MAT1830

Lecture 19: Selections and arrangements

Counting - why should you care?

Counting is really important in computer science for

- working out the run-time of algorithms
- working out memory/disk space requirements of processes
- designing systems and protocols
- understanding probability and random algorithms.

Two basic counting rules

Addition principle: if there are a ways of doing something and b ways of doing another thing, then there are $a + b$ ways to do one of the two things.

Multiplication principle: if there are a ways of doing something and b ways of doing another thing, then there are $a \times b$ ways to do both things.

Example. If a shelf contains 20 fiction books and 6 non-fiction books then there are

- $20 + 6 = 26$ ways to choose a book from the shelf
- $20 \times 6 = 120$ ways to choose a fiction book and a non-fiction book from the shelf.

19.1 Ordered selections without repetition

A reviewer is going to compare ten phones and list, in order, a top three. In how many ways can she do this? More generally, how many ways are there to arrange r objects chosen from a set of n objects?

In our example, the reviewer has 10 options for her favourite, but then only 9 for her second-favourite, and 8 for third-favourite. So there are $10 \times 9 \times 8$ ways she could make her list.

For an ordered selection without repetition of r elements from a set of n elements there are

$$\begin{array}{rl} n & \text{options for the 1st element} \\ n-1 & \text{options for the 2nd element} \\ n-2 & \text{options for the 3rd element} \\ \vdots & \qquad\qquad\vdots \\ n-r+1 & \text{options for the } r\text{th element.} \end{array}$$

So we have the following formula.

> The number of ordered selections without repetition of r elements from a set of n elements ($0 \leqslant r \leqslant n$) is
>
> $$n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

When $r = n$ and all the elements of a set S are ordered, we just say that this is a *permutation of S*. Our formula tells us there are $n!$ such permutations. For example, there are $3! = 6$ permutations of the set $\{a, b, c\}$:

$$(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a).$$

Question There are 7 athletes in an olympic event. How many ways can the gold, silver and bronze medals be awarded?

Answer There are 7 people the gold could go to, after that there are 6 the silver could go to and then 5 the bronze could go to.
So $7 \times 6 \times 5 = 210$. (Or equivalently $\frac{7!}{(7-3)!} = 210$.)

Question There are 6 kids at a party and I have 6 different presents. In how many ways can I give one present to each kid?

Answer Imagine the kids in a line. Then the question is really asking how many ways I can arrange the 6 presents in a line.
So $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$.

19.2 Unordered selections without repetition

What if our reviewer instead chose an unordered top three? In how many ways could she do that? More generally, how many ways are there to choose (without order) r objects from a set of n objects?

> A *combination* of r elements from a set S is a subset of S with r elements.

The ordered selections of 3 objects from a set of 4 objects:

$$
\begin{array}{cccccc|c}
(a,b,c) & (a,c,b) & (b,a,c) & (b,c,a) & (c,a,b) & (c,b,a) & \{a,b,c\} \\
(a,b,d) & (a,d,b) & (b,a,d) & (b,d,a) & (d,a,b) & (d,b,a) & \{a,b,d\} \\
(a,c,d) & (a,d,c) & (c,a,d) & (c,d,a) & (d,a,c) & (d,c,a) & \{a,c,d\} \\
(b,c,d) & (b,d,c) & (c,b,d) & (c,d,b) & (d,b,c) & (d,c,b) & \{b,c,d\}
\end{array}
$$

These are written so each row corresponds to an unordered selection.

Each row has to have $3! = 6$ ordered arrangements in it.

And we know there are $4 \times 3 \times 2 = 24$ ordered arrangements in total.

So there must be $\frac{4\times3\times2}{3!} = 4$ rows.

In general this logic shows that we can arrange the $\frac{n!}{(n-r)!}$ ordered selections of r objects from a set of n objects into rows, each with $r!$ ordered selections in it, so that each row corresponds to an unordered selection.

So there must be $\frac{n!}{r!(n-r)!}$ unordered selections.

