Senior 2 Math Part I

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Contents

Chapter 12

Sequence and Series

12.1 Sequence and Series

12.1.1 Practice 1

1. Find the first 5 terms of the sequence $a_n = \frac{2^n}{n+1}$.

   Sol. $a_1 = \frac{2}{2} = 1, a_2 = \frac{4}{3}, a_3 = \frac{8}{4}, a_4 = \frac{16}{5}, a_5 = \frac{32}{6}$

2. Write the general term of the sequence 1, 8, 27, 64, ...

   Sol. $a_n = n^3$

12.1.2 Practice 2

1. Express the series $\sum_{n=1}^{10} n^2 + 1$ in the form of numbers.

   Sol. $$\sum_{n=1}^{10} n^2 + 1$$
   $$= (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1)$$
   $$+ (5^2 + 1) + (6^2 + 1) + (7^2 + 1)$$
   $$+ (8^2 + 1) + (9^2 + 1) + (10^2 + 1)$$
   $$= 2 + 5 + 10 + 17 + 26 + 37 + 50 + 65$$
   $$+ 82 + 101$$

2. Write the first term, last term and the number of terms of the series $\sum_{n=1}^{10} (3^n - 2^n)$.

   Sol. $$First\ term = (3^1 - 2^1) = 1$$
   $$Last\ term = (3^{10} - 2^{10}) = 59049$$
   $$Number\ of\ terms = 10$$

3. Express the series $2\times5+3\times7+4\times9+\ldots+15\times31$ in the form of $\sum$.

   Sol.
   $$a_1 = 2 \times 5 = 10$$
   $$a_2 = 3 \times 7 = 21$$
   $$a_3 = 4 \times 9 = 36$$
   $$a_4 = 5 \times 11 = 55$$
   $$\vdots$$
   $$a_{15} = 15 \times 31 = 465$$
   $$\therefore 2 \times 5 + 3 \times 7 + 4 \times 9 + \ldots + 15 \times 31$$
   $$= \sum_{n=1}^{15} a_n$$

12.1.3 Exercise 12.1

1. Find the general term of the following sequences.

   (a) 5, 8, 11, 14, ...
   Sol. $a_n = 3n + 2$

   (b) 2, 4, 8, 16, ...
   Sol. $a_n = 2^n$

   (c) $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$
   Sol. $a_n = \frac{n+1}{n}$

   (d) $\frac{2}{5}, \frac{4}{7}, \frac{6}{9}, \frac{8}{11}, \ldots$
   Sol. $a_n = \frac{2n}{2n+1}$

2. Find the first 5 terms of the following sequences.

   (a) $a_n = 2n + 3$
   Sol. $a_1 = 2 \times 1 + 3 = 5, a_2 = 2 \times 2 + 3 = 7, a_3 = 2\times3+3 = 9, a_4 = 2\times4+3 = 11, a_5 = 2 \times 5 + 3 = 13$

   (b) $a_n = n(n - 2)$
   Sol. $a_1 = 1\times(-1) = -1, a_2 = 2\times0 = 0, a_3 = 3 \times 1 = 3, a_4 = 4 \times 2 = 8, a_5 = 5 \times 3 = 15$

   (c) $a_n = \frac{n}{2n+1}$
   Sol. $a_1 = \frac{1}{2\times1+1} = \frac{1}{3}, a_2 = \frac{2}{2\times2+1} = \frac{2}{5}, a_3 = \frac{3}{2\times3+1} = \frac{3}{7}, a_4 = \frac{4}{2\times4+1} = \frac{4}{9}, a_5 = \frac{5}{2\times5+1} = \frac{5}{11}$

   (d) $a_n = (-3)^n$
   Sol. $a_1 = (-3)^1 = -3, a_2 = (-3)^2 = 9, a_3 = (-3)^3 = -27, a_4 = (-3)^4 = 81, a_5 = (-3)^5 = -243$

3. Express the following series in the form of numbers.

(a) $\sum_{n=1}^{5} n(n+3)$

Sol. $$\sum_{n=1}^{5} n(n+3)$$
$$= (1\times 4) + (2\times 5) + (3\times 6) + (4\times 7)$$
$$+ (5\times 8)$$
$$= 4 + 10 + 18 + 28 + 40$$

(b) $\sum_{n=2}^{6} \frac{1}{3^n}$

Sol. $$\sum_{n=2}^{6} \frac{1}{3^n}$$
$$= \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \frac{1}{3^6}$$
$$= \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729}$$

(c) $\sum_{n=1}^{6} \frac{1}{n(2n+1)}$

Sol. $$\sum_{n=1}^{6} \frac{1}{n(2n+1)}$$
$$= \frac{1}{1(2\times 1+1)} + \frac{1}{2(2\times 2+1)}$$
$$+ \frac{1}{3(2\times 3+1)} + \frac{1}{4(2\times 4+1)}$$
$$+ \frac{1}{5(2\times 5+1)} + \frac{1}{6(2\times 6+1)}$$
$$= \frac{1}{3} + \frac{1}{10} + \frac{1}{21} + \frac{1}{36} + \frac{1}{55} + \frac{1}{78}$$

(d) $\sum_{n=2}^{5} \frac{1}{n^2+2}$

Sol. $$\sum_{n=2}^{5} \frac{1}{n^2+2}$$
$$= \frac{1}{4+2} + \frac{1}{9+2} + \frac{1}{16+2} + \frac{1}{25+2}$$
$$= \frac{1}{6} + \frac{1}{11} + \frac{1}{18} + \frac{1}{27}$$

4. Find the first term, last term and the number of terms of the following series.

(a) $\sum_{n=3}^{10} 2^2$
Sol. $a_3 = 2^2 = 4, a_{10} = 2^2 = 4, n = 10 - 3 + 1 = 8$

(b) $\sum_{n=1}^{8} \frac{n+2}{n}$
Sol. $a_1 = \frac{1+2}{1} = \frac{3}{1} = 3, a_8 = \frac{8+2}{8} = \frac{10}{8} = \frac{5}{4}, n = 8 - 1 + 1 = 8$

(c) $\sum_{n=1}^{10} 3n^2 - n$
Sol. $a_1 = 3\times 1^2 - 1 = 2, a_{10} = 3\times 10^2 - 10 = 290, n = 10 - 1 + 1 = 10$

(d) $\sum_{n=9}^{14} n^2(n-7)$
Sol. $a_9 = 9^2(9-7) = 9^2\times 2 = 162, a_{14} = 14^2(14-7) = 14^2\times 7 = 2744, n = 14-9+1 = 6$

5. Express the following series in the form of $\sum$.

(a) $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{30}$

Sol.

$$a_1 = 1$$
$$a_2 = \frac{1}{2}$$
$$a_3 = \frac{1}{3}$$
$$\vdots$$
$$a_{30} = \frac{1}{30}$$
$$\therefore 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{30} = \sum_{n=1}^{30} \frac{1}{n}$$

(b) $1^3 + 2^3 + 3^3 + \ldots + 50^3$

Sol.

$$a_1 = 1^3$$
$$a_2 = 2^3$$
$$a_3 = 3^3$$
$$\vdots$$
$$a_{50} = 50^3$$
$$\therefore 1^3 + 2^3 + 3^3 + \ldots + 50^3 = \sum_{n=1}^{50} n^3$$

(c) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16}$

Sol.

$$
\begin{aligned}
a_1 &= (-\frac{1}{2})^{1-1} \\
a_2 &= (-\frac{1}{2})^{2-1} \\
a_3 &= (-\frac{1}{2})^{3-1} \\
a_4 &= (-\frac{1}{2})^{4-1} \\
a_5 &= (-\frac{1}{2})^{5-1} \\
\therefore 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} & \\
&= \sum_{n=1}^{5} (-\frac{1}{2})^{n-1}
\end{aligned}
$$

(d) $2 \times 4 + 4 \times 7 + 6 \times 10 + 8 \times 13 + 10 \times 16$

Sol.

$$
\begin{aligned}
a_1 &= 2 \times 1 \times (3 \times 1 + 1) \\
a_2 &= 2 \times 2 \times (3 \times 2 + 1) \\
a_3 &= 2 \times 3 \times (3 \times 3 + 1) \\
a_4 &= 2 \times 4 \times (3 \times 4 + 1) \\
a_5 &= 2 \times 5 \times (3 \times 5 + 1) \\
\therefore 2 \times 4 + 4 \times 7 + 6 \times 10 + 8 \times 13 & \\
+ 10 \times 16 &= \sum_{n=1}^{5} 2n(3n+1)
\end{aligned}
$$

12.2 Arithmetic Progression

General term of an Arithmetic Progression (AP) is given by

$$a_n = a_1 + (n-1)d$$

where a_1 is the first term, d is the common difference and n is the number of terms.

