Mathematics

Senior 3 Part I

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Introduction

Why this book?

Disclaimer

Acknowledgements

Contents

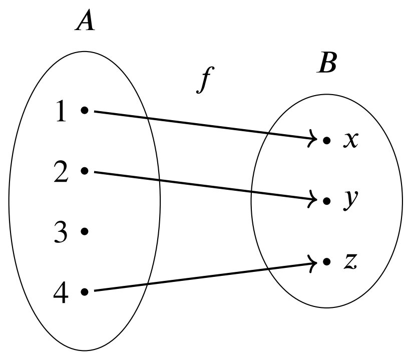
Chapter 1

Function

1.1 Definition of a Function

Mapping, Preimage and Image

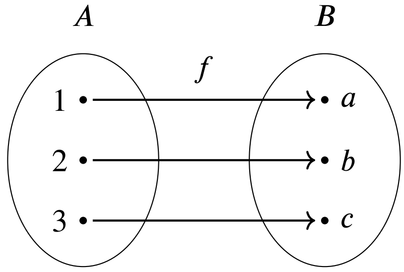
For two non-empty sets A and B, If an element a inside set A has a corresponding element b inside set B, denoted as $a \to b$, then we say that a is mapped to b or a and b are paired. The mapping between two sets is normally denoted as f, g, h, etc. The mapping shown in the diagram below can be denoted as $f : 1 \to x,\ 2 \to y,\ 4 \to z$.



Let $f : A \to B$ is a mapping, a is an element in A. If a is mapped to b under the mapping f, then b is said to be the image of a under the mapping f, denoted as $b = f(a)$; a is said to be the preimage of b under the mapping f. In the diagram above, under the mapping f, the image of 1, 2, and 4 are x, y, and z respectively, while the preimage of x, y, and z are 1, 2, and 4 respectively.

Let A and B be two non-empty sets, f is a mapping from A to B such that for all elements in A, there is a unique corresponding element in B, then f is a function or a mapping from A to B, denoted as $f : A \to B$.

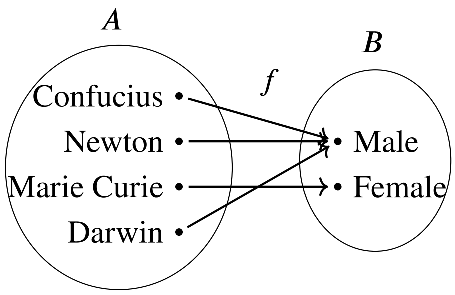
The mapping shown in the diagram below is a function.



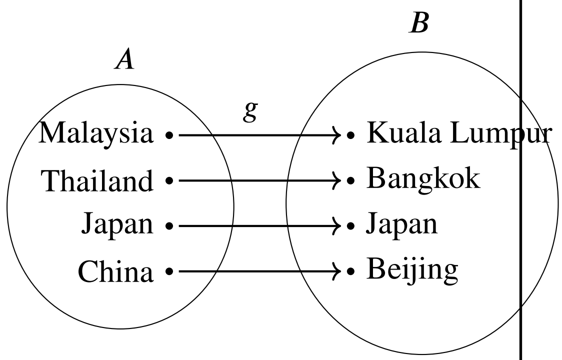
Practice 1

1. For the following mappings, list the image of each element in A and the preimage of each element in B, and determine whether the mapping is a function or not:

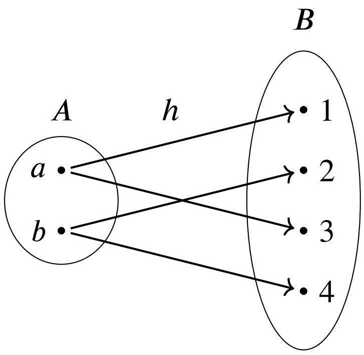
(a)



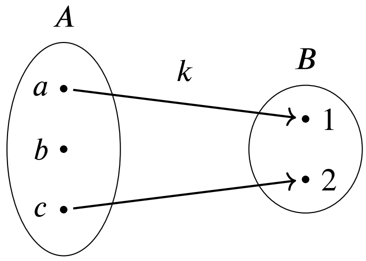
(b)



(c)



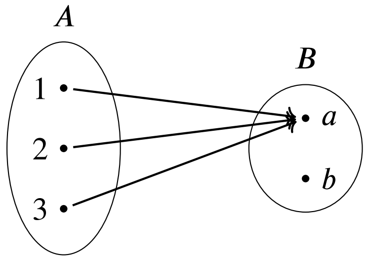
(d)



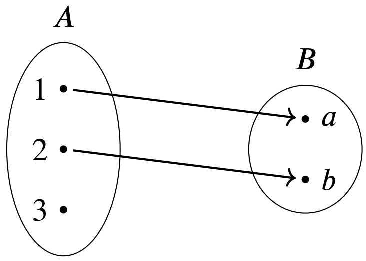
2. Given a mapping $g : x \to x + 3$, $x \in \{-2, -1, 0, 1, 2, 3\}$, find the image of each x.

