Praktis 3
Integration

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Praktis Formatif

3.1 Integration as the Inverse of Differentiation

1. (a) Given $\frac{d}{dx}(2x^3+5x^2-7x)=6x^2+10x-7$, find $\int 6x^2+10x-7\,dx$.

   Sol.

$$\int 6x^2+10x-7\,dx = 2x^3+5x^2-7x \quad \square$$

   (b) Given $\frac{d}{dx}(5x^4+3x^2+x)=20x^3+6x+1$, find $\int 20x^3+6x+1\,dx$.

   Sol.

$$\int 20x^3+6x+1\,dx = 5x^4+3x^2+x \quad \square$$

2. (a) Given $\frac{d}{dx}(4x-5x^2+2x^3)=4-10x+6x^2$, find $\int 2-5x+3x^2\,dx$.

   Sol.

$$\begin{aligned}\int 2-5x+3x^2\,dx &= \frac{2}{2}\int 2-5x+3x^2\,dx\\ &= \frac{1}{2}\int 4-10x+6x^2\,dx\\ &= \frac{1}{2}(4x-5x^2+2x^3)\\ &= 2x-\frac{5}{2}x^2+x^3 \quad \square\end{aligned}$$

   (b) Given $\frac{d}{dx}\left(2x-\frac{3}{x^4}\right)=2+\frac{12}{x^5}$, find $\int 6+\frac{36}{x^5}\,dx$

   Sol.

$$\begin{aligned}\int 6+\frac{36}{x^5}\,dx &= 6\int 1+\frac{6}{x^5}\,dx\\ &= 3\int 2+\frac{12}{x^5}\,dx\\ 3\left(2x-\frac{3}{x^4}\right)\\ &= 6x-\frac{9}{x^4} \quad \square\end{aligned}$$

   (c) Given $f(x)=\frac{d}{dx}[g(x)]$, find $\int 2f(x)\,dx$.

   Sol.

$$\begin{aligned}\int 2f(x)\,dx &= 2\int f(x)\,dx\\ &= 2g(x) \quad \square\end{aligned}$$

   (d) Differentiate $\frac{2x^2}{3x-1}$ with respect to x and hence, find $\int \frac{6x(3x-2)}{(3x-1)^2}\,dx$.

   Sol.

$$\begin{aligned}\frac{d}{dx}\left(\frac{2x^2}{3x-1}\right) &= \frac{4x(3x-1)-3(2x^2)}{(3x-1)^2}\\ &= \frac{12x^2-4x-6x^2}{(3x-1)^2}\\ &= \frac{6x^2-4x}{(3x-1)^2}\\ &= \frac{2x(3x-2)}{(3x-1)^2}\end{aligned}$$

$$\begin{aligned}\int \frac{6x(3x-2)}{(3x-1)^2}\,dx &= 3\int \frac{2x(3x-2)}{(3x-1)^2}\,dx\\ &= 3\left(\frac{2x^2}{3x-1}\right)\\ &= \frac{6x^2}{3x-1} \quad \square\end{aligned}$$

3. The daily production of bread of a bakery shop is given by the function $R(x)=-50(x^2-12x)$, where x represents the number of bakers who work in the shop with condition x is not more than 6.

   (a) Find the rate of daily production of bread in terms of x.

   Sol.

$$R'(x) = -100x+600 \quad \square$$

   (b) If the rate of daily production of bread becomes $300-50x$ on a particular day, calculate the revenue of the bakery shop if all the loaves of bread baked by three bakers on that day are sold out at a price of RM5.50 for each loaf.

   Sol.

$$\begin{aligned}\int 300-50x\,dx &= \frac{1}{2}\int (600-100x)\,dx\\ &= \frac{1}{2}(-50x^2+600x)\\ &= -25x^2+300x\\ R(3) &= -25(3)^2+300(3)\\ &= -225+900\\ &= 675\end{aligned}$$

$$\begin{aligned}\text{Revenue} &= 675\times 5.50\\ &= \text{RM}3712.50 \quad \square\end{aligned}$$

4. Given $f(x)=x^4-2x^3$ and $f'(x)=4x^3-6x^2$. Express $f'(x)\int f'(x)\,dx$ in factored form.

   Sol.

$$\begin{aligned}f'(x)\int f'(x)\,dx &= (4x^3-6x^2)(x^4-2x^3)\\ &= 2x^5(2x-3)(x-2) \quad \square\end{aligned}$$

5. Given $y = \dfrac{2x-6}{x}$.

   (a) Find $\dfrac{dy}{dx}$.
   Sol.
   $$\begin{aligned}\frac{dy}{dx} &= \frac{2x-2x-6}{x^2}\\ &= -\frac{6}{x^2} \quad \square\end{aligned}$$

   (b) Solve $4 + \displaystyle\int \left(\frac{dy}{dx}\right)\, dx = 0$.
   Sol.
   $$\begin{aligned}4 + \int \left(\frac{dy}{dx}\right)\, dx &= 0\\ 4 + \int \left(-\frac{6}{x^2}\right)\, dx &= 0\\ 4 + \frac{2x-6}{x} &= 0\\ 4x + 2x - 6 &= 0\\ 6x &= 6\\ x &= 1 \quad \square\end{aligned}$$

6. Given $f'(x) = g(x)$. Find $\dfrac{3f(x)}{\displaystyle\int g(x)dx}$.

   Sol.
   $$\begin{aligned}f'(x) &= g(x)\\ f(x) &= \int g(x)dx\\ \frac{3f(x)}{\displaystyle\int g(x)dx} &= \frac{3f(x)}{f(x)}\\ &= 3 \quad \square\end{aligned}$$

7. The population of town A is given by a function $P(t) = \frac{5}{6}(2.72^{1.2t}) - t^2 + 1495$ and the population continues to increase at the rate of $2.72^{1.2t} - 2t$ people per year where t is the number of years. Given that the population of town b increases at twice the rate of the population of town A based on the same model, find, to the nearest integer,

   (a) the rate of increase of the population of town B at $t = 5$ years.
   Sol.
   $$\begin{aligned}P_B'(5) &= 2[2.72^{1.2(5)} - 2(5)]\\ &= 2[404.96 - 10]\\ &= 2(394.96)\\ &= 789.92\\ &= 790 \text{ people per year} \quad \square\end{aligned}$$

   (b) the population of town B after 5 years.
   Sol.
   $$\begin{aligned}P_B(5) &= 2\left[\frac{5}{6}(2.72^{1.2\cdot 5}) - (5)^2 + 1495\right]\\ &= \frac{5}{3}(2.72^6) - 50 + 2990\\ &= 3614.93\\ &= 3615 \text{ people} \quad \square\end{aligned}$$

3.2 Indefinite Integral

8. By using the indefinite integral formula, find the integral of each of the following constants or algebraic functions.

   (a) $\displaystyle\int 3\, dx$
   Sol.
   $$\int 3\, dx = 3x + C \quad \square$$

   (b) $\displaystyle\int 24x\, dx$
   Sol.
   $$\int 24x\, dx = 12x^2 + C \quad \square$$

   (c) $\displaystyle\int 6x^2\, dx$
   Sol.
   $$\int 6x^2\, dx = 2x^3 + C \quad \square$$

   (d) $\displaystyle\int 3x^2 + 4x\, dx$
   Sol.
   $$\int 3x^2 + 4x\, dx = x^3 + 2x^2 + C \quad \square$$

   (e) $\displaystyle\int \frac{2}{x^4}\, dx$
   Sol.
   $$\int \frac{2}{x^4}\, dx = -\frac{2}{x^3} + C \quad \square$$

   (f) $\displaystyle\int x^2(x-3)\, dx$
   Sol.
   $$\begin{aligned}\int x^2(x-3)\, dx &= \int x^3 - 3x^2\, dx\\ &= \frac{1}{4}x^4 - x^3 + C \quad \square\end{aligned}$$

   (g) $\displaystyle\int (x+2)(2x^4-1)\, dx$
   Sol.
   $$\begin{aligned}&\int (x+2)(2x^4-1)\, dx\\ &= \int 2x^5 - x + 4x^4 - 2\\ &= \frac{1}{3}x^6 + \frac{4}{5}x^5 - \frac{1}{2}x^2 - 2x + C \quad \square\end{aligned}$$

(h) $\int \frac{x^2+3x+2}{x+2}\,dx$

Sol.

$$\begin{aligned}\int \frac{x^2+3x+2}{x+2}\,dx &= \int \frac{(x+2)(x+1)}{x+2}\,dx\\ &= \int x+1\,dx\\ &= \frac{1}{2}x^2+x+C \quad \square\end{aligned}$$

9. Find the indefinite integral for each of the following by using

(a) the substitution method.

(b) the indefinite integral formula.

i. $\int \frac{2}{(x+2)^5}\,dx$

Sol.

(a) Let $v=(x+2)$.

$$\begin{aligned}\int \frac{2}{(x+2)^5}\,dx &= \int \frac{2}{v^5}\,dv\\ &= \int 2v^{-5}\,dv\\ &= -\frac{1}{2}v^{-4}+C\\ &= -\frac{1}{2v^4}+C\\ &= -\frac{1}{2(x+2)^4}+C \quad \square\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{2}{(x+2)^5}\,dx &= \int 2(x+2)^{-5}\,dx\\ &= 2\int (x+2)^{-5}\,dx\\ &= 2\left[\frac{(x+2)^{-4}}{-4}\right]+C\\ &= -\frac{1}{2(x+2)^4}+C \quad \square\end{aligned}$$

ii. $\int \frac{3}{5}(3x+2)^8\,dx$

Sol.

