Queensland University of Technology

EGB242

Signal Analysis

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Contents

Chapter 1

Signal Sampling and Quantisation

1.1 Continuous and Discrete Time Representations

- An analogue signal is continuous in time and amplitude
- A discrete-time signal is discrete in time but continuous in amplitude/value
- A digital signal is discrete in time and amplitude

1.2 The Impulse Function

The Dirac Delta function $\delta(t)$, is intuitively

- The limit of a pulse of fixed area as its width tends to zero
- Area is equal to one
- Also called the impulse function

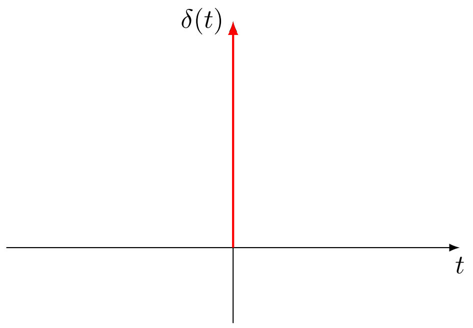


Figure 1.1: Impulse Function

$$
\begin{aligned}
\delta(t) &= 0 && t \neq 0 \\
\delta(t) &\to \infty && t = 0 \\
\int_{-\infty}^{\infty} \delta(t)dt &= 1
\end{aligned}
$$

1.3 Sifting Property of the Impulse

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0) \qquad (f(t) \text{ is continuous})$$

This follows as the impulse function is zero everywhere except at $t = t_0$. So if $t_0 = T_s$ where T_s is the sampling interval, then:

$$\int_{\infty}^{-\infty} f(t)\delta(t - nT_s)dt = f(nT_s)$$

Produces the signals at the following intervals/samples $f(0), f(T_s), f(2T_s), f(3T_s), \ldots f(nT_s)$

1.4 Analogue to Digital Conversion

Consists of 2 steps

- Sampling

- Quantisation

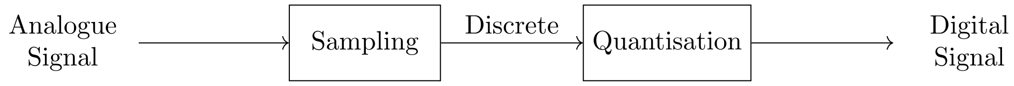


Figure 1.2: Analogue to Digital Conversion

1.5 Sampling Theory

A continuous signal with a maximum frequency of f_m can be recovered back grom its samples if the sampling frequency f_s is greater than twice the maximum frequency of the continuous signal.
That is $f_s > 2f_m$

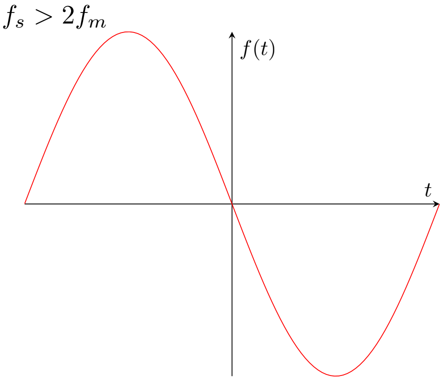


Figure 1.3: Continuous Signal

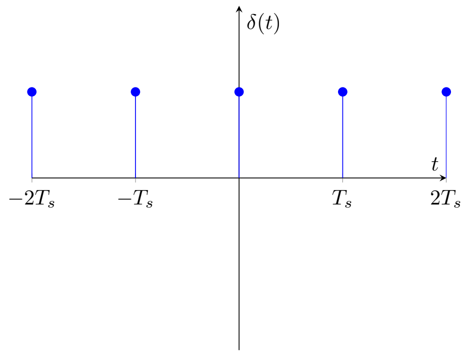


Figure 1.4: Sampling

1.6 From Pulse Train to Impulse Train

When evaluating the Fourier series of a pulse train with height A, width τ (area $A\tau$), and period T_s

$$c_n = \frac{A\tau}{T_s}\frac{\sin(\pi n f_s \tau)}{\pi n f_s \tau}$$

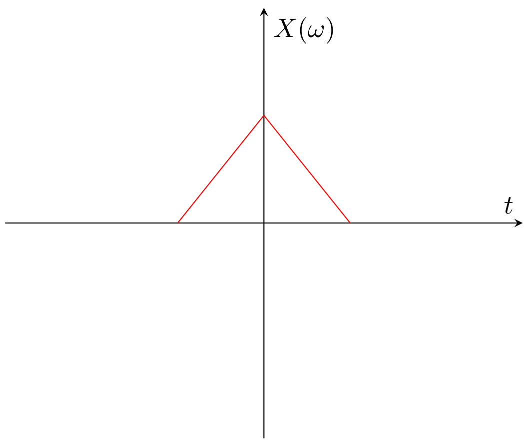
Taking the limit of $\tau \to 0$:

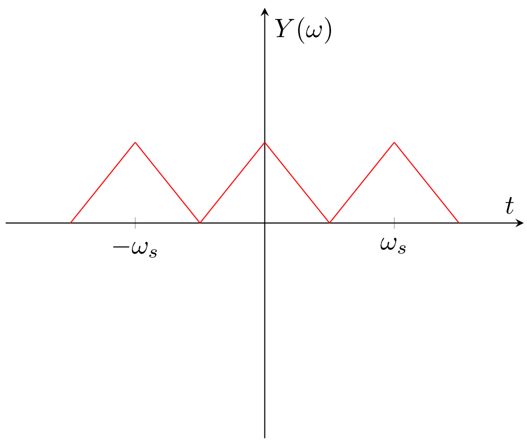
$$\lim_{\tau\to 0}\frac{\sin(\pi n f_s \tau)}{\pi n f_s \tau} = \frac{1}{T_s}\lim_{\tau\to 0}\frac{\sin(\pi n f_s \tau)}{\pi n f_s \tau} = \frac{1}{T_s}$$

$$\begin{aligned}\delta(t)_{train} &= \sum_{n=-\infty}^{\infty}\frac{1}{T_s}e^{j2\pi n f_s t}\\ &= \frac{1}{T_s} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{2}{T_s}\cos\left(2\pi n f_s t\right)\end{aligned}$$

1.7 Sampling Process

Sampling of a discrete signal can be obtained by multiplying the signal by an impulse train $\delta(t)$ of period T_s. The output of the multiplier is a discrete signal called the sample signal
The spectra of the input and sampled signals are represented as





1.8 Anti-Aliasing

Hardware constraints may not allow sampling at the Nyquist frequency.
By using an anti-aliasing (low-pass) filter before the sampling stage, we can eliminate spectral overlap.
This lowpass filter must have a bandwidth at or below $\frac{f_s}{2}$
Example: If an analog signal has a bandwidth of 2 kHz and must be sampled at 3 kHz, then the anti-aliasing filter must have a bandwidth of 1 kHz or less.

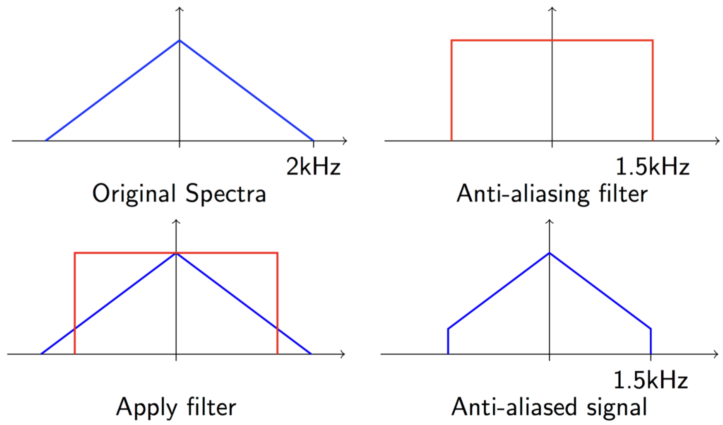


Figure 1.6: Anti-Aliasing Spectra

1.9 Anti-Aliasing Spectrum

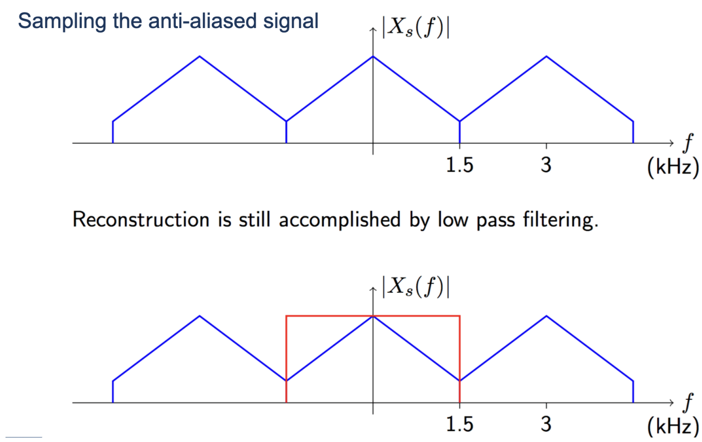


Figure 1.7: Anti-Aliasing Spectrum

1.10 Discrete to Digital

Digital signals are discrete in time AND amplitude. The quantizer transforms each sampled value to take one of L distinct levels.

