Chapter 1

Limits

Exercise 1.1

In Problems 1 to 4, find Δy for x and Δx as given for each function.

1. $y(x) = 2x - 3$; $x = 2$, $\Delta x = 1$; $x = 5$, $\Delta x = 1$
2. $y(x) = 3x + 1$; $x = -1$, $\Delta x = 2$; $x = 7$. $\Delta x = 2$
3. $y(x) = x^2 - x + 1$; $x = -2$, $\Delta x = 1$; $x = 3$, $\Delta x = 1$
4. $y(x) = x^2 + 3x - 5$; $x = -5$, $\Delta x = 2$; $x = 6$, $\Delta x = 2$
5. If x is near 3, what number is the value of the function $3x^2 - 2$ near? Is it true that the values of the function are arbitrarily near this number for all values of s that are sufficiently near 3? Is this number the limit of the function as $x \to 3$?
6. What number is the value of the function $x^2 - x - 1$ near if x is near 5? Are the values of the function arbitrarily near this number 14 for all replacements for x that are sufficiently near 5? Is this number the limit of the function as $x \to 5$?
7. What number is the value of the function $\dfrac{x^2+6}{x-4}$ near if x is near 6? Is the value of the function arbitrarily near this number for all replacements for x that are sufficiently near 6? Is this number the limit of the function as $x \to 6$?
8. What number is the value of the function $\dfrac{x^3-5}{x+2}$ near if x is near 2? Is it true that the value of this function is arbitrarily near this number if x is sufficiently near 2? Is this number the limit of the function as $x \to 2$?
9. Is there a number L such that the values of the function $\dfrac{x^2+1}{x-2}$ are near L for all value of x near 2? What statement can you make about the limit of this function as $x \to 2$?
10. Is there a number L such that the values of the function $\dfrac{x^2+7x+10}{x^2-9}$ are near L for all values of x near 3? What statement can you make about the limit of this function as $x \to 3$?
11. Find the value of the function $\dfrac{x^2-9}{x-3}$ for $x = 2.9, 2.99, 3.1, 3.01$. These results indicate that a number L may be the limit of this function as $x \to 3$. WHat is this number? Prove that this number is actually the limit.
12. Find the values of the function $\dfrac{x^2-3x-10}{x-5}$ for $x = 4.9, 4.99, 5.1, 5.01$. These results indicate that what number may be the limit of this function as $x \to 5$? Prove that this number is actually the limit.

In each of Problems 13 to 16, find the limit of the given functions as $h \to 0$.

13. $\dfrac{(3+h)^2-9}{h}$

14. $\dfrac{(2+h)^3-8}{h}$

15. $\dfrac{\dfrac{1}{2+h}-\dfrac{1}{2}}{h}$

16. $\dfrac{\sqrt{4+h}-2}{h}$

17. Evaluate $\lim\limits_{x\to 2}\dfrac{4x}{3x-5}$. Find the value of this function corresponding to $x=2/$ Is the function continuous for $x=2$? Is there a value of x for which the function is not continuous? Sketch the graph.

18. Evaluate $\lim\limits_{x\to 1}\dfrac{3x}{x^2+1}$. Find the value of this function corresponding to $x=1$. Is the function not continuous? Sketch the graph.

19. Evaluate $\lim\limits_{x\to -3}\dfrac{4x+1}{(2+x^2)}$. Find the value of the function for $x=-4$. Is the function continuous for $x=-3$? Is there any value of x for which it is not continuous? Sketch the graph.

Find each of the following limits if it exists.

21. $\lim\limits_{x\to 2}\dfrac{x^3-8}{x=2}$

22. $\lim\limits_{x\to -1}\dfrac{x^3+1}{x+1}$

23. $\lim\limits_{x\to \frac{1}{2}}\dfrac{2x^2+7x-4}{4x-2}$

24. $\lim\limits_{x\to -5}\dfrac{3x+15}{x^2-25}$

25. $\lim\limits_{x\to 2}\dfrac{2-x}{x+2}$

26. $\lim\limits_{h\to 0}\dfrac{h^2+3h}{h-1}$

27. $\lim\limits_{h\to 0}\dfrac{1}{h}[(6+h)^2-36]$

28. $\lim\limits_{x\to -a}\dfrac{x^2-a^2}{x+a}$

29. $\lim\limits_{x\to 2}\dfrac{x^3-4x}{x^3-2x^2}$

30. $\lim\limits_{h\to 0}\dfrac{h^3-4h}{h^3-2h^2}$

31. $\lim\limits_{x\to x_1}\dfrac{4x^2 \; - \; 4{x_1}^2}{x-x_1}$

32. $\lim\limits_{x\to x_1}\dfrac{x^3-{x_1}^3}{x-x_1}$

33. $\lim\limits_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$

34. $\lim\limits_{h\to 0}\dfrac{(x+h)^3-x^3}{h}$

35. $\lim\limits_{h\to 0}\dfrac{1}{h}\left[\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}\right]$

