Math 485 HW 1

1.1)

1) Response variables are:

a) Attitude toward gun control

b) Heart disease

c) Vote for President

d) Quality of life.

1.2)

2) a) nominal c) ordinal e) nominal

b) ordinal d) nominal f) ordinal

1.3)

3) a) let X: the # of correct student's answers on the test

$\Rightarrow X \sim \text{Binomial}(n = 100, \pi = .25)$

b) the mean, $\mu = E[X] = n\pi = 100 \times .25 = 25$

and the standard deviation, $s.d.(x) = \sqrt{Var(X)} = \sqrt{n\pi(1-\pi)}$

$= \sqrt{100(.25)(.75)} = 4.33$

$$P(X \geq 50) = P\left(Z > \frac{50-25}{4.33}\right) = P(Z > 5.8) \simeq 0$$

Yes, at least 50 correct responses would be a big surprise, because its probability is almost zero!

1.4)

4) a) Y = the # of heads out of two trials

$Y \sim \text{Binomial}(n=2, \pi=.5)$

$p(Y=0) = \binom{2}{0}(.5)^0(1-.5)^2 = .25$

$p(Y=1) = \binom{2}{1}(.5)(1-.5) = .5$

$p(Y=2) = \binom{2}{2}(.5)^2(1-.5)^0 = .25$

Note
$p(X=0)+p(X=1)+p(X=2) = 1$

$\mu = E[Y] = n\pi = 2(.5) = 1$

$s.d. = \sqrt{Var(Y)} = \sqrt{n\pi(1-\pi)} = \sqrt{2(.5)(.5} = .707$

b) $\pi = .6$

$p(Y=0) = \binom{2}{0}(.6)^0(1-.6)^2 = .16$

$p(Y=1) = \binom{2}{1}.6(.4) = .48$

$p(Y=2) = \binom{2}{2}(.6)^2(.4)^0 = .36$

Note:
$.16 + .48 + .36 = 1$

$\pi = .4$

$p(Y=0) = \binom{2}{0}.4^0(1-.4)^2 = .36$

$p(Y=1) = \binom{2}{1}.4(.6) = .48$

$p(Y=2) = \binom{2}{2}.4^2(.6)^0 = .16$

c) if $Y=1$

$l(\pi) = P_\pi(Y=1) = \binom{2}{1}\pi(1-\pi) = 2\pi(1-\pi)$

d) $\frac{\partial}{\partial\pi} l(\pi) = 2(1-2\pi) = 0 \Rightarrow \boxed{\hat{\pi} = .5}$

$\frac{\partial^2}{\partial\pi^2} l(\pi) = -4 < 0$, so $\hat{\pi}$ is the maximizer.

2

5) 1.8)

a) $\hat{\pi} = P = \frac{344}{1170} = 0.294$

where π is the proportion of the people who willing to accept cuts to protect the envirovment.

b) $H_0: \pi = .5$

$H_A: \pi < .5$

$$Z_0 = \frac{\hat{\pi} - \pi_0}{\sqrt{\frac{\pi_0(1-\pi_0)}{n}}} = \frac{.294 - .5}{\sqrt{\frac{.5(.5)}{1170}}} = -14.9$$

$\text{P-value} = P(Z < Z_0) = P(Z < -14.9) \simeq 0$

$\alpha >$ P-value $\Rightarrow$ Reject H_0 in favor of H_A.

That is, minority of population would say "yes".

c) $(1-\alpha)\%$ C.I. for π is given by: $\left(P \pm Z_{\alpha/2}\sqrt{\frac{P(1-P)}{n}}\right)$

$\alpha = .01$

$\alpha/2 = .005$ $\Rightarrow$ $\left(.294 \pm 2.57\sqrt{\frac{.294(1-.294)}{1170}}\right)$

$Z_{.005} = 2.57$ or $(0.2597, 0.3282)$

(1.9)

6)

a) $H_0: \pi = .5$

$H_A: \pi > .5$

where π = the proportion of population having greater relief with new analgesic.

$$\hat{\pi} = p = \frac{60}{100} = .6$$

$$Z_0 = \frac{\hat{\pi} - \pi_0}{\sqrt{\frac{\pi_0(1-\pi_0)}{n}}} = \frac{.6 - .5}{\sqrt{\frac{.5(.5)}{100}}} = 2$$

p-value $= P(Z > Z_0) = P(Z > 2) = .0227$

$\alpha = .05 >$ p-value $= .0227 \Rightarrow$ reject H_0 in favor of H_A.