For every unordered list our reviewer could make there are $3! = 6$ corresponding possible ordered lists. And we've seen that she could make $10 \times 9 \times 8$ ordered lists. So the number of unordered lists she could make is $\frac{10\times9\times8}{6}$.

For every combination of r elements from a set of n elements there are $r!$ corresponding permutations. So, using our formula for the number of permutations we have the following.

> The number of combinations of r elements from a set of n elements ($0 \leqslant r \leqslant n$) is
> $$\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

Notice that the notation $\binom{n}{r}$ is used for $\frac{n!}{r!(n-r)!}$. Expressions like this are called *binomial coefficients*. We'll see why they are called this in the next lecture.

Question How many ways are there to pick a team of 4 from a group of 9 employees?

Answer This is an unordered selection of 4 people from a set of 9 people.
So $\binom{9}{4} = \frac{9!}{4! \times 5!} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 126.$

Flux Exercise

Elodie buys a pizza from a shop by choosing exactly 4 toppings from a list of 11 options. How many ways can she do this? (Assume "double" toppings aren't allowed)

A. $\frac{11!}{7!}$

B. $11!$

C. $\binom{11}{4}$

D. $\frac{11!}{4!}$

Answer It's reasonable to assuming that Elodie doesn't care about the order in which the toppings are put on the pizza. So this is an unordered section without repetition. So she has

$$\binom{11}{4} = \frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = 330$$

options. So C.

19.3 Ordered selections with repetition

An ordered selection of r elements from a set X is really just a sequence of length r with each term in X. If X has n elements, then there are n possibilities for each term and so:

The number of sequences of r terms, each from some set of n elements, is

$$\underbrace{n \times n \times \cdots \times n}_{r} = n^r.$$

Note: for selections with repetition we might have $r > n$.

Questions

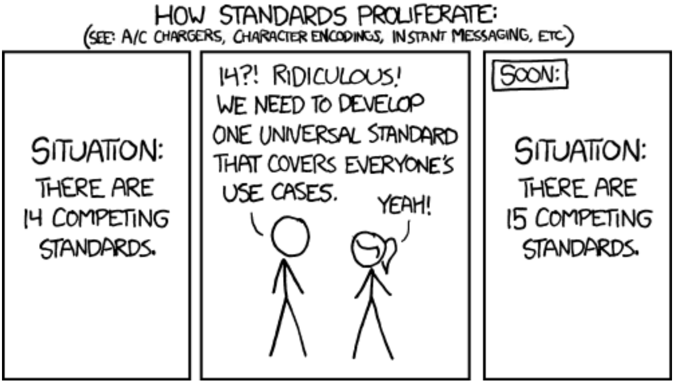
19.1 A bank requires a PIN that is a string of four decimal digits. How many such PINs are there? How many are made of four different digits?

A PIN is an ordered selection with repetition of four elements from the set $\{0, \ldots, 9\}$. So there are $10^4 = 10000$ possible PINs.

A PIN with four *different* digits is an ordered selection without repetition of four elements from the set $\{0, \ldots, 9\}$. So there are $\frac{10!}{6!} = 5040$ possible PINs with four different digits.

19.2 How many binary strings of length 5 are there? How many of these contain exactly two 1s?

There are $2^5 = 32$ binary strings of length 5.
There are $\binom{5}{2} = 10$ that contain exactly two 1s.
(To find a binary string of length 5 with two 1s we must choose two of the five positions for the 1s to appear in. While the string is ordered the selection of these two positions is unordered.)


HOW STANDARDS PROLIFERATE:
(SEE: A/C CHARGERS, CHARACTER ENCODINGS, INSTANT MESSAGING, ETC.)
SITUATION:
THERE ARE 14 COMPETING STANDARDS.
14?! RIDICULOUS! WE NEED TO DEVELOP ONE UNIVERSAL STANDARD THAT COVERS EVERYONE'S USE CASES.
YEAH!
SOON:
SITUATION:
THERE ARE 15 COMPETING STANDARDS.

