MAT1830

Lecture 11: Sets

Sets - why should you care?

- Sets are an important data structure when programming.
- Sets are very important concepts CS and maths.
- Set notation is used a lot in writing about CS and maths.
- The standard approach to building maths up from logic is based on sets. (Caring optional here.)

Sets are vital in expressing mathematics formally and are also very important data structures in computer science.

A set is basically just an unordered collection of distinct objects, which we call its *elements* or *members*. Note that there is no notion of order for a set, even though we often write down its elements in some order for convenience. Also, there is no notion of multiplicity: an object is either in a set or not – it cannot be in the set multiple times.

Sets A and B are equal when every element of A is an element of B and vice-versa.

11.1 Set notation

- $x \in S$ means x is an element of set S.
- $\{x_1, x_2, x_3, \ldots\}$ is the set with elements $x_1, x_2, x_3, \ldots$.
- $\{x : P(x)\}$ is the set of all x with property P.

Example.
$17 \in \{x : x \text{ is prime}\} = \{2, 3, 5, 7, 11, 13, \ldots\}$
$\{1, 2, 3\} = \{3, 1, 2\}$
$\{1, 1, 1\} = \{1\}$

Sometimes it is more convenient to use a slight variation on the colon notation given above. For a set S and a property P, we sometimes write $\{x \in S : P(x)\}$ instead of $\{x : x \in S \text{ and } P(x)\}$.

For a finite set S, we write $|S|$ for the number of elements of S.

$\{\text{eraser}, \text{pen}, \{\text{stapler}, \{\}\}\}$


WHAT ARE YOU DOING?
MAKING THE WORLD A WEIRDER PLACE.
MRROWLLL
STARTING WITH MY EBAY FEEDBACK PAGE.
COMMENTS:
INSTEAD OF OFFICE CHAIR PACKAGE CONTAINED BOBCAT.
WOULD NOT BUY AGAIN.

Questions Let $S = \{a, \{a\}, \{b\}, \{a, b, c\}\}$.

Is $a \in S$? Yes

Is $b \in S$? No

Flux Exercise

Is $\{a\} \in S$? Is $\{a, b\} \in S$?

A. Yes, yes
B. Yes, no
C. No, yes
D. No, no

Questions

Let $R = \{a, b, c\}$. What is $|R|$? 3

Let $S = \{a, \{a\}, \{b\}, \{a, b, c\}\}$. What is $|S|$? 4

Let $T = \{0, 1, 2, \ldots, 100\}$. What is $|T|$? 101

What is $|\{\}|$? 0

Question 11.1

$E(x)$: "x is even"
$F(x)$: "5 divides x"
(Assume we're working in the integers ≥ 0.)

What is the set $\{x : E(x) \land F(x)\}$?

The set containing all multiples of 10, that is $\{0, 10, 20, 30, \ldots\}$.

Write a formula for the set $\{5, 15, 25, 35, \ldots\}$.

$\{x : \neg E(x) \land F(x)\}$.

11.2 Universal set

The idea of a "set of all sets" leads to logical difficulties. Difficulties are avoided by always working within a local "universal set" which includes only those objects under consideration.

For example, when discussing arithmetic it might be sufficient to work just with the numbers $0, 1, 2, 3, \ldots$. Our universal set could then be taken as

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\},$$

and other sets of interest, e.g. $\{x : x \text{ is prime}\}$, are parts of $\mathbb{N}$.

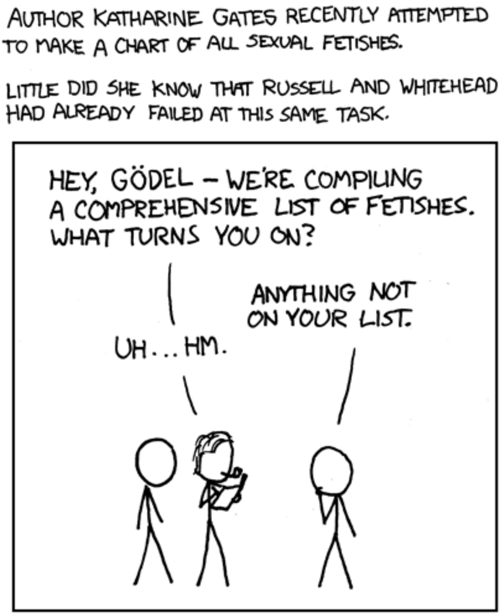
Russel's paradox (* not assessable)

Cantor's Set Building Rule For every property $P(x)$ there exists a set $\{x : P(x)\}$.

IS WRONG.

Russel's Paradox Consider the set $R = \{x : x \notin x\}$. Is $R \in R$?

If $R \in R$ then $R \notin R$ by definition of R.
If $R \notin R$ then $R \in R$ by definition of R.


AUTHOR KATHARINE GATES RECENTLY ATTEMPTED TO MAKE A CHART OF ALL SEXUAL FETISHES.
LITTLE DID SHE KNOW THAT RUSSELL AND WHITEHEAD HAD ALREADY FAILED AT THIS SAME TASK.
HEY, GÖDEL – WE'RE COMPILING A COMPREHENSIVE LIST OF FETISHES. WHAT TURNS YOU ON?
UH... HM.
ANYTHING NOT ON YOUR LIST.

