MAT1830 Practice Exam

(1) Find the greatest common divisor of 504 and 385. $\boxed{}$ **[2]**

(2) Is there an integer y such that $504y \equiv 10 \pmod{385}$ and $0 \leqslant y < 385$? If there is, enter such an integer y. If there isn't, enter NA. $\boxed{}$ [1]

(3) Is there an integer z such that $504z \equiv 7 \pmod{385}$ and $0 \leqslant z < 385$? If there is, enter such an integer z. If there isn't, enter NA. $\boxed{}$ **[3]**

(4) Consider the statement "If $x \equiv 3 \pmod{12}$ and $y \equiv 7 \pmod{18}$, then $x + y \equiv 4 \pmod{6}$". Which of the following is true?

(A) The statement is true for all integers x and y.
(B) The statement is true for some integers x and y but not for all.
(C) The statement is false for all integers x and y.

[2]

(5) Consider the sentences $\neg((p \to \neg q) \wedge r)$ and $(p \wedge q) \vee \neg r$. There are 8 possible interpretations for these sentences according to the truth values of p, q and r. Under how many of these interpretations do the two sentences have the same truth value? $\boxed{}$ **[3]**

(6) Let A and B be sets. If p is the proposition "$x \in A$" and q is the proposition "$x \in B$", which of the following is logically equivalent to "$x \in A \cup B$"?

(A) $p \veebar q$
(B) $p \wedge q$
(C) $p \vee q$
(D) $p \leftrightarrow q$
(E) $p \wedge \neg q$

[1]

(7) Consider the sentence $\big(\exists x(P(x) \wedge Q(x))\big) \leftrightarrow \big((\exists x P(x)) \wedge (\exists x Q(x))\big)$. Which of the following is true?

(A) The sentence is valid.
(B) The sentence is false under the interpretation where x ranges over the integers, $P(x)$ is "$x \leqslant 0$" and $Q(x)$ is "$x \geqslant 6$".
(C) The sentence is false under the interpretation where x ranges over the integers, $P(x)$ is "$x \leqslant 6$" and $Q(x)$ is "$x \geqslant 0$".
(D) Both (B) and (C) are true.

(If (D) is true, you must answer (D) and not (B) or (C).) **[2]**

Under the interpretation where x and y range over the positive integers and $P(x, y)$ is "$x \leqslant y$", state whether each sentence is true or false.

(8) $\exists x \forall y P(x, y)$

True
False

[1]

(9) $\forall y \exists x P(x, y)$

True
False

[1]

Let P be the set of all prime numbers, let T be the set of all natural numbers which are divisible by 3, and let $A = \{3, 4, 5, 6\}$.

(10) What is $|P \cap T|$? $\boxed{}$ **[1]**

(11) What is $T \cap A$?

(A) $\{\{3, 4, 5, 6\}\}$
(B) $\{3, 4, 5, 6\}$
(C) $\{\{3, 6\}\}$
(D) $\{3, 6\}$
(E) T

[1]

(12) What is $T \cup \mathbb{N}$?

(A) T
(B) $\mathbb{N}$
(C) $\mathbb{Z}$
(D) P
(E) None of the above

[1]

(13) What is $|\mathcal{P}(A \times \{1, 2\})|$? $\boxed{}$ **[1]**

Let A be the set of all non-empty subsets of $\{1, 2, \ldots, 10\}$ and let f and g be the following functions.
$f : A \to \mathbb{N}$ defined by $f(X) = a - b$, where a is the largest element of X and b is the smallest element of X.
$g : A \to A$ defined by $g(X) = X \cup \{1, 2\}$.

(14) What is $f(\{2, 3, 6\})$? $\boxed{}$ **[1]**

(15) What is $|g(\{2, 7, 10\})|$? $\boxed{}$ **[1]**

(16) Is f onto? Is f one-to-one?

(A) yes, yes
(B) yes, no
(C) no, yes
(D) no, no

[1]

(17) Is g onto? Is g one-to-one?

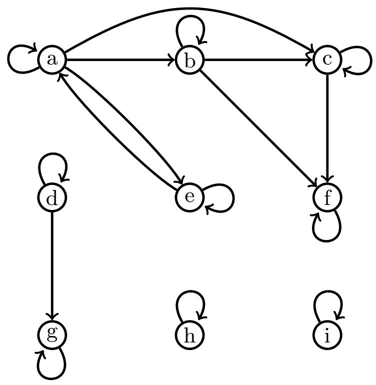
(A) yes, yes
(B) yes, no
(C) no, yes
(D) no, no

[1]

(18) If $f \circ g$ exists then enter $f \circ g(\{9\})$. If it doesn't exist, enter NA. **[1]**

Let R and S be binary relations defined as follows.