(a) Let $v=3x+2$, $\frac{dv}{dx}=3$.

$$\begin{aligned}\int \frac{3}{5}(3x+2)^8\,dx &= \int \frac{3}{5}v^8\,dv\\ &= \int \frac{3}{5}v^8\frac{dv}{3}\\ &= \int \frac{1}{5}v^8\,dv\\ &= \frac{1}{45}v^9+C\\ &= \frac{(3x+2)^9}{45}+C \quad \square\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{3}{5}(3x+2)^8\,dx &= \frac{3}{5}\int (3x+2)^8\,dx\\ &= \frac{3}{5}\left[\frac{(3x+2)^9}{27}\right]+C\\ &= \frac{(3x+2)^9}{45}+C \quad \square\end{aligned}$$

10. Determine the equation of a curve based on the following information.

(a) The gradient function of the curve is $\frac{dy}{dx}=3x^2+x-2$ and it passes through the point $p(2,15)$.

Sol.

$$\begin{aligned}\frac{dy}{dx} &= 3x^2+x-2\\ y &= \int 3x^2+x-2\,dx\\ &= x^3+\frac{x^2}{2}-2x+C\end{aligned}$$

When $x=2$, $y=15$,

$$\begin{aligned}15 &= 2^3+\frac{2^2}{2}-2(2)+C\\ 15 &= 8+2-4+C\\ 15 &= 6+C\\ C &= 9\end{aligned}$$

Hence, the equation of the curve is $y=x^3+\frac{x^2}{2}-2x+9$. $\square$

(b) The gradient function of the curve is $f'(x)=2x+9$ and $f(3)=21$.

Sol.

$$\begin{aligned}f'(x) &= 2x+9\\ f(x) &= \int 2x+9\,dx\\ &= x^2+9x+C\\ f(3) &= 3^2+9(3)+C\\ 21 &= 9+27+C\\ C &= -15\end{aligned}$$

Hence, the equation of the curve is $f(x)=x^2+9x-15$. $\square$

(c) The gradient function of the curve is given by $g(t)=\frac{5t^2-6t+1}{t^3(t-1)}$ and it passes through the point $(1,3)$.

Sol.

$$\begin{aligned}
g(t) &= \frac{5t^2-6t+1}{t^3(t-1)} \\
&= \frac{(5t-1)(t-1)}{t^3(t-1)} \\
&= \frac{5t-1}{t^3} \\
&= \frac{5}{t^2}-\frac{1}{t^3} \\
&= 5t^{-2}-t^{-3} \\
f(t) &= \int 5t^{-2}-t^{-3}\,dt \\
&= -\frac{5}{t}+\frac{1}{2t^2}+C
\end{aligned}$$

When $t=1$, $f(1)=3$,

$$\begin{aligned}
3 &= -5+\frac{1}{2}+C \\
3 &= -\frac{9}{2}+C \\
C &= \frac{15}{2}
\end{aligned}$$

Hence, the equation of the curve is $f(t)=-\frac{5}{t}+\frac{1}{2t^2}+\frac{15}{2}$. ▯

11. Tommy moves in his roller skates at the rate of change in displacement, $\frac{ds}{dt}=t^2+9$ metres per second, where t is the time in seconds. At $t=3$ seconds, Tommy is 4 metres away from his starting place. Find the displacement, s metres, when $t=10$ seconds.

Sol.

$$\begin{aligned}
\frac{ds}{dt} &= t^2+9 \\
s &= \int t^2+9\,dt \\
&= \frac{t^3}{3}+9t+C
\end{aligned}$$

When $t=3$, $s=4$,

$$\begin{aligned}
4 &= \frac{3^3}{3}+9(3)+C \\
4 &= 9+27+C \\
4 &= 54+C \\
C &= -32 \\
s &= \frac{t^3}{3}+9t-32
\end{aligned}$$

When $t=10$,

$$\begin{aligned}
s &= \frac{10^3}{3}+9(10)-32 \\
&= 333+90-32 \\
&= 391\frac{1}{3}m
\end{aligned}$$
▯

12. Given the gradient function of a curve is $\frac{dy}{dx}=kx^2+2x$ where k is a constant. The curve passes through point $A(1,6)$ and point $B(-2,0)$. Determine the equation of the curve.

Sol.

$$\begin{aligned}
\frac{dy}{dx} &= kx^2+2x \\
y &= \int kx^2+2x\,dx \\
&= \frac{kx^3}{3}+x^2+C
\end{aligned}$$

When $x=1$, $y=6$,

$$\begin{aligned}
6 &= \frac{k}{3}+1+C \\
k+3C &= 15 \quad (1)
\end{aligned}$$

When $x=-2$, $y=0$,

$$\begin{aligned}
0 &= -\frac{8k}{3}+4+C \\
8k-3C &= 12 \quad (2)
\end{aligned}$$

$$\begin{aligned}
(2)+(1)\cdot 8:\ 9k &= 27 \\
k &= 3 \\
C &= 4
\end{aligned}$$

Hence, the equation of the curve is $y=x^3+x^2+4$ ▯

3.3 Definite Integral

13. Calculate each of the following.

(a) $\int_2^1 \left(\sqrt{x}+\frac{1}{x}\right)$

Sol.

$$\begin{aligned}
\int_2^1 \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right) &= \int_1^2 \left(x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right) \\
&= \left[\frac{2\sqrt{x^3}}{3}+2\sqrt{x}\right]_1^2 \\
&= \left[\frac{4\sqrt{2}}{3}+2\sqrt{2}\right]-\left[\frac{2}{3}+2\right] \\
&= \frac{10\sqrt{2}}{3}-\frac{8}{3} \\
&= \frac{10\sqrt{2}-8}{3} \\
&\approx 2.0474
\end{aligned}$$
▯

(b) $\int_0^3 \left(\frac{x^4+3x}{x}\right)dx$

Sol.

$$\begin{aligned}\int_0^3 \left(\frac{x^4+3x}{x}\right)\,dx &= \int_0^3 \left(x^3+3\right)\,dx \\ &= \left[\frac{1}{4}x^4+3x\right]_0^3 \\ &= \left[\frac{1}{4}(3^4)+3\cdot 3\right]-0 \\ &= \frac{81}{4}+9 \\ &= \frac{117}{4} \\ &= 29.25 \quad \square\end{aligned}$$

(c) $\displaystyle\int_{-2}^{-1} \left(\frac{(4-x)(3-x)}{x^5}\right)\,dx$

Sol.

$$\begin{aligned}&\int_{-2}^{-1} \left(\frac{(4-x)(3-x)}{x^5}\right)\,dx \\ &= \int_{-2}^{-1} \left(\frac{x^2-7x+12}{x^5}\right)\,dx \\ &= \int_{-2}^{-1} \left(\frac{1}{x^3}-\frac{7}{x^4}+\frac{12}{x^5}\right)\,dx \\ &= \left[-\frac{1}{2x^2}+\frac{7}{3x^3}-\frac{3}{x^4}\right]_{-2}^{-1} \\ &= \left[-\frac{1}{2}-\frac{7}{3}-3\right]-\left[-\frac{1}{8}-\frac{7}{24}-\frac{3}{16}\right] \\ &= -\frac{35}{6}+\frac{29}{48} \\ &= -5\frac{11}{48} \quad \square\end{aligned}$$

14. Given $\displaystyle\int_a^b f(x)\,dx = 5$, $\displaystyle\int_b^c f(x)\,dx = 8$ and $\displaystyle\int_a^b g(x)\,dx = 2$. Find each of the following.
[answer can be in terms of a and/or b.]

(a) $\displaystyle\int_a^b 3f(x)\,dx$

Sol.

$$\begin{aligned}\int_a^b 3f(x)\,dx &= 3\int_a^b f(x)\,dx \\ &= 3\cdot 5 \\ &= 15 \quad \square\end{aligned}$$

(b) $\displaystyle\int_a^c f(x)\,dx$

Sol.

$$\begin{aligned}\int_a^c f(x)\,dx &= \int_a^b f(x)\,dx+\int_b^c f(x)\,dx \\ &= 5+8 \\ &= 13 \quad \square\end{aligned}$$

(c) $\displaystyle\int_a^b [f(x)+g(x)]\,dx$

Sol.

$$\begin{aligned}\int_a^b [f(x)+g(x)]\,dx &= \int_a^b f(x)\,dx-\int_a^b g(x)\,dx \\ &= 5-2 \\ &= 3 \quad \square\end{aligned}$$

(d) $\displaystyle\int_c^a f(x)\,dx$

Sol.

$$\begin{aligned}\int_c^a f(x)\,dx &= -\int_a^c f(x)\,dx \\ &= -13 \quad \square\end{aligned}$$

(e) $\displaystyle\int_a^b [g(x)+3]\,dx$

Sol.

$$\begin{aligned}\int_a^b [g(x)+3]\,dx &= \int_a^b g(x)\,dx+\int_a^b 3\,dx \\ &= 2+3(b-a) \\ &= 3b-3a+2 \quad \square\end{aligned}$$

(f) $\displaystyle\int_a^a f(x)\,dx$

Sol.

$$\int_a^a f(x)\,dx = 0 \quad \square$$

(g) The value of k such that $\displaystyle\int_b^a [f(x)+kx]\,dx = 25$ if $a=1$ and $b=4$.

