1} \\ &= 243 \cdot (-\frac{128}{2187}) \\ &= -\frac{128}{9} \end{aligned}$$

3. Find the number of terms of the following GP.

(a) $8, 4, 2, 1, \ldots, \frac{1}{64}$

Sol.

$$\begin{aligned} a_1 &= 8 \\ r &= \frac{4}{8} \\ &= \frac{1}{2} \\ \frac{1}{64} &= 8 \cdot (\frac{1}{2})^{n-1} \\ \frac{1}{512} &= (\frac{1}{2})^{n-1} \\ \frac{1}{2^9} &= (\frac{1}{2})^{n-1} \\ n-1 &= 9 \\ n &= 10 \end{aligned}$$

(b) $6, -18, 54, \ldots, -13122$

Sol.

$$\begin{aligned} a_1 &= 6 \\ r &= \frac{-18}{6} \\ &= -3 \\ -13122 &= 6 \cdot (-3)^{n-1} \\ -2187 &= (-3)^{n-1} \\ (-3)^7 &= (-3)^{n-1} \\ n-1 &= 7 \\ n &= 8 \end{aligned}$$

(c) $54, 36, 24, \ldots, 3\frac{13}{81}$

Sol.

$$\begin{aligned} a_1 &= 54 \\ r &= \frac{36}{54} \\ &= \frac{2}{3} \\ \frac{256}{81} &= 54 \cdot (\frac{2}{3})^{n-1} \\ \frac{256}{81} \cdot \frac{1}{54} &= (\frac{2}{3})^{n-1} \\ \frac{128}{2187} &= (\frac{2}{3})^{n-1} \\ (\frac{2}{3})^7 &= (\frac{2}{3})^{n-1} \\ n-1 &= 7 \\ n &= 8 \end{aligned}$$

4. Given that the 2nd term of a GP is 12, and its 4th term is 108, find the first term and the common ratio of it.

Sol.

$$\begin{aligned} a_2 &= ar = 12 \\ a_4 &= ar^3 = 109 \\ r^2 &= 9 \\ r &= \pm 3 \\ a_1 &= \pm 4 \\ \therefore a_1 &= 4, r = 3 \text{ or } a_1 = -4, r = -3 \end{aligned}$$

5. Given that the 3rd term of an GP is $1\frac{1}{3}$, and its 8th term is $-10\frac{1}{8}$. Find the 5th term of this AP.

Sol.

$$\begin{aligned}
a_3 &= ar^2 = \frac{4}{3}\\
a_8 &= ar^7 = -\frac{81}{8}\\
r^5 &= -\frac{81}{8} \cdot \frac{3}{4}\\
&= -\frac{243}{32}\\
&= (-\frac{3}{2})^5\\
r &= -\frac{3}{2}\\
a &= \frac{4}{3} \cdot \frac{4}{9}\\
&= \frac{16}{27}\\
a_5 &= \frac{16}{27} \cdot (\frac{3}{2})^4\\
&= \frac{16}{27} \cdot \frac{81}{16}\\
&= 3
\end{aligned}$$

6. Find the geometric mean of 2 and 18.

Sol.

$$\begin{aligned}
G &= \pm\sqrt[2]{2 \cdot 18}\\
&= \pm\sqrt[2]{36}\\
&= \pm 6
\end{aligned}$$

7. Given that x+12, x+4 and x-2 are in GP, find the value of x and the common ratio of this GP.

Sol.

$$\begin{aligned}
x + 4 &= \pm\sqrt{(x + 12)(x - 2)}\\
x^2 + 8x + 16 &= x^2 + 10x - 24\\
2x &= 40\\
x &= 20\\
a_1 &= 20 + 12 = 32\\
a_2 &= 20 + 4 = 24\\
r &= \frac{24}{32}\\
&= \frac{3}{4}
\end{aligned}$$

8. Find 3 numbers between 14 and 224 such that these 5 numbersare in GP.

Sol.

$$\begin{aligned}
a_1 &= 14\\
a_5 &= 224\\
244 &= 14 \cdot r^4\\
16 &= r^4\\
(\pm 2)^4 &= r^4\\
r &= \pm 2
\end{aligned}$$

$$\begin{aligned}
&\therefore These\ 3\ numbers\ are\ 28,\ 56,\ 112\\
&\quad or\ -28, 56,\ -112
\end{aligned}$$

9. Calculate the sum of the first 6 terms of the GP $2+6+18+\cdots$

Sol.

$$\begin{aligned}
a_1 &= 2\\
r &= \frac{6}{2}\\
&= 3\\
S_6 &= \frac{2(1 - 3^6)}{1 - 3}\\
&= \frac{2(1 - 729)}{-2}\\
&= 728
\end{aligned}$$

10. Calculate the sum of the first 8 terms of the GP $32 - 16 + 8 - \cdots$

Sol.

$$\begin{aligned}
a_1 &= 32\\
r &= \frac{-16}{32}\\
&= -\frac{1}{2}\\
S_8 &= \frac{32(1 - (\frac{1}{2})^8)}{1 + \frac{1}{2}}\\
&= \frac{32(1 - \frac{1}{256})}{\frac{3}{2}}\\
&= 32 \cdot \frac{255}{256} \cdot \frac{2}{3}\\
&= \frac{85}{4}
\end{aligned}$$

11. Find the sum of the GP $14 - 28 + 56 - \cdots + 3584$

Sol.

$$\begin{aligned}
a_1 &= 14\\
r &= \frac{-28}{14} = -2\\
3584 &= 14 \cdot (-2)^{n-1}\\
256 &= (-2)^{n-1}\\
(-2)^8 &= (-2)^{n-1}\\
n-1 &= 8\\
n &= 9\\
S_9 &= \frac{14(1-(-2)^9)}{1-(-2)}\\
&= \frac{14(1+512)}{3}\\
&= \frac{14 \cdot 513}{3}\\
&= 2394
\end{aligned}$$

12. If the first term of a GP is 7, its common ratio is 3, and the sum of its terms is 847, find the number of terms and the last term of this GP.

Sol.

$$\begin{aligned}
a_1 &= 7\\
r &= 3\\
S_n &= \frac{7(1-3^n)}{1-3} = 847\\
7(1-3^n) &= -1694\\
1-3^n &= -242\\
3^n &= 243\\
3^n &= 3^5\\
n &= 5\\
a_5 &= 7 \cdot 3^4 = 567
\end{aligned}$$

13. Find the sum of the following infinite GP.

(a) $24 + 18 + 13\frac{1}{2} + \cdots$

Sol.

$$\begin{aligned}
a_1 &= 24\\
r &= \frac{18}{24} = \frac{3}{4}\\
S_\infty &= \frac{24}{1-\frac{3}{4}}\\
&= \frac{24}{\frac{1}{4}}\\
&= 96
\end{aligned}$$

(b) $27 - 9 + 3 - 1 + \cdots$

Sol.

$$\begin{aligned}
a_1 &= 27\\
r &= \frac{-9}{27} = -\frac{1}{3}\\
S_\infty &= \frac{27}{1+\frac{1}{3}}\\
&= \frac{27}{\frac{4}{3}}\\
&= \frac{81}{4}
\end{aligned}$$

(c) $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \cdots$

Sol.

$$\begin{aligned}
a_1 &= 2\\
r &= \frac{-\frac{1}{2}}{2} = -\frac{1}{4}\\
S_\infty &= \frac{2}{1+\frac{1}{4}}\\
&= \frac{2}{\frac{5}{4}}\\
&= \frac{8}{5}
\end{aligned}$$

14. Given an infinite GP which has a sum of 24 and first term of 30, find the common difference.

Sol.

$$\begin{aligned}
a_1 &= 30\\
S_\infty &= 24\\
24 &= \frac{30}{1-r}\\
24(1-r) &= 30\\
24 - 24r &= 30\\
-24r &= 6\\
r &= -\frac{1}{4}
\end{aligned}$$

15. Convert the following recurrring decimals into fractions.

(a) $0.\overline{45}$

Sol.

$$\begin{aligned} a_1 &= 0.45 \\ r &= 0.01 \\ S_\infty &= \frac{0.45}{1-0.01} \\ &= \frac{0.45}{0.99} \\ &= \frac{45}{99} \\ &= \frac{5}{11} \end{aligned}$$

$$\therefore 0.\overline{45} = \frac{5}{11}$$

(b) $0.\overline{037}$

Sol.

$$\begin{aligned} a_1 &= 0.037 \\ r &= 0.001 \\ S_\infty &= \frac{0.037}{1-0.001} \\ &= \frac{0.037}{0.999} \\ &= \frac{37}{999} \\ &= \frac{1}{27} \end{aligned}$$

$$\therefore 0.\overline{037} = \frac{1}{27}$$

(c) $0.2\overline{18}$

Sol.

$$\begin{aligned} a_1 &= 0.018 \\ r &= 0.01 \\ S_\infty &= \frac{0.018}{1-0.01} \\ &= \frac{0.018}{0.99} \\ &= \frac{18}{990} \\ &= \frac{1}{55} \end{aligned}$$

$$\begin{aligned} \therefore 0.2\overline{18} &= \frac{1}{5} + \frac{1}{55} \\ &= \frac{12}{55} \end{aligned}$$

(d) $1.\overline{3}$

Sol.

$$\begin{aligned} a_1 &= 0.3 \\ r &= 0.1 \\ S_\infty &= \frac{0.3}{1-0.1} \\ &= \frac{0.3}{0.9} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \therefore 1.\overline{3} &= 1 + \frac{1}{3} \\ &= \frac{4}{3} \end{aligned}$$

16. Three integers are in GP, summing up to 42 while accumulating up to 512, find these three integers.

Sol.

$$\begin{aligned} a_1 + a_2 + a_3 &= 42 \\ a_1 a_2 a_3 &= 512 \\ a_2 &= \pm\sqrt{a_1 a_3} \\ a_1 a_3 &= a_2^2 \\ a_2^3 &= 512 \\ a_2 &= \sqrt[3]{512} \\ &= 8 \\ a_1 a_3 &= 64 \\ a_3 &= \frac{64}{a_1} \\ a_1 + 8 + \frac{64}{a_1} &= 42 \\ a_1 + \frac{64}{a_1} &= 34 \\ a_1^2 + 64 &= 34a_1 \\ a_1^2 - 34a_1 + 64 &= 0 \\ (a_1 - 32)(a_1 - 2) &= 0 \\ a_1 &= 32 \; or \; a_1 = 2 \end{aligned}$$

$$\therefore These\ three\ integers\ are\ 2,\ 8,\ 32$$

17. The sum of first 6 term of a GP is 9 times the sum of first 3 terms. Find the common ratio.

Sol.

$$\begin{aligned} S_6 &= 9S_3 \\ \frac{a(1-r^6)}{1-r} &= 9\cdot\frac{a(1-r^3)}{1-r} \\ a(1-r^6) &= 9a(1-r^3) \\ 1-r^6 &= 9(1-r^3) \\ &= 9-9r^3 \\ r^6-9r^3+8 &= 0 \\ (r^3-8)(r^3-1) &= 0 \\ r^3 &= 8 \; or \; r^3 = 1 \\ r &= 1 \; (invalid) \\ r &= 2 \end{aligned}$$

18. Given a GP, its first term is 16, last term is $\frac{1}{2}$ and its sum is $31\frac{1}{2}$, find its common ratio and number of terms.

Sol.

$$\begin{aligned} a_1 &= 16 \\ \frac{1}{2} &= 16r^{n-1} \\ \frac{1}{32} &= r^{n-1} \\ &= r^n\cdot\frac{1}{r} \\ r^n &= \frac{r}{32} \\ \frac{63}{2} &= \frac{16(1-r^n)}{1-r} \\ 63(1-r) &= 32(1-r^n) \\ 63-63r &= 32-32r^n \\ -31 &= 32r^n-63r \\ -31 &= r-63r \\ -31 &= -62r \\ r &= \frac{1}{2} \\ (\frac{1}{2})^{n-1} &= \frac{1}{32} \\ &= (\frac{1}{2})^5 \\ n-1 &= 5 \\ n &= 6 \end{aligned}$$

19. Given a GP, its 3rd term is 6 less than its 2nd term, ant its 2nd term is 9 less than its 1st term. Find the 4th term and the sum of the first 4 terms.

Sol.

$$\begin{aligned} Let\ x &= a_2 \\ a_3 &= x-6 \\ a_1 &= x+9 \\ x &= \pm\sqrt{(x-6)(x+9)} \\ x^2 &= x^2+3x-54 \\ 3x-54 &= 0 \\ x &= 18 \\ a_2 &= 18 \\ a_1 &= 27 \\ r &= \frac{12}{18} \\ &= \frac{2}{3} \\ a_4 &= 27\cdot(\frac{2}{3})^3 \\ &= 8 \\ S_4 &= \frac{27(1-(\frac{16}{3})^4)}{1-\frac{2}{3}} \\ &= \frac{27(1-\frac{8}{81})}{\frac{1}{3}} \\ &= 81\cdot\frac{65}{81} \\ &= 65 \end{aligned}$$

20. GIven an infinite GP, its common ratio is positive and the sum of it is 9. The sum of the first two terms is 5, find the 4th term.

Sol.

$$
\begin{aligned}
S_\infty &= \frac{a}{1-r} = 9 \\
a &= 9(1-r) \\
&= 9 - 9r \\
S_2 &= \frac{a(1-r^2)}{1-r} = 5 \\
a - ar^2 &= 5 - 5r \\
9 - 9r - (9-9r)r^2 &= 5 - 5r \\
9 - 9r - 9r^2 + 9r^3 &= 5 - 5r \\
4 - 4r - 9r^2 + 9r^3 &= 0 \\
4(1-r) - 9r^2(1-r) &= 0 \\
(4-9r^2)(1-r) &= 0 \\
(9r^2-4)(r-1) &= 0 \\
(3r^2+2)(3r^2-2)(r-1) &= 0 \\
r &= 1\ (invalid) \\
r &= -\frac{2}{3}(invalid) \\
r &= \frac{2}{3} \\
a &= 9(1-\frac{2}{3}) \\
&= 3 \\
a_4 &= 3(\frac{2}{3})^3 \\
&= 3 \cdot \frac{8}{27} \\
&= \frac{8}{9}
\end{aligned}
$$

21. If $x+1, x-2, \frac{1}{2}x$ are the first three terms of an infinite GP, find:

    (a) The value of x

    Sol.

$$
\begin{aligned}
x - 2 &= \pm\sqrt{(x+1)(\frac{1}{2}x)} \\
x^2 - 4x + 4 &= \frac{1}{2}x(x+1) \\
2x^2 - 8x + 8 &= x^2 + x \\
x^2 - 9x + 8 &= 0 \\
(x-8)(x-1) &= 0 \\
x &= 8\ or\ x = 1
\end{aligned}
$$

    (b) The common ratio

    Sol.

$$
\begin{aligned}
&When\ x = 8, \\
r &= \frac{8-2}{8+1} \\
&= \frac{6}{9} \\
&= \frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
&When\ x = 1, \\
r &= \frac{1-2}{1+1} \\
&= -\frac{1}{2}
\end{aligned}
$$

    (c) The sum of the GP

    Sol.

$$
\begin{aligned}
&When\ x = 8, \\
S_\infty &= \frac{a}{1-r} \\
&= \frac{9}{1-\frac{2}{3}} \\
&= 9 \cdot 3 \\
&= 27
\end{aligned}
$$

$$
\begin{aligned}
&When\ x = 1, \\
S_\infty &= \frac{a}{1-r} \\
&= \frac{2}{1+\frac{1}{2}} \\
&= 2 \cdot \frac{2}{3} \\
&= \frac{4}{3}
\end{aligned}
$$

12.4 Simple Summation of Special Series

Sum formula of natural number:

$$\sum_{i=1}^{n} k = \frac{n(n+1)}{2}$$

Sum formula of square of natural number:

$$\sum_{i=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum formula of cube of natural number:

$$\sum_{i=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

12.4.1 Practice 10

1. Find the sum of the following series.

   (a) $\sum_{k=1}^{8} 3k$
   Sol.

$$\begin{aligned}\sum_{k=1}^{8} 3k &= 3\sum_{k=1}^{8} k \\ &= 3 \cdot \frac{8(8+1)}{2} \\ &= 3 \cdot \frac{8 \cdot 9}{2} \\ &= 3 \cdot \frac{72}{2} \\ &= 3 \cdot 36 \\ &= 108\end{aligned}$$

   (b) $\sum_{k=1}^{12} k^2$
   Sol.

$$\begin{aligned}\sum_{k=1}^{12} k^2 &= \frac{12(12+1)(2 \cdot 12+1)}{6} \\ &= \frac{12 \cdot 13 \cdot 25}{6} \\ &= 650\end{aligned}$$

   (c) $\sum_{k=3}^{10}(2k-3)$
   Sol.

$$\begin{aligned}&\sum_{k=3}^{10}(2k-3) \\ &= 2\sum_{k=3}^{10} k - \sum_{k=3}^{10} 3 \\ &= 2\left[\sum_{k=1}^{10} k - \sum_{k=1}^{2} k\right] - (30-6) \\ &= 2\left[\frac{10(10+1)}{2} - \frac{2(2+1)}{2}\right] - 8 \\ &= 2(55-3) - 24 \\ &= 2 \cdot 52 - 24 \\ &= 104 - 24 \\ &= 80\end{aligned}$$

   (d) $\sum_{k=7}^{13} 3k^2$

   Sol.

$$\begin{aligned}&\sum_{k=7}^{13} 3k^2 \\ &= 3\left[\sum_{k=1}^{13} k^2 - \sum_{k=1}^{6} k^2\right] \\ &= 3 \cdot \left[\frac{13(13+1)(2 \cdot 13+1)}{6}\right. \\ &\quad \left. - \frac{6(6+1)(2 \cdot 6+1)}{6}\right] \\ &= 3 \cdot \left[\frac{13 \cdot 14 \cdot 27}{6} - \frac{6 \cdot 7 \cdot 13}{6}\right] \\ &= 3 \cdot \left[\frac{4914}{6} - \frac{546}{6}\right] \\ &= 3 \cdot \frac{4368}{6} \\ &= 3 \cdot 728 \\ &= 2184\end{aligned}$$

2. Given that the nth term of a series is n (n+3), find the sum of the first 20 terms of the series.

   Sol.

$$\begin{aligned}&\sum_{k=1}^{20} k(k+3) \\ &= \sum_{k=1}^{20} k^2 + 3k \\ &= \sum_{k=1}^{20} k^2 + 3\sum_{k=1}^{20} k \\ &= \frac{20(20+1)(2 \cdot 20+1)}{6} + 3 \cdot \frac{20(20+1)}{2} \\ &= \frac{20 \cdot 21 \cdot 41}{6} + 3 \cdot \frac{20 \cdot 21}{2} \\ &= 2870 + 630 \\ &= 3500\end{aligned}$$

3. Find the sum of series $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2)$.

Sol.

$$
\begin{aligned}
&\sum_{k=1}^{n} k(k+2) \\
&= \sum_{k=1}^{n} k^2 + 2k \\
&= \sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k \\
&= \frac{n(n+1)(2n+1)}{6} + 2 \cdot \frac{n(n+1)}{2} \\
&= \frac{n(n+1)(2n+1)}{6} + n(n+1) \\
&= \frac{n(n+1)(2n+1) + 6n(n+1)}{6} \\
&= \frac{n(n+1)(2n+7)}{6}
\end{aligned}
$$

12.4.2 Exercise 12.4

1. Find the sum of the following series.

(a) $\sum k = 1^8 5k^2$

Sol.

$$
\begin{aligned}
\sum_{k=1}^{8} 5k^2 = 5\sum_{k=1}^{8} k^2 &= 5 \cdot \frac{8(8+1)(2 \cdot 8 + 1)}{6} \\
&= 5 \cdot \frac{8 \cdot 9 \cdot 17}{6} \\
&= 5 \cdot \frac{1368}{6} \\
&= 5 \cdot 204 \\
&= 1020
\end{aligned}
$$

(b) $\sum_{k=1}^{9} k^3$

Sol.

$$
\begin{aligned}
\sum_{k=1}^{9} k^3 &= \left[\frac{9(9+1)}{2}\right]^2 \\
&= 45^2 \\
&= 2025
\end{aligned}
$$

(c) $\sum_{n=1}^{10} (3n-5)$

Sol.

$$
\begin{aligned}
\sum_{n=1}^{10} (3n-5) &= 3\sum_{n=1}^{10} n - 5\sum_{n=1}^{10} 1 \\
&= 3 \cdot \frac{10(10+1)}{2} - 5 \cdot 10 \\
&= 3 \cdot \frac{10 \cdot 11}{2} - 5 \cdot 10 \\
&= 3 \cdot 55 - 50 \\
&= 3 \cdot 5 - 50 \\
&= 165 - 50 \\
&= 115
\end{aligned}
$$

(d) $\sum_{k=3}^{6} 2k^3$

Sol.

$$
\begin{aligned}
\sum_{k=3}^{6} 2k^3 &= 2\sum_{k=3}^{6} k^3 \\
&= 2\left(\sum_{k=1}^{6} k^3 - \sum_{k=1}^{2} k^3\right) \\
&= 2\left\{\left[\frac{6(6+1)}{2}\right]^2 \right. \\
&\qquad \left. - \left[\frac{2(2+1)}{2}\right]^2\right\} \\
&= 2(21^2 - 3^2) \\
&= 2(441 - 9) \\
&= 2 \cdot 432 \\
&= 864
\end{aligned}
$$

