MAT1830

Lecture 16: Relations

Relations - why should you care?

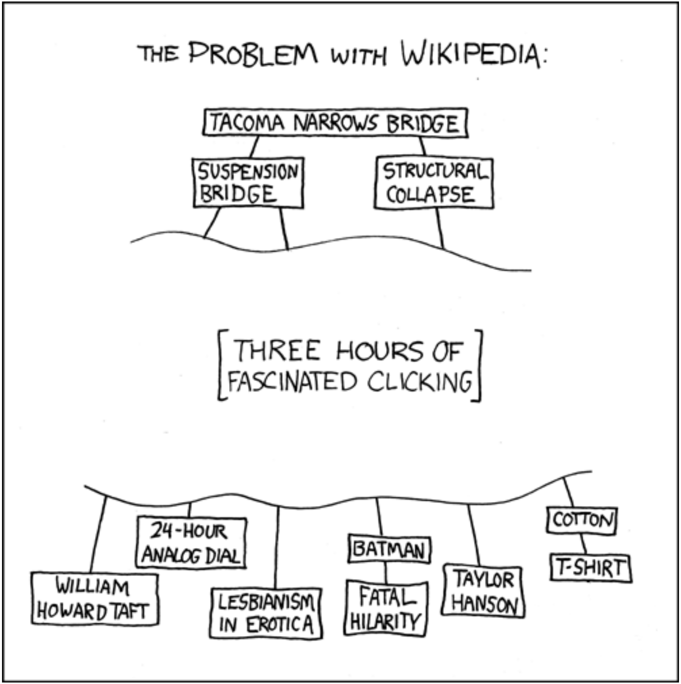
- Relations are used heavily in database theory in computer science.
- They are also used in theories of object orientation in programming.
- Relations can be thought of as a generalisation of functions.
- Like there is a functional programming paradigm there's a relational programming paradigm.

Roughly speaking, a binary relation on a set is something that tells us, for any two things in the set, that they are related or they are not related. (Order *is* important.)

You already know lots of examples of binary relations.

For example:

- $\leq$ on real numbers
- "divides" on natural numbers
- "is consecutive with" on integers


THE PROBLEM WITH WIKIPEDIA:
TACOMA NARROWS BRIDGE
SUSPENSION BRIDGE
STRUCTURAL COLLAPSE
[THREE HOURS OF FASCINATED CLICKING]
WILLIAM HOWARD TAFT
24-HOUR ANALOG DIAL
LESBIANISM IN EROTICA
BATMAN
FATAL HILARITY
TAYLOR HANSON
COTTON
T-SHIRT

We could make a set $\{(x, y) : x \text{ links to } y\}$.

So, for example, $(\text{Tacoma Narrows Bridge}, \text{Suspension bridge})$ would be in the set and so would $(\text{Tacoma Narrows Bridge}, \text{Structural collapse})$, etc.

Mathematical objects can be related in various ways, and any particular way of relating objects is called a *relation* on the set of objects in question.

(This also applies to relations in the everyday sense. For example, "parent of" is a relation on the set of people.)

A *binary relation* R on a set A consists of A and a set of ordered pairs from $A \times A$.

When (a, b) is in this set we write aRb.

Similarly, a *ternary* relation on A would be defined by a set of ordered triples from $A \times A \times A$, and so on. (A *unary* relation on A is just a subset of A.)

A binary relation R on a set A consists of the set A together with a set of ordered pairs from $A \times A$.

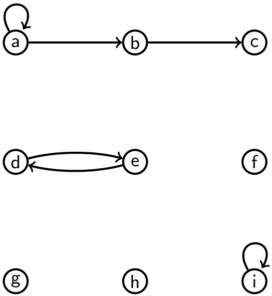
If (x, y) is in the set then we write xRy and say "x is R-related to y".

If (x, y) is not in the set then we write $x \not R y$.

Order matters. It might be that xRy but $y \not R x$.

Arrow diagrams

Example Let R be the relation on $\{a, b, c, d, e, f, g, h, i\}$ given by the set $\{(a, a), (a, b), (b, c), (d, e), (e, d), (i, i)\}$.



Is bRc? Yes.
Is eRf? No.
Is aRc? No.
Is dRe? Yes.
Is iRi? Yes.

16.1 Relations and functions

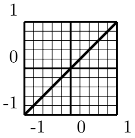
Any function $f : X \to Y$ can be viewed as a relation R on $X \cup Y$. The relation is defined by

$$xRy \text{ if and only if } y = f(x).$$

However, not every relation is a function. Remember that a function must have exactly one output y for each input x in its domain. In a relation, on the other hand, an element x may be related to many elements y, or to none at all.

16.2 Examples

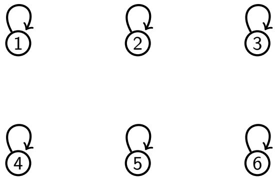
1. Equality on $\mathbb{R}$.
   This is the relation consisting of the pairs (x, x) for $x \in \mathbb{R}$. Thus it is the following subset of the plane.



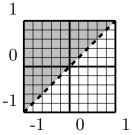
This relation is also a function (the identity function on $\mathbb{R}$), since there is exactly one pair for each $x \in \mathbb{R}$.

