Chapter2

SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

Linear differential equations are obtained in modeling various applications such as electrical circuits, vibration systems mixture of substances etc. Consider for example a mass and spring system in the Figure 2.1.

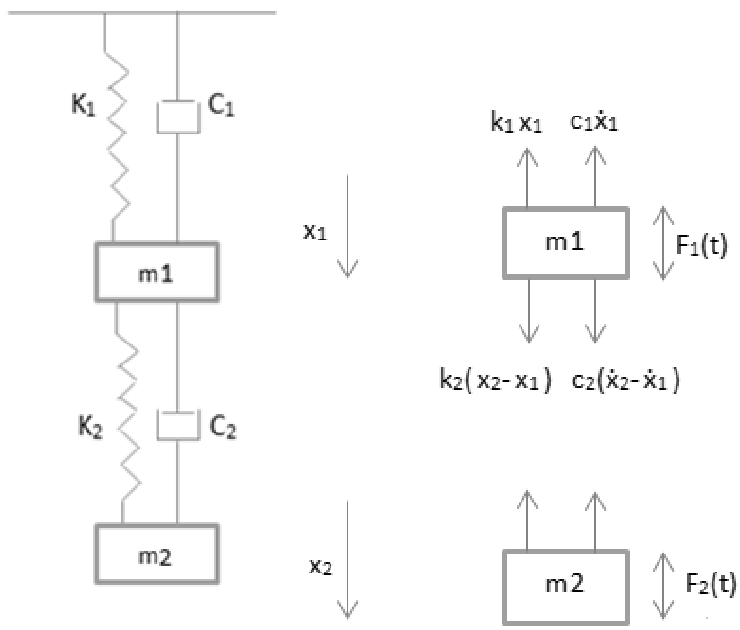

Figure 2.1 Modelling a spring and mass system.

Newton's 2nd law states that; $\sum \mathrm{F} = \mathrm{m.a}$ which implies for two bodies respectively

$$m_1\ddot{x}_1 = k_2(x_2 - x_1) + C_2(\dot{x}_2 - \dot{x}_1) - k_1x_1 - C_1x_1 + F_1(t),$$

and

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - C_2(\dot{x}_2 - \dot{x}_1) + F_2(t).$$

Manipulations yield that;

$$m_1\ddot{x}_1 + (C_1 + C_2)\dot{x}_1 - C_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1(t)$$

$$m_2\ddot{x}_2 - C_2\dot{x}_1 + C_2\dot{x}_2 - k_2x_1 + k_2x_2 = F_2(t)$$

This is a system of two second order linear differential equations. There have been many other physical applications that produce similar systems of linear equations.

There are various solution techniques for such linear systems of differential equations. One approach is to decouple the system, and obtain a number of differential equations with only one dependent variable. This approach inevitably increases the order of differential equations. Another approach is to decrease the order of differential equations to first order which increases the number of dependent variables, and equations in the set. Some of these techniques will be explained in this section.

2.1 Decoupling Method

This is probably one of the most popular techniques that decouple the system of coupled equations into several higher order differential equations in each of the dependent variables. These higher order linear differential equations are then solved individually.

Example 2.1

Solve the system of the two differential equations

$$\frac{dy}{dt} + z = e^t;$$

$$y + \frac{dz}{dt} = e^{-t}$$

$$\frac{dy}{dt} + z = e^t \rightarrow \frac{dy}{dt} = -z + e^t \rightarrow \frac{d^2y}{dt^2} = -\frac{dz}{dt} + e^t$$

$$y + \frac{dz}{dt} = e^{-t} \rightarrow \frac{dz}{dt} = -y + e^{-t} \rightarrow \frac{d^2z}{dt^2} = -\frac{dy}{dt} - e^{-t}$$

then,

$$\frac{d^2y}{dt^2} = \mathrm{y} - e^{-t} + e^t \rightarrow \frac{d^2y}{dt^2} - \mathrm{y} = -e^{-t} + e^t$$
$$\frac{d^2z}{dt^2} = \mathrm{z} - e^t - e^{-t} \rightarrow \frac{d^2z}{dt^2} - \mathrm{z} = -e^t - e^{-t}$$