3. Determine whether the following mappings are functions.

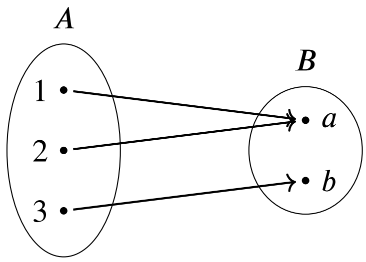
(a)



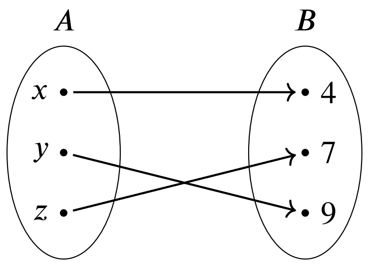
(b)



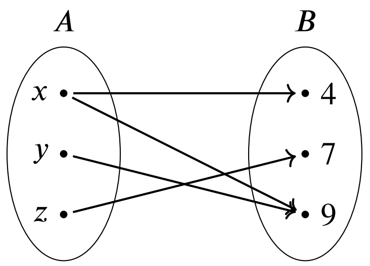
(c)



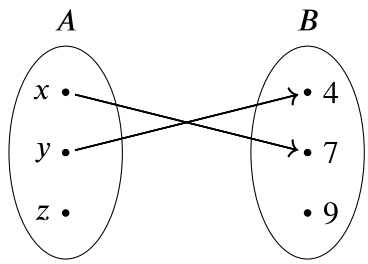
(d)



(e)



(f)



The function $f : A \to B$ can be written as $y = f(x)$, x is the element of A and y is the element of B. When x changes, y changes as well. x is called independent variable, while y is called dependent variable.

Keep in mind that $f(x)$ is NOT the product of f and x.

Representation of Functions

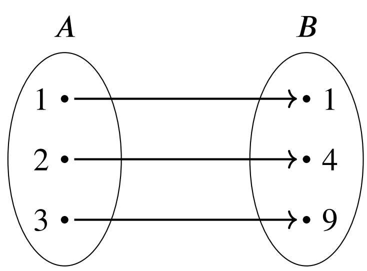
Generally speaking, there are a few ways to represent a function:

1. **Narrative Form**: express the function of two sets in words. For example, Let $A = \{1, 2, 3\}$ and $B = \{1, 4, 9\}$, f is a function from A to B, its definition is that for any element x in A, its corresponding element is x^2 in B.

2. **Arrow Method**: draw an arrow to connect the preimage and image of a function such that the preimage is corresponding to the image. To express the example above, we express it as $f : 1 \to 1,\ 2 \to 4,\ 3 \to 9$.

3. **Analytical Method**: express the function in the form of mathematical expression to represent the relationship between the independent variable and the dependent variable. For example, $f(x) = x^2, x \in A$.

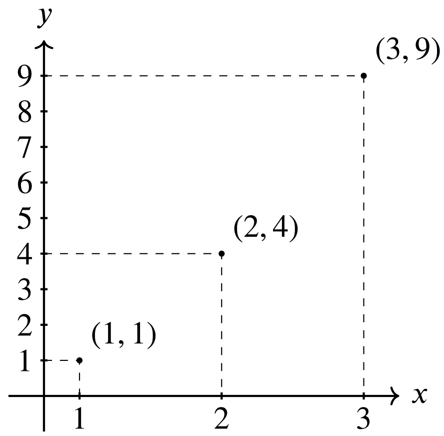
4. **Venn Diagram**: draw arrows between the venn diagram of two sets to represent the function, as shown below:



5. **Table Method**: express the function in the form of table, showing the relationship of the chosen value between independent variable x and the value of its corresponding dependent variable y, as shown below:

x	1	2	3
y	1	4	9

6. **Graphical Method**: draw a graph to represent the function of the two variables, as shown below:



Practice 2

Express the following functions using analytical method, venn diagram, table method and graphical method.

(a) f mapping each integers from -3 to 3 to its squares plus 4.

(b) g mapping each natural numbers from 1 to 4 to its cubes.

