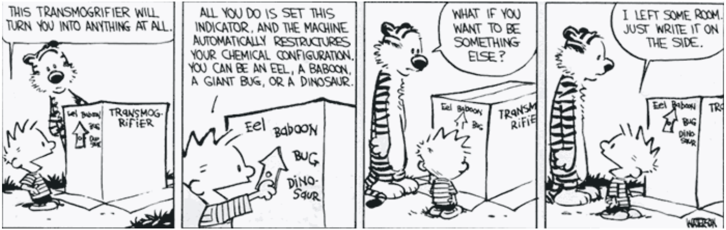
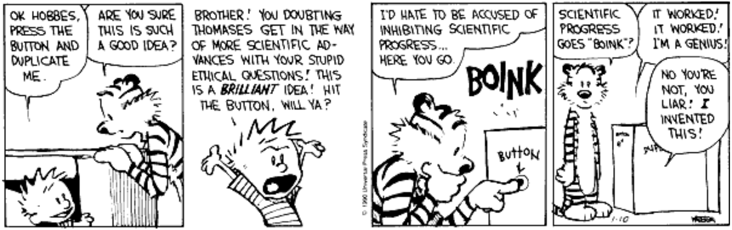
MAT1830

Lecture 15: Composition and Inversion


THIS TRANSMOGRIFIER WILL TURN YOU INTO ANYTHING AT ALL.
EEL BABOON BUG DINOSAUR
TRANSMOG-RIFIER
ALL YOU DO IS SET THIS INDICATOR, AND THE MACHINE AUTOMATICALLY RESTRUCTURES YOUR CHEMICAL CONFIGURATION. YOU CAN BE AN EEL, A BABOON, A GIANT BUG, OR A DINOSAUR.
EEL
BABOON
BUG
DINO-SAUR
WHAT IF YOU WANT TO BE SOMETHING ELSE?
I LEFT SOME ROOM. JUST WRITE IT ON THE SIDE.
WATTERSON


OK HOBBES, PRESS THE BUTTON AND DUPLICATE ME.
ARE YOU SURE THIS IS SUCH A GOOD IDEA?
BROTHER! YOU DOUBTING THOMASES GET IN THE WAY OF MORE SCIENTIFIC AD-VANCES WITH YOUR STUPID ETHICAL QUESTIONS! THIS IS A BRILLIANT IDEA! HIT THE BUTTON, WILL YA?
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I'D HATE TO BE ACCUSED OF INHIBITING SCIENTIFIC PROGRESS... HERE YOU GO.
BOINK
BUTTON
SCIENTIFIC PROGRESS GOES "BOINK"?
IT WORKED! IT WORKED! I'M A GENIUS!
NO YOU'RE NOT, YOU LIAR! I INVENTED THIS!
1-10
WATTERSON

Complicated functions are often built from simple parts. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x^2 + 1)^3$ is computed by doing the following steps in succession:

- square,
- add 1,
- cube.

We say that $f(x) = (x^2 + 1)^3$ is the composite of the functions (from $\mathbb{R}$ to $\mathbb{R}$)

- square(x)=x^2,
- successor(x)=$x + 1$,
- cube(x)=x^3.

15.1 Notation for composite functions

In the present example we write

$$f(x) = \text{cube}(\text{successor}(\text{square}(x))),$$

or

$$f = \text{cube} \circ \text{successor} \circ \text{square}.$$

In general, if $f(x) = g(h(x))$ we write $f = g \circ h$ and say f is the *composite* of g and h.

> **Warning:** Remember that $g \circ h$ means "do h first, then g." $g \circ h$ is usually different from $h \circ g$.

Example.
$\text{square}(\text{successor}(x)) = (x + 1)^2 = x^2 + 2x + 1$
$\text{successor}(\text{square}(x)) = x^2 + 1$

Question 15.1 Let f, m and s be functions on the set of people defined by

$m(x) =$ mother of x
$f(x) =$ father of x
$s(x) =$ spouse of x.

What are the following?

(Note s is not actually a valid function on the set of people.)