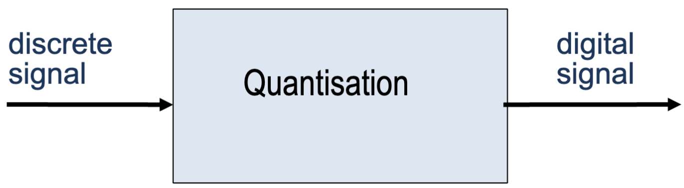


Figure 1.8: Quantiser

These leves are allocated over the entire dynamic range of the analog signal.

$$x_{\min} \leq x(t) \leq x_{\max}$$

This is lossy and non-invertible.

1.10.1 Uniform Quantiser

Levels are **uniformly** allocated across the dynamic range
The step size Δ between quantization levels is given by

$$\Delta = \frac{x_{\max} - x_{\min}}{L}$$

Many applications have $x_{\max} = -x_{\min}$. Which gives

$$\Delta = \frac{2x_{\max}}{L}$$

The number of levels is often chosen to be a power of two to better use binary words

$$L = 2^n$$

Where n is the number of bits used to encode L levels.

Example

A signal with a dynamic range between $\pm 4V$ is to be uniformly quantiszed with each level encoded by 4 bits. What is the step size between bits.

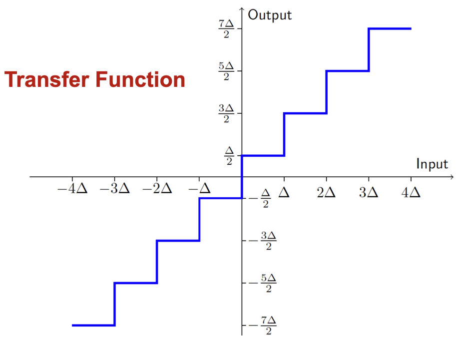
- $n = 4$ bits
- $L = 2^n = 2^4 = 16$ levels

$$\Delta = \frac{2x_{\max}}{L} = \frac{2(4)}{16} = 0.5V$$

- 4 bits allow for encoding 16 unique code words
- Each code word represents a single quantization level

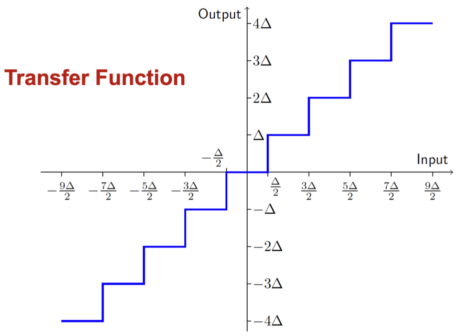
Uniform Quantizer Types

– **Mid-tread** quantizer



– Enforces a quantisation step at zero
– L is odd if $x_{\min} = -x_{\max}$
– May waste one level

– **Mid-rise** quantizer



– L levels are evenly distributed about 0
– Can't guarantee 0
– Uses all levels when $x_{\min} = -x_{\max}$

1.10.2 Quantisation Error

Quantisation error is the difference between the quantised signal and the sampled signal

$$e_q(n) = x(n) - \hat{x}(n)$$

This is a function of n, where n is the sample number.

1.10.3 Signal to Quantisation Noise Ratio

SQNR represents the amount of distortion due to quantisation defined as the ratio of signal power to the quantisation noise power

$$\text{SQNR} = \frac{\text{Signal Power}}{\text{Quantisation Noise Power}}$$

The difference between the sampled signal and quantised signal is the quantisation error.