36. $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

Exercise 1.2

1. If $\lim\limits_{x\to\alpha} f(x)=0$ and $\lim\limits_{x\to\alpha} g(x)=k\neq 0$, what can be said about $\lim\limits_{x\to\alpha}\dfrac{f(x)}{g(x)}$?
2. If $\lim\limits_{x\to\alpha} f(x)\neq 0$ and $\lim\limits_{x\to\alpha} g(x)=0$, what can be said about $\lim\limits_{x\to\alpha}\dfrac{f(x)}{g(x)}$?
3. If $\lim\limits_{x\to\alpha} f(h)=0$ and $\lim\limits_{x\to\alpha} g(x)=k$, what can be said about $\lim\limits_{x\to\alpha} f(x)g(x)$?
4. If $\lim\limits_{x\to\alpha} f(x)=A$ and $\lim\limits_{x\to\alpha} g(x)=A$, what can be said about $\lim\limits_{x\to\alpha}[f(x)-g(x)]$?
5. Sketch the graph of the function $4^{-\frac{1}{x}}$. Discuss the continuity of the function.

6. Sketch the graph of the function $\dfrac{1}{1+2^{\frac{1}{x}}}$. Discuss the continuity of the function.

7. Sketch the graph of the function $\dfrac{2^{\frac{1}{x}}}{1+2^{\frac{1}{x}}}$. Discuss the continuity of the function.

8. If $\lim\limits_{x\to\alpha} f(x) = L$ and $\lim\limits_{x\to\alpha} g(x) = M \neq 0$, prove that $\lim\limits_{x\to\alpha} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$.

In Problems 9 to 16 prove ny use of the ϵ, δ definition that each function is continuous for the given value of x.

9. $f(x) = x^2 + 2$, $x = 3$

10. $f(x) = 3x = 7$, $x = a$

11. $f(x) = x^3$, $x = 2$

12. $f(x) = x^3 - x$, $x = 4$

13. $f(x) = \sqrt{2x-1}$, $x = 5$

14. $f(x) = \sqrt{3x+1}$, $x = 5$

15. $f(x) = \dfrac{x+2}{x+1}$, $x = a \neq -1$

16. $f(x) = \dfrac{3x-1}{2x+3}$, $x = a \neq -1.5$

In Problems 17 to 20, assume that $f(x)$ and $g(x)$ are continuous for $x = a$.

17. Prove that the sum of two continuous functions is continuous.

18. Prove that the product of a constant and a continuous function is continuous.

19. Prove that the product of two continuous functions is continuous.

20. Prove that $\dfrac{f(x)}{g(x)}$ is continuous at $x = a$ unless $g(a) = 0$.

Exercise 2.1

1. In Fig. 2.1 assume that $x = 4$ in. so that $A = 16$ sq in. Go through the steps of Example 1 using 4 and x_1 instead of x and x_1, and thus find the rate of change of the area relative to x for the instant that $x = 4$ in. Compute the average rate of change of the area with respect to x over the interval from $x = 4$ to 4.5 in.

2. If the edge of a cube is 2 in., its surface area is $6(2)^2$ or 24 sq in. If the edge is increased to x in., the new surface area will be $6x^2$ sq in.

   Find the value of the fraction
   $$\frac{6x^2-24}{x-2}$$
   if $x = 2.5$ and interpret the result. Find the limit of the fraction as $x \to 2$ and interpret this result.

3. Find the average rate of change of the volume of a cube relative to its edge x over the interval from $x = 2.4$ to 2.6 in. Find the instantaneous rate of $x = 2.4$ in.

4. Find the average rate of change of the surface area of a cube relative to its edge x over the interval from $x = 4$ to 4.5 in. Find the instantaneous rate if $x = 4$ in.

5. Find the rate of change of the area of a circle relative to its radius.

6. Find the rate of change of the surface area fo a sphere relative to its radius.

7. Find the rate of change of the volume of a cube relative to its edge.

8. Find the rate of change of the area of an equilateral triangle relative to a side.

9. A rectangular box has a square base of side x ft and its height is $5x$ ft. Find the rate of change of its volume relative to x.

10. A right pyramid has a square base of side x in. and its height is $6x$ in. Find the rate of change of its volume relative to x.

11. Find the rate of change of the volume of a cube relative to its surface area. Hint: If the edge is x_1 then the volume is ${x_1}^3$ and the surface area is $6{x_1}^2$. If the edge changes to a new value x, the volume becomes x^3 and the surface area becomes $6x^2$. Consequently,

$$\frac{V - V_1}{S - S_1} = \frac{x^3 - {x_1}^3}{6x^2 - 6{x_1}^2}$$

12. Find the rate of change of the volume of a sphere relative to the area πr^2 of a great circle.

13. Find the rate of chang eof the volume of a right circular cylinder of radius r and height h relative to the cross-sectional area.

