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Expanding the parameter space approach for LTV-systems

Bachelor Thesis
of
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I declare that this work made in the ordinary course of self-care at the Institute used to have, and no other than the specified sources.

(Mohamed Essam Mohamed Kassem Abdelaziz)

Aachen, 31.08.2013

Contents

List Of Symbols

$\phi(t, t_0)$ State Transition Matrix
K_P Proportional Gain
K_D Derivative Gain
K_I Integral Gain
$G_H(s)$ Sensor's transfer function
$G_C(s)$ Controller's transfer function
$G_S(s)$ System's transfer function
A_k Closed loop system matrix
H Hamiltonian Matrix
$T(t)$ Transformation Matrix

1

Introduction

1.1 Motivation

After the development of the negative feedback amplifier in 1927 by H. Black, it had taken 45 years until the term robust control appeared in control literature. Today, in the field of robust control, two main approaches for the controller synthesis exist. One deals with the H_∞ approach for unstructured uncertainties (refer e.g. [1]), the second approach is based on the idea of calculating all stabilizing controller parameter spaces (in consideration of structured uncertainties). The following thesis will focus on the latter idea.

The parameter space approach is a method used to compute the stable regions in the controller parameter spaces for ideal, real and discrete PID controllers for LTI and LTI time delay systems, refer to [2] and [3], and state feedback controllers for LTI systems [4]. The resulting parameter provides a good possibility for the control engineer to analyse the resulting controller (e.g. size of the stability region, stability margin, performance portraits, etc.). In addition, there are many controller fine tuning methods available, see [5]. Inspired by [6] and [7], this approach can also handle some types of parameter uncertainties.

Easy controller structures (such as PID and state feedback controllers) are the most commonly used controllers in industrial practices. Even though a great variety of different methods for tuninig these control systems exist, many surveys show that a high percentage of them are not tuned optimally. One of the main reasons why this occurs is because, there exists no general method for tuning these controllers. The main limitations of today's tuning methods are: the controller design methods rely on restrictions of the system model (concerning the model order, pole and zero locations). Moreover, they are confined to delay free systems and do not support essential specifications. Not to mention, robustness requirements are often not incorporated in the design process.

In some cases, easy controller structures for LTI systems may not be sufficient or effective enough to achive the control goal. In such situations, switch to LTV systems can improve the control results. The goal of the presented approach is to extend the synthesis step of the classical parameter space approach introduced in [6] for LTV systems. Thereby, a quite analogous procedure to the classical PID controller design process for the calculation of the stability boundaries will be performed.

The results of this thesis are provided in the open source PIDrobust MATLAB toolbox. A preliminary version of the toolbox can be downloaded from http://www.irt.rwth-aachen.de/pidrobust/.

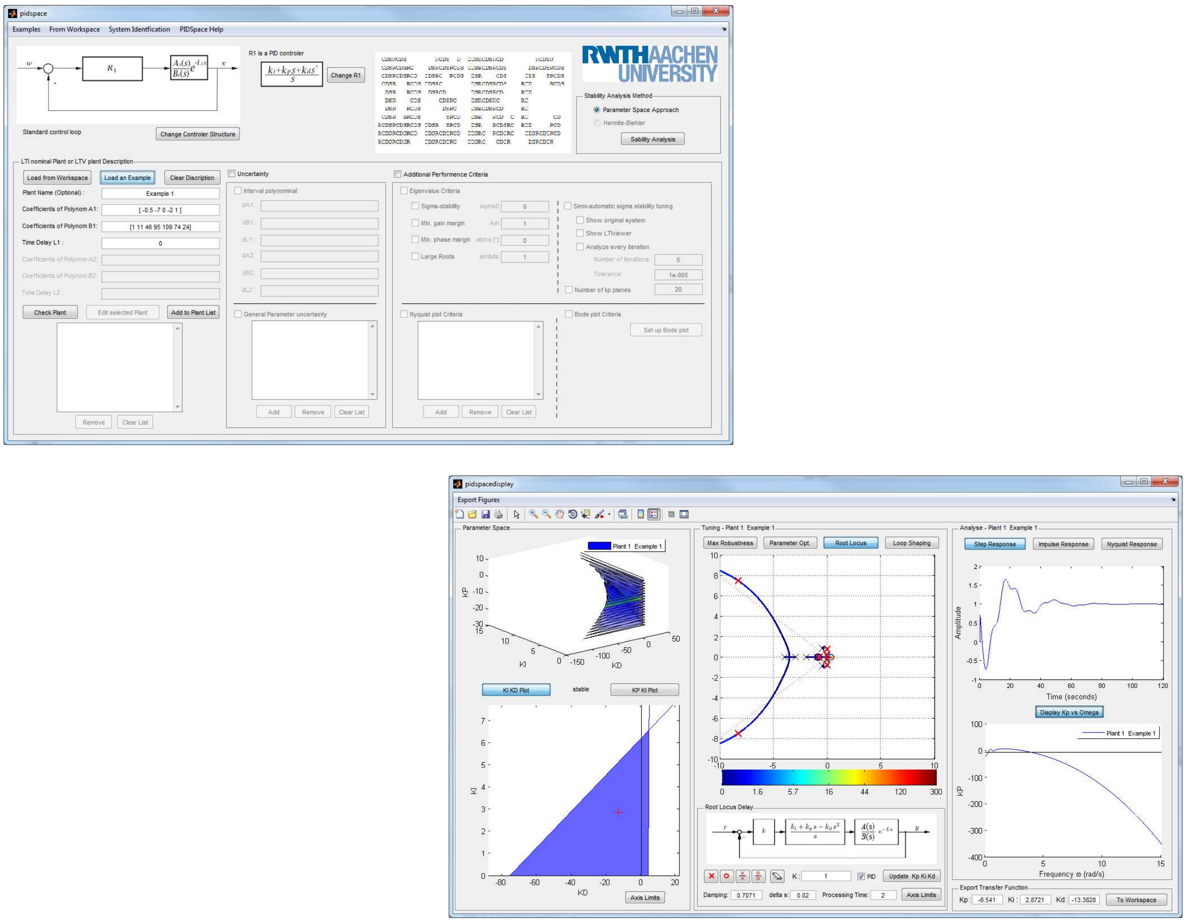


Figure 1.1: PID Robust Toolbox

1.2 Task Description

The concept of the parameter space approach which enables a robust design of controller parameters is extended for the Linear Time Varying (LTV) Systems. Numerous research works have shown that the stability of LTV systems cannot be determined by the Frozen Time Eigenvalues (FTEs) and in order to tackle this problem, several attemps are made using the following concepts: Dynamic Eigenvalues, Differential Riccati Equation Mapping and the Transformation of Linear Time Varying (LTV) Systems to their equivalent Linear Time Invariant (LTI) System.

To begin with, dynamic eigenvalues were developed by Van der Kloet and Neerhoff to provide a means to formulate a general solution for LTV systems. However, in this thesis these values are assumed to be analogous to the eigenvalues of LTI Systems. Based on this assumption the expansion of the parameter space approach for LTV systems using the dynamic eigenvalues is proposed. The procedure goes as follows: a state feedback is performed on the given LTV System. Next, the dynamic eigenvalues are calculated using the closed loop system. Followed by, obtaining a characteristic polynomial using these values. Based on the parameter space approach, the three famous substitutions are perofmed in order to get the RRB (Real Root Boundary), CRB (Complex Root Boundary) and IRB (Infinite Root Boundary) limits. It is important to note that the obtained limits are functions of time and the controller parameters. Next for every time t, the limits are plotted, and the stable areas are identified.

In [8] the author defines a method for mapping the corresponding Algebraic Riccati Equation (ARE) of LTI Systems which is similar to the Lyapunov Equation, into the parameter space using four mapping equations. Under the same definition, a second attempt is made to map the corresponding Riccati Equation (Differential Riccati Equation (DRE)) for LTV systems into the parameter space.

In the final attempt, the works of [9] are applied. Using a suitable algebraic transformation only (or) an algebraic transformation plus the t $\leftrightarrow \tau$ transformation, the LTV system could be transformed into a LTI System. With this outcome, the defined methods for LTI systems can be utilised. Hence, the PSA is extended for LTV systems.

1.3 Structure Of The Thesis

In Chapter 2, the theoretical basis used in the thesis are presented. Starting off with a general overview of the different linear frameworks used to describe linear/nonlinear systems. Followed by a summarised overview on the stability of LTI-Systems, based on the concept of eigenvalues.

In subsection 2.3.1, the limitations of analysing the stability of LTV using the (FTEs) are discussed with an example. In the same subsection, the problems that arise due to time variations are presented. Based on the conclusion reached about the ineffectiveness of FTEs in the stability analysis of LTV systems, additional conditions are reviewed in subseciton 2.3.2, in order to determine whether the LTV system is exponentially stable or not [1]. In an attempt to find a substitue for eigenvalues of LTI systems, the concept of dynamic eigenvalues of LTV systems is introduced in subsection 2.3.3. Added to that, a step by step algorithm for calculating these values is presented.

[1]The majority of works presented in this subsection are related to exponential stability. However, some works were introduced conditions to check for instability are presented.

In section 2.4, the framework of Lyapunov stability is presented. Here, a summary about the main concepts defined by the Russian Mathematician is given. Applications for both LTI and LTV systems are illustrated. In the next section, the PSA for LTI systems is introduced. This section is divided into two subsections, based on the type of system representation. In 2.5.1, the LTI system is represented by the Transfer Function and the type of controller used is PID (K_P, K_D and K_I). The basic ideas and steps of calculating the three limits (RRB, CRB and IRB), which form the PID controller space are established in this subsection. In the following subsection, the PSA is extended for LTI systems represented in the state space form, and two different techniques are applied to determine the state feedback $(k_1, k_2, ..k_n)$ controller space. Where n represents the order of the system.

In chapter 3, the three extension attemps of the PSA for LTV Systems are presented. The first attempt is made based on the concept of t eigenvalues described in 2.3.3. Next, the Differential Riccati Equation (DRE) of LTV systems and the mapping equations presented in 2.5.2.2 are both utilised to obtain the controller parameter space. And finally, special forms of LTV systems are transformed into their equivalent LTI systems. Subsequently, the general approaches defined for LTI systems described in 2.5.1, are implemented on the transformed systems.

Chapter 4 gives conclusions and suggests further possibilities of improving the results.

2

Theoretical Basis

2.1 Overview of LTI, LTV and LPV Systems

In this section, a comparative study between the different types of linear dynamical system representations are discussed. The study is mainly based on the works of [10].

LTI (Linear Time Invariant) systems consist of constant (stationary) parameters. The parameters do not vary with time. The state space representation of LTI systems is:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

On the other hand, LPV (Linear Parameter Varying) systems are simply non-stationary , i.e. the exogenous parameter does vary with time. Exogenous parameter simply means independently of the states (x_1, x_2,..., x_n). Therefore, abstracting away the non-linear dependency on the states resulting in a linear system with non-stationary dynamics - which considered as the middle ground between linear and non-linear dynamics.

$$\dot{x}(t) = A(\theta(t))x(t) + B(\theta(t))u(t)$$
$$y(t) = C(\theta(t))x(t) + D(\theta(t))u(t)$$

The difference between LTV (Linear Time Varying) systems with the state space representation:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

and LPV systems is less apparent, because if a trajectory of the parameter $\theta(.)$ [1] is taken into consideration, the dynamics of the LPV system will act as a LTV system. LPV and LTV systems mainly differ from each other in the perspective of analysis (such as stability analysis, disturbance rejection, tracking, etc.) and control design or synthesis.

When it comes to analysis, the bounds of the exogenous parameter (magnitude: $\theta(t)$ and the rate of variation: $\dot{\theta}(t)$) , for all $t \geq 0$ are considered as follows:

$$-\alpha \leq \theta(t) \leq \alpha$$

$$-\beta \leq \dot{\theta}(t) \leq \beta$$

This characterization is defined for a family of admissible parameter trajectories Γ. In simpler terms, these bounds are defined for all the acceptable trajectories of the exogenous varying parameter $\theta(t)$. This is a typical assumption on the exogenous parameter. What makes LPV systems differ from LTV systems is the fact that when LPV systems are analyzed, properties such as the above mentioned (stability, etc) are assessed for a family of trajectories i.e. a group of LTV systems.

In the case of control synthesis, the state feedback controller $u = K(\theta(t))x(t)$ is considered where the feedback gain K is a function of the exogenous parameter $\theta(t)$.

For LPV systems,the control is constrained to be a *causal* function of the parameter. Causal function means that the current control values cannot depend on the future parameter values (i.e. it depends on the entire history of the parameter including the current value). It could be represented as follows :

$$u(t) = K(\theta|_{[0,t]})x(t)$$

In the case of LTV systems, the control depends exclusively on future parameter trajectories. Thereby, violating the causality on the control (It does not depend on the history of the parameter values). A good example for a time varying linear state feedback law is the ***optimal controller***. It could be represented as follows:

$$u(t) = K(\theta|_{[t,\infty)})x(t)$$

For a better understanding of the difference between LPV and LTV systems, a simplified model of a Rocket is taken as an example. In general, rockets lose significant amounts of mass as fuel during flight i.e. mass does not remain constant with respect to time. Therefore, the mass of the rocket is taken as the exogenous varying parameter $m(t)$.

For simplicity, 3 trajectories $m_1(.), m_2(.)$ and $m_3(.)$ of the exogenous parameter $m(t)$ are considered. As it can be seen from 2.1, for each trajectory of $m(t)$, there is a corresponding LTV system.

[1] For a better understanding of the term $\theta(.)$ refer to the rocket example presented in figure 2.1

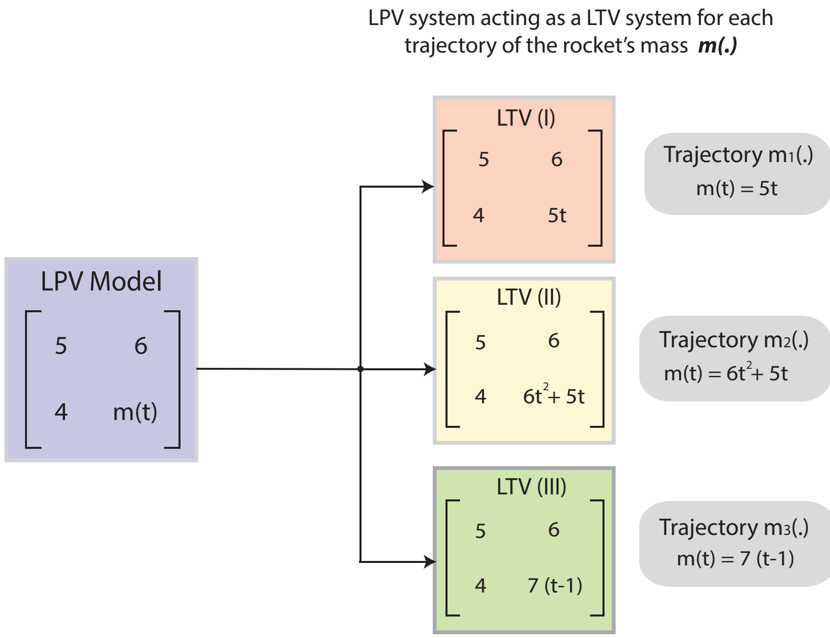


Figure 2.1: LPV vs LTV systems

In this thesis, LTI and LTV systems are mainly considered.

2.2 LTI-System Stability (Eigenvalue Concept)

Stability is a desirable characteristic for linear dynamical systems. If a system is unstable, it is usually of no use in practice. Furthermore, the stability of LTI systems can be analyzed through several methods. Some of these methods include *Routh-Hurwitz Criterion*, *Nyquist Stability Criterion*, *Lyapunov Stability* (Discussed in Section 2.4 of the thesis) and using the *poles/eigenvalues* of LTI systems. In this section, the stability analysis of LTI systems using the poles/eigenvalues will be discussed. For more information about Routh-Hurwitz and Nyquist Criterion, refer to [11] and [12].

In general, LTI systems can be represented by Transfer Functions (Output/Input) or state space equations. The Transfer Function of a LTI system is considered:

$$G_{CL} = \frac{N(s)}{D(s)} \tag{2.1}$$

where $N(s)$ is short for Numerator and $D(s)$ is short for Denominator. To study the stability of this system, the characteristic equation is generated by equating $(D(s) = 0)$ and solving for the roots. In figure 2.16 of Section 2.5, a systematic approach to derive the characteristic equation is presented. The roots of the characteristic equation are known as the *poles* of the system. The poles are then plotted in the complex s-domain (Imaginary vs. Real). In order to determine whether the system is stable, marginally stable or unstable refer to figure 2.2.

For a given state space equation of a LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.2}$$

where A is the system dynamics matrix and B is the input matrix, the stability of this system can analyzed by obtaining the eigenvalues. In order to get the eigenvalues of the system, equation 2.3 must be solved for λ:

$$\det(A - \lambda I) = 0 \tag{2.3}$$

Followed by plotting the obtained eigenvalues in the complex s-domain. It is important to mention that the poles and the eigenvalues of a system are the same, provided that the system is fully observable and controllable.

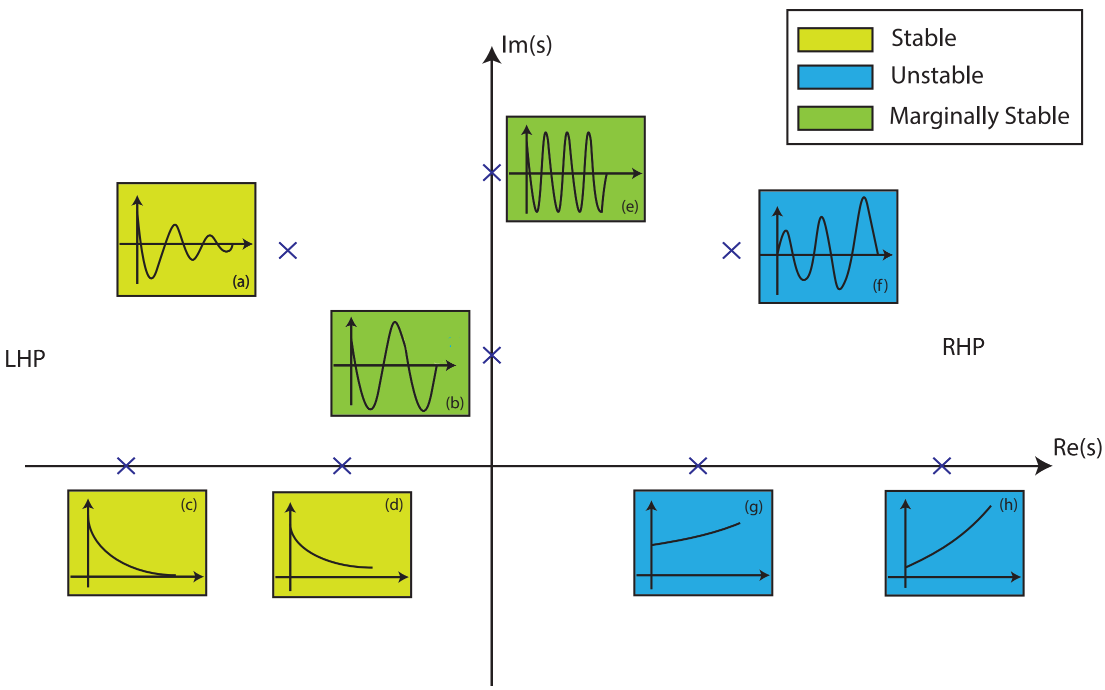


Figure 2.2: System's Impulse response vs pole placement

Figure 2.2 shows the different types of stability depending on the pole/eigenvalue (P/E) placement. If the P/E lies in the LHP (Left Half Plane) of the complex s-domain, cases: (a), (c) and (d), then the system is considered to be *stable*. On the other hand, if the P/E lies on the imaginary axis, cases: (b) and (e), then the system is said to be *marginally stable*. In short, a marginally stable system exhibits oscillatory bounded output. Bounded means that the system's response does not grow infinitely in amplitude. Lastly, if the P/E lies on the RHP

(Right Half Plane) of the complex s-domain, cases: (f), (g) and (h), then the system is said to be *unstable*. When the terms RHP and LHP are used, the imaginary axis is not taken into consideration.
Important remarks to be considered while analyzing the stability of a LTI system using this method:

I. The differences between cases (a, b, c, d, e, f, g and h), refer to figure 2.2:
Firstly, if the poles are complex conjugate pairs ($s = \sigma \pm j\omega$), the response will consist of oscillations such as in the case of (a), (b), (e) and (f). If the poles are located on the real axis i.e. ($s = \sigma, j\omega = 0$), cases(c), (d), (g) and (h), the response will not consist of any oscillations. Secondly, the difference between (c) and (d) is: moving along the negative real axis, the system reaches the equilibrium point faster (c reaches the equilibrium point faster than d). In case of (g) and (h) , moving along the positive real axis, the system becomes unstable by approaching an infinite amplitude faster (h grows to infinity faster than g). Lastly, moving along the imaginary axis, the number of oscillations increase in time (e has more oscillations than b).

II. When more than one P/E are obtained, refer to figure 2.3:
(i) If there exists any P/E in the RHP, then the system is unstable.
(ii) If (case i) is not satisfied and there exists any P/E on the imaginary axis, then the system is considered to be marginally stable.
(iii) If (cases i and ii) are not satisified, i.e. all P/E are placed in the LHP, then the system is stable.