Exercise 22.1

1. Express the mapping from set A to set B, and determine which of the following mappings are functions.

	Set A	Set B	Mapping
(a)	{0, 3, 9, 12}	{0, 1, 2, 3}	Divide by 3
(b)	{-2, -1, 0, 1, 2}	{0, 1, 4, 9, 16}	Power of 4
(c)	{-2, -1, 0, 1, 2}	{0, 1, 4}	Square
(d)	{30°, 45°, 60°}	$\left\{\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}\right\}$	Sine
(e)	{-1, 0, 1, 2}	{-1, 0, 1}	Cube

2. Let function $f(x) = 3x^2 + 1$.

(a) Find the image of the following elements:

i. -3

ii. -2

iii. 0

iv. 2

v. 5

(b) Find the preimage of the following elements:

i. 13

ii. 28

iii. 1

iv. 0

v. 4

3. Let function $g(x) = 5x - 2$. Find:

(a) $g(-2)$

(b) $g(-1)$

(c) $g(0)$

4. Let function $f(x) = \begin{cases} 2x, & x \leq -1 \\ x-1, & -1 \leq x < 3 \\ 4x+2, & x \geq 3 \end{cases}$,

find

(a) $f(-5)$

(b) $f(-2)$

(c) $f(0)$

(d) $f(2)$

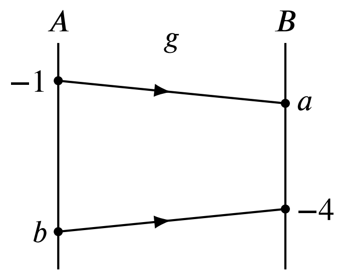
(e) $f(10)$

5. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$. Find the image of -1, 0, 1, and 2 under f.

6. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^4$. Find the preimage of 0, 1, and 4 under f.

In $\mathbb{R}$, which element does not have a preimage?

7. In the diagram below, given that function $g : A \to B$ is defined as $g : x \to 2x - 8$. Find the value of a and b.

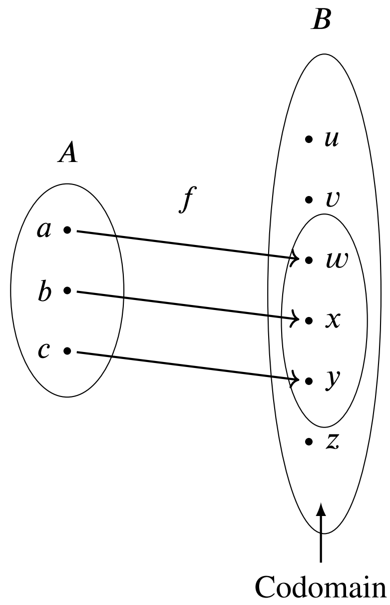


8. Using narrative form, arrow method, venn diagram, table method and graphical method, express the function $f(x) = 2x$, $x \in \{-2, -1, 0, 1, 2\}$.

1.2 Domain and Range

Let f is a function from set A to set B, then set A is called the domain of f, denoted by D_f; set B is called the codomain of f; the set of the images of all elements of A under f is called the range of f, denoted by R_f.

If the domain A and range B of function $f : A \to B$ are both subsets of real number set $\mathbb{R}$, then this function is called real valued function / real function. This book primarily discusses about real valued functions. When the domain of a real function is not mentioned and only the mapping rule is given, its domain is assumed to be the set of all real numbers that yield defined values $f(x)$. After the domain and the mapping rule are determined, the range of a function will then be determined.



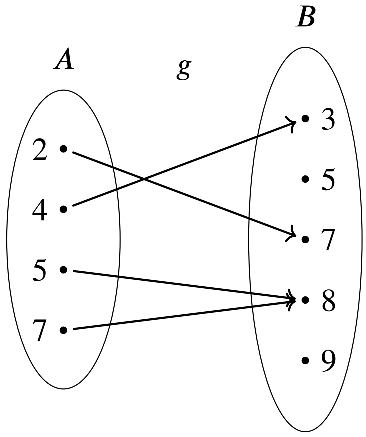
Interval Notation

Let a and b be two real number, $a < b$.

Intervals	Set Notations
(a, b)	$\{x \mid x \in \mathbb{R}, a < x < b\}$
$[a, b)$	$\{x \mid x \in \mathbb{R}, a \leq x < b\}$
$(a, b]$	$\{x \mid x \in \mathbb{R}, a < x \leq b\}$
$[a, b]$	$\{x \mid x \in \mathbb{R}, a \leq x \leq b\}$
(a, ∞)	$\{x \mid x \in \mathbb{R}, x > a\}$
$[a, \infty)$	$\{x \mid x \in \mathbb{R}, x \leq a\}$
$(-\infty, a)$	$\{x \mid x \in \mathbb{R}, x < a\}$
$(-\infty, a]$	$\{x \mid x \in \mathbb{R}, x \leq a\}$

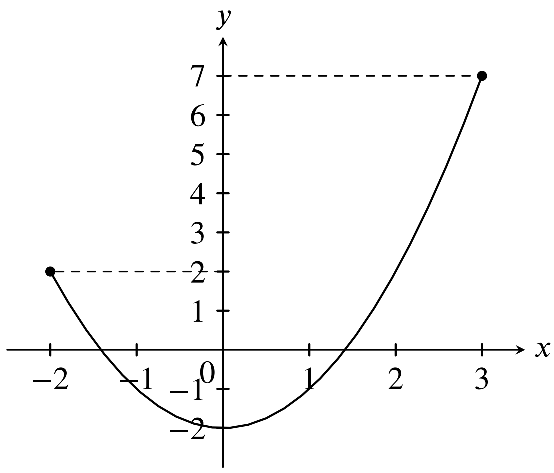
Practice 3

1. Let $A = \{2, 4, 5, 7\}$ and $B = \{3, 5, 7, 8, 9\}$, the definition of function g is given by the diagram below. Find the domain, codomain and range of function g.



