40. By differentiating with respect to t, show that the result of eliminating x from

$$3\frac{dx}{dt} + \frac{dy}{dt} + 2x = k; \;\; \frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0,$$

is

$$11\frac{d^2y}{dt^2} + 17\frac{dy}{dt} + 6y = 0.$$

Hence, solve the original equations given $x = 0$, $y = 0$ when $t = 0$.

Sol.

$$\begin{aligned}
3\frac{dx}{dt} + \frac{dy}{dt} + 2x &= k \;\cdots\; (1) \\
\frac{dx}{dt} + 4\frac{dy}{dt} + 3y &= 0 \;\cdots\; (2)
\end{aligned}$$

$$\begin{aligned}
\text{From (2)}: \frac{dx}{dt} &= -4\frac{dy}{dt} - 3y \;\cdots\; (3) \\
\frac{d^2x}{dt^2} &= -4\frac{d^2y}{dt^2} - 3\frac{dy}{dt} \;\cdots\; (4) \\
\text{From (2)}: 4\frac{dy}{dt} &= -\frac{dx}{dt} - 3y \\
\frac{dy}{dt} &= -\frac{1}{4}\frac{dx}{dt} - \frac{3}{4}y \;\cdots\; (5)
\end{aligned}$$

$$\begin{aligned}
\text{Substituting (5) into (1)}: 3\frac{dx}{dt} + \left(-\frac{1}{4}\frac{dx}{dt} - \frac{3}{4}y\right) + 2x &= k \\
\frac{11}{4}\frac{dx}{dt} - \frac{3}{4}y + 2x &= k \\
\frac{11}{4}\frac{d^2x}{dt^2} - \frac{3}{4}\frac{dy}{dt} + 2\frac{dx}{dt} &= 0 \;\cdots\; (6)
\end{aligned}$$

$$\begin{aligned}
\text{Substituting (3) and (4) into (6)}: \frac{11}{4}\left(-4\frac{d^2y}{dt^2} - 3\frac{dy}{dt}\right) - \frac{3}{4}\frac{dy}{dt} + 2\left(-4\frac{dy}{dt} - 3y\right) &= 0 \\
-44\frac{d^2y}{dt^2} - 33\frac{dy}{dt} - 3\frac{dy}{dt} - 32\frac{dy}{dt} - 24y &= 0 \\
-44\frac{d^2y}{dt^2} - 68\frac{dy}{dt} - 24y &= 0 \\
11\frac{d^2y}{dt^2} + 17\frac{dy}{dt} + 6y &= 0 \text{ (shown)} \quad \blacksquare
\end{aligned}$$

$$\text{Auxiliary equation:} \quad \begin{aligned} 11m^2 + 17m + 6 &= 0 \\ (11m+6)(m+1) &= 0 \\ m &= -1, -\frac{6}{11} \end{aligned}$$

$\therefore$ General solution is $y = Ae^{-t} + Be^{-\frac{6}{11}t}$

$$\frac{dy}{dt} = -Ae^{-t} - \frac{6}{11}Be^{-\frac{6}{11}t} \quad \cdots \ (7)$$

When $t = 0$, $y = 0$,

$$\begin{aligned} 0 &= Ae^0 + Be^0 \\ 0 &= A + B \quad \cdots \ (8) \end{aligned}$$

$$\begin{aligned} \text{Substituting (2) into (1)} : 3\left(-4\frac{dy}{dt} - 3y\right) + \frac{dy}{dt} + 2x &= k \\ -12\frac{dy}{dt} - 9y + \frac{dy}{dt} + 2x &= k \\ -11\frac{dy}{dt} - 9y + 2x &= k \quad \cdots \ (9) \end{aligned}$$

$$\begin{aligned} \text{Substituting (7) into (9)} : \ -11\left(-Ae^{-t} - \frac{6}{11}Be^{-\frac{6}{11}t}\right) - 9(Ae^{-t} + Be^{-\frac{6}{11}t}) + 2x &= k \\ 11Ae^{-t} + 6Be^{-\frac{6}{11}t} - 9Ae^{-t} - 9Be^{-\frac{6}{11}t} + 2x &= k \\ 2Ae^{-t} - 3Be^{-\frac{6}{11}t} + 2x &= k \end{aligned}$$

When $t = 0$, $x = 0$, $y = 0$,

$$2A - 3B = k \quad \cdots \ (10)$$

$$\begin{aligned} (8) \times 2 : 2A + 2B &= 0 \quad \cdots \ (11) \\ (10) + -(11) : \ -5B &= k \\ B &= -\frac{k}{5} \\ A &= \frac{k}{5} \end{aligned}$$

$\therefore \ y = \dfrac{k}{5}\left(e^{-t} - e^{-\frac{6}{11}t}\right) \quad \blacksquare$

$$
\begin{aligned}
\text{From (3)}: \frac{dx}{dt} &= -4\left(-\frac{k}{5}e^{-t} + \frac{6k}{55}e^{-\frac{6}{11}t}\right) - \frac{3k}{5}\left(e^{-t} - e^{-\frac{6}{11}t}\right) \\
\frac{dx}{dt} &= \frac{4k}{5}e^{-t} - \frac{24k}{55}e^{-\frac{6}{11}t} - \frac{3k}{5}e^{-t} + \frac{3k}{5}e^{-\frac{6}{11}t} \\
\frac{dx}{dt} &= \frac{k}{5}e^{-t} + \frac{9k}{55}e^{-\frac{6}{11}t} \quad \cdots \\
x &= -\frac{k}{5}e^{-t} - \frac{9k}{55} \cdot \frac{11}{6}e^{-\frac{6}{11}t} + C \\
x &= -\frac{k}{5}e^{-t} - \frac{3k}{10}e^{-\frac{6}{11}t} + C
\end{aligned}
$$

When $t = 0$, $x = 0$,

$$
\begin{aligned}
0 &= -\frac{k}{5} - \frac{3k}{10} + C \\
C &= \frac{k}{2}
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ x &= -\frac{k}{5}e^{-t} - \frac{3k}{10}e^{-\frac{6}{11}t} + \frac{k}{2} \\
&= \frac{k}{2}\left(1 - \frac{2}{5}e^{-t} - \frac{3}{5}e^{-\frac{6}{11}t}\right) \quad \blacksquare
\end{aligned}
$$