1.10.4 Quantisation Noise Power

Quantisation noise power is the variance of the quantisation error
Can be treated as a uniformly distruibuted random variable between $\frac{\Delta}{2}$ and $\frac{\Delta}{2}$

$$\text{P}_{\text{noise}} = \int_{-\infty}^{\infty} e^2 f(e) de = \frac{\Delta}{12} \tag{1.1}$$

1.10.5 Noise Power as a Function of n

Expressing the noise power as a function of n gives

$$\begin{aligned} \text{P}_{\text{noise}} &= \frac{\Delta^2}{12} = \frac{1}{12} \times \left(\frac{x_{\text{max}} - x_{\text{min}}}{L}\right)^2 \\ &= \frac{x_{\text{max}}^2 - x_{\text{min}}^2}{12n^2} \end{aligned} \tag{1.2}$$

Increasing the number of quantisation levels improves SQNR. The number of representation levels is determined by the number of bits used to encode the sample $L = 2^n$. PCM for example encodes each sample with equal length binary words.

1.10.6 Limitations

While qunatisation noise power is equal across the input signal range, the input signal's power changes across decision levels. Resulting in differing SQNR across the input signal range.
For example with human speech, louder sounds carry higher signal power than softer sounds.
Uniform quantisation would result in different SQNR for different sounds.

1.10.7 Non-Uniform Quantisers

A constant SQNR can be achieved using a non uniform quantiser. These have different step sizes over their dynamic range, allocating more levels to the range of signals with smaller power.

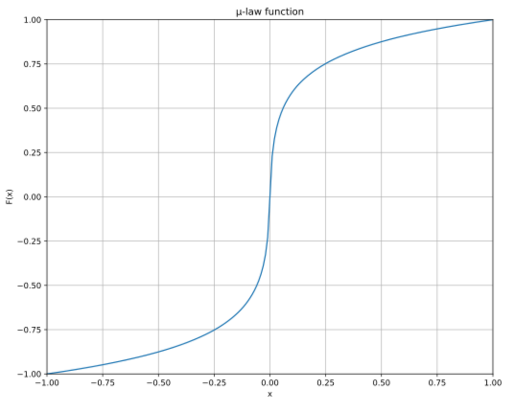


Figure 1.9: μ Law Non-Uniform Quantiser Transfer Function

1.11 Encoding

1.11.1 Waveform Encoding

Encodes each quantized sample as a string of bits

- Pulse Code Modulation (PCM)
- Differential Pulse Code Modulation (DPCM)
- Shannon-Fano Encoding
- Huffman Encoding

Pulse Code Modulation

Each sample is an equal length code binary word with length n bits, allowing for 2^n encoding levels and is the standard for analogue to digital (A/D) conversion.

Chapter 2

Linear Time Invariant Systems

2.1 Transfer Functions

- Time domain output $\longrightarrow$ convolution of time domain input with impulse response of the system

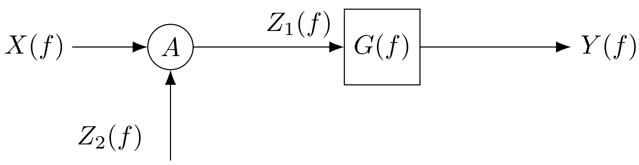
  $y(t) = x(t) * h(t)$
- Spectrum of the output $\longrightarrow$ product of the Fourier transform of the input with the Fourier transform of the impulse response

  $Y(f) = X(f)H(f)$
- Fourier Transform of the impulse response $\longrightarrow$ Transfer Function

  $H(f) = \frac{Y(f)}{X(f)}$

  Provides the system's gain and phase change information

2.2 Transfer Functions from Block Diagrams



From the block diagram we can see that:

$$Z_1(f) = X(f) - Z_2(f) \tag{1}$$
$$Z_2(f) = kY(f) \tag{2}$$

Substituting (2) into (1) gives:

$$Z_1(f) = X(f) - kY(f)$$

Note that:

$$Y(f) = G(f)X(f) \tag{3}$$

Substituting this into the above equation gives:

$$\begin{aligned} Y(f) &= G(f)\left(X(f) - kY(f)\right) \\ &= G(f)X(f) - kG(f)Y(f) \end{aligned} \tag{4}$$

Therefore:

$$\begin{aligned} Y(f) + kG(f)Y(f) &= G(f)X(f) \\ Y(f)\left(1 + kG(f)\right) &= G(f)X(f) \\ \frac{Y(f)}{X(f)} &= \frac{G(f)}{1 + kG(f)} \end{aligned} \tag{5}$$

