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ECE463

HW9

04/07/2021

WEDNESDAY

SOAP FILM

1) calculate the shape of a soap film connecting two rings around the x-axis:

$$Y(0) = 6 \quad , \quad Y(2) = 5$$

Here our functional would be $F = y\sqrt{1 + x'(y)^2}$

$$\frac{\partial F}{\partial x} = 0 \quad , \quad \frac{\partial F}{\partial \dot{x}} = \frac{y x'}{\sqrt{1 + \dot{x}'^2}} \quad \Rightarrow \quad \cancel{\frac{\partial F}{\partial x}}^{0} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0$$

$$\Downarrow$$

$$y\frac{dx}{dy} = \alpha\sqrt{1+\dot{x}^2} \quad \Leftarrow \quad \frac{y x'}{\sqrt{1+\dot{x}^2}} = \alpha$$

$$\dot{x} = \frac{\alpha}{\sqrt{y^2 - \alpha^2}}$$

Using $\cosh(\theta) = \frac{y}{\alpha} \Rightarrow dy = \alpha \sinh\theta\, d\theta$

$$\Rightarrow \left(\frac{y}{\alpha}\right)^2 - 1 = \cosh^2\theta - 1 = \sinh^2\theta$$

$$x = \int \frac{1}{\sqrt{\left(\frac{y}{\alpha}\right)^2 - 1}}\, dt$$

$$= \int \frac{(\alpha \cancel{\sinh\theta}\, d\theta)}{\cancel{\sinh\theta}} = \int \alpha\, d\theta \overset{!}{=} \boxed{\alpha\theta + \beta}$$

$$\theta = \cosh^{-1}\left(\frac{y}{\alpha}\right)$$

$$x = \alpha\cosh^{-1}\left(\frac{y}{\alpha}\right) + \beta$$

$$\Downarrow$$

$$\boxed{Y = \alpha\cosh\left(\frac{x-\beta}{\alpha}\right)} \Rightarrow \begin{aligned} Y(0) &= \alpha\cosh\left(\frac{-\beta}{\alpha}\right) = 6 \\ Y(5) &= \alpha\cosh\left(\frac{5-\beta}{\alpha}\right) = 5 \end{aligned}$$

(see next page)

$$\Rightarrow \begin{aligned} &\cancel{\alpha = 1.0868} \\ &\cancel{\beta = 2.6011} \end{aligned} \Rightarrow \boxed{Y = \cancel{1.0868}\cosh\left(\frac{x - \cancel{2.6011}}{\cancel{1.0868}}\right)}$$

$$\begin{aligned} a &= 4.7255 \\ b &= 3.3970 \end{aligned} \Rightarrow \boxed{Y = 4.7255\cosh\left(\frac{x - 3.3970}{4.7255}\right)}$$

Problem 1 – Soap Film

The cost function I used:

```
function [E] = soapFilmCost(Z)
c1 = 6; c2 = 5;

a = Z(1); b = Z(2);

E1 = abs(a*cosh(-b/a) - c1);
E2 = abs(a*cosh((5-b)/a) - c2);

E = E1 + E2;

end
```

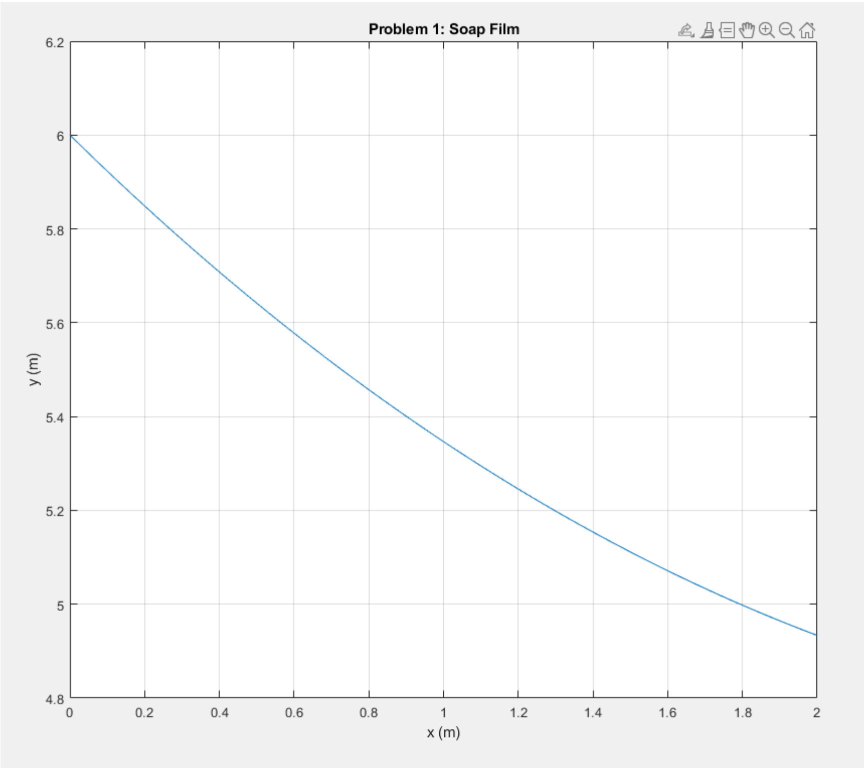
```
>> [m,n] = fminsearch('soapFilmCost',[1,1])

m =

    4.7255    3.3970


n =

   4.9984e-05
```