These two 2nd order linear differential equations are to be solved individually. Let's solve homogeneous equations first;

$$\begin{array}{lll} y'' - y = 0 & \text{Let} & Y_h = e^{mt} \quad Y_h'' = m^2 e^{mt} \quad (m^2 - 1)e^{mt} = 0 \\ z'' - z = 0 & & Z_h = e^{mt} \quad Z_h'' = m^2 e^{mt} \quad\quad m_{1,2} = \pm 1 \end{array}$$

$$Y_h = C_1 e^t + C_2 e^{-t}$$
$$Z_h = C_3 e^t + C_4 e^{-t}$$

The general solution of a system of the two first order linear differential equations contains exactly two arbitrary constants. The relations between the four arbitrary constants in the homogeneous solution are obtained from homogeneous form of the given set of equations

$$\frac{dy}{dt} + \mathrm{z} = 0;$$

$$y + \frac{dz}{dt} = 0$$

Substitution yields that

$$C_1 e^t - C_2 e^{-t} + C_3 e^t + C_4 e^{-t} = 0 \xrightarrow{yields} (C_1 + C_3)e^t + (-C_2 + C_4)e^{-t} = 0$$
$$C_1 e^t + C_2 e^{-t} + C_3 e^t - C_4 e^{-t} = 0 \xrightarrow{yields} (C_1 + C_3)e^t + (C_2 - C_4)e^{-t} = 0$$

$C_3 = -C_1 \& C_4 = C_2$.

So homogeneous solutions with two arbitrary constants is obtained;

$$y_h = C_1 e^t + C_2 e^{-t}$$
$$z_h = -C_1 e^t + C_2 e^{-t}$$

Particular solution can be proposed based on RHS of the equations as seen in Chapter 1.

$$y_p = t(A_1 e^t + A_2 e^{-t})$$

$$y_p' = A_1 e^t + A_2 e^{-t} + t(A_1 e^t - A_2 e^{-t})$$

$$y_p'' = A_1 e^t - A_2 e^{-t} + A_1 e^t - A_2 e^{-t} + t(A_1 e^t + A_2 e^{-t})$$

Substitution of these into $y'' - y = e^t - e^{-t}$ yields that

$$(A_1 + A_1 + A_1 t - A_1 t - 1)e^t + (-A_2 - A_2 + A_2 t - A_2 t + 1)e^{-t} = 0$$

$$A_1 = \frac{1}{2} \& A_2 = \frac{1}{2}$$

$y_p = \frac{1}{2} t(e^t + e^{-t})$ Since

$$y_{gen} = y_h + y_p = C_1 e^t + C_2 e^{-t} + \frac{1}{2} t(e^t + e^{-t})$$

Similarly, particular solution for *z* is proposed as

$$z_p = t(B_1 e^t + B_2 e^{-t})$$

$$z_p'' = B_1 e^t - B_2 e^{-t} + B_1 e^t - B_2 e^{-t} + t(B_1 e^t + B_2 e^{-t})$$

Substitution of these into $z'' - z = -e^t - e^{-t}$ yields that

$$(B_1 + B_1 + B_1 t - B_1 t + 1)e^t + (-B_2 - B_2 + B_2 t - B_2 t + 1)e^{-t} = 0$$

$B_1 = -\frac{1}{2} \& B_2 = \frac{1}{2}$ then $z_p = \frac{1}{2} t(e^{-t} - e^t)$

$$z_{gen} = -C_1 e^t + C_2 e^{-t} + \frac{1}{2} t(e^{-t} - e^t)$$

Another strategy to find z_{gen} is to use y_{gen} in

$$\frac{dy_{gen}}{dt} + z_{gen} = e^t.$$

$$z_{gen} = e^t - \frac{dy_{gen}}{dt} = -C_1 e^t + C_2 e^{-t} + \frac{1}{2} t(e^{-t} - e^t).$$

Example 2.2

Solve the system of the two first order linear differential equations

$$\frac{dx}{dt} + \mathrm{y} = 2$$
$$x + \frac{dy}{dt} = \cos t$$

$$\frac{dx}{dt} + \mathrm{y} = 2, \qquad x' = -\mathrm{y} + 2 \qquad x'' = -y', \qquad x'' - x = -\cos t$$

$$x + \frac{dy}{dt} = \cos t, \quad \mathrm{y}' = -\mathrm{x} + \cos t \quad y'' = -x' - \sin t, \qquad y'' - y = -2 - \sin t$$

Let's solve them individually.

