# Solution Book of Mathematic

*Ssnior 2 Part I*

MELVIN CHIA

Written on 9 October 2022

# Contents

## 16.1 Linear Programming

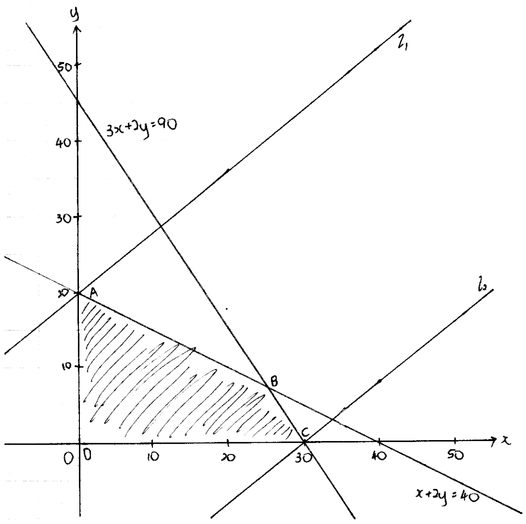
### 16.1.1 Practice 12

Find the maximum and minimum value of $z = 8x - 10y$ subject to the following constraints:

$$\begin{cases} x + 2y \le 40 \\ 3x + 2y \le 90 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

**Sol.**

$$\begin{aligned} \text{Objective function: } z &= 8x - 10y \\ 10y &= 8x - z \\ y &= \frac{4}{5}x - \frac{z}{10} \end{aligned}$$




When $y = \frac{4}{5}x - \frac{z}{10}$ translates towards bottom right of the feasible region, the value of $z$ increases. Therefore, the maximum value of the objective function is the value of $z$ in $l_2$. The point of intersection $C$ of $l_2$ and the feasible region makes the objective function to have its maximum value. Since $C$ is also the point of intersection of $3x + 2y = 90$ and $y = 0$,

$$\begin{cases} 3x + 2y = 90 \\ y = 0 \end{cases}$$

$D = (30, 0)$

$z_{\text{max}} = 8(30) - 0 = 240$

When $y = x - z$ translates towards top left of the feasible region, the value of $z$ decreases. Therefore, the minimum value of the objective function is the value of $z$ in $l_1$. The point of intersection $A$ of $l_1$ and the feasible region makes the objective function to have its minimum value. Since $A$ is also the point of intersection of $x + 2y = 40$ and $x = 0$,

$$\begin{cases} x + 2y = 40 \\ x = 0 \end{cases}$$

$A = (0, 20)$

$z_{\text{min}} = 0 - 10(20) = -200$

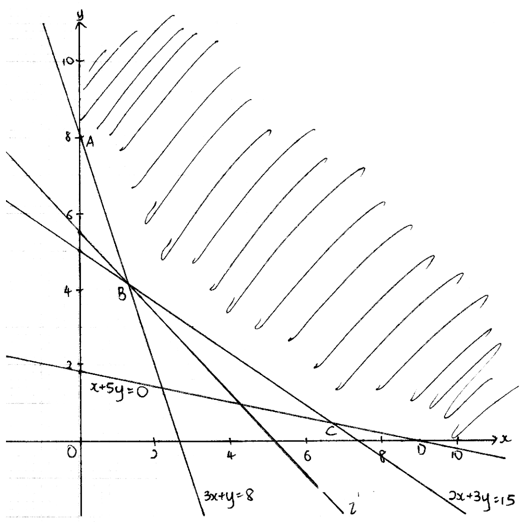
### 16.1.2 Exercise 15.8

1. Find the minimum value of $z = 10x + 12y$, subject to the following constraints:

$$\begin{cases} 3x + y \ge 8 \\ 2x + 3y \ge 15 \\ x + 5y \ge 9 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

**Sol.**

$$\begin{aligned} \text{Objective function: } z &= 10x + 12y \\ 12y &= -10x + z \\ y &= -\frac{5}{6}x + \frac{z}{12} \end{aligned}$$




The minimum value of the objective function is the value of $z$ in $l$. The point of intersection $B$ of $l$ and the feasible region makes the objective function to have its minimum value. Since $B$ is also the point of intersection of $3x + y = 8$ and $2x + 3y = 15$,

$$\begin{cases} 3x + y = 8 \\ 2x + 3y = 15 \end{cases}$$

$$2x + 3(8 - 3x) = 15$$
$$2x + 24 - 9x = 15$$
$$-7x = -9$$
$$x = \frac{9}{7}$$
$$\frac{27}{7} + y = 8$$
$$y = 8 - \frac{27}{7} = \frac{29}{7}$$
$$B = (\frac{9}{7}, \frac{29}{7})$$
$$z_{\text{min}} = 10(\frac{9}{7}) + 12(\frac{29}{7})$$
$$= 62\frac{4}{7}$$

2. A housing developer owns a tract of land that is $2,400m^2$ in area and a construction capital of \$4,600,000. The developer wishes to build two types of houses: type A and type B. Given that each type A house requires $150m^2$ of land and \$250,000 of construction fees, can earn \$55,000 in profit; and each type B house requires $200m^2$ of land and \$400,000 of construction fees, can earn \$80,000 in profit. Assume that all houses built can be sold, how many of each type of house should be built to maximize the profit? Find the maximum profit.

**Sol.**

Let $x$ be the number of type A houses and $y$ be the number of type B houses.

| | **A ($x$ units)** | **B ($y$ units)** | **Limit** |
|---|---|---|---|
| Area ($m^2$) | $150x$ | $200x$ | $2,400$ |
| Cost(\$) | $250,000x$ | $400,000y$ | $4,600,000$ |
| Profit(\$) | $55,000x$ | $80,000y$ | |

The total profit is $z = 55,000x + 80,000y$, this is the objective function. According to the descriptions above, we find the maximum value of it.

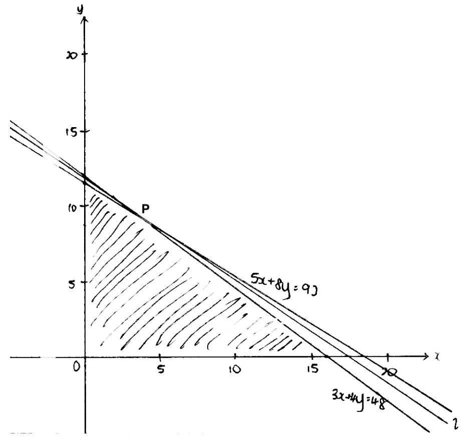
The constraints are:

$$\begin{cases} 150x + 200y \leq 2,400 \\ 250,000x + 400,000y \leq 4,600,000 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

After simplifying the constraints, we get:

$$\begin{cases} 3x + 4y \leq 48 \\ 5x + 8y \leq 92 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The feasible region is as follows:




Let $l$ : $55,000x + 80,000y = z$.

