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An invariant-based theory of composites

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ABSTRACT

A novel invariant-based approach to describe elastic properties and failure of composite plies and laminates is proposed. The approach is based on the trace of the plane stress stiffness matrix as a material property, which can be used to reduce the number of tests and simplify the design of laminates. Omni strain failure envelopes are proposed as the minimum inner failure envelope in strain space, which defines the failure of a given composite material for all ply orientations. The proposed approach is demonstrated using various carbon/epoxy composites and offers radically new scaling to improve design and manufacturing.

1. Introduction

Carbon fiber reinforced composites have been increasingly used in the aircraft industry due to their unique combination of properties, which includes high strength and modulus to weight ratio and high corrosion resistance. Besides the well-known weight savings, with direct implications in fuel efficiency and emissions, other important benefits include reduced maintenance and higher passenger comfort level. The highest performance commercially available composite materials are the multi-directionally reinforced laminates of carbon fibers with polymer matrix.

The inherent anisotropy and complicated failure mechanisms of composite materials, which are fundamental to the design flexibility and to their superior properties, make the mechanical characterization more complex and time consuming. For unidirectional plies, there are four independent stiffness parameters to be measured; *i.e.*, longitudinal, transverse and shear moduli and Poisson's ratio; and five strengths; *i.e.*, longitudinal and transverse tensile and compressive, and shear. These parameters have been treated as separate entities, as reflected by non-interacting failure criteria such as maximum strain, Hashin [1], and Puck and Schurmann [2].

An experimental program to generate the mechanical properties and design allowable of composite materials for aircraft structures may cost millions of dollars and years of work [3]. Therefore, approaches leading to the characterization of composite materials using simulations and a reduced number of tests have always been a topic of great interest [4]. In this previous investigation, the local stress distribution was used to predict shear strength and stiffness as a function of the material properties of the constituents and their geometry. Later collaborative efforts such as the World Wide Failure Exercise [5] have demonstrated that failure criteria capable of predicting strength of composite laminates under biaxial loading conditions are still a subject of great debate.

A conservative failure envelope that is independent of fiber orientation has been proposed in the literature [6]. Two equations are presented for the failure envelopes: a second-order and fourth-order equation with respect to the strains. The equations are functions of two strain invariants and are independent of stacking sequence. The inner envelope to be used as the conservative failure envelope is represented by one of the two equations, depending upon the material properties.

While the concept of invariants to describe mechanical properties of anisotropic materials has been known for a long time, there has been recent renewed interest in the topic. In 1967, Tsai and Pagano presented stiffness transformation equations for ply rotation in a laminate as a function of invariants [7]. The concept of invariance was proven useful in the design of laminates because the invariants are not affected by ply orientation. Years later, a method for the measurement of invariants that describe the elastic response of anisotropic plates in bending was investigated by other authors [8].

In this work, a novel invariant-based approach to describe elastic properties and failure of composite plies and laminates is

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proposed. The trace of the plane stress stiffness matrix is evaluated as a material property. In addition, "omni strain" failure envelopes are proposed as the minimum inner envelope in strain space, which defines the first-ply-failure (FPF) of a given composite material for all ply orientations. Ultimately, it is demonstrated that invariants would be the most salient properties for carbon composites that can be powerful scaling for testing, design and understanding of composites in general.

2. Background

The on-axis stress–strain relations for a unidirectional tape in terms of stiffness and compliance are given in Eq. (1) using engineering (contracted) notation.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} & 0 \\ Q_{yx} & Q_{yy} & 0 \\ 0 & 0 & Q_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix} \text{ and } \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_s \end{Bmatrix} = \begin{bmatrix} S_{xx} & S_{xy} & 0 \\ S_{yx} & S_{yy} & 0 \\ 0 & 0 & S_{ss} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} \tag{1}$$

where σ_i and ε_i are the stress and strain components, respectively, and $[Q]$ and $[S]$ are the plane stress stiffness and compliance, respectively. If tensorial notation is used, instead of engineering notation, the stress–strain relations are given as shown in

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} S_{1111} & S_{1122} & 0 \\ S_{2211} & S_{2222} & 0 \\ 0 & 0 & 2S_{1212} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \tag{2}$$

or, back to the contracted notation,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{\varepsilon_s}{2} \end{Bmatrix} = \begin{bmatrix} S_{xx} & S_{xy} & 0 \\ S_{yx} & S_{yy} & 0 \\ 0 & 0 & \frac{S_{ss}}{2} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} \text{ and } \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_s \end{Bmatrix} = \begin{bmatrix} Q_{xx} & Q_{xy} & 0 \\ Q_{yx} & Q_{yy} & 0 \\ 0 & 0 & 2Q_{ss} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{\varepsilon_s}{2} \end{Bmatrix} \tag{3}$$

While stiffness and compliance are tensors, engineering constants are not. As with all tensors, there are invariant scalar sums and products associated with coordinate transformation. Trace is an invariant of stiffness and compliance tensors associated with coordinate transformation. For laminates, the in-plane and flexural stiffness components can be normalized according to

$$[A^*] = \frac{[A]}{h} \quad \text{and} \quad [D^*] = \frac{12}{h^3}[D] \tag{4}$$

where $[A]$ and $[D]$ are the in-plane and flexural laminate stiffness.

