1. Kinematics

ME 707 Advanced Dynamics

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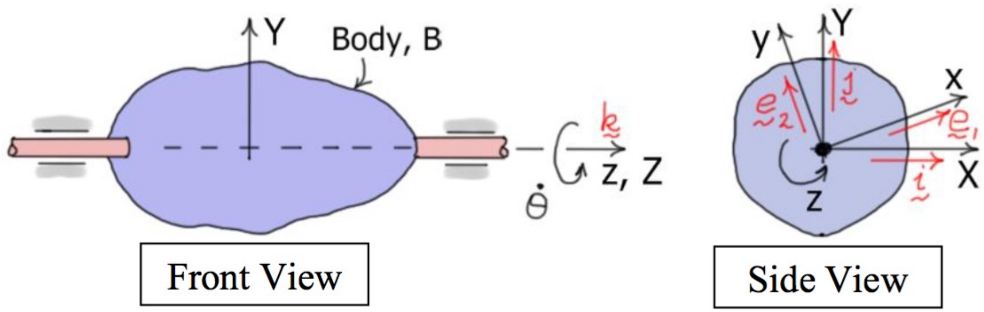
Simple Angular Motion

Simple Angular Velocity

The rigid body B shown in the diagram below rotates about the Z-axis. The XYZ reference frame is a fixed frame, while the xyz reference frame is fixed in (and rotates with) the body. The XYZ reference frame is represented by the unit vector set $R:(\underset{\sim}{i},\ \underset{\sim}{j},\ \underset{\sim}{k})$, and the xyz reference frame is represented by the unit vector set $B:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{k})$. Note that each unit vector set is a *right-handed* set, that is $\underset{\sim}{i} \times \underset{\sim}{j} = \underset{\sim}{k}$ and $\underset{\sim}{e}_1 \times \underset{\sim}{e}_2 = \underset{\sim}{k}$.

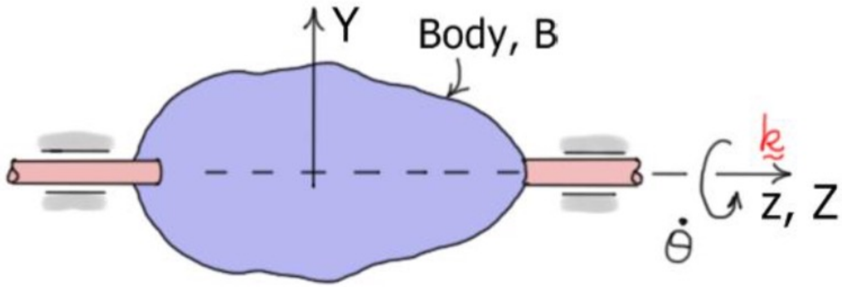


The unit vectors fixed in the body B can be differentiated by using the concept of *angular velocity*. It can be shown that

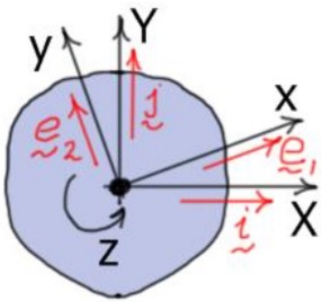
$$\frac{{}^R d\underset{\sim}{e}_i}{dt} = {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{e}_i \qquad (i = 1,2)$$

where $\frac{{}^R d\underset{\sim}{e}_i}{dt}$ represents the derivative of the unit vector $\underset{\sim}{e}_i$ in the reference frame R, and ${}^R\underset{\sim}{\omega}_B = \dot{\theta}\underset{\sim}{k}$ is the angular velocity of the body B in the reference frame R.

Simple Angular Motion



Front View



Side View

Aside:

$$\frac{{}^{R}d\underset{\sim}{e}_1}{dt} = \frac{{}^{R}d}{dt}(C_\theta \underset{\sim}{i} + S_\theta \underset{\sim}{j})$$

$$= \dot{\theta}(-S_\theta \underset{\sim}{i} + C_\theta \underset{\sim}{j})$$

$$= \dot{\theta}\underset{\sim}{e}_2$$

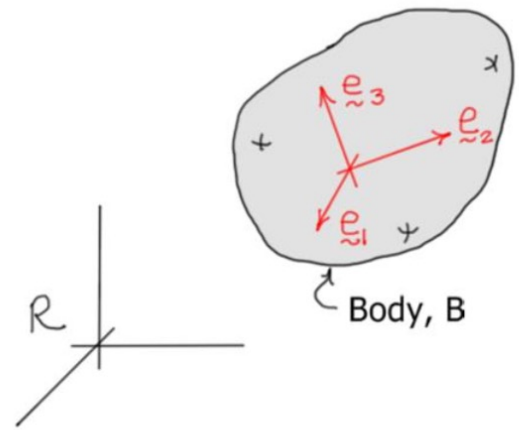
$$= \dot{\theta}(\underset{\sim}{k} \times \underset{\sim}{e}_1)$$

$$= {}^{R}\underset{\sim}{\omega}_B \times \underset{\sim}{e}_1$$

Simple Angular Motion

Differentiation of Unit Vectors – General Case

Consider now a rigid body B moving in three dimensional space. In general, given a set of unit vectors $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ fixed in B, it can be shown that



$$\frac{{}^R d\underset{\sim}{e}_i}{dt} = {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{e}_i \qquad (i = 1,2,3)$$

where, as before, $\frac{{}^R d\underset{\sim}{e}_i}{dt}$ represents the derivative of the unit vector $\underset{\sim}{e}_i$ in the reference frame R, and ${}^R\underset{\sim}{\omega}_B$ is the angular velocity of the body B in the reference frame R. What we are presently lacking is a means of calculating ${}^R\underset{\sim}{\omega}_B$, unless the body has simple angular motion.

Simple Angular Acceleration

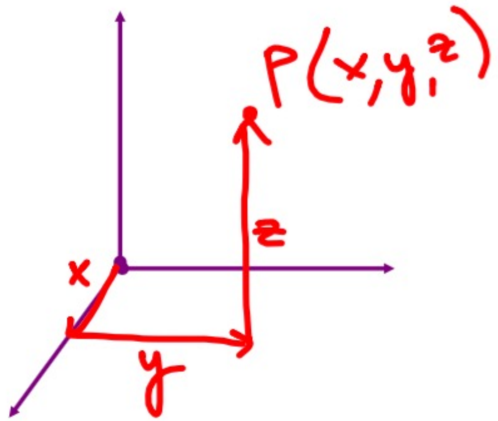
The angular acceleration of B in R is found by ***differentiating*** the angular velocity vector. That is,

$${}^R\underset{\sim}{\alpha}_B = \frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_B\right) = \ddot{\theta}\underset{\sim}{k}$$

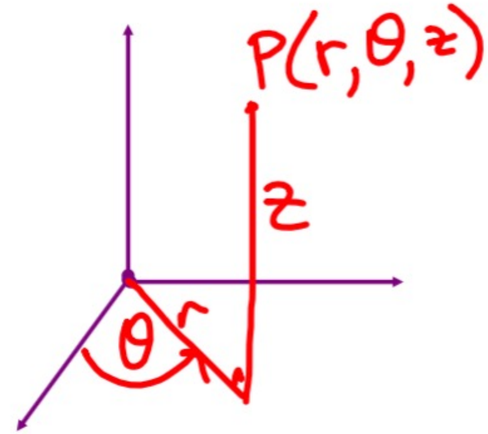
Coordinate Systems

We can describe a point, *P*, in three different ways.