2.3 Filters

Attenuate certain ranges of frequencies in a signal In the following forms

- Lowpass

  Only allows low frequencies to pass
- Highpass

  Only allows high frequencies to pass
- Bandpass

  Only allows a band of frequencies to pass
- Bandstop

  Only blocks a band of frequencies

Note that:

- Pass band is the range of frequencies that are allowed to pass
- Stop band is the range of frequencies that are blocked

2.3.1 Ideal Filters

They are unrealisable (cannot exist in the real world) and have the following characteristics

- Completely attenuate signals in their stop bands
- Completely pass signals in their pass bands
- Immediately transition between pass and stop bands
- Have no effect on phase

2.3.2 Practical Filters

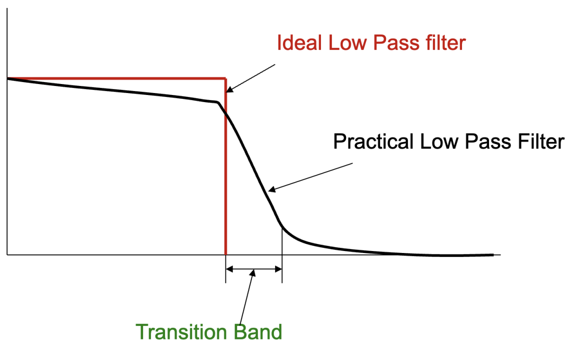


Figure 2.1: Ideal Filter

Chapter 3

Fourier Transform

3.1 Euler's Identity

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

3.2 Exponential Fourier Series

$$s(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$
$$c_n = \frac{1}{T}\int_0^T s(t)e^{-j2\pi n f_0 t}dt$$

Multiplying by $e^{-j2\pi m f_0 t}$ and integrating over one period removes all frequency excepts those at nf_0.

3.3 Transforms

3.3.1 Fourier Transform

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}dt$$

3.3.2 Inverse Fourier Transform

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi f t}df$$

3.4 Linearity

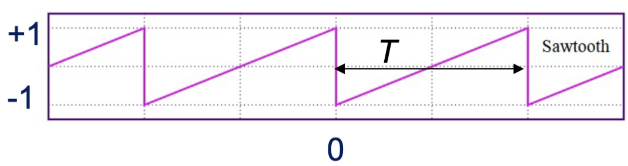


Figure 3.1: Sawtooth Wave

$$
\begin{aligned}
c_n &= \frac{1}{T}\int_0^T s(t)e^{-j2\pi n f_0 t}dt, f_0\frac{1}{T} \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}} te^{-j2\pi nt}dt - \int_{\frac{T}{2}}^T te^{-j2\pi nt}dt\right) \\
&= \frac{1}{T}\left(\int_0^{\frac{T}{2}} te^{-j2\pi n\frac{1}{T}t}dt - \int_{\frac{T}{2}}^T te^{-j2\pi n\frac{1}{T}t}dt\right)
\end{aligned}
$$

3.5 Alternate Expression of the Fourier Series

$$
\begin{aligned}
c_0 &= \frac{1}{T}\int_0^T s(t)dt \\
c_n &= \frac{1}{T}\int_0^T s(t)e^{-j2\pi n f_0 t}dt \\
s(t) &= c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n e^{j2\pi n f_0 t}
\end{aligned}
$$

In this form, c_0 is the DC component of the signal and c_n is the amplitude of the nth harmonic.