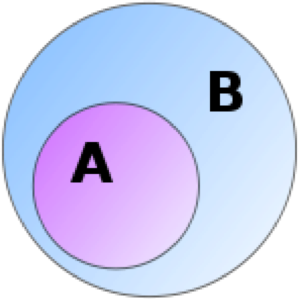
Important sets

$\mathbb{N}$	natural numbers	$\{0, 1, 2, \ldots\}$
$\mathbb{Z}$	integers	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Q}$	rational numbers	$\{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$
$\mathbb{R}$	real numbers	
$\emptyset$	empty set	$\{\}$

11.3 Subsets

We say that A is a *subset* of B and write $A \subseteq B$ when each element of A is an element of B.

Example. The set of primes forms a *subset* of $\mathbb{N}$, that is $\{x : x \text{ is prime}\} \subseteq \mathbb{N}$.



Formally, $A \subseteq B$ if $\forall x(x \in A \rightarrow x \in B)$.

That is, A is a subset of B if every element of A is also an element of B.

Notes:
Every set is a subset of itself.
$\{\}$ is a subset of every set.

Questions Let $S = \{a, \{a\}, \{b\}, \{a, b, c\}\}$.

Is $\{b\} \subseteq S$? No, because $b \notin S$.

Is $\{a, \{a, b, c\}\} \subseteq S$? Yes, because $a \in S$ and $\{a, b, c\} \in S$.

Flux Exercise

Is $\{a, b\} \subseteq S$? Is $\{\{a\}, \{b\}\} \subseteq S$?

A. Yes, yes
B. Yes, no
C. No, yes
D. No, no

11.4 Characteristic functions

A subset A of B can be specified by its *characteristic function* χ_A, which tells which elements of B are in A and which are not.

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Example. The subset $A = \{a, c\}$ of $B = \{a, b, c\}$ has the characteristic function χ_A with

$$\chi_A(a) = 1, \quad \chi_A(b) = 0, \quad \chi_A(c) = 1.$$

We also write this function more simply as

$$\begin{array}{ccc} a & b & c \\ 1 & 0 & 1 \end{array}$$

In fact we can list all characteristic functions on $\{a, b, c\}$, and hence all subsets of $\{a, b, c\}$, by listing all sequences of three binary digits:

characteristic function			subset
a	b	c	
0	0	0	$\{\}$
0	0	1	$\{c\}$
0	1	0	$\{b\}$
0	1	1	$\{b, c\}$
1	0	0	$\{a\}$
1	0	1	$\{a, c\}$
1	1	0	$\{a, b\}$
1	1	1	$\{a, b, c\}$

We could similarly list all the subsets of a four-element set, and there would be $2^4 = 16$ of them, corresponding to the 2^4 sequences of 0s and ls.

In the same way, we find that an n-element set has 2^n subsets, because there are 2^n binary sequences of length n. (Each of the n places in the sequence can be filled in two ways.)

11.5 Power set

The set of all subsets of a set U is called the *power set* $\mathcal{P}(U)$ of U.

Example. We see from the previous table that $\mathcal{P}(\{a, b, c\})$ is the set

$$\{\{\}, \{c\}, \{b\}, \{b, c\}, \{a\}, \{a, c\}, \{a, b\}, \{a, b, c\}\}.$$

If U has n elements, then $\mathcal{P}(U)$ has 2^n elements.

(The reason $\mathcal{P}(U)$ is called the "power" set is probably that the number of its elements is this power of 2. In fact, the power set of U is sometimes written 2^U.)

Question 11.2 How many subsets does $\{2, 5, 10, 20\}$ have?

$2^4 = 16$

Question What is $\mathcal{P}(\{2, 5, 10, 20\})$?

$$
\begin{aligned}
&\{ \quad \{\}, \\
&\{2\}, \{5\}, \{10\}, \{20\}, \\
&\{2, 5\}, \{2, 10\}, \{2, 20\}, \{5, 10\}, \{5, 20\}, \{10, 20\}, \\
&\{2, 5, 10\}, \{2, 5, 20\}, \{2, 10, 20\}, \{5, 10, 20\}, \\
&\{2, 5, 10, 20\} \quad \}.
\end{aligned}
$$

11.6 Sets and properties

We mentioned at the beginning that $\{x : P(x)\}$ stands for the set of objects x with property P. Thus sets correspond to properties.

Properties of the natural numbers $0, 1, 2, 3, \ldots$, for example, correspond to subsets of the set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Thus the subset

$$\{0, 2, 4, 6, \ldots\} = \{n \in \mathbb{N} : n \text{ is even}\},$$

corresponds to the property of being even. Similarly, the set

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots\}$$

corresponds to the property of being prime. The power set $\mathcal{P}(\mathbb{N})$ corresponds to all possible properties of natural numbers.

Question 11.3 Consider the sets

$\{x : 0 < x < 1\}$
$\{x : 0 < x < \frac{1}{2}\}$
$\{x : 0 < x < \frac{1}{3}\}$
$\{x : 0 < x < \frac{1}{4}\}$
$\vdots$

Do they have an element in common?

No.

Suppose they had r in common. Then $r > 0$, so there is a (big) natural number n such that $r > \frac{1}{n}$. But then r is not in the set $\{x : 0 < x < \frac{1}{n}\}$. Contradiction.

11.7* What are numbers?

"Everything is a set" in mathematics. This claim can illustrated by defining the numbers $0, 1, 2, 3, \ldots$ as particular sets, starting with the empty set. This definition is due to von Neumann.

$$
\begin{array}{rcl}
0 & = & \{\} \\
1 & = & \{0\} \\
2 & = & \{0, 1\} \\
 & \vdots & \\
n + 1 & = & \{0, 1, 2, \ldots, n\}
\end{array}
$$

We are not going to use this definition in this course. Still, it is interesting that numbers *can* be defined in such a simple way.

(* not assessable)