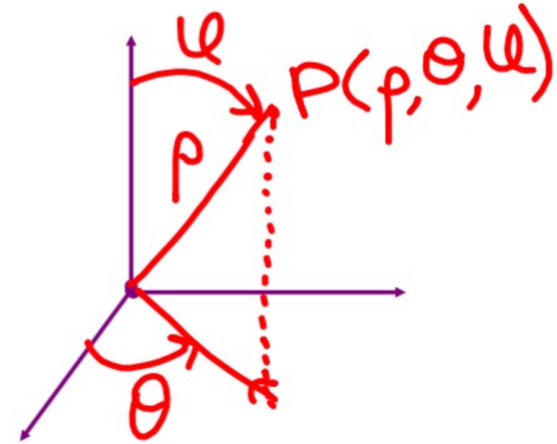
<u>Cartesian</u>



<u>Cylindrical</u>

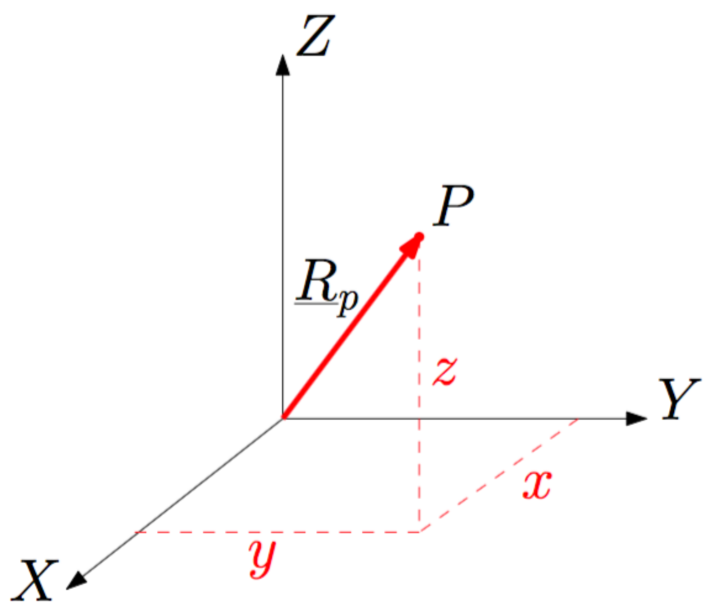


<u>Spherical</u>



Coordinate Systems

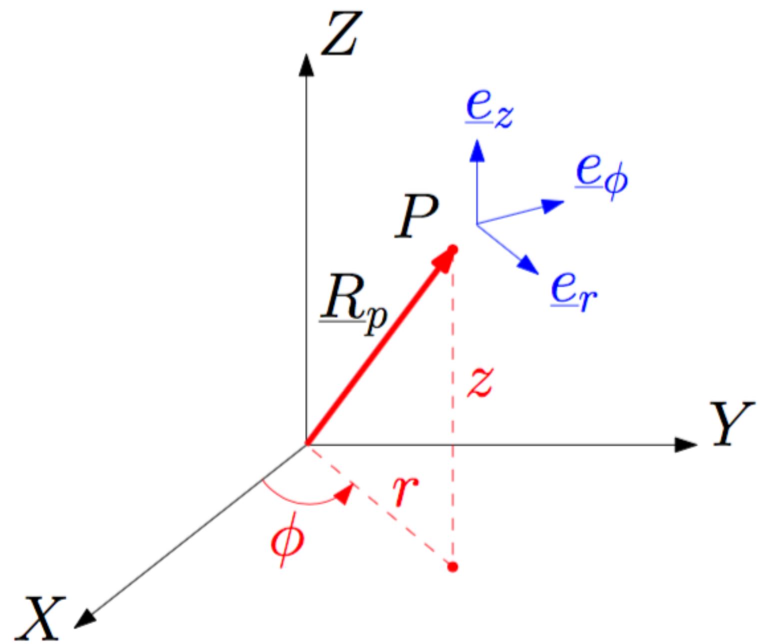
Cartesian Coordinates



$$\underline{R} = x\underline{I} + y\underline{J} + z\underline{K}$$
$$\Rightarrow \dot{\underline{R}} = \dot{x}\underline{I} + \dot{y}\underline{J} + \dot{z}\underline{K}$$
$$\Rightarrow \ddot{\underline{R}} = \ddot{x}\underline{I} + \ddot{y}\underline{J} + \ddot{z}\underline{K}$$

Coordinate Systems

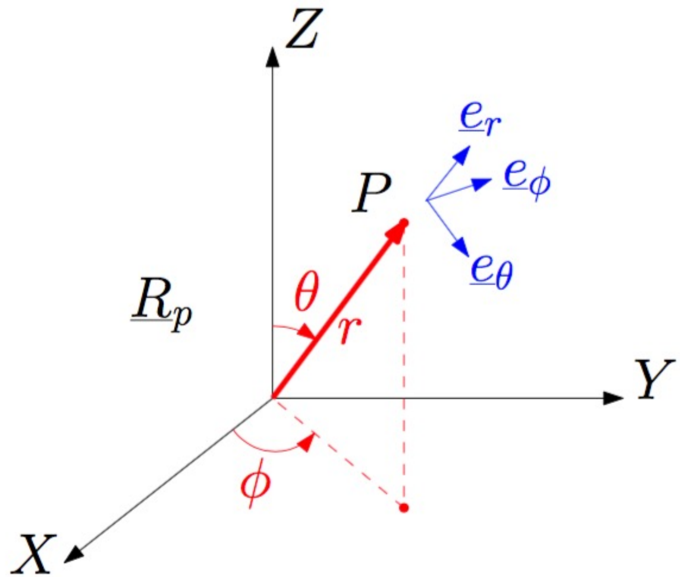
Cylindrical Coordinates



$$
\begin{aligned}
\underline{R} &= r\underline{e}_r + z\underline{e}_z \\
\Rightarrow \dot{\underline{R}} &= \dot{r}\underline{e}_r + r\dot{\underline{e}}_r + \dot{z}\underline{e}_z \\
&= \dot{r}\underline{e}_r + r\dot{\phi}\underline{e}_z \times \underline{e}_r + \dot{z}\underline{e}_z \\
&= \dot{r}\underline{e}_r + r\dot{\phi}e_\phi + \dot{z}\underline{e}_z \\
\Rightarrow \ddot{\underline{R}} &= \ddot{r}\underline{e}_r + \dot{r}\dot{\underline{e}}_r + \dot{r}\dot{\phi}e_\phi + r\ddot{\phi}e_\phi + r\dot{\phi}\dot{\underline{e}}_\phi + \ddot{z}\underline{e}_z \\
&= \ddot{r}\underline{e}_r + \dot{r}\dot{\phi}\underline{e}_z \times \underline{e}_r + \dot{r}\dot{\phi}e_\phi + r\ddot{\phi}e_\phi + r\dot{\phi}^2\underline{e}_z \times e_\phi + \ddot{z}\underline{e}_z \\
&= \ddot{r}\underline{e}_r + 2\dot{r}\dot{\phi}\underline{e}_\phi + r\ddot{\phi}e_\phi + r\dot{\phi}^2(-\underline{e}_r) + \ddot{z}\underline{e}_z \\
&= (\ddot{r} - r\dot{\phi}^2)\underline{e}_r + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\underline{e}_\phi + \ddot{z}\underline{e}_z.
\end{aligned}
$$

Coordinate Systems

Spherical Coordinates



$$\underline{r} = r\underline{e}_r$$
$$\Rightarrow \dot{\underline{r}} = \dot{r}\underline{e}_r + r\dot{\underline{e}}_r$$
$$= \dot{r}\underline{e}_r + r(\dot{\phi}\underline{k} + \dot{\theta}\underline{e}_\phi) \times \underline{e}_r$$
$$= \dot{r}\underline{e}_r + r[\dot{\phi}(\cos\theta\underline{e}_r - \sin\theta\underline{e}_\theta) + \dot{\theta}\underline{e}_\phi] \times \underline{e}_r$$
$$= \ldots\ldots$$

Coordinate Systems

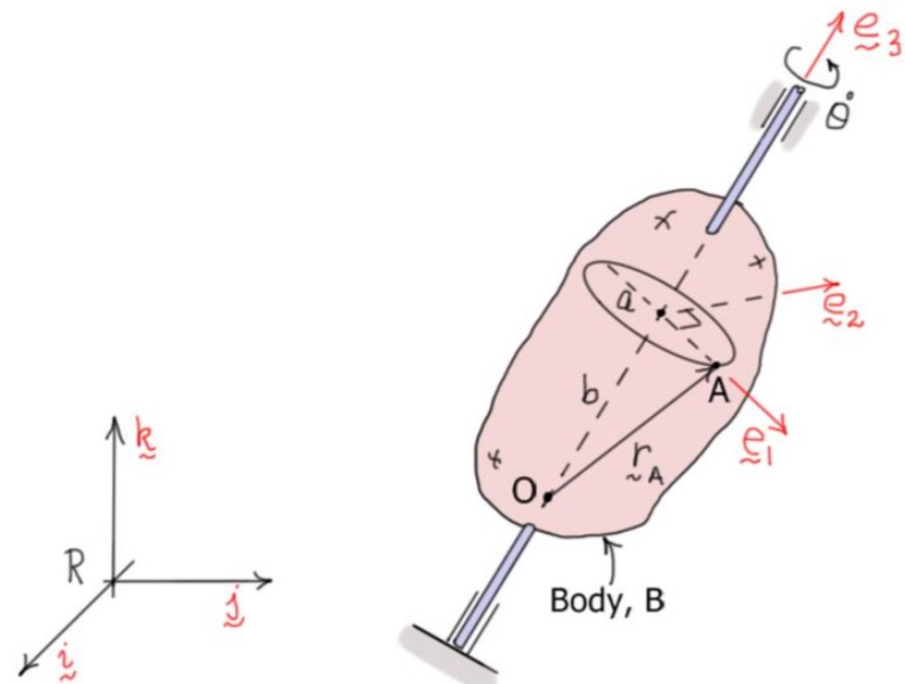
Transformations between Coordinate Systems:

Cylindrical to Cartesian coordinates	Cartesian to cylindrical coordinates
$x = r\cos\theta$ $y = r\sin\theta$ $z = z$	$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}(y/x)$ $z = z$
Spherical to Cartesian coordinates	**Cartesian to spherical coordinates**
$x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$	$\rho = \sqrt{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}(y/x)$ $\phi = \tan^{-1}\left(\sqrt{x^2 + y^2}/z\right)$
Spherical to cylindrical coordinates	**Cylindrical to spherical coordinates**
$r = \rho\sin\phi$ $z = \rho\cos\phi$ $\theta = \theta$	$\rho = \sqrt{r^2 + z^2}$ $\phi = \tan^{-1}(r/z)$ $\theta = \theta$