(e) $\sum_{k=6}^{10} (2k^2 + 3)$

Sol.

$$\sum_{k=6}^{10}(2k^2+3)$$
$$= 2\sum_{k=6}^{10}k^2 + 3\sum_{k=6}^{10}1$$
$$= 2\left(\sum_{k=1}^{10}k^2 - \sum_{k=1}^{5}k^2\right)$$
$$+ 3\cdot(10-5)$$
$$= 2\cdot\left[\frac{10\cdot 11\cdot 21}{6} - \frac{5\cdot 6\cdot 11}{6}\right]$$
$$+ 3\cdot 5$$
$$= 2\cdot\left[\frac{2310}{6} - \frac{330}{6}\right] + 3\cdot 5$$
$$= 2\cdot\frac{1980}{6} + 3\cdot 5$$
$$= 2\cdot 330 + 3\cdot 5$$
$$= 660 + 15$$
$$= 675$$

(f) $\sum_{n=11}^{15}(n^2+2n)$

Sol.

$$\sum_{n=11}^{15}(n^2+2n)$$
$$= \sum_{n=11}^{15}n^2 + 2\sum_{n=11}^{15}n$$
$$= \left[\sum_{n=1}^{15}n^2 - \sum_{n=1}^{10}n^2\right]$$
$$+ 2\left[\sum_{n=1}^{15}n - \sum_{n=1}^{10}n\right]$$
$$= \left[\frac{15\cdot 16\cdot 31}{6} - \frac{10\cdot 11\cdot 21}{6}\right]$$
$$+ 2\left[\frac{15\cdot 16}{2} - \frac{10\cdot 11}{2}\right]$$
$$= 985$$

(g) $\sum_{n=2}^{6} n(n^2-n+1)$

Sol.

$$\sum_{n=2}^{6} n(n^2-n+1)$$
$$= \sum_{n=2}^{6}n^3 - \sum_{n=2}^{6}n^2 + \sum_{n=2}^{6}n$$
$$= \left[\sum_{n=1}^{6}n^3 - \sum_{n=1}^{1}n^3\right] - \left[\sum_{n=1}^{6}n^2 - \sum_{n=1}^{1}n^2\right]$$
$$+ \left[\sum_{n=1}^{6}n - \sum_{n=1}^{1}n\right]$$
$$= \left[\left(\frac{6\cdot 7}{2}\right)^2 - \left(\frac{1\cdot 2}{2}\right)^2\right]$$
$$- \left(\frac{6\cdot 7\cdot 13}{6} - \frac{1\cdot 2\cdot 3}{6}\right)$$
$$+ \left(\frac{6\cdot 7}{2} - \frac{1\cdot 2}{2}\right)$$
$$= 21^2 - 1^2 - (7\cdot 13 - 1) + (3\cdot 7 - 1)$$
$$= 440 - 90 + 20$$
$$= 370$$

2. Fiven that the nth term of a series is $3n^2 + n$, find the sum of the first 10 terms of the series.

Sol.

$$\sum_{n=1}^{10} 3n^2 + n = 3\sum_{n=1}^{10}n^2 + \sum_{n=1}^{10}n$$
$$= 3\left(\frac{10\cdot 11\cdot 21}{6}\right) + \left(\frac{10\cdot 11}{2}\right)$$
$$= 3\cdot\frac{2310}{6} + \frac{110}{2}$$
$$= 3\cdot 385 + 55$$
$$= 1210$$

3. Find the sum of first nth term of series $1\cdot 3 + 2\cdot 7 + 3\cdot 11 + \cdots$

Sol.

$$\begin{aligned}
&\sum_{n=1}^{n} n\cdot(4n-1)\\
&= 4\sum_{n=1}^{n} n^2 - \sum_{n=1}^{n} n\\
&= 4\left(\frac{n(n+1)(2n+1)}{6}\right) - \left(\frac{n(n+1)}{2}\right)\\
&= \frac{4n(n+1)(2n+1)-3n(n+1)}{6}\\
&= \frac{n(n+1)(8n+1)}{6}
\end{aligned}$$

4. Find the sum fo the series $1^2+3^2+5^2+\cdots+15^2$

Sol.

$$\begin{aligned}
\sum_{n=1}^{8}(2n-1)^2 = \sum_{n=1}^{8}(4n^2-4n+1)\\
&= 4\sum_{n=1}^{8} n^2 - 4\sum_{n=1}^{8} n + \sum_{n=1}^{8} 1\\
&= 4\left(\frac{8\cdot 9\cdot 17}{6}\right) - 4\left(\frac{8\cdot 9}{2}\right) + 8\\
&= 4\cdot 204 - 4\cdot 36 + 8\\
&= 816 - 144 + 8\\
&= 680
\end{aligned}$$

12.5 Revision Exerise 12

1. Express the following series in form of $\sum$.

(a) $\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\cdots+\frac{49}{50}$

Sol.

$$\begin{aligned}
a_1 &= \frac{2\cdot 1-1}{2\cdot 1}\\
a_2 &= \frac{2\cdot 2-1}{2\cdot 2}\\
a_3 &= \frac{2\cdot 3-1}{2\cdot 3}\\
&\vdots\\
a_{25} &= \frac{2\cdot 25-1}{2\cdot 25}\\
\therefore \frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\cdots+\frac{49}{50} &= \sum_{n=1}^{25}\frac{2n-1}{2n}
\end{aligned}$$

(b) $6-7+8-9+\cdots$

Sol.

$$\begin{aligned}
a_1 &= (-1)^6\cdot 6\\
a_2 &= (-1)^7\cdot 7\\
a_3 &= (-1)^8\cdot 8\\
&\vdots\\
a_n &= (-1)^n n \therefore \quad 6-7+8-9+\cdots = \sum_{n=1}^{\infty}(-1)^n n
\end{aligned}$$

(c) $2\cdot 5+3\cdot 7+4\cdot 9+\cdots+15\cdot 31$

Sol.

$$\begin{aligned}
a_1 &= (1+1)(2\cdot 1+3)\\
a_2 &= (2+1)(2\cdot 2+3)\\
a_3 &= (3+1)(2\cdot 3+3)\\
&\vdots\\
a_{14} &= (14+1)(2\cdot 14+3)\\
&\therefore 2\cdot 5+3\cdot 7+4\cdot 9+\cdots+15\cdot 31\\
&= \sum_{n=1}^{14}(n+1)(2n+3)
\end{aligned}$$

2. Given a general formula $a_n = \frac{3^n}{2n-3}$, state the first 5 terms of the sequence.

Sol.

$$\begin{aligned}
a_1 &= \frac{3^1}{2\cdot 1-3} = -3\\
a_2 &= \frac{3^2}{2\cdot 2-3} = 9\\
a_3 &= \frac{3^3}{2\cdot 3-3} = 9\\
a_4 &= \frac{3^4}{2\cdot 4-3} = \frac{81}{5}\\
a_5 &= \frac{3^5}{2\cdot 5-3} = \frac{243}{7}
\end{aligned}$$

3. Express the series $\sum_{k=1}^{10}(2k^2-3)$

Sol.

$$\begin{aligned}&\sum_{k=1}^{10}(2k^2-3)\\&=(2\cdot 1^2-3)+(2\cdot 2^2-3)+(2\cdot 3^2-3)\\&\quad+(2\cdot 4^2-3)+(2\cdot 5^2-3)+(2\cdot 6^2-3)\\&\quad+(2\cdot 7^2-3)+(2\cdot 8^2-3)+(2\cdot 9^2-3)\\&\quad+(2\cdot 10^2-3)\\&=-1+5+15+29+47+69+95+125\\&\quad+159+197\end{aligned}$$

4. State the first term, last term and the number of terms of theh series $\sum_{k=3}^{7}(3^k-2^k-k)$

Sol.

$$\begin{aligned}a_3&=3^3-2^3-3=27-8-3=16\\a_7&=3^7-2^7-7=2187-128-7=2052\\n&=5\end{aligned}$$

5. Find the number of terms of the AP $-4-2\frac{3}{4}-112-\frac{1}{4}+\cdots+16$

Sol.

$$\begin{aligned}a&=-4\\d&=\frac{5}{4}\\16&=-4+(n-1)\frac{5}{4}\\20&=\frac{5}{4}(n-1)\\5n-5&=80\\5n&=85\\n&=17\end{aligned}$$

6. If x+1, 2x+1, x-3 are the first 3 terms of AP, find:

   (a) The value of x

   Sol.

$$\begin{aligned}2x+1&=\frac{x+1+x-3}{2}\\4x+2&=2x-2\\2x&=-4\\x&=-2\end{aligned}$$

   (b) Sum from the 10th term to the 20th term

   Sol.

$$\begin{aligned}a_1&=-1\\a_2&=-3\\r&=-2\\S&=S_{20}-S_9\\&=\frac{20}{2}(-2+(20-1)(-2))\\&\quad-\frac{9}{2}(-2+(9-1)(-2))\\&=10\cdot(-40)-9\cdot(-9)\\&=-400+81\\&=-319\end{aligned}$$

7. Find 4 numbers between 28 and -12 such that these 6 numbers form an AP.

Sol.

$$\begin{aligned}a_1&=28\\a_n&=-12\\n&=6\\-12&=28+5d\\5d&=40\\d&=8\end{aligned}$$

$$\therefore These\ 4\ numbers\ are\ -4,4,12,20$$

8. Find the sum of the following AP.

   (a) $7+11+15+\cdots$ up to the 10th term

   Sol.

$$\begin{aligned}a_1&=7\\d&=4\\n&=10\\S_{10}&=\frac{10}{2}(2\cdot 7+(10-1)4)\\&=5(14+36)\\&=250\end{aligned}$$

   (b) $20+18\frac{1}{2}+17+\cdots$ up to the 16tm term

Sol.

$$\begin{aligned} a_1 &= 20 \\ d &= -\frac{3}{2} \\ n &= 16 \\ S_{16} &= \frac{16}{2}(2 \cdot 20 + (16-1)(-\frac{3}{2})) \\ &= 8(40 - \frac{45}{2}) \\ &= 8 \cdot \frac{35}{2} \\ &= 140 \end{aligned}$$

(c) $2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} + \cdots + 13\sqrt{2}$

Sol.

$$\begin{aligned} a_1 &= 2\sqrt{2} \\ d &= \sqrt{2} \\ n &= 12 \\ S_{12} &= \frac{12}{2}(2 \cdot 2\sqrt{2} + (12-1)\sqrt{2}) \\ &= 6(4\sqrt{2} + 11\sqrt{2}) \\ &= 6 \cdot 15\sqrt{2} \\ &= 90\sqrt{2} \end{aligned}$$

9. Given an AP which the sum of the first n terms $S_n = n(1+2n)$, find:

(a) First term

Sol.

$$\begin{aligned} \frac{n}{2}(2a + (n-1)d) &= n(1+2n) \\ n(2a + (n-1d)) &= 2n(1+2n) \\ 2an + dn^2 - dn &= 2n - 4n^2 \\ (2a-d)n + dn^2 &= 2n - 4n^2 \end{aligned}$$

$$\begin{aligned} &\textit{Comparing both sides,} \\ &a = 3 \\ &d = 4 \end{aligned}$$

(b) Common Difference

Sol.

$$\begin{aligned} &\textit{According to the sol. of } (a), \\ &d = 4 \end{aligned}$$

(c) Sum of the first 20 terms.

Sol.

$$\begin{aligned} &\textit{According to the sol. of } (a), \\ &a = 3 \\ &d = 4 \\ &n = 20 \\ S_{20} &= \frac{20}{2}(2 \cdot 3 + (20-1)4) \\ &= 10(6 + 76) \\ &= 10 \cdot 82 \\ &= 820 \end{aligned}$$

10. Given an AP $33 + 27 + 21 + \cdots$

(a) If the first sum of the first n terms is 105, find the value of n.

Sol.

$$\begin{aligned} a_1 &= 33 \\ d &= -6 \\ 105 &= \frac{n}{2}(2 \cdot 33 + (n-1) \cdot (-6)) \\ 210 &= n(66 - (n-1)6) \\ 35 &= 11n - n^2 + n \\ n^2 - 12n + 35 &= 0 \\ (n-7)(n-5) &= 0 \\ n &= 7 \textit{ or } n = 5 \end{aligned}$$

(b) If the sum of the first n terms is negative value, find the minimum value of n.

Sol.

$$\begin{aligned} a_1 &= 33 \\ d &= -6 \\ \frac{n}{2}(2 \cdot 33 + (n-1) \cdot (-6)) &< 0 \\ n(66 - 6n + 6) &< 0 \\ 12n - n^2 &< 0 \\ n(12-n) &< 0 \\ n &> 12 \end{aligned}$$

$$\therefore \textit{The minimum value of } n \textit{ is } 13$$

11. Find the sum of the numbers between 150 and 300 that are multiple of both 5 and 3.

Sol.

$$\begin{aligned} a_1 &= 165 \\ a_n &= 285 \\ d &= 15 \\ 285 &= 165 + (n-1)\cdot 15 \\ 8 &= n-1 \\ n &= 9 \end{aligned}$$

$$\begin{aligned} S_9 &= \frac{9}{2}(2\cdot 165 + (9-1)\cdot 15) \\ &= \frac{9}{2}\cdot 450 \\ &= 2025 \end{aligned}$$

12. Find the sum of all the numbers between 100 and 200 that can be divided by 2 or 3.

Sol.

$$\begin{aligned} a_1 &= 102 \\ a_n &= 198 \end{aligned}$$

When $d = 2$,

$$\begin{aligned} 198 &= 102 + (n-1)\cdot 2 \\ 48 &= n-1 \\ n &= 49 \\ S_{49} &= \frac{49}{2}(2\cdot 102 + (49-1)\cdot 2) \\ &= \frac{49}{2}\cdot(204+96) \\ &= 7350 \end{aligned}$$

When $d = 3$,

$$\begin{aligned} 198 &= 102 + (n-1)\cdot 3 \\ 32 &= n-1 \\ n &= 33 \\ S_{33} &= \frac{33}{2}(2\cdot 102 + (33-1)\cdot 3) \\ &= \frac{33}{2}\cdot(204+96) \\ &= 4950 \end{aligned}$$

When $d = 6$,

$$\begin{aligned} 198 &= 102 + (n-1)\cdot 6 \\ 16 &= n-1 \\ n &= 17 \\ S_{17} &= \frac{17}{2}(2\cdot 102 + (17-1)\cdot 6) \\ &= \frac{17}{2}\cdot(204+96) \\ &= 2550 \end{aligned}$$

$$\begin{aligned} \therefore S &= 7350 + 4950 - 2550 \\ &= 9750 \end{aligned}$$

13. Find the sum of the numbers between 50 and 100 that cannot be divided by 5.

Sol.

When $d = 1$,

$$\begin{aligned} a_1 &= 51 \\ a_n &= 99 \\ 99 &= 51 + (n-1)\cdot 1 \\ 48 &= n - 1 \\ n &= 49 \\ S_{49} &= \frac{49}{2}(2\cdot 51 + (49-1)\cdot 1) \\ &= \frac{49}{2}\cdot(102+48) \\ &= 3675 \end{aligned}$$

When $d = 5$,

$$\begin{aligned} a_1 &= 55 \\ a_n &= 95 \\ 95 &= 55 + (n-1)\cdot 5 \\ 8 &= n - 1 \\ n &= 9 \\ S_9 &= \frac{9}{2}(2\cdot 55 + (9-1)\cdot 5) \\ &= \frac{9}{2}\cdot(110+40) \\ &= 675 \end{aligned}$$

$$\begin{aligned} \therefore S &= 3675 - 675 \\ &= 3000 \end{aligned}$$

14. Which term is the first negative term of the AP $20 + 16\frac{1}{4} + 12\frac{1}{2} + \cdots$?

Sol.

$$\begin{aligned} a_1 &= 20 \\ d &= -\frac{15}{4} \\ a_n = 20 - (n-1)\cdot\frac{15}{4} &< 0 \\ 80 - 15(n-1) &< 0 \\ 16 - 3n + 3 &< 0 \\ 3n &> 19 \\ n &> 6\frac{1}{3} \end{aligned}$$

$$\therefore The\ first\ negative\ term\ is\ 7$$

15. Three numbers are in AP, thier sum is 15 while the sum of the square of these numbers is 83. Find this three numbers.

Sol.

$$\begin{aligned} a_1 + a_2 + a_3 &= 15 \\ a_1^2 + a_2^2 + a_3^2 &= 83 \\ a_2 - a_1 &= a_3 - a_2 \\ a_1 + a_3 &= 2a_2 \\ 3a_2 &= 15 \\ a_2 &= 5 \\ a_3 &= 10 - a_1 \\ a_1^2 + a_3^2 &= 83 - 25 \\ &= 58 \\ a_1^2 + (10 - a_1)^2 &= 58 \\ a_1^2 + 100 - 20a_1 + a_1^2 &= 58 \\ 2a_1^2 - 20a_1 + 100 &= 58 \\ 2a_1^2 - 20a_1 + 42 &= 0 \\ a_1^2 - 10a_1 + 21 &= 0 \\ (a_1 - 7)(a_1 - 3) &= 0 \\ a_1 &= 7\ or\ a_1 = 3 \end{aligned}$$

$$\therefore The\ three\ numbers\ are\ 7, 5, 3$$

16. Find the sum of the series $18^2 - 17^2 + 16^2 - 15^2 + 14^2 - 13^2 + \cdots + 2^2 - 1^2$

Sol.

$$\begin{aligned} &18^2 - 17^2 + 16^2 - 15^2 + \cdots + 2^2 - 1^2 \\ &= (18^2 - 17^2) + (16^2 - 15^2) + \cdots + (2^2 - 1^2) \\ &= ((2\cdot 9)^2 - (2\cdot 9 - 1)^2) + ((2\cdot 8)^2 - (2\cdot 8 - 1)^2) \\ &\quad + \cdots + ((2\cdot 1)^2 - (2\cdot 1 - 1)^2) \\ &= \sum_{n=1}^{9}\left[(2n)^2 - (2n-1)^2\right] \\ &= \sum_{n=1}^{9}(4n - 1) \\ &= 4\sum_{n=1}^{9} n - \sum_{n=1}^{9} 1 \\ &= 4\cdot\frac{9\cdot 10}{2} - 9 \\ &= 180 - 9 \\ &= 171 \end{aligned}$$

17. State the general formula of the series $20, -10, 5, -2\frac{1}{2}, \cdots$

Sol.

$$a_1 = 20$$
$$r = -\frac{1}{2}$$
$$a_n = 20(-\frac{1}{2})^{n-1}$$

18. Given three integers x-3, x+1, 4x-2 that are in GP. If the sum of this GP is S, common ratio is r, find the value of S+r.

Sol.

$$x + 1 = \pm\sqrt{(x-3)(4x-2)}$$
$$x^2 + 2x + 1 = 4x^2 - 14x + 6$$
$$3x^2 - 16^x + 5 = 0$$
$$(3x-1)(x-5) = 0$$
$$x = 5 \; or \; x = \frac{1}{3}$$

$$a_1 = x - 3 = 5 - 3 = 2$$
$$a_2 = x + 1 = 5 + 1 = 6$$
$$a_3 = 4x - 2 = 4(5) - 2 = 18$$

$$S = a_1 + a_2 + a_3$$
$$= 2 + 6 + 18$$
$$= 26$$

$$r = \frac{a_3}{a_2} = \frac{18}{6} = 3$$

$$\therefore S + r = 26 + 3$$
$$= 29$$

19. Find the geometric mean of $\frac{1}{3}$ and $\frac{1}{5}$

Sol.

$$G = \pm\sqrt{\frac{1}{3} \cdot \frac{1}{5}}$$
$$= \pm\sqrt{\frac{1}{15}}$$
$$= \pm\frac{1}{\sqrt{15}}$$
$$= \pm\frac{\sqrt{15}}{15}$$

20. Find 5 numbers between $-\frac{1}{4}$ and $-\frac{1}{256}$ such that these 7 numbers form a GP.

Sol.

$$a_1 = -\frac{1}{4}$$
$$n = 7$$
$$-\frac{1}{256} = -\frac{1}{4}r^6$$
$$\frac{1}{64} = r^6$$
$$\left(\pm\frac{1}{2}\right)^6 = r^6$$
$$r = \pm\frac{1}{2}$$

When $r = \frac{1}{2}$,

These 5 numbers are

$\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$

When $r = -\frac{1}{2}$,

These 5 numbers are

$\frac{1}{8}, -\frac{1}{16}, \frac{1}{32}, -\frac{1}{64}, \frac{1}{128}$

21. Find the sum of the series $\sum_{n=5}^{15} n^2(3n+1)$

Sol.

$$\sum_{n=5}^{15} n^2(3n+1) = \sum_{n=5}^{15} n^3 + \sum_{n=5}^{15} 3n^2$$
$$= 3\sum_{n=5}^{15} n^3 + \sum_{n=5}^{15} n^2$$
$$= 3\left[\sum_{n=1}^{15} n^3 - \sum_{n=1}^{4} n^3\right]$$
$$+ \left[\sum_{n=1}^{15} n^2 - \sum_{n=1}^{4} n^2\right]$$
$$= 3\left[\left(\frac{15\cdot 16}{2}\right)^2 - \left(\frac{4\cdot 5}{2}\right)^2\right]$$
$$+ \left[\frac{15\cdot 16\cdot 31}{6} - \frac{4\cdot 5\cdot 9}{6}\right]$$
$$= 3\left[(15\cdot 8)^2 - (2\cdot 5)^2\right]$$
$$+ 1240 - 30$$
$$= 3(14400 - 100) + 1210$$
$$= 42900 + 1210$$
$$= 44110$$

22. Find the sum of the series $5^2 + 7^2 + 9^2 + \cdots + 25^2$

Sol.

$$\sum_{n=1}^{11} (2n+3)^2$$
$$= \sum_{n=1}^{11} 4n^2 + 12n + 9$$
$$= 4\sum_{n=1}^{11} n^2 + 12\sum_{n=1}^{11} n + 11$$
$$= 4\left[\frac{11\cdot 12\cdot 23}{6}\right] + 12\left[\frac{11\cdot 12}{2}\right] + 99$$
$$= 2024 + 792 + 99$$
$$= 2915$$

23. Find the sum of the series $2\cdot 3 + 3\cdot 12 + 4\cdot 27 + \cdots + (n+1)\cdot 3n^2$

Sol.

$$\sum_{n=1}^{n}(n+1)3n^2$$
$$= \sum_{n=1}^{n} 3n^3 + \sum_{n=1}^{n} 3n^2$$
$$= 3\left[\sum_{n=1}^{n} n^3 + \sum_{n=1}^{n} n^2\right]$$
$$= 3\left[\left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6}\right]$$
$$= 3\left[\frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}\right]$$
$$= 3\left[\frac{3n^2(n+1)^2 + 2n(n+1)(2n+1)}{12}\right]$$
$$= \frac{n(n+1)\left[3n^2+3n+4n+2\right]}{4}$$
$$= \frac{n(n+1)\left[3n^2+7n+2\right]}{4}$$
$$= \frac{n(n+1)(n+2)(3n+1)}{4}$$

Chapter 13

System of Equations

13.1 System of Equations with Two Variables

13.1.1 Practice 1

Solve the following system of equations.

