# Chapter 13

# Sequence and Series

An array of numbers arranged according to a certain rule is called a **sequence**. For example:

(a) $2, 4, 6, 8, 10, \cdots$

(b) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$

Each number in the sequence is called a term.

- In (a), the **first term** of the sequence is 2, the second term is 4, the third term is 6, and so on. This sequence has infinitely many terms, making it an **infinite sequence**.
- In (b), the first term is $\frac{1}{2}$, the second term is $\frac{2}{3}$, and so on. It has 6 terms, making it a **finite sequence**. The last item in the sequence is $\frac{6}{7}$, which is the **last term**.

Generally, the terms of a sequence can be denoted as $a_1, a_2, \cdots, a_n, \cdots$, abbreviated as $\{a_n\}$, where $a_n$ represents the $n$th term. We call $a_n$ the general term of the sequence, and the relationship between $a_n$ and the term number $n$ is called the general term formula of the sequence.

**Example 1**

Write down the sequence of integers from 1 to 8, each multiplied by 10 and then added by 1. Hence, find the first term, last term, and general term formula of this sequence.

**Solution:**

The sequence is: $11, 21, 31, 41, 51, 61, 71, 81$.

The first term of the sequence is 11, the last term is 81, and the general term formula is $a_n = 10n + 1$.

**Example 2**

The sequence of $\{a_n\}$ is given by the formula $a_n = n(n+1)$. Write down the first 5 terms of this sequence.

**Solution:**

$a_1 = 1(1+1) = 2$

$a_2 = 2(2+1) = 6$

$a_3 = 3(3+1) = 12$

$a_4 = 4(4+1) = 20$

$a_5 = 5(5+1) = 30$

$\therefore$ The first 5 terms of the sequence are 2, 6, 12, 20, and 30.

**Example 3**

The sequence $\{a_n\}$ is defined as $a_1 = 3$, and for $n \geq 2$, $a_n = a_{n-1} + 2n$. Write down the first 5 terms of this sequence.

**Solution:**

$$
\begin{aligned}
a_2 &= a_1 + 2(2) = 3 + 4 = 7 \\
a_3 &= a_2 + 2(3) = 7 + 6 = 13 \\
a_4 &= a_3 + 2(4) = 13 + 8 = 21 \\
a_5 &= a_4 + 2(5) = 21 + 10 = 31 \\
a_6 &= a_5 + 2(6) = 31 + 12 = 43
\end{aligned}
$$

$\therefore$ The first 5 terms of the sequence are 3, 7, 13, 21, and 31.

**Keep in Mind:**

$a_{n-1}$ is the term before $a_n$.

**Example 4**

Write down the general formula for the sequence $3, 7, 11, 15, \ldots$.

**Solution:**

$$
\begin{aligned}
a_1 &= 3 \\
a_2 &= 3 + 4(2 - 1) \\
a_3 &= 3 + 4(3 - 1) \\
a_4 &= 3 + 4(4 - 1) \\
&\vdots \\
a_n &= 3 + 4(n - 1) = 4n - 1
\end{aligned}
$$

$\therefore$ The general formula is $a_n = 4n - 1$.

**Example 5**

Write down the general formula for the sequence $1, 0, 1, 0, \ldots$.

**Solution:**

In this case, $a_n = \dfrac{1 - (-1)^n}{2}$, $a_n = \sin^2 \dfrac{n\pi}{2}$, or $a_n = \dfrac{1 - \cos n\pi}{2}$ can all serve as general formulas for the sequence $1, 0, 1, 0, \ldots$.

From Example 5, it is evident that there could be more than one general formula for a sequence.

## Practice 13.1a

1. Write down the sequence of reciprocals of integers from 1 to 10, and find the first term, last term, and general formula for this sequence.
2. The sequence $\{a_n\}$ is given by the formula $a_n = \dfrac{2^n}{n+1}$. Write down the first 5 terms of this sequence.
3. Write down a general formula for the following sequences:
   (a) $1, -1, 1, -1, 1, -1, \ldots$
   (b) $1, 8, 27, 64, \ldots$

If $\{a_n\}$ is a sequence, the expression obtained by adding up all the terms of the sequence is called a series.

We use $\sum_{k=i}^{n} a_k$ to represent the sum of terms from the $i$-th term to the $n$-th term in the sequence $\{a_n\}$: $\sum_{k=i}^{n} a_k = a_i + a_{i+1} + \cdots + a_{n-1} + a_n$. The symbol "$\Sigma$" is the Greek letter **sigma**.

For example: $\sum_{n=4}^{10} n^2 = 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2$.

**Example 6**

Write down the following series:

(a) $\sum_{n=1}^{6} n(n-1)$

(b) $\sum_{n=2}^{5} (-1)^n(2n+1)$

**Solution:**

(a)
$$\begin{aligned}\sum_{n=1}^{6} n(n-1) &= 1(1-1) + 2(2-1) + 3(3-1) + 4(4-1) + 5(5-1) + 6(6-1)\\ &= 0 + 2 + 6 + 12 + 20 + 30\end{aligned}$$

(b)
$$\begin{aligned}\sum_{n=2}^{5} (-1)^n(2n+1) &= (-1)^2[2(2)+1] + (-1)^3[2(3)+1] + (-1)^4[2(4)+1]\\ &\quad + (-1)^5[2(5)+1]\\ &= 5 - 7 + 9 - 11\end{aligned}$$

**Example 7**

Find the first term, last term, and the number of terms in the series $\sum_{n=3}^{7} (2n+3)$.

**Solution:**

First term $= 2(3) + 3 = 9$

Last term $= 2(7) + 3 = 17$

Number of terms $= 7 - 3 + 1 = 5$

**Example 8**

Express the following series in sigma notation:

(a) $2 + 2 + 2 + 2 + 2 + 2$

(b) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

(c) $3 \times 5 + 4 \times 7 + 5 \times 9 + 6 \times 11$

(d) $3^2 + 2 \times 3^3 + 3 \times 3^4 + 4 \times 3^5 + 5 \times 3^6 + 6 \times 3^7$

**Solution:**

(a) $2 + 2 + 2 + 2 + 2 + 2 = \sum_{n=1}^{6} 2$

(b) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n}$

(c) $3 \times 5 + 4 \times 7 + 5 \times 9 + 6 \times 11 = \sum_{n=3}^{6} n(2n-1)$

(d) $3^2 + 2 \times 3^3 + 3 \times 3^4 + 4 \times 3^5 + 5 \times 3^6 + 6 \times 3^7 = \sum_{n=1}^{6} n \times 3^{n+1}$

**Note:** The series in Example 8(c) can also be written as $\sum_{n=1}^{4}(n+2)(2n+3)$

## Practice 13.1b

1. Write down the following series:

   (a) $\sum_{n=2}^{7} \frac{n}{n+1}$

   (b) $\sum_{n=1}^{5}(-1)^n(2n-1)$

2. Express the following series using $\Sigma$ notation:

   (a) $3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3$

   (b) $6 + 12 + 20 + \cdots + (n+1)(n+2) + \cdots + 10100$

   (c) $3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \cdots$

## Exercise 13.1

**1.** In the following questions, given the general formula of the sequence, write down its first 5 terms:

(a) $a_n = 2n - 1$

(b) $a_n = (-3)^n$

(c) $a_n = n(n+3)$

(d) $a_n = \dfrac{n}{2n+1}$

**2.** Write down a general formula for the following sequences:

(a) $4, 7, 10, 13, 16, \cdots$

(b) $5, -11, 17, -23, 29, \cdots$

(c) $\dfrac{3}{7}, \dfrac{5}{10}, \dfrac{7}{13}, \dfrac{9}{16}, \dfrac{11}{19}, \cdots$

(d) $-\dfrac{2}{3}, \dfrac{3}{9}, -\dfrac{4}{27}, \dfrac{5}{81}, -\dfrac{6}{243}, \cdots$

(e) $3, 6, 11, 18, 27, \cdots$

(f) $3, 6, 12, 24, 48, \cdots$

**3.** Given the sequence $\{a_n\}$ defined as $a_1 = -5$, and for $n \geq 2, a_n = a_{n-1} + 3$.

(a) Write down the first 6 terms of the sequence.

(b) Write down the general formula of the sequence. Hence, find the 99th term of the sequence.

**4.** Given that the sequence $\{a_n\}$ is defined as $a_1 = 2$, and for $n \geq 2, a_n = 3a_{n-1}$.

(a) Write down the first 4 terms of the sequence.

(b) Write down the general formula of the sequence. Hence, find the 8th term of the sequence.

**5.** Given the sequence $\{a_n\}$ defined as $a_1 = 2, a_2 = 5$, and for $n \geq 3, a_n = a_{n-1} + a_{n-2}$, write down the 10th term of the sequence.

**6.** Write down the following series:

(a) $\displaystyle\sum_{n=2}^{7} n(n+3)$

(b) $\displaystyle\sum_{n=4}^{9} \frac{(-1)^n}{n^2+3}$

**7.** Express the following series using the $\Sigma$ notation:

(a) $1 + \dfrac{1}{3} + \dfrac{1}{5} + \cdots + \dfrac{1}{31}$

(b) $4^3 + 5^3 + 6^3 + \cdots + 27^3$

(c) $2 \times 4 + 4 \times 7 + 6 \times 10 + 8 \times 13 + \cdots + 50 \times 76$

(d) $1 - \dfrac{2}{4} + \dfrac{3}{7} - \dfrac{4}{10} + \cdots + \dfrac{19}{55}$

## 13.1 Arithmetic Sequence and Series

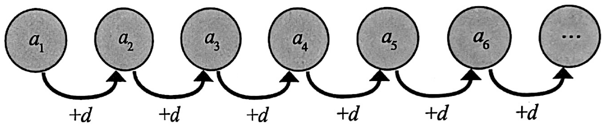
When each term of a sequence $\{a_n\}$ subtracted by its preceding term yields a constant value, this sequence is called an **arithmetic sequence**, and this constant difference is called the **common difference**, typically denoted by $d$.