12.2.1 Practice 3

1. Find the number of terms of the AP $-4 - 2\frac{3}{4} - 1\frac{1}{2} - \frac{1}{4} + \ldots + 16$.

$$
\begin{aligned}
a_1 &= -4 \\
a_n &= 16 \\
d &= -2\frac{3}{4} - (-4) \\
&= -2\frac{3}{4} + 4 \\
&= \frac{5}{4} \\
16 &= -4 + (n-1)\frac{5}{4} \\
20 &= \frac{5}{4}(n-1) \\
80 &= 5(n-1) \\
n - 1 &= 16 \\
n &= 17
\end{aligned}
$$

2. Given that $a_2 = 4$ and $a_6 = -8$, find the 10th term of the AP.

Sol.

$$
\begin{aligned}
a_2 &= 4 \\
a + (2-1)d &= 4 \\
a_6 &= -8 \\
a + (6-1)d &= -8
\end{aligned}
$$

$$
\begin{cases} a + d &= 4 \quad (12.1) \\ a + 5d &= \text{-8} \quad (12.2) \end{cases}
$$

$$
\begin{aligned}
(2) - (1) : 4d &= -12 \\
d &= -3 \\
a + (-3) &= 4 \\
a &= 7 \\
\therefore a_{10} &= 7 + (10-1)(-3) \\
&= 7 - 27 \\
&= -20
\end{aligned}
$$

3. How many multiples of 7 are there between 50 and 500?

Sol.

$$
\begin{aligned}
a_1 &= 56 \\
a_n &= 497 \\
d &= 7 \\
497 &= 56 + (n-1)7 \\
441 &= 7(n-1) \\
n - 1 &= 63 \\
n &= 64
\end{aligned}
$$

4. Find 5 numbers between 30 and 54 such that these numbers form an AP.

Sol.

$$\begin{aligned} a_1 &= 30 \\ a_7 &= 54 \\ 54 &= 30 + (7-1)d \\ 24 &= 6d \\ d &= 4 \end{aligned}$$

$$\therefore \textit{These } 5 \textit{ numbers are } 34,\ 38,\ 42,\ 46, \textit{ and } 50.$$

Arithmetic mean

If A is in between x and y, and x, A, y are in AP, then

$$A = \frac{x+y}{2}$$

12.2.2 Practice 4

1. If 9, x, 17 are in AP, find x.

Sol.

$$\begin{aligned} x &= \frac{9+17}{2} \\ &= \frac{26}{2} \\ &= 13 \end{aligned}$$

2. Find the arithmetic mean of 26 and -11.

Sol.

$$\begin{aligned} A &= \frac{26-11}{2} \\ &= \frac{15}{2} \end{aligned}$$

3. Find x and y when 3, x, 12, y, 21 are in AP.

Sol.

$$\begin{aligned} x &= \frac{3+12}{2} \\ &= \frac{15}{2} \\ y &= \frac{12+21}{2} \\ &= \frac{33}{2} \end{aligned}$$

Summation of Arithmetic Progression

The summation formula for AP is given by

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

or

$$S_n = \frac{n}{2}(a_1 + a_n)$$

12.2.3 Practice 5

1. Find the sum of the first 16 terms of the AP $22 + 18 + 14 + 10 + \ldots$

Sol.

$$\begin{aligned} a_1 &= 22 \\ n &= 16 \\ d &= -4 \\ S_n &= \frac{16}{2}(2 \times 22 + (-4)(16-1)) \\ &= \frac{16}{2}(44 + (-4)(15)) \\ &= \frac{16}{2}(44 - 60) \\ &= \frac{16}{2}(-16) \\ &= -128 \end{aligned}$$

2. If the sum of AP $23 + 19 + 15 + \ldots$ is 72, find the number of terms.

Sol.

$$\begin{aligned} a_1 &= 23 \\ S_n &= 72 \\ d &= -4 \\ 72 &= \frac{n}{2}(2 \times 23 + (-4)(n-1)) \\ 72 &= \frac{n}{2}(46 + (-4)(n-1)) \\ 144 &= n(46 + (-4)(n-1)) \\ 144 &= n(46 - 4n + 4) \\ 144 &= n(50 - 4n) \\ 144 &= 50n - 4n^2 \\ 72 &= 25n - 2n^2 \\ 2n^2 - 25n + 72 &= 0 \\ (n-8)(2n-9) &= 0 \\ n &= 8 \end{aligned}$$

3. Given that $S_n = 2n + 3n^2$, find the first term and the common difference of the AP.

   Sol.

$$\begin{aligned} S_n &= 2n + 3n^2 \\ 2n + 3n^2 &= \frac{n}{2}(2a + (n-1)d) \\ 4n + 6n^2 &= n(2a + (n-1)d) \\ 4n + 6n^2 &= 2na + (n-1)nd \\ 4n + 6n^2 &= 2na + n^2d - nd \\ 4n + 6n^2 &= (2a-d)n + dn^2 \end{aligned}$$

$$\begin{aligned} &\textit{Comparing both sides,} \\ &2a - d = 4 \\ &a = 6 \\ &d = 2 \end{aligned}$$

12.2.4 Exercise 12.2

1. Find the 10th terms of the AP $5, 13, 21, \ldots$

   Sol.

$$\begin{aligned} a_1 &= 5 \\ n &= 10 \\ d &= 8 \\ a_{10} &= 5 + (10-1) \times 8 \\ &= 5 + 72 \\ &= 77 \end{aligned}$$

2. Find the 8th term of the AP $5, 4\frac{1}{4}, 3\frac{1}{2}, 2\frac{3}{4}, \ldots$

   Sol.

$$\begin{aligned} a_1 &= 5 \\ n &= 8 \\ d &= -\frac{3}{4} \\ a_8 &= 5 + (8-1) \times -\frac{3}{4} \\ &= 5 - \frac{3}{4} \times 7 \\ &= 5 - \frac{21}{4} \\ &= -\frac{1}{4} \end{aligned}$$

3. Find the number of terms of the following AP.

   (a) $4, 9, \ldots, 64$

   Sol.

$$\begin{aligned} a_1 &= 4 \\ a_n &= 64 \\ d &= 5 \\ 64 &= 4 + (n-1) \times 5 \\ 60 &= 5(n-1) \\ 12 &= n - 1 \\ n &= 13 \end{aligned}$$

   (b) $4\frac{1}{3}, 3\frac{2}{3}, 3, \ldots, -10\frac{1}{3}$

   Sol.

$$\begin{aligned} a_1 &= 4\frac{1}{3} \\ a_n &= -10\frac{1}{3} \\ d &= -\frac{2}{3} \\ -10\frac{1}{3} &= 4\frac{1}{3} + (n-1) \times -\frac{2}{3} \\ -\frac{31}{3} &= \frac{13}{3} - \frac{1}{3}(n-1) \\ -31 &= 13 - 2n + 2 \\ -46 &= 2n \\ n &= 23 \end{aligned}$$