Then, the traces of the in-plane and flexural normalized stiffness are constraint by the same value of the same invariant as shown in

$$Tr[Q] = Tr[A^*] = Tr[D^*] \tag{5}$$

While the terms of $[A^*]$ in Eq. (5) are not dependent on stacking sequence, those of $[D^*]$ are. Although their components vary in value, the traces of both have the same value, thus invariant to stacking sequence. Trace represents the total and upper bound of the stiffness property of a material in an explicit form as defined by mathematics. It is independent of the loading condition (in-plane versus flexural), stacking sequence (mid-plane symmetric versus asymmetric), and material symmetry: isotropic, orthotropic and anisotropic. These invariants have practical values that are discussed in this paper. First, for a given material, its value of trace is a material property as much as its Young's and shear moduli. Since the traces are linear combinations of three terms, if any two are determined, the third is simply what is left in the trace. Thus, testing for data can be simplified by having fewer properties to be tested, and also simpler tests can be selected over more difficult tests. An example is the shear test, which is more difficult to perform than uniaxial tensile test. Thus shear tests can be avoided all together.

3. Master ply for carbon/polymer composites

Carbon fiber/epoxy and thermoplastic composite materials were found to share common stiffness properties if they are normalized by their respective trace of the stiffness matrix, $Tr[Q]$. In Table 1, there are 10 different carbon fiber composites and their trace-normalized stiffness factors are very close, particularly in the longitudinal stiffness, parallel to the fiber. In fact, their coefficient of variation is 1.5%. The median values of these factors define a "master ply" used to gain understanding of the laminate behavior of these composites.

Longitudinal properties are known to be fiber dominated while transverse and shear properties are matrix dominated. Although the contributions of the matrix dominated stiffness moduli are small in carbon/epoxy composites, their variation is large because different matrices and curing processes are used. However, even with large dispersion, their effect on laminate properties is small, as it will be shown later in Table 3. If the uncertainty of the results is expressed as standard deviation, it is much smaller for the matrix dominated parameters (Table 1). Thus, errors introduced by them in terms of trace are very small because their absolute values are small. Laminates normally include angles other than 0 and 90 deg and, for these angles, the trace normalized plane stress stiffness component Q^*_{11} is bounded by the values at 0 and 90 deg. For intermediate angles such as 45 deg, the standard deviation proved to be even smaller than that of 90 deg (Table 1), which indicates that the trace-based theory improves accuracy for multidirectional laminates, as compared to unidirectional plies.

The stiffness along the fiber is responsible for about 88% of the trace for the unidirectional carbon/epoxy composites presented in Table 1. Thus, the matrix related properties are responsible for 12%. While their coefficients of variation are high (up to 16.4% in Table 1) they are related to the particular matrix, interface and processing conditions. Their contribution to the transverse stiffness is complex and can vary significantly although their relative importance to trace is a few percentage points. Shear modulus suffers from the same matrix and processing variations, but also the difficulty in its experimental measurement and nonlinear stress–strain curves. However, when compared with most test coupons, the presence of plies in the 0 deg direction in the laminate makes stress–strain curves linear to failure, with no observable nonlinearity. Thus, uncertainty from matrix related property is masked by the dominant fiber property, which has a small variation of 1.5%. Hence, the matrix contribution and its wider variation are of small consequence when ply stiffness is converted to laminate stiffness.

In order to measure the trace for each material, it is better to measure the longitudinal stiffness components as opposed to measuring the transverse and shear moduli, which have less than 10% of the longitudinal moduli and therefore with less certainty.

Available data on the longitudinal stiffness of various E-glass/thermoplastic, carbon/thermoplastic and carbon/epoxy are plotted normalized by the trace in Fig. 1. The materials considered were: (a) E-glass/thermoplastic: E-glass/PA6, E-glass/PA46, E-glass/PPS, E-glass/PEI, and E-glass/PEEK; (b) carbon/thermoplastic: T700/PPS, IM7/PPS, AS4/PPS, CF SM/PPS, AS4/PEKK, CF SM/PEKK, CF SM/PEEK, IM7/PEEK Cytec, and IM7/PEEK TenCate; (c) carbon/epoxy: IM7/977-3, T800/Cytec, T700 C-Ply 55, T700 C-Ply 64, AS4/3501, IM6/epoxy, AS4/F937, T300/N5208, IM7/977-3, IM7/8552, and IM7/MTM45. It can be observed that, while a very good agreement is observed for carbon/epoxy data, the coefficient of variation increases for carbon/thermoplastics and, for E-glass/thermoplastics, the agreement is not as good. This can be explained by the role played by the matrix in these composites.

