CH-231-A

Algorithms and Data Structures

ADS

Lecture 23

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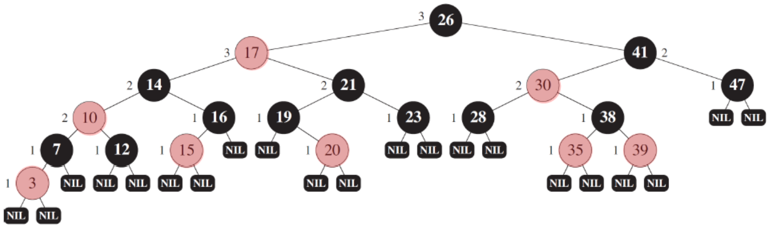
Red-Black Trees: Definition

- A red-black tree is a BST that besides the attributes about parent, left child, right child, and key holds the attribute of a color (**red** or **black**), which is encoded in one additional bit.
- Special convention: All leaves have NIL as key.
- The node colors are used to impose constraints on the nodes such that no path from the root to a leaf is more than twice as long as any other path.
- Hence, the tree is approximately balanced.

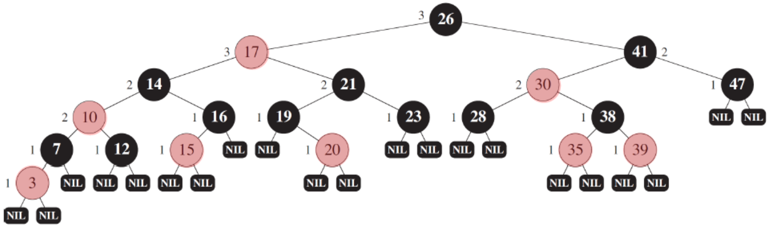
Property 1 (Duh Property)

Every node is either red or black.



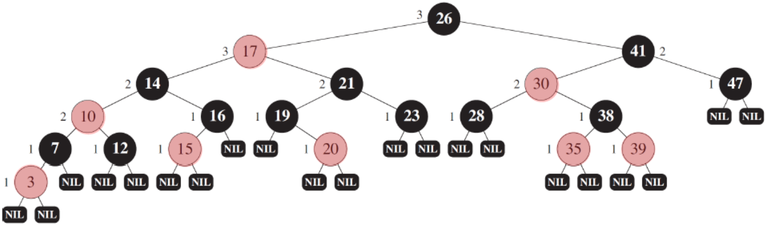
Property 2 (RooB Property)

The root is black.



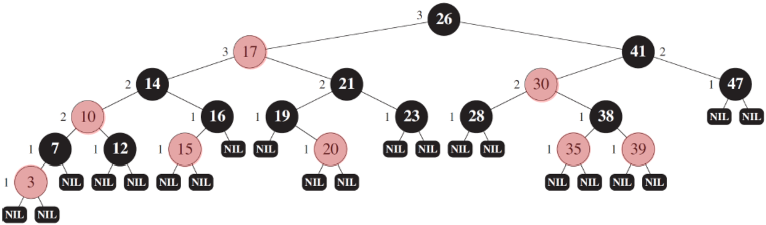
Property 3 (LeaB Property)

All leaves (NIL) are black.



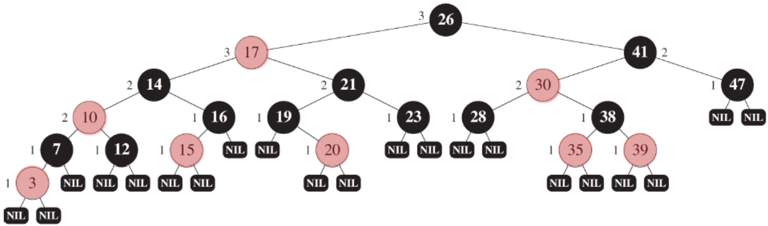
Property 4 (BredB Property)

If a node is red, then both children are black.



Property 5 (BH Property)

For each node all paths from the node to a leaf have the same number of black nodes.

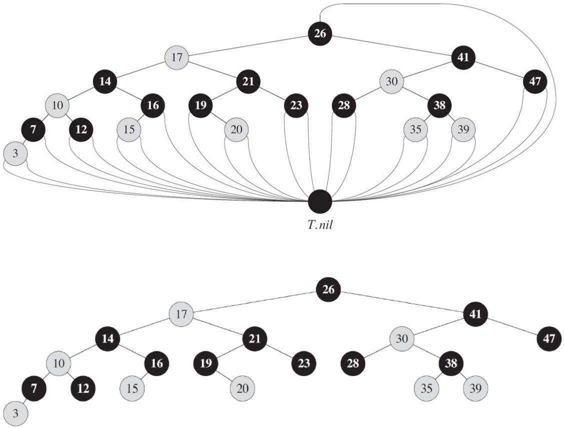


For each node x, we can define a unique black height $bh(x)$.

Properties

1. Every node is either red or black (Duh)
2. The root is black (RooB)
3. All leaves are black (LeaB)
4. If a node is red, then both children are black (BredB)
5. For each node all paths from the node to a leaf have the same number of black nodes (BH)

NIL Sentinel



Number of Nodes vs. Black-Height

Lemma 1:
Let $n(x)$ be the number of non-leaf nodes of a red-black subtree rooted at x. Then, $n(x) \geq 2^{bh(x)} - 1$.
Proof (by induction on height $h(x)$ of node x):

- $h(x) = 0$: x is a leaf. $bh(x) = 0$. $2^{bh(x)} - 1 = 0$. $n(x) \geq 0$. True.
- $h(x) > 0$: x is a non-leaf node. It has two children c_1 and c_2.
  If c_i is red, then $bh(c_i) = bh(x)$, else $bh(c_i) = bh(x) - 1$.
  Use assumption, since $h(c_i) < h(x)$,
  $n(c_i) \geq 2^{bh(c_i)} - 1 \geq 2^{bh(x)-1} - 1$.
  Thus,
  $n(x) = n(c_1) + n(c_2) + 1 \geq 2(2^{bh(x)-1} - 1) + 1 = 2^{bh(x)} - 1$.

Height vs. Black-Height

Lemma 2:
Let h be the height of a red-black tree with root r. Then, $bh(r) \geq h/2$.
Proof:

- Let $r, v_1, v_2, ..., v_h$ be the longest path in the tree.
- The number of black nodes in the path is $bh(r)$.
- Thus, the number of red nodes is $h - bh(r)$.
- Since v_h is black (LeaB property) and every red node in the path must be followed by a black one (BredB property), we have $h - bh(r) \leq bh(r)$.
- Hence, $bh(r) \geq h/2$.

Height of a Red-Black Tree

Theorem:
A red-black tree with n non-leaf nodes has height $h \leq 2\lg(n+1)$.
Proof:

- Lemma 1: $n \geq 2^{bh(r)} - 1$ (r being the root).
- Lemma 2: $bh(r) \geq h/2$.
- Thus, $n \geq 2^{h/2} - 1$.
- So, $h \leq 2\lg(n+1)$.

Corollary:
The height of a red-black tree is $O(\lg n)$.
All dynamic set operations can be performed in $O(\lg n)$, if we maintain the red-black tree properties.

Operations

- Querying
  - Search/Minimum & Maximum/Successor & Predecessor
  - Just as in normal BST
  - $O(\lg n)$
- Modifying
  - Tree-Insert/Tree-Delete $\rightarrow$ $O(\lg n)$
  - But, need to guarantee red-black tree properties:
    - must change color of some nodes
    - change pointer structure through rotation