4. The 6th term of an AP is 43, and its 10th term is 75. Find the first term and common difference of this AP.

   Sol.

$$\begin{aligned} a_6 &= 43 \\ a_{10} &= 75 \\ 43 &= a + (6-1)d \\ 75 &= a + (10-1)d \\ 32 &= 4d \\ d &= 8 \\ 43 &= a + 5 \times 8 \\ 43 &= a + 40 \\ 3 &= a \\ a &= 3 \\ &\therefore a_1 = 3,\ d = 8 \end{aligned}$$

5. The 7th term of an AP is -10, and the 12th term -25, find the 15th term of this AP.

Sol.

$$
\begin{aligned}
a_7 &= -10 \\
a_{12} &= -25 \\
-10 &= a + (7-1)d \\
-25 &= a + (12-1)d \\
-15 &= 5d \\
d &= -3 \\
-10 &= a + 6 \times -3 \\
-10 &= a - 18 \\
a &= 8 \\
a_{15} &= 8 + (15-1) \times -3 \\
&= 8 - 42 \\
&= -34
\end{aligned}
$$

6. How many multiples of 7 are there between 100 and 200?

   Sol.

$$
\begin{aligned}
a &= 105 \\
d &= 7 \\
a_n &= 196 \\
196 &= 105 + (n-1) \times 7 \\
91 &= 7(n-1) \\
13 &= n - 1 \\
n &= 14
\end{aligned}
$$

7. Find the arithmetic mean o fthe following number pairs.

   (a) $(8, 20)$

   Sol.

$$\frac{8+20}{2} = 14$$

   (b) $(-9, 17)$

   Sol.

$$\frac{-9+17}{2} = 4$$

8. Find 5 numbers between 22 and 58 such that these 7 numbers are in AP.

   Sol.

$$
\begin{aligned}
&a_1 = 22 \\
&a_7 = 58 \\
&58 = 22 + (7-1)d \\
&36 = 6d \\
&d = 6 \\
&\therefore \textit{These } 5 \textit{ numbers are } 22, 28, 34, 40, 46
\end{aligned}
$$

9. Find the sum of first 20 terms of AP $12 + 15 + 18 + \ldots$

   Sol.

$$
\begin{aligned}
a_1 &= 12 \\
n &= 20 \\
d &= 3 \\
S_{20} &= \frac{20}{2}(2 \times 12 + (20-1) \times 3) \\
&= 10(24 + 57) \\
&= 10(81) \\
&= 810
\end{aligned}
$$

10. Find the sum of first 12 terms of the AP $18 + 10 + 2 - 6 - \ldots$

    Sol.

$$
\begin{aligned}
a_1 &= 18 \\
n &= 12 \\
d &= -8 \\
S_{12} &= \frac{12}{2}(2 \times 18 + (12-1) \times -8) \\
&= 6(36 - 88) \\
&= 6(-52) \\
&= -312
\end{aligned}
$$

11. Find the sum of first 14 terms of the AP $\frac{1}{6} + \frac{4}{3} + \frac{5}{2} + \ldots$

Sol.

$$\begin{aligned}
a_1 &= \frac{1}{6}\\
n &= 14\\
d &= \frac{7}{6}\\
S_{14} &= \frac{14}{2}(2\times\frac{1}{6}+(14-1)\times\frac{7}{6})\\
&= 7(\frac{1}{3}+\frac{91}{6})\\
&= 7\times\frac{93}{6}\\
&= 7\times\frac{31}{2}\\
&= \frac{217}{2}
\end{aligned}$$

12. Find the sum of all the multiples of 13 in between 200 and 800.

Sol.

$$\begin{aligned}
a_1 &= 208\\
a_n &= 793\\
d &= 13\\
793 &= 208+(n-1)\times 13\\
585 &= 13(n-1)\\
45 &= n-1\\
n &= 46
\end{aligned}$$

$$\begin{aligned}
S_{46} &= \frac{46}{2}(2\times 208+(46-1)\times 13)\\
&= 23(416+585)\\
&= 23(1001)\\
&= 23023
\end{aligned}$$

13. If the sum of first n terms of the AP $-3, -7, -11, \ldots$ is -903, find the value of n.

Sol.

$$\begin{aligned}
a_1 &= -3\\
d &= -4\\
-903 &= \frac{n}{2}(2\times(-3)-4(n-1))\\
-1806 &= -2n-4n^2\\
4n^2+2n-1806 &= 0\\
2n^2+n-903 &= 0\\
(n-21)(2n+43) &= 0\\
n &= 21, -43(invalid)\\
\therefore\ n &= 21
\end{aligned}$$

14. Given that the first 3 terms of an AP are $x,\ 3x-4,\ 2x+7$, find:

(a) The value of x

Sol.

$$\begin{aligned}
3x-4 &= \frac{x+2x+7}{2}\\
6x-8 &= 3x+7\\
3x &= 15\\
x &= 5
\end{aligned}$$

(b) The common difference

Sol.

$$\begin{aligned}
a_1 &= x = 5\\
a_2 &= 3x-4 = 3\times 5-4 = 11\\
d &= 11-5\\
&= 6
\end{aligned}$$

(c) The sum of first 10 terms.

Sol.

$$\begin{aligned}
a_1 &= x = 5\\
n &= 10\\
d &= 6\\
S_{10} &= \frac{10}{2}(2\times 5+(10-1)\times 6)\\
&= 5(10+54)\\
&= 5(64)\\
&= 320
\end{aligned}$$

15. Let the sum of the first n terms of an AP to be $S_n = \frac{n(n+1)}{4}$, find:

(a) The first term

Sol.

$$\begin{aligned}\frac{n(n+1)}{4} &= \frac{n}{2}(2a+(n-1)d)\\ n(n+1) &= 2n(2a+dn-d)\\ n^2+n &= 4na+2dn^2-2nd\\ n^2+n &= 2dn^2+(4a-2d)n\end{aligned}$$

$$\begin{aligned}&\mathit{Comparing\ both\ sides,}\\ 2d &= 1\\ d &= \frac{1}{2}\\ 4a-2d &= 1\\ 4a-1 &= 1\\ 4a &= 2\\ a &= \frac{1}{2}\end{aligned}$$

(b) The common difference

Sol.

$$d = \frac{1}{2}$$

gg

(c) The 6th terms

Sol.

$$\begin{aligned}a_1 &= \frac{1}{2}\\ n &= 6\\ d &= \frac{1}{2}\\ a_6 &= \frac{1}{2}+(6-1)\times\frac{1}{2}\\ &= \frac{1}{2}+\frac{5}{2}\\ &= 3\end{aligned}$$

(d) The sum from 6th term to 10th term

Sol.

$$\begin{aligned}a &= \frac{1}{2}\\ d &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}S_{10} &= \frac{10}{2}(2\times\frac{1}{2}+(10-1)\times\frac{1}{2})\\ &= \frac{10}{2}(1+\frac{9}{2})\\ &= 5\times\frac{11}{2}\\ &= \frac{55}{2}\end{aligned}$$

$$\begin{aligned}S_5 &= \frac{5}{2}(2\times\frac{1}{2}+(5-1)\times\frac{1}{2})\\ &= \frac{5}{2}(1+2)\\ &= \frac{15}{2}\end{aligned}$$

$$\begin{aligned}S_{10}-S_6 &= \frac{55}{2}-\frac{15}{2}\\ &= \frac{40}{2}\\ &= 20\end{aligned}$$

16. Given three numbers in an AP, the sum of these three numbers is 30, and the sum of square of these numbers is 318, find these three numbers.

