CH-231-A

Algorithms and Data Structures

ADS

Lecture 41

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General: Linear Programming[1]

- Important tool for optimal allocation of scarce resources, among a number of competing activities.
- Powerful and general problem-solving method.
- Applications:
  - Computer science: Compiler register allocation, data mining.
  - Electrical engineering: VLSI design, optimal clocking.
  - Economics: Equilibrium theory, two-person zero-sum games.
  - Environment: Water quality management.
  - Logistics: Supply-chain management, Berlin airlift.
  - Manufacturing: Production line balancing, cutting stock.
  - Telecommunication: Network design, Internet routing.

[1]Source of slides: Kevin Wayne: Algorithms and Data Structures, Spring 2004, Princeton University

Brewery Problem: A Toy LP Example

Small brewery produces ale and beer.

- Production limited by scarce resources: corn, hops, malt.
- Recipes for ale and beer require different proportions of resources.

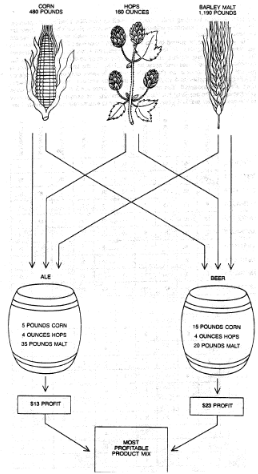
Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale	5	4	35	13
Beer	15	4	20	23
Quantity	480	160	1190	

How can brewer maximize profits?

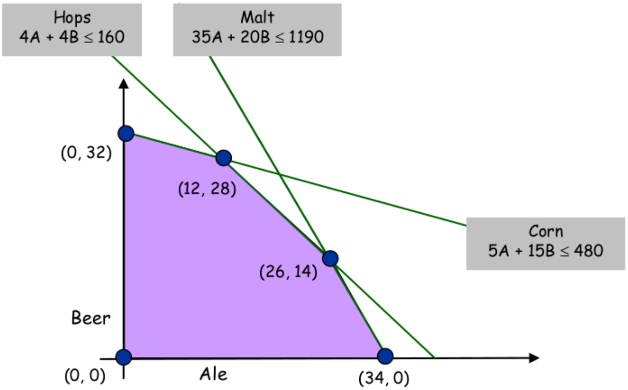
- Devote all resources to ale: 34 barrels of ale $\rightarrow$ \$442.
- Devote all resources to beer: 32 barrels of beer $\rightarrow$ \$736.
- 7.5 barrels of ale, 29.5 barrels of beer $\rightarrow$ \$776.
- 12 barrels of ale, 28 barrels of beer $\rightarrow$ \$800.

Brewery Problem

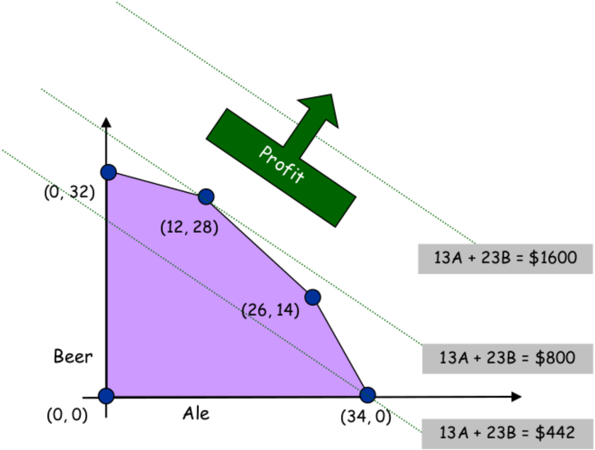
$$
\begin{array}{lrcrcrl}
 & \text{Ale} & & \text{Beer} & & & \\
\max & 13A & + & 23B & & & \text{Profit} \\
\text{s.t.} & 5A & + & 15B & \leq & 480 & \text{Corn} \\
 & 4A & + & 4B & \leq & 160 & \text{Hops} \\
 & 35A & + & 20B & \leq & 1190 & \text{Malt} \\
 & A & , & B & \geq & 0 &
\end{array}
$$



Brewery Problem: Feasible Region

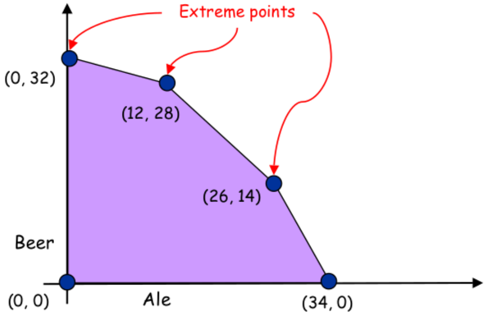


Brewery Problem: Objective Function



Brewery Problem: Geometry

Observation: Regardless of objective function coefficients, an optimal solution occurs at an extreme point.



