

MAT1830

Lecture 14: Examples of Functions

The functions discussed in the last lecture were familiar functions of real numbers. Many other examples occur elsewhere, however.

14.1 Functions of several variables

We might define a function

$$\text{sum} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \quad \text{by} \quad \text{sum}(x, y) = x + y.$$

Because the domain of this function is $\mathbb{R} \times \mathbb{R}$, the inputs to this function are ordered pairs (x, y) of real numbers. Because its codomain is \mathbb{R} , we are guaranteed that each output will be a real number. This function can be thought of as a function of two variables x and y .

Similarly we might define a function

$$\text{binomial} : \mathbb{R} \times \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{R}$$

by

$$\text{binomial}(a, b, n) = (a + b)^n.$$

Here the inputs are ordered triples (x, y, n) such that x and y are real numbers and n is a natural number. We can think of this as a function of three variables.

Question What are the ordered pairs which define the function $\text{sum} : \{1, 2\} \times \{1, 2\} \rightarrow \mathbb{N}$ defined by $\text{sum}(x, y) = x + y$?

Answer

We have $\text{sum}((1, 1)) = 2$, $\text{sum}((1, 2)) = 3$, $\text{sum}((2, 1)) = 3$, and $\text{sum}((2, 2)) = 4$.

So $\{ ((1, 1), 2), ((1, 2), 3), ((2, 1), 3), ((2, 2), 4) \}$.

Note We often abbreviate $f((x, y))$ to $f(x, y)$ and so on when dealing with multivariable functions.

Question 14.1 Suggest domains and codomains for the following functions.

gcd domain: $\mathbb{Z} \times \mathbb{Z} - \{(0,0)\}$ codomain: \mathbb{N}

reciprocal domain: $\mathbb{R} - \{0\}$ codomain: $\mathbb{R} - \{0\}$

Flux Exercise

Suggest a domain and a codomain for a \cap (intersection) function for sets of real numbers.

- A. domain: $\mathbb{R} \times \mathbb{R}$, codomain: \mathbb{R}
- B. domain: $\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R})$, codomain: $\mathcal{P}(\mathbb{R})$
- C. domain: $\mathcal{P}(\mathbb{R})$, codomain: $\mathcal{P}(\mathbb{R})$
- D. domain: $\mathbb{R} \times \mathbb{R}$, codomain: $\mathcal{P}(\mathbb{R})$

Example Input: $(\{1, 2, 3, 4\}, \{2, 3, \pi\})$ Output: $\{2, 3\}$

14.2 Sequences

An infinite sequence of numbers, such as

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots,$$

can be viewed as the function $f : \mathbb{N} \rightarrow \mathbb{R}$ defined by $f(n) = 2^{-n}$. In this case, the inputs to f are natural numbers, and its outputs are real numbers.

Any infinite sequence $a_0, a_1, a_2, a_3, \dots$ can be viewed as a function $g(n) = a_n$ from \mathbb{N} to some set containing the values a_n .

Question For each of the following sequences, find a function f such that the sequence is $f(0), f(1), f(2), \dots$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \dots$$

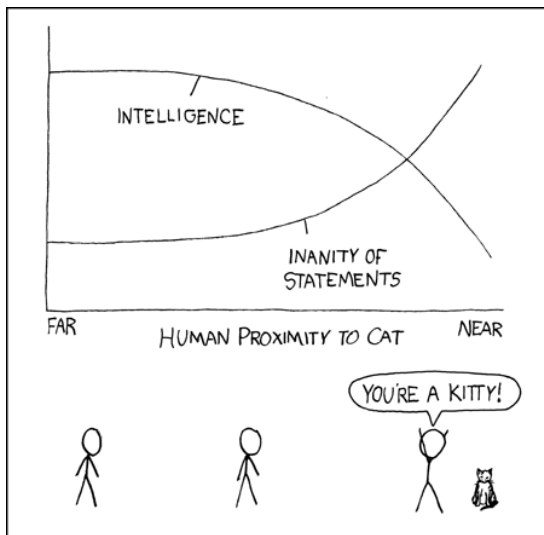
$$f : \mathbb{N} \rightarrow \mathbb{Q}, f(n) = \frac{1}{n+1}$$