Sol.

$$\begin{aligned}
a_1 + a_2 + a_3 &= 30 \\
a_1^2 + a_2^2 + a_3^2 &= 318 \\
a_2 - a_1 &= a_3 - a_2 \\
a_1 - 2a_2 + a_3 &= 0 \\
3a_2 &= 30 \\
a_2 &= 10 \\
a_1 - 20 + a_3 &= 0 \\
a_1 + a_3 &= 20 \\
a_3 &= 20 - a_1 \\
a_1^2 + 100 + (20 - a_1)^2 &= 318 \\
a_1^2 + 100 + 400 + a_1^2 - 40a_1 &= 318 \\
2a_1^2 - 40a_1 + 182 &= 0 \\
a_1^2 - 20a_1 + 91 &= 0 \\
(a_1 - 7)(a_1 - 13) &= 0 \\
a_1 &= 7 or a_1 = 13
\end{aligned}$$

$$\therefore\ These\ three\ numbers\ are\ 7,\ 10,\ and\ 13$$

17. Find the sum of all the numbers between 100 and 200 that are both the multiples of 2 and 3.

Sol.

$$\begin{aligned}
a_1 &= 102 \\
d &= 6 \\
a_n &= 198 \\
198 &= 102 + (n-1) \times 6 \\
96 &= 6(n-1) \\
6n - 6 &= 96 \\
6n &= 102 \\
n &= 17
\end{aligned}$$

$$\begin{aligned}
S_{17} &= \frac{17}{2}(2 \times 102 + (17-1) \times 6) \\
&= \frac{17}{2}(204 + 96) \\
&= \frac{17}{2}(300) \\
&= 150 \times 17 \\
&= 2550
\end{aligned}$$

18. Given an AP $-100 - 96 - 92 - \ldots$:

(a) Find the term where the number become positive.

Sol.

$$\begin{aligned}
a_1 &= -100 \\
d &= 4 \\
a_n &= -100 + (n-1) \times 4 > 0 \\
-100 + 4n - 4 &> 0 \\
4n &> 104 \\
n &> 26
\end{aligned}$$

$$\therefore\ n = 27$$

(b) Find the term where the sum of this AP becomes positive.

Sol.

$$\begin{aligned}
S_n &= \frac{n}{2}(2(-100) + (n-1) \times (4)) > 0 \\
\frac{n}{2}(-200 + 4n - 4) &> 0 \\
\frac{n}{2}(-204 + 4n) &> 0 \\
n(2n - 102) &> 0 \\
n(n - 51) &> 0 \\
n &> 51
\end{aligned}$$

$$\therefore\ n = 52$$

19. Find the first negative term of the AP $20, 19\frac{1}{5}, 18\frac{2}{5}, \ldots$

Sol.

$$\begin{aligned}
a_1 &= 20 \\
d &= -\frac{4}{5} \\
a_n &= 20 + (n-1) \times (-\frac{4}{5}) < 0 \\
100 - 4n + 4 &< 0 \\
4n &> 104 \\
n &> 26
\end{aligned}$$

$$\therefore\ n = 27$$

20. Given an AP $10 + 9\frac{1}{5} + 8\frac{2}{5} + \ldots$, what is the first negative term? When will the sum of the terms become negative, and what's the value of it?

Sol.

$$\begin{aligned}
a_n &= 10 + (n-1) \times (-\frac{4}{5}) < 0 \\
10 - \frac{4}{5}(n-1) &< 0 \\
50 - 4n + 4 &< 0 \\
-4n &< -54 \\
n &> 13\frac{1}{2} \\
\\
\therefore\ n &= 14
\end{aligned}$$

$$\begin{aligned}
S_n = \frac{n}{2}(2 \times 10 + (n-1) \times (-\frac{4}{5})) &< 0 \\
\frac{n}{2}(20 - \frac{4}{5}(n-1)) &< 0 \\
20n - \frac{4}{5}(n^2 - n) &< 0 \\
100n - 4n^2 + 4n &< 0 \\
25n - n^2 + n &< 0 \\
26n - n^2 &< 0 \\
n(n-26) &> 0 \\
n &> 26 \\
\\
\therefore\ n &= 27
\end{aligned}$$

$$\begin{aligned}
S_{27} &= \frac{27}{2}(2 \times 10 + (27-1) \times (-\frac{4}{5})) \\
&= \frac{27}{2}(20 - \frac{4}{5}(27-1)) \\
&= \frac{27}{2}(20 - \frac{4}{5}(26)) \\
&= \frac{27}{2} \times (-\frac{4}{5}) \\
&= -\frac{54}{5}
\end{aligned}$$

$\therefore$ *The first negative term is the 14th term*

$\therefore$ *The first term where the sum of the terms becomes negative is the 27th term*

$\therefore$ *The value of the sum of the terms when it becomes negative is* $-\frac{54}{5}$

21. Given a polygon which all their internal angles are in AP. The common difference of this AP is 6°, the largest angle is 135°. How many sides does this polygon have?

Sol.

$$\begin{aligned}
a_1 &= 135 \\
d &= -6 \\
\frac{n}{2}(2 \times 135 + (n-1) \times (-6)) &= 180(n-2) \\
n(270 - 6(n-1)) &= 360(n-2) \\
n(276 - 6n) &= 360n - 720 \\
276n - 6n^2 &= 360n - 720 \\
46n - n^2 &= 60n - 120 \\
n^2 + 14n - 120 &= 0 \\
(n+20)(n-6) &= 0 \\
n &= -20\ (invalid) \\
n &= 6 \\
\therefore\ \textit{The number of sides is } 6
\end{aligned}$$

22. Given an AP which its 5th term is 3 and the sum of its first 10 terms is $26\frac{1}{4}$. Whcih term in this AP is 0?

Sol.

$$\begin{aligned}
a_5 = a + (5-1)d &= 3 \\
a + 4d &= 3 \\
S_{10} = \frac{10}{2}(2a + (10-1)d) &= 26\frac{1}{4} \\
5(2a + 9d) &= 26\frac{1}{4} \\
20(2a + 9d) &= 105 \\
4(2a + 9d) &= 21 \\
8a + 36d &= 21 \\
8a + 32d &= 24 \\
4d &= -3 \\
d &= -\frac{3}{4} \\
a &= 3 + \frac{3}{4} \times 4 \\
&= 6 \\
a_n = 6 + (n-1) \times (-\frac{3}{4}) &= 0 \\
6 - \frac{3}{4}(n-1) &= 0 \\
24 - 3n + 3 &= 0 \\
3n &= 27 \\
n &= 9
\end{aligned}$$

23. Given that the sum of the first 6 terms of an AP is 96, and the sum of the first 20 terms is 3 times the sum of the first 10 terms of this AP. Find the first term and the 10th term of it.

Sol.

$$
\begin{aligned}
S_6 = \frac{6}{2}(2a + (6-1)d) &= 96 \\
3(2a+5d) &= 96 \\
2a+5d &= 32 \\
S_{20} &= 3S_{10} \\
\frac{20}{2}(2a + (20-1)d) &= 3 \times \frac{10}{2}(2a \\
&\quad + (10-1)d) \\
10(2a+19d) &= 15(2a+9d) \\
2(2a+19d) &= 3(2a+9d) \\
4a+38d &= 6a+27d \\
2a-11d &= 0 \\
16d &= 32 \\
d &= 2 \\
a &= \frac{11 \times 2}{2} \\
&= 11 \\
a_{10} &= 11 + (10-1) \times 2 \\
&= 29
\end{aligned}
$$

24. Given that $5^2 \times 5^4 \times 5^6 \times \ldots \times 5^{2n} = (0.04)^{-28}$, find the value of n.

Sol.

$$
\begin{aligned}
(0.04)^{-28} &= \frac{1}{25}^{-28} \\
&= (5^{-2})^{-28} \\
&= 5^{56} \\
\because n^a \times n^b &= n^{a+b} \\
2+4+6+\ldots+2n &= 56 \\
S_n = \frac{n}{2}(2 \times 2 + (n-1) \times 2) &= 56 \\
n(4+2(n-1)) &= 112 \\
n(2+2n) &= 112 \\
2n^2+2n &= 112 \\
n^2+n-56 &= 0 \\
(n+8)(n-7) &= 0 \\
n &= -8 \ (invalid) \\
n &= 7
\end{aligned}
$$

25. Given that the 9th term of an AP is double the 5th term of it. Find the ratio of the sum of first 9 terms and the sum of first 5 terms of the AP.

