Spectral Shape Analysis for 3D matching

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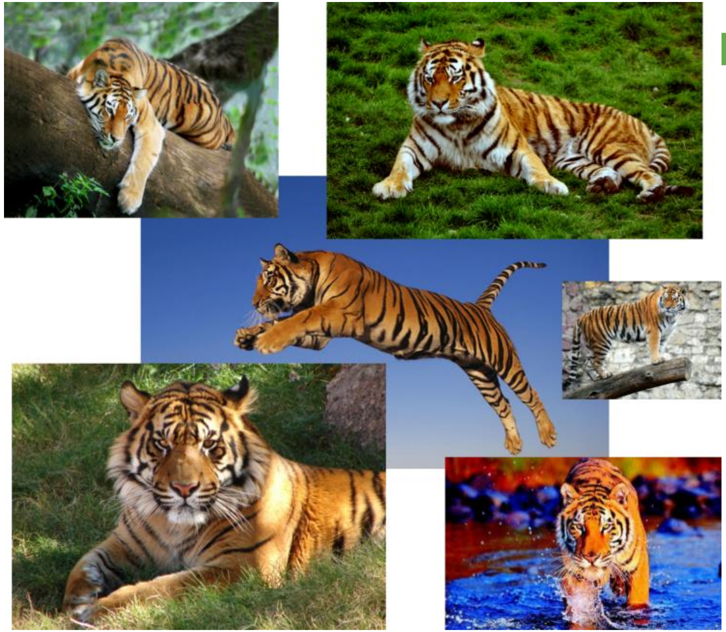
Differential Geometry

Motivations

Study and Handle 3D object from real word that we generally denote with $\mathcal{X}$



$\mathcal{X}$ is a geometric object

$\mathcal{X}$ undergoes several deformations

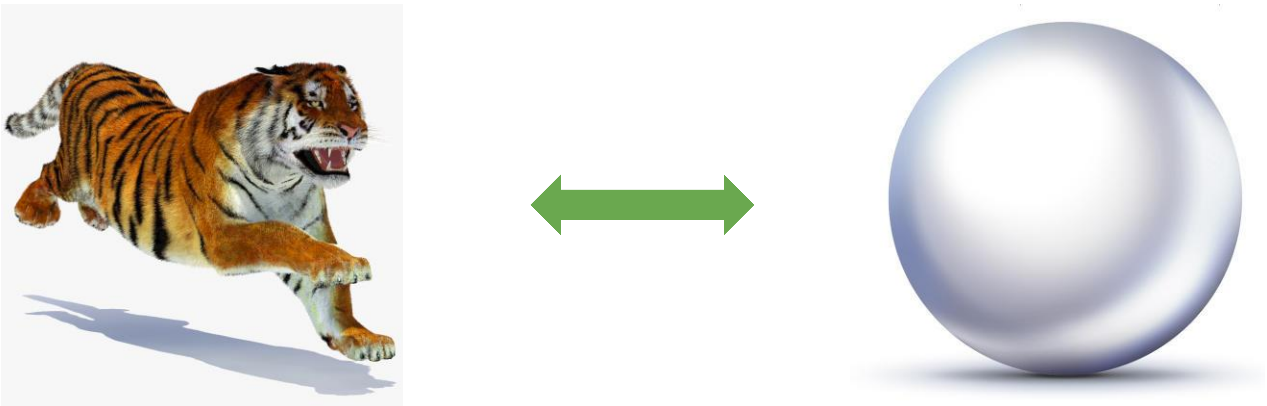
$\mathcal{X}$ is fully represented by its external surface

The perfect tool to describe $\mathcal{X}$ is:

DIFFERENTIAL GEOMETRY

Overview

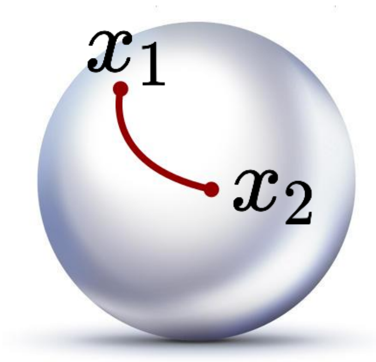
The main goal of this part of the course is: **to provide the general idea on the differential geometry necessary to study $\mathcal{X}$**



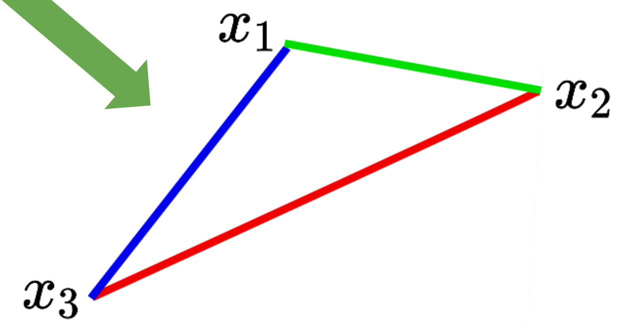
All the definitions and constructions that we introduce in this course are valid on both the surfaces

metric on a space $\mathcal{X}$

a metric on a set $\mathcal{X}$ is defined as a function $d : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$ such that:



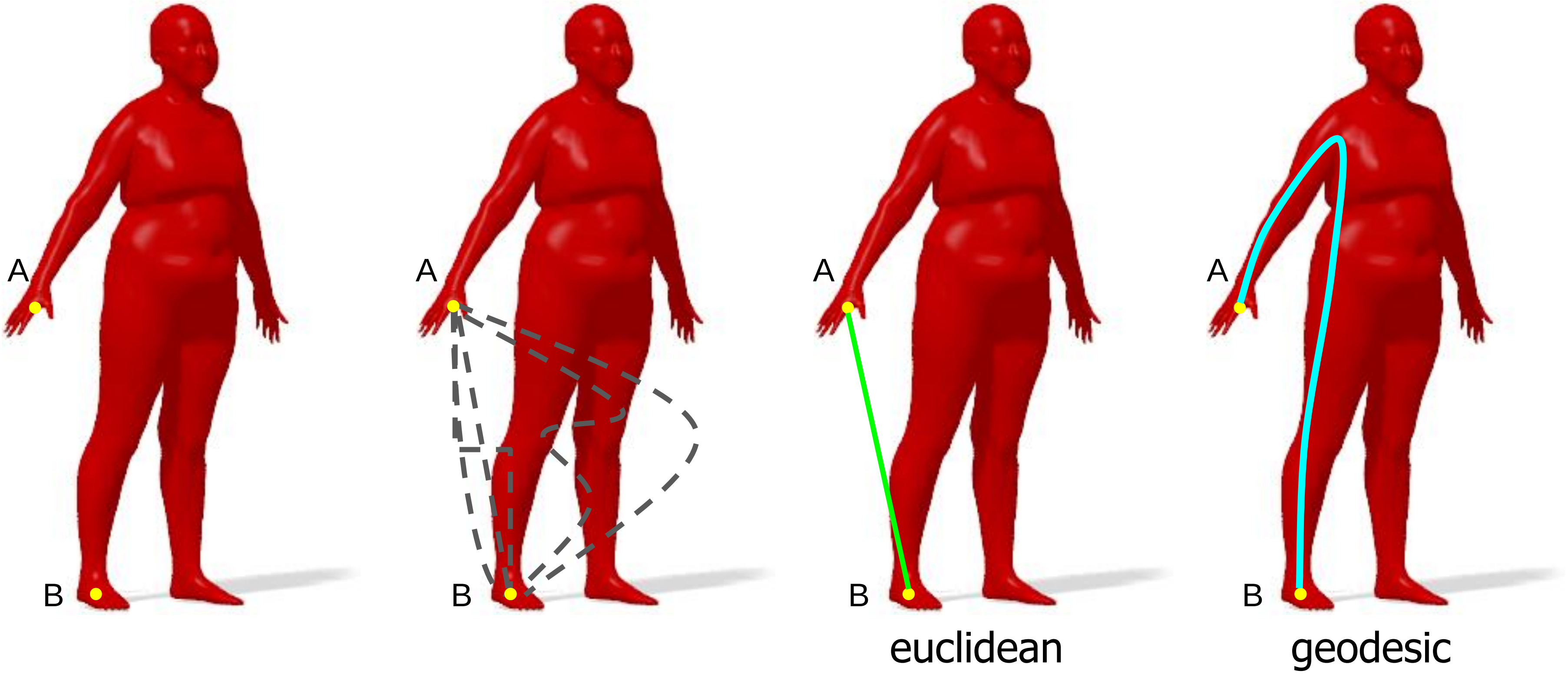
1) **non negativity:** $d(x_1, x_2) \geq 0$
2) **indiscernability:** $d(x_1, x_2) = 0 \iff x_1 = x_2$
3) **symmetry:** $d(x_1, x_2) = d(x_2, x_1)$
4) **triangle inequality:** $d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)$



and if all these hold for all $x_1, x_2, x_3 \in \mathcal{X}$

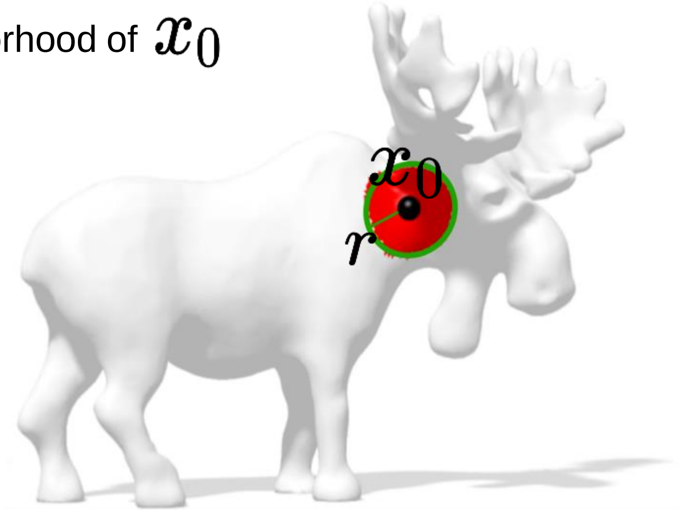
$(\mathcal{X}, d)$ is a **metric space**

distances



metric neighborhood of a point $x_0 \in \mathcal{X}$

Given a point $x_0 \in \mathcal{X}$ we define a open/close neighborhood of x_0

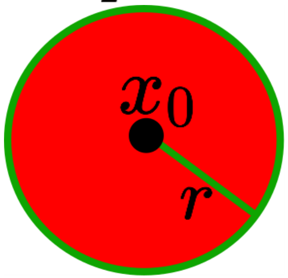


open: $$B_r^{x_0} := \{x \in \mathcal{X} | d(x_0, x) < r\}$$

close: $$\bar{B}_r^{x_0} := \{x \in \mathcal{X} | d(x_0, x) \leq r\}$$

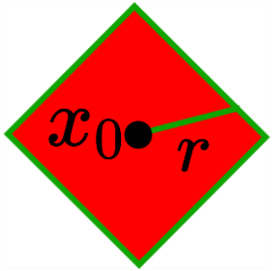
In Euclidean spaces $(\mathbb{R}^2)$ we have the norms:

L$_2$ ball



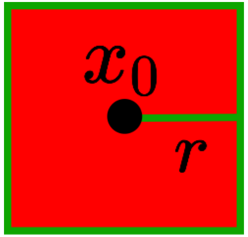
$\|x - x_0\|_2 < r$

L$_1$ ball



$\|x - x_0\|_1 < r$

L$_\infty$ ball



$\|x - x_0\|_\infty < r$

$$\|x - x_0\|_p = \left(\sum_k |x^k - x_0^k|^p\right)^{\frac{1}{p}}$$

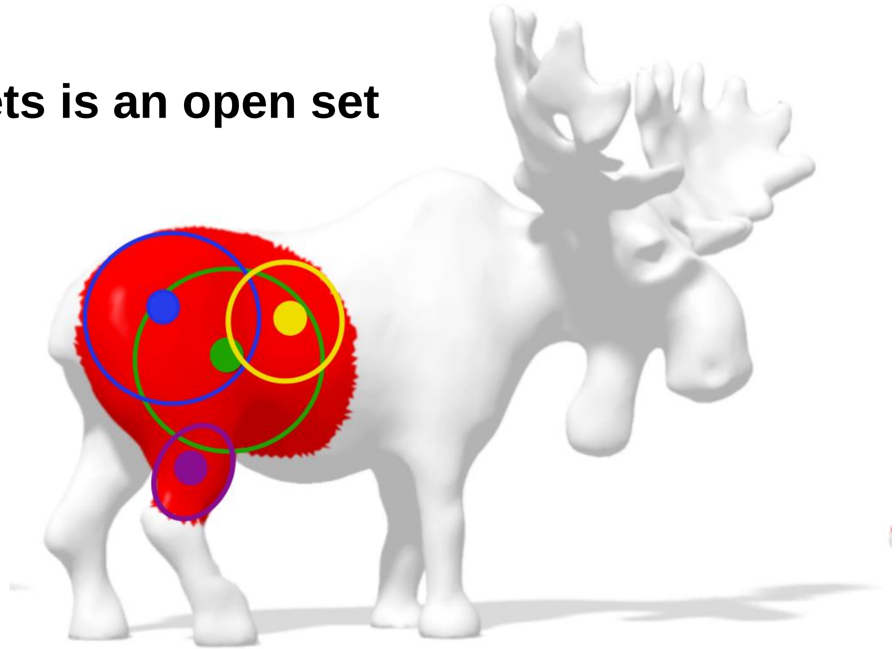
$$\|x - x_0\|_\infty = max|x^k - x_0^k|$$

open set

A set $A \subseteq \mathcal{X}$ is an **open set** of $\mathcal{X}$ if: $\forall x \in A$ there exists $r > 0$ $s.t.\ B_r^x \subseteq A$

Properties of open sets:

1) **the union of any number of open sets is an open set**
2) **the empty set is an open set**
3) **the intersection of a finite number of open sets is an open set**



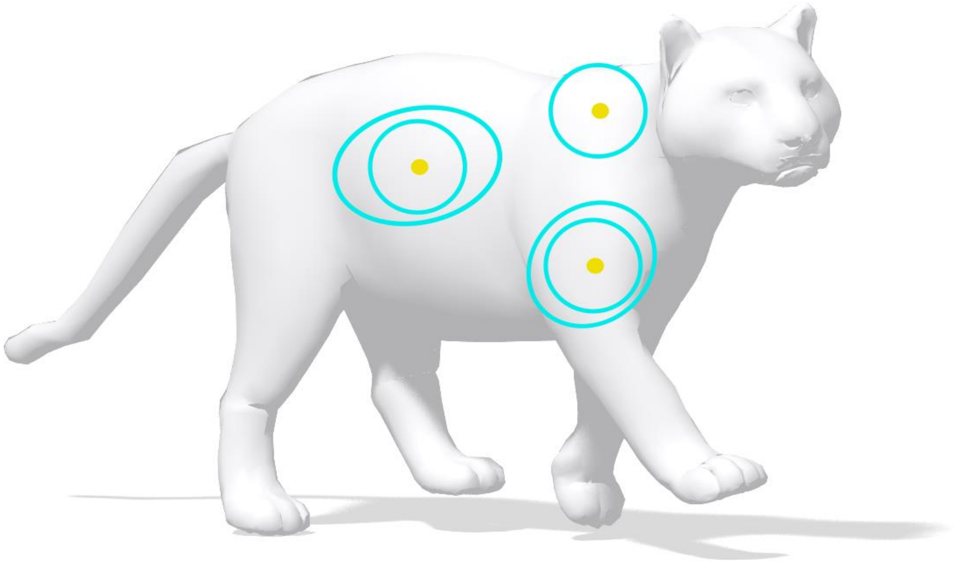
A topology for $\mathcal{X}$

A **topology** on the set $\mathcal{X}$ is the collection of all its open subsets.

Remark: **metric** ⟹ **open sets** ⟹ **topology**

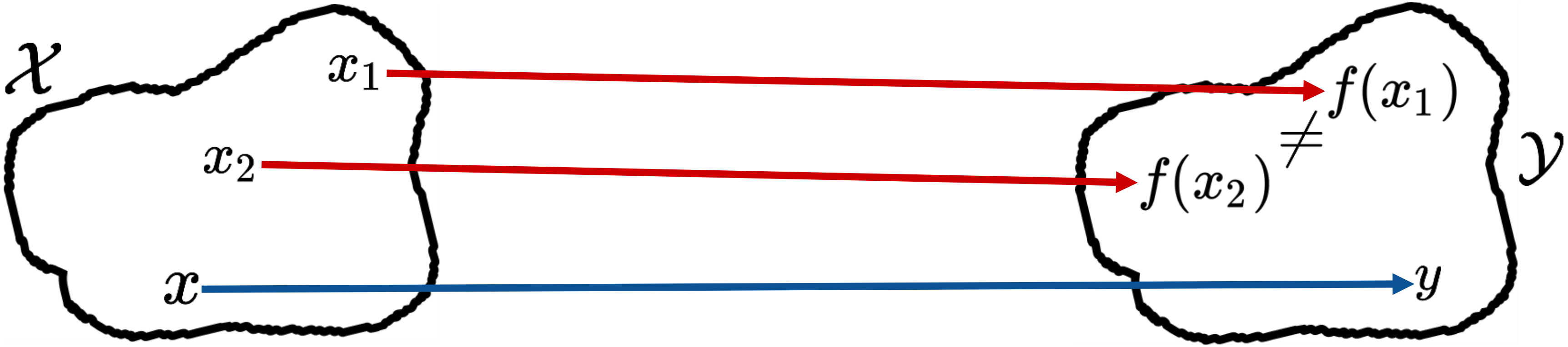
A **topology** can be also defined independently from a metric through an axiomatic definition of the open sets.

The standard **topology** on shape is the one defined by the geodesic metric



function between sets

Given two sets $\mathcal{X} = \{x_1, \dots, x_n\}$ and $\mathcal{Y} = \{y_1, \dots, y_m\}$, a function $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a list of associations of every element from $\mathcal{X}$ with an element of $\mathcal{Y}$



- f is **injective** if $f(x_1) \neq f(x_2)$, $\forall x_1, x_2 \in \mathcal{X}$ s.t. $x_1 \neq x_2$.
- f is **surjective** if $\forall y \in \mathcal{Y} \Rightarrow \exists$ at least one $x \in \mathcal{X}$ s.t. $f(x) = y$.
- f is **bijective** if $\forall y \in \mathcal{Y} \Rightarrow \exists! x \in \mathcal{X}$ s.t. $f(x) = y$. ($=$ **injective** $+$ **surjective**)

continuity

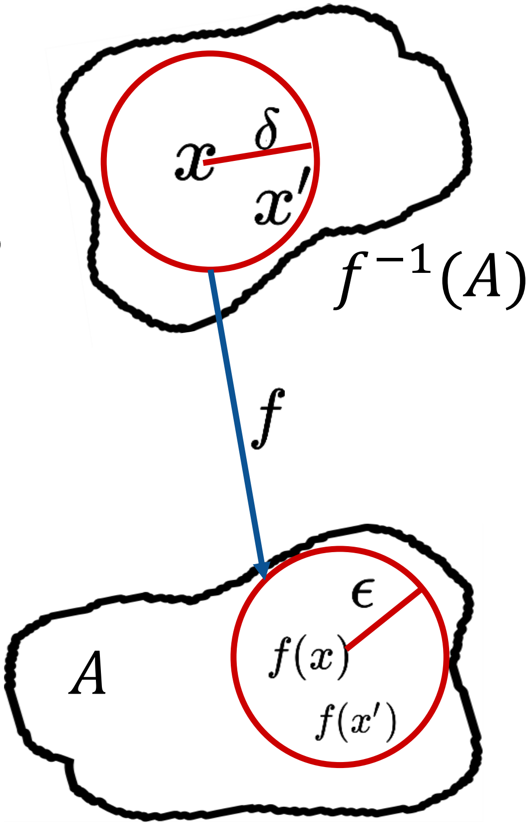
Definition: A function $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is **continuous** if

Topological

for any open subset $A \subseteq \mathcal{Y}$ its preimage via f,
$f^{-1}(A) = \{x \in \mathcal{X} \text{ s.t. } f(x) \in \mathcal{Y}\}$, is an open subset of $\mathcal{X}$

Metric

$\forall \epsilon > 0$ exists a $\delta > 0$ s.t.
$d_{\mathcal{Y}}(f(x), f(x')) < \epsilon \;\; \forall x, x' \in \mathcal{X}$ and $d_{\mathcal{X}}(x, x') < \delta$



the previous 2 definitions are equivalent

continuity is a **local** property

homeomorphisms

A **bijective** function is a function for which exists *one and only one* **image** for each point in the **domain** and *one and only one* **preimage** for each point in the **codomain**