2. Let $A = \{-2, -1, 0, 1, 2\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = x^2 - 1$. Find the domain and range of f.

3. The curve in the diagram below represents the function $y = f(x)$, $-2 \leq x \leq 3$. Find the domain and range of f.



4. Find the domain and range of the following functions:

   (a) $f(x) = -4x + 5$

   (b) $g(x) = x^2 - 1$

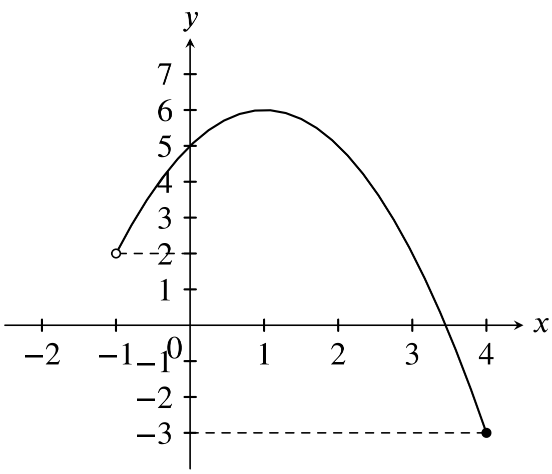
   (c) $h(x) = \dfrac{1}{4x + 7}$

   (d) $k(x) = \sqrt{6 - x}$

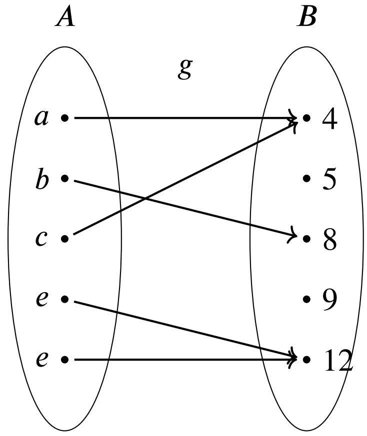
Exercise 22.2

1. Let $X = \{a, b, c, d\}$ and $Y = \{-1, 2, 9, 11\}$, function $f : X \to Y$ is defined by $f(a) = 2$, $f(b) = -1$, $f(c) = 2$, $f(d) = 9$. Find the domain and range of the f.

2. The curve in the diagram below represents the function $y = f(x)$, $-1 < x \leq 4$. Find the domain and range of f.



3. Let $A = \{a, b, c, d, e\}$ and $B = \{4, 5, 8, 9, 12\}$, the definition of function $g : A \to B$ is given by the digram below. Find the domain, codomain and range of function g.



4. Let $A = \{-1, 0, 1, 2\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = 3x^2 - 2$, find the domain and range of f.

5. Let $A = \{-1, 0, 2, 5, 11\}$, function $f : A \to \mathbb{R}$ is defined by $f(x) = x^2 - x - 2$, find the domain and range of f.

6. Find the domain and range of the following functions:

   (a) $f(x) = x^3$

   (b) $g(x) = \sqrt{1 - x^2}$

   (c) $h(x) = \dfrac{1}{2x + 3}$

   (d) $k(x) = x^2 - 2x + 4$

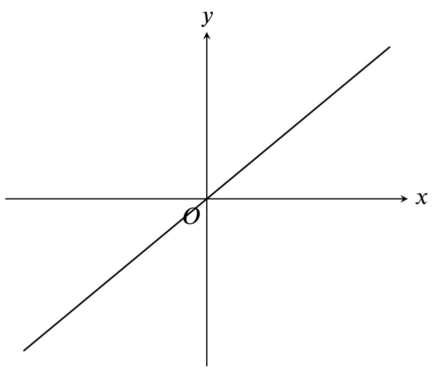
1.3 Graphs of Functions and Their Transformations

Graphs of Simple Functions

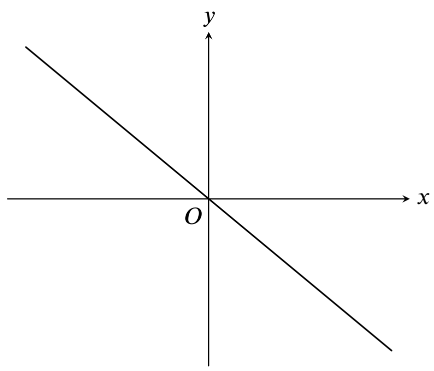
On a Cartesian plane, the graphs formed by all the point (x, y) that satisfied the equation $y = f(x)$ are called graphs of function f. Below are some examples of graphs of simple functions.