Sol.

$$
\begin{aligned}
a_9 &= 2a_5 \\
a + (9-1)d &= 2(a + (5-1)d) \\
a+8d &= 2a+8d \\
a &= 0 \\
S_9 : S_5 &= \frac{9}{2}(2a+a_9) : \frac{5}{2}(2a+a_5) \\
&= \frac{9}{2}(2a+2a_5) : \frac{5}{2}(2a+a_5) \\
&= 9(a+a_5) : \frac{5}{2}(2a+a_5) \\
\frac{S_9}{S_5} &= \frac{9(a+a_5)}{\frac{5}{2}(2a+a_5)} \\
&= \frac{18(a+a_5)}{5(2a+a_5)} \\
&= \frac{18 \times a_5}{5 \times a_5} \\
&= \frac{18}{5} \\
\therefore S_9 : S_5 &= 18 : 5
\end{aligned}
$$

12.3 Geometric Progression

The general formula of a geometric progression (GP) is given by

$$a_n = a_1 \times r^{n-1}$$

where a_1 is the first term, r is the common ratio, and n is the number of terms.

12.3.1 Practice 6

1. Find the 6th term of the GP $12, -18, 27, \ldots$

Sol.

$$
\begin{aligned}
a_1 &= 12 \\
r &= \frac{-18}{12} \\
&= -\frac{3}{2} \\
a_6 &= 12 \times (-\frac{3}{2})^{6-1} \\
&= 12 \times (-\frac{3}{2})^5 \\
&= 12 \times (-\frac{243}{32}) \\
&= -\frac{729}{8}
\end{aligned}
$$

2. Find the number of terms of GP $\frac{1}{64} - \frac{1}{32} + \frac{1}{16} - \frac{1}{8} + \ldots - 512$

   Sol.

$$
\begin{aligned}
a_1 &= \frac{1}{64} \\
r &= \frac{-\frac{1}{32}}{\frac{1}{64}} \\
&= -2 \\
-512 &= \frac{1}{64}(-2)^{n-1} \\
(-2)^9 &= \frac{1}{2^6}(-2)^{n-1} \\
(-2)^{15} &= (-2)^{n-1} \\
n-1 &= 15 \\
n &= 16
\end{aligned}
$$

3. The 5th term of a GP is 3, and its 9th term is $\frac{1}{27}$, find the first term and the common ratio of this GP.

   Sol.

$$
\begin{aligned}
a_5 &= ar^4 = 3 \\
a_9 &= ar^8 = \frac{1}{27} \\
r^4 &= \frac{1}{27} \times \frac{1}{3} \\
&= \frac{1}{81} \\
r &= \frac{1}{3} \\
a_1 &= 3 \times 81 \\
&= 243
\end{aligned}
$$

4. Find 5 numbers between $\frac{1}{2}$ and $frac1128$ such that these 7 numbers are in GP.

   Sol.

$$
\begin{aligned}
a_1 &= \frac{1}{2} \\
n &= 7 \\
\frac{1}{128} &= \frac{1}{2}r^{7-1} \\
r^6 &= \frac{1}{64} \\
r &= \frac{1}{2}
\end{aligned}
$$

$$\therefore\ \textit{These } 5 \textit{ numbers are } \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$$

Geometric Mean

The geometric mean G of two numbers x and y is given by

$$
\begin{aligned}
\frac{G}{x} &= \frac{G}{y} \\
G^2 &= xy \\
G &= \mp \sqrt[2]{xy}
\end{aligned}
$$

12.3.2 Practice 7

Find the geometric mean of $\frac{27}{8}$ and $\frac{2}{3}$.

Sol.

$$
\begin{aligned}
G &= \pm \sqrt[2]{\frac{27}{8} \times \frac{2}{3}} \\
&= \pm \sqrt[2]{\frac{9}{4}} \\
&= \pm \frac{3}{2}
\end{aligned}
$$

Summation of Geometric Progression

The sum of n terms of a GP is given by

$$S_n = \frac{a_1(1-r^n)}{1-r} \ (r \neq 1)$$

12.3.3 Practice 8

1. Find the sum of the first 8 terms of GP $3 + 6 + 12 + \ldots$

Sol.

$$\begin{aligned}
a_1 &= 3 \\
r &= \frac{6}{3} \\
&= 2 \\
n &= 8 \\
S_n &= \frac{3(1-2^8)}{1-2} \\
&= \frac{3(1-256)}{1-2} \\
&= 3 \times 255 \\
&= 765
\end{aligned}$$

2. Find the sum of the GP $1 + \sqrt{3} + 3 + \ldots + 81$

Sol.

$$\begin{aligned}
a_1 &= 1 \\
r &= \sqrt{3} \\
81 &= 1 \times (\sqrt{3})^{n-1} \\
3^4 &= (\sqrt{3})^{n-1} \\
(\sqrt{3})^8 &= (\sqrt{3})^{n-1} \\
n-1 &= 8 \\
n &= 9 \\
S_n &= \frac{1(1-(\sqrt{3})^9)}{1-\sqrt{3}} \\
&= \frac{1-81\sqrt{3}}{1-\sqrt{3}} \\
&= \frac{(1-81\sqrt{3})(1+\sqrt{3})}{-2} \\
&= \frac{1-81\sqrt{3}+\sqrt{3}-243}{-2} \\
&= \frac{-242-80\sqrt{(3)}}{-2} \\
&= 121 + 40\sqrt{3}
\end{aligned}$$

3. Given that the sum of the first n terms of GP $4\frac{4}{5}, 1\frac{3}{5}, \frac{8}{15}, \ldots$ is $7\frac{145}{729}$, find n.

Sol.

$$\begin{aligned}
a_1 &= \frac{24}{5} \\
r &= \frac{8}{5} \times \frac{5}{24} \\
&= \frac{1}{3} \\
S_n &= \frac{24}{5} \times \frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}} \\
\frac{5248}{729} &= \frac{24}{5} \times \frac{1-(\frac{1}{3})^n}{\frac{2}{3}} \\
\frac{5248}{729} \times \frac{5}{24} \times \frac{2}{3} &= 1-(\frac{1}{3})^n \\
\frac{6560}{6561} &= 1-(\frac{1}{3})^n \\
-\frac{1}{6561} &= -(\frac{1}{3})^n \\
(\frac{1}{3})^8 &= (\frac{1}{3})^n \\
n &= 8
\end{aligned}$$

Summation of Infinite Geometric Progression

The sum of infinite GP is given by

$$S_\infty = \frac{a_1}{1-r} \ (-1 < r < 1)$$

12.3.4 Practice 9

1. Find the sum of the following infinite GP.

(a) $16 + 8 + 4 + \ldots$

Sol.

$$\begin{aligned}
a_1 &= 16 \\
r &= \frac{8}{16} \\
&= \frac{1}{2} \\
S_\infty &= \frac{16}{1-\frac{1}{2}} \\
&= \frac{16}{\frac{1}{2}} \\
&= 32
\end{aligned}$$

(b) $18 - 12 + 8 + \ldots$

Sol.

$$\begin{aligned} a_1 &= 18 \\ r &= \frac{8}{-12} \\ &= -\frac{2}{3} \\ S_\infty &= \frac{18}{1+\frac{2}{3}} \\ &= \frac{18}{\frac{5}{3}} \\ &= \frac{54}{5} \end{aligned}$$