When the line is at $l$, the value of $z$ is at its maximum. The point of intersection $P$ of $l$ and the feasible region makes the objective function to have its maximum value. Since $P$ is also the point of intersection of $3x + 4y = 48$ and $5x + 8y = 92$,

$$\begin{cases} 3x + 4y = 48 & (1) \\ 5x + 8y = 92 & (2) \end{cases}$$

$$(1) \times 2 : 6x + 8y = 96$$
$$(1) - (2) : x = 4$$
$$\text{Sub } x = 4 \text{ into } (1) : 12 + 4y = 48$$
$$y = 9$$
$$P = (4, 9)$$
$$z_{\text{max}} = 55,000(4) + 80,000(9)$$
$$= 940,000$$

Thus, the maximum profit of \$940,000 can be obtained by building 4 type A houses and 9 type B houses.

3. One has a building lot that is $180m^2$ in area. He plans to pay \$7,000 to split the lot into two type of rooms and rent them out to students: each bigger room is $20m^2$ in area and can accommodate 5 students with a monthly rent of \$225 per student; each smaller room is $15m^2$ in area and can accommodate 3 students with a monthly rent of \$250 per student. The renovation cost for each bigger room is \$700 and for each smaller room is \$600. Assume that the source of tenants is stable, how many of each type of room should be divided into to maximize the profit? Find the maximum profit.

**Sol.**

Let $x$ be the number of bigger rooms and $y$ be the number of smaller rooms.

| | **Big ($x$ unit)** | **Small ($y$ unit)** | **Limit** |
|---|---|---|---|
| Area ($m^2$) | $20x$ | $15y$ | $180$ |
| Cost(\$) | $700x$ | $600y$ | $7,000$ |
| Profit(\$) | $1125x$ | $750y$ | |

The total profit is $z = 1125x + 750y$, this is the objective function. According to the descriptions above, we find the maximum value of it.

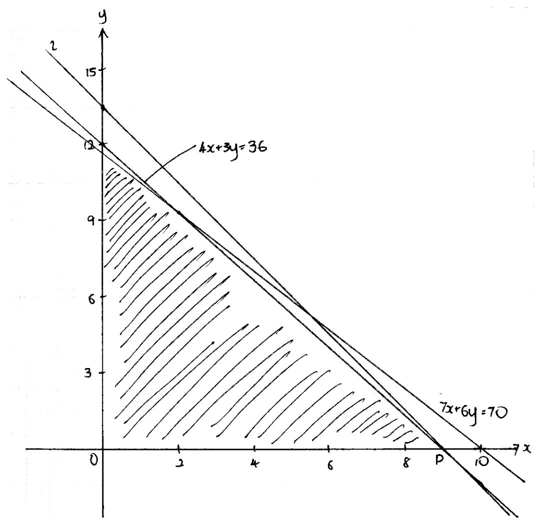
The constraints are:

$$\begin{cases} 20x + 15y \leq 180 \\ 700x + 600y \leq 7,000 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

After simplifying the constraints, we get:

$$\begin{cases} 4x + 3y \leq 36 \\ 7x + 6y \leq 70 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The feasible region is as follows:




Let $l$ : $1125x + 750y = z$.

When the line is at $l$, the value of $z$ is at its maximum. The point of intersection $P$ of $l$ and the feasible region makes the objective function to have its maximum value. Since $P$ is also the point of intersection of $4x + 3y = 36$ and $y = 0$,

$$\begin{cases} 4x + 3y = 36 \\ y = 0 \end{cases}$$

$P = (9, 0)$

$z_{\max} = 1125(9) + 750(0) = 10,125$

Thus, the maximum profit of \$10, 125 can be obtained by spliting the building lot into 9 bigger rooms.

4. Ms. Tan is a tuition teacher who teaches Mathematic subject to junior 3 and senior 3 students. There are a total of 5 students in each junior 3 class, each student pays tuition fees of \$50 per month, and each class is held for 4 hours per week. There are a total of 3 students in each senior 3 class, each student pays tuition fees of \$120 per month, and each class is held for 6 hours per week. Assume that here is a stable source of students, but the number of junior 3 students cannot exceed 2 times the number of senior 3 students. If Ms. Tan is wiling to earn at least \$6, 600 per month, how many junior 3 and senior 3 classes should she held per week to minimize the number hours she has to teach? What's the mimimum number of hours she has to teach?

**Sol.**

Let $x$ be the number of junior 3 classes and $y$ be the number of senior 3 classes.

The number of hours she has to teach is $z = 4x + 6y$, this is the objective function. According to the descriptions above, we find the minimum value of it.

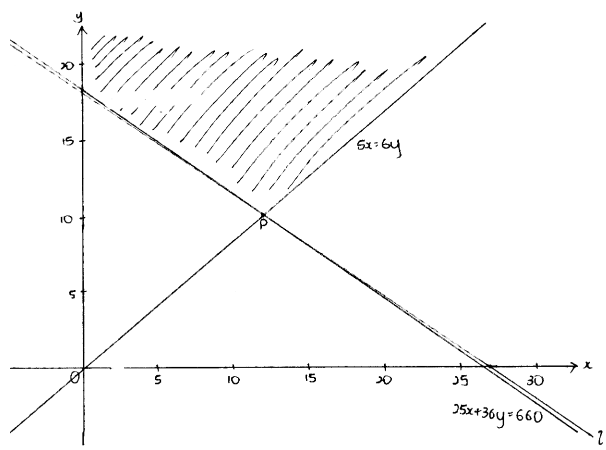
The constraints are:

$$\begin{cases} 5 \times 50 \times x + 3 \times 120 \times y \geq 6,600 \\ 5x \leq 2 \times 3y \\ x \geq 0 \\ y \geq 0 \end{cases}$$

After simplifying the constraints, we get:

$$\begin{cases} 25x + 36y \geq 660 \\ 5x \leq 6y \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The feasible region is as follows:




Let $l$ : $4x + 6y = z$.

When the line is at $l$, the value of $z$ is at its minimum. The point of intersection $P$ of $l$ and the feasible region makes the objective function to have its minimum value. Since $P$ is also the point of intersection of $25x + 36y = 660$ and $5x = 6y$,

$$\begin{cases} 25x + 36y = 660 & (1) \\ 5x = 6y & (2) \end{cases}$$

$$\begin{aligned} \text{Sub (2) into (1)}: 25x + 30x &= 660 \\ 55x &= 660 \\ x &= 12 \\ \text{Sub } x = 12 \text{ into (2)}: 60 &= 6y \\ y &= 10 \end{aligned}$$

$$\begin{aligned} P &= (12, 10) \\ z_{\text{min}} &= 4(12) + 6(10) = 108 \end{aligned}$$

Thus, Ms. Tan should hold 12 junior 3 classes and 10 senior 3 classes per week, and she has to teach for at lest 108 hours per week.