That is, the proportion of population having a greater relief with new analgesic is higher.

b) $(1-\alpha)\%$ C.I for π: $\quad p \pm Z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}}$

$\alpha = .05$

$\alpha/2 = .025$

$Z_{.025} = 1.96$

$$.6 \pm 1.96\sqrt{\frac{.6(.4)}{100}}$$

$\Rightarrow (.50398,\ 0.696) \Rightarrow$ This is Wald C.I.

Score C.I. can be obtained by solving $\frac{|.6 - \pi|}{\sqrt{\frac{\pi(1-\pi)}{100}}} < 1.96$ with respect to π, which gives:

$(0.502,\ 0.691)$. ← Please cross check it.

(4)

7) 1.10)

Notice that the C.I. for π is given by $((1-\alpha)\%\ C.I)$

$$\hat{\pi} \pm Z_{\alpha/2}\, S.E.(\hat{\pi})$$

which is: $\hat{\pi} \pm Z_{\alpha/2}\sqrt{\frac{\pi(1-\pi)}{n}}$

or $\hat{\pi} \pm m$

where $m = Z_{\alpha/2}\sqrt{\frac{\pi(1-\pi)}{n}}$ is the margin of error.

or, $m^2 = Z_{\alpha/2}^2\,\frac{\pi(1-\pi)}{n}$

$$\Rightarrow n = \left(\frac{Z_{\alpha/2}}{m}\right)^2 \pi(1-\pi)$$

the m is given, $m = .08$, $Z_{\alpha/2} = 1.96$ and, $\pi = .75$, so

$$n = \left(\frac{1.96}{.08}\right)^2 .75(1-.75) = 112.54 \Rightarrow \boxed{n = 113}$$

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8) 1.12

a) since $p = \frac{0}{25} = 0$, we have $S.E = \sqrt{\frac{p(1-p)}{n}} = 0$

or equivalently $Z = \frac{p-\pi_0}{\sqrt{\frac{p(1-p)}{n}}} = \frac{p-\pi_0}{0} = \infty$.

b) C.I. for π: $(0,0)$

This is not acceptable, because in the population we expect some vegetarians, even the propotion is small.

(1.12) continue

8) continue

c) $$Z_0 = \frac{p - \pi_0}{\sqrt{\frac{\pi_0(1-\pi_0)}{n}}} = \frac{0 - .5}{\sqrt{\frac{.5(.5)}{25}}} = -5$$

$$\text{P-value} = 2P(Z > |z_0|) = 2P(Z > 5) \simeq 0$$

d) $H_0: \pi = .133$

$H_A: \pi \neq .133$

$$Z_0 = \frac{0 - 0.133}{\sqrt{\frac{.133(1-.133)}{25}}} = -1.96$$, so, .133 is the value

of π_0 which has the P-value = .05

$$\text{P-value} = 2P(Z > |z_0|)$$
$$= 2P(Z > 1.96) = 2(.025) = .05$$

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9) $Y_1, \dots, Y_n \sim MN(10, (\pi_1, \pi_2, \pi_3))$

$$l(\pi_1, \pi_2, \pi_3 | \underset{\sim}{Y}) = \prod_{i=1}^{n} \binom{10}{n_{1i}, n_{2i}, n_{3i}} \pi_1^{n_{1i}} \pi_2^{n_{2i}} \pi_3^{n_{3i}}$$

$$= \left(\prod_{i=1}^{n} \frac{10!}{n_{1i}!\, n_{2i}!\, n_{3i}!} \right) \pi_1^{\sum n_{1i}} \pi_2^{\sum n_{2i}} \pi_3^{\sum n_{3i}}$$

where $\pi_1 + \pi_2 + \pi_3 = 1$, $n_{i1} + n_{i2} + n_{i3} = 10$

and $\sum n_{i1} + \sum n_{i2} + \sum n_{i3} = 10n$

9) continue let $\sum n_{i1} = N_1$, $\sum n_{i2} = N_2$, $\sum n_{i3} = N_3$

& $10n = N$.