19.4 Unordered selections with repetition

A shop has a special deal on any four cans of soft drink. Cola, lemonade and sarsaparilla flavours are available. In how many ways can you select four cans?

We can write a selection in a table, for example,

$$\begin{array}{c|c|c} \mathrm{C} & \mathrm{L} & \mathrm{S} \\ \hline \bullet & \bullet\bullet & \bullet \end{array} \quad \text{and} \quad \begin{array}{c|c|c} \mathrm{C} & \mathrm{L} & \mathrm{S} \\ \hline & \bullet & \bullet\bullet\bullet \end{array}.$$

We can change a table like this into a string of zeroes and ones, by moving from left to right reading a "$\bullet$" as a 0 and a column separator as a 1. The tables above would be converted into

$$0\,1\,0\,0\,1\,0 \quad \text{and} \quad 1\,0\,1\,0\,0\,0$$

Notice that each string has four zeroes (one for each can selected) and two ones (one fewer than the number of flavours). We can choose a string like this by beginning with a string of six ones and then choosing four ones to change to zeroes. There are $\binom{6}{4}$ ways to do this and so there are $\binom{6}{4}$ possible can selections.

An unordered selection of r elements, with repetition allowed, from a set X of n elements can be thought of as a multiset with r elements, each in X. As in the example, we can represent each such multiset with a string of r zeroes and $n-1$ ones. We can choose a string like this by beginning with a string of $n+r-1$ ones and then choosing r ones to change to zeroes.

> The number of multisets of r elements, each from a set of n elements, is
> $$\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!(n-1)!}.$$

A multiset is like a set, except we allow elements to appear more than once and we keep track of how many times they occur. Like with a normal set, order is unimportant.

Questions

19.3 In a game, each of ten players holds red, blue and green marbles, and places one marble in a bag. How many possibilities are there for the colours of marbles in the bag? If each player chooses their colour at random are all of these possibilities equally likely?

This is an unordered selection with repetition of 10 marbles chosen from 3 colors. So there are $\binom{n+r-1}{r} = \binom{3+10-1}{10} = \binom{12}{10} = \frac{12!}{10!2!} = 66$ possibilities.

Remember: for selections with repetition we might have $r > n$.

The possibilities are *not* equally likely. For example, the "all red" case is less likely than having 3 red, 3 blue and 4 green.

This is because the "all red" can only happen by choosing RRRRRRRRR whereas 3 red, 3 blue and 4 green can happen by picking RRRBBBGGGG or RBRBRBGGGG or GRBGRBGRBG or GBRGGBRBRG, etc

Flux Exercise

Sam buys 3 pizzas from a shop that sells 7 different pizzas. How may ways can she do this?

A. $\binom{9}{3}$

B. $\binom{10}{3}$

C. $\frac{7!}{3!}$

D. $\binom{7}{3}$

Hint She might buy two supreme and one meatlovers, say.

Answer The same pizza may be picked more than once, but the order doesn't matter. So this is an unordered section with repetition. There are

$$\binom{7+3-1}{3} = \binom{9}{3} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84$$

options. So A.

19.5 The pigeonhole principle

The pigeonhole principle is a reasonably obvious statement, but can still be very useful.

> If n items are placed in m containers with $n > m$, then at least one container has at least two items.

Example. If a drawer contains only blue, black and white socks and you take out four socks without looking at them, then you are guaranteed to have two of the same colour.

We can generalise the pigeonhole principle as follows.

> If n items are placed in m containers, then at least one container has at least $\lceil \frac{n}{m} \rceil$ items.

In the above $\lceil \frac{n}{m} \rceil$ means the smallest integer greater than or equal to $\frac{n}{m}$ (or $\frac{n}{m}$ "rounded up").

Example. If 21 tasks have been distributed between four processor cores, the busiest core must have been assigned at least 6 tasks.