2) Same but now $Y(0) = 6$, $Y(1) = $ free.

Like before,

$$Y = a\cosh\left(\frac{x-b}{a}\right) \Rightarrow Y(0) = a\cosh\left(\frac{-b}{a}\right) = 6$$

$$a = 5.9152$$

We also have

$$F_{y'} = 0 \Rightarrow y' = 0, \text{ also } y' = -a\sinh\left(\frac{x-b}{a}\right) = 0$$

$$Y'(1) = -a\sinh\left(\frac{1-b}{a}\right) = 0$$

$$\Rightarrow b = 1$$

$$\boxed{Y = 5.9152\cosh\left(\frac{x-1}{5.9152}\right)}$$

HANGING CHAIN

3) Calculate the shape of a chain: $L = 4m$, $(0,6) \to (2,5)$

The ~~functional~~ LaGrange here is: $F = x\sqrt{1+\dot{Y}^2}$ subject to

$Y(0) = 6$, $Y(2) = 5$

$$\int_0^2 \sqrt{1+\dot{Y}^2}\,dx = 4$$

~~From Calc 3, we're optimizing same~~

Using a Lagrange multiplier: $F = x\sqrt{1+\dot{Y}^2} + M\sqrt{1+\dot{Y}^2}$

$$F_Y = 0 \Rightarrow F_{Y'} = a \Rightarrow \frac{(x+M)\dot{Y}}{\sqrt{1+\dot{Y}^2}} = a$$

$$\dot{Y} = \frac{a}{\sqrt{(x+M)^2 - a^2}}$$

$$Y = a\cosh^{-1}\left(\frac{x+M}{a}\right) + b \Leftarrow (\text{Integrate}) \Leftarrow$$

$$\cancel{x = a\cosh\left(\frac{Y-b}{a}\right) - M} \Rightarrow Y = a\cosh\left(\frac{x-b}{a}\right) - M$$

$$Y(0) = a\cosh\left(\frac{-b}{a}\right) - M = 6$$

$$Y(2) = a\cosh\left(\frac{2-b}{a}\right) - M = 5$$

$$\int\sqrt{1+\dot{Y}^2}\,dx = a\sinh\left(\frac{2-b}{a}\right) - a\sinh\left(\frac{-b}{a}\right) = 2$$