1.

$$\begin{cases} 2x - 3y &= 11 \\ xy &= -5 \end{cases}$$

Sol.

$$\begin{cases} 2x - 3y = 11 & (1) \\ xy = -5 & (2) \end{cases}$$

$$\begin{aligned} (2) &\Rightarrow y = -\frac{5}{x} && (3) \\ \text{Sub (3) into (1)} &\Rightarrow 2x - \frac{15}{x} = 11 \\ & 2x^2 - 15 = 11x \\ & 2x^2 - 11x - 15 = 0 \\ & (2x-5)(x-3) = 0 \\ & x = 3 \text{ or } x = \frac{5}{2} \\ \text{Sub x} = 3 \text{ into (2)} &\Rightarrow y = -\frac{5}{3} \\ \text{Sub x} = \frac{5}{2} \text{ into (2)} &\Rightarrow y = -\frac{5}{\frac{5}{2}} \\ &\Rightarrow y = -\frac{5}{5} \\ &\Rightarrow y = -1 \end{aligned}$$

$$\therefore \begin{cases} x = 3 \\ y = -\frac{5}{3} \end{cases} or \begin{cases} x = \frac{5}{2} \\ y = -1 \end{cases}$$

2.

$$\begin{cases} 3x + y &= 5 \\ x^2 - 2xy &= 8 \end{cases}$$

Sol.

$$\begin{cases} 3x + y = 5 & (1) \\ x^2 - 2xy = 8 & (2) \end{cases}$$

$$\begin{aligned} 3(1) &\Rightarrow y = 5 - 3x && (3) \\ \text{Sub (3) into (2)} &\Rightarrow x^2 - 2x(5-3x) = 8 \\ & x^2 - 10x + 6x^2 = 8 \\ & 7x^2 - 10x + 8 = 0 \\ & (7x+4)(x-2) = 0 \\ & x = -\frac{4}{7} \text{ or } x = 2 \\ \text{Sub } x = -\frac{4}{7} \text{ into (1)} &\Rightarrow y = 5 - 3\left(-\frac{4}{7}\right) \\ &\Rightarrow y = \frac{47}{7} \\ \text{Sub } x = 2 \text{ into (1)} &\Rightarrow y = -1 \end{aligned}$$

$$\therefore \begin{cases} x = -\frac{4}{7} \\ y = \frac{47}{7} \end{cases} or \begin{cases} x = 2 \\ y = -1 \end{cases}$$

13.1.2 Exercise 13.1

Solve the following system of equations.

1.

$$\begin{cases} x - y &= 1 \\ xy &= 6 \end{cases}$$

Sol.

$$\begin{cases} x - y = 1 & (1) \\ xy = 6 & (2) \end{cases}$$

$$(1) \Rightarrow y = x - 1 \qquad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow x(x-1) = 6$$

$$x^2 - x = 6$$

$$x^2 - x - 6 = 0$$

$$(x+2)(x-3) = 0$$

$$x = -2 \text{ or } x = 3$$

$$\text{Sub } x = -2 \text{ into (1)} \Rightarrow y = -2 - 1$$

$$\Rightarrow y = -3$$

$$\text{Sub } x = 3 \text{ into (1)} \Rightarrow y = 3 - 1$$

$$\Rightarrow y = 2$$

$$\therefore \begin{cases} x = -2 \\ y = -3 \end{cases} \text{or} \begin{cases} x = 3 \\ y = 2 \end{cases}$$

2.

$$\begin{cases} 3x - y = 4 \\ xy = 4 \end{cases}$$

Sol.

$$\begin{cases} 3x - y = 4 & (1) \\ xy = 4 & (2) \end{cases}$$

$$(1) \Rightarrow y = 3x - 4 \qquad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow x(3x-4) = 4$$

$$3x^2 - 4x = 4$$

$$3x^2 - 4x - 4 = 0$$

$$(3x+2)(x-2) = 0$$

$$x = -\frac{2}{3} \text{ or } x = 2$$

$$\text{Sub } x = -\frac{2}{3} \text{ into (1)} \Rightarrow y = 3\left(-\frac{2}{3}\right) - 4$$

$$\Rightarrow y = -6$$

$$\text{Sub } x = 2 \text{ into (1)} \Rightarrow y = 3(2) - 4$$

$$\Rightarrow y = 2$$

$$\therefore \begin{cases} x = -\frac{2}{3} \\ y = -6 \end{cases} \text{or} \begin{cases} x = 2 \\ y = 2 \end{cases}$$

3.

$$\begin{cases} 3x + 4y = -39 \\ xy = 30 \end{cases}$$

Sol.

$$\begin{cases} 3x + 4y = -39 & (1) \\ xy = 30 & (2) \end{cases}$$

$$(2) \Rightarrow y = \frac{30}{x} \qquad (3)$$

$$\text{Sub (3) into (1)} \Rightarrow 3x + 4\frac{30}{x} = -39$$

$$3x^2 + 120 = -39x$$

$$3x^2 + 39x + 120 = 0$$

$$x^2 + 13x + 40 = 0$$

$$(x+5)(x+8) = 0$$

$$x = -5 \text{ or } x = -8$$

$$\text{Sub } x = -5 \text{ into (1)} \Rightarrow y = \frac{30}{-5} - 39$$

$$\Rightarrow y = -6$$

$$\text{Sub } x = -8 \text{ into (1)} \Rightarrow y = \frac{30}{-8} - 39$$

$$\Rightarrow y = -\frac{15}{4}$$

$$\therefore \begin{cases} x = -5 \\ y = -6 \end{cases} \text{or} \begin{cases} x = -8 \\ y = -\frac{15}{4} \end{cases}$$

4.

$$\begin{cases} y = 2x + 3 \\ y = x^2 - 2x + 1 \end{cases}$$

Sol.

$$\begin{cases} y = 2x + 3 & (1) \\ y = x^2 & (2) \end{cases}$$

$$(1) = (2) \Rightarrow 2x + 3 = x^2$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow (x+1)(x-3) = 0$$

$$\Rightarrow x = -1 \text{ or } x = 3$$

$$\text{Sub } x = -1 \text{ into (1)} \Rightarrow y = 2(-1) + 3$$

$$\Rightarrow y = 1$$

$$\text{Sub } x = 3 \text{ into (1)} \Rightarrow y = 2(3) + 3$$

$$\Rightarrow y = 9$$

$$\therefore \begin{cases} x = -1 \\ y = 1 \end{cases} \text{or} \begin{cases} x = 3 \\ y = 9 \end{cases}$$

5.

$$\begin{cases} x - y = 1 \\ x^2 + y^2 = 25 \end{cases}$$

Sol.

$$\begin{cases} x - y = 1 & (1) \\ x^2 + y^2 = 25 & (2) \end{cases}$$

$$\begin{aligned}
(1) &\Rightarrow x = y + 1 && (3) \\
\text{Sub (3) into (2)} &\Rightarrow (y+1)^2 + y^2 = 25 \\
&\Rightarrow y^2 + 2y + 1 + y^2 = 25 \\
&\Rightarrow 2y^2 + 2y = 24 \\
&\Rightarrow y^2 + y = 12 \\
&\Rightarrow y^2 + y - 12 = 0 \\
&\Rightarrow (y+4)(y-3) = 0 \\
&\Rightarrow y = -4 \text{ or } y = 3 \\
\text{Sub } y = -4 \text{ into (1)} &\Rightarrow x = -4 + 1 \\
&\Rightarrow x = -3 \\
\text{Sub } y = 3 \text{ into (1)} &\Rightarrow x = 3 + 1 \\
&\Rightarrow x = 4
\end{aligned}$$

$$\therefore \begin{cases} x = -3 \\ y = -4 \end{cases} \text{ or } \begin{cases} x = 4 \\ y = 3 \end{cases}$$

6.

$$\begin{cases} 5x - y = 3 \\ y^2 - 6x^2 = 25 \end{cases}$$

Sol.

$$\begin{cases} 5x - y = 3 & (1) \\ y^2 - 6x^2 = 25 & (2) \end{cases}$$

$$\begin{aligned}
(1) &\Rightarrow y = 5x - 3 && (3) \\
\text{Sub (3) into (2)} &\Rightarrow (5x-3)^2 - 6x^2 = 25 \\
&\Rightarrow 25x^2 - 30x + 9 \\
&\quad - 6x^2 = 25 \\
&\Rightarrow 19x^2 - 30x + 16 = 0 \\
&\Rightarrow (19x+8)(x-2) = 0 \\
&\Rightarrow x = -\frac{8}{19} \text{ or } x = 2 \\
\text{Sub } x = -\frac{8}{19} \text{ into (1)} &\Rightarrow y = 5(-\frac{8}{19}) - 3 \\
&\Rightarrow y = -\frac{97}{19} \\
\text{Sub } x = 2 \text{ into (1)} &\Rightarrow y = 7
\end{aligned}$$

$$\therefore \begin{cases} x = -\frac{8}{19} \\ y = -\frac{97}{19} \end{cases} \text{ or } \begin{cases} x = 2 \\ y = 7 \end{cases}$$

7.

$$\begin{cases} x + y = 3 \\ (x+2)(y+3) = 12 \end{cases}$$

Sol.

$$\begin{cases} x + y = 3 & (1) \\ (x+2)(y+3) = 12 & (2) \end{cases}$$

$$\begin{aligned}
(1) &\Rightarrow x = 3 - y && (3) \\
\text{Sub (3) into (2)} &\Rightarrow (3 - y + 2)(y+3) = 12 \\
&\Rightarrow (5-y)(y+3) = 12 \\
&\Rightarrow 5y + 15 - y^2 - 3y = 12 \\
&\Rightarrow 2y - y^2 = -3 \\
&\Rightarrow y^2 - 2y - 3 = 0 \\
&\Rightarrow (y+1)(y-3) = 0 \\
&\Rightarrow y = -1 \text{ or } y = 3 \\
\text{Sub } y = -1 \text{ into (1)} &\Rightarrow x = 4 \\
\text{Sub } y = 3 \text{ into (1)} &\Rightarrow x = 0
\end{aligned}$$

$$\therefore \begin{cases} x = 4 \\ y = -1 \end{cases} \text{ or } \begin{cases} x = 0 \\ y = 3 \end{cases}$$

8.

$$\begin{cases} 5x - 6y = -1 \\ 25x^2 + 36y^2 = 61 \end{cases}$$

Sol.

$$\begin{cases} 5x - 6y = -1 & (1) \\ 25x^2 + 36y^2 = 61 & (2) \end{cases}$$

$$(1) \Rightarrow y = \frac{5x+1}{6} \qquad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow 25x^2 + 36\left(\frac{5x+1}{6}\right)^2 = 61$$

$$\Rightarrow 25x^2 + 36\left(\frac{5x+1}{6}\right)^2 + 36 = 61$$

$$\Rightarrow 25x^2 + 25x^2 + 10x + 1 = 61$$

$$\Rightarrow 50x^2 + 10x = 60$$

$$\Rightarrow 5x^2 + x - 6 = 0$$

$$\Rightarrow (5x+6)(x-1) = 0$$

$$\Rightarrow x = -\frac{6}{5} \text{ or } x = 1$$

$$\text{Sub } x = -\frac{6}{5} \text{ into (1)} \Rightarrow y = \frac{5(-\frac{6}{5})+1}{6}$$

$$\Rightarrow y = -\frac{5}{6}$$

$$\text{Sub } x = 1 \text{ into (1)} \Rightarrow y = \frac{5(1)+1}{6}$$

$$\Rightarrow y = \frac{6}{6}$$

$$\Rightarrow y = 1$$

$$\therefore \begin{cases} x = -\frac{6}{5} \\ y = -\frac{5}{6} \end{cases} \text{ or } \begin{cases} x = 1 \\ y = 1 \end{cases}$$

9.

$$\begin{cases} x + 4y = 5 \\ 2x^2 + 21xy + 27y^2 = 0 \end{cases}$$

Sol.

$$\begin{cases} x + 4y = 5 & (1) \\ 2x^2 + 21xy + 27y^2 = 0 & (2) \end{cases}$$

$$(1) \Rightarrow x = 5 - 4y \qquad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow 2(5-4y)^2 + 21(5-4y)y + 27y^2 = 0$$

$$\Rightarrow 2\left(25 - 40y + 16y^2\right) + 105y - 84y^2 + 27y^2 = 0$$

$$\Rightarrow 50 - 80y + 32y^2 + 105y - 57y^2 = 0$$

$$\Rightarrow 25y^2 - 25y - 50 = 0$$

$$\Rightarrow y^2 - y - 2$$

$$\Rightarrow (y+1)(y-2) = 0$$

$$\Rightarrow y = -1 \text{ or } y = 2$$

$$\text{Sub } y = -1 \text{ into (1)} \Rightarrow x = 5 - 4(-1) = 9$$

$$\text{Sub } y = 2 \text{ into (1)} \Rightarrow x = 5 - 4(2) = -3$$

$$\therefore \begin{cases} x = 9 \\ y = -1 \end{cases} \text{ or } \begin{cases} x = -3 \\ y = 2 \end{cases}$$

10.

$$\begin{cases} \frac{x}{3} - \frac{y}{10} = \frac{5}{6} \\ x(y-2) = 2y + 3 \end{cases}$$

Sol.

$$\begin{cases} \frac{x}{3} - \frac{y}{10} = \frac{5}{6} & (1) \\ x(y-2) = 2y + 3 & (2) \end{cases}$$

$$\begin{aligned}
(1) &\Rightarrow 10x - 3y = 25 && (3)\\
(2) &\Rightarrow x = \frac{2y+3}{y-2} && (4)\\
\text{Sub (4) int 3} &\Rightarrow 10\left(\frac{2y+3}{y-2}\right) - 3y = 25\\
&\Rightarrow 10(2y+3) - 3y(y-2)\\
&\quad = 25(y-2)\\
&\Rightarrow 20y + 30 - 3y^2 + 6y\\
&\quad = 25y - 50\\
&\Rightarrow 3y^2 - y - 80 = 0\\
&\Rightarrow (y+5)(3y-16) = 0\\
&\Rightarrow y = -5 \text{ or } y = \frac{16}{3}\\
\text{Sub } y = -5 \text{ into (1)} &\Rightarrow 10x - 3(-5) = 25\\
&\Rightarrow 10x + 15 = 25\\
&\Rightarrow 10x = 10\\
&\Rightarrow x = 1\\
\text{Sub } y = \frac{16}{3} \text{ into (1)} &\Rightarrow 10x - 3\left(\frac{16}{3}\right) = 25\\
&\Rightarrow 10x = 41\\
&\Rightarrow x = \frac{41}{10}
\end{aligned}$$

$$\therefore \begin{cases} x = 1 \\ y = -5 \end{cases} or \begin{cases} x = \frac{41}{10} \\ y = \frac{16}{3} \end{cases}$$

13.2 System of Equations with Three Variables

13.2.1 Practice 2

Solve the system of equation

$$\begin{cases} x + 2y - z = -5 \\ 2x - y + z = 6 \\ x - y - 3z = -3 \end{cases}$$

Sol.

$$\begin{cases} x + 2y - z = -5 & (1)\\ 2x - y + z = 6 & (2)\\ x - y - 3z = -3 & (3) \end{cases}$$

$$\begin{aligned}
(1)\cdot 3 &\Rightarrow 3x + 6y - 3z = -15 && (4)\\
(2)\cdot 3 &\Rightarrow 6x - 3y + 3z = 18 && (5)\\
(3)+(5) &\Rightarrow 7x - 4y = 15 && (6)\\
(4)+(5) &\Rightarrow 9x + 3y = 3 && (7)\\
(6)\cdot 3 &\Rightarrow 21x - 12y = 45 && (8)\\
(7)\cdot 4 &\Rightarrow 36x + 12y = 12 && (9)\\
(8)+(9) &\Rightarrow 57x = 57 && (10)\\
&\Rightarrow x = 1\\
\text{Sub } x = 1 \text{ into (7)} &\Rightarrow -4y = 8\\
&\Rightarrow y = -2\\
\text{Sub } y = -2 \text{ and } x = 1 \text{ into (1)} &\Rightarrow -z = -2\\
&\Rightarrow z = 2
\end{aligned}$$

$$\therefore x = 1, y = -2, z = 2$$

13.2.2 Exercise 13.2

Solve the following system of equations.

1.

$$\begin{cases} x + y - z = 1 \\ 2x - 3y + z = 0 \\ 2x + y + 2z = 5 \end{cases}$$

Sol.

$$\begin{cases} x + y - z = 1 & (1)\\ 2x - 3y + z = 0 & (2)\\ 2x + y + 2z = 5 & (3) \end{cases}$$

$$\begin{aligned}
(1)\cdot 2 &\Rightarrow 2x + 2y - 2z = 2 && (4)\\
(4)-(3) &\Rightarrow y - 4z = -3 && (5)\\
(3)-(2) &\Rightarrow 4y + z = 5 && (6)\\
(5)\cdot 4 &\Rightarrow 4y - 16z = -12 && (7)\\
(6)-(7) &\Rightarrow 17z = 17\\
&\Rightarrow z = 1\\
\text{Sub } z = 1 \text{ into (5)} &\Rightarrow y = 1\\
\text{Sub } y = 1 \text{ and } z = 1 \text{ into (1)} &\Rightarrow z = 1
\end{aligned}$$

$$\therefore x = 1, y = 1, z = 1$$

2.

$$\begin{cases} x - 2y = 5 \\ 2x + y - 3z = 8 \\ x + 4y - z = 0 \end{cases}$$

Sol.

$$\begin{cases} x - 2y = 5 & (1) \\ 2x + y - 3z = 8 & (2) \\ x + 4y - z = 0 & (3) \end{cases}$$

$$\begin{aligned} (3)\cdot 3 &\Rightarrow 3x + 12y - 3z = 0 \quad (4) \\ (4) - (2) &\Rightarrow x + 11y = -8 \quad (5) \\ (5) - (1) &\Rightarrow 13y = -13 \\ &\Rightarrow y = -1 \\ \text{Sub } y = -1 \text{ into } (1) &\Rightarrow x + 2 = 5 \\ &\Rightarrow x = 3 \\ \text{Sub } x = 3 & \\ \text{and } y = -1 \text{ into } (2) &\Rightarrow -3z = 3 \\ &\Rightarrow z = -1 \end{aligned}$$

$\therefore x = 3, y = -1, z = -1$

3.

$$\begin{cases} x + y = z - 5 \\ y + z = x - 3 \\ z + x = y + 1 \end{cases}$$

Sol.

$$\begin{cases} x + y = z - 5 & (1) \\ y + z = x - 3 & (2) \\ z + x = y + 1 & (3) \end{cases}$$

$$\begin{aligned} (1) &\Rightarrow x + y - z = -5 \quad (4) \\ (2) &\Rightarrow -x + y + z = -3 \quad (5) \\ (3) &\Rightarrow x - y + z = 1 \quad (6) \\ (4) + (5) &\Rightarrow 2y = -8 \\ &\Rightarrow y = -4 \\ (5) + (6) &\Rightarrow 2z = -2 \\ &\Rightarrow z = -1 \\ \text{Sub } y = -4 & \\ \text{and } z = -1 \text{ into } (2) &\Rightarrow x - 3 = -5 \\ &\Rightarrow x = -2 \end{aligned}$$

$\therefore x = -2, y = -4, z = -1$

4.

$$\begin{cases} x + 4y + 2z = 4 \\ 2x - 2y + z = 4 \\ x - 2y + 3z = 3 \end{cases}$$

Sol.

