CH-231-A

Algorithms and Data Structures

ADS

Lecture 33

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Graph Representations: Directed and Undirected Graphs

Definition:

- A directed graph (digraph) $G = (V, E)$ is an ordered pair consisting of
  - a set V of vertices and
  - a set $E \subset V \times V$ of edges.
- In an undirected graph $G = (V, E)$, the edge set E consists of unordered pairs of vertices.

Number of Edges and Vertices

- In a graph, the number of edges is bound by $|E| = O(|V|^2)$.
- If G is connected, then $|E| \geq |V| - 1$.
- Hence, for a connected graph we get $\lg |E| = \Theta(\lg |V|)$.

Adjacency Matrices

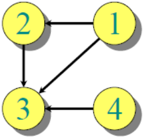
Definition:

The adjacency matrix of a graph $G = (V, E)$ with $V = \{1, ..., n\}$ is the $n \times n$ matrix A given by

$$A[i,j] = \begin{cases} 1 & \text{if } (i,j) \in \mathrm{E}, \\ 0 & \text{if } (i,j) \notin \mathrm{E}. \end{cases}$$

Dense representation: Storage requirements are $\Theta(|V|^2)$.

Example:



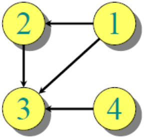
A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

Adjacency List

Definition:

An adjacency list of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to v.

Example:

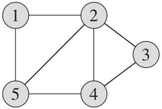


$Adj[1] = \{2, 3\}$
$Adj[2] = \{3\}$
$Adj[3] = \{\}$
$Adj[4] = \{3\}$

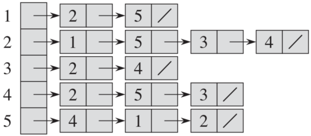
Sparse representation:

- Storage requirements for $Adj[v]$ is $\Theta(|\text{outgoing edges from } v|)$.
- Storage requirement for $Adj[v]$ for all $v \in V$ is $\Theta(|E|)$.
- Overall storage requirement is $\Theta(|V| + |E|)$.

Examples for Undirected & Directed Graphs



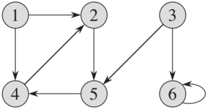
(a)



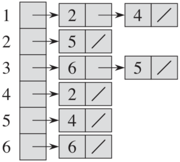
(b)

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

(c)



(a)

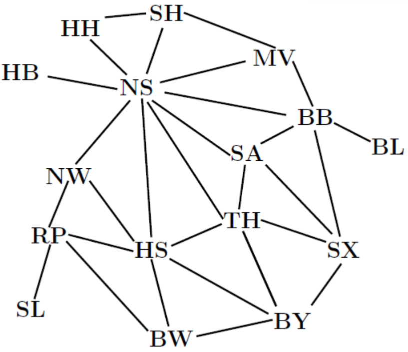


(b)

	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

(c)

Application Example: Neighboring States



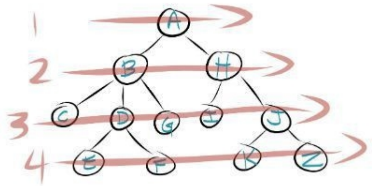
Breadth-First Search (BFS)

Problem:

- Given (directed or undirected) graph $G = (V, E)$ and a starting vertex $s \in V$.
- Systematically explore all vertices reachable from s.

BFS strategy:

- First find all vertices of distance 1 from s, then of distance 2, then of distance 3, etc.

