Case 2: If a pair of the distinct eigenvalues is complex conjugate numbers, the corresponding eigenvectors are also complex conjugate vectors. Let

$$\lambda_1 = \alpha + i\beta, \text{and } \lambda_2 = \alpha - i\beta$$

Are complex conjugate eigenvalues, then

$v^1 = u^1 + iu^2$ and $v^2 = u^1 - iu^2$

are complex conjugate eigenvectors. Then the two linearly independent real solutions are

$$x^1 = \frac{\left[(u^1 + iu^2)e^{(\alpha+i\beta)t} + (u^1 - iu^2)e^{(\alpha-i\beta)t}\right]}{2} = \text{RealPart}\left((u^1 + iu^2)e^{(\alpha+i\beta)t}\right)$$

$$= e^{\alpha t}\text{RealPart}\big((u^1 + iu^2)(\cos\beta t + i\sin\beta t)\big)$$

Hence

$$x^1 = e^{\alpha t}(u^1 \cos\beta t - u^2 \sin\beta t)$$

And

$$x^2 = \frac{\left[(u^1 + iu^2)e^{(\alpha+i\beta)t} - (u^1 - iu^2)e^{(\alpha-i\beta)t}\right]}{2\text{i}} = \text{ImPart}\left((u^1 + iu^2)e^{(\alpha+i\beta)t}\right)$$

$$= e^{\alpha t}\text{ImPart}\big((u^1 + iu^2)(\cos\beta t + i\sin\beta t)\big)$$

Hence

$$x^2 = e^{\alpha t}(u^1 \sin\beta t + u^2 \cos\beta t).$$

Case 3: In the case of repeated eigenvalues, determinant of a different solution corresponding to the all eigenvalues may require a different procedure. In the state equation, $x' = Ax$, let the state matrix A be an *nxn* real constant matrix; and λ_1 has a multiplicity m where $1 < m \leq n$; and the other eigenvalues λ_{m+1}, $\lambda_{m+2}, \dots, \lambda_n$ are distinct. Under these considerations two possible sub cases may arise;

a. Repeated eigenvalue λ_1 with the multiplicity *m* may have equal number of independent solution (*p* = *m*, where *p* is the number of independent solution). In this sub case the number of independent solutions are complete, so the general solution are obtained in terms of the linear combination of them similar to before

b. Repeated eigenvalue λ, with the multiplicity *m* may have smaller number independent solution ($p<m$) where *p* is a number of independent solutions. For the sake of simplicity let $m=2$ and $p=1$. As known, in the case of distinct eigenvalues, a solution proposal is made as

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} e^{\lambda t} \Longrightarrow x = \alpha e^{\lambda t} \quad \text{where } x \text{ and } \alpha \text{ are vectors.}$$

This is equally valid for the repeated eigenvalue. For the missing solutions, however, a different proposal should be made as

$$x = (\gamma t + \beta)e^{\lambda t}$$

Substitution of this proposal into the state equation, $\dot{x} = Ax$, yields;

$$(\gamma t + \beta)\lambda e^{\lambda t} + \gamma e^{\lambda t} = A(\gamma t + \beta)e^{\lambda t}$$

After grouping and cancellations,

$$(\lambda\gamma - A\gamma)t + (\lambda\beta + \gamma - A\beta) = 0, \quad \text{yields two equations}$$

$$\lambda\gamma - A\gamma = 0 \implies (A - \lambda I)\gamma = 0\; and \implies (A - \lambda I) = 0$$
$$\lambda\beta + \gamma - A\beta = 0$$

The first expression implies that γ in fact is an eigenvector a corresponding to the eigenvalue λ. Substitution of this feature ($\gamma = \alpha$) into the second equation;

$$\lambda\beta + \alpha - A\beta = 0 \xrightarrow{yields} (A\beta - \lambda\beta) = \alpha \quad \text{so} \quad (A - \lambda I)\beta = \alpha$$

As the equation for the determination of β’ It can be shown that the solutions $x = \alpha\, e^{\lambda t}$ and $x = (\alpha t + \beta)e^{\lambda t}$ are linearly independent. Also, notice that in the case of further missing independent solutions; the next solution can be proposed as:

$$x = (\theta t^2 + \tau t + \emptyset)e^{\lambda t} \qquad \text{and so on.}$$

Example 2.7

Solve the given homogeneous linear system;

$$\begin{aligned} x_1' &= 3x_1 + x_2 - x_3 \\ x_2' &= x_1 + 3x_2 - x_3 \\ x_3' &= 3x_1 + 3x_2 - x_3 \end{aligned}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad x' = Ax \quad \text{where} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Eigen values are obtained using $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3 & 3 & -1-\lambda \end{vmatrix} = 0 \xrightarrow{yields} \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0 \Longrightarrow (\lambda - 1)(\lambda - 2)^2 = 0$$