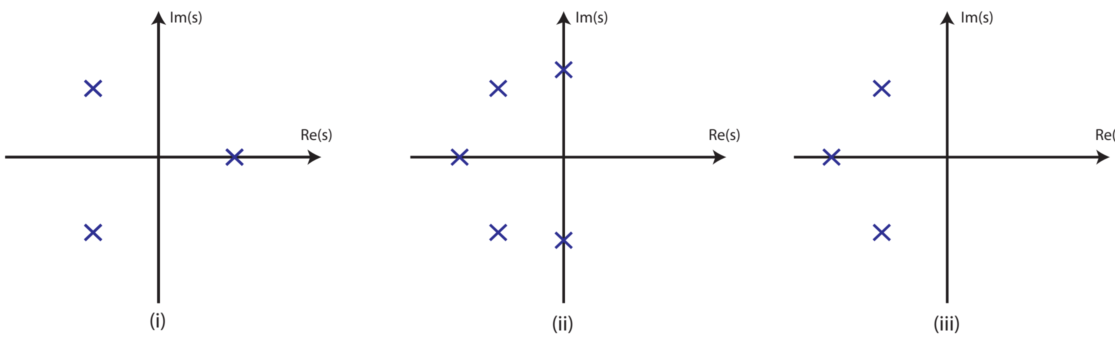


Figure 2.3: Multiple poles

2.3 LTV-System Stability (Eigenvalue Concept)

2.3.1 Problem Description

There has been considerable interest in the stability analysis of LTV systems. In this subsection, the stability analysis of LTV systems using the concept of eigenvalues is discussed. Followed by an overview of the problems that arise due to the time variation aspect.

Based on references [13], [14], [15], [16], several attempts were made to study the stability and instability of LTV-systems using the Frozen Time Eigenvalues (FTE) of their system A-matrix $A(t)$, but they failed to be sufficient. For example:

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2(t) & -1 + 1.5\sin(t)\cos^2(t) \\ -1 - 1.5\sin(t)\cos(t) & -1 + 1.5\sin^2(t) \end{bmatrix}$$

Using equation 2.3:

$$\lambda^2 - \frac{3}{2}\cos^2(t)\lambda - \frac{3}{2}\sin^2(t)\lambda + 2\lambda - \frac{3}{2}\cos^2(t) - \frac{3}{2}\sin^2(t) + 2\lambda = 0$$

Knowing that $\cos^2(t) + \sin^2(t) = 1$, the simplified characteristic equation is:

$$\lambda^2 + \frac{1}{2}\lambda + \frac{1}{2} = 0$$

This gives the FTE:

$$\lambda_{1,2} = \frac{-1 \pm j\sqrt{7}}{4} \tag{2.4}$$

In this example, the FTE are independent of time, i.e. for all ***t***>0 ,the system's FTE are constant and are located in the LHP [2], suggesting that the system is asymptotically stable. In order to verify this result, the state transition matrix $\Phi(t, t_0)$ is calculated. (refer to the next page for a better understanding of the state transition matrix $\Phi(t, t_0)$). The state transition matrix for $A(t)$ is:

$$\Phi(t, 0) = \begin{bmatrix} e^{0.5t}\cos(t) & e^{-t}\sin(t) \\ -e^{0.5t}\sin(t) & e^{-t}\cos(t) \end{bmatrix}$$

Due to the presence of $e^{0.5t}$ in the state transition matrix, the system is unstable. And this clearly contradicts with the result of the FTE.

Despite the failure, it is still possible to use the FTE to study the stability and instability of LTV systems; but under certain conditions. These conditions are described thoroughly in subsection 2.3.2.

[2] 2.4 could have also been in terms of time ***(t)***. In such a case, the FTE for every ***t*** are obtained and the beviour of the FTE are observed with time

State Transition Matrix

The state transition matrix is defined as the matrix whose product with the state vector x at an initial time t_0 gives x at later time t.

$$x(t) = \Phi(t, t_0)x(t_0)$$

It can also be used to obtain the general solution of linear dynamical systems (LTI and LTV systems). For LTI systems, it is relatively easy to determine the state transition matrix using several techniques such as: Laplace Transform, Cayley Hamilton (Vandermonde matrix) or Diagonalisation (Jordan Form) technique. For a good review, refer to [17]. On the other hand, for LTV systems, $\Phi(t, t_0)$ is often very difficult to obtain. However, several formulations exist to find $\Phi(t, t_0)$ for LTV systems. In references [18],[19],[20] and references within, the calculations of $\Phi(t, t_0)$ for several classes of LTV systems are presented. In the recent paper [20], the authors extend the Cayley Hamilton technique to find the general solution of n-dimensional continuous LTV systems. In parallel to the researches in the development of $\Phi(t, t_0)$ for LTV systems, several other theories were proposed in order to obtain the general solution of LTV systems. One of these theories is discussed in subsection 2.3.3.

For LTV systems, certain problems arise due to the variations with time. Two important problems will be discussed: 1) Induced Instability 2) Induced Non-Minimum Phasedness.

1) Induced Intability:

In linear dynamical system analysis, time variations tend to induce instability. To explore this phenomenon more, a periodic LTV system is given below, whereby the time varying element periodically switches between two values -1 and -3 for all $t \geq 0$.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \underbrace{-\text{sgn}(\sin(\pi t)) - 2}_{\text{Time Varying Element}} & 0 \end{bmatrix} x(t) \tag{2.5}$$

where $A(t) = \begin{bmatrix} 0 & 1 \\ -\text{sgn}(\sin(\pi t)) - 2 & 0 \end{bmatrix}$ and $u(t) = 0$.

Sign Function (Signum Function)

$$sgn(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

To begin with, the system above is considered as a LTI system by taking two fixed (frozen) time values of the time varying element (For e.g. $t = 1$ and $t = 2$).

$$\text{LTI System 1} : A(t=1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t)$$

$$\text{LTI System 2} : A(t=2) = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix} \Rightarrow \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix} x(t)$$

Followed by, implementing the method of calculating the eigenvalues of the LTI systems above (using equation 2.3), in order to determine whether the systems are stable:

LTI System 1:

$$det\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = det\left(\begin{bmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{bmatrix}\right) = 0$$

$$(0-\lambda)(0-\lambda) - (1)(-1) = 0 \Rightarrow \lambda^2 + 1 = 0$$
$$\lambda = \sqrt{-1} = 0 \pm j$$

Similarly, for LTI System 2:

$$det\left(\begin{bmatrix} 0 & 1 \\ -3 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = det\left(\begin{bmatrix} 0-\lambda & 1 \\ -3 & 0-\lambda \end{bmatrix}\right) = 0$$

$$(0-\lambda)(0-\lambda) - (1)(-3) = 0 \Rightarrow \lambda^2 + 3 = 0$$
$$\lambda = \sqrt{-3} = 0 \pm \sqrt{3}j$$

For both LTI Systems 1 and 2, the system is said to be *marginally stable* due to the zero real part poles and no positive real poles, i.e. the unrepeated eigenvalues are placed on the imaginary axis. Cases (b) and (e) of figure 2.2 represent a marginally stable system. For a brief recap about this type of stability review section 2.2.

In order to verify these results, two different graphical representations are used: ***Phase Plane*** and ***states vs time*** representations.

Firstly, a phase plane of LTI System 1 and LTI System 2 are presented in figure 2.5. Phase plane is a graphical tool used to visualise how the solutions of a given system of differential equations would behave in the long run. From which, a statement about stability can be obtained by interpreting the plot. The type of phase plane that both system exhibit is known as the *Center Phase Plane*. In this type of phase plane the trajectories neither converge to the equilibrium point nor move infinite distant away. Rather, they stay in constant elliptical orbits as shown in figure 2.5. The type of stability that could be interpreted for both systems from figure 2.5 is marginal stability. This verifies the result obtained from evaluating the eigenvalues of the system.

Secondly, figures 2.7 and 2.8 for LTI System 1 and figures 2.9 and 2.10 for LTI System 2, illustrate the behaviour of each state (x_1 and x_2) against time. The results obtained are similar to cases (b) and (e) of figure 2.2, proving that the both systems in hand are *marginally stable*.

In order to study the effect of induced instability due to time variations, the normal functioning of the LTV system 2.5 is considered. The behaviour of the time varying element with respect to time is shown in the figure below:

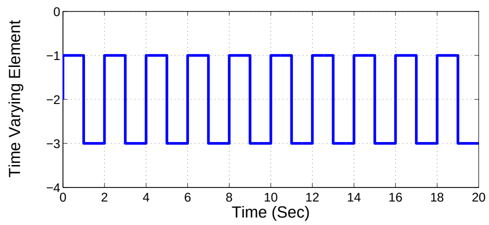


Figure 2.4: Time Varying Element vs Time

As shown in figure 2.4, the time varying element ($-\text{sgn}(\sin(\pi t)) - 2$) switches between -1 and -3 with a time time step of 1. [3] Similar to the LTI Systems, the phase plane and states vs time graphical representations are used to verify the effect of induced instability due to time variations. A phase plane of the LTV system is obtained and shown in figure 2.6. The type of phase plane that the LTV exhibits is known as a *Spiral Source Phase Plane*. In this type of phase plane, the trajectories are in the form of an outgoing spiral i.e. the trajectory starts from the intial point and grows outwards infintely in the form of a spiral as indicated in figure 2.6. In this case, it could be interpreted from figure 2.6 that the system is unstable.

Added to that, the states (x_1(t) and x_2(t)) are plotted against time in figures 2.11 and 2.12. Based on the similarity between the resulting figures and case (f) of figure 2.2, it could be concluded that time variations of LTV systems tend to induce instability.

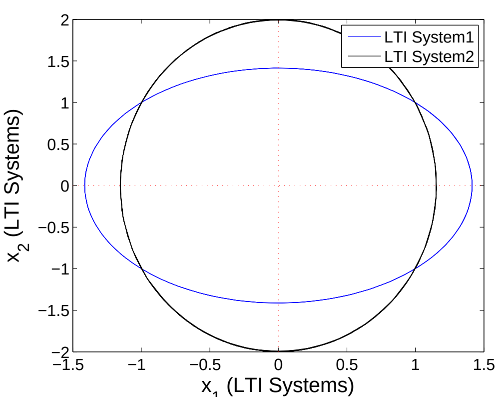


Figure 2.5: Phase Plane (x_2 vs x_1) LTI Systems

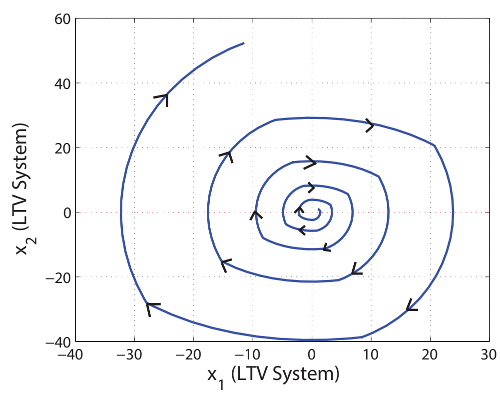


Figure 2.6: Phase Plane (x_2 vs x_1) LTV System

[3] Normaly LTV systems switch more smoothly than this example system.

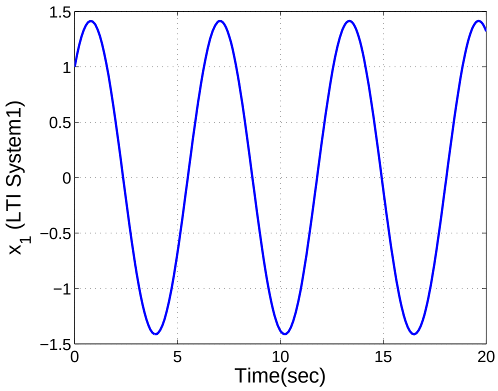


Figure 2.7: x_1 State vs Time (LTI System1)

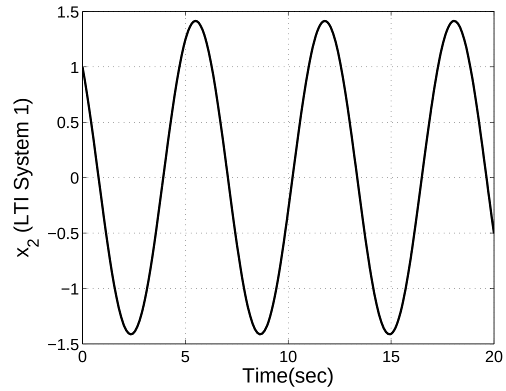


Figure 2.8: x_2 State vs Time (LTI System1)

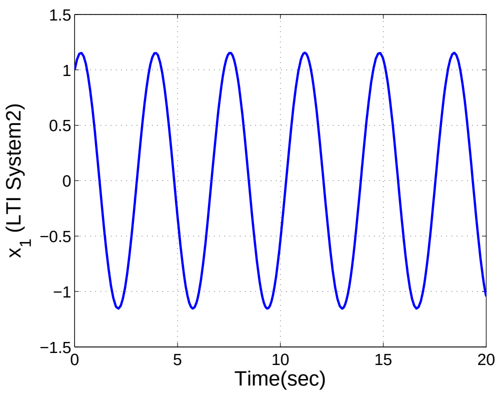


Figure 2.9: x_1 State vs Time (LTI System2)

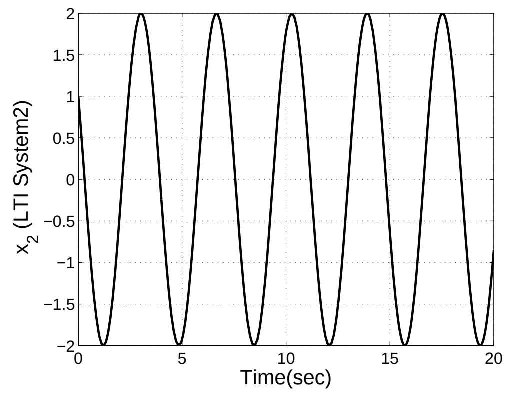


Figure 2.10: x_2 State vs Time (LTI System2)

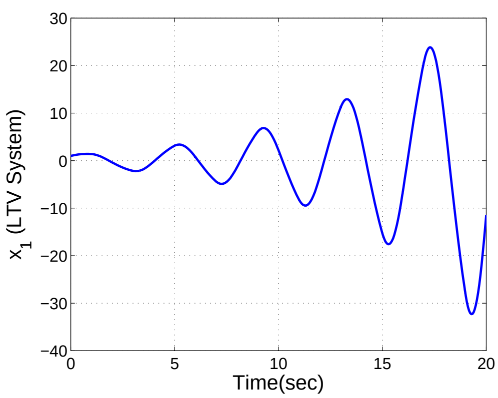


Figure 2.11: x_1 State vs Time (LTV System)

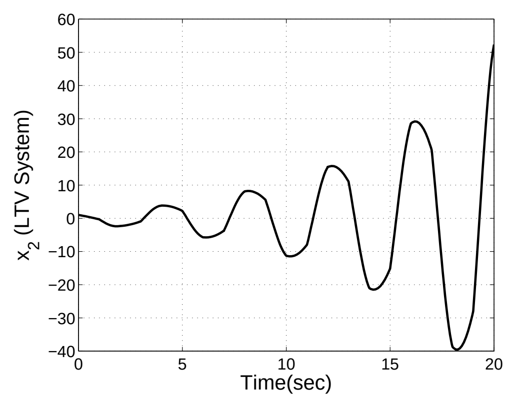


Figure 2.12: x_1 State vs Time (LTV System)

2) Induced Non-Minimum Phasedness:

Based on reference [10], LTV systems can be minimum phase for constant parameter values, but non-minimum phase under time variations. In the world of LTI systems, a minimum phase system is a system with poles and zeros (when equating the Numerator part of the Transfer function $N(s) = 0$, the zeros of the system are obtained) in the LHP, i.e. poles and zeros have negative real parts. Whereas, a non-minimum phase system is a system with one or more zeros in the Right Half Plane (RHP) complex s-domain. Right Half Plane zeros (Non-minimum phasedness) impose fundamental limitations in achievable performance for e.g. Undershoot. However, time varying systems do not have right half plane zeros per se, but its time variations result in an induced non-minimum phasedness, thereby having fundmanetal limits of achievable performance.

2.3.2 Slow Time Variations - Approximation Methods

As it was pointed out in subsection 2.3.1, the FTE (Frozen Time Eigenvalues) fails to be sufficient to analyse the stability of LTV systems. However, references [21] and [22] showed that it is possible to study the stability of LTV systems using FTEs, provided that the rate of variation of the $A(t)$ matrix is sufficiently slow [4]:

$$\sup_{t\geq 0} \|\dot{A}(t)\| < \delta \text{ (for sufficiently small } \delta) \tag{2.6}$$

To illustrate this result more clearly, the following frozen time system (LTI system) is taken into consideration:

$$\dot{x}(t) = Ax(t)$$

Assuming that the system is *exponentially stable* (refer to 2.15 for a brief recap about this type of stability) and satisfies the following solution :

$$|x(t)| \leq me^{-\lambda t}|x(0)| \tag{2.7}$$

where the parameter λ is the rate of convergence to the equilibrium position and parameter m is the peaking constant. *Note*: Let $m \geq 1$ and $\lambda > 0$.

The peaking constant m implies that the state $|x(t)|$ may increase in magnitude before decaying exponentially. In the case of LTV systems, time variations result in a period of peaking before exponentially decaying back to the equlibrium position. However, as long as the time variations do not occur rapidly, stability can be achieved. It is important to mention that the variations in the $A(t)$ matrix may be large provided that the system dynamics change slowly.

[4] This was the very first basic condition in order to find out whether the LTV system in hand is exponentially stable or not. Refer to the following page for an overview.

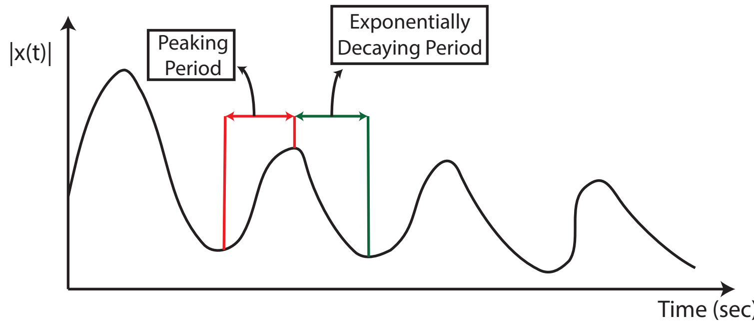


Figure 2.13: State magnitude |x(t)| (with parameter variation) vs time

Based on the chronological order of the references [21], [22], [23], [24], Lyapunov and Bellman-Gronwall arguments were used to develop sufficient conditions (In addition to the condition 2.6) for the exponential stability of slowly time varying linear systems. For a brief summary of these conditions, refer to [16]. These conditions are mainly related to the time variations of the matrix $A(t)$. Furthermore, it is not a trivial task, when it comes to specifying an upper bound (δ) that satisfies these conditions. However, quantitative conditions for δ to guarantee exponential stability of the LTV system have been obtained in [22] and [24].

On the other hand, in [25], the author developed a condition which relies on the FTEs. He performed some modifications to the techniques used in [22] and [23] and concluded that it is possible for LTV systems to be exponentially stable even if the FTE wander into the RHP. Provided that ***on average*** they are strcitly in the LHP.

As presented above, many researches have worked on ensuring exponential stability (asymptotic stability) for slowly time varying linear systems. Whereas, the instability criteria obtained is not as general as its counterpart. Speaking of which, in an attempt to derive an instability version of [21], [22] early criterion for the exponential stability of slowly time varying linear systems, [26] stated that if the matrix $A(t)$ has some FTEs in the RHP and all the FTEs are bounded away from the imaginary axis (i.e. The FTEs do not cross the imaginary axis) and $\|\dot{A}(t)\|$ is sufficiently small then the system is said to be unstable. Moreover, based on the analysis of [27], the requirement of no crossing of the imaginary axis by the eigenvalues of the system is irrelevant. Refer to [27], for a detailed reasoning.

References [21],[22], [23], [24] and [25] are considered to be the building blocks of the much more recent *numerous* works. Examples of such works include [28], [29] where the author uses Times Scales (which is considered to be the link between discrete time and continuous time systems) in order to come up with conditions that ensure stability and instability of slowly time varying linear systems. More examples include [30], [31] (for continuous and discrete time systems), [32] where authors present more sufficient and relaxed conditions to ensure that the LTV system in hand is exponentially stable.

2.3.3 Dynamic Eigenvalues

In parallel to the researches in the development of $\Phi(t, t_0)$ for LTV systems, several other theories were proposed in order to obtain the general solution of LTV systems. The timeline presented in the appendix illustrates the ideas (developed over the past 30 years) which have led to the formulation of *Dynamic Eigenvalues/Eigenvectors*. To learn more about these ideas, refer to the timeline presented in the appendix.

In this thesis, the concept of dynamic digenvalues is considered. In this subsection, a short introduction about dynamic digenvalues and a detailed step by step algorithm that calculates the dynamic eigenvalues are presented. Moreover, a detailed example is illustrated in the appendix. These values are used later on in section 3.1 to expand the parameter space approach (discussed in section 2.5) for LTV systems.

Introduction:

This theory was developed by Van der Kloet and Neerhoff. The main aim behind the development of dynamic eigenvalues is to provide a means to formulate the general solution of n^{th} order homogenuous LTV systems of the form:

$$\dot{x}(t) = A(t)x(t) \tag{2.8}$$

This is done by decoupling 2.8 into n different scalar LTV systems of the the first order via coordinate transformation, where each of the first-order systems take the general form:

$$\dot{x_t}(t) = \lambda(t)x_t(t) \tag{2.9}$$

where x_t is a new state variable related to the state variables of the original LTV system via a coordinate transformation and $\lambda(t)$ is the dynamic eigenvalue. For a better understanding of how dynamic eigenvalues are calculated, refer to the next part of this subsection. Each of these scalar systems can be solved easily using the dynamic eigenvalues. The general solution is given by:

$$x_t(t) = Ce^{\int \lambda(t)dt}$$

where C is an arbitrary constant which depends on the initial conditions given for the original problem. As a result, the system's general solution can be obtained and the LTV system's stability could be determined. Another application of dynamic eigenvalues (which will not be discussed) is determining the Lyapunov exponents of LTV using the *mean value of the dynamic eigenvalues* [5] [33],[34] and [35].

