MAT1830 Practice Exam

(1) Find the greatest common divisor of 504 and 385. $\boxed{}$ **[2]**

ANS: $\boxed{7}$ Calculate

$$\begin{array}{rcrcrcr} 504 &=& 1 &\times& 385 &+& 119\\ 385 &=& 3 &\times& 119 &+& 28\\ 119 &=& 4 &\times& 28 &+& 7\\ 28 &=& 4 &\times& 7 &+& 0 \end{array}$$

So $\gcd(504, 385) = 7$.

(2) Is there an integer y such that $504y \equiv 10 \pmod{385}$ and $0 \leqslant y < 385$? If there is, enter such an integer y. If there isn't, enter NA. $\boxed{}$ **[1]**

ANS: $\boxed{\text{NA}}$ If there was an integer y such that $504y \equiv 10 \pmod{385}$, then we would have, for some integer k, $504y = 385k + 10$ or, equivalently, $504y - 385k = 10$. This is impossible because we have just seen that 7 divides both 504 and 385, but obviously 7 does not divide 10.

(3) Is there an integer z such that $504z \equiv 7 \pmod{385}$ and $0 \leqslant z < 385$? If there is, enter such an integer z. If there isn't, enter NA. $\boxed{}$ **[3]**

ANS: $\boxed{13}$ Calculate

$$\begin{array}{rcrcr} 7 &=& 119 - (4 \times 28) && \\ 7 &=& 119 - (4 \times (385 - 3 \times 119)) &=& -4 \times 385 + 13 \times 119\\ 7 &=& -4 \times 385 + 13 \times (504 - 1 \times 385) &=& 13 \times 504 - 17 \times 385 \end{array}$$

So $504 \times 13 + 385 \times -17 = 7$ or, equivalently, $504 \times 13 = 385 \times 17 + 7$. So for $z = 13$ we have $504z \equiv 7 \pmod{385}$.

(4) Consider the statement "If $x \equiv 3 \pmod{12}$ and $y \equiv 7 \pmod{18}$, then $x + y \equiv 4 \pmod{6}$". Which of the following is true?

(A) The statement is true for all integers x and y.

(B) The statement is true for some integers x and y but not for all.

(C) The statement is false for all integers x and y. **[2]**

ANS: $\boxed{\text{(A)}}$ Suppose $x \equiv 3 \pmod{12}$ and $y \equiv 7 \pmod{18}$.
Because $x \equiv 3 \pmod{12}$, we have that $x = 12k + 3$ for some integer k.
Because $y \equiv 7 \pmod{18}$, we have that $y = 18l + 7$ for some integer l.
So $x + y = 12k + 3 + 18l + 7 = 6(2k + 3l) + 10 = 6(2k + 3l + 1) + 4$. Because $2k + 3l + 1$ is an integer, this means that $x + y \equiv 4 \pmod{6}$.

(5) Consider the sentences $\neg((p \to \neg q) \wedge r)$ and $(p \wedge q) \vee \neg r$. There are 8 possible interpretations for these sentences according to the truth values of p, q and r. Under how many of these interpretations do the two sentences have the same truth value? $\boxed{}$ **[3]**

ANS: $\boxed{8}$ Each line of the truth table below corresponds to one interpretation.

p	q	r	$\neg q$	$p \to \neg q$	$(p \to \neg q) \wedge r$	$\neg((p \to \neg q) \wedge r)$	$p \wedge q$	$\neg r$	$(p \wedge q) \vee \neg r$
T	T	T	F	F	F	T	T	F	T
T	T	F	F	F	F	T	T	T	T
T	F	T	T	T	T	F	F	F	F
T	F	F	T	T	F	T	F	T	T
F	T	T	F	T	T	F	F	F	F
F	T	F	F	T	F	T	F	T	T
F	F	T	T	T	T	F	F	F	F
F	F	F	T	T	F	T	F	T	T

The columns for $\neg((p \to \neg q) \wedge r)$ and $(p \wedge q) \vee \neg r$ agree in all 8 rows.

(6) Let A and B be sets. If p is the proposition "$x \in A$" and q is the proposition "$x \in B$", which of the following is logically equivalent to "$x \in A \cup B$"?