According to the definition,

$$a_n - a_{n-1} = d$$

Hence,

$$a_n = a_{n-1} + d$$




Let $a_1 = a$,

| | |
|---|---|
| $a_1$ | $a$ |
| $a_2$ | $a_1 + d = a + d$ |
| $a_3$ | $a_2 + d = a + 2d$ |
| $a_4$ | $a_3 + d = a + 3d$ |
| $a_5$ | $a_4 + d = a + 4d$ |
| $\cdots$ | $\cdots$ |

From the table above, it can be deduced that if an arithmetic sequence $\{a_n\}$ has a first term $a$ and a common difference $d$, then its general term formula is

**General Term Formula for Arithmetic Sequence**

$$a_n = a + (n-1)d$$

**Example 9**

Find the 8th term of the arithmetic sequence $1, 5, 9, \cdots$.

**Solution:**

First term $a = 1$, common difference $d = 5 - 1 = 4$

$$\begin{aligned}\therefore \text{ the 8th term } a_8 &= a + 7d \\ &= 1 + 7 \times 4 \\ &= 29\end{aligned}$$

**Example 10**

Given that the third term of an arithmetic sequence is $-1$ and the seventh term is $-17$. Find the first term, common difference, and the ninth term of this sequence.

**Solution:** $a_3 = a + 2d = -1 \cdots (1)$

$a_7 = a + 6d = -17 \cdots (2)$

From (1) and $(2)$, we get $d = -4, a = 7$

$a_9 = a + 8d = 7 + 8(-4) = -25$

$\therefore$ the first term of this sequence is 7, the common difference is $-4$, and the ninth term is $-25$.

**Example 11**

Which term of the arithmetic sequence $-4, -\frac{11}{4}, -\frac{3}{2}, \cdots$ is 16?

**Solution:**

$a = -4$

$d = a_2 - a = -\frac{11}{4} - (-4) = \frac{5}{4}$

Let's the $n$th term of this arithmetic sequence be 16, then

$$\begin{aligned} a_n = a + (n-1)d = -4 + \frac{5}{4}(n-1) &= 16 \\ \frac{5}{4}(n-1) &= 20 \\ n - 1 &= 16 \\ n &= 17 \end{aligned}$$

$\therefore$ the 17th term of this arithmetic sequence is 16.

**Example 12**

How many multiples of 6 are there between 100 and 300?

**Solution:**

The multiples of 6 from 100 to 300 are $102, 108, 114, \cdots, 300$.

They form an arithmetic sequence such that $a = 102$ and $d = 6$.

Let the number of terms in this sequence be $n$, then

$$\begin{aligned} a_n &= a + (n-1)d \\ 300 &= 102 + 6(n-1) \end{aligned}$$

$\therefore n = 34$

$\therefore$ there are 34 multiples of 6 between 100 and 300.

**Keep in Mind:**

In Example 12, the number of multiples of 6 does not equal to (last term − first term) ÷ 6 + 1.

The arithmetic mean $A$ of two numbers $x$ and $y$ is the number that forms an arithmetic sequence with $x$ and $y$. Therefore,

$$A - x = y - A$$
$$\therefore A = \frac{x+y}{2}$$

In other words, the arithmetic mean of two numbers is their arithmetic average.

**Example 13**

Find the arithmetic mean of 3 and 15.

**Solution:**

The arithmetic mean of 3 and 15 is $\dfrac{3+15}{2} = 9$.

**Example 14**

Find four numbers between 18 and 33 such that these six numbers form an arithmetic sequence.

**Solution:**

In the targeted arithmetic sequence, $a = 18$

$$_6 = a + 5d = 33$$
$$5d = 33 - 18$$
$$d = 3$$

$\therefore$ the four targeted numbers are 21, 24, 27, and 30.

**Example 15**

If the lengths of the sides of a right triangle form an arithmetic sequence, prove that their ratio is $3 : 4 : 5$.

**Proof:**

**Approach 1:**

Suppose $a < b < c$, then $c$ is the hypotenuse. According to the Pythagorean theorem,

$$a^2 + b^2 = c^2 \cdots (1)$$

Since $a, b, c$ form an arithmetic sequence,

$$b = \frac{a+c}{2} \cdots (2)$$

Substituting (2) into (1), we get

$$\begin{aligned} a^2 + \frac{a^2 + 2ac + c^2}{4} &= c^2 \\ 5a^2 + 2ac - 3c^2 &= 0 \\ (5a - 3c)(a + c) &= 0 \\ \because a + c &\neq 0 \\ \therefore 5a - 3c &= 0 \\ a &= \frac{3}{5}c \end{aligned}$$

Substituting $a = \frac{3}{5}c$ into (2), we get: $b = \frac{4}{5}c$.
$\therefore$ the ratio of the lengths of the sides is $3 : 4 : 5$.

**Approach 2:**

Let the lengths of the sides be $a, a + d$, and $a + 2d$, where $d > 0$, then $a + 2d$ is the length of the hypotenuse. By the Pythagorean theorem,

$$\begin{aligned} a^2 + (a + d)^2 &= (a + 2d)^2 \\ a^2 - 2ad - 3d^2 &= 0 \\ (a - 3d)(a + d) &= 0 \\ \because a + d &\neq 0 \\ \therefore a - 3d &= 0 \\ a &= 3d \end{aligned}$$

$\because$ the lengths of the sides are $3d, 4d$, and $5d$.

$\therefore$ the ratio of the lengths of the sides is $3 : 4 : 5$.

## Practice 13.2a

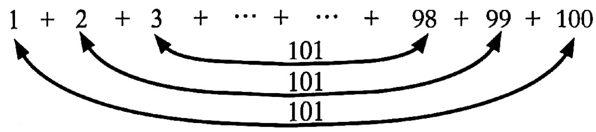
1. Given that the 4th term and the 9th term of an arithmetic sequence are -2 and 28 respectively, find the first term, common difference, and the 12th term of this sequence.
2. Find the arithmetic mean of -9 and 21.
3. Find five numbers between -40 and 56 such that these seven numbers form an arithmetic sequence.
4. How many multiples of 7 are there between 100 and 1000?

## Sum of the First $n$ Terms

If $a_1, a_2, a_3, \cdots, a_n$ is an arithmetic sequence, the expression obtained by adding the terms of the sequence, $a_1 + a_2 + \cdots + a_n$, is called an **arithmetic series**. We usually denote the sum of the first $n$ terms of a sequence $\{a_n\}$ and its corresponding series $a_1 + a_2 + \cdots + a_n + \cdots$ as $S_n$, that is,

$$S_n = a_1 + a_2 + \cdots + a_n$$

The great mathematician Gauss, at the age of 8, already knew a quick method to find the sum of the arithmetic sequence $1, 2, 3, \cdots, 100$. His idea was as follows:




Pairing $1, 2, 3, \cdots, 100$ from the beginning and end, we can divide them into 50 pairs, and the sum of each pair is 101. Therefore,

$$1 + 2 + 3 + \cdots + 98 + 99 + 100 = 50 \times 101 = 5050$$

By generalizing this idea, we can derive the formula for the sum of an arithmetic sequence. Let $S_n$ be the sum of the first $n$ terms of an arithmetic sequence $\{a_n\}$,

$$\begin{array}{rcccccccc} S_n &=& a &+& a+d &+\cdots+& a+(n-2)d &+& a+(n-1)d \\ S_n &=& a+(n-1)d &+& a+(n-2)d &+\cdots+& a+d &+& a \\ \hline 2S_n &=& \multicolumn{7}{l}{\underbrace{2a+(n-1)d + 2a+(n-1)d + \cdots + 2a+(n-1)d + 2a+(n-1)d}_{\text{n terms}}} \end{array}$$

$$2S_n = n[2a + (n-1)d]$$

Hence,

**Sum of the First $n$ Terms of an Arithmetic Sequence**

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

Since $2a + (n - 1)d$ is the sum of the first and last terms, this formula for the sum can also be written as

**Sum of the First $n$ Terms of an Arithmetic Sequence**

$$S_n = \frac{n}{2}(a_1 + a_n)$$

**Example 16**

Find the sum of all multiples of 4 from 50 to 500.

**Solution:**

From 50 to 500, all multiples of 4 are $52, 56, \cdots, 500$. They form an arithmetic sequence.
$a = 52, d = 4,$
There are a total of $n = \dfrac{500 - 52}{4} + 1 = 113$ terms
$\therefore$ the sum of all multiples of 4 from 50 to 500 is $S_{113} = \dfrac{113}{2}(52 + 500) = 31188.$

**Example 17**

If the first term of an arithmetic sequence is 13, the third term is 29, and the sum of the first $n$ terms is 910, find $n$.

