CHAPTER 1

1.1. Show that

$$\textbf{(i)}\ \frac{\partial x_i}{\partial x_j} = \delta_{ij} \ \textbf{and (ii)}\ R = \sqrt{x_i x_i}\ ,$$

where $R = |\boldsymbol{R}|$ is the distance from the origin. Hence find $\partial R/\partial x_j$ in index notation. Confirm your result by finding $\partial R/\partial x$ in x, y, z notation.

For an orthogonal coördinate system,

$$\frac{\partial x}{\partial y} = 0$$

(this is what is meant by orthogonality) and

$$\frac{\partial x}{\partial x} = 1\ .$$

In index notation, these results can be combined as

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}\ .$$

The distance from the origin is

$$R = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{x_i x_i}\ .$$

Combining these results, we have

$$\begin{aligned}
\frac{\partial R}{\partial x_j} &= \frac{\partial}{\partial x_j}\sqrt{x_i x_i} = \frac{1}{2\sqrt{x_i x_i}}\left(\frac{\partial x_i}{\partial x_j}x_i + x_i\frac{\partial x_i}{\partial x_j}\right) \\
&= \frac{2x_i\delta_{ij}}{2\sqrt{x_i x_i}} \\
&= \frac{x_j}{\sqrt{x_i x_i}}\ .
\end{aligned}$$

In x, y, z notation, we would have $R = \sqrt{x^2 + y^2 + z^2}$ and hence

$$\frac{\partial R}{\partial x} = \frac{(2x)}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{R}\ ,$$

which agrees.

1.2. Prove that the partial derivatives $\partial^2 f/\partial x^2; \partial^2 f/\partial x \partial y; \partial^2 f/\partial y^2$ of the scalar function $f(x, y)$ transform into the rotated coördinate system x', y' by rules similar to equations (1.15–1.17).

We first note from equation (1.43) that

$$\frac{\partial}{\partial x'} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$

and by a similar argument

$$\begin{aligned}
\frac{\partial}{\partial y'} &= \boldsymbol{\nabla}.\boldsymbol{j'} \\
&= \boldsymbol{i}.\boldsymbol{j'} \frac{\partial}{\partial x} + \boldsymbol{j}.\boldsymbol{j'} \frac{\partial}{\partial y} \\
&= -\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y} \ .
\end{aligned}$$

We then have

$$\begin{aligned}
\frac{\partial^2 f}{\partial x'^2} &= \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right)\left(\cos\theta \frac{\partial f}{\partial x} + \sin\theta \frac{\partial f}{\partial y}\right) \\
&= \cos^2\theta \frac{\partial^2 f}{\partial x^2} + \sin^2\theta \frac{\partial^2 f}{\partial y^2} + 2\sin\theta\cos\theta \frac{\partial^2 f}{\partial x \partial y} \\
\frac{\partial^2 f}{\partial x' \partial y'} &= \left(-\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}\right)\left(\cos\theta \frac{\partial f}{\partial x} + \sin\theta \frac{\partial f}{\partial y}\right) \\
&= (\cos^2\theta - \sin^2\theta) \frac{\partial^2 f}{\partial x \partial y} + \sin\theta\cos\theta \left(\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x^2}\right) \\
\frac{\partial^2 f}{\partial y'^2} &= \left(-\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}\right)\left(-\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}\right) \\
&= \cos^2\theta \frac{\partial^2 f}{\partial y^2} + \sin^2\theta \frac{\partial^2 f}{\partial x^2} - 2\sin\theta\cos\theta \frac{\partial^2 f}{\partial x \partial y}
\end{aligned}$$

and these equations are clearly of the same form as (1.15–1.17).

1.3. Show that the direction cosines defined in (1.19) satisfy the identity

$$l_{ij}l_{ik} = \delta_{jk} \ .$$

Hence or otherwise, show that the product $\sigma_{ij}\sigma_{ij}$ is invariant under coördinate transformation.

For a given value of j, l_{ij} defines the components in x'_i coördinates of a unit vector in the direction of the x_j-axis. It follows that

$$l_{ij}l_{ik} \ ,$$

is the dot product between two unit vectors defined in the x'_i-system. One of these vectors represents the x_j-axis and the other the x_k-axis. This dot product is unity if the axes are identical and zero if they are not, since the three axes are orthogonal. Hence

$$l_{ij}l_{ik} = \delta_{jk} \ .$$

Now consider

$$\sigma'_{ij} = l_{ip}l_{jq}\sigma_{pq} \ ,$$

from equation (1.22). We can write another version of the same quantity using different dummy indices as

$$\sigma'_{ij} = l_{ir}l_{js}\sigma_{rs} \ .$$

We need to do this because otherwise when we take the product the same index would appear more than twice which leads to an ambiguity in terms of the summation convention.

Taking the product of these quantities, including the implied summations, we then have

$$\sigma'_{ij}\sigma'_{ij} = l_{ip}l_{jq}l_{ir}l_{js}\sigma_{pq}\sigma_{rs}$$

and using the identity we proved above, this gives

$$\sigma'_{ij}\sigma'_{ij} = \delta_{pr}\delta_{qs}\sigma_{pq}\sigma_{rs} = \sigma_{pq}\sigma_{pq} \ ,$$

showing that the product is invariant under coördinate transformation.

1.4. By restricting the indices i, j etc. to the values 1,2 only, show that the two-dimensional stress transformation relations (1.15–1.17) can be obtained from (1.22) using the two-dimensional direction cosines (1.20).

From (1.20), we have

$$l_{11} = \cos\theta \; ; \quad l_{12} = \sin\theta \; ; \quad l_{21} = -\sin\theta \; ; \quad l_{22} = \cos\theta$$

and from (1.22),

$$\sigma'_{ij} = l_{ip} l_{jq} \sigma_{pq} \; .$$

Thus,

$$\begin{aligned}
\sigma'_{11} &= l_{11}l_{11}\sigma_{11} + l_{12}l_{11}\sigma_{21} + l_{11}l_{12}\sigma_{12} + l_{12}l_{12}\sigma_{22} \\
&= \sigma_{11}\cos^2\theta + 2\sigma_{12}\sin\theta\cos\theta + \sigma_{22}\sin^2\theta \\
\sigma'_{12} &= l_{11}l_{21}\sigma_{11} + l_{12}l_{21}\sigma_{21} + l_{11}l_{22}\sigma_{12} + l_{12}l_{22}\sigma_{22} \\
&= -\sigma_{11}\sin\theta\cos\theta - \sigma_{21}\sin^2\theta + \sigma_{12}\cos^2\theta + \sigma_{22}\sin\theta\cos\theta \\
\sigma'_{22} &= l_{21}l_{21}\sigma_{11} + l_{22}l_{21}\sigma_{21} + l_{21}l_{22}\sigma_{12} + l_{22}l_{22}\sigma_{22} \\
&= \sigma_{11}\sin^2\theta - 2\sigma_{12}\sin\theta\cos\theta + \sigma_{22}\cos^2\theta
\end{aligned}$$

and these are identical to (1.15–1.17).

1.5. Use the index notation to develop concise expressions for the three stress invariants I_1, I_2, I_3 and the equivalent tensile stress σ_E.

In index notation,

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{ii} \ ,$$

see equation (1.3).

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 \ .$$

This is clearly a product of stresses. Two simple product terms in index notation are

$$I_1^2 = \sigma_{ii}\sigma_{jj} = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{11}\sigma_{22} + 2\sigma_{22}\sigma_{33} + 2\sigma_{33}\sigma_{11}$$

and

$$\sigma_{ij}\sigma_{ij} = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2\sigma_{12}^2 + 2\sigma_{23}^2 + 2\sigma_{31}^2 \ .$$

Comparing these expressions, we see that I_2 can be written

$$I_2 = \frac{\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}}{2} \ .$$

The third stress invariant

$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{31}^2 - \sigma_{33}\sigma_{12}^2 + 2\sigma_{12}\sigma_{23}\sigma_{31}$$

is the determinant of the stress matrix σ_{ij}. It can be written

$$I_3 = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr} \ .$$

From the above expressions,

$$I_1^2 - 3I_2 = \sigma_{ii}\sigma_{jj} - \frac{3(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij})}{2} = \frac{(3\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj})}{2} \ .$$

It follows that

$$\sigma_E = \sqrt{\frac{(3\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj})}{2}} \ ,$$

from equation (1.31).

1.6. Choosing a local coördinate system x_1, x_2, x_3 aligned with the three principal axes, determine the tractions on the *octahedral plane* defined by the unit vector

$$\boldsymbol{n} = \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\}^T$$

which makes equal angles with all three principal axes, if the principal stresses are $\sigma_1, \sigma_2, \sigma_3$. Hence show that the magnitude of the resultant shear stress on this plane is $\sqrt{2}\sigma_E/3$, where σ_E is given by equation (1.31).

The traction on the octahedral plane is given by equation (1.21) as

$$t_i = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{Bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{Bmatrix} = \begin{Bmatrix} \sigma_1/\sqrt{3} \\ \sigma_2/\sqrt{3} \\ \sigma_3/\sqrt{3} \end{Bmatrix}$$

or

$$t_i = \frac{\sigma_i}{\sqrt{3}} \, .$$

To find the resultant shear stress, we first find the *normal* compont of the vector $\boldsymbol{t}$ as

$$\boldsymbol{t} \cdot \boldsymbol{n} = t_i n_i = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \bar{\sigma} \, ,$$

from equation (1.75).

The vector corresponding to this component is

$$(\boldsymbol{t} \cdot \boldsymbol{n})\, \boldsymbol{n} = \bar{\sigma} n_i$$

and the shear stress on the octahedral plane is a vector defined by

$$t_i - \bar{\sigma} n_i = \left\{ \frac{\sigma_1 - \bar{\sigma}}{\sqrt{3}}, \frac{\sigma_2 - \bar{\sigma}}{\sqrt{3}}, \frac{\sigma_3 - \bar{\sigma}}{\sqrt{3}} \right\}^T \, .$$

To find the magnitude τ_{oct} of this resultant, we take the square root of the sum the squares of the compontents, obtaining

$$\begin{aligned} \tau_{\text{oct}} &= \sqrt{\frac{(\sigma_1 - \bar{\sigma})^2 + (\sigma_2 - \bar{\sigma})^2 + (\sigma_3 - \bar{\sigma})^2}{3}} \\ &= \sqrt{\frac{(2\sigma_1 - \sigma_2 - \sigma_3)^2 + (2\sigma_2 - \sigma_3 - \sigma_1)^2 + (2\sigma_3 - \sigma_1 - \sigma_2)^2}{27}} \, . \end{aligned}$$

Expanding and simplifying, we obtain

$$\tau_{\text{oct}} = \frac{1}{3}\sqrt{2\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1\right)} \, .$$

Comparing this with equation (1.31), we see that

$$\tau_{\text{oct}} = \frac{\sqrt{2}\sigma_E}{3} \, .$$

1.7. A rigid body is subjected to a small rotation $\omega_z = \Omega \ll 1$ about the z-axis. If the displacement of the origin is zero, find expressions for the three displacement components u_x, u_y, u_z as functions of x, y, z.

Since the body is rigid, the strains are zero and hence

$$\frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial y} = \frac{\partial u_z}{\partial z} = 0 \ , \tag{1}$$

$$\frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \ . \tag{2}$$

Also, from the given rotations, we have

$$\frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \Omega \quad ; \quad \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) = 0 \ . \tag{3}$$

From equations (2,3), we conclude that

$$\frac{\partial u_y}{\partial x} = \Omega \ ; \ \frac{\partial u_x}{\partial y} = -\Omega \ ; \ \frac{\partial u_z}{\partial y} = \frac{\partial u_y}{\partial z} = \frac{\partial u_x}{\partial z} = \frac{\partial u_z}{\partial x} = 0 \tag{4}$$

and hence u_z is independent of x, y, z, u_x is a function of y only and u_y is a function of x only. Integrating the first two of equation (4) we then have

$$u_y = \Omega x + A \ ; \ u_x = -\Omega y + B \tag{5}$$

where A, B are constants. Since the origin has zero displacement, the only permissible displacement field is therefore

$$u_x = -\Omega y \ ; \ u_y = \Omega x \ ; \ u_z = 0 \ .$$

An alternative proof

A shorter and more physical way of reaching these results is to note that the rigid body rotating through a small clockwise angle Ω about the z-axis will experience the displacements

$$u_r = 0 \ ; \ u_\theta = r\Omega \ ; \ u_z = 0$$

in polar coördinates r, θ, z.

We then use the transformation rules

$$u_x = u_r \cos\theta - u_\theta \sin\theta \quad ; \quad u_y = u_r \sin\theta + u_\theta \cos\theta$$

to obtain

$$\begin{aligned} u_x &= -r\Omega \sin\theta = -\Omega y \\ u_y &= r\Omega \cos\theta = \Omega x \ . \end{aligned}$$

1.8. Use the index notation to develop a general expression for the derivative

$$\frac{\partial u_i}{\partial x_j}$$

in terms of strains and rotations.

It's probably easier to start this question in x, y, z notation. If the indices are the same, we have (for example)

$$\frac{\partial u_x}{\partial x} = e_{xx}$$

If they are different and in cyclic order, we have

$$\frac{\partial u_x}{\partial y} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) - \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = e_{xy} - \omega_z \ .$$

If they are in reverse cyclic order, we have

$$\frac{\partial u_y}{\partial x} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = e_{xy} + \omega_z \ .$$

We can generalize these expressions using the index notation by writing

$$\frac{\partial u_i}{\partial x_j} = e_{ij} - \epsilon_{ijk}\omega_k \ .$$

Recall that if the indices are the same, $\epsilon_{ijk} = 0$ and hence the second term will be zero. If they are in cyclic order, $\epsilon_{ijk} = 1$, whereas for reverse cyclic order, $\epsilon_{ijk} = -1$.

1.9. Use the three-dimensional vector transformation rule (1.19) and the index notation to prove that the strain components (1.51) transform according to the equation

$$e'_{ij} = l_{ip} l_{jq} e_{pq} \ .$$

Hence show that the dilatation e_{ii} is invariant under coördinate transformation.

From (1.51) in x'_i coördinates, we have

$$e'_{ij} = \frac{1}{2}\left(\frac{\partial u'_i}{\partial x'_j} + \frac{\partial u'_j}{\partial x'_i}\right)$$

Also, since $\boldsymbol{u}$ is a vector,

$$u'_i = l_{ip} u_p \ ; \quad u'_j = l_{jq} u_q \ ,$$

from (1.19). The derivatives are components of the gradient operator which is also a vector, so

$$\frac{\partial}{\partial x'_i} = l_{ip} \frac{\partial}{\partial x_p} \ ; \quad \frac{\partial}{\partial x'_j} = l_{jq} \frac{\partial}{\partial x_q} \ .$$

Using these results, we then have

$$e'_{ij} = \frac{1}{2}\left(l_{ip} l_{jq} \frac{\partial u_p}{\partial x_q} + l_{jq} l_{ip} \frac{\partial u_q}{\partial x_p}\right) = \frac{l_{ip} l_{jq}}{2}\left(\frac{\partial u_p}{\partial x_q} + \frac{\partial u_q}{\partial x_p}\right) = l_{ip} l_{jq} e_{pq} \ .$$

Using this result, the dilatation is

$$e'_{ii} = l_{ip} l_{iq} e_{pq} \ .$$

Now for a given value of p, l_{ip} defines the components in x'_i coördinates of a unit vector in the direction of the x_p-axis. It follows that

$$l_{ip} l_{iq} = \delta_{pq} \ ,$$

since this defines the dot product between two unit vectors along two of a set of three orthogonal axes. It follows that

$$e'_{ii} = \delta_{pq} e_{pq} = e_{qq}$$

and hence that the dilatation is invariant under coördinate transformation.

1.10 Find an index notation expression for the compliance tensor s_{ijkl} of equation (1.55) for the isotropic elastic material in terms of the elastic constants E, ν.

We start in x, y, z coördinates and write

$$e_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} - \frac{\nu\sigma_{zz}}{E} = \frac{(1+\nu)\sigma_{xx}}{E} - \frac{\nu(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})}{E}$$

and

$$e_{xy} = \frac{(1+\nu)\sigma_{xy}}{E} \ .$$

These results can be combined in index notation as

$$e_{ij} = \frac{(1+\nu)\sigma_{ij}}{E} - \frac{\nu\delta_{ij}\sigma_{mm}}{E} \ .$$

Comparing with equation $(1.55)_2$ and getting a hint from the transition from (1.71) to (1.72), we then see that

$$s_{ijkl} = \frac{(1+\nu)}{2E}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) - \frac{\nu\delta_{ij}\delta_{kl}}{E} \ .$$

To verify this, we substitute it into (1.55), obtaining

$$\begin{aligned} e_{ij} &= \frac{(1+\nu)}{2E}\delta_{ik}\delta_{jl}\sigma_{kl} + \frac{(1+\nu)}{2E}\delta_{il}\delta_{jk}\sigma_{kl} - \frac{\nu\delta_{ij}\delta_{kl}}{E}\sigma_{kl} \\ &= \frac{(1+\nu)\sigma_{ij}}{2E} + \frac{(1+\nu)\sigma_{ji}}{2E} - \frac{\nu\delta_{ij}\sigma_{kk}}{E} \\ &= \frac{(1+\nu)\sigma_{ij}}{E} - \frac{\nu\delta_{ij}\sigma_{kk}}{E} \ , \end{aligned}$$

since $\sigma_{ij} = \sigma_{ji}$.