R is defined on $A = \{a, b, c, d, e, f, g, h, i\}$ by the following arrow diagram.



S is defined on $\mathbb{Z} \times \mathbb{Z}$ by $(w, x)S(y, z)$ if and only if $w + x - y - z$ is even.

(19) Is R reflexive? Is R symmetric?

- (A) yes, yes
- (B) yes, no
- (C) no, yes
- (D) no, no

[1]

(20) Is R antisymmetric? Is R transitive?

- (A) yes, yes
- (B) yes, no
- (C) no, yes
- (D) no, no

[1]

(21) Is S reflexive? Is S symmetric?

- (A) yes, yes
- (B) yes, no
- (C) no, yes
- (D) no, no

[1]

(22) Is S antisymmetric? Is S transitive?

- (A) yes, yes
- (B) yes, no
- (C) no, yes
- (D) no, no

[1]

(23) Which of the following is true of S?

- (A) S is not an equivalence relation.
- (B) S is an equivalence relation and one of its equivalence classes is $\{(w, x) : (w, x) \in \mathbb{Z} \times \mathbb{Z}$ and $w + x$ is odd$\}$.
- (C) S is an equivalence relation and one of its equivalence classes is $\{(w, x) : (w, x) \in \mathbb{Z} \times \mathbb{Z}$ and both w and x are odd$\}$.

(D) S is an equivalence relation and one of its equivalence classes is $\{(w, x) : (w, x) \in \mathbb{Z} \times \mathbb{Z}$ and both w and x are even$\}$.

(E) S is an equivalence relation and one of its equivalence classes is $\{(w, x) : (w, x) \in \mathbb{Z} \times \mathbb{Z}$ and at least one of w and x is even$\}$. **[2]**

For $n \geqslant 1$, let s_n be the number of ways of writing n as a sum of 1s, 3s and 4s (order being important). For example, $s_4 = 4$ because 4 can be written in four ways:

$$1+1+1+1, \quad 1+3, \quad 3+1, \quad 4.$$

(24) What is s_3? $\boxed{}$ **[1]**

(25) Which of the following is a recurrence for s_n that holds for all integers $n \geqslant 5$?

(A) $s_n = s_{n-1} + s_{n-2} + s_{n-3} - s_{n-4}$

(B) $s_n = s_{n-1} + s_{n-2} + s_{n-4}$

(C) $s_n = s_{n-1} + s_{n-3} + s_{n-4}$

(D) $s_n = 2s_{n-2} + s_{n-3} + s_{n-4}$

(E) $s_n = s_{n-1} + s_{n-2}$

[2]

Let X_1 and X_2 be independent random variables that are each selected uniformly at random from the set $\{1, 2, 3\}$. Let $Y = \max(X_1, X_2)$.

(26) What is $\Pr(X_1 = 2 \text{ and } Y = 2)$? $\boxed{}$ **[1]**

(27) What is $\mathrm{E}[Y]$? $\boxed{}$ **[2]**

(28) What would $\mathrm{E}[Y]$ be if X_1 and X_2 were instead selected uniformly at random from the set $\{1, 2, \ldots, 100\}$?

(A) $\frac{1}{10000} \sum_{i=1}^{100} i^2$

(B) $\frac{1}{10000} \sum_{i=1}^{100} i(i-1)$

(C) $\frac{1}{10000} \sum_{i=1}^{100} i(2i-1)$

(D) $\frac{1}{100} \sum_{i=1}^{100} i^2$

(E) $\frac{1}{100} \sum_{i=1}^{100} i(i-1)$

[2]

The number of calls received by a call center forms a Poisson distribution. An average of 3 call per minute are received.
(Remember that if X is a Poisson random variable with $E(X) = \lambda$, then $\Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$.)

(29) Which of the following is the probability that no calls come in a certain one minute period?