So the eigenvalues are $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 2$ and the corresponding eigenvector is obtain using

$$A\alpha = \lambda\alpha \quad \text{where} \quad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\text{For } \lambda = 1 \Longrightarrow \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 1 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

This corresponds to the equation set

$$\begin{aligned} 2\alpha_1 + \alpha_2 - \alpha_3 &= 0 \\ \alpha_1 + 2\alpha_2 - \alpha_3 &= 0 \\ 3\alpha_1 + 3\alpha_2 - 2\alpha_3 &= 0 \end{aligned}$$

Let's check the rank

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 0 & -3 & 1 \\ 1 & 2 & -1 \\ 0 & -3 & -1 \end{vmatrix} = -\begin{vmatrix} -3 & 1 \\ -3 & 1 \end{vmatrix} = 0$$

$$\Delta_{\text{sub}} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 13 \neq 0 \Longrightarrow \text{rank} = 2$$

So, select 3-2 =1 unknown as known (say $\alpha_1 = 1$) and solve the others using any two equation, solution; $\alpha_1 = 1,\ \alpha_2 = 1,\ \alpha_3 = 3$

Therefore, the eigenvector corresponding to λ=1 is; $\alpha = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

Since the solution is $x = \alpha e^{\lambda t}$ $\quad x = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^{t}$ $\quad x = \begin{bmatrix} e^{t} \\ e^{t} \\ 3e^{t} \end{bmatrix}$

Let's obtain the eigenvector for the repeated eigenvalue λ=2;

$$A\alpha = \lambda\alpha \Longrightarrow \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

The corresponding equation set;

$$\begin{array}{rrrl} \alpha_1 & +\alpha_2 & -\alpha_3 & = 0 \\ \alpha_1 & +\alpha_2 & -\alpha_3 & = 0 \\ 3\alpha_1 & +3\alpha_2 & -3\alpha_3 & = 0 \end{array}$$

The first one is redundant equation.

$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{vmatrix} \Longrightarrow$ **Rank=1** So 3-2 two of them must be selected arbitrary.

Let's select $\alpha_1 = 1,\ \alpha_3 = 0 \Longrightarrow \alpha_2 = -1$ or $\alpha_1 = 1,\ \alpha_2 = 0 \Longrightarrow \alpha_3 = 1$

$$\alpha = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Longrightarrow x = \alpha e^{\lambda t} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ -e^{2t} \\ 0 \end{bmatrix}$$ and

$$\alpha = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow x = \alpha e^{\lambda t} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}$$

Therefore, the general solution is the linear combination of these three solution as follows;

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C_1 \begin{bmatrix} e^{t} \\ e^{t} \\ 3e^{t} \end{bmatrix} + C_2 \begin{bmatrix} e^{2t} \\ -e^{2t} \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}$$

Then the solution of the given homogeneous set is;

$$x_1 = C_1 e^{t} + (C_2 + C_3) e^{2t}$$

$$x_2 = C_1 e^{t} - C_2 e^{2t}$$

$$x_3 = 3C_1 e^{t} + C_3 e^{2t}$$

Example 2.8

Solve the homogeneous linear system

$$\begin{aligned} x_1' &= 4x_1 \quad +3x_2 \quad +x_3 \\ x_2' &= -4x_1 \quad -4x_2 \quad -2x_3 \\ x_3' &= 8x_1 \quad +12x_2 \quad +6x_3 \end{aligned}$$

The matrix representation of this set,

$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ Assuming a solution of the form, $x = \alpha e^{\lambda t}$ where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$;

$|A - \lambda I| = \begin{bmatrix} 4-\lambda & 3 & 1 \\ -4 & -4-\lambda & -2 \\ 8 & 12 & 6-\lambda \end{bmatrix} = 0$ Expanding the characteristic equation;

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0 \Longrightarrow (\lambda - 2)^3 = 0 \Longrightarrow \lambda_1 = \lambda_2 = \lambda_3 = 2$$

Let's obtain the eigenvector of the eigenvalue λ=2;

$$A\alpha = \lambda\alpha \Longrightarrow \begin{bmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