$\Rightarrow$ Using cost3.m, $\boxed{a = 0.4717,\ b = 1.1205,\ M = -3.4415}$

So, $\boxed{Y(x) = 0.4717\cosh\left(\frac{x - 1.1205}{0.4717}\right) + 3.4415}$

Problem 2 – Soap Film

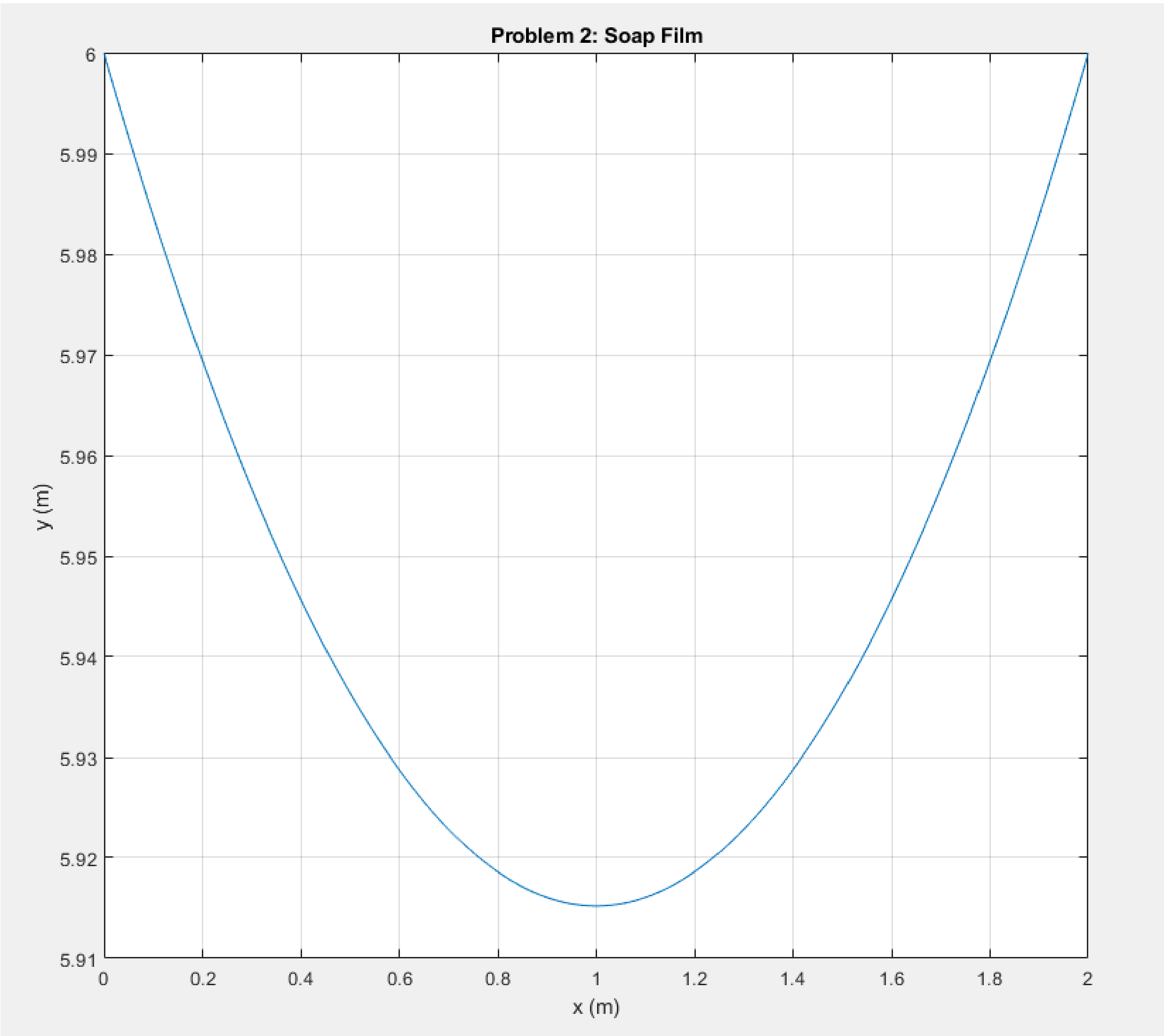
```
>> f = @(a) a.*cosh(-1./a);
>> e = @(a) abs(f(a) - 6);
>> [m,n] = fminsearch(e,4)

m =

    5.9152


n =

   3.6615e-05
```



Problem 3 – Hanging Chain

```
function J = cost3(z)
% ECE 463 lecture #23
% Calculate the shape of a soap film

    a = z(1);
    b = z(2);
    M = z(3);
 % assume gravity is in the -y direction
 % y = f(x)
    Length = 4;
    x1 = 0;
    y1 = 6;
    x2 = 2;
    y2 = 5;

    e1 = a*cosh((x1-b)/a) - M - y1;
    e2 = a*cosh((x2-b)/a) - M - y2;
    e3 = a*sinh((x2-b)/a) - a*sinh((x1-b)/a) - Length;

    x = [x1:0.001:x2]';
    y = a*cosh( (x-b)/a ) - M;
%     plot(x,y);
%     xlim([x1,x2]);
%     ylim([0,2]);
%     pause(0.01);
    J = e1^2 + e2^2 + e3^2;

 End
```