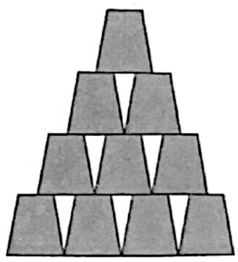
**Solution:**

$$\begin{aligned} a &= 13 \\ a_3 &= a + 2d = 29 \\ 2d &= 29 - 13 \\ d &= 8 \\ S_n &= \frac{n}{2}[2 \times 13 + 8(n - 1)] = 910 \\ n(4n + 9) &= 910 \\ 4n^2 + 9n - 910 &= 0 \\ (n - 14)(4n + 65) &= 0 \end{aligned}$$

$\because n > 0$
$\therefore n = 14$

## Think about It:

If you precisely stack all 100 cups into a pyramid shape, one by one, starting from the bottom and moving up, will you use up all of them?



**Example 18**

At which term of the arithmetic progression $-56, -50, -44, \ldots$ does the sum become positive when adding from the first term?

**Solution:**

$$\begin{aligned} a &= -56, d = -50-(-56) = 6 \\ S_n &= \frac{n}{2}[2(-56)+6(n-1)] > 0 \\ & n(3n-59) > 0 \\ & \because n > 0, \therefore 3n-59 > 0 \\ & n > 19\frac{2}{3} \end{aligned}$$

$\therefore$ you must add from the first term to the $20^{th}$ term for the sum to become positive.

**Example 19**

Given a sequence $\{a_n\}$ with the sum of its first $n$ terms being $S_n = 2n^2 + 3n$,

(a) Find the first term of the sequence $a_1$.

(b) Find the general formula for the sequence $a_n$.

(c) Prove that $\{a_n\}$ is an arithmetic sequence.

(d) Find the sum of the sequence from the 5th term to the 12th term.

**Solution:**

(a) $a_1 = S_1 = 2(1)^2 + 3(1) = 5$

(b)
$$\begin{aligned} S_n &= a_1 + a_2 + \cdots + a_{n-1} + a_n \\ S_{n-1} &= a_1 + a_2 + \cdots + a_{n-1} \\ \therefore a_n &= S_n - S_{n-1} \\ &= 2n^2 + 3n - \left[2(n-1)^2 + 3(n-1)\right] \\ &= 4n+1 \end{aligned}$$

(c) $a_{n-1} = 4(n-1) + 1 = 4n - 3$

$\therefore a_n - a_{n-1} = 4$ is a constant

$\therefore \{a_n\}$ is an arithmetic sequence

(d) Sum of the sequence from the 5th term to the 12th term

$$\begin{aligned} &= a_5 + a_6 + \cdots + a_{12} \\ &= (a_1 + a_2 + \cdots + a_{12}) - (a_1 + a_2 + a_3 + a_4) \\ &= S_{12} - S_4 \\ &= 2(12)^2 + 3(12) - \left[2(4)^2 + 3(4)\right] \\ &= 280 \end{aligned}$$

**Additional Information:**

From the solution to Example 19, it can be observed that if the sum of the first $n$ terms of a sequence $\{a_n\}$ is $S_n$, then:

- The $n$th term is given by $a_n = S_n - S_{n-1}$.
- The sum of the terms from the $k$th term to the $l$th term is $S_l - S_{k-1}$.

**Example 20**

Given that the first term of an arithmetic sequence is 13, and the sum of its first 4 terms equals the sum of its first 10 terms, find the value of $n$ when the sum of the first $n$ terms reaches its maximum.

**Solution:**

$$\begin{aligned}
a &= 13 \\
S_4 &= S_{10} \\
\frac{4}{2}(2 \times 13 + 3d) &= \frac{10}{2}(2 \times 13 + 9d) \\
52 + 6d &= 130 + 45d \\
39d &= -78 \\
d &= -2
\end{aligned}$$

$$\begin{aligned}
S_n &= \frac{n}{2}[2 \times 13 - 2(n-1)] \\
&= n(14 - n) \\
&= -\left(n^2 - 14n + 49\right) + 49 \\
&= 49 - (n-7)^2
\end{aligned}$$

$\therefore$ When $n = 7$, the sum of the first $n$ terms reaches its maximum.

## Think about It:

In Example 20, do you expect the answer to be 7 before any calculation? Why would you have such a thought?

## Practice 13.2b

1. On the "365-Day Savings Plan," you start by saving RM 0.10 on the first day, RM 0.20 on the second day, RM 0.30 on the third day, increasing by RM 0.10 each day, for a total of 365 days. How much money will you have saved after 365 days?

2. Given that the $3^{rd}$ term of an arithmetic sequence is 24, the $15^{th}$ term is $-12$, and the sum of the first $n$ terms is 30, find the value of $n$.

3. Given that the sum of the first $n$ terms of a sequence $\{a_n\}$ is $S_n = \dfrac{4n - 3n^2}{2}$.

   (a) Find the $n$th term $a_n$ of the sequence.

   (b) Find the sum of the sequence from the $8^{th}$ term to the $15^{th}$ term.

## Exercise 13.2

1. Find the $10^{th}$ term and the $n$th term of the arithmetic sequence $5, 13, 21, \cdots$.

2. Given that the first term of an arithmetic sequence is $\dfrac{13}{3}$, the $3^{rd}$ term is 3, and the last term is $-\dfrac{31}{3}$, find the number of terms.

3. Given that the $7^{th}$ term of an arithmetic sequence is $-10$, and the $12^{th}$ term is $-25$, find the $17^{th}$ term of this sequence.

4. Given that the $6^{th}$ term of an arithmetic sequence is 43, the $10^{th}$ term is 75, find which term is 155.

5. Find five numbers between 18 and 30 such that these seven numbers form an arithmetic sequence, then find the sum of these 7 numbers.

6. Given that the first term of an arithmetic sequence is 12, and the $2^{nd}$ term is 15, find the sum of the first 20 terms.

7. Given that the first term of an arithmetic sequence is 20, and the third term is $\dfrac{92}{5}$, find which term is the first negative term.

8. Given that three numbers form an arithmetic sequence, and their sum is 30, and the sum of their squares is 318, find these three numbers.

9. Find the sum of the first 12 terms of the arithmetic series $18, 10, 2, -6, \cdots$.

10. If the $4^{th}$ term of an arithmetic sequence is 9, and the 8th term is $-7$, find the sum of its first 10 terms.

11. Find the sum of all integers between 200 and 800 that are divisible by 11.

12. Given that the first three terms of an arithmetic sequence are $x, 3x - 4, 2x + 7$.

    (a) Find the value of $x$.

    (b) Find the sum of the first 10 terms of this sequence.

13. Given that the first term of an arithmetic sequence is 12, the common difference is $-3$, and the sum of all terms is 21, find the number of terms.

14. Given that the $3^{rd}$ term of an arithmetic sequence is $8$, and the $6^{th}$ term is $4$. Starting from the first term, up to which term must we add to get a negative sum?

15. Given that the sum of the first $n$ terms of a sequence is $S_n = \dfrac{n(n+3)}{4}$.

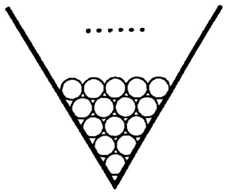
    (a) Find the first term $a_1$ of the sequence.

    (b) Find the general formula $a_n$ of the sequence.

    (c) Prove that $\{a_n\}$ is an arithmetic sequence.

    (d) Find the sum of the sequence from the $8^{th}$ term to the $22^{nd}$ term.

16. Refer to the image below:



    (a) If there are $50$ pencils in the top layer, how many pencils are there in total?

    (b) If there are $990$ pencils in total, how many layers are there?

17. Given that the first term of an arithmetic sequence is positive, and the $20^{th}$ term is $0$. If the sum of the first $n$ terms of this sequence is $0$, find the value of $n$.

18. Given that the $5^{th}$ term of an arithmetic sequence is $3$, and the sum of the first $10$ terms is $26\frac{1}{4}$, find which term has the value $0$.

19. Let $S_n$ be the sum of the first $n$ terms of an arithmetic sequence $\{a_n\}$. If $S_{10} = 465$ and $9S_3 = 4S_6$, find $a_5$ and $S_5$.

20. Find $38^2 - 37^2 + 36^2 - 35^2 + \cdots + 2^2 - 1^2$.

21. Given that the first term of an arithmetic sequence is $45$, and the common difference is $-6$, for what value of $n$ does the sum $S_n$ of the first $n$ terms reach its maximum?

22. Given that the interior angles of a convex polygon form an arithmetic sequence with a common difference of 6, and the largest interior angle is $135°$. How many sides does this polygon have?

23. Given that the sum of the first 6 terms of an arithmetic sequence is 96, and the sum of the first 10 terms is one-third of the sum of the first 20 terms, find the first term and the 10th term of this sequence.

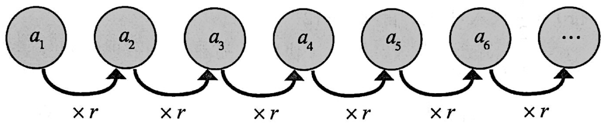
## 13.2 Geometric Sequence and Series

When each term of a sequence $\{a_n\}$ is equal to the ratio of the term and its preceding term, this sequence is called a geometric sequence, and this equal ratio is called the common ratio, generally denoted by $r$. According to the definition,

$$\frac{a_n}{a_{n-1}} = r$$

Hence,

$$a_n = a_{n-1} \times r$$




Let $a_1 = a$,

| | |
|---|---|
| $a_1$ | $a$ |
| $a_2$ | $a_1 \times r = ar$ |
| $a_3$ | $a_2 \times r = ar^2$ |
| $a_4$ | $a_3 \times r = ar^3$ |
| $a_5$ | $a_4 \times r = ar^4$ |
| $\cdots$ | $\cdots$ |

From the table above, it can be deduced that if a geometric sequence $\{a_n\}$ has a first term of $a$ and a common ratio of $r$, then its general term formula is

$$a_n = ar^{n-1}$$

**Example 21**

Given that the second term and the fifth term of a geometric sequence are 6 and $\frac{3}{4}$ respectively, find its first term and the tenth term.