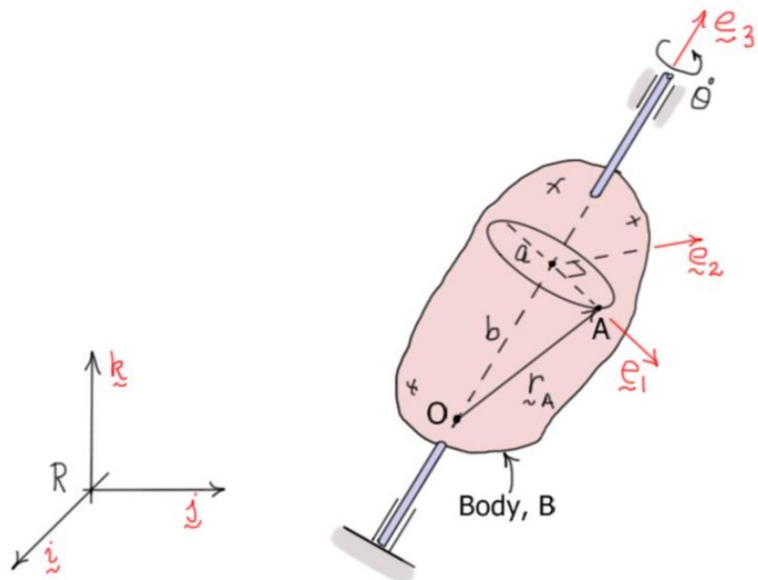
Fixed Axis Rotation

Kinematics of Fixed Axis Rotation

Consider the rigid body B shown in the diagram below. The fixed reference frame is represented by the unit vector set $R:(\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k})$, and the rotating reference frame is represented by the unit vector set $B:(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$. All points of B travel in a circular path around the fixed axis. The ***velocity*** and ***acceleration*** of any point within the body can be determined by differentiating (with respect to time) its position vector $\underset{\sim}{r}_A$ relative to any point on the fixed axis.



Fixed Axis Rotation



For example, the velocity of point A may be calculated as follows

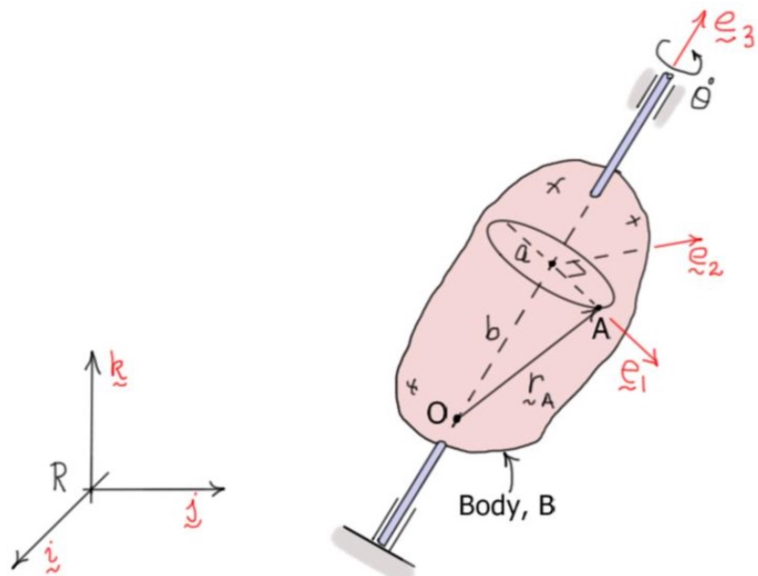
$$\underset{\sim}{v}_A = \frac{{}^R d}{dt}(a\underset{\sim}{e}_1 + b\underset{\sim}{e}_3) = a\frac{{}^R d\underset{\sim}{e}_1}{dt} + b\frac{{}^R d\underset{\sim}{e}_3}{dt}$$

$$= a({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{e}_1) + b({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{e}_3)$$

$$= {}^R\underset{\sim}{\omega}_B \times (a\underset{\sim}{e}_1 + b\underset{\sim}{e}_3)$$

$$= {}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_A$$

Fixed Axis Rotation



Performing the cross product in the last equation gives $\underset{\sim}{v}_A = a\dot{\theta}\underset{\sim}{e}_2$. Note that the velocity is *tangent* to the circular path. Similarly, the acceleration of A may be calculated as follows

$$
\begin{aligned}
{}^R\underset{\sim}{a}_A &= \frac{{}^R d}{dt}({}^R\underset{\sim}{v}_A) = \frac{{}^R d}{dt}({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_A) \\
&= ({}^R\underset{\sim}{\alpha}_B \times \underset{\sim}{r}_A) + ({}^R\underset{\sim}{\omega}_B \times {}^R\underset{\sim}{v}_A)
\end{aligned}
$$

Performing the operations in this last equation gives ${}^R\underset{\sim}{a}_A = -a\dot{\theta}^2\underset{\sim}{e}_1 + a\ddot{\theta}\underset{\sim}{e}_2$. Note the acceleration has both *normal* and *tangential* components.

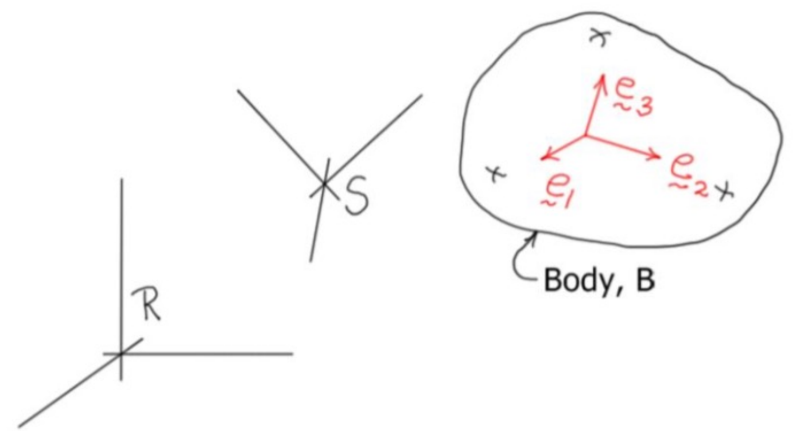
Fixed Axis Rotation

Summation Rule for Angular Velocities

Consider a rigid body B undergoing three dimensional motion as shown in the diagram below. R and S represent two reference frames that are *rotating* relative to each other. The angular velocity of the body B relative to the reference frame R (${}^R\underset{\sim}{\omega}_B$) may be found by using the *summation rule* for angular velocities to work through the intermediate reference frame S as follows

$$\boxed{{}^R\underset{\sim}{\omega}_B = {}^S\underset{\sim}{\omega}_B + {}^R\underset{\sim}{\omega}_S}$$

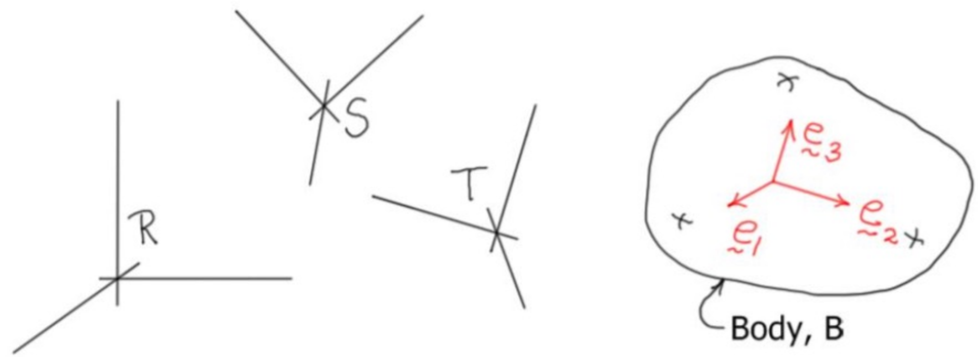
Here, ${}^S\underset{\sim}{\omega}_B$ represent the angular velocity of B relative to the reference frame S, and ${}^R\underset{\sim}{\omega}_S$ represents the angular velocity of frame S relative to R.