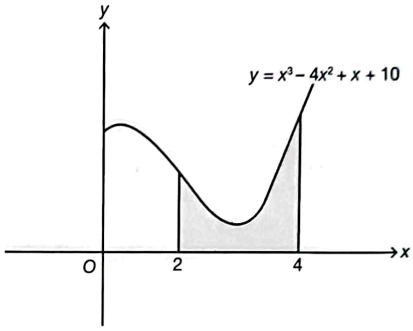
Sol.

$$\begin{aligned}\int_b^a [f(x)+kx]\,dx &= \int_b^a f(x)\,dx+\int_b^a kx\,dx \\ &= -5+\int_b^a kx\,dx\end{aligned}$$

$$\begin{aligned}-5+\int_1^4 kx\,dx &= 25 \\ \int_4^1 kx\,dx &= 30 \\ k\left[\frac{x^2}{2}\right]_4^1 &= 30 \\ k\left(\frac{1}{2}-\frac{16}{2}\right) &= 30 \\ -\frac{15k}{2} &= 30 \\ -15k &= 60 \\ k &= -4 \quad \square\end{aligned}$$

15. Find the area of the shaded region for each of the following diagrams.

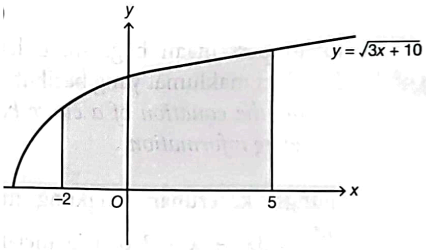
(a)



Sol.

$$
\begin{aligned}
A &= \int_2^4 (x^3 - 4x^2 + x + 10)\,dx \\
&= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{1}{2}x^2 + 10x\right]_2^4 \\
&= \left(64 - \frac{256}{3} + 8 + 40\right) - \left(4 - \frac{32}{3} + 2 + 20\right) \\
&= \frac{80}{3} - \frac{46}{3} \\
&= 11\frac{1}{3}\text{ units}^2 \quad \square
\end{aligned}
$$

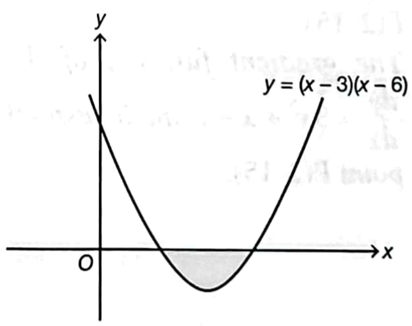
(b)



Sol.

$$
\begin{aligned}
A &= \int_{-2}^{5} \sqrt{3x+10}\,dx \\
&= \int_{-2}^{5} (3x+10)^{\frac{1}{2}}\,dx \\
&= \left[\frac{2(3x+10)^{\frac{3}{2}}}{9}\right]_{-2}^{5} \\
&= \frac{2(25)^{\frac{3}{2}}}{9} - \frac{2(4)^{\frac{3}{2}}}{9} \\
&= \frac{250}{9} - \frac{16}{9} \\
&= \frac{234}{9} \\
&= 26\text{ units}^2 \quad \square
\end{aligned}
$$

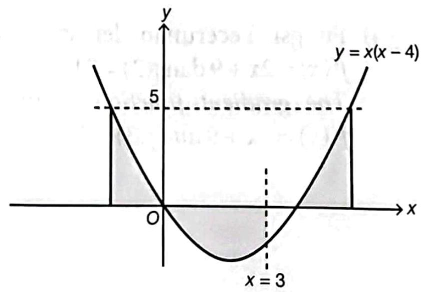
(c)



Sol.

$$
\begin{aligned}
A &= \left|\int_3^6 (x-3)(x-6)\,dx\right| \\
&= \left|\int_3^6 (x^2 - 9x + 18)\,dx\right| \\
&= \left|\left[\frac{1}{3}x^3 - \frac{9}{2}x^2 + 18x\right]_3^6\right| \\
&= \left|(72 - 162 + 108) - \left(9 - \frac{81}{2} + 54\right)\right| \\
&= \left|18 - \frac{45}{2}\right| \\
&= 4.5\text{ units}^2 \quad \square
\end{aligned}
$$

(d)



Sol.

When $y = 5$,

$$
\begin{aligned}
x(x-4) &= 5 \\
x^2 - 4x - 5 &= 0 \\
(x-5)(x+1) &= 0 \\
x &= -1 \text{ or } x = 5
\end{aligned}
$$

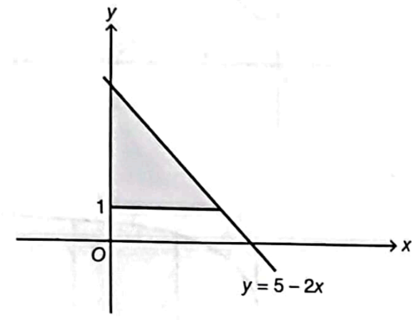
When $y = 0$,

$$
\begin{aligned}
x(x-4) &= 0 \\
x &= 0 \text{ or } x = 4
\end{aligned}
$$

$$
\begin{aligned}
A &= \int_{-1}^{0} x(x-4)\,dx + \left|\int_0^3 x(x-4)\,dx\right| \\
&\quad + \int_4^5 x(x-4)\,dx \\
&= \int_{-1}^{0} (x^2-4x)\,dx + \left|\int_0^3 (x^2-4x)\,dx\right| \\
&\quad + \int_4^5 (x^2-4x)\,dx \\
&= \left[\frac{1}{3}x^3 - 2x^2\right]_{-1}^{0} + \left|\left[\frac{1}{3}x^3 - 2x^2\right]_0^3\right| \\
&\quad + \left[\frac{1}{3}x^3 - 2x^2\right]_4^5 \\
&= 0 - \left(-\frac{1}{3} - 2\right) + |(9-18) - 0| \\
&\quad + \left(\frac{125}{3} - 50\right) - \left(\frac{64}{3} - 32\right) \\
&= 13\frac{2}{3}\text{ units}^2 \quad \square
\end{aligned}
$$

16. Determine the area bounded by the curve, the horizontal line(s) and the y-axis.

(a)



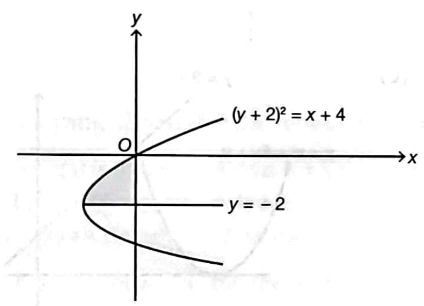
Sol. When $x = 0$,

$$\begin{aligned} y &= 5 - 2x \\ y &= 5 - 2(0) \\ &= 5 \end{aligned}$$

$$\begin{aligned} y &= 5 - 2x \\ 2x &= 5 - y \\ x &= \frac{5-y}{2} \end{aligned}$$

$$\begin{aligned} A &= \int_1^5 \frac{5-y}{2}\,dy \\ &= \int_1^5 \left(\frac{5}{2} - \frac{1}{2}y\right)\,dy \\ &= \left[\frac{5}{2}y - \frac{1}{4}y^2\right]_1^5 \\ &= \left(\frac{25}{2} - \frac{25}{4}\right) - \left(\frac{5}{2} - \frac{1}{4}\right) \\ &= \frac{25}{4} - \frac{9}{4} \\ &= 4 \text{ units}^2 \quad \square \end{aligned}$$

(b)

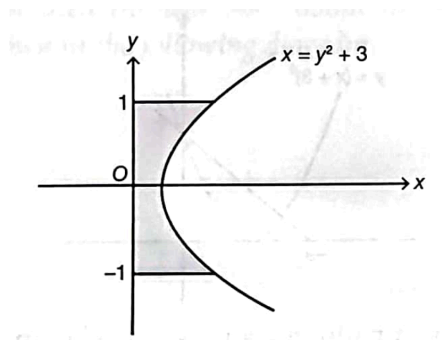


Sol.

$$\begin{aligned} (y+2)^2 &= x + 4 \\ x &= (y+2)^2 - 4 \\ &= y^2 + 4y + 4 - 4 \\ &= y^2 + 4y \end{aligned}$$

$$\begin{aligned} A &= \int_{-2}^{0} (y^2 + 4y)\,dy \\ &= \left[\frac{1}{3}y^3 + 2y^2\right]_{-2}^{0} \\ &= 0 - \left(-\frac{8}{3} + 8\right) \\ &= 5\frac{1}{3} \text{ units}^2 \quad \square \end{aligned}$$

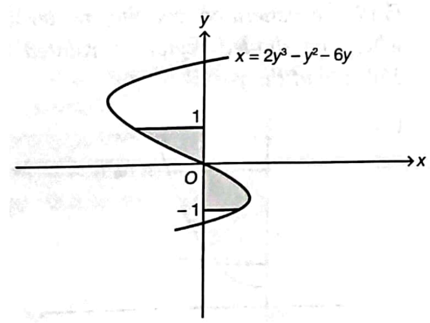
(c)



Sol.

$$\begin{aligned} A &= \int_{-1}^{1} (y^2 + 3)\,dy \\ &= \left[\frac{1}{3}y^3 + 3y\right]_{-1}^{1} \\ &= \left(\frac{1}{3} + 3\right) - \left(-\frac{1}{3} - 3\right) \\ &= 6\frac{2}{3} \text{ units}^2 \quad \square \end{aligned}$$

(d)

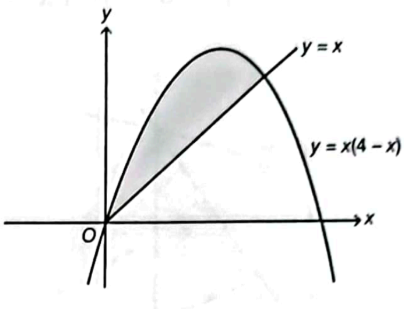


Sol.

