

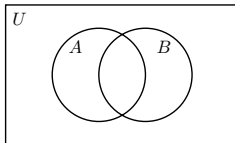
# MAT1830

## Lecture 12: Operations on Sets

There is an “arithmetic” of sets similar to ordinary arithmetic. There are operations similar to addition, subtraction and multiplication.

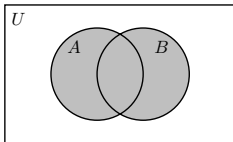
### 12.1 Venn diagrams

The simple operations on sets can be visualised with the help of *Venn diagrams*, which show sets  $A, B, C, \dots$  as disks within a rectangle representing the universal set  $U$ .



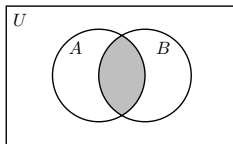
### 12.2 Union $A \cup B$

The union  $A \cup B$  of sets  $A$  and  $B$  consists of the elements in  $A$  *or*  $B$ , and is indicated by the shaded region in the following Venn diagram.



### 12.3 Intersection $A \cap B$

The intersection  $A \cap B$  of sets  $A$  and  $B$  consists of the elements in  $A$  *and*  $B$ , indicated by the shaded region in the following Venn diagram.



## Questions

What is  $\{1, 2, 3\} \cup \{2, 5, 7\}$ ?  $\{1, 2, 3, 5, 7\}$

What is  $\{1, 2, 3\} \cap \{2, 3, 6, 7\}$ ?  $\{2, 3\}$

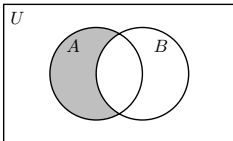
What is  $\{1, 2, 3\} \cup \{2, 3\}$ ?  $\{1, 2, 3\}$

What is  $\{1, 2, 3\} \cap \{3\}$ ?  $\{3\}$

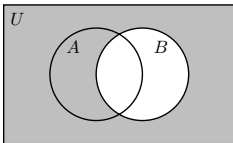
What is  $\{1, 2, 3\} \cap \{7, 8\}$ ?  $\{\}$  (or  $\emptyset$  if you prefer)

## 12.4 Difference $A - B$

The difference  $A - B$  of sets  $A$  and  $B$  consists of the elements in  $A$  and *not* in  $B$ , indicated by the shaded region in the following Venn diagram.

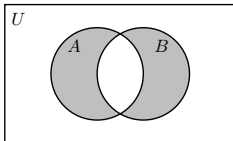


The difference  $U - B$  relative to the universal set  $U$  is called the *complement*  $\overline{B}$  of  $B$ . Here is the Venn diagram of  $\overline{B}$ .



## 12.5 Symmetric difference $A \triangle B$

The union of  $A - B$  and  $B - A$  is called the *symmetric difference*  $A \triangle B$  of  $A$  and  $B$ .



$A \triangle B$  consists of the elements of *one of*  $A, B$  but not the other.

It is clear from the diagram that we have not only

$$A \triangle B = (A - B) \cup (B - A),$$

but also

$$A \triangle B = (A \cup B) - (A \cap B).$$

## Questions

What is  $\{1, 2, 3\} - \{2, 5, 7\}$ ?  $\{1, 3\}$

What is  $\{1, 2, 3\} \triangle \{2, 3, 6, 7\}$ ?  $\{1, 6, 7\}$

What is  $\{2, 3\} - \{1, 2, 3\}$ ?  $\{\}$

What is  $\{1, 2, 3\} \triangle \{3\}$ ?  $\{1, 2\}$

What is  $\{1, 2, 3\} - \{7, 8\}$ ?  $\{1, 2, 3\}$

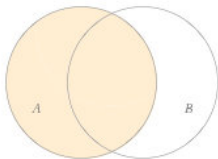
## Flux Exercise

Let  $S = \{-2, -1, 0, 1, 2\} \cap \mathbb{N}$ . If we know that  $S \subseteq (\{-1, 0, 1\} \cup T)$ , what can we say about  $T$ ?