1.11. Show that equations (1.58–1.60, 1.64) can be written in the concise form

$$e_{ij} = \frac{(1+\nu)\sigma_{ij}}{E} - \frac{\nu\delta_{ij}\sigma_{mm}}{E} \ . \tag{1.77}$$

This can be demonstrated by considering two particular cases, one in which the indices i, j are the same and one in which they are different.

We first note that because of the summation convention

$$\sigma_{mm} \equiv \sigma_{11} + \sigma_{22} + \sigma_{33} \ . \tag{1}$$

Now suppose $i = j = 1$. The right hand side of (1.77) then takes the form

$$\frac{(1+\nu)\sigma_{11}}{E} - \frac{\nu\delta_{11}(\sigma_{11} + \sigma_{22} + \sigma_{33})}{E} \ . \tag{2}$$

Noting that $\delta_{11} = 1$ and simplifying, we obtain

$$e_{11} = \frac{\sigma_{11}}{E} - \frac{\nu(\sigma_{22} + \sigma_{33})}{E} \ , \tag{3}$$

which agrees with (1.58).

Next consider the case $i = 1, j = 2$. The right hand side of (1.77) is then

$$\frac{(1+\nu)\sigma_{12}}{E} - \frac{\nu\delta_{12}(\sigma_{11} + \sigma_{22} + \sigma_{33})}{E} \tag{4}$$

and since $\delta_{12} = 0$, this reduces to

$$\frac{(1+\nu)\sigma_{12}}{E} \ , \tag{5}$$

agreeing with (1.64).

CHAPTER 2

2.1. Show that, if there are no body forces, the dilatation e must satisfy the condition

$$\nabla^2 e = 0 \ .$$

The only condition imposed upon the displacement $\boldsymbol{u}$ is the equilibrium condition (2.17), which in the absence of body forces takes the form

$$\boldsymbol{\nabla}\text{div } \boldsymbol{u} + (1-2\nu)\nabla^2\boldsymbol{u} = 0 \ . \tag{1}$$

Taking the divergence of this equation, we obtain

$$\nabla^2\text{div } \boldsymbol{u} + (1-2\nu)\text{div } \boldsymbol{\nabla}^2\boldsymbol{u} = 0 \ , \tag{2}$$

since div $\boldsymbol{\nabla} \equiv \nabla^2$.

Also, since div $\boldsymbol{\nabla}^{\mathbf{2}} \equiv \nabla^2$div , the two terms can be added to give

$$2(1-\nu)\nabla^2\text{div } \boldsymbol{u} = 0 \tag{3}$$

and hence

$$\nabla^2\text{div } \boldsymbol{u} \equiv \nabla^2 e = 0 \ , \tag{4}$$

since $\nu \neq 1$.

2.2. Show that, if there are no body forces, the rotation $\boldsymbol{\omega}$ must satisfy the condition

$$\nabla^2\boldsymbol{\omega} = 0 \ .$$

We first note that

$$\boldsymbol{\omega} = \frac{1}{2}\text{curl } \boldsymbol{u} \tag{1}$$

and hence

$$\nabla^2\boldsymbol{\omega} = \frac{1}{2}\nabla^2\text{curl } \boldsymbol{u} = \frac{1}{2}\text{curl } \nabla^2\boldsymbol{u} \ . \tag{2}$$

If we take the curl of equation (2.17) with $\boldsymbol{p} = 0$, we obtain

$$\text{curl } \boldsymbol{\nabla}\text{div } \boldsymbol{u} + (1-2\nu)\text{curl } \nabla^2\boldsymbol{u} = 0 \tag{3}$$

and since curl $\boldsymbol{\nabla}$ of any scalar function is zero, the first term is zero and we have

$$(1-2\nu)\text{curl } \nabla^2\boldsymbol{u} = (1-2\nu)\nabla^2\text{curl } \boldsymbol{u} = 2(1-2\nu)\nabla^2\boldsymbol{\omega} = 0 \ , \tag{4}$$

proving the required result except in the limiting case $\nu = 0.5$. Notice that a separate proof could be developed for this case, starting with the modified equations of elasticity appropriate to incompressible materials. However, it is sufficient for our purpose to note that the above proof is satisfactory for ν arbitrarily close to 0.5.

To prove that curl $\boldsymbol{\nabla} = 0$, we write it in index notation, obtaining

$$\text{curl } \boldsymbol{\nabla}\phi = \epsilon_{ijk}\frac{\partial^2\phi}{\partial x_i \partial x_j} \ ,$$

from (1.10,1.12). For each set of indices giving a non-zero contribution to this sum, there is another giving an equal and opposite contribution. For example, if $i = 1, j = 2, k = 3$ we get

$$\epsilon_{123}\frac{\partial^2\phi}{\partial x_1 \partial x_2} = \frac{\partial^2\phi}{\partial x_1 \partial x_2} \ ,$$

whereas for $i = 2, j = 1, k = 3$ we get

$$\epsilon_{213}\frac{\partial^2\phi}{\partial x_1 \partial x_2} = -\frac{\partial^2\phi}{\partial x_1 \partial x_2} \ ,$$

since this time the indices in the alternating tensor are in reverse cyclic order. Thus, we conclude that curl $\boldsymbol{\nabla} = 0$.

2.3. One way of satisfying the compatibility equations in the absence of rotation is to define the components of displacement in terms of a potential function ψ through the relations

$$u_x = \frac{\partial \psi}{\partial x} \;\; ; \;\; u_y = \frac{\partial \psi}{\partial y} \;\; ; \;\; u_z = \frac{\partial \psi}{\partial z} \; .$$

Use the stress-strain relations to derive expressions for the stress components in terms of ψ.

Hence show that the stresses will satisfy the equilibrium equations in the absence of body forces if and only if

$$\nabla^2 \psi = \textbf{constant} \; .$$

Using the above definitions, and the strain-displacement relations (1.51), we obtain

$$e_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial^2 \psi}{\partial x^2} \;\; ; \;\; e_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) = \frac{\partial^2 \psi}{\partial x \partial y} \;\; ; \;\; e_{xz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) = \frac{\partial^2 \psi}{\partial x \partial z} \; . \tag{1}$$

Similar relations can be obtained for the other strain components. We also note that the dilatation

$$e \equiv e_{xx} + e_{yy} + e_{zz} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \equiv \nabla^2 \psi \; . \tag{2}$$

Substituting (1,2) in the Lamé equations (1.71), we obtain

$$\sigma_{xx} = \lambda \nabla^2 \psi + 2\mu \frac{\partial^2 \psi}{\partial x^2} \quad ; \quad \sigma_{xy} = 2\mu \frac{\partial^2 \psi}{\partial x \partial y} \quad ; \quad \sigma_{xz} = 2\mu \frac{\partial^2 \psi}{\partial x \partial z} \tag{3}$$

etc.

Finally, substituting these results into the equilibrium equation (2.2) without body forces ($p_x = 0$), we obtain

$$\lambda \frac{\partial}{\partial x} \nabla^2 \psi + 2\mu \frac{\partial^3 \psi}{\partial x^3} + 2\mu \frac{\partial^3 \psi}{\partial x \partial y^2} + 2\mu \frac{\partial^3 \psi}{\partial x \partial z^2} = 0 \tag{4}$$

and hence

$$(\lambda + 2\mu) \frac{\partial}{\partial x} \nabla^2 \psi = 0 \; . \tag{5}$$

By a similar procedure we can develop two similar equations for equilibrium in the y and z directions, all of which can be summarized in the vector equation

$$\boldsymbol{\nabla} \nabla^2 \psi = 0 \; . \tag{6}$$

The most general solution of this equation is

$$\nabla^2 \psi = C \ , \tag{7}$$

where C is a constant.

Notice from (2) that this implies that the dilatation $e = C$ is constant and the stress components (3) take the form

$$\sigma_{xx} = \lambda C + 2\mu \frac{\partial^2 \psi}{\partial x^2} \quad ; \quad \sigma_{xy} = 2\mu \frac{\partial^2 \psi}{\partial x \partial y} \quad ; \quad \sigma_{xz} = 2\mu \frac{\partial^2 \psi}{\partial x \partial z} \tag{8}$$

etc.

2.4. Plastic deformation during a manufacturing process generates a state of residual stress in the large body $z > 0$. If the residual stresses are functions of z only and the surface $z = 0$ is not loaded, show that the stress components $\sigma_{yz}, \sigma_{zx}, \sigma_{zz}$ must be zero everywhere.

If the stresses are functions of z only, the equilibrium equations (2.2–2.4) without body force simplify to

$$\frac{\partial \sigma_{xz}}{\partial z} = 0 \tag{1}$$

$$\frac{\partial \sigma_{yz}}{\partial z} = 0 \tag{2}$$

$$\frac{\partial \sigma_{zz}}{\partial z} = 0\,. \tag{3}$$

Thus, $\sigma_{yz}, \sigma_{zx}, \sigma_{zz}$ must each be constant (independent of z) and since these stresses also correspond to the three traction components on the free surface $z = 0$, the resulting three constants must be zero. Notice that the stresses $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ need not be zero.

2.5. By considering the equilibrium of a small element of material similar to that shown in Figure 1.2, derive the three equations of equilibrium in cylindrical polar coördinates r, θ, z.

Summing forces in the r-direction, we have

$$\begin{aligned}(\sigma_{rr} + \delta\sigma_{rr})(r + \delta r)\delta\theta\delta z + (\sigma_{\theta r} + \delta\sigma_{\theta r})\delta r\delta z + (\sigma_{zr} + \delta\sigma_{zr})r\delta\theta\delta r \\ -\sigma_{rr}r\delta\theta\delta z - \sigma_{\theta r}\delta r\delta z - \sigma_{zr}r\delta\theta\delta r - 2\sigma_{\theta\theta}\delta r\delta z\sin(\delta\theta/2) + p_r r\delta\theta\delta r\delta z = 0\ .\end{aligned}$$

Notice in particular that (i) the outer curved surface of the element has a larger circumferential length $(r + \delta r)\delta\theta$ than the inner surface and (ii) the stress component $\sigma_{\theta\theta}$ acting on the θ surfaces has a component in the radial direction, because the two opposing θ-surfaces are not parallel — they are inclined to each other at an angle $\delta\theta$.

Dividing through by $r\delta\theta\delta r\delta z$ and letting the small quantities tend to zero, we obtain

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{(\sigma_{rr} - \sigma_{\theta\theta})}{r} + p_r = 0\ .$$

In the θ-direction, we have

$$\begin{aligned}(\sigma_{r\theta} + \delta\sigma_{r\theta})(r + \delta r)\delta\theta\delta z + (\sigma_{\theta\theta} + \delta\sigma_{\theta\theta})\delta r\delta z + (\sigma_{z\theta} + \delta\sigma_{z\theta})r\delta\theta\delta r \\ -\sigma_{r\theta}r\delta\theta\delta z - \sigma_{\theta\theta}\delta r\delta z - \sigma_{z\theta}r\delta\theta\delta r + 2\sigma_{\theta r}\delta r\delta z\sin(\delta\theta/2) + p_\theta r\delta\theta\delta r\delta z = 0\end{aligned}$$

and hence

$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{z\theta}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + p_\theta = 0\ .$$

Finally, in the z-direction,

$$\begin{aligned}(\sigma_{rz} + \delta\sigma_{rz})(r + \delta r)\delta\theta\delta z + (\sigma_{\theta z} + \delta\sigma_{\theta z})\delta r\delta z + (\sigma_{zz} + \delta\sigma_{zz})r\delta\theta\delta r \\ -\sigma_{rz}r\delta\theta\delta z - \sigma_{\theta z}\delta r\delta z - \sigma_{zz}r\delta\theta\delta r + p_z r\delta\theta\delta r\delta z = 0\end{aligned}$$

and hence

$$\frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + p_z = 0\ .$$

2.6. In cylindrical polar coördinates, the strain-displacement relations for the 'in-plane' strains are

$$e_{rr} = \frac{\partial u_r}{\partial r} \quad ; \quad e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \quad ; \quad e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \ .$$

Use these relations to obtain a compatibility equation that must be satisfied by the three strains.

The displacement u_θ appears only in the second and third equations. Multiplying the second equation by $2r$ and differentiating with respect to θ, we obtain

$$2r\frac{\partial e_{r\theta}}{\partial \theta} = \frac{\partial^2 u_r}{\partial \theta^2} + r\frac{\partial^2 u_\theta}{\partial \theta \partial r} - \frac{\partial u_\theta}{\partial \theta} \ . \tag{1}$$

The third equation can be written

$$\frac{\partial u_\theta}{\partial \theta} = r e_{\theta\theta} - u_r \tag{2}$$

and differentiating it with respect to r we get

$$\frac{\partial^2 u_\theta}{\partial \theta \partial r} = e_{\theta\theta} + r\frac{\partial e_{\theta\theta}}{\partial r} - \frac{\partial u_r}{\partial r} \ . \tag{3}$$

Using (2,3) to eliminate u_θ from (1), we have

$$\begin{aligned} 2r\frac{\partial e_{r\theta}}{\partial \theta} &= \frac{\partial^2 u_r}{\partial \theta^2} + r e_{\theta\theta} + r^2\frac{\partial e_{\theta\theta}}{\partial r} - r\frac{\partial u_r}{\partial r} - r e_{\theta\theta} + u_r \\ &= \frac{\partial^2 u_r}{\partial \theta^2} + r^2\frac{\partial e_{\theta\theta}}{\partial r} - r e_{rr} + u_r \ . \end{aligned} \tag{4}$$

Differentiating this equation with respect to r and using the first strain-displacement equation to eliminate u_r, we obtain

$$\begin{aligned} 2\frac{\partial}{\partial r}\left(r\frac{\partial e_{r\theta}}{\partial \theta}\right) &= \frac{\partial^3 u_r}{\partial \theta^2 \partial r} + 2r\frac{\partial e_{\theta\theta}}{\partial r} + r^2\frac{\partial^2 e_{\theta\theta}}{\partial r^2} - e_{rr} - r\frac{\partial e_{rr}}{\partial r} + \frac{\partial u_r}{\partial r} \\ &= \frac{\partial^2 e_{rr}}{\partial \theta^2} + 2r\frac{\partial e_{\theta\theta}}{\partial r} + r^2\frac{\partial^2 e_{\theta\theta}}{\partial r^2} - r\frac{\partial e_{rr}}{\partial r} \ , \end{aligned} \tag{5}$$

which can be written

$$\frac{1}{r^2}\frac{\partial^2 e_{rr}}{\partial \theta^2} + \frac{2}{r}\frac{\partial e_{\theta\theta}}{\partial r} + \frac{\partial^2 e_{\theta\theta}}{\partial r^2} - \frac{1}{r}\frac{\partial e_{rr}}{\partial r} = \frac{2}{r^2}\frac{\partial e_{r\theta}}{\partial \theta} + \frac{2}{r}\frac{\partial^2 e_{r\theta}}{\partial \theta \partial r} \ . \tag{6}$$

2.7. If no stresses occur in a body, an increase in temperature T causes unrestrained thermal expansion defined by the strains

$$e_{xx} = e_{yy} = e_{zz} = \alpha T \ ; \ e_{xy} = e_{yz} = e_{zx} = 0 \ .$$

Show that this is possible only if T is a linear function of x, y, z and that otherwise stresses must be induced in the body, regardless of the boundary conditions.

Substituting the thermal strains into the compatibility equation (2.8), we obtain

$$\alpha \frac{\partial^2 T}{\partial x^2} + \alpha \frac{\partial^2 T}{\partial y^2} = 0 \ .$$

Two similar equations can be obtained by permuting suffices and hence we conclude

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial y^2} = \frac{\partial^2 T}{\partial z^2} = 0 \ . \tag{1}$$

A similar procedure with the three equations like (2.9) yields

$$\frac{\partial^2 T}{\partial x \partial y} = \frac{\partial^2 T}{\partial y \partial z} = \frac{\partial^2 T}{\partial z \partial x} = 0 \ . \tag{2}$$

From the first of (1) and the first and third of (2), we have

$$\boldsymbol{\nabla} \frac{\partial T}{\partial x} = 0$$

and hence

$$\frac{\partial T}{\partial x} = \text{constant} \ .$$

Similar operations on the remaining equations yields the conclusion that $\boldsymbol{\nabla} T$ is constant and hence that the most general solution for T that involves no stresses is a linear function of x, y, z.