Fixed Axis Rotation

Consider next the body B in the the diagram below. Here, there are three reference frames, R, S, and T, all rotating relative to each other. In this case, ${}^R\underset{\sim}{\omega}_B$ the angular velocity of B relative to R may be found using the summation rule for angular velocities to work through the intermediate frames S and T as follows

$$\boxed{\begin{aligned} {}^R\underset{\sim}{\omega}_B &= {}^T\underset{\sim}{\omega}_B + {}^R\underset{\sim}{\omega}_T \\ &= {}^T\underset{\sim}{\omega}_B + {}^S\underset{\sim}{\omega}_T + {}^R\underset{\sim}{\omega}_S \end{aligned}}$$

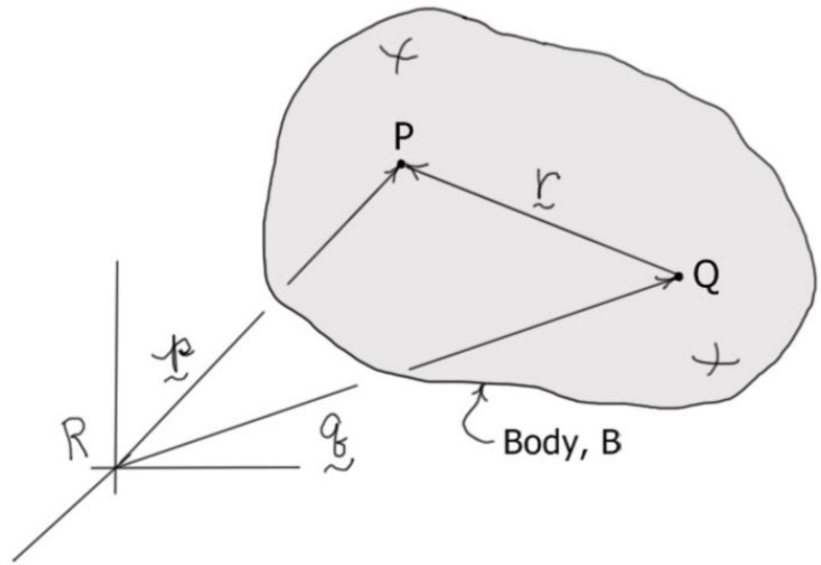


In fact, this rule may be extended to as many frames as necessary.

The ***summation rule*** may be used to compute the angular velocity of a body (undergoing three-dimensional motion) by introducing a set of reference frames whose relative angular motions may be described using simple angular velocities. Then, the angular velocity of the body is found by summing the simple angular velocities.

Note: There is ***no*** corresponding summation rule for ***angular accelerations***. The ***angular acceleration*** of a body is found by ***direct differentiation*** of the angular velocity vector. That is, ${}^R\underset{\sim}{\alpha}_B = \dfrac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_B\right)$.

Derivatives of a Vector in Multiple Reference Frames



It is often convenient to express vectors in terms of local (or rotating) unit vector sets (reference frames). For example, consider the position vector $\underset{\sim}{r}_{P/Q}$ shown in the diagram at the right. This vector describes the relative position of P and Q, two points fixed in the rigid body B. As such, it is most easily described in terms of the unit vector set $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ that is fixed in (and rotates with) B.

To describe the relative motion of P and Q, the position vector $\underset{\sim}{r}_{P/Q}$ must be differentiated in the fixed reference frame (unit vector set $(\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k})$). This can be done in one of two ways:

1) express $\underset{\sim}{r}_{P/Q}$ in terms of $(\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k})$, and then *differentiate*, or
2) express $\underset{\sim}{r}_{P/Q}$ in terms of $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$, and then *use the derivative rule*.

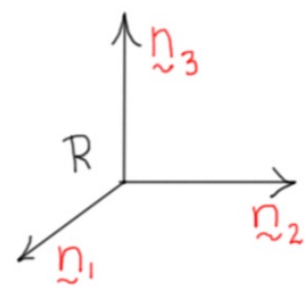
Derivatives of a Vector in Multiple Reference Frames

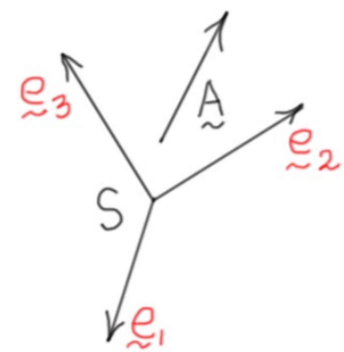
The Derivative Rule

Given the two reference frames

R: $(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$ (rotating frame)

S: $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ (rotating frame),





the derivatives of *any* vector $\underset{\sim}{A}$ in the two reference frames are related by the following rule

$$\boxed{\frac{{}^{R}d\underset{\sim}{A}}{dt} = \frac{{}^{S}d\underset{\sim}{A}}{dt} + ({}^{R}\underset{\sim}{\omega}_S \times \underset{\sim}{A})}$$

where ${}^{R}\underset{\sim}{\omega}_S$ is the angular velocity of frame S relative to the frame R.)

Derivatives of a Vector in Multiple Reference Frames

Derivation

Suppose for convenience $\underset{\sim}{A}$ is expressed in terms of the unit vectors of frame S. That is,

$$\underset{\sim}{A} = a_1\underset{\sim}{e}_1 + a_2\underset{\sim}{e}_2 + a_3\underset{\sim}{e}_3$$

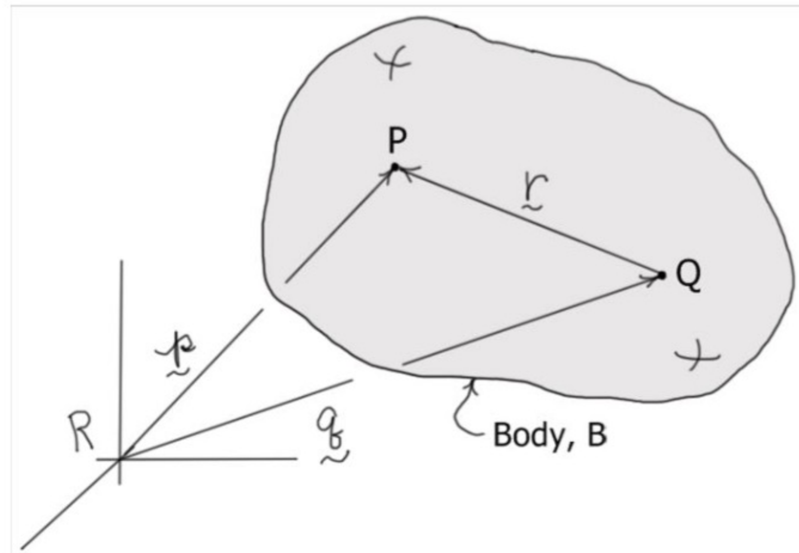
Then, the derivative of $\underset{\sim}{A}$ in the reference frame R can be computed as follows

$$\begin{aligned}
\frac{{}^R d\underset{\sim}{A}}{dt} &= (\dot{a}_1\underset{\sim}{e}_1 + \dot{a}_2\underset{\sim}{e}_2 + \dot{a}_3\underset{\sim}{e}_3) + a_1\frac{{}^R d\underset{\sim}{e}_1}{dt} + a_2\frac{{}^R d\underset{\sim}{e}_2}{dt} + a_3\frac{{}^R d\underset{\sim}{e}_3}{dt} \\
&= \frac{{}^S d\underset{\sim}{A}}{dt} + a_1({}^R\underset{\sim}{\omega}_S \times \underset{\sim}{e}_1) + a_2({}^R\underset{\sim}{\omega}_S \times \underset{\sim}{e}_2) + a_3({}^R\underset{\sim}{\omega}_S \times \underset{\sim}{e}_3) \\
&= \frac{{}^S d\underset{\sim}{A}}{dt} + {}^R\underset{\sim}{\omega}_S \times (a_1\underset{\sim}{e}_1 + a_2\underset{\sim}{e}_2 + a_3\underset{\sim}{e}_3) \\
&= \frac{{}^S d\underset{\sim}{A}}{dt} + ({}^R\underset{\sim}{\omega}_S \times \underset{\sim}{A})
\end{aligned}$$

Here ${}^R\underset{\sim}{\omega}_S$ is the angular velocity of the reference frame S $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ relative to the reference frame R $(\underset{\sim}{n}_1, \underset{\sim}{n}_2, \underset{\sim}{n}_3)$.

Relative Kinematics of Two Points Fixed on a Rigid Body

Consider the three dimensional motion of a rigid body B as shown in the diagram at the right. The points P and Q represent two points that are fixed in the body. The velocities and accelerations of P and Q in the reference frame R are related as follows



$$\boxed{{}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_Q + {}^R\underset{\sim}{v}_{P/Q} = {}^R\underset{\sim}{v}_Q + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)}$$

and

$$\boxed{{}^R\underset{\sim}{a}_P = {}^R\underset{\sim}{a}_Q + {}^R\underset{\sim}{a}_{P/Q} = {}^R\underset{\sim}{a}_Q + \left({}^R\underset{\sim}{\alpha}_B \times \underset{\sim}{r}\right) + {}^R\underset{\sim}{\omega}_B \times \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)}$$

These equations are easily verified using "the derivative rule."

