4 Continuous-time Linear Dynamical Systems

4.1 Scope

We want to study homogenuous differential equations with constant coefficients that can be written succintly in matrix form as

$$\dot{\mathbf{x}}(t) = F\mathbf{x}(t) \tag{50}$$

4.2 First order linear differential equations

Consider the following first order homogenuous linear ODE with constant coefficients

$$\dot{x}(t) + ax(t) = 0 \tag{51}$$

$$x(0) = k \tag{52}$$

To solve the differential equation means to find the function $x(t)$ which passes through k at time $t = 0$ and has a derivative, which irregardless of time, is the function evaluation scaled by a. Lets begin by making the formal rearrangement

$$\frac{dx}{x} = -a\,dt \tag{53}$$

$$\Longrightarrow x(t) = \exp(-at) \tag{54}$$

If we were not given the initial condition, $x(0) = k$, this is as far as we could go in determining a solution. Fortunately, we can solve for c_1 by using the initial condition, so that $c_1 = k$. Thus, the full solution to the differential equation is just $x(t) = k\exp(-at)$.

Remark: The system this differential equation describes is not very interesting. If $a < 0$ then $x(t) \to \infty$, whereas if $a > 0$ then $x(t) \to 0$.

49. Draw the phase portrait for this first order system.

4.3 N-th order linear differential equations

Lets consider a more interesting example where we consider the solution to an N-th order linear differential equation with constant coefficients given by

$$\sum_{i=0}^{N} a_i x^{(i)}(t) = 0 \tag{55}$$

$$x^{(i)}(0) = k_i \qquad i = 0, \ldots, N-1 \tag{56}$$

without loss of generality $a_N = 1$. If we define

$$\mathbf{x}(t) = \begin{bmatrix} x(t) & x^{(1)}(t) & \cdots & x^{(N-1)} \end{bmatrix}^\top \tag{57}$$

then we can transform this N-th order equation into a first order one below

$$\dot{\mathbf{x}}(t) = F\mathbf{x}(t) \tag{58}$$

$$\mathbf{x}(0) = \mathbf{k} \tag{59}$$

by overloading the differential operator so that $\dot{\mathbf{x}}(t) = \begin{bmatrix} x^{(1)}(t) & x^{(2)}(t) & \cdots & x^{(N)} \end{bmatrix}^\top$, and setting

$$F = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & -a_2 & -a_3 & \cdots & -a_{N-1} \end{pmatrix} \tag{60}$$

Remark: A matrix in this form is said to be a companion form matrix. They arise when we transform an N-th order differential equation into a first order system as we did above.

4.4 Fundamental matrix solution

How can we actually solve this matrix differential equation? That is, how can we find the vector that satisfies Eq. (58) and the associated initial condition? To answer this question, we have to take a slight detour and discuss the *fundamental matrix solution*

Lets begin by assuming that we can find the solution to Eq. (58), then if we denote the N solutions as $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$ then we can write

$$a_0\phi_1(t) + a_1\phi_1^{(1)}(t) + \cdots + \phi_1^{(N)}(t) = 0 \tag{61}$$

$$a_0\phi_2(t) + a_1\phi_2^{(1)}(t) + \cdots + \phi_2^{(N)}(t) = 0 \tag{62}$$

$$\vdots \tag{63}$$

$$a_0\phi_N(t) + a_1\phi_N^{(1)}(t) + \cdots + \phi_N^{(N)}(t) = 0 \tag{64}$$

which if take $\Phi_i(t) = \begin{bmatrix} \phi_i(t) & \phi_i^{(1)}(t) & \cdots & \phi_i^{(N)}(t) \end{bmatrix}^\top$, and set

$$\mathbf{\Phi}(t,0) = \begin{pmatrix} | & | & & | \\ \Phi_1(t) & \Phi_2(t) & \cdots & \Phi_N(t) \\ | & | & & | \end{pmatrix} \tag{65}$$

