CH-231-A

Algorithms and Data Structures

ADS

Lecture 36

Dr. Kinga Lipskoch

Spring 2020

Complexity Analysis (1)

$\Theta(V)$ total:

$Q \leftarrow V$
$key[v] \leftarrow \infty$ for all $v \in V$
$key[s] \leftarrow 0$ for some arbitrary $s \in V$

$|V|$ times:

while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)

$degree(u)$ times:

    for each $v \in Adj[u]$
      do if $v \in Q$ and $w(u, v) < key[v]$
        then $\boxed{key[v] \leftarrow w(u, v)}$
          $\pi[v] \leftarrow u$

$\Theta(E)$ implicit DECREASE-KEY's.

Notation $\Theta(V)$ means $\Theta(|V|)$.

Complexity Analysis (2)

$$\text{Time} = \Theta(V) \cdot T_{\text{EXTRACT-MIN}} + \Theta(E) \cdot T_{\text{DECREASE-KEY}}$$

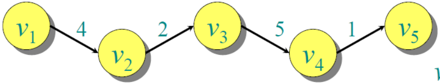
Q	$T_{\text{EXTRACT-MIN}}$	$T_{\text{DECREASE-KEY}}$	Total
min-heap	$O(\lg V)$	$O(\lg V)$	$O(E \lg V)$
array	$O(V)$	$O(1)$	$O(V^2)$

Definition: Path

- Consider a directed graph $G = (V, E)$, where each edge $e \in E$ is assigned a non-negative weight $w : E \to \mathbb{R}^+$.
- A path is a sequence of vertices in the graph, where two consecutive vertices are connected by a respective edge.
- The weight of a path $p = (v_1, \ldots, v_k)$ is defined by

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}).$$

- Example:



$w(p) = 12$

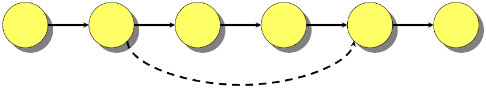
Definition: Shortest Path

- A shortest path from a vertex u to a vertex v in a graph G is a path of minimum weight.
- The weight of a shortest path from u to v is defined as $\delta(u, v) = \min\{w(p) : p \text{ is a path from } u \text{ to } v\}$.
- Note that $\delta(u, v) = \infty$, if no path from u to v exists.
- Why of interest?
  One example is finding a shortest route in a road network.

Optimal Substructure

Theorem:
A subpath of a shortest path is a shortest path.
Proof:

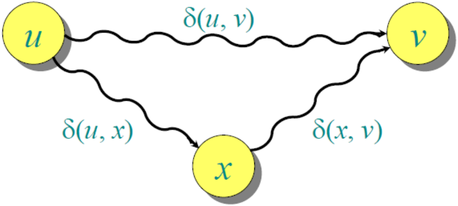
- Let $p = (v_1, ..., v_k)$ be a shortest path and $q = (v_i, ..., v_j)$ a subpath of p.
- Assume that q is not a shortest path.
- Then, there exists a shorter path from v_i to v_j than q.
- But then, there is also a shorter path from v_1 to v_k than p. Contradiction.



Triangle Inequality

Theorem:
For all $u, v, x \in V$, we have that $\delta(u, v) \leq \delta(u, x) + \delta(x, v)$.

Proof:



(Single-Source) Shortest Paths

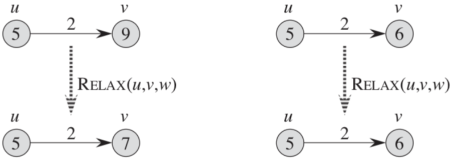
Problem:
Given a source vertex $s \in V$, find for all $v \in V$ the shortest-path weights $\delta(s, v)$.

Idea: Greedy approach.

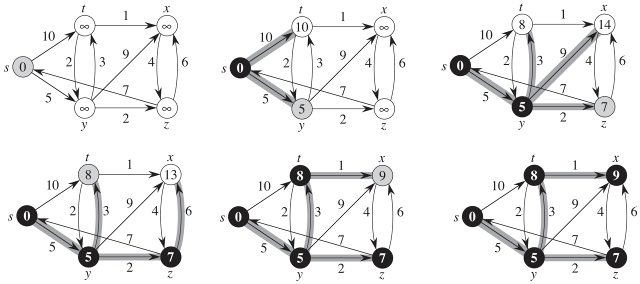
1. Maintain a set S of vertices whose shortest-path distances from s are known.
2. At each step, add to S the vertex $v \in V \setminus S$ whose distance estimate from s is minimal.
3. Update the distance estimates of vertices adjacent to v.