(c) $1 + \frac{3}{4} + \frac{9}{16} + \dots$

Sol.

$$\begin{aligned} a_1 &= 1 \\ r &= \frac{9}{16} \times \frac{16}{9} \\ &= \frac{3}{4} \\ S_\infty &= \frac{1}{1-\frac{3}{4}} \\ &= \frac{1}{\frac{1}{4}} \\ &= 4 \end{aligned}$$

(d) $\sqrt{2} + 1 + \frac{1}{\sqrt{2}} + \dots$

Sol.

$$\begin{aligned} a_1 &= \sqrt{2} \\ r &= \frac{1}{\sqrt{2}} \\ S_\infty &= \frac{\sqrt{2}}{1-\frac{1}{\sqrt{2}}} \\ &= \frac{\sqrt{2}}{\frac{\sqrt{2}-1}{\sqrt{2}}} \\ &= \frac{2}{\sqrt{2}-1} \\ &= 2(\sqrt{2}+1) \end{aligned}$$

2. Convert the following recurring decimals to fraction using the summation of inifinite geometric series.

(a) $0.\overline{3}$

Sol.

$$\begin{aligned} a_1 &= 0.3 \\ r &= 0.1 \\ S_\infty &= \frac{0.3}{1-0.1} \\ &= \frac{0.3}{0.9} \\ &= \frac{1}{3} \end{aligned}$$

$$\therefore 0.\overline{3} = \frac{1}{3}$$

(b) $0.5\overline{3}$

Sol.

$$\begin{aligned} a_1 &= 0.03 \\ r &= 0.01 \\ S_\infty &= \frac{0.03}{1-0.01} \\ &= \frac{0.03}{0.99} \\ &= \frac{3}{99} \end{aligned}$$

$$\begin{aligned} \therefore 0.5\overline{3} &= \frac{5}{10} + \frac{3}{99} \\ &= \frac{53}{99} \end{aligned}$$

12.3.5 Exercise 12.3

1. Find the 10th term of the GP $2, 4, 8, \dots$

Sol.

$$\begin{aligned} a_1 &= 2 \\ r &= \frac{4}{2} \\ &= 2 \\ a_{10} &= 2 \times 2^{10-1} \\ &= 2 \times 512 \\ &= 1024 \end{aligned}$$

2. Find the 8th term of the GP $243, -162, 108, \dots$

Sol.

$$
\begin{aligned}
a_1 &= 243 \\
r &= \frac{-162}{243} \\
&= -\frac{2}{3} \\
a_8 &= 243 \times (-\frac{2}{3})^{8-1} \\
&= 243 \times (-\frac{128}{2187}) \\
&= -\frac{128}{9}
\end{aligned}
$$

3. Find the number of terms of the following GP.

   (a) $8, 4, 2, 1, \ldots, \frac{1}{64}$

   Sol.

$$
\begin{aligned}
a_1 &= 8 \\
r &= \frac{4}{8} \\
&= \frac{1}{2} \\
\frac{1}{64} &= 8 \times (\frac{1}{2})^{n-1} \\
\frac{1}{512} &= (\frac{1}{2})^{n-1} \\
\frac{1}{2^9} &= (\frac{1}{2})^{n-1} \\
n-1 &= 9 \\
n &= 10
\end{aligned}
$$

   (b) $6, -18, 54, \ldots, -13122$

   Sol.

$$
\begin{aligned}
a_1 &= 6 \\
r &= \frac{-18}{6} \\
&= -3 \\
-13122 &= 6 \times (-3)^{n-1} \\
-2187 &= (-3)^{n-1} \\
(-3)^7 &= (-3)^{n-1} \\
n-1 &= 7 \\
n &= 8
\end{aligned}
$$

   (c) $54, 36, 24, \ldots, 3\frac{13}{81}$

   Sol.

$$
\begin{aligned}
a_1 &= 54 \\
r &= \frac{36}{54} \\
&= \frac{2}{3} \\
\frac{256}{81} &= 54 \times (\frac{2}{3})^{n-1} \\
\frac{256}{81} \times \frac{1}{54} &= (\frac{2}{3})^{n-1} \\
\frac{128}{2187} &= (\frac{2}{3})^{n-1} \\
(\frac{2}{3})^7 &= (\frac{2}{3})^{n-1} \\
n-1 &= 7 \\
n &= 8
\end{aligned}
$$

4. Given that the 2nd term of a GP is 12, and its 4th term is 108, find the first term and the common ratio of it.

   Sol.

$$
\begin{aligned}
a_2 &= ar = 12 \\
a_4 &= ar^3 = 109 \\
r^2 &= 9 \\
r &= \pm 3 \\
a_1 &= \pm 4 \\
\therefore a_1 &= 4, r = 3 \text{ or } a_1 = -4, r = -3
\end{aligned}
$$

5. Given that the 3rd term of an GP is $1\frac{1}{3}$, and its 8th term is $-10\frac{1}{8}$. Find the 5th term of this AP.

Sol.

$$a_3 = ar^2 = \frac{4}{3}$$
$$a_8 = ar^7 = -\frac{81}{8}$$
$$r^5 = -\frac{81}{8} \times \frac{3}{4}$$
$$= -\frac{243}{32}$$
$$= (-\frac{3}{2})^5$$
$$r = -\frac{3}{2}$$
$$a = \frac{4}{3} \times \frac{4}{9}$$
$$= \frac{16}{27}$$
$$a_5 = \frac{16}{27} \times (\frac{3}{2})^4$$
$$= \frac{16}{27} \times \frac{81}{16}$$
$$= 3$$

6. Find the geometric mean of 2 and 18.

Sol.

$$G = \pm\sqrt[2]{2 \times 18}$$
$$= \pm\sqrt[2]{36}$$
$$= \pm 6$$

7. Given that x+12, x+4 and x-2 are in GP, find the value of x and the common ratio of this GP.

Sol.

$$x + 4 = \pm\sqrt{(x+12)(x-2)}$$
$$x^2 + 8x + 16 = x^2 + 10x - 24$$
$$2x = 40$$
$$x = 20$$
$$a_1 = 20 + 12 = 32$$
$$a_2 = 20 + 4 = 24$$
$$r = \frac{24}{32}$$
$$= \frac{3}{4}$$

8. Find 3 numbers between 14 and 224 such that these 5 numbersare in GP.

Sol.

$$a_1 = 14$$
$$a_5 = 224$$
$$244 = 14 \times r^4$$
$$16 = r^4$$
$$(\pm 2)^4 = r^4$$
$$r = \pm 2$$

$$\therefore\ These\ 3\ numbers\ are\ 28,\ 56,\ 112$$
$$or\ \ -28,\ 56,\ -112$$

9. Calculate the sum of the first 6 terms of the GP $2 + 6 + 18 + \ldots$

Sol.

$$a_1 = 2$$
$$r = \frac{6}{2}$$
$$= 3$$
$$S_6 = \frac{2(1-3^6)}{1-3}$$
$$= \frac{2(1-729)}{-2}$$
$$= 728$$

10. Calculate the sum of the first 8 terms of the GP $32 - 16 + 8 - \ldots$

Sol.

$$a_1 = 32$$
$$r = \frac{-16}{32}$$
$$= -\frac{1}{2}$$
$$S_8 = \frac{32(1-(\frac{1}{2})^8)}{1+\frac{1}{2}}$$
$$= \frac{32(1-\frac{1}{256})}{\frac{3}{2}}$$
$$= 32 \times \frac{255}{256} \times \frac{2}{3}$$
$$= \frac{85}{4}$$

11. Find the sum of the GP $14 - 28 + 56 - \ldots + 3584$

Sol.

$$\begin{aligned}
a_1 &= 14\\
r &= \frac{-28}{14} = -2\\
3584 &= 14 \times (-2)^{n-1}\\
256 &= (-2)^{n-1}\\
(-2)^8 &= (-2)^{n-1}\\
n-1 &= 8\\
n &= 9\\
S_9 &= \frac{14(1-(-2)^9)}{1-(-2)}\\
&= \frac{14(1+512)}{3}\\
&= \frac{14 \times 513}{3}\\
&= 2394
\end{aligned}$$

12. If the first term of a GP is 7, its common ratio is 3, and the sum of its terms is 847, find the number of terms and the last term of this GP.