[5] The chronoligical order of the publications should be followed

Algorithm:

The algorithm for calculating the dynamic eigenvalues presented here, is mainly based on references [36] and [37].

LTV system described by a n^{th} order linear differential equation with time-varying coefficients is given by:

$$D^n x + a_1(t)D^{n-1}x + ...a_{n-1}(t)Dx + a_n(t)x = 0 \tag{2.10}$$

The equivalent state space description is the following:

$$\dot{x} = A(t)x(t) = \begin{bmatrix} I_{n-1}^{+} & e_{n-1} \\ q_n^T(t) & \tilde{q_n}(t) \end{bmatrix} x(t) \tag{2.11}$$

where:

I_{n-1}^{+} represents an upward shifted identity matrix with dimensions $(n-1) \times (n-1)$:

$$I_{n-1}^{+} = \begin{bmatrix} 0 & 1 & ... & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & ... & ... & 0 \end{bmatrix}$$

e_{n-1} represents the $(n-1)$ dimensional unit vector:

$$e_{n-1} = \begin{bmatrix} 0 & ... & 0 & 1 \end{bmatrix}^T$$

And $q_n^T(t)$ and $\tilde{q_n}(t)$ represent the coefficients of the differential equation:

$$q_n^T(t) = -[a_n(t)......a_2(t)]$$

$$\tilde{q_n}(t) = -a_1(t)$$

Let the system 2.11 be subjected to a time dependent coordinate transformation:

$$x(t) = R(t)y(t)$$

where $y(t)$ is the new unknown. In simpler terms, the $R(t)$ matrix (also known as Riccati matrix) transforms the system from coordinate system $x(t)$ to $y(t)$. Resulting in the new LTV system:

$$\dot{y}(t) = B(t)y(t)$$

in which $B(t) = R^{-1}(t)A(t)R(t) - R^{-1}\dot{R}$. The proof of $(B(t) = R^{-1}(t)A(t)R(t) - R^{-1}(t)\dot{R}(t))$ is shown below:

$$
\begin{aligned}
x(t) &= R(t)y(t) \\
\dot{x}(t) &= \dot{R}(t)y(t) + R(t)\dot{y}(t) \\
\text{Substituting } \dot{x}(t) = A(t)x(t) \text{ in the equation above:}& \\
A(t)x(t) &= \dot{R}(t)y(t) + R(t)\dot{y}(t) \\
R(t)\dot{y}(t) &= A(t)x(t) - \dot{R}(t)y(t) \\
\text{Substituting } x(t) = R(t)y(t) \text{ in the equation above:}& \\
[R(t)\dot{y}(t) &= A(t)R(t)y(t) - \dot{R}(t)y(t)] \cdot R^{-1}(t) \\
\text{Multiplying both sides of the equation by } R^{-1}(t):& \\
\textit{(to cancel out R(t) in the left hand side of the equation)}& \\
\underbrace{R^{-1}(t)R(t)}_{1}\dot{y}(t) &= R^{-1}(t)A(t)R(t)y(t) - R^{-1}(t)\dot{R}(t)y(t) \\
\dot{y}(t) &= R^{-1}(t)A(t)R(t)y(t) - R^{-1}(t)\dot{R}(t)y(t) \\
\text{Taking } y(t) \text{ as the common factor of the right hand side:}& \\
\dot{y}(t) &= \underbrace{[R^{-1}(t)A(t)R(t) - R^{-1}(t)\dot{R}(t)]}_{B(t)}y(t)
\end{aligned}
$$

The goal is to obtain an upper triangular matrix $B(t)$. This is achieved by iteratively performing coordinate transformations. In figure 2.14, the iterative coordinate transformation algorithm is illustrated with the help of a flowchart. Prior to proceeding with figure 2.14, the following must be noted:

1. The iterations are coordinate transformations performed on the original LTV-system's $A(t)$ matrix. The counter is the letter k, and the initial value of $k = n$ where n is the size of the initial LTV-system's $A(t)$ matrix. i.e. if the dimensions of the $A(t)$ matrix are 3×3 then $n = 3$. The counter decrements until $k = 1$. After each iteration, a new $A(t)$ matrix is formulated (For e.g: After the first iteration $A_{n-1}(t)$ is obtained from performing the coordinate transformation on $A_n(t)$). Once $A_1(k = 1)$ is formulated, an upper triangle matrix B(t) is obtained.

$$
\begin{aligned}
A(t) = A_{(k)}(t) = A_{(3)}(t) &\xrightarrow{1^{st}\text{coordinate transformation}} A_{(2)}(t) \\
A_{(2)}(t) &\xrightarrow{2^{nd}\text{ coordinate transformation}} A_{(1)}(t) = B(t)
\end{aligned}
\tag{2.12}
$$

   Note: The number of coordinate transformation iterations is always $n - 1$.

2. The Riccati transformation matrix $R(t)$ is taken as $P_{(k)}(t)$ for every iteration.

$$
R(t) = P_{(k)}(t) = \begin{bmatrix} I & 0 \\ p_k^T & 1 \end{bmatrix}
$$

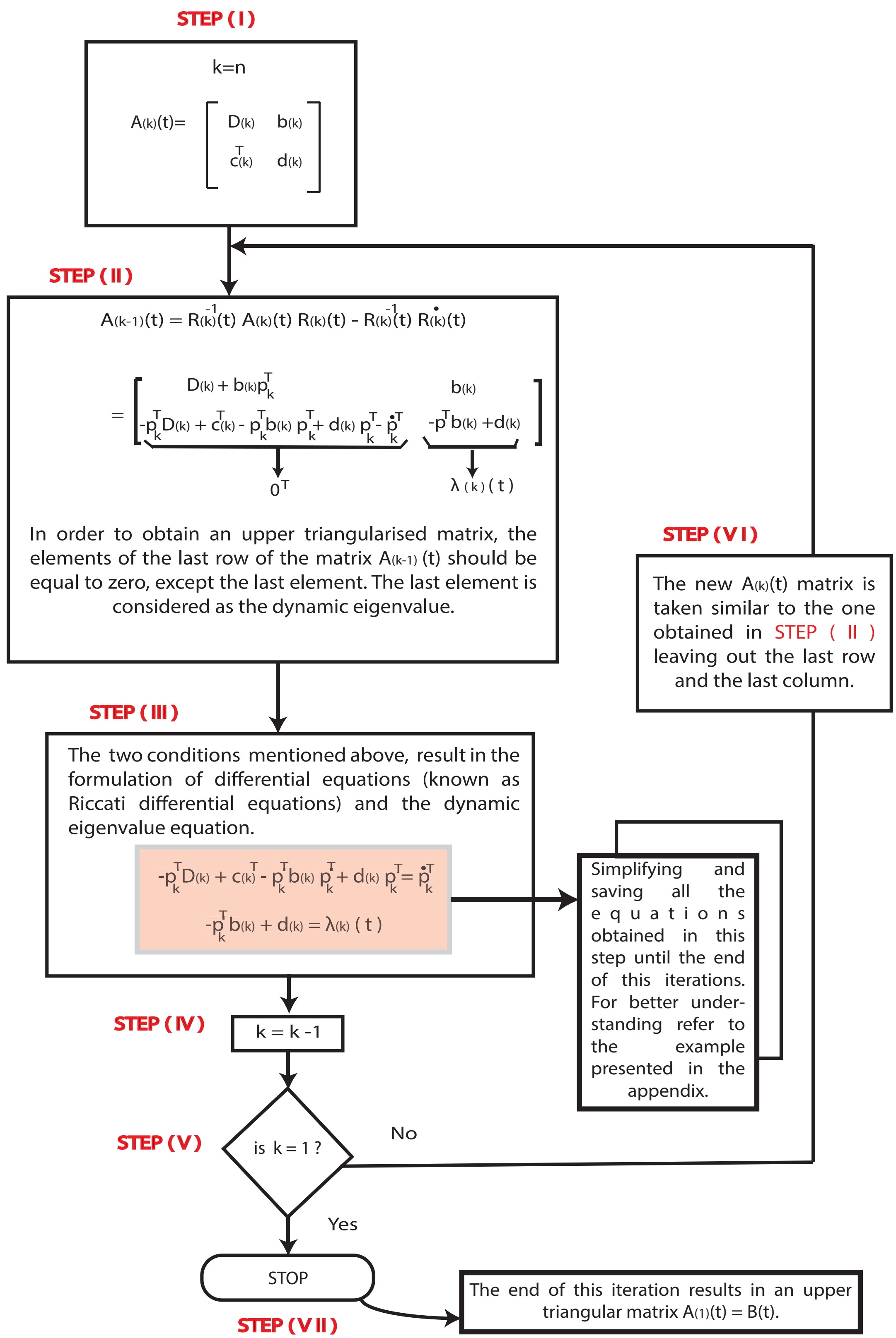


Figure 2.14: Algorithm for deriving the Dynamic Eigenvalues

where I is the identity matrix of the size $(k-1)\times(k-1)$ and p_k^T is a vector of the size $1\times(k-1)$. Similarly, if n=3:

a) First iteration $P_3(t) \Rightarrow k = n = 3$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } p_3^T = \begin{bmatrix} p_{3,1}^T & p_{3,2}^T \end{bmatrix}$$

b) Second/Last iteration $P_2(t) \Rightarrow k = 2$

$$I = \begin{bmatrix} 1 \end{bmatrix} \text{ and } p_2^T = \begin{bmatrix} p_{2,1}^T \end{bmatrix}$$

3. The matrix $A_k(t)$ is considered in the following format:

$$\begin{bmatrix} D_k & b_k \\ p_k^T & 1 \end{bmatrix}$$

where D_k is of the size $(k-1)\times(k-1)$, b_k is of the size $(k-1)\times 1$ and c_k^T is of the size $1\times(k-1)$.

For a better understanding of the algorithm explained and illustrated in figure 2.14, a detailed example is presented in the appendix.

2.4 Lyapunov Stability

As mentioned in the previous section 2.2, there exists several stability criteria for LTI-Systems. However, if the system is nonlinear or linear but time varying (LTV), then such stability is not applicable.

In the late 19^{th} century, the Russian mathematician Alexander Mikhailovich Lyapunov formulated and developed a number of definitions of stability. Figure 2.15 shows the various types of stability in a summarised manner. From there two founding methodologies on the investigation of the stability of nonlinear and LTV systems were presented, namely:

(1) Lyapunov's First (Indirect) method
(2) Lyapunov's Second (Direct) method

In short, Lyapunov's first method is a technique which uses the idea of system linearization (lowest order approximation) around a given point which means that only local stability results with small stability regions can be achieved. This method is not the scope of this thesis, as it deals with Linear systems.

On the other hand, Lyapunov's direct method is a *qualitative, potentially global*, *stability analysis technique* with the following basic concept:

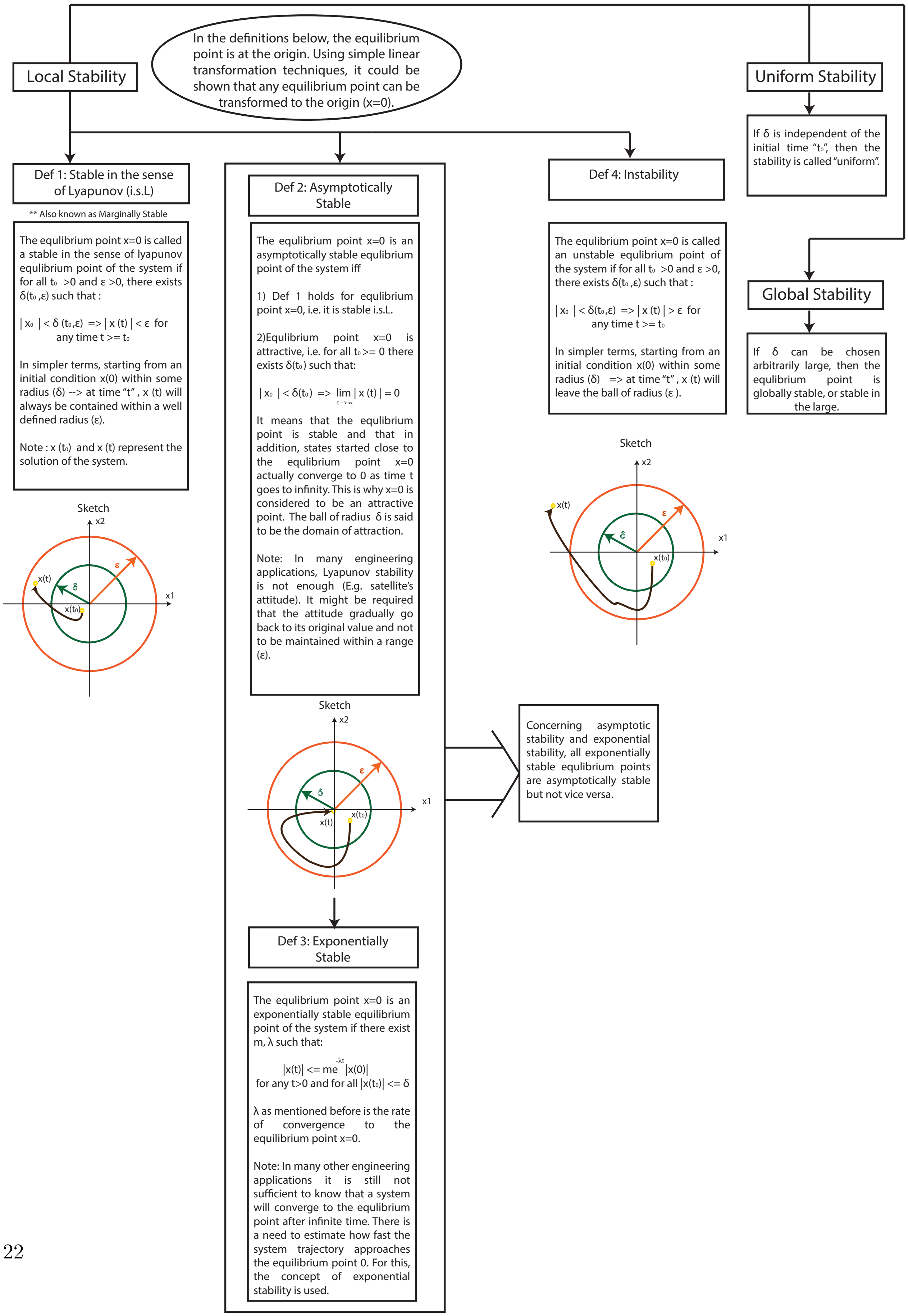


Figure 2.15: Stability Recap

"If the total energy of a system (Electrical, Mechanical,etc) and (Linear or Nonlinear), is continuously dissipating, then the system will eventually reach an equlibrium point and remain at that point."

Lyapunov's direct method is divided into two steps:

- Find an appropriate scalar function, referred to as **Lyapunov function**. In most references it is denoted by the capital letter *V(x)*.

- Evaluate its first-order **time** derivative along the trajectory of the system ($\dot{V}(x)$). If the derivative is decreasing along the system trajectory as time increases, then the system energy is dissipating and thus the system will eventually settle down.

In a more formal statement the above can be described. Given any set of differential equations, if any function V(x) [countinuously differentiable function] can be defined within these two well defined properties:

- *V(x)* > 0 [Positive Definite]

- $\dot{V}(\mathbf{x})$ [with respect to time] ≤ 0 [Negative Semidefinite $\Rightarrow$ the system is stable i.s.L]
  OR
  $\dot{V}(\mathbf{x})$ [with respect to time] < 0 [Negative definite $\Rightarrow$ the system is asymptotically stable]

then the system is said to be stable i.s.L (in the sense of Lyapunov) / asymptotically stable without having the differential equations solved. This is an advantage because solving nonlinear and/or time varying state equations is very difficult.

However, for every aspect there are two sides. Eventhough, this method possesses the above mentioned important advantage, it can also be tricky to construct a Lyapunov function for a nonlinear (or) LTV system. On the other hand, for LTI systems, there exists a class of functions where the check of positive definiteness is easy. This class of functions is known as ***Quadratic Lyapunov Function***. It can be written as:

$$V(x) = x^T P x$$

with P being real, symmetric, and positive definite. Next, an unforced LTI System (where $u(t) = 0$ in 2.2) is considered:

$$\dot{x} = Ax(t)$$

Now the derivative is computed :

$$\begin{aligned}
\dot{V}(x) = \frac{\partial V}{\partial t} &= \frac{\partial V(x(t)^T, P, x(t))}{\partial t} \\
&= \dot{x}^T P x + x^T P \dot{x} \\
&= (Ax)^T P x + x^T P (Ax) \\
&= x^T A^T P x + x^T P A x \\
&= x^T (A^T P + PA) x
\end{aligned} \tag{2.13}$$

For $\dot{V}(x)$ to be a negative/semi-definite function to ensure Asymptotic stability/i.s.L stability, the following equation is derived:

$$A^T P + PA = -Q \tag{2.14}$$

where Q is a positive definite matrix. This famous equation is known as the *Lyapunov Equation*. So the P matrix can be guessed and Q computed to see whether it is positive definite (asymptotically stable) or positive semidefinite (Stable i.s.L). A more common (and reliable) technique which is used, is choosing a positive definite Q and solving the Lyapunov equation for P matrix (which is not an easy task). Any Q will do, so $Q = I$ is often used. Note that any pre-defined Q will show whether a LTI system is stable, whereas choosing a P and solving for Q will show neither stability nor instability if Q does not turn out to be positive definite.

While Quadratic Lyapunov Functions need not characterise stability under time variations i.e. for LTV Systems, sufficient conditions for exponential stability in connection with the Lyapunov Equations were obtained in the recent paper [38]. The results are presented here are in a simplified form.

An unforced LTV System is considered:

$$\dot{x}(t) = A(t)x(t) \tag{2.15}$$

The Riccati Differential Equation (RDE) [6] of a LTV system is written in the following form:

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + P(t)B(t)B^T(t)P(t) + Q = 0 \tag{2.16}$$

Since 2.15 is an unforced LTV system, equation 2.16 can be rewritten as:

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + Q = 0 \tag{2.17}$$

Based on [38], this equation is known as *Time Varying Lyapunov Equation*. For 2.15 to be exponentially stable, the following condition must be satisfied: The time varying Lyapunov

[6] For an overview of the existing literature on the Riccati Equation refer to [39], [40] and [41]

Equation has a solution $P(t)$ (Positive Semi-Definite) for some Q (Symmetric Positive Definite Constant Matrix). Provided that:

$$Q - [A(t) + A^T(t)] > 0 \tag{2.18}$$

Refer to pp.7 of [38], for an example. It is important to note that LTI systems, are represented by the Algebraic Riccati Equation (ARE):

$$A^T P + PA + Q = 0$$

which is similar to the Lyapunov Equation 3.2. Refer to section 2.5.2.2 for an application of this note.

2.5 Parameter Space Approach

This section presents the parameter space approach method. The goal of this approach is to calculate the whole space of stabilizing controller parameters. The presented approach to guarantee robust stability was first introduced in [6] for LTI systems, and was then extended to time delay LTI systems in [2] and [5][7].

In section 2.5.1, the basics of the parameter space approach are presented for Transfer Function representations of LTI systems. The type of controller used in this section is the PID controller (Control parameters: K_P, K_D and K_I). In the following section 2.5.2, the results of [4] are utilised, whereby the parameter space approach is applied to the state space representations of LTI systems controlled/stabilized by state feedback controllers (Control parameters: k_1, k_2,..., k_n, where n represents the order of the system). Based on this representation, two different approaches for obtaining the stabilizing space of the controller parameters are illustrated in sections 2.5.2.1 and 2.5.2.2.

2.5.1 Transfer Function Represention - LTI Systems

To identify the stable area in the space of controller parameters, it is required to obtain the characteristic polynomial of the system in hand. Figure 2.16 briefly describes the derivation of the characteristic polynomial *P(s,q,k)* of an LTI (Linear Time Invariant) system. The characteristic polynomial depends on the parameters s, q and k, where s is the complex argument of the laplace transform, k represents the controller parameters (In the case of PID Controllers: K_I,K_D and K_P) and q represents the system parameters.

The first step of deriving the characteristic polynomial is obtaining the system's equation of motion (differential equation) using : e.g. Newton's Second Law (or) Euler's Rotational Law. Followed by performing the Laplace Transform to the equation of motion obtained, to

[7] Time delay free LTI systems are considered in this seciton.

get the System / Plant's Transfer function $G(s)$. Laplace transform method is used, because the transform turns differential equations to polynomial equations, which are much easier to solve. In figure 2.16(a), Laplace transformations which are applied to differential equations are illustrated.

Using the System / Plant's transfer function $G(s)$, a Feedback Control Loop (closed loop system) of the corresponding system is represented. To get the characteristic polynomial, the overall transfer function of the closed loop system must be obtained. This step is done using the block diagram transformation shown in figure 2.16(b). This step is considered to be the final step, because the characteristic polynomial *P(s,q,k)* is simply the denominator *D(s,q,k)* of the overall transfer function.

The next step in identifying the controller parameters' stable areas is obtaining the three different limits (RRB, CRB and IRB). In the following sections 3.1.1 to 3.1.3, all three limits are explained and derived. Finally, once the limits are obtained, the stable area of the controller parameters is identified by testing the eigenvlaues.

For a better understanding of this state of the art control approach, refer to the SISO (Single Input Single Output) example system presented in the appendix.