$$5, 1, -3, -7, -11, -15, \dots$$

$$f : \mathbb{N} \rightarrow \mathbb{Z}, f(n) = 5 - 4n$$

$$4, 12, 36, 108, 324, 972, \dots$$

$$f : \mathbb{N} \rightarrow \mathbb{Z}, f(n) = 4(3^n)$$



14.3 Characteristic functions

A subset of $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ can be represented by its characteristic function. For example, the set of squares is represented by the function $\chi : \mathbb{N} \rightarrow \{0, 1\}$ defined by

$$\chi(n) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{if } n \text{ is not a square} \end{cases}$$

which has the following sequence of values

110010000100000010000000010000000000100...

(with 1s at the positions of the squares 0, 1, 4, 9, 16, 25, 36, ...).

Any property of natural numbers can likewise be represented by a characteristic function. For example, the function χ above represents the property of being a square.

Thus any set or property of natural numbers is represented by a function

$$\chi : \mathbb{N} \rightarrow \{0, 1\}.$$

Characteristic functions of two or more variables represent relations between two or more objects. For example, the relation $x \leq y$ between real numbers x and y has the characteristic function $\chi : \mathbb{R} \times \mathbb{R} \rightarrow \{0, 1\}$ defined by

$$\chi(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise.} \end{cases}$$

Question 14.2 If A and B are subsets of \mathbb{N} with characteristic functions $\chi_A(n)$ and $\chi_B(n)$, then what set does the function $\chi_A(n)\chi_B(n)$ represent?

Answer

If $n \in A$ and $n \in B$ then $\chi_A(n)\chi_B(n) = 1 \times 1 = 1$.

If $n \in A$ and $n \notin B$ then $\chi_A(n)\chi_B(n) = 1 \times 0 = 0$.

If $n \notin A$ and $n \in B$ then $\chi_A(n)\chi_B(n) = 0 \times 1 = 0$.

If $n \notin A$ and $n \notin B$ then $\chi_A(n)\chi_B(n) = 0 \times 0 = 0$.

So $\chi_A(n)\chi_B(n)$ is the characteristic function of $A \cap B$.

Question Let d be a positive integer. If $\chi_d : \mathbb{N} \rightarrow \{0, 1\}$ is a function defined by

$$\chi_d(x) = \begin{cases} 1, & \text{if } x \text{ divides } d; \\ 0, & \text{if } x \text{ does not divide } d. \end{cases}$$

then what is $1\chi_d(1) + 2\chi_d(2) + 3\chi_d(3) + \cdots + d\chi_d(d)$?

Answer

The sum of the positive divisors of d .

Question How many functions are there with domain $\{1, 2, 3, 4\}$ and codomain $\{-1, 0, 1\}$?

Answer

The domain has 4 elements. (There are 4 possible inputs.)
For each input, we can decide if it's mapped to -1 or 0 or 1 .
We can do this in $\underbrace{3 \times 3 \times \cdots \times 3}_4 = 3^4 = 81$ ways.

Question How many functions are there with domain X and codomain Y ?

Answer

The domain has $|X|$ elements. (There are $|X|$ possible inputs.)
For each input, we have $|Y|$ options for where it's mapped to.
We can do this in $\underbrace{|Y| \times |Y| \times \cdots \times |Y|}_{|X|} = |Y|^{|X|}$ ways.

14.4 Boolean functions

The connectives \wedge , \vee and \neg are functions of variables whose values come from the set $\mathbb{B} = \{\mathsf{T}, \mathsf{F}\}$ of Boolean values (named after George Boole).

\neg is a function of one variable, so

$$\neg : \mathbb{B} \rightarrow \mathbb{B}$$

and it is completely defined by giving its values on T and F , namely

$$\neg \mathsf{T} = \mathsf{F} \quad \text{and} \quad \neg \mathsf{F} = \mathsf{T}.$$

This is what we previously did by giving the truth table of \neg .