Table 1
Trace normalized plane stress stiffness components and engineering constants.

Material	E_x (GPa)	E_y (GPa)	ν_x	E_s (GPa)	$Q_{xx}{}^*$	$Q_{yy}{}^*$	$Q_{xy}{}^*$	$Q_{ss}{}^*$	*Tr* (GPa)	Q^*_{11} (45)
IM7/977-3	191	9.94	0.35	7.79	0.8825	0.0459	0.0161	0.0358	217.8	0.2759
T800/Cytec	162	9.0	0.4	5.0	0.8955	0.0497	0.0199	0.0274	182.5	0.2736
T700 C-Ply 55	121	8.0	0.3	4.7	0.8746	0.0578	0.0173	0.0338	139.2	0.2756
T700 C-Ply 64	141	9.3	0.3	5.8	0.8713	0.0575	0.0172	0.0356	162.8	0.2764
AS4/3501	138	8.96	0.3	7.1	0.8567	0.0556	0.0167	0.0438	162	0.2803
IM6/epoxy	203	11.2	0.32	8.4	0.8791	0.0485	0.0155	0.0362	232.2	0.2758
AS4/F937	148	9.65	0.3	4.55	0.8878	0.0579	0.0174	0.0271	167.7	0.2723
T300/N5208	181	10.3	0.28	7.17	0.8805	0.0501	0.0140	0.0347	206.5	0.2744
IM7/8552	171	9.08	0.32	5.29	0.8972	0.0476	0.0152	0.0276	191.6	0.2714
IM7/MTM45	175	8.2	0.33	5.5	0.9014	0.0422	0.0139	0.0282	195.1	0.2711
Std dev	25.9	1.0	0.034	1.4	0.013	0.0056	0.0018	0.0054	28.5	0.0028
Coeff var %	15.9	10.2	10.7	22.4	1.5	10.9	10.9	16.4	15.4	1.0
Master ply					0.8815	0.0499	0.0164	0.0342	185.7	

Note: Each trace-normalized parameter is defined as the respective parameter divided by the trace of the plane stress stiffness matrix.

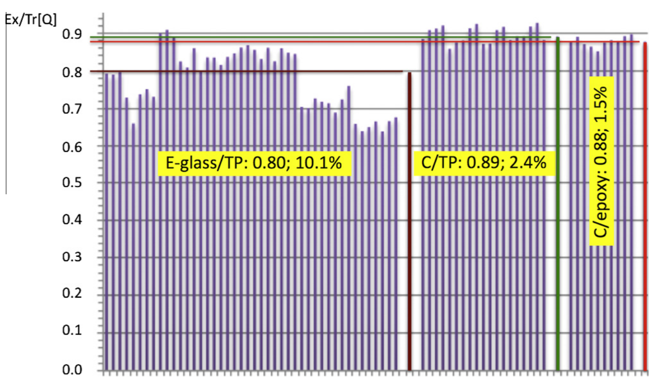


Fig. 1. Trace-normalized longitudinal stiffness of various E-glass/thermoplastic, carbon/thermoplastic and carbon/epoxy composites. Median and coefficient of variation also shown for each group.

For E-glass/polymer composites, the dominance of the fiber on the trace of a ply is less than that of carbon composites. Also the glass fiber volume fractions are between 27% and 55%. They are lower than carbon composites, which are usually higher than 55%. Thus, matrix property for glass composites will have a larger effect than in carbon composites.

The difference observed in the trace-normalized longitudinal stiffness of unidirectional composites between those with epoxy versus thermoplastic may be traced to the higher processing temperature of the latter. The epoxies used in the plies considered in this study were more uniform in properties than those used in thermoplastics, which included PEEK, PEKK, and PBS.

4. Composite laminates

Classical laminated theory can be used to generate trace-normalized factors for stiffness components of multi-directional laminates. The trace-normalized plane stress stiffness components of the master ply $Q_{xx}{}^*$, $Q_{yy}{}^*$, $Q_{xy}{}^*$ and $Q_{ss}{}^*$, can used to calculate the stiffness matrix of a given laminate [*A*], if the stacking sequence is specified. Then, the laminate compliance matrix [*a*] can be determined taking the inverse of the stiffness matrix. From the compliance matrix, the trace-normalized engineering constants for the laminate composed of master plies ($E_1{}^o$, $E_2{}^o$ and $E_6{}^o$) can be calculated.