In this section, a PID controller is considered:

$$G_H(s) = 1$$
$$G_C(s) = K_P + K_D s + \frac{K_I}{s}$$
$$G_S(s) = \frac{A(s)}{R(s)}$$

2.5.1.1 Real Root Boundary (RRB)

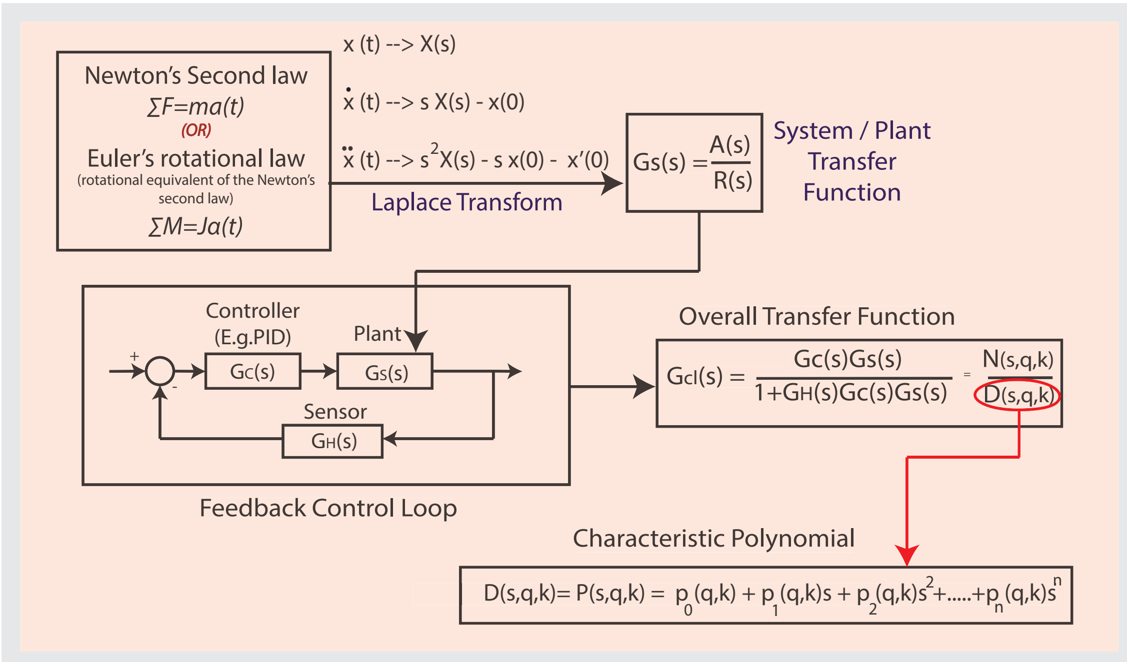
In this limit, the polynomial has a root at the origin. While crossing the RRB in the parameter space, a real root crosses the imaginary axis at the origin (as shown in figure 2.17 - represented in green). In order to derive this limit, the characteristic polynomial $P(s = 0, q, k) = 0$ is solved for the parameters q and k.

The following is the derivation of the RRB limit:

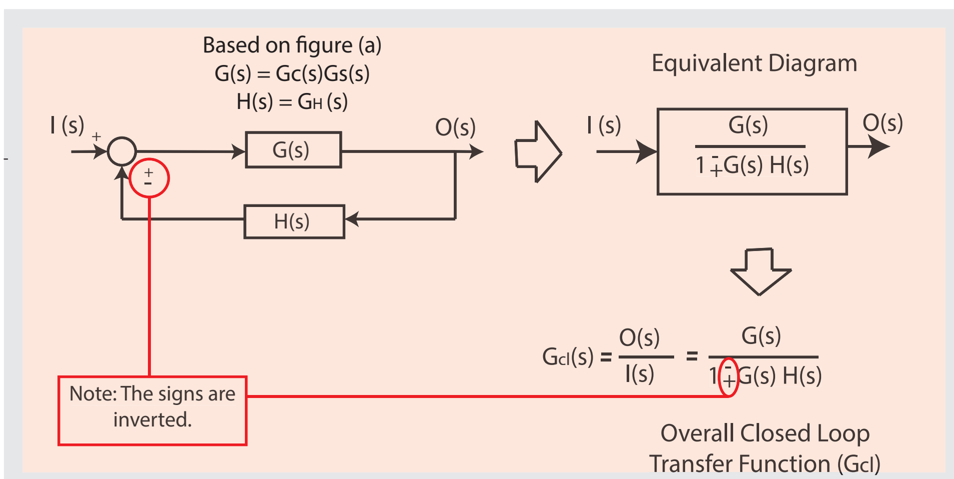
$$0 = 1 + G_H(s)\; G_C(s)\; G_S(s)$$

$$0 = 1 + \left(\frac{K_P s + K_D s^2 + K_I}{s}\right)\left(\frac{A(s)}{R(s)}\right)$$

$$0 = 1 + \left(\frac{\overbrace{(K_P s + K_D s^2 + K_I)\; A(s)}^{A^*(s)}}{\underbrace{s\; R(s)}_{B(s)}}\right)$$



(a)



(b)

Figure 2.16: Derivation of characteristic polynomial

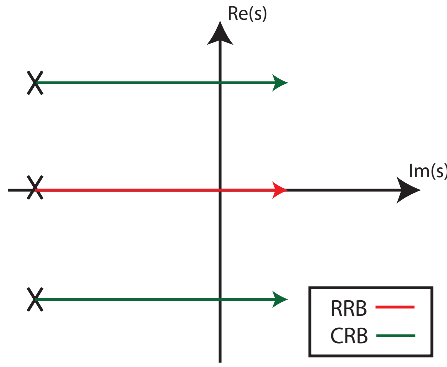


Figure 2.17: RRB and CRB

$$B(s) \cdot \left(0 = 1 + \frac{A^*(s)}{B(s)}\right)$$
$$0 = B(s) + A^*(s) \tag{2.19}$$

$$0 = \left(b_n s^n + b_{n-1} s^{n-1} + ... + b_2 s^2 + b_1 s + b_0\right) + \left(K_P s + K_D s^2 + K_I\right)\left(a_m s^m + a_{m-1} s^{m-1} + ... + a_2 s^2 + a_1 s + a_0\right) \tag{2.20}$$

Substituting $s = 0$ in the equation above yields:

$$b_0 + K_I(a_0) = 0,$$

where b_0 is always equal to 0. The resulting equation is:

$$K_I = -\frac{b_0}{a_0} = -\frac{0}{a_0} = 0 \tag{2.21}$$

2.5.1.2 Infinite Root Boundary (IRB)

To understand this type of limit, refer to [42]. In order to derive the IRB limit, $s = \infty$ is substituted in equation 2.20.

Note: ∞^n is $\gg$ $\infty^{n-1}, \infty^{n-2}, \infty^{n-1} .... \infty^3, \infty^2, \infty$ and as a result, the coeffecient with the highest power of s is taken and the rest are eliminated.

Taking the note above into consideration results in:

$$0 = b_n s^n + K_D s^2 (a_m s^m)$$

$$K_D = -\frac{b_n s^n}{a_m s^m s^2} = -\frac{b_n s^n}{a_m s^{(m+2)}}$$

To evaluate the derived equation and get the IRB limit, it is required to get the limits as s tends to infinity.

$$\lim_{s \to \infty} K_D = -\frac{b_n s^n}{a_m s^{(m+2)}}$$

In order to achieve a reasonable and correct result, the powers of both complex arguments s in the numerator (n in the case of s^n) and the denominator ($m+2$ in the case of s^{m+2}) should be equal.

$$n = m + 2 \tag{2.22}$$

If condition 2.22 is satisfied, the resulting equation is:

$$K_D = -\frac{b_n}{a_m} \tag{2.23}$$

2.5.1.3 Complex Root Boundary (CRB)

In this limit, the polynomial has a complex conjugate pair of roots on the imaginary axis. While crossing the CRB in the parameter space, the complex conjugate pair cross the imaginary axis as shown in figure 2.17). In order to derive this limit, the characteristic polynomial $P(s = j\omega, q, k) = 0$ is solved for the parameters q and k.

The following is the derivation of the CRB limit:

Equation 2.19 can be rewritten in the form:

$$0 = B(s) + (K_P s + K_D s^2 + K_I)\, A(s) \tag{2.24}$$

Substituting $s = j\omega$ in polynomials $A(s)$ and $B(s)$, results in:

$$A(s) = A(j\omega) = R_A + jI_A$$

$$B(s) = B(j\omega) = R_B + jI_B$$

$$0 = R_B + jI_B + (K_P(j\omega) + K_D(j\omega)^2 + K_I)(R_A + jI_A)$$

$$Note : j^2 = -1 \Rightarrow 0 = R_B + jI_B + (K_P(j\omega) - K_D\omega^2 + K_I)(R_A + jI_A)$$

$$0 = R_B + jI_B + j\omega R_A K_P - \omega^2 R_A K_D + R_A K_I + j^2\omega I_A K_P - j\omega^2 K_D I_A + jI_A K_I$$

$$0 = R_B + jI_B + j\omega R_A K_P - \omega^2 R_A K_D + R_A K_I - \omega I_A K_P - j\omega^2 K_D I_A + jI_A K_I = \hat{p}(\omega)$$

Seperating the equation above into real and imaginary parts:

$$\mathrm{Re}(\hat{p}(\omega)) = R_B - \omega^2 R_A K_D + R_A K_I - \omega I_A K_P = 0 \tag{2.25}$$

$$\mathrm{Im}(\hat{p}(\omega)) = I_B + \omega R_A K_P - \omega^2 K_D I_A + I_A K_I = 0 \tag{2.26}$$

In order to obtain the CRB limits, the equations above are to be fromulated to get the imaginary part as a function in terms of (K_P and ω) and the real part as a function in terms of (K_D, K_I and ω). From equation 2.26:

$$I_B + \omega R_A K_P + I_A \underbrace{(K_I - \omega^2 K_D)}_{x} = 0 \tag{2.27}$$

Similarly, from equation 2.25:

$$R_B + R_A \underbrace{(K_I - \omega^2 K_D)}_{x} - \omega I_A K_P = 0$$

$$R_B + R_A(x) - \omega I_A K_P = 0$$

$$x = \frac{\omega I_A K_P - R_B}{R_A} \tag{2.28}$$

Substituting 2.28 in 2.27:

$$I_B + \omega R_A K_P + I_A \left(\frac{\omega I_A K_P - R_B}{R_A} \right) = 0$$

$$I_B + \omega R_A K_P + \frac{{I_A}^2 \omega K_P}{R_A} - \frac{I_A R_B}{R_A} = 0$$

$$I_A{}^2\omega K_P + \omega R_A{}^2 K_P = I_A R_B - I_B R_B$$

$$K_P(I_A{}^2\omega + \omega R_A{}^2) = I_A R_B - I_B R_A$$

The above expression is reformulated to get:

$$K_P = \frac{I_A R_B - I_B R_A}{\omega(I_A{}^2 + R_A^2)} \tag{2.29}$$

Given a constant K_P value, the values of ω are evaluated. To get the CRB limits, the K_P and ω values calculated are substituted in equation 2.30:

Note: Equation 2.30 is a simple reformulation of 2.25.

$$K_I = \omega^2 K_D + \frac{\omega I_A K_P - R_B}{R_A} \tag{2.30}$$

Refer to the appendix for an example.

2.5.2 State Space Representation - LTI Systems

In the previous section, LTI systems were represented in the form of transfer functions (Output/Input). In this section, the state space representation of an LTI system is considered. When it comes to the type of controller implemented, state feedback controllers are used instead of the PID Controllers used in the previous section.

In this type of controller design, the main aim is to control the location of all closed loop poles by the feedback of all states to the control $u(t)$ through a gain k (where k can be adjusted to yield the required closed loop poles). Similarly, the parameter space approach is adopted to obtain the RRB, IRB and CRB limits, in order to identify the stable areas of the state feedback controller parameters (k_i) where $i = 1, 2, 3...n$. This can be applied to systems which are completely controllable only. The three limits can be obtained using one of the two aprroaches presented in sections 2.5.2.1and 2.5.2.2.

However, prior to proceeding with these two approaches, a general overview about state feedback is presented. To begin with, an LTI state space equation is considered:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.31}$$

which represents the open loop system to be controlled. Suppose "state feedback" is implemented:

$$u(t) = -Kx(t) + r(t)$$

where $r(t)$ is the reference input (i.e. the equilibrium point to be reached). Refer to figure 2.18 for a block diagram representation of a state feedback closed loop system. Next, the equilibrium point is considered to be 0. $\therefore r(t) = 0$.

$$u(t) = -Kx(t) \tag{2.32}$$

Substituting 2.32 in 2.31 to get the closed loop system:

$$\dot{x}(t) = Ax(t) + B(-Kx(t))$$
$$\dot{x}(t) = Ax(t) - BKx(t)$$

The resulting closed loop state equation:

$$\dot{x}(t) = (A - BK)x(t) \tag{2.33}$$

where $A - BK = A_k$.

$$\dot{x}(t) = A_k x(t)$$

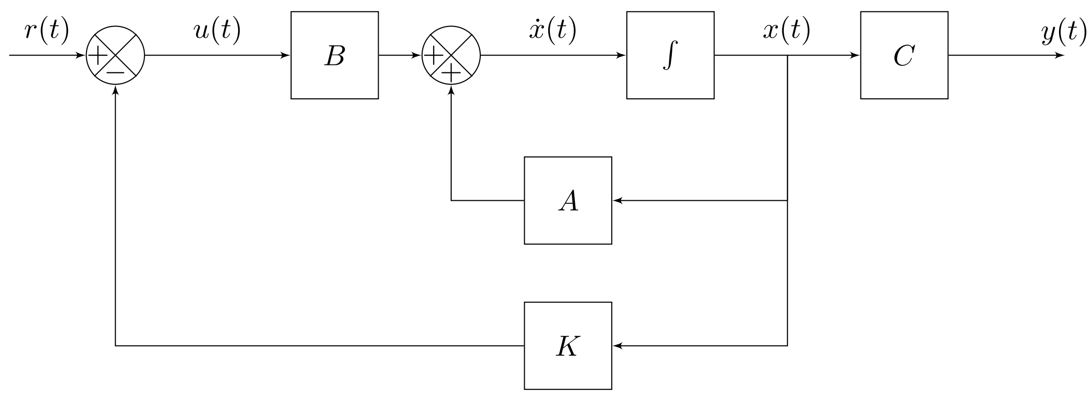


Figure 2.18: State feedback closed Loop

In this thesis, LTI systems in the Control Canonical Form (CCF) are considered [8]:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u(t) \tag{2.34}$$

[8] The reason why this representation is considered, is because the system is fully controllable + almost all system can be transformed into the CCF.

The state feedback gain matrix K is considered:

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \tag{2.35}$$

The closed loop state equation (by simply substituting in equation 2.33) is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -q_1 - k_1 & -q_2 - k_2 & -q_3 - k_3 \end{bmatrix}}_{A_k = A - BK} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{2.36}$$

With equation 2.36, the overview concludes. Next, two approaches are presented to obtain the three limits (RRB,CRB and IRB) which form the state feedback controller space.

2.5.2.1 Direct Approach

The following results are based on the works of [4], where the author extends the parameter space approach to state feedback controllers.

The step by step procedure of obtaining the controller parameter space of 2.36 is illustrated as follows. Firslty, the characteristic equation is obtained by finding the determinant of $(A_k - sI)$ and equating it to 0. The characteristic equation $p(s, q, k)$ is:

$$p(s, q, k) = s^3 + (q_3 + k_3)s^2 + (q_2 + k_2)s + (q_1 + k_1) = 0 \tag{2.37}$$

In case of RRB , where $s = 0$, the limit is:

$$q_1 + k_1 = 0$$

$$k_1 = -q_1 \tag{2.38}$$

Secondly, the IRB where $s = \infty$, the limit is: [9]

$$\text{Coefficient of}(s^3) = 1 \neq 0 \tag{2.39}$$

Finally, the CRB where $s = j\omega$, the limit is:

[9] Refer to section 2.5.1.2

$$(j\omega)^3 + (q_3 + k_3)(j\omega)^2 + (q_2 + k_2)j\omega + (q_1 + k_1) = 0$$

$$-j\omega^3 - (q_3 + k_3)\omega^2 + (q_2 + k_2)j\omega + (q_1 + k_1) = 0$$

Taking the Real part:

$$-(q_3 + k_3)\omega^2 + q_1 + k_1 = 0$$

$$k_1 = (q_3 + k_3)\omega^2 - q_1 \tag{2.40}$$

Taking the Imaginary part:

$$-\omega^3 + (q_2 + k_2)\omega = 0$$

$$k_2 = \omega^2 - q_2 \tag{2.41}$$

2.5.2.2 Algebraic Riccati Equation Approach (Lyapunov Equation)

This approach is based mainly on the results of [8]. However, some amendments were made to the approach used in this thesis in order to comply with obtaining the three limits which form the controller parameter space instead of mapping systems with uncertain parameters (implemented in the thesis).

Instead of directly applying $\det(sI - A_k) = 0$ and obtaining the characteristic equation as illustrated in the previous approach, the closed loop LTI system is reformulated using the Algebraic Riccati Equation (ARE) and the characteristic equation is obtained through the equation.

An Algebraic Riccati Equation (ARE) is a matrix equation that is quadratic in an unknown Hermitian matrix P. The general ARE for the unknown matrix P is given by:

$$PRP - A_k^T P + PA_k = -Q \tag{2.42}$$

where $R = BB^T$.

Hermitian Matrix

A square matrix P is called Hermitian when :

$$P = P^H$$

where P^H denotes the conjugate transpose. In case P is a real matrix, then P^H is basically the transpose of matrix P.
Example:

$$P = \begin{bmatrix} 5 & 6 \\ 6 & 8 \end{bmatrix} \text{ and } P^H = \begin{bmatrix} 5 & 6 \\ 6 & 8 \end{bmatrix}$$

Since $P = P^H$, then P matrix is called Hermitian.

Many important problems in dynamics and control of systems can be formulated as AREs. In this thesis, the ARE equation associated with LTI system is studied in order to map the closed loop LTI, into the controller parameter space using four mapping equations to be discussed below.

Prior to proceeding with any explanation, note that for $R = 0$, the ARE reduces to an affine matrix equation in P. The resulting equation is the *Lyapunov Equation*, similar to equation 3.2. As explained earlier in section 2.4, the Lyapunov Equation is a useful tool in analysing the stability of systems.

Associated with 2.42 is a $2n \times 2n$ Hamiltonian matrix :

$$H = \begin{bmatrix} -A_k & R \\ Q & A_k^T \end{bmatrix} \tag{2.43}$$

The matrix H can be used to obtain the solutions of 2.42. For the mapping of Lyapunov equations to the parameter space, these solutions are irrelevant, but some properties of 2.43 are used. Namely, the set of all eigenvalues of H is symmetric about the imaginary axis, as shown in figure 2.19.

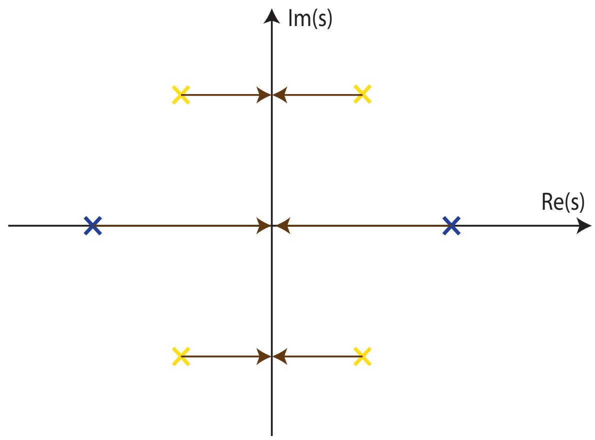


Figure 2.19: Eigenvalues of the Hamiltonian Matrix - symmetric about the imaginary axis

To check whether matrix H satisfies the Hamiltonian Matrix Form:

1. Begin with defining the skew-symmetric matrix J:

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \tag{2.44}$$

   where I_n is the $n \times n$ identity matrix.

2. Followed by calculating:
   a) $J^{-1}HJ$
   b) $-JHJ$
   c) $-H^T$

3. H is Hamiltonian, if and only if the following condition is satisfied:

$$J^{-1}HJ = -JHJ = -H^T$$

Once the condition above is satisfied, the 3 limits RRB, IRB and CRB are retrieved using the mapping equations 2.45, 2.46, 2.47. Refer to sections 2.5.1.1, 2.5.1.2 and 2.5.1.3 for an introduction to the three limits RRB, IRB and CRB. For the RRB limit, the corresponding mapping equation is:

$$\det(sI - H)_{s=0} = 0 \tag{2.45}$$

Similarly, for the IRB limit:

$$\det(sI - H)_{s=\infty} = 0 \tag{2.46}$$

In case of the CRB limit, a double eigenpair crosses the imaginary axis. When a polynomial has a multiple root, its derivate also shares that root. In simpler terms, if a polynomial $f(w)$ has a double root, then one of the roots is obtained when $f(w) = 0$ and the other root is obtained when $\frac{df(w)}{dw} = 0$. The reason behind the existence of double roots is because of the appearance of A_k twice in the diagonal of 2.43. Subsequently, these two mapping CRB equations are given:

$$\det(sI - H)_{s=j\omega} = 0 \tag{2.47a}$$

$$\frac{\partial}{\partial w}\det(sI - H)_{s=j\omega} = 0 \tag{2.47b}$$

An example is presented in the appendix, showing the calculations of all three limits clearly and in a step by step manner. Added to that the resulting plots are also illustrated.

2.5.3 Summary of the characteristic equation derivations

In this section, a summary of the characteristic equation derivations for LTI systems illustrated in previous sections 2.5.1 and 2.5.2 is presented.

