Equation 2: $\partial_x^2 f(x,y) + \partial_y^2 f(x,y) - \gamma f(x,y) = 0$
First method: Cemetery

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1 Equation

In this part we consider a second equation:

$$\partial_x^2 f(x,y) + \partial_y^2 f(x,y) - \gamma f(x,y) = 0$$

with the following Dirichlet border conditions:

$$\forall (x,y) \in \partial\mathbb{D}, f(x,y) = \phi(x,y)$$

where:

- $\mathbb{D} = [0;1]^2$ is the square of side length 1.
- $\partial\mathbb{D} = \{0;1\} \times [0;1] \cup [0;1] \times \{0;1\}$ is the border of $\mathbb{D}$.
- ϕ is a function $\partial\mathbb{D} \to \mathbb{R}$.

Let $L \in \mathbb{N}^*$: our goal is to build $u : \{0,...,L-1\} \times \{0,...,L-1\} \to \mathbb{R}$ such that for every $0 \leq i < L$ and $0 \leq j < L$, $u(i,j)$ approaches $f(\frac{i}{L}, \frac{j}{L})$. We call:

- $\mathcal{D}$ the lattice $\{0,...,L\}^2$
- $\partial\mathcal{D} = \{0,L\} \times \{0,...,L\} \sqcup \{0,...,L\} \times \{0,L\}$ its border

2 Random Walk

Let $(i,j) \in \mathcal{D}$, and $\alpha = \gamma/L^2$: L must be chosen so that $0 \leq \alpha \leq 1$. We define a random walk $(X_n)_{n\in\mathbb{N}}$ on $\mathcal{D} \cup \{\dagger\}$, where $\dagger$ is a separate state called the *cemetery*, and such that $X_0 = (i,j)$. For every $n \in \mathbb{N}$:

- If $X_n = (k,l)$ is **on the border $\partial\mathcal{D}$ of the lattice**, *i.e.* $k \in \{0,L\}$ or $l \in \{0,L\}$, then it stays put with probability 1:
$$\mathbb{P}[X_{n+1} = (k,l) | X_n = (k,l)] = 1$$
- If $X_n = (k,l)$ is **inside the lattice**, *i.e.* $0 < k < L$ and $0 < l < L$, then it can:
  - **Die** with probability α:
  $$\mathbb{P}[X_{n+1} = \dagger | X_n = (k,l)] = \alpha$$
  - **Move** with equal probability $\frac{1}{4}(1-\alpha)$ to one of the 4 neighbours of (k,l): for every neighbour (k',l') of (k,l),
  $$\mathbb{P}[X_{n+1} = (k',l') | X_n = (k,l)] = \frac{1}{4}(1-\alpha)$$

3 Approximation

3.1 Defining $u(i,j)$

We now introduce the stopping times:

- $T = \inf\{n \in \mathbb{N} | X_n = \dagger\}$ is the time of **death**.
- $\tau = \inf\{n \in \mathbb{N} | X_n \in \partial\mathcal{D}\}$ is the time the **border is reached**.

$u(i,j)$ is then defined as the expectation $\mathbb{E}_{(i,j)}[\tilde{\phi}(X_\tau)\mathbf{1}_{\tau<T}] = \mathbb{E}[\tilde{\phi}(X_\tau)\mathbf{1}_{\tau<T}|\ X_0 = (i,j)]$, where $\tilde{\phi}$ sends $(i,j) \in \partial\mathcal{D}$ to $\phi(i/L, j/L)$.