If the longitudinal modulus of a carbon fiber composite laminate ($E_1{}^o$) is measured, the trace can be determined according to Eq. (6) assuming the master ply represents the material.

$$Tr = \frac{E_1^o(laminate)}{E_1^o(trace\ of\ normalized\ master\ ply)} \quad (6)$$

Then, based on the plane stress stiffness components of the master ply $Q_{xx}{}^*$, $Q_{yy}{}^*$, $Q_{xy}{}^*$ and $Q_{ss}{}^*$, the plane stress stiffness components of the composite ply (Q_{xx}, Q_{yy}, Q_{xy} and Q_{ss}) are determined according to

$$Q_{ij} = Q_{ij}^* \cdot Tr \quad (7)$$

Taking the inverse of the plane stress stiffness matrix [*Q*], the compliance matrix is calculated and therefore, the engineering constants of the carbon composite ply (E_x, E_y, ν_x and E_s) are determined.

In principle, other invariants could be used for the determination of the normalized stiffness properties. However, trace is more fiber controlled, while other invariants such as ($Q_{xy} - Q_{ss}$) are more matrix controlled. Being so small in value, they have small influence on test data from laminates. In strength, matrix dependent invariants are expected to play a more prominent role.

Looking at the governing equations of plane elasticity and bending of laminated plates [9] (Eqs. (8) and (9)), the coefficients of these equations are components of compliance and stiffness of the laminate.

$$a_{22}\frac{\partial^4 F}{\partial x^4} - 2a_{26}\frac{\partial^4 F}{\partial x^3 \partial y} + (2a_{12} + a_{66})\frac{\partial^4 F}{\partial x^2 \partial y^2} - 2a_{16}\frac{\partial^4 F}{\partial x \partial y^3} + a_{11}\frac{\partial^4 F}{\partial y^4} = 0 \quad (8)$$

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 4D_{26}\frac{\partial^4 w}{\partial x \partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} = 0 \quad (9)$$

where $F(x, y)$ is a stress function and $w(x, y)$ is the deflection function of the mid-plane.

In particular, for orthotropic plates, in-plane shear-extension coupling coefficients a_{26} and a_{16} and bend–twist coupling coefficients D_{16} and D_{26} are all zero. If the laminate is composed by a large number of repeated sub-laminates, it becomes homogenized [3]. Some of the benefits of homogenized laminates include increased toughness and strength and simpler manufacturing and optimization. The benefits of increased ply dispersion to the mechanical properties such as toughness and strength of laminates with and without hole have been demonstrated under static, fatigue and impact loadings [10]. The normalized flexural stiffness [D^*] approaches that of the in-plane [A^*] [3]. Thus, for a homogenized orthotropic laminate, Eqs. (8) and (9) can be simplified to:

$$a_{22}^*\frac{\partial^4 F}{\partial x^4} + (2a_{12}^* + a_{66}^*)\frac{\partial^4 F}{\partial x^2 \partial y^2} + a_{11}^*\frac{\partial^4 F}{\partial y^4} = 0 \quad (10)$$

$$A_{11}^*\frac{\partial^4 w}{\partial x^4} + 2(A_{12}^* + 2A_{66}^*)\frac{\partial^4 w}{\partial x^2 \partial y^2} + A_{22}^*\frac{\partial^4 w}{\partial y^4} = 0 \quad (11)$$

When the coefficients of Eqs. (10) and (11) are normalized by the trace, the results obtained for various [0/±45] carbon fiber

Table 2
Coefficients of the governing equation of plane elasticity of [0/±45]s laminated plates of various carbon/thermoset composites.

Material	$\frac{2a^*_{12}+a^*_{66}}{a^*_{22}}$	$\frac{a^*_{11}}{a^*_{22}}$	$\frac{2(A^*_{12}+2A^*_{66})}{A^*_{11}}$	$\frac{A^*_{22}}{A^*_{11}}$	*Tr* (GPa)
IM7/977-3	0.38	0.42	1.94	0.42	217.8
T800/Cytec	0.31	0.41	1.97	0.41	182.5
T700 C-Ply 55	0.38	0.43	1.96	0.43	139.2
T700 C-Ply 64	0.40	0.43	1.96	0.43	162.8
AS4/3501	0.47	0.43	1.93	0.43	162.0
IM6/epoxy	0.39	0.42	1.94	0.42	232.2
AS4/F937	0.32	0.42	1.98	0.42	167.7
T300/N5208	0.38	0.42	1.95	0.42	206.5
IM7/8552	0.31	0.41	1.97	0.41	191.6
IM7/MTM45	0.31	0.40	1.96	0.40	195.1
Std dev	0.052	0.010	0.016	0.010	
Coeff var %	14.3	2.4	0.8	2.4	
Master ply	0.37	0.42	1.95	0.42	

laminates are presented in Table 2. The highest coefficient of variation is observed for the matrix dominated properties ratio shown in the first column due to difference in matrices and curing processes, as previously discussed.