$$\begin{cases} x + 4y + 2z = 4 & (1) \\ 2x - 2y + z = 4 & (2) \\ x - 2y + 3z = 3 & (3) \end{cases}$$

$$\begin{aligned} (1)\cdot 2 &\Rightarrow 2x + 8y + 4z = 8 \quad (4) \\ (3)\cdot 2 &\Rightarrow 2x - 4y + 6z = 6 \quad (5) \\ (4) - (2) &\Rightarrow 10y + 3z = 4 \quad (6) \\ (5) - (4) &\Rightarrow -12y + 2z = -2 \quad (7) \\ (6)\cdot 2 &\Rightarrow 20y + 6z = 8 \quad (8) \\ (7)\cdot 3 &\Rightarrow -36y + 6z = -6 \quad (9) \\ (8) - (9) &\Rightarrow 56y = 14 \\ &\Rightarrow y = \frac{1}{4} \\ \text{Sub } y = \frac{1}{4} \text{ into } (6) &\Rightarrow 6z = 3 \\ &\Rightarrow z = \frac{1}{2} \\ \text{Sub } y = \frac{1}{4} \text{ and } z = \frac{1}{2} \text{ into } (1) &\Rightarrow x + 1 + 1 = 4 \\ &\Rightarrow x = 2 \end{aligned}$$

$\therefore x = 2, y = \frac{1}{4}, z = \frac{1}{2}$

5.

$$\begin{cases} x - y - z = 0 \\ 3x + 2y = 13 \\ y - 3z = -1 \end{cases}$$

Sol.

$$\begin{cases} x - y - z = 0 & (1) \\ 3x + 2y = 13 & (2) \\ y - 3z = -1 & (3) \end{cases}$$

$$(3) \Rightarrow y = 3z - 1 \quad (4)$$
$$\text{Sub (4) into (1)} \Rightarrow x - (3z - 1) - z = 0$$
$$\Rightarrow x - 4z = -1 \quad (5)$$
$$\text{Sub (4) into (2)} \Rightarrow 3x + 2(3z - 1) = 13$$
$$\Rightarrow 3x + 6z = 15 \quad (6)$$
$$(5) \cdot 3 \Rightarrow 3x - 12z = -3 \quad (7)$$
$$(6) - (7) \Rightarrow 18z = 18$$
$$\Rightarrow z = 1$$
$$\text{Sub } z = 1 \text{ into (4)} \Rightarrow y = 2$$
$$\text{Sub } z = 1 \text{ into (5)} \Rightarrow x - 4 = -1$$
$$\Rightarrow x = 3$$

$\therefore x = 3, y = 2, z = 1$

6.

$$\begin{cases} 2x + 2y - z = -1 \\ x + 3y + z = -8 \\ 3x - 2y + 3z = 9 \end{cases}$$

Sol.

$$\begin{cases} 2x + 2y - z = -1 & (1) \\ x + 3y + z = -8 & (2) \\ 3x - 2y + 3z = 9 & (3) \end{cases}$$

$$(1) \cdot 3 \Rightarrow 6x + 6y - 3z = -3 \quad (4)$$
$$(2) \cdot 3 \Rightarrow 3x + 9y + 3z = -24 \quad (5)$$
$$(3) + (4) \Rightarrow 9x + 4y = 6 \quad (6)$$
$$(4) + (5) \Rightarrow 9x + 15y = -27 \quad (7)$$
$$(7) - (6) \Rightarrow 11y = -33$$
$$\Rightarrow y = -3$$
$$\text{Sub } y = -3 \text{ into (6)} \Rightarrow 9x = 18$$
$$\Rightarrow x = 2$$
$$\text{Sub } x = 2$$
$$\text{and } y = -3 \text{ into (2)} \Rightarrow -7 + z = -8$$
$$\Rightarrow z = -1$$

$\therefore x = 2, y = -3, z = -1$

7.

$$\begin{cases} \frac{3}{x} + \frac{1}{y} + \frac{4}{z} = 0 \\ \frac{1}{x} + \frac{4}{y} - \frac{2}{z} = 4 \\ \frac{2}{x} - \frac{3}{y} - \frac{1}{z} = -11 \end{cases}$$

Sol.

$$\begin{cases} \dfrac{3}{x} + \dfrac{1}{y} + \dfrac{4}{z} = 0 & (1) \\ \dfrac{1}{x} + \dfrac{4}{y} - \dfrac{2}{z} = 4 & (2) \\ \dfrac{2}{x} - \dfrac{3}{y} - \dfrac{1}{z} = -11 & (3) \end{cases}$$

$$\text{Let } u = \frac{1}{x}, v = \frac{1}{y}, w = \frac{1}{z}$$
$$(1) \Rightarrow 3u + v + 4w = 0 \quad (4)$$
$$(2) \Rightarrow u + 4v - 2w = 4 \quad (5)$$
$$(3) \Rightarrow 2u - 3v - w = -11 \quad (6)$$
$$(5) \cdot 2 \Rightarrow 2u + 8v - 4w = 8 \quad (7)$$
$$(6) \cdot 4 \Rightarrow 8u - 12v - 4w = -44 \quad (8)$$
$$(4) + (7) \Rightarrow 5u + 9v = 8 \quad (9)$$
$$(4) + (8) \Rightarrow 11u - 11v = -44$$
$$\Rightarrow u - v = -4 \quad (10)$$
$$(10) \cdot 5 \Rightarrow 5u - 5v = -20 \quad (11)$$
$$(9) - (11) \Rightarrow 14v = 28 \quad (12)$$
$$\Rightarrow v = 2$$
$$\text{Sub } v = 2 \text{ into (10)} \Rightarrow u = -2$$
$$\text{Sub } u = -2$$
$$\text{and } v = 2 \text{ into (4)} \Rightarrow -4 + 4w = 0$$
$$\Rightarrow w = 1$$

$\therefore u = -2, v = 2, w = 1$

$\therefore x = -\dfrac{1}{2}, y = \dfrac{1}{2}, z = 1$

13.3 Revision Exercise 13

Solve the following system of equations.

1.

$$\begin{cases} 3x + 4y = 24 \\ xy = 12 \end{cases}$$

Sol.

$$\begin{cases} 3x + 4y = 24 & (1) \\ xy = 12 & (2) \end{cases}$$

$$(2) \Rightarrow y = \frac{12}{x} \quad (3)$$

$$\text{Sub (3) into (1)} \Rightarrow 3x + 4(\frac{12}{x}) = 24$$
$$\Rightarrow 3x^2 + 48 = 24x$$
$$\Rightarrow x^2 - 8x + 16 = 0$$
$$\Rightarrow (x-4)^2 = 0$$
$$\Rightarrow x = 4, x = -4$$

$$\text{Sub } x = 4 \text{ into (3)} \Rightarrow y = \frac{12}{4} = 3$$
$$\text{Sub } x = -4 \text{ into (3)} \Rightarrow y = \frac{12}{-4} = -3$$

$$\therefore \begin{cases} x = 4 \\ y = 3 \end{cases} \text{ or } \begin{cases} x = -4 \\ y = -3 \end{cases}$$

2.

$$\begin{cases} x + 2y = 5 \\ 5x^2 + 4y^2 + 12x = 29 \end{cases}$$

Sol.

$$\begin{cases} x + 2y = 5 & (1) \\ 5x^2 + 4y^2 + 12x = 29 & (2) \end{cases}$$

$$(1) \Rightarrow x = 5 - 2y \quad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow 5(5-2y)^2 + 4y^2 + 12(5-2y) = 29$$
$$\Rightarrow 5(25 - 20y + 4y^2) + 4y^2 + 60 - 24y = 29$$
$$\Rightarrow 125 - 100y + 20y^2 + 4y^2 + 60 - 24y = 29$$
$$\Rightarrow 24y^2 + 124y + 156 = 0$$
$$\Rightarrow 6y^2 + 31y + 39 = 0$$
$$\Rightarrow (y-3)(6y-13) = 0$$
$$\Rightarrow y = 3, y = \frac{13}{6}$$

$$\text{Sub } y = 3 \text{ into (1)} \Rightarrow x = 5 - 2(3) = -1$$
$$\text{Sub } y = \frac{13}{6} \text{ into (1)} \Rightarrow x = 5 - 2(\frac{13}{6}) = \frac{2}{3}$$

$$\therefore \begin{cases} x = -1 \\ y = 3 \end{cases} \text{ or } \begin{cases} x = \frac{2}{3} \\ y = \frac{13}{6} \end{cases}$$

3.

$$\begin{cases} 2x + y = 7 \\ x^2 - xy + y^2 = 7 \end{cases}$$

Sol.

$$\begin{cases} 2x + y = 7 & (1) \\ x^2 - xy + y^2 = 7 & (2) \end{cases}$$

$$(1) \Rightarrow y = 7 - 2x \quad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow x^2 - x(7-2x) + (7-2x)^2 = 7$$
$$\Rightarrow x^2 - 7x + 2x^2 - 28x + 49 + 4x^2 = 7$$
$$\Rightarrow 7x^2 - 35x + 42 = 0$$
$$\Rightarrow x^2 - 5x + 6 = 0$$
$$\Rightarrow (x-2)(x-3) = 0$$
$$\Rightarrow x = 2, x = 3$$

$$\text{Sub } x = 2 \text{ into (3)} \Rightarrow y = 7 - 2(2) = 3$$
$$\text{Sub } x = 3 \text{ into (3)} \Rightarrow y = 7 - 2(3) = 1$$

$$\therefore \begin{cases} x = 2 \\ y = 3 \end{cases} \text{ or } \begin{cases} x = 3 \\ y = 1 \end{cases}$$

4.

$$\begin{cases} 2x + 3y = 7 \\ x^2 + xy + y^2 = 7 \end{cases}$$

Sol.

$$\begin{cases} 2x + 3y = 7 & (1) \\ x^2 + xy + y^2 = 7 & (2) \end{cases}$$

$$(1) \Rightarrow y = \frac{7-2x}{3} \qquad (3)$$

$$\text{Sub (3) into (2)} \Rightarrow x^2 + x(\frac{7-2x}{3})$$

$$+(\frac{7-2x}{3})^2 = 7$$

$$\Rightarrow x^2 + \frac{7x-2x^2}{3}$$

$$+\frac{49-28x+4x^2}{9} = 7$$

$$\Rightarrow 9x^2 + 21x - 6x^2 + 49$$

$$-28x + 4x^2 = 63$$

$$\Rightarrow 7x^2 - 7x - 14 = 0$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x+1)(x-2) = 0$$

$$\Rightarrow x = -1, x = 2$$

$$\text{Sub } x = -1 \text{ into (3)} \Rightarrow y = \frac{7-2(-1)}{3} = 3$$

$$\text{Sub } x = 2 \text{ into (3)} \Rightarrow y = \frac{7-2(2)}{3} = 1$$

$$\therefore \begin{cases} x = -1 \\ y = 3 \end{cases} \quad \text{or} \quad \begin{cases} x = 2 \\ y = 1 \end{cases}$$

5.

$$\begin{cases} 4x - 3y + 2 = 0 \\ 2y + 5z - 19 = 0 \\ 5x - 7z + 16 = 0 \end{cases}$$

Sol.

$$\begin{cases} 4x - 3y + 2 = 0 & (1) \\ 2y + 5z - 19 = 0 & (2) \\ 5x - 7z + 16 = 0 & (3) \end{cases}$$

$$(1)\cdot 2 \Rightarrow 8x - 6y + 4 = 0 \qquad (4)$$

$$(2)\cdot 3 \Rightarrow 6y + 15z - 57 = 0 \qquad (5)$$

$$(4)+(5) \Rightarrow 8x + 15z - 53 = 0 \qquad (6)$$

$$(3)\cdot 8 \Rightarrow 40x - 56z + 128 = 0 \qquad (7)$$

$$(6)\cdot 5 \Rightarrow 40x + 75z - 265 = 0 \qquad (8)$$

$$(7)-(8) \Rightarrow -131z + 393 = 0 \qquad (9)$$

$$\Rightarrow 131z = 393$$

$$\Rightarrow z = 3$$

$$\text{Sub } z = 3 \text{ into (8)} \Rightarrow 40x + 75(3) - 265 = 0$$

$$\Rightarrow 40x + 225 - 265 = 0$$

$$\Rightarrow 40x - 40 = 0$$

$$\Rightarrow x = 1$$

$$\text{Sub } z = 3 \text{ into (2)} \Rightarrow 6y - 12 = 0$$

$$\Rightarrow y = 2$$

$$\therefore x = 1,\ y = 2,\ z = 3$$

6.

$$\begin{cases} x + y + z = 9 \\ 3x + y - 2z = 1 \\ x - 2y + z = 0 \end{cases}$$

Sol.

$$\begin{cases} x + y + z = 9 & (1) \\ 3x + y - 2z = 1 & (2) \\ x - 2y + z = 0 & (3) \end{cases}$$

$$(1) \Rightarrow x + z = 9 - y \qquad (4)$$

$$\text{Sub (4) into (3)} \Rightarrow 9 - y - 2y = 0$$

$$\Rightarrow 3y = 9$$

$$\Rightarrow y = 3$$

$$\text{Sub } y = 3 \text{ into (2)} \Rightarrow 3x - 2z = -2 \qquad (5)$$

$$\text{Sub } y = 3 \text{ into (3)} \Rightarrow x + z = 6 \qquad (6)$$

$$(6)\cdot 2 \Rightarrow 2x + 2z = 12 \qquad (7)$$

$$(5)+(7) \Rightarrow 5x = 10$$

$$\Rightarrow x = 2$$

$$\text{Sub } x = 2 \text{ into (6)} \Rightarrow z = 4$$

$$\therefore x = 2,\ y = 3,\ z = 4$$

7.

$$\begin{cases} 2x - 3y - z = 4 \\ 4x + y + 2z = 3 \\ x - 4y - 3z = 2 \end{cases}$$

Sol.

$$\begin{cases} 2x - 3y - z = 4 & (1) \\ 4x + y + 2z = 3 & (2) \\ x - 4y - 3z = 2 & (3) \end{cases}$$

$$\begin{aligned}
(1)\cdot 2 &\Rightarrow 4x - 6y - 2z = 8 && (4)\\
(3)\cdot 4 &\Rightarrow 4x - 16y - 12z = 8 && (5)\\
(2)-(4) &\Rightarrow 7y + 4z = -5 && (6)\\
(4)-(5) &\Rightarrow 10y + 10z = 0\\
&\Rightarrow y + z = 0\\
&\Rightarrow y = -z\\
\text{Sub } y = -z \text{ into } (6) &\Rightarrow 7(-z) + 4z = -5\\
&\Rightarrow 3z = 5\\
&\Rightarrow z = \frac{5}{3}\\
y = -z &\Rightarrow y = -\frac{5}{3}\\
\text{Sub } y = -\frac{5}{3}&\\
\text{and } z = \frac{5}{3} \text{ into } (1) &\Rightarrow 2x - 3(-\frac{5}{3}) - \frac{5}{3} = 4\\
&\Rightarrow 2x - \frac{5}{3} = -1\\
&\Rightarrow 2x = \frac{2}{3}\\
&\Rightarrow x = \frac{1}{3}
\end{aligned}$$

$$\therefore x = \frac{1}{3},\ y = -\frac{5}{3},\ z = \frac{5}{3}$$

8.

$$\begin{cases} \frac{3}{x+1} - \frac{1}{y+2} + \frac{1}{z-1} = 2 \\ \frac{2}{x+1} - \frac{3}{y+2} - \frac{1}{z-1} = 7 \\ \frac{1}{x+1} + \frac{1}{y+2} - \frac{4}{z-1} = 8 \end{cases}$$

Sol.

$$\begin{cases} \dfrac{3}{x+1} - \dfrac{1}{y+2} + \dfrac{1}{z-1} = 2 & (1) \\ \dfrac{2}{x+1} - \dfrac{3}{y+2} - \dfrac{1}{z-1} = 7 & (2) \\ \dfrac{1}{x+1} + \dfrac{1}{y+2} - \dfrac{4}{z-1} = 8 & (3) \end{cases}$$

Let $u = \dfrac{1}{x+1}, v = \dfrac{1}{y+2}, w = \dfrac{1}{z-1}$

$$\begin{aligned}
(1) &\Rightarrow 3u - v + w = 2 && (4)\\
(2) &\Rightarrow 2u - 3v - w = 7 && (5)\\
(3) &\Rightarrow u + v - 4w = 8 && (6)\\
(4)\cdot 3 &\Rightarrow 9u - 3v + 3w = 6 && (7)\\
(6)\cdot 3 &\Rightarrow 3u + 3v - 12w = 24 && (8)\\
(5)+(8) &\Rightarrow 5u - 13w = 31 && (9)\\
(7)+(8) &\Rightarrow 12u - 9w = 30\\
&\Rightarrow 4u - 3w = 10 && (10)\\
(9)\cdot 4 &\Rightarrow 20u - 52w = 124 && (11)\\
(10)\cdot 5 &\Rightarrow 20u - 15w = 50 && (12)\\
(12)-(11) &\Rightarrow 37w = -74 && (13)\\
&\Rightarrow w = -2\\
\text{Sub } w = -2 \text{ into } (10) &\Rightarrow 4u = 4\\
&\Rightarrow u = 1\\
\text{Sub } u = 1&\\
\text{and } w = -2 \text{ into } (6) &\Rightarrow 9 + v = 8\\
&\Rightarrow v = -1
\end{aligned}$$

$$\therefore u = 1, v = -1,\ w = -2$$

$$\therefore x = 0, y = -3,\ z = \frac{1}{2}$$

Chapter 14

Marix and Determinant

14.1 Matrix

Definition of Matrix

A matrix is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. It is generally denoted as:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where m is the number of rows and n is the number of columns.

Each number in the matrix is called *an entry of the matrix*, the number in the i^{th} row and j^{th} column is denoted as a_{ij}. Thus, a matrix can also be denoted as $A = (a_{ij})$, or $A = (a_{ij})_{mn}$ where m is the number of rows and n is the number of columns.

A matrix with m rows and n columns is called an $m \cdot n$ matrix, where $m \cdot n$ is called the *order of the matrix*. For example, the following matrix is a **$3 \cdot 4$ matrix**:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

When $m = n$, the matrix is called a *square matrix*. For example, the following matrix is a **third-order square matrix**:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

When $m = 1$, the matrix is called a *row matrix*. For example, the following matrix is a **row matrix**:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

When $n = 1$, the matrix is called a *column matrix*. For example, the following matrix is a **column matrix**:

$$A = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

Equal Matrices

Two matrices A and B are equal if they have the same order and the same entries. That is, $A = B$ if and only if $A_{ij} = B_{ij}$ for all i and j.

Zero Matrix

The matrix with all entries equal to zero is called the *zero matrix* and is denoted as O. Zero matrix can be in any order. For exmaple, the matrix below is a **2 · 2 zero matrix** or a **second-order square zero matrix**:

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Identity Matrix

The matrix with all entries equal to zero except the entries on the main diagonal, which are equal to one, is called the *identity matrix* and is denoted as I. Identity matrix can be in any order. The form of an identity matrix is:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Transpose Matrix

The transpose of a matrix A is denoted as A', A^t or A^T and is obtained by interchanging the rows and columns of A. For example, given the matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

The transpose of A is:

$$A' = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

Thus, we know that the transpose matrix of $m \cdot n$ matrix is a $n \cdot m$ matrix.

14.1.1 Exercise 14.1

1. State the order of the following matrices.

   (a) $A = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

   Sol. A is a matrix with order $\mathbf{3 \cdot 1}$.

   (b) $B = \begin{pmatrix} 2 & 1 & 3 & 0 \\ 4 & 2 & 1 & 5 \end{pmatrix}$

   Sol. B is a matrix with order $\mathbf{2 \cdot 4}$.

   (c) $C = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 3 & 0 \\ 2 & 1 & -1 \end{pmatrix}$

   Sol. C is a matrix with order $\mathbf{3 \cdot 3}$.

2. Given $A = \begin{pmatrix} 1 & 5 & -2 & 4 \\ 2 & -4 & 3 & 1 \\ 0 & 6 & 4 & 7 \end{pmatrix}$, what is a_{23} and a_{34}?

   Sol. $a_{23} = 3$ and $a_{34} = 7$.

3. If $\begin{pmatrix} 2 & 0 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & x \end{pmatrix}$, what is x?

   Sol. $x = -4$.

14.2 Matrix Addition and Substraction

Given two matrices A and B of the same order, the sum of A and B is defined as the matrix $A + B$ whose (i, j)-th entry is the sum of the (i, j)-th entries of A and B. That is:

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

The difference of A and B is defined as the matrix $A - B$ whose (i, j)-th entry is the difference of the (i, j)-th entries of A and B. That is:

$$A - B = \begin{pmatrix} a_{11} - b_{11} & a_{12} - b_{12} & \cdots & a_{1n} - b_{1n} \\ a_{21} - b_{21} & a_{22} - b_{22} & \cdots & a_{2n} - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} - b_{m1} & a_{m2} - b_{m2} & \cdots & a_{mn} - b_{mn} \end{pmatrix}$$

Note that the order of A and B must be the same. For example, the following metrices cannot be added or subtracted:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

The addition of matrices has the following properties:

- Commutative: $A + B = B + A$.
- Associative: $(A + B) + C = A + (B + C)$.
- Identity: $A \pm O = A$.
- Inverse: $A + (-A) = O$.
- Transpose: $(A \pm B)' = A' \pm B'$.

where A, B, C are matrices of the same order and O is the zero matrix of the same order as A.