Let's start with the two corresponding homogeneous equations;

$$\begin{array}{lll} x'' - x = 0 & \text{Let} & x_h = e^{mt} \Longrightarrow x_h'' = m^2 e^{mt} \quad \Longrightarrow \quad m_{1,2} \quad = \pm 1 \\ y'' - y = 0 & & y_h = e^{mt} \Longrightarrow y_h'' = m^2 e^{mt} \quad \Longrightarrow \quad m_{1,2} \quad = \pm 1 \end{array}$$

$$x_h = C_1 e^t + C_2 e^{-t}$$
$$y_h = C_3 e^t + C_4 e^{-t}$$

For the first second order equation

$$x'' - x = -\cos t$$

since the right side is {-cost} let a particular solution be;

$$x_p = A \cos t + B \sin t$$

Then

$$x_p' = -A \sin t + B \cos t$$

$$x_p'' = -A \cos t - B \sin t$$

Substitution of these into $\quad x'' - x = -\cos t \quad$ yields that

$$-A \cos t - B \sin t - A \cos t - B \sin t = -\cos t$$

$$-2A \cos t - 2B \sin t = -\cos t \xrightarrow{yields} \begin{array}{l} A = 1/2 \\ B = 0 \end{array} \xrightarrow{yields} x_p = \frac{1}{2} \cos t$$

Hence

$x_{gen} = x_h + x_p = C_1 e^t + C_2 e^{-t} + \frac{1}{2}\cos t.$

Consider the other equation; $\quad y'' - y = -\sin t - 2,$

let a particular solution be;

$y_p = C\cos t + D\sin t + E.$

Then

$y_p' = -C\sin t + D\cos t$

$y_p'' = -C\cos t - D\sin t$

Substitution of these into $\quad y'' - y = -\sin t - 2 \quad$ yields that

$-C\cos t - D\sin t - Ccost - D\sin t - E + \sin t + 2 = 0$

$-2C\cos t + (-2D + 1)\sin t + (-E + 2) = 0$

$C = 0, \qquad D = \frac{1}{2}\ \&\ \ E = 2 \xrightarrow{yields} y_p = \frac{1}{2}\sin t + 2$

Hence

$y_{gen} = y_h + y_p = C_3 e^t + C_4 e^{-t} + \frac{1}{2}\sin t + 2$

The general solution of a linear system of two first order differential equations contains exactly two arbitrary constants. Now we will substitute the two solutions into one of the equations

$$x + \frac{dy}{dt} = \cos t$$

in the given system to eliminate extra arbitrary constants.

Recall

$$x_{gen} = C_1 e^t + C_2 e^{-t} + \frac{1}{2}\cos t$$

$$\frac{dy_{gen}}{dt} = C_3 e^t - C_4 e^{-t} + \frac{1}{2}\cos t$$

Substitution yields that

$$C_1e^t + C_2e^{-t} + \frac{1}{2}\cos t + C_3e^t - C_4e^{-t} + \frac{1}{2}\cos t - \cos t = 0$$

$$(C_1 + C_3)e^t + (C_2 - C_4)e^{-t} + \left(\frac{1}{2} + \frac{1}{2} - 1\right) cost = 0 \xrightarrow{yields} \begin{matrix} C_3 = -C_1 \\ C_4 = C_2 \end{matrix}$$

Therefore general solution of the two unknowns are:

$$x_{gen} = C_1e^t + C_2e^{-t} + \frac{1}{2}\cos t$$

$$y_{gen} = -C_1e^t + C_2e^{-t} + \frac{1}{2}\sin t + 2$$

This result may be verified using the other differential equation $\frac{dy}{dt} + x = 2$. The same result will be obtained.