**Solution:**

$a_2 = ar = 6 \cdots (1)$

$a_5 = ar^4 = \frac{3}{4} \cdots (2)$

$\frac{(2)}{(1)}$ gives $\quad r^3 = \frac{1}{8}$ Substituting $r = \frac{1}{2}$ into (1), we get $a = 12$.

$r = \frac{1}{2}$

The $10^{th}$ term $a_{10} = ar^9 = 12\left(\frac{1}{2}\right)^9 = \frac{3}{128}$

**▸ Example 22**

The $2^{nd}$ of a geometric sequence is $-8$, and the $3^{rd}$ is $4$. What is the term number where the value is $\frac{1}{1024}$?

**Solution:**

$a_2 = ar = -8 \cdots (1)$

$a_3 = ar^2 = 4 \cdots (2)$

$\frac{(2)}{(1)}$ gives $\quad r = -\frac{1}{2}, \; a \; = 16$

Let the $n^{th}$ term be $\frac{1}{1024}$, then

$$
\begin{aligned}
a_n = ar^{n-1} &= 16\left(-\frac{1}{2}\right)^{n-1} \\
\frac{1}{1024} &= \frac{16}{(-2)^{n-1}} \\
(-2)^{n-1} &= (-2)^{14} \\
n &= 15 \\
\therefore \text{ the } 15^{th} &\text{ term is } \frac{1}{1024}.
\end{aligned}
$$

The geometric mean $G$ of two numbers $x$ and $y$ is a number that forms a geometric sequence with $x$ and $y$. Hence,

$$
\begin{aligned}
\frac{G}{x} &= \frac{y}{G} \\
G^2 &= xy \\
\therefore G &= \pm\sqrt{xy}
\end{aligned}
$$

If $x$ and $y$ are both positive numbers, then $\sqrt{xy}$ is their geometric mean.

### Think about It:

Two numbers $x$ and $y$ must have the same sign to have a geometric mean. Why?

**▸ Example 23**

Find three numbers between $\frac{1}{6}$ and 216 such that these five numbers form a geometric sequence.

**Solution:**

In the targeted geometric sequence, $a = \frac{1}{6}$

$$
\begin{aligned}
a_5 &= ar^4 = 216 \\
r^4 &= 216 \times 6 = 6^4 \\
r &= \pm 6
\end{aligned}
$$

$\therefore$ the three targeted numbers are $1, 6, 36$ or $-1, 6, -36$.

**Example 24**

Given that the population of Malaysia was 28.33 million people in 2010 and 29.72 million people in 2013:

(a) Find the annual growth rate of Malaysia's population during this period.

(b) Estimate the population of Malaysia in 2020.

**Solution:**

(a) Let $a_1$ be the population in 2010 (in tens of millions of people), $a_2$ be the population in 2011, and so on.

$$\begin{aligned} a_1 &= a = 2833 \\ a_4 &= ar^3 = 2972 \\ r^3 &= \frac{2972}{2833} \\ r &= \sqrt[3]{\frac{2972}{2833}} \\ &= 1.0161 \end{aligned}$$

$\therefore$ the annual growth rate of Malaysia's population is $(1.0161 - 1) \times 100\% = 1.61\%$.

(b) In the year 2020, the population of Malaysia will be $a_{11} = ar^{10} = 2833 \times 1.0161^{10} = 33.24$ million

## Practice 13.3a

1. The first term of a geometric sequence is 18, and the second term is 12. Find the sixth term.
2. The first term of a geometric sequence is 8, and the fourth term is $-27$. What term is $-\frac{2187}{16}$ located at?
3. Find the geometric mean of $\frac{27}{8}$ and $\frac{2}{3}$.

## Exercise 13.3a

1. Given that the first term of a geometric sequence is 8, the second term is 4, and the last term is $\frac{1}{128}$, find the number of terms.
2. Given that the second term of a geometric sequence is 12 and the fourth term is 108, find the seventh term.
3. Insert three numbers between 14 and 224 to form a geometric sequence with these five numbers.
4. Given that the seventh term of a geometric sequence is 18 and the common ratio is $-\frac{3}{2}$, find the fourth term.
5. If $x + 12$, $x + 4$, and $x - 2$ form a geometric sequence, find the value of $x$ and the common ratio of the sequence.
6. Given that the second, sixth, and eighth terms of an arithmetic sequence form a geometric sequence, find the common ratio of this geometric sequence.
7. Three different numbers 2, $x$, and $y$ form a geometric sequence. If these three numbers are the first, second, and twelfth terms of an arithmetic sequence, find the values of $x$ and $y$.

## Sum of the First $n$ Terms

If $a_1, a_2, a_3, \cdots, a_n$ is a geometric sequence, then the expression obtained by adding all the terms in the sequence $a_1 + a_2 + \cdots + a_n$ is called a geometric series. Let $S_n$ be the sum of the first $n$ terms of a geometric sequence $\{a_n\}$.

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $S_n$ | $=$ | $a$ | $+$ | $ar$ | $+$ | $ar^2$ | $+$ | $\cdots$ | $+$ | $ar^{n-2}$ | $+$ | $ar^{n-1}$ | | |
| $-)\ rS_n$ | $=$ | | | $ar$ | $+$ | $ar^2$ | $+$ | $\cdots$ | $+$ | $ar^{n-2}$ | $+$ | $ar^{n-1}$ | $+$ | $ar^n$ |
| $(1-r)S_n$ | $=$ | $a$ | $+$ | $0$ | $+$ | $0$ | $+$ | $0$ | $+$ | $0$ | $+$ | $0$ | $-$ | $ar^n$ |

$\therefore\ (1-r)S_n = a\left(1-r^n\right)$

From this, it follows that when $r \neq 1$, the sum formula for a geometric series is:

**Sum of the First $n$ Terms of a Geometric Sequence**

$$S_n = \frac{a(1-r^n)}{1-r}$$

When $r = 1$, the resulting geometric sequence is a constant sequence $a, a, a, \cdots$. Clearly, in this case, the sum of the first $n$ terms of the sequence is $S_n = na$.

**Example 25**

Find the sum of the first 7 terms of the geometric series $2 - 3 + \frac{9}{2} - \cdots$.

**Solution:**

$a = 2,\ r = \frac{-3}{2} = -\frac{3}{2}$

$\therefore$ Sum of the first 7 terms $S_7 = \dfrac{2\left[1-\left(-\frac{3}{2}\right)^7\right]}{1-\left(-\frac{3}{2}\right)} = \dfrac{463}{32}$.

**Example 26**

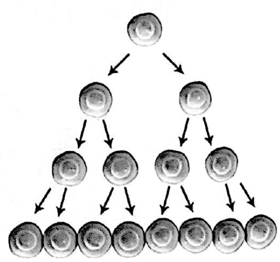
If during the process of cell division, one cell divides into two, find the total number of cells after 8 rounds of cell division.

**Solution:**

$a = 1,\ r = 2,\ n = 8$

The total number of cells after 8 rounds of cell division is

$$S_9 = \frac{1\times\left(1-2^8\right)}{1-2}$$
$$= 255$$



**▸ Example 27**

Given that all terms of a geometric sequence are positive, the second term is 72, and the fourth term is 32. Find the value of $n$ such that the sum of the first $n$ terms is $281\frac{1}{3}$.

**Solution:**

$a_2 = ar = 72 \ \cdots \ (1)$

$a_4 = ar^3 = 32 \ \cdots \ (2)$

$\frac{(2)}{(1)}$ gives $\qquad r^2 = \frac{32}{72} = \frac{4}{9}$

Since all the terms in this geometric sequence are positive, $r > 0$,

$$\therefore \ r = \frac{2}{3}, \ a = \frac{72}{\frac{2}{3}} = 108$$

$$S_n = \frac{108\left[1-\left(\frac{2}{3}\right)^n\right]}{1-\frac{2}{3}} = \frac{844}{3}$$

$$1-\left(\frac{2}{3}\right)^n = \frac{211}{243}$$

$$\left(\frac{2}{3}\right)^n = \frac{32}{243} = \left(\frac{2}{3}\right)^5$$

$$\therefore n = 5$$

**▸ Example 28**

Given that the first term of an arithmetic sequence is $-3$, and its $4^{th}$, $6^{th}$, and $10^{th}$ terms are the $2^{nd}$, $3^{rd}$, and $4^{th}$ terms of a geometric sequence, respectively. If the arithmetic sequence is not a constant sequence,

(a) Find the common difference of the arithmetic sequence.

(b) Find the sum of the first 10 terms of the geometric sequence.

**Solution:**

(a) The $4^{th}$, $6^{th}$, and $10^{th}$ terms of the arithmetic sequence are $-3+3d$, $-3+5d$, and $-3+9d$ respectively.