14. Find the rate of change of th volume of a sphere relative to its surface area.

15. The radius r of a right circular cylinder is increasing and its height h is a constant. Find the rate at which its total surface area increases relative to r.

16. Find the rate of change of the volume of a right circular cone relative to its radius r if its height h is a constant.

17. Show that when $x = 25$, the square root of x is increasing one-tenth as fast as x.

18. Find, for any positive value of x, the rate of change of the square root of x relative to x.

19. Find the average rate of change of the value of the function $x^2 - 4x$ relative to x over the interval from $x = 7$ to 9. Find the instantaneous rate when $x = 7$.

20. Find the average rate of change of the value of the function $x^3 + 6$ relative to x over the interval from $x = 5$ to 7. Find the instantaneous rate when $x = 6$.

21. Find the rate of chang eof the value of the function $\dfrac{1}{x}$ relative to x. What is the significance of the negative sign?

22. Find the rate of change of the value of the function $\dfrac{x+1}{x-1}$ relative to x. IN particular, show that if x has the value 3 and is increasing, the value of this function is decreasing one-half as fast as x is increasing.

23. Show that for $x = 1$ the value of the function $x^2 + \sqrt{x}$ increases $2\frac{1}{2}$ times as fast as x.

24. Show that for $x = 4$, the value of the function $\dfrac{1}{\sqrt{x}}$ decreases one-sixteenth as fast as x increases.

Exercise 2.2

In each of Problems 1 to 20 find the derivative with respect to x of the given function using form (2) of the definition.

1. $3x^2 + 2$
2. $4x^2 + 3x$
3. $x^2 - 5x$
4. $\dfrac{x^3}{2}$
5. $x^3 - 2x$
6. $2x^3 + 3x$
7. $2x^2 - 7x$
8. $x^4 - 3x^2$
9. $\dfrac{3}{x}$
10. $\dfrac{4}{x+2}$
11. $\dfrac{x}{2x+1}$
12. $\dfrac{x+2}{x+3}$
13. $\dfrac{8}{x^2+4}$
14. $\dfrac{1}{\sqrt{x}}$
15. $\dfrac{3}{x^2-2}$
16. $\dfrac{x}{x^2+2}$
17. $\dfrac{8x}{x^2-16}$
18. $\dfrac{x^2}{x^2+4}$
19. $\dfrac{x^2-1}{x^2+1}$
20. $\dfrac{x^2+4}{x^2-4}$
21. Given $y = 4t^3 + 3$, find $D_t y$.
22. Given $w = u^2 - 3u$, find $D_u w$.
23. Given $S = 4\sqrt{w}$, find $\dfrac{dS}{dw}$.
24. Given $A = 4\pi r^2$, find $\dfrac{dA}{dr}$.
25. Given $y = \sqrt{x+2}$, find $D_x y$.
26. Given $Q = t(2t+1)$, find $\dfrac{dQ}{dt}$.
27. Given $S = \dfrac{t-1}{t^2+4}$, find $D_t S$.
28. Given $T = \dfrac{u-3}{u^2+2}$, find $D_u T$.

In each of Problems 29 to 36 sketch the graph of the given equation, and find the slope of the tangent line to the graph at the points indicated. Draw teh corresponding tangent line.

29. $y = \dfrac{x^2}{2+x}$; $(-2, 0)$, $(3, 2.5)$
30. $y = \dfrac{2x}{3+1}$; $(-1, \frac{1}{3})$, $(3, 3)$
31. $y = \dfrac{2}{\sqrt{x}}$; $(1, 2)$, $(4, 4)$
32. $y = \dfrac{x^3}{2}$; $\left(1, \dfrac{1}{2}\right)$, $(2, 4)$
33. $y = \dfrac{4}{x}$; $(2, 2)$, $\left(8, \dfrac{1}{2}\right)$
34. $y = \dfrac{x+2}{x-1}$; $(0, -2)$, $(-2, 0)$
35. $y = \dfrac{3x-6}{x+2}$; $(2, 0)$, $(-5, 7)$
36. $y = \dfrac{4x}{x+4}$; $(-2, -4)$, $(0, 0)$

Exercise 2.3

1. Given an opinion as the truth of each of the following assertions:

   (a) If $f(x)$ is a constant for $a \leq x \leq b$, then the average rate of change of $f(x)$ is over this interval is zero.

   (b) If the average rate of change of $f(x)$ over the interval $a \leq x \leq b$ is zero, then $f(x)$ is a constant over this interval.

2. Show that the average rate of change of $f(x)$ over the interval $a \leq x \leq b$ is zero, then $f(x)$ is constant over this interval.