Relative Kinematics of Two Points Fixed on a Rigid Body

Derivation

1. The position vector of point P relative to the reference frame R can be written as $\underset{\sim}{p} = \underset{\sim}{q} + \underset{\sim}{r}$. Differentiating this equation and using "the derivative rule" gives

$$
\begin{aligned}
{}^{R}\underset{\sim}{v}_P &= \frac{{}^{R}d\underset{\sim}{p}}{dt} = \frac{{}^{R}d\underset{\sim}{q}}{dt} + \frac{{}^{R}d\underset{\sim}{r}}{dt} \\
&= {}^{R}\underset{\sim}{v}_Q + \frac{{}^{B}d\underset{\sim}{r}}{dt} + {}^{R}\underset{\sim}{\omega}_B \times \underset{\sim}{r} \\
&= {}^{R}\underset{\sim}{v}_Q + {}^{R}\underset{\sim}{\omega}_B \times \underset{\sim}{r}
\end{aligned}
\quad \Rightarrow \quad \boxed{\underset{\sim}{v}_P = \underset{\sim}{v}_Q + \underset{\sim}{v}_{P/Q} = \underset{\sim}{v}_Q + (\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/Q})}
$$

where

$$\frac{{}^{R}d\underset{\sim}{r}}{dt} = {}^{R}\underset{\sim}{v}_{P/Q} = {}^{R}\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{P/Q} \qquad \text{(velocity of } P \text{ relative to } Q \text{ in } R,\ {}^{R}\underset{\sim}{v}_{P/Q})$$

Relative Kinematics of Two Points Fixed on a Rigid Body

2. Differentiating the velocity equation and again using "the derivative rule" gives

$$
\begin{aligned}
{}^{R}\underset{\sim}{a}_{P} &= \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{v}_{P}\right) = \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{v}_{Q}\right) + \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right) \\
&= {}^{R}\underset{\sim}{a}_{Q} + \left(\frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{\omega}_{B}\right) \times \underset{\sim}{r}\right) + \left({}^{R}\underset{\sim}{\omega}_{B} \times \frac{{}^{R}d\underset{\sim}{r}}{dt}\right) \\
&= {}^{R}\underset{\sim}{a}_{Q} + \left({}^{R}\underset{\sim}{\alpha}_{B} \times \underset{\sim}{r}\right) + {}^{R}\underset{\sim}{\omega}_{B} \times \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right)
\end{aligned}
$$

or

$$
\boxed{{}^{R}\underset{\sim}{a}_{P} = {}^{R}\underset{\sim}{a}_{Q} + {}^{R}\underset{\sim}{a}_{P/Q} = {}^{R}\underset{\sim}{a}_{Q} + \left({}^{R}\underset{\sim}{\alpha}_{B} \times \underset{\sim}{r}\right) + {}^{R}\underset{\sim}{\omega}_{B} \times \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right)}\,.
$$

Here ${}^{R}\underset{\sim}{a}_{P/Q}$ is the acceleration of P with respect to Q in R, and by inspection of the above equation, it is defined to be

$$
{}^{R}\underset{\sim}{a}_{P/Q} = \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{v}_{P/Q}\right) = \left({}^{R}\underset{\sim}{\alpha}_{B} \times \underset{\sim}{r}\right) + {}^{R}\underset{\sim}{\omega}_{B} \times \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right).
$$

Relative Kinematics of Two Points Fixed on a Rigid Body

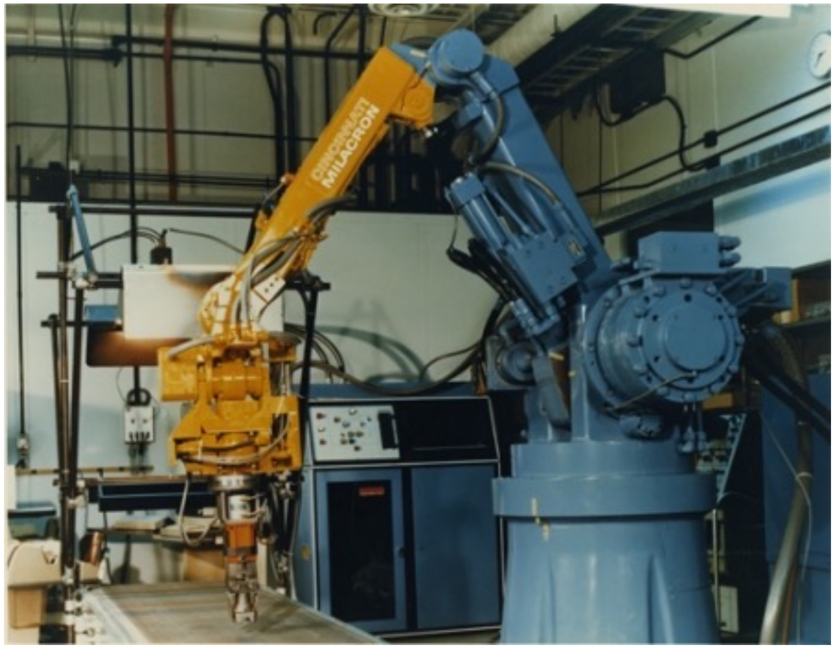
$$
{}^{R}\underset{\sim}{a}_{P/Q} = \frac{{}^{R}d}{dt}\left({}^{R}\underset{\sim}{v}_{P/Q}\right) = \left({}^{R}\underset{\sim}{\alpha}_{B} \times \underset{\sim}{r}\right) + {}^{R}\underset{\sim}{\omega}_{B} \times \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right).
$$

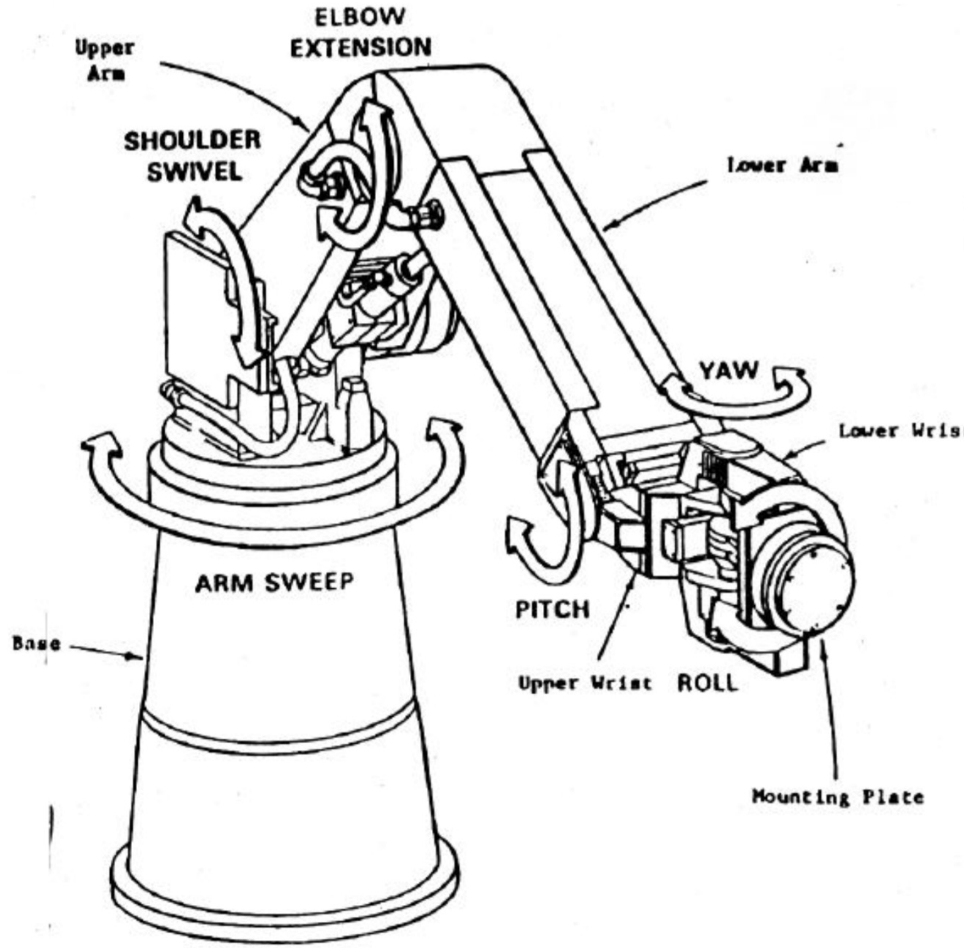
Notes

1. The above formulae may be applied *recursively* to calculate motions of remote points within a mechanical system.

2. Calculation of velocities and accelerations does not require differentiation, only multiplication and addition.

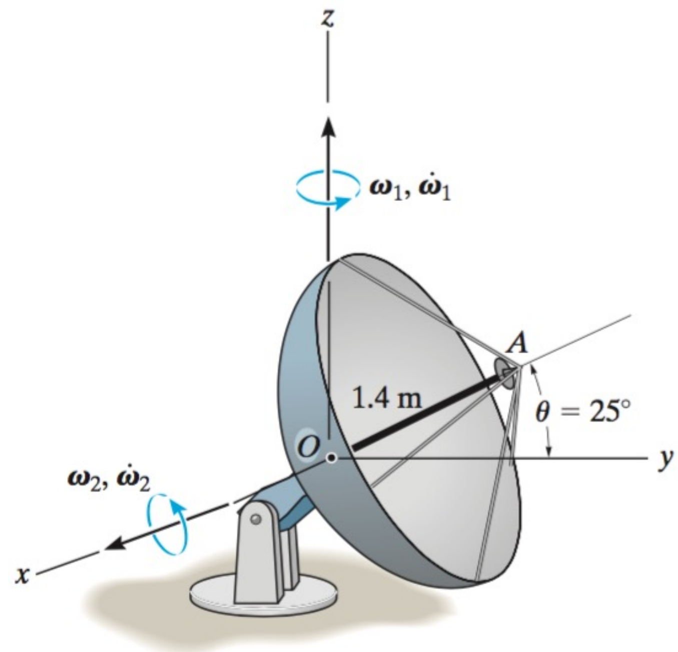
Mechanical Configuration of T3 Robot





Example

At a given instant, the satellite dish has an angular motion $\omega_1 = 6$ rad/s and $\dot{\omega}_1 = 3$ rad/s^2 about the z axis. At this same instant $\theta = 25°$, the angular motion about the x axis is $\omega_2 = 2$ rad/s, and $\dot{\omega}_2 = 1.5$ rad/s^2. Determine the velocity and acceleration of the signal horn A at this instant.