After finding the solution x_{gen}, a simpler way to find y_{gen} is to substitute x_{gen} in the first equation $\frac{dx}{dt} + \mathrm{y} = 2$ which leads

$$y_{gen} = 2 - \frac{dx_{gen}}{dt} = -C_1e^t + C_2e^{-t} + \frac{1}{2}\sin t + 2.$$

Example 2.3

Solve the system of the two first order linear differential equations

$$\frac{dx}{dt} - \mathrm{y} = \frac{1}{\sin t}$$
$$x + \frac{dy}{dt} = \frac{1}{\cos t}$$

$$x' = y + \frac{1}{\sin t}, \quad x'' = y' - \frac{\cos t}{\sin^2 t}, \quad x'' = -x + \frac{1}{\cos t} - \frac{\cos t}{\sin^2 t}$$

$$y' = -x + \frac{1}{\cos t}, \quad y'' = -x' + \frac{\sin t}{cos^2 t}, \quad y'' = -y' - \frac{1}{\sin t} + \frac{\sin t}{cos^2 t}$$

The two uncoupled equations are

$$x'' + x = \frac{1}{\cos t} - \frac{\cos t}{\sin^2 t}$$

and

$$y'' + y' = -\frac{1}{\sin t} + \frac{\sin t}{cos^2 t}.$$

Homogeneous solutions;

$$\begin{array}{lll} x'' + x = 0 & \text{Let} & x_h = e^{mt} \Longrightarrow x_h'' = m^2 e^{mt} \Longrightarrow m_{1,2} = \pm i \\ y'' + y = 0 & & y_h = e^{mt} \Longrightarrow y_h'' = m^2 e^{mt} \Longrightarrow m_{1,2} = \pm i \end{array}$$

$$x_h = C_1 \cos t + C_2 \sin t$$
$$y_h = C_3 \cos t + C_4 \sin t$$

These should satisfy homogeneous equations. Therefore one has

$$C_4 = -C_1$$
$$C_3 = C_2$$

2.2 Operator Method:

This is probably one of the most popular techniques that eliminate the number of dependent parameters to a single variable of higher order differential equations. In operator method each differentiation is denoted by a linear operator, D, and the resulting algebraic equation set is solved in terms of D using linear algebra rules. The unknowns are solved in terms of D. Since D is a differential operator, higher order independent linear differential equations are then obtained and solved individually.

Example

Solve the differential equation system

$$\frac{dy}{dt} + z = e^t$$
$$y + \frac{dz}{dt} = e^{-t}$$

Let $D = \frac{d}{dt}$ then,

$$\begin{matrix} Dy + z = e^t \\ y + Dz = e^{-t} \end{matrix} \quad \Longrightarrow \quad \Delta = \begin{vmatrix} D & 1 \\ 1 & D \end{vmatrix} = \mathrm{D}^2 - 1$$

$$\Delta y = \begin{vmatrix} e^t & 1 \\ e^{-t} & D \end{vmatrix} = De^t - e^{-t} = e^t - e^t$$

$$\Delta z = \begin{vmatrix} D & e^t \\ 1 & e^{-t} \end{vmatrix} = De^{-t} - e^{-t} = -e^{-t} - e^t$$

$$y = \frac{\Delta y}{\Delta} = \frac{e^t - e^t}{D^2 - 1} \xrightarrow{yields} (D^2 - 1)y = e^t - e^{-t} \xrightarrow{yields} y'' - y = e^t - e^{-t}$$

$$z = \frac{\Delta z}{\Delta} = \frac{-e^{-t} - e^t}{D^2 - 1} \xrightarrow{yields} (D^2 - 1)z = -e^{-t} - e^t \xrightarrow{yields} z'' - z = -e^t - e^{-t}$$