Question Give the set of ordered pairs for the relation "$=$" on $\{1, 2, 3, 4, 5, 6\}$ and draw an arrow diagram for it.

$\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$



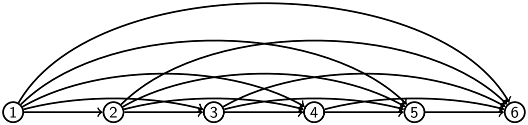
2. The $<$ relation on $\mathbb{R}$.
This relation consists of all the pairs (x, y) with $x < y$. It is the following shaded subset of the plane.

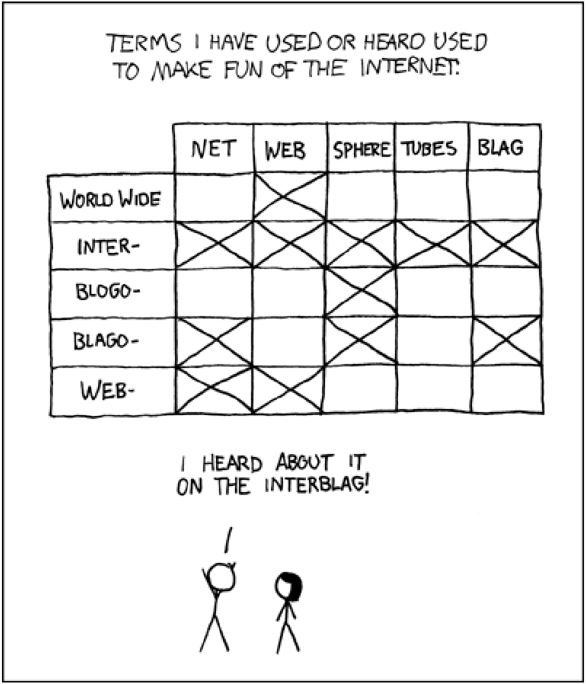


(The dashed line indicates that the points where $x = y$ are omitted.)

Question Give the set of ordered pairs for the relation "$<$" on $\{1,2,3,4,5,6\}$ and draw an arrow diagram for it.

$$\{(1,2),(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),\\ (3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\}$$




TERMS I HAVE USED OR HEARD USED TO MAKE FUN OF THE INTERNET:
NET
WEB
SPHERE
TUBES
BLAG
WORLD WIDE
INTER-
BLOGO-
BLAGO-
WEB-
I HEARD ABOUT IT ON THE INTERBLAG!

Flux Exercise

Which of the following binary relations R satisfy $\forall x \exists y (xRy)$?

(1) R defined on $\mathbb{B}$ by xRy if and only if $x \wedge y \equiv \mathrm{T}$

(2) R defined on $\mathcal{P}(\mathbb{N})$ by xRy if and only if $x \subseteq y$

(3) R defined on $\mathbb{R}$ by xRy if and only if $x > y$

(4) R defined on $\mathbb{N}$ by xRy if and only if x divides y

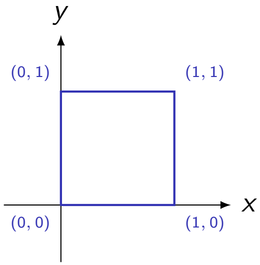
A. (1), (2) and (3) but not (4)
B. (2), (3) and (4) but not (1)
C. (2) and (3) but not (1) and (4)
D. None of them

Hint $\forall x \exists y (xRy)$ means roughly "everything is R-related to something".

Answer

To show $\forall x \exists y (xRy)$ is true we must find, for each x, a y such that xRy.
To show $\forall x \exists y (xRy)$ is false we must find one specific x such that $x \not R y$ for all y.
False for (1). If $x = \mathrm{F}$ then $x \not R y$ for all y.
True for (2). For each x, xRx for example.
True for (3). For each x, $xR(x-1)$ for example.
True for (4). For each x, $xR(2x)$ for example.
So B.

Question 16.2 Use logic symbols and the $\leq$ relation to write a relation between real numbers x and y which says that the point (x, y) lies in the square with corners $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$.



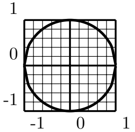
Answer

$(0 \leq x) \wedge (x \leq 1) \wedge (0 \leq y) \wedge (y \leq 1)$

3. Algebraic curves.

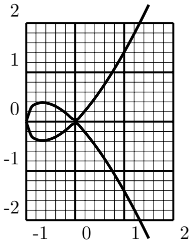
An algebraic curve consists of the points (x, y) satisfying an equation $p(x, y) = 0$, where p is a polynomial.

E.g. unit circle $x^2 + y^2 - 1 = 0$.



Notice that this relation is not a function, because there are two pairs with the same x, e.g. $(0, 1)$ and $(0, -1)$.

Likewise, the curve $y^2 = x^2(x + 1)$ is not a function.