5. A company can produce a product with two types of raw materials. Each ton of the first type of raw material cost \$300, freight cost \$50, and can produce $90kg$ of the product; each ton of the second type of raw material cost \$700, freight cost \$40, and can produce $100kg$ of the product. If the company has a total of \$2,100 to spend on raw materials and \$200 to spend on freight every day, what's the maximum amount of product that can be produced every day? How many tons of each type of raw material should be used?

**Sol.**

Let $x$ be the number of tons of the first type of raw material and $y$ be the number of tons of the second type of raw material.

| | **M1 ($x$ t)** | **M2 ($y$ t)** | **Limit** |
|---|---|---|---|
| Cost (\$) | $300x$ | $700y$ | $2,100$ |
| Freight(\$) | $50x$ | $40y$ | $200$ |
| Product($kg$) | $90x$ | $100y$ | |

The objective function is $z = 90x + 100y$, which is the amount of product that can be produced every day. According to the descriptions above, we find the maximum value of it.

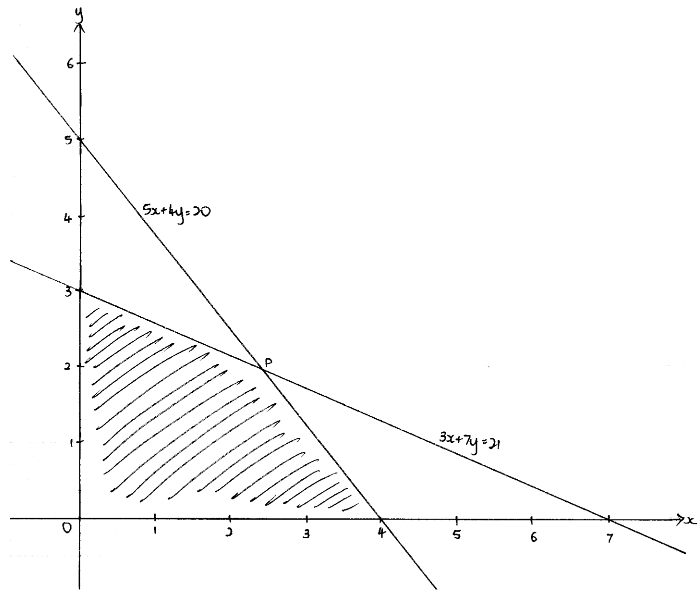
The constraints are:

$$\begin{cases} 300x + 700y \leq 2,100 \\ 50x + 40y \leq 200 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

After simplifying the constraints, we get:

$$\begin{cases} 3x + 7y \leq 21 \\ 5x + 4y \leq 20 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The feasible region is as follows:




Let $l : 90x + 100y = z$.

When the line is at $l$, the value of $z$ is at its maximum. The point of intersection $P$ of $l$ and the feasible region makes the objective function to have its maximum value. Since $P$ is also the point of intersection of $3x + 7y = 21$ and $5x + 4y = 20$,

$$\begin{cases} 3x + 7y = 21 & (1) \\ 5x + 4y = 20 & (2) \end{cases}$$

$$
\begin{aligned}
(1)\times 5 &: 15x+35y=105\\
(2)\times 3 &: 15x+12y=60\\
(1)-(2) &: 23y=45\\
y &= \frac{45}{23}\\
&= 1.96\\
\text{Sub } y=\frac{45}{23} \text{ into (2)} &: 5x+\frac{180}{23}=20\\
5x &= \frac{280}{23}\\
x &= \frac{56}{23}\\
&= 2.43
\end{aligned}
$$

$$
\begin{aligned}
P &= (2.43, 1.96)\\
z_{\max} &= 90\left(\frac{56}{23}\right)+100\left(\frac{45}{23}\right)\\
&= 414.78
\end{aligned}
$$

Thus, the company should use 2.43 tons of the first type of raw material and 1.96 tons of the second type of raw material, and the maximum amount of product that can be produced every day is $414.78kg$.

6. A factory uses four types of raw materials: $a$, $b$, $c$, and $d$ to produce two types of products: $A$ and $B$, the stock of raw materials $a$, $b$, $c$, and $d$ are 22, 14, 15, and 18 units respectively. Given that the required amount of raw materials $a$, $b$, $c$, and $d$ for producing one unit of product $A$ is 3, 2, 0, 3 units respectively, and the required amount of raw materials $a$, $b$, $c$, and $d$ for producing one unit of product $B$ is 2, 1, 3, 0 units respectively. If each product $A$ can make a profit of \$7,000 and each product $B$ can make a profit of \$5,000, how many units of each product should be produced to maximize the profit with the current stock of raw materials?

7. Mr. Wong is willing to mix two types of drinks: $A$ and $B$ to produce a new drink. Drink $A$ cost \$2 per litre, contains $20mg$ of vitamin $C$, $3mg$ of coloring agent, and $150g$ of sugar; drink $B$ cost \$4 per litre, contains $35mg$ of vitamin $C$, $2mg$ of coloring agent, and $100g$ of sugar. Mr. Tan is willing to mix at least 50 litres of the new drink, but each litre of the new drink has to contain at least $30mg$ of vitamin $C$, the total amount of sugar cannot exceed $6kg$, and the total cost cannot exceed \$180. How many litres of each type of drink should be mixed to minimize the amount of coloring agent?

8. A bakery bakes two types of cake: $A$ and $B$. The ingredients required for baking one cake of type $A$ is $1kg$ of flour, 5 eggs, and $300g$ of sugar; the ingredients required for baking one cake of type $B$ is $800g$ of flour, 8 eggs, and $200g$ of sugar. The bakery has 3 bakers, each of them works for at least 8 hours per day, and the total time required for each baker to bake one cake of type $A$ and $B$ is 40 minutes and 50 minutes respectively. If the bakery has to bake at least 32 cakes every day, and the everyday supply of ingredients is limited to 220 eggs and $9kg$ of sugar. Due to the shortage of flour, the bakery needs to lower the usage of it. How many cakes of each type should be baked to minimize the usage of flour? What's the minimum amount of flour used?