Linear Programming: Standard Form

Standard form:

- Input: real numbers c_j, b_i, a_{ij}.
- Output: real numbers x_j.
- $n = \#$ nonnegative variables, $m = \#$ constraints.
- Maximize linear objective function subject to linear inequalities.

$$
\begin{aligned}
(P)\quad \max\ & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.}\ & \sum_{j=1}^{n} a_{ij} x_j = b_i \quad 1 \le i \le m \\
& x_j \ge 0 \quad 1 \le j \le n
\end{aligned}
$$

$$
\begin{aligned}
(P)\quad \max\ & c^T x \\
\text{s.t.}\ & Ax = b \\
& x \ge 0
\end{aligned}
$$

Linear: No x^2 , xy, $\arccos(x)$, etc.

Programming: Planning (term predates computer programming).

Brewery Problem: Converting to Standard Form

Original input:

$$
\begin{array}{lrcrcl}
\max & 13A & + & 23B & & \\
\text{s.t.} & 5A & + & 15B & \leq & 480 \\
 & 4A & + & 4B & \leq & 160 \\
 & 35A & + & 20B & \leq & 1190 \\
 & A & , & B & \geq & 0
\end{array}
$$

Standard form:

- Add slack variable for each inequality.
- Now a 5-dimensional problem.

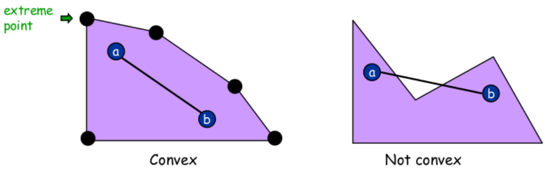
$$
\begin{array}{lrcrcccccccl}
\max & 13A & + & 23B & & & & & & & & \\
\text{s.t.} & 5A & + & 15B & + & S_C & & & & & = & 480 \\
 & 4A & + & 4B & & & + & S_H & & & = & 160 \\
 & 35A & + & 20B & & & & & + & S_M & = & 1190 \\
 & A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0
\end{array}
$$

Geometry (1)

Geometry:

- Inequalities : halfplanes (2D), hyperplanes.
- Bounded feasible region: convex polygon (2D), (convex) polytope.

Convex: if a and b are feasible solutions, then so is $(a + b)/2$.
Extreme point: feasible solution x that cannot be written as $(a + b)/2$ for any two distinct feasible solutions a and b.



Geometry (2)

Extreme point property: If there exists an optimal solution to (P), then there exists one that is an extreme point.

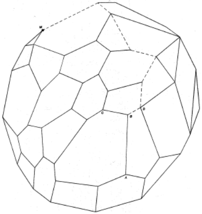
- Only need to consider finitely many possible solutions.

Challenge: Number of extreme points can be exponential.

- Consider n-dimensional hypercube.

Greedy: Local optima are global optima.

- Extreme point is optimal if no neighboring extreme point is better.



Simplex Algorithm

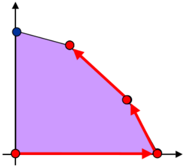
Simplex algorithm: (George Dantzig, 1947)

- Developed after WWII in response to logistical problems.
- Used for 1948 Berlin airlift.

Generic algorithm:

- Start at some extreme point.
- Pivot from one extreme point to a neighboring one. (never decrease objective function)
- Repeat until optimal.

How to implement? Linear algebra.



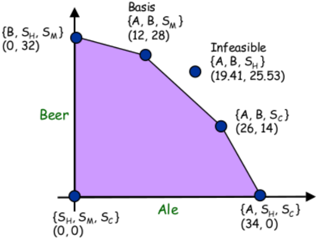
Simplex Algorithm: Basis

Basis: Subset of m of the n variables.

Basic feasible solution (BFS): Set $n - m$ nonbasic variables to 0, solve for remaining m variables.

- Solve m equations in m unknowns.
- If unique and feasible solution $\to$ BFS.
- BFS corresponds to extreme point.
- Simplex only considers BFS.

$$
\begin{array}{llllllllll}
\max & 13A & + & 23B & & & & & & \\
\text{s.t.} & 5A & + & 15B & + S_C & & & = & 480 \\
& 4A & + & 4B & & + S_H & & = & 160 \\
& 35A & + & 20B & & & + S_M & = & 1190 \\
& A & , & B & , S_C & , S_H & , S_M & \geq & 0
\end{array}
$$



Simplex Algorithm: Pivot 1 (1)

max Z subject to

$$
\begin{array}{rcrcrcrcrcrcr}
13A & + & 23B & & & & & & & - & Z & = & 0 \\
\hline
5A & + & 15B & + & S_C & & & & & & & = & 480 \\
4A & + & 4B & & & + & S_H & & & & & = & 160 \\
35A & + & 20B & & & & & + & S_M & & & = & 1190 \\
A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0
\end{array}
$$

Basis = $\{S_C, S_H, S_M\}$
$A = B = 0$
$Z = 0$
$S_C = 480$
$S_H = 160$
$S_M = 1190$

