Equation 3: $\partial_x^2 f(x,y) + \partial_y^2 f(x,y) - 2f(x,y) + f(x,y)^2 = 0$

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1 Equation

Let's now look at a nonlinear equation:

$$\partial_x^2 f(x,y) + \partial_y^2 f(x,y) - 2f(x,y) + f(x,y)^2 = 0$$

with the usual Dirichlet border conditions:

$$\forall (x,y) \in \partial\mathbb{D}, f(x,y) = \phi(x,y)$$

where:

- $\mathbb{D} = [0;1]^2$ is the square of side length 1.
- $\partial\mathbb{D} = \{0;1\} \times [0;1] \cup [0;1] \times \{0;1\}$ is the border of $\mathbb{D}$.
- ϕ is a function $\partial\mathbb{D} \to \mathbb{R}$.

Let $L \in \mathbb{N}^*$: our goal is to build $u : \{0,...,L-1\} \times \{0,...,L-1\} \to \mathbb{R}$ such that for every $0 \leq i < L$ and $0 \leq j < L$, $u(i,j)$ approaches $f(\frac{i}{L}, \frac{j}{L})$. We call:

- $\mathcal{D}$ the lattice $\{0,...,L\}^2$
- $\partial\mathcal{D} = \{0,L\} \times \{0,...,L\} \sqcup \{0,...,L\} \times \{0,L\}$ its border

2 Random Walk

In order to handle the quadratic extra term, we're going to add a layer of complexity by allowing our process to branch. Not only does our random walk move and die: it can also duplicate!
This begs a more rigourous explanation: let $(i,j) \in \mathcal{D}$ be a position in the lattice. We introduce a sequence $(X_n)_{n\in\mathbb{N}}$ of random vectors of variable dimension, the components of which stand for the branches of our process that haven't died yet (the heads of the hydra that haven't been cut). The (X_n) are defined such that:

- The sequence starts at (i,j): $X_0 = [(i,j)]$
- For every $n \in \mathbb{N}$, every branch that hasn't reached the border can:
  - **Die** with probability $\alpha = \frac{1}{L^2}$, in which case there is no corresponding component in the subsequent vector X_{n+1}.

– **Live** with probability $\beta = \alpha = \frac{1}{L^2}$: move up down, left or right with probability 1/4. The corresponding component in X_{n+1} contains the new position.

– **Duplicate**: don't move, but the next vector will have (at least) an extra component containing the branche's present coordinates.

- Every branch that has reached the border stays put.

3 Approximation

3.1 Defining $u(i, j)$

Let N_n be the number of components in the vector X_n. We now introduce the stopping times:

- $T = \inf\{n \in \mathbb{N} | X_n = [\ ]\}$ is the time at which **all the branches have died**.
- $\tau = \inf\{n \in \mathbb{N} |\ \forall 1 \leq i \leq N_n,\ X_n^{(i)} \in \partial\mathcal{D}\}$ is the time at which **all the branches have reached the border**.

We define $u(i,j) = \mathbb{E}\big[\prod_{i=1}^{N_n} \tilde{\phi}(X_\tau^{(i)})\mathbf{1}_{\tau<T}|\ X_0 = [(i,j)]\big]$, where $\tilde{\phi}$ sends $(i,j) \in \partial\mathcal{D}$ to $\phi(i/L, j/L)$.

3.2 What equation does $u(i, j)$ follow?

Let's show that u defined as above does indeed provide us with a reasonable approximation for our differential equation. For this, we condition on the first step in the random walk, *i.e.* whether the walk dies, duplicates or marches on on its first step:

$$\begin{aligned}
u(i,j) &= \mathbb{E}\big[\prod_{i=1}^{N_\tau} \tilde{\phi}(X_\tau^{(i)})\mathbf{1}_{\tau<T}|\ X_0 = [(i,j)]\big] \\
&= \mathbb{E}\big[\prod_{i=1}^{N_\tau} \tilde{\phi}(X_\tau^{(i)})\mathbf{1}_{\tau<T}\mathbf{1}_{N_1=0}|\ X_0 = [(i,j)]\big] \quad \text{(die)} \\
&+ \mathbb{E}\big[\prod_{i=1}^{N_\tau} \tilde{\phi}(X_\tau^{(i)})\mathbf{1}_{\tau<T}\mathbf{1}_{N_1=1}|\ X_0 = [(i,j)]\big] \quad \text{(keep walking)} \\
&+ \mathbb{E}\big[\prod_{i=1}^{N_\tau} \tilde{\phi}(X_\tau^{(i)})\mathbf{1}_{\tau<T}\mathbf{1}_{N_1=2}|\ X_0 = [(i,j)]\big] \quad \text{(duplicate)}
\end{aligned}$$