These two 2nd order differential equations are to be solved individually. Let's solve homogeneous equations first;

$$\begin{matrix} y'' - y = 0 \\ z'' - z = 0 \end{matrix} \qquad \text{Let} \qquad \begin{matrix} Y_h = e^{mt} & Y_h'' = m^2 e^{mt} & (m^2 - 1)e^{mt} = 0 \\ Z_h = e^{mt} & Z_h'' = m^2 e^{mt} & m_{1,2} = \pm 1 \end{matrix}$$

$$\begin{matrix} Y_h = C_1 e^t + C_2 e^{-t} \\ Z_h = C_3 e^t + C_4 e^{-t} \end{matrix}$$

The relation between coefficients are obtained from homogeneous form of the given equation set

$$\begin{matrix} Dy_h = C_1 e^t - C_2 e^{-t} \\ Dz_h = C_3 e^t - C_4 e^{-t} \end{matrix} \qquad \text{Substitution yields that}$$

$$\begin{matrix} Dy + z = e^t & \xrightarrow{yields} & C_1 e^t - C_2 e^{-t} + C_3 e^t + C_4 e^{-t} = 0 & \xrightarrow{yields} & (C_1 + C_3)e^t + (-C_2 + C_4)e^{-t} = 0 \\ y + Dz = e^{-t} & \xrightarrow{yields} & C_1 e^t + C_2 e^{-t} + C_3 e^t - C_4 e^{-t} = 0 & \xrightarrow{yields} & (C_1 + C_3)e^t + (C_2 - C_4)e^{-t} = 0 \end{matrix}$$

$C_3 = -C_1$ & $C_4 = C_3$.

So homogeneous solutions;

$$\begin{matrix} Y_h = C_1 e^t + C_2 e^{-t} \\ Z_h = -C_1 e^t + C_2 e^{-t} \end{matrix}$$

Particular solution can be proposed based on RHS of the equation;

$$Y_p = t(A_1 e^t + A_2 e^{-t})$$

$$Y_p' = A_1 e^t + A_2 e^{-t} + t(A_1 e^t - A_2 e^{-t})$$

$Y_p'' = A_1e^t - A_2e^{-t} + A_1e^t - A_2e^{-t} + t(A_1e^t + A_2e^{-t})$

Substitution of these into $y'' - y = e^t - e^{-t}$ yields that

$(A_1 + A_1 + A_1t - A_1t - 1)e^t + (-A_2 - A_2 + A_2t - A_2t + 1)e^{-t} = 0$

$A_1 = \dfrac{1}{2}$ & $A_2 = \dfrac{1}{2}$

$Y_p = \frac{1}{2}t(e^t + e^{-t})$ Since

$Y_{gen} = Y_h + Y_p = C_1e^t + C_2e^{-t} + \dfrac{1}{2}t(e^t + e^{-t})$

Similarly, particular solution for Z is proposed to be

$Z_p = t(B_1e^t + B_2e^{-t})$

$Z_p'' = B_1e^t - B_2e^{-t} + B_1e^t - B_2e^{-t} + t(B_1e^t + B_2e^{-t})$

Substitution of these into $z'' - z = -e^t - e^{-t}$ yields that

$(B_1 + B_1 + B_1t - B_1t + 1)e^t + (-B_2 - B_2 + B_2t - B_2t + 1)e^{-t} = 0$

$B_1 = -\frac{1}{2}$ & $B_2 = \frac{1}{2}$ then $Z_p = \frac{1}{2}t(e^{-t} - e^t)$

$Z_{gen} = -C_1e^t + C_2e^{-t} + \dfrac{1}{2}t(e^{-t} - e^t)$

<u>Example</u>

Solve the given differential equations $\begin{array}{l} \frac{dx}{dt} + y = 2 \\ x + \frac{dy}{dt} = \cos t \end{array}$

$D = \frac{d}{dt}$, substituting D into the system;

$\begin{array}{l} Dx + y = 2 \\ x + Dy = \cos t \end{array} \Rightarrow \Delta = \begin{vmatrix} D & 1 \\ 1 & D \end{vmatrix} = D^2 - 1$