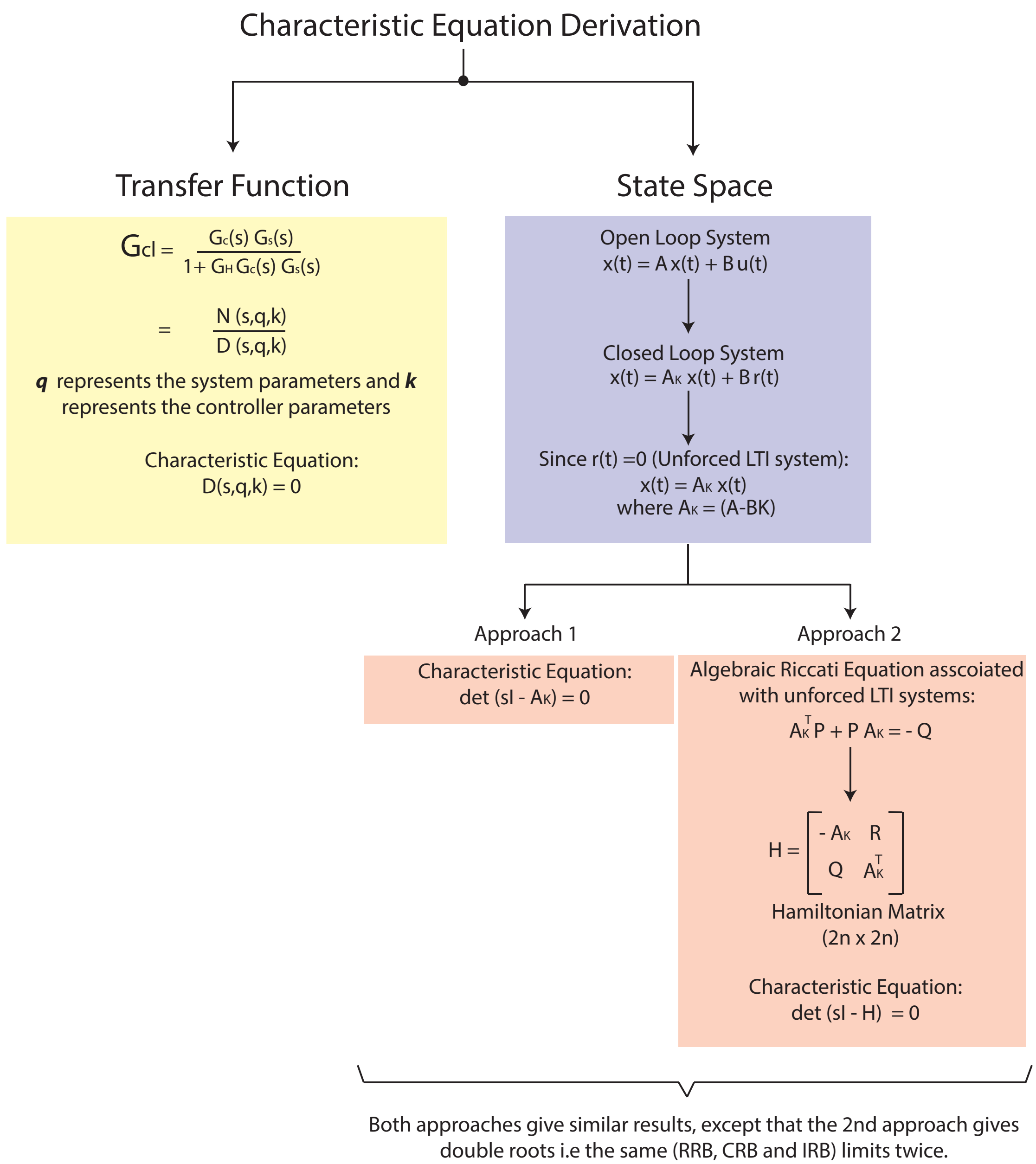


Figure 2.20: Summary of characteristic equation derivation of section 2.5

3

Expansion of the Parameter Space Approach for LTV Systems

3.1 Dynamic Eigenvalues Approach

As mentioned earlier in section 2.3.3 of chapter two, the aim of the thesis is to study the stability of LTV systems using the dynamic eigenvalues in a similar manner to the eigenvalues of LTI systems. i.e. treating the dynamic eigenvalues as time-varying poles in the complex s-domain.[1]

For an LTI systems, the eigenvalues (λ_1, λ_2 and λ_3) of the A-matrix are sketched below in figure 3.1:

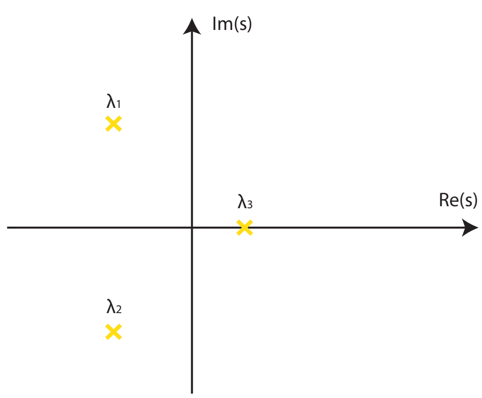


Figure 3.1: LTI Eigenvalue Plot

[1] complex s-domain, frequency domain, Laplace domain all mean the same

Based on subsection 2.2, the LTI system is said to be unstable. Similarly, the dynamic eigenvalues of an LTV system ($\lambda_1(t)$, $\lambda_2(t)$ and $\lambda_3(t)$) are plotted in the complex s-domain. For simplicity, let t vary from $t = 1$ to $t = 5$. For every time t (with step size = 1), the dynamic eigenvalues are plotted. This is shown clearly in sketch 3.2.

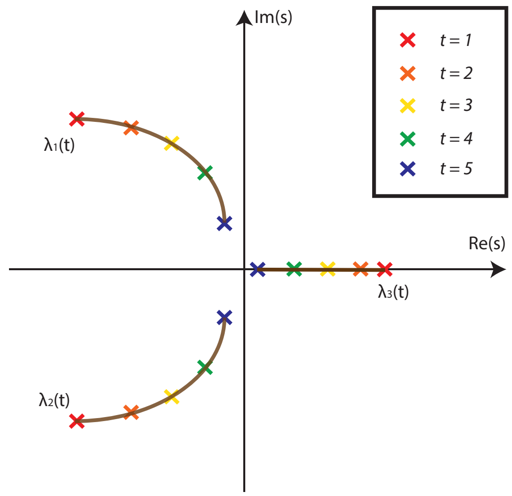


Figure 3.2: LTV Dynamic Eigenvalue Plot

With this idea in mind, a step by step procedure for the expansion of the parameter space approach for LTV systems using the Dynamic Eigenvalues is presented in figure 3.3.

Results and Misconceptions:

Multiple trials of the proposed idea presented in figure 3.3 were held. Throughout the trials, it was observed that the computation of the RRB, CRB and IRB limits equations were extremely lengthy and time consuming. Added to that, wrong results were obtained *(For example: The controller parameters obtained were imaginary and not real numbers)*. These observations have led to the questioning of the technique proposed for expanding the Parameter Space Approach for LTV systems using the Dynamic Eigenvalues.

After an extended literature review of the dynamic eigenvalues concept, it was determined that considering "The dynamic eigenvalues as quantities similar to the eigenvalues of LTI systems which wander in time thorugh the complex s-domain as stated earlier; is a common misconception and the proposed method fails". This conclusion comes from the works of de Anda. In [43], de Anda presents a complete overview of the relation between the dynamic eigenvalues and the frequency domain. To study the the role of the dynamic eigenvalue in the frequency behaviour of a scalar LTV system, he considered two different hyphotheses.

In this thesis, the analysis of the first hypothesis is mainly considered. In LTI systems, the notion of poles is directly, connected to the concept of a transfer function for these systems. Based on section 2.2, the poles of a LTI system are basically the roots of $D(s) = 0$ where $D(s)$ is the denominator of the transfer function 2.1. For LTI systems, the poles of the transfer function are equal to the eigenvalues of the same system, provided that that system is fully

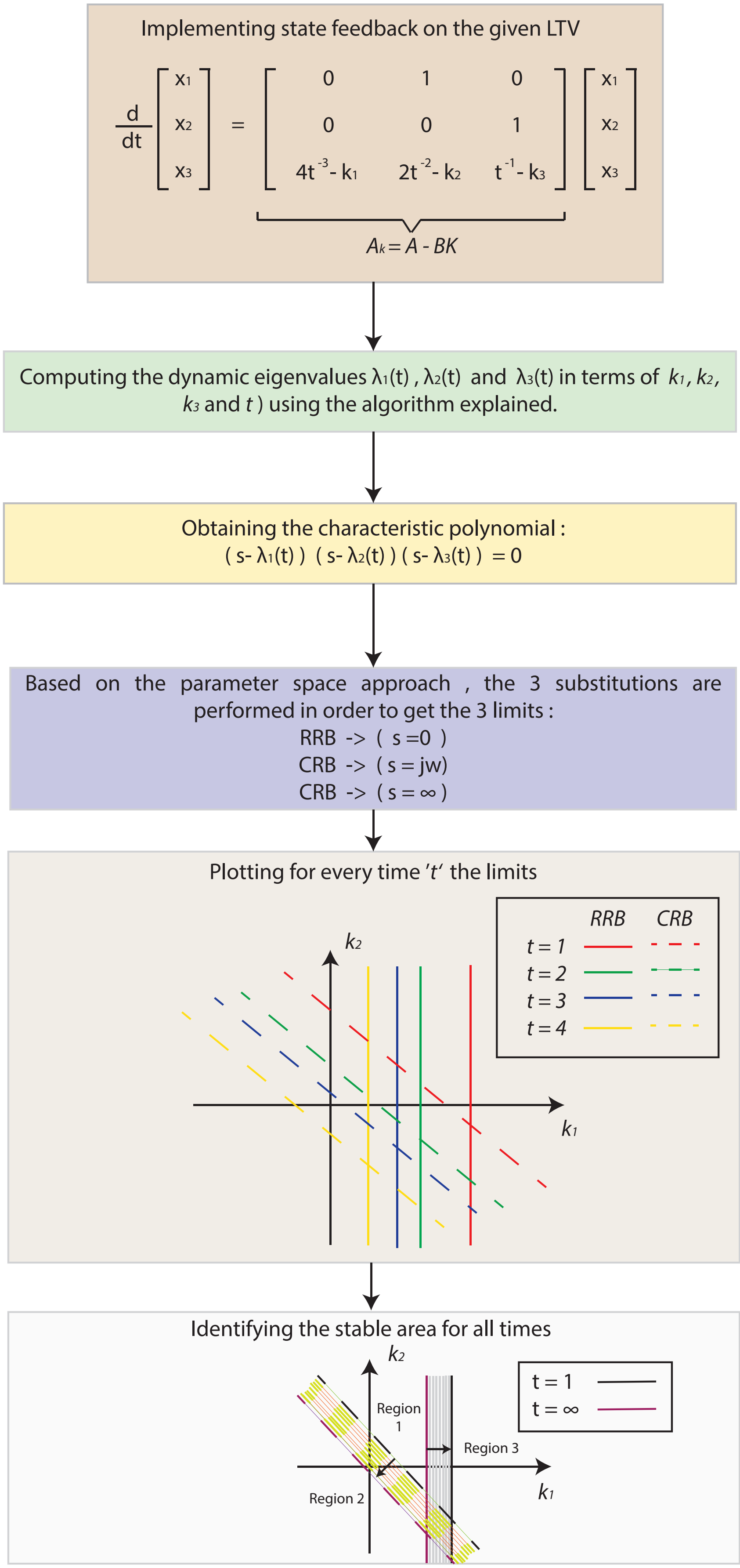


Figure 3.3: Dynamic Eigenvalues Proposed Idea

controllable and observable. On the other hand, for LTV systems, a clear and direct connection between the dynamic eigenvalues and the definition chosen for the transfer function of a scalar LTV system does not exist in the general case.

For detailed analysis of this hypothesis refer to Chapter 7 in [43].

3.2 Differential Riccati Equation (DRE) Approach

In this seciton, the aim is to extend the approach described in 2.5.2.2 for LTV systems.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

Implementing state feedback for LTV systems:

$$\dot{x}(t) = (A(t) - B(t)K)x(t) = A_k(t)$$

The Riccati Equation associated with the above LTV systems (Differential Riccati Equation (DRE)):

$$\dot{P}(t) + A_k^T(t)P(t) + P(t)A_k(t) + P(t)R(t)P(t) + Q = 0$$

where $R(t) = 0$. The equation above reduces to the following:

$$\dot{P}(t) + A_k^T(t)P(t) + P(t)A_k(t) + Q = 0 \tag{3.1}$$

The Hamiltonian Matrix associated with 3.1 is given by:

$$H(t) = \begin{bmatrix} -A(t)_k & R(t) \\ Q(t) & A(t)_k^T \end{bmatrix}$$

with the knownledge of the corresponding Hamiltonian matrix, the mapping equations 2.45, 2.46, 2.47a and 2.47b presented previously in 2.5.2.2, are applied to the time varying Hamiltonian matrix, in order to find the limits that form the controller parameter space. To illustrate this idea an example is given below:

Example:

$$A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & 5t & 6t \end{bmatrix}$$

$$B(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

$$A_k(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t - k_1 & 5t - k_2 & 6t - k_3 \end{bmatrix}$$

Taking $Q(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the corresponding Hamiltonian matrix is given by:

$$H = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ k_1 - t & k_2 - 5t & k_3 - 6t & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & t - k_1 \\ 0 & 1 & 0 & 1 & 0 & 5t - k_2 \\ 0 & 0 & 1 & 0 & 1 & 6t - k3 \end{bmatrix}$$

The corresponding characteristic equation:

$$\begin{gathered} -k_1^2 - 2k_1k_3s^2 + 12k_1s^2t + 2k_1t + k_2^2s^2 + 2k_2s^4 - 10k_2s^2t - k_3^2s^4 + \\ 12k_3s^4t + 2k_3s^2t + s^6 - 36s^4t^2 - 10s^4t + 13s^2t^2 - t^2 = 0 \end{gathered} \tag{3.2}$$

For the RRB limit:

$$k_1 = t$$

$$k_1 = t$$

For the IRB limit:

Coefficient of $(s^6) = 1 \neq 0$. In this case, the CRB limit does not exist.

For the CRB limits (Acceptable limits):

$$k_1 = t - 6t\omega^2 + \omega^2$$
$$k_1 = 6t\omega^2 + \omega^2$$

$$k_2 = \omega^2 + 5t$$

$$k_2 = \omega^2 + 5t$$

With these results, the idea of extending the approach in 2.5.2.2 to the DRE associated with LTV systems clearly fails. The reason is because the limits obtained from this approach are identical to the results obtained from directly using $\det(sI - A_k(t)) = 0$ as the characteristic equation. Based on 2.3.1, taking $\det(sI - A_k(t)) = 0$ as the characteristic equation indicates that the Frozen Time Eigenvalues (FTE) of the LTV system are being studied. Since the FTE are ineffective for analysing the stability of LTV systems, this method fails to be sufficient.

3.3 Transformation of LTV to LTI systems

The analysis and synthesis of LTI systems have been quite developed. However, general transparent methods of analysing LTV systems are still lacking. Therefore, it becomes apparent that if a LTV system can be transformed into a LTI system, then transparent results for time-varying systems can be obtained. According to [9], any LTV system can be transformed into an equivalent LTI system, provided the state transition matrix $\phi(t, t_0)$ is known. Since it is a very difficult task to obtain $\phi(t, t_0)$ for LTV systems, the author illustrates explicit methods of obtaining the transformation for special classes of LTV systems.

A LTV system $\dot{x} = A(t)x$ is said to be invariable if it can be transformed into a LTI system of the form $\dot{\overline{x}} = A\overline{x}$ by some valid transformation such as the algebraic transformation and the $t \leftrightarrow \tau$ to be discussed below. An algebraic transformation is a transformation of the states defined by $x(t) = T(t)\overline{x}(t)$, where $T(t)$ is a non-singular matrix for all t (matrix $T(t)$has an inverse) and $\dot{T}$ (t) exists. Moreover, a $t \leftrightarrow \tau$ transformation is a transformation of time scale from t into τ.

In [9], the author divides the special classes of LTV systems into two different classes: algebraically invariable systems and $\tau-$algebraically invariable systems. Algebraically invariable systems are LTV system that can be transformed into LTI systems by the use of algebraic transformation alone. Whereas a $\tau-$algebraically invariable systems, are systems which can be transformed into LTI systems by the use of the algebraic transformation plus the $t \leftrightarrow \tau$. For each type of system, an example is illustrated. [2]

For algebraically invariable systems:

Corollary

Commutative system:

$$\dot{x}(t) = A(t)x(t)$$

[2] Note that these definitions are completely based on reference [9]

If $A(t)$ can be written as:

$$A(t) = F(t) + R$$

where R is a constant matrix, then the LTV system is algebraically invariable. The corresponding algebraic transformation:

$$F(t) = \exp\left[\int_0^t F(\tau)d\tau\right]\overline{x}(t) = \underbrace{\exp\left[\sum_{i=1}^{n} F_i\beta_i(t)\right]}_{T(t)}\overline{x}(t)$$

$$F(t) = \sum_{i=1}^{n} \alpha_i(t)F_i, \text{where } F_i\text{'s are constant matrices, and}$$

$$\beta_i(t) = \int_0^t \alpha_i(\tau)d\tau$$

Example

$$A(t) = \begin{bmatrix} -5+\cos\omega t & \sin\omega t \\ -\sin\omega t & -5+\cos\omega t \end{bmatrix}$$

$A(t)$ is a commutative system because:

$$A(t)\left[\int_{t_0}^t A(\tau)d\tau\right] = \left[\int_{t_0}^t A(\tau)d\tau\right]A(t)$$

$A(t)$ can be written as:

$$A(t) = \underbrace{\cos\omega t}_{\alpha_1(t)}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{F_1} + \underbrace{\sin\omega t}_{\alpha_2(t)}\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{F_2} + \underbrace{\begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}}_{R}$$

$$\beta_1(t) = \int_0^1 \cos\omega t = \frac{1}{\omega}\sin\omega t$$
$$\beta_2(t) = \int_0^1 \sin\omega t = \frac{1}{\omega}(1-\cos\omega t)$$

$$T(t) = \exp\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\beta_1(t) + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\beta_2(t)\right] = \exp\begin{bmatrix} \beta_1(t) & \beta_2(t) \\ -\beta_2(t) & \beta_1(t) \end{bmatrix}$$

Exponential Matrix Calculation

$$\exp[M] = \sum_{n=0}^{\infty} \frac{M^n}{n!} = I + M + \frac{MM}{2!} + \frac{MMM}{3!} + ...$$

where I is the indetity matrix. Note: the taylor series expansion of several functions such as $\sin(t)$, $\cos(t)$ and $\exp(t)$, etc are needed to simplify the final expressions. However, it is much more effecient to use programs such as Mathematica in order to evaluate the exponential of matrices. Mathematica Command: MatrixExp[].

$$T(t) = \exp[\beta_1(t)] \begin{bmatrix} \cos\beta_2(t) & \sin\beta_2(t) \\ -\sin\beta_2(t) & \cos\beta_2(t) \end{bmatrix}$$

Applying :

$$A = T^{-1}(t)A(t)T(t) - T^{-1}(t)\dot{T}(t) \tag{3.3}$$

Results in the equivalent LTI system :

$$\dot{\overline{x}}(t) = \underbrace{\begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}}_{A=R} \overline{x}(t)$$

For τ- algebraically invariable systems:

Corollary

The LTV system is considered:

$$\dot{x}(t) = A(t)x(t) \tag{3.4}$$

$A(t)$ is τ-algebraically invariable if it satisfies the following conditions:

1. $\dot{A}(t)$ exists,

2. There exist a scalar time function $h(t)$ and a constant matrix A_1 such that:

$$A_1A(t) - A(t)A_1 = \frac{\dot{A}(t)}{h(t)} - \frac{\dot{h}(t)}{h^2(t)}A(t) \tag{3.5}$$

The corresponding algebraic transformation:

$$x(t) = \underbrace{\exp[A_1 g(t, t_0)]}_{T(t)} \overline{x}(t)$$

along with the $t \leftrightarrow \tau$ transformation:

$$\tau = g(t, t_0) = \int_{t_0}^{t} h(\tau) d\tau$$

transform 3.4 into the LTI system:

$$\dot{\overline{x}}(t) = A_2 \overline{x}(t)$$

Example

$$A(t) = \begin{bmatrix} -3t^2 & 0 \\ 6t^5 & -6t^2 \end{bmatrix}$$

$\dot{A}(t)$ exists :

$$\dot{A}(t) = \begin{bmatrix} -6t & 0 \\ 30t^4 & -12t \end{bmatrix}$$

and $h(t) = 3t^2$

where

$$h^2(t) = 9t^4 \text{ and } \dot{h}(t) = 6t$$

Assuming

$$A1 = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

Applying 3.7, results in :

$$\begin{bmatrix} 6t^5 y & -3t^2 y \\ 6t^5 w - 6t^5 x + 3t^2 z & -6t^5 y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 6t^2 & 0 \end{bmatrix}$$

$$6t^5 y = 0 \text{ and } -3t^2 y = 0$$

result in:

$$y = 0$$

$$6t^5 w - 6t^5 x + 3t^2 z = 6t^3$$

is solvesolved by assuming $z = 2$ in order to cancel out the Right Hand Side of the equation, resulting in :

$$x = w$$

Therefore:

$$A_1 = \begin{bmatrix} \alpha & 0 \\ 2 & \alpha \end{bmatrix}$$

$$\tau = g(t, t_0) = \int_0^t 3t^2 d\tau = t^3$$

Assuming $\alpha = 0$,

$$T(t) = \exp \begin{bmatrix} 0 & 0 \\ 2t^3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2t^3 & 1 \end{bmatrix}$$

where:

Applying 3.3, results in:

$$A = \begin{bmatrix} -3t^2 & 0 \\ -6t^2 & -6t^2 \end{bmatrix}$$

Applying the $t \leftrightarrow \tau$ transformation by taking $h(t) = \frac{d\tau}{dt} = 3t^2$ as the common factor. Resulting in the LTI system:

$$\dot{\overline{x}}(t) = \begin{bmatrix} -1 & 0 \\ -2 & 2 \end{bmatrix} \overline{x}(t)$$

With these examples, the introductory part of this section concludes. Next, the idea of transformation is adopted in order to extend the parameter space approach for LTV systems. To present this idea, an example is illustrated below:

Given a LTV system:

$$\begin{bmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{bmatrix} = \begin{bmatrix} 5t & 1 \\ -1 & -4+5t \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

The aim is to obtain the stablizing space of the controller parameters (in this case, state feedback controllers) for this LTV system for all times t. First step is to apply state feedback to the open loop system:

$$\dot{x}(t) = \underbrace{(A(t) - B(t)K)}_{A_k(t)} x$$

where $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. Resulting in:

$$A_k(t) = \begin{bmatrix} 5t & 1 \\ -1-k_1 & 5t-4-k_2 \end{bmatrix}$$

Based on the first corollary presented earlier:

$$A_k(t) = \underbrace{t}_{\alpha_1(t)} \underbrace{\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}}_{F_1} + \underbrace{\begin{bmatrix} 0 & 1 \\ -1-k_1 & -k2 \end{bmatrix}}_{R}$$

$$\beta_1(t) = \int_0^t t d\tau = \frac{t^2}{2}$$

$$T(t) = \exp[F_1\beta_1(t)] = \exp\begin{bmatrix} \frac{5t^2}{2} & 0 \\ 0 & \frac{5t^2}{2} \end{bmatrix} = \begin{bmatrix} \exp[\frac{5t^2}{2}] & 0 \\ 0 & \exp[\frac{5t^2}{2}] \end{bmatrix}$$

$$T^{-1}(t) = \begin{bmatrix} \exp[-\frac{5t^2}{2}] & 0 \\ 0 & \exp[-\frac{5t^2}{2}] \end{bmatrix} \text{ and } \dot{T}(t) = \begin{bmatrix} 5t\exp[\frac{5t^2}{2}] & 0 \\ 0 & 5t\exp[\frac{5t^2}{2}] \end{bmatrix}$$

Applying 3.3, results in the equivalent closed loop LTI system:

$$A_k = \begin{bmatrix} 0 & 1 \\ -1-k_1 & -4-k_2 \end{bmatrix}$$

with this result, the characteristic equation is obtained using equation:

$$\det(sI - A_k) = 0$$

The characteristic Equation:

$$s^2 + (4+k_2)s + (k_1+1) = 0$$

For the RRB limit:

$$k_1 = -1$$

For the CRB limit:

$$-\omega^2 + (4+k_2)\omega j + (k_1+1)$$

Imaginary part:

$$k_2 = -4$$

Real part:

$$k_1 = \omega^2 - 1$$

For the IRB limit: Coefficient of$(s^2) = 1 \neq 0$. Therefore, there exists no IRB limit.