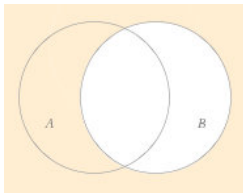
- A.  $T$  must equal  $\{2\}$
- B.  $T$  must equal  $\mathbb{N}$
- C.  $T$  can be any set such that  $-2 \in T$
- D.  $T$  can be any set such that  $2 \in T$



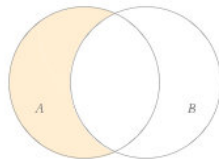
**Question 12.1** Draw a Venn diagram for  $A \cap \overline{B}$ . What is another name for this set?



$A$

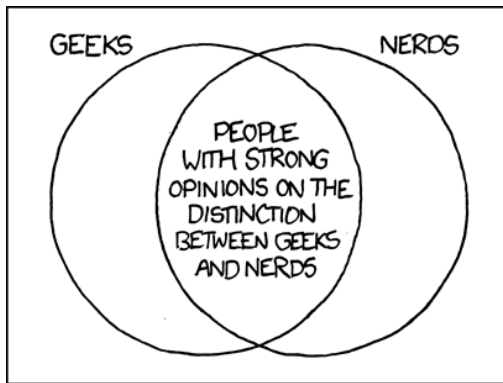


$\overline{B}$



$A \cap \overline{B}$

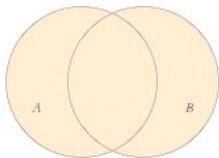
So  $A \cap \overline{B} = A - B$ .



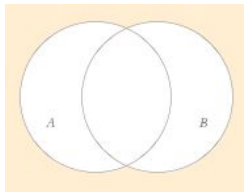
## Set operations and logic operations

$x \in A \cup B$	if and only if	$(x \in A) \vee (x \in B)$
$x \in A \cap B$	if and only if	$(x \in A) \wedge (x \in B)$
$x \in A - B$	if and only if	$(x \in A) \wedge (x \notin B)$
$x \in A \triangle B$	if and only if	$(x \in A) \underline{\vee} (x \in B)$

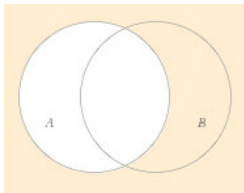
**Question 12.2** Show that  $\overline{A \cup B} = \overline{A} \cap \overline{B}$  is true using Venn diagrams.



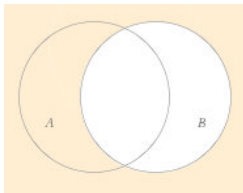
$A \cup B$



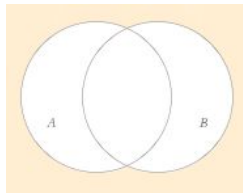
$\overline{A \cup B}$



$\overline{A}$



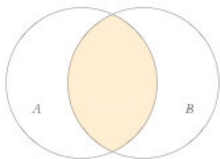
$\overline{B}$



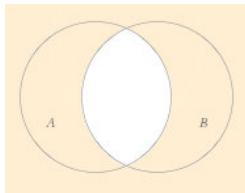
$\overline{A} \cap \overline{B}$

So  $\overline{A \cup B} = \overline{A} \cap \overline{B}$ .

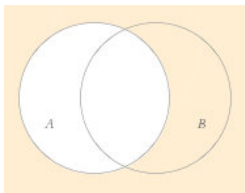
**Question 12.2 (cont)** Show that  $\overline{A \cap B} = \overline{A} \cup \overline{B}$  is true using Venn diagrams.