$\because$ they are three consecutive terms of a geometric sequence

$\therefore \ (-3+3d)(-3+9d) = (-3+5d)^2$

$$2d^2 - 6d = 0$$

$$2d(d-3) = 0$$

$\because$ the arithmetic sequence is not a constant sequence, $d \neq 0$

$\therefore \ d = 3$

**Example 28**

(b) The $2^{nd}$, $3^{rd}$, and $4^{th}$ terms of the geometric sequence are 6, 12, and 24 respectively.

$$\text{Common ratio } r = \frac{12}{6} = 2$$
$$\text{First term } a = \frac{6}{2} = 3$$

$\therefore$ The sum of the first 10 terms $S_10 = \dfrac{3(1-2^{10})}{1-2} = 3069$.

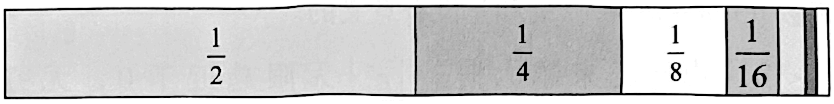
## Practice 13.3b

1. Find the sum of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$.
2. Given that the first term of a geometric series is 2, the fourth term is $-128$, and the last term is 8192, find the sum of this series.
3. A ball falls from a height of 18 m. After each bounce, it rebounds to $\frac{2}{3}$ of its previous height. Find the total distance traveled by the ball from the start to the fourth bounce.
4. In the Chinese mathematical masterpiece "Suanfa Tongzong," there is the following problem:

   "Looking afar, a towering seven-story pagoda, dots of red light increasingly bright, a total of 381 lamps, please tell me how many lamps are at the top?"

   Meaning: A seven-story pagoda has a total of 381 lamps. The number of lamps on the next lower level is twice the number on the level above. Find the number of lamps at the top level of the pagoda. Can you figure it out?

# Sum of an Infinite Geometric Series




We can imagine the ribbon in the figure above as a ribbon of length 1. Initially, we cut off $\frac{1}{2}$ of the ribbon, then we cut off $\frac{1}{4}$ from the remaining $\frac{1}{2}$, which is $\frac{1}{4}$ of the original length, and so on. Each time we cut off a ribbon whose length is $\frac{1}{2}$ of the remaining ribbon. Therefore, after the $n$th cut, the length of the ribbon cut off is $\frac{1}{2^n}$, leaving $\frac{1}{2^n}$ of the ribbon.

Hence, the total length of ribbon cut off after $n$ cuts is the sum of the infinite series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$. Obviously, this sum equals 1, i.e.,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1$$

Mathematically, we say that the sum of the first $n$ terms of this infinite series is $1 - \frac{1}{2^n}$. As $n$ approaches infinity, since $\frac{1}{2^n}$ tends to 0, the sum of the first $n$ terms approaches 1. Therefore, we say that 1 is the sum of the infinite series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$.

For a geometric series $a + ar + ar^2 + \cdots + ar^n + \cdots$,

- When $r \neq 1$, the sum of its first $n$ terms is $S_n = \frac{a(1-r^n)}{1-r}$.
- When $r = 1$, $S_n = na$.
- If $a \neq 0$, when $r = -1$, $S_n = \begin{cases} a, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$.

### Think about It:

Does it make sense to sum an infinite series with an arithmetic progression?

For an infinite geometric series $a+ar+ar^2+\cdots$, when $|r| \geq 1$, as more terms are added, the sum does not converge to any constant value. Therefore, we say the sum of the infinite geometric series $a + ar + ar^2 + \cdots$ is undefined.

If $|r| < 1$, as $n$ becomes larger, $|r^n|$ approaches 0, and the sum of the first $n$ terms of the infinite geometric series $a + ar + ar^2 + \cdots$, denoted as $S_n = \frac{a(1-r^n)}{1-r}$, approaches $\frac{a}{1-r}$. Therefore, the sum of the infinite geometric series $a + ar + ar^2 + \cdots$ is

**Sum of an Infinite Geometric Series**

$$S = \frac{a}{1-r} \qquad (-1 < r < 1)$$

### Additional Information:

In "Zhuangzi," there is a saying:

"A one-foot stick, if you take half of it each day, will never be exhausted for ten thousand generations."

**▸ Example 29**

Given that the $3^{rd}$ and $6^{th}$ terms of an infinite geometric series are $^{21}$ and $-\dfrac{56}{9}$ respectively, find the sum of this series.

**Solution:**

$$a_3 = ar^2 = \frac{21}{9} \ \cdots\ (1)$$
$$a_6 = ar^5 = -\frac{56}{9} \ \cdots\ (2)$$

$$\frac{(2)}{(1)} \text{ gives} \qquad r^3 = -\frac{8}{27} = \left(-\frac{2}{3}\right)^3$$
$$r = -\frac{2}{3}$$
$$a = \frac{21}{\left(-\frac{2}{3}\right)^2} = \frac{189}{4}$$

$$\therefore \text{ the sum of this series } S = \frac{a}{1-r} = \frac{\frac{189}{4}}{1-\left(-\frac{2}{3}\right)} = \frac{567}{20}.$$

Back in junior high, we learned that any fraction can be expressed as a finite decimal or a repeating decimal. For example:

$$\frac{15}{8} = 1.875$$
$$\frac{1}{6} = 0.1666\cdots = 0.16$$

Clearly, any finite decimal can be converted into a fraction. As for repeating decimals, we can use the formula for the sum of an infinite geometric series to convert them into fractions, as demonstrated in the following examples.

**▸ Example 30**

Convert the repeating decimal $0.2\dot{1}\dot{3}$ into a fraction.

**Solution:**

$$0.2\dot{1}\dot{3} = 0.2 + (0.013 + 0.00013 + 0.0000013 + \cdots)$$

$0.013 + 0.00013 + 0.0000013 + \cdots$ is an infinite geometric series with the first term $a = 0.013$, and the common ratio $r = 00.1$.

$$\begin{aligned}\therefore\ 0.2\dot{1}\dot{3} &= 0.2 + \frac{0.013}{1-0.01}\\ &= 0.2 + \frac{0.013}{0.99}\\ &= \frac{2}{10} + \frac{13}{990}\\ &= \frac{211}{990}\end{aligned}$$

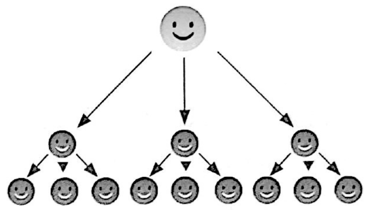
## Practice 13.3c

1. Find the sum of the infinite geometric series $48 - 36 + 27 - \cdots$.
2. Convert the repeating decimal $0.\dot{2}\dot{3}$ into a fraction.

## Exercise 13.3b

1. Given that the $3^{rd}$ and $8^{th}$ terms of a geometric sequence are $\frac{4}{3}$ and $-\frac{81}{8}$ respectively.
   (a) Find the $6^{th}$ term.
   (b) Find the sum of the first 6 terms.
2. Given that the $2^{nd}$ term of an infinite geometric series is $-16$, and the $5^{th}$ term is $2$.
   (a) Find the sum of the first 10 terms.
   (b) Find the sum of the infinite series.
3. Given the first term of a geometric series is $7$, the common ratio is $3$, and the sum of the series is $847$. Find the number of terms and the last term of this series.
4. If the sum of an infinite geometric series is $\frac{25}{3}$, and the 2nd term is $-2$. Find the first term and the common ratio.
5. Convert the following recurring decimals into fractions:
   (a) $0.\dot{4}\dot{5}$
   (b) $2.1\dot{3}1\dot{4}$
6. Three numbers form a geometric sequence, their sum is $42$, and their product is $512$. Find these three numbers.
7. Given the first term of a geometric series is $16$, the last term is $\frac{1}{2}$, and the sum is $\frac{63}{2}$. Find the common ratio and the number of terms.
8. Given the sum of the first $6$ terms of a geometric sequence is $9$ times the sum of its first $3$ terms. Find the common ratio.
9. Given the first term of a geometric series is $\frac{81}{8}$, and the $7^{th}$ term is $\frac{8}{9}$. Find the sum of this series to infinity.
10. Given that the 2nd term of a geometric series is $9$ less than the $1^{st}$ term, and the $3^{rd}$ term is $6$ less than the $2^{nd}$ term. Find:
    (a) the $4^{th}$ term.
    (b) the sum of the first $6$ terms.
    (c) the sum of this series to infinity.
11. Given that the common ratio of an infinite geometric series is negative, the sum is $81$, and the sum of the first two terms is $45$. Find the $4^{th}$ term.
12. If $x + 1, x - 2, \frac{1}{2}x$ form the first $3$ terms of an infinite geometric series, find the sum of this series.
13. Given three positive numbers forming an arithmetic sequence with a sum of $36$, and after adding $1$, $4$, and $43$ to them respectively, they form a geometric sequence. Find these three numbers.

14. Suppose a person receives a piece of information and then passes it on to 3 different friends (called the $1^{st}$ round of transmission). Each friend, upon receiving the information, passes it on to 3 different friends (called the 2nd round of transmission), and so on. Assuming that the information is passed to different people during the transmission process, if the information is transmitted in this way for 12 rounds, how many people know this information in total?



15. Sweetheart Sugar Factory produced $50,000$ tons of sugar this year (the first year). If the production increases by $10\%$ each year compared to the previous year, find the production for the 5th year and the total production for these 5 years.

16. When Xiaohua was in high school, her mother gave her RM $200$ pocket money each month. Xiaohua's mother asked her to consider the following two plans:

    First plan: In each subsequent year, the monthly pocket money increases by RM $20$ compared to the previous year.

    Second plan: In each subsequent year, the monthly pocket money increases by $8.8\%$ compared to the previous year.