Given a matrix A, if $A = A'$, then A is called a *symmetric matrix*. If $A = -A'$, then A is called an *antisymmetric matrix*.

For any given matrix A, $A + A'$ is symmetric, and $A - A'$ is antisymmetric.

14.2.1 Practice 1

Let $A = \begin{pmatrix} -4 & 2 & -7 \\ 5 & 4 & 0 \\ 3 & -2 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & -5 \\ 4 & -1 & 1 \end{pmatrix}$. Find the following:

1. $A + B'$.

   Sol.

$$B' = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 1 & -1 \\ 2 & -5 & 1 \end{pmatrix}$$

$$A + B' = \begin{pmatrix} -3 & 5 & -3 \\ 8 & 5 & -1 \\ 5 & -7 & -2 \end{pmatrix}$$

2. (A-B)'

   Sol.

$$A - B = \begin{pmatrix} -5 & -1 & -9 \\ 2 & 3 & 5 \\ -1 & -1 & -4 \end{pmatrix}$$

$$(A - B)' = \begin{pmatrix} -5 & 2 & -1 \\ -1 & 3 & -1 \\ -9 & 5 & -4 \end{pmatrix}$$

14.2.2 Exercise 14.2

Let $P = \begin{pmatrix} -5 & 4 & 2 \\ 6 & -4 & 3 \\ -2 & 1 & 6 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & 1 \\ 0 & 0 & 4 \end{pmatrix}$. Evaluate the following:

1. $(P + Q)'$

   Sol.

$$P + Q = \begin{pmatrix} -4 & 2 & 2 \\ 9 & -2 & 4 \\ -2 & 1 & 10 \end{pmatrix}$$

$$\therefore (P + Q)' = \begin{pmatrix} -4 & 9 & -2 \\ 2 & -2 & 1 \\ 2 & 4 & 10 \end{pmatrix}$$

2. $Q' - P'$

   Sol.

$$Q - P = \begin{pmatrix} 6 & -6 & 2 \\ -3 & 6 & -2 \\ 2 & -1 & -2 \end{pmatrix}$$

$$\therefore Q' - P' = (Q - P)' = \begin{pmatrix} 6 & -3 & 2 \\ -6 & 6 & -1 \\ 2 & -2 & -2 \end{pmatrix}$$

3. $(P' - Q)'$

Sol.

$$P' = \begin{pmatrix} -5 & 6 & -2 \\ 4 & -4 & 1 \\ 2 & 3 & 6 \end{pmatrix}$$

$$P' - Q = \begin{pmatrix} -6 & 8 & -2 \\ 1 & -6 & 0 \\ 2 & 3 & 2 \end{pmatrix}$$

$$\therefore (P' - Q)' = \begin{pmatrix} -6 & 1 & 2 \\ 8 & -6 & 3 \\ -2 & 0 & 2 \end{pmatrix}$$

4. $P' - (I - Q)'$

   Sol.

$$\begin{aligned} P' - (I - Q)' &= P' - I' + Q' \\ &= (P + Q)' - I' \\ &= (P + Q - I)' \end{aligned}$$

$$P + Q - I = \begin{pmatrix} -4 & 2 & 2 \\ 9 & -2 & 4 \\ -2 & 1 & 10 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 2 & 2 \\ 9 & -3 & 4 \\ -2 & 1 & 9 \end{pmatrix}$$

$$\begin{aligned} \therefore P' - (I - Q)' &= (P + Q - I)' \\ &= \begin{pmatrix} -5 & 9 & -2 \\ 2 & -3 & 1 \\ 2 & 4 & 9 \end{pmatrix} \end{aligned}$$

14.3 Scalar Product of Matrices

Let $A = (a_{ij})_{m \cdot n}$ be an $m \cdot n$ matrix, k be any real number, then $kA = (ka_{ij})_{m \cdot n}$. This is called scalar product of a matrix A and scalar k. For example:

$$k \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} k & 2k & 3k \\ 4k & 5k & 6k \end{pmatrix}$$

The scalar product of a matrix has the following properties:

- $r(A + B) = rA + sB$
- $(r + s)A = rA + sA$
- $(rs)A = r(sA)$

14.3.1 Practice 2

Let $A = \begin{pmatrix} 2 & 0 \\ -3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 2 & -4 \end{pmatrix}$. Evaluate the following:

1. $3A + B$

   Sol.

$$\begin{aligned} 3A + B &= \begin{pmatrix} 6 & 0 \\ -9 & 15 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 2 & -4 \end{pmatrix} \\ &= \begin{pmatrix} 7 & -1 \\ -7 & 11 \end{pmatrix} \end{aligned}$$

2. $2A - 3B$

   Sol.

$$\begin{aligned} 2A - 3B &= \begin{pmatrix} 4 & 0 \\ -6 & 10 \end{pmatrix} - \begin{pmatrix} 3 & -3 \\ 6 & -12 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3 \\ -12 & 22 \end{pmatrix} \end{aligned}$$

3. $4B - 2A$

   Sol.

$$\begin{aligned} 4B - 2A &= 2(2B - A) \\ &= 2\left(\begin{pmatrix} 2 & -2 \\ 4 & -8 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ -3 & 5 \end{pmatrix}\right) \\ &= 2\begin{pmatrix} 0 & -2 \\ 7 & -13 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -4 \\ 14 & -26 \end{pmatrix} \end{aligned}$$

4. $A' - 2B'$

Sol.

$$\begin{aligned} A' - 2B' &= (A - 2B)' \\ &= \left(\begin{pmatrix} 2 & 0 \\ -3 & 5 \end{pmatrix} - \begin{pmatrix} 2 & -2 \\ 4 & -8 \end{pmatrix}\right)' \\ &= \left(\begin{pmatrix} 0 & 2 \\ -7 & 13 \end{pmatrix}\right)' \\ &= \begin{pmatrix} 0 & -7 \\ 2 & 13 \end{pmatrix} \end{aligned}$$

14.3.2 Exercise 14.3

1. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 1 \\ 3 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$, Calculate the following:

   (a) $2A - 3B + 4C$

   Sol.

$$\begin{aligned} &2A - 3B + 4C \\ &= \begin{pmatrix} 4 & 6 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} 18 & 3 \\ 9 & 6 \end{pmatrix} + \begin{pmatrix} 12 & 4 \\ 4 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 16 & 10 \\ 6 & 0 \end{pmatrix} - \begin{pmatrix} 18 & 3 \\ 9 & 6 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 7 \\ -3 & -6 \end{pmatrix} \end{aligned}$$

   (b) $4A' - (C + B)'$

   Sol.

$$\begin{aligned} &4A' - (C + B') \\ &= 4\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} - \left(\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 6 & 3 \\ 1 & 2 \end{pmatrix}\right) \\ &= \begin{pmatrix} 8 & 4 \\ 12 & 0 \end{pmatrix} - \begin{pmatrix} 9 & 4 \\ 2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 10 & -2 \end{pmatrix} \end{aligned}$$

2. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 4 & -1 \\ 3 & 1 \\ 2 & -3 \end{pmatrix}$, evaluate the following:

   (a) $3A + B - 2C$

   Sol.

$$\begin{aligned} 3A + B - 2C &= \begin{pmatrix} 3 & 6 \\ 0 & 3 \\ 9 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 8 & -2 \\ 6 & 2 \\ 4 & -6 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 7 \\ 4 & 6 \\ 10 & 3 \end{pmatrix} - \begin{pmatrix} 8 & -2 \\ 6 & 2 \\ 4 & -6 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 9 \\ -2 & 4 \\ 6 & -3 \end{pmatrix} \end{aligned}$$

   (b) $3(A + C)' - B'$

   Sol.

$$\begin{aligned} &3(A + C)' - B' \\ &= 3\left(\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ 3 & 1 \\ 2 & -3 \end{pmatrix}\right)' - \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{pmatrix}' \\ &= 3\left(\begin{pmatrix} 5 & 1 \\ 3 & 2 \\ 5 & -2 \end{pmatrix}\right)' - \begin{pmatrix} 2 & 4 & 1 \\ 1 & 3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 15 & 9 & 15 \\ 3 & 6 & -6 \end{pmatrix} - \begin{pmatrix} 2 & 4 & 1 \\ 1 & 3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 13 & 5 & 14 \\ 2 & 3 & -6 \end{pmatrix} \end{aligned}$$

3. Given $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \end{pmatrix}$, Find the matrix X in the following expression:

   (a) $X + 4A = 3(X + B) - A$

   Sol.

$$\begin{aligned} X + 4A &= 3(X + B) - A \\ &= 3X + 3B - A \\ 2X &= 5A - 3B \\ 2X &= 5\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix} - 3\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 10 & 15 \\ 0 & 5 & 5 \end{pmatrix} - \begin{pmatrix} 6 & 3 & 9 \\ 3 & 6 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 7 & 6 \\ -3 & -1 & 5 \end{pmatrix} \\ x &= \begin{pmatrix} -\frac{1}{2} & \frac{7}{2} & 3 \\ -\frac{3}{2} & -\frac{1}{2} & \frac{5}{2} \end{pmatrix} \end{aligned}$$

   (b) $2A - B + X' = B$

Sol.

$$\begin{aligned}
2A - B + X' &= B \\
X' &= -2A + 2B \\
&= -2(A - B) \\
&= -2\left(\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}\right. \\
&\quad \left. - \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \end{pmatrix}\right) \\
&= -2\begin{pmatrix} -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -2 & 0 \\ 2 & 2 & -2 \end{pmatrix} \\
X &= \begin{pmatrix} 2 & 2 \\ -2 & 2 \\ 0 & -2 \end{pmatrix}
\end{aligned}$$

14.4 Multiplication of Matrices

Let A and B be matrices of order $m \cdot n$ and $n \cdot p$ respectively. Then the product of A and B is defined as the matrix AB of order $m \cdot p$ such that the $(i, j)^{th}$ element of AB is given by

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

The multiplication of matrices has the following properties:

- Associative: $A(BC) = (AB)C$
- Distributive: $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$
- $k(AB) \neq (kA)B$ for $k \neq 0$

14.4.1 Practice 3

Evaluate the following:

1. $\begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

   Sol.

$$\begin{aligned}
&\begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} -1(-1) + (-1)(2) & -1(2) + (-1)(1) \\ 2(2) + 3(-1) & 2(1) + 3(2) \end{pmatrix} \\
&= \begin{pmatrix} 1 & -3 \\ 1 & 8 \end{pmatrix}
\end{aligned}$$

2. $\begin{pmatrix} 3 & 4 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 \\ 3 & -3 & 2 \end{pmatrix}$

   Sol.

$$\begin{aligned}
&\begin{pmatrix} 3 & 4 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 \\ 3 & -3 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 3(0) + 4(3) & 3(1) + 4(-3) & 3(2) + 4(2) \\ -1(0) + 1(3) & -1(1) + 1(-3) & -1(2) + 1(2) \end{pmatrix} \\
&= \begin{pmatrix} 12 & -9 & 14 \\ 3 & -4 & 0 \end{pmatrix}
\end{aligned}$$

3. $\begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} 6 & 1 & 5 \\ -2 & 3 & 2 \end{pmatrix}$

   Sol.

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ 2 & 4 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} 6 & 1 & 5 \\ -2 & 3 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 1(6) + 0(-2) & 1(1) + 0(3) & 1(5) + 0(2) \\ 2(6) + 4(-2) & 2(1) + 4(3) & 2(5) + 4(2) \\ 3(6) + (-5)(-2) & 3(1) + (-5)(3) & 3(5) + (-5)(2) \end{pmatrix} \\
&= \begin{pmatrix} 6 & 1 & 5 \\ 4 & 14 & 18 \\ 28 & -12 & 5 \end{pmatrix}
\end{aligned}$$

4. $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 1 \\ -1 & 3 \end{pmatrix}$

   Sol.

$$\begin{aligned}
&\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 1 \\ -1 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 1(1) + 3(2) + 2(-1) & 1(3) + 3(1) + 2(3) \\ 0(1) + 1(2) + 5(-1) & 0(3) + 1(1) + 5(3) \end{pmatrix} \\
&= \begin{pmatrix} 5 & 12 \\ -3 & 16 \end{pmatrix}
\end{aligned}$$

14.4.2 Exercise 14.4

Calculate the following products (Question 1 to 8):

1. $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

   Sol.

$$\begin{aligned}
&\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} 1(1) + 2(2) + 3(3) \end{pmatrix} \\
&= \begin{pmatrix} 14 \end{pmatrix}
\end{aligned}$$

2. $\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix}$

Sol.

$$\begin{aligned}&\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix}\\ =&\begin{pmatrix}1 & 2 & 3\\2 & 4 & 6\\3 & 6 & 9\end{pmatrix}\end{aligned}$$

3. $\begin{pmatrix}2 & -3\\1 & 5\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

Sol.

$$\begin{aligned}&\begin{pmatrix}2 & -3\\1 & 5\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\\ =&\begin{pmatrix}2(1)+(-3)(0) & 2(0)+(-3)(1)\\1(1)+5(0) & 1(0)+5(1)\end{pmatrix}\\ =&\begin{pmatrix}2 & -3\\1 & 5\end{pmatrix}\end{aligned}$$

4. $\begin{pmatrix}-6 & -4 & 2\\7 & 8 & -5\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix}$

Sol.

$$\begin{aligned}&\begin{pmatrix}-6 & -4 & 2\\7 & 8 & -5\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix}\\ =&\begin{pmatrix}-6(1)+(-4)(2)+2(3)\\7(1)+8(2)+(-5)(3)\end{pmatrix}\\ =&\begin{pmatrix}-8\\8\end{pmatrix}\end{aligned}$$

5. $\begin{pmatrix}2 & 3 & 4\\0 & 1 & 2\end{pmatrix}\begin{pmatrix}2 & 0\\3 & 1\\4 & 2\end{pmatrix}$

Sol.

$$\begin{aligned}&\begin{pmatrix}2 & 3 & 4\\0 & 1 & 2\end{pmatrix}\begin{pmatrix}2 & 0\\3 & 1\\4 & 2\end{pmatrix}\\ =&\begin{pmatrix}2(2)+3(3)+4(4) & 2(0)+3(1)+4(2)\\0(2)+1(3)+2(4) & 0(0)+1(1)+2(2)\end{pmatrix}\\ =&\begin{pmatrix}27 & 11\\11 & 5\end{pmatrix}\end{aligned}$$

6. $\begin{pmatrix}6 & 4 & 2\\5 & -2 & 0\\0 & 3 & 1\end{pmatrix}\begin{pmatrix}5\\2\\3\end{pmatrix}$

Sol.

$$\begin{aligned}&\begin{pmatrix}6 & 4 & 2\\5 & -2 & 0\\0 & 3 & 1\end{pmatrix}\begin{pmatrix}5\\2\\3\end{pmatrix}\\ =&\begin{pmatrix}6(5)+4(2)+2(3)\\5(5)+(-2)(2)+0(3)\\0(5)+3(2)+1(3)\end{pmatrix}\\ =&\begin{pmatrix}44\\21\\9\end{pmatrix}\end{aligned}$$

7. $\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}$

Sol.

$$\begin{aligned}&\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}\\ =&\begin{bmatrix}0(0)+1(1)+0(0) & 0(1)+1(0)+0(0) & 0(0)+1(0)+0(1)\\1(0)+0(1)+0(0) & 1(1)+0(0)+0(0) & 1(0)+0(0)+0(1)\\0(0)+0(1)+1(0) & 0(1)+0(0)+1(0) & 0(0)+0(0)+1(1)\end{bmatrix}\\ =&\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}\end{aligned}$$

8. $\begin{pmatrix}1 & -1 & 1\\-3 & 2 & -1\\-2 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\2 & 4 & 6\\1 & 2 & 3\end{pmatrix}$

Sol.

$$\begin{aligned}&\begin{pmatrix}1 & -1 & 1\\-3 & 2 & -1\\-2 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\2 & 4 & 6\\1 & 2 & 3\end{pmatrix}\\ =&\begin{bmatrix}1+(-2)+1 & 2+(-4)+2 & 3+(-6)+3\\(-3)+4+(-1) & (-6)+8+(-2) & (-9)+12+(-3)\\(-2)+2+0 & (-4)+4+0 & (-6)+6+0\end{bmatrix}\\ =&\begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}\end{aligned}$$

14.5 Determinants

The determinant of an n-order matrix $A = (a_{ij})_{n\cdot n}$ is denoted as $\det(A)$. When $n \leq 2$, the deternimant can also be denoted as $|A|$. The determinant is a value.

When $n = 1$, the determinant is the value of the only element in the matrix.

Determinant of a 2x2 matrix

For a 2x2 matrix, the determinant is defined as:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

14.5.1 Practice 4

Find the value of the following determinants.

1. $\begin{vmatrix} 2 & -3 \\ 5 & 7 \end{vmatrix}$

   Sol.

$$\begin{aligned} &\begin{vmatrix} 2 & -3 \\ 5 & 7 \end{vmatrix} \\ &= 2(7) - (-3)(5) \\ &= 14 + 15 \\ &= 29 \end{aligned}$$

2. $\begin{vmatrix} -6 & -7 \\ -8 & -9 \end{vmatrix}$

   Sol.

$$\begin{aligned} &\begin{vmatrix} -6 & -7 \\ -8 & -9 \end{vmatrix} \\ &= (-6)(-9) - (-7)(-8) \\ &= 54 - 56 \\ &= -2 \end{aligned}$$

3. $\begin{vmatrix} 12 & -20 \\ -21 & 35 \end{vmatrix}$

   Sol.

$$\begin{aligned} &\begin{vmatrix} 12 & -20 \\ -21 & 35 \end{vmatrix} \\ &= 12(35) - (-20)(-21) \\ &= 420 - 420 \\ &= 0 \end{aligned}$$

Determinant of a 3x3 matrix

For a 3x3 matrix, the determinant is defined as:

$$\begin{aligned} \det(A) &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - a_3 b_2 c_1 - b_3 c_2 a_1 \\ &\quad - c_3 a_2 b_1 \end{aligned}$$

A 3x3 matrix can be expanded using the Sarrus method. The Sarrus method is defined as:

$$\begin{array}{ccc:cc} + & + & + & & \\ a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \\ - & - & - & & \end{array}$$

Note that the Sarrus method is only applicable to 3x3 matrices.

14.5.2 Practice 5

Calculate the value of the following determinants.

1. $\begin{vmatrix} 1 & 5 & 1 \\ 1 & 6 & 3 \\ 9 & 8 & 9 \end{vmatrix}$

   Sol.

$$\begin{array}{ccc:cc} + & + & + & & \\ 1 & 5 & 1 & 1 & 5 \\ 1 & 6 & 3 & 1 & 6 \\ 9 & 8 & 9 & 9 & 8 \\ - & - & - & & \end{array}$$

$$\begin{aligned} \begin{vmatrix} 1 & 5 & 1 \\ 1 & 6 & 3 \\ 9 & 8 & 9 \end{vmatrix} &= 54 + 135 + 8 - 54 - 24 - 45 \\ &= 74 \end{aligned}$$

2. $\begin{vmatrix} 3 & 1 & -2 \\ 0 & -1 & 1 \\ 4 & 2 & 5 \end{vmatrix}$

   Sol.

$$\begin{array}{ccc:cc} + & + & + & & \\ 3 & 1 & -2 & 3 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ 4 & 2 & 5 & 4 & 2 \\ - & - & - & & \end{array}$$

$$\begin{aligned} \begin{vmatrix} 3 & 1 & -2 \\ 0 & -1 & 1 \\ 4 & 2 & 5 \end{vmatrix} &= -15 + 4 - 0 - 8 - 6 - 0 \\ &= -25 \end{aligned}$$

Minor and Cofactor

The minor of an element in a matrix is the determinant of the matrix obtained by deleting the row and column containing the element. Take $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ as an example. The minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, the minor of c_2 is $\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$, and so on.

The cofactor of an element in a matrix is the minor of the element multiplied by $(-1)^{i+j}$, where i and j are the row and column indices of the element. The cofactor of a_1 is $(-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, the cofactor of c_2 is $(-1)^{3+2}\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$, and so on.

Let A_1, B_1, C_1 are the cofactors of a_1, b_1, c_1 respectively. Then

$$A_1 = (-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

$$B_1 = (-1)^{1+2}\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

$$C_1 = (-1)^{1+3}\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Thus,

$$|A| = a_1A_1 + a_2B_1 + a_3C_1$$

That is, the value of the determinant is the elements of the first row multiplied by the cofactors of the elements of the first row.

The sign of the cofactor is determined by the sum of the row and column indices of the element. If the sum is even, the cofactor is positive; if the sum is odd, the cofactor is negative.

Generally, a 3x3 determinant has the following theorem:

Theorem 1 *The determinant of a 3x3 matrix is the sum of the elements of any row or column multiplied by the cofactors of the elements of that row or column.*

That is, we can use the cofactor expansion to calculate the determinant of a 3x3 matrix.

$$\begin{aligned} |A| &= a_1A_1 + b_1B_1 + c_1C_1 \\ &= a_2B_2 + b_2B_2 + c_2C_2 \\ &= a_3C_3 + b_3C_3 + c_3C_3 \\ &= a_1A_1 + a_2A_2 + a_3A_3 \\ &= b_1B_1 + b_2B_2 + b_3B_3 \\ &= c_1C_1 + c_2C_2 + c_3C_3 \end{aligned}$$

The determinant of any order matrix can also be calculated by the cofactor expansion.