Dijkstra's Algorithm

```
d[s] := 0
for each v ∈ V\{s}
  d[v] := infinity
S := Ø
Q := V   // min-priority queue maintaining V \ S.
while Q != Ø
    u := Extract-Min(Q)
    S := S U {u}
    for each v ∈ Adj[u]
        if d[v] > d[u] + w(u,v)      // *****
        then d[v] := d[u] + w(u,v)   // Relaxation
            pi[v] := u                // *****
```



Example Dijkstra's Algorithm



```
while Q != Ø
    u := Extract-Min(Q)
    S := S U {u}
    for each v ∈ Adj[u]
        if d[v] > d[u] + w(u,v)
        then d[v] := d[u] + w(u,v)
            pi[v] := u
```

S = {s, y, z, t, x}

Correctness of Dijkstra's Algorithm

Correctness can be shown in 3 steps:

(i) $d[v] \geq \delta(s, v)$ at all steps (for all v)

(ii) $d[v] = \delta(s, v)$ after relaxation from u,

(iii) if (u, v) on shortest path (for all v) algorithm terminates with $d[v] = \delta(s, v)$

Correctness (i)

Lemma:

- Initializing $d[s] = 0$ and $d[v] = \infty$ for all $v \in V \setminus \{s\}$ establishes $d[v] \geq \delta(s, v)$ for all $v \in V$.
- This invariant is maintained over any sequence of relaxation steps.

Proof:

Suppose the Lemma is not true, then let v be the first vertex for which $d[v] < \delta(s, v)$ and let u be the vertex that caused $d[v]$ to change by $d[v] = d[u] + w(u, v)$. Then,

$$
\begin{aligned}
d[v] &< \delta(s, v) && \text{supposition} \\
&\leq \delta(s, u) + \delta(u, v) && \text{triangle inequality} \\
&\leq \delta(s,u) + w(u, v) && \text{sh. path} \leq \text{specific path} \\
&\leq d[u] + w(u, v) && v \text{ is first violation}
\end{aligned}
$$

Contradiction.

Correctness (ii)

Lemma:

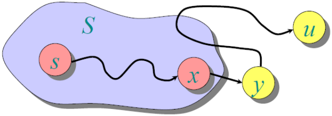
- Let u be v's predecessor on a shortest path from s to v.
- Then, if $d[u] = \delta(s, u)$, we have $d[v] = \delta(s, v)$ after the relaxation of edge (u, v).

Proof:

- Observe that $\delta(s, v) = \delta(s, u) + w(u, v)$.
- Suppose that $d[v] > \delta(s, v)$ before relaxation (else: done).
- Then, $d[v] > \delta(s, v) = \delta(s, u) + w(u, v) = d[u] + w(u, v)$ (if clause in the algorithm).
- Thus, the algorithm sets $d[v] = d[u] + w(u, v) = \delta(s, v)$.

Correctness (iii)

Theorem:
Dijkstra's algorithm terminates with $d[v] = \delta(s, v)$ for all $v \in V$.



Proof:

- It suffices to show that $d[v] = \delta(s, v)$ for every $v \in V$ when v is added to S.
- Suppose u is the first vertex added to S with $d[u] > \delta(s, u)$.
- Let y be the first vertex in $V \setminus S$ along the shortest path from s to u, and let x be its predecessor.
- Then, $d[x] = \delta(s, x)$ and $d[y] = \delta(s, y) \leq \delta(s, u) < d[u]$.
- But we chose u such that $d[u] \leq d[y]$. Contradiction.

Complexity Analysis

$|V|$ times:

while $Q \neq \emptyset$
  do $u \leftarrow \text{EXTRACT-MIN}(Q)$
    $S \leftarrow S \cup \{u\}$

$degree(u)$ times:

    for each $v \in Adj[u]$
      do if $d[v] > d[u] + w(u, v)$
        then $d[v] \leftarrow d[u] + w(u, v)$

- Similar to Prim's minimum spanning tree algorithm, we get the computation time

$$\Theta(V \cdot T_{\text{EXTRACT-MIN}} + E \cdot T_{\text{DECREASE-KEY}})$$

- Hence, depending on what data structure we use, we get the same computation times as for Prim's algorithm.

Unweighted Graphs

- Suppose that we have an unweighted graph, i.e., the weights $w(u, v) = 1$ for all $(u, v) \in E$.
- Can we improve the performance of Dijkstra's algorithm?
- Observation: The vertices in our data structure Q are processed following the FIFO principle.
- Hence, we can replace the min-priority queue with a queue.
- This leads to a breadth-first search.

BFS Algorithm

```
d[s] := 0
for each v ∊ V\{s}
  d[v] := infinity
Enqueue (Q,s)
while Q != Ø
  u := Dequeue(Q)
  for each v ∊ Adj[u]
      if d[v] = infinity
      then d[v] := d[u] + 1
           pi[v] :=u
           Enqueue(Q,v)
```

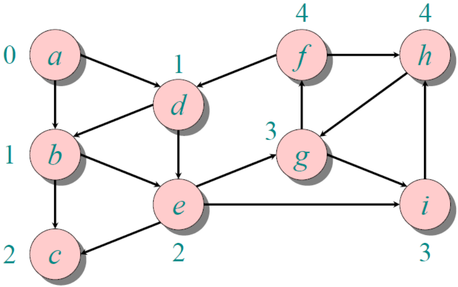
Analysis: BFS Algorithm

Correctness:

- The FIFO queue Q mimics the min-priority queue in Dijkstra's algorithm.
- Invariant:
  If v follows u in Q, then $d[v] = d[u]$ or $d[v] = d[u] + 1$.
- Hence, we always dequeue the vertex with smallest d.

Time complexity:
$O(|V| T_{Dequeue} + |E| T_{Enqueue}) = O(|V| + |E|)$

Example: BFS Algorithm



Q: a b d c e g i f h