## 16.2 Revision Exercise 15

Compare the algebraic expressions in the following questions (Question 1 to 2):

1. $(x-3)(4-x)$ and $(6-x)(x-1)$

   **Sol.**

$$
\begin{aligned}
&(x-3)(4-x)-(6-x)(x-1)\\
&= -x^2+7x-12-(-x^2+7x-6)\\
&= -x^2+7x-12+x^2-7x+6\\
&= -6<0\\
&\therefore (x-3)(4-x)<(6-x)(x-1)
\end{aligned}
$$

2. $6-x^2$ and $4x-2x^2$

   **Sol.**

$$
\begin{aligned}
&6-x^2-(4x-2x^2)\\
&= 6x-x^2-4x+2x^2\\
&= x^2-2x\\
&= (x-1)^2+1\\
&\because (x-1)^2+1>0\\
&\therefore (x-1)^2+1>0\\
&\therefore 6-x^2>4x-2x^2
\end{aligned}
$$

Solve the following inequalities (Question 3 to 16):

3. $4(x-1)>x+6$

   **Sol.**

$$
\begin{aligned}
4(x-1) &> x+6\\
4x-4 &> x+6\\
3x &> 10\\
x &> \frac{10}{3}
\end{aligned}
$$

4. $3(3-x)\geq 2(x+3)$

   **Sol.**

$$
\begin{aligned}
3(3-x) &\geq 2(x+3)\\
9-3x &\geq 2x+6\\
-5x &\geq -3\\
5x &\leq 3\\
x &\leq \frac{3}{5}
\end{aligned}
$$

5. $3 - \frac{x-1}{4} \geq 2 + \frac{3(x+1)}{8}$

**Sol.**

$$
\begin{aligned}
3 - \frac{x-1}{4} &\geq 2 + \frac{3(x+1)}{8} \\
24 - 2(x-1) &\geq 16 + 3(x+1) \\
24 - 2x + 2 &\geq 16 + 3x + 3 \\
26 - 2x &\geq 19 + 3x \\
-5x &\geq -7 \\
5x &\leq 7 \\
x &\leq \frac{7}{5}
\end{aligned}
$$

6. $x - \frac{x-1}{2} \leq \frac{2x-1}{3} + \frac{x+1}{2}$

**Sol.**

$$
\begin{aligned}
x - \frac{x-1}{2} &\leq \frac{2x-1}{3} + \frac{x+1}{2} \\
6x - 3(x-1) &\leq 2(2x-1) + 3(x+1) \\
6x - 3x + 3 &\leq 4x - 2 + 3x + 3 \\
3x + 3 &\leq 7x + 1 \\
-4x &\leq -2 \\
4x &\geq 2 \\
x &\geq \frac{1}{2}
\end{aligned}
$$

7. $-1 < \frac{1}{2}x + 3 < 7$

**Sol.**

$$
\begin{aligned}
-1 < \frac{1}{2}x + 3 < 7 \\
-2 < x + 6 < 14 \\
-8 < x < 8
\end{aligned}
$$

8. $-\frac{3}{2} < 1 - 3x \leq 8$

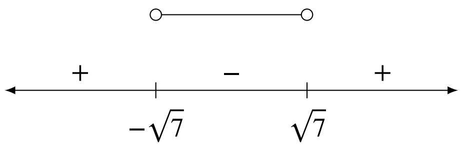
**Sol.**

$$
\begin{aligned}
-\frac{3}{2} < 1 - 3x \leq 8 \\
-3 < 2 - 6x \leq 16 \\
-5 < -6x \leq 14 \\
-14 \leq 6x < 5 \\
-\frac{7}{3} \leq x < \frac{5}{6}
\end{aligned}
$$

9. $x^2 < 7$

**Sol.**

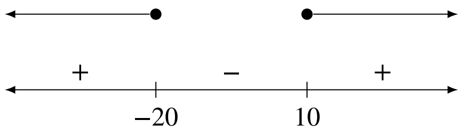
$$
\begin{aligned}
x^2 &< 7 \\
x^2 - 7 &< 0 \\
(x + \sqrt{7})(x - \sqrt{7}) &< 0 \\
-\sqrt{7} < x &< \sqrt{7}
\end{aligned}
$$




10. $x^2 + 10x - 200 \geq 0$

**Sol.**

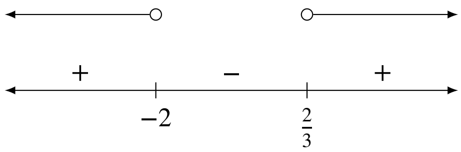
$$
\begin{aligned}
x^2 + 10x - 200 &\geq 0 \\
(x + 20)(x - 10) &\geq 0 \\
x \leq -20 \text{ or } x &\geq 10
\end{aligned}
$$




11. $4 < 3x^2 + 4x$

**Sol.**

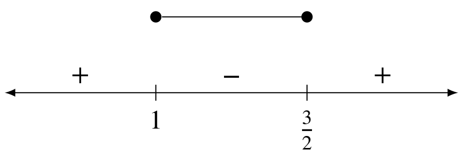
$$
\begin{aligned}
4 &< 3x^2 + 4x \\
3x^2 + 4x - 4 &> 0 \\
(3x - 2)(x + 2) &> 0 \\
x < -2 \text{ or } x &> \frac{2}{3}
\end{aligned}
$$




12. $5x - 3 \geq 2x^2$

**Sol.**

$$
\begin{aligned}
5x - 3 &\geq 2x^2 \\
2x^2 - 5x + 3 &\leq 0 \\
(2x - 3)(x - 1) &\leq 0 \\
1 \leq x &\leq \frac{3}{2}
\end{aligned}
$$




13. $x^2 - x(x - 6) > 5(x - 1)$

**Sol.**

$$
\begin{aligned}
x^2 - x(x - 6) &> 5(x - 1) \\
x^2 - x^2 + 6x &> 5x - 5 \\
6x &> 5x - 5 \\
x &> -5
\end{aligned}
$$

14. $(2x + 1)^2 + 5 \leq 4(x + 2)^2$

**Sol.**

$$
\begin{aligned}
(2x+1)^2+5 &\le 4(x+2)^2 \\
4x^2+4x+1+5 &\le 4(x^2+4x+4) \\
4x^2+4x+6 &\le 4x^2+16x+16 \\
-12x &\le 10 \\
12x &\ge -10 \\
x &\ge -\frac{5}{6}
\end{aligned}
$$

15. $9x^2+2 \le 12x-2$

**Sol.**

$$
\begin{aligned}
9x^2+2 &\le 12x-2 \\
9x^2-12x+4 &\le 0 \\
(3x-2)^2 &\le 0 \\
x &= \frac{2}{3}
\end{aligned}
$$

16. $4(x^2+7) > 3-20x$

**Sol.**

$$
\begin{aligned}
4(x^2+7) &> 3-20x \\
4x^2+28 &> 3-20x \\
4x^2+20x-25 &> 0 \\
(2x+5)^2 &> 0 \\
x \in \mathbb{R}, x &\ne -\frac{5}{2}
\end{aligned}
$$

Solve the following system of inequalities (Question 17 to 28):

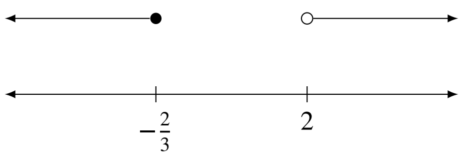
17.

$$
\begin{cases}
3x+2 \le 0 & (1) \\
4-x < x & (2)
\end{cases}
$$

**Sol.**

$$
\begin{aligned}
(1):\ & 3x \le -2 \\
& x \le -\frac{2}{3} \\
(2):\ & -2x < -4 \\
& x > 2
\end{aligned}
$$

$\therefore$ No solution.