Fig. 2 shows many exact solutions of plane elasticity that can be found in Lehknitskii's *Anisotropic Plates* [8]. The most important parameters shown in Fig. 2 are n and k. They control the stress distribution in all of the solutions listed. In fact, the maximum stress around a circular hole, open or filled, can be solved with combinations of n and k that control maximum stress at the periphery of the opening. Table 3 shows these combinations using normalized properties for various carbon fiber composites for a given laminate of [0/±45]. It can be observed that the trace normalized parameters are insensitive to the particular composite material within a maximum error of 1.4% (the pressure case). For other laminates, the coefficients of variation vary, but are no worse than the commonly encountered experimental error.

Thus, the solution of the governing equation using the master ply for carbon fiber composites for each laminate should be more accurate that what can be expected from experimental data. Even more important are common examples of the stress around an open hole (by plane elasticity), and bending, buckling and vibration of composite laminates can be described by two separate parameters: (1) the geometric factors reflected by each laminate, and (2) the material property as represented by trace.

5. Failure criterion

The invariant concept can also be applied to a failure criterion based on a tensor polynomial. The failure envelope can be defined as a material property independent of laminate layup composition. Maximum strain, Hashin and nearly all other failure criteria are not tensor-based, have no established transformation relation, thus cannot establish a strain envelope with ply independence.

The use of tensor polynomial is preferred for failure criteria for several reasons [3]: transformation relations for tensor polynomial failure criteria are known naturally; mathematical manipulations are possible with well-behaved, single-value functions that piece-wise failure criteria cannot do.

For highly anisotropic materials such as carbon composites, the use of strain space for the representation of failure envelopes is preferred over stress space because failure envelopes in strain space are invariant; *i.e.*, their shapes remain the same independent of the presence of other plies. Thus, simply superimposing the envelopes of the constituent plies forms the failure envelope for a given laminate [11].

The formulation of this tensor polynomial failure criterion is often referred to as Tsai-Wu [3].

$$F_{ij}\sigma_i\sigma_j + F_i\sigma_i = 1 \tag{12}$$

$$G_{ij}\varepsilon_i\varepsilon_j + G_i\varepsilon_i = 1 \tag{13}$$

where $[F]$ and $\{F\}$ are strength parameters in stress space and $[G]$ and $\{G\}$ are strength parameters in strain space. $[F]$ and $\{F\}$ are based on plane stress, and $[G]$ and $\{G\}$ are the corresponding strain from plane stress, not plane strain. There is a unique relation between them [3], as shown in Eqs. (14) and (15)

$$G_{ij} = F_{kl}Q_{ki}Q_{lj} \tag{14}$$

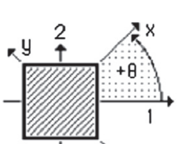
$$G_j = F_jQ_{ij} \tag{15}$$

The components of $[F]$ and $\{F\}$ are determined from strength data according to Eq. (16)

$$\begin{aligned} &F_{xx} = \frac{1}{XX'} \quad F_x = \frac{1}{X} - \frac{1}{X'} \\ &F_{yy} = \frac{1}{YY'} \quad F_y = \frac{1}{Y} - \frac{1}{Y'} \\ &F_{ss} = \frac{1}{S^2} \end{aligned} \tag{16}$$

where X, X' are longitudinal tensile and compressive strength; Y, Y', transverse tensile and compressive strength; and S, shear strength. Five experiments are needed for the measurement of these strength parameters.

The transformation of $[G]$ and $\{G\}$ are shown in



$$\begin{Bmatrix} G_{11} \\ G_{22} \\ G_{12} \\ G_{66} \\ G_{16} \\ G_{26} \end{Bmatrix} = \begin{bmatrix} U_1 & U_2 & U_3 & 0 & 0 \\ U_1 & -U_2 & U_3 & 0 & 0 \\ U_4 & 0 & -U_3 & 0 & 0 \\ U_5 & 0 & -U_3 & 0 & 0 \\ 0 & 0 & 0 & \frac{U_2}{2} & U_3 \\ 0 & 0 & 0 & \frac{U_2}{2} & -U_3 \end{bmatrix} \begin{Bmatrix} 1 \\ cos2\theta \\ cos4\theta \\ sin2\theta \\ sin4\theta \end{Bmatrix} \tag{17}$$

$$\begin{Bmatrix} G_1 \\ G_2 \\ G_6 \end{Bmatrix} = \begin{bmatrix} U_6 & U_7 & 0 \\ U_6 & -U_7 & 0 \\ 0 & 0 & U_7 \end{bmatrix} \begin{Bmatrix} 1 \\ cos2\theta \\ sin2\theta \end{Bmatrix}$$

