CH-231-A

Algorithms and Data Structures

ADS

Lecture 16

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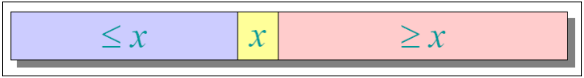
Quicksort: Divide & Conquer

1. Divide:
   Partition the array into two subarrays around a pivot x such that the elements in lower subarray $\leq x \leq$ the elements in upper subarray.



2. Conquer:
   Recursively sort the two subarrays
3. Combine:
   Nothing to be done.

Key observation: Linear-time partitioning subroutine.

Quicksort: Divide (1)

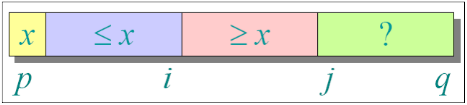
In the literature there are two popular division (partition) methods:

- Nico Lomuto's partition used in the textbook of Cormen and other textbooks
- C.A.R. Hoare's partition
- Hoare's partition is more efficient than Lomuto's partition because it does three times fewer swaps on average and it creates efficient partitions even when all values are equal
- But the time complexities are the same

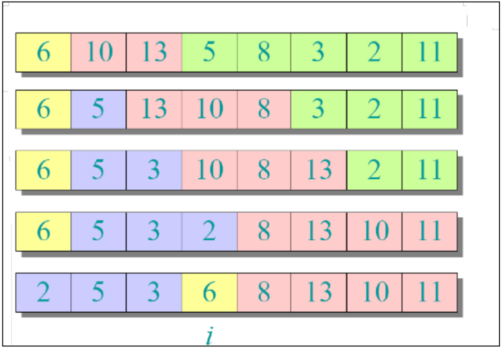
Quicksort: Divide (2)

```
Partition(A, p, q)     //A[p. . q]
  x = A[p]             //pivot = A[p]
  i = p
  for j = p + 1 to q
    if A[j] <= x then
      i = i + 1
      exchange A[i] and A[j]
  exchange A[p] and A[i]
  return i
```

Invariant:



Quicksort: Partition Example



Quicksort: Divide Complexity

```
Partition(A, p, q)      //A[p. . q]
  x = A[p]              //pivot = A[p]
  i = p
  for j = p + 1 to q
    if A[j] <= x then
      i = i + 1
      exchange A[i] and A[j]
  exchange A[p] and A[i]
  return i
```

Time complexity:

For $n = q - p + 1$ elements: $T(n) = \Theta(n)$

Quicksort: Conquer

```
QuickSort(A, p, r)
  if p < r
    q = Partition(A, p, r)
    QuickSort(A, p, q - 1)
    QuickSort(A, q + 1, r)
```

Initial call: `QuickSort(A, 1, n)`

Runtime Analysis (1)

- Assume all input elements are distinct.
  - In practice, there are better partitioning algorithms for when duplicate input elements may exist.
- Let $T(n)$ be the worst-case running time for n elements.
- Worst case:
  - Input sorted or reverse sorted.
  - Partition around min or max element.
  - One side of partition always has no elements.

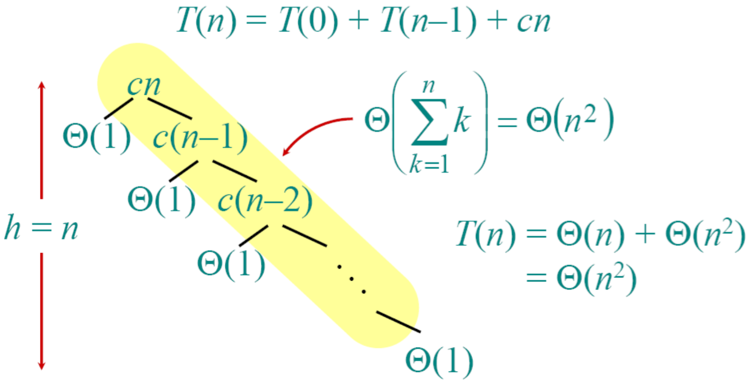
Runtime Analysis (2)

Worst case:

$$
\begin{aligned}
T(n) &= T(0) + T(n-1) + \Theta(n) \\
&= \Theta(1) + T(n-1) + \Theta(n) \\
&= T(n-1) + \Theta(n) \\
&= \Theta(n^2) \qquad \textit{(arithmetic series)}
\end{aligned}
$$

Runtime Analysis (3)

Worst-case recursion tree:



Runtime Analysis (4)