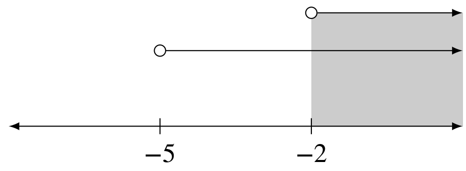

18.

$$
\begin{cases}
x+4 > -x & (1) \\
\dfrac{3x-1}{2} < 2(x+1) & (2)
\end{cases}
$$

**Sol.**

$$
\begin{aligned}
(1):\ & 2x > -4 \\
& x > -2 \\
(2):\ & 3x-1 < 4(x+1) \\
& 3x-1 < 4x+4 \\
& -x < 5 \\
& x > -5
\end{aligned}
$$

$\therefore x > -2$



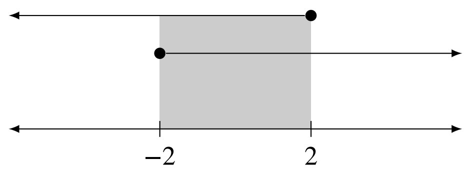

19.

$$
\begin{cases}
x-3 \le 5-3x & (1) \\
4+(2x-1) \le 4x+7 & (2)
\end{cases}
$$

**Sol.**

$$
\begin{aligned}
(1):\ & 4x \le 8 \\
& x \le 2 \\
(2):\ & 3+2x \le 4x+7 \\
& -2x \le 4 \\
& x \ge -2
\end{aligned}
$$

$\therefore -2 \le x \le 2$



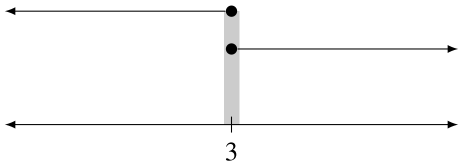

20.

$$
\begin{cases}
4x-5 \ge 2x+1 & (1) \\
x+\dfrac{2}{3} \le \dfrac{2x+5}{3} & (2)
\end{cases}
$$

**Sol.**

$$
\begin{aligned}
(1):\ & 2x \ge 6 \\
& x \ge 3 \\
(2):\ & 3x+2 \le 2x+5 \\
& x \le 3
\end{aligned}
$$

$\therefore x = 3$




21. $5 < 2x - 7 < x + 1$

**Sol.**

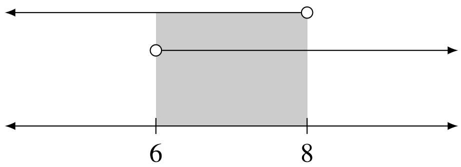
$$\begin{cases} 5 < 2x - 7 & (1) \\ 2x - 7 < x + 1 & (2) \end{cases}$$

$(1):\ 12 < 2x$

$x > 6$

$(2):\ x < 8$

$\therefore 6 < x < 8$




22. $4 < 6 + 2x \leq 4x$

**Sol.**

$$\begin{cases} 4 < 6 + 2x & (1) \\ 6 + 2x \leq 4x & (2) \end{cases}$$

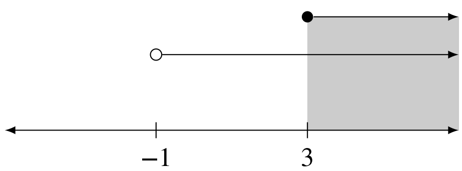
$(1):\ -2 < 2x$

$x > -1$

$(2):\ 6 \leq 2x$

$x \geq 3$

$\therefore x \geq 3$




23.

$$\begin{cases} x - \dfrac{1}{2} \geq 1 - \dfrac{x}{2} & (1) \\ 2 - \dfrac{x}{3} < \dfrac{2x}{3} - 3 & (2) \\ \dfrac{x}{3} + \dfrac{1}{4} \geq \dfrac{x}{2} - \dfrac{3}{4} & (3) \end{cases}$$

**Sol.**

$(1):\ 2x - 1 \geq 2 - x$

$3x \geq 3$

$x \geq 1$

$(2):\ 6 - x < 2x - 9$

$-3x < -15$

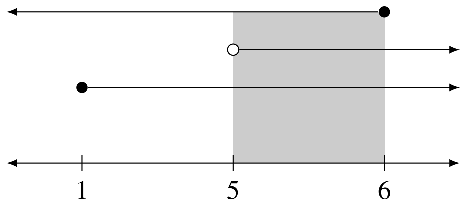
$x > 5$

$(3):\ 4x + 3 \geq 6x - 9$

$-2x \geq -12$

$x \leq 6$

$\therefore 5 < x \leq 6$




24.

$$\begin{cases} x + \dfrac{13}{2} > \dfrac{7 - x}{2} & (1) \\ 2\left(x + \dfrac{1}{3}\right) < 2 - x & (2) \\ x^2 \geq \dfrac{5x}{2} & (3) \end{cases}$$

**Sol.**

$(1):\ 2x + 13 > 7 - x$

$3x > -6$

$x > -2$

$(2):\ 2x + \dfrac{2}{3} < 2 - x$

$6x + 2 < 6 - 3x$

$9x < 4$

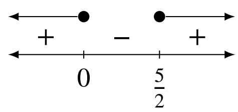
$x < \dfrac{4}{9}$

$(3):\ 2x^2 \geq 5x$

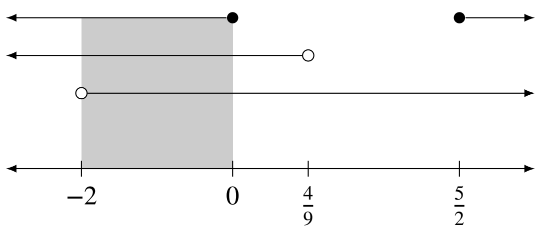
$2x^2 - 5x \geq 0$

$x(2x - 5) \geq 0$

$x \leq 0$ or $x \geq \dfrac{5}{2}$




$\therefore -2 < x \leq 0$



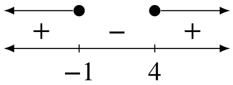
25.