$$\begin{aligned} A &= \int_{-1}^{0} (2y^3 - y^2 - 6y)\,dy + \left|\int_0^1 (2y^3 - y^2 - 6y)\,dy\right| \\ &= \left[\frac{1}{2}y^4 - \frac{1}{3}y^3 - 3y^2\right]_{-1}^{0} + \left|\left[\frac{1}{2}y^4 - \frac{1}{3}y^3 - 3y^2\right]_0^1\right| \\ &= 0 - \left(\frac{1}{2} + \frac{1}{3} - 3\right) + \left|\left(\frac{1}{2} - \frac{1}{3} - 3\right) - 0\right| \\ &= \frac{13}{6} + \frac{17}{6} \\ &= 5 \text{ units}^2 \quad \square \end{aligned}$$

17. Find the area of the shaded region for each of the following.

(a)

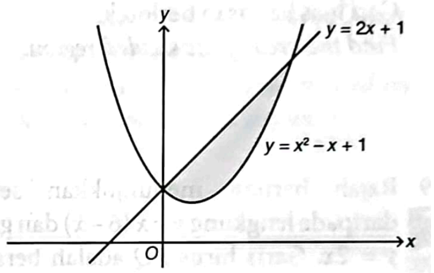


Sol.

$$\begin{aligned} x &= x(4-x) \\ x &= 4x - x^2 \\ x^2 - 3x &= 0 \\ x(x-3) &= 0 \\ x &= 0 \text{ or } x = 3 \end{aligned}$$

$$\begin{aligned} A &= \int_0^3 [x(4-x) - x]\, dx \\ &= \int_0^3 \left[4x - x^2 - x\right]\, dx \\ &= \int_0^3 \left[3x - x^2\right]\, dx \\ &= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3 \\ &= \left(\frac{27}{2} - 9\right) - 0 \\ &= 4.5 \text{ units}^2 \quad \square \end{aligned}$$

(b)

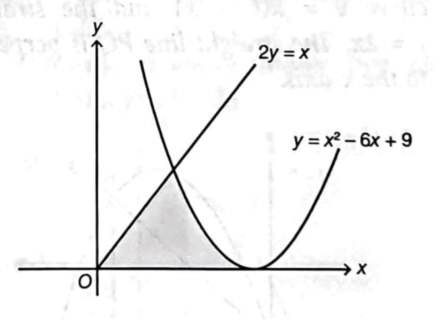


Sol.

$$\begin{aligned} 2x + 1 &= x^2 - x + 1 \\ x^2 - 3x &= 0 \\ x(x-3) &= 0 \\ x &= 0 \text{ or } x = 3 \end{aligned}$$

$$\begin{aligned} A &= \int_0^3 \left[2x + 1 - x^2 + x - 1\right]\, dx \\ &= \int_0^3 \left[-x^2 + 3x\right]\, dx \\ &= \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2\right]_0^3 \\ &= \left(-9 + \frac{27}{2}\right) - 0 \\ &= 4.5 \text{ units}^2 \quad \square \end{aligned}$$

(c)

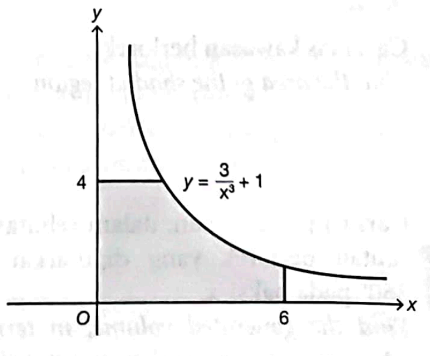


Sol.

$$\begin{aligned} 2y &= x \\ y &= \frac{1}{2}x \\ \frac{1}{2}x &= x^2 - 6x + 9 \\ x &= 2x^2 - 12x + 18 \\ 2x^2 - 13x + 18 &= 0 \\ (2x-9)(x-2) &= 0 \\ x &= 2 \text{ or } x = 4.5 \\ x^2 - 6x + 9 &= 0 \\ (x-3)^2 &= 0 \\ x &= 3 \end{aligned}$$

$$\begin{aligned} A &= \int_0^2 \frac{x}{2}\, dx + \int_2^3 (x^2 - 6x + 9)\, dx \\ &= \left[\frac{1}{4}x^2\right]_0^2 + \left[\frac{1}{3}x^3 - 3x^2 + 9x\right]_2^3 \\ &= 1 - 0 + (9 - 27 + 27) - \left(\frac{8}{3} - 12 + 18\right) \\ &= 10 - \frac{26}{3} \\ &= 1\frac{1}{3} \text{ units}^2 \quad \square \end{aligned}$$

(d)

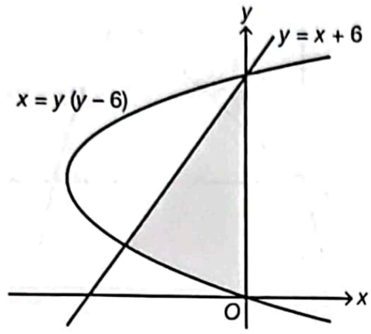


Sol.

$$\begin{aligned} \frac{3}{x^3} + 1 &= 4 \\ \frac{3}{x^3} &= 3 \\ x^3 &= 1 \\ x &= 1 \end{aligned}$$

$$
\begin{aligned}
A &= 1 \cdot 4 + \int_1^6 \left(\frac{3}{x^3} + 1\right) dx \\
&= 4 + \left[-\frac{3}{2x^2} + x\right]_1^6 \\
&= 4 + \left(-\frac{1}{24} + 6\right) - \left(-\frac{3}{2} + 6\right) \\
&= 4 + \frac{143}{24} + \frac{9}{2} \\
&= 14\frac{11}{24} \text{ units}^2 \quad \square
\end{aligned}
$$

18. The following diagram shows a part of the curve $x = y(y-6)$ and the straight line $y = x + 6$.



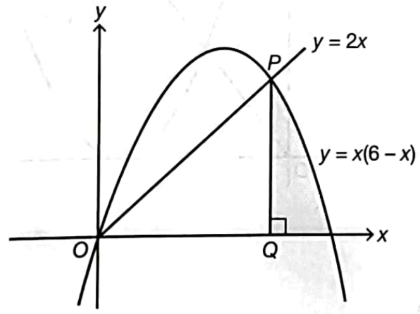
Find the area of the shaded region.

Sol.

$$
\begin{aligned}
y &= x + 6 \\
x &= y - 6 \\
y - 6 &= y(y-6) \\
&= y^2 - 6y \\
y^2 - 7y + 6 &= 0 \\
(y-6)(y-1) &= 0 \\
y &= 6 \text{ or } y = 1 \\
1 &= x + 6 \\
x &= -5
\end{aligned}
$$

$$
\begin{aligned}
A &= \left|\int_0^1 y(y-6)\, dy\right| + \frac{1}{2}(5)(5) \\
&= \left|\int_0^1 (y^2 - 6y)\, dy\right| + \frac{25}{2} \\
&= \left|\left[\frac{1}{3}y^3 - 3y^2\right]_0^1\right| + \frac{25}{2} \\
&= \left|\frac{1}{3} - 3 - 0\right| + \frac{25}{2} \\
&= \frac{8}{3} + \frac{25}{2} \\
&= 15\frac{1}{6}\ \textit{units}^2 \quad \square
\end{aligned}
$$

19. The following diagram shows a part of the curve $y = x(6-x)$ and a straight line $y = 2x$. The straight line PQ is perpendicular to the x-axis.



Find the area of the shaded region.

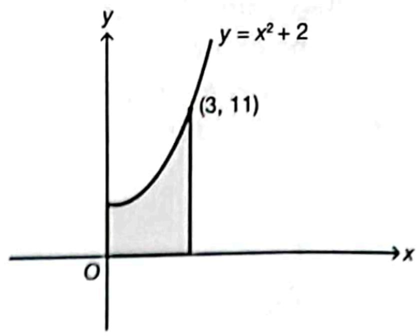
Sol.

$$
\begin{aligned}
x(6-x) &= 0 \\
x &= 0 \text{ or } x = 6 \\
2x &= x(6-x) \\
2x &= 6x - x^2 \\
x^2 - 4x &= 0 \\
x(x-4) &= 0 \\
x &= 0 \text{ or } x = 4
\end{aligned}
$$

$$
\begin{aligned}
A &= \int_4^6 x(6-x)\, dx \\
&= \int_4^6 (6x - x^2)\, dx \\
&= \left[3x^2 - \frac{1}{3}x^3\right]_4^6 \\
&= (108 - 72) - \left(48 - \frac{64}{3}\right) \\
&= 36 - \frac{80}{3} \\
&= 9\frac{1}{3}\ \textit{units}^2 \quad \square
\end{aligned}
$$

20. Find the generated volume, in terms of π, when the shaded region is rotated through $360°$ about the x-axis.

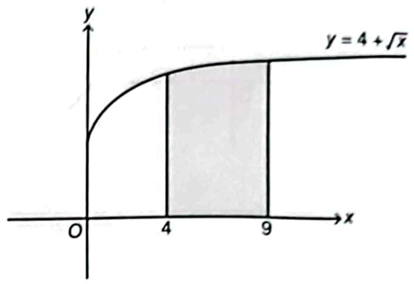
(a)



Sol.

$$
\begin{aligned}
V_x &= \int_0^3 \pi y^2 \, dx \\
&= \pi \int_0^3 (x^2+2)^2 \, dx \\
&= \pi \int_0^3 (x^4 + 4x^2 + 4) \, dx \\
&= \pi \left[\frac{1}{5}x^5 + \frac{4}{3}x^3 + 4x \right]_0^3 \\
&= \left[\left(\frac{243}{5} + 36 + 12 \right) - 0 \right] \pi \\
&= 96.6\pi \text{ units}^3 \quad \square
\end{aligned}
$$