3.2 What equation does $u(i,j)$ follow?

Let's take a closer look at the $u(i,j)$ defined as above:

$$\begin{aligned}
u(i,j) &= \mathbb{E}_{(i,j)}[\tilde{\phi}(X_\tau)\mathbf{1}_{\tau<T}] \\
&= \mathbb{E}_{(i,j)}[\tilde{\phi}(X_\tau)\mathbf{1}_{\tau<T}\mathbf{1}_{X_1=(i+1,j)}] \\
&\quad + \mathbb{E}_{(i,j)}[\tilde{\phi}(X_\tau)\mathbf{1}_{\tau<T}\mathbf{1}_{X_1=(i-1,j)}] \\
&\quad + \mathbb{E}_{(i,j)}[\tilde{\phi}(X_\tau)\mathbf{1}_{\tau<T}\mathbf{1}_{X_1=(i,j-1)}] \\
&\quad + \mathbb{E}_{(i,j)}[\tilde{\phi}(X_\tau)\mathbf{1}_{\tau<T}\mathbf{1}_{X_1=(i,j+1)}] \\
&\quad + \mathbb{E}_{(i,j)}[\tilde{\phi}(X_\tau)\mathbf{1}_{\tau<T}\mathbf{1}_{X_1=\dagger}]
\end{aligned}$$

Now let's focus on $\mathbb{E}_{(i,j)}[\tilde{\phi}(X_\tau)\mathbf{1}_{\tau<T}\mathbf{1}_{X_1=(i+1,j)}]$, the first term:

$$\begin{aligned}
\mathbb{E}_{(i,j)}[\tilde{\phi}(X_\tau)\mathbf{1}_{\tau<T}\mathbf{1}_{X_1=(i+1,j)}] &= \mathbb{P}_{(i,j)}[X_1 = (i+1,j)] \cdot \mathbb{E}_{(i,j)}[\tilde{\phi}(X_\tau)\mathbf{1}_{\tau<T}|X_1 = (i+1,j)] \\
&= \mathbb{P}[X_1 = (i+1,j)|X_0 = (i,j)] \cdot \mathbb{E}[\tilde{\phi}(X_\tau)\mathbf{1}_{\tau<T}|X_1 = (i+1,j), X_0 = (i,j)] \\
&= \frac{1-\alpha}{4} \cdot \mathbb{E}[\tilde{\phi}(X_\tau)\mathbf{1}_{\tau<T}|X_1 = (i+1,j)] \\
&= \frac{1-\alpha}{4} \cdot \mathbb{E}_{(i+1,j)}[\tilde{\phi}(X_\tau)\mathbf{1}_{\tau<T}] \\
&= \frac{1-\alpha}{4} u(i+1,j)
\end{aligned}$$

Proceeding similarly with the four other terms, we end up with:

$$u(i,j) = \frac{1-\alpha}{4}[u(i+1,j) + u(i-1,j) + u(i,j+1) + u(i,j-1)] \quad (*)$$

This can be rewritten as $(1-\alpha) \cdot \tilde{\Delta} u(i,j) = \alpha \cdot u(i,j)$, where $\tilde{\Delta}$ is the discretised Laplacian, sending every function $u : \mathcal{D} \to \mathbb{R}$ to the function

$$\tilde{\Delta} u : (i,j) \mapsto \frac{1}{4} \sum_{(k,l)\ \ (i,j)} [u(k,l) - u(i,j)]$$

3.3 Back to continuous functions

Let's now show how all of this relates to the original partial differential equation: define $f : \mathbb{D} \to \mathbb{R}$ by $f(x, y) = u(\lfloor xL \rfloor, \lfloor yL \rfloor)$. Then for every $(x, y) \in \mathbb{D}$:

$$f(x + 1/L, y) + f(x - 1/L, y) - 2f(x, y) = u(\lfloor xL \rfloor + 1, \lfloor yL \rfloor) + u(\lfloor xL \rfloor - 1, \lfloor yL \rfloor) - 2u(\lfloor xL \rfloor, \lfloor yL \rfloor)$$

$$f(x, y + 1/L) + f(x, y - 1/L) - 2f(x, y) = u(\lfloor xL \rfloor, \lfloor yL \rfloor + 1) + u(\lfloor xL \rfloor, \lfloor yL \rfloor - 1) - 2u(\lfloor xL \rfloor, \lfloor yL \rfloor)$$

Therefore:

$$\frac{1}{4(1/L)^2}[f(x + 1/L, y) + f(x - 1/L, y) + f(x, y + 1/L) + f(x, y - 1/L) - 4f(x, y)f(x, y)] = 0$$