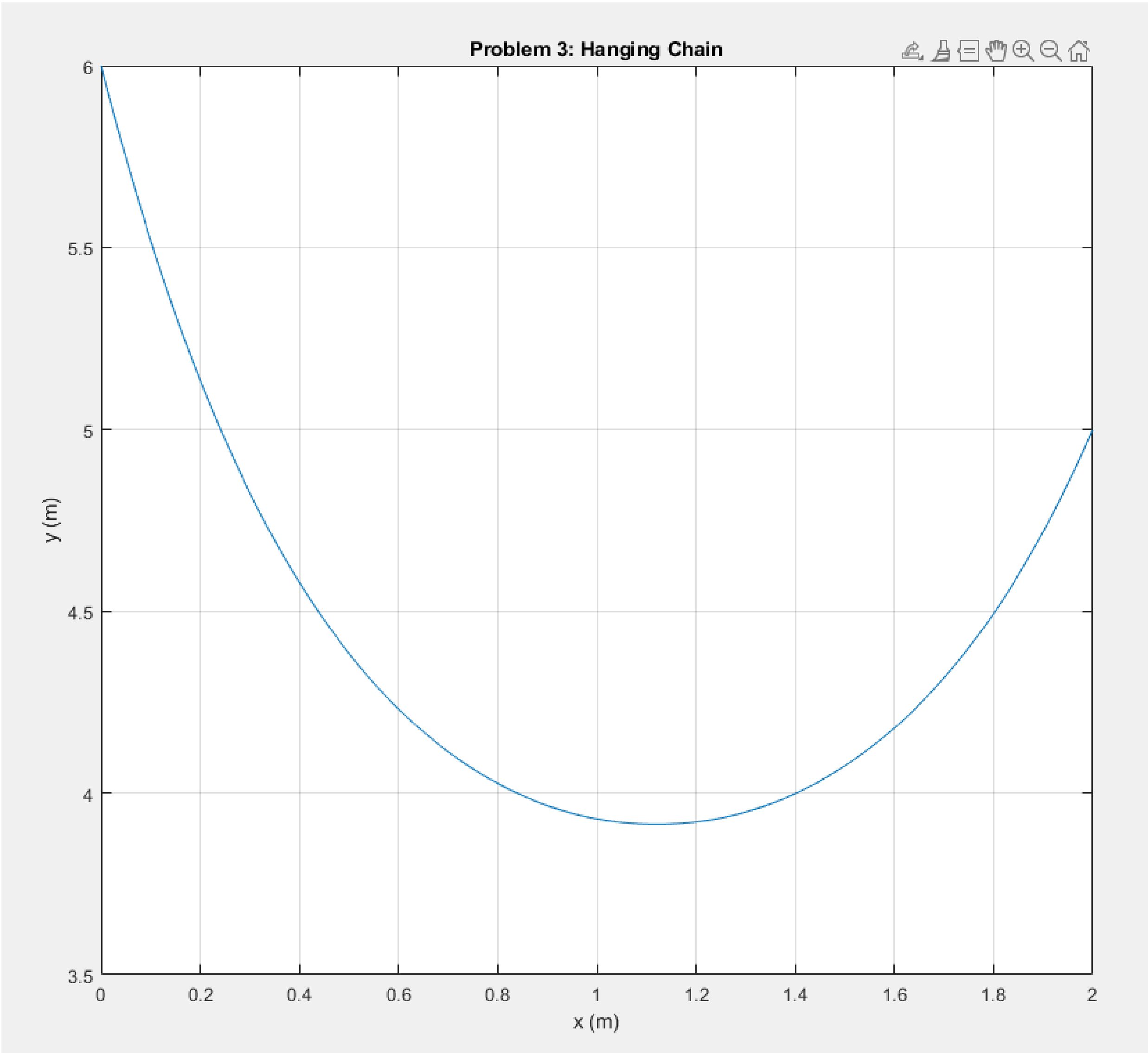
```
>> [m,n] = fminsearch('cost3',[1,2,3]);

m =

    0.4717    1.1205   -3.4415


n =

   1.2132e-08
```


Problem 3: Hanging Chain
y (m)
x (m)
6
5.5
5
4.5
4
3.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

RICATTI EQUATION

4) Find the function that minimizes

$$J = \int_0^{10} (x^2 + 9\dot{x}^2)\,dt \quad , \; x(0) = 6, \; x(10) = 4$$

$$F = x^2 + 9\dot{x}^2 \implies \underbrace{\frac{\partial F}{\partial x} = 2x \; , \quad \frac{\partial F}{\partial \dot{x}} = 18\dot{x} \; , \quad \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 18\ddot{x}}$$

$$\Downarrow$$

$$(2x) - (18\ddot{x}) = 0$$

$$\implies x - 9\ddot{x} = 0$$

$$\implies x = 0, \; e^{-3t}, \; e^{3t}$$

$$\implies x(t) = ae^{-3t} + be^{3t}$$

$$x(0) = a + b = 6$$

$$x(10) = ae^{-30} + be^{30} = 4$$

$$\implies a = 5.999\ldots \simeq 6$$
$$b = 3.743 \times 10^{-13}$$

$$\boxed{x(t) = 6e^{-3t} + 3.743 \times 10^{-13} e^{3t}}$$

5) " " $J = \int_0^8 (4x^2 + 9u^2)\,dt \; , \; \dot{x} = -0.2x + u, \; x(0) = 6, \; x(10) = 4$

$$\implies F = (4x^2 + 9u^2) + m(\dot{x} + 0.2x - u)$$

$$8x + 0.2m + \dot{m} = 0$$

$$\cancel{m = -40x}$$

$x: \implies F_x = 8x + 0.2m \; , \quad F_{\dot{x}} = m, \quad \frac{d}{dt}F_{\dot{x}} = \cancel{0}\;\dot{m} \implies \cancel{8x + 0.2m = 0}$