Some manipulations yield the homogenous set,

$$\begin{aligned} 2\alpha_1 \quad +3\alpha_2 \quad +\alpha_3 &= 0 \\ -4\alpha_1 \quad -6\alpha_2 \quad -2\alpha_3 &= 0 \\ 8\alpha_1 \quad +12\alpha_2 \quad +4\alpha_3 &= 0 \end{aligned}$$

The rank of the coefficient matrix $\begin{bmatrix} 2 & 3 & 1 \\ -4 & -6 & -2 \\ 8 & 12 & 4 \end{bmatrix}$ is 1

because all of the 2x2 matrices are all zero, too, so, two arbitrary selections are made

Let's select $\alpha_1 = 1,\ \alpha_2 = 0$ then $\alpha_3 = -2$ or selected; $\alpha_1 = 0,\ \alpha_2 = 1$ then $\alpha_3 = -3$

Therefore, two of the eigenvectors and independent solutions corresponding to λ=2

$$\alpha = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ 0 \\ -2e^{2t} \end{bmatrix} \quad \& \quad \alpha = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} e^{2t} = \begin{bmatrix} 0 \\ e^{2t} \\ -3e^{2t} \end{bmatrix}$$

Since the state matrix 3x3 then the total number of independent solutions is 3; so, one of the solutions is missing.

The third solution proposal for λ=2 is made as $(\alpha t+\beta)e^{2t}$ where

α satisfies $(A-2I)\alpha=0$ and β satisfies $(A-2I)\beta=\alpha.$

Notice that α above is the linear combinations of the eigenvector obtained before

$$\alpha = k_1\begin{bmatrix}1\\0\\-2\end{bmatrix}+k_2\begin{bmatrix}0\\1\\-3\end{bmatrix}=\begin{bmatrix}k_1\\k_2\\-2k_1-3k_2\end{bmatrix}$$

and $\beta=\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}$ under these considerations, since $(A-2I)\beta=\alpha$ then,

$$\left[\begin{bmatrix}4&3&1\\-4&-4&-2\\8&12&6\end{bmatrix}-2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right]\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}=\begin{bmatrix}k_1\\k_2\\-2k_1-3k_2\end{bmatrix}$$

Manipulations yields that

$$\begin{array}{llll}2\beta_1 & +3\beta_2 & +\beta_3 & =k_1\\ -4\beta_1 & -6\beta_2 & -2\beta_3 & =k_2\\ 8\beta_1 & 12\beta_2 & +4\beta_3 & =-2k_1-3k_2\end{array} \quad \Longrightarrow k_2=-2k_1$$

Since the rank of the coefficient matrix is 2, 3-2=1 free selection is made; say

$k_1=1 \Longrightarrow k_2=-2$; substitution yields,

$$\begin{array}{lllll}2\beta_1 & +3\beta_2 & +\beta_3 & = & 1\\ -4\beta_1 & -6\beta_2 & -2\beta_3 & = & -2\\ 8\beta_1 & 12\beta_2 & +4\beta_3 & = & 4\end{array}$$ Rank=1, so proposal should be made

Let $\beta_1=\beta_2=0 \Longrightarrow \beta_3=1 \Longrightarrow \beta=\begin{bmatrix}0\\0\\1\end{bmatrix}$

Since $\alpha=\begin{bmatrix}k_1\\k_2\\-2k_1-3k_2\end{bmatrix}=\begin{bmatrix}1\\-2\\4\end{bmatrix}$ and $\beta=\begin{bmatrix}0\\0\\1\end{bmatrix}$

Substitution of these into $x=[\alpha t+\beta]e^{\lambda t}$

$$x = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} te^{2t} \\ -2te^{2t} \\ (4t+1)e^{2t} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C_1 \begin{bmatrix} e^{2t} \\ 0 \\ -2e^{2t} \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ e^{2t} \\ -3e^{2t} \end{bmatrix} + C_3 \begin{bmatrix} te^{2t} \\ -2te^{2t} \\ (4t+1)e^{2t} \end{bmatrix}$$

Therefore;

$x_1 = C_1 e^{2t} + C_3 t e^{2t}$

$x_2 = C_2 e^{2t} - 2C_3 t e^{2t}$

$x_3 = -2C_1 e^{2t} - 3C_2 e^{2t} + C_3 (4t+1) e^{2t}$

Non-homogeneous Systems of Linear Differential Equations:

Let $f(t) \neq 0$, and consider the system,

$$x' = Ax + f, \qquad f \neq 0.$$

Let solutions of the homogeneous system

$$x' = Ax$$

be;

$$x_h = C_1 x^1 + C_2 x^2 + \cdots + C_n x^n$$

If a particular solution of the non-homogeneous system is x_p, then the general solution of the non-homogeneous system will be

$$x_{gen} = x_h + x_p$$

As in the case of scalar equations we'll distinguish two cases, and apply techniques similar to the method of undetermined coefficients, and variation of parameters accordingly.