Definition: A **homeomorphism** is a **bijective** and **continuous** function with an **inverse** that is also **continuous**

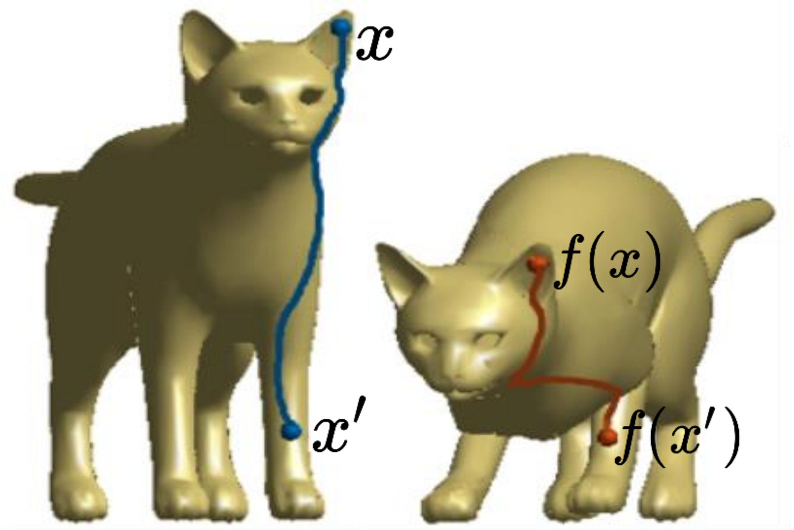
Definition: Two domains for which exists an homeomorphism are **topologically equivalent**

isometry

Definition: A function $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an **isometric embedding** or a **distance preserving function** if it preserves the distances or equivalently if the following condition holds:

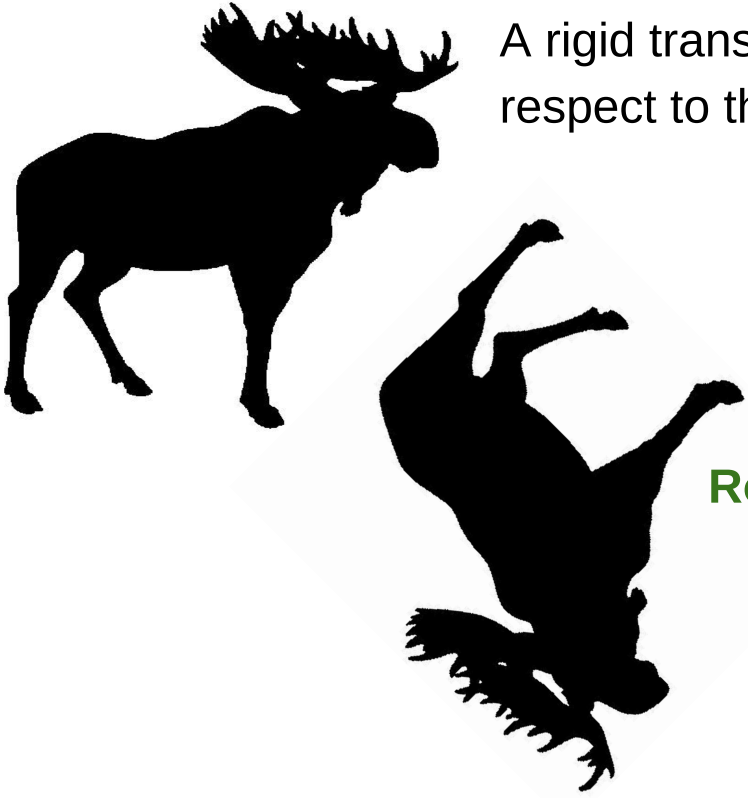
$$d_{\mathcal{X}}(x, x') = d_{\mathcal{Y}}(f(x), f(x')),\ \forall\, x, x' \in \mathcal{X}$$

Definition: A **distance preserving function** that is also **bijective** is an **isometry**



Two domains for which exists an isometry are **metrically equivalent** **=** they have the same metric.

Euclidean isometries



A rigid transform is an **isometry** in the Euclidean space with respect to the euclidean distance (all distances are preserved)

Translation: $$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

Rotation: $$R_\theta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Reflection: $$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Near-isometry

A function $f : (\mathcal{X}, d_{\mathcal{X}}) \longrightarrow (\mathcal{Y}, d_{\mathcal{Y}})$ is a **near-isometry** or an ϵ**-isometry** if:

f is ϵ-distance preserving:

$$d_{\mathcal{X}}(x, x') - d_{\mathcal{Y}}(f(x), f(x')) \leq \epsilon, \ \forall x, x' \in \mathcal{X}$$

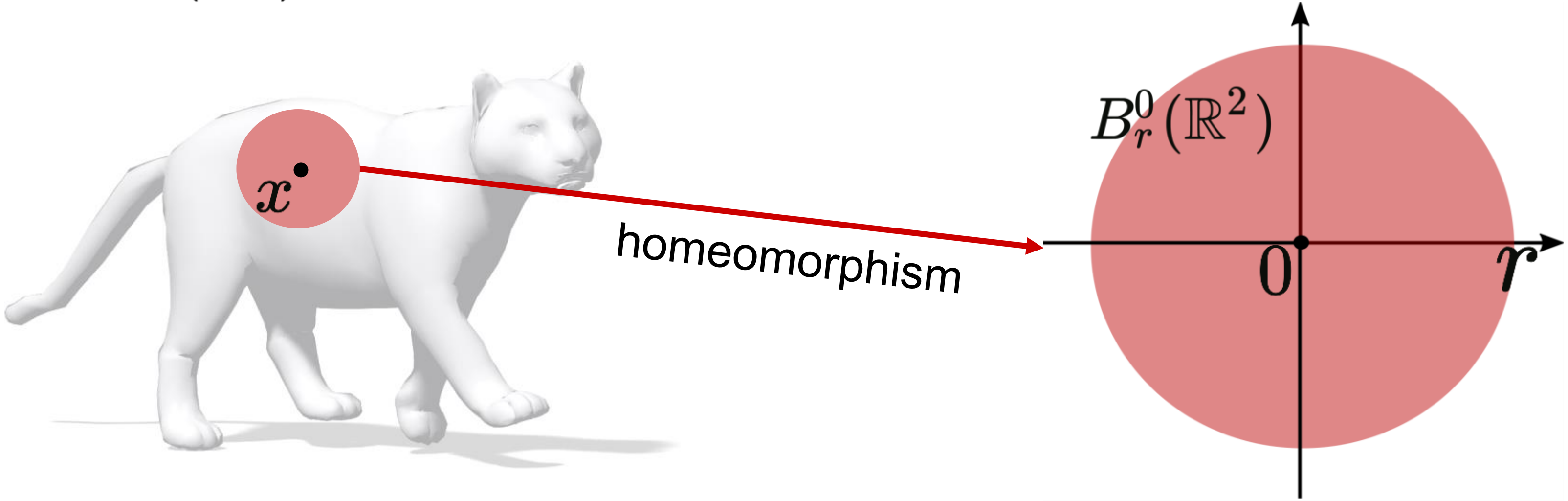
f is ϵ-surjective:

$$\forall y \in \mathcal{Y} \ \exists x \in \mathcal{X} \ \text{s.t.} \ d_{\mathcal{Y}}(y, f(x)) \leq \epsilon$$

For a fixed $\epsilon > 0 \in \mathbb{R}$, if $\epsilon = 0 \implies f$ is an **isometry**.

manifolds

A topological space in which every point has a neighborhood *homeomorphic* to $B_r^0(\mathbb{R}^n)$ (**topological disc** in $\mathbb{R}^n$) is an n **- dimensional manifold**



$$B_r^0(\mathbb{R}^n) := \{x \in \mathbb{R}^n | d(x_0, x) \leq r\}$$

charts and atlases

Definition: Given a point $x \in \mathcal{X}$, a **chart** is a homeomorphism α from an open neighborhood $U_\alpha \subseteq \mathcal{X}$ of x and $\mathbb{R}^n$.

$$\alpha : U_\alpha \subseteq \mathcal{X} \longrightarrow \mathbb{R}^n \text{ s.t. } x \in U_\alpha \text{ and } \alpha \text{ is an homeomorphism.}$$

Definition: An **atlas** is a collection of charts $\{(U_{\alpha_i}, \alpha_i)\}_{i \in I}$ such that:

$$\mathcal{X} \subseteq \bigcup_{i \in I} U_{\alpha_i}$$

where I is the set of the indices in the collection of charts.

smooth manifold

Definition: Given two charts (U_α, α) and (U_β, β) such that $U_\alpha \bigcap U_\beta \neq \emptyset$ a **transition function** $\beta \circ \alpha^{-1}$(a **change of coordinates**) is defined as:

$$\beta \circ \alpha^{-1} : \alpha(U_\alpha \bigcap U_\beta) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

Definition: A manifold $\mathcal{X}$ with an atlas $\{(U_{\alpha_i}, \alpha_i)\}_{i \in I}$ for which all the transition functions are $\mathcal{C}^k$ is said a **manifold** $\mathcal{C}^k$.

