CH-231-A

Algorithms and Data Structures

ADS

Lecture 19

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Spring 2020

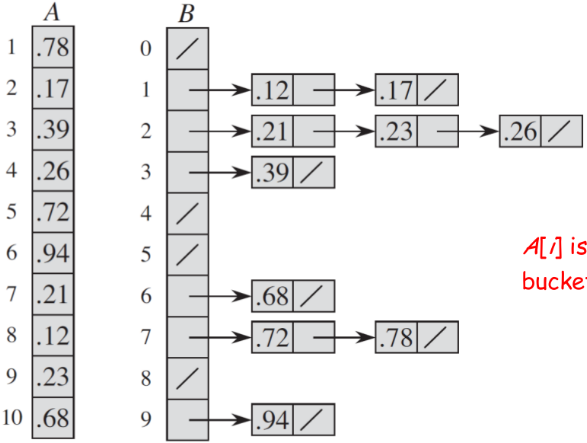
Bucket Sort: Motivation

- Can we use the idea of Radix Sort to sort any numbers, i.e., without assuming them to be integers?
- In order to do this efficiently, we make a new assumption:
  - The to-be-sorted elements shall distribute uniformly and independently over the interval $[0, 1)$.
- Remark:
  - Interval $[0, 1)$ is not a real restriction, as we can normalize the elements to this interval in linear time.
  - However, uniform distribution and independence are restrictions and we will see that we need this to assure good expected running time.

Bucket Sort: Idea

- Assuming that we have to sort n numbers, we split the interval $[0, 1)$ into n subintervals or buckets.
- Then, we can distribute the n numbers to the n buckets.
- Assuming uniform distribution, we can conclude that we have only few numbers falling into each bucket.

Bucket Sort: Example $n = 10$



$A[i]$ is put into bucket $\lfloor n A[i] \rfloor$

Bucket Sort: Pseudocode

```
BUCKET-SORT(A)
let B[0..n − 1] be a new array
n = A.length
for i = 0 to n − 1
    make B[i] an empty list
for i = 1 to n
    insert A[i] into list B[⌊n A[i]⌋]
for i = 0 to n − 1
    sort list B[i] with insertion sort
concatenate the lists B[0], B[1], ..., B[n − 1] together in order
```

Bucket Sort: Time Complexity

BUCKET-SORT(A)

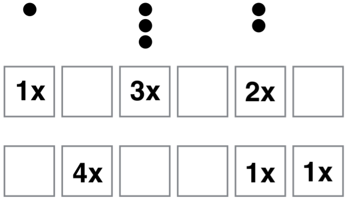
let $B[0\,..\,n-1]$ be a new array
$n = A.length$
for $i = 0$ **to** $n-1$
    make $B[i]$ an empty list
for $i = 1$ **to** n
    insert $A[i]$ into list $B[\lfloor n A[i] \rfloor]$
for $i = 0$ **to** $n-1$
    sort list $B[i]$ with insertion sort
concatenate the lists $B[0], B[1], \ldots, B[n-1]$ together in order

Time complexity:

$$T(n) = \Theta(n) + \sum_{i=0}^{n-1} O({n_i}^2),$$

where n_i denotes the number of elements in bucket i.

Bucket Sort: Average Case



might fall into boxes like that

or that

...

Bucket Sort: Expected Time Complexity (1)

- $E[T(n)] = \Theta(n) + \sum_{i=0}^{n-1} O(E[{n_i}^2])$
- What is $E[{n_i}^2]$?
- Let X_{ij} be the event that $A[j]$ falls into bucket i.
- Then, $n_i = \sum_{j=1}^{n} X_{ij}$
- Use assumptions of uniform distribution and independence.

Bucket Sort: Estimate $E[{n_i}^2]$ (1)

$$E[n_i^2] = E\left[\left(\sum_{j=1}^{n} X_{ij}\right)^2\right] = E\left[\sum_{j=1}^{n}\sum_{k=1}^{n} X_{ij}X_{ik}\right]$$

$$= E\left[\sum_{j=1}^{n} X_{ij}^2 + \sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} X_{ij}X_{ik}\right]$$

$$= \sum_{j=1}^{n} E[X_{ij}^2] + \sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} E[X_{ij}X_{ik}]$$

$$= \sum_{j=1}^{n} E[X_{ij}^2] + \sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} E[X_{ij}]\,E[X_{ik}]$$

Bucket Sort: Estimate $E[{n_i}^2]$ (2)

$$E[X_{ij}]\,E[X_{ik}] = \tfrac{1}{n}\cdot\tfrac{1}{n} = \tfrac{1}{n^2}.$$

$$E[X_{ij}^2] = 1^2\cdot\tfrac{1}{n} + 0^2\cdot\left(1-\tfrac{1}{n}\right) = \tfrac{1}{n}.$$

$$\begin{aligned}
E[n_i^2] &= \sum_{j=1}^{n} E[X_{ij}^2] + \sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} E[X_{ij}]\,E[X_{ik}] \\
&= \sum_{j=1}^{n}\frac{1}{n} + \sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n}\frac{1}{n^2} \\
&= \frac{n}{n} + n\,(n-1)\frac{1}{n^2} \\
&= 2 - \frac{1}{n}.
\end{aligned}$$

Bucket Sort: Expected Time Complexity (2)

- $E[T(n)] = \Theta(n) + \sum_{i=0}^{n-1} O(E[{n_i}^2])$
- Based on the previous estimation we have the following
- $E[T(n)] = \Theta(n) + n \cdot O(2 - 1/n) = \Theta(n)$

Searching Problem

- Given a sorted sequence.
- Find an element in that sequence.
- Example:
  - Sequence

3	5	7	8	9	12	15

  - Find element 9.
  - Brute-force approach (going through the sequence from start until we find the 9) runs in $O(n)$.