$u: \implies F_u = \overset{18u}{-m}, \quad F_{\dot{u}} = 0 \implies 18u - m = 0 \implies 18u + 40x = 0 \implies \cancel{u = -\frac{20}{9}x}$

$m: \implies F_m = \dot{x} + 0.2x - u; \quad F_{\dot{m}} = 0 \implies \dot{x} + 0.2x - u = 0$

$$\Downarrow$$

$$\cancel{\dot{x} + 0.2x + \frac{20}{9}x = 0}$$

$$\cancel{\implies \dot{x} + \frac{109}{45}x = 0}$$

$$\cancel{\implies X\left(s + \frac{109}{45}\right) = 0}$$

$$\cancel{\implies s = -\frac{109}{45}}$$

$$\implies 8X + (0.2 + s)M = 0 \implies M = \frac{-8}{s + 0.2}X$$

$$18U - M = 0 \implies 18U + \frac{8}{s+0.2}X = 0$$

$$U = -\frac{4}{9(s+0.2)}X$$

$$(s+0.2)X + \frac{4}{9(s+0.2)} = 0 \implies (s^2 + 0.4s + 0.04)X + 4/9 = 0$$

$$\cancel{X = \frac{-4/9}{s^2 + 0.4s + 0.04} = \frac{-4/9}{(s+0.2)^2} \Rightarrow \boxed{x(t) = \frac{-4}{9} t e^{-0.2t}}}$$

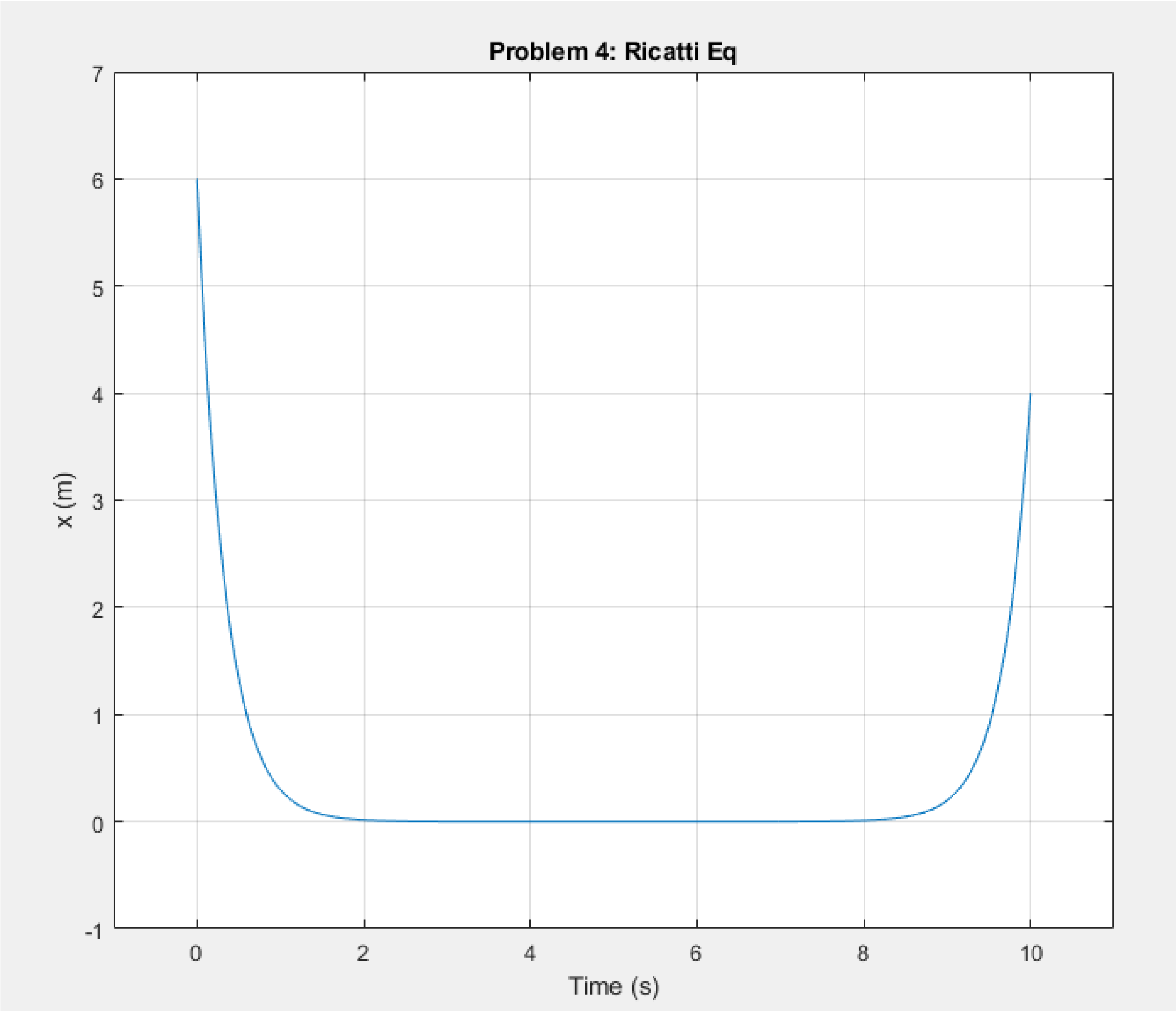
$\Rightarrow$ Can't see where I went wrong... $x(t) = -\frac{4}{9} c t e^{-0.2t}$