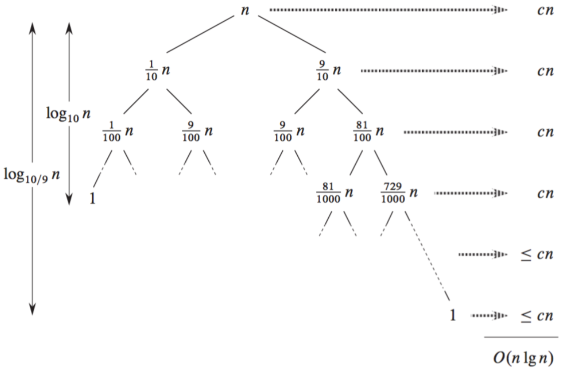
Best case:

- In best case partition splits the array evenly.
- $T(n) = 2T(n/2) + \Theta(n) = \Theta(n \lg n)$
- This is the same as Merge Sort.

Runtime Analysis (5)

What if the split is $1/10 : 9/10$?

$$T(n) = T\left(\frac{1}{10}n\right) + T\left(\frac{9}{10}n\right) + \Theta(n)$$



Runtime Analysis

- What if we alternate between lucky and unlucky choices
  - $L(n) = 2U(n/2) + \Theta(n)$ lucky
  - $U(n) = L(n-1) + \Theta(n)$ unlucky
- Solving:
  - $L(n) = 2(L(n/2-1) + \Theta(n/2)) + \Theta(n)$
    $= 2L(n/2-1) + \Theta(n)$
    $= \Theta(n \lg n)$
- How can we make sure that this is usually happening?

Randomized Quicksort (1)

- Idea: Partition around a random element.
- Running time is independent of the input order.
- No assumptions need to be made about the input distribution.
- No specific input elicits the worst-case behavior.
- The worst case is determined only by the output of a random-number generator.

Randomized Quicksort (2)

```
RANDOMIZED-PARTITION(A, p, r)
  i = RANDOM(p, r)
  exchange A[p] with A[i]
  return PARTITION(A, p, r)
```

```
RANDOMIZED-QUICKSORT(A, p, r)
  if p < r
      q = RANDOMIZED-PARTITION(A, p, r)
      RANDOMIZED-QUICKSORT(A, p, q − 1)
      RANDOMIZED-QUICKSORT(A, q + 1, r)
```

Randomized Quicksort (3)

- Let $T(n)$ be the random variable for the running time of the randomized quicksort on an input of size n (assuming random numbers are independent).

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n{-}k{-}1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

- For $k = 0, 1, ..., n - 1$, define indicator random variable
- $E[X_k] = Pr\{X_k = 1\} = 1/n$, since all splits are equally likely (assuming elements are distinct).

Randomized Quicksort (4)

Recurrence:

$$T(n) = \begin{cases} T(0) + T(n-1) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(1) + T(n-2) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \vdots & \\ T(n-1) + T(0) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \left(T(k) + T(n-k-1) + \Theta(n) \right)$$

Randomized Quicksort (5)

Calculating expectations:

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\right]$$

$$= \sum_{k=0}^{n-1} E\big[X_k\big(T(k) + T(n-k-1) + \Theta(n)\big)\big]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(k) + T(n-k-1) + \Theta(n)]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E[T(k)] + \frac{1}{n}\sum_{k=0}^{n-1} E[T(n-k-1)] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

$$= \frac{2}{n}\sum_{k=2}^{n-1} E[T(k)] + \Theta(n)$$

Randomized Quicksort (6)

- Use substitution method to solve recurrence.
- Guess: $E[T(n)] = \Theta(n \lg n)$.
- Prove: $E[T(n)] \leq an \lg n$ for constant $a > 0$.
- Use:

$$\sum_{k=2}^{n-1} k \lg k \leq \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$$

(proof by induction)

Randomized Quicksort (7)

Proof:

$$E[T(n)] \leq \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n)$$

$$\begin{aligned}
E[T(n)] &\leq \frac{2}{n}\sum_{k=2}^{n-1} ak\lg k + \Theta(n) \\
&= \frac{2a}{n}\left(\frac{1}{2}n^2\lg n - \frac{1}{8}n^2\right) + \Theta(n) \\
&= an\lg n - \left(\frac{an}{4} - \Theta(n)\right) \\
&\leq an\lg n,
\end{aligned}$$

if a is chosen large enough.

Quicksort: Conclusion

- Quicksort is a great general-purpose sorting algorithm.
- Quicksort is often the best practical choice because its expected runtime is $\Theta(n \lg n)$ and the constant is quite small.
- Quicksort is typically over twice as fast as MergeSort.
- Quicksort is an in-situ sorting algorithm (debatable).
- Quicksort has a worst-case runtime of $\Theta(n^2)$ when the array is already sorted.
- Visualization Randomized Quicksort: `http://www.sorting-algorithms.com/quick-sort`