Since the z transform of the exponential function is

$$\mathcal{Z}[e^{-at}] = \frac{1}{1 - e^{-aT}z^{-1}}$$

we have

$$\begin{aligned} X(z) = \mathcal{Z}[\sin \omega t] &= \mathcal{Z}\left[\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\right] \\ &= \frac{1}{2j}\left(\frac{1}{1 - e^{j\omega T}z^{-1}} - \frac{1}{1 - e^{-j\omega T}z^{-1}}\right) \\ &= \frac{1}{2j}\frac{(e^{j\omega T} - e^{-j\omega T})z^{-1}}{1 - (e^{j\omega T} + e^{-j\omega T})z^{-1} + z^{-2}} \\ &= \frac{z^{-1}\sin \omega T}{1 - 2z^{-1}\cos \omega T + z^{-2}} \\ &= \frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1} \end{aligned}$$

Example 2–1

Obtain the z transform of the cosine function

$$x(t) = \begin{cases} \cos \omega t, & 0 \le t \\ 0, & t < 0 \end{cases}$$

If we proceed in a manner similar to the way we treated the z transform of the sine function, we have

$$\begin{aligned} X(z) = \mathcal{Z}[\cos \omega t] &= \tfrac{1}{2}\mathcal{Z}[e^{j\omega t} + e^{-j\omega t}] \\ &= \frac{1}{2}\left(\frac{1}{1 - e^{j\omega T}z^{-1}} + \frac{1}{1 - e^{-j\omega T}z^{-1}}\right) \\ &= \frac{1}{2}\frac{2 - (e^{-j\omega T} + e^{j\omega T})z^{-1}}{1 - (e^{j\omega T} + e^{-j\omega T})z^{-1} + z^{-2}} \\ &= \frac{1 - z^{-1}\cos \omega T}{1 - 2z^{-1}\cos \omega T + z^{-2}} \\ &= \frac{z^2 - z\cos \omega T}{z^2 - 2z\cos \omega T + 1} \end{aligned}$$

Example 2–2

Obtain the z transform of

$$X(s) = \frac{1}{s(s+1)}$$

Whenever a function in s is given, one approach for finding the corresponding z transform is to convert $X(s)$ into $x(t)$ and then find the z transform of $x(t)$. Another approach is to expand $X(s)$ into partial fractions and use a z transform table to find the z transforms of the expanded terms. Still other approaches will be discussed in Section 3–3.

The inverse Laplace transform of $X(s)$ is

$$x(t) = 1 - e^{-t}, \qquad 0 \le t$$

Hence,

$$\begin{aligned} X(z) = \mathcal{Z}[1 - e^{-t}] &= \frac{1}{1 - z^{-1}} - \frac{1}{1 - e^{-T}z^{-1}} \\ &= \frac{(1 - e^{-T})z^{-1}}{(1 - z^{-1})(1 - e^{-T}z^{-1})} \\ &= \frac{(1 - e^{-T})z}{(z - 1)(z - e^{-T})} \end{aligned}$$

Comments. Just as in working with the Laplace transformation, a table of z transforms of commonly encountered functions is very useful for solving problems in the field of discrete-time systems. Table 2–1 is such a table.

TABLE 2–1 TABLE OF z TRANSFORMS

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	—	—	Kronecker delta $\delta_0(k)$ $1, \quad k = 0$ $0, \quad k \ne 0$	1
2.	—	—	$\delta_0(n - k)$ $1, \quad n = k$ $0, \quad n \ne k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1 - z^{-1}}$
4.	$\frac{1}{s + a}$	e^{-at}	e^{-akT}	$\frac{1}{1 - e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1 - z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}$
8.	$\frac{a}{s(s + a)}$	$1 - e^{-at}$	$1 - e^{-akT}$	$\frac{(1 - e^{-aT})z^{-1}}{(1 - z^{-1})(1 - e^{-aT}z^{-1})}$
9.	$\frac{b - a}{(s + a)(s + b)}$	$e^{-at} - e^{-bt}$	$e^{-akT} - e^{-bkT}$	$\frac{(e^{-aT} - e^{-bT})z^{-1}}{(1 - e^{-aT}z^{-1})(1 - e^{-bT}z^{-1})}$
10.	$\frac{1}{(s + a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s + a)^2}$	$(1 - at)e^{-at}$	$(1 - akT)e^{-akT}$	$\frac{1 - (1 + aT)e^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$