$x(10) = -0.6015C = 4$

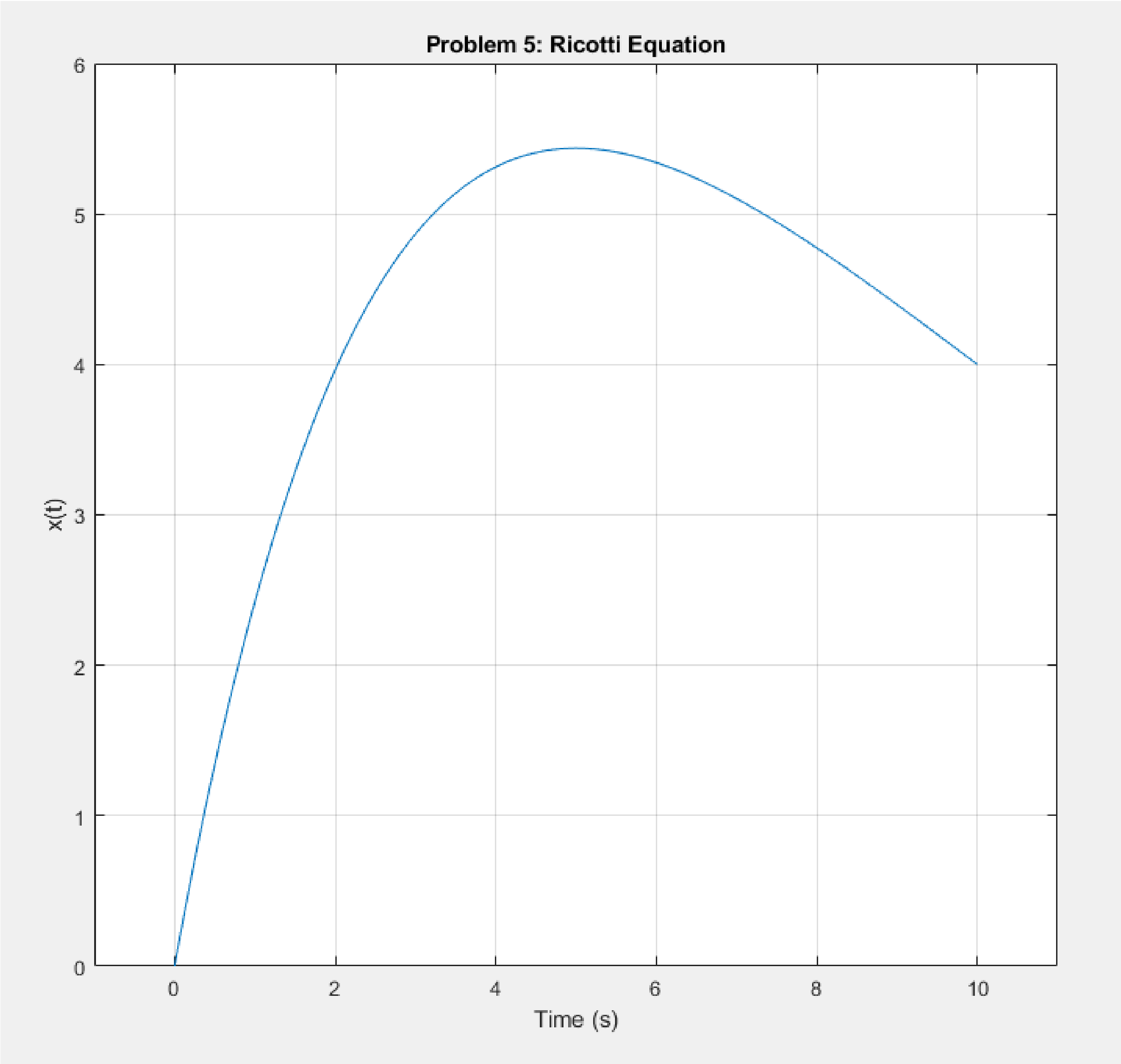
$\Rightarrow C = -6.6502$

$$\boxed{x(t) = 2.9556\, t e^{-0.2t}}$$

Problem 4 – Ricatti Equation



Problem 5 – Ricatti Equation



LQG CONTROL

6) CART & PENDULUM Design a FSF $u = K_r R - K_x X$ for the Cart & Pend. from HW#6 using LQC so that DC Gain = 1, 2% Ts = 6s, 10% OS.

$\Downarrow$
$\xi = 0.5911$
$\Downarrow$
$\theta = 53.8°$
$\Downarrow$
$\tan\theta = 1.36$

(a) Where are the closed-loop poles w/ pole placement & w/ LQC?
(b) Are the feedback gains larger or smaller w/ LQC?
(c) Which works better?

Recall, $m_{CART} = m_c = 1kg$, $m_{BALL} = m_B = 4kg$, $L = 1m$

$\Rightarrow$ Linearized model:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -39.2 & 0 & 0 \\ 0 & 49 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Pole placement: I'll place poles at $\cancel{[-0.667, -2, -3, -4]}$ $\left[-\frac{2}{3} \pm j0.9097, -3, -4\right]$

$$\boxed{K_x = [-1.5575, -73.1629, -2.5412, -10.8746] \quad K_r = -1.5575}$$

$\Leftarrow$

$$\cancel{K_x = [-1.6327, -39.3673, 3.1293, -0.5374]}$$
$$-82.6327 \quad -4.2177 \quad -13.8849$$
$$\cancel{K_r = -1.6327}$$

$\hookrightarrow$ Checked w/ eig(A − BK_x) ✓

Recall:

$$K_r = \frac{-1}{C(A - BK_x)^{-1} B}$$

LQC: Fine tune Q until I get desired response.