$\Delta x = \begin{vmatrix} 2 & 1 \\ cost & D \end{vmatrix} = D^2 - \cos t = -\cos t$

$$\Delta y = \begin{vmatrix} D & 2 \\ 1 & \cos t \end{vmatrix} = D\cos t - \ 2 = -\sin t - 2$$

$$\mathrm{x} = \frac{\Delta x}{\Delta} = \frac{-\cos t}{\mathrm{D}^2\ -1} \xrightarrow{yields} (\mathrm{D}^2 - 1)\mathrm{x} = -\cos t \xrightarrow{yields} x'' - x = -\cos t$$

$$\mathrm{y} = \frac{\Delta y}{\Delta} = \frac{-\sin t - 2}{\mathrm{D}^2\ -1} \xrightarrow{yields} (\mathrm{D}^2 - 1)\mathrm{y} = -\sin t - 2 \xrightarrow{yields} y'' - y = -\sin t - 2$$

Let’s solve them individually. Let’s start with the corresponding homogeneous system;

$$\begin{matrix} x'' - x = 0 \\ y'' - y = 0 \end{matrix} \qquad \text{Let} \qquad \begin{matrix} x_h = e^{mt} \Rightarrow x_h'' = m^2 e^{mt} & \Rightarrow & m^2 - 1 & = 0 \\ y_h = e^{mt} \Rightarrow y_h'' = m^2 e^{mt} & \Rightarrow & m_{1,2} & = \pm 1 \end{matrix}$$

$$x_h = C_1 e^t + C_2 e^{-t}$$
$$y_h = C_3 e^t + C_4 e^{-t}$$

Since the right side is {-cost} let a particular solution be;

$$x_p = A\cos t + B\sin t$$

$$x_p' = -A\sin t + B\cos t$$

$$x_p'' = -A\cos t - B\sin t$$

Substitution of these into $\qquad x'' - x = -\cos t \qquad$ yields that

$$-A\cos t - B\sin t - A\cos t + B\sin t = -\cos t$$

$$-2A\cos t - 2B\sin t = -\cos t \xrightarrow{yields} \begin{matrix} A = 1/2 \\ B = 0 \end{matrix} \xrightarrow{yields} x_p = \frac{1}{2}\cos t$$

$$x_{gen} = x_h + x_p = C_1 e^t + C_2 e^{-t} + \frac{1}{2}\cos t$$

Consider the other equation; $\qquad y'' - y = -\sin t - 2, \qquad$ let a particular solution be;

$$y_p = C\cos t + D\sin t + E$$

$$y_p' = -C\sin t + D\cos t$$

$$y_p'' = -C\cos t - D\sin t$$

Substitution of these into $\qquad y'' - y = -\sin t - 2 \qquad$ yields that

$$-C\cos t - D\sin t - Ccost + D\sin t + E + \sin t + 2 = 0$$

$$-2C\cos t + (-2D+1)\sin t + (-E+2) = 0$$

$$C = 0, \qquad D = \frac{1}{2} \;\; \& \;\; E = 2 \xrightarrow{yields} y_p = \frac{1}{2}\sin t + 2$$

$$y_{gen} = y_h + y_p = C_3 e^t + C_4 e^{-t} + \frac{1}{2}\sin t + 2$$

Now we should substitute the solution into the given set to eliminate the dependent parameters.

$$x + \frac{dy}{dt} = \cos t \qquad \text{where} \qquad \begin{array}{l} x = C_1 e^t + C_2 e^{-t} + \frac{1}{2}\cos t \\ \frac{dy}{dt} = C_3 e^t - C_4 e^{-t} + \frac{1}{2}\cos t \end{array}$$

Substitution yields that

$$C_1 e^t + C_2 e^{-t} + \frac{1}{2}\cos t + C_3 e^t - C_4 e^{-t} + \frac{1}{2}\cos t - \cos t = 0$$

$$(C_1 + C_3)e^t + (C_2 - C_4)e^{-t} + \left(\frac{1}{2} + \frac{1}{2} - 1\right) cost = 0 \xrightarrow{yields} \begin{array}{l} C_3 = -C_1 \\ C_4 = C_2 \end{array}$$

$$x_{gen} = C_1 e^t + C_2 e^{-t} + \frac{1}{2}\cos t$$

$$y_{gen} = -C_1 e^t + C_2 e^{-t} + \frac{1}{2}\sin t + 2$$

This result may be verified using the other differential equation $\frac{dy}{dt} + x = 2$. The same result will be obtained.