$$\begin{cases} x^2 - 3x - 4 \geq 0 & (1) \\ 2x^2 - x - 6 > 0 & (2) \end{cases}$$

**Sol.**

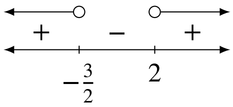
(1) : $(x-4)(x+1) \geq 0$

$x \leq -1$ or $x \geq 4$

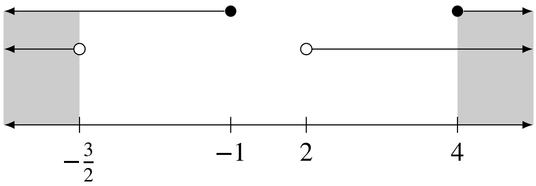



(2) : $(2x+3)(x-2) > 0$

$x < -\dfrac{3}{2}$ and $x > 2$




$\therefore x < -\dfrac{3}{2}$ or $x \geq 4$




26.

$$\begin{cases} (2x-1)(x-2) \leq 8x - 9 & (1) \\ 3(x^2-2) < 7x & (2) \end{cases}$$

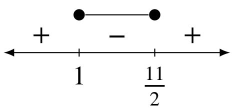
**Sol.**

(1) : $2x^2 - 5x + 2 \leq 8x - 9$

$2x^2 - 13x + 11 \leq 0$

$(2x-11)(x-1) \leq 0$

$1 \leq x \leq \dfrac{11}{2}$

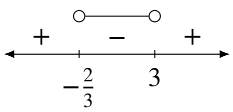



(2) : $3x^2 - 6 < 7x$

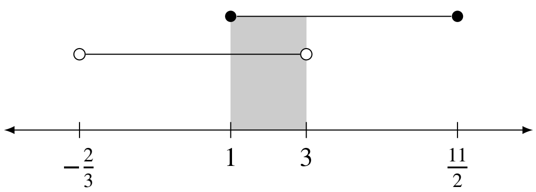
$3x^2 - 7x - 6 < 0$

$(3x+2)(x-3) < 0$

$-\dfrac{2}{3} < x < 3$




$\therefore 1 \leq x < 3$




27.

$$\begin{cases} 2(x^2+3) \geq 7 - x & (1) \\ (x+3)^2 > 1 & (2) \end{cases}$$

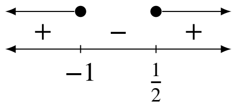
**Sol.**

(1) : $2x^2 + 6 \geq 7 - x$

$2x^2 + x - 1 \geq 0$

$(2x-1)(x+1) \geq 0$

$x \leq -1$ or $x \geq \dfrac{1}{2}$

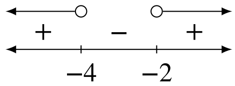



(2) : $x^2 + 6x + 9 > 1$

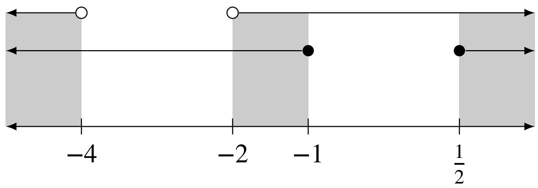
$x^2 + 6x + 8 > 0$

$(x+4)(x+2) > 0$

$x < -4$ or $x > -2$




$\therefore x < -4$ or $-2 < x \leq -1$ or $x \geq \dfrac{1}{2}$




28.

$$\begin{cases} x(x-1) \leq 2 & (1) \\ x(x+1) \geq 6 & (2) \end{cases}$$

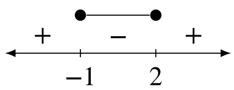
**Sol.**

(1) : $x^2 - x \leq 2$

$x^2 - x - 2 \leq 0$

$(x-2)(x+1) \leq 0$

$-1 \leq x \leq 2$

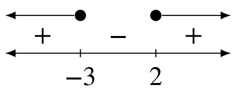



(2) : $x^2 + x \geq 6$

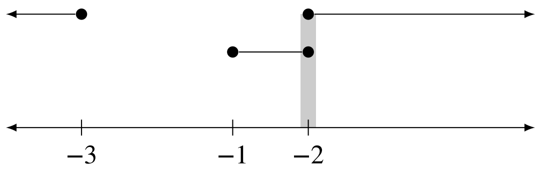
$x^2 + x - 6 \geq 0$

$(x+3)(x-2) \geq 0$

$x \leq -3$ or $x \geq 2$




$\therefore x = 2$

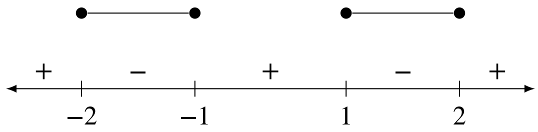



Solve the following inequalities (Question 29 to 40):

29. $x^4 - 5x^2 + 4 \leq 0$

**Sol.**

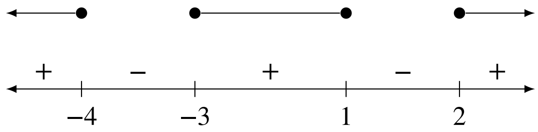
$$\begin{aligned} x^4 - 5x^2 + 4 &\leq 0 \\ (x-1)(x^3 + x^2 - 4x - 4) &\leq 0 \\ (x-1)(x+1)(x^2-4) &\leq 0 \\ (x-1)(x+1)(x-2)(x+2) &\leq 0 \\ -2 \leq x \leq -1 \text{ or } 1 &\leq x \leq 2 \end{aligned}$$




30. $(x^2 + 2x - 8)(x^2 + 2x - 3) > 0$

**Sol.**

$$\begin{aligned} (x+4)(x-2)(x+3)(x-1) &> 0 \\ x < -4 \text{ or } -3 < x < -1 &\text{ or } x > 2 \end{aligned}$$