(A) $3e^{-3}$
(B) e^{-3}
(C) 0
(D) $\frac{1}{3}e^{-3}$
(E) $\frac{1}{3}e$

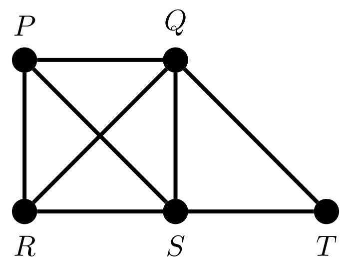
[1]

(30) Which of the following is the probability that at least two calls will arrive in a certain two minute period?

(A) 1
(B) $4e^{-3}$
(C) $7e^{-6}$
(D) $1 - 4e^{-3}$
(E) $1 - 7e^{-6}$

[2]

Consider the following graph.



(31) How many edges does the graph have? $\boxed{}$ **[1]**

(32) Which of the following is true?

(A) The graph has a closed Euler trail.
(B) If the edge PR is removed the resulting graph has a closed Euler trail.
(C) If the edge PT is added the resulting graph has a closed Euler trail.
(D) (A) and (B) are both true
(E) (A) and (C) are both true
(F) (B) and (C) are both true

(If (D), (E) or (F) are true you must give one of them as your answer and not (A), (B) or (C).) **[1]**

(33) How many spanning trees of the graph contain both the edges QS and RS? $\boxed{}$ **[1]**

Consider a graph whose adjacency matrix is

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

where V_1, V_2 V_3 and V_4 are the vertices corresponding to row/column 1,2,3 and 4 respectively.

(34) What is the degree of the vertex V_4? $\boxed{}$ **[1]**

(35) Which of the following gives the number of walks of length 6 starting from V_3 in the graph?

(A) The entry in row 3 and column 2 of M
(B) The entry in row 3 and column 2 of M^6
(C) The sum of the entries in row 3 of M
(D) The sum of the entries in row 3 of M^6
(E) The entry in row 3 and column 2 of M^3

[1]

(36) Does there exist a simple graph with 100 vertices and 98 edges? If so, what is the maximum number of spanning trees such a graph can contain?

(A) Such a graph doesn't exist.
(B) Such a graph exists. No such graph contains a spanning tree.
(C) Such a graph exists. The maximum number of spanning trees such a graph can have is 1.
(D) Such a graph exists. The maximum number of spanning trees such a graph can have is 2.
(E) Such a graph exists. The maximum number of spanning trees such a graph can have is 3 or more.

[1]

(37) Which of the following is true?

(A) Every simple graph with no odd degree vertices has a closed Euler trail.
(B) Some, but not all, simple graphs with no odd degree vertices have closed Euler trails.
(C) Every simple graph with no odd degree vertices does not have a closed Euler trail.

[1]

(38) Which of the following sequences can be degrees of the vertices of a simple graph (on 5 vertices)?

(A) $(5, 3, 2, 2, 2)$
(B) $(4, 2, 2, 2, 0)$
(C) $(2, 2, 2, 2, 2)$
(D) Both (A) and (B)
(E) Both (A) and (C)
(F) Both (B) and (C)

(If (D), (E) or (F) are true you must give one of them as your answer and not (A), (B) or (C).) **[1]**

(39) Which of the following sequences can be degrees of a tree (on 6 vertices)?

(A) $(2, 2, 1, 1, 1, 1)$
(B) $(3, 2, 2, 1, 1, 1)$
(C) $(2, 2, 2, 2, 2, 2)$
(D) Both (A) and (B)
(E) Both (A) and (C)
(F) Both (B) and (C)

(If (D), (E) or (F) are true you must give one of them as your answer and not (A), (B) or (C).) **[1]**

(40) Which of the following is true for all integers n and r such that $n > r \geqslant 1$?

(A) $\binom{n}{r} = \frac{n}{n-r-1}\binom{n-1}{r}$
(B) $\binom{n}{r} = \frac{n-r-1}{n}\binom{n-1}{r}$
(C) $\binom{n}{r} = \frac{n}{n-r}\binom{n-1}{r}$
(D) $\binom{n}{r} = \frac{n-r}{n}\binom{n-1}{r}$
(E) None of the above is true.

[1]

(41) How many ternary strings (that is, strings made up of 0s, 1s and 2s) of length 5 contain at most two 0s, at most two 1s and at most two 2s? $\boxed{}$ **[3]**

(42) Prove using simple induction that 7 divides $19^n - 12^n$ for all integers $n \geqslant 1$.

Write your answer on paper (in a real exam you would get a chance to upload a photo of your work after the exam). **[5]**