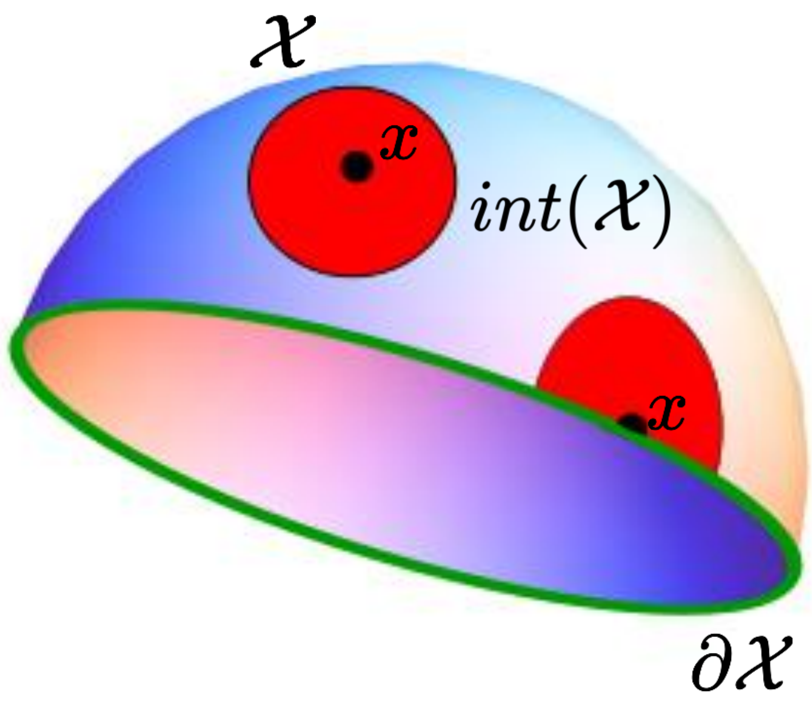
A $\mathcal{C}^\infty$ manifold is said a **smooth manifold**.

manifold with boundary

A **topological space** in which every point x has an open neighborhood **homeomorphic** to either:

- to a topological disk (i.e. $\mathbb{R}^n$)
- to a topological half disk (i.e. $[0, +\infty) \times \mathbb{R}^{n-1}$)

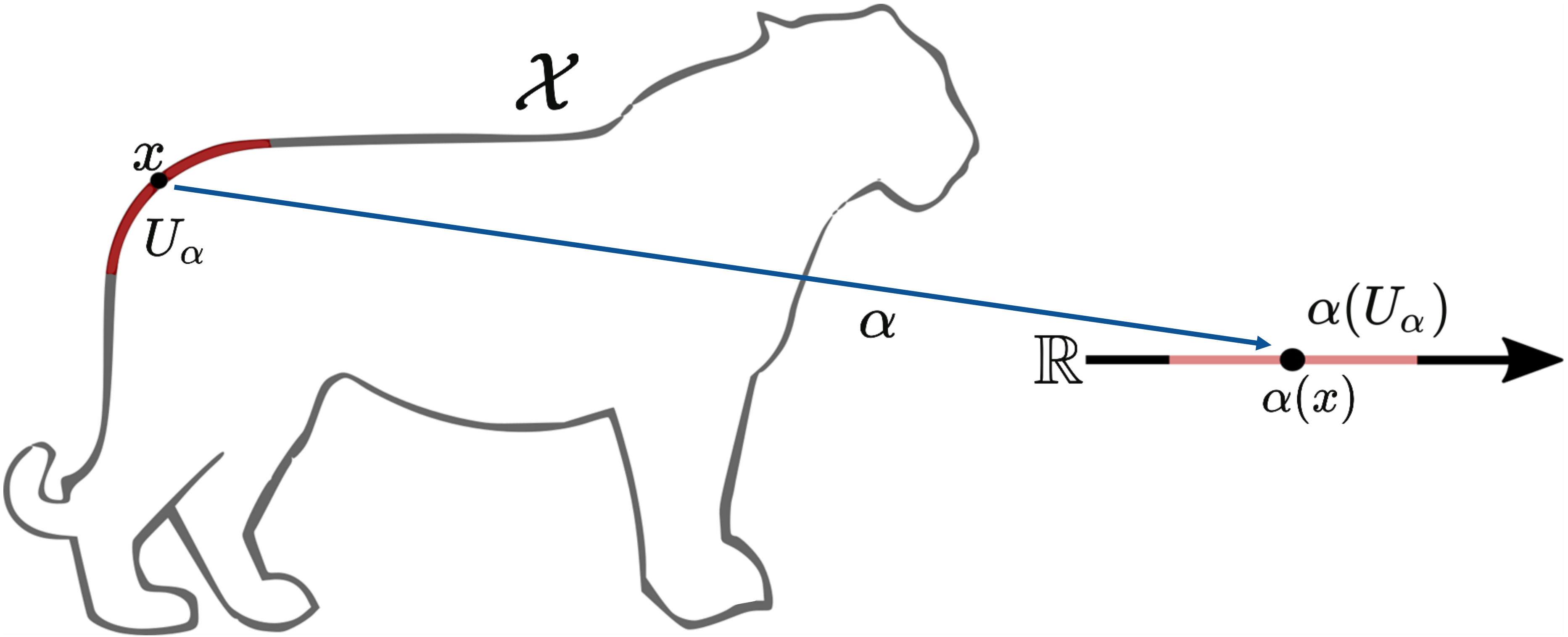
is called a **manifold with boundary**



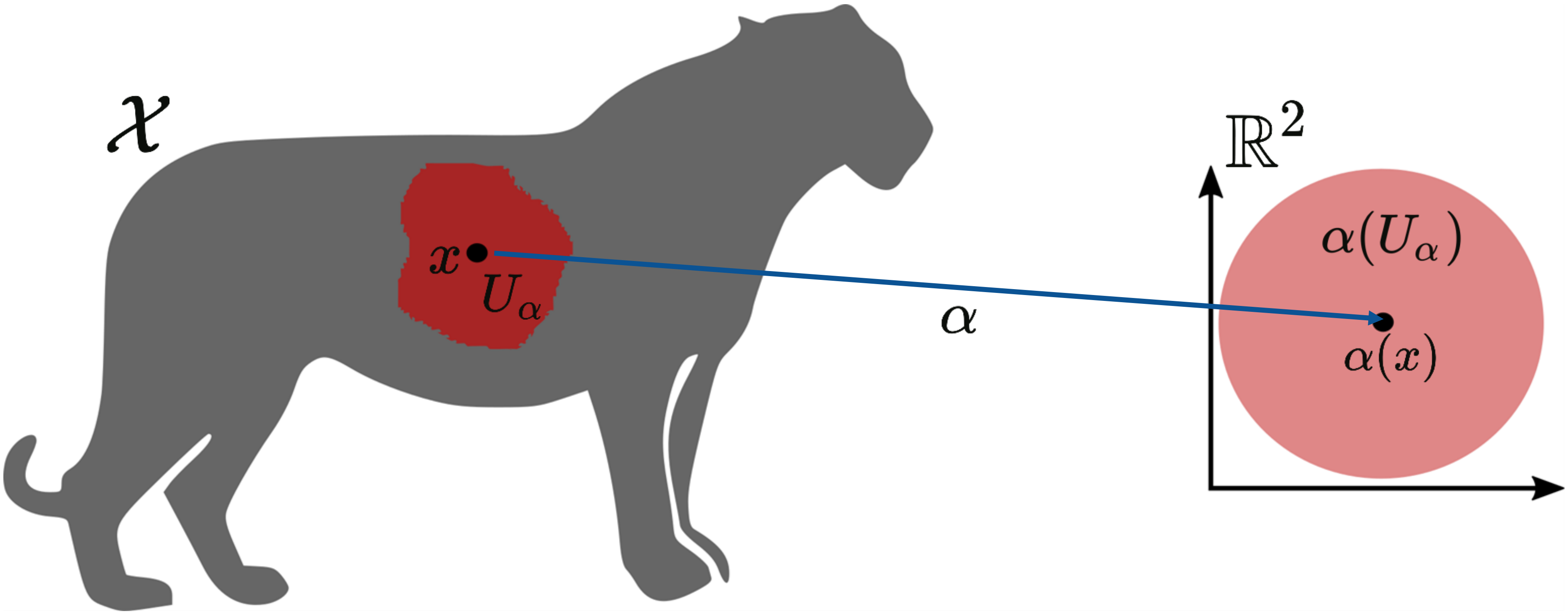
$int(\mathcal{X})$ is defined as the set of point for wich exist a disk-like open neighboood

$\partial\mathcal{X}$ is defined as the set of point for wich exist a half disk-like open neighboood