3.6 Gate Function

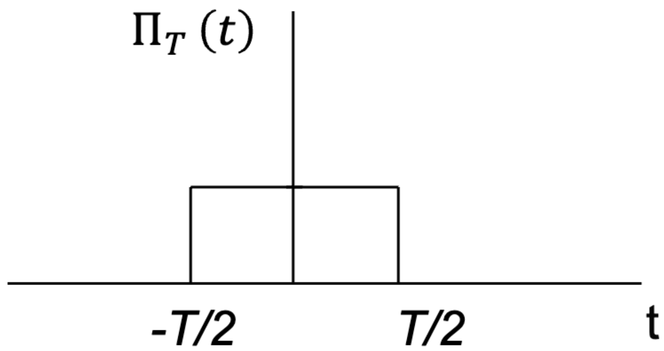


Figure 3.2: Gate Function

$$
\Pi_T(t)
$$
$$
rect\left(\frac{t}{T}\right)
$$

3.6.1 Fourier Transform

$$
\begin{aligned}
\mathcal{F}\{\Pi_T(t)\} &= \int_{-\infty}^{\infty} \Pi_T(t)\, e^{-j2\pi ft} dt \\
&= \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi ft} dt \\
&= \frac{1}{-j2\pi f}\left[e^{-j2\pi ft}\right]_{-\frac{T}{2}}^{\frac{T}{2}} \\
&= \frac{1}{-j2\pi f}\left[e^{-j2\pi f\frac{T}{2}} - e^{j2\pi f\frac{T}{2}}\right] \\
&= \frac{1}{-j2\pi f}\left[e^{-j\pi fT} - e^{j\pi fT}\right] \\
&= \frac{1}{\pi f}\left[\frac{e^{j\pi fT} - e^{-j\pi fT}}{2j}\right] \\
&= \frac{T}{T\pi f}\left[\frac{e^{j\pi fT} - e^{-j\pi fT}}{2j}\right] \\
&= T\,\mathrm{sinc}(Tf)
\end{aligned}
$$

3.6.2 Time Shifted Gate Function

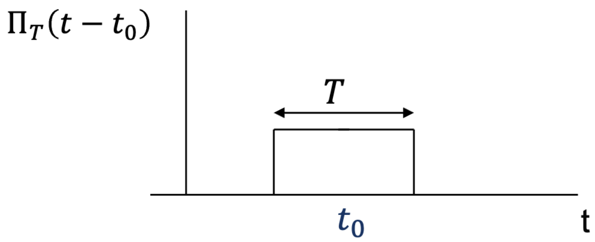


Figure 3.3: Time-Shifted Gate Function

$$
\begin{aligned}
\mathcal{F}\{\Pi_T(t)\} &= \int_{-\infty}^{\infty} \Pi_T(t)\, e^{-j2\pi ft} dt \\
&= \int_{t_0-\frac{T}{2}}^{\frac{t_0+T}{2}} e^{-j2\pi ft} dt \\
&= \frac{1}{-j2\pi f}\left[e^{-j2\pi ft}\right]_{t_0-\frac{T}{2}}^{\frac{t_0+T}{2}} \\
&= \frac{1}{-j2\pi f}\left[e^{-j2\pi f\frac{t_0+T}{2}} - e^{j2\pi f\frac{t_0+T}{2}}\right] \\
&= \frac{e^{-2j\pi ft_0}}{\pi f}\left[\frac{e^{j\pi fT} - e^{-j\pi fT}}{2j}\right] \\
&= \frac{e^{-2j\pi ft_0}}{\pi f}\left[\frac{T\left(e^{j\pi fT} - e^{-j\pi fT}\right)}{2Tj}\right] \\
&= e^{-j2\pi ft_0}\,T\,\mathrm{sinc}\,(fT)
\end{aligned}
$$

Chapter 4

Laplace Transform

4.1 Relationship with the Fourier Transform

$$
\begin{aligned}
G(s) &= \int_{-\infty}^{\infty} g(t)e^{-(\sigma+j\omega t)}dt \\
&= \int_{-\infty}^{\infty} g(t)e^{-\sigma t}e^{-j\omega t}dt
\end{aligned}
$$

Where σ is the real part of s and ω is the imaginary part of s

for $\sigma = 0$

$$
\begin{aligned}
G(j\omega) &= \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt \\
G(j\omega) &= \int_{-\infty}^{\infty} g(t)e^{-j2\pi n f t}dt = G(f)
\end{aligned}
$$

4.2 Region of Convergence (ROC)

The Laplace Transform exists for all complex numbers if $a < Re\{s\} < b$ where a and b are real numbers.
The set of values of s for which the Laplace Transform converges is called the Region of Convergence (ROC).