We will now consider these three terms separately, assuming here that initially $(i,j) \in \partial\mathcal{D}$ (we shall examine this case later):

- If $N_1 = 0$ then $T = 1$ and since (i,j) is not on the border $\partial\mathcal{D}$, $\tau > 0$: it cannot be the case that $\tau < T$, therefore the first term in the sum is always 0.

- If $N_1 = 1$, then conditioning further with respect to the neighbour reached yields:

$$\begin{aligned}\mathbb{E}[\prod_{i=1}^{N_\tau}\tilde{\phi}(X_\tau^{(i)})\mathbf{1}_{\tau<T}\mathbf{1}_{N_1=1}|\ X_0=[(i,j)]] &= \mathbb{E}[\prod_{i=1}^{N_\tau}\tilde{\phi}(X_\tau^{(i)})\mathbf{1}_{\tau<T}\mathbf{1}_{N_1=1}\mathbf{1}_{X_1=[(i+1,j)]}|\ X_0=[(i,j)]]\\ &+\mathbb{E}[\prod_{i=1}^{N_\tau}\tilde{\phi}(X_\tau^{(i)})\mathbf{1}_{\tau<T}\mathbf{1}_{N_1=1}\mathbf{1}_{X_1=[(i-1,j)]}|\ X_0=[(i,j)]]\\ &+\mathbb{E}[\prod_{i=1}^{N_\tau}\tilde{\phi}(X_\tau^{(i)})\mathbf{1}_{\tau<T}\mathbf{1}_{N_1=1}\mathbf{1}_{X_1=[(i,j+1)]}|\ X_0=[(i,j)]]\\ &+\mathbb{E}[\prod_{i=1}^{N_\tau}\tilde{\phi}(X_\tau^{(i)})\mathbf{1}_{\tau<T}\mathbf{1}_{N_1=1}\mathbf{1}_{X_1=[(i,j-1)]}|\ X_0=[(i,j)]]\end{aligned}$$

  Therefore:

$$\begin{aligned}\mathbb{E}[\prod_{i=1}^{N_\tau}\tilde{\phi}(X_\tau^{(i)})\mathbf{1}_{\tau<T}\mathbf{1}_{N_1=1}|\ X_0=[(i,j)]] &= \frac{(1-\alpha-\beta)}{4}[u(i+1,j)+u(i-1,j)+u(i,j+1)+u(i,j-1)]\\ &= (1-\alpha-\beta)\tilde{\Delta}u(i,j)+(1-\alpha-\beta)u(i,j)\end{aligned}$$

- Finally, if $N_2 = 1$, then the process has split into two hereby independent branches. All subsequent vectors X_n can be split into two parts, which each part corresponding to the components stemming from either branch: let m_τ be the number of components stemming from the first branch at time τ. Then:

$$\mathbb{E}[\prod_{i=1}^{N_\tau}\tilde{\phi}(X_\tau^{(i)})\mathbf{1}_{\tau<T}\mathbf{1}_{N_1=2}|\ X_0=[(i,j)]] = \mathbb{P}[N_1=2]\cdot\mathbb{E}_{[(i,j)]}[\prod_{i=1}^{m_\tau}\tilde{\phi}(X_\tau^{(i)})\prod_{i=m_\tau+1}^{N_\tau}\tilde{\phi}(X_\tau^{(i)})\mathbf{1}_{\tau<T}|X_1=[(i,j),(i,j)]]$$

  We now consider the random sequences $(Y_n)_{n\in\mathbb{N}}$ and $(Z_n)_{n\in\mathbb{N}}$ of truncated vectors:

$$\forall n\in\mathbb{N},\ Y_n=[X_{n+1}^{(i)}]_{1\le i\le m_n}\quad\text{and}\quad Z_n=[X_{n+1}^{(i+m_n)}]_{1\le i\le N_n-m_n}$$