    (a) Under each plan, calculate the monthly pocket money Xiaohua receives when she is in the third year of high school. Which one is more?

    (b) Under each plan, calculate the total pocket money Xiaohua receives during six years of secondary school. Which one is more?

17. A ball falls vertically from a height of $2$ meters to the ground. After hitting the ground, it rebounds vertically, reaching $80\%$ of its original height, and then falls back down. This process repeats, with each subsequent rebound reaching $80\%$ of the previous height. How far does the ball travel in total?

## 13.3 Compound Interest and Annuities

Back in junior high, we learned about simple interest calculations, but in real life, interest is typically calculated using compound interest. Suppose you open a bank account and deposit RM $p$, then leave it without withdrawing. The bank's annual interest rate is $r\%$, compounded annually. After one year, you'll receive interest of RM $p \times \frac{r}{100}$. Adding this to the principal RM $p$, your total deposit (principal plus interest) becomes:

$$\text{RM } p \times \left(1 + \frac{r}{100}\right)$$

When settling interest after two years, it's calculated based on the deposit from one year earlier, i.e., RM $p \times \left(1 + \frac{r}{100}\right)$ as the principal. Therefore, the deposit after two years becomes:

$$\text{RM } p \times \left(1 + \frac{r}{100}\right) \times \left(1 + \frac{r}{100}\right)$$

Similarly, the deposit after three years becomes:

$$\text{RM } p \times \left(1+\frac{r}{100}\right) \times \left(1+\frac{r}{100}\right) \times \left(1+\frac{r}{100}\right)$$

And so on. The deposit after $n$ years becomes:

$$\text{RM } p \times \underbrace{\left(1+\frac{r}{100}\right) \times \left(1+\frac{r}{100}\right) \times \cdots \times \left(1+\frac{r}{100}\right)}_{n \text{ terms}} = \text{RM } p \times \left(1+\frac{r}{100}\right)^n$$

If RM $a_n$ is the deposit after $n$ years, then $\{a_n\}$ forms a geometric sequence with common ratio $\left(1+\frac{r}{100}\right)$.

Suppose the total deposit is $A$, with an annual interest rate of $r\%$. Then, the total deposit $A = p\left(1+\frac{r}{100}\right)^n$ or $p(1+r\%)^n$, where $p$ is also known as the present value.

For some deposits, interest is not calculated annually but rather semi-annually, quarterly, monthly, or daily, corresponding to 2, 4, 12, or 365 times per year respectively. If the principal is RM $p$, the annual interest rate is $r\%$, and the interest is compounded $m$ times per year, then each time it's calculated using an interest rate of $\frac{r}{m}\%$. In total, interest is compounded $nm$ times after $n$ years. Therefore, the **accumulated value** after $n$ years is:

**Compound Interest Formula**

$$\text{Accumulated value} = \text{RM } p\left(1+\frac{\frac{r}{100}}{m}\right)^{nm}$$

**Example 31**

Xiao Hua deposited RM $50,000$ into the bank, with an annual interest rate of $4\%$, compounded annually. If he does not withdraw the principal or interest, find the accumulated value after 15 years.

**Solution:**

Given that $p = 50,000$, $r = 4$, $n = 15$,

$$\text{Accumulated value} = 50,000 \times \left(1+\frac{4}{100}\right)^{15} = \text{RM } 90,047.18$$

**Example 32**

Given that the principal is RM $100,000$, with an annual interest rate of $5\%$, compounded semi-annually, calculate the interest after $5$ years.

**Solution:**

$$\begin{aligned}
\text{5 years of interest} &= \text{Accumulated value} - \text{Principal} \\
&= 100000 \times \left(1+\frac{2.5}{100}\right)^{10} - 100000 \\
&= 128008.45 - 100000 \\
&= \text{RM } 28,008.45
\end{aligned}$$

**Example 33**

Xiao Fang deposits RM 5,000 every year. If the annual interest rate is 3.5%, compounded annually, calculate the total amount after 20 years.

**Solution:**

The accumulated value after 20 years

$$
\begin{aligned}
&= 5000\left(1+\frac{3.5}{100}\right)^{20} + 5000\left(1+\frac{3.5}{100}\right)^{19} + 5000\left(1+\frac{3.5}{100}\right) + \cdots + 5000\left(1+\frac{3.5}{100}\right) \\
&= 5000\left(1.035 + 1.035^2 + 1.035^3 + \cdots + 1.035^{20}\right) \\
&= 5000 \times \frac{1.035\left(1-1.035^{20}\right)}{1-1.035} \\
&= \text{ RM } 146,347.35
\end{aligned}
$$

**Example 34**

Given that the principal is RM 100,000, the annual interest rate is 5.5%, compounded annually. How many years will it take for the total amount to exceed RM 150,000?

**Solution:**

Let the number of years required be $n$ for the total amount to exceed RM 150,000.

$$
\begin{aligned}
100000\left(1+\frac{5.5}{100}\right)^{n} &> 150000 \\
1.055^{n} &> 1.5 \\
\lg(1.055)^{n} &> \lg 1.5 \\
n &> \frac{\lg 1.5}{\lg 1.055} \\
n &> 7.573
\end{aligned}
$$

$\therefore$ It will take at least 8 years for the total amount to exceed RM 150,000.

**Example 35**

Given an annual interest rate of 6%, compounded annually, with interest settled annually, and interest of RM 15,281.28 after 3 years, find the principal amount.

**Solution:**

Let the principal be $p$.

$$
\begin{aligned}
p\left(1+\frac{6}{100}\right)^{3} - p &= 15281.28 \\
1.191016p - p &= 15281.28 \\
p &= 80000
\end{aligned}
$$

$\therefore$ The principal amount is RM 80,000.

## Annuities and Present Value

An annuity refers to a series of equal payments made or received at regular intervals under a certain contract. Examples include various instalment payments, rent, insurance premiums, instalment loan repayments, etc. The present value of an annuity is the total present value of all payments after putting the annual interest rate and the number of periods into consideration.

If the annual interest rate is $r\%$, and the annuity is RM $A$, with payments made annually, using the formula $A = p(1+r\%)^n$, we can determine that the present value at year $n$ is $A(1+r\%)^{-n}$, where $n$ represents the number of periods. Therefore, the present value of the payment in the first year is $\frac{A}{1+r\%}$, in the second year is $\frac{A}{(1+r\%)^2}$, and so on.

The present value of an annuity paid for $n$ years starting from the current year is:

$$\begin{aligned}
\text{Present value of annuity} &= \frac{A}{1+r\%} + \frac{A}{(1+r\%)^2} + \cdots + \frac{A}{(1+r\%)^n} \\
&= A\left[\frac{1}{1+r\%} + \frac{1}{(1+r\%)^2} + \cdots + \frac{1}{(1+r\%)^n}\right] \\
&= A\left[\frac{1}{1+r\%} \times \frac{1-\frac{1}{(1+r\%)^n}}{1-\frac{1}{1+r\%}}\right] \\
&= \frac{A}{r\%}\left[1-\frac{1}{(1+r\%)^n}\right]
\end{aligned}$$

An annuity is not limited to paying only once per year.

**Additional Information:**

As $n$ approaches infinity, in the case of perpetuity, where payments are made continuously without end, $\frac{1}{(1+r\%)^n}$ approaches 0. Thus, the present value formula for perpetuity is $\frac{A}{r\%}$.

**Example 36**

Given an annuity of RM 1, 500, an annual interest rate of 5%, payments made annually, and continuous payments for 20 years, we want to find the present value.

**Solution:**

Given that $n = 20$, $A =$ RM 1, 500, $r = 5$,

$$\begin{aligned}
\text{Present value} &= \frac{1500}{5\%}\left(1-\frac{1}{(1+5\%)^{20}}\right) \\
&= \frac{1500}{0.05}(0.62311) \\
&= \text{RM } 18,693.32
\end{aligned}$$

**Example 37**

Given an annuity of RM $2,000$, an annual interest rate of $6\%$, and payments made once per year, how many years are needed for the present value to exceed RM $20,000$?

**Solution:**

Let the number of years required be $n$ for the present value to exceed RM $20,000$.

$$\begin{aligned}
\frac{2000}{6\%}\left[1-\frac{1}{(1+6\%)^n}\right] &> 20000\\
\left(1-\frac{1}{1.06^n}\right) &> 0.6\\
\frac{1}{1.06^n} &< 0.4\\
1.06^n &> 2.5\\
n\lg 1.06 &> \lg 2.5\\
n &> \frac{\lg 2.5}{\lg 1.06}\\
n &> 15.73
\end{aligned}$$

$\therefore$ It will take at least $16$ years for the present value to exceed RM $20,000$.

## Practice 13.4

1. Xiaofang deposited RM $2,000$ in the bank at an annual interest rate of $4\%$, compounded annually. She withdrew all the savings after ten years. How much money can she withdraw?
2. Xiaoyu deposited RM $1,000$ in the bank with an annual interest rate of $3.6\%$. Calculate Xiaoyu's total deposit after one year in the following scenarios:
   (a) Interest compounded annually;
   (b) Interest compounded semi-annually;
   (c) Interest compounded quarterly.