2.8. If there are no body forces, show that the equations of equilibrium and compatibility imply that

$$(1+\nu)\frac{\partial^2\sigma_{ij}}{\partial x_k\partial x_k}+\frac{\partial^2\sigma_{kk}}{\partial x_i\partial x_j}=0\ .$$

The compatibility equations are satisfied identically if the stresses are expressed in terms of displacements. Using equations (1.51, 1.71), we obtain

$$\sigma_{ij}=\lambda\delta_{ij}\frac{\partial u_p}{\partial x_p}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\ . \tag{1}$$

Substituting this result into the equilibrium equation (2.5) without body force, we have

$$\lambda\delta_{ij}\frac{\partial^2 u_p}{\partial x_p\partial x_j}+\mu\left(\frac{\partial^2 u_i}{\partial x_j\partial x_j}+\frac{\partial^2 u_j}{\partial x_i\partial x_j}\right)=0\ . \tag{2}$$

The differentials in the first and last terms of this equation are actually identical and hence

$$(\lambda+\mu)\frac{\partial^2 u_j}{\partial x_i\partial x_j}+\mu\frac{\partial^2 u_i}{\partial x_j\partial x_j}=0\ , \tag{3}$$

as in equation (2.14).

Using equation (1), we have

$$\begin{aligned}(1+\nu)\frac{\partial^2\sigma_{ij}}{\partial x_k\partial x_k}+\frac{\partial^2\sigma_{kk}}{\partial x_i\partial x_j} &= \lambda(1+\nu)\delta_{ij}\frac{\partial^3 u_p}{\partial x_k\partial x_k\partial x_p}+\mu(1+\nu)\frac{\partial^3 u_i}{\partial x_j\partial x_k\partial x_k}\\ &\quad+\mu(1+\nu)\frac{\partial^3 u_j}{\partial x_i\partial x_k\partial x_k}+\lambda(1+\nu)\delta_{kk}\frac{\partial^3 u_p}{\partial x_p\partial x_i\partial x_j}\\ &\quad+\mu\frac{\partial^3 u_k}{\partial x_k\partial x_i\partial x_j}+\mu\frac{\partial^3 u_k}{\partial x_k\partial x_i\partial x_j} \qquad (4)\\ &= \lambda(1+\nu)\delta_{ij}\frac{\partial^3 u_p}{\partial x_k\partial x_k\partial x_p}+\mu(1+\nu)\frac{\partial^3 u_i}{\partial x_j\partial x_k\partial x_k}\\ &\quad+\mu(1+\nu)\frac{\partial^3 u_j}{\partial x_i\partial x_k\partial x_k}+(3\lambda+2\mu)\frac{\partial^3 u_p}{\partial x_p\partial x_i\partial x_j}\ , \qquad (5)\end{aligned}$$

since $\delta_{kk}=3$.

Differentiating the equilibrium equation (3) with respect to x_i, we have

$$(\lambda+\mu)\frac{\partial^3 u_j}{\partial x_i\partial x_i\partial x_j}+\mu\frac{\partial^3 u_i}{\partial x_i\partial x_j\partial x_j}=(\lambda+2\mu)\frac{\partial^3 u_j}{\partial x_i\partial x_i\partial x_j}=0 \tag{6}$$

and hence the first term in equation (5) is zero.

Also, differentiating (3) with respect to x_k, we have

$$(\lambda+\mu)\frac{\partial^3 u_j}{\partial x_i \partial x_j \partial x_k} + \mu \frac{\partial^3 u_i}{\partial x_j \partial x_j \partial x_k} = 0 \tag{7}$$

and hence

$$\frac{\partial^3 u_i}{\partial x_j \partial x_j \partial x_k} = -\left(\frac{\lambda+\mu}{\mu}\right)\frac{\partial^3 u_j}{\partial x_i \partial x_j \partial x_k} \ . \tag{8}$$

Using this result with a change of notation to substitute for the second and third terms in (5), we obtain

$$(1+\nu)\frac{\partial^2 \sigma_{ij}}{\partial x_k \partial x_k} + \frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_j} = 0 - 2\mu(1+\mu)\left(\frac{\lambda+\mu}{\mu}\right)\frac{\partial^3 u_k}{\partial x_i \partial x_j \partial x_k} + (3\lambda+2\mu)\frac{\partial^3 u_p}{\partial x_p \partial x_i \partial x_j} \ . \tag{9}$$

The two differential terms on the right-hand side of this equation are actually identical and the combined multiplier is

$$3\lambda + 2\mu - 2(\lambda+\mu)(1+\nu) = 2\mu\left(\frac{3\nu}{(1-2\nu)} + 1 - (1+\nu)\left[\frac{2\nu}{(1-2\nu)} + 1\right]\right) = 0 \ . \tag{10}$$

In other words,

$$(1+\nu)\frac{\partial^2 \sigma_{ij}}{\partial x_k \partial x_k} + \frac{\partial^2 \sigma_{kk}}{\partial x_i \partial x_j} = 0 \ . \tag{11}$$

2.9 Using the strain-displacement relation (1.51), show that an alternative statement of the compatibility condition is that the tensor

$$\mathcal{C}_{ijkl} \equiv \frac{\partial^2 e_{ij}}{\partial x_k \partial x_l} + \frac{\partial^2 e_{kl}}{\partial x_i \partial x_j} - \frac{\partial^2 e_{il}}{\partial x_j \partial x_k} - \frac{\partial^2 e_{jk}}{\partial x_i \partial x_l} = 0 \ .$$

The strain-displacement equations are

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

It follows that

$$\begin{aligned} 2\mathcal{C}_{ijkl} \quad = \quad & \frac{\partial^3 u_i}{\partial x_k \partial x_l \partial x_j} + \frac{\partial^3 u_j}{\partial x_k \partial x_l \partial x_i} + \frac{\partial^3 u_k}{\partial x_i \partial x_j \partial x_l} + \frac{\partial^3 u_l}{\partial x_i \partial x_j \partial x_k} \\ & - \frac{\partial^3 u_i}{\partial x_j \partial x_k \partial x_l} - \frac{\partial^3 u_l}{\partial x_j \partial x_k \partial x_i} - \frac{\partial^3 u_j}{\partial x_i \partial x_l \partial x_k} - \frac{\partial^3 u_k}{\partial x_i \partial x_l \partial x_j} \end{aligned}$$

Since these terms cancel in pairs, we conclude that $\mathcal{C}_{ijkl} = 0$.

CHAPTER 3

3.1. The plane strain solution for the stresses in the rectangular block $0 < x < a, -b < y < b, -c < z < c$ with a given loading is

$$\sigma_{xx} = \frac{3Fxy}{2b^3} \ ; \ \sigma_{xy} = \frac{3F(b^2 - y^2)}{4b^3} \ ; \ \sigma_{yy} = 0 \ .$$

Find the tractions on the surfaces of the block and illustrate the results on a sketch of the block.

We wish to use this solution to solve the corresponding problem in which the surfaces $z = \pm c$ are traction-free. Determine an approximate corrective solution for this problem by offloading the unwanted force and moment resultants using the elementary bending theory. Find the maximum error in the stress σ_{zz} in the corrected solution and compare it with the maximum tensile stress in the plane strain solution.

In plane strain, $e_{zz} = 0$ and hence

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) = \frac{3\nu Fxy}{2b^3} \ .$$

The tractions acting on the block are, on $x = 0$,

$$\sigma_{xx} = 0 \ ; \ \sigma_{xy} = \frac{3F(b^2 - y^2)}{4b^3} \ ; \ \sigma_{xz} = 0 \ .$$

On $x = a$,

$$\sigma_{xx} = \frac{3Fay}{2b^3} \ ; \ \sigma_{xy} = \frac{3F(b^2 - y^2)}{4b^3} \ ; \ \sigma_{xz} = 0 \ .$$

On $y = \pm b$,

$$\sigma_{yx} = \sigma_{yy} = \sigma_{yz} = 0 \ .$$

On $z = \pm c$,

$$\sigma_{zx} = \sigma_{zy} = 0 \ ; \ \sigma_{zz} = \frac{3\nu Fxy}{2b^3} \ .$$

The tractions on the surface $z = c$ are illustrated in the Figure 1. There are equal tractions on the back surface $z = -c$.

The loading on the surfaces $z = \pm c$ is antisymmetric in y and hence there is no net force (since for every place where there is a tensile traction, there is a symmetric point with an equal compressive traction). Similarly, there is no moment about the y-axis. However, there is a moment about the x axis. To eliminate it, we superpose the bending stress distribution $\sigma_{zz} = Cy$, where C is a constant chosen so as to make the total bending moment zero.

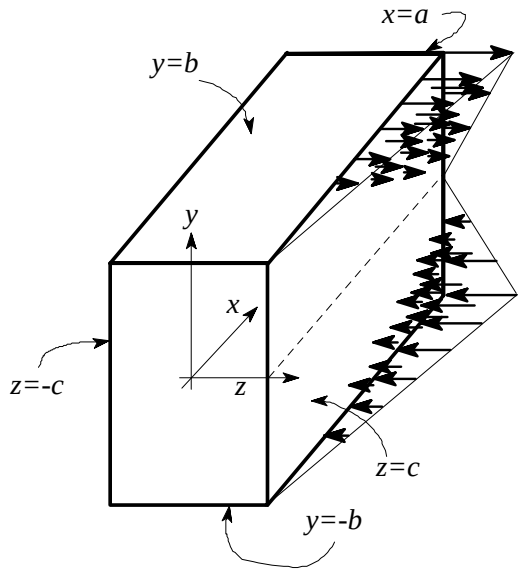


Figure 1

Thus,

$$\sigma_{zz} = \frac{3\nu F x y}{2b^3} + Cy$$

and the total bending moment is

$$\begin{aligned} M &= \int_{-b}^{b}\int_{0}^{a}\left(\frac{3\nu F x y}{2b^3} + Cy\right) y\,dx\,dy \\ &= \int_{-b}^{b}\left(\frac{3\nu F a^2 y}{4b^3} + Cay\right) y\,dy \\ &= \frac{\nu F a^2}{2} + \frac{2Cab^3}{3}\ . \end{aligned}$$

Equating this expression to zero, we find

$$C = -\frac{3\nu F a}{4b^3}$$

and

$$\sigma_{zz} = \frac{3\nu F x y}{2b^3} - \frac{3\nu F a^2 y}{4b^3} = \frac{3\nu F(2x-a)y}{4b^3}\ .$$

Ideally, we would like this stress to be zero. The maximum error occurs at the four corners. For example, at $x = 0, y = b$, we have $\sigma_{zz}(0,b) = -3\nu F a/4b^2$. For comparison, the maximum tensile stress is

$$\sigma_{xx}(a,b) = \frac{3Fa}{2b^2}\ ,$$

which is greater than the error in the ratio $2/\nu$ (=6.7 if $\nu = 0.3$).

3.2. For a solid in a state of plane stress, show that if there are body forces p_x, p_y per unit volume in the direction of the axes x, y respectively, the compatibility equation can be expressed in the form

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -(1+\nu)\left(\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y}\right) .$$

Hence deduce that the stress distribution for any particular case is independent of the material constants and the body forces, provided the latter are constant.

Substituting the plane stress constitutive relations (3.11, 3.12, 1.44) into the compatibility equation (2.8) and cancelling a common factor $1/E$, we obtain

$$\frac{\partial^2\sigma_{xx}}{\partial y^2} - \nu\frac{\partial^2\sigma_{yy}}{\partial y^2} - 2(1+\nu)\frac{\partial^2\sigma_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_{yy}}{\partial x^2} - \nu\frac{\partial^2\sigma_{xx}}{\partial x^2} = 0 . \tag{1}$$

Differentiating (2.2) with respect to x and (2.3) with respect to y and adding the resulting equations, we obtain

$$\frac{\partial^2\sigma_{xx}}{\partial x^2} + 2\frac{\partial^2\sigma_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_{yy}}{\partial y^2} = -\left(\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y}\right) . \tag{2}$$

Now use this equation to eliminate σ_{xy} from equation (1), with the result

$$\frac{\partial^2\sigma_{xx}}{\partial y^2} - \nu\frac{\partial^2\sigma_{yy}}{\partial y^2} + (1+\nu)\left(\frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial y^2}\right) + \frac{\partial^2\sigma_{yy}}{\partial x^2} - \nu\frac{\partial^2\sigma_{xx}}{\partial x^2} = -(1+\nu)\left(\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y}\right) , \tag{3}$$

which can be rearranged to give

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -(1+\nu)\left(\frac{\partial p_x}{\partial x} + \frac{\partial p_y}{\partial y}\right) . \tag{4}$$

We conclude that, if p_x is independent of x and p_y is independent of y, the equation satisfied by $\sigma_{xx} + \sigma_{yy}$ is independent of the material constants and the body forces. To proceed to the stronger conclusion that the stress distribution is independent of these quantities, we would need to impose the condition that the boundary conditions be also so independent. For example, if the displacement is prescribed at some point on the boundary, we should expect dependence on the material constants to be reintroduced.

3.3.(i) Show that the compatibility equation (2.8) is satisfied by unrestrained thermal expansion ($e_{xx} = e_{yy} = \alpha T, e_{xy} = 0$), provided that the temperature, T, is a two-dimensional harmonic function — i.e.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \ .$$

(ii) Hence deduce that, subject to certain restrictions which you should explicitly list, no thermal stresses will be induced in a thin body with a steady-state, two-dimensional temperature distribution and no boundary tractions.

(iii) Show that an initially straight line on such a body will be distorted by the heat flow in such a way that its curvature is proportional to the local heat flux across it.

(i) Substituting $e_{xx} = e_{yy} = \alpha T, e_{xy} = 0$ into (2.8), we obtain

$$\alpha \frac{\partial^2 T}{\partial x^2} + \alpha \frac{\partial^2 T}{\partial y^2} = 0 \ . \tag{1}$$

(ii) Equation (1) is the steady-state heat conduction equation if there are no internal heat sources and hence the stress state $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0$ — i.e. a stress-free state — satisfies the compatibility equation. It also satisfies the boundary conditions if the body is traction-free. This is therefore the solution of the thermoelastic problem **unless** the body is multiply-connected (see §2.2.1). For multiply-connected bodies, the compatibility condition must be supplemented by the conditions that Cesaro integrals around each hole should be zero and these lead to additional constraints on the temperature field. These are derived by J.Dundurs[1].

(iii) We can choose a Cartesian coordinate system such that the straight line in question coincides with the x-axis, without loss of generality. The slope of this line after deformation is then

$$\phi = \frac{\partial u_y}{\partial x} \tag{2}$$

and its curvature is

$$\frac{1}{R} = \frac{\partial \phi}{\partial x} = \frac{\partial^2 u_y}{\partial x^2} \ . \tag{3}$$

If this expression is unfamiliar, recall the equation

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} = E\frac{d^2 u}{dx^2} \tag{4}$$

[1] J.Dundurs, Distortion of a body caused by free thermal expansion, *Mechanics Research Communications,* Vol. 1 (1974), pp.121-124.

from elementary beam theory, in which $1/R = d^2u/dx^2$ represents the curvature of an initially straight beam.

Now, since the shear strain

$$e_{xy} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) = 0 \ , \tag{5}$$

we can differentiate with respect to x to obtain

$$\frac{\partial^2 u_y}{\partial x^2} = -\frac{\partial^2 u_x}{\partial x \partial y} = -\frac{\partial e_{xx}}{\partial y} = -\alpha \frac{\partial T}{\partial y} \ . \tag{6}$$

Finally, noting that the heat flux is proportional to the temperature gradient — i.e.

$$\boldsymbol{q} = -K \boldsymbol{\nabla} T \ , \tag{7}$$

or

$$q_y = -K \frac{\partial T}{\partial y} \ , \tag{8}$$

we have

$$\frac{\partial^2 u_y}{\partial x^2} = \frac{\alpha q_y}{K} \ . \tag{9}$$

In other words, the curvature of the line is proportional to the local heat flux across it. Notice that the sign conventions used imply that the curvature is such that a surface into which heat is flowing will become convex. This effect is further discussed in §14.3.1.

3.4. Find the inverse relations to equations (3.18) — i.e. the substitutions that should be made for the elastic constants E, ν in a plane strain solution if we want to recover the solution of the corresponding plane stress problem.

The second equation can be solved for ν'. We have

$$\nu' = \nu - \nu\nu'$$

and hence

$$\nu' = \frac{\nu}{(1+\nu)} \ .$$

Substituting this result into the first equation, we have

$$E' = E\left(1 - \frac{\nu^2}{(1+\nu)^2}\right) = \frac{E(1+2\nu+\nu^2-\nu^2)}{(1+\nu)^2}$$

and hence

$$E' = \frac{E(1+2\nu)}{(1+\nu)^2} \ .$$

If the solution is known in plane strain with prescribed values of E, ν, we therefore need to substitute

$$\nu = \frac{\nu'}{(1+\nu')} \quad ; \quad E = \frac{E'(1+2\nu')}{(1+\nu')^2}$$

and then remove the primes in order to obtain the plane stress solution.