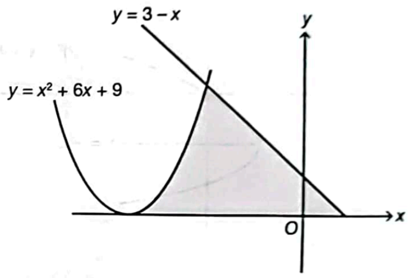
(b)



Sol.

$$
\begin{aligned}
V_x &= \int_4^9 \pi (4+\sqrt{x})^2 \, dx \\
&= \pi \int_4^9 (16 + 8\sqrt{x} + x) \, dx \\
&= \pi \left[16x + \frac{16x^{\frac{3}{2}}}{3} + \frac{1}{2}x^2 \right]_4^9 \\
&= \pi \left[\left(144 + 144 + \frac{81}{2} \right) - \left(64 + \frac{128}{3} + 8 \right) \right] \\
&= 213.83\pi \text{ units}^3 \quad \square
\end{aligned}
$$

(c)

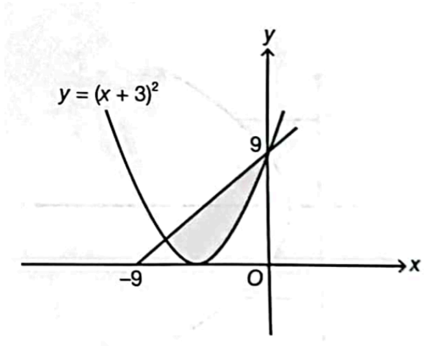


Sol.

$$
\begin{aligned}
x^2 + 6x + 9 &= 3 - x \\
x^2 + 7x + 6 &= 0 \\
(x+6)(x+1) &= 0 \\
x &= -6 \text{ or } x = -1 \\
x^2 + 6x + 9 &= 0 \\
(x+3)^2 &= 0 \\
x &= -3 \\
0 &= 3 - x \\
x &= 3
\end{aligned}
$$

$$
\begin{aligned}
V_x &= \int_{-3}^{-1} \pi (x+3)^4 \, dx + \int_{-1}^{3} \pi (3-x)^2 \, dx \\
&= \pi \int_{-3}^{-1} (x+3)^4 \, dx + \pi \int_{-1}^{3} (3-x)^2 \, dx \\
&= \pi \left\{ \left[\frac{(x+3)^5}{5} \right]_{-3}^{-1} + \left[-\frac{(3-x)^3}{3} \right]_{-1}^{3} \right\} \\
&= \pi \left[\left(\frac{32}{5} - 0 \right) + \left(0 + \frac{64}{3} \right) \right] \\
&= 27\frac{11}{15}\pi \text{ units}^3 \quad \square
\end{aligned}
$$

(d)



Sol.

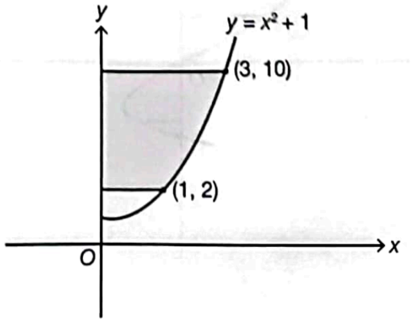
Let the line be l. Since l passes through $(-9, 0)$ and $(0, 9)$,

$$
\begin{aligned}
m_l &= \frac{9}{9} = 1 \\
y - 9 &= 1(x - 0) \\
y &= x + 9 \\
x + 9 &= (x+3)^2 \\
&= x^2 + 6x + 9 \\
x^2 + 5x &= 0 \\
x(x+5) &= 0 \\
x &= 0 \text{ or } x = -5
\end{aligned}
$$

$$
\begin{aligned}
V_x &= \int_{-5}^{0} \pi (x+9)^2 \, dx - \int_{-5}^{0} \pi (x+3)^4 \, dx \\
&= \pi \int_{-5}^{0} (x+9)^2 \, dx - \pi \int_{-5}^{0} (x+3)^4 \, dx \\
&= \pi \left\{ \left[\frac{(x+9)^3}{3} \right]_{-5}^{0} - \left[\frac{(x+3)^5}{5} \right]_{-5}^{0} \right\} \\
&= \pi \left[\left(243 - \frac{64}{3} \right) - \left(\frac{243}{5} + \frac{32}{5} \right) \right] \\
&= 116\frac{2}{3}\pi \text{ units}^3 \quad \square
\end{aligned}
$$

21. Find the generated volume, in terms of π, when the shaded region is rotated through 360° about the y-axis.

(a)

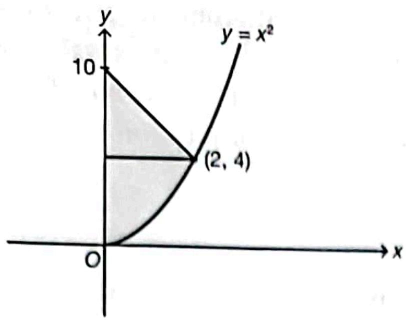


Sol.

$$\begin{aligned} y &= x^2+1 \\ x^2 &= y-1 \\ x-\sqrt{y-1} &= 0 \end{aligned}$$

$$\begin{aligned} V_y &= \int_2^{10} \pi(\sqrt{y-1})^2\,dy \\ &= \pi\int_2^{10} y-1\,dy \\ &= \pi\left[\frac{y^2}{2}-y\right]_2^{10} \\ &= \pi\left[(50-10)-(2-2)\right] \\ &= 40\pi\ \textit{units}^3 \quad \square \end{aligned}$$

(b)

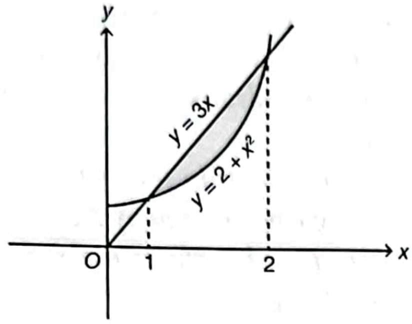


Sol.

$$\begin{aligned} y &= x^2 \\ x &= \sqrt{y} \end{aligned}$$

$$\begin{aligned} V_y &= \int_0^4 \pi(\sqrt{y})^2\,dy + \frac{1}{3}\pi\cdot 4\cdot 6 \\ &= \pi\int_0^4 y\,dy + 8\pi \\ &= \pi\left[\frac{y^2}{2}\right]_0^4 + 8\pi \\ &= 8\pi + 8\pi \\ &= 16\pi\ \textit{units}^3 \quad \square \end{aligned}$$

(c)

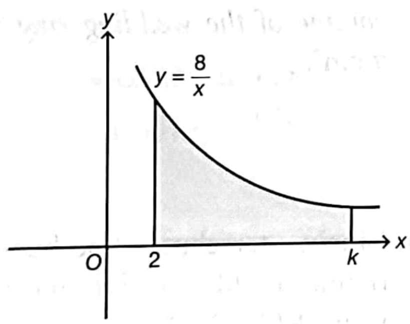


Sol.

$$\begin{aligned} y &= 3x \\ x &= \frac{y}{3} \\ y &= 2+x^2 \\ x^2 &= y-2 \\ x &= \sqrt{y-2} \\ x = 1,\ y &= 3(1) = 3 \\ x = 2,\ y &= 3(2) = 6 \end{aligned}$$

$$\begin{aligned} V_y &= \int_3^6 \pi\left(\sqrt{y-2}\right)^2 dy - \int_3^6 \pi\left(\frac{y}{3}\right)^2 dy \\ &= \pi\int_3^6 (y-2)\,dy - \pi\int_3^6 \frac{y^2}{9}\,dy \\ &= \pi\left[\frac{y^2}{2}-2y\right]_3^6 - \pi\left[\frac{y^3}{27}\right]_3^6 \\ &= \pi\left\{\left[(18-12)-\left(\frac{9}{2}-6\right)\right]-(8-1)\right\} \\ &= \pi\left(\frac{15}{2}-7\right) \\ &= \frac{1}{2}\pi\ \textit{units}^3 \quad \square \end{aligned}$$

22. The region bounded by the curve $y = \dfrac{8}{x}$, the x-axis, and the straight line $x = 2$ and $x = k$ is rotated through $360°$ about the x-axis as shown in the following diagram.



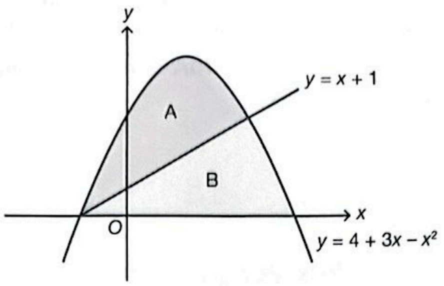
Express the volume generated by the region in terms of k. If the value of k becomes extremely large, deduce the nearest value of volume.

$$
\begin{aligned}
V_x &= \int_2^k \pi\left(\frac{8}{x}\right)^2 dx \\
&= \pi \int_2^k \frac{64}{x^2}\, dx \\
&= \pi\left[-\frac{64}{x}\right]_2^k \\
&= -\frac{64\pi}{k} + 32\pi \quad \square
\end{aligned}
$$

$$k \to \infty \Rightarrow \frac{1}{k} \approx 0$$

$\therefore V_x \approx 32$ *units*^{3} $\square$

23. The following diagram shows a part of the curve $y = 4 + 3x - x^2$ and the straight line $y = x + 1$.



Find the ratio of the area of the shaded region A to the area of the shaded region B.