$$Q = \alpha(C^T C) + \beta(A^T C^T C A)$$

Let's try $\alpha = 5$, $\beta = 1 \Rightarrow$

Resulting Poles
- $-0.49 \pm j0.45$
- -6.6
- -7.4

Speed up a little... $\alpha = 10 \Rightarrow$
- $-0.58 \pm j0.5375$
- ⋮

A little more... $\alpha = 15 \Rightarrow$

Seems like $\alpha = 17$ is the spot for getting 2% Ts...
However $\theta \approx 42° \Rightarrow \xi \approx 0.74 \Rightarrow OS \approx 3.2\%$ ...

$K_x = [-4.1231, -121.9602, -7.8989, -23.2831]$
$\Rightarrow K_r = -\cancel{1}4.1231$

/w Pole Placement : $[-\frac{2}{3} \pm j\, 0.9097, -3, -4]$

(a)

LQC : $[-0.6693 \pm j\, 0.6108, -6.6988, -7.3469]$

(b) It seems pole placement resulting in smaller feedback gains overall!

(c) I think here pole placement wins.

7) Ball & Beam "Same"

des_poles : $[-\frac{2}{3} \pm j\, 0.9097, -3, -4]$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -7 & 0 & 0 \\ -1.96 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.4 \end{bmatrix}$$

Pole Placement

$$\boxed{\begin{aligned} K_x &= [-10.3514, 56.5133, -8.8943, 20.8333] \\ K_r &= -5.4514 \end{aligned}}$$

LQC: ⟶ Ok this was really difficult. There was no way of meeting BOTH OS spec and 2% T_s... It seemed there was gonna be OS no matter what...

$$\boxed{\begin{aligned} K_x &= [-10.2497, 36.4808, -9.3060, 13.5057] \\ K_r &= -7.3491 \end{aligned}}$$