$m \circ s(x)$	mother in law of x
$f \circ s(x)$	father in law of x
$m \circ m(x)$	grandmother (maternal) of x
$f \circ m(x)$	grandfather (maternal) of x
$s \circ s(x)$	x

Question 15.2 Write the following as composites of square(x), sqrt(x), successor(x) and cube(x).

(Assume that all of these have domain and codomain $\{x : x \in \mathbb{R} \text{ and } x \geq 0\}$.)

$$\begin{aligned}
\sqrt{1+x^3} &= \text{sqrt}(\text{successor}(\text{cube}(x))) = \text{sqrt} \circ \text{successor} \circ \text{cube}(x) \\
x^{\frac{3}{2}} &= \text{sqrt}((\text{cube}(x)) = \text{sqrt} \circ \text{cube}(x) \\
(1+x)^3 &= \text{cube}(\text{successor}(x)) = \text{cube} \circ \text{successor}(x) \\
(1+x^3)^2 &= \text{square}(\text{successor}(\text{cube}(x))) = \text{square} \circ \text{successor} \circ \text{cube}(x)
\end{aligned}$$

Note Composition of functions is associative: $(f \circ g) \circ h = f \circ (g \circ h)$. So we don't bother with the brackets.

15.2 Conditions for composition

Composite functions do not always exist.

Example. If reciprocal : $\mathbb{R} - \{0\} \to \mathbb{R}$ is defined by reciprocal$(x) = \frac{1}{x}$ and predecessor : $\mathbb{R} \to \mathbb{R}$ is defined by predecessor$(x) = x - 1$, then reciprocal $\circ$ predecessor does not exist, because predecessor$(1) = 0$ is not a legal input for reciprocal.

To avoid this problem, we demand that the codomain of h be equal to the domain of g for $g \circ h$ to exist. This ensures that each output of h will be a legal input for g.

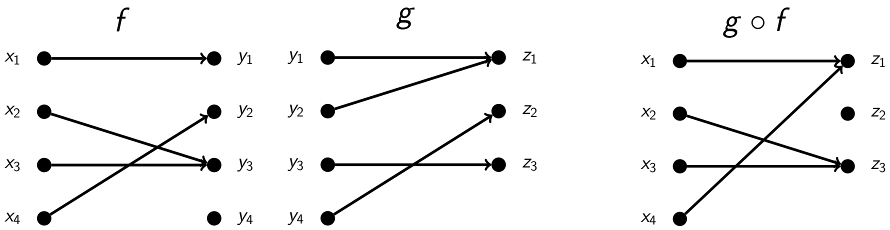
Let $h : A \to B$ and $g : C \to D$ be functions. Then $g \circ h : A \to D$ exists if and only if $B = C$.

Let $g : C \to D$ and $h : A \to B$ be functions.

The function $g \circ h$ exists if and only if $C = B$.

If it exists, $g \circ h : A \to D$ and is defined by $g \circ h(x) = g(h(x))$.

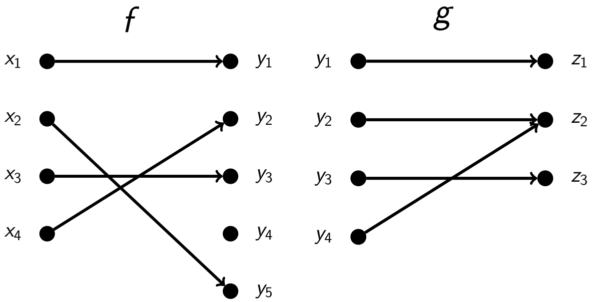
Question Let $f : A \to B$ and $g : C \to D$ be the functions pictured below.



Does $g \circ f$ exist?

codomain$(f) = \{y_1, y_2, y_3, y_4\}$ and domain$(g) = \{y_1, y_2, y_3, y_4\}$.
So $g \circ f$ does exist because codomain$(f) =$ domain(g).
$g \circ f : A \to D$

Question Let $f : A \to B$ and $g : C \to D$ be the functions pictured below.