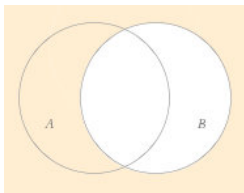
$A \cap B$



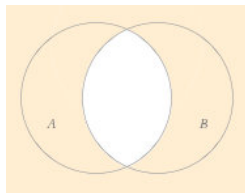
$\overline{A \cap B}$



$\overline{A}$



$\overline{B}$



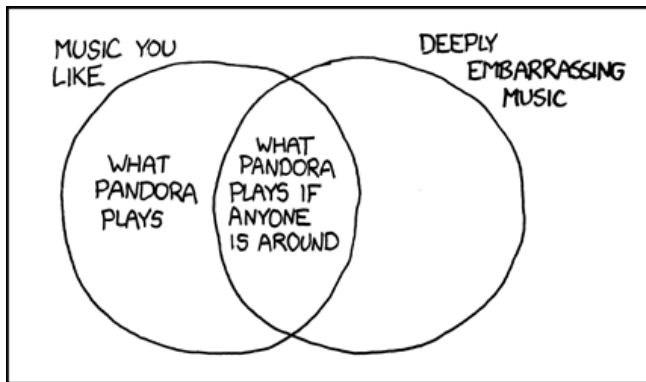
$\overline{A} \cup \overline{B}$

So  $\overline{A \cap B} = \overline{A} \cup \overline{B}$ .

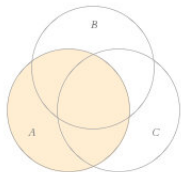
**Question** Show that  $\overline{A \cup B} = \overline{A} \cap \overline{B}$  is true using logic.

$$\begin{aligned}x \in \overline{A \cup B} &\equiv \neg(x \in A \cup B) \\&\equiv \neg((x \in A) \vee (x \in B)) \\&\equiv \neg(x \in A) \wedge \neg(x \in B) \\&\equiv (x \in \overline{A}) \wedge (x \in \overline{B}) \\&\equiv x \in \overline{A} \cap \overline{B}\end{aligned}$$

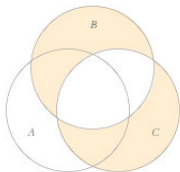
So  $\overline{A \cup B} = \overline{A} \cap \overline{B}$ .



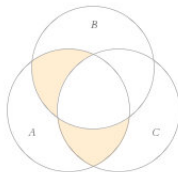
**Question 12.3** Find whether  $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$  is true using Venn diagrams.



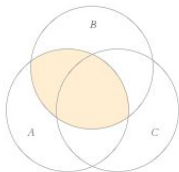
$A$



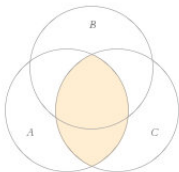
$B \triangle C$



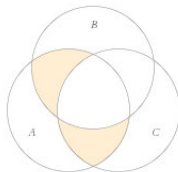
$A \cap (B \triangle C)$



$A \cap B$



$A \cap C$

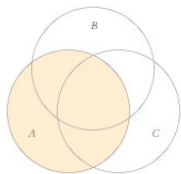


$(A \cap B) \triangle (A \cap C)$

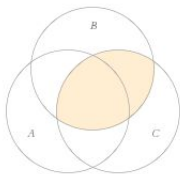
So  $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$ .



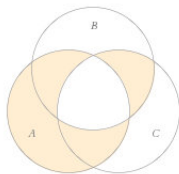
**Question 12.3 (cont)** Find whether  $A \Delta (B \cap C) = (A \Delta B) \cap (A \Delta C)$  is true using Venn diagrams.



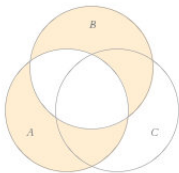
$A$



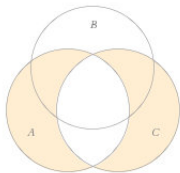
$B \cap C$



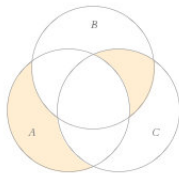
$A \Delta (B \cap C)$



$A \Delta B$

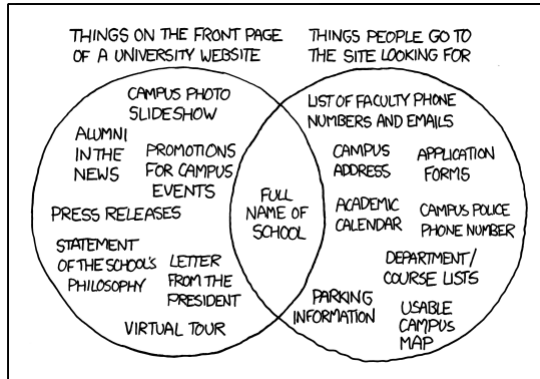


$A \Delta C$



$(A \Delta B) \cap (A \Delta C)$

So  $A \Delta (B \cap C) = (A \Delta B) \cap (A \Delta C)$  is not true \*in general\*.  
(But it will be true for some choices of  $A$ ,  $B$  and  $C$ .)