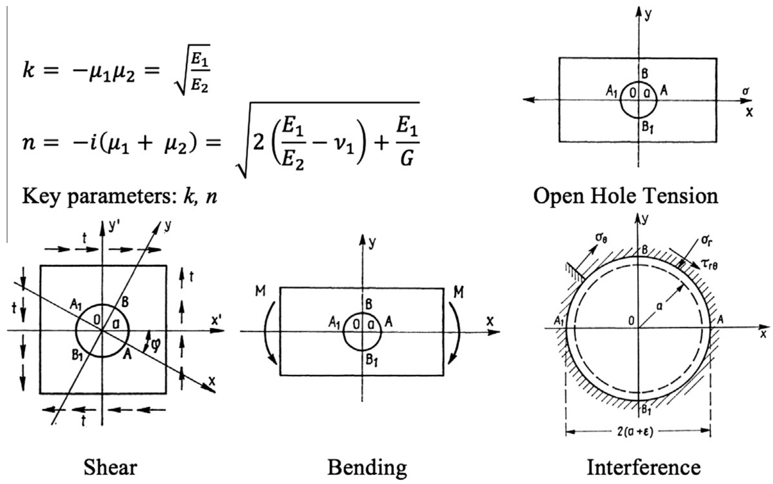


Fig. 2. Exact solutions of plane elasticity [8].

Table 3
Combinations of n and k for various exact solutions of plane elasticity for a [0/±45]s laminate.

Parameters Material	Open hole tension $n + 1$	Pressure at 0 deg $(n - 1)/k$	Shear $(1 + k + n)n$	Bending $2k + n$	Interference $k(1 + n) - \nu_1$	Tr (GPa)
IM7/977-3	3.39	0.90	11.81	5.49	4.54	217.8
T800/Cytec	3.36	0.88	11.61	5.47	4.46	182.5
T700 C-Ply 55	3.36	0.89	11.55	5.42	4.43	139.2
T700 C-Ply 64	3.37	0.89	11.57	5.42	4.44	162.8
AS4/3501	3.38	0.91	11.69	5.42	4.48	162.0
IM6/epoxy	3.39	0.90	11.77	5.48	4.53	232.2
AS4/F937	3.35	0.87	11.48	5.43	4.41	167.7
T300/N5208	3.38	0.90	11.75	5.47	4.52	206.5
IM7/8552	3.37	0.88	11.71	5.50	4.51	191.6
IM7/MTM45	3.39	0.88	11.84	5.53	4.57	195.1
Std dev	0.014	0.012	0.120	0.039	0.053	
Coeff var %	0.4	1.4	1.0	0.7	1.2	
Master ply	3.38	0.89	11.72	5.47	4.50	

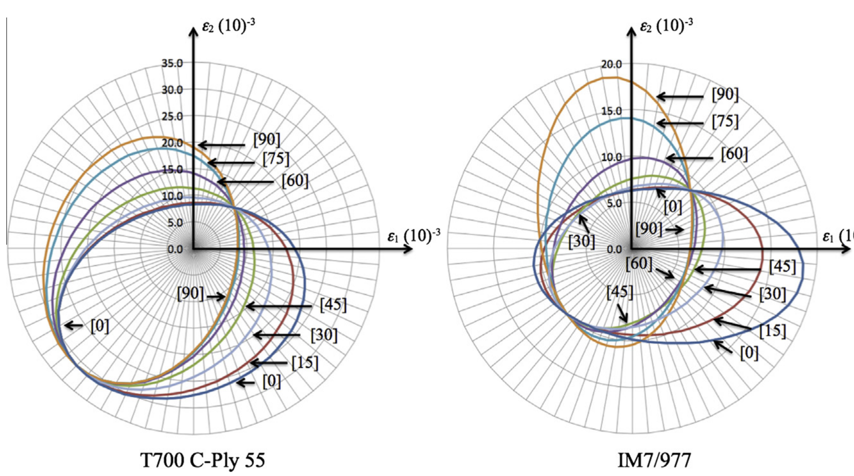


Fig. 3. Failure envelopes in strain space for T700 C-Ply 55 and IM7-977 [π/12] laminates.