TABLE 2-1 (continued)

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
12.	$\frac{2}{(s+a)^3}$	$t^2 e^{-at}$	$(kT)^2 e^{-akT}$	$\frac{T^2 e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at - 1 + e^{-at}$	$akT - 1 + e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T + z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T + z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T + e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T + e^{-2aT}z^{-2}}$
18.			a^k	$\frac{1}{1-az^{-1}}$
19.			a^{k-1} $k = 1, 2, 3, \ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20.			ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.			$k^2 a^{k-1}$	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.			$k^3 a^{k-1}$	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.			$k^4 a^{k-1}$	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.			$a^k \cos k\pi$	$\frac{1}{1+az^{-1}}$
25.			$\frac{k(k-1)}{2!}$	$\frac{z^{-2}}{(1-z^{-1})^3}$
26.			$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}$	$\frac{z^{-m+1}}{(1-z^{-1})^m}$
27.			$\frac{k(k-1)}{2!}a^{k-2}$	$\frac{z^{-2}}{(1-az^{-1})^3}$
28.			$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}a^{k-m+1}$	$\frac{z^{-m+1}}{(1-az^{-1})^m}$

$x(t) = 0$, for $t < 0$.
$x(kT) = x(k) = 0$, for $k < 0$.
Unless otherwise noted, $k = 0, 1, 2, 3, \ldots$.

2-4 IMPORTANT PROPERTIES AND THEOREMS OF THE z TRANSFORM

The use of the z transform method in the analysis of discrete-time control systems may be facilitated if theorems of the z transform are referred to. In this section we present important properties and useful theorems of the z transform. We assume that the time function $x(t)$ is z-transformable and that $x(t)$ is zero for $t < 0$.

Multiplication by a Constant. If $X(z)$ is the z transform of $x(t)$, then

$$\mathcal{Z}[ax(t)] = a\,\mathcal{Z}[x(t)] = aX(z)$$

where a is a constant.

To prove this, note that by definition

$$\mathcal{Z}[ax(t)] = \sum_{k=0}^{\infty} ax(kT)z^{-k} = a\sum_{k=0}^{\infty} x(kT)z^{-k} = aX(z)$$

Linearity of the z Transform. The z transform possesses an important property: linearity. This means that, if $f(k)$ and $g(k)$ are z-transformable and α and β are scalars, then $x(k)$ formed by a linear combination

$$x(k) = \alpha f(k) + \beta g(k)$$

has the z transform

$$X(z) = \alpha F(z) + \beta G(z)$$

where $F(z)$ and $G(z)$ are the z transforms of $f(k)$ and $g(k)$, respectively.

The linearity property can be proved by referring to Equation (2-2) as follows:

$$\begin{aligned} X(z) &= \mathcal{Z}[x(k)] = \mathcal{Z}[\alpha f(k) + \beta g(k)] \\ &= \sum_{k=0}^{\infty}[\alpha f(k) + \beta g(k)]z^{-k} \\ &= \alpha \sum_{k=0}^{\infty} f(k)z^{-k} + \beta \sum_{k=0}^{\infty} g(k)z^{-k} \\ &= \alpha\mathcal{Z}[f(k)] + \beta\mathcal{Z}[g(k)] \\ &= \alpha F(z) + \beta G(z) \end{aligned}$$

Multiplication by a^k. If $X(z)$ is the z transform of $x(k)$, then the z transform of $a^k x(k)$ can be given by $X(a^{-1}z)$:

$$\mathcal{Z}[a^k x(k)] = X(a^{-1}z) \tag{2-6}$$

This can be proved as follows:

$$\begin{aligned} \mathcal{Z}[a^k x(k)] &= \sum_{k=0}^{\infty} a^k x(k)z^{-k} = \sum_{k=0}^{\infty} x(k)(a^{-1}z)^{-k} \\ &= X(a^{-1}z) \end{aligned}$$

Shifting Theorem. The shifting theorem presented here is also referred to as the real translation theorem. If $x(t) = 0$ for $t < 0$ and $x(t)$ has the z transform $X(z)$, then