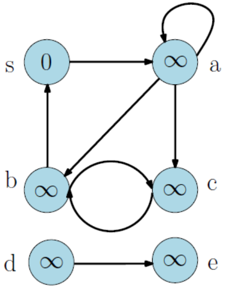


BFS Approach

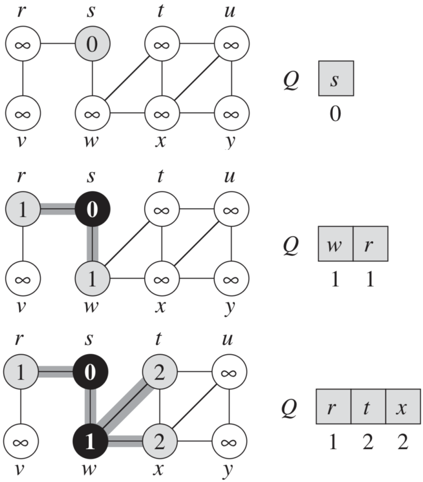
- Use adjacency-list representation.
- Use a color attribute for each *vertex* $\in \{\text{white}, \text{gray}, \text{black}\}$.
  - white: not detected yet
  - gray: just detected, waiting for us to explore their adjacency lists
  - black: done, all neighbors have been visited
- Store all gray vertices in a queue (FIFO principle).
- In addition, store for each vertex an attribute with the (topological) distance to starting vertex s.
- Finally, also store a pointer to the predecessor.

BFS Algorithm

```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```



BFS Example (1)




```
BFS(G, s)
for each vertex u ∈ G.V − {s}
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            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```
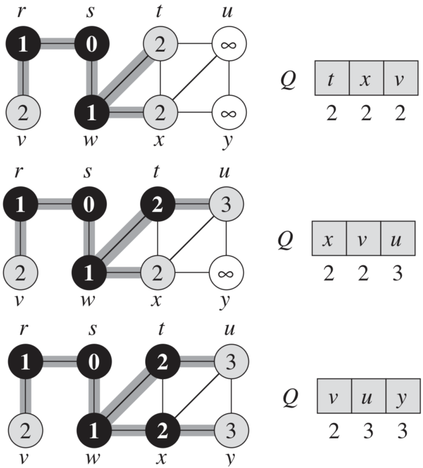

BFS Example (2)




```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
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Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```
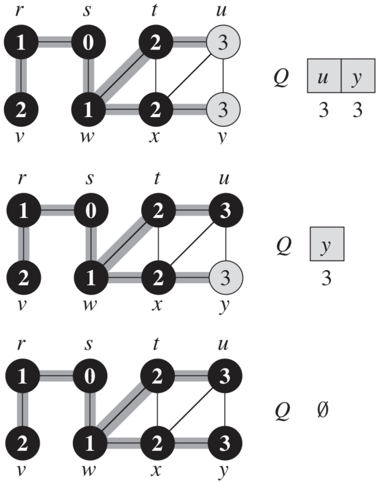

BFS Example (3)




```
BFS(G, s)
for each vertex u ∈ G.V − {s}
    u.color = WHITE
    u.d = ∞
    u.π = NIL
s.color = GRAY
s.d = 0
s.π = NIL
Q = ∅
ENQUEUE(Q, s)
while Q ≠ ∅
    u = DEQUEUE(Q)
    for each v ∈ G.Adj[u]
        if v.color == WHITE
            v.color = GRAY
            v.d = u.d + 1
            v.π = u
            ENQUEUE(Q, v)
    u.color = BLACK
```

BFS Analysis

- Each vertex is enqueued and dequeued once.
- Each queue operation is O(1).
- Total time for queue operations is $O(|V|)$.
- Loop over adjacency list of all vertices is in total $\Theta(|E|)$.
- Together, we get a time complexity of $O(|V| + |E|)$.

Breadth-First Tree

- When storing the predecessors, we can construct the predecessor subgraph $G_\pi = (V_\pi, E_\pi)$ of G with

$$V_\pi = \{v \in V \mid v.\pi \neq NIL\} \cup \{s\}$$
$$E_\pi = \{(v.\pi, v) \mid v \in V_\pi - \{s\}\}$$

- This subgraph represents a tree structure.
- It is called the breadth-first tree.
- It contains a unique path from s to every vertex in V_π.
- All these paths are shortest paths in G.