Equation 2: $\partial_x^2 f(x,y) + \partial_y^2 f(x,y) - \gamma f(x,y) = 0$
Second method: Feynman-Kac

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1 Equation

In this part we consider a second equation:

$$\partial_x^2 f(x,y) + \partial_y^2 f(x,y) - \gamma f(x,y) = 0$$

with the following Dirichlet border conditions:

$$\forall (x,y) \in \partial\mathbb{D}, f(x,y) = \phi(x,y)$$

where:

- $\mathbb{D} = [0;1]^2$ is the square of side length 1.
- $\partial\mathbb{D} = \{0;1\} \times [0;1] \cup [0;1] \times \{0;1\}$ is the border of $\mathbb{D}$.
- ϕ is a function $\partial\mathbb{D} \to \mathbb{R}$.

Let $L \in \mathbb{N}^*$: our goal is to build $u : \{0,...,L-1\} \times \{0,...,L-1\} \to \mathbb{R}$ such that for every $0 \leq i < L$ and $0 \leq j < L$, $u(i,j)$ approaches $f(\frac{i}{L}, \frac{j}{L})$. We call:

- $\mathcal{D}$ the lattice $\{0,...,L\}^2$
- $\partial\mathcal{D} = \{0,L\} \times \{0,...,L\} \sqcup \{0,...,L\} \times \{0,L\}$ its border

2 Random Walk

Define $\beta = \ln(\gamma/L^2 + 1)$. For every $(i,j) \in \mathcal{D}$, let $(X_n)_{n\in\mathbb{N}}$ be the two-dimensional random walk initially in (i,j), such that for every $n \in \mathbb{N}$:

- if $X_n = (k,l)$ is **on the border** $\partial\mathcal{D}$, the walk **stays put** with probability 1: $\mathbb{P}[X_{n+1} = (k,l)] = 1$.
- else, $X_n = (k,l) \in \mathcal{D} \setminus \partial\mathcal{D}$, we **chose a neighbour** of (k,l) with **uniform** probability:
  - $\mathbb{P}[X_{n+1} = (k+1,l)] = 1/4$
  - $\mathbb{P}[X_{n+1} = (k-1,l)] = 1/4$
  - $\mathbb{P}[X_{n+1} = (k,l+1)] = 1/4$
  - $\mathbb{P}[X_{n+1} = (k,l-1)] = 1/4$

3 Approximation

3.1 Defining $u(i,j)$

Let $T = \inf\{n \in \mathbb{N}, X_n \in \partial\mathcal{D}\}$ be the first time the border of $\mathcal{D}$ is reached. Consider the function $u : \mathcal{D} \to \mathbb{R}$ defined on every $(i,j) \in \mathcal{D}$ by:

$$u(i,j) = \mathbb{E}_{(i,j)}\big[\phi(X_T) exp\big(-\beta T\big)\big]$$

3.2 What equation does $u(i,j)$ follow?

Since the walk is symmetric, conditionning with respect to the first step yields for every $(i,j) \notin \partial\mathbb{D}$:

$$\begin{aligned} u(i,j) = & \frac{1}{4}\mathbb{E}\big[\phi(X_T)exp\big(-\beta T\big)|X_0 = (i,j), X_1 = (i+1,j)\ \big] \\ & + \frac{1}{4}\mathbb{E}\big[\phi(X_T)exp\big(-\beta T\big)|X_0 = (i,j), X_1 = (i-1,j)\ \big] \\ & + \frac{1}{4}\mathbb{E}\big[\phi(X_T)exp\big(-\beta T\big)|X_0 = (i,j), X_1 = (i,j+1)\ \big] \\ & + \frac{1}{4}\mathbb{E}\big[\phi(X_T)exp\big(-\beta T\big)|X_0 = (i,j), X_1 = (i,j-1)\ \big] \end{aligned}$$

Then Markov's property implies that the sequence $(X_{n+1})_{n\ \in\mathbb{N}}$ is also a Markov chain with same transition as $(X_n)_{n\ \in\mathbb{N}}$, and with initial position X_1. Therefore:

$$u(i,j) = \frac{e^{-\beta}}{4}[u(i+1,j) + u(i,j+1) + u(i-1,j) + u(i,j-1)]$$

$$\frac{e^{-\beta}}{4}[u(i+1,j) + u(i,j+1) + u(i-1,j) + u(i,j-1) - 4u(i,j)] - (1 - e^{-\beta})u(i,j) = 0$$

Let $\tilde{\Delta}$ be the discretised Laplacian, sending every function $f : \mathbb{D} \to \mathbb{R}$ to the function

$$\tilde{\Delta} f : (i,j) \mapsto \frac{1}{4} \sum_{(k,l)\sim(i,j)} [f(k,l) - f(i,j)]$$

The previous equality can therefore be rewritten as follows:

$$\tilde{\Delta} u(i,j) - (e^{\beta} - 1)u(i,j) = 0 \quad (*)$$

3.3 Back to continuous functions

Intuitively, v_n checks a discretised version of the following equation:

$$\Delta f(x,y) - \gamma f(x,y)$$

Let's make this statement more precise: define $f : \mathbb{D} \to \mathbb{R}$ by $f(x,y) = u(\lfloor xL \rfloor, \lfloor yL \rfloor)$. Then for every $(x,y) \in \mathbb{D}$, multiplying both sides of $(*)$ by L^2 yields:

$$\frac{1}{4(1/L)^2}[f(x+1/L,y) + f(x-1/L,y) + f(x,y+1/L) + f(x,y-1/L) - 4f(x,y)f(x,y)] - \gamma f(x,y) = 0$$

