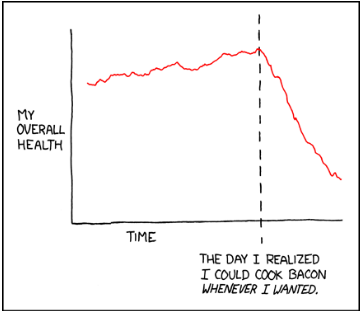
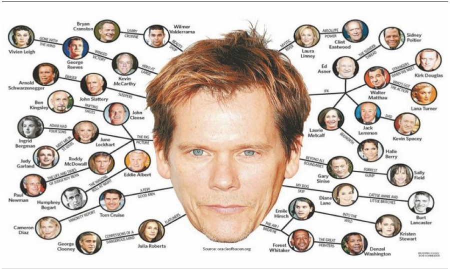
MAT1830

Lecture 29: Graphs



not a graph



graph

Graphs - why should you care?

Graph theory is used extensively in computer science for:

- analysing networks:
  - computer
  - social
  - transport
  - etc. etc.
- representing and algorithmically solving problems
  - especially scheduling and allocation problems

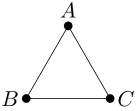
A *graph* consists of a set of objects called *vertices* together with a set of unordered pairs of vertices, called *edges*.

Graphs are normally represented by pictures, with vertex A represented by a dot labelled A and each edge $\{A, B\}$ represented by a line joining A and B. Sometimes we do not include the vertex labels when the names of the vertices are not important.

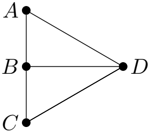
Such pictures are helpful for displaying data or relationships, and they make it easy to recognise properties which might otherwise not be noticed.

The description by sets of vertices and edges is useful when graphs have to be manipulated by computer. It is also a useful starting point for precise definitions of graph concepts.

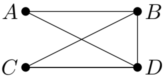
29.1 Examples of graphs

Description	**Picture**
vertex set: $\{A, B, C\}$ edge set: $\{\{A, B\}, \{B, C\}, \{C, A\}\}$	

To save space, an edge $\{A, B\}$ is often written simply as AB (or, equivalently, BA). We will do this from now on.

Description	**Picture**
vertex set: $\{A, B, C, D\}$ edge set: $\{AB, BC, AD, BD, CD\}$	

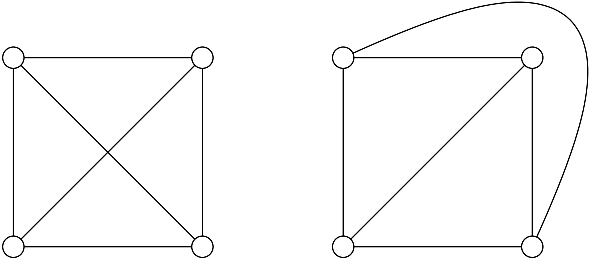
Warning: A graph can be represented by pictures that look very different. This last example could be redrawn as:



Edge crossings

When we draw a graph, the lines that represent edges can be straight or curved.

If they cross over each other it doesn't mean anything.

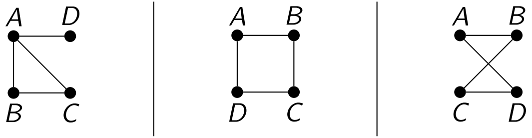


These drawings are two different ways to represent the **same** graph.

There is **no** vertex in the middle of the left hand picture.

Question

29.1 Write the vertex sets and edge sets for the graphs corresponding to the following pictures



Answer for left picture
vertex set: $\{A, B, C, D\}$
edge set: $\{AB, AC, AD, BC\}$

Answer for centre picture
vertex set: $\{A, B, C, D\}$
edge set: $\{AB, AD, BC, CD\}$

Answer for right picture
vertex set: $\{A, B, C, D\}$
edge set: $\{AB, AD, BC, CD\}$

The centre and right pictures are of the same graph!

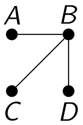
Question

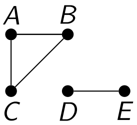
29.2 Draw pictures of graphs with the following vertex and edge sets.

