MAT1830

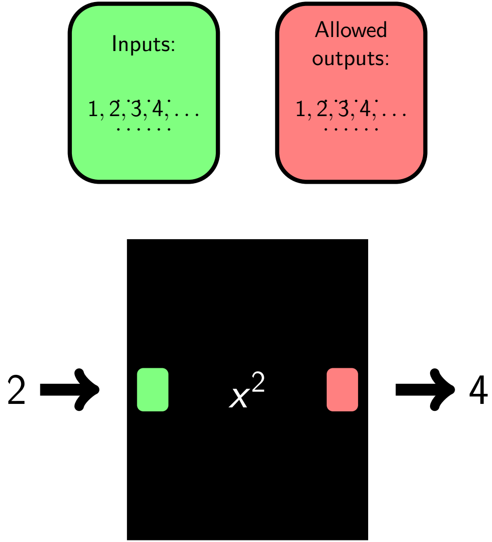
Lecture 13: Functions

Functions - why should you care?

The concept of a function is extremely important in both computer science and maths.

- Functions (subroutines) in programming are closely related to functions in the mathematical sense.
- In the case of functional programming languages (eg. Lisp, Haskell, Rust) they are exactly functions in the mathematical sense.
- Functions are used to define a lot of important concepts in maths and theoretical computer science.

A function can be thought of as a “black box” which accepts inputs and, for each input, produces a single output.



- Each input produces exactly one output.
  (Always the same output for a given input.)

Domain:

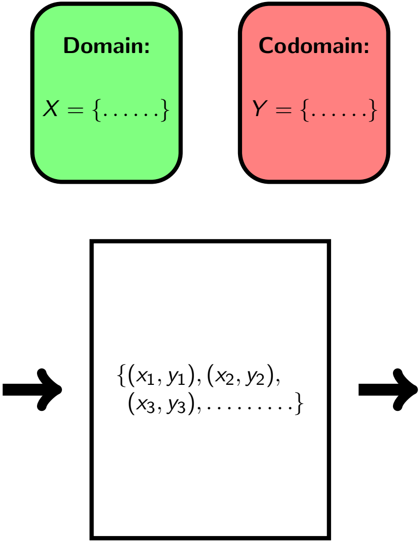
$X = \{\ldots\ldots\}$

Codomain:

$Y = \{\ldots\ldots\}$

input	**output**
x_1	y_1
x_2	y_2
x_3	y_3
$\vdots$	$\vdots$

- Each input produces exactly one output.
  (Always the same output for a given input.)



A set of ordered pairs from $X \times Y$ that contains exactly one ordered pair (x, y) for each $x \in X$.

Remember: The domain and codomain are part of the function and must always be defined.

THIS TRANSMOGRIFIER WILL TURN YOU INTO ANYTHING AT ALL.
TRANSMOG-RIFIER
ALL YOU DO IS SET THIS INDICATOR, AND THE MACHINE AUTOMATICALLY RESTRUCTURES YOUR CHEMICAL CONFIGURATION. YOU CAN BE AN EEL, A BABOON, A GIANT BUG, OR A DINOSAUR.
Eel
Baboon
BUG
DINO-SAUR
WHAT IF YOU WANT TO BE SOMETHING ELSE?
I LEFT SOME ROOM. JUST WRITE IT ON THE SIDE.
WATTERSON

13.1 Defining functions via sets

Formally we represent a function f as a set X of possible inputs, a set Y so that every output of f is guaranteed to be in Y, and a set of (input,output) pairs from $X \times Y$. The vital property of a function is that each input gives exactly one output.

A function f consists of a *domain* X, a *codomain* Y, and a set of ordered pairs from $X \times Y$ which has exactly one ordered pair (x, y) for each $x \in X$.

When (a, b) is in this set we write $f(a) = b$.

The set of y values occurring in these pairs is the *image* of f.

Note that the image of a function is always a subset of its codomain but they may or may not be equal.

If the image of a function is equal to its codomain, we say the function is *onto*.

Formally, a function consists of a domain X, a codomain Y, and a set of ordered pairs from $X \times Y$ which has exactly one ordered pair (x, y) for each $x \in X$.

The set of y values occurring in these ordered pairs is called the *image* of the function.

The image is always a subset of the codomain but they may not be equal. If they are equal we say the function is *onto*.