(a) Pole Placement: $[-\frac{2}{3} \pm j\, 0.9097, -3, -4]$

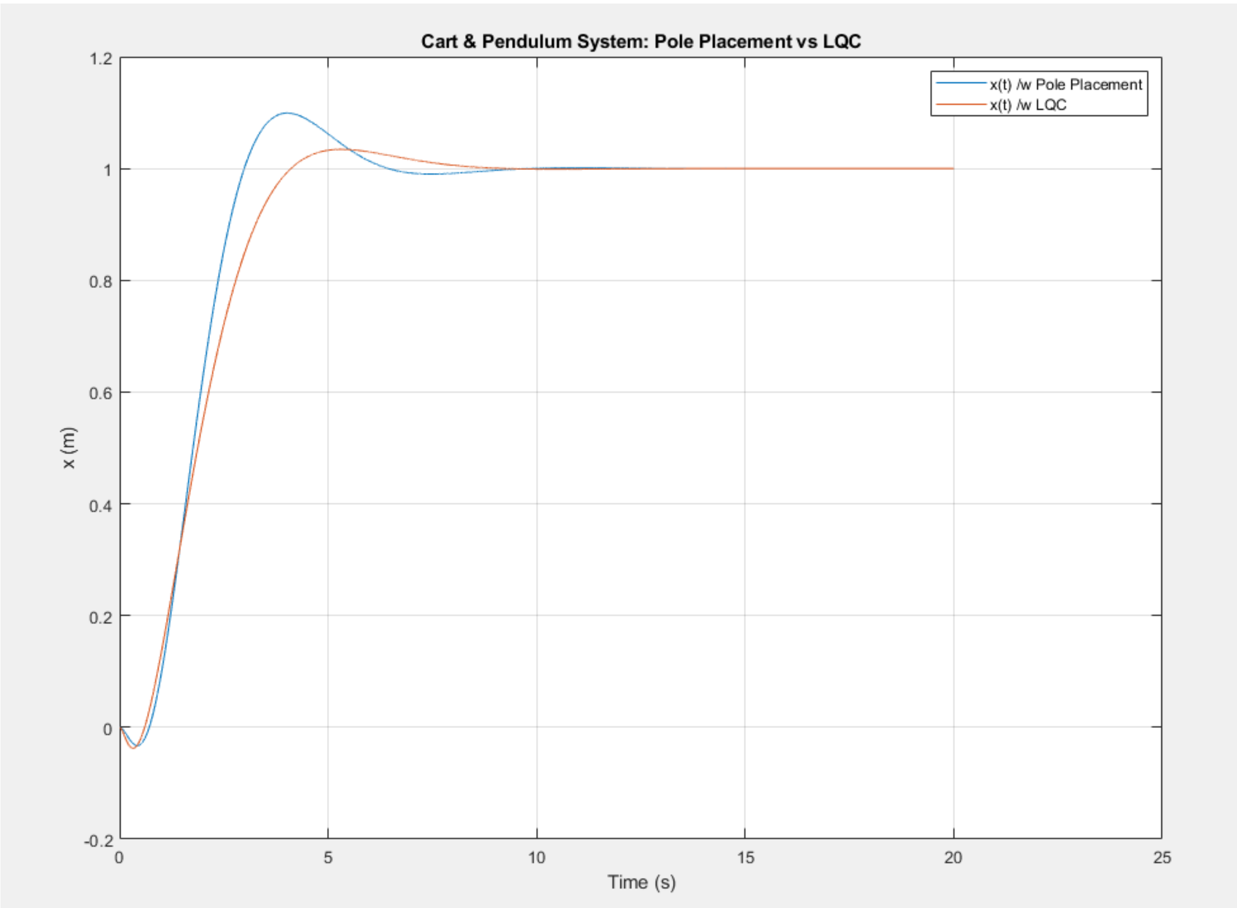
LQC: $[-0.6734 \pm j\, 2.1509, -2.2748, -1.7808]$

(b) Kinda similar.

(c) Pole placement

Problem 6 – LQG Control – Cart & Pendulum

```
function [A,B] = linearizedCartPend(mcart, mball, L)
g = 9.8;
A = [0 0 1 0; 0 0 0 1; 0 (-mball*g/mcart) 0 0; 0 ((mcart+mball)*g/(mcart*L))
0 0];
B = [0;0;1/mcart;-1/(mcart*L)];
end
```



Problem 6 – LQG Control – Cart & Pendulum

```
function [A,B] = linearizedCartPend(mcart, mball, L)
g = 9.8;
A = [0 0 1 0; 0 0 0 1; 0 (-mball*g/mcart) 0 0; 0 ((mcart+mball)*g/(mcart*L))
0 0];
B = [0;0;1/mcart;-1/(mcart*L)];
end
```