3.5. Show that in a state of plane stress without body forces, the in-plane displacements must satisfy the equations

$$\nabla^2 u_x + \frac{(1+\nu)}{(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) = 0 \ ; \ \nabla^2 u_y + \frac{(1+\nu)}{(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) = 0 \ .$$

In plane stress, we have

$$e_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} \ ; \ e_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu\sigma_{xx}}{E} \ ; \ e_{xy} = \frac{(1+\nu)\sigma_{xy}}{E} \ .$$

Solving for the stresses, we obtain

$$\sigma_{xx} = \frac{E(e_{xx} + \nu e_{yy})}{(1-\nu^2)} \ ; \ \sigma_{yy} = \frac{E(e_{yy} + \nu e_{xx})}{(1-\nu^2)} \ ; \ \sigma_{xy} = \frac{E e_{xy}}{(1+\nu)} \ .$$

Multiplying the equilibrium equation

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0$$

by $E/(1-\nu^2)$ and substituting for the stress components, we obtain

$$\frac{\partial e_{xx}}{\partial x} + \nu\frac{\partial e_{yy}}{\partial x} + (1-\nu)\frac{\partial e_{xy}}{\partial y} = 0$$

and hence

$$\frac{\partial^2 u_x}{\partial x^2} + \nu\frac{\partial^2 u_y}{\partial x \partial y} + \frac{(1-\nu)}{2}\frac{\partial^2 u_y}{\partial x \partial y} + \frac{(1-\nu)}{2}\frac{\partial^2 u_x}{\partial y^2} = 0 \ .$$

We can regroup this equation to expose $\nabla^2 u_x$ by splitting the first term using

$$1 = \frac{(1-\nu)}{2} + \frac{(1+\nu)}{2} \ .$$

We then obtain

$$\frac{(1-\nu)}{2}\frac{\partial^2 u_x}{\partial x^2} + \frac{(1+\nu)}{2}\frac{\partial^2 u_x}{\partial x^2} + \nu\frac{\partial^2 u_y}{\partial x \partial y} + \frac{(1-\nu)}{2}\frac{\partial^2 u_y}{\partial x \partial y} + \frac{(1-\nu)}{2}\frac{\partial^2 u_x}{\partial y^2} = 0$$

and hence

$$\frac{(1-\nu)}{2}\nabla^2 u_x + \frac{(1+\nu)}{2}\frac{\partial^2 u_x}{\partial x^2} + \frac{(1+\nu)}{2}\frac{\partial^2 u_y}{\partial x \partial y} = \frac{(1-\nu)}{2}\nabla^2 u_x + \frac{(1+\nu)}{2}\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y}\right) = 0 \ .$$

Multiplying through by $2/(1-\nu)$ then gives the required result

$$\nabla^2 u_x + \frac{(1+\nu)}{(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) = 0 \ .$$

The second equation can be obtained in the same way, using the second equilibrium equation

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \ ,$$

or more simply by permuting suffices on the first equation.

3.6. Show that in a state of plane strain without body forces,

$$\frac{\partial e}{\partial x} = \left(\frac{1-2\nu}{1-\nu}\right)\frac{\partial \omega_z}{\partial y} \;\; ; \;\; \frac{\partial e}{\partial y} = -\left(\frac{1-2\nu}{1-\nu}\right)\frac{\partial \omega_z}{\partial x} \; .$$

In plane strain, we have

$$u_z = 0 \;\; ; \;\; e = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \; .$$

It follows that

$$\frac{\partial e}{\partial x} = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial y \partial x} \;\; ; \;\; \frac{\partial e}{\partial y} = \frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_y}{\partial y^2}$$

$$2\frac{\partial \omega_z}{\partial y} = \frac{\partial^2 u_y}{\partial x \partial y} - \frac{\partial^2 u_x}{\partial y^2} \;\; ; \;\; 2\frac{\partial \omega_z}{\partial x} = \frac{\partial^2 u_y}{\partial x^2} - \frac{\partial^2 u_x}{\partial y \partial x} \; .$$

From these equations, we have

$$\frac{\partial e}{\partial x} - 2\frac{\partial \omega_z}{\partial y} = \nabla^2 u_x \;\; ; \;\; \frac{\partial e}{\partial y} + 2\frac{\partial \omega_z}{\partial x} = \nabla^2 u_y \; .$$

Now the first equilibrium equation (2.17) without body force states that

$$(\lambda + \mu)\frac{\partial e}{\partial x} + \mu \nabla^2 u_x = 0 \; .$$

Using the above equation to eliminate $\nabla^2 u_x$, we obtain

$$(\lambda + \mu)\frac{\partial e}{\partial x} + \mu\left(\frac{\partial e}{\partial x} - 2\frac{\partial \omega_z}{\partial y}\right) = 0$$

and hence

$$(\lambda + 2\mu)\frac{\partial e}{\partial x} = 2\mu\frac{\partial \omega_z}{\partial y} \; ,$$

or

$$\frac{\partial e}{\partial x} = \frac{2\mu}{(\lambda + 2\mu)}\frac{\partial \omega_z}{\partial y} = \left(\frac{1-2\nu}{1-\nu}\right)\frac{\partial \omega_z}{\partial y} \; ,$$

using (1.69).

Similarly, eliminating $\nabla^2 u_x$ from the second equilibrium equation

$$(\lambda + \mu)\frac{\partial e}{\partial y} + \mu \nabla^2 u_y = 0$$

yields

$$\frac{\partial e}{\partial y} = -\left(\frac{1-2\nu}{1-\nu}\right)\frac{\partial \omega_z}{\partial x} \; .$$

3.7. If a material is incompressible ($\nu = 0.5$), a state of hydrostatic stress $\sigma_{xx} = \sigma_{yy} = \sigma_{zz}$ produces no strain. One way to write the corresponding stress-strain relations is

$$\sigma_{ij} = 2\mu e_{ij} - q\delta_{ij} \ ,$$

where q is an unknown hydrostatic pressure which will generally vary with position. Also, the condition of incompressibility requires that the dilatation

$$e \equiv e_{kk} = 0 \ .$$

Show that under plain strain conditions, the stress components and the hydrostatic pressure q must satisfy the equations

$$\nabla^2 q = \textbf{div } \boldsymbol{p} \ \textbf{ and } \ \sigma_{xx} + \sigma_{yy} = -2q \ ,$$

where $\boldsymbol{p}$ is the body force.

In a state of plane strain, $e_{zz} = 0$ and hence the condition $e = 0$ implies that

$$e_{xx} + e_{yy} = 0 \ .$$

We then have

$$\sigma_{xx} + \sigma_{yy} = 2\mu(e_{xx} + e_{yy}) - 2q$$

and hence

$$\sigma_{xx} + \sigma_{yy} = -2q \ . \tag{1}$$

To complete the proof, we need to use the compatibility equation (2.8). Solving the stress-strain relations for the strains, we have

$$2\mu e_{xx} = \sigma_{xx} + q \ \ ; \ \ 2\mu e_{yy} = \sigma_{yy} + q \ \ ; \ \ 2\mu e_{xy} = \sigma_{xy}$$

and hence

$$2\mu \left(\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} - 2\frac{\partial^2 e_{xy}}{\partial x \partial y} \right) = \frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} - 2\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \nabla^2 q = 0 \ . \tag{2}$$

However, from the equilibrium equations,

$$\frac{\partial \sigma_{xy}}{\partial x} = -\frac{\partial \sigma_{yy}}{\partial y} - p_y \ \ ; \ \ \frac{\partial \sigma_{xy}}{\partial y} = -\frac{\partial \sigma_{xx}}{\partial x} - p_x$$

and hence

$$\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_{xy}}{\partial y^2} - \frac{\partial p_y}{\partial y} = -\frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial p_x}{\partial x} \ .$$

Using this result in equation (2), we obtain

$$\nabla^2(\sigma_{xx} + \sigma_{yy} + q) + \text{div } \boldsymbol{p} = 0$$

and hence, using equation (1) to eliminate the stress components

$$\nabla^2 q = \text{div } \boldsymbol{p} \ .$$

CHAPTER 4

4.1. Newton's law of gravitation states that two heavy particles of mass m_1, m_2 respectively will experience a mutual attractive force

$$F = \frac{\gamma m_1 m_2}{R^2} ,$$

where R is the distance between the particles and γ is the gravitational constant. Use an energy argument to show that the force acting on a particle of mass m_0 can be written

$$\boldsymbol{F} = -\gamma m_0 \boldsymbol{\nabla} V ,$$

where

$$V(x, y, z) = -\int\int\int_\Omega \frac{\rho(\xi, \eta, \zeta) d\xi d\eta d\zeta}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} ,$$

Ω represents the volume of the universe and ρ is the density of material in the universe, which will generally be a function of position (ξ, η, ζ).

Could a similar method have been used if Newton's law had been of the more general form

$$F = \frac{\gamma m_1 m_2}{R^\lambda} .$$

If so, what would have been the corresponding expression for V? If not, why not?

Suppose there are only two particles, one at (x, y, z) of mass m_0 and the other at the origin of mass m_1. Particle m_0 will then be attracted towards the origin by a force

$$F_R = -\frac{\gamma m_0 m_1}{R^2} ,$$

where

$$R^2 = x^2 + y^2 + z^2 .$$

This force can be written in the form

$$\boldsymbol{F} = -\gamma m_0 \boldsymbol{\nabla} V ,$$

where

$$V = -\frac{m_1}{R} .$$

since the vector $\boldsymbol{\nabla} V$ then has magnitude

$$\frac{m_1}{R^2}$$

and direction away from the origin.

If we now move the origin so that the mass m_1 is at the point (ξ, η, ζ), the same argument will apply, except that

$$R = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2} \ .$$

Now suppose there is a distribution of mass of density $\rho(\xi, \eta, \zeta)$ in the universe. The mass contained in the small element $d\xi d\eta d\zeta$ will be $dm = \rho(\xi, \eta, \zeta) d\xi d\eta d\zeta$ and it will cause a force to act on the mass m_0 obtained by replacing m_1 by dm. This force is described by the potential

$$V = -\frac{\rho(\xi, \eta, \zeta) d\xi d\eta d\zeta}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} \ .$$

We then use superposition (integration) to sum the effect of all the mass in the universe, obtaining

$$V(x, y, z) = -\int\int\int_\Omega \frac{\rho(\xi, \eta, \zeta) d\xi d\eta d\zeta}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} \ .$$

If the force law had been

$$F = \frac{\gamma m_1 m_2}{R^\lambda} \ ,$$

an exactly similar procedure shows that the potential for a single particle at the origin can be written

$$V = -\frac{m_1}{(\lambda - 1) R^{(\lambda-1)}}$$

and hence for the more general distribution

$$V(x, y, z) = -\frac{1}{(\lambda - 1)} \int\int\int_\Omega \frac{\rho(\xi, \eta, \zeta) d\xi d\eta d\zeta}{[(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{(\lambda-1)/2}} \ .$$

In the special case where $\lambda = 1$ we obtain

$$V(x, y, z) = \frac{1}{2} \int\int\int_\Omega \ln((x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2) \rho(\xi, \eta, \zeta) d\xi d\eta d\zeta \ .$$

4.2. An ionized liquid in an electric field experiences a body force $\boldsymbol{p}$. Show that the liquid can be in equilibrium only if $\boldsymbol{p}$ is a conservative vector field. Hint: Remember that a stationary liquid must be everywhere in a state of hydrostatic stress.

The stresses must be everywhere hydrostatic, so

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma \ \ ; \ \ \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \ .$$

The first equilibrium equation

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + p_x = 0$$

then requires that

$$\frac{\partial \sigma}{\partial x} + p_x = 0$$

and the complete equilibrium condition is therefore

$$\boldsymbol{\nabla} \sigma + \boldsymbol{p} = 0 \ .$$

Thus, the body force

$$\boldsymbol{p} = -\boldsymbol{\nabla} \sigma \ ,$$

which defines a conservative vector field for all functions σ.

4.3. An *antiplane* state of stress is one for which the only non-zero stress components are σ_{zx}, σ_{zy} and these are independent of z. Show that two of the three equilibrium equations are then satisfied identically if there is no body force. Use a technique similar to that of §4.3 to develop a representation of the non-zero stress components in terms of a scalar function, such that the remaining equilibrium equation is satisfied identically.

The first equilbrium equation is

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0 \ .$$

The first two terms are identically zero and although σ_{zx} is non-zero, it is independent of z, so the third term is also zero. A similar argument works for the second equation.

The non-zero terms in the third equation are

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0 \ ,$$

which we write as

$$\frac{\partial \sigma_{xz}}{\partial x} = -\frac{\partial \sigma_{yz}}{\partial y} = \frac{\partial^2 \phi}{\partial x \partial y} \ ,$$

where we define a new function ϕ, taking the lowest common denominator of the two derivatives. We can then construct a solution in the form

$$\sigma_{xz} = \frac{\partial \phi}{\partial y} \ ; \quad \sigma_{yz} = -\frac{\partial \phi}{\partial x} \ .$$

This is *Prandtl's stress function*, introduced in §16.1.

4.4. If a body of fairly general axisymmetric shape is loaded in torsion, the only non-zero stress components in cylindrical polar coördinates are $\sigma_{\theta r}, \sigma_{\theta z}$ and these are required to satisfy the equilibrium equation

$$\frac{\partial \sigma_{\theta r}}{\partial r} + \frac{2\sigma_{\theta r}}{r} + \frac{\partial \sigma_{\theta z}}{\partial z} = 0 \ .$$

Use a technique similar to that of §4.3 to develop a representation of these stress components in terms of a scalar function, such that the equilibrium equation is satisfied identically.

Separating terms with $\sigma_{\theta r}$ and $\sigma_{\theta z}$, we have

$$\frac{\partial \sigma_{\theta r}}{\partial r} + \frac{2\sigma_{\theta r}}{r} = -\frac{\partial \sigma_{\theta z}}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial r} + \frac{2\psi}{r}\right) \ ,$$

where we have defined a new function $\psi(r, z)$ in such a way that the right-hand side contains both the differential operators of the two other terms. It follows that

$$\sigma_{\theta r} = -\frac{\partial \psi}{\partial z} \ ; \quad \sigma_{\theta z} = \frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \ .$$

A rather more convenient form is obtained by noting that

$$\frac{\partial \psi}{\partial r} + \frac{2\psi}{r} \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \psi\right) \ .$$

Thus, if we define

$$\phi = r^2 \psi \ ,$$

we have

$$\sigma_{\theta r} = -\frac{1}{r^2}\frac{\partial \phi}{\partial z} \ ; \quad \sigma_{\theta z} = \frac{1}{r^2}\frac{\partial \phi}{\partial r} \ .$$

We shall encounter this stress function in Chapter 27.

4.5.(i) Show that the function

$$\phi = y\omega + \psi$$

satisfies the biharmonic equation provided that ω, ψ are both *harmonic* (i.e. $\nabla^2\omega = 0, \nabla^2\psi = 0$).

(ii) Develop expressions for the stress components in terms of ω, ψ, based on the use of ϕ as an Airy stress function.

(iii) Show that a solution suitable for the half-plane $y > 0$ subject to normal surface tractions only (i.e. $\sigma_{xy} = 0$ on $y = 0$) can be obtained by writing

$$\omega = -\frac{\partial\psi}{\partial y}$$

and hence that under these conditions the normal stress σ_{xx} near the surface $y = 0$ is equal to the applied traction σ_{yy}.

(iv) Do you think this is a rigorous proof? Can you think of any exceptions? If so, at what point in your proof of section (iii) can you find a lack of generality?

(i) Differentiating by parts, we have

$$\frac{\partial\phi}{\partial y} = \omega + y\frac{\partial\omega}{\partial y} + \frac{\partial\psi}{\partial y} \quad ; \quad \frac{\partial^2\phi}{\partial y^2} = 2\frac{\partial\omega}{\partial y} + y\frac{\partial^2\omega}{\partial y^2} + \frac{\partial^2\psi}{\partial y^2} \tag{1}$$

and hence

$$\begin{aligned} \nabla^2\phi &= y\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\psi}{\partial x^2} + 2\frac{\partial\omega}{\partial y} + y\frac{\partial^2\omega}{\partial y^2} + \frac{\partial^2\psi}{\partial y^2} + y\frac{\partial^2\omega}{\partial z^2} + \frac{\partial^2\psi}{\partial z^2} \\ &= y\nabla^2\omega + \nabla^2\psi + 2\frac{\partial\omega}{\partial y} \\ &= 2\frac{\partial\omega}{\partial y}\,, \end{aligned} \tag{2}$$

since $\nabla^2\omega = \nabla^2\psi = 0$.

It follows that

$$\nabla^4\phi = 2\nabla^2\frac{\partial\omega}{\partial y} = 2\frac{\partial}{\partial y}\nabla^2\omega = 0 \tag{3}$$

and hence that ϕ is biharmonic.

(ii) If we use ϕ as an Airy stress function, we have

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = 2\frac{\partial\omega}{\partial y} + y\frac{\partial^2\omega}{\partial y^2} + \frac{\partial^2\psi}{\partial y^2}\,; \tag{4}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -y\frac{\partial^2 \omega}{\partial x \partial y} - \frac{\partial \omega}{\partial x} - \frac{\partial^2 \psi}{\partial x \partial y} \; ; \tag{5}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = y\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2} \; , \tag{6}$$

using (1,2) where appropriate.

(iii) On the surface $y = 0$, we have

$$\sigma_{yy} = \frac{\partial^2 \psi}{\partial x^2} \quad ; \quad \sigma_{xy} = -\frac{\partial \omega}{\partial x} - \frac{\partial^2 \psi}{\partial x \partial y} \; . \tag{7}$$

Thus, the boundary condition $\sigma_{xy} = 0$ can be satisfied by writing

$$\omega = -\frac{\partial \psi}{\partial y} \; . \tag{8}$$

Notice that ω so defined will be harmonic if ψ is harmonic.