where U's, the linear combination of components of [G] and {G}, are defined in

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{3}{8} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{4} & -\frac{1}{2} \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{4} & -\frac{1}{2} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{Bmatrix} G_{xx} \\ G_{yy} \\ G_{xy} \\ G_{ss} \end{Bmatrix} \text{ and } \begin{Bmatrix} U_6 \\ U_7 \end{Bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} G_x \\ G_y \end{Bmatrix} \tag{18}$$

The quadratic failure criterion for a particular ply orientation is defined according to

$$G_{11}\varepsilon_1^2 + G_{22}\varepsilon_2^2 + 2G_{12}\varepsilon_1\varepsilon_2 + 2G_{16}\varepsilon_1\varepsilon_6 + 2G_{26}\varepsilon_2\varepsilon_6 + G_{66}\varepsilon_6^2 + G_1\varepsilon_1 + G_2\varepsilon_2 + G_6\varepsilon_6 = 1 \tag{19}$$

The failure envelope for a laminate with many ply orientations will be limited by the controlling ply in each orientation of the strain space, as shown in Fig. 3 for two carbon fiber reinforced

Table 4
Mechanical properties for IM7/977, T700 C-Ply 55 and T800/Cytec.

Material	E_x	E_y	ν_x	E_s	X	X'	Y	Y'	S
IM7/977-3	191	9.94	0.35	7.79	3.25	1.60	0.062	0.098	0.075
T700 C-Ply 55	121	8.0	0.30	4.7	2.53	1.70	0.066	0.022	0.093
T800/Cytec	162	9.0	0.40	5.0	3.77	1.66	0.056	0.150	0.098

All properties in (GPa).

composites: IM7/977 and T700 C-Ply 55. The properties used are given in Table 4. The controlling ply varies with the applied strain and the composite material. Plies transverse to the applied strain are not necessarily the controlling ply, as shown in Fig. 3.

The inner envelope can be determined by finding the controlling ply that would fail first for unit loading strain vectors from 0 to 2π. This envelope, herein named omni strain failure envelope, is independent of layup composition since it covers all controlling plies from 0 to 2π. Thus, the omni strain failure envelope is an invariant that can represent any laminate constructed with this material.

Thus, the omni strain envelope for a given composite material can be determined assuming that the four independent elastic constants (E_x, E_y, E_s, and ν_x) and five strengths (X, X', Y, Y', and S) that characterize a ply are known. Omni strain envelopes for two carbon composites are plotted in Fig. 4. All these envelopes are based on an interaction term $F_{xy}{}^* = -1/2$. It is straightforward to go to the other limit when this interaction term is zero. The envelope represents the controlling ply that would fail first for unit loading strain vectors from 0 to 2π.

In Fig. 5 the omni strain failure envelope for T800/epoxy is shown. The properties for this material are given in Table 4. A failure envelope expanded 1.5 times is also shown for comparison with strains from uniaxial tensile loads applied to various laminates. Thus, for each laminate it is possible to determine how much the applied load can be increased or the laminate thickness reduced before failure occurs, as predicted by the omni strain envelope. It can be seen that many laminates can carry more load beyond first ply failure. From Fig. 5, it can be verified not only that the omni envelope is safe for all laminates, but also all laminate data can be displayed on one graph in strain space, a very valuable and concise display of the strength of a given composite material. Laminates with various Poisson's ratios were selected for validation and Poisson's ratios ranging from 0 to 1.5 covered 56 deg in the 4th quadrant. Compression test data of the same test coupons will cover the same 56 deg in the 2nd quadrant.

Thus, the omni strain failure envelope for a given material is invariant since it is independent of laminate layup composition or ply orientation. It is a material property that defines the minimum first-ply failure (FPF) failure envelope for all ply orientations. Thus far, the extension of the omni strain failure envelope to the ultimate envelope has not yet been developed, since modeling of post-FPF behavior by matrix degradation involves assumptions, which cannot be easily supported rationally. In spite of that, the omni strain will represent a safe design regardless of laminate layup composition. The actual strain levels proposed are still