(a) vertex set: $\{A, B, C, D\}$
edge set: $\{AB, BC, BD\}$

(b) vertex set: $\{A, B, C, D, E\}$
edge set: $\{AB, BC, CA, DE\}$

Answer

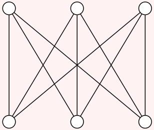




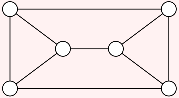
Flux Exercise

Three of the following pictures represent the same graph. The odd one out is:

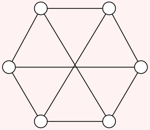
A.



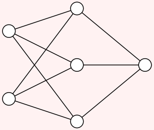
B.



C.



D.



Answer B. It's the only one that contains a triangle.

29.2 Variants of graphs

Many different variants of graphs are used in different contexts. For example *multigraphs* are allowed to have multiple edges between the same pair of vertices and also edges called loops joining a vertex to itself. (A normal graph is sometimes called a *simple graph* to emphasise it is not a multigraph.) In *directed graphs* the edges have a direction associated with them. In *hypergraphs* edges may join more that two vertices. For various problems it is useful to consider graphs where the vertices and/or edges have weights, labels or colours etc. etc. We will focus on the basic definition here, but will occasionally mention these variants.

For us a *graph* is not allowed to include multiple edges between the same vertices or edges from vertices to themselves.

A *multigraph* is allowed to include these things (but we will only mention these briefly).

We sometimes say a graph is *simple* to emphasise it is not allowed these things.

So for us 'graph' = 'simple graph'.

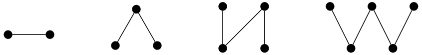
29.3 Important kinds of graphs

A *complete graph* is a graph in which every pair of vertices is joined by an edge.

Below are pictures of complete graphs with 2, 3, 4 and 5 vertices.

A *path of length ℓ* is a graph whose vertices can be renamed so that its vertex set is $\{V_1, \ldots, V_{\ell+1}\}$ and its edge set is $\{V_1V_2, V_2V_3, \ldots, V_\ell V_{\ell+1}\}$.

We say this is a path *from* V_1 *to* $V_{\ell+1}$ (or, equivalently, from $V_{\ell+1}$ to V_1). Note that the length of a path refers to its number of edges, not its number of vertices. Below are pictures of paths of lengths 1, 2, 3 and 4.

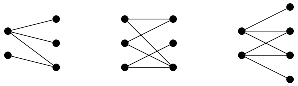


A *cycle of length* ℓ (for $\ell \geqslant 3$) is a graph whose vertices can be renamed so that its vertex set is $\{V_1, \ldots, V_\ell\}$ and its edge set is $\{V_1V_2, V_2V_3, \ldots, V_{\ell-1}V_\ell, V_\ell V_1\}$.

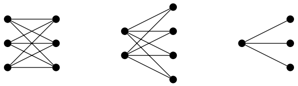
Below are pictures of cycles of lengths 3, 4, 5 and 6.

A graph is *bipartite* if its vertices can be renamed so that its vertex set is $\{U_1, U_2, \ldots, U_i, V_1, V_2, \ldots, V_j\}$ and each of its edges joins a vertex in $\{U_1, U_2, \ldots, U_i\}$ to a vertex in $\{V_1, V_2, \ldots, V_j\}$.

The graph is a *complete bipartite graph with parts of sizes i and j* if every vertex in $\{U_1, U_2, \ldots, U_i\}$ is joined by an edge to every vertex in $\{V_1, V_2, \ldots, V_j\}$. Below are pictures of some bipartite graphs. The vertices have been arranged to the left and right to make it obvious the graphs are bipartite.



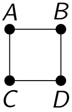
Below are pictures of some complete bipartite graphs.

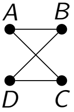


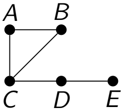
Notice that all paths are bipartite and cycles of even length are bipartite. Cycles of odd length are not bipartite, however.

Roughly speaking a graph is bipartite if **we can** divide its vertices into 'left' and 'right' parts so that there are no edges between two left vertices and no edges between two right vertices.