Remember, $\alpha = \gamma/L^2$, so that:

$$\frac{1}{4(1/L)^2}[f(x + 1/L, y) + f(x - 1/L, y) + f(x, y + 1/L) + f(x, y - 1/L) - 4f(x, y)f(x, y)] = 0$$

When $L \to \infty$, the left hand side converges to $\Delta f(x, y)$, hence:

$$\boxed{\forall (x, y) \in \mathbb{D}, \Delta f(x, y) = 0}$$

3.4 Border conditions

Let $(x, y) \in \partial\mathbb{D}$: then $(\lfloor xL \rfloor, \lfloor yL \rfloor) \in \partial\mathcal{D}$. Let $i = \lfloor xL \rfloor$ and $j = \lfloor yL \rfloor$, and consider once again the random walk $(X_n)_{n\in\mathbb{N}}$ initially in (i, j) defined above. Since $X_0 = (i, j) \in \partial\mathcal{D}$, the border is reached immediately, therefore the expectation $\mathbb{E}_{(i,j)}[\tilde{\phi}(X_\tau)]$ is simply the value $\tilde{\phi}(X_0) = \tilde{\phi}(i, j) = \phi(x, y)$.

Since this is true for every $L \in \mathbb{N}^*$, by taking the limit as $L \to \infty$ we also have:

$$\boxed{\forall (x, y) \in \partial\mathbb{D}, f(x, y) = \phi(x, y)}$$

4 Watching the process unfold through time

4.1 Defining $v_n(i, j)$

Define for every $n \in \mathbb{N}$ the function $v_n : \mathcal{D} \to \mathbb{R}$ sending every $(i, j) \in \mathcal{D}$ to:

$$v_n(i, j) = \mathbb{E}_{(i,j)}[\tilde{\phi}(X_n)]$$

4.2 What equation does $v_n(i,j)$ follow?

Since the walk is symmetric, a calculation very similar to the one above leads to:

$$\begin{aligned} v_{n+1}(i,j) &= \mathbb{E}_{(i,j)}[\tilde{\phi}(X_{n+1})] \\ &= \mathbb{E}_{(i,j)}[\tilde{\phi}(X_{n+1})\mathbf{1}_{X_1=(i+1,j)}] \\ &\quad + \mathbb{E}_{(i,j)}[\tilde{\phi}(X_{n+1})\mathbf{1}_{X_1=(i-1,j)}] \\ &\quad + \mathbb{E}_{(i,j)}[\tilde{\phi}(X_{n+1})\mathbf{1}_{X_1=(i,j-1)}] \\ &\quad + \mathbb{E}_{(i,j)}[\tilde{\phi}(X_{n+1})\mathbf{1}_{X_1=(i,j+1)}] \end{aligned}$$

Therefore:

$$v_{n+1}(i,j) - v_n(i,j) = \frac{1}{4}[v_n(i+1,j) + v_n(i,j+1) + v_n(i-1,j) + v_n(i,j-1) - 4v_n(i,j)] \quad (**)$$

4.3 Back to continuous functions

Define $f : \mathbb{R}_+^* \times \mathbb{D} \to \mathbb{R}$ by $f(t,x,y) = u_{\lfloor nL^2 \rfloor}(\lfloor xL \rfloor, \lfloor yL \rfloor)$. Then for every $(t,x,y) \in \mathbb{R}_+^* \times \mathbb{D}$, multiplying both sides of $(**)$ by L^2 yields:

$$\frac{f(t+1/L^2,x,y) - f(t,x,y)}{(1/L^2)} = \frac{f(t,x+1/L,y) + f(t,x-1/L,y) + f(t,x,y+1/L) + f(t,x,y-1/L) - 4f(t,x,y)}{4(1/L)^2}$$

Therefore when $L \to \infty$:

$$\boxed{\forall (t,(x,y)) \in \mathbb{R}_+^* \times \mathbb{D}, \partial_t f(t,x,y) = \Delta f(t,x,y)}$$

The border conditions are also verified:

$$\boxed{\forall (t,(x,y)) \in \mathbb{R}_+^* \times \partial\mathbb{D}, f(t,x,y) = \phi(x,y)}$$