Angular Velocity: The coordinate axes for the fixed frame (X, Y, Z) and rotating frame (x, y, z) at the instant shown are set to be coincident. Thus, the angular velocity of the satellite at this instant (with reference to X, Y, Z) can be expressed in terms of **i, j, k** components.

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = \{2\mathbf{i} + 6\mathbf{k}\} \text{ rad/s}$$

Example

Angular Acceleration: The angular acceleration α will be determined by investigating separately the time rate of change of *each angular velocity component* with respect to the fixed XYZ frame. ω_2 is observed to have a *constant direction* from the rotating xyz frame if this frame is rotating at $\Omega = \omega_1 = \{6\mathbf{k}\}$ rad/s. $(\dot{\omega}_2)_{xyz} = \{1.5\mathbf{i}\}$ rad/s^2. we have

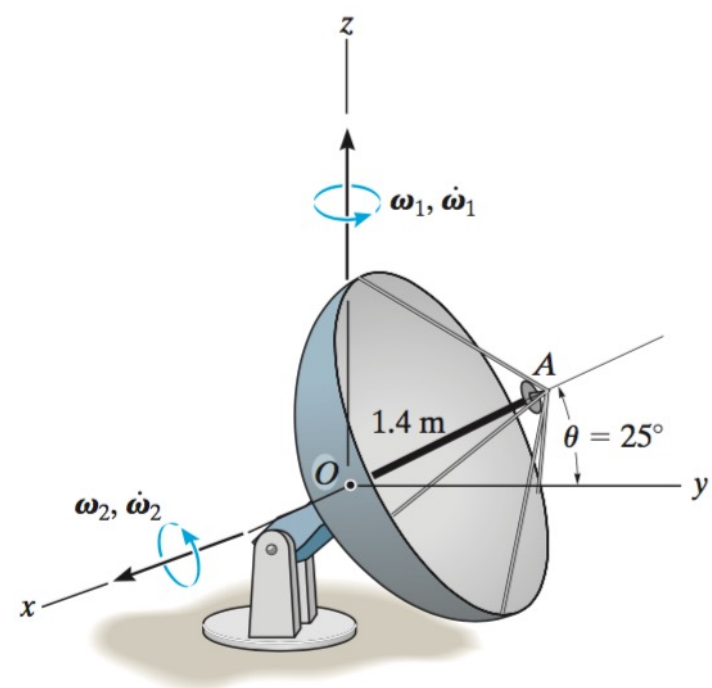
$$\dot{\omega}_2 = (\dot{\omega}_2)_{xyz} + \omega_1 \times \omega_2 = 1.5\mathbf{i} + 6\mathbf{k} \times 2\mathbf{i} = \{1.5\mathbf{i} + 12\mathbf{j}\}\ \text{rad/s}^2$$

Since ω_1 is always directed along the Z axis ($\Omega = \mathbf{0}$), then

$$\dot{\omega}_1 = (\dot{\omega}_1)_{xyz} + \mathbf{0} \times \omega_1 = \{3\mathbf{k}\}\ \text{rad/s}^2$$

Thus, the angular acceleration of the satellite is

$$\alpha = \dot{\omega}_1 + \dot{\omega}_2 = \{1.5\mathbf{i} + 12\mathbf{j} + 3\mathbf{k}\}\ \text{rad/s}^2$$



Example

Velocity and Acceleration: with the ω and α obtained above and $\mathbf{r}_A = \{1.4\cos 25°\mathbf{j} + 1.4\sin 25°\mathbf{k}\}$ m $= \{1.2688\mathbf{j} + 0.5917\mathbf{k}\}$ m, we have

$$\mathbf{v}_A = \omega \times \mathbf{r}_A = (2\mathbf{i} + 6\mathbf{k}) \times (1.2688\mathbf{j} + 0.5917\mathbf{k})$$

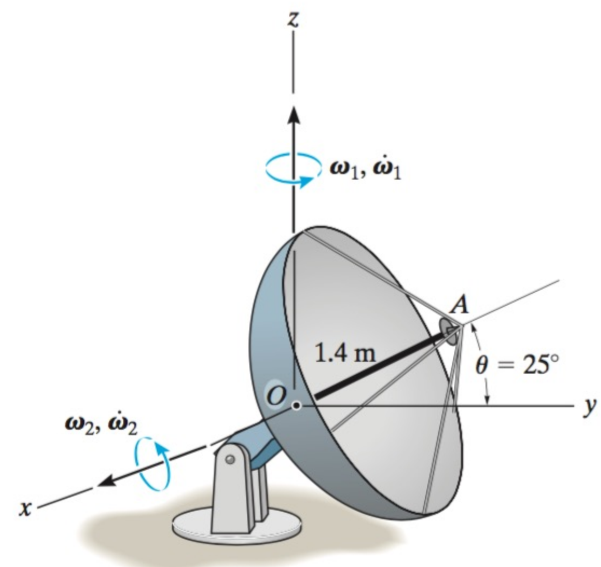
$$= \{-7.61\mathbf{i} - 1.18\mathbf{j} + 2.54\mathbf{k}\} \text{ m/s}$$

$$\mathbf{a}_A = \alpha \times \mathbf{r}_A + \omega \times (\omega \times \mathbf{r}_A)$$

$$= (1.3\mathbf{i} + 12\mathbf{j} + 3\mathbf{k}) \times (1.2688\mathbf{j} + 0.5917\mathbf{k})$$

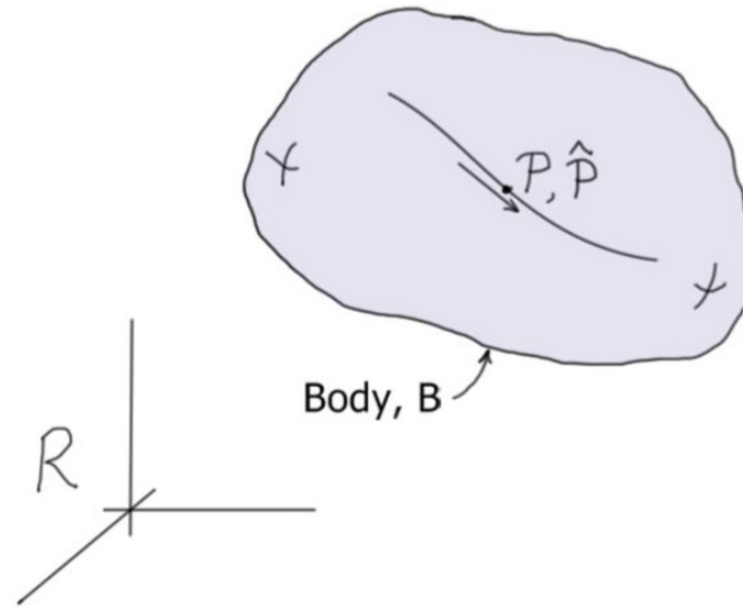
$$+ (2\mathbf{i} + 6\mathbf{k}) \times [(2\mathbf{i} + 6\mathbf{k}) \times (1.2688\mathbf{j} + 0.5917\mathbf{k})]$$

$$= \{10.4\mathbf{i} - 51.6\mathbf{j} - 0.463\mathbf{k}\} \text{ m/s}^2$$



Motion of a Point Moving on a Rigid Body

We now extend our kinematic analysis to include systems where interconnected bodies may *rotate and translate* relative to each other. In this case, we have a need to describe the kinematics of points that are moving on (relative to) a rotating body. To analyze this motion, consider the figure shown at the right. Here, we have



R: a fixed reference frame,
B: a rigid body,
P: a point *moving* on B, and
$\hat{P}$: a point *fixed* on B that coincides with P at this instant of time.