Sol.

$$\begin{aligned}
a_1 &= 7\\
r &= 3\\
S_n &= \frac{7(1-3^n)}{1-3} = 847\\
7(1-3^n) &= -1694\\
1-3^n &= -242\\
3^n &= 243\\
3^n &= 3^5\\
n &= 5\\
a_5 &= 7 \times 3^4 = 567
\end{aligned}$$

13. Find the sum of the following infinite GP.

(a) $24 + 18 + 13\frac{1}{2} + \dots$

Sol.

$$\begin{aligned}
a_1 &= 24\\
r &= \frac{18}{24} = \frac{3}{4}\\
S_\infty &= \frac{24}{1-\frac{3}{4}}\\
&= \frac{24}{\frac{1}{4}}\\
&= 96
\end{aligned}$$

(b) $27 - 9 + 3 - 1 + \dots$

Sol.

$$\begin{aligned}
a_1 &= 27\\
r &= \frac{-9}{27} = -\frac{1}{3}\\
S_\infty &= \frac{27}{1+\frac{1}{3}}\\
&= \frac{27}{\frac{4}{3}}\\
&= \frac{81}{4}
\end{aligned}$$

(c) $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \dots$

Sol.

$$\begin{aligned}
a_1 &= 2\\
r &= \frac{-\frac{1}{2}}{2} = -\frac{1}{4}\\
S_\infty &= \frac{2}{1+\frac{1}{4}}\\
&= \frac{2}{\frac{5}{4}}\\
&= \frac{8}{5}
\end{aligned}$$

14. Given an infinite GP which has a sum of 24 and first term of 30, find the common difference.

Sol.

$$\begin{aligned}
a_1 &= 30\\
S_\infty &= 24\\
24 &= \frac{30}{1-r}\\
24(1-r) &= 30\\
24 - 24r &= 30\\
-24r &= 6\\
r &= -\frac{1}{4}
\end{aligned}$$

15. Convert the following recurrring decimals into fractions.

(a) $0.\overline{45}$

Sol.

$$\begin{aligned} a_1 &= 0.45 \\ r &= 0.01 \\ S_\infty &= \frac{0.45}{1-0.01} \\ &= \frac{0.45}{0.99} \\ &= \frac{45}{99} \\ &= \frac{5}{11} \end{aligned}$$

$$\therefore 0.\overline{45} = \frac{5}{11}$$

(b) $0.\overline{037}$

Sol.

$$\begin{aligned} a_1 &= 0.037 \\ r &= 0.001 \\ S_\infty &= \frac{0.037}{1-0.001} \\ &= \frac{0.037}{0.999} \\ &= \frac{37}{999} \\ &= \frac{1}{27} \end{aligned}$$

$$\therefore 0.\overline{037} = \frac{1}{27}$$

(c) $0.2\overline{18}$

Sol.

$$\begin{aligned} a_1 &= 0.018 \\ r &= 0.01 \\ S_\infty &= \frac{0.018}{1-0.01} \\ &= \frac{0.018}{0.99} \\ &= \frac{18}{990} \\ &= \frac{1}{55} \end{aligned}$$

$$\begin{aligned} \therefore\ 0.2\overline{18} &= \frac{1}{5} + \frac{1}{55} \\ &= \frac{12}{55} \end{aligned}$$

(d) $1.\overline{3}$

Sol.

$$\begin{aligned} a_1 &= 0.3 \\ r &= 0.1 \\ S_\infty &= \frac{0.3}{1-0.1} \\ &= \frac{0.3}{0.9} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} \therefore\ 1.\overline{3} &= 1 + \frac{1}{3} \\ &= \frac{4}{3} \end{aligned}$$

16. Three integers are in GP, summing up to 42 while accumulating up to 512, find these three integers.

Sol.

$$\begin{aligned} a_1 + a_2 + a_3 &= 42 \\ a_1 a_2 a_3 &= 512 \\ a_2 &= \pm\sqrt{a_1 a_3} \\ a_1 a_3 &= a_2^2 \\ a_2^3 &= 512 \\ a_2 &= \sqrt[3]{512} \\ &= 8 \\ a_1 a_3 &= 64 \\ a_3 &= \frac{64}{a_1} \\ a_1 + 8 + \frac{64}{a_1} &= 42 \\ a_1 + \frac{64}{a_1} &= 34 \\ a_1^2 + 64 &= 34a_1 \\ a_1^2 - 34a_1 + 64 &= 0 \\ (a_1 - 32)(a_1 - 2) &= 0 \\ a_1 &= 32 \ or \ a_1 = 2 \end{aligned}$$

$$\therefore \ \textit{These three integers are } 2,\ 8,\ 32$$

17. The sum of first 6 term of a GP is 9 times the sum of first 3 terms. Find the common ratio.

Sol.

$$
\begin{aligned}
S_6 &= 9S_3 \\
\frac{a(1-r^6)}{1-r} &= 9 \times \frac{a(1-r^3)}{1-r} \\
a(1-r^6) &= 9a(1-r^3) \\
1-r^6 &= 9(1-r^3) \\
&= 9-9r^3 \\
r^6-9r^3+8 &= 0 \\
(r^3-8)(r^3-1) &= 0 \\
r^3 &= 8 \ or \ r^3 = 1 \\
r &= 1 \ (invalid) \\
r &= 2
\end{aligned}
$$

18. Given a GP, its first term is 16, last term is $\frac{1}{2}$ and its sum is $31\frac{1}{2}$, find its common ratio and number of terms.

Sol.

$$
\begin{aligned}
a_1 &= 16 \\
\frac{1}{2} &= 16r^{n-1} \\
\frac{1}{32} &= r^{n-1} \\
&= r^n \times \frac{1}{r} \\
r^n &= \frac{r}{32} \\
\frac{63}{2} &= \frac{16(1-r^n)}{1-r} \\
63(1-r) &= 32(1-r^n) \\
63-63r &= 32-32r^n \\
-31 &= 32r^n - 63r \\
-31 &= r-63r \\
-31 &= -62r \\
r &= \frac{1}{2} \\
(\frac{1}{2})^{n-1} &= \frac{1}{32} \\
&= (\frac{1}{2})^5 \\
n-1 &= 5 \\
n &= 6
\end{aligned}
$$

19. Given a GP, its 3rd term is 6 less than its 2nd term, ant its 2nd term is 9 less than its 1st term. Find the 4th term and the sum of the first 4 terms.

Sol.

$$
\begin{aligned}
Let \ x &= a_2 \\
a_3 &= x-6 \\
a_1 &= x+9 \\
x &= \pm\sqrt{(x-6)(x+9)} \\
x^2 &= x^2+3x-54 \\
3x-54 &= 0 \\
x &= 18 \\
a_2 &= 18 \\
a_1 &= 27 \\
r &= \frac{12}{18} \\
&= \frac{2}{3} \\
a_4 &= 27 \times (\frac{2}{3})^3 \\
&= 8 \\
S_4 &= \frac{27(1-(\frac{16}{3})^4)}{1-\frac{2}{3}} \\
&= \frac{27(1-\frac{8}{81})}{\frac{1}{3}} \\
&= 81 \times \frac{65}{81} \\
&= 65
\end{aligned}
$$

20. GIven an infinite GP, its common ratio is positive and the sum of it is 9. The sum of the first two terms is 5, find the 4th term.