Theorem 2 *The product of the elements of any row or column and the cofactor of corresponding elements of another row or column of a determinant is 0.*

For example, the product of the elements of the second row and the corresponding element of the cofactor of first row of the determinant is 0. That is,

$$\begin{aligned} &a_2B_1 + b_2B_1 + c_2C_1 \\ &= a_2\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_2\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_2\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\ &= a_2b_2c_3 + a_2b_3c_2 - a_2b_2c_3 + a_3b_2c_2 + a_2b_3c_2 - a_3b_2c_2 \\ &= 0 \end{aligned}$$

14.5.3 Practice 6

Find the value of the following 3x3 determinants.

1. $\begin{vmatrix} 4 & -2 & 1 \\ 1 & -3 & 0 \\ 2 & 7 & -1 \end{vmatrix}$

   Sol.

$$\begin{aligned} &\begin{vmatrix} 4 & -2 & 1 \\ 1 & -3 & 0 \\ 2 & 7 & -1 \end{vmatrix} \\ &= 4\begin{vmatrix} -3 & 0 \\ 7 & -1 \end{vmatrix} - \begin{vmatrix} -2 & 1 \\ 7 & -1 \end{vmatrix} + 2\begin{vmatrix} -2 & 1 \\ -3 & 0 \end{vmatrix} \\ &= 4(3-0) - (2-7) + 2(0+3) \\ &= 12 + 5 + 6 \\ &= 23 \end{aligned}$$

2. $\begin{vmatrix} 5 & -4 & 2 \\ 1 & 0 & -3 \\ 1 & -1 & 2 \end{vmatrix}$

   Sol.

$$\begin{aligned} &\begin{vmatrix} 5 & -4 & 2 \\ 1 & 0 & -3 \\ 1 & -1 & 2 \end{vmatrix} \\ &= 5\begin{vmatrix} 0 & -3 \\ -1 & 2 \end{vmatrix} - \begin{vmatrix} -4 & 2 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} -4 & 2 \\ 0 & -3 \end{vmatrix} \\ &= 5(0-3) - (-8+2) + (12+0) \\ &= -15 + 6 + 12 \\ &= 3 \end{aligned}$$

3. $\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & -1 \end{vmatrix}$$
$$= 2\begin{vmatrix} 2 & 0 \\ 0 & -1 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix}$$
$$= 2(-2-0) + 2(-2)$$
$$= -4 - 4$$
$$= -8$$

14.5.4 Exercise 14.5a

Find the value of the following determinants.

1. $\begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix}$$
$$= 3(-4) - 2(1)$$
$$= -12 - 2$$
$$= -14$$

2. $\begin{vmatrix} 35 & -2 \\ -11 & 5 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 35 & -2 \\ -11 & 5 \end{vmatrix}$$
$$= 35(5) - (-2)(-11)$$
$$= 175 - 22$$
$$= 153$$

3. $\begin{vmatrix} 1 & a \\ -a & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 1 & a \\ -a & 1 \end{vmatrix}$$
$$= 1(1) - a(-a)$$
$$= 1 + a^2$$

4. $\begin{vmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{vmatrix}$

Sol.

$$\begin{vmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{vmatrix}$$
$$= \sin x \sin x - (-\cos x)(\cos x)$$
$$= \sin^2 x + \cos^2 x$$
$$= 1$$

5. $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -4 \\ 3 & -2 & 5 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -4 \\ 3 & -2 & 5 \end{vmatrix}$$
$$= 1\begin{vmatrix} 3 & -4 \\ -2 & 5 \end{vmatrix} - 2\begin{vmatrix} -2 & 3 \\ -2 & 5 \end{vmatrix} + 3\begin{vmatrix} -2 & 3 \\ 3 & -4 \end{vmatrix}$$
$$= 1(15-8) - 2(-10+6) + 3(8-9)$$
$$= 7 + 8 - 3$$
$$= 12$$

6. $\begin{vmatrix} 1 & -3 & 4 \\ 2 & 0 & -5 \\ 3 & -1 & 7 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 1 & -3 & 4 \\ 2 & 0 & -5 \\ 3 & -1 & 7 \end{vmatrix}$$
$$= \begin{vmatrix} 0 & -5 \\ -1 & 7 \end{vmatrix} - 2\begin{vmatrix} -3 & 4 \\ -1 & 7 \end{vmatrix} + 3\begin{vmatrix} -3 & 4 \\ 0 & -5 \end{vmatrix}$$
$$= (0-5) - 2(-21+4) + 3(15-0)$$
$$= -5 + 34 + 45$$
$$= 74$$

7. $\begin{vmatrix} -1 & 3 & -2 \\ -3 & 2 & 0 \\ 4 & 0 & 5 \end{vmatrix}$

Sol.

$$\begin{vmatrix} -1 & 3 & -2 \\ -3 & 2 & 0 \\ 4 & 0 & 5 \end{vmatrix}$$
$$= -1\begin{vmatrix} 2 & 0 \\ 0 & 5 \end{vmatrix} + 3\begin{vmatrix} 3 & -2 \\ 0 & 5 \end{vmatrix} + 4\begin{vmatrix} 3 & -2 \\ 2 & 0 \end{vmatrix}$$
$$= -1(10) + 3(-15+0) + 4(0+4)$$
$$= -10 + 45 + 16$$
$$= 51$$

8. $\begin{vmatrix} 0 & -q & -r \\ q & 0 & -s \\ r & s & 0 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 0 & -q & -r \\ q & 0 & -s \\ r & s & 0 \end{vmatrix}$$
$$= 0\begin{vmatrix} 0 & -s \\ s & 0 \end{vmatrix} - q\begin{vmatrix} -q & -r \\ s & 0 \end{vmatrix} + r\begin{vmatrix} -q & -r \\ 0 & -s \end{vmatrix}$$
$$= 0 - q(0 + sr) + r(0 + qs)$$
$$= 0$$

9. $\begin{vmatrix} p & -q & r \\ q & r & -s \\ -r & s & p \end{vmatrix}$

Sol.

$$\begin{vmatrix} p & -q & r \\ q & r & -s \\ -r & s & p \end{vmatrix}$$
$$= p\begin{vmatrix} r & -s \\ s & p \end{vmatrix} - q\begin{vmatrix} -q & r \\ s & p \end{vmatrix} + r\begin{vmatrix} -q & r \\ r & -s \end{vmatrix}$$
$$= p(rp + s^2) - q(-qp - sr) - r(qs - r^2)$$
$$= rp^2 + ps^2 + q^2p + qsr - qsr + r^3$$
$$= rp^2 + s^2p + q^2p - r^3$$

10. $\begin{vmatrix} 1 & x & a \\ 1 & y & b \\ 1 & z & c \end{vmatrix}$

Sol.

$$\begin{vmatrix} 1 & x & a \\ 1 & y & b \\ 1 & z & c \end{vmatrix}$$
$$= \begin{vmatrix} y & b \\ z & c \end{vmatrix} - \begin{vmatrix} x & a \\ z & c \end{vmatrix} + \begin{vmatrix} x & a \\ y & b \end{vmatrix}$$
$$= (yc - bz) - (xc - az) + (xb - ay)$$
$$= bx + cy + az - cx - ay - bz$$

Identities of Determinants

Theorem 1 *The value of a determinant is the same as the value of its transpose, aka $|A| = |A'|$.*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

Theorem 2 *Switching any two rows or columns of a determinants results in the opposite value.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

14.5.5 Practice 7

Given $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 10$, find $\begin{vmatrix} b & c & a \\ e & f & d \\ h & i & g \end{vmatrix}$.

Sol.

$$\begin{vmatrix} b & c & a \\ e & f & d \\ h & i & g \end{vmatrix} = -\begin{vmatrix} a & c & b \\ d & f & e \\ g & i & h \end{vmatrix}$$
$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$
$$= 10$$

Theorem 3 *If two rows or cols of a determinant are identical, the value of the determinant is zero.*

$$\begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix} = 0$$

Theorem 4 *If all elements of a row (or column) of a determinant are multiplied by some scalar number k, the value of* *the new determinant is k times of the given determinant.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

14.5.6 Practice 8

Using the identities of determinants, prove that $\begin{vmatrix} 10 & -12 & 2 \\ -15 & 18 & 3 \\ 5 & 6 & -1 \end{vmatrix} = 180 \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$.

Sol.

$$\begin{aligned} &\begin{vmatrix} 10 & -12 & 2 \\ -15 & 18 & 3 \\ 5 & 6 & -1 \end{vmatrix} \\ &= 5 \cdot 6 \begin{vmatrix} 2 & -2 & 2 \\ -3 & 3 & 3 \\ 1 & 1 & -1 \end{vmatrix} \\ &= 5 \cdot 6 \cdot 2 \cdot 3 \cdot \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \\ &= 180 \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \end{aligned}$$

Theorem 5 *In a determinant each element in any row (or column) consists of the sum of two terms, then the determinant can be expressed as sum of two determinants of same order.*

$$\begin{vmatrix} a_1 + d_1 & b_1 & c_1 \\ a_2 + d_2 & b_2 & c_2 \\ a_3 + d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Theorem 6 *If a determinant is obtained by adding a row or column multiplied by a some scalar number k to a different row or column, then the value of the new determinant is the same as the original determinant.*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

14.5.7 Practice 9

Prove that $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$.

Sol.

$$\begin{aligned} &\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} && \text{(Adding row 1 multiplied by -1 to row 2 and 3)} \\ &= 2 \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} && \text{(Theorem 4)} \\ &= 0 && \text{(Theorem 3)} \end{aligned}$$

Theorem 7 *The determinant of product of two matrices of equal size is equal to the product of determinants of each matrix, aka $|AB| = |A||B|$.*

14.5.8 Practice 10

Let $A = \begin{vmatrix} -2 & 1 \\ 4 & 3 \end{vmatrix}$ and $B = \begin{vmatrix} 1 & x \\ 2 & 3 \end{vmatrix}$. Given that $|AB| = -18$, find x.

Sol.

$$\begin{aligned} \because |AB| = |A||B| &= -18 \\ \therefore \begin{vmatrix} -2 & 1 \\ 4 & 3 \end{vmatrix} \begin{vmatrix} 1 & x \\ 2 & 3 \end{vmatrix} &= -18 \\ -2(3 - 2x) &= -18 \\ 3 - 2x &= 9 \\ -2x &= 6 \\ x &= -3 \end{aligned}$$

14.5.9 Exercise 14.5b

1. Given $\begin{vmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix} = -1$, Find the value of the following determinants.

   (a) $\begin{vmatrix} 2 & 0 & -1 \\ -2 & -1 & 2 \\ 3 & -2 & 1 \end{vmatrix}$

   Sol.

$$\begin{aligned} &\begin{vmatrix} 2 & 0 & -1 \\ -2 & -1 & 2 \\ 3 & -2 & 1 \end{vmatrix} \\ &= \left| \begin{pmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{pmatrix}' \right| \\ &= -1 && \text{(Theorem 1)} \end{aligned}$$

(b) $\begin{vmatrix} 2 & 0 & -1 \\ 3 & -2 & 1 \\ -2 & -1 & 2 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 2 & 0 & -1 \\ 3 & -2 & 1 \\ -2 & -1 & 2 \end{vmatrix}$$

Sol.

$$\begin{vmatrix} 2 & 0 & -1 \\ 3 & -2 & 1 \\ -2 & -1 & 2 \end{vmatrix}$$

$$= - \begin{vmatrix} 2 & 0 & -1 \\ -2 & -1 & 2 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= - \left| \begin{pmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{pmatrix}' \right|$$

$$= 1 \qquad \text{(Theorem 1)}$$

(c) $\begin{vmatrix} 4 & -2 & 3 \\ 0 & -2 & -4 \\ -2 & 2 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 4 & -2 & 3 \\ 0 & -2 & -4 \\ -2 & 2 & 1 \end{vmatrix}$$

$$= 2 \cdot 2 \begin{vmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= -4$$

(d) $\begin{vmatrix} -3 & -2 & -2 \\ 2 & -1 & 0 \\ -1 & 2 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} -3 & -2 & -2 \\ 2 & -1 & 0 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -2 & 2 \\ -2 & -1 & 0 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 & 3 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= 1$$

(e) $\begin{vmatrix} 0 & 2 & 5 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 0 & 2 & 5 \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2+(2\cdot(-1)) & -2+(-2\cdot 2) & 3+(2\cdot 1) \\ 0 & -1 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= -1$$

(f) $\begin{vmatrix} 2 & 4 & 3 \\ 0 & -5 & -2 \\ -1 & 4 & 1 \end{vmatrix}$

Sol.

$$\begin{vmatrix} 2 & 4 & 3 \\ 0 & -5 & -2 \\ -1 & 4 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2+(2\cdot 3) & 3 \\ 0 & -1+(2\cdot(-2)) & -2 \\ -1 & 2+(2\cdot 1) & 1 \end{vmatrix}$$

$$= -1$$

2. Prove the following equations using identities of determinants without expanding them.

(a) $\begin{vmatrix} 1 & 0 & 3 \\ 4 & 6 & 12 \\ 3 & 3 & 9 \end{vmatrix} = 0$

Proof.

$$L.H.S. = \begin{vmatrix} 1 & 0 & 3 \\ 4 & 6 & 12 \\ 3 & 3 & 9 \end{vmatrix}$$

$$= 2 \cdot 3 \begin{vmatrix} 1 & 0 & 3 \\ 2 & 3 & 6 \\ 1 & 1 & 3 \end{vmatrix}$$

$$= 2 \cdot 3 \cdot 3 \begin{vmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 0 = R.H.S. \qquad \text{(Theorem 3)}$$

(b) $\begin{vmatrix} 2 & -1 & 3 \\ -1 & 1 & 2 \\ 4 & -2 & 6 \end{vmatrix} = 0$

Proof.

$$L.H.S. = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 1 & 2 \\ 4 & -2 & 6 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & -1 & 3 \\ -1 & 1 & 2 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 0 = R.H.S. \qquad \text{(Theorem 3)}$$

(c) $\begin{vmatrix} 3 & 2 & 2 \\ 9 & 6 & 5 \\ 12 & 8 & 8 \end{vmatrix} = 0$

Proof.

$$L.H.S. = \begin{vmatrix} 3 & 2 & 2 \\ 9 & 6 & 5 \\ 12 & 8 & 8 \end{vmatrix}$$

$$= 4\begin{vmatrix} 3 & 2 & 2 \\ 9 & 6 & 5 \\ 3 & 2 & 2 \end{vmatrix}$$

$$= 0 = R.H.S. \qquad \text{(Theorem 3)}$$

(d) $\begin{vmatrix} 10 & 8 & 2 \\ 15 & 12 & 3 \\ 20 & 32 & 12 \end{vmatrix} = 0$

Proof.

$$L.H.S. = \begin{vmatrix} 10 & 8 & 2 \\ 15 & 12 & 3 \\ 20 & 32 & 12 \end{vmatrix}$$

$$= 2 \cdot 3 \cdot 4\begin{vmatrix} 5 & 4 & 1 \\ 5 & 4 & 1 \\ 5 & 8 & 3 \end{vmatrix}$$

$$= 2 \cdot 3 \cdot 4 \cdot 5 \cdot 4\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= 0 = R.H.S. \qquad \text{(Theorem 3)}$$

(e) $\begin{vmatrix} 1 & -4 & 5 \\ 2 & 3 & -2 \\ 7 & 3 & 0 \end{vmatrix} = \begin{vmatrix} -4 & 3 & 3 \\ 5 & -2 & 0 \\ 1 & 2 & 7 \end{vmatrix}$

Proof.

$$L.H.S = \begin{vmatrix} 1 & -4 & 5 \\ 2 & 3 & -2 \\ 7 & 3 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} 5 & -4 & 1 \\ -2 & 3 & 2 \\ 0 & 3 & 7 \end{vmatrix}$$

$$= \begin{vmatrix} -4 & 5 & 1 \\ 3 & -2 & 2 \\ 3 & 0 & 7 \end{vmatrix} \qquad \text{(Theorem 2)}$$

$$= \left|\begin{pmatrix} -4 & 5 & 1 \\ 3 & -2 & 2 \\ 3 & 0 & 7 \end{pmatrix}'\right| \qquad \text{(Theorem 1)}$$

$$= \begin{vmatrix} -4 & 3 & 3 \\ 5 & -2 & 0 \\ 1 & 2 & 7 \end{vmatrix} = R.H.S.$$

(f) $\begin{vmatrix} -6 & 6 & 3 \\ 0 & -9 & -3 \\ 3 & -3 & -6 \end{vmatrix} = -27\begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ -2 & 2 & -1 \end{vmatrix}$

Proof.

$L.H.S.$

$$= \begin{vmatrix} -6 & 6 & 3 \\ 0 & -9 & -3 \\ 3 & -3 & -6 \end{vmatrix}$$

$$= 3 \cdot 3 \cdot 3\begin{vmatrix} -2 & 2 & 1 \\ 0 & -3 & -1 \\ 1 & -1 & -2 \end{vmatrix}$$

$$= -27\begin{vmatrix} 1 & -1 & -2 \\ 0 & -3 & -1 \\ -2 & 2 & 1 \end{vmatrix} \qquad \text{(Theorem 2)}$$

$$= 27\begin{vmatrix} -1 & 1 & -2 \\ -3 & 0 & -1 \\ 2 & -2 & 1 \end{vmatrix} \qquad \text{(Theorem 2)}$$

$$= -27\begin{vmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ -2 & 2 & -1 \end{vmatrix} = R.H.S. \qquad \text{(Theorem 4)}$$

(g) $\begin{vmatrix} 1 & 0 & -3 \\ 3 & -2 & 4 \\ 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ -1 & 2 & 4 \\ 7 & 3 & -2 \end{vmatrix}$

Proof.

$L.H.S.$

$$= \begin{vmatrix} 1 & 0 & -3 \\ 3 & -2 & 4 \\ 1 & 3 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+(2\cdot 0) & 0 & -3 \\ 3+(2\cdot(-2)) & -2 & 4 \\ 1+(2\cdot 3) & 3 & -2 \end{vmatrix} \quad \text{(Theorem 6)}$$

$$= \begin{vmatrix} 1 & 0 & -3 \\ -1 & 2 & 4 \\ 7 & 3 & -2 \end{vmatrix} = R.H.S.$$

(h) $\begin{vmatrix} 5 & 1 & -1 \\ 2 & -1 & -1 \\ 1 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 0 \\ 4 & -1 & -2 \\ 5 & -2 & 2 \end{vmatrix}$

Proof.

$L.H.S.$

$$= \begin{vmatrix} 5 & 1 & -1 \\ 2 & -1 & -1 \\ 1 & -2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 5+(-2\cdot 1) & 1 & -1+1 \\ 2+(-2\cdot(-1)) & -1 & -2-1 \\ 1+(-2\cdot(-2)) & -2 & 4-2 \end{vmatrix} \quad \text{(Theorem 6)}$$

$$= \begin{vmatrix} 3 & 1 & 0 \\ 4 & -1 & -2 \\ 5 & -2 & 2 \end{vmatrix} = R.H.S.$$

3. Let $A = \begin{pmatrix} 7 & -4 \\ -3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2x+1 & -2 \\ x & 1 \end{pmatrix}$. Given that $|AB| = -22$, find the value of x.

Sol.

$$\because |AB| = |A||B| = -22$$

$$\therefore \begin{vmatrix} 7 & -4 \\ -3 & 2 \end{vmatrix}\begin{vmatrix} 2x+1 & -2 \\ x & 1 \end{vmatrix} = -22$$

$$2(2x+1+2x) = -22$$

$$4x+1 = -11$$

$$4x = -12$$

$$x = -3$$

4. Let $P = \begin{pmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $Q = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$. Given that $PQ = \begin{pmatrix} 30 & -18 & -33 \\ -6 & 4 & 6 \\ -11 & 6 & 14 \end{pmatrix}$, find the value of $|Q|$.

Sol.

$$\because |P||Q| = |PQ|$$

$$\therefore \begin{vmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = \begin{vmatrix} 30 & -18 & -33 \\ -6 & 4 & 6 \\ -11 & 6 & 14 \end{vmatrix}$$

$$3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} 30 & -18 & -33 \\ -6 & 4 & 6 \\ -11 & 6 & 14 \end{vmatrix}$$

$$3\left(\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\right)^2 = 3\begin{vmatrix} 10 & -6 & -11 \\ -6 & 4 & 6 \\ -11 & 6 & 14 \end{vmatrix}$$

$$\left(\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\right)^2 = \begin{vmatrix} 10 & -6 & -11 \\ -6 & 4 & 6 \\ -11 & 6 & 14 \end{vmatrix}$$

$$= 4$$

$$\therefore |Q| = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$

$$= \left|\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}'\right|$$

$$= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= \pm 2$$

Find the value of x in the following equations.