"f is a function with domain X and codomain Y" is shortened to

$$f : X \to Y.$$

Example Let $f : \{0, 1, 2, 3\} \to \{0, 1, 2, 3, 4, 5, 6\}$ be defined by $f(x) = 2x$.

x	$f(x)$
0	0
1	2
2	4
3	6

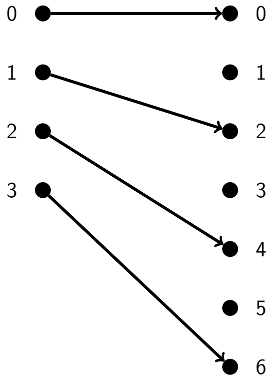
The set of ordered pairs defining f is $\{(0, 0), (1, 2), (2, 4), (3, 6)\}$.

Example Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x$.

The set of ordered pairs defining f is $\{(x, 2x) : x \in \mathbb{R}\}$.

Arrow diagrams

Example Let $f : \{0, 1, 2, 3\} \to \{0, 1, 2, 3, 4, 5, 6\}$ be defined by $f(x) = 2x$.



The image of f is $\{0, 2, 4, 6\}$. (So f is not onto.)

Why don't we always set the codomain equal to the image?

Think about $f : \mathbb{R} \to \mathbb{R}$ defined by
$f(x) = x^8 + 102x^7 - 7x^5 + 20x^4 - 100x + 7$.

We've set the codomain to $\mathbb{R}$ and that's fine - certainly $f(x) \in \mathbb{R}$.

What is image of f?   Hard to find and probably ugly.

Another reason is that "$\mathbb{R} \to \mathbb{R}$ functions", for example, make a nice class to consider.

Question What set of ordered pairs does $f : \{0, 1, 2, 3\} \to \mathbb{N}$ defined by $f(x) = x^2$ correspond to?

$\{(0, 0), (1, 1), (2, 4), (3, 9)\}$.

Flux Exercise

Which of the following sets of ordered pairs correspond to functions from $\{0, 1, 2\}$ to $\mathbb{R}$?

$$\begin{aligned} S &= \{(0,7),(2,\pi)\} \\ T &= \{(0,7),(1,1),(2,\pi)\} \\ U &= \{(0,7),(1,4),(2,\pi),(2,3)\} \end{aligned}$$

A. Just T

B. S and T

C. T and U

D. All of them

Answer:

Not S – it doesn't have an ordered pair with first coordinate 1.

Not U – it has two ordered pairs with first coordinate 2.

But T is fine.

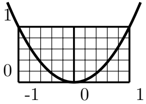
So A.

Examples.

1. The squaring function square(x)= x^2 with domain $\mathbb{R}$, codomain $\mathbb{R}$, and pairs

$$\{(x, x^2)\ :\ x \in \mathbb{R}\},$$

which form what we usually call the *plot* of the squaring function.

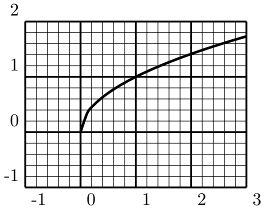


The image of this function (the set of y values) is the set $\mathbb{R}^{\geqslant 0}$ of real numbers $\geqslant 0$.

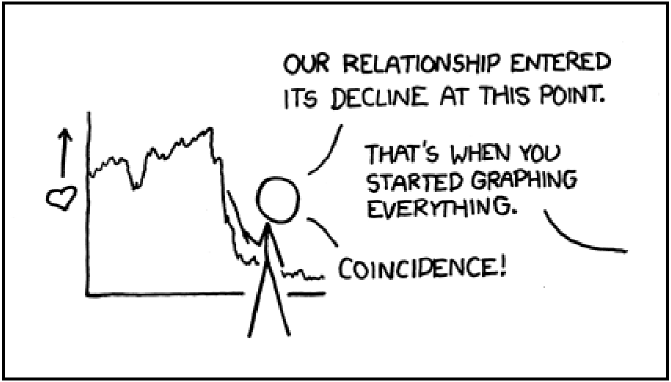
sqrt : $\mathbb{R}^{\geq 0} \to \mathbb{R}$

2. The square root function $\mathrm{sqrt}(x) = \sqrt{x}$ with domain $\mathbb{R}^{\geqslant 0}$, codomain $\mathbb{R}$, and pairs

$$\{(x, \sqrt{x}) \ : \ x \in \mathbb{R} \text{ and } x \geqslant 0\}.$$

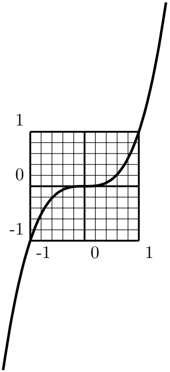


The image of this function (the set of y values) is the set $\mathbb{R}^{\geqslant 0}$.