Sol.

$$\begin{aligned}
S_\infty &= \frac{a}{1-r} = 9\\
a &= 9(1-r)\\
&= 9-9r\\
S_2 &= \frac{a(1-r^2)}{1-r} = 5\\
a - ar^2 &= 5-5r\\
9-9r-(9-9r)r^2 &= 5-5r\\
9-9r-9r^2+9r^3 &= 5-5r\\
4-4r-9r^2+9r^3 &= 0\\
4(1-r)-9r^2(1-r) &= 0\\
(4-9r^2)(1-r) &= 0\\
(9r^2-4)(r-1) &= 0\\
(3r^2+2)(3r^2-2)(r-1) &= 0\\
r &= 1\ (invalid)\\
r &= -\frac{2}{3}(invalid)\\
r &= \frac{2}{3}\\
a &= 9(1-\frac{2}{3})\\
&= 3\\
a_4 &= 3(\frac{2}{3})^3\\
&= 3\times\frac{8}{27}\\
&= \frac{8}{9}
\end{aligned}$$

21. If $x+1, x-2, \frac{1}{2}x$ are the first three terms of an infinite GP, find:

(a) The value of x

Sol.

$$\begin{aligned}
x-2 &= \pm\sqrt{(x+1)(\frac{1}{2}x)}\\
x^2-4x+4 &= \frac{1}{2}x(x+1)\\
2x^2-8x+8 &= x^2+x\\
x^2-9x+8 &= 0\\
(x-8)(x-1) &= 0\\
x &= 8\ or\ x = 1
\end{aligned}$$

(b) The common ratio

Sol.

$$\begin{aligned}
&When\ x = 8,\\
r &= \frac{8-2}{8+1}\\
&= \frac{6}{9}\\
&= \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
&When\ x = 1,\\
r &= \frac{1-2}{1+1}\\
&= -\frac{1}{2}
\end{aligned}$$

(c) The sum of the GP

Sol.

$$\begin{aligned}
&When\ x = 8,\\
S_\infty &= \frac{a}{1-r}\\
&= \frac{9}{1-\frac{2}{3}}\\
&= 9\times 3\\
&= 27
\end{aligned}$$

$$\begin{aligned}
&When\ x = 1,\\
S_\infty &= \frac{a}{1-r}\\
&= \frac{2}{1+\frac{1}{2}}\\
&= 2\times\frac{2}{3}\\
&= \frac{4}{3}
\end{aligned}$$

12.4 Simple Summation of Special Series

Sum formula of natural number:

$$\sum_{i=1}^{n} k = \frac{n(n+1)}{2}$$

Sum formula of square of natural number:

$$\sum_{i=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Sum formula of cube of natural number:

$$\sum_{i=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

12.4.1 Practice 10

1. Find the sum of the following series.

   (a) $\sum_{k=1}^{8} 3k$

   Sol.

$$\begin{aligned}\sum_{k=1}^{8} 3k &= 3\sum_{k=1}^{8} k \\ &= 3 \times \frac{8(8+1)}{2} \\ &= 3 \times \frac{8 \times 9}{2} \\ &= 3 \times \frac{72}{2} \\ &= 3 \times 36 \\ &= 108\end{aligned}$$

   (b) $\sum_{k=1}^{12} k^2$

   Sol.

$$\begin{aligned}\sum_{k=1}^{12} k^2 &= \frac{12(12+1)(2 \times 12+1)}{6} \\ &= \frac{12 \times 13 \times 25}{6} \\ &= 650\end{aligned}$$

   (c) $\sum_{k=3}^{10}(2k-3)$

   Sol.

$$\begin{aligned}&\sum_{k=3}^{10}(2k-3) \\ &= 2\sum_{k=3}^{10} k - \sum_{k=3}^{10} 3 \\ &= 2\left[\sum_{k=1}^{10} k - \sum_{k=1}^{2} k\right] - (30-6) \\ &= 2\left[\frac{10(10+1)}{2} - \frac{2(2+1)}{2}\right] - 8 \\ &= 2(55-3) - 24 \\ &= 2 \times 52 - 24 \\ &= 104 - 24 \\ &= 80\end{aligned}$$

   (d) $\sum_{k=7}^{13} 3k^2$

   Sol.

$$\begin{aligned}&\sum_{k=7}^{13} 3k^2 \\ &= 3\left[\sum_{k=1}^{13} k^2 - \sum_{k=1}^{6} k^2\right] \\ &= 3 \times \left[\frac{13(13+1)(2 \times 13+1)}{6}\right. \\ &\quad \left. - \frac{6(6+1)(2 \times 6+1)}{6}\right] \\ &= 3 \times \left[\frac{13 \times 14 \times 27}{6} - \frac{6 \times 7 \times 13}{6}\right] \\ &= 3 \times \left[\frac{4914}{6} - \frac{546}{6}\right] \\ &= 3 \times \frac{4368}{6} \\ &= 3 \times 728 \\ &= 2184\end{aligned}$$

2. Given that the nth term of a series is n(n+3), find the sum of the first 20 terms of the series.

   Sol.

$$\begin{aligned}&\sum_{k=1}^{20} k(k+3) \\ &= \sum_{k=1}^{20} k^2 + 3k \\ &= \sum_{k=1}^{20} k^2 + 3\sum_{k=1}^{20} k \\ &= \frac{20(20+1)(2 \times 20+1)}{6} + 3 \times \frac{20(20+1)}{2} \\ &= \frac{20 \times 21 \times 41}{6} + 3 \times \frac{20 \times 21}{2} \\ &= 2870 + 630 \\ &= 3500\end{aligned}$$

3. Find the sum of series $1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n+2)$.

Sol.

$$\sum_{k=1}^{n} k(k+2)$$
$$= \sum_{k=1}^{n} k^2 + 2k$$
$$= \sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k$$
$$= \frac{n(n+1)(2n+1)}{6} + 2 \times \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)(2n+1)}{6} + n(n+1)$$
$$= \frac{n(n+1)(2n+1) + 6n(n+1)}{6}$$
$$= \frac{n(n+1)(2n+7)}{6}$$

4. Find the sum of the following series.

(a) $\sum k = 1^8 5k^2$

Sol.

$$\sum_{k=1}^{8} 5k^2 = 5\sum_{k=1}^{8} k^2$$
$$= 5 \times \frac{8(8+1)(2\times 8+1)}{6}$$
$$= 5 \times \frac{8 \times 9 \times 17}{6}$$
$$= 5 \times \frac{1368}{6}$$
$$= 5 \times 204$$
$$= 1020$$

(b) $\sum_{k=1}^{9} k^3$

Sol.

$$\sum_{k=1}^{9} k^3 = \left[\frac{9(9+1)}{2}\right]^2$$
$$= 45^2$$
$$= 2025$$

(c) $\sum_{n=1}^{10} (3n-5)$

Sol.

$$\sum_{n=1}^{10} (3n-5) = 3\sum_{n=1}^{10} n - 5\sum_{n=1}^{10} 1$$
$$= 3 \times \frac{10(10+1)}{2} - 5 \times 10$$
$$= 3 \times \frac{10 \times 11}{2} - 5 \times 10$$
$$= 3 \times 55 - 50$$
$$= 3 \times 5 - 50$$
$$= 165 - 50$$
$$= 115$$

(d) $\sum_{k=3}^{6} 2k^3$

Sol.

$$\sum_{k=3}^{6} 2k^3 = 2\sum_{k=3}^{6} k^3$$
$$= 2\left(\sum_{k=1}^{6} k^3 - \sum_{k=1}^{2} k^3\right)$$
$$= 2\left\{\left[\frac{6(6+1)}{2}\right]^2 - \left[\frac{2(2+1)}{2}\right]^2\right\}$$
$$= 2(21^2 - 3^2)$$
$$= 2(441 - 9)$$
$$= 2 \times 432$$
$$= 864$$