\wedge and \vee are functions of two variables, so

$$\wedge : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$$

and

$$\vee : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B}$$

They are completely defined by giving their values on the pairs $\{\mathsf{T}, \mathsf{T}\}$, $\{\mathsf{T}, \mathsf{F}\}$, $\{\mathsf{F}, \mathsf{T}\}$, $\{\mathsf{F}, \mathsf{F}\}$ in $\mathbb{B} \times \mathbb{B}$, which is what their truth tables do.

Flux Exercise

Let $\mathbb{B} = \{0, 1\}$. How many functions are there with domain $\underbrace{\mathbb{B} \times \mathbb{B} \times \cdots \times \mathbb{B}}_n$ and codomain \mathbb{B} ?

- A. n^2
- B. $2^{(n^2)}$
- C. 2^n
- D. $2^{(2^n)}$

Example (Hamming distance)

Let B_n be the set of all binary strings of length n .

Hamming distance is a function $h : B_n \times B_n \rightarrow \mathbb{N}$ defined by $h(s, t)$ equals the number of places in which s and t disagree.

For example,

$$h(000, 101) = 2,$$

$$h(011, 010) = 1,$$

$$h(10111, 01000) = 5.$$

A set of binary strings of length n such that any two different strings in the set have Hamming distance at least d is called a *binary error correcting code of length n and distance d* .

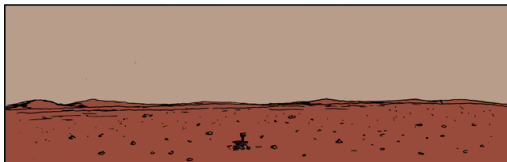
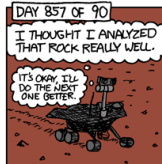
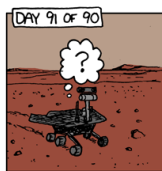
These are useful in sending information across noisy channels.

$\{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111\}$
is a binary code of length 4 and distance 2.

If we only send strings in this set across a channel and at most one error occurs in each string then we will be able to detect the errors.

$\{0000000, 1110000, 1001100, 0111100, 0101010, 1011010, 1100110, 0010110, 1101001, 0011001, 0100101, 1010101, 1000011, 0110011, 0001111, 1111111\}$
is a binary code of length 7 and distance 3.

If we only send strings in this set across a channel and at most one error occurs in each string then we will be able to *correct* the errors on the fly.



14.5* Characteristic functions and subsets of \mathbb{N}

Mathematicians say that two (possibly infinite) sets A and B have the same *cardinality* (size) if there is a one-to-one and onto function from A to B . This function associates each element of A with a unique element of B and vice-versa. With this definition, it is not too hard to show that, for example, \mathbb{N} and \mathbb{Z} have the same cardinality (they are both “countably infinite”).

It turns out, though, that $\mathcal{P}(\mathbb{N})$ has a strictly greater cardinality than \mathbb{N} . We can prove this by showing: *no sequence $f_0, f_1, f_2, f_3, \dots$ includes all characteristic functions for subsets of \mathbb{N} .* (This shows that there are more characteristic functions than natural numbers.)

In fact, for any infinite list $f_0, f_1, f_2, f_3, \dots$ of characteristic functions, we can define a characteristic function f which is *not* on the list. Imagine each function given as the infinite sequence of its values, so the list might look like this:

f_0	values	<u>0</u> 101010101...
f_1	values	0 <u>0</u> 000111101...
f_2	values	11 <u>1</u> 1111111...
f_3	values	000 <u>0</u> 000000...
f_4	values	1001 <u>0</u> 01001...
		\vdots

Now if we switch each of the underlined values to its opposite, we get a characteristic function

$$f(n) = \begin{cases} 1 & \text{if } f_n(n) = 0 \\ 0 & \text{if } f_n(n) = 1 \end{cases}$$

which is *different* from each function on the list. In fact, it has a different value from f_n on the number n .

For the given example, f has values

$$11011\dots$$

The construction of f is sometimes called a “diagonalisation argument”, because we get its values by switching values along the diagonal in the table of values of $f_0, f_1, f_2, f_3, \dots$