With this definition, we have

$$\sigma_{xx} = -y\frac{\partial^3 \psi}{\partial y^3} - \frac{\partial^2 \psi}{\partial y^2} \; ; \tag{9}$$

$$\sigma_{xy} = y\frac{\partial^3 \psi}{\partial x \partial y^2} \; ; \tag{10}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = -y\frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^2 \psi}{\partial x^2} \; ; \tag{11}$$

and at $y = 0$,

$$\sigma_{xx} = -\frac{\partial^2 \psi}{\partial y^2} \quad ; \quad \sigma_{xy} = 0 \quad ; \quad \sigma_{yy} = \frac{\partial^2 \psi}{\partial x^2} \; . \tag{12}$$

Finally, since ψ is harmonic, we have $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$ — i.e. $\frac{\partial^2 \psi}{\partial x^2} = -\frac{\partial^2 \psi}{\partial y^2}$ and hence $\sigma_{xx} = \sigma_{yy}$ at $y = 0$.

(iv) We seek a simple counter-example, where $\sigma_{xx} \neq \sigma_{yy}$ at $y = 0$. Consider the case of the half-space in uniaxial tension $\sigma_{xx} = S, \sigma_{xy} = \sigma_{yy} = 0$. This is clearly a permissible state of stress and it satisfies the condition that $\sigma_{xy} = 0$ at y=0. However, it violates the 'theorem' we have just proved.

To see how this happens, construct the Airy stress function for this case and try to decompose it as in (i,ii,iii) above.

We can obtain uniaxial tension from the stress function $\phi = Sy^2/2$. This in turn can be written in terms of ω, ψ using

$$\omega = \frac{Sy}{2} \quad ; \quad \psi = 0 \; . \tag{13}$$

You can see that this works by substituting it into equations (9–11) above. However, it doesn't satisfy the condition (8).

The flaw in the argument occurs when we deduce (8) from the condition that $\sigma_{xy} = 0$ on $y = 0$. In fact the latter condition only requires that

$$\frac{\partial \omega}{\partial x} = -\frac{\partial^2 \psi}{\partial x \partial y} \tag{14}$$

and integrating this relation, we obtain

$$\omega = -\frac{\partial \psi}{\partial y} + f(y) \ , \tag{15}$$

where $f(y)$ is an arbitrary function of y. Of course, $f(y)$ cannot be chosen totally arbitrarily, since the resulting function ω must be harmonic and hence $f(y)$ must be harmonic. However, this still permits us to add in the terms $f(y) = Ay + B$ where A, B are arbitrary constants. The term B leads to a null stress field, but the term Ay (with $A = S/2$) defines the uniform state of stress described above.

Thus, we can conclude that

$$\sigma_{xx}(x, 0) = \sigma_{yy}(x, 0) + S \ , \tag{16}$$

where S is an arbitrary state of uniaxial stress, but is not necessarily zero. Alternatively, we can conclude that

$$\frac{\partial \sigma_{xx}}{\partial x}(x, 0) = \frac{\partial \sigma_{yy}}{\partial x} = 0 \ . \tag{17}$$

4.6. The constitutive law for an orthotropic elastic material in plane stress can be written

$$e_{xx} = s_{11}\sigma_{xx} + s_{12}\sigma_{yy} \ ; \ e_{yy} = s_{12}\sigma_{xx} + s_{22}\sigma_{yy} \ ; \ e_{xy} = s_{44}\sigma_{xy} \ ,$$

where $s_{11}, s_{12}, s_{22}, s_{44}$ are elastic constants.

Using the Airy stress function ϕ to represent the stress components, find the equation that must be satisfied by ϕ.

Writing

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} \ ; \ \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} \ ; \ \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \ ,$$

we have

$$e_{xx} = s_{11}\frac{\partial^2\phi}{\partial y^2} + s_{12}\frac{\partial^2\phi}{\partial x^2} \ ; \ e_{yy} = s_{12}\frac{\partial^2\phi}{\partial y^2} + s_{22}\frac{\partial^2\phi}{\partial x^2} \ ; \ e_{xy} = -s_{44}\frac{\partial^2\phi}{\partial x\partial y} \ .$$

Substituting in the compatibility equation

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} - 2\frac{\partial^2 e_{xy}}{\partial x\partial y} = 0 \ ,$$

we obtain the governing equation

$$s_{11}\frac{\partial^4\phi}{\partial y^4} + s_{12}\frac{\partial^4\phi}{\partial x^2\partial y^2} + s_{12}\frac{\partial^4\phi}{\partial x^2\partial y^2} + s_{22}\frac{\partial^4\phi}{\partial x^4} + 2s_{44}\frac{\partial^4\phi}{\partial x^2\partial y^2} = 0$$

i.e.

$$s_{22}\frac{\partial^4\phi}{\partial x^4} + 2(s_{12}+s_{44})\frac{\partial^4\phi}{\partial x^2\partial y^2} + s_{11}\frac{\partial^4\phi}{\partial y^4} = 0 \ . \tag{1}$$

Notice that equation (1) can be factorized as

$$s_{11}\left(\frac{\partial^2}{\partial x^2} + \lambda_1\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\phi}{\partial x^2} + \lambda_2\frac{\partial^2\phi}{\partial y^2}\right) = 0 \ ,$$

where λ_1, λ_2 are the two roots of the algebraic equation

$$s_{22}\lambda^2 - 2(s_{12}+s_{44})\lambda + s_{11} = 0 \ .$$

It follows that the general solution of equation (1) can be written as the sum of the general solution of the two distinct equations

$$\frac{\partial^2\phi}{\partial x^2} + \lambda_1\frac{\partial^2\phi}{\partial y^2} = 0 \ ; \ \frac{\partial^2\phi}{\partial x^2} + \lambda_2\frac{\partial^2\phi}{\partial y^2} = 0 \ .$$

In fact a further factorization can be performed, writing for example

$$\frac{\partial^2\phi}{\partial x^2} + \lambda_1 \frac{\partial^2\phi}{\partial y^2} = \left(\frac{\partial}{\partial x} + \imath\sqrt{\lambda_1}\frac{\partial}{\partial y}\right)\left(\frac{\partial\phi}{\partial x} - \imath\sqrt{\lambda_1}\frac{\partial\phi}{\partial y}\right) .$$

From this relation and a similar decomposition of the factor containing λ_2, we conclude that we can write a general solution of equation (1) in the form

$$\phi = f_1\left(x + \imath\sqrt{\lambda_1}y\right) + f_2\left(x - \imath\sqrt{\lambda_1}y\right) + f_3\left(x + \imath\sqrt{\lambda_2}y\right) + f_4\left(x - \imath\sqrt{\lambda_2}y\right) ,$$

where f_1, f_2, f_3, f_4 are arbitrary functions.

This procedure fails for isotropic materials, since the roots λ_1, λ_2 become identical[1]. However, a similar decomposition of the biharmonic equation can be shown to lead then to a rather more complicated solution in terms of the complex variable $\zeta = x + \imath y$. This solution is developed in a different way in §19.6.

[1]This also happens for non-isotropic orthotropic materials if $(s_{12} + s_{44})^2 = s_{11}s_{22}$.

4.7. Show that if the two-dimensional function $\omega(x, y)$ is harmonic ($\nabla^2\omega = 0$), the function

$$\phi = (x^2 + y^2)\omega$$

will be biharmonic.

Differentiating by parts, we have

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= 2x\omega + (x^2 + y^2)\frac{\partial\omega}{\partial x} && (1)\\
\frac{\partial^2\phi}{\partial x^2} &= 2\omega + 4x\frac{\partial\omega}{\partial x} + (x^2 + y^2)\frac{\partial^2\omega}{\partial x^2} && (2)\\
\frac{\partial^2\phi}{\partial y^2} &= 2\omega + 4y\frac{\partial\omega}{\partial y} + (x^2 + y^2)\frac{\partial^2\omega}{\partial y^2}\ , && (3)
\end{aligned}$$

so

$$\nabla^2\phi = 4\omega + 4x\frac{\partial\omega}{\partial x} + 4y\frac{\partial\omega}{\partial y} + (x^2 + y^2)\nabla^2\omega \qquad (4)$$

and the last term is zero, since ω is harmonic.

Writing

$$\psi = \nabla^2\phi = 4\omega + 4x\frac{\partial\omega}{\partial x} + 4y\frac{\partial\omega}{\partial y}\ ,$$

we have

$$\begin{aligned}
\frac{\partial\psi}{\partial x} &= 4\frac{\partial\omega}{\partial x} + 4\frac{\partial\omega}{\partial x} + 4x\frac{\partial^2\omega}{\partial x^2} + 4y\frac{\partial^2\omega}{\partial y\partial x} && (5)\\
\frac{\partial^2\psi}{\partial x^2} &= 8\frac{\partial^2\omega}{\partial x^2} + 4\frac{\partial^2\omega}{\partial x^2} + 4x\frac{\partial^3\omega}{\partial x^3} + 4y\frac{\partial^3\omega}{\partial x^2\partial y} && (6)\\
\frac{\partial^2\psi}{\partial y^2} &= 12\frac{\partial^2\omega}{\partial y^2} + 4y\frac{\partial^3\omega}{\partial y^3} + 4x\frac{\partial^3\omega}{\partial y^2\partial x} && (7)
\end{aligned}$$

and hence

$$\begin{aligned}
\nabla^2\phi &= 12\frac{\partial^2\omega}{\partial x^2} + 12\frac{\partial^2\omega}{\partial y^2} + 4x\frac{\partial^3\omega}{\partial x^3} + 4y\frac{\partial^3\omega}{\partial x^2\partial y} + 4y\frac{\partial^3\omega}{\partial y^3} + 4x\frac{\partial^3\omega}{\partial y^2\partial x}\\
&= 12\nabla^2\omega + 4x\frac{\partial}{\partial x}\nabla^2\omega + 4y\frac{\partial}{\partial y}\nabla^2\omega\\
&= 0\ , && (8)
\end{aligned}$$

since $\nabla^2\omega = 0$.

4.8. The constitutive law for an incompressible elastic material can be written

$$\sigma_{ij} = \bar{\sigma}\delta_{ij} + 2\mu e_{ij} \ ,$$

where

$$\bar{\sigma} = \frac{\sigma_{kk}}{3}$$

represents an arbitrary hydrostatic stress field. Some soils can be approximated as an incompressible material whose modulus varies linearly with depth, so that

$$\mu = Mz$$

for the half space $z > 0$.

Use the displacement function representation

$$\boldsymbol{u} = \boldsymbol{\nabla}\phi$$

to develop a potential function solution for the stresses in such a body. Show that the functions $\phi, \bar{\sigma}$ must satisfy the relations

$$\nabla^2\phi = 0 \ \ ; \ \ \bar{\sigma} = -2M\frac{\partial\phi}{\partial z}$$

and hence obtain expressions for the stress components in terms of the single harmonic function ϕ.

If the half-space is loaded by a normal pressure

$$\sigma_{zz}(x, y, 0) = -p(x, y) \ \ ; \ \ \sigma_{zx}(x, y, 0) = \sigma_{zy}(x, y, 0) = 0 \ ,$$

show that the corresponding normal surface displacement $u_z(x, y, 0)$ is linearly proportional to the local pressure $p(x, y)$ and find the constant of proportionality[2].

If $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, we have

$$e_{ij} = \frac{\partial^2\phi}{\partial x_i \partial x_j} \tag{1}$$

and since the material is incompressible, the dilatation

$$e \equiv e_{kk} = \frac{\partial^2\phi}{\partial x_k \partial x_k} = \nabla^2\phi = 0 \ . \tag{2}$$

The stresses are given by

$$\sigma_{ij} = \bar{\sigma}\delta_{ij} + 2Mx_3\frac{\partial^2\phi}{\partial x_i \partial x_j} \tag{3}$$

[2]For an alternative proof of this result see C.R.Calladine and J.A.Greenwood, Line and point loads on a non-homogeneous incompressible elastic half-space, *Q.J.Mech.Appl.Math.,* Vol 31 (1978), 507–529.

and substituting in the equilibrium equation (2.5), we have

$$\delta_{ij}\frac{\partial\bar{\sigma}}{\partial x_j} + 2Mx_3\frac{\partial^3\phi}{x_i x_j x_j} + 2M\delta_{3j}\frac{\partial^2\phi}{\partial x_i\partial x_j} = 0\ . \tag{4}$$

The second term in this equation is zero because of (2) and hence on performing the implied summations we have

$$\frac{\partial\bar{\sigma}}{\partial x_i} + 2M\frac{\partial^2\phi}{\partial x_i\partial x_3} = 0 \tag{5}$$

or

$$\boldsymbol{\nabla}\left(\bar{\sigma} + 2M\frac{\partial\phi}{\partial x_3}\right) = 0\ . \tag{6}$$

Apart from an arbitrary constant, this equation has the solution

$$\bar{\sigma} = -2M\frac{\partial\phi}{\partial x_3}\ . \tag{7}$$

Using these results, we can now write the stress components as

$$\sigma_{ij} = -2M\frac{\partial\phi}{\partial x_3}\delta_{ij} + 2Mx_3\frac{\partial^2\phi}{\partial x_i\partial x_j}\ , \tag{8}$$

which expands in x, y, z notation as

$$\sigma_{xx} = 2M\left(-\frac{\partial\phi}{\partial z} + z\frac{\partial^2\phi}{\partial x^2}\right)\ ;\ \sigma_{xy} = 2Mz\frac{\partial^2\phi}{\partial x\partial y}\ ;\ \sigma_{yy} = 2M\left(-\frac{\partial\phi}{\partial z} + z\frac{\partial^2\phi}{\partial y^2}\right) \tag{9}$$

$$\sigma_{xz} = 2Mz\frac{\partial^2\phi}{\partial x\partial z}\ ;\ \sigma_{yz} = 2Mz\frac{\partial^2\phi}{\partial y\partial z}\ ;\ \sigma_{zz} = 2M\left(-\frac{\partial\phi}{\partial z} + z\frac{\partial^2\phi}{\partial z^2}\right)\ . \tag{10}$$

From these results, we see that the shear tractions σ_{zx}, σ_{zy} on the surface $z = 0$ are identically zero, whereas the normal contact pressure is

$$p(x, y) = -\sigma_{zz}(x, y, 0) = 2M\frac{\partial\phi}{\partial z}\ . \tag{11}$$

The normal surface displacement is

$$u_z = \frac{\partial\phi}{\partial z} \tag{12}$$

and hence

$$u_z = \frac{p(x, y)}{2M}\ . \tag{13}$$

In other words, the half-space acts as a linear elastic (Winkler) foundation of modulus $2M$.

4.9. Show that Dundurs' constant $\beta \to 0$ for plane strain in the limit where $\nu_1 = 0.5$ and $\mu_1/\mu_2 \to 0$ — i.e. material 1 is incompressible and has a much lower shear modulus[3] than material 2. What is the value of α in this limit?

Dundurs constants are given by equations (4.12, 4.13) as

$$\alpha = \left(\frac{\kappa_1+1}{\mu_1} - \frac{\kappa_2+1}{\mu_2}\right) \Big/ \left(\frac{\kappa_1+1}{\mu_1} + \frac{\kappa_2+1}{\mu_2}\right) \tag{1}$$

$$\beta = \left(\frac{\kappa_1-1}{\mu_1} - \frac{\kappa_2-1}{\mu_2}\right) \Big/ \left(\frac{\kappa_1+1}{\mu_1} + \frac{\kappa_2+1}{\mu_2}\right) . \tag{2}$$

Multiplying numerator and denominator of these expressions by μ_1, we obtain the alternative forms

$$\alpha = \left(\kappa_1 + 1 - \frac{\mu_1}{\mu_2}(\kappa_2+1)\right) \Big/ \left(\kappa_1 + 1 + \frac{\mu_1}{\mu_2}(\kappa_2+1)\right) \tag{3}$$

$$\beta = \left(\kappa_1 - 1 - \frac{\mu_1}{\mu_2}(\kappa_2-1)\right) \Big/ \left(\kappa_1 + 1 + \frac{\mu_1}{\mu_2}(\kappa_2+1)\right) . \tag{4}$$

If $\mu_1/\mu_2 \to 0$, we have the further simplification

$$\alpha = (\kappa_1+1)/(\kappa_1+1) = 1 \tag{5}$$

$$\beta = (\kappa_1-1)/(\kappa_1+1) \tag{6}$$

and in plane strain, $\kappa = 3 - 4\nu$, so if $\nu_1 = 0.5$,

$$\kappa_1 - 1 = 2 - 4\nu_1 = 0 , \tag{7}$$

giving $\beta = 0$. Thus for this material combination, we have $\alpha = 1, \beta = 0$.

[3] This is a reasonable approximation for the important case of rubber (material 1) bonded to steel (material 2).

4.10. Solve Problem 3.7 for the case where there is no body force, using the Airy stress function ϕ to represent the stress components. Hence show that the governing equation is $\nabla^4\phi = 0$, as in the case of compressible materials.