Sol.

$$
\begin{aligned}
x + 1 &= 4 + 3x - x^2 \\
-x^2 + 2x + 3 &= 0 \\
x^2 - 2x - 3 &= 0 \\
(x-3)(x+1) &= 0 \\
x = 3 &\text{ or } x = -1 \\
x = 3,\ y &= 3 + 1 = 4
\end{aligned}
$$

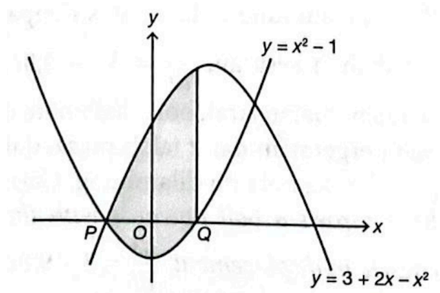
$$
\begin{aligned}
A_A &= \int_{-1}^3 (4 + 3x - x^2)\, dx - \frac{1}{2}\cdot 4 \cdot 4 \\
&= \left[4x + \frac{3}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^3 - 8 \\
&= \left(12 + \frac{27}{2} - 9\right) - \left(-4 + \frac{3}{2} + \frac{1}{3}\right) - 8 \\
&= \frac{33}{2} + \frac{13}{6} - 8 \\
&= \frac{32}{3} \textit{ units}^2
\end{aligned}
$$

$$
\begin{aligned}
4 + 3x - x^2 &= 0 \\
x^2 - 3x - 4 &= 0 \\
(x-4)(x+1) &= 0 \\
x = 4 &\text{ or } x = -1
\end{aligned}
$$

$$
\begin{aligned}
A_B &= \int_3^4 (4 + 3x - x^2)\, dx + \frac{1}{2}\cdot 4 \cdot 4 \\
&= \left[4x + \frac{3}{2}x^2 - \frac{1}{3}x^3\right]_3^4 + 8 \\
&= \left(16 + 24 - \frac{64}{3}\right) - \left(12 + \frac{27}{2} - 9\right) + 8 \\
&= \frac{56}{3} - \frac{33}{2} + 8 \\
&= \frac{61}{6} \textit{ units}^2
\end{aligned}
$$

$$
\begin{aligned}
\therefore A_A : A_B &= \frac{32}{3} : \frac{61}{6} \\
&= \frac{64}{61} \\
&= 64 : 61 \quad \square
\end{aligned}
$$

24. The following diagram shows two curves $y = x^2 - 1$ and $y = 3 + 2x - x^2$.



Find the coordinate of the points P and Q. Hence, calculate the area of the shaded region.

Sol.

$$
\begin{aligned}
x^2 - 1 &= 0 \\
(x+1)(x-1) &= 0 \\
x = -1 &\text{ or } x = 1
\end{aligned}
$$

$$
\begin{aligned}
3 + 2x - x^2 &= 0 \\
(x-3)(x+1) &= 0 \\
x = 3 &\text{ or } x = -1
\end{aligned}
$$

Since $y = x^2 - 1$ and $y = 3 + 2x - x^2$ intersect at $x = -1$, $P(-1, 0)$. $\square$

Since another root of $y = x^2 - 1$ is 1, $Q(1, 0)$. $\square$

$$
\begin{aligned}
A &= \left|\int_{-1}^1 (x^2 - 1)\, dx\right| + \int_{-1}^1 (3 + 2x - x^2)\, dx \\
&= \left|\left[\frac{1}{3}x^3 - x\right]_{-1}^1\right| + \left[3x + x^2 - \frac{1}{3}x^3\right]_{-1}^1 \\
&= \left|\left(\frac{1}{3} - 1\right) - \left(-\frac{1}{3} + 1\right)\right| + \left(3 + 1 - \frac{1}{3}\right) \\
&\quad - \left(-3 + 1 + \frac{1}{3}\right) \\
&= \frac{4}{3} + \frac{11}{3} + \frac{5}{3} \\
&= 6\frac{2}{3} \textit{ units}^2 \quad \square
\end{aligned}
$$

25. Calculate the volume generated, in terms of π, when the region bound by the curve $y = 3x^2$, the straight line $x = 1$ and $x = 4$ and the x-axis is rotated through two right angles about the x-axis.

Sol.

$$
\begin{aligned}
V_x &= \frac{1}{2}\int_1^4 \pi(3x^2)^2\,dx \\
&= \frac{1}{2}\pi\int_1^4 (9x^4)\,dx \\
&= \frac{1}{2}\pi\left[\frac{9}{5}x^5\right]_1^4 \\
&= \frac{1}{2}\pi\left(\frac{9216}{5} - \frac{9}{5}\right) \\
&= 920\frac{7}{10}\pi\ \textit{units}^3 \quad \square
\end{aligned}
$$

3.4 Application of Integration

27. A container is filled with water. After t seconds, the height of the water, hcm, in the containers increases at the rate of $0.56tcms^{-1}$. Given that the container is empty when $t = 0$, find the value of t when $h = 28$.

Sol.

$$
\begin{aligned}
\frac{dh}{dt} &= 0.56t \\
\int \frac{dh}{dt}\,dt &= \int 0.56t\,dt \\
h &= 0.28t^2 + C \\
\because\ t &= 0 \text{ when } h = 0 \\
\therefore\ C &= 0 \\
h &= 0.28t^2 \\
0.28t^2 &= 28 \\
0.01t^2 &= 1 \\
t^2 &= 100 \\
t &= 10s \quad (t > 0) \quad \square
\end{aligned}
$$

28. Raja throws a ball upwards with the rate of change in displacement, $\frac{ds}{dt} = 3 - 9.8t$, where s is the displacement, in m, of the ball from the initial point and t is the time, in seconds, the moment the ball is thrown upwards. Find

Sol.

$$
\begin{aligned}
\frac{ds}{dt} &= 3 - 9.8t \\
\int \frac{ds}{dt}\,dt &= \int (3 - 9.8t)\,dt \\
s &= 3t - 4.9t^2 + C \\
\because\ t &= 0,\ s = 0 \\
\therefore\ C &= 0 \\
s &= 3t - 4.9t^2
\end{aligned}
$$

(a) the maximum height, in m, achieved by the ball.

Sol. $\frac{ds}{dt} = 0$ when the ball is at its maximum height.

$$
\begin{aligned}
3 - 9.8t &= 0 \\
9.8t &= 3 \\
t &= \frac{15}{49}s \\
s &= 3\left(\frac{15}{49}\right) - 4.9\left(\frac{15}{49}\right)^2 \\
&= \frac{45}{98} \quad \square
\end{aligned}
$$

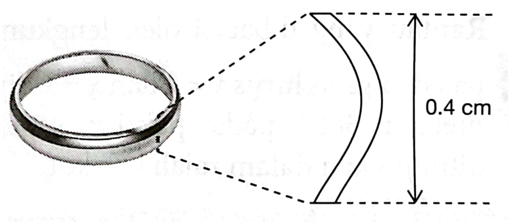
(b) the time taken, in seconds, for the ball to return to its initial point.

Sol.

$$
\begin{aligned}
s &= 0 \\
3t - 4.9t^2 &= 0 \\
t(3 - 4.9t) &= 0 \\
t &= 0 \text{ or } t = \frac{30}{49}s
\end{aligned}
$$

Therefore, the time taken for the ball to return to its initial point is $\frac{30}{49}$ seconds.

29. The following diagram shows the cross section of a wedding ring ordered by Azmin. The inner and outer curved surfaces are represented by the quadratic equations of $x = 0.8 - 0.5y^2$ and $x = 1 - y^2$ respectively.



(a) By using the calculus method, find the volume of the wedding ring in terms of πcm^3.

Sol.

$$
\begin{aligned}
V_x &= \int_{-0.2}^{0.2} \pi \left[(1-y^2)^2 - (0.8-0.5y^2)^2 \right] dy \\
&= \pi \int_{-0.2}^{0.2} \left[(1-2y^2+y^4) - (0.64-0.8y^2 \right. \\
&\quad \left. +0.25y^4) \right] dy \\
&= \pi \int_{-0.2}^{0.2} \left[0.36 - 1.2y^2 + 0.75y^4 \right] dy \\
&= \pi \int_{-\frac{1}{5}}^{\frac{1}{5}} \left[\frac{9}{25} - \frac{6}{5}y^2 + \frac{3}{4}y^4 \right] dy \\
&= \pi \left[\frac{9}{25}y - \frac{2}{5}y^3 + \frac{3}{20}y^5 \right]_{-\frac{1}{5}}^{\frac{1}{5}} \\
&= \pi \left[\left(\frac{9}{125} - \frac{2}{625} + \frac{3}{62500} \right) - \left(-\frac{9}{125} + \frac{2}{625} \right. \right. \\
&\quad \left. \left. - \frac{3}{62500} \right) \right] \\
&= \pi \left(\frac{4303}{62500} + \frac{4303}{62500} \right) \\
&= 0.1377\pi cm^3 \quad \square
\end{aligned}
$$

(b) The ring is made of titanium. If the rate of price of titanium is RM153.49 per gram and the rate of titanium mass per unit volume is $4.51gcm^{-3}$, calculate the price needed to be paid by Azmin. (Use $\pi = 3.142$).