## Ordered pairs

For sets we have  $\{a, b\} = \{b, a\}$ .

But sometimes order is important.

So we define ordered pairs  $(a, b)$ , where the order is significant:  
 $(a, b) \neq (b, a)$ .

We can of course also define ordered triples  $(a, b, c)$  etc.

## 12.6 Ordered Pairs

Sometimes we do want order to be important. In computer science arrays are ubiquitous examples of ordered data structures. In maths, *ordered pairs* are frequently used. An ordered pair  $(a, b)$  consists simply of a first object  $a$  and a second object  $b$ . The objects  $a$  and  $b$  are sometimes called the *entries* or *coordinates* of the ordered pair.

Two ordered pairs  $(a, b)$  and  $(c, d)$  are equal if and only if  $a = c$  and  $b = d$ .

**Example.**  $\{0, 1\} = \{1, 0\}$  but  $(0, 1) \neq (1, 0)$ .

There's no reason we need to stop with pairs. We can similarly define ordered triples, quadruples, and so on. When there are  $k$  coordinates, we call the object an *ordered  $k$ -tuple*. Two ordered  $k$ -tuples are equal if and only if their  $i$ th coordinates are equal for  $i = 1, 2, \dots, k$ .

## How can we represent ordered pairs using sets?\*

If we want to define ordered pairs just using sets, how can we introduce an order?

One way is to define  $(a, b) = \{\{a, b\}, \{a\}\}$ .

The 2-element set tells us what the two things in the ordered pair are (but not their order).

The 1-element set tells us which of them comes first.

\* this method of defining of ordered pairs is not assessable, but your ability to work with ordered pairs is.

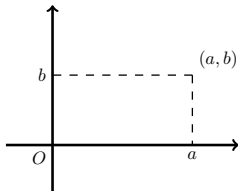
## 12.7 Cartesian product $A \times B$

The set of ordered pairs

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

is the *cartesian product* of sets  $A$  and  $B$ .

The commonest example is where  $A = B = \mathbb{R}$  (the set of real numbers, or the number line). Then the pairs  $(a, b)$  are points in the plane, so  $\mathbb{R} \times \mathbb{R}$  is the plane.



Because Descartes used this idea in geometry, the cartesian product is named after him.

Remember the elements of  $A \times B$  are always *ordered pairs*.

**Question** If  $A = \{0, 1\}$  and  $B = \{1, 2, 3\}$  what is  $A \times B$ ?

$\{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3)\}$

**Question** If  $A = \{0, 1\}$  what is  $A \times \mathbb{N}$ ?

$\{(0, 0), (1, 0), (0, 1), (1, 1), (0, 2), (1, 2), \dots\}$

## Flux Exercise

Let  $S = \{-1, 1\} \times \{0, 1, 2\}$ . Is  $(-1, 1) \in S$ ? Is  $(0, 1) \in S$ ?

- A. Yes, yes
- B. Yes, no
- C. No, yes
- D. No, no

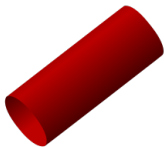


## 12.8 $A \times B$ and multiplication

If  $A$  has  $|A|$  elements and  $B$  has  $|B|$  elements, then  $A \times B$  has  $|A| \times |B|$  elements.

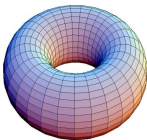
Similarly, if  $L$  is a line of length  $l$ , and  $W$  is a line of length  $w$ , then  $L \times W$  is a rectangle of area  $l \times w$ . In fact, we call it an “ $l \times w$  rectangle.” This is probably the reason for using the  $\times$  sign, and for calling  $A \times B$  a “product.”

**Question 12.4** If  $\text{line} \times \text{line} = \text{plane}$ , what is  $\text{line} \times \text{circle}$ ?



A cylinder.

**Question 12.4 (cont)** What is  $\text{circle} \times \text{circle}$ ?



A torus.