4.3 Circuit Analysis

Element	Time Domain	Laplace Domain
Resistor	R	R
Capacitor	$\frac{1}{C}\int i(t)\,dt$	$\frac{1}{Cs}$
Inductor	$L\frac{di(t)}{dt}$	Ls

Figure 4.1: Circuit Analysis

Done in the following steps

1. Convert circuit components, voltages, and currents to Laplace domain
2. Obtain the transfer function in the Laplace domain using mesh, nodal or any other circuit analysis technique (may need to be simplified using Partial Fraction Expansion)
3. Convert the transfer function back to the time domain using the inverse Laplace Transform

4.4 Partial Fraction Expansion

Form of the Rational Function	Partial Fraction Expansion
$\frac{ps+q}{(s+a)(s+b)}$	$\frac{A}{s+a} + \frac{B}{s+b}$
$\frac{ps+q}{(s+a)^2}$	$\frac{A}{s+a} + \frac{B}{(s+a)^2}$
$\frac{ps^2+qs+r}{(s+a)(s+b)(s+c)}$	$\frac{A}{s+a} + \frac{B}{s+b} + \frac{C}{s+c}$
$\frac{ps^2+qs+r}{(s+a)^2(s+b)}$	$\frac{A}{s+a} + \frac{B}{(s+a)^2} + \frac{C}{s+b}$
$\frac{ps^2+qs+r}{(s+a)(s^2+bs+c)}$	$\frac{A}{s+a} + \frac{Bs+C}{s^2+bs+c}$

Figure 4.2: Partial Fraction Expansion Forms

1. Factorise the denominator of the transfer function
2. Write the transfer function as a sum of fractions with the factors of the denominator as the denominators of the fractions
3. Write the numerator of the transfer function as the numerator of the fractions
4. Solve for the unknown coefficients
5. Convert the fractions back to the time domain using the inverse Laplace Transform

Example:

$$
\begin{aligned}
H(s) &= \frac{1}{s^2+2s+1} \\
&= \frac{1}{(s+1)^2} \\
&= \frac{A}{s+1} + \frac{B}{(s+1)^2} \\
&= \frac{A(s+1)+B}{(s+1)^2} \\
&= \frac{As+A+B}{(s+1)^2}
\end{aligned}
$$

Comparing coefficients gives:

$$
\begin{aligned}
A &= 1 \\
A+B &= 0 \\
B &= -1
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
H(s) &= \frac{1}{s+1} - \frac{1}{(s+1)^2} \\
&= \frac{1}{s+1} - \frac{1}{s+1} + \frac{1}{(s+1)^2} \\
&= \frac{1}{s+1} - \frac{1}{(s+1)^2} \\
&= e^{-t} - te^{-t}
\end{aligned}
$$

Chapter 5

System Characterisation

5.1 Second Order Systems

Systems in the form of

$$\frac{Y(s)}{X(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

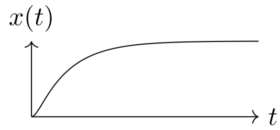
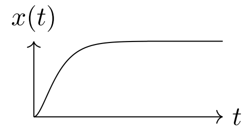
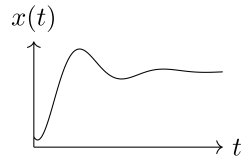
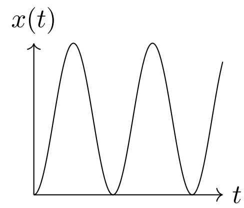
System Type	Nature of Roots	Condition	Roots	System Figure
Overdamped	Two Real Roots	$\zeta > 1$	$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$	
Critically Damped	One Real Root	$\zeta = 1$	$s_1 = -\zeta\omega_n$	
Underdamped	Two Complex Roots	$0 < \zeta < 1$	$s_1 = -\zeta\omega_n \pm j\omega_n\sqrt{\zeta^2 - 1}$	
Undamped	Two Imaginary Roots	$\zeta = 0$	$s_1 = \pm j\omega_n$	

Figure 5.1: Second Order Systems

- ω_n is the natural frequency
- ζ is the damping ratio

5.2 Natural Response By Inspection

- Overdamped, poles at $-\sigma_1, -\sigma_2$
  $y(t) = A_1 e^{-\sigma_1 t} + A_2 e^{-\sigma_2 t}$
- Underdamped, poles at $-\sigma \pm j\omega_d$
  $y(t) = Ae^{-\sigma t}\cos(\omega_d t + \phi)$
- Undamped, poles at $\pm j\omega_n$
  $y(t) = A\cos(\omega_n t + \phi)$
- Critically damped, poles at $-\sigma$
  $y(t) = (A_1 + A_2 t)e^{-\sigma t}$

5.3 Poles of general 2nd order systems

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.1}$$

The roots of the denominator polynomial are the poles of the system

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{5.2}$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{5.3}$$

