MAT1830

Lecture 2: Divisors and Primes

Number theory - why should you care?

Number theory is used in tonnes of places across computer science:

- pseudorandom number generation
- hash functions
- memory management
- error correction
- fast arithmetic operations
- cryptography and authentication

Definition An *integer* is a "whole number". It may be positive or negative or zero.

So the integers are the numbers

$$\ldots\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ldots.$$

Is 12 an integer? Yes.

Is -6 an integer? Yes.

Is $\frac{1}{2}$ an integer? No.

The set of all the integers is often written as $\mathbb{Z}$.

We say that integer a *divides* integer b if
$$b = qa \quad \text{for some integer } q.$$

Example. 2 divides 6 because $6 = 3 \times 2$.

This is the same as saying that division with remainder gives remainder 0. Thus a does *not* divide b when the remainder is $\neq 0$.

Example. 3 does not divide 14 because it leaves remainder 2: $\quad 14 = 4 \times 3 + 2$.

When a divides b we also say:

- a is a *divisor* of b,
- a is a *factor* of b,
- b is *divisible* by a,
- b is a *multiple* of a.

Does 7 divide 21?	Yes (because $21 = 3 \times 7$).
Does 8 divide 12?	No (because $12 = 1 \times 8 + 4$).
Does 25 divide 5?	No (because $5 = 0 \times 25 + 5$).

Flux Exercise (AVT95F)

Does 10 divide 60? Does 11 divide 11?

2.1 Primes

A positive integer $p > 1$ is a prime if its only positive integer divisors are 1 and p. Thus the first few prime numbers are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \ldots$$

The number 1 is not counted as a prime, as this would spoil the

Fundamental Theorem of Arithmetic. *Each integer > 1 can be expressed in exactly one way, up to order, as a product of primes.*

Example. $210 = 2 \times 3 \times 5 \times 7$, and this is the only product of primes which equals 210.

This would not be true if 1 was counted as a prime, because many factorisations involve 1. E.g.

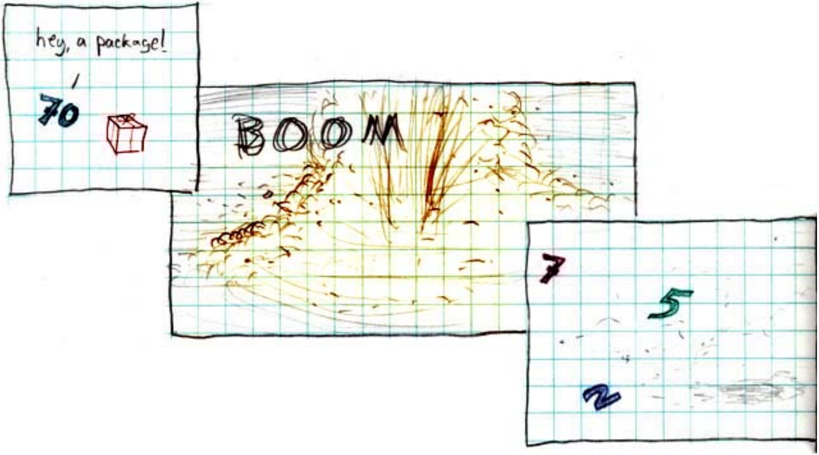
$$210 = 1 \times 2 \times 3 \times 5 \times 7 = 1^2 \times 2 \times 3 \times 5 \times 7 = \ldots$$

Question 2.4 What are the prime factorisations of 999 and 1000?

Answer

$$\begin{aligned} 999 &= 9 \times 111 \\ &= 3^2 \times (3 \times 37) \\ &= 3^3 \times 37 \end{aligned}$$

$$\begin{aligned} 1000 &= 10^3 \\ &= (2 \times 5)^3 \\ &= 2^3 \times 5^3 \end{aligned}$$



hey, a package!
70
BOOM
7
5
2

2.2 Recognising primes

If an integer $n > 1$ has a divisor, it has a divisor $\leqslant \sqrt{n}$, because for any divisor $a > \sqrt{n}$ we also have the divisor n/a, which is $< \sqrt{n}$.

