MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #10 and Additional Practice Questions

Tutorial Questions

1. Find recursive definitions for the following.

   (a) The sequence $a_0, a_1, a_2, \ldots$ where $a_n = 2^n$ for $n \geq 0$.

   (b) The sequence $b_0, b_1, b_2, \ldots$ where $b_n = n^2$ for $n \geq 0$. (Your recurrence may involve n, but not n^2.)

2. For each of the following graphs, give the following.

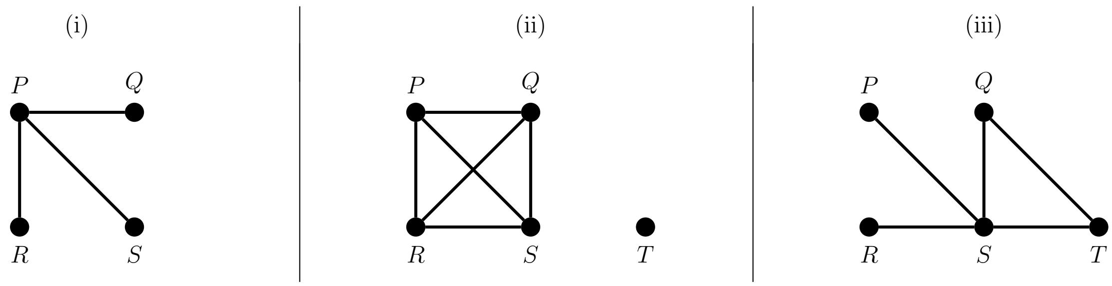
   (a) The graph's vertex set and edge set.

   (b) The length of a longest path in the graph and an example of such a path.

   (c) The length of a longest trail in the graph and an example of such a trail.

   (d) Whether the graph is bipartite.

   (e) The number of edges in a largest matching in the graph. A *matching* is defined as a subgraph in which each vertex is in exactly one edge.



3. (a) Draw the simple graph with adjacency matrix

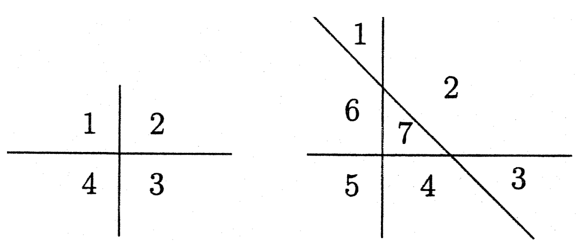
$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

   using V_1, V_2, V_3, V_4 as the names for the vertices corresponding to columns 1, 2, 3, 4 respectively.

   (b) Find the number of walks of length 3 from V_1 to V_2 in the graph.

   (c) Without any calculation show that the top row of M^n for any even $n \geq 2$ is "1 0 0 0".

4. Let r_n be the number of regions created when the plane is divided by n straight (infinite) lines, with no two lines parallel and no three meeting in a single point. For example,



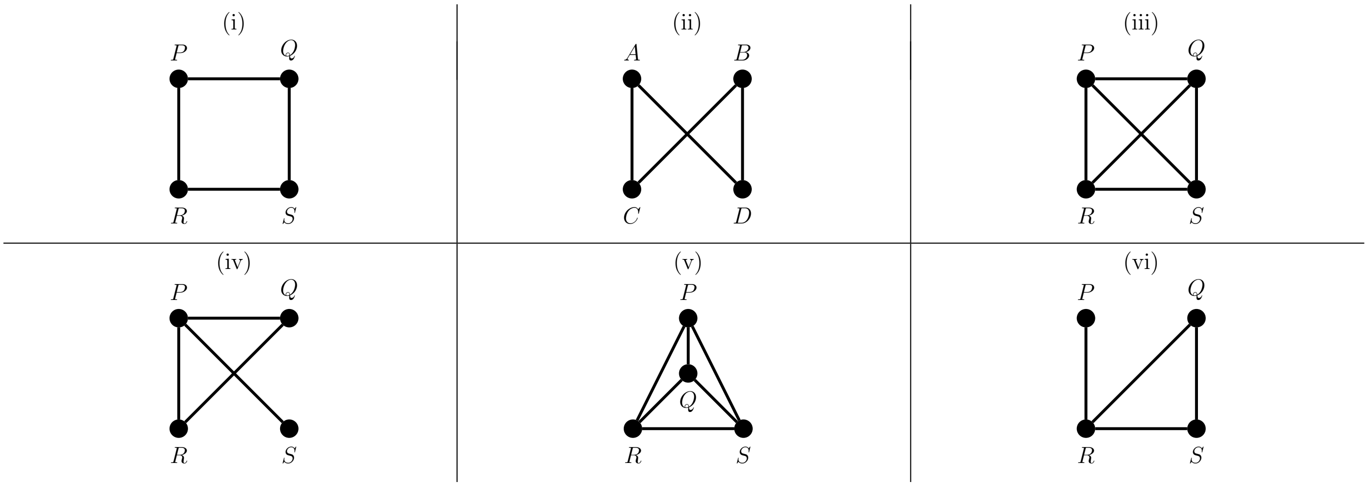
$r_2 = 4$ $\qquad$ $r_3 = 7$

   Find a recurrence for r_n.

(See over for practice questions.)

Practice Questions

1. Two graphs are equal if they have exactly the same vertex and edge sets. They are isomorphic if we can "rename" the vertices of one graph to make it equal to the other. Which of the following graphs are equal? Which are isomorphic? How would you prove this?



2. How would you change the definition of isomorphic graphs given above to make it more formal?

3. (a) What would you do if you saw someone drive past in a car that was isomorphic to yours?
   (b) What would you do if you saw someone drive past in a car that was equal to yours?

4. Suppose you want to network some computers together in such a way that

   - each computer is directly joined to at most three others; and
   - any two computers are either directly joined or are both directly joined to some third computer.

   Can you find a way to network 7 computers like this? 8? 10?

   (For a way to do this for 10 computers, google "Petersen graph". It turns out there's no way to do this with 9 computers – after you've seen lecture 31, try to prove this.)

5. Let $S(n, k)$ be the number of equivalence relations on the set $\{1, 2, \ldots, n\}$ with exactly k (non-empty) equivalence classes. Prove that $S(n, k) = kS(n-1, k) + S(n-1, k-1)$ for all integers n and k such that $n > k > 1$.

   ($S(n, k)$ are sometimes called *Stirling numbers of the second kind.*)