OUR RELATIONSHIP ENTERED ITS DECLINE AT THIS POINT.
THAT'S WHEN YOU STARTED GRAPHING EVERYTHING.
COINCIDENCE!

3. The cubing function $\text{cube}(x) = x^3$ with domain $\mathbb{R}$, codomain $\mathbb{R}$, and pairs

$$\{(x, x^3) \ : \ x \in \mathbb{R}\},$$



The image of this function is the whole of the codomain $\mathbb{R}$, so it is onto.

Question 13.1 Which of the following rules define functions?

- For each non-empty set S of natural numbers, let $f(S)$ be the least member of S.

  Yes.

- For each set X of real numbers between 0 and 1, let $g(X)$ be the least member of X.

  No - $g(\{x : x \in \mathbb{R} \text{ and } \frac{1}{2} < x < 1\})$ is not defined.

- For each circle C in the (x, y) plane, let $h(C)$ be the minimum distance from C to the x axis.

  Yes.

Question 13.1 (cont.) Which of the following rules define functions?

- For a pair A, B of sets of real numbers let $s(A, B)$ be the smallest set which has both A and B as subsets.

  Yes (depending on your interpretation of "smallest").
  $s(A, B) = A \cup B$.

- For a pair A, B of sets of real numbers let $t(A, B)$ be the largest set which is a subset of both A and B.

  Yes (depending on your interpretation of "largest").
  $t(A, B) = A \cap B$.

13.2 Arrow notation

If f is a function with domain A and codomain B we write

$$f : A \to B,$$

and we say that f is from A to B.

For example, we could define

$$\text{square} : \mathbb{R} \to \mathbb{R}.$$

We could also define

$$\text{square} : \mathbb{R} \to \mathbb{R}^{\geqslant 0}.$$

Likewise, we could define

$$\text{cube} : \mathbb{R} \to \mathbb{R}.$$

However we could not define

$$\text{cube} : \mathbb{R} \to \mathbb{R}^{\geqslant 0},$$

because for some $x \in \mathbb{R}$, $\text{cube}(x)$ is negative. For example, $\text{cube}(-1) = -1$.

Question 13.2 Which of the following functions can be defined on the whole of $\mathbb{R}$, so that the function values also lie in $\mathbb{R}$?

(In other words, which can be $\mathbb{R} \to \mathbb{R}$ functions?)

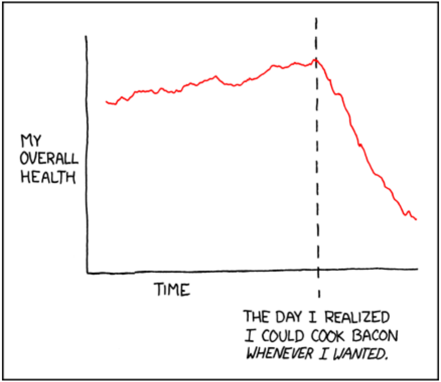
x^2 Yes.

$\frac{1}{x}$ No - undefined for $x = 0$.

$\log(x)$ No - undefined for $x \leq 0$ (because $e^x > 0$ for all $x \in \mathbb{R}$).

$\sqrt{x}$ No - undefined for $x < 0$.

$\sqrt[3]{x}$ Yes.


MY
OVERALL
HEALTH
TIME
THE DAY I REALIZED
I COULD COOK BACON
WHENEVER I WANTED.

13.3 One-to-one functions

A function $f : X \to Y$ is *one-to-one* if for each y in the image of f there is only one $x \in X$ such that $f(x) = y$.

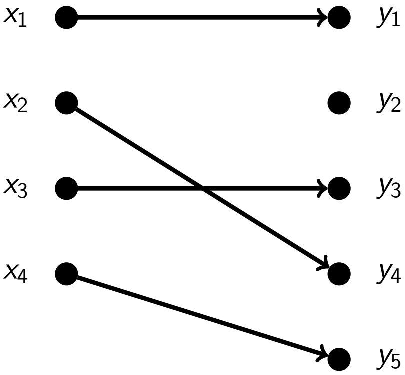
For example, the function cube(x) is one-to-one because each real number y is the cube of exactly one real number x.