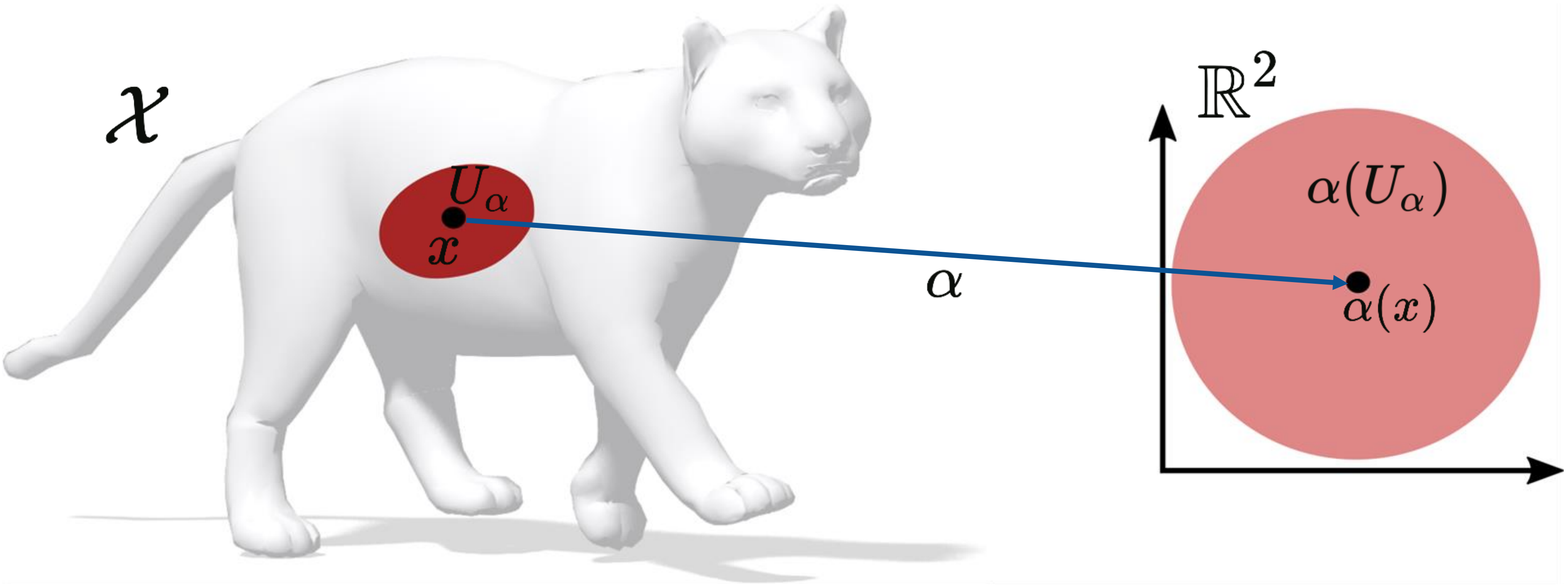
1D manifold



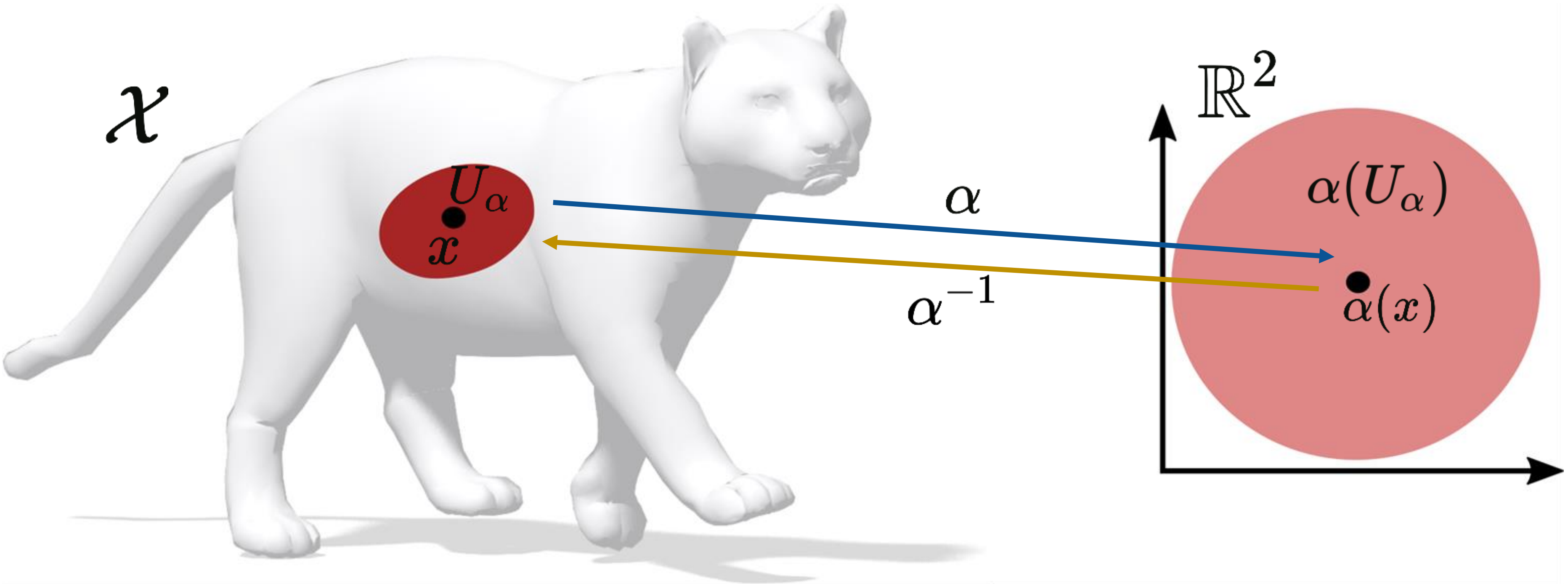
2D planar manifold



2D manifold in 3D space



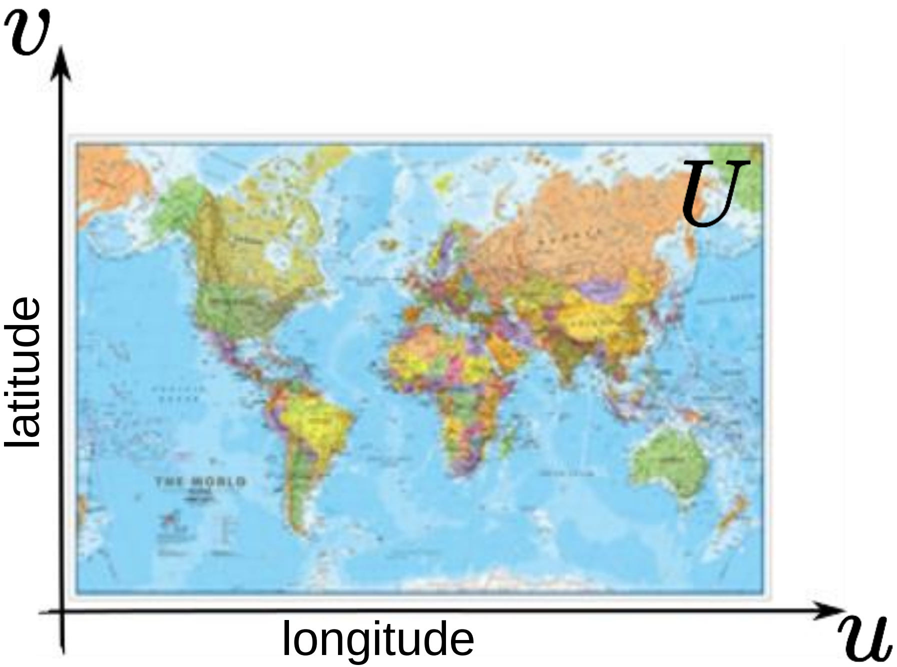
2D manifold in 3D space



α is an homeomorphism so it is a **bijective** and **continuous** function with an **inverse** that is also **continuous, we refer to this maps as** α^{-1}

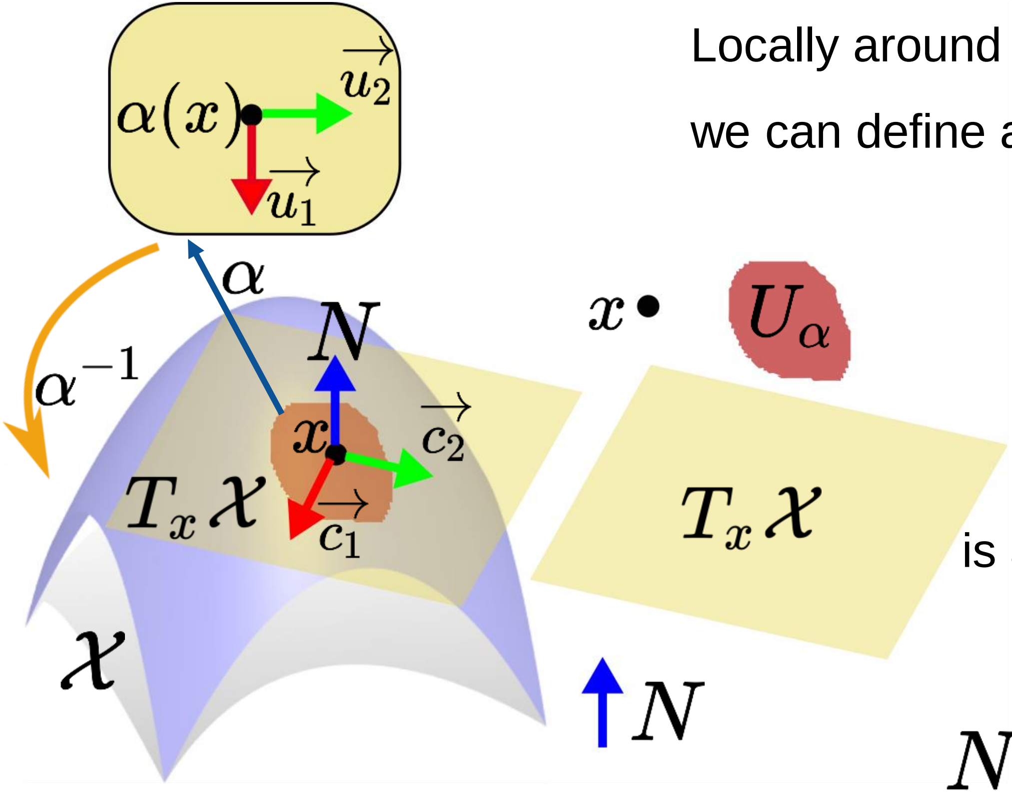
parametrization: a well-known example

$$U = [\frac{-\pi}{2}, \frac{\pi}{2}] \times [-\pi, \pi]$$



$$x = r \cdot cos(v) \cdot cos(u)$$
$$y = r \cdot sin(v) \cdot cos(u)$$
$$z = r \cdot sin(v)$$

tangent plane



Locally around each point $x \in \mathcal{X}$ thanks to the chart we can define a **local system of coordinates** (U_α, α)

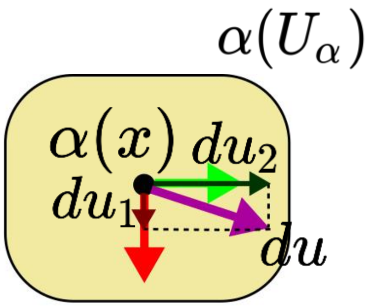
$$\vec{c_1} = \frac{\partial \alpha^{-1}}{\partial \vec{u_1}} \qquad \vec{c_2} = \frac{\partial \alpha^{-1}}{\partial \vec{u_2}}$$

The plane $T_x\mathcal{X} = span(\vec{c_1}, \vec{c_2})$ is the **tangent plane** at $x \in \mathcal{X}$ is a **local Euclidean approximation** of $\mathcal{X}$

N is the **normal vector** and is orthogonal to $\vec{c_1}, \vec{c_2}$ and then to $T_x\mathcal{X}$

The Jacobian

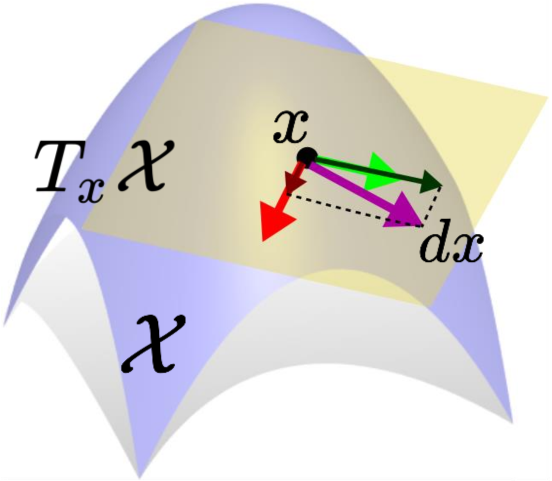
we can consider an infinitesimal du displacement on the chart $\alpha(U_\alpha)$ and look at its image dx on $T_x\mathcal{X}$



$$dx = \alpha^{-1}(\alpha(x) + du) - \alpha^{-1}(\alpha(x))$$
$$= du_1\,\vec{c_1} + du_2\,\vec{c_2} = Jdu$$