Sol.

$$
\begin{aligned}
\text{Mass} &= 4.51gcm^{-3} \times 0.1377\pi cm^3 \\
&= 1.9513g \\
\text{Price} &= \text{RM}153.49g^{-1} \times 1.9513g \\
&= \text{RM}299.51 \quad \square
\end{aligned}
$$

Praktis Summatif

3.1 Kertas 1

1. Given that $\int_m^2 (2x+3)\,dx = -8$ where $m > 0$, find the value of m.

   Sol.

$$\begin{aligned}
\int_m^2 (2x+3)\,dx &= \left[x^2+3x\right]_m^2 \\
&= 4+6-m^2-3m \\
&= 10-3m-m^2 \\
10-3m-m^2 &= -8 \\
m^2+3m-10 &= 8 \\
m^2+3m-18 &= 0 \\
(m+6)(m-3) &= 0 \\
m &= 3 \quad (m>0) \quad \square
\end{aligned}$$

2. Given that $\frac{dy}{dx} = 10(5x+3)^2$ and $y = 4$ when $x = 0$. Express y in terms of x.

3. Given $\int_5^m f(t)\,dx = \frac{7}{3}$, find

   (a) $\int_m^5 3f(t)\,dx = \frac{7}{3}$.

   Sol.

$$\begin{aligned}
\int_m^5 3f(t)\,dx &= -\int_5^m 3f(t)\,dx \\
&= -3\int_5^m f(t)\,dx \\
&= -3\frac{7}{3} \\
&= -7 \quad \square
\end{aligned}$$

   (b) the value of m, where $\int_5^m [4-f(t)]\,dx = 7$.

   Sol.

$$\begin{aligned}
\int_5^m [4-f(t)]\,dx &= 7 \\
\int_5^m 4\,dx - \int_5^m f(t)\,dx &= 7 \\
[4x]_5^m - \frac{7}{3} &= 7 \\
4m-20 &= \frac{28}{3} \\
12m-60 &= 28 \\
12m &= 88 \\
m &= 7\frac{1}{3}
\end{aligned}$$

4. Given that $\int \frac{3}{(3x-2)^n}\,dx = a(3x-2)^{1-n} + C$,

   (a) State the impossible value of n.

   Sol.

$$\begin{aligned}
\int \frac{3}{(3x-2)^n}\,dx &= 3\int (3x-2)^{-n}\,dx \\
&= 3\left[\frac{(3x-2)^{1-n}}{3(1-n)}\right] + C \\
&= \frac{(3x-2)^{1-n}}{1-n} + C \\
a(3x-2)^{1-n} + C &= \frac{1}{1-n}(3x-2)^{1-n} + C
\end{aligned}$$

   Comparing both sides,

$$a = \frac{1}{1-n}$$

   a is undefined when $1-n=0$.

$$\begin{aligned}
1-n &= 0 \\
n &= 1
\end{aligned}$$

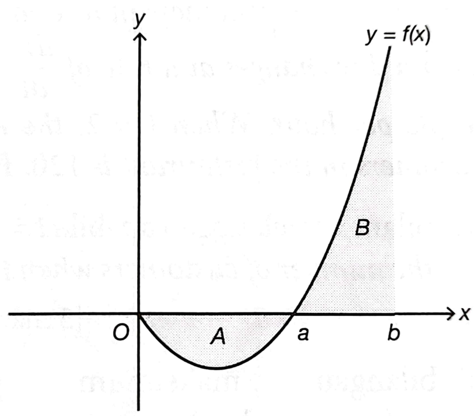
   Therefore, $n = 1$ is impossible. $\square$

   (b) Hence, express n in terms of a.

   Sol.

$$\begin{aligned}
a &= \frac{1}{1-n} \\
a(1-n) &= 1 \\
a-an &= 1 \\
an &= a-1 \\
n &= \frac{a-1}{a} \quad \square
\end{aligned}$$

5. Diagram below shows a curve $y = f(x)$.



   Given area of region B is three times the area of region A and $\int_0^b f(x)\,dx = 20$, find the area of region B.

Sol.

$$\begin{aligned}
\int_0^b f(x)\,dx &= \int_0^a f(x)\,dx + \int_a^b f(x)\,dx \\
20 &= -\int_0^a f(x)\,dx + \int_a^b f(x)\,dx \\
20 &= -\frac{1}{3}\int_a^b f(x)\,dx + \int_a^b f(x)\,dx \\
20 &= -\frac{1}{3}A_B + A_B \\
20 &= \frac{2}{3}A_B \\
2A_B &= 60 \\
A_B &= 30 \quad \square
\end{aligned}$$

3.2 Kertas 2

1. Differentiate $2x^4\sqrt{4x-3}$ with respect to x. Hence, find $\int \frac{3x^4-2x^3}{\sqrt{4x-3}}\,dx$.

   Sol.

$$\begin{aligned}
\frac{dy}{dx}(2x^4\sqrt{4x-3}) &= 8x^3\sqrt{4x-3} + 2x^4\left(\frac{4}{2\sqrt{4x-3}}\right) \\
&= 8x^3\sqrt{4x-3} + \frac{4x^4}{\sqrt{4x-3}} \\
&= \frac{8x^3(4x-3)+4x^4}{\sqrt{4x-3}} \\
&= \frac{4x^3(8x-6+x)}{\sqrt{4x-3}} \\
&= \frac{12(3x^4-2x^3)}{\sqrt{4x-3}} \quad \square
\end{aligned}$$

$$\begin{aligned}
\int \frac{3x^4-2x^3}{\sqrt{4x-3}}\,dx &= \frac{1}{12}\int \frac{12(3x^4-2x^3)}{\sqrt{4x-3}}\,dx \\
&= \frac{1}{12}\cdot 2x^4\sqrt{4x-3} \\
&= \frac{x^4\sqrt{4x-3}}{6} \quad \square
\end{aligned}$$

2. The number of customers in a restaurant on a certain day changes at a rate of $\frac{dB}{dt} = 70 - 10t$ people per hour. When $t = 2$, the number of customers in the restaurant is 120. Find,

   Sol.

$$\begin{aligned}
\frac{dB}{dt} &= 70 - 10t \\
\int \frac{dB}{dt}\,dt &= \int (70-10t)\,dt \\
B &= 70t - 5t^2 + C \\
\because\ t = 2,\ B &= 120, \\
120 &= 70(2) - 5(2)^2 + C \\
C &= 120 - 140 + 20 = 0 \\
\therefore\ B &= 70t - 5t^2
\end{aligned}$$

   (a) the number of customers when $t = 10$.

   Sol.

$$\begin{aligned}
B &= 70(10) - 5(10)^2 \\
&= 700 - 500 \\
&= 200 \quad \square
\end{aligned}$$

   (b) the maximum number of customers at a certain time on that day. Hence, find the income of the restaurant at that moment if each customer spends an average of RM25.

   Sol.

$$\begin{aligned}
70 - 10t &= 0 \\
10t &= 70 \\
t &= 7
\end{aligned}$$

   When $t = 7$, the number of customers is

$$\begin{aligned}
B &= 70(7) - 5(7)^2 \\
&= 490 - 245 \\
&= 245 \quad \square
\end{aligned}$$

   Hence, the income of the restaurant at that moment is

$$\begin{aligned}
\text{Income} &= 245 \times 25 \\
&= RM6,125 \quad \square
\end{aligned}$$

3. The gradient function of a curve is given by $\frac{dy}{dx} = kx - 6$, where k is a constant. The gradient of normal to the curve at point $(2, -5)$ is $\frac{1}{2}$. Find the equation of the curve.

   Sol.

   The gradient of the tangent to the curve at point $(2, -5)$ is -2.

$$\begin{aligned}
-2 &= k(2) - 6 \\
2k &= 4 \\
k &= 2
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= 2x - 6 \\
y &= \int \frac{dy}{dx}\,dx \\
&= \int (2x-6)\,dx \\
&= x^2 - 6x + C \\
\because\ x = 2,\ y &= -5, \\
-5 &= 2^2 - 6(2) + C \\
C &= -5 - 4 + 12 = 3
\end{aligned}$$

   Hence, the eq. of the curve is $y = x^2 - 6x + 3$. $\square$

4. The curve with gradient function $f'(x) = 3x^2 + mx + n$ where m and n are constants, has stationary points at $(1, -3)$ and $(-3, 29)$. Find

(a) the values of m and n.

Sol.

$$
\begin{aligned}
f'(1) &= 3(1)^2 + m + n \\
0 &= 3 + m + n \\
m + n &= -3 \quad \cdots \quad (1) \\
f'(-3) &= 3(-3)^2 - 3m + n \\
&= 27 - 3m + n \\
-3m + n &= -27 \quad \cdots \quad (2)
\end{aligned}
$$

$$
\begin{aligned}
(2) - (1):\ 4m &= 24 \\
m &= 6 \quad \square \\
n &= -3 - 6 = -9 \quad \square
\end{aligned}
$$

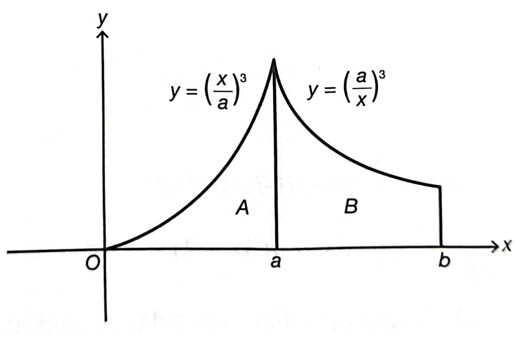
(b) the equation of the curve.