In the figure below 3.4, the RRB and CRB limits are plotted and the stable area is identified as shown in figure 3.5 using one of the two procedures described in sections 2 and 4 of the appendix.

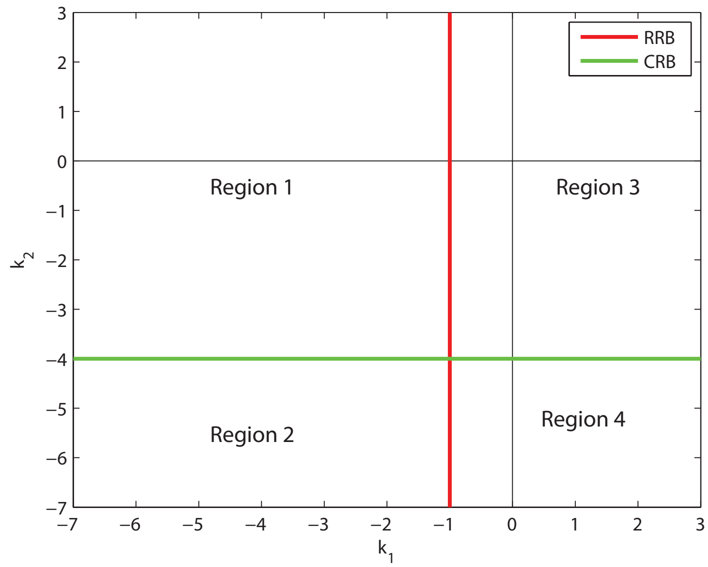


Figure 3.4: RRB and CRB limits

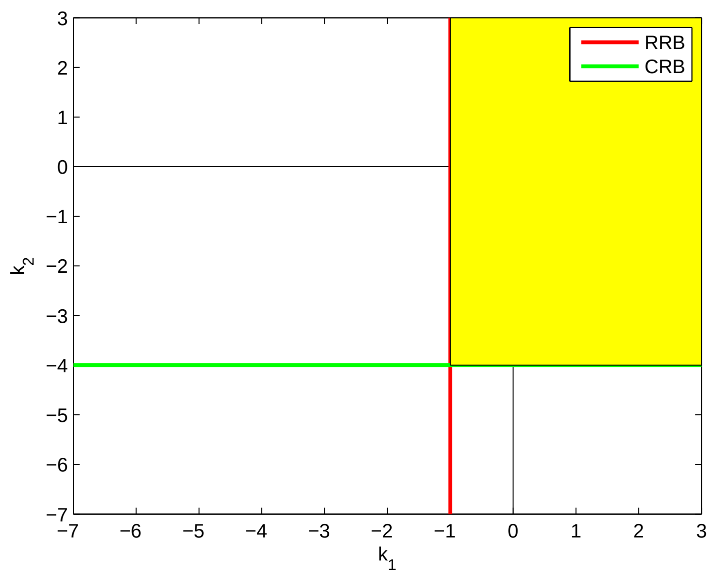


Figure 3.5: Stable Area

Hence, the space of stabilizing controller parameters for all times 't', of the LTV system is indentified. It is important to note that, not all five transformation techniques described in [9] could be applied to closed loop LTV systems ($A_k(t)$). Since the transformations described were intended to homogenous LTV systems i.e. $B(t) = 0$. In order to tackle this, an assumption is made regarding the $B(t)$, based on the transformation matrix $T(t)$ defined for the homogenous euler systems presented in corollary 5 of [9]. Therefore, the $A(t)$ and $B(t)$ are transformed into their LTI equivalent matrices and then state feedback is applied to this system. This assumption is not practical because, normally the $B(t)$ is not assumed or specified, it is part of the original system's representation. The modified corollary 5 is illustrated below: [3]

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{3.6}$$

$$A(t) = \begin{bmatrix} 0 & 1 & ... & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ -\frac{a_n}{t^n} & -\frac{-a_{n-1}}{t^{n-1}} & ... & \frac{-a_1}{t} \end{bmatrix}, B(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \frac{b}{t^n} \end{bmatrix}$$

The algebraic transformation is:

$$x(t) = T(t)\overline{x}(t) \tag{3.7}$$

and the corresponding $t \leftrightarrow \tau$ transformation is:

$$\tau = \ln t \qquad (t_0 = 1)$$

where

$$T(t) = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & \frac{1}{t} & 0 & .. & 0 \\ 0 & -\frac{1}{t^2} & \frac{1}{t^2} & ... & 0 \\ ... & ... & ... & ... & ... \end{bmatrix}$$

$$\dot{x}(t) = \dot{T}(t)\overline{x}(t) + T(t)\dot{\overline{x}}(t)$$

Substituting 3.6 in the equation above:

$$A(t)x(t) + B(t)u(t) = \dot{T}(t)\overline{x}(t) + T(t)\dot{\overline{x}}(t)$$

$$T(t)\dot{\overline{x}}(t) = A(t)x(t) + B(t)u(t) - \dot{T}(t)\overline{x}(t)$$

Substituting 3.7 in the equation above:

$$T(t)\dot{\overline{x}}(t) = A(t)T(t)\overline{x}(t) + B(t)u(t) - \dot{T}(t)\overline{x}(t)$$

Multiplying by T^{-1} to both sides:

$$T^{-1}(t)T(t)\dot{\overline{x}}(t) = T^{-1}(t)A(t)T(t)\overline{x}(t) + T^{-1}(t)B(t)u(t) - T^{-1}(t)\dot{T}(t)\overline{x}(t)$$

[3] Note that the $B(t)$ matrix cannot have other forms. And this is the main disadvantage of this approach.

$$\begin{aligned}\dot{\overline{x}}(t) &= T^{-1}(t)A(t)T(t)\overline{x}(t) - T^{-1}(t)\dot{T}(t)\overline{x}(t) + T^{-1}(t)B(t)u(t)\\ \dot{\overline{x}}(t) &= \underbrace{\left(T^{-1}(t)A(t)T(t) - T^{-1}(t)\dot{T}(t)\right)}_{A}\overline{x}(t) + \underbrace{T^{-1}(t)B(t)}_{B}u(t)\end{aligned} \tag{3.8}$$

$$A = T^{-1}(t)A(t)T(t) - T^{-1}(t)\dot{T}(t) \tag{3.9}$$

$$B = T^{-1}(t)B(t) \tag{3.10}$$

Now, the idea is to transform the LTV system ($A(t)$ and $B(t)$) into its LTI system equivalent (A and B), followed by implementing state feedback to the transformed LTI system and obtaining the characteristic equation and the 3 limits using the approaches explained in section 2 for LTI systems. An example is shown below:

3rd order system example:

$$A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{5}{t^3} & -\frac{9}{t^2} & -\frac{2}{t} \end{bmatrix} \qquad B(t) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{t^3} \end{bmatrix}$$

the corresponding algebraic transformation:

$$T(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{t} & 0 \\ 0 & -\frac{1}{t^2} & \frac{1}{t^2} \end{bmatrix}, \qquad T^{-1}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & t & t^2 \end{bmatrix} \qquad \dot{T}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{t^2} & 0 \\ 0 & \frac{2}{t^3} & -\frac{2}{t^3} \end{bmatrix}$$

Applying 3.9:

$$A = \begin{bmatrix} 0 & \frac{1}{t} & 0 \\ 0 & 0 & \frac{1}{t} \\ -\frac{5}{t} & -\frac{9}{t} & \frac{1}{t} \end{bmatrix}$$

Applying 3.10:

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{t} \end{bmatrix}$$

Finally, applying the $t \leftrightarrow \tau$ transformation by taking the $\frac{d\tau}{dt} = \frac{1}{t}$ as the common factor. Resulting in the LTI system:

$$\dot{\overline{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -9 & 1 \end{bmatrix}\overline{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(t)$$

Now, the direct approach explained in 2.5.2.1 is implemented to the transformed LTI system. where: [4]

$$K = \begin{bmatrix} k1 & k2 & k3 \end{bmatrix}$$

Using 2.33, the closed loop A_k is:

$$A_k = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5-k_1 & -9-k_2 & 1-k_3 \end{bmatrix} \overline{x}(t)$$

The characteristic equation accroding to 2.37 is:

$$p(s,q,k) = s^3 + (-1+k_3)s^2 + (9+k_2)s + (5+k_1) = 0$$

Therefore the RRB limit referring to (2.38):

$$k1 = -5$$

The CRB limits, referring to (2.40) and (2.41):

$$k_1 = (-1+k_3)\omega^2 - 5$$
$$k_2 = \omega^2 - 9$$

The IRB limit doesnt exist.

The two figures below illustrate the stabilizing controller parameters area for a specific value of $k_3 = 0.2$ (Figure 3.6) and for all values of $k_3 = 1$ (Figure 3.8). Refer to 4 for the method of finding the stable area.

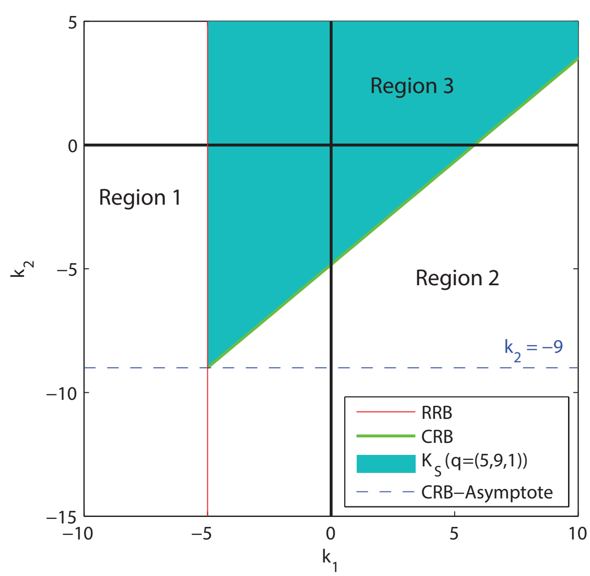


Figure 3.6: 2D Plot $k_3 = 0.2$

[4] The K matrix in this approach, is not transformed.

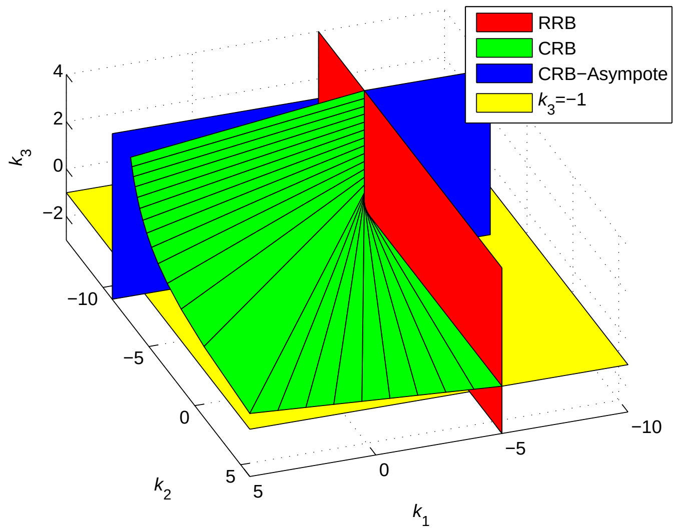


Figure 3.7: 3D Plot

A summary of this section's results, including a possible improvement is presented in the figure below:

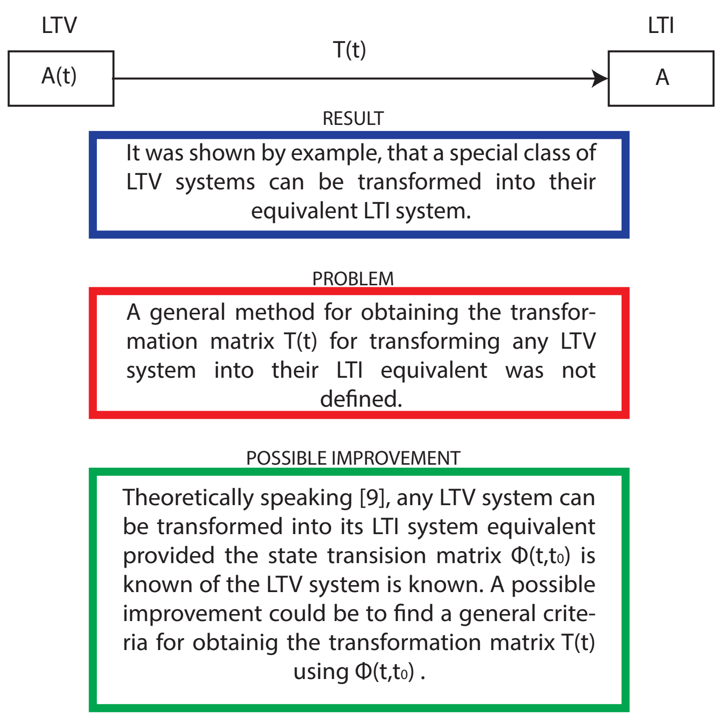


Figure 3.8: A brief summary and outlook

4

Outlook and Conclusion

The aim of this thesis was to expand the parameter space approach for LTV systems, by finding the whole space of stabilizing controller parameters for all times t. It was intriguing initially to ask whether the idea of Frozen Time Eigenvalues (FTEs) could be utilised to analyse the stability of LTV systems. However, this idea was proven to be insufficient and ineffective, as presented in section 2.3.1.

Moreover, three other techniques were considered to obtain the governing characteristic equation that could be utilised to obtain the three famous limits (RRB, CRB and IRB) which form the stable area. In the first attempt of expanding the parameter space approach for LTV, the concept of dynamic eigenvalues was considered in section 3.1. It was found out that treating the dynamic eigenvalues as quantities, similar to the eigenvalues for LTI systems which wander in time thorugh the complex s-domain is a common misconception and the proposed attempt described in section 3.1, could not be applied. In a second attempt, the idea of mapping the corresponding Riccati Equation of LTV systems (i.e. Differential Riccati Equation) was tackled. Unfortunately, the results of mapping the DRE of LTV systems (as illustrated in section 3.2) were similar to the results obtained using the FTEs. This had led to the conclusion that this method cannot be applied as well. In the final attempt, the concept of eliminating the time dependency in LTV systems using algebraic transformations was implemented. Furthermore, the results obtained from transforming LTV systems into their corresponding LTI systems, are considered to be the most rewarding. However, this is applicable for very specific classes of LTV systems. Thus, there is still some room for improvement.

A fundamental improvement of this method could be based on finding a generic method for transforming any LTV system into its equivalent LTI system. As it was pointed out in [9] this is possible, provided that the state transition matrix $\phi(t, t_0)$ of the LTV system is known. However, a clear method for deriving the algebraic transformation matrix $T(t)$ using $\phi(t, t_0)$

for any generic case was not illustrated, due to the difficulty in obtaining the state transition matrices of LTV systems back in the 1970s.

With the numerous researches done over the past 30 years in calculating the $\phi(t, t_0)$ for LTV systems, (for e.g. [20]), it is suggested that such results could be utilised along with the above mentioned note, in finding a general transformation criteria for more classes of LTV systems.

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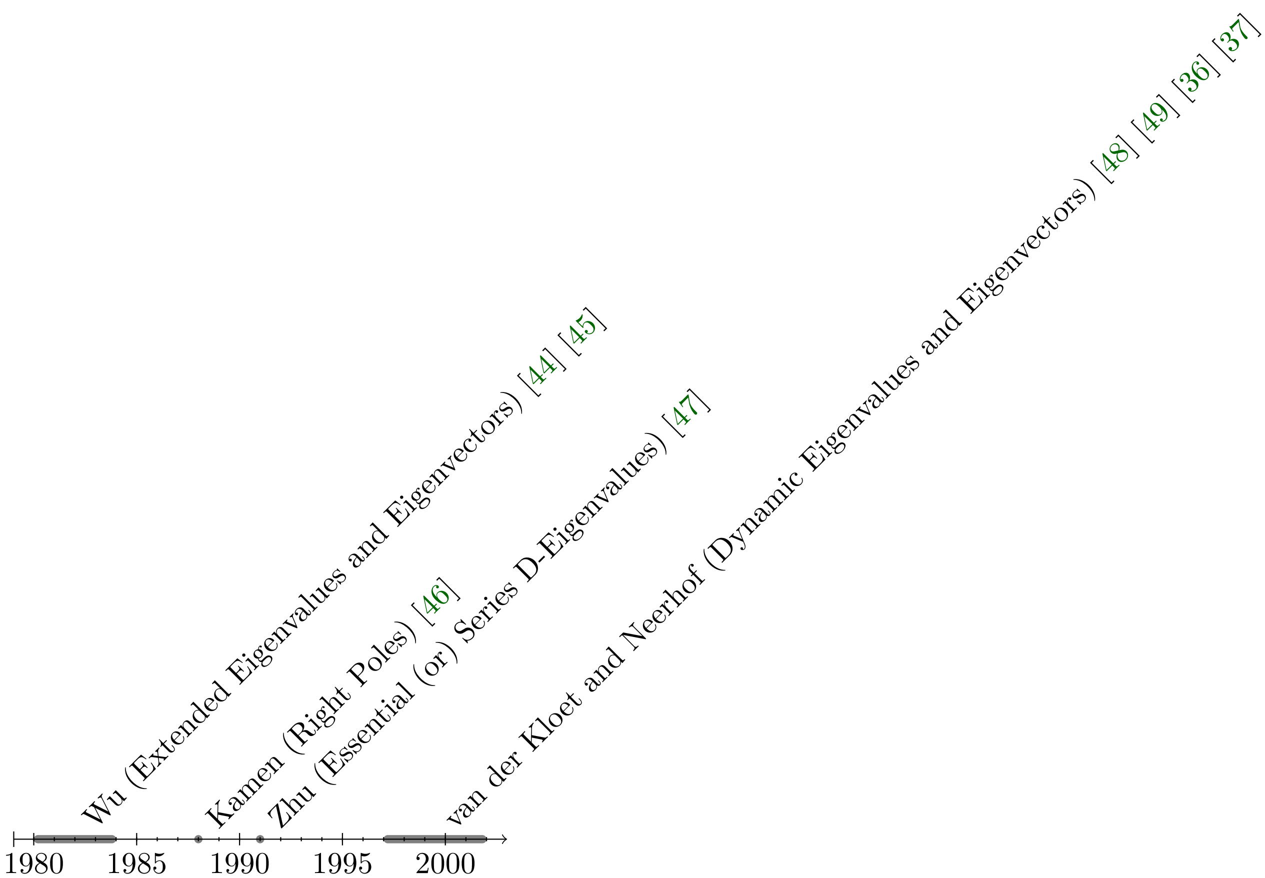
Appendix

1 Dynamic Eigenvalues

Timeline:



Example:

Similar to the example used in reference [37], the third order Euler differential equation is considered:

$$(t^3D^3 - t^2D^2 - 2tD - 4)x = 0$$

Dividing by t^{-3} to get an euqation with the same form shown in 2.10:

$$(D^3 - t^{-1}D^2 - 2t^{-2}D - 4t^{-3})x = 0$$

The equivalent state space representation is:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4t^{-3} & 2t^{-3} & t^{-1} \end{bmatrix} x(t)$$

Since the dimensions of the $A(t)$ matrix are 3×3:

$$n = 3$$

The iterations are started with $k = n = 3$:

$$A(t) = A_3(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4t^{-3} & 2t^{-3} & t^{-1} \end{bmatrix} \tag{1}$$