Example

Solve the differential equations

$$\begin{array}{l} \frac{dx}{dt} - \mathrm{y} = \frac{1}{\sin t} \\ x + \frac{dy}{dt} = \frac{1}{\cos t} \end{array}$$

$$\begin{array}{l} Dx - \mathrm{y} = \dfrac{1}{\sin t} \\ x + Dy = \dfrac{1}{\cos t} \end{array} \Longrightarrow \Delta = \begin{vmatrix} D & -1 \\ 1 & D \end{vmatrix} = D^2 + 1$$

$$\Delta x = \begin{vmatrix} \dfrac{1}{\sin t} & -1 \\ \dfrac{1}{\cos t} & D \end{vmatrix} = D\frac{1}{\sin t} + \frac{1}{\cos t} = -\frac{\cos t}{sin^2 t} + \frac{1}{\cos t}$$

$$\Delta y = \begin{vmatrix} \mathrm{D} & \dfrac{1}{\sin t} \\ 1 & \dfrac{1}{\cos t} \end{vmatrix} = D\frac{1}{\cos t} + \frac{1}{\sin t} = -\frac{\sin t}{cos^2 t} - \frac{1}{\sin t}$$

$$\mathrm{x} = \frac{\Delta x}{\Delta} = \frac{\Delta x}{\mathrm{D}^2 + 1} \xrightarrow{yields} (\mathrm{D}^2 + 1)\mathrm{x} = \Delta x \xrightarrow{yields} x'' + x = -\frac{\cos t}{sin^2 t} + \frac{1}{\cos t}$$

$$\mathrm{y} = \frac{\Delta y}{\Delta} = \frac{\Delta y}{\mathrm{D}^2 + 1} \xrightarrow{yields} (\mathrm{D}^2 + 1)\mathrm{y} = \Delta y \xrightarrow{yields} y'' + y = -\frac{\sin t}{cos^2 t} - \frac{1}{\sin t}$$

Homogeneous solutions;

$$\begin{array}{lcllll} x'' + x = 0 & \text{Let} & x_h = e^{mt} \Rightarrow x_h'' = m^2 e^{mt} & \Rightarrow & m^2 + 1 & = 0 \\ y'' + y = 0 & & y_h = e^{mt} \Rightarrow y_h'' = m^2 e^{mt} & \Rightarrow & m_{1,2} & = \pm i \end{array}$$

$$x_h = C_1 \cos t + C_2 \sin t$$
$$y_h = C_3 \cos t + C_4 \sin t$$

Example 2.4 Consider two masses connected to the walls by springs, and they are connected to each other by massless springs. Suppose that $m_1 = m_2 = k_1 = k_2 = k_3 = 1$. Find $x_1(t)$ and $x_2(t)$ are the deviation from equilibrium for both masses.

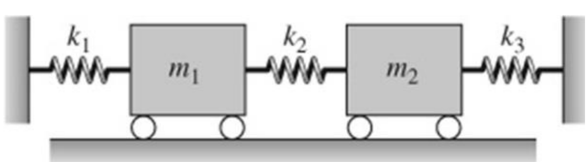

Figure.2.2 Spring and mass system.

Newton's Second Law of Motion yields

Example 2.5

Solve the differential equations

$$\begin{aligned} \dot{x} - x - y &= 3t \\ \dot{x} + \dot{y} - 5x - 2y &= 5 \end{aligned}$$

$\dot{x} - x = y + 3t$ (1)

$\dot{x} - 5x = 2y - \dot{y} + 5$ (2)

From (1) solve y, compute $\dot{y}$

$y = \dot{x} - x - 3t \rightarrow \dot{y} = \ddot{x} - \dot{x} - 3$

Substitute in (2) get the decoupled differential equation for x

$\dot{x} - 5x = 2y - \dot{y} + 5 = 2(\dot{x} - x - 3t) - (\ddot{x} - \dot{x} - 3) + 5$