The velocity and acceleration of P may be written as

$$\boxed{{}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_{\hat{P}} + {}^B\underset{\sim}{v}_P}$$

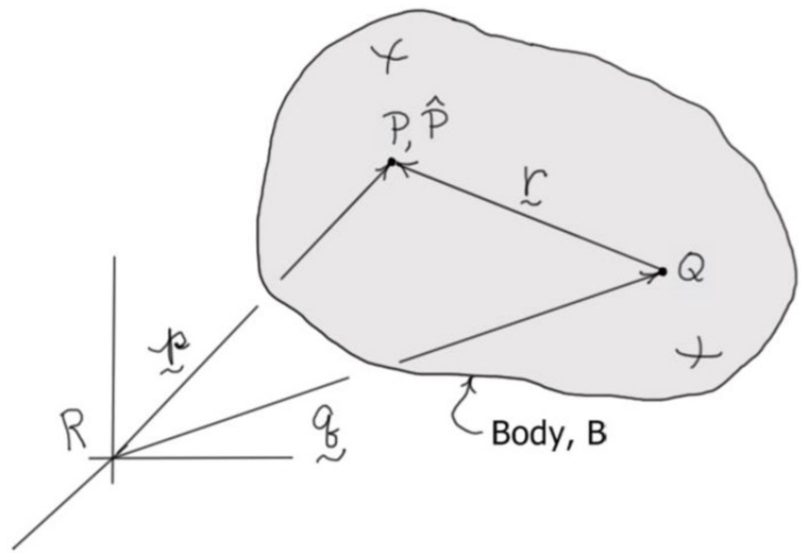
$$\boxed{{}^R\underset{\sim}{a}_P = {}^R\underset{\sim}{a}_{\hat{P}} + {}^B\underset{\sim}{a}_P + 2\left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right)}$$

$2\left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right)$: Coriolis acceleration of P

Motion of a Point Moving on a Rigid Body

Derivation

The results shown above can easily be shown by using the "derivative rule." Consider the rigid body shown in the diagram at the right. Here, we have



R : fixed reference frame,
B : rigid body,
P : point *moving* on B
$\hat{P}$: point *fixed* on B that coincides with P
Q : point *fixed* on B.

The velocity of P may be found as follows:

$$
\begin{aligned}
{}^{R}\underset{\sim}{v}_{P} &= \frac{{}^{R}d\underset{\sim}{p}}{dt} = \frac{{}^{R}d}{dt}\left(\underset{\sim}{q}+\underset{\sim}{r}\right) \\
&= \frac{{}^{R}d\underset{\sim}{q}}{dt} + \frac{{}^{R}d\underset{\sim}{r}}{dt} \quad = {}^{R}\underset{\sim}{v}_{Q} + \frac{{}^{B}d\underset{\sim}{r}}{dt} + \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right) \\
&= {}^{R}\underset{\sim}{v}_{Q} + {}^{B}\underset{\sim}{v}_{P} + \left({}^{R}\underset{\sim}{\omega}_{B} \times \underset{\sim}{r}\right)
\end{aligned}
$$

Now, letting $\underset{\sim}{r} \rightarrow \underset{\sim}{0}$ (that is, letting Q be $\hat{P}$) the desired result is obtained.

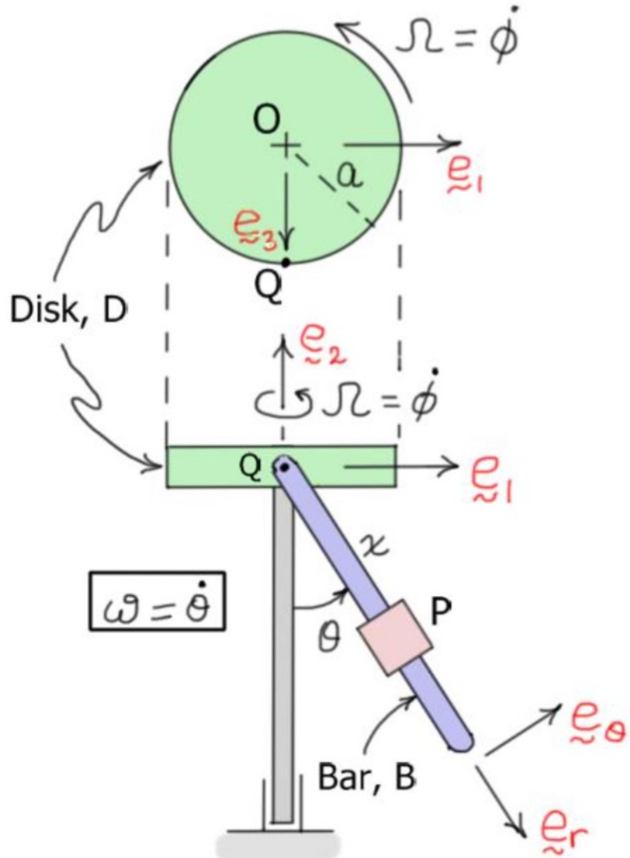
Motion of a Point Moving on a Rigid Body

The acceleration of P may be found as follows:

$$
\begin{aligned}
{}^R\underset{\sim}{a}_P &= \frac{{}^R d}{dt}\left({}^R\underset{\sim}{v}_Q + {}^B\underset{\sim}{v}_P + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)\right) \\
&= \frac{{}^R d}{dt}\left({}^R\underset{\sim}{v}_Q\right) + \left\{\frac{{}^R d}{dt}\left({}^B\underset{\sim}{v}_P\right)\right\} + \left\{\frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)\right\} \\
&= {}^R\underset{\sim}{a}_Q + \left\{\frac{{}^B d}{dt}\left({}^B\underset{\sim}{v}_P\right) + \left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right)\right\} + \left\{\left({}^R\underset{\sim}{\alpha}_B \times \underset{\sim}{r}\right) + {}^R\underset{\sim}{\omega}_B \times \left(\frac{{}^B d\underset{\sim}{r}}{dt} + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)\right)\right\} \\
&= {}^R\underset{\sim}{a}_Q + \left\{{}^B\underset{\sim}{a}_P + \left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right)\right\} + \left\{\left({}^R\underset{\sim}{\alpha}_B \times \underset{\sim}{r}\right) + \left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right) + {}^R\underset{\sim}{\omega}_B \times \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}\right)\right\}
\end{aligned}
$$

Now, letting $\underset{\sim}{r} \to \underset{\sim}{0}$ (that is, letting Q be $\hat{P}$) the desired result is obtained.

Example: Slider on a Rotating Bar



<u>Problem</u>: Given the position coordinates (x, ϕ, θ), find $\underset{\sim}{v}_P$ the velocity of P and $\underset{\sim}{a}_P$ the acceleration of P.

<u>Solution</u>:

Reference frames: $\begin{cases} D : (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3) \\ B : (\underset{\sim}{e}_r, \underset{\sim}{e}_\theta, \underset{\sim}{e}_3) \end{cases}$

Angular Velocity of B:

$$\boxed{\begin{aligned} {}^R\underset{\sim}{\omega}_B &= {}^R\underset{\sim}{\omega}_D + {}^D\underset{\sim}{\omega}_B \\ &= \Omega\underset{\sim}{e}_2 + \omega\underset{\sim}{e}_3 \end{aligned}}$$

Angular Acceleration of B:

$${}^R\underset{\sim}{\alpha}_B = \frac{{}^R d}{dt}\left({}^R\underset{\sim}{\omega}_B\right) = \dot\Omega\underset{\sim}{e}_2 + \dot\omega\underset{\sim}{e}_3 + \omega\dot{\underset{\sim}{e}}_3 = \dot\Omega\underset{\sim}{e}_2 + \dot\omega\underset{\sim}{e}_3 + \omega\left(\Omega\underset{\sim}{e}_2 \times \underset{\sim}{e}_3\right)$$

$$\boxed{{}^R\underset{\sim}{\alpha}_B = \omega\,\Omega\,\underset{\sim}{e}_1 + \dot\Omega\,\underset{\sim}{e}_2 + \dot\omega\,\underset{\sim}{e}_3}$$

Example: Slider on a Rotating Bar

Velocity of P:

$$\boxed{{}^R\underset{\sim}{v}_P = {}^R\underset{\sim}{v}_{\hat{P}} + {}^B\underset{\sim}{v}_P}$$

