MAT1830

Lecture 17: Equivalence Relations

An *equivalence relation* R on a set A is a binary relation with the following three properties.

1. Reflexivity.

$$aRa$$

for all $a \in A$.

2. Symmetry.

$$aRb \Rightarrow bRa$$

for all $a, b \in A$.

3. Transitivity.

$$aRb \text{ and } bRc \Rightarrow aRc$$

for all $a, b, c \in A$.

Equality and congruence mod n (for fixed n) are examples of equivalence relations.

Reflexivity   (For a binary relation R on a set A.)

Everywhere I see:

I actually see:
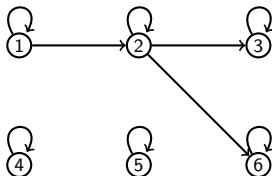
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To prove R is reflexive, show that...

For all $x \in A$, xRx.

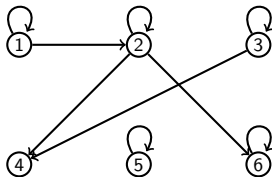To prove R is not reflexive, show that...

There is an $x \in A$ such that $x \not{R} x$.

Question   Let R be the relation on A pictured below. Is R reflexive?

Yes. xRx for all $x \in A$.

Question   Let S be the relation on A pictured below. Is S reflexive?

No. $4 \not S 4$.

Symmetry (For a binary relation R on a set A.)

Everywhere I see:

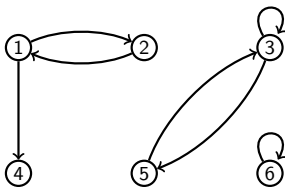I actually see:

To prove R is symmetric, show that...

For all $x, y \in A$, if xRy then yRx.
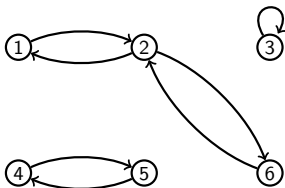

To prove R is not symmetric, show that...

There are some $x, y \in A$ such that xRy but $y \not R x$.

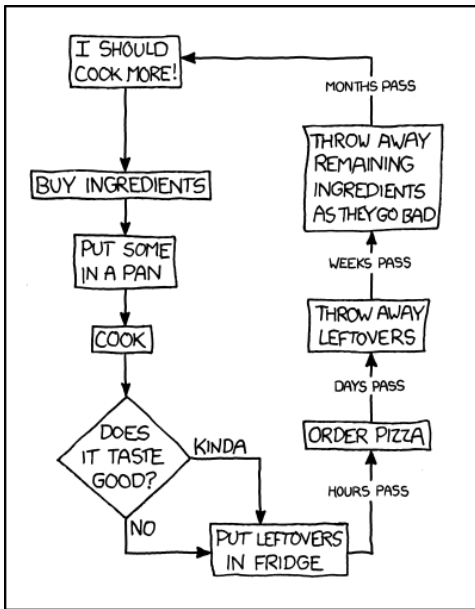**Question**   Let R be the relation on A pictured below. Is R symmetric?

No. $1R4$ but $4\not{R}1$.

Question   Let S be the relation on A pictured below. Is S symmetric?

Yes. For all $x, y \in A$ if xSy then ySx.

Transitivity   (For a binary relation R on a set A.)

Everywhere I see:

I actually see:

Everywhere I see:

I actually see:

To prove R is transitive, show that...

For all $x, y, z \in A$, if xRy and yRz then xRz.

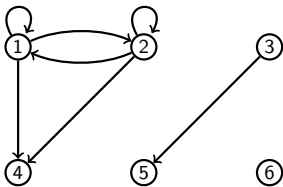To prove R is not transitive, show that...

There are some $x, y, z \in A$ such that xRy and yRz but $x \not\mathrel{R} z$.
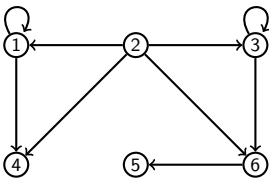
Question   Let R be the relation on A pictured below. Is R transitive?

Yes. For all $x, y, z \in A$, if xRy and yRz then xRz.

Question   Let S be the relation on A pictured below. Is S transitive?

No, because $3S6$ and $6S5$ but $3 \not{S} 5$.

17.1 Other equivalence relations

1. Equivalence of fractions.

Two fractions are equivalent if they reduce to the same fraction when the numerator and denominator of each are divided by their gcd. E.g. $\frac{2}{4}$ and $\frac{3}{6}$ are equivalent because both reduce to $\frac{1}{2}$.