Note that any line that is parallel to the y-axis intersects the graph of a function at most once.

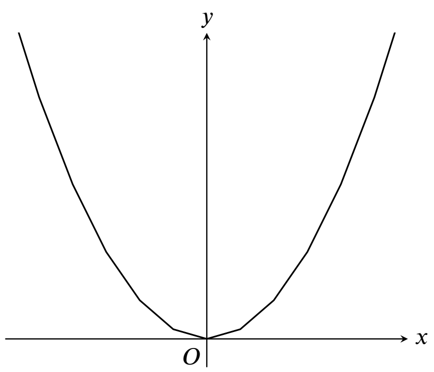
(a) $y = x$



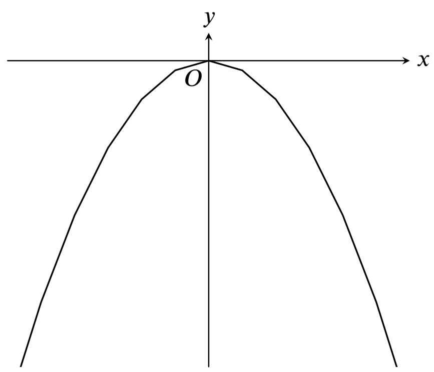
(b) $y = -x$



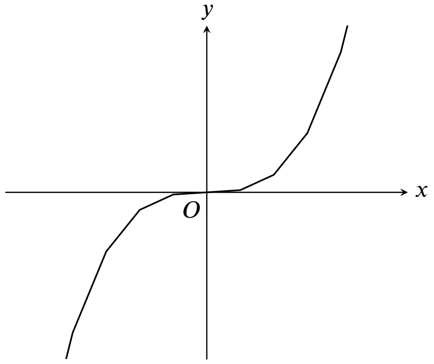
(c) $y = x^2$



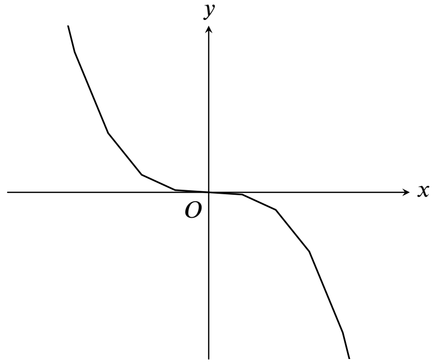
(d) $y = x^2$



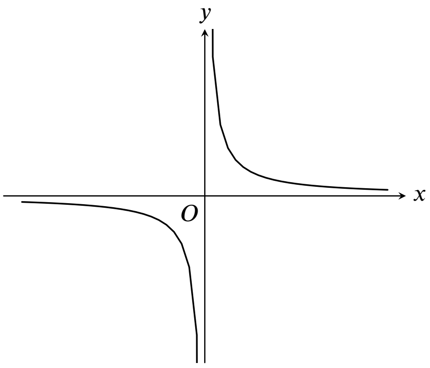
(e) $y = x^3$



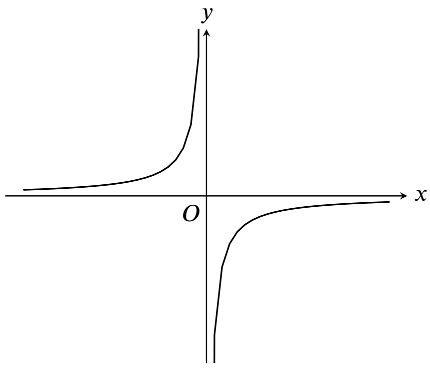
(f) $y = -x^3$



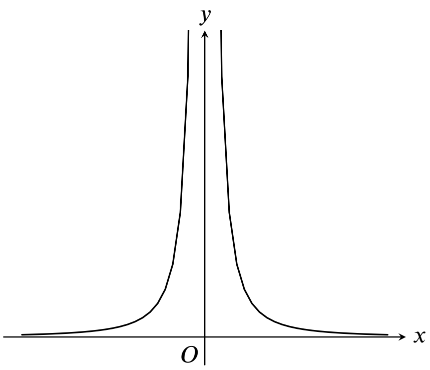
(g) $y = \dfrac{1}{x}$



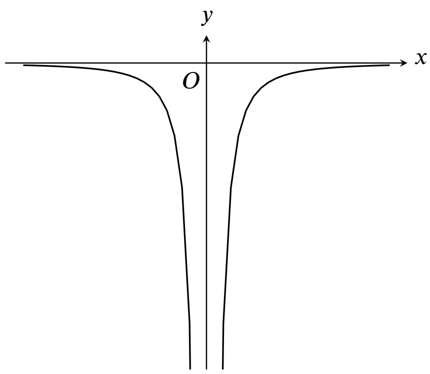
(h) $y = -\dfrac{1}{x}$



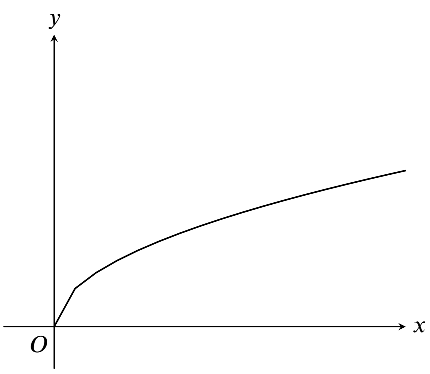
(i) $y = \dfrac{1}{x^2}$



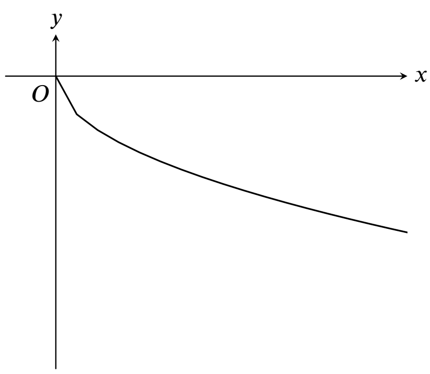
(j) $y = -\dfrac{1}{x^2}$



(k) $y = \sqrt{x}$



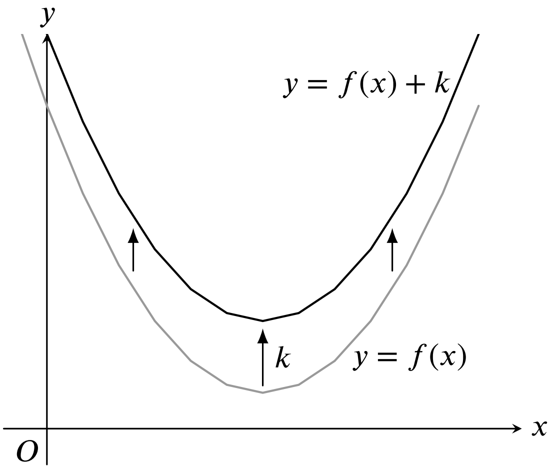
(l) $y = -\sqrt{x}$

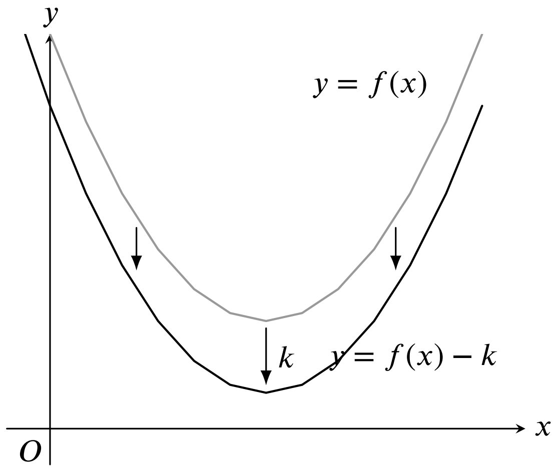


Transformations of Graphs

- If $k > 0$, translate the graph of $y = f(x)$ vertically upwards by k units, the graph of $y = f(x) + k$ is obtained.
- If $k > 0$, translate the graph of $y = f(x)$ vertically downwards by k units, the graph of $y = f(x) - k$

is obtained.





- If $h > 0$, translate the graph of $y = f(x)$ horizontally to the right by h units, the graph of $y = f(x+h)$ is obtained.
- If $h > 0$, translate the graph of $y = f(x)$ horizontally to the left by h units, the graph of $y = f(x-h)$ is obtained.

- If $k > 0$, reflect the graph of $y = f(x)$ about the x-axis, the graph of $y = -f(x)$ is obtained.
- If $k > 0$, reflect the graph of $y = f(x)$ about the y-axis, the graph of $y = f(-x)$ is obtained.

If $a > 0$, zooming (when $a > 1$) or shrinking (when $0 < a < 1$) the graph of $y = f(x)$ by a factor of a in the y-direction, the graph of $y = af(x)$ is obtained.

If $a > 0$, shrinking (when $a > 1$) or zooming (when $0 < a < 1$) the graph of $y = f(x)$ by a factor of $\frac{1}{a}$ in the x-direction, the graph of $y = f(ax)$ is obtained.