The function square: $\mathbb{R} \to \mathbb{R}$ is *not* one-to-one because the real number 1 is the square of two different real numbers, 1 and -1. (In fact each real $y > 0$ is the square of two different real numbers, $\sqrt{y}$ and $-\sqrt{y}$)

On the other hand, square : $\mathbb{R}^{\geqslant 0} \to \mathbb{R}$ *is* one-to-one because each real number y in $\mathbb{R}^{\geqslant 0}$ is the square of only one real number in $\mathbb{R}^{\geqslant 0}$, namely $\sqrt{y}$.

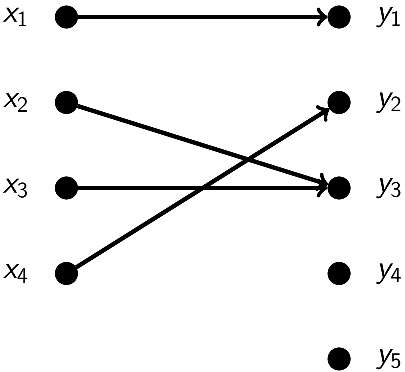
The last example shows that the domain of a function is an important part of its description, because changing the domain can change the properties of the function.

Question Is the function pictured below one-to-one?



Yes.

Question Is the function pictured below one-to-one?



No. $f(x_2) = f(x_3)$.

13.4 Proving a function is one-to-one

There is an equivalent way of phrasing the definition of one-to-one: a function $f : X \to Y$ is one-to-one when, for all $x_1, x_2 \in X$,

$$f(x_1) = f(x_2) \;\Rightarrow\; x_1 = x_2.$$

This can be useful for proving that some functions are or are not one-to-one.

Example. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 6x + 2$ is one-to-one because

$$\begin{aligned} & f(x_1) = f(x_2) \\ \Rightarrow \quad & 6x_1 + 2 = 6x_2 + 2 \\ \Rightarrow \quad & 6x_1 = 6x_2 \\ \Rightarrow \quad & x_1 = x_2. \end{aligned}$$

Example. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 1$ is not one-to-one because $f(-1) = 2$ and $f(1) = 2$ and so

$$f(-1) = f(1).$$

To show that a function $f : X \to Y$ is one-to-one we must show that, for all $x_1, x_2 \in X$,

$$\text{if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

To show that a function $f : X \to Y$ is *not* one-to-one we must show that there exist $x_1, x_2 \in X$ such that

$$f(x_1) = f(x_2) \text{ and } x_1 \neq x_2.$$

Question Is $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2 + 1$ one-to-one?

No. $f(1) = 2$ and $f(-1) = 2$ (and obviously $1 \neq -1$)

Flux Exercise

Which of the following functions are one-to-one?

$f : \mathbb{N} \to \mathbb{Z}$ defined by $f(x) = y$ where y is the least even integer greater than x.

$g : \mathbb{N} \to \mathbb{Z}$ defined by $g(x) = (x + 6)^2 + 1$.

A. Neither

B. Just f

C. Just g

D. Both

Examples for f: $f(0) = 2$, $f(1) = 2$, $f(2) = 4$, $f(3) = 4$, $f(4) = 6$, $f(5) = 6$, ...

Examples for g: $g(0) = 37$, $g(1) = 50$, $g(2) = 65$, $g(3) = 82$, $g(4) = 101$, ...

Answer:

f isn't one-to-one because $f(2) = f(3)$.

g is one-to-one. Full proof on next slide.

So C.

Example Show $g : \mathbb{N} \to \mathbb{Z}$ defined by $g(x) = (x+6)^2 + 1$ is one-to-one.

Suppose that

$$\begin{array}{lrcll} & g(x_1) & = & g(x_2) & \quad \text{for some } x_1, x_2 \in \mathbb{N}. \\ \text{Then} & (x_1+6)^2+1 & = & (x_2+6)^2+1. & \\ \text{So} & (x_1+6)^2 & = & (x_2+6)^2. & \\ \text{So} & x_1+6 & = & x_2+6. & \\ \multicolumn{5}{l}{\text{(Two positive integers with equal squares are equal.)}} \\ \text{So} & x_1 & = & x_2. & \end{array}$$

This shows that g is one-to-one.