then we can write that

$$\dot{\mathbf{\Phi}}(t,0) = F\mathbf{\Phi}(t,0) \tag{66}$$

Now, if we were to make an *ansatz* and guess that the solution is $\mathbf{x}(t) = \mathbf{\Phi}(t,0)\mathbf{x}(0)$ then we can easily verify whether or not this is true

$$\dot{\mathbf{x}}(t) = F\mathbf{x}(t) \tag{67}$$
$$\implies \dot{\mathbf{\Phi}}(t,0)\mathbf{x}(0) = F\mathbf{\Phi}(t,0)\mathbf{x}(0) \tag{68}$$
$$\implies F\mathbf{\Phi}(t,0)\mathbf{x}(0) = F\mathbf{\Phi}(t,0)\mathbf{x}(0) \tag{69}$$

which also implies that $\mathbf{\Phi}(0,0) = I$.

Fundamental Solution

Theorem 1 (Series representation of fundamental solution)**.** *The series of matrices, $M_0, M_1, M_2, \ldots$, defined recursively as*

$$M_0 = I \tag{70}$$
$$M_k = I + F\int_0^t M_{k-1}(\sigma)d\sigma \tag{71}$$

converges uniformly to $\Phi(t,0)$

If we expand the recursion above (which we could do with the aid of induction), we see something interesting

$$\mathbf{\Phi}(t,0) = I + Ft + F^2\frac{t^2}{2!} + F^3\frac{t^3}{3!} + \cdots \tag{72}$$

If F were a scalar, then this would exactly be the series representation of $\exp(Ft)$. In fact, this infinite series is exactly how we define the *matrix exponential.*

Properties of the matrix exponential

Diagonal matrices: If $D = \mathrm{diag}(d_1, d_2, \ldots, d_N)$ then $\exp(D) = \mathrm{diag}(e^{d_1}, e^{d_2}, \ldots, e^{d_N})$.

Commuting matrices: If A and B commute ($AB = BA$), then we have that $\exp(A+B) = \exp(A)\exp(B) = \exp(B)\exp(A)$.

Inverse: $\exp(A)\exp(-A) = I$

50. Prove the property about commuting matrices.

In many cases $\mathbf{\Phi}(t,0)$ is unwieldly to actually compute. However, we are fortunate in that we are working with differential equations of constant coefficients; in this particular case,

$$\mathbf{\Phi}(t,0) = \exp(Ft) \tag{73}$$

which we can verify satisfies both $\mathbf{\Phi}(0,0) = I$ as well as

$$\frac{d}{dt}\exp(Ft) = \frac{d}{dt}\left(I + Ft + F^2\frac{t^2}{2!} + F^3\frac{t^3}{3!}\cdots\right) \tag{74}$$
$$= F + F^2 t + F^3\frac{t^2}{2!} + \cdots \tag{75}$$
$$= F\exp(Ft) \tag{76}$$

4.5 Jordan form

Since the matrix exponential is defined in terms of an infinite series, we are left wondering whether it can be computed analytically. For this, we have to introduce the concept of a *Jordan matrix*. A matrix $J \in \mathbb{R}^{M \times M}$ is said to be a Jordan matrix if it takes the form

$$J = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda \end{pmatrix} \tag{77}$$

which means we could write that $J = \lambda I + N$; this matrix N is said to be nilpotent.

Nilpotent matrices

A matrix N is said to be nilpotent if there exists an integer k such that $N^k = 0$

51. Is the following matrix nilpotent? Support your claim either way.

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{78}$$

As we have shown, N and λI commute. Combining the first two properties of the matrix exponential, this allows us to write that $\exp(Jt) = \exp(\lambda t) \exp(Nt)$. However, we still have more work to do as we have not yet computed $\exp(Nt)$. Recalling that N is nilpotent, we can write

$$\exp(Nt) = \sum_{i=0}^{\infty} N^i \frac{t^i}{i!} \tag{79}$$

$$= \sum_{i=0}^{M-1} N^i \frac{t^i}{i!} \tag{80}$$

$$= \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \cdots & \frac{t^{M-1}}{(M-1)!} \\ 0 & 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{M-2}}{(M-2)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{81}$$

Great! Now we have the ability to easily calculate the matrix exponential of any Jordan form matrix. Now we may ask, what about the matrix exponential of more general matrices? As it turns out, any matrix can be written in Jordan canonical form.