Practice 4

Find the line of symmetry and vertex of the following parabola, and sketch its graph. (Question 1 to 2):

1. $y = 2x^2 + 8x + 11$
2. $y = -3x^2 + 18x - 7$

Sketch the graph of the following functions.

(Question 3 to 4):

3. $y = \dfrac{4}{(x+2)^2}$

4. $y = \sqrt{x-1} + 3$

Exercise 22.3

Find the line of symmetry and vertex of the following parabola, and sketch its graph.

1. $y = 2x^2 + 4x + 5$
2. $y = -3x^2 + 12x - 4$
3. $y = 4x^2 - 20x + 19$
4. $y = -3x^2 - 6x - 4$

Sketch the graph of the following functions.

5. $y = (x+2)^3 - 5$
6. $y = \sqrt{x-5}$
7. $y = \dfrac{1}{(x+2)^2}$
8. $y = -\dfrac{1}{2(x-1)^2}$
9. $y = 3\sqrt{x+1} - 4$
10. $y = \dfrac{4}{2x+3}$
11. $y = \begin{cases} 4x+9, & x \le 0 \\ 9-2x, & x > 0 \end{cases}$
12. $y = \begin{cases} x, & x < -1 \\ \sqrt{x+1}, & x \ge -1 \end{cases}$
13. Sketch the graph for the function $f(x) = x^2 - 6x + 12$, $-2 \le x \le 8$, and find its domain and range.
14. Sketch the graph for the function $g(x) = -x^2 - 4x - 7$, $-2 \le x \le 5$, and find its domain and range.
15. Sketch the graph for the function $f(x) = -x^2 + 2x + 10$, and find its domain and range.
16. Sketch the graph of the function $y = \sqrt{x}$, and transform it according the following steps. Sketch the graph of each function after each step on the same diagram, and write down the corresponding function.

    Step 1: Translate 4 units to the left;

    Step 2: Scale up by a factor of 2 in the x-direction;

    Step 3: Reflect about the y-axis;

    Step 4: Translate 3 units downwards.

    Step 5: Scale down by half in the y-direction.

1.4 Composite Functions

Let A, B, and C be three non-empty sets, $f : A \to B$ and $g : B \to C$ be two functions, an element x in set A is mapped to an element $f(x)$ in set B by function f, and $f(x)$ is mapped to an element $g\big(f(x)\big)$ in set C by function g. In other words, x in set A is mapped to an element $g\big(f(x)\big)$ in C after two mappings. That is:

$$x \xrightarrow{f} f(x) \xrightarrow{g} g\big(f(x)\big)$$

The combination of these two mappings are a function from set A to set C, this function is called the *composite function* of f and g, denoted by $g \circ f$. When defining the composite function $g \circ f$, the range of f must be a subset of the domain of g, that is, $R_f \subseteq D_g$.

Note that $D_{g \circ f} = D_f$, $R_{g \circ f} \subseteq R_g$.

$\forall n \in \mathbb{N}$, $f^{n+1} = f \circ f^n$.

Generally speaking, $g \circ f \neq f \circ g$.

If $f \circ (g \circ h)$ is defined, then $(f \circ g) \circ h$ is also defined, and $f \circ (g \circ h) = (f \circ g) \circ h$. Therefore, we can write $f \circ g \circ h$ without ambiguity.

Practice 5

1. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x+3$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 5 - x$. Find $(g \circ f)(x)$ and $(f \circ g)(x)$.

2. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 2x + 3$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 3x - 4$. Find

   (a) $g \circ f$ and $f \circ g$;

   (b) $g\big(f(2)\big)$, $f\big(g(2)\big)$, $(g \circ f)(2)$, and $(f \circ g)(2)$.

3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4 - x^2$ and $g : \{x | x \leq 4\} \to \mathbb{R}$, $g(x) = \sqrt{4 - x}$. Prove the existence of $f \circ g$ and $g \circ f$ respectively.

1.5 One to One Function, Onto Function and One to One Onto Function

One to One Function

Let $f : A \to B$ be a function, if there is at most one preimage in set A for each element in set B, then f is called a *one to one function*.

As shown in the diagram above, each element in the codomain B of the function $f : A \to B$ has at most one preimage in the domain A of the function, thus f is a one to one function; while the element b_2 in the codomain B of the function $g : A \to B$ has two preimages a_2 and a_3, thus g is not a one to one function.

A function $y = f(x)$ is a one to one function, if and only if any line parallel to the x-axis intersects the graph of the function at most once.

Onto Function

If each element in the codomain B of the function $f : A \to B$ has at least one preimage under the function f, then f is said to be an *onto function*.

As shown in the diagram above, each element in the codomain B of the function $f : A \to B$ has at least one preimage under the function f, therefore f is an onto function; while the element b_3 in the codomain B of the function $g : A \to B$ has no preimage under the function g, therefore g is not an onto function.