Thus to test whether 10001 is prime, say, we only have to see whether any of the numbers $2, 3, 4, \ldots \leqslant 100$ divide 10001, since $\sqrt{10001} < 101$. (The least divisor found is in fact 73, because $10001 = 73 \times 137$.)

This explains a common algorithm for recognising whether n is prime: try dividing n by $a = 2, 3, \ldots$ while $a \leqslant \sqrt{n}$.

The algorithm is written with a boolean variable *prime*, and n is prime if $prime = \mathsf{T}$ (true) when the algorithm terminates.

```
assign a the value 2.
assign prime the value T.
while a ⩽ √n and prime= T
    if a divides n
        give prime the value F
    else
        increase the value of a by 1.
```

Divisors come in pairs!

For example, the divisors of 40 are paired as follows:

- 1, 40
- 2, 20
- 4, 10
- 5, 8

Fact If $n = ab$ for integers n, a and b, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Otherwise $a > \sqrt{n}$ and $b > \sqrt{n}$, and then $ab > n$.

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2.3 Finding divisors

This algorithm also finds a prime divisor of n.
Either
the least $a \leqslant \sqrt{n}$ which divides n,
or,
if we do not find a divisor among the $a \leqslant \sqrt{n}$, n itself is prime.

Definition Suppose m and n are positive integers. Then a *common divisor* of m and n is an integer which divides both m and n.

Example The common divisors of 30 and 45 are 1,3,5,15 (and their negatives).

Definition Suppose m and n are positive integers. Then the *greatest common divisor* (or gcd) of m and n is the greatest integer which is a common divisor of m and n.

Examples
$\gcd(30, 45) = 15$
$\gcd(13, 21) = 1$
$\gcd(15, 21) = 3$

Note $\gcd(a, b)=\gcd(b, a)$ for any integers a and b

2.4 The greatest common divisor of two numbers

It is remarkable that we can find the greatest common divisor of positive integers m and n, $\gcd(m, n)$, without finding their prime divisors.

This is done by the famous *Euclidean algorithm*, which repeatedly divides the greater number by the smaller, keeping the smaller number and the remainder.

Euclidean Algorithm.
Input: positive integers m and n with $m \geqslant n$
Output: $\gcd(m, n)$

$a := m$, $b := n$
$r :=$ remainder when a is divided by b
while $r \neq 0$ **do**
    $a := b$
    $b := r$
    $r :=$ remainder when a is divided by b
end
return b

Example. $m = 237, n = 105$

The first values are $a = 237, b = 105$,
so $r = 237 - 2 \times 105 = 27$.
The next values are $a = 105, b = 27$,
so $r = 105 - 3 \times 27 = 24$.
The next values are $a = 27, b = 24$,
so $r = 27 - 1 \times 24 = 3$.
The next values are $a = 24, b = 3$,
so $r = 24 - 8 \times 3 = 0$.
Thus the final value of b is 3, which is $\gcd(237, 105)$.

This can be set out more neatly:

$$\begin{array}{rcrcrcr} 237 & = & 2 & \times & 105 & + & 27 \\ 105 & = & 3 & \times & 27 & + & 24 \\ 27 & = & 1 & \times & 24 & + & \mathbf{3} \\ 24 & = & 8 & \times & 3 & + & 0 \end{array}$$

Example

Find $\gcd(165, 120)$.

$$\begin{array}{rcrcrcr} 165 & = & 1 & \times & 120 & + & 45 \\ 120 & = & 2 & \times & 45 & + & 30 \end{array}$$

Flux: What's the next line?

Fact $\gcd(a - kb, b) = \gcd(a, b)$ for any positive integers a, b, k.

Proof If d is a common divisor of a and b then d is a common divisor of $a - kb$ and b.