(A) $p \veebar q$

(B) $p \wedge q$

(C) $p \vee q$

(D) $p \leftrightarrow q$

(E) $p \wedge \neg q$

[1]

ANS: $\boxed{\text{(C)}}$

(7) Consider the sentence $\big(\exists x(P(x) \wedge Q(x))\big) \leftrightarrow \big((\exists x P(x)) \wedge (\exists x Q(x))\big)$. Which of the following is true?

(A) The sentence is valid.

(B) The sentence is false under the interpretation where x ranges over the integers, $P(x)$ is "$x \leqslant 0$" and $Q(x)$ is "$x \geqslant 6$".

(C) The sentence is false under the interpretation where x ranges over the integers, $P(x)$ is "$x \leqslant 6$" and $Q(x)$ is "$x \geqslant 0$".

(D) Both (B) and (C) are true.

(If (D) is true, you must answer (D) and not (B) or (C).) **[2]**

ANS: $\boxed{\text{(B)}}$ The sentence is false under the interpretation given in (B) because $\exists x(P(x) \wedge Q(x))$ is false (there does not exist a single integer that is at most 0 and at least 6) but $(\exists x P(x)) \wedge (\exists x Q(x))$ is true (there does exist an integer that is at most 0 and there does exist an integer which is at least 6). Since the sentence is false under some interpretation (A) cannot be true.

The sentence is true under the interpretation given in (C) because $\exists x(P(x) \wedge Q(x))$ is true (there is a single integer that is at least 0 and at most 6, for example 3) and $(\exists x P(x)) \wedge (\exists x Q(x))$ is true (there does exist an integer that is at least 0 and there does exist an integer which is at most 6). So (C) is wrong.

Under the interpretation where x and y range over the positive integers and $P(x, y)$ is "$x \leqslant y$", state whether each sentence is true or false.

(8) $\exists x \forall y P(x, y)$

True

False

[1]

ANS: $\boxed{\text{True}}$ Because there is a positive integer, namely 1, which is less than or equal to every positive integer.

(9) $\forall y \exists x P(x, y)$

True

False

[1]

ANS: $\boxed{\text{True}}$ Because for any positive integer, say y, there is a positive integer, for example $x = y$, which is less than or equal to y.

Let P be the set of all prime numbers, let T be the set of all natural numbers which are divisible by 3, and let $A = \{3, 4, 5, 6\}$.

(10) What is $|P \cap T|$? $\boxed{}$ **[1]**

ANS: $\boxed{1}$ $|P \cap T| = |\{3\}| = 1$

(11) What is $T \cap A$?

(A) $\{\{3, 4, 5, 6\}\}$

(B) $\{3, 4, 5, 6\}$

(C) $\{\{3, 6\}\}$

(D) $\{3, 6\}$

(E) T

[1]

ANS: $\boxed{\text{(D)}}$

(12) What is $T \cup \mathbb{N}$?

(A) T

(B) $\mathbb{N}$

(C) $\mathbb{Z}$

(D) P

(E) None of the above

[1]

ANS: $\boxed{\text{(B)}}$

(13) What is $|\mathcal{P}(A \times \{1, 2\})|$? $\boxed{}$ **[1]**

ANS: $\boxed{256}$ $A \times \{1, 2\}$ has $4 \times 2 = 8$ elements, because A has four elements and $\{1, 2\}$ has two. Thus $\mathcal{P}(A \times \{1, 2\})$ has $2^8 = 256$ elements.

Let A be the set of all non-empty subsets of $\{1, 2, \ldots, 10\}$ and let f and g be the following functions.
$f : A \to \mathbb{N}$ defined by $f(X) = a - b$, where a is the largest element of X and b is the smallest element of X.
$g : A \to A$ defined by $g(X) = X \cup \{1, 2\}$.

(14) What is $f(\{2, 3, 6\})$? $\boxed{}$ **[1]**

ANS: $\boxed{4}$ $f(\{2, 3, 6\}) = 6 - 2 = 4$.

(15) What is $|g(\{2, 7, 10\})|$? $\boxed{}$ **[1]**

ANS: $\boxed{4}$ $|g(\{2, 7, 10\})| = |\{2, 7, 10\} \cup \{1, 2\}| = |\{1, 2, 7, 10\}| = 4$.