J is the **Jacobian matrix** whose columns correspond to the 2D vectors $\vec{c_1}$ and $\vec{c_2}$

$$J = [\ \vec{c_1}\ ,\ \vec{c_2}\ ]$$

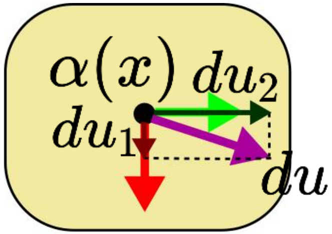


metric and first fundamental form

We can measure the length l of the displacement dx as follows:

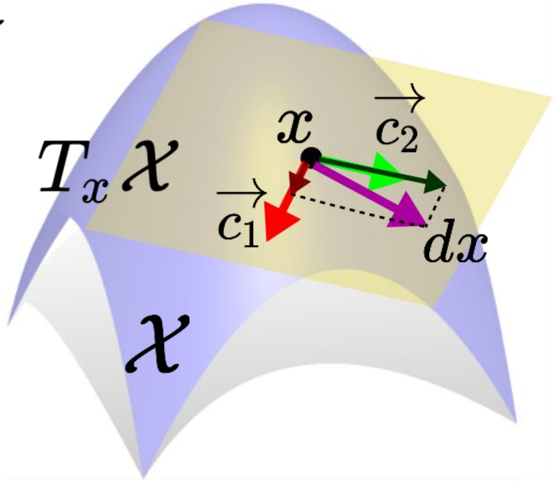
$$l^2 = \|dx\|_F^2 = du^\top J^\top J du = du^\top G du$$

$G = J^\top J$ is a positive definite symmetric 2×2 matrix



G through its entries defines the inner product on $T_x\mathcal{X}$

$$g_{i,j} = < \vec{c_i}, \vec{c_j} > \text{ with } i,j \in \{1,2\}$$



It defines a quadratic form: $G: T_x\mathcal{X} \times T_x\mathcal{X} \rightarrow \mathbb{R}$

$$l^2 = du^\top G du$$

namely this quadratic form is the **I fundamental form**

curvature on a plane

Let $\gamma : [0,1] \longrightarrow \mathbb{R}^2$ be a smooth curve parameterized by arclength:

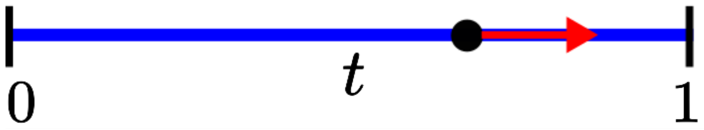
$\int_a^b \|\gamma'\| dt = |a-b|, \ \forall a,b \in [0,1]$ where $\gamma' = \dfrac{\partial \gamma(t)}{\partial t}, \ t \in [0,1]$

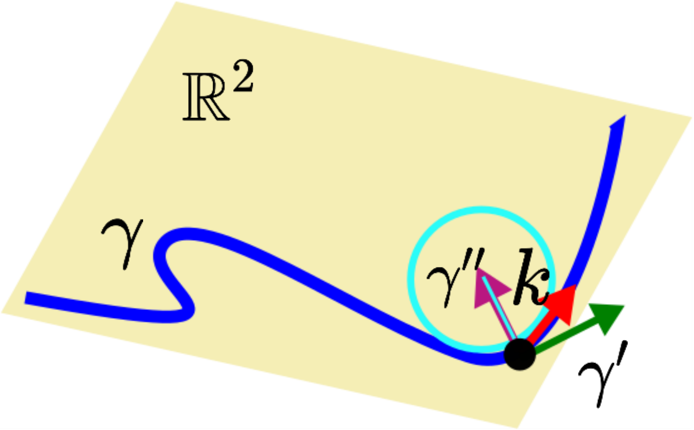
γ **trajectory** = a line on the plane

γ' **velocity** = vector that indicates the rate of change of position

γ'' **accelaration** = vector (curvature)

k **curvature** = measures of the rate of rotation of the velocity vector





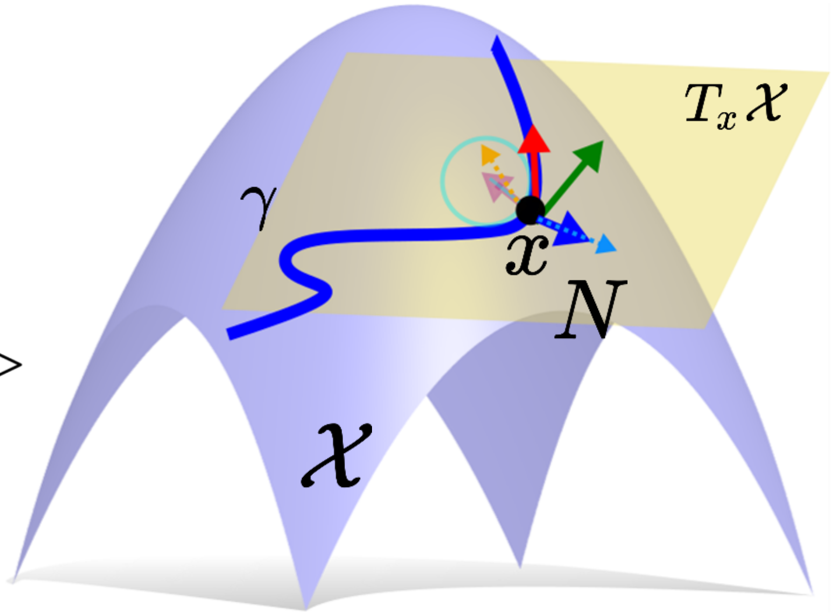
curvature on a surface

Let $\gamma : [0,1] \longrightarrow \mathcal{X}$ be a smooth curve defined on $\mathcal{X}$

$$x \in \mathcal{X}, x = \gamma(t),\ t \in [0,1]$$

The curvature vector γ'' decomposes into:

- **geodesic curvature =** $k_g = \gamma''|_{T_x\mathcal{X}}$
- **normal curvature =** $k_n = < \gamma'', N >$

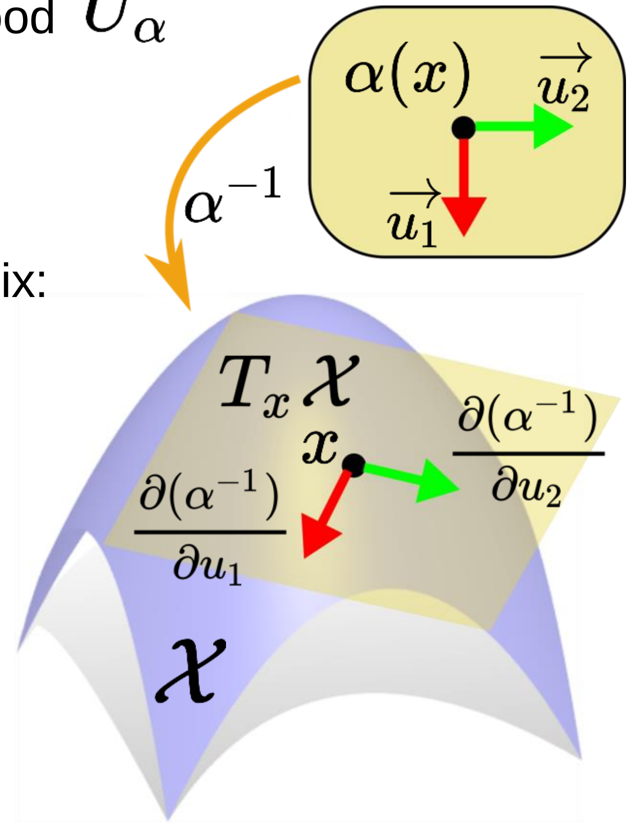


for a point $x \in \mathcal{X}$ there is not a unique curvature

Curves passing at x in different directions have different curvatures

II fundamental form

$\alpha^{-1} : \mathbb{R}^2 \longrightarrow U_\alpha \ni x$ provides the neighborhood U_α of x of a regular parametrization



We define the **second fundamental form** as the quadratic form defined on $T_x\mathcal{X}$ by the following matrix:

$$II = \begin{bmatrix} \frac{\partial^2 \alpha^{-1}}{\partial u_1 \partial u_1} \cdot N & \frac{\partial^2 \alpha^{-1}}{\partial u_1 \partial u_2} \cdot N \\ \frac{\partial^2 \alpha^{-1}}{\partial u_1 \partial u_2} \cdot N & \frac{\partial^2 \alpha^{-1}}{\partial u_2 \partial u_2} \cdot N \end{bmatrix}$$

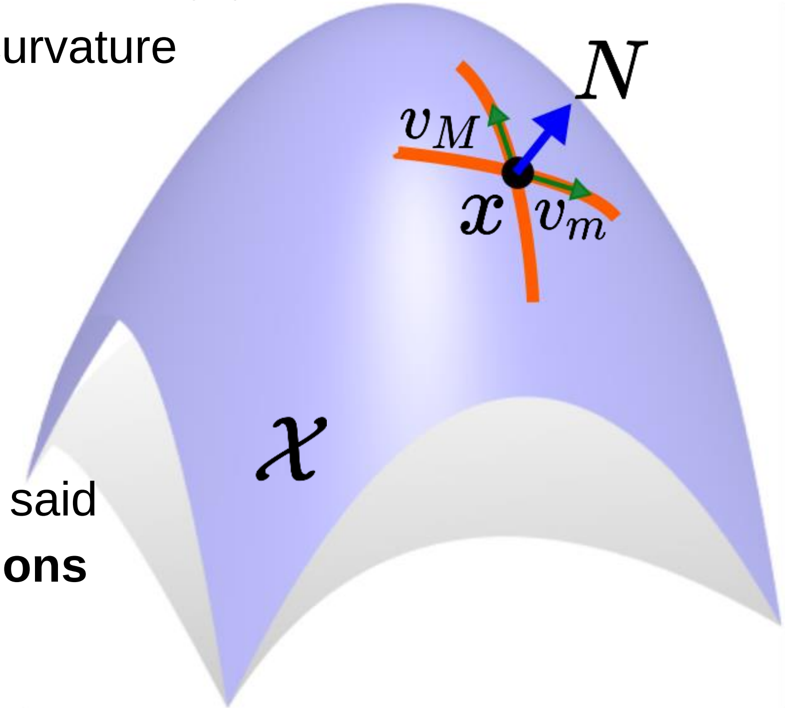
It gives us an idea about the local curvature of the surface!