Substitute: $B = 1/15\,(480 - 5A - S_C)$

max Z subject to

$$
\begin{array}{rcrcrcrcrcrcr}
\frac{16}{3}A & & & - & \frac{23}{15}S_C & & & & & - & Z & = & -736 \\
\hline
\frac{1}{3}A & + & B & + & \frac{1}{15}S_C & & & & & & & = & 32 \\
\frac{8}{3}A & & & - & \frac{4}{15}S_C & + & S_H & & & & & = & 32 \\
\frac{85}{3}A & & & - & \frac{4}{3}S_C & & & + & S_M & & & = & 550 \\
A & , & B & , & S_C & , & S_H & , & S_M & & & \geq & 0
\end{array}
$$

Basis = $\{B, S_H, S_M\}$
$A = S_C = 0$
$Z = 736$
$B = 32$
$S_H = 32$
$S_M = 550$

Simplex Algorithm: Pivot 1 (2)

max Z subject to

$$
\begin{array}{rcrcccccccl}
13A & + & 23B & & & & & - & Z & = & 0 \\
\hline
5A & + & 15B & + & S_C & & & & & = & 480 \\
4A & + & 4B & & & + & S_H & & & = & 160 \\
35A & + & 20B & & & & & + & S_M & = & 1190 \\
A & , & B & , & S_C & , & S_H & , & S_M & \geq & 0
\end{array}
$$

Basis = $\{S_C, S_H, S_M\}$
$A = B = 0$
$Z = 0$
$S_C = 480$
$S_H = 160$
$S_M = 1190$

Why pivot on column 2?

- Each unit increase in B increases objective value by \$23.
- Pivoting on column 1 also OK.

Why pivot on row 2?

- Preserves feasibility by ensuring $RHS \geq 0$.
- Minimum ratio rule: $\min\{480/15, 160/4, 1190/20\}$.

Simplex Algorithm: Pivot 2

max Z subject to

$$
\begin{array}{rrrrrrl}
\frac{16}{3}A & & -\frac{23}{15}S_C & & & -Z & = -736 \\
\hline
\frac{1}{3}A & +B & +\frac{1}{15}S_C & & & & = 32 \\
\frac{8}{3}A & & -\frac{4}{15}S_C & +S_H & & & = 32 \\
\frac{85}{3}A & & -\frac{4}{3}S_C & & +S_M & & = 550 \\
A, & B, & S_C, & S_H, & S_M & & \geq 0
\end{array}
$$

Basis = $\{B, S_H, S_M\}$
$A = S_C = 0$
$Z = 736$
$B = 32$
$S_H = 32$
$S_M = 550$

Substitute: $A = 3/8\,(32 + 4/15\, S_C - S_H)$

max Z subject to

$$
\begin{array}{rrrrrrl}
 & & -S_C & -2S_H & & -Z & = -800 \\
\hline
 & B & +\frac{1}{10}S_C & +\frac{1}{8}S_H & & & = 28 \\
A & & -\frac{1}{10}S_C & +\frac{3}{8}S_H & & & = 12 \\
 & & -\frac{25}{6}S_C & -\frac{85}{8}S_H & +S_M & & = 110 \\
A, & B, & S_C, & S_H, & S_M & & \geq 0
\end{array}
$$

Basis = $\{A, B, S_M\}$
$S_C = S_H = 0$
$Z = 800$
$B = 28$
$A = 12$
$S_M = 110$

Simplex Algorithm: Optimality

When to stop pivoting?

- If all coefficients in top row are non-positive.

Why is resulting solution optimal?

- Any feasible solution satisfies system of equations in tableaux.
  - in particular: $Z = 800 - S_C - 2S_H$
- Thus, optimal objective value $Z^* \leq 800$ since $S_C, S_H \geq 0$.
- Current BFS has value 800 $\rightarrow$ optimal.

max Z subject to

$$
\begin{array}{rcl}
- S_C - 2\,S_H - Z &=& -800 \\
\hline
B + \frac{1}{10} S_C + \frac{1}{8} S_H &=& 28 \\
A - \frac{1}{10} S_C + \frac{3}{8} S_H &=& 12 \\
- \frac{25}{6} S_C - \frac{85}{8} S_H + S_M &=& 110 \\
A\ ,\ B\ ,\ S_C\ ,\ S_H\ ,\ S_M &\geq& 0
\end{array}
$$

Basis = $\{A, B, S_M\}$
$S_C = S_H = 0$
$Z = 800$
$B = 28$
$A = 12$
$S_M = 110$

Simplex Algorithm: Issues

Remarkable property: In practice, simplex algorithm typically terminates in at most $2(m+n)$ pivots.

- No polynomial pivot rule known.
- Most pivot rules known to be exponential in worst-case.

Issues: Which neighboring extreme point?

Degeneracy: New basis, same extreme point.

- "Stalling" is common in practice.

Cycling: Get stuck by cycling through different bases that all correspond to same extreme point.

- Does not occur in the wild.
- Bland's least index rule $\rightarrow$ finite # of pivots.