If e is a common divisor of $a - kb$ and b then e is a common divisor of a and b (note $a = (a - kb) + kb$).

So the list of common divisors of $a - kb$ and b is exactly the same as the list of common divisors of a and b.

So the greatest common divisor of $a - kb$ and b is equal to the greatest common divisor of a and b. $\square$

2.5 The Euclidean algorithm works!

We start with the precondition $m \geqslant n > 0$. Then the division theorem tells us there is a remainder $r < b$ when $a = m$ is divided by $b = n$. Repeating the process gives successively smaller remainders, and hence the algorithm eventually returns a value.

That the value returned value is actually $\gcd(m, n)$ relies on the following fact.

Fact. If a, b and k are integers, then

$$\gcd(a - kb, b) = \gcd(a, b).$$

By using this fact repeatedly, we can show that after each execution of the while loop in the algorithm $\gcd(b, r) = \gcd(m, n)$. When the algorithm terminates, this means $b = \gcd(b, 0) = \gcd(m, n)$. (Equivalently, in the neat set out given above, the gcd of the numbers in the last two columns is always $\gcd(m, n)$.)

2.6 Extended Euclidean algorithm

If we have used the Euclidean algorithm to find that $\gcd(m, n) = d$, we can "work backwards" through its steps to find integers a and b such that $am + bn = d$.

Example. For our $m = 237$, $n = 105$ example above:

$$\begin{aligned} 3 &= & 27 - 1\times 24 & & \\ 3 &= & 27 - 1(105 - 3\times 27) &= & -105 + 4\times 27 \\ 3 &= & -105 + 4(237 - 2\times 105) &= & 4\times 237 - 9\times 105 \end{aligned}$$

So we see that $a = 4$ and $b = -9$ is a solution in this case.

Our first line above was a rearrangement of the second last line of our original Euclidean algorithm working. In the second line we made a substitution for 24 based on the second line of our original Euclidean algorithm working. In the third line we made a substitution for 27 based on the first line of our original Euclidean algorithm working.

Question. Find integers a and b such that $353a + 78b = 1$.

We first use the Euclidean algorithm to find $\gcd(353, 78)$:

$$\begin{array}{rcrcrcr} 353 & = & 4 & \times & 78 & + & 41 \\ 78 & = & 1 & \times & 41 & + & 37 \\ 41 & = & 1 & \times & 37 & + & 4 \\ 37 & = & 9 & \times & 4 & + & 1 \\ 4 & = & 4 & \times & 1 & + & 0 \end{array}$$

Then we use the extended Euclidean algorithm:

$$\begin{array}{rcrcr} 1 & = & 37 - 9 \times 4 & & \\ 1 & = & 37 - 9(41 - 37) & = & -9 \times 41 + 10 \times 37 \\ 1 & = & -9 \times 41 + 10(78 - 41) & = & 10 \times 78 - 19 \times 41 \\ 1 & = & 10 \times 78 - 19(353 - 4 \times 78) & = & -19 \times 353 + 86 \times 78 \end{array}$$

So $-19 \times 353 + 86 \times 78 = 1$. One solution is $a = -19$, $b = 86$.

Question 2.1 (hint) Try something similar to what we did on the last slide.

Question 2.2 Can a multiple of 15 and a multiple of 21 differ by 1?

Answer

Note $\gcd(15, 21) = 3$.

So 3 divides $x \times 21$
and 3 divides $y \times 15$.

So 3 divides $x \times 21 - y \times 15$.

The answer is no. The difference between a multiple of 15 and a multiple of 21 is always divisible by 3.

Earlier we saw that

$$\begin{aligned} 999 &= 3^3 \times 37 \\ 1000 &= 2^3 \times 5^3 \end{aligned}$$

So 999 and 1000 have no prime factors in common.

Question 2.5 Is there a way of seeing this without factoring the numbers?

Answer Yes. If d divides 999 and d divides 1000 then d must divide $1000 - 999 = 1$. So $d = 1$ and d is not prime.