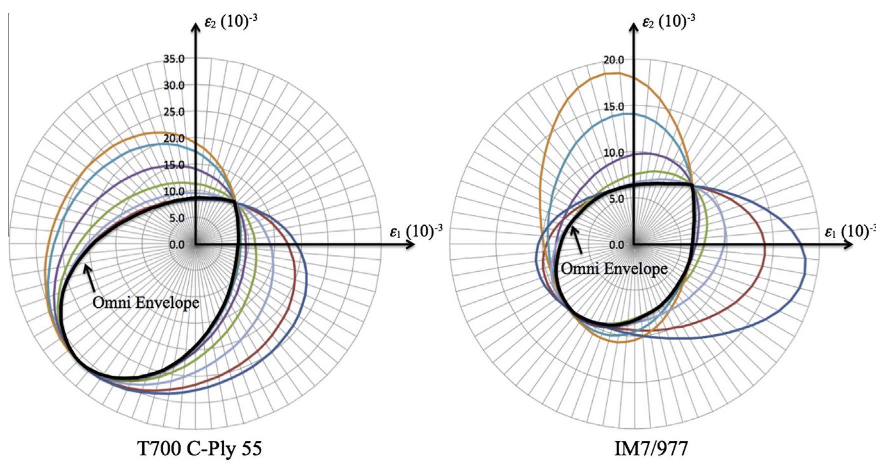


Fig. 4. Omni strain failure envelopes for T700 C-Ply 55 and IM7-977.

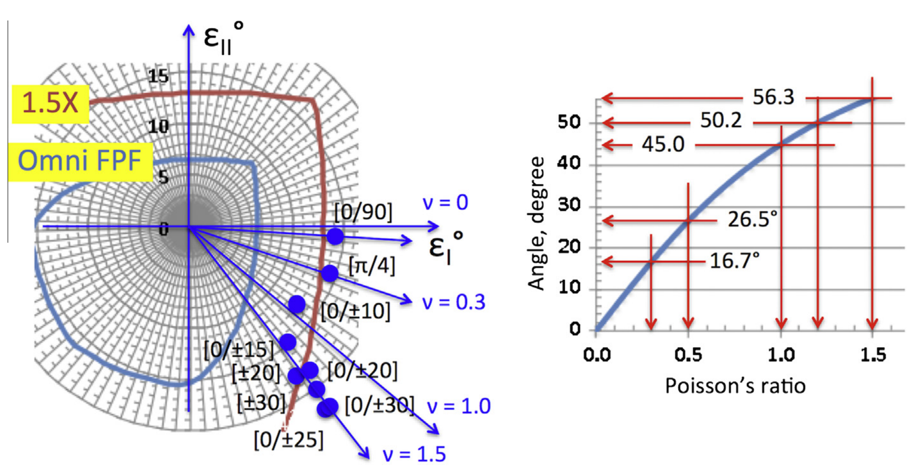


Fig. 5. Omni strain failure envelope for T800-Cytec and strains from uniaxial tensile loads applied to various laminates.

higher than the current practice by aerospace industry of a fixed 0.4% strain for all carbon fiber epoxy matrix composites.

The authors believe that the omni strain envelopes are practical for laminate design for a number of reasons:

(1) They are invariant material properties covering all ply orientations from 0 to 2π. Thus, data from all different laminates can be shown in a single envelope.
(2) Laminate selection can be based on the preferred stiffness, which will also allow the search for layups that will result in higher productivity. The omni strain envelopes cover all of them.
(3) The omni strain envelope is for principal strain. Thus the actual strain must be rotated to a principal strain axes for failure determination for all laminates. This approach should bring designing composites to the same comfort level as that with metals.

6. Conclusions

An invariant-based approach to describe elastic properties and failure of composite plies and laminates has been proposed. It was demonstrated that the trace-normalized stiffness parameters for unidirectional carbon fiber composites are very close, particularly in the longitudinal stiffness, parallel to the fibers. Thus, the median values of these factors were used to define a "master ply", which was found to properly describe the behavior of these composites. With the "master ply" concept, testing for data can be simplified by having fewer properties to be tested, and also simpler tests can be selected over more difficult tests. Testing multidirectional laminates such as $[\pi/4]$ is also possible to generate materials properties. These tests offer advantages over unidirectional coupons since many processing variables that include lamination (by hand, automated tape laying or fiber placement), curing (by autoclave, vacuum bagging, or RTM), residual stress, in situ properties, defects such as resin rich areas, voids, ply misalignment and waviness, and uncertainties of constituent properties are all built in. Laminate data obtained will have all these imperfections and uncertainties included and are closer to what a structure will be when it is built. Thus, simple uniaxial tensile tests in laminates can produce all necessary stiffness data for certification while the more difficult shear tests including their nonlinearity can be avoided.

An omni strain envelope was also proposed as an invariant failure criterion. This envelope describes the failure of a laminate by considering infinite number of fiber orientations. These omni strain failure envelopes are invariant since they are independent of laminate layup composition. Thus, they are essentially a material property.

In summary, the proposed use of invariants puts all the stiffness and strength measurements in a consistent and easy-to-implement framework. Not only testing and design allowable generation can be improved in their accuracy and speed, but also the design and use of composites can be done with higher quality and confidence. The performance of all materials can be revealed through their invariants. Therefore, the proposed approach is intended to make these invariants useful and practical for design, testing and certification.

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