2. Congruence of triangles.

Triangles ABC and $A'B'C'$ are congruent if $AB = A'B'$, $BC = B'C'$ and $CA = C'A'$. E.g. the following triangles are congruent.
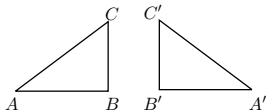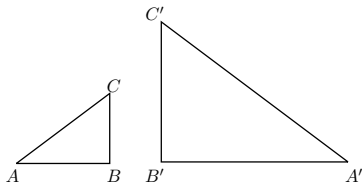
3. Similarity of triangles.

   Triangles ABC and $A'B'C'$ are similar if
   $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

E.g the following triangles are similar

4. Parallelism of lines.

   The relation $L\|M$ (L is parallel to M) is an equivalence relation.

Remark

In all these cases the relation is an equivalence because it says that objects are the *same* in some respect.

1. Equivalent fractions have the same re-
   duced form.

2. Congruent triangles have the same side
   lengths.

3. Similar triangles have the same shape.

4. Parallel lines have the same direction.

Sameness is always reflexive (a is the same as a), symmetric (if a is the same as b, then b is the same as a) and transitive (if a is the same as b and b is the same as c, then a is the same as c).

Flux Exercise

Which of the following relations are equivalence relations on $\mathbb{Z}$?

(1) R defined by xRy if and only if $|x| = |y|$
(2) S defined by xSy if and only if $x^3 - y^3 = 1$
(3) T defined by xTy if and only if x divides y
(4) U defined by xUy if and only if 5 divides $x - y$

A. Just (1) and (3)
B. Just (1)
C. Just (1) and (4)
D. Just (1), (2) and (4)

(1) R defined on $\mathbb{Z}$ by xRy if and only if $|x| = |y|$

Reflexive: Yes. $|a| = |a|$ for all $a \in \mathbb{Z}$.
Symmetric: Yes. If $|a| = |b|$, then $|b| = |a|$ for all $a, b \in \mathbb{Z}$.
Transitive: Yes. If $|a| = |b|$ and $|b| = |c|$, then $|a| = |c|$ for all $a, b, c \in \mathbb{Z}$.
So it is an equivalence relation.

(2) S defined on $\mathbb{Z}$ by xSy if and only if $x^3 - y^3 = 1$

Reflexive: No. $1^3 - 1^3 \neq 1$ so $1 \not S 1$.
Symmetric: No. $1^3 - 0^3 = 1$ but $0^3 - 1^3 \neq 1$, so $1S0$ but $0 \not S 1$.
Transitive: No. $1^3 - 0^3 = 1$ and $0^3 - (-1)^3 = 1$ but $1^3 - (-1)^3 \neq 1$, so $1S0$ and $0S(-1)$ but $1 \not S (-1)$.
So it is not an equivalence relation.

(3) T defined on $\mathbb{Z}$ by xTy if and only if x divides y

Reflexive: Yes. a divides a for all $a \in \mathbb{Z}$.
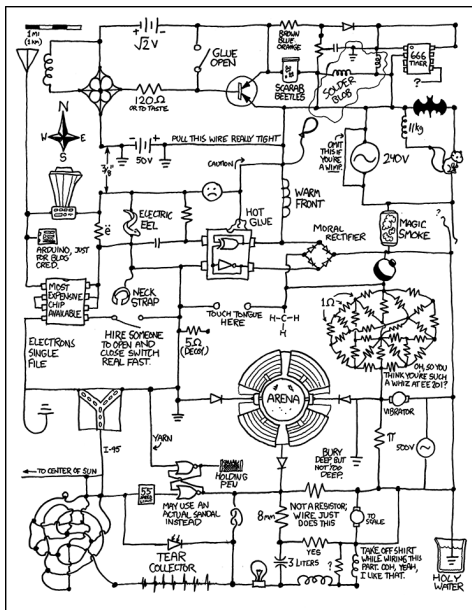Symmetric: No. 3 divides 6 but 6 does not divide 3, so $3T6$ but $6 \not{T} 3$.
Transitive: Yes. If a divides b and b divides c, then a divides c for all $a, b, c \in \mathbb{Z}$.
So it is not an equivalence relation.

(4) U defined on $\mathbb{Z}$ by xUy if and only if 5 divides $x - y$

Yes. This relation is the same as $x \equiv y \pmod 5$ and we know that's an equivalence relation.

Question What is the same about the equivalent objects for the equivalence relations (1) and (4)?

(1) R defined on $\mathbb{Z}$ by xRy if and only if $|x| = |y|$

x and y have the same "magnitude".

(4) U defined on $\mathbb{Z}$ by xUy if and only if 5 divides $x - y$

x and y have the same remainder when divided by 5.