In the first iteration, $A_2(t)$ is obtained from performing a coordinate transformation on the $A_3(t)$ matrix:

$$A_2(t) = P_3^{-1}(t)A_3(t)P_3(t) - P_3^{-1}(t)\dot{P}_3(t)$$

where:

$$P_3(t) = \begin{bmatrix} I & 0 \\ p_3^T & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_{3,1}^T & p_{3,2}^T & 1 \end{bmatrix}, P_3^{-1}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p_{3,1}^T & -p_{3,2}^T & 1 \end{bmatrix} \text{ and } \dot{P}_3(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dot{p}_{3,1}^T & \dot{p}_{3,2}^T & 0 \end{bmatrix}$$

Now, $A_2(t)$ is calculated:

$$A_2(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p_{3,1}^T & -p_{3,2}^T & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4t^{-3} & 2t^{-3} & t^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p_{3,1}^T & p_{3,2}^T & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -p_{3,1}^T & -p_{3,2}^T & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dot{p}_{3,1}^T & \dot{p}_{3,2}^T & 0 \end{bmatrix}$$

$$A_2(t) = \begin{bmatrix} 0 & 1 & 0 \\ p^T_{3,1} & p^T_{3,2} & 1 \\ 4t^{-3} + (-p^T_{3,2} + t^{-1})p^T_{3,1} & -p^T_{3,1} + 2t^{-2} + (-p^T_{3,2} + t^{-1})p^T_{3,2} & -p^T_{3,2} + t^{-1} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \dot{p}^T_{3,1} & \dot{p}^T_{3,2} & 0 \end{bmatrix}$$

$$A_2(t) = \begin{bmatrix} 0 & 1 & 0 \\ p^T_{3,1} & p^T_{3,2} & 1 \\ \underbrace{4t^{-3} + (-p^T_{3,2} + t^{-1})p^T_{3,1} - \dot{p}^T_{3,1}}_{0} & \underbrace{-p^T_{3,1} + 2t^{-2} + (-p^T_{3,2} + t^{-1})p^T_{3,2} - \dot{p}^T_{3,2}}_{0} & \underbrace{-p^T_{3,2} + t^{-1}}_{\lambda_3(t)} \end{bmatrix} \tag{2}$$

Based on the conditions mentioned in the explanation of the previous subsection i.e. all the elements of the final row should be equal to zero, except the last element, which is considered as the dynamic eigenvalue, the following 2 Riccati differential equations and the dynamic eigenvalue equation are derived:

$$4t^{-3} + (-p^T_{3,2} + t^{-1})p^T_{3,1} - \dot{p}^T_{3,1} = 0 \tag{3}$$

$$-p^T_{3,1} + 2t^{-2} + (-p^T_{3,2} + t^{-1})p^T_{3,2} - \dot{p}^T_{3,2} = 0 \tag{4}$$

$$\lambda_3(t) = -p^T_{3,2} + t^{-1} \tag{5}$$

Substituting 5 in 3 and 4 results in:

$$\dot{p}^T_{3,1} = \lambda_3(t)p^T_{3,1} + 4t^{-3} \tag{6}$$

$$\dot{p}^T_{3,2} = \lambda_3(t)p^T_{3,2} - p^T_{3,1} + 2t^{-2} \tag{7}$$

For $k = 2$:

Leaving out the last row and column of the $A_2(t)$ in 2, results in the new $A_2(t)$ matrix:

$$A_2(t) = \begin{bmatrix} 0 & 1 \\ p^T_{3,1} & p^T_{3,2} \end{bmatrix}$$

In the second iteration, $A_1(t)$ is obtained from performing a coordinate transformation on the $A_2(t)$ matrix:

$$A_1(t) = P_2^{-1}(t)A_2(t)P_2(t) - P_2^{-1}(t)\dot{P}_2(t)$$

where:
$P_2(t) = \begin{bmatrix} I & 0 \\ p_2^T & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ p_{2,1}^T & 1 \end{bmatrix}$, $P_2^{-1}(t) = \begin{bmatrix} 1 & 0 \\ -p_{2,1}^T & 1 \end{bmatrix}$ And $\dot{P}_2(t) = \begin{bmatrix} 0 & 0 \\ \dot{p}_{2,1}^T & 0 \end{bmatrix}$

Now, $A_1(t)$ is calculated:

$$A_1(t) = \begin{bmatrix} 1 & 0 \\ -p_{2,1}^T & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ p_{3,1}^T & p_{3,2}^T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ p_{2,1}^T & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -p_{2,1}^T & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \dot{p}_{2,1}^T & 0 \end{bmatrix}$$

$$A_1(t) = \begin{bmatrix} p_{2,1}^T & 1 \\ p_{3,1}^T + p_{2,1}^T(-p_{2,1} + p_{3,2}^T) & -p_{2,1}^T + p_{3,2}^T \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \dot{p}_{2,1}^T & 0 \end{bmatrix}$$

$$A_1(t) = \begin{bmatrix} p_{2,1}^T & 1 \\ \underbrace{p_{3,1}^T + p_{2,1}^T(-p_{2,1} + p_{3,2}^T) - \dot{p}_{2,1}^T}_{0} & \underbrace{-p_{2,1}^T + p_{3,2}^T}_{\lambda_2(t)} \end{bmatrix} \tag{8}$$

Similarly based on the two conditions:

$$p_{3,1}^T + p_{2,1}^T(-p_{2,1}^T + p_{3,2}^T) - \dot{p}_{2,1}^T = 0 \tag{9}$$

$$\lambda_2(t) = -p_{2,1}^T + p_{3,2}^T \tag{10}$$

Substituting 10 in 9 to get:

$$\dot{p}_{2,1}^T = p_{3,1}^T + p_{2,1}^T \lambda_2(t) \tag{11}$$

After deriving 5 , 6, 7, 10 and 11, the next step is eliminating and reformulating the obtained equations to come up with 3 equations consisting of only 3 unknowns ($\lambda_1(t), \lambda_2(t)$ and $\lambda_3(t)$). The following elimination method is a general method and the results obtained are applicable to all Scalar LTV-systems with order 3. Therefore, a general form of the $A(t)$ matrix is used [1]:

$$A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3(t) & -a_2(t) & -a_1(t) \end{bmatrix}$$

[1] In this elimination method, note that, in equations 5, 6 and 7, $4t^{-3}$ is substituted by $-a_3(t)$, $2t^{-2}$ by $-a_2(t)$ and t^{-1} by $-a_1(t)$

- Prior to proceeding with the elimination method, the following is defined: The Riccati transformation is trace preserving. The trace of an n-by-n square matrix $A(t)$ is defined to be the sum of the elements on the main diagonal (the diagonal from the upper left to the lower right) of the $A(t)$ matrix. This means that the summation of the main diagonal prior to performing the Riccati transformation is equal to the summation of the main diagonal after it. In this case:
  Summation of the main diagonal of the original $A(t)$ matrix= $0 + 0 - a_1 = -a_1$
  Summation of the main diagonal of the transformed $B(t)$ matrix = summation of all dynamic eigenvalues = $\lambda_1(t) + \lambda_2(t) + \lambda_3(t)$
  Based on the trace preserving principle the first equation is derived:
$$\lambda_1(t) + \lambda_2(t) + \lambda_3(t) = -a_1 \tag{12}$$
- The next step is the elimination of $\lambda_3(t)$ using equations 12 and 5:
$$\lambda_1(t) + \lambda_2(t) - p^T_{3,2} - a_1 = -a_1$$
$$p^T_{3,2} = \lambda_1(t) + \lambda_2(t) \tag{13}$$
- Followed by a substitution of 13 in 10:
$$\lambda_2(t) = \lambda_1(t) + \lambda_2(t) - p^T_{2,1}$$
  Resulting in:
$$\lambda_1(t) = p^T_{2,1} \tag{14}$$
- Substituting 14 in 11 results in:
$$p^T_{3,1} = \dot{\lambda}_1(t) - \lambda_2(t)\lambda_1(t) \tag{15}$$
- The next step is substituting 13 and 15 in 7, taking into consideration 12:
$$\dot{\lambda}_1(t) + \dot{\lambda}_2(t) = (-a_1 - \lambda_2(t) - \lambda_1(t))(\lambda_1(t) + \lambda_2(t)) - \dot{\lambda}_1(t) + \lambda_2(t)\lambda_1(t) - a_2$$
$$\dot{\lambda}_1(t)+\dot{\lambda}_2(t) = -a_1(\lambda_1(t)+\lambda_2(t))-\lambda_2(t)(\lambda_1(t)+\lambda_2(t))-\lambda_1(t)(\lambda_1(t)+\lambda_2(t))-\dot{\lambda}_1(t)+\lambda_2(t)\lambda_1(t)-a_2$$
$$\dot{\lambda}_1(t)+\dot{\lambda}_2(t) = -a_1(\lambda_1(t)+\lambda_2(t))-\lambda_2(t)\lambda_1(t)-\lambda_2^2(t)-\lambda_1^2(t)-\lambda_1(t)\lambda_2(t)-\dot{\lambda}(t)+\lambda_2(t)\lambda_1(t)-a_2$$
  Rearranging the above equation to get the second equation:
$$2\dot{\lambda_1(t)} + a_1(\lambda_1(t) + \lambda_2(t)) + a_2 + \dot{\lambda}_2(t) + \lambda_2^2(t) + \lambda_1^2(t) + \lambda_2(t)\lambda_1(t) = 0 \tag{16}$$
- From equation 16 , $\lambda_2(t)$ can be solved provided that $\lambda_1(t)$ can be solved first. To do so, equation 15 is substituted in 6. Similarly, 12 is taken into cosideration to eliminate $\lambda_3(t)$ from the result.

$$\begin{aligned}\ddot{\lambda}_1(t) - \dot{\lambda}_2(t)\lambda_1(t) - \lambda_2(t)\dot{\lambda}_1(t) = -a_1(\dot{\lambda}_1(t) - \lambda_2(t)\lambda_1(t)) - \\ \lambda_2(t)(\dot{\lambda}_1(t) - \lambda_2(t)\lambda_1(t)) - \lambda_1(t)(\dot{\lambda}_1(t) - \lambda_2(t)\lambda_1(t)) - a_3\end{aligned}$$

$$\ddot{\lambda_1}(t) - \dot{\lambda_2}(t)\lambda_1(t) - \lambda_2(t)\dot{\lambda_1}(t) + a_1\dot{\lambda_1}(t) - a_1\lambda_2(t)\lambda_1(t) + \\ \lambda_2(t)\dot{\lambda_1}(t) - \lambda_2^2(t)\lambda_1(t) + \lambda_1(t)\dot{\lambda_1}(t) - \lambda_2(t)\lambda_1^2(t) + a_3 = 0 \tag{17}$$

Finally, 16 is multiplied by $\lambda_1(t)$ and then summed together with 17 to obtain the final equation in terms of only $\lambda_1(t)$:

$$\ddot{\lambda_1}(t) - \dot{\lambda_2}(t)\lambda_1(t) - \lambda_2(t)\dot{\lambda_1}(t) + a_1\dot{\lambda_1}(t) - a_1\lambda_2(t)\lambda_1(t) + \lambda_2(t)\dot{\lambda_1}(t) - \lambda_2^2(t)\lambda_1(t) \\ + \lambda_1(t)\dot{\lambda_1}(t) - \lambda_2(t)\lambda_1^2(t) + a_3 + 2\dot{\lambda_1}(t)\lambda_1(t) \\ + a_1\lambda_1(t)(\lambda_1(t) + \lambda_2(t)) + a_2\lambda_1(t) + \dot{\lambda_2}(t)\lambda_1(t) + \lambda_2^2(t)\lambda_1(t) + \lambda_1^3(t) + \lambda_2(t)\lambda_1^2(t) = 0$$

$$\ddot{\lambda_1}(t) + 3\lambda_1\dot{(t)}\lambda_1(t) + \lambda_1^3(t) + a_1(\lambda_1^2 + \dot{\lambda}(t)) + a_2\lambda_1 + a_3 = 0 \tag{18}$$

- Using the three equations 18, 16 and 12, the three dynamic eigenvalues ($\lambda_1(t), \lambda_2(t)$ and $\lambda_1(t)$) can be obtained for our example.

To solve for $\lambda_1(t)$ in 18, the following substitution is performed:

$$\lambda_1(t) = \frac{\alpha}{t} \tag{19}$$

The first and second derivates of 26 are:

$$\dot{\lambda_1}(t) = \frac{-\alpha}{t^2} \tag{20}$$

$$\ddot{\lambda_1}(t) = \frac{2\alpha}{t^3} \tag{21}$$

The resulting equation:

$$\frac{2\alpha}{t^3} + 3\left(\frac{-\alpha}{t^2}\right)\left(\frac{\alpha}{t}\right) + \frac{\alpha^3}{t^3} + \left(-\frac{1}{t}\right)\left(\frac{\alpha^2}{t^2} - \frac{\alpha}{t^2}\right) + \left(-\frac{2}{t^2}\right)\left(\frac{\alpha}{t}\right) - \frac{4}{t^3} = 0 \tag{22}$$

Solving for α results in the three solutions:

$$\alpha = 4$$

$$\alpha = i$$

$$\alpha = -i$$

And consequently, the dynamic eigenvalue $\lambda_1(t)$

$$\lambda_1(t) = \frac{4}{t} \tag{23}$$

$$\lambda_1(t) = \frac{i}{t} \tag{24}$$

$$\lambda_1(t) = \frac{-i}{t} \tag{25}$$

To solve for $\lambda_2(t)$ in 16, a similar substitution is done:

$$\lambda_2(t) = \frac{\gamma}{t} \tag{26}$$

$$2\dot{\lambda_1}(t) + \left(-\frac{1}{t}\right)\left(\lambda_1(t) + \frac{\gamma}{t}\right) + \left(-\frac{2}{t^2}\right) + \left(-\frac{\gamma}{t^2}\right) + \frac{\gamma^2}{t^2} + \lambda_1^2(t) + \lambda_1(t)\left(\frac{\gamma}{t}\right) = 0 \tag{27}$$

Using 23:

$$\gamma = -1 - i$$

$$\lambda_2(t) = \frac{-1-i}{t} \tag{28}$$

and:

$$\gamma = -1 + i$$

$$\lambda_2(t) = \frac{-1+i}{t} \tag{29}$$

Using 24:

$$\gamma = 3$$

$$\lambda_2(t) = \frac{3}{t} \tag{30}$$

and:

$$\gamma = -1 - i$$

$$\lambda_2(t) = \frac{-1-i}{t} \tag{31}$$

Using 25:

$$\gamma = 3$$

$$\lambda_2(t) = \frac{3}{t} \tag{32}$$

and:

$$\gamma = -1 + i$$

$$\lambda_2(t) = \frac{-1+i}{t} \tag{33}$$

Using 12 and the $\lambda_1(t)$ and $\lambda_2(t)$ obtained, $\lambda_3(t)$ is calculated.

Using 23 and 28:

$$\lambda_3(t) = \frac{-2+i}{t} \tag{34}$$

Using 23 and 29:

$$\lambda_3(t) = \frac{-2-i}{t} \tag{35}$$

Using 24 and 30:

$$\lambda_3(t) = \frac{-2-i}{t} \tag{36}$$

Using 24 and 31:

$$\lambda_3(t) = \frac{2}{t} \tag{37}$$

Using 25 and 32:

$$\lambda_3(t) = \frac{-2+i}{t} \tag{38}$$

Using 25 and 33:

$$\lambda_3(t) = \frac{2}{t} \tag{39}$$

$\lambda_1(t)$	$\lambda_2(t)$	$\lambda_3(t)$
$\frac{4}{t}$	$\frac{-1-i}{t}$	$\frac{-2+i}{t}$
	$\frac{-1+i}{t}$	$\frac{-2-i}{t}$
$\frac{i}{t}$	$\frac{3}{t}$	$\frac{-2-i}{t}$
	$\frac{-1-i}{t}$	$\frac{2}{t}$
$\frac{-i}{t}$	$\frac{3}{t}$	$\frac{-2+i}{t}$
	$\frac{-1+i}{t}$	$\frac{2}{t}$

Table 1: Summary of all the possible sets of Dynamic Eigenvalues

2 Transfer Function Representation - LTI Systems

Example:

For the single-input single-output rotational mechanical system shown, the single input is an externally applied torque $\tau(t)$, and the output is the angular displacement $\theta(t)$. The constant parameters are motor shaft inertia J, rotational viscous damping coefficient b, and torsional spring constant k_R (provided by the flexible shaft). [2]

where : $J = 3, b = 1, k_R = 2$ and $K_P = 0.5$

Deriving the system equation of motion (EOM):

Applying Eurler's rotational law (the rotational equivalent of Newton's second law):

$$\sum M = J\alpha$$

$$\sum M = J\ddot{\theta}(t) = \tau(t) - b\dot{\theta}(t) - k_R\theta(t)$$

[2] This example system was adopted from [50].

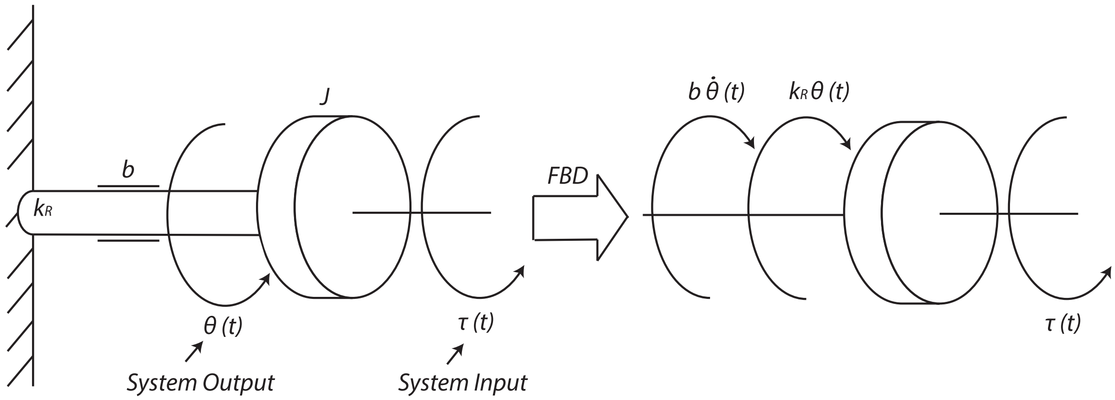


Figure 1: Example

$$\tau(t) = J\ddot{\theta}(t) + b\dot{\theta}(t) + k_R\theta(t)$$

Applying Laplace Transform:

$$\mathrm{T}(s) = Js^2\Theta(s) + bs\Theta(s) + k_R\Theta(s)$$

Yields the Transfer Function:

$$G_s(s) = \frac{\Theta(s)}{\mathrm{T}(s)} = \frac{\overbrace{1}^{A}}{\underbrace{Js^2 + bs + k_R}_{R}}$$

For the A polynomial :

$$A = 1$$

Since there is no s term in the A polynomial, it is not needed to substitute $s = j\omega$. The real part :

$$R_A = 1 \tag{40}$$

The imaginary part:

$$I_A = 0 \tag{41}$$

For the R and B polynomial :

$$R = Js^2 + bs^2 + k_R$$

$$B = R.s = \underbrace{Js^3}_{b_3} + \underbrace{bs^2}_{b_2} + \underbrace{k_R s}_{b_1} + \underbrace{0}_{b_0}$$

Substituting $s = j\omega$ in order to get the real and imaginary parts.

$$B = -Jj\omega^3 - b\omega^2 + jk_R\omega$$

The real part :

$$R_B = -b\omega^2 \tag{42}$$

The imaginary part:

$$I_B = -J\omega^3 + k_R\omega \tag{43}$$

Substituting equations 40 , 41, 42, and 43 in 2.21, 2.23, 2.29 and 2.30:

$$K_I = 0 \tag{44}$$

$$\omega = \sqrt{\frac{5}{6}} \tag{45}$$

$$K_I = \frac{5}{6}K_D + \frac{25}{36} \tag{46}$$

Based on the above results, the RRB and CRB limits are plotted, as shown in figure 2.

