MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #9 and Additional Practice Questions

Tutorial Questions

1. Captain Kirk fires photon torpedoes, one at a time, at a Romulan ship until he hits it. Each torpedo has a 10% chance of hitting the ship (independent of any other torpedoes). Let X be the number of torpedos that *miss* the ship.
   (a) What's the name for the probability distribution of X?
   (b) What is the probability that $X \leq 2$?
   (c) Give an explanation for your why your answer to (b) is what it is. (Without referring to formulas.)
   (d) What would be the (likely) effect on X if Mr Spock fires the torpedoes instead? Spock is a much better shot than Kirk.

2. Write down the next four values of each of the following recursive sequences.
   (a) $r_0 = 3, \qquad r_n = 2r_{n-1} - 1 \quad$ for all integers $n \geq 1$.
   (b) $t_0 = 1, t_1 = 1, t_2 = -2, \qquad t_n = t_{n-1}t_{n-3} \quad$ for all integers $n \geq 3$.

3. Rewrite the following expressions without using $\sum$ or $\prod$.

   (a) $\displaystyle\sum_{i=6}^{10} \frac{1}{2i+1}$ (b) $\displaystyle\prod_{i=4}^{6} \left(\frac{x^i}{2i} + i\right)$ (c) $\displaystyle\sum_{i=0}^{3} \frac{(-1)^i}{(2i+1)!} x^{2i+1}$

4. Let s_n be the number of ways (order being important) of writing n as a sum of 1s and 2s. For example $s_4 = 5$ because 4 can be written in five ways:

   $$1+1+1+1, \quad 1+1+2, \quad 1+2+1, \quad 2+1+1, \quad 2+2.$$

   Find a recurrence for s_n.

5. Aperture Labs has a research division of 400 scientists, each of whom has a 2.5% chance of making a major breakthrough each year. Let Y be the number major breakthroughs made by the Aperture research division in a given year.
   (a) What's the name for the probability distribution of Y?
   (b) What is $\mathrm{E}[Y]$? What is $\mathrm{Var}[Y]$?
   (c) What's the probability that exactly $\mathrm{E}[Y]$ breakthroughs are made?
   (d) Aperture's annual report confidently predicts that between 5 and 20 major breakthroughs will be made this year. Write an expression for the probability of this occurring.

   Umbrella Corp has a research division of 25 scientists, each of whom has a 40% chance of making a major breakthrough each year. Let Z be the number major breakthroughs made by the Umbrella research division in a given year.
   (e) What is $\mathrm{E}[Z]$? What is $\mathrm{Var}[Z]$?
   (f) Which row in the following table would you guess belongs to each company? Why?

company	Number of breakthroughs in year									
	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
?	8	7	16	13	14	10	10	9	7	6
?	13	10	9	7	8	11	10	12	7	10

Practice Questions

1. Write down the next four values of each of the following recursive sequences.

   (a) $s_0 = 0,\ s_1 = 2, \quad s_n = 2s_{n-1} - 3s_{n-2}$ for $n \geq 2$.

   (b) $u_0 = 2, \quad u_n = 3u_{n-1} + n$ for $n \geq 1$.

2. Consider the following pseudo code of a function "foo".

```
function foo(x) (input: a positive integer)
    if x = 0 then
        return 1
    else
        return x×foo(x − 1)
    end if
end function
```

   (a) What will foo return when given input 4?

   (b) What is the recurrence relation corresponding to foo?

   (c) What function of x does foo calculate?

3. The Fibonacci sequence is defined recursively by

$$t_0 = 0,\ t_1 = 1, \quad t_k = t_{k-1} + t_{k-2} \text{ for } k \geq 2.$$

   Use strong induction to prove that, for all $n \geq 0$,

$$t_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}.$$

4. (a) Let P be a Poisson random variable with $\lambda = 10$. What is $E(P)$? What is $\mathrm{Var}(P)$?

   (b) Pick a value of k between 0 and 20 and calculate $\Pr(P) = k$.

   (c) Now let B_1, B_2, B_3 be binomial random variables with $(n, p) = (100, \frac{1}{10}), (1000, \frac{1}{100}), (10000, \frac{1}{1000})$. What are the expected values and variances of these variables?

   (d) For the same value of k you used in (b), calculate $\Pr(B_1) = k$, $\Pr(B_2) = k$ and $\Pr(B_3) = k$.

   (e) Is something going on here? If so what and why?