(16) Is f onto? Is f one-to-one?

(A) yes, yes
(B) yes, no
(C) no, yes
(D) no, no

[1]

ANS: $\boxed{\text{(D)}}$ $11 \notin \text{image}(f)$ but $11 \in \mathbb{N}$, so f is not onto.
f is not one-to-one. For example, $f(\{1, 2\}) = 2 - 1 = 1$ and $f(\{6, 7\}) = 7 - 6 = 1$.

(17) Is g onto? Is g one-to-one?

(A) yes, yes
(B) yes, no
(C) no, yes
(D) no, no

[1]

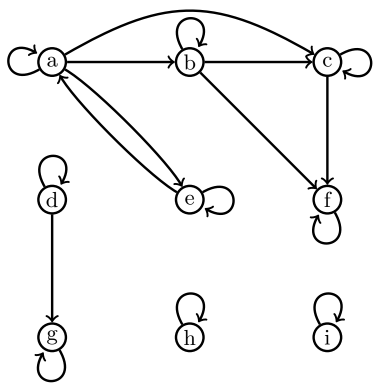
ANS: $\boxed{\text{(D)}}$
$\{4\} \notin \text{image}(g)$ but $\{4\} \in A$, so g is not onto.
g is not one-to-one. For example, $g(\{1\}) = \{1\} \cup \{1, 2\} = \{1, 2\}$ and $g(\{1, 2\}) = \{1, 2\} \cup \{1, 2\} = \{1, 2\}$.

(18) If $f \circ g$ exists then enter $f \circ g(\{9\})$. If it doesn't exist, enter NA. **[1]**

ANS: $\boxed{8}$ $f \circ g$ exists because $\text{codomain}(g) = A = \text{domain}(f)$.
$f \circ g(\{9\}) = f(g(\{9\})) = f(\{1, 2, 9\}) = 9 - 1 = 8$.

Let R and S be binary relations defined as follows.

R is defined on $A = \{a, b, c, d, e, f, g, h, i\}$ by the following arrow diagram.



S is defined on $\mathbb{Z} \times \mathbb{Z}$ by $(w, x)S(y, z)$ if and only if $w + x - y - z$ is even.

(19) Is R reflexive? Is R symmetric?

(A) yes, yes
(B) yes, no
(C) no, yes
(D) no, no

[1]

ANS: $\boxed{\text{(B)}}$ R is reflexive because, for all $x \in A$, xRx (there is a loop on every vertex).
R is not symmetric because bRf but $f \not R b$, for example.

(20) Is R antisymmetric? Is R transitive?

(A) yes, yes
(B) yes, no
(C) no, yes
(D) no, no

[1]

ANS: $\boxed{\text{(D)}}$ R is not antisymmetric because aRe and eRa, for example.
R is not transitive because, aRb and bRf but $a \not R f$, for example.

(21) Is S reflexive? Is S symmetric?

(A) yes, yes
(B) yes, no
(C) no, yes
(D) no, no

[1]

ANS: $\boxed{\text{(A)}}$ S is reflexive because, for all $(w, x) \in \mathbb{Z}\times\mathbb{Z}$, $w+x-w-x = 0$ and 0 is even, so $(w, x)S(w, x)$.
S is symmetric because, for all $(w, x), (y, z) \in \mathbb{Z} \times \mathbb{Z}$, if $(w, x)S(y, z)$, then $w + x - y - z$ is even and so $y + z - w - x = -(w + x - y - z)$ is even, which means $(y, z)S(w, y)$.

(22) Is S antisymmetric? Is S transitive?

(A) yes, yes

(B) yes, no

(C) no, yes

(D) no, no

[1]

ANS: $\boxed{\text{(C)}}$ S is not antisymmetric because $(1,1)S(2,2)$ and $(2,2)S(1,1)$, for example. S is transitive because, for all $(u,v),(w,x),(y,z) \in \mathbb{Z}\times\mathbb{Z}$.

(23) Which of the following is true of S?

(A) S is not an equivalence relation.