Then, we have $N_1 + N_2 + N_3 = N$.

The natural log of the likelihood function can be written as

$$\log L(\pi_1, \pi_2, \pi_3 | \underset{\sim}{Y}) = \log\left(\prod_{i=1}^{n} \frac{10!}{n_{1i}!\, n_{2i}!\, n_{3i}!}\right)$$
$$+ N_1 \log \pi_1 + N_2 \log \pi_2 + N_3 \log \pi_3$$
$$= \sum_{i=1}^{n} \log \frac{10!}{n_{1i}!\, n_{2i}!\, n_{3i}!} + N_1 \log \pi_1 + N_2 \log \pi_2$$
$$+ (N - N_1 - N_2) \log(1 - \pi_1 - \pi_2)$$

$$\frac{\partial \log L}{\partial \pi_1} = \frac{N_1}{\pi_1} - \frac{N - N_1 - N_2}{1 - \pi_1 - \pi_2} = 0 \Rightarrow \pi_1 = \frac{N_1 - \pi_2 N_1}{N - N_2} = \frac{N_1(1 - \pi_2)}{N - N_2}$$

$$\frac{\partial \log L}{\partial \pi_2} = \frac{N_2}{\pi_2} - \frac{N - N_1 - N_2}{1 - \pi_1 - \pi_2} = 0 \Rightarrow \pi_2 = \frac{N_2 - \pi_1 N_2}{N - N_1}$$

$$\Rightarrow 1 - \pi_2 = 1 - \frac{N_2(1 - \pi_1)}{N - N_1}$$

$$\Rightarrow \pi_1 = \frac{N_1}{N - N_2}\left(1 - \frac{N_2 - N_2 \pi_1}{N - N_1}\right)$$

$$\Rightarrow \pi_1 \left(1 - \frac{N_1 N_2}{(N - N_2)(N - N_1)}\right) = \frac{N_1(N - N_1 - N_2)}{(N - N_2)(N - N_1)}$$

$\vdots$

$\boxed{\hat{\pi}_1 = \frac{N_1}{N}}$ or $\boxed{\hat{\pi}_1 = \frac{\sum n_{1i}}{10n}}$ & $\boxed{\hat{\pi}_2 = \frac{N_2}{N} = \frac{\sum n_{2i}}{10n}}$

10)

a) $L(\pi|\underline{y}) = \prod_{i=1}^{n} \pi(1-\pi)^{y_i}$

$= \pi^{n}(1-\pi)^{\sum y_i}$

$\log L(\pi|y_1,\dots,y_n) = n\log\pi + \sum y_i \log(1-\pi)$

$\frac{\partial}{\partial\pi}\log L(\pi|\underline{y}) = \frac{n}{\pi} - \frac{\sum y_i}{1-\pi} = 0$

$n - n\pi - \pi\sum y_i = 0 \Rightarrow \hat{\pi} = \frac{n}{n+\sum y_i}$

$= \frac{1}{1+\bar{y}}$

$\sum y_i = 77$

so, $\hat{\pi} = \frac{20}{20+77} = \boxed{0.2061}$

b) $L(\lambda|\underline{y}) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{y_i}}{y_i!} = \frac{e^{-n\lambda}\lambda^{\sum y_i}}{\prod_{i=1}^{n} y_i!}$

$\log L(\lambda|y_1,\dots,y_n) = -n\lambda + \sum y_i \log\lambda - \sum_{i=1}^{n}\log y_i!$

$\frac{\partial}{\partial\lambda}\log L(\lambda|\underline{y}) = -n + \frac{\sum y_i}{\lambda} = 0 \Rightarrow \hat{\lambda} = \frac{\sum y_i}{n} = \bar{y}$

$\frac{\partial^2}{\partial\lambda^2} = -\frac{\sum y_i}{\lambda^2} < 0$

$\hat{\lambda} = \frac{45}{10} = \boxed{4.5}$