## Exercise 13.4

1. Given a principal of RM80, 000 and an annual interest rate of 5%, compounded annually, find the total amount of principal and interest after 10 years.
2. Someone deposits a sum of money with a compound interest rate of 6%, compounded annually. After 3 years, the interest earned is RM 955.08. Find the amount of the deposit.
3. Deposit RM 80, 000 into a financial company with an annual interest rate of 5.5%, compounded quarterly. Find the accumulated value after 5 years.
4. Prove that with a compound interest rate of 5%, compounded annually, the accumulated value will exceed twice the principal after 15 years.
5. Given a principal of RM 20, 000 and an annual interest rate of 6%, compounded annually, how many years are needed for the accumulated value to exceed RM 200, 000?
6. Given a principal of RM 120, 000 and a compound interest rate of 4.5% compounded annually, how many years are needed to earn RM 50, 652.07 in interest?
7. Someone deposits RM 2, 500 annually. If the annual interest rate is 4%, compounded annually, find the total principal and interest after 15 years.
8. If the present value is RM 15, 443.46 and the annual interest rate is 5%, find the continuous annuity for 10 years.
9. Given an annuity of RM 5, 000, an annual interest rate of 5%, with annual payments for 25 years, find the present value of the annuity and the present value of if it is a perpetuity.
10. Given an annuity of RM 2, 500, an annual interest rate of 4.5%, with annual payments, how many years are needed for the present value to exceed RM 30, 000?

## 13.4 Special Summation of Series

Let's start by observing the formula for the sum of the first $n$ natural numbers:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Now, let's examine the sum of squares of the first $n$ natural numbers, denoted as $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$.

First, we notice that:

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$
$$(n+1)^3 - n^3 = 3n^2 + 3n + 1$$

$$\begin{array}{cccccccccccc}
3 & \times & 1^2 & + & 3 & \times & 1 & + & 1 & = & 2^3 & - & 1^3 \\
3 & \times & 2^2 & + & 3 & \times & 2 & + & 1 & = & 3^3 & - & 2^3 \\
3 & \times & 3^2 & + & 3 & \times & 3 & + & 1 & = & 4^3 & - & 3^3 \\
 & & \cdots & & & & \cdots & & \cdots & & & & \\
3 & \times & n^2 & + & 3 & \times & n & + & 1 & = & (n+1)^3 & - & n^3
\end{array}$$

Adding these $n$ equations together, we get:

$$\begin{aligned}
3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 &= (n+1)^3 - 1 \\
3\sum_{k=1}^{n} k^2 + 3\times\frac{n(n+1)}{2} + n &= n^3 + 3n^2 + 3n \\
3\sum_{k=1}^{n} k^2 &= n^3 + 3n^2 + 2n - \frac{3n(n+1)}{2} \\
&= n(n+1)(n+2) - \frac{3n(n+1)}{2} \\
&= \frac{n(n+1)(2n+4-3)}{2} \\
\therefore \sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

**Sum of the First $n$ Squares**

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Next, let's inspect $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3$. From

$$\begin{aligned}
(n+1)^4 &= n^4 + 4n^3 + 6n^2 + 4n + 1 \\
(n+1)^4 - n^4 &= 4n^3 + 6n^2 + 4n + 1
\end{aligned}$$

$$\begin{array}{ccccccccccccccc}
4 & \times & 1^3 & + & 6 & \times & 1^2 & + & 4 & \times & 1 & + & 1 & = & 2^4 & - & 1^4 \\
4 & \times & 2^3 & + & 6 & \times & 2^2 & + & 4 & \times & 2 & + & 1 & = & 3^4 & - & 2^4 \\
4 & \times & 3^3 & + & 6 & \times & 3^2 & + & 4 & \times & 3 & + & 1 & = & 4^4 & - & 3^4 \\
 & & \cdots & & & & \cdots & & & & \cdots & & \cdots & & & & \\
4 & \times & n^3 & + & 6 & \times & n^2 & + & 4 & \times & n & + & 1 & = & (n+1)^4 & - & n^4
\end{array}$$

Adding these $n$ equations together, we get:

$$\begin{aligned}
4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 &= (n+1)^4 - 1 \\
4\sum_{k=1}^{n} k^3 + 6\times\frac{n(n+1)(2n+1)}{6} + 4\times\frac{n(n+1)}{2} + n &= n^4 + 4n^3 + 6n^2 + 4n \\
4\sum_{k=1}^{n} k^3 &= n^4 + 4n^3 + 6n^2 + 4n - \left(2n^3 + 3n^2 + n\right) - \left(2n^2 + 2n\right) - n \\
&= n^4 + 2n^3 + n^2 \\
&= n^2(n+1)^2 \\
\therefore \sum_{k=1}^{n} k^3 &= \frac{n^2(n+1)^2}{4} \\
&= \left[\frac{n(n+1)}{2}\right]^2
\end{aligned}$$

**Sum of the First $n$ Cubes**

$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$$

**Example 38**

Find $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1)$.

**Solution:**

$$\begin{aligned}
1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1) &= \sum_{k=1}^{n} k(k+1) \\
&= \sum_{k=1}^{n} \left(k^2 + k\right) \\
&= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\
&= \frac{n(n+1)(2n+1+3)}{6} \\
&= \frac{n(n+1)(n+2)}{3}
\end{aligned}$$

**Example 39**

Find the sum of the first $n$ terms of the series $1^3 + 3^3 + 5^3 + \cdots$.

**Solution:**

The $n$th term of the series $1^3 + 3^3 + 5^3 + \cdots$ is given by $a_n = (2n-1)^3$.

**Approach 1:**

$$\begin{aligned}
&1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 \\
&= \sum_{k=1}^{n} (2k-1)^3 \\
&= \sum_{k=1}^{n} \left(8k^3 - 12k^2 + 6k - 1\right) \\
&= 8\sum_{k=1}^{n} k^3 - 12\sum_{k=1}^{n} k^2 + 6\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \\
&= 8 \times \frac{n^2(n+1)^2}{4} - 12 \times \frac{n(n+1)(2n+1)}{6} + 6 \times \frac{n(n+1)}{2} - n \\
&= n\left(2n^3 + 4n^2 + 2n - 4n^2 - 6n - 2 + 3n + 3 - 1\right) \\
&= n\left(2n^3 - n\right) \\
&= n^2\left(2n^2 - 1\right)
\end{aligned}$$

**▸ Example 39**

**Approach 2:**

$$\begin{aligned}2^3+4^3+6^3+\cdots+(2n)^3 &= 2^3\left(1^3+2^3+3^3+\cdots+n^3\right)\\ &= 8\times\frac{n^2(n+1)^2}{4}\\ &= 2n^2(n+1)^2\end{aligned}$$

$$\begin{aligned}&1^3+3^3+5^3+\cdots+(2n-1)^3\\ &=\left[1^3+2^3+3^3+4^3+\cdots+(2n-1)^3+(2n)^3\right]-\left[2^3+4^3+\cdots+(2n)^3\right]\\ &=\frac{(2n)^2(2n+1)^2}{4}-2n^2(n+1)^2\\ &=n^2\left(4n^2+4n+1-2n^2-4n-2\right)\\ &=n^2\left(2n^2-1\right)\end{aligned}$$

**▸ Example 40**

Given that $\dfrac{1}{n(n+1)}=\dfrac{1}{n}-\dfrac{1}{n+1}$. Find $\displaystyle\sum_{k=1}^{n}\frac{1}{k(k+1)}$.

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k(k+1)} &= \frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{n(n+1)}\\ &=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)\\ &=1-\frac{1}{n+1}\\ &=\frac{n}{n+1}\end{aligned}$$

**▸ Example 41**

Find sum of the series $\dfrac{1}{2}+\dfrac{3}{4}+\dfrac{5}{8}+\dfrac{7}{16}+\cdots+\dfrac{2n-1}{2^n}$.

**Solution:**

This series is formed by the product of corresponding terms of an arithmetic sequence $1,3,5,7,\cdots$ and a geometric sequence $\dfrac{1}{2},\dfrac{1}{4},\dfrac{1}{8},\dfrac{1}{16},\cdots$. Series like this are usually summed using the following method.

$$\begin{aligned}\text{Let}\quad S_n &= \frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\frac{7}{2^4}+\cdots+\frac{2n-1}{2^n}\quad\cdots\ (1)\\ \frac{1}{2}S_n &= \frac{1}{2^2}+\frac{3}{2^3}+\frac{5}{2^4}+\frac{7}{2^5}+\cdots+\frac{2n-1}{2^{n+1}}\quad\cdots\ (2)\end{aligned}$$

**Example 41**

$$
\begin{aligned}
(1)-(2) \text{ gives } \frac{1}{2}S_n &= \frac{1}{2}+\left(\frac{2}{2^2}+\frac{2}{2^3}+\frac{2}{2^4}+\cdots+\frac{2}{2^n}\right)-\frac{2n-1}{2^{n+1}}\\
&= \frac{1}{2}+\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}\right)-\frac{2n-1}{2^{n+1}}\\
&= \frac{1}{2}+1-\frac{1}{2^{n-1}}-\frac{2n-1}{2^{n+1}}\\
S_n &= 3-\frac{2n+3}{2^n}
\end{aligned}
$$

## Practice 13.5

1. Find the sum of $10^2+11^2+12^2+\cdots+50^2$.
2. Find the sum of $1\times2\times3+2\times3\times4+3\times4\times5+\cdots+n(n+1)(n+2)$.
3. Find the sum of the first $n$ terms of the series $1+3+5+7+\cdots$.
4. Find the sum of the first $n$ terms of the series $1^2+3^2+5^2+7^2+\cdots$.