(B) S is an equivalence relation and one of its equivalence classes is $\{(w,x) : (w,x) \in \mathbb{Z}\times\mathbb{Z} \text{ and } w+x \text{ is odd}\}$.

(C) S is an equivalence relation and one of its equivalence classes is $\{(w,x) : (w,x) \in \mathbb{Z}\times\mathbb{Z} \text{ and both } w \text{ and } x \text{ are odd}\}$.

(D) S is an equivalence relation and one of its equivalence classes is $\{(w,x) : (w,x) \in \mathbb{Z}\times\mathbb{Z} \text{ and both } w \text{ and } x \text{ are even}\}$.

(E) S is an equivalence relation and one of its equivalence classes is $\{(w,x) : (w,x) \in \mathbb{Z}\times\mathbb{Z} \text{ and at least one of } w \text{ and } x \text{ is even}\}$.

[2]

ANS: $\boxed{\text{(B)}}$ S is an equivalence relation (it is reflexive, symmetric and transitive). S has two equivalence classes, as given below.

$$\{(w,x) : (w,x) \in \mathbb{Z}\times\mathbb{Z} \text{ and } w+x \text{ is even}\}$$
$$\{(w,x) : (w,x) \in \mathbb{Z}\times\mathbb{Z} \text{ and } w+x \text{ is odd}\}$$

For $n \geqslant 1$, let s_n be the number of ways of writing n as a sum of 1s, 3s and 4s (order being important). For example, $s_4 = 4$ because 4 can be written in four ways:

$$1+1+1+1, \quad 1+3, \quad 3+1, \quad 4.$$

(24) What is s_3? $\boxed{}$ **[1]**

ANS: $\boxed{2}$ The two possible sums are $1+1+1$ and 3.

(25) Which of the following is a recurrence for s_n that holds for all integers $n \geqslant 5$?

(A) $s_n = s_{n-1} + s_{n-2} + s_{n-3} - s_{n-4}$

(B) $s_n = s_{n-1} + s_{n-2} + s_{n-4}$

(C) $s_n = s_{n-1} + s_{n-3} + s_{n-4}$

(D) $s_n = 2s_{n-2} + s_{n-3} + s_{n-4}$

(E) $s_n = s_{n-1} + s_{n-2}$

[2]

ANS: $\boxed{\text{(C)}}$ Call an ordered sum "legal" if every term in it is a 1, 3 or 4.
If we add a "+1" to the end of a legal sum adding to $n-1$, then we obtain a legal sum adding to n. If we add a "+3" to the end of a legal sum adding to $n-3$, then we obtain a legal sum adding to n. If we add a "+4" to the end of a legal sum adding to $n-4$, then we obtain a legal sum adding to n. Furthermore, every legal sum adding to n can be obtained by exactly one of these three methods (because every legal sum ends in a "+1", a "+3" or a "+4"). So we can see that $s_n = s_{n-1} + s_{n-3} + s_{n-4}$ for all $n \geqslant 5$.

Let X_1 and X_2 be independent random variables that are each selected uniformly at random from the set $\{1, 2, 3\}$. Let $Y = \max(X_1, X_2)$.

(26) What is $\Pr(X_1 = 2 \text{ and } Y = 2)$? $\boxed{}$ **[1]**

ANS: $\boxed{2/9}$ If $X_1 = 2 \wedge Y = 2$, then $(X_1, X_2) \in \{(2, 2), (2, 1)\}$. So,

$$\Pr(X_1 = 2 \wedge Y = 2) = \Pr(X_1 = 2 \wedge X_2 = 2) + \Pr(X_1 = 2 \wedge X_2 = 1) = \tfrac{1}{3} \times \tfrac{1}{3} + \tfrac{1}{3} \times \tfrac{1}{3} = \tfrac{2}{9}.$$

(27) What is $E[Y]$? $\boxed{}$ **[2]**

ANS: $\boxed{22/9}$ (X_1, X_2) takes each value in $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$ with probability $\frac{1}{9}$.
$Y = 1$ if and only if $(X_1, X_2) = (1, 1)$. So $\Pr(Y = 1) = \frac{1}{9}$.
$Y = 2$ if and only if $(X_1, X_2) \in \{(1, 2), (2, 1), (2, 2)\}$. So $\Pr(Y = 2) = \frac{3}{9}$.
$Y = 3$ if and only if $(X_1, X_2) \in \{(1, 3), (2, 3), (3, 1), (3, 2), (3, 3)\}$. So $\Pr(Y = 3) = \frac{5}{9}$.
So $E[Y] = 1 \times \frac{1}{9} + 2 \times \frac{3}{9} + 3 \times \frac{5}{9} = \frac{22}{9}$.