5. $\begin{vmatrix} x & x \\ -2x & -1 \end{vmatrix} = 6$

Sol.

$$\begin{vmatrix} x & x \\ -2x & -1 \end{vmatrix} = 6$$

$$x\begin{vmatrix} 1 & 1 \\ -2x & -1 \end{vmatrix} = 6$$

$$x(-1+2x) = 6$$

$$-x+2x^2 = 6$$

$$2x^2 - x - 6 = 0$$

$$(x-2)(2x+3) = 0$$

$$x = 2 \text{ or } x = -\frac{3}{2}$$

6. $\begin{vmatrix} 2 & 4 & 0 \\ 2 & 5 & 6 \\ 3 & x & 9 \end{vmatrix} = 0$

Sol.

$$\begin{vmatrix} 2 & 4 & 0 \\ 2 & 5 & 6 \\ 3 & x & 9 \end{vmatrix} = 0$$

$$2 \cdot 3 \begin{vmatrix} 1 & 2 & 0 \\ 2 & 5 & 2 \\ 3 & x & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 0 \\ 2 & 5 & 2 \\ 3 & x & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 5 & 2 \\ x & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ x & 3 \end{vmatrix} + 3\begin{vmatrix} 2 & 0 \\ 5 & 2 \end{vmatrix} = 0$$

$$15 - 2x - 12 + 12 = 0$$

$$-2x = -15$$

$$x = \frac{15}{2}$$

7. $\begin{vmatrix} 1 & 1 & x \\ 1 & x & 1 \\ x & 1 & 1 \end{vmatrix} = 0$

Sol.

$$\begin{vmatrix} 1 & 1 & x \\ 1 & x & 1 \\ x & 1 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & 1 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & x \\ 1 & 1 \end{vmatrix} + x\begin{vmatrix} 1 & x \\ x & 1 \end{vmatrix} = 0$$

$$x - 1 - 1 + x + x - x^3 = 0$$

$$-x^3 + 3x - 2 = 0$$

$$x^3 - 3x + 2 = 0$$

$$(x+2)(x^2 - 2x + 1) = 0$$

$$x = -2 \text{ or } x = 1$$

8. $\begin{vmatrix} 2x-7 & 6 & 9 \\ 3x-5 & 5 & 4 \\ x-3 & 0 & 1 \end{vmatrix} = 0$

Sol.

$$\begin{vmatrix} 2x-7 & 6 & 9 \\ 3x-5 & 5 & 4 \\ x-3 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2x & 6 & 9 \\ 3x & 5 & 4 \\ x & 0 & 1 \end{vmatrix} + \begin{vmatrix} -7 & 6 & 9 \\ -5 & 5 & 4 \\ -3 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2x & 6 & 9 \\ 3x & 5 & 4 \\ x & 0 & 1 \end{vmatrix} = -58$$

$$x\begin{vmatrix} 6 & 9 \\ 5 & 4 \end{vmatrix} + \begin{vmatrix} 2x & 6 \\ 3x & 5 \end{vmatrix} = -58$$

$$-21x + 10x - 18x = -58$$

$$-29x = -58$$

$$x = 2$$

9. $\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0$

Sol.

$$\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0$$

$$\begin{vmatrix} 15 & 11 & 10 \\ 11 & 17 & 16 \\ 7 & 14 & 13 \end{vmatrix} + \begin{vmatrix} -2x & 11 & 10 \\ -3x & 17 & 16 \\ -x & 14 & 13 \end{vmatrix} = 0$$

$$\begin{vmatrix} -2x & 11 & 10 \\ -3x & 17 & 16 \\ -x & 14 & 13 \end{vmatrix} = 36$$

$$-2x\begin{vmatrix} 17 & 16 \\ 14 & 13 \end{vmatrix} + 3x\begin{vmatrix} 11 & 10 \\ 14 & 13 \end{vmatrix} - x\begin{vmatrix} 11 & 10 \\ 17 & 16 \end{vmatrix} = 36$$

$$x\left(2\begin{vmatrix} 17 & 16 \\ 14 & 13 \end{vmatrix} - 3\begin{vmatrix} 11 & 10 \\ 14 & 13 \end{vmatrix} + \begin{vmatrix} 11 & 10 \\ 17 & 16 \end{vmatrix}\right) = -36$$

$$(-6 - 9 + 6)x = -36$$

$$-9x = -36$$

$$x = 4$$

10. $\begin{vmatrix} x-1 & 0 & x-3 \\ 1 & x-2 & 1 \\ 2 & x-2 & 2 \end{vmatrix} = 0$

Sol.

$$
\begin{vmatrix} x-1 & 0 & x-3 \\ 1 & x-2 & 1 \\ 2 & x-2 & 2 \end{vmatrix} = 0
$$
$$
(x-2)\begin{vmatrix} x-1 & 0 & x-3 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = 0
$$
$$
(x-2)\left(-\begin{vmatrix} x-1 & x-3 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} x-1 & x-3 \\ 1 & 1 \end{vmatrix}\right) = 0
$$
$$
(x-2)\left[-(2x-2-2x+6)+(x-1-x-3)\right] = 0
$$
$$
(x-2) = 0
$$
$$
x = 2
$$

14.6 Inverse Matrix

If two square matrices A and B are of the same order such that $AB = BA = I$, while I is an identity matrix that has the same order as A and B, then A and B are said to be inverse matrices of each other, and can be denoted as $B = A^{-1}$ and $A = B^{-1}$.

Note that only square matrix have inverse matrix. If a matrix has an inverse matrix, then it is said to be invertible, and the inverse matrix is unique.

Inverse Matrix of a 2x2 Matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2x2 matrix. Then

$$
A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (ad-bc \neq 0)
$$

If $|A| = ad - bc = 0$, then A is said to be non-invertible.

14.6.1 Practice 11

Determine if the following matrices are invertible. If they are, find their inverse matrices.

1. $\begin{pmatrix} 6 & 3 \\ 7 & 5 \end{pmatrix}$

   Sol.

   $$|A| = 6\cdot 5 - 3\cdot 7 = 9 \neq 0$$
   $$\therefore A \text{ is invertible.}$$
   $$A^{-1} = \frac{1}{9}\begin{pmatrix} 5 & -3 \\ -7 & 6 \end{pmatrix} = \begin{pmatrix} \frac{5}{9} & -\frac{1}{1} \\ -\frac{7}{9} & \frac{2}{3} \end{pmatrix}$$

2. $\begin{pmatrix} -3 & -2 \\ 6 & 4 \end{pmatrix}$

   Sol.

   $$|A| = -3\cdot 4 - (-2)\cdot 6 = 0$$
   $$\therefore A \text{ is non-invertible.}$$

3. $\begin{pmatrix} 2 & -6 \\ 3 & -5 \end{pmatrix}$

   Sol.

   $$|A| = 2\cdot -5 - (-6)\cdot 3 = 8 \neq 0$$
   $$\therefore A \text{ is invertible.}$$
   $$A^{-1} = \frac{1}{8}\begin{pmatrix} -5 & 6 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{5}{8} & \frac{3}{4} \\ -\frac{3}{8} & \frac{1}{4} \end{pmatrix}$$

4. If $\begin{pmatrix} 2b+1 & 2 \\ -3b-3 & -4 \end{pmatrix}$ is non-invertible, find the value of b.

   Sol.

   $$\because \text{The matrix is non-invertible}$$
   $$\therefore \begin{vmatrix} 2b+1 & 2 \\ -3b-3 & -4 \end{vmatrix} = 0$$
   $$-8b - 4 + 6b + 6 = 0$$
   $$-2b + 2 = 0$$
   $$b = 1$$

Inverse Matrix of a 3x3 Matrix

Let a 3x3 matrix A be of the form $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$. Arrange all the cofactors of elements in A into a matrix:

$$
\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}
$$

Then the transpose of the matrix is the adjoint matrix of A, and can be denoted as adj A. That is:

$$
\text{adj } A = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}
$$

The inverse matrix of A is:

$$
A^{-1} = \frac{1}{|A|}\text{adj } A \quad (|A| \neq 0)
$$

14.6.2 Practice 12

Find the inverse matrix of the following matrices.

1. $\begin{pmatrix} -1 & 2 & 3 \\ 3 & 0 & 1 \\ 2 & -1 & -2 \end{pmatrix}$

Sol.

$$|A| = \begin{vmatrix} -1 & 2 & 3 \\ 3 & 0 & 1 \\ 2 & -1 & -2 \end{vmatrix} = 6$$

$$\begin{pmatrix} \begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 2 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 2 & -2 \end{vmatrix} & -\begin{vmatrix} -1 & 2 \\ 2 & -1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 3 & 0 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 8 & -3 \\ 1 & -4 & 3 \\ 2 & 10 & -6 \end{pmatrix}$$

$$\text{adj } A = \begin{pmatrix} 1 & 1 & 2 \\ 8 & -4 & 10 \\ -3 & 3 & -6 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{6}\begin{pmatrix} 1 & 1 & 2 \\ 8 & -4 & 10 \\ -3 & 3 & -6 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{4}{3} & -\frac{2}{3} & \frac{5}{3} \\ -\frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}$$

2. $\begin{pmatrix} 1 & -2 & -1 \\ -1 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix}$

Sol.

$$|A| = \begin{vmatrix} 1 & -2 & -1 \\ -1 & 2 & -3 \\ 1 & 0 & 1 \end{vmatrix} = 8$$

$$\begin{pmatrix} \begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & -3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} -2 & -1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} -2 & -1 \\ 2 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ -1 & -3 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -2 & -2 \\ 2 & 2 & -2 \\ 8 & 4 & 0 \end{pmatrix}$$

$$\text{adj } A = \begin{pmatrix} 2 & 2 & 8 \\ -2 & 2 & 4 \\ -2 & -2 & 0 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{8}\begin{pmatrix} 2 & 2 & 8 \\ -2 & 2 & 4 \\ -2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 1 \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & 0 \end{pmatrix}$$

Solving Systems of Linear Equations

Binary and ternary systems of linear equations can be solved by using the inverse matrix of the coefficient matrix. Note that the coefficient matrix must be invertible for this method to work.

14.6.3 Practice 13

Solve the following systems of linear equations using the inverse matrix method.

1. $\begin{cases} 3x - 2y = 12 \\ 7x + 5y = -1 \end{cases}$

Sol.

$$
\begin{aligned}
\text{Let } A &= \begin{pmatrix} 3 & -2 \\ 7 & 5 \end{pmatrix} \\
A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 12 \\ -1 \end{pmatrix} \\
A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} &= A^{-1}\begin{pmatrix} 12 \\ -1 \end{pmatrix} \\
\begin{pmatrix} x \\ y \end{pmatrix} &= A^{-1}\begin{pmatrix} 12 \\ -1 \end{pmatrix} \\
&= \frac{1}{29}\begin{pmatrix} 5 & 2 \\ -7 & 3 \end{pmatrix}\begin{pmatrix} 12 \\ -1 \end{pmatrix} \\
&= \frac{1}{29}\begin{pmatrix} 58 \\ -87 \end{pmatrix} \\
&= \begin{pmatrix} 2 \\ -3 \end{pmatrix} \\
\therefore x &= 2,\ y = -3
\end{aligned}
$$

2. $\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ x - y - 2z = -7 \end{cases}$

Sol.

$$
\begin{aligned}
\text{Let } A &= \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & -2 \end{pmatrix} \\
A\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 6 \\ 3 \\ -7 \end{pmatrix} \\
A^{-1}A\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= A^{-1}\begin{pmatrix} 6 \\ 3 \\ -7 \end{pmatrix} \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= A^{-1}\begin{pmatrix} 6 \\ 3 \\ -7 \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{7} & \frac{1}{7} & \frac{2}{7} \\ \frac{5}{7} & -\frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} & -\frac{3}{7} \end{pmatrix}\begin{pmatrix} 6 \\ 3 \\ -7 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \\
\therefore x &= 1,\ y = 2,\ z = 3
\end{aligned}
$$

14.6.4 Exercise 14.6

Determine if the following second-order matrices are invertible. If they are, find their inverse matrix.

1. $\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$

   Sol.

   $|A| = 5 \cdot 3 - 2 \cdot 7 = 1 \neq 0$
   $\therefore A$ is invertible
   $A^{-1} = \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$

2. $\begin{pmatrix} 4 & -8 \\ -1 & 2 \end{pmatrix}$

   Sol.

   $|A| = 4 \cdot 2 - (-8) \cdot (-1) = 0$
   $\therefore A$ is not invertible

3. $\begin{pmatrix} 10 & 5 \\ -6 & -3 \end{pmatrix}$

   Sol.

   $|A| = 10 \cdot (-3) - 5 \cdot (-6) = 0$
   $\therefore A$ is not invertible

4. $\begin{pmatrix} 4 & -5 \\ -7 & 9 \end{pmatrix}$

   Sol.

   $|A| = 4 \cdot 9 - (-5) \cdot (-7) = 1 \neq 0$
   $\therefore A$ is invertible
   $A^{-1} = \begin{pmatrix} 9 & 5 \\ 7 & 4 \end{pmatrix}$

5. $\begin{pmatrix} -2 & -1 \\ 6 & 3 \end{pmatrix}$

   Sol.

   $|A| = (-2) \cdot 3 - (-1) \cdot 6 = 0$
   $\therefore A$ is not invertible

6. $\begin{pmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{pmatrix}$

Sol.

$$|A| = \sin\alpha \cdot \sin\alpha - (-\cos\alpha)\cdot\cos\alpha = 1 \neq 0$$
$\therefore A$ is invertible
$$A^{-1} = \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix}$$

7. Given that the inverse matrix of matrix $\begin{pmatrix} -2 & 5 \\ 1 & x \end{pmatrix}$ is $\begin{pmatrix} x & y \\ -1 & -2 \end{pmatrix}$, find the value of x and y.

Sol.

$$\begin{vmatrix} -2 & 5 \\ 1 & x \end{vmatrix} = -2x - 5$$
$$(-2x-5)\begin{pmatrix} x & -5 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} x & y \\ -1 & -2 \end{pmatrix}$$
$$\begin{pmatrix} -2x^2 - 10x & 10x + 25 \\ 2x + 5 & 4x + 10 \end{pmatrix} = \begin{pmatrix} x & y \\ -1 & -2 \end{pmatrix}$$
Comparing coefficients,
$$\begin{cases} -2x^2 - 10x = x \\ 10x + 25 = y \\ 2x + 5 = -1 \\ 4x + 10 = -2 \end{cases}$$
$$2x = -6$$
$$x = -3$$
$$-30 + 25 = y$$
$$y = -5$$
$$\therefore x = -3,\ y = -5$$

8. If the matrix $\begin{pmatrix} 3 & x \\ -2 & 4 \end{pmatrix}$ is not invertible, find the value of x.

Sol.

$$\begin{vmatrix} 3 & x \\ -2 & 4 \end{vmatrix} = 3 \cdot 4 - x \cdot (-2) = 0$$
$$12 + 2x = 0$$
$$x = -6$$

9. Given the matrix $\begin{pmatrix} y^2 - 7 & -2 \\ 6 & 2y \end{pmatrix}$, find the range of y such that the matrix is invertible.

Sol.

$$\begin{vmatrix} y^2 - 7 & -2 \\ 6 & 2y \end{vmatrix} = (y^2 - 7)\cdot 2y + 12 \neq 0$$
$$y^3 - 7y + 6 \neq 0$$
$$(y-1)(y+3)(y-2) \neq 0$$
$$y \in \mathbb{R},\ y \neq -3, 1, 2$$

10. Given the matrix $\begin{pmatrix} x & 2 & 1 \\ -1 & x-1 & -2 \\ 1-x & 1 & 1 \end{pmatrix}$, find the range of x such that the matrix is not invertible.

Sol.

$$\begin{vmatrix} x & 2 & 1 \\ -1 & x-1 & -2 \\ 1-x & 1 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} -1 & x-1 \\ 1-x & 1 \end{vmatrix} + 2\begin{vmatrix} x & 2 \\ 1-x & 1 \end{vmatrix} + \begin{vmatrix} x & 2 \\ -1 & x-1 \end{vmatrix}$$
$$= 1 + x^2 - 2x + 1 + 2x - 4 + 4x + x^2 - x + 2$$
$$= 2x^2 + 3x - 4 = 0$$
$$(x+2)(2x-1) = 0$$
$$x = -2 \text{ or } x = \frac{1}{2}$$

11. Given an identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $A = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$. If $AJA = J$, and $A + A^{-1} = 3I$, find A.

Sol.

$$\begin{aligned}
AJA &= J\\
A^{-1}AJA &= A^{-1}J\\
JA &= A^{-1}J\\
A^{-1} &= 3I - A\\
&= \begin{pmatrix} 3-a & -1 \\ -1 & 3-b \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} = \begin{pmatrix} 3-a & -1 \\ -1 & 3-b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ -a & -1 \end{pmatrix} = \begin{pmatrix} 1 & 3-a \\ -3+b & -1 \end{pmatrix}$$

$$b = 3-a$$

$$\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} + \frac{1}{ab-1}\begin{pmatrix} b & -1 \\ -1 & a \end{pmatrix} = 3I$$

$$\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} + \begin{pmatrix} \frac{b}{ab-1} & \frac{-1}{ab-1} \\ \frac{-1}{ab-1} & \frac{a}{ab-1} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{a^2b-a+b}{ab-1} & \frac{ab-2}{ab-1} \\ \frac{ab-2}{ab-1} & \frac{ab^2-b+a}{ab-1} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\begin{aligned}
\frac{ab-2}{ab-1} &= 0\\
a(3-a) - 2 &= 0\\
a^2 - 3a + 2 &= 0\\
(a-2)(a-1) &= 0\\
a = 2 &\text{ or } a = 1
\end{aligned}$$

When $a = 2,\ b = 1$, and when $a = 1,\ b = 2$

$$\therefore A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ or } A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

12. Given that $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{pmatrix}$, if $AB = C$, find A.

Sol.

$$\begin{aligned}
AB &= C\\
ABB^{-1} &= CB^{-1}\\
A &= CB^{-1}\\
A &= \begin{pmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -9 & -2 \\ 0 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix}\\
&= \begin{pmatrix} 0 & 0 & 1 \\ 1 & -4 & 0 \\ -7 & 20 & 5 \end{pmatrix}
\end{aligned}$$

Find the inverse matrix of the following matrices.

13. $\begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & 3 \\ 2 & 1 & 0 \end{pmatrix}$

Sol.

$$\begin{vmatrix} 1 & 0 & 2 \\ 4 & 1 & 3 \\ 2 & 1 & 0 \end{vmatrix} = 1$$

$$\begin{pmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -6 & 2 \\ -2 & -4 & -1 \\ -2 & 5 & 1 \end{pmatrix}$$

$$\text{adj}\, A = \begin{pmatrix} 3 & -2 & -2 \\ -6 & -4 & 5 \\ 2 & -1 & 1 \end{pmatrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 3 & -2 & -2 \\ -6 & -4 & 5 \\ 2 & -1 & 1 \end{pmatrix}$$

14. $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -1 & 0 & -2 \end{pmatrix}$

Sol.

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -1 & 0 & -2 \end{vmatrix} = 9$$

$$\begin{pmatrix} \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} & -\begin{vmatrix} 3 & 0 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -1 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 5 & 1 \\ 4 & -3 & -2 \\ 1 & -3 & -5 \end{pmatrix}$$

$$\text{adj } A = \begin{pmatrix} -2 & 4 & 1 \\ 6 & -3 & -3 \\ 1 & -2 & -5 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{9}\begin{pmatrix} -2 & 4 & 1 \\ 6 & -3 & -3 \\ 1 & -2 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{2}{5} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{9} & -\frac{2}{9} & -\frac{5}{9} \end{pmatrix}$$

15. $\begin{pmatrix} 1 & -1 & 3 \\ 0 & -4 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

Sol.

$$\begin{vmatrix} 1 & -1 & 3 \\ 0 & -4 & 3 \\ 2 & 3 & 1 \end{vmatrix} = 5$$

$$\begin{pmatrix} \begin{vmatrix} -4 & 3 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & -4 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} -1 & 3 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} -1 & 3 \\ -4 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 0 & -4 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -13 & 6 & 8 \\ 10 & -5 & -5 \\ 9 & -3 & -4 \end{pmatrix}$$

$$\text{adj } A = \begin{pmatrix} -13 & 10 & 9 \\ 6 & -5 & -3 \\ 8 & -5 & -4 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{5}\begin{pmatrix} -13 & 10 & 9 \\ 6 & -5 & -3 \\ 8 & -5 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{13}{5} & \frac{10}{5} & \frac{9}{5} \\ \frac{6}{5} & -\frac{1}{5} & -\frac{3}{5} \\ \frac{8}{5} & -\frac{1}{5} & -\frac{4}{5} \end{pmatrix}$$

Solve the following systems of linear equations using the inverse matrix method.