One to One Onto Function

If a function is both a one to one function and an onto function, then it is a *one to one onto function*, as shown in the diagram above.

Practice 7

Determine whether the following functions are one to one functions or onto functions.

Exercise 22.5

1. Let $A = \{1, 2, 3\}$, $f : A \to A$ is defined by $f : 1 \to 1, 2 \to 3, 3 \to 2$. Determine if f is a one to one function or an onto function.

2. Let $A = \{a, b, c, d\}$ and $B = \{x, y, z\}$, $f : A \to B$ is defined by $f : a \to y, b \to x, c \to z, d \to y$. Determine if f is a one to one function or an onto function.

3. Let the function $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 2x + 1$. Determine if g is a one to one function or an onto function.

4. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x^3 - 3$. Determine if f is a one to one function or an onto function.

5. Let the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = \frac{1}{x}$. Determine if f is a one to one function or an onto function.

6. Let the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $f(x) = \sqrt{x}$. Determine if f is a one to one function or an onto function.

7. Determine whether the following functions are one to one, onto or one to one onto functions.

(a) $A = \{a, b, c\}$, $B = \{x, y, z\}$, $f : A \to B$, $f : a \to x, b \to x, c \to y$

(b) $A = \{a, b, c\}$, $B = \{x, y, z\}$, $g : A \to B$, $g : a \to x, b \to y, c \to z$

(c) $A = \{a, b, c\}$, $B = \{x, y\}$, $h : A \to B$, $h : a \to x, b \to y, c \to y$

(d) $A = \{a, b, c\}$, $B = \{x, y\}$, $k : A \to B$, $k : a \to x, a \to y, c \to y$

(e) $A = \{a, b, c\}$, $B = \{x, y\}$, $f : A \to B$, $f : a \to x, a \to y, b \to x, c \to y$

(f) $A = \{a, b, c, d\}$, $B = \{u, v, x, y, z\}$, $g : A \to B$, $g : a \to u, b \to v, c \to x, d \to y$

8. determine whether the following functions mapping A to B are one to one functions or onto functions.

1.6 Inverse Functions

If $f : A \to B$ is a one to one onto function, then there exist a function $g : B \to A$, such that if $y = f(x)$, then $g(y) = x$. The function g is called the *inverse function* of f, and is denoted by f^{-1}.

from the diagram above, we can conclude the following:

$$x \xrightarrow{f} y = f(x) \xrightarrow{f^{-1}} f^{-1}\big(f(x)\big) = f^{-1}(y)$$

or

$$y \xrightarrow{f^{-1}} x = f^{-1}(y) \xrightarrow{f} f\big(f^{-1}(y)\big) = f(x)$$

If both $(g \circ f)^-1$ and $f^{-1} \circ g^{-1}$ exist, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Practice 8

Exercise 22.5

1. Find the inverse function of the following functions:

   (a) $f : x \to 7x - 3$

   (b) $g : x \to \frac{1}{2}x + 9$

   (c) $h : x \to \frac{x+1}{x-8}$, $x \neq 8$

   (d) $k : x \to \frac{x-1}{2x}$, $x \neq 0$

2. Given the function $f : x \to 2x + 1$ and $g : x \to \frac{1}{x-4}$, $x \neq 4$. Find:

   (a) f^{-1}

   (b) g^{-1}

   (c) $f^{-1} \circ g^{-1}$

   (d) $g^{-1} \circ f^{-1}$

   (e) $(f \circ g)^{-1}$

   (f) $(g \circ f)^{-1}$

Graph of Inverse Functions

If f is a one to one function, then the graph of f^{-1} is the reflection of the graph of f about the line $y = x$.

Practice 9

Given the function $g : \mathbb{R}^+ \cup 0 \to \mathbb{R}^+ \cup 0$, $g : x \to x^2$. On the same set of axes, draw the graph of the function g and its inverse function g^{-1}.

Chapter 2

Exponents and Logarithms

2.1 Exponents

2.2 Logarithms

2.3 Arithmetic Properties of Logarithms and Base Changing Formula

2.4 Exponential Equations

2.5 Logarithmic Equations

2.6 Compound Interest and Annuity

Chapter 3

Limits

3.1 Concept of Limits

3.2 Limits of Functions

3.3 Arithmetic Properties of Limits of Functions

Chapter 4

Differentiation

4.1 Gradient of Tangent Line on a Curve

4.2 Gradient of Tangent Line and Derivative

4.3 Law of Differentiation

4.4 Chain Rule - Differentiation of Composite Functions

4.5 Higher Order Derivatives

4.6 Implicit Differentiation

4.7 Two Basic Limits

4.8 Derivatives of Trigonometric Functions

4.9 Derivatives of Exponential Functions

4.10 Derivatives of Logarithmic Functions