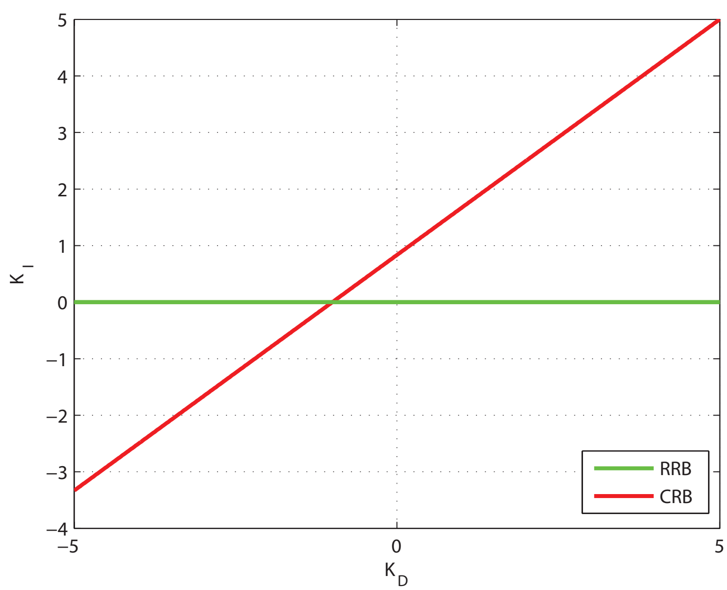


Figure 2: RRB and CRB Limits ($K_P = 0.5$)

Next, in order to identify the stable area, an arbitrary point is taken in each area as shown in 3 (Denoted by the blue 'X' mark). Knowing the K_P, K_D and K_I values, the eigenvalues are calculated for each point. Using 2.24, the characteristic equation is:

$$Js^3 + bs^2 + k_R s + (K_P s + K_D s^2 + K_I)1 = 0$$

$$3s^3 + s^2 + 2s + (K_P s + K_D s^2 + K_I) = 0$$

Case(1) : $K_P = 0.5, K_D = -2$ and $K_I = 3$

$$s = -0.667 \qquad s = 0.5 \pm j1.11803$$

Case(2) : $K_P = 0.5, K_D = -3.5$ and $K_I = -1$

$$s = 0.5 \qquad s = 0.1667 \pm j0.799305$$

Case(3) : $K_P = 0.5, K_D = 1$ and $K_I = -2$

$$s = 0.481142 \qquad s = -0.573903 \pm j1.02773$$

Case(4) : $K_P = 0.5, K_D = 3$ and $K_I = 1$

$$s = -0.81262 \qquad s = -0.260354 \pm j0.585157$$

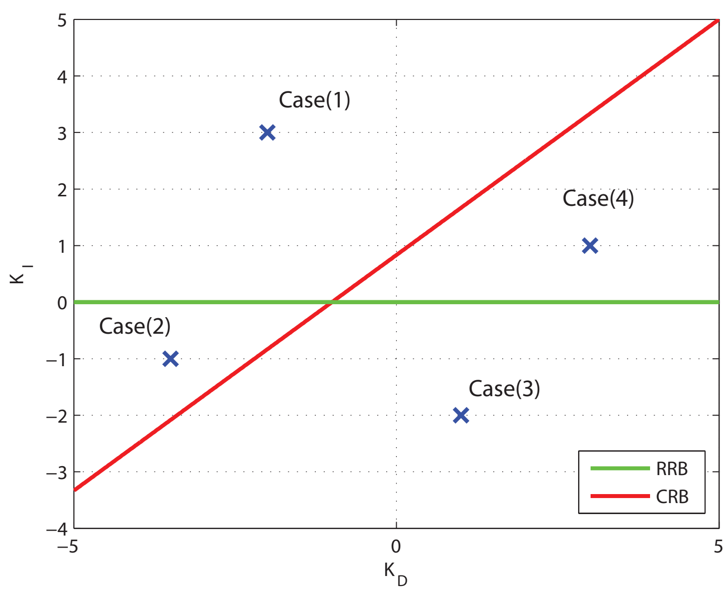


Figure 3: RRB and CRB Limits with test points

Based on the eigenvalues calculated, the stable area is identified to be the area where case(4) is located. Hence, the whole space of stabilizing controller parameters are shown in the figure below:

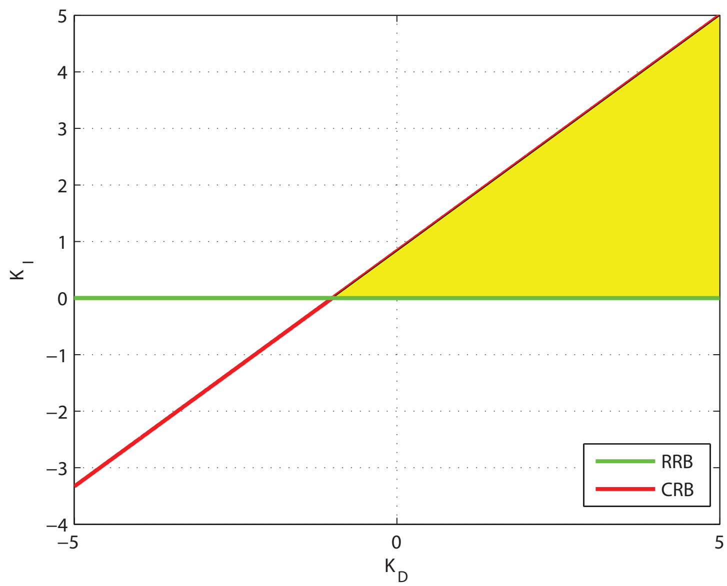


Figure 4: Stable Area

Similarly, using the PIDrobust Matlab Toolbox, the whole space of stabilizing controller parameters are obtained. Figure 5 represents the K_I vs. K_D plot for $K_P = 0.5$ [3]. And figure 6, shows a 3D plot for the stabilizing controller parameter space for several K_P values.

[3] Note: The difference between 4 and 5 is because of the scale used.

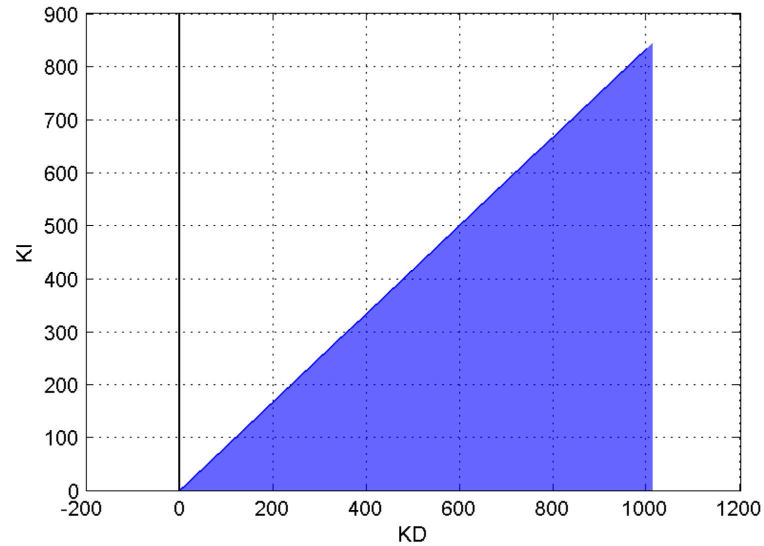


Figure 5: 2D plot- obtained using the PID Robust Toolbox

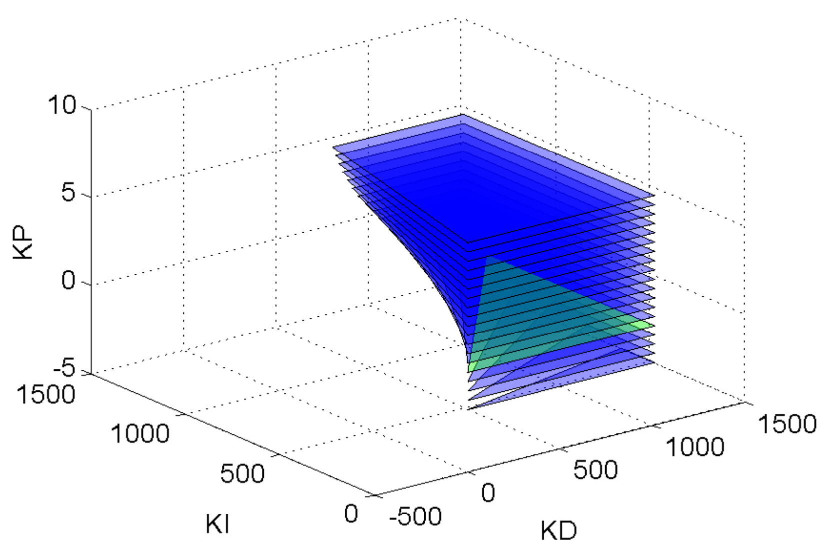


Figure 6: 3D plot- obtained using the PID Robust Toolbox

3 State Space Representation - LTI Systems (Direct Approach)

Example:

For a better understanding of this approach, the following example is presented:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -3 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

The closed loop state equation based on 2.36:

$$A_k = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 - k_1 & -3 - k_2 & -1 - k_3 \end{bmatrix}$$

where k_3 is assumed to be 0.2 .

Using 2.38, the RRB limit:

$$k_1 = -3 \tag{47}$$

Based on 2.39, there exists no IRB. Lastly, the CRB limits based on the results 2.40 and 2.41 are:

$$k_1 = \frac{6}{5}\omega^2 - 3 \tag{48}$$

$$k_2 = \omega^2 - 3 \tag{49}$$

The plots concerning this example are illustrated at the end of section 4.

4 State Space Representation - LTI Systems (Algebraic Riccati Equation Approach)

Example:

Given the A_k matrix of a closed loop LTI system :

$$A_k = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3-k_1 & -3-k_2 & -1-k_3 \end{bmatrix} \tag{50}$$

Since $n = 3$:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{51}$$

Using 2.43 results in the Hamitlonian matrix:

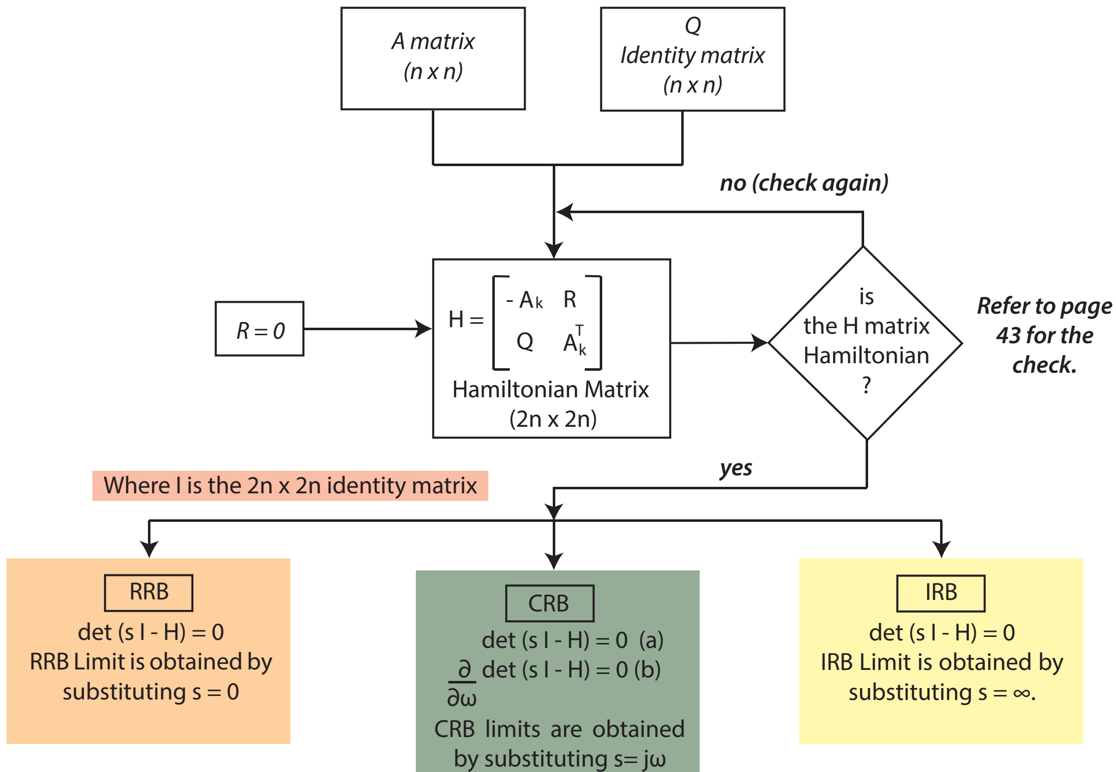
$$H = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 3+k_1 & 3+k_2 & 1+k_3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -3-k_1 \\ 0 & 1 & 0 & 1 & 0 & -3-k_2 \\ 0 & 0 & 1 & 0 & 1 & -1-k_3 \end{bmatrix} \tag{52}$$

Checking whether 53 satisfies the Hamiltonian Matrix form. Using 2.44 :

$$J = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{53}$$

Followed by applying the three calculations and checking for the condition *(mentioned in page 43)*:

$$J^{-1}HJ = -JHJ = -H^T = \begin{bmatrix} 0 & 0 & -3-k_1 & -1 & 0 & 0 \\ 1 & 0 & -3-k_2 & 0 & -1 & 0 \\ 0 & 1 & -1-k_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3+k_1 & 3+k_2 & 1+k_3 \end{bmatrix}$$



The CRB limit calculations differ depending on the size of the ***A_k matrix*** i.e. (n x n) and the number of controllers ***k*** assumed. In this section and based on the previous examples (PSA introduction and Steady State Feedback), a (3 x 3) ***A_k matrix*** is take into consideration and controller ***k_3*** is assumed. ***[The following steps are applicable to any (n x n) A matrices with all k controllers assumed except 2 k's].***

Step 1: Dividing (b) by ω, (Equation (b) contains the factor ω, which can be neglected since the solution ω=0 is independently mapped using the RRB limit. Now both equations contain only terms with even powers of ω). (c)

Step 2: Rearranging equation (c) , to get k_1 in terms of k_2 and ω . (d)

Step 3: Substituting equation (d) in equation (a), to get an equation in terms of only k_2 and ω . (e)

Step 4: Solving (e) for k_2 to get the first CRB Limit . *(Notice the repetition of each solution twice)* (f)

Step 5: Taking the real (accepted) limits of equation (e) and substitute them in equation (b) to get an equation in terms of k_1 and ω. (g)

Step 6: Solving (g) for k_1 to get the 2nd CRB Limit. *(Similarly, the repetition as in step 4)* (h)

Figure 7: Algebraic Riccati Equation Approach

The next step is obtaining the determinant of $[sI - H]$ and equating it to 0 :

$$\det\left[\begin{bmatrix} s & 0 & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 3+k_1 & 3+k_2 & 1+k_3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -3-k_1 \\ 0 & 1 & 0 & 1 & 0 & -3-k_2 \\ 0 & 0 & 1 & 0 & 1 & -1-k_3 \end{bmatrix}\right] = 0$$

The resulting determinant is:

$$-k_1^2 - 2k_1k_3s^2 - 2k_1s^2 - 6k_1 + k_2^2s^2 + 2k_2s^4 + 6k_2s^2 - k_3^2s^4 - 2k_3s^4 - 6k_3s^2 + s^6 + 5s^4 + 3s^2 - 9 = 0 \tag{54}$$

Now, the three limits are obtained as presented in figure 7. Firstly, the RRB Limit :

$$-k_1^2 - 6k_1 - 9 = 0 \tag{55}$$

Solving 55 for k_1 results in the RRB limits:

$$k_1 = -3 \tag{56a}$$

$$k_1 = -3 \tag{56b}$$

Secondly, the IRB limit: [4]

$$\text{Coefficient of}(s^6) = 1 \neq 0 \tag{57}$$

In this case, the IRB limit does not exist. For the CRB limit, $s = j\omega$ is substituted in 54:

$$-k_1^2 + 2k_1k_3\omega^2 + 2k_1\omega^2 - 6k_1 - k_2^2\omega^2 + 2k_2\omega^4 - 6k_2\omega^2 - k_3^2\omega^4 - 2k_3\omega^4 + 6k_3\omega^2 - \omega^6 + 5\omega^4 - 3\omega^2 - 9 = 0 \tag{58}$$

Followed by differentiating 58 in terms of ω:

$$4k_1\omega - 4k_3^2\omega^3 - 6\omega - 12k_2\omega + 12k_3\omega - 2k_2^2\omega + 8k_2\omega^3 - 8k_3\omega^3 + 20\omega^3 - 6\omega^5 + 4k_1k_3\omega = 0 \tag{59}$$

[4] Refer to section 2.5.1.2

Substituting $k_3 = 0.2$ in 58:

$$-k_1^2 + \frac{12}{5}k_1\omega^2 - 6k_1 - k_2^2\omega^2 + 2k_2\omega^4 - 6k_2\omega^2 - \omega^6 + \frac{114}{25}\omega^4 - \frac{9}{5}\omega^2 - 9 = 0 \tag{60}$$

and 59:

$$\frac{24}{5}k_1\omega - \frac{18}{5}\omega - 12k_2\omega - 2k_2^2\omega + 8k_2\omega^3 + \frac{456}{25}\omega^3 - 6\omega^5 = 0 \tag{61}$$

Step results:

Step 1:

$$\frac{24}{5}k_1 - \frac{18}{5} - 12k_2 - 2k_2^2 + 8k_2\omega^2 + \frac{456}{25}\omega^2 - 6\omega^4 = 0$$

Step 2:

$$k_1 = \frac{5}{12}k_2^2 - \frac{5}{3}k_2\omega^2 + \frac{5}{2}k_2 + \frac{5}{4}\omega^4 - \frac{19}{5}\omega^2 + \frac{3}{4} = 0$$

Step 3:

$$\frac{12\omega^2(\frac{5}{12}k_2^2 - \frac{5}{3}k_2\omega^2 + \frac{5}{2}k_2 + \frac{5}{4}\omega^4 - \frac{19}{5}\omega^2 + \frac{3}{4})}{5} - k_2^2\omega^2 - 15k_2 + 4k_2\omega^2 + 2k_2\omega^4 -$$

$$\left(\frac{5}{12}k_2^2 - \frac{5}{3}k_2\omega^2 + \frac{5}{2}k_2 + \frac{5}{4}\omega^4 - \frac{19}{5}\omega^2 + \frac{3}{4}\right)^2 - \frac{5}{2}k_2^2 + 21\omega^2 - \frac{147}{50}\omega^4 - \omega^6 - \frac{27}{2} = 0$$

Step 4:

$$k_2 = \omega^2 - 3 \tag{62a}$$

$$k_2 = \omega^2 - 3 \tag{62b}$$

$$k_2 = 3\omega^2 - \frac{12}{5}\omega j - 3 \tag{62c}$$

$$k_2 = 3\omega^2 - \frac{12}{5}\omega j - 3 \tag{62d}$$

Step 5:

$$\frac{24}{5}k_1 + 8\omega^2(\omega^2 - 3) - 2(\omega^2 - 3)^2 + \frac{156}{25}\omega^2 - 6\omega^4 + \frac{162}{5} = 0$$

Step 6:

$$k_1 = \frac{6}{5}\omega^2 - 3 \tag{63a}$$

$$k_1 = \frac{6}{5}\omega^2 - 3 \tag{63b}$$

Figure 8 shows the stabilizing controller parameters area (Region 3) for $k_3 = 0.2$. In this case, the stable area was identified using a more effecient technique than the one already described in section 2. This method was presented in [4], where the author uses the concept of Hurwitz Criterion to enhance the speed of determining the stability area (instead of calculating the eigenvalues in each region). The Hurwitz criterion states:

If any of the coefficients of equation 2.37 are either negative or not present, then the system is said to be unstable. For the system to be stable the following is obtained:

$$q_1 + k_1 > 0 \rightarrow 3 + k_1 > 0 \rightarrow k_1 > -3$$

$$q_2 + k_2 > 0 \rightarrow 3 + k_2 > 0 \rightarrow k_2 > -3$$

$$q_3 + k_3 > 0 \rightarrow 1 + k_3 > 0 \rightarrow k_3 > -1$$

For the 2D plot shown below, the condition $k_2 > -3$ is used. As shown in the plot, a horizontal line ($k_2 = -3$) is drawn. Knowing that k_2 should be greater than -3, results in the elimination of regions 1 and 2. Accordingly, region 3 is the stable area. A check could be performed by picking any arbitrary point of this region, and substituting these values in 2.37, and solving the resulting equation to obtain the eigenvalues.

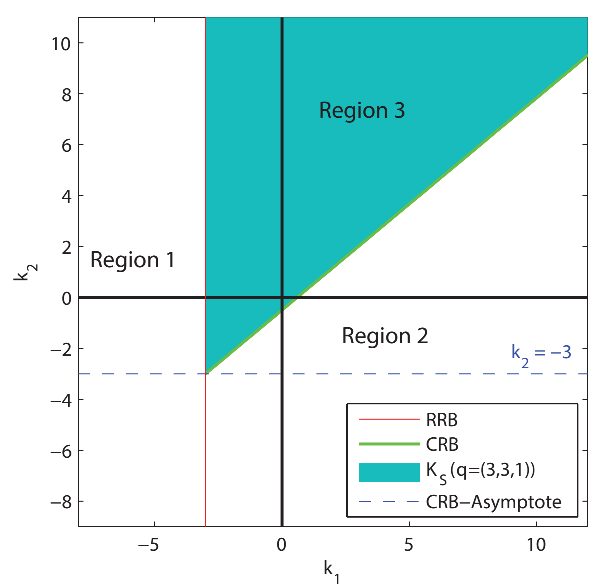


Figure 8: 2D Plot $k_3 = 0.2$

Similarly for the 3D plot, the condition $k_3 > -1$ is utilised in order to identify the stabilizing controller parameters area for all values of k_3.

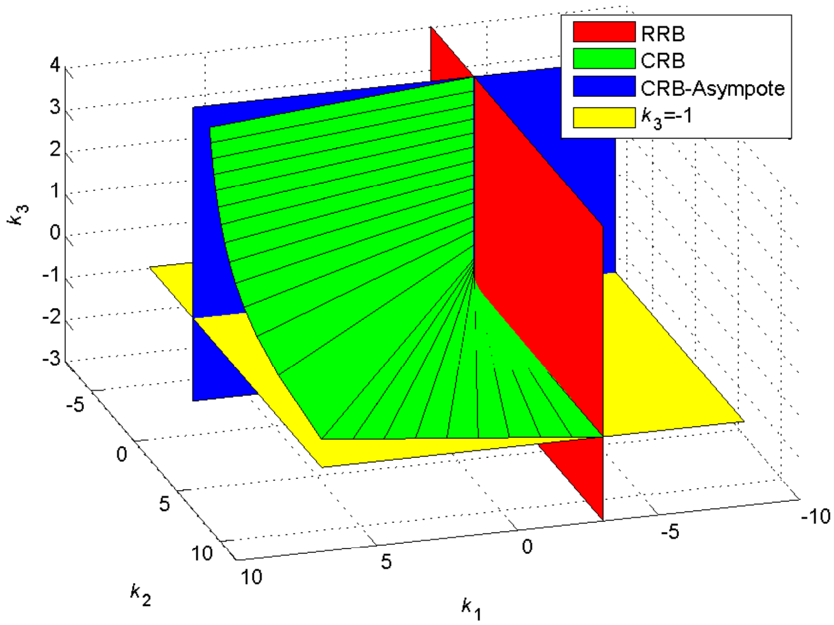


Figure 9: 3D Plot