31. $(2x+1)^2(x^2 + 3x - 10) < 0$

**Sol.**

For all real numbers $x$,

$$\begin{aligned} (2x+1)^2 > 0 \text{ when } x &\neq -\frac{1}{2} \\ x^2 + 3x - 10 &< 0 \\ (x+5)(x-2) &< 0 \\ \therefore -5 < x &< -2 \\ \text{When } x &= -\frac{1}{2}, \\ (2x+1)^2(x^2+3x-10) &= 0 \\ \therefore x = -\frac{1}{2} &\text{ is not a solution.} \end{aligned}$$

$$\therefore -5 < x < -2,\ x \neq -\frac{1}{2}$$

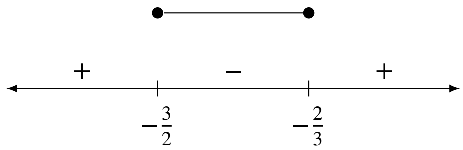

32. $(x-1)^2(6x^2 + 13x + 6) \leq 0$

**Sol.**

For all real numbers $x$,

$$\begin{aligned} (x-1)^2 > 0 \text{ when } x &\neq 1 \\ 6x^2 + 13x + 6 &\leq 0 \\ (3x+2)(2x+3) &\leq 0 \\ \therefore -\frac{3}{2} \leq x &\leq -\frac{2}{3} \\ \text{When } x &= 1, \\ (x-1)^2(6x^2+13x+6) &= 0 \\ \therefore x = 1 &\text{ is a solution.} \end{aligned}$$

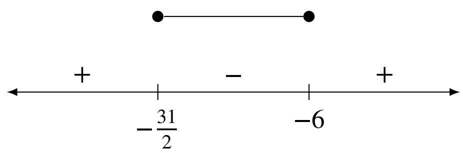
$$\therefore -\frac{3}{2} \leq x \leq -\frac{2}{3} \text{ or } x = 1$$




33. $\frac{2x-7}{x+6} \geq 4$

**Sol.**

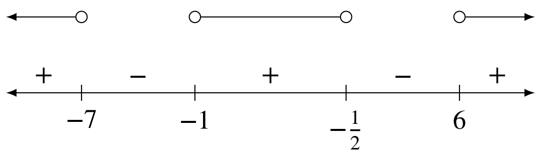
$$\begin{aligned} \frac{2x - 7 - 4(x+6)}{x+6} &\geq 0 \\ \frac{2x - 7 - 4x - 24}{x+6} &\geq 0 \\ \frac{-2x-31}{x+6} &\geq 0 \\ -\frac{2x+31}{x+6} &\geq 0 \\ \frac{2x+31}{x+6} &\leq 0 \\ -\frac{31}{2} \leq x &\leq -6 \end{aligned}$$




34. $\frac{x}{2x+1} > \frac{6}{x+7}$

**Sol.**

$$\begin{aligned}
\frac{x}{2x+1} &> \frac{6}{x+7}\\
\frac{x}{2x+1} - \frac{6}{x+7} &> 0\\
\frac{x(x+7)-6(2x+1)}{(2x+1)(x+7)} &> 0\\
\frac{x^2+7x-12x-6}{(2x+1)(x+7)} &> 0\\
\frac{x^2-5x-6}{(2x+1)(x+7)} &> 0\\
\frac{(x-6)(x+1)}{(2x+1)(x+7)} &> 0\\
x < -7 \text{ or } -1 < x < -\frac{1}{2} &\text{ or } x > 6
\end{aligned}$$




35. $\frac{(x+3)(x-2)^2}{x^2-1} \le 0$

**Sol.**

For all real number x,

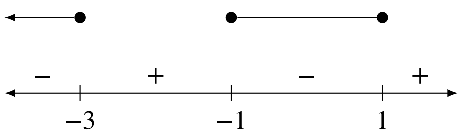
$(x-2)^2 > 0$ when $x \neq 2$

$$\begin{aligned}
\frac{x+3}{x^2-1} &\le 0\\
\frac{x+3}{(x+1)(x-1)} &\le 0\\
\therefore x \le -3 \text{ or } -1 \le x &\le 1
\end{aligned}$$

When $x = 2$, $\frac{(x+3)(x-2)^2}{x^2-1} = 0$

$\therefore x = 2$ is a solution.

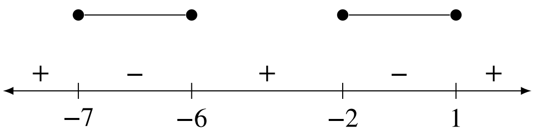
$\therefore x \le -3$ or $-1 \le x \le 1$ or $x = 2$




36. $4 + \frac{7}{x+6} \le \frac{15}{x+2}$

**Sol.**

$$\begin{aligned}
4 + \frac{7}{x+6} &\le \frac{15}{x+2}\\
4 + \frac{7}{x+6} - \frac{15}{x+2} &\le 0\\
\frac{4(x+6)(x+2)+7(x+2)-15(x+6)}{(x+6)(x+2)} &\le 0\\
\frac{4(x^2+8x+12)+7x+14-15x-90}{(x+6)(x+2)} &\le 0\\
\frac{4x^2+32x+48+7x+14-15x-90}{(x+6)(x+2)} &\le 0\\
\frac{4x^2+24x-28}{(x+6)(x+2)} &\le 0\\
\frac{x^2+6x-7}{(x+6)(x+2)} &\le 0\\
\frac{(x+7)(x-1)}{(x+6)(x+2)} &\le 0\\
-7 \le x \le -6 \text{ or } -2 \le x &\le 1\\
\because x \neq -6 \text{ and } x &\neq 2,\\
\therefore -7 \le x < -6 \text{ or } -2 < x &\le 1
\end{aligned}$$




37. $|3-5x| \ge 7$

**Sol.**

$$\begin{aligned}
&|3-5x| \ge 7\\
&3-5x \ge 7 \text{ or } 3-5x \le -7\\
&-5x \ge 4 \text{ or } -5x \le -10\\
&5x \le -4 \text{ or } 5x \ge 10\\
&x \le -\frac{4}{5} \text{ or } x \ge 2
\end{aligned}$$

38. $2 < |x-5| < 9$

**Sol.**

$$\begin{cases} |x-5| > 2 & (1)\\ |x-5| < 9 & (2) \end{cases}$$

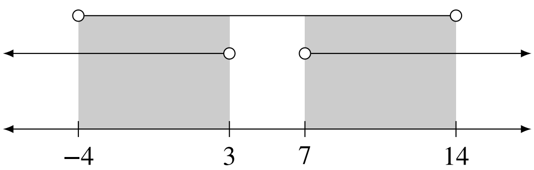
$(1): x-5 < -2$ or $x-5 > 2$

$x < 3$ or $x > 7$

$(2): -9 < x-5 < 9$

$-4 < x < 14$

$\therefore -4 < x < 3$ or $7 < x < 14$



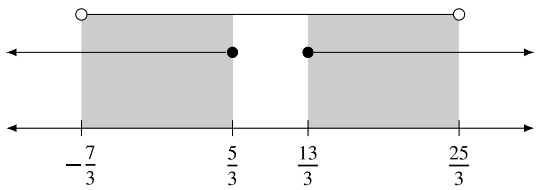
39. $1 \le \left|\frac{3x-1}{4} - 2\right| < 4$