Does $g \circ f$ exist?

codomain$(f) = \{y_1, y_2, y_3, y_4, y_5\}$ and domain$(g) = \{y_1, y_2, y_3, y_4\}$.
So $g \circ f$ does not exist because codomain$(f) \neq$ domain(g).

Flux Exercise

Let f, g and h be the functions

$f : \mathbb{R} \to \mathbb{Z}$ defined by $f(x) = \lfloor x \rfloor$. ("x rounded down")

$g : \mathbb{Z} \to \mathbb{R}$ defined by $g(x) = \frac{x}{2}$.

$h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^2 + 7$.

Which of the following statements is *false*?

A. $g \circ f$ exists, $g \circ f : \mathbb{R} \to \mathbb{R}$

B. $f \circ g$ exists, $f \circ g : \mathbb{Z} \to \mathbb{Z}$

C. $g \circ h$ does not exist

D. $g \circ f \circ g$ does not exist

15.3 The identity function

On each set A the function $i_A : A \to A$ defined by

$$i_A(x) = x,$$

is called the *identity function* (on A).

15.4 Inverse functions

Functions $f : A \to A$ and $g : A \to A$ are said to be inverses (of each other) if

$$f \circ g = g \circ f = i_A.$$

Example. square and sqrt are inverses of each other on the set $\mathbb{R}^{\geqslant 0}$ of reals $\geqslant 0$.

$$\mathrm{sqrt}(\mathrm{square}(x)) = x \text{ and } \mathrm{square}(\mathrm{sqrt}(x)) = x.$$

In fact, this is exactly what sqrt is supposed to do – reverse the process of squaring. However, this works only if we restrict the domain to $\mathbb{R}^{\geqslant 0}$. On $\mathbb{R}$ we do not have $\mathrm{sqrt}(\mathrm{square}(x)) = x$ because, for example,

$$\mathrm{sqrt}(\mathrm{square}(-1)) = \mathrm{sqrt}(1) = 1.$$

This problem arises whenever we seek an inverse for a function which is not one-to-one. The squaring function on $\mathbb{R}$ sends both 1 and -1 to 1, but we want a single value 1 for sqrt(1). Thus we have to restrict the squaring function to $\mathbb{R}^{\geqslant 0}$.

15.5 Conditions for inversion

A function f can have an inverse without its domain and codomain being equal.

The inverse of a function $f : A \to B$ is a function $f^{-1} : B \to A$ such that

$$f^{-1} \circ f = i_A \quad \text{and} \quad f \circ f^{-1} = i_B.$$

Note that $f^{-1} \circ f$ and $f \circ f^{-1}$ are both identity functions but they have different domains.

Not every function has an inverse, but we can neatly classify the ones that do.

Let $f : A \to B$ be a function. Then $f^{-1} : B \to A$ exists if and only if f is one-to-one and onto.

Let $f : A \to B$.

The function $f^{-1} : B \to A$ exists if and only if f is one-to-one and onto.

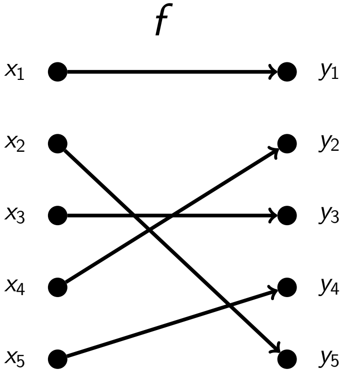
(Remember onto means image$(f) = B$.)

If it exists, $f^{-1} : B \to A$ is defined by $f^{-1}(y)$ equals the unique $x \in A$ such that $f(x) = y$.

We have $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

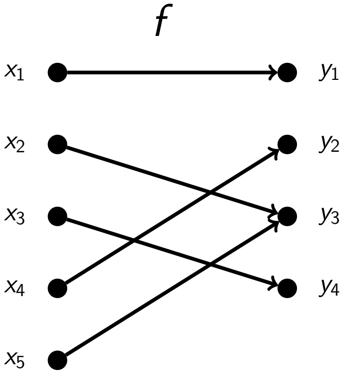
Note f^{-1} is just a notation for "the inverse function of f".
It is *not* an exponential.