  Then (Y_n) and (Z_n) are independent Markov Chains initially in (i,j), with same transition as (X_n). Letting τ^Y and τ^Z be the associated stopping times, we have $\{\tau<T\}=\{\tau^Y<T\}\cap\{\tau^Z<T\}$ hence $\mathbf{1}_{\{\tau<T\}}=\mathbf{1}_{\{\tau^Y<T\}}\cdot\mathbf{1}_{\{\tau^Z<T\}}$. Moreover $N_{\tau^Y}=m_\tau$ and $N_{\tau^Z}=N_\tau-m_\tau$. By independence:

$$\begin{aligned}\mathbb{E}[\prod_{i=1}^{N_\tau}\tilde{\phi}(X_\tau^{(i)})\mathbf{1}_{\tau<T}\mathbf{1}_{N_1=2}|\ X_0=[(i,j)]] &= \beta\cdot\mathbb{E}[\prod_{i=1}^{N_{\tau^Y}}\tilde{\phi}(Y_\tau^{(i)})\mathbf{1}_{\tau^Y<T}|Y_0=[(i,j)]]\\ &\quad\times\mathbb{E}[\prod_{i=1}^{N_{\tau^Z}}\tilde{\phi}(Z_\tau^{(i)})\mathbf{1}_{\tau^Z<T}|Z_0=[(i,j)]]\\ &= \beta u(i,j)^2\end{aligned}$$

We end up with the following equation for $u(i,j)$ (when $(i,j)\notin\partial\mathcal{D}$):

$$u(i,j)=(1-\alpha-\beta)\tilde{\Delta}u(i,j)+(1-\alpha-\beta)u(i,j)+\beta u(i,j)^2$$

Or equivalently:

$$(1-\alpha-\beta)\tilde{\Delta}u(i,j)-(\alpha+\beta)u(i,j)+\beta u(i,j)^2=0\quad(*)$$

3.3 Back to continuous functions

Once again, define $f : \mathbb{D} \to \mathbb{R}$ by $f(x,y) = u(\lfloor xL \rfloor, \lfloor yL \rfloor)$, and recall that $\alpha = \beta = \frac{1}{L^2}$. Then for every $(x,y) \in \mathbb{D}$, multiplying both sides of $(*)$ by L^2 yields:

$$(1 - \frac{2}{L^2}) \cdot \frac{\tilde{\Delta} u(\lfloor xL \rfloor, \lfloor yL \rfloor)}{(1/L)^2} - 2u(\lfloor xL \rfloor, \lfloor yL \rfloor) + u(\lfloor xL \rfloor, \lfloor yL \rfloor)^2 = 0$$

When $L \to \infty$:

$$\frac{\tilde{\Delta} u(\lfloor xL \rfloor, \lfloor yL \rfloor)}{(1/L)^2} \to \Delta f(x,y) \quad \text{and} \quad u(\lfloor xL \rfloor, \lfloor yL \rfloor) \to f(x,y)$$

Therefore:

$$\boxed{\forall (x,y) \in \mathbb{D}, \Delta f(x,y) - 2f(x,y) + f(x,y)^2 = 0}$$

3.4 Border conditions

Let $(x,y) \in \partial\mathbb{D}$: then $(\lfloor xL \rfloor, \lfloor yL \rfloor) \in \partial\mathcal{D}$. Let $i = \lfloor xL \rfloor$ and $j = \lfloor yL \rfloor$, and consider once again the random walk $(X_n)_{n\in\mathbb{N}}$ initially in (i,j) defined above. Since $(x,y) \in \partial\mathbb{D}$, $X_0 = [(i,j)]$ is such that $(i,j) \in \partial\mathcal{D}$. This implies that $\mathbb{P}[X_n = [(i,j)]]$ almost surely for every $n \in \mathbb{N}$. In particular $\tau = 0$ and $T = +\infty$, so that $u(i,j) = \tilde{\phi}(i,j)$.
Since this is true for every $L \in \mathbb{N}^*$, by taking the limit as $L \to \infty$ we also have:

$$\boxed{\forall (x,y) \in \partial\mathbb{D}, f(x,y) = \phi(x,y)}$$