The stress-strain relation

$$\sigma_{ij} = 2\mu e_{ij} - q\delta_{ij} \tag{1}$$

can be solved to give

$$2\mu e_{ij} = q\delta_{ij} + \sigma_{ij} \ . \tag{2}$$

Using the Airy stress function to represent the in-plane stresses, we have

$$2\mu e_{xx} = q + \frac{\partial^2\phi}{\partial y^2} \ ; \ 2\mu e_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} \ ; \ 2\mu e_{yy} = q + \frac{\partial^2\phi}{\partial x^2} \ . \tag{3}$$

The compatibility equation (2.8) then requires

$$\frac{\partial^2 q}{\partial y^2} + \frac{\partial^4\phi}{\partial y^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^2 q}{\partial x^2} + \frac{\partial^4\phi}{\partial x^4} = 0 \tag{4}$$

and hence

$$\nabla^4\phi + \nabla^2 q = 0 \ . \tag{5}$$

Also, the material is incompressible so $e_{11} + e_{22} = 0$, leading to

$$\nabla^2\phi + 2q = 0 \ , \tag{6}$$

or

$$q = -\frac{1}{2}\nabla^2\phi \ . \tag{7}$$

Eliminating q between (5,7) then yields

$$\frac{1}{2}\nabla^4\phi = 0 \ , \tag{8}$$

showing that the stress function is biharmonic. It then follows immediately from (7) that

$$\nabla^2 q = -\frac{1}{2}\nabla^4\phi = 0 \tag{9}$$

and that

$$\sigma_{xx} + \sigma_{yy} = \nabla^2\phi = -2q \ . \tag{10}$$

CHAPTER 5

5.1. The beam $-b < y < b,\ 0 < x < L$, is built-in at the end $x = 0$ and loaded by a uniform shear traction $\sigma_{xy} = S$ on the upper edge, $y = b$, the remaining edges, $x = L,\ y = -b$ being traction-free. Find a suitable stress function and the corresponding stress components for this problem, using the weak boundary conditions on $x = L$.

The problem is illustrated in Figure 1. Some advantage can be gained from moving the origin to the free end $x = L$, since the resulting problem would then be antisymmetric in x. However, it is important that the student should recognize that such problems can be solved without the extra insight gained from symmetry. We therefore give two solutions — the first does not use symmetry and the second does.

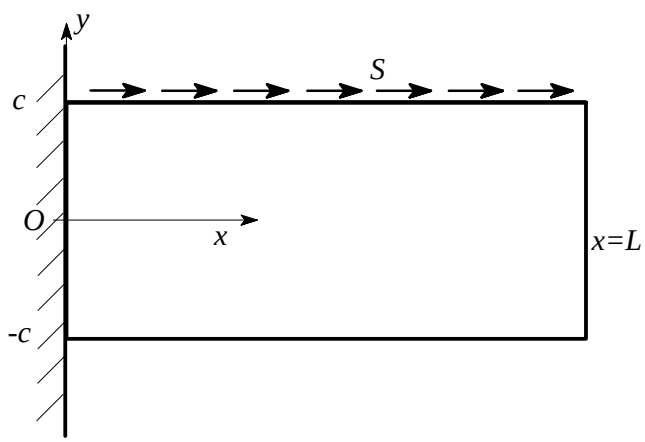


Figure 1: Rectangular beam loaded by shear tractions

Strength of Materials considerations suggest that the bending moment will vary linearly with x and hence that σ_{xx} will vary with xy, requiring a 4th order stress function. The most general such function is

$$\begin{aligned}\phi &= C_1x^4 + C_2x^3y + C_3x^2y^2 + C_4xy^3 + C_5y^4 + C_6x^3 + C_7x^2y \\ &\quad + C_8xy^2 + C_9y^3 + C_{10}x^2 + C_{11}xy + C_{12}y^2 \,. \end{aligned} \tag{1}$$

We shall impose strong boundary conditions on the upper and lower edges — i.e.

$$\sigma_{xy} = S \quad ; \quad y = b \,; \tag{2}$$
$$= 0 \quad ; \quad y = -b \,; \tag{3}$$
$$\sigma_{yy} = 0 \quad ; \quad y = b \,; \tag{4}$$
$$= 0 \quad ; \quad y = -b \,. \tag{5}$$

The stress component

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 12C_1 x^2 + 6C_2 xy + 2C_3 y^2 + 6C_6 x + 2C_7 y + 2C_{10} \ . \tag{6}$$

The boundary condition (4) therefore requires that

$$12C_1 x^2 + 6C_2 xb + 2C_3 b^2 + 6C_6 x + 2C_7 b + 2C_{10} = 0 \tag{7}$$

for all x and this is only possible if the coefficients of $x^2, x, 1$ are all separately zero — i.e.

$$C_1 = 0 \quad ; \quad C_2 b + C_6 = 0 \quad ; \quad C_3 b^2 + C_7 b + C_{10} = 0 \ . \tag{8}$$

A similar agrgument applied to condition (5) yields

$$C_1 = 0 \quad ; \quad -C_2 b + C_6 = 0 \quad ; \quad C_3 b^2 - C_7 b + C_{10} = 0 \ . \tag{9}$$

We therefore conclude that

$$C_1 = C_2 = C_6 = C_7 = 0 \tag{10}$$

and

$$C_3 b^2 + C_{10} = 0 \ . \tag{11}$$

We next consider the stress component σ_{xy}, taking advantage of the results (10) to simplify the algebra. We have

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -4C_3 xy - 3C_4 y^2 - 2C_8 y - C_{11} \ . \tag{12}$$

The boundary conditions (2,3) require that

$$4C_3 bx + 3C_4 b^2 + 2C_8 b + C_{11} = -S \ ; \tag{13}$$

$$-4C_3 bx + 3C_4 b^2 - 2C_8 b + C_{11} = 0 \ ; \tag{14}$$

and hence

$$C_3 = 0 \ ; \tag{15}$$

$$3C_4 b^2 + 2C_8 b + C_{11} = -S \ ; \tag{16}$$

$$3C_4 b^2 - 2C_8 b + C_{11} = 0 \ . \tag{17}$$

Equations (15,11) require that $C_{10} = 0$ and (16,17) that

$$C_8 = -\frac{S}{4b} \ ; \tag{18}$$

$$6C_4 b^2 + 2C_{11} = -S \ . \tag{19}$$

One additional equation is obtained from the biharmonic equation $\nabla^4\phi = 0$ from which, using (10,15), we obtain

$$C_5 = 0 \ . \tag{20}$$

There remain four constants to be determined — C_4, C_9, C_{11}, C_{12} — from equation (19) and the three weak conditions

$$\int_{-b}^{b} \sigma_{xy}(L,y)dy = 0 \ ; \tag{21}$$

$$\int_{-b}^{b} \sigma_{xx}(L,y)dy = 0 \ ; \tag{22}$$

$$\int_{-b}^{b} y\sigma_{xx}(L,y)dy = 0 \ . \tag{23}$$

Substituting the non-zero terms in (12) into (21), we obtain

$$\int_{-b}^{b} (-3C_4y^2 - 2C_8y - C_{11})dy = 0 \tag{24}$$

and hence

$$C_4b^2 + C_{11} = 0 \ . \tag{25}$$

It follows from (19,25) that

$$C_4 = -\frac{S}{4b^2} \quad ; \quad C_{11} = \frac{S}{4} \ . \tag{26}$$

The remaining stress component

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = 6C_4xy + 2C_8x + 6C_9y + 2C_{12} \ . \tag{27}$$

Substituting into (22,23), we obtain

$$\int_{-b}^{b} (6C_4Ly + 2C_8L + 6C_9y + 2C_{12})dy = 0 \ ; \tag{28}$$

$$\int_{-b}^{b} (6C_4Ly^2 + 2C_8Ly + 6C_9y^2 + 2C_{12}y)dy = 0 \ ; \tag{29}$$

and hence

$$C_8L + C_{12} = 0 \ ; \tag{30}$$

$$C_4L + C_9 = 0 \ , \tag{31}$$

from which

$$C_{12} = \frac{SL}{4b} \quad ; \quad C_9 = \frac{SL}{4b^2} \ . \tag{32}$$

The complete stress function is therefore

$$\phi = \frac{S}{4b^2}(-xy^3 - bxy^2 + Ly^3 + b^2xy + LCy^2) \tag{33}$$

and the stress field is

$$\sigma_{xx} = \frac{S(3y+b)(L-x)}{2b^2} \; ; \tag{34}$$

$$\sigma_{xy} = \frac{S(3y-b)(y+b)}{4b^2} \; ; \tag{35}$$

$$\sigma_{yy} = 0 \; . \tag{36}$$

We notice from the final result that $\sigma_{xx} = 0$ on $x = L$ for all y — in other words, this boundary condition is satisfied in the strong sense. The only approximation involved in the solution is that σ_{xy} is not zero on that boundary, being

$$\sigma_{xy}(L, y) = \frac{S(3L-b)(L+b)}{4b^2} \; . \tag{37}$$

An exact solution would require a correction in which these unintended but self-equilibrated tractions were removed.

Solution using symmetry

We now repeat the solution making use of the antisymmetry of the problem in x.

Suppose we were to extend the body to length $2L$ as shown in Figure 2 and apply a boundary condition at the right end that retains the zero slope for the beam but permits unrestricted vertical motion.

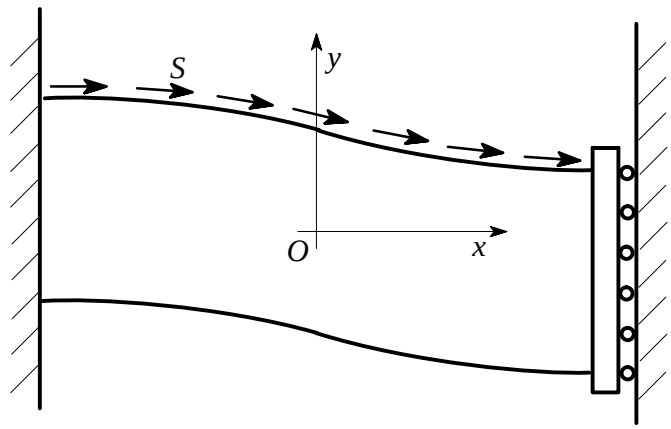


Figure 2: Extension of the body to demonstrate antisymmetry

The vertical shear force in the beam will therefore be zero everywhere and the new problem is antisymmetric in x. This immediately implies that $\sigma_{xx} = 0$ in the strong sense on the mid-plane. This satisfies two of the weak conditions on that plane ((22,23) in the above solutions) and the third condition (21) is equivalent to the condition that the shear force be zero, which we have already seen to be the case by equilibrium considerations. Thus, the left half of the antisymmetric problem in Figure 2 is equivalent to the problem of Figure 1 and it follows that the appropriate stress function will be odd in x if the origin is moved to the free end of the beam (which corresponds to the mid-plane (symmetry plane) in Figure 2).

For this problem, we again use a fourth order polynomial, but we only need to keep the terms

$$\phi = Axy^3 + Bx^3y + Cxy^2 + Dx^3 + Exy \ . \tag{38}$$

This function satisfies the biharmonic equation $\nabla^4\phi = 0$ identically.

The stress components are

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = 6Bxy + 6Dx \ ; \tag{39}$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -3Ay^2 - 3Bx^2 - 2Cy - E \ ; \tag{40}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 6Axy + 2Cx \ . \tag{41}$$

The two conditions $\sigma_{yy} = 0$ on $y = \pm b$ yield the equations

$$6Bb + 6D = 0 \quad ; \quad -6Bb + 6D = 0 \ , \tag{42}$$

which have only the trivial solution $B = D = 0$.

The conditions $\sigma_{xy}(x, b) = S$, $\sigma_{xy}(x, -b) = 0$ yield the equations

$$-3Ab^2 - 2Cb - E = S \quad ; \quad -3Ab^2 + 2Cb - E = 0 \ , \tag{43}$$

from which

$$C = -\frac{S}{4b} \ ; \tag{44}$$

$$6Ab^2 + 2E = -S \ . \tag{45}$$

As explained above, two of the three weak boundary conditions are satisfied identically by virtue of the symmetry, leaving the condition

$$\int_{-b}^{b} \sigma_{xy}(0, y) dy = 0 \ , \tag{46}$$

from which

$$\int_{-b}^{b} (3Ay^2 + 2Cy + E) dy = 0 \tag{47}$$

and hence

$$Ab^2 + E = 0 \ . \tag{48}$$

Equations (45,48) can be solved to give

$$A = -\frac{S}{4b^2} \quad ; \quad E = \frac{S}{4} \ , \tag{49}$$

completing the solution. The final stress function is

$$\phi = \phi = \frac{S}{4b^2}(-xy^3 - bxy^2 + b^2xy) \tag{50}$$

and the stress field is

$$\sigma_{xx} = -\frac{Sx(3y + b)}{2b^2} \ ; \tag{51}$$

$$\sigma_{xy} = \frac{S(3y - b)(y + b)}{4b^2} \ ; \tag{52}$$

$$\sigma_{yy} = 0 \ . \tag{53}$$

The solution in the original coordinate system is easily recovered by replacing x in the equations (50–53) by $(x - L)$ leading directly to equations (33–36) (except for a null term $SLy/4$ in ϕ that yields no stresses).

5.2. The beam $-b<y<b, -L<x<L$ is simply supported at the ends $x=\pm L$ and loaded by a shear traction $\sigma_{xy}=Sx/L$ on the lower edge, $y=-b$, the upper edge being traction-free. Find a suitable stress function and the corresponding stress components for this problem, using the weak boundary conditions on $x=\pm L$.

The shear traction varies with x^1, suggesting a fifth order stress function ($m=1, m+4=5$). Also, the problem is symmetric about $x=0$, so we include only the terms with even powers of x, giving

$$\phi = C_1x^4y + C_2x^2y^3 + C_3y^5 + C_4x^4 + C_5x^2y^2 + C_6y^4 + C_7x^2y + C_8y^3 + C_9x^2 + C_{10}y^2 \ . \quad (1)$$

The corresponding stress components are

$$\begin{aligned} \sigma_{xx} &= 6C_2x^2y + 20C_3y^3 + 2C_5x^2 + 12C_6y^2 + 6C_8y + 2C_{10} && (2)\\ \sigma_{xy} &= -4C_1x^3 - 6C_2xy^2 - 4C_5xy - 2C_7x && (3)\\ \sigma_{yy} &= 12C_1x^2y + 2C_2y^3 + 12C_4x^2 + 2C_5y^2 + 2C_7y + 2C_9 \ . && (4) \end{aligned}$$

The boundary conditions

$$\sigma_{yy} = 0 \quad ; \quad y = \pm b \quad (5)$$

yield the equations

$$\begin{aligned} 12C_1x^2b + 2C_2b^3 + 12C_4x^2 + 2C_5b^2 + 2C_7b + 2C_9 &= 0 && (6)\\ -12C_1x^2b - 2C_2b^3 + 12C_4x^2 + 2C_5b^2 - 2C_7b + 2C_9 &= 0 && (7) \end{aligned}$$

and hence

$$C_1 = C_4 = 0 \ ; \quad C_2b^2 + C_7 = 0 \ ; \quad C_5b^2 + C_9 = 0 \ . \quad (8)$$

For the shear tractions, we require

$$\begin{aligned} -4C_1x^3 - 6C_2xb^2 - 4C_5xb - 2C_7x &= 0 && (9)\\ -4C_1x^3 - 6C_2xb^2 + 4C_5xb - 2C_7x &= Sx/L && (10) \end{aligned}$$

and hence

$$C_1 = 0 \ ; \quad -12C_2b^2 - 4C_7 = S/L \ ; \quad C_5 = \frac{S}{8bL} \ . \quad (11)$$

The second of (8) and the second of (11) can be solved for C_2, C_7, giving

$$C_2 = -\frac{S}{8Lb^2} \ ; \ C_7 = \frac{S}{8L} \quad (12)$$

and the third of (8) and (11) gives

$$C_9 = -\frac{Sb}{8L} \ . \quad (13)$$

The biharmonic condition (after dropping the known zero terms) gives

$$24C_2y + 120C_3y + 8C_5 + 24C_6 = 0 \quad (14)$$

and hence

$$C_3 = -\frac{C_2}{5} = \frac{S}{40Lb^2} \quad ; \quad C_6 = -\frac{C_5}{3} = -\frac{S}{24bL} \ . \tag{15}$$

Finally, we impose the weak boundary conditions

$$\int_{-b}^{b} \sigma_{xx}(L,y)dy = 0 \quad ; \quad \int_{-b}^{b} \sigma_{xx}(L,y)ydy = 0 \ . \tag{16}$$

The third weak condition is satisfied identically because of symmetry and equilibrium considerations. These yield

$$4C_5L^2b + 8C_6b^3 + 2C_{10}b = 0 \ ; \ 4C_2L^2b^3 + 8C_3b^5 + 4C_8b^3 = 0 \tag{17}$$

and the remaining constants are determined as

$$C_8 = \frac{(5L^2 - 2b^2)S}{40Lb^2} \quad ; \quad C_{10} = \frac{(2b^2 - 3L^2)S}{24bL} \ . \tag{18}$$

The final stress function is

$$\begin{aligned} \phi \ &= \ S(-15x^2y^3 + 3y^5 + 15bx^2y^2 - 5by^4 + 15b^2x^2y + 15L^2y^3 - 6b^2y^3 \\ &\quad -15b^3x^2 + 10b^3y^2 - 15L^2by^2)/120Lb^2 \end{aligned} \tag{19}$$

and the stress field is

$$\begin{aligned} \sigma_{xx} &= \frac{S}{60b^2L}(10b^3 - 15L^2b - 18b^2y + 45L^2y + 15bx^2 - 30by^2 - 45x^2y + 30y^3) \\ \sigma_{xy} &= -\frac{Sx(b-y)(b+3y)}{4b^2L} \\ \sigma_{yy} &= -\frac{S(b-y)^2(y+b)}{4b^2L} \ . \end{aligned} \tag{20}$$

A Maple solution of this problem ('P52') is available from the author.