Question Let $f : A \to B$ be the function pictured below.



Does f^{-1} exist? Yes.

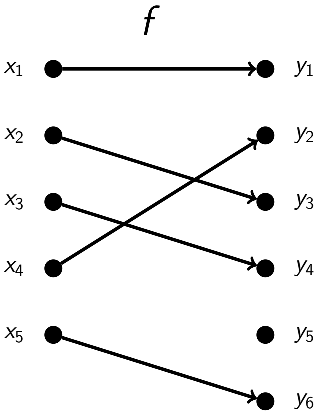
Question Let $f : A \to B$ be the function pictured below.



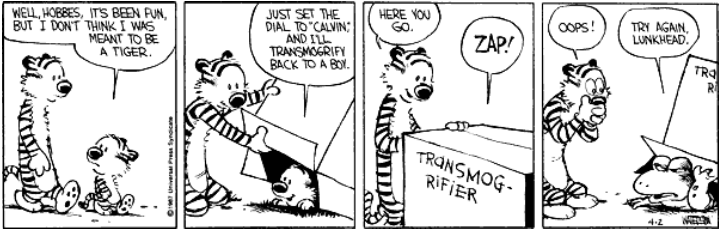
Does f^{-1} exist? No.

f is not one-to-one.

Question Let $f : A \to B$ be the function pictured below.



Does f^{-1} exist? No.
f is not onto.


WELL, HOBBES, IT'S BEEN FUN, BUT I DON'T THINK I WAS MEANT TO BE A TIGER.
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JUST SET THE DIAL TO "CALVIN," AND I'LL TRANSMOGRIFY BACK TO A BOY.
HERE YOU GO.
ZAP!
TRANSMOG-RIFIER
OOPS!
TRY AGAIN, LUNKHEAD.
4-2

Question 15.4 What feature do

$\neg : \mathbb{B} \to \mathbb{B}$ defined by $\neg(x) = \neg x$;

$f(x) : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ defined by $f(x) = \frac{1}{x}$; and

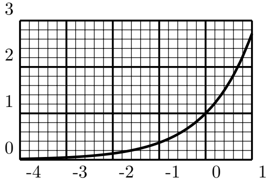
$g(x) : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ defined by $g(x) = \frac{x}{x-1}$;

have in common?

They are their own inverses.

Example: e^x and $\log$

Consider $f : \mathbb{R} \to \mathbb{R}^{\geqslant 0} - \{0\}$ defined by $f(x) = e^x$. We know that e^x is one-to-one (e.g. because it is strictly increasing), and onto. So it has an inverse f^{-1} on $\mathbb{R}^{\geqslant 0} - \{0\}$.



Plot of $y = e^x$.

In fact, $f^{-1} = \log(y)$ where

$$\log : \mathbb{R}^{\geqslant 0} - \{0\} \to \mathbb{R}.$$

Now

$$e^{\log x} = x \quad \text{and} \quad \log(e^x) = x,$$

so $e^{\log x}$ and $\log(e^x)$ are both identity functions, but they have different domains.

The domain of $e^{\log x}$ is $\mathbb{R}^{\geqslant 0} - \{0\}$ (note log is defined only for reals > 0). The domain of $\log(e^x)$ is $\mathbb{R}$.

Question Let $f : \{x : x \text{ is a Monash student}\} \to \mathbb{N}$ be the function defined by $f(x)$ equals the ID number of x. Does f^{-1} exist?

Answer

No. f is not onto. (E.g. there is no student with ID number 10^{200}.)

Flux Exercise

Let g and h be the functions

$g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $g(a, b) = ab$

$h : \{C : C \text{ is a circle in the plane with centre } (0, 0)\} \to \mathbb{R}$ defined by $h(C)$ is the area of C.

Does g^{-1} exist? Does h^{-1} exist?

A. Yes, yes

B. Yes, no

C. No, yes

D. No, no