## Exercise 13.5

1. Find the sum of the first 20 terms of the series $2\frac{1}{2}+4\frac{1}{4}+8\frac{1}{8}+\cdots$.
2. Given the sequence $\{a_n\}$ where the $n$th term is $a_n = 2n-1+\left(\frac{2}{3}\right)^n$, find the sum of the first $n$ terms of this sequence.
3. Find the sum of the first $n$ terms of the series $1\frac{1}{3}+4\frac{1}{9}+7\frac{1}{27}+\cdots$.
4. Find the sum of the infinite series $\frac{1}{3}+\frac{4}{9}+\frac{7}{27}+\frac{10}{81}+\cdots$.
5. Find the sum of the first $n$ terms of the series $x+5x^2+9x^3+13x^4+\cdots$.

   Hence, if $|x|<1$, find the sum of this series to infinity.
6. Find the sum of the first $n$ terms of the series $\frac{1}{1\times4}+\frac{1}{4\times7}+\frac{1}{7\times10}+\cdots$.
7. Given $\frac{1}{(3n-2)(3n+1)(3n+4)}=\frac{1}{18(3n-2)}-\frac{1}{9(3n+1)}+\frac{1}{18(3n+4)}$, find the sum of the first $n$ terms of the series $\frac{1}{1\times4\times7}+\frac{1}{4\times7\times10}+\frac{1}{7\times10\times13}+\cdots$, and find the sum of this series to infinity.
8. Without using a calculator, find the value of $1-2\times3+3\times3^2-4\times3^3+5\times3^4-\cdots+11\times3^{10}$.
9. Find the sum of the series $\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}+\frac{1}{\sqrt{5}+\sqrt{7}}+\cdots+\frac{1}{\sqrt{119}+\sqrt{121}}$.

10. Find the value of the following expressions:

(a) $\sum_{k=1}^{8}(2k-1)^2$

(b) $\sum_{k=6}^{12}k(k+2)$

(c) $\sum_{k=1}^{10}(2k-1)^3$

(d) $\sum_{k=3}^{10}k\left(k^2-k+1\right)$

11. Given that the $n$th term of a sequence is $a_n = 3n^2 + n$, find the sum of the first $n$ terms of this sequence.

12. Find the sum of the first $n$ terms of the series $1\times 3 + 2\times 7 + 3\times 11 + 4\times 15 + \cdots$.

13. Find the sum of the first $n$ terms of the series $1\times 4\times 7 + 2\times 5\times 8 + 3\times 6\times 9 + \cdots$.

14. Given a sequence $\{a_n\}$ defined by $a_1 = 3$ and for $n \geq 2$, $a_n = a_{n-1} + 2n$,

    (a) Find the general formula for this sequence.

    (b) Find the sum of the first $n$ terms of this sequence.

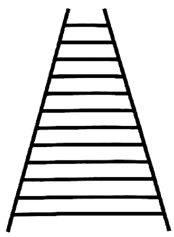
## Revision Exercise 13

1. Given that the sum of the first 4 terms of an arithmetic series is 28, and the sum of the first 8 terms is 48, find the sum of the first 12 terms.

2. Find the sum of all integers from 1 to 1000 that are not divisible by 7.

3. Find the sum of all integers from 150 to 300 that are divisible by both 3 and 5.

4. Find the sum of all natural numbers less than 300 that are divisible by 6 but not by 8.

5. 

   A ladder has a total of 12 steps, with the top step being 33 cm wide and the bottom step being 110 cm wide. The widths of the steps form an arithmetic sequence.

   (a) Find the width of the $8^{th}$ step counting from the bottom.

   (b) If the material for the horizontal beams of these ladders costs RM 2 per meter, how much money is needed to manufacture all 12 steps of the ladder?

6. In the arithmetic series $10 + 9\frac{1}{5} + 8\frac{2}{5} + \cdots$, what is the position of the first negative term? Summing the terms from the first term up to which term will start to result in a negative sum?

7. It is known that the $10^{th}$ term of an arithmetic series is $-23$, and the $25^{th}$ term is 22.

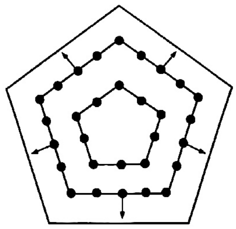
   (a) From which term onwards are the terms positive?

   (b) Up to which term, starting from the first term, is the sum positive?

8. In a certain place, the rice yield was $100,000$ kilograms in $2015$, and it increased by $10\%$ every year thereafter. In which year will the cumulative yield reach $771,561$ kilograms?

9. It is known that the interior angles of a convex polygon form an arithmetic sequence with a common difference of $5$, and the smallest angle is $120^\circ$. Find the number of sides of this polygon.

10. 



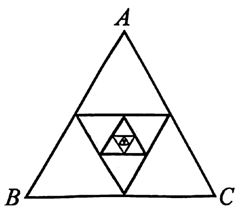
As shown in the diagram, there are several students forming a cheerleading formation in the shape of a regular pentagon, with the number of students per side increasing as you move outward from the center. It is known that there are $3$ students per side in the innermost circle, and the number of students per side increases by $2$ for each successive circle (i.e. the second circle has $5$ students per side, the third circle has $7$ students per side, and so on). How many students are there in the outermost circle (the $10^{th}$ circle)? How many students are there in total participating in the cheerleading performance?

11. Given that the second term of an arithmetic sequence is $10$, and the sum of the first $3$ terms equals the sum of the first $10$ terms, determine from which term onwards the sequence becomes negative.

12. Given that $w, x, y, z$ form a geometric sequence and $w+z = 9, x+y = 6$, find the common ratio of this sequence.

13. Given that $a, b, c$ are all positive numbers, in the sequence $4, a, b, c, 64$, $4, a, b$ form a geometric sequence, $b, c, 64$ form a geometric sequence, but $a, b, c$ form an arithmetic sequence. Find the values of $a, b, c$.

14. 




In the diagram, $\triangle ABC$ is an equilateral triangle with area $S$. Connecting the midpoints of its sides forms another triangle. Continuing this process infinitely, find the sum of the areas of all the triangles.

15. If $a, b, c$ are positive numbers and $\log a, \log b, \log c$ form an arithmetic sequence, prove that $a, b, c$ form a geometric sequence.

16. Given that $x, y, z$ form an arithmetic sequence, $x, y, z + 8$ form a geometric sequence, and $x + y + z = 18$, find the values of $x, y, z$.

17. Given that $a, b, c$ form a geometric sequence and $a + 1, b + 5, c + 25$ also form a geometric sequence, find the ratio of $a : b : c$.

18. Given that $x - 3, x + 1, 4x - 2$ form a geometric sequence with a common ratio of $r$, and the sum of these three numbers is $S$, find $r + S$.

19. Given that the sum of the first $n$ terms of a geometric series is $5$, and the sum of the first $2n$ terms is $650$, find the sum of the first $3n$ terms.

20. Starting from $2005$, Xiaotian deposits RM $6,000$ into a bank at the beginning of each year with an annual interest rate of $3\%$, compounded semi-annually. If Xiaotian does not withdraw any principal or interest, what is the total amount of his deposit in the bank after depositing money at the beginning of $2020$?

21. Given a principal of RM 75,000 with an annual interest rate of $4.5\%$, compounded quarterly, find the accumulated value after $10$ years.

22. Xiaoxiang deposited a sum of money with an annual interest rate of $5.5\%$, compounded annually. After 5 years, the deposit increased by RM $2455.68$. Find the amount of the deposit.

23. Given a principal of RM $120,000$ with an annual interest rate of $5\%$, compounded annually, how many years will it take for the accumulated value to exceed RM 250,000?

24. If the present value is RM $22,939.84$ and the annual interest rate is $6\%$, find the annuity payable continuously for $20$ years.

25. Given an annuity of RM 8,000 with an annual interest rate of $4.5\%$, paid annually for $15$ years, find the present value and the present value of perpetuity.

26. Given an annuity of RM $4,500$ with an annual interest rate of $4\%$, paid annually, how many years will it take for the present value to exceed RM $50,000$?

27. Xiao Ming, who has just entered university, plans to buy a computer worth RM $3,000$ using a loan and then repay it in instalments by working part-time. If Xiao Ming repays the loan once a month, clearing all the debt in $12$ instalments, and the monthly interest rate of the loan is $0.5\%$ compounded monthly, find the monthly instalment amount (rounded to the nearest integer).

28. A brand laptop worth RM $2,500$ can be purchased in two instalment payment options:

    Option 1: Payment in $3$ installments, with one payment every $4$ months.

    Option 2: Payment in $6$ installments, with one payment every $2$ months.

    If the monthly interest rate is $0.6\%$, calculated on a compound interest basis, and the payment amount is the same each time, determine which payment option is more cost-effective.

29. Find the sum of the first $n$ terms of the series $2\frac{1}{2} + 5\frac{1}{4} + 8\frac{1}{8} + 11\frac{1}{16} + \cdots$.

30. Given that $S_n = [n(n+1)]^2$ is the sum of the first $n$ terms of the sequence $\{a_n\}$, find:

    (a) $a_n$

    (b) $\displaystyle\sum_{n=5}^{10} a_n$

31. Given that the sequence $\{a_n\}$ satisfies $a_1 + 2a_2 + 3a_3 + \cdots + na_n = n^2(n+1)$, find $a_n$ and $a_{200}$.

32. Given that the $n$th term of a sequence is $a_n = n(n+1)(n+3)$, find the sum of the first $n$ terms of this sequence.

33. Find the sum of the first $n$ terms of the series $2 \times 5 + 5 \times 8 + 8 \times 11 + 11 \times 17 + \cdots$.