5.3. The beam $-b<y<b,\ 0<x<L$, is built-in at the end $x=L$ and loaded by a linearly-varying compressive normal traction $p(x)=Sx/L$ on the upper edge, $y=b$, the remaining edges, $x=0,\ y=-b$ being traction-free. Find a suitable stress function and the corresponding stress components for this problem, using the weak boundary conditions on $x=0$.

The normal loading varies linearly with x, so $n=1$ and we start with a polynomial of degree 6 ($n+5=6$).

$$\begin{aligned}\phi \;=\;& C_1x^2+C_2xy+C_3y^2+C_4x^3+C_5x^2y+C_6xy^2+C_7y^3+C_8x^4+C_9x^3y+C_{10}x^2y^2\\ &+C_{11}xy^3+C_{12}y^4+C_{13}x^5+C_{14}x^4y+C_{15}x^3y^2+C_{16}x^2y^3+C_{17}xy^4+C_{18}y^5+C_{19}x^6\\ &+C_{20}x^5y+C_{21}x^4y^2+C_{21}x^3y^3+C_{22}x^2y^4+C_{23}xy^5+C_{25}y^6\ . \end{aligned}\tag{1}$$

Substituting (1) into the biharmonic equation (5.1), we obtain

$$\begin{aligned}&(360C_{19}+48C_{21}+24C_{23})x^2+(120C_{20}+72C_{22}+120C_{24})xy+(24C_{21}+48C_{23}+360C_{25})y^2\\ &\quad+(120C_{13}+24C_{15}+24C_{17})x+(24C_{14}+24C_{16}+120C_{18})y+(24C_8+8C_{10}+24C_{12})=0\end{aligned}$$

and this must be zero for all x,y leading to the six constraint equations

$$360C_{19}+48C_{21}+24C_{23} = 0 \tag{2}$$
$$120C_{20}+72C_{22}+120C_{24} = 0 \tag{3}$$
$$24C_{21}+48C_{23}+360C_{25} = 0 \tag{4}$$
$$120C_{13}+24C_{15}+24C_{17} = 0 \tag{5}$$
$$24C_{14}+24C_{16}+120C_{18} = 0 \tag{6}$$
$$24C_8+8C_{10}+24C_{12} = 0\ . \tag{7}$$

The stresses are obtained by substituting (1) into (5.14–5.16) with the result

$$\begin{aligned}\sigma_{xx} \;=\;& 2C_3+2C_6x+6C_7y+2C_{10}x^2+6C_{11}xy+12C_{12}y^2+2C_{15}x^3+6C_{16}x^2y\\ &+12C_{17}xy^2+20C_{18}y^3+2C_{21}x^4+6C_{22}x^3y+12C_{23}x^2y^2+20C_{24}xy^3+30C_{25}y^4\\ \sigma_{xy} \;=\;& -C_2-2C_5x-2C_6y-3C_9x^2-4C_{10}xy-3C_{11}y^2-4C_{14}x^3-6C_{15}x^2y\\ &-6C_{16}xy^2-4C_{17}y^3-5C_{20}x^4-8C_{21}x^3y-9C_{22}x^2y^2-8C_{23}xy^3-5C_{24}y^4\\ \sigma_{yy} \;=\;& 2C_1+6C_4x+2C_5y+12C_8x^2+6C_9xy+2C_{10}y^2+20C_{13}x^3+12C_{14}x^2y\\ &+6C_{15}xy^2+2C_{16}y^3+30C_{19}x^4+20C_{20}x^3y+12C_{21}x^2y^2+6C_{22}xy^3+2C_{23}y^4\ .\end{aligned}$$

The tractions on $y=b$ are therefore

$$\begin{aligned}\sigma_{yx} \;=\;& -5C_{20}x^4-(4C_{14}+8C_{20}b)x^3-(3C_9+6C_{15}b+9C_{22}b^2)x^2\\ &-(2C_5+4C_{10}b+6C_{16}b^2+8C_{23}b^3)x-(C_2+2C_6b+3C_{11}b^2+4C_{17}b^3+5C_{24}b^4)\\ \sigma_{yy} \;=\;& 30C_{19}x^4+(20C_{13}+20C_{20}b)x^3+(12C_8+12C_{14}b+12C_{21}b^2)x^2\\ &+(6C_4+6C_9b+6C_{15}b^2+6C_{22}b^3)x+(2C_1+2C_5b+2C_{10}b^2+2C_{16}b^3+2C_{23}b^4)\end{aligned}$$

and these must satisfy the equations

$$\sigma_{yy}(x,b) = -Sx/L \; ; \quad \sigma_{yx}(x,b) = 0 \tag{8}$$

for all x, giving

$$\begin{aligned}
5C_{20} &= 0 && (9)\\
4C_{14} + 8C_{20}b &= 0 && (10)\\
3C_9 + 6C_{15}b + 9C_{22}b^2 &= 0 && (11)\\
2C_5 + 4C_{10}b + 6C_{16}b^2 + 8C_{23}b^3 &= 0 && (12)\\
C_2 + 2C_6b + 3C_{11}b^2 + 4C_{17}b^3 + 5C_{24}b^4 &= 0 && (13)\\
30C_{19} &= 0 && (14)\\
20C_{13} + 20C_{20}b &= 0 && (15)\\
12C_8 + 12C_{14}b + 12C_{21}b^2 &= 0 && (16)\\
6C_4 + 6C_9b + 6C_{15}b^2 + 6C_{22}b^3 &= -S/L && (17)\\
2C_1 + 2C_5b + 2C_{10}b^2 + 2C_{16}b^3 + 2C_{23}b^4 &= 0\,. && (18)
\end{aligned}$$

A similar procedure for the traction-free edge $y=-b$ yields the additional equations

$$\begin{aligned}
4C_{14} - 8C_{20}b &= 0 && (19)\\
3C_9 - 6C_{15}b + 9C_{22}b^2 &= 0 && (20)\\
2C_5 - 4C_{10}b + 6C_{16}b^2 - 8C_{23}b^3 &= 0 && (21)\\
C_2 - 2C_6b + 3C_{11}b^2 - 4C_{17}b^3 + 5C_{24}b^4 &= 0 && (22)\\
20C_{13} - 20C_{20}b &= 0 && (23)\\
12C_8 - 12C_{14}b + 12C_{21}b^2 &= 0 && (24)\\
6C_4 - 6C_9b + 6C_{15}b^2 - 6C_{22}b^3 &= -S/L && (25)\\
2C_1 - 2C_5b + 2C_{10}b^2 - 2C_{16}b^3 + 2C_{23}b^4 &= 0\,. && (26)
\end{aligned}$$

On $x=0$, we have

$$\begin{aligned}
\sigma_{xx} &= 2C_3 + 6C_7y + 12C_{12}y^2 + 20C_{18}y^3 + 30C_{25}y^4 && (27)\\
\sigma_{xy} &= -C_2 - 2C_6y - 3C_{11}y^2 - 4C_{17}y^3 - 5C_{24}y^4\,. && (28)
\end{aligned}$$

Substituting into the weak conditions

$$\begin{aligned}
F_x(0) = \int_{-b}^{b} \sigma_{xx}(0,y)dy &= 0 && (29)\\
F_y(0) = \int_{-b}^{b} \sigma_{xy}(0,y)dy &= 0 && (30)\\
M(0) = \int_{-b}^{b} \sigma_{xx}(0,y)ydy &= 0 && (31)
\end{aligned}$$

and evaluating the integrals, we obtain the three additional equations

$$
\begin{aligned}
4C_3 b + 8C_{12} b^3 + 12C_{25} b^5 &= 0 && (32)\\
-2C_2 b - 2C_{11} b^3 - 2C_{24} b^5 &= 0 && (33)\\
4C_7 b^3 + 8C_{18} b^5 &= 0\ . && (34)
\end{aligned}
$$

Solving equations (2–7, 9–26, 32–34) for the unknown constants, we obtain

$$
C_2 = -\frac{Sb}{40L}\ ;\ \ C_4 = -\frac{S}{12L}\ ;\ \ C_9 = -\frac{S}{8bL}\ ;\ \ C_{11} = \frac{S}{20bL}\ ;\ \ C_{22} = \frac{S}{24b^3L}\ ;\ \ C_{24} = -\frac{S}{40b^3L}\ ,
$$

all the remaining constants being zero. The final stress function[1] is

$$
\phi = \frac{S}{120b^3L}(-3b^4xy - 10b^3x^3 - 15b^2x^3y + 6b^2xy^3 + 5x^3y^3 - 3xy^5) \qquad (35)
$$

and the stress components are

$$
\begin{aligned}
\sigma_{xx} &= \frac{S}{20b^3L}(6b^2xy + 5x^3y - 10xy^3) && (36)\\
\sigma_{xy} &= \frac{S}{40b^3L}(b^2 - y^2)(b^2 + 15x^2 - 5y^2) && (37)\\
\sigma_{yy} &= \frac{S}{4b^3L}(-2b^3x - 3b^2xy + xy^3)\ . && (38)
\end{aligned}
$$

[1]It is clear at this point that the stress function is odd in x. This could have been argued from the beginning, as suggested in §5.2.3, in which case the number of terms in the initial stress function would have been halved.

5.4. The beam $-b<y<b, -L<x<L$ is simply supported at the ends $x=\pm L$ and loaded by a compressive normal traction

$$p(x) = S\cos\left(\frac{\pi x}{2L}\right)$$

on the upper edge, $y=b$, the lower edge being traction-free. Find a suitable stress function and the corresponding stress components for this problem.

If the beam were infinitely long, a solution could be obtained using the biharmonic stress function

$$\phi_1 = \{(A+By)\cosh(\lambda y) + (C+Dy)\sinh(\lambda y)\}\cos(\lambda x)\ , \tag{1}$$

where

$$\lambda = \frac{\pi}{2L}\ . \tag{2}$$

Our strategy will therefore be to determine the coefficients A, B, C, D in this function required to satisfy the strong boundary conditions on the edges $y=\pm b$ and then to correct the boundary conditions on the ends $x=\pm L$ in the weak sense using an appropriate low order polynomial.

The stress components due to ϕ_1 are

$$\sigma_{xx} = \lambda\{(A\lambda + B\lambda y + 2D)\cosh(\lambda y) + (C\lambda + D\lambda y + 2B)\sinh(\lambda y)\}\cos(\lambda x) \tag{3}$$

$$\sigma_{xy} = \lambda\{(A\lambda + B\lambda y + D)\sinh(\lambda y) + (B + C\lambda + D\lambda y)\cosh(\lambda y)\}\sin(\lambda x) \tag{4}$$

$$\sigma_{yy} = -\lambda^2\{(A+By)\cosh(\lambda y) + (C+Dy)\sinh(\lambda y)\}\cos(\lambda x) \tag{5}$$

and the boundary conditions on $y=\pm b$ therefore require that

$$(A+Bb)\cosh(\lambda b) + (C+Db)\sinh(\lambda b) = \frac{S}{\lambda^2} \tag{6}$$

$$(A-Bb)\cosh(\lambda b) - (C-Db)\sinh(\lambda b) = 0 \tag{7}$$

$$(A\lambda + B\lambda b + D)\sinh(\lambda b) + (B + C\lambda + D\lambda b)\cosh(\lambda b) = 0 \tag{8}$$

$$-(A\lambda - B\lambda b + D)\sinh(\lambda b) + (B + C\lambda - D\lambda b)\cosh(\lambda b) = 0 \tag{9}$$

The matrix can be partitioned by taking sums and differences of these equations in pairs, resulting in the simpler set

$$A\cosh(\lambda b) + Db\sinh(\lambda b) = \frac{S}{2\lambda^2} \tag{10}$$

$$Bb\cosh(\lambda b) + C\sinh(\lambda b) = \frac{S}{2\lambda^2} \tag{11}$$

$$B\lambda b\sinh(\lambda b) + (B + C\lambda)\cosh(\lambda b) = 0 \tag{12}$$

$$(A\lambda + D)\sinh(\lambda b) + D\lambda b\cosh(\lambda b) = 0\ . \tag{13}$$

The solution of these equations is

$$A = \frac{S(\sinh(\lambda b) + \lambda b \cosh(\lambda b))}{\lambda^2[2\lambda b + \sinh(2\lambda b)]} \tag{14}$$

$$B = \frac{S \cosh(\lambda b)}{\lambda[2\lambda b - \sinh(2\lambda b)]} \tag{15}$$

$$C = -\frac{S(\lambda b \sinh(\lambda b) + \cosh(\lambda b)}{\lambda^2[2\lambda b - \sinh(2\lambda b)]} \tag{16}$$

$$D = -\frac{S \sinh(\lambda b)}{\lambda[2\lambda b + \sinh(2\lambda b)]} \tag{17}$$

The tractions on $x = L$ are

$$\sigma_{xx} = 0 \tag{18}$$

$$\sigma_{xy} = \lambda\{(A\lambda + B\lambda y + D)\sinh(\lambda y) + (B + C\lambda + D\lambda y)\cosh(\lambda y)\} , \tag{19}$$

so there is no moment or axial force at $x = L$ and two of the three weak conditions are satisfied (in fact in the strong sense). The remaining condition is also satisfied, since the vertical forces at $x = \pm L$ must be equal by symmetry and they must satisfy global equilibrium for the bar. Thus, the final stress function for this problem is given by equations (1,2,14–17) and the stress field by (3-5,14–17).

5.5. The beam $-b<y<b,\ 0<x<L$, is built-in at the end $x=L$ and loaded by a compressive normal traction

$$p(x) = S\sin\left(\frac{\pi x}{2L}\right)$$

on the upper edge, $y=b$, the remaining edges, $x=0,\ y=-b$ being traction-free. Use a combination of the stress function (5.97) and an appropriate polynomial to find the stress components for this problem, using the weak boundary conditions on $x=0$.

If the beam were infinitely long, a solution could be obtained using the biharmonic stress function

$$\phi_1 = \{(A+By)\cosh(\lambda y) + (C+Dy)\sinh(\lambda y)\}\sin(\lambda x)\ , \tag{1}$$

where

$$\lambda = \frac{\pi}{2L}\ . \tag{2}$$

Our strategy will therefore be to determine the coefficients A, B, C, D in this function required to satisfy the strong boundary conditions on the edges $y=\pm b$ and then to correct the boundary conditions on the end $x=0$ in the weak sense using an appropriate low order polynomial.

The stress components due to ϕ_1 are

$$\sigma_{xx} = \lambda\{(A\lambda + B\lambda y + 2D)\cosh(\lambda y) + (C\lambda + D\lambda y + 2B)\sinh(\lambda y)\}\sin(\lambda x) \tag{3}$$

$$\sigma_{xy} = -\lambda\{(A\lambda + B\lambda y + D)\sinh(\lambda y) + (B + C\lambda + D\lambda y)\cosh(\lambda y)\}\cos(\lambda x) \tag{4}$$

$$\sigma_{yy} = -\lambda^2\{(A+By)\cosh(\lambda y) + (C+Dy)\sinh(\lambda y)\}\sin(\lambda x) \tag{5}$$

and the boundary conditions on $y=\pm b$ therefore require that

$$(A+Bb)\cosh(\lambda b) + (C+Db)\sinh(\lambda b) = -S \tag{6}$$

$$(A-Bb)\cosh(\lambda b) - (C-Db)\sinh(\lambda b) = 0 \tag{7}$$

$$(A\lambda + B\lambda b + D)\sinh(\lambda b) + (B + C\lambda + D\lambda b)\cosh(\lambda b) = 0 \tag{8}$$

$$-(A\lambda - B\lambda b + D)\sinh(\lambda b) + (B + C\lambda - D\lambda b)\cosh(\lambda b) = 0 \tag{9}$$

The matrix can be partitioned by taking sums and differences of these equations in pairs, resulting in the simpler set

$$A\cosh(\lambda b) + Db\sinh(\lambda b) = -\frac{S}{2} \tag{10}$$

$$Bb\cosh(\lambda b) + C\sinh(\lambda b) = -\frac{S}{2} \tag{11}$$

$$B\lambda b\sinh(\lambda b) + (B + C\lambda)\cosh(\lambda b) = 0 \tag{12}$$

$$(A\lambda + D)\sinh(\lambda b) + D\lambda b\cosh(\lambda b) = 0\ . \tag{13}$$

The solution of these equations is

$$A = -\frac{S(\sinh(\lambda b) + \lambda b \cosh(\lambda b))}{2\lambda b \cosh(2\lambda b) + \sinh(2\lambda b)} \tag{14}$$

$$B = -\frac{S\lambda \cosh(\lambda b)}{2\lambda b \cosh(2\lambda b) - \sinh(2\lambda b)} \tag{15}$$

$$C = \frac{S(\lambda b \sinh(\lambda b) + \cosh(\lambda b)}{2\lambda b \cosh(2\lambda b) - \sinh(2\lambda b)} \tag{16}$$

$$D = \frac{S\lambda \sinh(\lambda b)}{2\lambda b \cosh(2\lambda b) + \sinh(2\lambda b)} \tag{17}$$

The tractions on $x = 0$ are

$$\sigma_{xx}(0, y) = 0 \tag{18}$$

$$\sigma_{xy}(0, y) = -\lambda\{(A\lambda + B\lambda y + D)\sinh(\lambda y) + (B + C\lambda + D\lambda y)\cosh(\lambda y)\} , \tag{19}$$

so there is no moment or axial force at $x = 0$ and two of the three weak conditions are satisfied (in fact in the strong sense).