$$\begin{cases} \left|\dfrac{3x-1}{4} - 2\right| \ge 1 & (1) \\ \left|\dfrac{3x-1}{4} - 2\right| < 4 & (2) \end{cases}$$

$$\begin{aligned} (1) :\ \frac{3x-1}{4} - 2 \le -1 &\text{ or } \frac{3x-1}{4} - 2 \ge 1 \\ \frac{3x-1}{4} \le 1 &\text{ or } \frac{3x-1}{4} \ge 3 \\ 3x - 1 \le 4 &\text{ or } 3x - 1 \ge 12 \\ 3x \le 5 &\text{ or } 3x \ge 13 \\ x \le \frac{5}{3} &\text{ or } x \ge \frac{13}{3} \end{aligned}$$

$$\begin{aligned} (2) :\ -4 < \frac{3x-1}{4} - 2 < 4 \\ -2 < \frac{3x-1}{4} < 6 \\ -8 < 3x - 1 < 24 \\ -7 < 3x < 25 \\ -\frac{7}{3} < x < \frac{25}{3} \end{aligned}$$

$$\therefore\ -\frac{7}{3} < x \le \frac{5}{3} \text{ or } \frac{13}{3} \le x < \frac{25}{3}$$




40. $\frac{4}{|x+3|} - 5 \le 3$

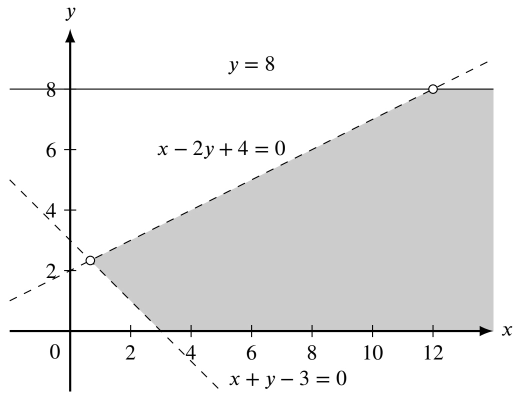
**Sol.**

$$\begin{aligned} \frac{4}{|x+3|} - 5 \le 3 \\ \frac{4}{|x+3|} \le 8 \\ 4 \le 8|x+3| \\ |x+3| \ge \frac{1}{2} \\ x + 3 \le -\frac{1}{2} \text{ or } x + 3 \ge \frac{1}{2} \\ x \le -\frac{7}{2} \text{ or } x \ge -\frac{5}{2} \end{aligned}$$

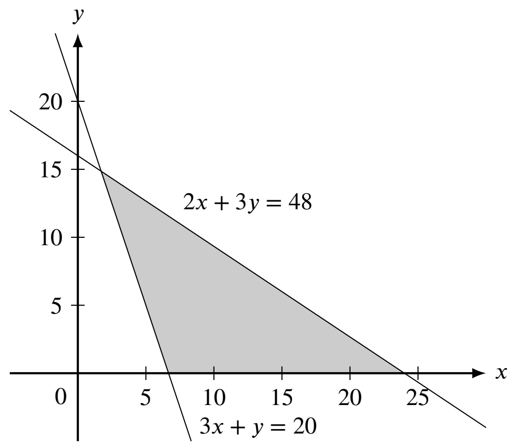
Solve the following system of inequalities with graphs (Question 41 to 42):

41. $\begin{cases} x + y - 3 > 0 \\ x - 2y + 4 > 0 \\ 0 \le y \le 8 \end{cases}$

**Sol.**




42. $\begin{cases} 3x + y \ge 20 \\ 2x + 3y \le 48 \\ x \ge 0 \\ y \ge 0 \end{cases}$



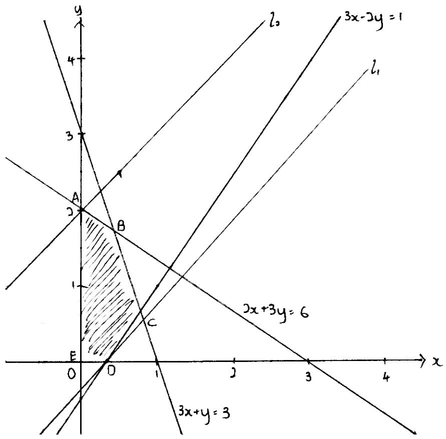

43. Find the maximum and minimum value of $z = x - y$, subject to the following constraints:

$$\begin{cases} 3x + y \le 3 \\ 2x + 3y \le 6 \\ 3x - 2y \le 1 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

**Sol.**

Objective function: $z = x - y$

$y = x - z$




When $y = x - z$ translates towards bottom right of the feasible region, the value of $z$ increases. Therefore, the maximum value of the objective function is the value of $z$ in $l_1$. The point of intersection $D$ of $l_1$ and the feasible region makes the objective function to have its maximum value. Since $D$ is also the point of intersection of $3x - 2y = 1$ and $y = 0$,

$$\begin{cases} 3x - 2y = 1 \\ y = 0 \end{cases}$$

$$D = \left(\frac{1}{3}, 0\right)$$

$$z_{\max} = \frac{1}{3} - 0 = \frac{1}{3}$$

When $y = x - z$ translates towards top left of the feasible region, the value of $z$ decreases. Therefore, the minimum value of the objective function is the value of $z$ in $l_2$. The point of intersection $A$ of $l_2$ and the feasible region makes the objective function to have its minimum the point of intersection of $2x + 3y = 6$ and $x = 0$,

$$\begin{cases} 2x + 3y = 6 \\ x = 0 \end{cases}$$

$$A = (0, 2)$$

$$z_{\min} = 0 - 2 = -2$$

44. A factory produces two types of products: $A$ and $B$. The ingredients used in each kilogram of these two products are as follows:

| Product (per kg) | Ingr. X (kg) | Ingr. Y (kg) |
|---|---|---|
| $A$ | 0.6 | 0.5 |
| $B$ | 0.3 | 0.7 |

The profit of each kilogram of product $A$ and $B$ is \$3 and \$5 respectively. The factory has $24kg$ of ingredient $X$ and $28kg$ of ingredient $Y$. How many kilograms of each product should be produced to maximize the profit?

45. An animal must consume three different kind of nutrients: $X$, $Y$ and $Z$ at least $11units$, $13units$ and $15units$ respectively every day. There are two types of animal food: $A$ and $B$ that contain the following nutrients:

| Food | X (unit) | Y (unit) | Z (unit) |
|---|---|---|---|
| $A$ | 1 | 3 | 2 |
| $B$ | 2 | 1 | 2 |

The animal food $A$ costs \$300 per kilogram and the animal food $B$ costs \$400 per kilogram. How many kilograms of each food should be consumed to meet the daily nutrient requirement at the minimum cost? Find the minimum cost.