$\rightarrow \ddot{x} - 2\dot{x} - 3x = 8 - 6t$

From these equations (1) and (2) we solve $x, \dot{x}$

$4x = \dot{y} - y + 3t - 5 \rightarrow 4\dot{x} = \ddot{y} - \dot{y} + 3$

$4\dot{x} = 3y + \dot{y} + 15t - 5$

Equating the two equations, we get the decoupled differential equation for y

$\ddot{y} - \dot{y} + 3 = \dot{y} + 3y - 5 + 15t$

$\rightarrow \ddot{y} - 2\dot{y} - 3y = -8 + 15t$

Solution of x-equation

$$\ddot{x} - 2\dot{x} - 3x = 8 - 6t$$

Homogeneous solution;

$r^2 - 2r - 3 = 0 \Longrightarrow \begin{matrix} r_1 = -1 \\ r_2 = 3 \end{matrix} \Longrightarrow x_h = c_1 e^{-t} + c_2 e^{3t}$

Particular solution;

$x_p = A + Bt \rightarrow \dot{x}_p = B \rightarrow \ddot{x}_p = 0$

$\ddot{x} - 2\dot{x} - 3x = 8 - 6t \rightarrow -2B - 3(A + Bt) = 8 - 6t$

$-3B = -6 \rightarrow \mathrm{B} = 2$

$-2B - 3A = 8 \rightarrow \mathrm{A} = -4$

$x_p = 2t - 4$

$x_{gen} = 2t - 4 + c_1 e^{-t} + c_2 e^{3t}$

Solution of y-equation

$\ddot{y} - 2\dot{y} - 3y = -8 + 15t$

Homogeneous solution;

$r^2 - 2r - 3 = 0 \Longrightarrow \begin{matrix} r_1 = -1 \\ r_2 = 3 \end{matrix} \Longrightarrow y_h = c_3 e^{-t} + c_4 e^{3t}$

Particular solution;

$y_p = \mathrm{A} + \mathrm{Bt} \rightarrow \dot{y}_p = \mathrm{B} \rightarrow \ddot{y}_p = 0$

$\ddot{y} - 2\dot{y} - 3y = -8 + 15t \rightarrow -2B - 3(\mathrm{A} + \mathrm{Bt}) = -8 + 15t$

$-3B = 15 \rightarrow \mathrm{B} = -5$

$-2B - 3A = -8 \rightarrow \mathrm{A} = 6$

$y_p = 6 - 5t$

$y_{gen} = 6 - 5t + c_3 e^{-t} + c_4 e^{3t}$

To remove extra arbitrary constants we use one of the equations in the system

$\dot{x} - x = y + 3t$

which leads

$2 - e^{-t} c_1 + 3e^{3t} c_2 - (2t - 4 + c_1 e^{-t} + c_2 e^{3t}) = 6 - 5t + c_3 e^{-t} + c_4 e^{3t} + 3t$

Therefore $C_3 = -2C_1$, $C_4 = 2C_2$, and hence

$y_{gen} = 6 - 5t - 2c_1 e^{-t} - 2c_2 e^{3t}$

Homeworks

1. $$\begin{aligned} x' + 3x &= \frac{2e^{-t} - 3t}{\sqrt{2}} \\ y' + y &= \frac{2e^{-t} + 3t}{\sqrt{2}} \end{aligned}$$

2. $$\begin{aligned} x' - 2x - y &= -1, \quad x(0) = 1 \\ y' + x - 2y &= 8, \quad y(0) = 1 \end{aligned}$$

3. $$\begin{aligned} x' - 3x &= -4\sin 2t, \quad x(0) = 2 \\ y' - 5x + 2y &= \cos 2t, \quad y(0) = -1 \end{aligned}$$

4. $$\begin{aligned} \dot{x} - x - 2y &= 2t \\ \dot{y} - 3x - 2y &= -4t \end{aligned}$$

5. $$\begin{aligned} x' + 5x - y &= 6e^{2t} \\ y' - 4x + 2y &= -e^{2t} - 4t \end{aligned}$$