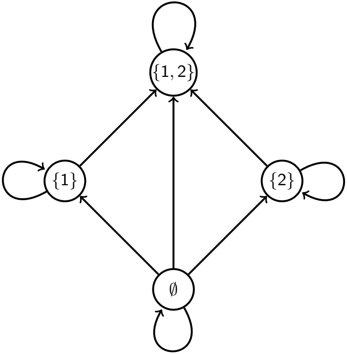


4. The subset relation $\subseteq$.
   This consists of the ordered pairs of sets (A, B) such that $A \subseteq B$. A and B must both be subsets of some universal set U.

Question Give the set of ordered pairs for the relation "$\subseteq$" on $\mathcal{P}(\{1,2\})$ and draw an arrow diagram for it.

Remember $\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

$$\{(\emptyset,\emptyset),(\emptyset,\{1\}),(\emptyset,\{2\}),(\emptyset,\{1,2\}),(\{1\},\{1\}),(\{1\},\{1,2\}),\\ (\{2\},\{2\}),(\{2\},\{1,2\}),(\{1,2\},\{1,2\})\}$$



Flux Exercise

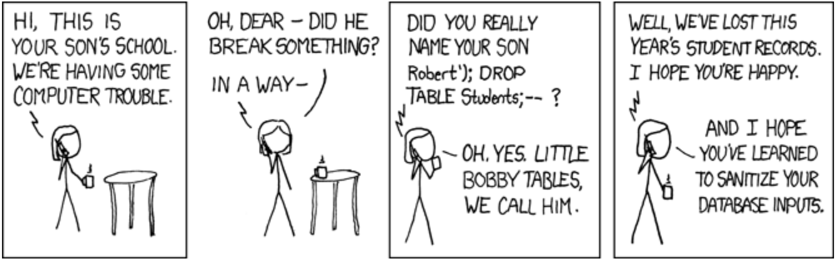
How many possible relations are there on a set X with $|X| = n$?

A. n^2
B. $2^{(n^2)}$
C. 2^n
D. $2^{(2^n)}$

Hint Think of the relation as a set of ordered pairs. How many possible ordered pairs are there? So how many possible relations?

Answer

A relation on X can be thought of as a subset of $X \times X$.
Every subset of $X \times X$ corresponds to a unique relation (and vice versa).
So the number of possible relations on X is the same as the number of subsets of $X \times X$.
$|X \times X| = n^2$. So the number of subsets of $X \times X$ is $2^{(n^2)}$.
So B.


HI, THIS IS YOUR SON'S SCHOOL. WE'RE HAVING SOME COMPUTER TROUBLE.
OH, DEAR – DID HE BREAK SOMETHING?
IN A WAY–
DID YOU REALLY NAME YOUR SON Robert'); DROP TABLE Students;-- ?
OH. YES. LITTLE BOBBY TABLES, WE CALL HIM.
WELL, WE'VE LOST THIS YEAR'S STUDENT RECORDS. I HOPE YOU'RE HAPPY.
AND I HOPE YOU'VE LEARNED TO SANITIZE YOUR DATABASE INPUTS.

5. Congruence modulo n.

   For a fixed n, congruence modulo n is a binary relation. It consists of all the ordered pairs of integers (a, b) such that n divides $a - b$.

Congruence modulo n (recap)

Remember $a \equiv b \pmod{n}$ means that a and b have the same remainder when you divide them by n.

Definition We say $a \equiv b \pmod{n}$ if n divides $a - b$.

For a fixed integer $n \geq 2$, congruence modulo n is a binary relation.

Question Which integers are congruent to 1 modulo 7?

Integers in the set $\{\ldots, -20, -13, -6, 1, 8, 15, 22, \ldots\}$.
This is the set $\{7k + 1 : k \in \mathbb{Z}\}$.

Question Which integers are congruent to 2 modulo 5?

Integers in the set $\{\ldots, -13, -8, -3, 2, 7, 12, 17, \ldots\}$.
This is the set $\{5k + 2 : k \in \mathbb{Z}\}$.

Numbers with the same parity (even or odd) are congruent modulo ???2.

Decimal numbers ending in the same digit are congruent modulo ???10.

The time (in hours) can be thought of as a number modulo ???24.

An angle (in degrees) can be thought of as a number modulo ???360.

16.3 Properties of congruence

As the symbol $\equiv$ suggests, congruence mod n is a lot like equality. Numbers a and b which are congruent mod n are not necessarily equal, but they are "equal up to multiples of n," because they have equal remainders when divided by n.

Because congruence is like equality, congruence $a \equiv b \pmod{n}$ behave a lot like equations. In particular, they have the following three properties.

1. Reflexive property.

$$a \equiv a \pmod{n}$$

for any number a.

2. Symmetric property.

$$a \equiv b \pmod{n} \Rightarrow b \equiv a \pmod{n}$$

for any numbers a and b.

3. Transitive property.

$$a \equiv b \pmod{n} \text{ and } b \equiv c \pmod{n} \Rightarrow$$
$$a \equiv c \pmod{n}$$

for any numbers a, b and c.

These properties are clear if one remembers that $a \equiv b \pmod{n}$ means a and b have the same remainder on division by n.

Question Let R be the binary relation on $\mathbb{Z}$ defined by xRy if and only if $x \equiv y \pmod 3$). Roughly, what would an arrow diagram for R look like?