3. Show that the function $\phi(x) = \sqrt{x}$ does not have a right-hand derivative at $x = 0$. *Hint:* Consider the fraction

$$\frac{\phi(x) - \phi(0)}{x - 0} = \frac{\sqrt{x} - 0}{x - 0} \quad x > 0$$

   Show that it does not have a limit as $x \to 0^+$.

4. Show that the function $f(x) = \sqrt{1 - x^2}$ does not have a left-hand derivative at the point for which $x = 1$. *Hint:* Consider the fraction

$$\frac{f(x) - f(1)}{x - 1} = \frac{\sqrt{1 - x^2} - 0}{x - 1} \quad 0 < x < 1$$

   Show that it does not have a limit as $x \to 1^-$.

In each of Problem 5 to 10 determine the x interval or intervals over which the value of y is increasing.

5. $y = x^2 - 8x$
6. $y = 4x - x^2$
7. $y = \dfrac{2}{x}$
8. $y = -\dfrac{5}{x + 1}$
9. $y = x^3 - 6x^2$
10. $y = 3x - x^3$

11. Show that the function $\dfrac{x + 1}{x - 1}$ is a decreasing function over any interval that does not include the point $x = 1$.

12. Let $y = \dfrac{1}{x^2 + 1}$. Show that y decreases as x increases if $x > 0$ and that y increases as x increases if $x < 0$.

13. If $y = \dfrac{4x}{x^2 + 4}$, show that y increases as x increases over the interval $-2 < x < 2$. Draw the corresponding graph. What slope does the curve have at the origin? At what points is the slope equal to zero?

14. Sketch the graph of the equation $y = \dfrac{8}{x^2 - 4}$. Use $D_x y$ to show that y increases as x increases if $x < -2$ and if $-2 < x < 0$. What is the slope of the tangent line to the graph at the point $\left(4, \dfrac{2}{3}\right)$?

15. If $y = x^3$, find the value of $D_x y$ for $x = 2$ by evaluating

$$\lim_{x \to 2} \frac{x^3 - 2^3}{x - 2} \qquad \text{or} \qquad \lim_{\Delta x \to 0} \frac{(2 + \Delta x)^3 - 2^3}{\Delta x}$$

16. If $y = \dfrac{4}{x}$, find the value of $D_x y$ for $x = 2$ by evaluating

$$\lim_{x \to 2} \frac{\dfrac{4}{x} - \dfrac{4}{2}}{x - 2} \qquad \text{or} \qquad \lim_{\Delta x \to 0} \frac{\dfrac{4}{2 + \Delta x} - \dfrac{4}{2}}{\Delta x}$$

17. If $y = \sqrt{x + 2}$, find $D_x y$ in terms of x by evaluating

$$\lim_{x \to x_1} \frac{\sqrt{x + 2} - \sqrt{x_1 + 2}}{x - x_1}$$

Check by writing down the corresponding fraction involving Δx and finding its limit as $\Delta x \to 0$.

18. A right pyramid has a rectangular base that is x ft wide and $2x$ ft long. The height of the pyramid is $3x$ ft. Express its volume V as a function of x. Find the value of $D_x V$ for $x = 2\frac{1}{2}$ ft. In what units is the rate expressed?

19. The number N of grams of a substance in solution varies with the time t in minutes that the substance has been in contact with the solvent in accordance with the formula $N = \dfrac{16}{t + 5}$. Find the value of $D_t N$ if $t = 3$. In what units is this rate expressed?

20. If an object si dropped from the top of a cliff and falls under the action of gravity alone, its distance S in feet below the top of the cliff at the end of t sec is given approximately by the formula $S = 16t^2$. Find the value of $D_t S$ when $t = 4$, and explain the meaning of the result. In what units is this rate expressed?

21. A cylinder contains 200 cu in. of air at a pressure of 15 lb per sq in. If the air is now compressed by moving a piston in the cylinder and if this is done under a condition of constant temperature, the pressure will increase as the volume decreases, the relation between them being $pv = 3,000$. Find the value of $D_v p$ for $v = 50$ cu in. In what units is this rate expressed?

22. A certain quantity Q varies with the time t in accordance with the formula $Q = 8t^2 - t^3$, $0 \le t \le 8$. Over what part of this time interval is Q increasing?

23. If r in. is the radius of a sphere, then the number of cubic inches in its volume and the number of square inches in its surface area are given, respectively, by the functions $\dfrac{4\pi r^3}{3}$ and $4\pi r^2$. Evaluate

$$\lim_{r \to r_1} \frac{4\pi \dfrac{r^3}{3} - \dfrac{4\pi {r_1}^3}{3}}{4\pi r^2 - 4\pi {r_1}^2}$$

What physical meaning can be attached to the result?