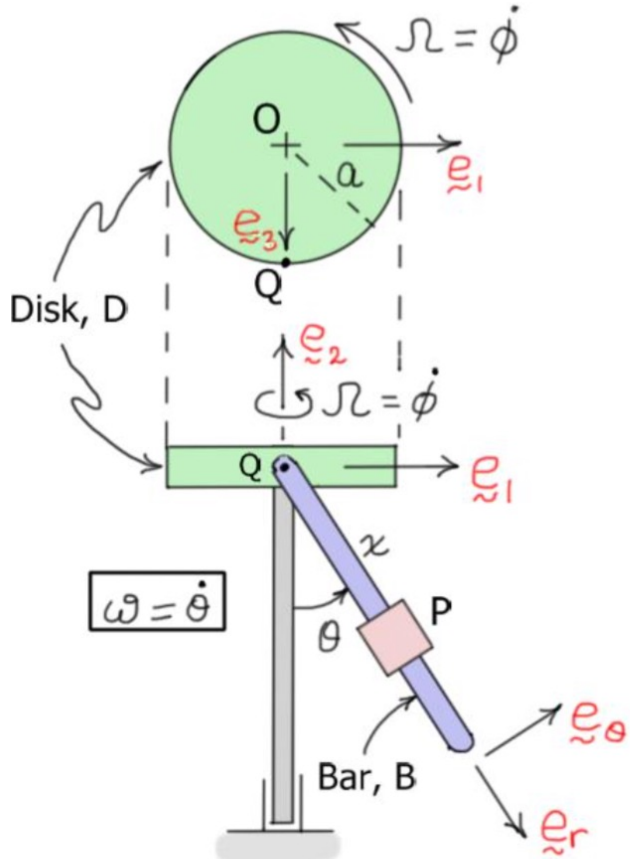
(P moves on B and $\hat{P}$ is fixed on B)

where

$$
\begin{aligned}
{}^R\underset{\sim}{v}_{\hat{P}} &= {}^R\underset{\sim}{v}_Q + {}^R\underset{\sim}{v}_{\hat{P}/Q} \\
&= a\Omega\underset{\sim}{e}_1 + \left({}^R\underset{\sim}{\omega}_B \times \underset{\sim}{r}_{\hat{P}/Q}\right) \\
&= a\Omega\underset{\sim}{e}_1 + \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ 0 & \Omega & \omega \\ xS_\theta & -xC_\theta & 0 \end{vmatrix}
\end{aligned}
$$

$$\boxed{{}^R\underset{\sim}{v}_{\hat{P}} = (a\Omega + x\omega C_\theta)\underset{\sim}{e}_1 + (x\omega S_\theta)\underset{\sim}{e}_2 - (x\Omega S_\theta)\underset{\sim}{e}_3}$$

$$\boxed{{}^B\underset{\sim}{v}_P = \dot{x}\underset{\sim}{e}_r = \dot{x}(S_\theta\underset{\sim}{e}_1 - C_\theta\underset{\sim}{e}_2)}$$



Example: Slider on a Rotating Bar

So,

$$\boxed{{}^R\underset{\sim}{v}_P = (a\Omega + x\omega C_\theta + \dot{x}S_\theta)\underset{\sim}{e}_1 + (x\omega S_\theta - \dot{x}C_\theta)\underset{\sim}{e}_2 - (x\Omega S_\theta)\underset{\sim}{e}_3}$$

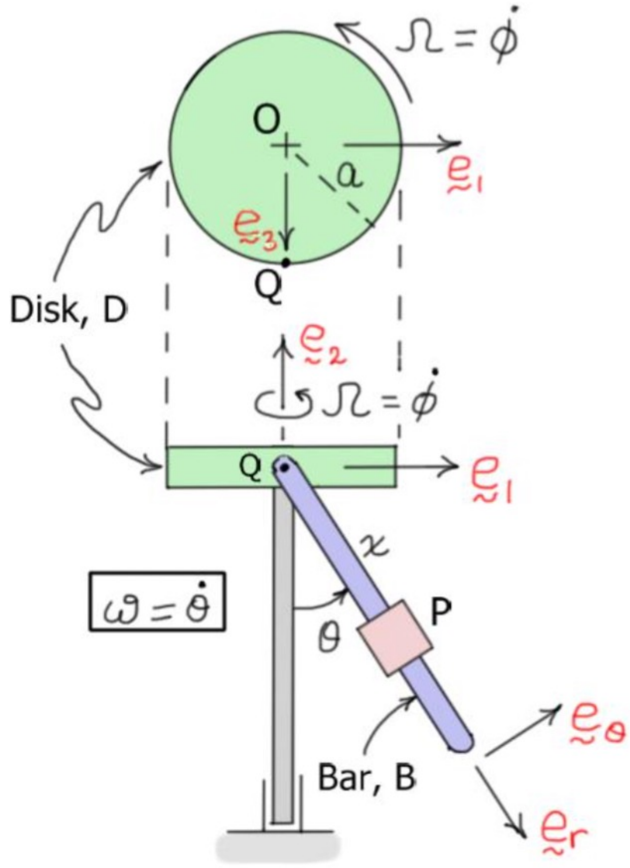
Acceleration of P:

$$\boxed{{}^R\underset{\sim}{a}_P = {}^R\underset{\sim}{a}_{\hat{P}} + {}^B\underset{\sim}{a}_P + 2\left({}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P\right)}$$

where

$${}^R\underset{\sim}{a}_{\hat{P}} = {}^R\underset{\sim}{a}_Q + {}^R\underset{\sim}{a}_{\hat{P}/Q}$$

$$\boxed{{}^R\underset{\sim}{a}_Q = a\dot{\Omega}\underset{\sim}{e}_1 - a\Omega^2\underset{\sim}{e}_3}$$



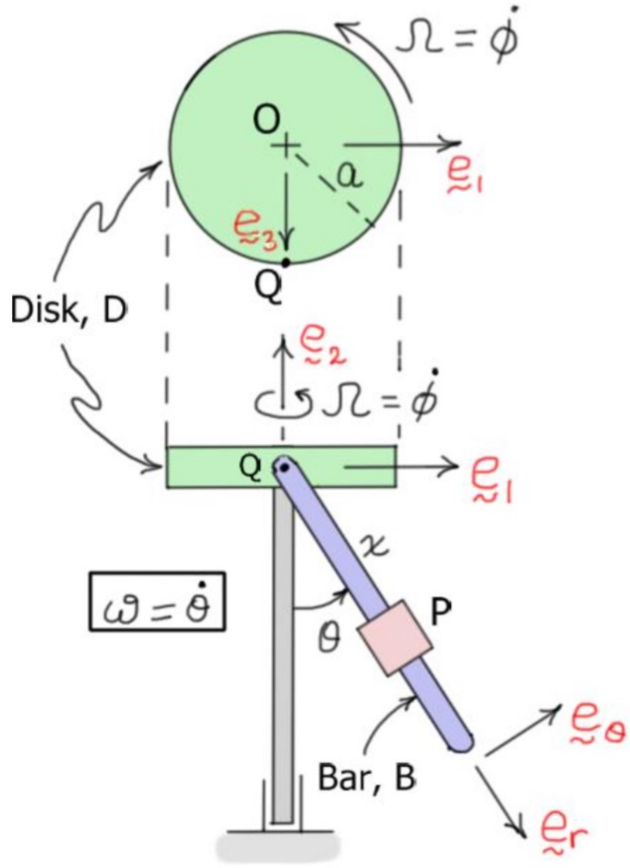
Example: Slider on a Rotating Bar

$$ {}^{R}\underset{\sim}{a}_{\hat{P}/Q} = ({}^{R}\underset{\sim}{\alpha}_{B} \times \underset{\sim}{r}_{\hat{P}/Q}) + ({}^{R}\underset{\sim}{\omega}_{B} \times {}^{R}\underset{\sim}{v}_{\hat{P}/Q}) $$

$$ = \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ \omega\Omega & \dot{\Omega} & \dot{\omega} \\ xS_\theta & -xC_\theta & 0 \end{vmatrix} + \begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ 0 & \Omega & \omega \\ x\omega C_\theta & x\omega S_\theta & -x\Omega S_\theta \end{vmatrix} $$

$$ \boxed{ {}^{R}\underset{\sim}{a}_{\hat{P}/Q} = \left(x\dot{\omega}C_\theta - x\Omega^2 S_\theta - x\omega^2 S_\theta\right)\underset{\sim}{e}_1 + \left(x\dot{\omega}S_\theta + x\omega^2 C_\theta\right)\underset{\sim}{e}_2 - \left(x\omega\Omega C_\theta + x\dot{\Omega}S_\theta + x\omega\Omega C_\theta\right)\underset{\sim}{e}_3 } $$

$$ \boxed{ {}^{B}\underset{\sim}{a}_{P} = \ddot{x}\underset{\sim}{e}_r = \ddot{x}\left(S_\theta \underset{\sim}{e}_1 - C_\theta \underset{\sim}{e}_2\right) } $$



Example: Slider on a Rotating Bar

$$2\,{}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P = 2\begin{vmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \\ 0 & \Omega & \omega \\ \dot{x}S_\theta & -\dot{x}C_\theta & 0 \end{vmatrix}$$

$$\boxed{2\,{}^R\underset{\sim}{\omega}_B \times {}^B\underset{\sim}{v}_P = 2\left[\left(\dot{x}\omega C_\theta\right)\underset{\sim}{e}_1 + \left(\dot{x}\omega S_\theta\right)\underset{\sim}{e}_2 - \left(\dot{x}\Omega S_\theta\right)\underset{\sim}{e}_3\right]}$$

Combining all the above terms gives:

$$\begin{aligned} {}^R\underset{\sim}{a}_P = &\left(a\dot{\Omega} + x\dot{\omega}C_\theta - xS_\theta(\omega^2+\Omega^2) + \ddot{x}S_\theta + 2\dot{x}\omega C_\theta\right)\underset{\sim}{e}_1 + \\ &\left(x\dot{\omega}S_\theta + x\omega^2 C_\theta - \ddot{x}C_\theta + 2\dot{x}\omega S_\theta\right)\underset{\sim}{e}_2 - \\ &\left(a\Omega^2 + x\dot{\Omega}S_\theta + 2x\omega\Omega C_\theta + 2\dot{x}\Omega S_\theta\right)\underset{\sim}{e}_3 \end{aligned}$$