16. $\begin{cases} 3x + 2y = 1 \\ 4x - y = 5 \end{cases}$

Sol.

$$\text{Let } A = \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}$$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$= A^{-1}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$= -\frac{1}{11}\begin{pmatrix} -1 & -2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$= -\frac{1}{11}\begin{pmatrix} -11 \\ 11 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\therefore x = 1,\ y = 1$$

17. $\begin{cases} 2x - 7y = 8 \\ 9x - 4y = -19 \end{cases}$

Sol.

$$\begin{aligned}
\text{Let } A &= \begin{pmatrix} 2 & -7 \\ 9 & -4 \end{pmatrix} \\
A\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 8 \\ -19 \end{pmatrix} \\
A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} &= A^{-1}\begin{pmatrix} 8 \\ -19 \end{pmatrix} \\
\begin{pmatrix} x \\ y \end{pmatrix} &= A^{-1}\begin{pmatrix} 8 \\ -19 \end{pmatrix} \\
&= \frac{1}{55}\begin{pmatrix} -4 & 7 \\ -9 & 2 \end{pmatrix}\begin{pmatrix} 8 \\ -19 \end{pmatrix} \\
&= \frac{1}{55}\begin{pmatrix} -165 \\ -110 \end{pmatrix} \\
&= \begin{pmatrix} -3 \\ -2 \end{pmatrix} \\
\therefore x &= -3,\ y = -2
\end{aligned}$$

18. $\begin{cases} 2x + 4y - 3z = 3 \\ 3x - 8y + 6z = 1 \\ 8x - 2y - 9z = 4 \end{cases}$

Sol.

$$\begin{aligned}
\text{Let } A &= \begin{pmatrix} 2 & 4 & -3 \\ 3 & -8 & 6 \\ 8 & -2 & -9 \end{pmatrix} \\
A\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= A^{-1}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} \frac{2}{7} & \frac{1}{7} & 0 \\ \frac{25}{98} & \frac{1}{49} & -\frac{1}{14} \\ \frac{29}{147} & \frac{6}{49} & -\frac{2}{21} \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{pmatrix} \\
\therefore x &= 1,\ y = \frac{1}{2},\ z = \frac{1}{3}
\end{aligned}$$

19. $\begin{cases} 3x - y + 4z = 0 \\ 5x + 4y - 3z = 0 \\ 2x - 3y - z = 0 \end{cases}$

Sol.

$$\begin{aligned}
\text{Let } A &= \begin{pmatrix} 3 & -1 & 4 \\ 5 & 4 & -3 \\ 2 & -3 & -1 \end{pmatrix} \\
A\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= A^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\therefore x &= 0,\ y = 0,\ z = 0
\end{aligned}$$

20. $\begin{cases} 3x - y = 14 \\ 2y + z = 5 \\ 5z - x = 10 \end{cases}$

Sol.

$$\begin{aligned}
\text{Let } A &= \begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ -1 & 0 & 5 \end{pmatrix} \\
A\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 14 \\ 5 \\ 10 \end{pmatrix} \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= A^{-1}\begin{pmatrix} 14 \\ 5 \\ 10 \end{pmatrix} \\
&= \begin{pmatrix} \frac{10}{31} & \frac{5}{31} & -\frac{1}{31} \\ -\frac{1}{31} & \frac{15}{31} & -\frac{3}{31} \\ \frac{2}{31} & \frac{1}{31} & \frac{6}{31} \end{pmatrix}\begin{pmatrix} 14 \\ 5 \\ 10 \end{pmatrix} \\
&= \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} \\
\therefore x &= 5,\ y = 1,\ z = 3
\end{aligned}$$

14.7 Gauss Elimination

The concept of Gauss elimination is to eliminate the variables in the equations one by one, through the use of elementary row operations. The elementary row operations are as follows:

1. Interchange two rows:
   $R_i \leftrightarrow R_j$: interchange row i and row j.
2. Multiply a row by a nonzero constant:

$R_i \to kR_i$: multiply row i by k, where k is a nonzero constant.

3. Add a multiple of one row to another row:

   $R_i \to R_i + kR_j$: add k times row j to row i.

14.7.1 Practice 14

Solve the following system of equations by Gauss elimination:

1. $\begin{cases} 3x - 2y - z = 4 \\ 2x + y - 4z = 4 \\ x + 2y - 3z = 4 \end{cases}$

   Sol.

$$\left(\begin{array}{ccc|c} 3 & -2 & -1 & 4 \\ 2 & 1 & -4 & 4 \\ 1 & 2 & -3 & 4 \end{array}\right) \xrightarrow{R_1 \to R_1 + R_3} \left(\begin{array}{ccc|c} 4 & 0 & -4 & 8 \\ 2 & 1 & -4 & 4 \\ 1 & 2 & -3 & 4 \end{array}\right)$$

$$\xrightarrow{R_1 \to \frac{1}{4}R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & 1 & -4 & 4 \\ 1 & 2 & -3 & 4 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & 1 & -4 & 4 \\ 0 & 2 & -2 & 2 \end{array}\right)$$

$$\xrightarrow{R_3 \to \frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & 1 & -4 & 4 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow[R_2 \to R_2 + 2R_3]{R_1 \to R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\therefore x = 3,\ y = 2,\ z = 1$$

2. $\begin{cases} 3x + y + 2z = 5 \\ 2x - 2y + 5z = 3 \\ x - 3y + 4z = 0 \end{cases}$

   Sol.

$$\left(\begin{array}{ccc|c} 3 & 1 & 2 & 5 \\ 2 & -2 & 5 & 3 \\ 1 & -3 & 4 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 + 2R_1} \left(\begin{array}{ccc|c} 3 & 1 & 2 & 5 \\ 8 & 0 & 9 & 13 \\ 10 & 0 & 10 & 15 \end{array}\right)$$

$$\xrightarrow[R_1 \leftrightarrow R_2]{R_3 \to \frac{1}{10}R_3} \left(\begin{array}{ccc|c} 8 & 0 & 9 & 13 \\ 3 & 1 & 2 & 5 \\ 1 & 0 & 1 & \frac{3}{2} \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 - 8R_3]{R_2 \to R_2 - 2R_3} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & \frac{3}{2} \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - R_3} \left(\begin{array}{ccc|c} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & \frac{3}{2} \\ 1 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

$$\therefore x = \frac{1}{2},\ y = \frac{3}{2},\ z = 1$$

Gauss elimination can also be used to find the inverse of a matrix. Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a invertible matrix, that is, $|A| \neq 0$. Now we arrange the matrix A and the identity matrix I into a 3 by 6 augmented matrix $A|I$ as follows:

$$\left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$

We then apply Gauss elimination to the augmented matrix $A|I$ to obtain the following matrix such that the left hand side of this matrix become an identity matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right)$$

where b_{ij} are constants, the right hand side of the augmented matrix is the inverse of A:

$$A^{-1} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

14.7.2 Practice 15

Using the method of Gauss elimination, find the inverse of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & -4 \end{pmatrix}$.

Sol.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 3 & -4 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & -6 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -8 & -1 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to -\frac{1}{8}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 + R_3]{R_2 \to R_2 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{17}{8} & -\frac{7}{8} & -\frac{1}{8} \\ 0 & 1 & 0 & -\frac{5}{4} & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{17}{8} & -\frac{7}{8} & -\frac{1}{8} \\ -\frac{5}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{8} \end{pmatrix}$$

14.7.3 Exercise 14.7

Solve the following system of linear equations using the method of Gauss elimination:

1. $\begin{cases} 3x - y - 14 = 0 \\ 2y + z - 5 = 0 \\ x - 5z + 10 = 0 \end{cases}$

Sol.

$$\begin{cases} 3x - y = 14 \\ 2y + z = 5 \\ x - 5z = -10 \end{cases}$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 0 & 14 \\ 0 & 2 & 1 & 5 \\ 1 & 0 & -5 & -10 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - 3R_3} \left(\begin{array}{ccc|c} 0 & -1 & 15 & 44 \\ 0 & 2 & 1 & 5 \\ 1 & 0 & -5 & -10 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 + 2R_1} \left(\begin{array}{ccc|c} 0 & -1 & 15 & 44 \\ 0 & 0 & 31 & 93 \\ 1 & 0 & -5 & -10 \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{1}{31}R_2} \left(\begin{array}{ccc|c} 0 & -1 & 15 & 44 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & -5 & -10 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 - 15R_2]{R_3 \to R_3 + 5R_2} \left(\begin{array}{ccc|c} 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 0 & 5 \end{array}\right)$$

$$\xrightarrow[R_1 \to -R_1]{R_3 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\therefore x = 5,\ y = 1,\ z = 3$$

2. $\begin{cases} x + y + z = 6 \\ x + 2y + 3z = 10 \\ 2x + 3y - 4z = 8 \end{cases}$

Sol.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 2 & 3 & -4 & 8 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & -6 & -4 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -8 & -8 \end{array}\right)$$

$$\xrightarrow{R_3 \to -\frac{1}{8}R_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 + R_3]{R_2 \to R_2 - 2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\therefore x = 3,\ y = 2,\ z = 1$$

3. $\begin{cases} -x + y + z = 5 \\ 2x - 7y + 4z = 1 \\ 2x - 5y + 3z = -2 \end{cases}$

Sol.

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 5 \\ 2 & -7 & 4 & 1 \\ 2 & -5 & 3 & -2 \end{array}\right)$$

$$\xrightarrow[R_2\to R_2+2R_1]{R_3\to R_3+2R_1} \left(\begin{array}{ccc|c} -1 & 1 & 1 & 5 \\ 0 & -5 & 6 & 11 \\ 0 & -3 & 5 & 8 \end{array}\right)$$

$$\xrightarrow{R_2\to R_2-R3} \left(\begin{array}{ccc|c} -1 & 1 & 1 & 5 \\ 0 & -2 & 1 & 3 \\ 0 & -3 & 5 & 8 \end{array}\right)$$

$$\xrightarrow[R_1\to R_1-R_2]{R_3\to R_3-5R_2} \left(\begin{array}{ccc|c} -1 & 3 & 0 & 2 \\ 0 & -2 & 1 & 3 \\ 0 & 7 & 0 & -7 \end{array}\right)$$

$$\xrightarrow[R_1\to -R_1]{R_3\to \frac{1}{7}R_3} \left(\begin{array}{ccc|c} 1 & -3 & 0 & -2 \\ 0 & -2 & 1 & 3 \\ 0 & 1 & 0 & -1 \end{array}\right)$$

$$\xrightarrow[R_1\to R_1+3R_3]{R_2\to R_2+2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{array}\right)$$

$$\xrightarrow{R_2\leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\therefore x=-5,\ y=-1,\ z=1$$

4. $\begin{cases} 4x-y-7z=0 \\ 5x-2y-z=1 \\ 3x+3y+5z=2 \end{cases}$

Sol.

$$\left(\begin{array}{ccc|c} 4 & -1 & -7 & 0 \\ 5 & -2 & -1 & 1 \\ 3 & 3 & 5 & 2 \end{array}\right)$$

$$\xrightarrow{R_2\to R_2-2R_1} \left(\begin{array}{ccc|c} 4 & -1 & -7 & 0 \\ -3 & 0 & 13 & 1 \\ 3 & 3 & 5 & 2 \end{array}\right)$$

$$\xrightarrow{R_3\to R_3+R_2} \left(\begin{array}{ccc|c} 4 & -1 & -7 & 0 \\ -3 & 0 & 13 & 1 \\ 0 & 3 & 18 & 3 \end{array}\right)$$

$$\xrightarrow{R_3\to \frac{1}{3}R_3} \left(\begin{array}{ccc|c} 4 & -1 & -7 & 0 \\ -3 & 0 & 13 & 1 \\ 0 & 1 & 6 & 1 \end{array}\right)$$

$$\xrightarrow{R_1\to R_1+R_3} \left(\begin{array}{ccc|c} 4 & 0 & -1 & 1 \\ -3 & 0 & 13 & 1 \\ 0 & 1 & 6 & 1 \end{array}\right)$$

$$\xrightarrow{R_3\to 4R_3} \left(\begin{array}{ccc|c} 4 & 0 & -1 & 1 \\ -12 & 0 & 52 & 4 \\ 0 & 1 & 6 & 1 \end{array}\right)$$

$$\xrightarrow{R_2\to R_2+3R_1} \left(\begin{array}{ccc|c} 4 & 0 & -1 & 1 \\ 0 & 0 & 49 & 7 \\ 0 & 1 & 6 & 1 \end{array}\right)$$

$$\xrightarrow{R_2\to \frac{1}{49}R_2} \left(\begin{array}{ccc|c} 4 & 0 & -1 & 1 \\ 0 & 0 & 1 & \frac{1}{7} \\ 0 & 1 & 6 & 1 \end{array}\right)$$

$$\xrightarrow[R_3\to R_3-6R_2]{R_1\to R_1+R_2} \left(\begin{array}{ccc|c} 4 & 0 & 0 & \frac{8}{7} \\ 0 & 0 & 1 & \frac{1}{7} \\ 0 & 1 & 0 & \frac{1}{7} \end{array}\right)$$

$$\xrightarrow[R_1\to \frac{1}{4}R_1]{R_2\leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{7} \\ 0 & 1 & 0 & \frac{1}{7} \\ 0 & 0 & 1 & \frac{1}{7} \end{array}\right)$$

$$\therefore x=\frac{2}{7},\ y=\frac{1}{7},\ z=\frac{1}{7}$$

Find the inverse of the following matrices using the method of Gauss Jordan elimination.

5. $\begin{pmatrix} 1 & -1 & 0 \\ 5 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}$

Sol.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 5 & 2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2\to R_2+2R_1]{R_3\to R_3-2R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 7 & 0 & 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2\to R_2-7R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 7 & 1 & -5 & 1 & 0 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2\to R_2-R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & -3 & 1 & -1 \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2\to \frac{1}{5}R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 2 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1\to R_1+R_2]{R_3\to R_3-2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -\frac{3}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{4}{5} & -\frac{2}{5} & \frac{7}{5} \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{3}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{4}{5} & -\frac{2}{5} & \frac{7}{5} \end{pmatrix}$$

6. $\begin{pmatrix} 3 & 14 & 0 \\ 2 & 5 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

Sol.

$$\left(\begin{array}{ccc|ccc} 3 & 14 & 0 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 - 3R_3]{R_2 \to R_2 - 2R_3} \left(\begin{array}{ccc|ccc} 0 & 8 & -3 & 1 & 0 & -3 \\ 0 & 1 & -1 & 0 & 1 & -2 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 - 3R_2]{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|ccc} 0 & 5 & 0 & 1 & -3 & 3 \\ 0 & 1 & -1 & 0 & 1 & -2 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_1 \to \frac{1}{5} R_1]{R_3 \to R_3 + R_2} \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{5} & -\frac{3}{5} & \frac{3}{5} \\ 0 & 1 & -1 & 0 & 1 & -2 \\ 1 & 3 & 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - 3R_1} \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{5} & -\frac{3}{5} & \frac{3}{5} \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{8}{5} & -\frac{13}{5} \\ 1 & 0 & 0 & -\frac{3}{5} & \frac{14}{5} & -\frac{14}{5} \end{array}\right)$$

$$\xrightarrow[R_1 \leftrightarrow R_3]{R_2 \to -R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & \frac{14}{5} & -\frac{14}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{8}{5} & \frac{13}{5} \\ 0 & 1 & 0 & \frac{1}{5} & -\frac{3}{5} & \frac{3}{5} \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & \frac{14}{5} & -\frac{14}{5} \\ 0 & 1 & 0 & \frac{1}{5} & -\frac{3}{5} & \frac{3}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{8}{5} & \frac{13}{5} \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} -\frac{3}{5} & \frac{14}{5} & -\frac{14}{5} \\ \frac{1}{5} & -\frac{3}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{8}{5} & \frac{13}{5} \end{pmatrix}$$

14.8 Cramer's Rule

When using this method, the determinant of the coefficient matrix is not zero.

Considering a ternary system of equations, we have the following:

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3 \end{cases}$$

The coefficient matrix of this system is

$$\Delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

Now we replace the coefficient of x, y and z in Δ with the constants d_1, d_2 and d_3 respectively, and we get the following:

$$\Delta_x = \begin{pmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{pmatrix} \Delta_y = \begin{pmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{pmatrix} \Delta_z = \begin{pmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{pmatrix}$$

The solution of the system of equations is

$$\begin{cases} x = \frac{\Delta_x}{\Delta} \\ y = \frac{\Delta_y}{\Delta} \\ z = \frac{\Delta_z}{\Delta} \end{cases} \qquad \Delta \neq 0$$

14.8.1 Practice 16

Solve the following system of equations using Cramer's Rule:

1. $\begin{cases} 2x + 3y + 4z = 5 \\ 3x + 4y + 5z = 2 \\ 4x + 5y + 2z = 3 \end{cases}$

Sol.

$$\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 2 \end{vmatrix} = 4$$

$$\Delta_x = \begin{vmatrix} 5 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 2 \end{vmatrix} = -60$$

$$\Delta_y = \begin{vmatrix} 2 & 5 & 4 \\ 3 & 2 & 5 \\ 4 & 3 & 2 \end{vmatrix} = 52$$

$$\Delta_z = \begin{vmatrix} 2 & 3 & 5 \\ 3 & 4 & 2 \\ 4 & 5 & 3 \end{vmatrix} = -4$$

$$\therefore x = \frac{-60}{4} = -15, \; y = \frac{52}{4} = 13, \; z = \frac{-4}{4} = -1$$

2. $\begin{cases} 3x - y + 2z = 4 \\ 2x + 3y - z = 0 \\ 3x - 2y + z = -1 \end{cases}$

Sol.

$$\Delta = \begin{vmatrix} 3 & -1 & 2 \\ 2 & 3 & -1 \\ 3 & -2 & 1 \end{vmatrix} = -18$$

$$\Delta_x = \begin{vmatrix} 4 & -1 & 2 \\ 0 & 3 & -1 \\ -1 & -2 & 1 \end{vmatrix} = 9$$

$$\Delta_y = \begin{vmatrix} 3 & 4 & 2 \\ 2 & 0 & -1 \\ 3 & -1 & 1 \end{vmatrix} = -27$$

$$\Delta_z = \begin{vmatrix} 3 & -1 & 4 \\ 2 & 3 & 0 \\ 3 & -2 & -1 \end{vmatrix} = -63$$

$$\therefore x = \frac{9}{-18} = -\frac{1}{2},\ y = \frac{-27}{-18} = \frac{3}{2},\ z = \frac{-63}{-18} = \frac{7}{2}$$

14.8.2 Exercise 14.8

Solve the following system of equations using Cramer's Rule:

1. $\begin{cases} x + 3y + 2z = -4 \\ 2x + y + 4z = -3 \\ 3x + 4y + z = -2 \end{cases}$

Sol.

$$\Delta = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{vmatrix} = 25$$

$$\Delta_x = \begin{vmatrix} -4 & 3 & 2 \\ -3 & 1 & 4 \\ -2 & 4 & 1 \end{vmatrix} = 25$$

$$\Delta_y = \begin{vmatrix} 1 & -4 & 2 \\ 2 & -3 & 4 \\ 3 & -2 & 1 \end{vmatrix} = -25$$

$$\Delta_z = \begin{vmatrix} 1 & 3 & -4 \\ 2 & 1 & -3 \\ 3 & 4 & -2 \end{vmatrix} = -25$$

$$\therefore x = \frac{25}{25} = 1,\ y = \frac{-25}{25} = -1,\ z = \frac{-25}{25} = -1$$

2. $\begin{cases} 2x + 3y - 5z = -4 \\ 4x - y + 3z = 2 \\ 3x + 2y + 4z = 1 \end{cases}$

Sol.

$$\Delta = \begin{vmatrix} 2 & 3 & -5 \\ 4 & -1 & 3 \\ 3 & 2 & 4 \end{vmatrix} = -96$$

$$\Delta_x = \begin{vmatrix} -4 & 3 & -5 \\ 2 & -1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 0$$

$$\Delta_y = \begin{vmatrix} 2 & -4 & -5 \\ 4 & 2 & 3 \\ 3 & 1 & 4 \end{vmatrix} = 48$$

$$\Delta_z = \begin{vmatrix} 2 & 3 & -4 \\ 4 & -1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = -48$$

$$\therefore x = \frac{0}{-96} = 0,\ y = \frac{48}{-96} = -\frac{1}{2},\ z = \frac{-48}{-96} = \frac{1}{2}$$

3. $\begin{cases} x + 2y - 3z = 4 \\ 2x + 3y - z = 5 \\ 3x - y + z = 6 \end{cases}$

Sol.

$$\Delta = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & -1 \\ 3 & -1 & 1 \end{vmatrix} = 25$$

$$\Delta_x = \begin{vmatrix} 4 & 2 & -3 \\ 5 & 3 & -1 \\ 6 & -1 & 1 \end{vmatrix} = 55$$

$$\Delta_y = \begin{vmatrix} 1 & 4 & -3 \\ 2 & 5 & -1 \\ 3 & 6 & 1 \end{vmatrix} = 0$$

$$\Delta_z = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & -1 & 6 \end{vmatrix} = -15$$

$$\therefore x = \frac{55}{25} = \frac{11}{5},\ y = \frac{0}{25} = 0,\ z = \frac{-15}{25} = -\frac{3}{5}$$

4. $\begin{cases} \frac{3}{x} + \frac{1}{y} - \frac{1}{z} = 3 \\ \frac{1}{x} - \frac{1}{y} + \frac{2}{z} = 13 \\ \frac{1}{x} + \frac{4}{y} - \frac{1}{z} = -9 \end{cases}$

Sol.

Let $a = \frac{1}{x},\ b = \frac{1}{y},\ c = \frac{1}{z}$

$$\begin{cases} 3a + b - c = 3 \\ a - b + 2c = 13 \\ a + 4b - c = -9 \end{cases}$$

$$\Delta = \begin{vmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = -23$$

$$\Delta_a = \begin{vmatrix} 3 & 1 & -1 \\ 13 & -1 & 2 \\ -9 & 4 & -1 \end{vmatrix} = -69$$

$$\Delta_b = \begin{vmatrix} 3 & 3 & -1 \\ 1 & 13 & 2 \\ 1 & -9 & -1 \end{vmatrix} = 46$$

$$\Delta_c = \begin{vmatrix} 3 & 1 & 3 \\ 1 & -1 & 13 \\ 1 & 4 & -9 \end{vmatrix} = -92$$

$$a = \frac{-69}{-23} = 3,\ b = \frac{46}{-23} = -2,\ c = \frac{-92}{-23} = 4$$

$$\therefore x = \frac{1}{3},\ y = -\frac{1}{2},\ z = \frac{1}{4}$$