Binary Search

Idea: Use a Divide & Conquer strategy.

1. Divide: Check middle element.
2. Conquer: Recursively search one subarray.
3. Combine: Nothing to be done.

Binary Search: Example (Find 9)

3	5	7	8	9	12	15

3	5	7	8	9	12	15

3	5	7	8	9	12	15

3	5	7	8	9	12	15

3	5	7	8	9	12	15

3	5	7	8	9	12	15

Binary Search: Time Complexity

$T(n) = 1T(n/2) + \Theta(1)$
$a = 1, b = 2$
$n^{log_b a} = n^{log_2 1} = 1$
$f(n) = \Theta(1)$
Case 2: $T(n) = \Theta(\lg n)$

Summary

- Sorting problem:
  - Comparison sorts:
    - InsertionSort: $\Theta(n)$ [best], $\Theta(n^2)$ [average & worst].
    - Merge Sort: $\Theta(n \lg n)$.
    - Heap Sort: $\Theta(n \lg n)$ – heap as a data structure
    - Quicksort: $\Theta(n \lg n)$ [best & average], $\Theta(n^2)$ [worst].
    - Decision trees: Worst case does not get better than $\Theta(n \lg n)$.
  - Sorting in linear time:
    - Counting Sort: small integers
    - Radix Sort: large integers
    - Bucket Sort: any numbers, but uniform distribution.
- Searching Problem:
  - Linear Search: $\Theta(1)$ [best], $\Theta(n)$ [average & worst]
  - Binary Search: $\Theta(1)$ [best], $\Theta(\lg n)$ [average & worst]

Data Structure

Definition:
A data structure is a way to store and organize data in order to facilitate access and modification.

Examples we have seen so far:

- Array
- Heap
- Max-priority queue
- Linked list

Array (1)

- Definition:
  An array is a random-access data structure consisting of a collection of elements, each identified by an index or key.
- The simplest type of data structure is a linear array, where the indices are one-dimensional.
- A dynamic array refers to an array which can change its size.

Array (2)

Examples of operations:

- Getting or setting the value at a particular index:
  - constant time
- Iterating over the elements in order:
  - linear time
- Inserting or deleting an element:
  - beginning – linear time
  - middle – linear time
  - end – constant time

Dynamic Set

- In the following, we assume that we are interested in storing and handling dynamic sets.
- Dynamic sets are sets of elements that can change their size.
- Elements are identified by a key from a totally ordered set.

Dynamic Set: Operations

Two categories of operations:

- Queries return the information of a stored object.
- Modify operations alter the set.

Examples for Queries

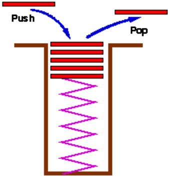
- *Search*(S, k):
  - returns element $x \in S$ with $key[x] = k$ (nil if not existent).
- *Minimum*(S):
  - returns element $x \in S$ with smallest $key[x]$.
- *Maximum*(S):
  - returns element $x \in S$ with largest $key[x]$.
- *Successor*(S, x):
  - returns for element $x \in S$ the next-larger element in S (nil if x is element with largest key).
- *Predecessor*(S, x):
  - returns for element $x \in S$ the next-smaller element in S (nil if x is element with smallest key).

Examples for Modify Operations

- *Insert*(S, x):
  - adds element x to dynamic set S (S grows).
- *Delete*(S, x):
  - deletes element x from dynamic set S (S shrinks).

Stack

- Elementary dynamic data structure.
- Implements idea of dynamic set.
- Idea follows that of a coin stacker.
- Delete operation is called `pop`.
- Insert operation is called `push`.
- LIFO principle (Last In First Out): The element that is returned by the `pop` operation is the last one that has been added (via `push`).



Stack Operations

- Queries:
  - *Stack-Empty*(S):
    True iff stack S is empty.
  - ...
- Modify operations:
  - *Push*(S, x):
    Add element x on top of stack S and push other elements down.
  - *Pop*(S):
    If stack is non-empty, remove top-most element and return it.

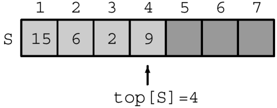
Stack: Implementation as an Array

S.top is the index of the top of the stack

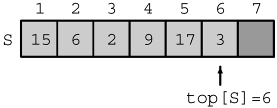
```
STACK-EMPTY(S)
if S.top == 0
    return TRUE
else return FALSE

PUSH(S, x)
S.top = S.top + 1
S[S.top] = x

POP(S)
if STACK-EMPTY(S)
    error "underflow"
else S.top = S.top - 1
    return S[S.top + 1]
```

Stack: Example (Array Implementation)

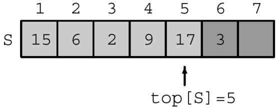
- Stack with four elements:



- Performing operations *Push*$(S, 17)$ and *Push*$(S, 3)$:



- Performing operation *Pop*(S) returning entry 3:



Stack Operations: Complexity

```
STACK-EMPTY(S)
if S.top == 0
    return TRUE
else return FALSE

PUSH(S, x)
S.top = S.top + 1
S[S.top] = x

POP(S)
if STACK-EMPTY(S)
    error "underflow"
else S.top = S.top − 1
    return S[S.top + 1]
```

Complexity:
when implemented as an array all operations are $O(1)$.

Stack Operations: Underflow and Overflow

- If we want to perform a *Pop*-operation on the empty stack, we have a stack-underflow situation.
- We may also have a stack-overflow situation, if we assume that the stack has a maximum amount of entries and then we try to perform a *Push*-operation (not considered in the array implementation).