Jordan normal form

Definition of Jordan Normal Form: We say a matrix $J \in \mathbb{R}^{N \times N}$ is in Jordan normal form if it can be written as follows:

$$J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ 0 & 0 & \cdots & J_K \end{pmatrix} \tag{82}$$

where each J_i is a Jordan block matrix, i.e.

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix} \tag{83}$$

The **Jordan decomposition** of a matrix, A, is a factorization that *almost* diagonalizes A. In fact, *any* square matrix has a Jordan decomposition, which means it can be factorized as

$$A = TJT^{-1} \tag{84}$$

where J is in Jordan normal form, and the columns of T contain the generalized eigenvectors of A.

4.6 Algebraic and geometric multiplicity

The **characteristic polynomial** of a matrix, A, is defined as

$$p_A(x) = \det(xI - A) \tag{85}$$

If A is non-singular, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$, and associated multiplicities $m_1, m_2, \ldots, m_s$, then its characterisitc polynomial will be of the form

$$p_A(x) = (x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2} \cdots (x - \lambda_s)^{m_s} \tag{86}$$

if we have already converted our matrix into Jordan form then finding the characteristic polynomial is a rudimentary excercise. The best way to see this is through a motivating example

From Jordan form to characteristic polynomial

Ex.1 Say we have a matrix A and we have factorized it into its Jordan decomposition TJT^{-1} where $J = \text{diag}(J_1, J_2, J_3)$ which we may sometimes write as $J = J_1 \bigoplus J_2 \bigoplus J_3$

$$J_1 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \qquad J_2 = \begin{pmatrix} \lambda_2 \end{pmatrix} \qquad J_3 = \begin{pmatrix} \lambda_3 & 1 & 0 \\ 0 & \lambda_3 & 1 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{87}$$

then the characteristic polynomail of A will be

$$p_A(x) = (x - \lambda_1)^2 (x - \lambda_2)(x - \lambda_3)^3 \tag{88}$$

Ex.2 Take a matrix A again, and let its Jordan decomposition be $J = J_1 \bigoplus J_2 \bigoplus J_3 \bigoplus J_4$ where

$$J_1 = \begin{pmatrix} \lambda_1 \end{pmatrix} \qquad J_2 = \begin{pmatrix} \lambda_1 \end{pmatrix} \qquad J_3 = \begin{pmatrix} \lambda_2 \end{pmatrix} \qquad J_4 = \begin{pmatrix} \lambda_3 & 1 & 0 \\ 0 & \lambda_3 & 1 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{89}$$

in which case the characteristic polynomial is

$$p_A(x) = (x - \lambda_1)^2 (x - \lambda_2)(x - \lambda_3)^3 \tag{90}$$

Although the two matrices we looked at have different Jordan decompositions, they have the same characteristic polynomial.

Lets overload notation as we have already done where we take the polynomial expression before i.e.

$$p(x) = \sum c_i x^i \tag{91}$$

so that if we *plug* a matrix into the argument then we would write

$$p(A) = \sum c_i A^i \tag{92}$$

It so happens that if $p(x)$ is the characteristic polynomial of A, then $p(A) = 0$ [5]. Is there a polynomial of lesser order than the characteristic polynomial, lets call it $\chi(x)$, that $\chi(A) = 0$? If there is, we call this polynomial the **minimal polynomial**.

Minimal polynomials

Looking at **Ex.1** from earlier, we have that the minimal polynomial is just the characteristic polynomial, so that $p(x) = \chi(x)$.

However, the matrix from **Ex.2** has a minimal polynomial that does not coincide with its characteristic polynomial. In fact, its minimal polynomial is

$$\chi(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)^3 \tag{93}$$