The Method of Undetermined Coefficients

Let

$$f(t) = \begin{bmatrix} a_1 \\ a_2 \\ . \\ a_n \end{bmatrix} \varphi(t),$$

and $\varphi(t)$ is a function of finite derivatives, where the set

$$D = \{\varphi_1(t), \varphi_2(t), \dots, \varphi_p(t)\}$$

is finite. And let functions in the solutions

$$x^i = \propto_i e^{\lambda_i t}, i = 1,2,..,n$$

of the homogeneous system be

$$H = \{f_1(t), f_2(t), \dots, f_n(t)\}$$

If $H \cap D = \emptyset$ empty set, a particular solution proposal be

$$x_p = c^1\varphi_1(t) + c^2\varphi_2(t) + \dots + c^p\varphi_p(t)$$

Where $\{c^1, c^2, \dots, c^p\}$ are n dimensional vectors to be specified.

If $H \cap D \neq \emptyset$ a non-empty set, D is multiplied by t till $H \cap t^n D = \emptyset$ is obtained. Then a particular solution proposal be a linear combination of contents of $t^n D$ with n dimensional vectors $\{c^0, c^1, c^2, \dots, c^p\}$ that are to be specified.

<u>Example 2.10 (Non-Homogeneous Version)</u>

Consider the homogenous linear system, and solve it using matrix method.

$$\begin{aligned} x_1' &= 7x_1 - x_2 + 6x_3 + \cos 2t \\ x_2' &= -10x_1 + 4x_2 - 12x_3 \\ x_3' &= -2x_1 + x_2 - x_3 \end{aligned}$$

In matrix notation

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos 2t \quad (1)$$

Along the previous Examples the homogeneous solution is found to be

$x_h = C_1 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} e^{3t} + C_3 \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix} e^{5t}$ then;

$$H = \{e^{2t}, e^{3t}, e^{5t}\}$$

and here

$$D = \{\cos 2t, \sin 2t\}$$

Since $H \cap D = \emptyset$ empty set, a particular solution proposal be

$x_p = c^1 \cos 2t + c^2 \sin 2t, \quad x_p{}' = -2c^1 \sin 2t + 2c^2 \cos 2t.$

Then Equation (1) implies

$$-2c^1 \sin 2t + 2c^2 \cos 2t = Ac^1 \cos 2t + Ac^2 \sin 2t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos 2t$$

Equating the coefficients of $\{\cos 2t, \sin 2t\}$ one has

$$\sin 2t: -2c^1 = Ac^2$$

$$\cos 2t: \; 2c^2 = Ac^1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to 2Ac^2 = A^2c^1 + \mathrm{A} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to -4c^1 = A^2c^1 + \mathrm{A} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\to (A^2 + 4I)c^1 = \mathrm{A} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \to c^1 = (A^2 + 4I)^{-1} \,\mathrm{x}\, \mathrm{A} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$c^1 = \{-0.25, -0.19, -0.05\}$

$c^2 = -2A^{-1}c^1$

$c^2 = \{0.24, 0.21, -0.17\}$

$$x_p = \{-0.25, -0.19, -0.05\} \cos 2t + \{0.24, 0.21, -0.17\} \sin 2t$$

$$x_{gen} = x_h + x_p$$

$x_1 = C_1 e^{2t} + C_2 e^{3t} + 3C_3 e^{5t} - 0.25 \cos 2t + 0.24 \sin 2t$

$$x_2 = -C_1e^{2t} - 2C_2e^{3t} - 6C_3e^{5t} - 0.19\cos 2t + 0.21\sin 2t$$

$$x_3 = -C_1e^{2t} - C_2e^{3t} - 2C_3e^{5t} - 0.05\cos 2t - 0.17\sin 2t$$

■

Exercises

1. $\begin{aligned} x' - 2x - y &= -1, \quad x(0) = 1 \\ y' + x - 2y &= 8, \quad\;\; y(0) = 1 \end{aligned}$

2. $\begin{aligned} x' - 3x &= -4\sin 2t\,, \quad x(0) = 2 \\ y' - 5x + 2y &= \cos 2t\,, \quad\;\; y(0) = -1 \end{aligned}$