principal curvatures

Given a point $x \in \mathcal{X}$ and a vector $v \in T_x\mathcal{X}$

we can consider a curve γ s.t. $\gamma(0) = x$ and $\gamma'(0) = v$

$\forall v \in T_x\mathcal{X},\ \gamma$ may have different normal curvature



minima principal curvature

$$k_m = \min_{v \in T_x\mathcal{X}} < \gamma''(0), N >$$

maxima principal curvature

$$k_M = \max_{v \in T_x\mathcal{X}} < \gamma''(0), N >$$

the vectors v_m, v_M that realize k_m, k_M are said the **minima** and the **maxima principal directions**

These are eigenvectors and eigenvalues of II

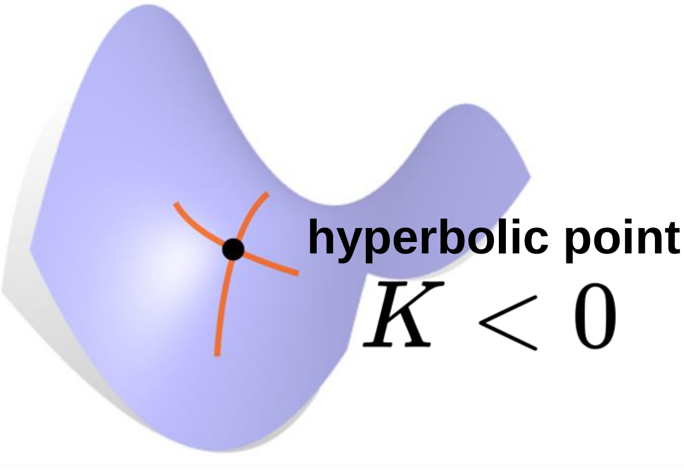
mean and gaussian Curvatures

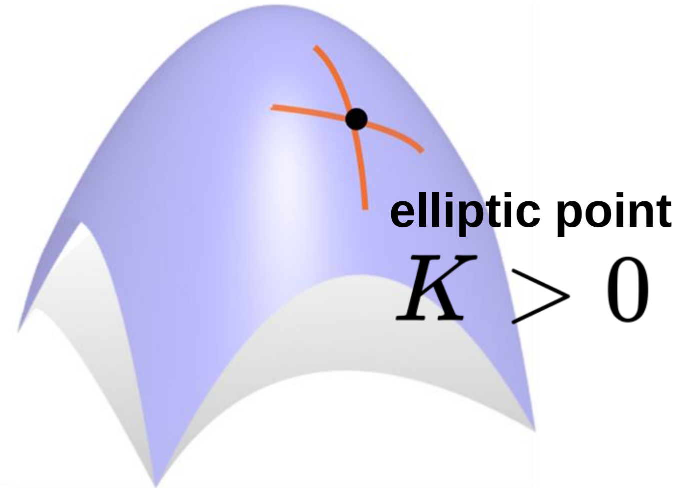
mean curvature

$$H = \frac{1}{2}(k_m + k_M)$$

Gaussian curvature

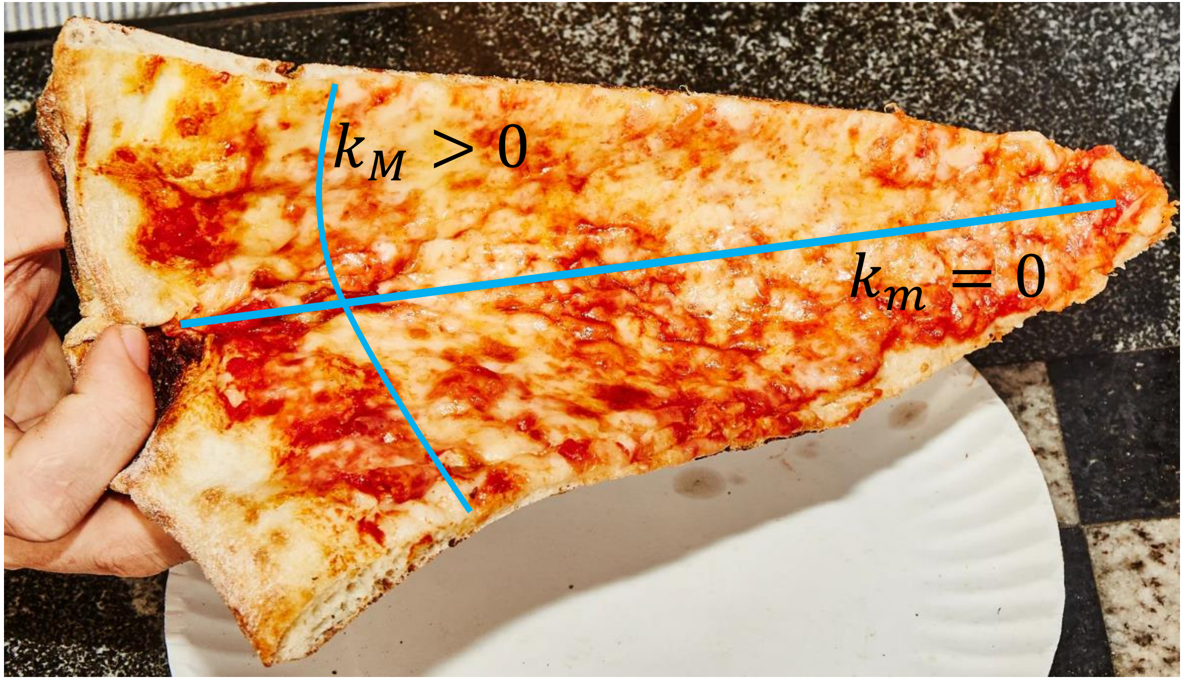
$$K = k_m \cdot k_M$$





What is $\boldsymbol{K}$ for a plane?

mean and gaussian Curvatures



What is $\boldsymbol{K}$ for a slice of pizza?

path on a surface

Let $\gamma : [0,1] \longrightarrow \mathcal{X}$ be a smooth curve defined on $\mathcal{X}$

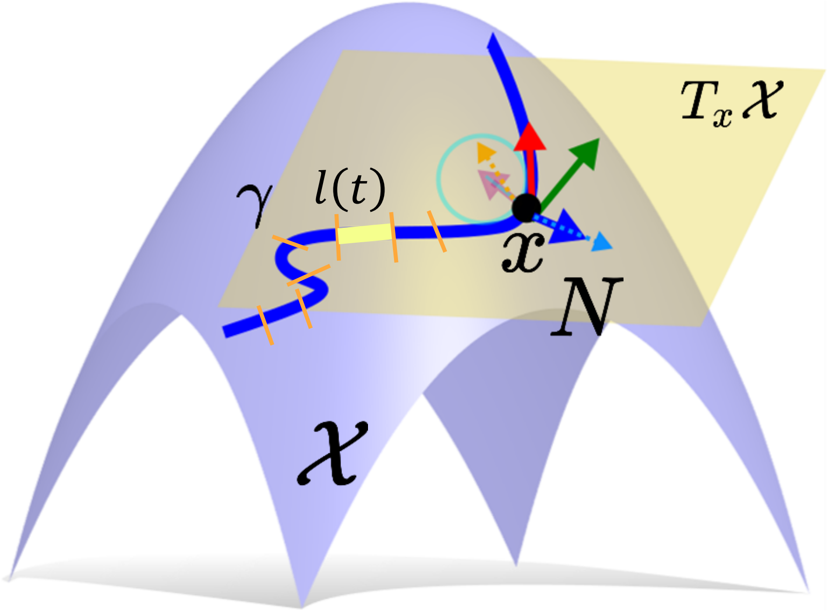
$$x \in \mathcal{X}, x = \gamma(t),\ t \in [0,1]$$

$$l^2 = \|dx\|_F^2 = du^\top J^\top J du = du^\top G du$$

$$l(t) = \sqrt{(\gamma'(t))^\top G \gamma'(t)}$$

$$L(\gamma) = \sum l_i$$

$$L(\gamma) = \int_a^b \sqrt{(\gamma'(t))^\top G \gamma'(t)} dt$$



path length

the **Riemannian metric** is strictly related with the length of the paths defined on $\mathcal{X}$

Let $\gamma : [0,1] \longrightarrow \mathcal{X}$ be a smooth curve defined on $\mathcal{X}$ the length of γ is:

$$L(\gamma) = \int_a^b \sqrt{(\gamma'(t))^\top G \gamma'(t)}dt$$

or

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}dt$$

where for a given point $x = \gamma(t)$ we denote with $g_{\gamma(t)}$ the Riemannian metric:

$$g_{\gamma(t)} = g_x = g : T_x\mathcal{X} \times T_x\mathcal{X} \longrightarrow \mathbb{R}$$

The Riemannian metric g can be seen as the map that associates at every point $x \in \mathcal{X}$ the bilinear form g_x on $T_x\mathcal{X}$

$$g : \mathcal{X} \longrightarrow \mathcal{F}(T_.\mathcal{X} \times T_.\mathcal{X}, \mathbb{R})$$

$$g : x \in \mathcal{X} \longmapsto g_x \ (g_x T_x\mathcal{X} \times T_x\mathcal{X} \to \mathbb{R})$$

Riemannian geometry

Given a surface $\mathcal{X}$ a **Riemannian metric** is a **bilinear symmetric positive definite form** g defined on the tangent space $T_x\mathcal{X}$

$$g : T_x\mathcal{X} \times T_x\mathcal{X} \longrightarrow \mathbb{R}$$

The **Riemannian metric** is completely independent from the 3D embedding in which we visualize $\mathcal{X}$

g can be represented as a matrix G that is related with the first fundamental form and its entries are $g_{i,j} =< \overrightarrow{c_i}, \overrightarrow{c_j} > \text{with}\, i, j \in \{1, 2\}$

path length and I fundamental form

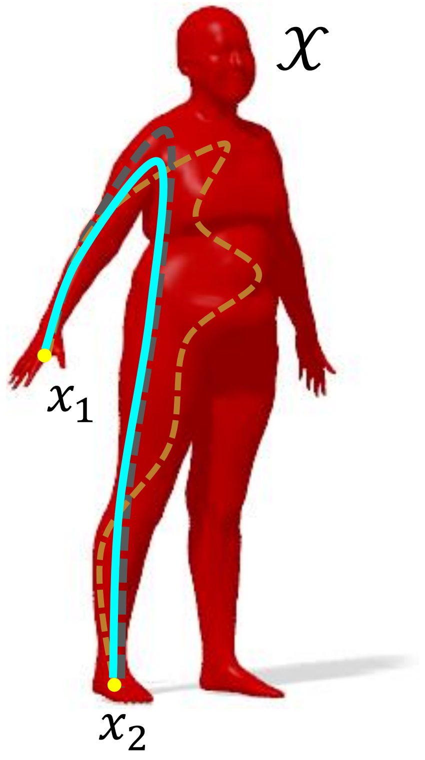
We have seen that the length l of the displacement dx is related with G:

Let $\gamma : [0,1] \longrightarrow \mathcal{X}$ its length is:

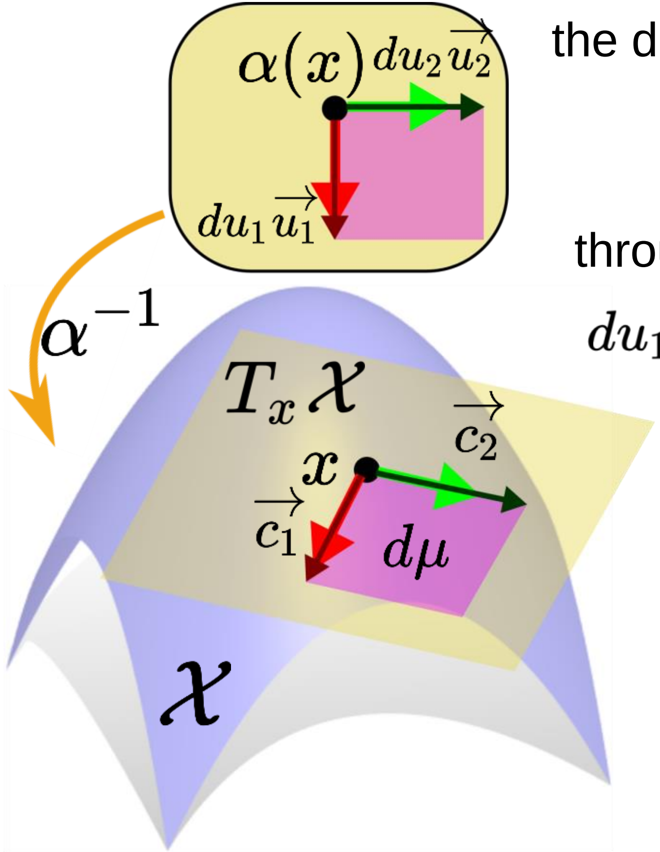
$$L(\gamma) = \int_a^b \sqrt{(\gamma'(t))^\top G \gamma'(t)} dt$$

G induces a metric on $\mathcal{X}$ namely the **intrinsic metric** $d_\mathcal{X}$ defined as:

$$d_\mathcal{X}(x_1, x_2) = \min_{\substack{\gamma : [0,1] \longrightarrow \mathcal{X} \\ \gamma(0) = x_1 \;\; \gamma(1) = x_2}} (L(\gamma))$$



area on the surface



the differential area element on $\mathbb{R}^2$ is the rectangle:

$$du_1\overrightarrow{u_1} \times du_2\overrightarrow{u_2} = du_1 du_2$$

through α^{-1} this rectangle is mapped on the parallelogram $du_1\overrightarrow{c_1} \times du_2\overrightarrow{c_2} \subset T_x\mathcal{X}$, the area of which is given by:

$$\begin{aligned} d\mu &= \|du_1\overrightarrow{c_1} \times du_2\overrightarrow{c_2}\| = \|\overrightarrow{c_1} \times \overrightarrow{c_2}\| du_1 du_2 \\ &= \sqrt{\|\overrightarrow{c_1}\|^2\|\overrightarrow{c_2}\|^2 - <\overrightarrow{c_1}, \overrightarrow{c_2}>^2} du_1 du_2 \\ &= \sqrt{g_{1,1}g_{2,2} - g_{1,2}^2}\, du_1 du_2 \\ &= \sqrt{det(G)} du_1 du_2 \end{aligned}$$

area and I fundamental form

We can consider a map $\rho : \Omega \subset \mathbb{R}^2 \longrightarrow U \subset \mathcal{X}$, and we compute the area

$$Area(U) = \int_U d\mu = \int_\Omega \sqrt{det(G)}\, du_1 du_2$$

We can be interested in compute a relative area for U, defined as:

$$A_{rel}(U) = \frac{Area(U)}{Area(\mathcal{X})}$$

$L, A_{rel}, Area$ are measures on $\mathcal{X}$ and with them we can analyze and study the properties of the surface.

gradient on the surface

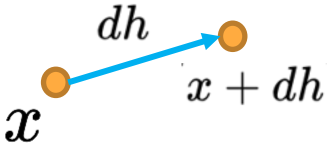
Given a vector v **the gradient** of a function f is the unique vector s.t. its product with a v gives the derivative of f in the direction of v.

The metric G induces a scalar product on $T_x\mathcal{X}$, $\forall dv, dw \in \mathbb{R}^2$ defined as:

$$dv^\top G dw$$

This scalar product induces a definition for the gradient of every $f : \mathcal{X} \longrightarrow \mathbb{R}$

$$f(x+dh) = f(x) + < \nabla_G f, dh > + o(\|dh\|)$$



$$= f(x) + (\nabla_G f)^\top G dh + o(\|dh\|)$$

$$= f(x+dh) = f(x) + (\nabla(f \cdot \alpha^{-1}))^\top dh + o(\|dh\|)$$

From this derivation we obtain: $\nabla_G(f) = G^{-1}(\nabla(f \cdot \alpha^{-1}))$

divergence on the surface

Given a vector field, its divergence at each point is the quantity of how much the vector is entering or exiting from the infinitesimal area around that point.

$$div\colon T\mathcal{X} \longrightarrow \mathcal{F}(\mathcal{X}, \mathbb{R})$$

Given a vector field $\vec{V} \in T_x\mathcal{X}$, we can define the **divergence** of $\vec{V}$ with respect to the metric G as:

$$div(\vec{V}) = \frac{1}{\sqrt{det(G)}} \sum_{i=1}^{2} \frac{\partial \sqrt{det(G)} V_i}{\partial \vec{c_i}}$$

where $\vec{V} = V_1 \vec{c_1} + V_2 \vec{c_2} \in T_x\mathcal{X} = span(\vec{c_1}, \vec{c_2})$

intrinsic geometry

➔ the first fundamental form fully represents the intrinsic geometry of $\mathcal{X}$

➔ this representation is completely independent from the embedding of $\mathcal{X}$

➔ this representation is independent on the embedding space of $\mathcal{X}$

➔ the intrinsic geometry provides an abstract representation of

To fix the first fundamental form on $\mathcal{X}$ we must have:

- $\forall x \in \mathcal{X}$ **the tangent space** $T_x\mathcal{X}$
- **the inner product** $g_{i,j} =< \vec{c_i}\,, \vec{c_j} >$

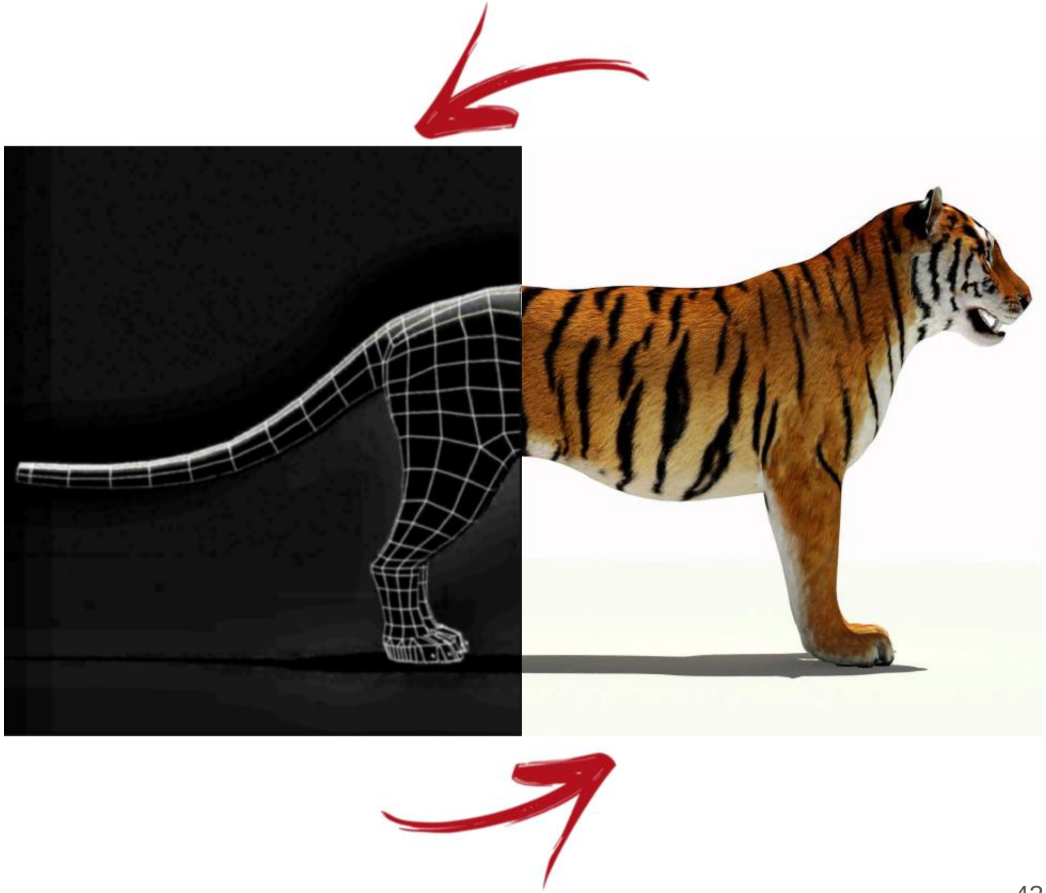
extrinsic and intrinsic geometry

- the **first fundamental form** completely represent the **intrinsic geometry**
- the **first fundamental form** is invariant to **isometries**
- the **second fundamental form** completely represent the **extrinsic geometry**
- the **second fundamental form** is invariant to **rigid transforms**

Theorem: Given two surfaces $\mathcal{X}$ and $\mathcal{Y}$ and a map $\pi : \mathcal{X} \longrightarrow \mathcal{Y}$ that preserves the **first and the second fundamental form** then the map π is a **congruence.**

Differential and discrete geometry

- Differential geometry is well studied from several centuries.
- Discrete geometry is relatively recent.
- In discrete geometry several tools and analysis are based on the differential geometry.
- Understand geometry is necessary to deal with computer graphics



questions?

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