TABLE 2–2 IMPORTANT PROPERTIES AND THEOREMS OF THE z TRANSFORM

	$x(t)$ or $x(k)$	$\mathcal{Z}[x(t)]$ or $\mathcal{Z}[x(k)]$
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t) + bx_2(t)$	$aX_1(z) + bX_2(z)$
3.	$x(t+T)$ or $x(k+1)$	$zX(z) - zx(0)$
4.	$x(t+2T)$	$z^2X(z) - z^2x(0) - zx(T)$
5.	$x(k+2)$	$z^2X(z) - z^2x(0) - zx(1)$
6.	$x(t+kT)$	$z^kX(z) - z^kx(0) - z^{k-1}x(T) - \cdots - zx(kT-T)$
7.	$x(t-kT)$	$z^{-k}X(z)$
8.	$x(n+k)$	$z^kX(z) - z^kx(0) - z^{k-1}x(1) - \cdots - zx(k-1)$
9.	$x(n-k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz\frac{d}{dz}X(z)$
11.	$kx(k)$	$-z\frac{d}{dz}X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$
13.	$e^{-ak}x(k)$	$X(ze^{a})$
14.	$a^kx(k)$	$X\left(\frac{z}{a}\right)$
15.	$ka^kx(k)$	$-z\frac{d}{dz}X\left(\frac{z}{a}\right)$
16.	$x(0)$	$\lim_{z\to\infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z\to 1}[(1-z^{-1})X(z)]$ if $(1-z^{-1})X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k) = x(k) - x(k-1)$	$(1-z^{-1})X(z)$
19.	$\Delta x(k) = x(k+1) - x(k)$	$(z-1)X(z) - zx(0)$
20.	$\sum_{k=0}^{n} x(k)$	$\frac{1}{1-z^{-1}}X(z)$
21.	$\frac{\partial}{\partial a}x(t,a)$	$\frac{\partial}{\partial a}X(z,a)$
22.	$k^mx(k)$	$\left(-z\frac{d}{dz}\right)^m X(z)$
23.	$\sum_{k=0}^{n} x(kT)y(nT-kT)$	$X(z)Y(z)$
24.	$\sum_{k=0}^{\infty} x(k)$	$X(1)$

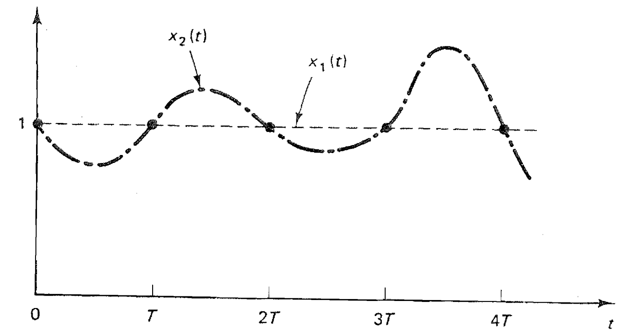

Figure 2–3 Two different continuous-time functions, $x_1(t)$ and $x_2(t)$, that have the same values at $t = 0, T, 2T, \ldots$.

1. Direct division method
2. Computational method
3. Partial-fraction-expansion method
4. Inversion integral method

In obtaining the inverse z transform, we assume, as usual, that the time sequence $x(kT)$ or $x(k)$ is zero for $k < 0$.

Before we present the four methods, however, a few comments on poles and zeros of the pulse transfer function are in order.

Poles and Zeros in the z Plane. In engineering applications of the z transform method, $X(z)$ may have the form

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n} \qquad (m \le n) \tag{2–17}$$

or

$$X(z) = \frac{b_0(z-z_1)(z-z_2)\cdots(z-z_m)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

where the p_i's ($i = 1, 2, \ldots, n$) are the poles of $X(z)$ and the z_j's ($j = 1, 2, \ldots, m$) the zeros of $X(z)$.

The locations of the poles and zeros of $X(z)$ determine the characteristics of $x(k)$, the sequence of values or numbers. As in the case of the s plane analysis of linear continuous-time control systems, we often use a graphical display in the z plane of the locations of the poles and zeros of $X(z)$.

Note that in control engineering and signal processing $X(z)$ is frequently expressed as a ratio of polynomials in z^{-1}, as follows:

$$X(z) = \frac{b_0 z^{-(n-m)} + b_1 z^{-(n-m+1)} + \cdots + b_m z^{-n}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \tag{2–18}$$