(28) What would $E[Y]$ be if X_1 and X_2 were instead selected uniformly at random from the set $\{1, 2, \ldots, 100\}$?

(A) $\frac{1}{10000} \sum_{i=1}^{100} i^2$
(B) $\frac{1}{10000} \sum_{i=1}^{100} i(i-1)$
(C) $\frac{1}{10000} \sum_{i=1}^{100} i(2i-1)$
(D) $\frac{1}{100} \sum_{i=1}^{100} i^2$
(E) $\frac{1}{100} \sum_{i=1}^{100} i(i-1)$

[2]

ANS: $\boxed{\text{(C)}}$ (X_1, X_2) takes each value in $\{1, 2, \ldots, 100\} \times \{1, 2, \ldots, 100\}$ with probability $\frac{1}{100^2} = \frac{1}{10000}$.
$Y = y$ if and only if $(X_1, X_2) \in \{(1, y), (2, y), \ldots, (y, y)\} \cup \{(y, 1), (y, 2), \ldots, (y, y-1)$, for each $y \in \{1, 2, \ldots, 100\}$. So $\Pr(Y = y) = \frac{2y-1}{10000}$.
So

$$\begin{aligned} E[Y] &= 1 \times \tfrac{2-1}{10000} + 2 \times \tfrac{4-1}{10000} + \cdots + 100 \times \tfrac{200-1}{10000} \\ &= \sum_{i=1}^{100} i \times \frac{2i-1}{10000} \\ &= \frac{1}{10000} \sum_{i=1}^{100} i(2i-1). \end{aligned}$$

The number of calls received by a call center forms a Poisson distribution. An average of 3 call per minute are received.
(Remember that if X is a Poisson random variable with $E(X) = \lambda$, then $\Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$.)

(29) Which of the following is the probability that no calls come in a certain one minute period?

(A) $3e^{-3}$
(B) e^{-3}
(C) 0
(D) $\frac{1}{3}e^{-3}$
(E) $\frac{1}{3}e$

[1]

ANS: $\boxed{\text{(B)}}$ Using the formula with $\lambda = 3$, the probability is $\frac{e^{-3}3^0}{0!} = e^{-3}$.

(30) Which of the following is the probability that at least two calls will arrive in a certain two minute period?

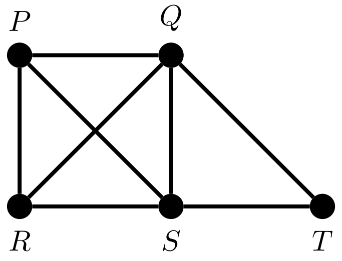
(A) 1
(B) $4e^{-3}$
(C) $7e^{-6}$
(D) $1 - 4e^{-3}$
(E) $1 - 7e^{-6}$

[2]

ANS: $\boxed{\text{(E)}}$ Let Y be the number of calls arriving in the two minute period. We know $E(Y) = 3+3 = 6$. So, using the formula with $\lambda = 6$, $\Pr(Y = y) = \frac{e^{-6}6^y}{y!}$. Then

$$\Pr(Y \geqslant 2) = 1 - \Pr(Y = 1) - \Pr(Y = 0) = 1 - \frac{e^{-6}6^1}{1!} - \frac{e^{-6}6^0}{0!} = 1 - 6e^{-6} - e^{-6} = 1 - 7e^{-6}.$$

Consider the following graph.



(31) How many edges does the graph have? $\boxed{}$ **[1]**

ANS: $\boxed{8}$

(32) Which of the following is true?

(A) The graph has a closed Euler trail.

(B) If the edge PR is removed the resulting graph has a closed Euler trail.

(C) If the edge PT is added the resulting graph has a closed Euler trail.