Sol.

$$
\begin{aligned}
f'(x) &= 3x^2 + 6x - 9 \\
f(x) &= \int f'(x)\,dx \\
&= \int (3x^2 + 6x - 9)\,dx \\
&= 3\int (x^2 + 2x - 3)\,dx \\
&= 3\left[\frac{x^3}{3} + x^2 - 3x\right] + C \\
&= x^3 + 3x^2 - 9x + C \\
\because\ x = 1,&\ y = -3, \\
-3 &= 1^3 + 3(1)^2 - 9(1) + C \\
C &= -3 - 1 - 3 + 9 = 2
\end{aligned}
$$

Hence, the equation of the curve is $y = x^3 + 3x^2 - 9x + 2$. $\square$

5. Diagram below shows two regions labelled as A and B respectively. Region A is bounded by the curve $y = \left(\frac{x}{a}\right)^3$, the straight line $x = a$ and the x-axis whereas region B is bounded by the curve $y = \left(\frac{a}{x}\right)^3$, the straight lines $x = a$ and $x = b$, and the x-axis.



(a) Find the area of the region A in terms of a.

Sol.

$$
\begin{aligned}
A_A &= \int_0^a \left(\frac{x}{a}\right)^3 dx \\
&= \int_0^a \frac{x^3}{a^3}\,dx \\
&= \left[\frac{x^4}{4a^3}\right]_0^a \\
&= \frac{a^4}{4a^3} \\
&= \frac{a}{4} \quad \square
\end{aligned}
$$

(b) Find the area of the region B in terms of a and b.

Sol.

$$
\begin{aligned}
A_B &= \int_a^b \left(\frac{a}{x}\right)^3 dx \\
&= a^3 \int_a^b \frac{1}{x^3}\,dx \\
&= a^3 \left[-\frac{1}{2x^2}\right]_a^b \\
&= a^3 \left(-\frac{1}{2b^2} + \frac{1}{2a^2}\right) \\
&= a^3 \left(\frac{-a^2 + b^2}{2a^2b^2}\right) \\
&= \frac{ab^2 - a^3}{2b^2} \quad \square
\end{aligned}
$$

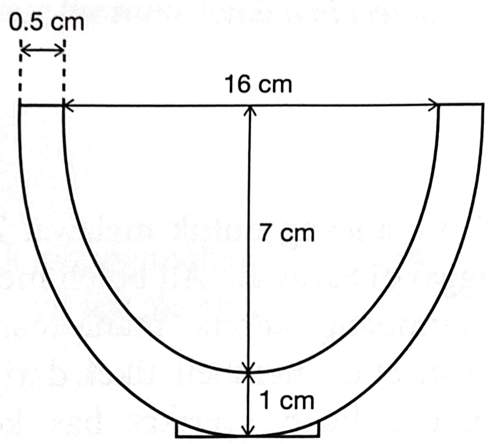
(c) Show that *the area of region* $A > \frac{1}{2}$ *area of region* B for all values of a and b where $0 < a < b$.

Sol.

$$
\begin{aligned}
A_A &= \frac{a}{4} \\
\frac{1}{2}A_B &= \frac{ab^2 - a^3}{4b^2} \\
&= \frac{ab^2}{4b^2} - \frac{a^3}{4b^2} \\
&= \frac{a}{4} - \frac{a^3}{4b^2}
\end{aligned}
$$

$$
\begin{aligned}
&\forall\, a, b \in \mathbb{R},\ 0 < a < b \\
&\because \frac{a}{4} > \frac{a}{4} - \frac{a^3}{4b^2} \\
&\therefore A_A > \frac{1}{2}A_B \quad (shown) \quad \square
\end{aligned}
$$

6. Diagram below shows the cross-section of an anti-heat bowl which is made of stainless steel. The bowl has two layers in which the space between the two layers is a vacuum which functions as a heat insulator.



The inner and the outer layers of the bowl are parabolic in shape which are represented by the equations $y = ax^2 + b$ and $y = \frac{32}{289}x^2$ respectively.

(a) Find the values of a and b.

Sol.

$$\begin{aligned} y &= ax^2 + b \\ b &= 1 \quad \text{(y-intercept)} \quad □ \\ y &= ax^2 + 1 \end{aligned}$$

The bowl is split into two equal parts by the y-axis, while the height of the bowl is $8cm$ Hence, the top-right corner of the bowl is $(8, 8)$.

$$\begin{aligned} 8 &= a(64) + 1 \\ 64a &= 7 \\ a &= \frac{7}{64} \quad □ \end{aligned}$$

(b) Anis wants to pour 1.5 litres of milk into the bowl. Identify whether the bowl can hold 1.5 litres of milk. Justify your answer.

Sol.

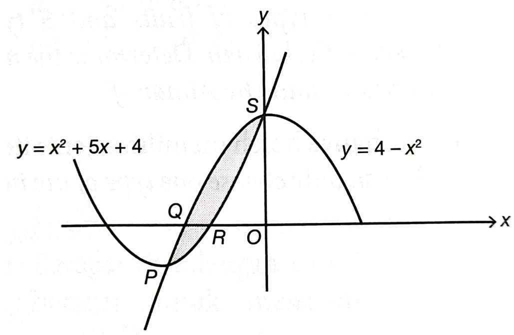
$$\begin{aligned} y &= \frac{7}{64}x^2 + 1 \\ 64y &= 7x^2 + 64 \\ 64y - 64 &= 7x^2 \\ 64(y-1) &= 7x^2 \\ x^2 &= \frac{64}{7}(y-1) \end{aligned}$$

The volume of the bowl is

$$\begin{aligned} V_x &= \int_1^7 \pi x^2 \, dy \\ &= \pi \int_1^7 \frac{64}{7}(y-1)\, dy \\ &= \frac{64}{7}\pi \int_1^8 (y-1)\, dy \\ &= \frac{64}{7}\pi \left[\frac{y^2}{2} - y\right]_1^8 \\ &= \frac{64}{7}\pi \left[\left(\frac{64}{2} - 8\right) - \left(\frac{1}{2} - 1\right)\right] \\ &= \frac{64}{7}\pi \left(24 + \frac{1}{2}\right) \\ &= 224\pi \\ &\approx 703.7168 cm^3 \\ &\approx 703.7168 ml \end{aligned}$$

Hence, the bowl is not capable of holding 1.5 litres of milk due to insufficient volume. □

7. Diagram below shows parts of the curves $y = x^2+5x+4$ and $y = 4 - x^2$.



Find

(a) the points of intersection P and S.

Sol.

$$\begin{aligned} x^2 + 5x + 4 &= 4 - x^2 \\ 2x^2 + 5x &= 0 \\ x(2x+5) &= 0 \\ x = 0 \text{ or } x &= -\frac{5}{2} \end{aligned}$$

$$\begin{aligned} x = 0,\ y &= 4 - 0^2 = 4 \\ x = -\frac{5}{2},\ y &= 4 - \left(-\frac{5}{2}\right)^2 \\ &= 4 - \frac{25}{4} \\ &= -\frac{9}{4} \end{aligned}$$

Hence, the points of intersection are $S(0, 4)$ and $P(-\frac{5}{2}, -\frac{9}{4})$. □

(b) the coordinates of the points Q and R.

Sol.

$$\begin{aligned} 4 - x^2 &= 0 \\ x^2 - 4 &= 0 \\ (x+2)(x-2) &= 0 \\ x &= -2 \text{ or } x = 2 \end{aligned}$$

$\because x_Q < 0,\ x_Q = -2$

$\therefore Q(-2, 0)$ ▯

$$\begin{aligned} x^2 + 5x + 4 &= 0 \\ (x+1)(x+4) &= 0 \\ x &= -1 \text{ or } x = -4 \end{aligned}$$

$\because x_R > x_Q,\ x_R = -1$

$\therefore R(-1, 0)$ ▯

(c) the area of the shaded region.

Sol.

Let the area above the x-axis be A_A and the area below the x-axis be A_B.

$$\begin{aligned} A_A &= \int_{-2}^{0} \left(4 - x^2\right) dx - \int_{-1}^{0} \left(x^2 + 5x + 4\right) dx \\ &= \left[4x - \frac{1}{3}x^3\right]_{-2}^{0} - \left[\frac{1}{3}x^3 + \frac{5}{2}x^2 + 4x\right]_{-1}^{0} \\ &= \left[0 - \left(-8 + \frac{8}{3}\right)\right] - \left[0 - \left(-\frac{1}{3} + \frac{5}{2} - 4\right)\right] \\ &= \frac{16}{3} - \frac{11}{6} \\ &= 3\frac{1}{2}\textit{units}^2 \end{aligned}$$

$$\begin{aligned} A_B &= \left|\int_{-2.5}^{-1} \left(x^2 + 5x + 4\right) dx\right| - \left|\int_{-2.5}^{-2} \left(4 - x^2\right) dx\right| \\ &= \left|\left[\frac{1}{3}x^3 + \frac{5}{2}x^2 + 4x\right]_{-2.5}^{-1}\right| - \left|\left[4x - \frac{1}{3}x^3\right]_{-2.5}^{-2}\right| \\ &= \left|\left[\left(-\frac{1}{3} + \frac{5}{2} - 4\right) - \left(-\frac{125}{24} + \frac{125}{8} - 10\right)\right]\right| \\ &\quad - \left|\left[\left(-8 + \frac{8}{3}\right) - \left(-10 + \frac{125}{24}\right)\right]\right| \\ &= \left|\left(-\frac{11}{6} - \frac{5}{12}\right)\right| - \left|\left(-\frac{16}{3} + \frac{115}{24}\right)\right| \\ &= \frac{9}{4} - \frac{13}{24} \\ &= 1\frac{17}{24}\textit{units}^2 \end{aligned}$$

$$\begin{aligned} A &= A_A + A_B \\ &= 3\frac{1}{2} + 1\frac{17}{24} \\ &= 5\frac{5}{24}\textit{units}^2 \end{aligned}$$ ▯