Questions

Is 

bipartite? Yes. 

Is 

bipartite? No. Two of A, B, C would be in the same part.

How many edges in a complete bipartite graph with parts of sizes i and j? ij

Question

29.3 What is the maximum number of edges that a bipartite graph with 6 vertices can have? What is the maximum number of edges that a bipartite graph with n vertices can have?

Answer

Because we're interested in the maximum we only need to think about complete bipartite graphs.

For 6 vertices we can have...
parts of sizes 1 and 5: 5 edges
parts of sizes 2 and 4: 8 edges
parts of sizes 3 and 3: 9 edges
So 9 edges is the maximum.

For n vertices when n is even, choose two parts of size $\frac{n}{2}$: $\frac{n^2}{4}$ edges.

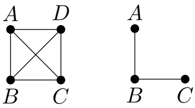
For n vertices when n is odd, choose parts of sizes $\frac{n-1}{2}$ and $\frac{n+1}{2}$: $\frac{n^2-1}{4}$ edges.

29.4 Subgraphs

A *subgraph* of a graph G is a graph whose vertex set is a subset of the vertex set of G and whose edge set is a subset of the edge set of G.

So a subgraph of a graph G is a graph that can be obtained from G by (possibly) deleting edges and/or vertices. Note that every graph is a subgraph of itself.

Example. the graph pictured on the right below is a subgraph of the graph pictured on the left.



Sometimes we say (for example):

"G contains a cycle of length 5"
to mean
"G has a subgraph that is a cycle of length 5"

"there is a path of length 6 in G"
to mean
"G has a subgraph that is a path of length 6"

Flux Exercise

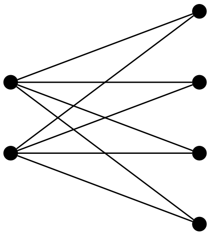
Let G be a complete bipartite graph with parts of sizes 2 and 4. Which of the following is true about the subgraphs of G?

A. G contains cycles of length 4 and 6

B. G contains paths of lengths 3 and 4

C. G contains a complete graph with 4 vertices

D. G contains a path of length 2 and a cycle of length 3



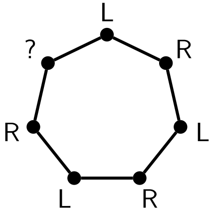
Answer B.

Sometimes important properties of a graph can be phrased in terms of its subgraphs. For example, the following is true.

> A graph is bipartite if and only if it has no subgraph that is an odd-length cycle.

Fact A graph is bipartite if and only if it has no subgraph that is an odd-length cycle.

It's fairly easy to see that a graph with an odd cycle cannot be bipartite:



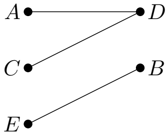
It's a bit harder but not too bad to show that a graph with no odd cycles must be bipartite. (We'll do it roughly soon.)

29.5 Connectivity

A graph G is *connected* if, for any two of its vertices A and B, it has a subgraph that is a path from A to B.

A graph that is not connected is *disconnected*. All the examples of graphs we have seen so far in this lecture have been connected.

Example. The graph pictured below is disconnected. For example it does not contain a path from D to E.



Fact. Every connected graph with no subgraph that is an odd-length cycle is bipartite.

Proof sketch Let $P(n)$ be the statement that "a connected graph with n vertices and no odd cycles is bipartite".

Base step. A graph with 2 vertices must be bipartite, so $P(2)$ is true.

Induction step. Let $k \geq 2$ be an integer. Suppose that $P(k)$ is true.

We want to prove that $P(k+1)$ is true: that a connected graph with $k+1$ vertices and no odd cycles is bipartite.

- Let G be a connected graph with $k+1$ vertices and no odd cycles.
- (Temporarily) delete a vertex A of G.
- The remaining graph has k vertices and no odd cycles. By $P(k)$ we can divide its vertices into left and right parts. In each connected 'piece' choose the left part so it has a vertex joined to A.
- G had no odd cycles, so A can't have been joined to any vertices on the right.
- So we can add A back into the right part. So $P(k+1)$ is true.

So $P(n)$ is true for each integer $n \geq 2$.