(D) (A) and (B) are both true

(E) (A) and (C) are both true

(F) (B) and (C) are both true

(If (D), (E) or (F) are true you must give one of them as your answer and not (A), (B) or (C).) **[1]**

ANS: $\boxed{\text{(B)}}$ In the original graph vertices P and R have odd degree. All vertices in a graph must have even degree (and the graph must be connected) for the graph to have a closed Euler trail. So the original graph and the graph obtained in (C) do not have closed Euler trails but the graph obtained in (B) does.

(33) How many spanning trees of the graph contain both the edges QS and RS? $\boxed{}$ **[1]**

ANS: $\boxed{6}$ A spanning tree of the graph containing the edges QS and RS, must contain exactly one of the edges in $\{QT, ST\}$ and exactly one of the edges in $\{PQ, PR, PS\}$. Any choice for the former and any choice for the latter produce a spanning tree. So there are $2 \times 3 = 6$ spanning trees containing the edges QS and RS.

Consider a graph whose adjacency matrix is

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

where V_1, V_2 V_3 and V_4 are the vertices corresponding to row/column 1,2,3 and 4 respectively.

(34) What is the degree of the vertex V_4? $\boxed{}$ **[1]**

ANS: $\boxed{2}$ Looking at the fourth row (or column) of the matrix we see that V_4 is adjacent to V_1 and V_2 but not to V_3.

(35) Which of the following gives the number of walks of length 6 starting from V_3 in the graph?

(A) The entry in row 3 and column 2 of M

(B) The entry in row 3 and column 2 of M^6

(C) The sum of the entries in row 3 of M

(D) The sum of the entries in row 3 of M^6

(E) The entry in row 3 and column 2 of M^3

[1]

ANS: $\boxed{\text{(D)}}$ The entry in row 3 and column i of M^6 gives the number of walks of length 6 in the graph from V_3 to V_i for $i \in \{1, 2, 3, 4\}$. Thus the sum of the entries in row 3 of M^6 is the number of walks of length 6 starting from V_3.

(36) Does there exist a simple graph with 100 vertices and 98 edges? If so, what is the maximum number of spanning trees such a graph can contain?

(A) Such a graph doesn't exist.

(B) Such a graph exists. No such graph contains a spanning tree.

(C) Such a graph exists. The maximum number of spanning trees such a graph can have is 1.

(D) Such a graph exists. The maximum number of spanning trees such a graph can have is 2.

(E) Such a graph exists. The maximum number of spanning trees such a graph can have is 3 or more. **[1]**

ANS: $\boxed{\text{(B)}}$ Such a graph exists: for example the graph with vertices $V_1, V_2, \ldots, V_{100}$ and edges $V_1V_2, V_2V_3, V_3V_4, \ldots, V_{98}V_{99}$. But any tree with 100 vertices has $100 - 1 = 99$ edges, so such a graph cannot contain a spanning tree.

(37) Which of the following is true?

(A) Every simple graph with no odd degree vertices has a closed Euler trail.

(B) Some, but not all, simple graphs with no odd degree vertices have closed Euler trails.

(C) Every simple graph with no odd degree vertices does not have a closed Euler trail. **[1]**

ANS: $\boxed{\text{(B)}}$ A cycle with three edges is an example of a simple graph with no odd degree vertices that has an Euler trail. A graph with six vertices and six edges made up of two cycles of length three that do not share any vertices is an example of a simple graph with no odd degree vertices that does not have an Euler trail.

(38) Which of the following sequences can be degrees of the vertices of a simple graph (on 5 vertices)?

(A) $(5, 3, 2, 2, 2)$

(B) $(4, 2, 2, 2, 0)$

(C) $(2, 2, 2, 2, 2)$

(D) Both (A) and (B)

(E) Both (A) and (C)

(F) Both (B) and (C)

(If (D), (E) or (F) are true you must give one of them as your answer and not (A), (B) or (C).) **[1]**

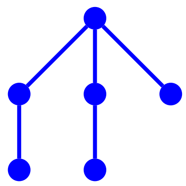
ANS: $\boxed{\text{(C)}}$ A cycle with 5 edges is a simple graph with degree sequence $(2, 2, 2, 2, 2)$. In a simple graph with 5 vertices, a vertex can be adjacent to at most 4 others, so $(5, 3, 2, 2, 2)$ is impossible because it contains a 5. Similarly, in a simple graph with 5 vertices and a vertex of degree 0, a vertex can be adjacent to at most 3 others, so $(4, 2, 2, 2, 0)$ is impossible because it contains a 4.