5.4 Step Response of Underdamped 2nd Order Systems

Specifications are

- Settling Time

  $T_s = \frac{4}{\zeta\omega_n}$

- % Overshoot

  $\%OS = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$

- Peak Time

  $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

5.5 Step Response of a 1st Order System

Specifications are

- Settling Time

  $T_s = \frac{4}{\alpha}$

- Rising Time

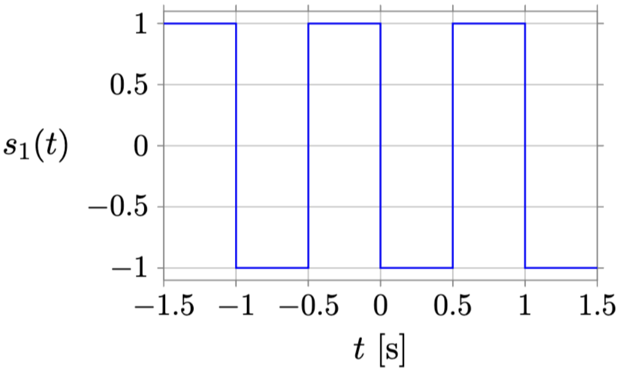
  $T_r = \frac{2.2}{\alpha}$

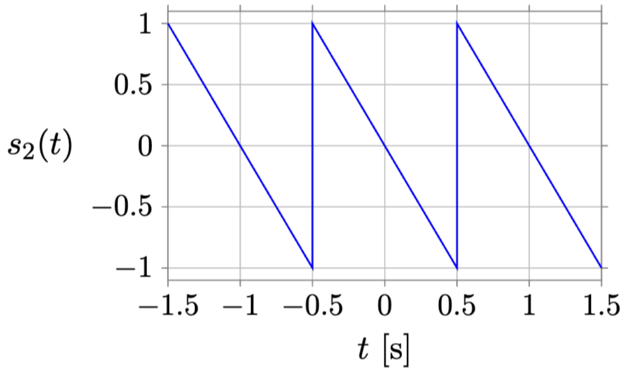
Chapter 6

Tutorial Questions

6.1 Tutorial 13

Question 1





Express $s_1(t)$ and $s_2(t)$ as complex Fourier series.

$$s_1(t) = \begin{cases} 1 & -\frac{1}{2} \leq t < 0 \\ -1 & 0 \leq t < \frac{1}{2} \end{cases}$$

Note that $T = 1$ and $f_0 = \frac{1}{T} = 1$

Part 1:

$$\begin{aligned} c_0 &= \int^{\infty} -\infty s_1(t)dt \\ &= \int^{0} -0.51dt + \int^{0.5} 0 - 1dt \\ &= [t]^0 -0.5 + [-t]^{0.5} 0 \\ &= 0.5 - 0.5 = 0 \end{aligned}$$

Part 2:

$$\begin{aligned} c_n &= \frac{1}{T}\int^{\infty} -\infty s_1(t)e^{-j2n\pi f_0 t}dt \\ &= \frac{1}{1}\int^{0} -0.5e^{-j2n\pi t}dt + \frac{1}{1}\int^{0.5} 0 - e^{-j2n\pi t}dt \\ &= \int^{0} -0.5e^{-j2n\pi t}dt + \int^{0} 0.5e^{-j2n\pi t}dt \\ &= \left[\frac{e^{-j2n\pi t}}{-j2n\pi}\right]^0 -0.5 + \left[\frac{e^{-j2n\pi t}}{-j2n\pi}\right]^{0.5} 0 \\ &= \frac{e^{-j2n\pi \times 0.5}}{-j2n\pi} - \frac{e^{-j2n\pi \times 0}}{-j2n\pi} + \frac{e^{-j2n\pi \times 0.5}}{-j2n\pi} - \frac{e^{-j2n\pi \times 0}}{-j2n\pi} \\ &= \frac{e^{-jn\pi}}{-j2n\pi} - \frac{1}{-j2n\pi} + \frac{e^{-jn\pi}}{-j2n\pi} - \frac{1}{-j2n\pi} \\ &= \frac{e^{-jn\pi}}{-jn\pi} - \frac{1}{-jn\pi} \\ &= \frac{j}{\pi n}\left((-1)^n - 1\right) \end{aligned}$$

Therefore:

$$s_1(t) = \sum_{n=-\infty}^{\infty} \frac{j}{\pi n}\left((-1)^n - 1\right)e^{j2n\pi t}$$