Therefore when $L \to \infty$:

$$\boxed{\forall (x,y) \in \mathbb{D}, \Delta f(x,y) - \gamma f(x,y) = 0}$$

3.4 Border conditions

Let $(x, y) \in \partial\mathbb{D}$: then $(\lfloor xL \rfloor, \lfloor yL \rfloor) \in \partial\mathcal{D}$. Let $i = \lfloor xL \rfloor$ and $j = \lfloor yL \rfloor$, and consider once again the random walk $(X_n)_{n\in\mathbb{N}}$ initially in (i, j) defined above. Since $X_0 = (i, j) \in \partial\mathcal{D}$, the border is reached immediately, and since the X_n stays constant, the sum $\sum_{j=0}^{n} \mathbf{1}_{\{X_j \notin \partial\mathbb{D}\}}$ equals 0 for every $n \in \mathbb{N}$, including for $n = T$. Therefore

$$u(i, j) = \tilde{\phi}(i, j) = \phi(x, y)$$

.
Since this is true for every $L \in \mathbb{N}^*$, by taking the limit as $L \to \infty$ we also have:

$$\boxed{\forall (x, y) \in \partial\mathbb{D}, f(x, y) = \phi(x, y)}$$

4 Watching the process unfold through time

4.1 Defining $v_n(i, j)$

Define for every $n \in \mathbb{N}$ the function $v_n : \mathcal{D} \to \mathbb{R}$ sending every $(i, j) \in \mathcal{D}$ to:

$$v_n(i, j) = \mathbb{E}_{(i,j)}\big[\phi(X_n) exp\big(-\beta \sum_{j=0}^{n} \mathbf{1}_{\{X_j \notin \partial\mathbb{D}\}}\big)\big]$$

4.2 What equation does $v_n(i, j)$ follow?

Since the walk is symmetric a calculation very similar to the one above leads to:

$$\begin{aligned}
v_{n+1}(i, j) = &\frac{1}{4}\mathbb{E}\big[\phi(X_{n+1}) exp\big(-\beta \sum_{i=0}^{n+1} \mathbf{1}_{X_i \notin \partial\mathbb{D}}\big) | X_0 = (i, j), X_1 = (i + 1, j)\ \big] \\
&+ \frac{1}{4}\mathbb{E}\big[\phi(X_{n+1}) exp\big(-\beta \sum_{i=0}^{n+1} \mathbf{1}_{X_i \notin \partial\mathbb{D}}\big) | X_0 = (i, j), X_1 = (i - 1, j)\ \big] \\
&+ \frac{1}{4}\mathbb{E}\big[\phi(X_{n+1}) exp\big(-\beta \sum_{i=0}^{n+1} \mathbf{1}_{X_i \notin \partial\mathbb{D}}\big) | X_0 = (i, j), X_1 = (i, j + 1)\ \big] \\
&+ \frac{1}{4}\mathbb{E}\big[\phi(X_{n+1}) exp\big(-\beta \sum_{i=0}^{n+1} \mathbf{1}_{X_i \notin \partial\mathbb{D}}\big) | X_0 = (i, j), X_1 = (i, j - 1)\ \big] \\
&= \frac{e^{-\beta}}{4}[v_n(i + 1, j) + v_n(i, j + 1) + v_n(i - 1, j) + v_n(i, j - 1)]
\end{aligned}$$

Therefore:

$$v_{n+1}(i, j) - v_n(i, j) = \frac{e^{-\beta}}{4}[v_n(i+1, j) + v_n(i, j+1) + v_n(i-1, j) + v_n(i, j-1) - 4v_n(i, j)] - (e^{\beta} - 1)v_n(i, j) = 0(**)$$

4.3 Back to continuous functions

Once again, intuitively, v_n checks a discretised version of the following equation:

$$\partial_t f(x,y,t) = \Delta f(x,y,t) - \gamma f(x,y,t)$$

Let's make this statement more precise: define $f : \mathbb{R}_+^* \times \mathbb{D} \to \mathbb{R}$ by $f(t,x,y) = v_{\lfloor nL^2 \rfloor}(\lfloor xL \rfloor, \lfloor yL \rfloor)$.
Then for every $(t,(x,y)) \in \mathbb{R}_+^* \times \mathbb{D}$, multiplying both sides of $(**)$ by L^2 yields:

$$\frac{f(t+1/L^2,x,y) - f(t,x,y)}{(1/L^2)} = \frac{f(t,x+1/L,y) + f(t,x-1/L,y) + f(t,x,y+1/L) + f(t,x,y-1/L) - 4f(t,x,y)}{4(1/L)^2} - \gamma f(t,x,y)$$

Therefore when $L \to \infty$:

$$\boxed{\forall (t,(x,y)) \in \mathbb{R}_+^* \times \partial\mathbb{D}, \partial_t f(t,x,y) = \Delta f(t,x,y) - \gamma f(t,x,y)}$$

The border conditions are also verified:

$$\boxed{\forall (t,(x,y)) \in \mathbb{R}_+^* \times \partial\mathbb{D}, f(x,y) = \phi(x,y)}$$