(39) Which of the following sequences can be degrees of a tree (on 6 vertices)?

(A) $(2, 2, 1, 1, 1, 1)$
(B) $(3, 2, 2, 1, 1, 1)$
(C) $(2, 2, 2, 2, 2, 2)$
(D) Both (A) and (B)
(E) Both (A) and (C)
(F) Both (B) and (C)

(If (D), (E) or (F) are true you must give one of them as your answer and not (A), (B) or (C).) **[1]**

ANS: $\boxed{\text{(B)}}$ A tree with degree sequence $(3, 2, 2, 1, 1, 1)$ is pictured below.



Any tree with 6 vertices must have 5 edges and so the sum of its degrees must be $2 \cdot 5 = 10$. This rules out the sequences $(2, 2, 1, 1, 1, 1)$ and $(2, 2, 2, 2, 2, 2)$ which have sums 8 and 12 respectively.

(40) Which of the following is true for all integers n and r such that $n > r \geqslant 1$?

(A) $\binom{n}{r} = \frac{n}{n-r-1}\binom{n-1}{r}$
(B) $\binom{n}{r} = \frac{n-r-1}{n}\binom{n-1}{r}$
(C) $\binom{n}{r} = \frac{n}{n-r}\binom{n-1}{r}$
(D) $\binom{n}{r} = \frac{n-r}{n}\binom{n-1}{r}$
(E) None of the above is true.

[1]

ANS: $\boxed{\text{(C)}}$ For all integers n and r such that $n > r \geqslant 1$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n}{n-r} \cdot \frac{(n-1)!}{r!(n-1-r)!} = \frac{n}{n-r}\binom{n-1}{r}.$$

(41) How many ternary strings (that is, strings made up of 0s, 1s and 2s) of length 5 contain at most two 0s, at most two 1s and at most two 2s? $\boxed{}$ **[3]**

ANS: $\boxed{90}$ Any ternary string of length 5 containing at most two 0s, at most two 1s and at most two 2s, must contain exactly one of the three symbols 0,1,2 and exactly two of each of the other two symbols. There are 3 ways to select the symbol that occurs once, 5 ways to select a position for that symbol, and then $\binom{4}{2}$ ways to select two positions for the smaller of the other two symbols. So the number ternary strings of length 5 containing at most two 0s, at most two 1s and at most two 2s is

$$3 \times 5 \times \binom{4}{2} = 90.$$

(42) Prove using simple induction that 7 divides $19^n - 12^n$ for all integers $n \geqslant 1$.

Write your answer on paper (in a real exam you would get a chance to upload a photo of your work after the exam). **[5]**

ANS: Let $P(n)$ be the statement "7 divides $19^n - 12^n$".

Base step. $19^1 - 12^1 = 19 - 12 = 7$ and 7 divides 7. So $P(1)$ is true.

Induction step. For some integer $k \geqslant 1$, assume that $P(k)$ is true, that is, assume 7 divides $19^k - 12^k$. So $19^k - 12^k = 7\ell$ for some integer ℓ.

Now we need to prove that $P(k+1)$ is true. So we must prove 7 divides $19^{k+1} - 12^{k+1}$.

We see that

$$\begin{aligned}
19^{k+1} - 12^{k+1} &= 19(19^k) - 12(12^k) \\
&= 7(19^k) + 12(19^k) - 12(12^k) \\
&= 7(19^k) + 12(19^k - 12^k) \\
&= 7(19^k) + 12(7\ell) \qquad \text{(using } P(k)\text{)} \\
&= 7(19^k + 12\ell).
\end{aligned}$$

So 7 divides $19^{k+1} - 12^{k+1}$ because $19^k + 12\ell$ is an integer. So $P(k+1)$ is true.

So we have proved by induction that $P(n)$ is true for each integer $n \geqslant 1$.