17.2   Equivalence classes

Conversely, we can show that if R is a reflexive, symmetric and transitive relation then aRb says that a and b are the same in some respect: *they have the same R-equivalence class.*

> If R is an equivalence relation we define the *R-equivalence class* of a to be
>
> $$[a] = \{s : sRa\}.$$

Thus $[a]$ consists of all the elements related to a. It can also be defined as $\{s : aRs\}$, because sRa if and only if aRs, by symmetry of R.

Examples
- The parallel equivalence class of a line L consists of all lines parallel to L.
- The equivalence class of 1 for congruence mod 2 is the set of all odd numbers.

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17.3   Equivalence class properties

Claim. *If two elements are related by an equivalence relation R on a set A, their equivalence classes are equal.*

Proof. Suppose $a, b \in A$ and aRb. Now

$$
\begin{aligned}
s \in [a] \quad &\Rightarrow \quad sRa \text{ by definition of } [a] \\
&\Rightarrow \quad sRb \text{ by transitivity of } R \\
&\qquad \text{since } sRa \text{ and } aRb \\
&\Rightarrow \quad s \in [b] \text{ by definition of } [b].
\end{aligned}
$$

Thus all elements of $[a]$ belong to $[b]$. Similarly, all elements of $[b]$ belong to $[a]$, hence $[a] = [b]$.
$\square$

Claim. *If R is an equivalence relation on a set A, each element of A belongs to exactly one equivalence class.*

Proof. Suppose $a, b, c \in A$, and $c \in [a] \cap [b]$.

$\qquad\qquad c \in [a]$ and $c \in [b]$

$\Rightarrow \quad cRa$ and cRb

$\qquad\qquad$ by definition of $[a]$ and $[b]$

$\Rightarrow \quad aRc$ and cRb by symmetry

$\Rightarrow \quad aRb$ by transitivity

$\Rightarrow \quad [a] = [b]$

$\qquad\qquad$ by the previous claim.

17.4 Partitions and equivalence classes

> A *partition* of a set S is a set of subsets of S such that each element of S is in exactly one of the subsets.

Using what we showed in the last section, we have the following.

> If R is an equivalence relation on a set A, then the equivalence classes of R form a partition of A. Two elements of A are related if and only if they are in the same equivalence class.

Example. Let R be the relation on $\mathbb{Z}$ defined by aRb if and only if $a \equiv b \pmod 3$. The three equivalence classes of R are

$$\{x : x \equiv 0 \pmod 3\} = \{3k : k \in \mathbb{Z}\}$$
$$\{x : x \equiv 1 \pmod 3\} = \{3k + 1 : k \in \mathbb{Z}\}$$
$$\{x : x \equiv 2 \pmod 3\} = \{3k + 2 : k \in \mathbb{Z}\}.$$

These partition the set $\mathbb{Z}$.

Examples of partitions

$\{\{1, 6\}, \{2\}, \{3, 4, 5\}\}$ is a partition of $\{1, 2, 3, 4, 5, 6\}$

$\{\{x : x \in \mathbb{Z} \text{ and } x \text{ is even}\}, \{x : x \in \mathbb{Z} \text{ and } x \text{ is odd}\}\}$ is a partition of $\mathbb{Z}$.

Question   What are the equivalence classes of the equivalence relations (1) and (4)?

(1) R defined on $\mathbb{Z}$ by xRy if and only if $|x| = |y|$

$\{0\}, \{1, -1\}, \{2, -2\}, \{3, -3\} \ldots$.

(4) U defined on $\mathbb{Z}$ by xUy if and only if 5 divides $x - y$

$\{\ldots, -10, -5, 0, 5, 10, \ldots\}$,
$\{\ldots, -9, -4, 1, 6, 11, \ldots\}$,
$\{\ldots, -8, -3, 2, 7, 12, \ldots\}$,
$\{\ldots, -7, -2, 3, 8, 13, \ldots\}$,
$\{\ldots, -6, -1, 4, 9, 14, \ldots\}$

Flux Exercise

Let X be the set $\{$there, was, a, wall, it, did, not, look, important$\}$.

Let R be the equivalence relation on X defined by xRy if and only if x and y contain the same number of 'o's.

What are the equivalence classes of R?

A. $\{$there, was, a, wall, it, did$\}$, $\{$not, look, important$\}$

B. $\{$there, was, a, wall$\}$, $\{$not, important$\}$, $\{$look$\}$

C. $\{$there, was, a, wall$\}$, $\{$it, did$\}$, $\{$not, important$\}$, $\{$look$\}$

D. $\{$there, was, a, wall, it, did$\}$, $\{$not, important$\}$, $\{$look$\}$