The remaining condition

$$F_y(0) = \int_{-b}^{b} \sigma_{xy}(0, y)dy = 0 . \tag{20}$$

is not satisfied. In fact, substituting for $\sigma_{xy}(0, y)$ from (19) and for the constants A, B, C, D from (14–17), we obtain

$$F_y(0) = -\frac{S}{\lambda} . \tag{21}$$

To satisfy the weak condition on $x = 0$ we must therefore superpose the solution for an equal and opposite force F_y on the end and this solution is given in §5.2.1.

The final solution involves the stress function

$$\phi = \{(A + By)\cosh(\lambda y) + (C + Dy)\sinh(\lambda y)\}\sin(\lambda x) + C_1(xy^3 - 3b^2xy) , \tag{22}$$

where the new constant

$$C_1 = \frac{S}{4\lambda b^3} \tag{23}$$

The final expressions for the stress components are

$$\begin{aligned}
\sigma_{xx} &= \lambda\{(A\lambda + B\lambda y + 2D)\cosh(\lambda y) + (C\lambda + D\lambda y + 2B)\sinh(\lambda y)\}\cos(\lambda x) + 6C_1xy \\
\sigma_{xy} &= \lambda\{(A\lambda + B\lambda y + D)\sinh(\lambda y) + (B + C\lambda + D\lambda y)\cosh(\lambda y)\}\sin(\lambda x) + 3C_1(b^2 - y^2) \\
\sigma_{yy} &= -\lambda^2\{(A + By)\cosh(\lambda y) + (C + Dy)\sinh(\lambda y)\}\cos(\lambda x) ,
\end{aligned}$$

where A, B, C, D, C_1 are given by (14–17, 23).

5.6. A large plate defined by $y>0$ is subjected to a sinusoidally varying load

$$\sigma_{yy} = S \sin \lambda x \quad ; \quad \sigma_{xy} = 0$$

at its plane edge $y = 0$.

Find the complete stress field in the plate and hence estimate the depth y at which the amplitude of the variation in σ_{yy} has fallen to 10% of S.

Hint: You might find it easier initially to consider the case of the layer $0<y<h$, with $y=h$ traction-free, and then let $h \to \infty$.

In view of the boundary conditions, it is reasonable to seek a stress function of the form

$$\phi = f(y) \sin(\lambda x) \ . \tag{1}$$

The biharmonic equation then requires that the function f satisfy the ordinary differential equation

$$\frac{d^4 f}{dy^4} - 2\lambda^2 \frac{d^2 f}{dy^2} + \lambda^4 f = 0 \tag{2}$$

the solution of which is

$$f(y) = Aye^{\lambda y} + Bye^{-\lambda y} + Ce^{\lambda y} + De^{-\lambda y} \ , \tag{3}$$

where A, B, C, D are four arbitrary constants (see §5.3).

We anticipate that the stresses will tend to zero as $y \to \infty$ and hence that the constants A, C must be zero. This can be verified by imposing traction-free conditions on a boundary at $y = h$ and then letting $h \to \infty$. We shall give this proof at the end of the present solution.

Using just the constants B, D in (1) and hence

$$\phi = (By + D)e^{-\lambda y} \sin(\lambda x) \ , \tag{4}$$

we obtain the stresses

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = (B\lambda^2 y - 2B\lambda + D\lambda^2)e^{-\lambda y} \sin(\lambda x) \ ; \tag{5}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\lambda(B - B\lambda y - D\lambda)e^{-\lambda y} \cos(\lambda x) \ ; \tag{6}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = -\lambda^2 (By + D)e^{-\lambda y} \sin(\lambda x) \ . \tag{7}$$

To satisfy the boundary conditions $\sigma_{yy}(x, 0) = S \sin(\lambda x)$, $\sigma_{xy}(x, 0) = 0$, we require

$$-\lambda^2 D = S \ ; \tag{8}$$

$$B - \lambda D = 0 \ ; \tag{9}$$

— i.e.

$$B = -\frac{S}{\lambda} \quad ; \quad D = -\frac{S}{\lambda^2} \ . \tag{10}$$

The resulting expressions for the stresses are then

$$\sigma_{xx} = S(1-\lambda y)e^{-\lambda y}\sin(\lambda x) \; ; \tag{11}$$

$$\sigma_{xy} = -S\lambda y e^{-\lambda y}\cos(\lambda x) \; ; \tag{12}$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = S(1+\lambda y)e^{-\lambda y}\sin(\lambda x) \; . \tag{13}$$

Notice incidentally that $\sigma_{xx}(x,0) = \sigma_{yy}(x,0)$ as proved in Problem 4.5.

To find the depth h at which σ_{yy} has decayed to 10% of its surface value we set

$$(1+\lambda y)e^{-\lambda y} = 0.1 \tag{14}$$

and solve for λy. This is conveniently done iteratively by writing equation (14) in the form

$$by = \log(10(1+\lambda y)) \; . \tag{15}$$

Starting with an initial guess $\lambda y = 0$ and iterating on equation (15), we obtain

λy	$\log(10(1+\lambda y))$
0	2.3
2.3	3.5
3.5	3.8
3.8	3.872
3.872	3.886
3.886	3.8899

Thus, the required depth is $\lambda y = 3.89$ which is a bit larger than a half wave of the sinusoid (a complete wave on the surface takes a distance $\lambda x = 2\pi$.

Solution for a finite layer thickness

We now consider the problem in which a layer of finite thickness h is subjected to sinusoidal loading on $y = 0$, the other edge $y = h$ being traction free.

Taking the full stress function of equations (1,3) above, we have

$$\phi = (Aye^{\lambda y} + Bye^{-\lambda y} + Ce^{\lambda y} + De^{-\lambda y})\sin(\lambda x) \tag{16}$$

and the corresponding stresses are

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = [(B\lambda^2 y - 2B\lambda + D\lambda^2)e^{-\lambda y} + (A\lambda^2 y + 2A\lambda + C\lambda^2)e^{\lambda y}]\sin(\lambda x) \; ; \tag{17}$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\lambda[(B - B\lambda y - D\lambda)e^{-\lambda y} + (A + A\lambda y + C\lambda)e^{\lambda y}]\cos(\lambda x) \; ; \tag{18}$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = -\lambda^2[(By + D)e^{-\lambda y} + (Ay + C)e^{\lambda y}]\sin(\lambda x) \; . \tag{19}$$

The boundary conditions $\sigma_{xy}(x,0) = 0$; $\sigma_{xy}(x,h) = 0$; $\sigma_{yy}(x,0) = S$; $\sigma_{yy}(x,h) = 0$ then yield the equations

$$\begin{aligned} B - D\lambda + A + C\lambda &= 0\,; && (20)\\ (B - B\lambda h - D\lambda)e^{-\lambda h} + (A + A\lambda h + C\lambda)e^{\lambda h} &= 0\,; && (21)\\ -\lambda^2(D + C) &= S\,; && (22)\\ (Bh + D)e^{-\lambda h} + (Ah + C)e^{\lambda h} &= 0\,. && (23) \end{aligned}$$

The simultaneous solution of these equations will give the stress field for a layer of arbitrary thickness. However, if $\lambda h \gg 1$, we can simplify the solution by dividing equations (21,23) by $\exp(\lambda h)$ leaving

$$\begin{aligned} (B - B\lambda h - D\lambda)e^{-2\lambda h} + (A + A\lambda h + C\lambda) &= 0\,; && (24)\\ (Bh + D)e^{-2\lambda h} + (Ah + C) &= 0\,. && (25) \end{aligned}$$

If $\lambda h \to \infty$, the exponential terms in these equations will tend to zero — in fact they will be extremely small even for $\lambda h > 10$ (say). Thus, the equations reduce to

$$\begin{aligned} (1 + \lambda h)A + C\lambda &= 0\,; && (26)\\ \lambda h A + C\lambda &= 0\,, && (27) \end{aligned}$$

whose solution is $A = C = 0$, justifying our restriction of the stress function in the original solution to the terms involving B, D only.

5.7. The beam $-a < x < a, -b < y < b$ is loaded by a uniform compressive traction p in the central region $-a/2 < x < a/2$ of both of the edges $y = \pm b$, as shown in Figure 5.5. The remaining edges are traction-free. Use a Fourier series with the appropriate symmetries to obtain a solution for the stress field, using the weak condition on σ_{xy} on the edges $x = \pm a$ and the strong form of all the remaining boundary conditions.

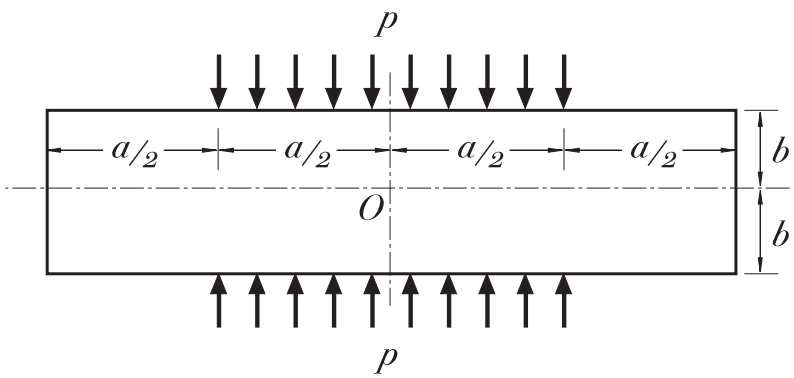


Figure 5.5

The problem is clearly even in both x and y. It follows that the weak condition on σ_{xy} on the ends must be satisfied identically, since σ_{xy} will be an odd function with average value zero at all cross sections.

A Fourier series solution with the appropriate symmetries is

$$\phi = \sum_{n=1}^{\infty} \{A_n \cosh(\lambda_n y) + B_n y \sinh(\lambda_n y)\} \cos(\lambda_n x) \ , \tag{1}$$

for which the stress components are

$$\sigma_{xx} = \sum_{n=1}^{\infty} \{A_n \lambda_n^2 \cosh(\lambda_n y) + B_n \lambda_n^2 y \sinh(\lambda_n y) + 2B_n \lambda_n \cosh(\lambda_n y)\} \cos(\lambda_n x) \tag{2}$$

$$\sigma_{xy} = \sum_{n=1}^{\infty} \{A_n \lambda_n^2 \sinh(\lambda_n y) + B_n \lambda_n^2 y \cosh(\lambda_n y) + B_n \lambda_n \sinh(\lambda_n y)\} \sin(\lambda_n x) \tag{3}$$

$$\sigma_{yy} = \sum_{n=1}^{\infty} \{A_n \lambda_n^2 \cosh(\lambda_n y) + B_n \lambda_n^2 y \sinh(\lambda_n y)\} \cos(\lambda_n x) \ . \tag{4}$$

The strong condition

$$\sigma_{xx} = 0 \ ; \quad x = \pm a \tag{5}$$

will be satisfied if we choose

$$\cos(\lambda_n a) = 0 \tag{6}$$

and hence

$$\lambda_n = \frac{(2n-1)\pi}{2a} \ . \tag{7}$$

The tractions on $y = b$ are

$$\sigma_{xy}(x,b) = \sum_{n=1}^{\infty}\{A_n\lambda_n^2\sinh(\lambda_n b)+B_n\lambda_n^2 b\cosh(\lambda_n b)+B_n\lambda_n\sinh(\lambda_n b)\}\sin(\lambda_n x) \tag{8}$$

$$\sigma_{yy}(x,b) = \sum_{n=1}^{\infty}\{A_n\lambda_n^2\cosh(\lambda_n b) + B_n\lambda_n^2 y\sinh(\lambda_n b)\}\cos(\lambda_n x) \ . \tag{9}$$

and hence the strong boundary conditions on these surfaces will be satisfied if

$$\begin{aligned}
\sum_{n=1}^{\infty}\{A_n\lambda_n^2\sinh(\lambda_n b)+B_n\lambda_n^2 b\cosh(\lambda_n b)+B_n\lambda_n\sinh(\lambda_n b)\}\sin(\lambda_n x) &= 0 \\
\sum_{n=1}^{\infty}\{A_n\lambda_n^2\cosh(\lambda_n b) + B_n\lambda_n^2 b\sinh(\lambda_n b)\}\cos(\lambda_n x) &= -p\ ;\ \ |x| < a/2 \\
&= 0\ ;\ \ |x| > a/2\ .
\end{aligned}$$

The first of these conditions will be satisfied for all x if

$$A_n\lambda_n^2\sinh(\lambda_n b)+B_n\lambda_n^2 b\cosh(\lambda_n b)+B_n\lambda_n\sinh(\lambda_n b) = 0 \ . \tag{10}$$

To invert the second condition, we multiply both sides of the equation by $\cos(\lambda_m x)$ and integrate in $-a < x < a$, obtaining

$$A_m a\lambda_m^2\cosh(\lambda_m b) + B_m ab\lambda_m^2\sinh(\lambda_m b) = -p\int_{-a/2}^{a/2}\cos(\lambda_m x)dx \tag{11}$$

$$= -\frac{2p\sin(\lambda_m a/2)}{\lambda_m} \tag{12}$$

$$= -\frac{2p\sin((2m-1)\pi/4)}{\lambda_m}\ , \tag{13}$$

using (7).

Solving these equations for the constants A, B, we obtain

$$A = -\frac{4p\sin((2m-1)\pi/4)((\lambda_m b)\cosh(\lambda_m b) + \sinh(\lambda_n b))}{\lambda_m^2 a\{(2\lambda_m b)\cosh(2\lambda_m b) + \sinh(2\lambda_m b)\}}$$

$$B = \frac{4p\sin((2m-1)\pi/4)\sinh(\lambda_m b)}{\lambda_m^2 a\{(2\lambda_m b)\cosh(2\lambda_m b) + \sinh(2\lambda_m b)\}}\ .$$

The final solution is then obtained by substituting for A, B into (1–4), using (7) for λ_m.

5.8. Use a Fourier series to solve the problem of Figure 5.4(a) in §5.2.3. Choose the terms in the series so as to satisfy the condition $\sigma_{xx}(\pm a, y) = 0$ in the strong sense.

If you are solving this problem in Maple or Mathematica, compare the solution with that of §5.2.3 by making a contour plot of the difference between the truncated Fourier series stress function and the polynomial stress function

$$\phi = \frac{p}{40b^3}(5x^2y^3 - y^5 - 15b^2x^2y - 5a^2y^3 + 2b^2y^3) \ .$$

Examine the effect of taking different numbers of terms in the series.

The problem is even in x and odd in y. A Fourier series solution with the appropriate symmetries is

$$\phi = \sum_{n=1}^{\infty} \{A_n y \cosh(\lambda_n y) + B_n \sinh(\lambda_n y)\} \cos(\lambda_n x) \ , \tag{1}$$

where A_n, B_n are arbitrary constants. The strong condition $\sigma_{xx} = 0$ on $x = \pm a$ can then be satisfied in every term by choosing

$$\lambda_n = \frac{(2n-1)\pi}{2a} \ . \tag{2}$$

The corresponding stresses are

$$\sigma_{xx} = \sum_{n=1}^{\infty} \{2A_n\lambda_n \sinh(\lambda_n y) + A_n\lambda_n^2 y \cosh(\lambda_n y) + B_n\lambda_n^2 \sinh(\lambda_n y)\} \cos(\lambda_n x) \tag{3}$$

$$\sigma_{xy} = \sum_{n=1}^{\infty} \{A_n\lambda_n \cosh(\lambda_n y) + A_n\lambda_n^2 y \sinh(\lambda_n y) + B_n\lambda_n^2 \cosh(\lambda_n y)\} \sin(\lambda_n x) \tag{4}$$

$$\sigma_{yy} = \sum_{n=1}^{\infty} \{A_n\lambda_n^2 y \cosh(\lambda_n y) + B_n\lambda_n^2 \sinh(\lambda_n y)\} \cos(\lambda_n x) \tag{5}$$

and hence the boundary conditions (5.82, 5.83) on $y = \pm b$ require that

$$\sum_{n=1}^{\infty} \{A_n\lambda_n \cosh(\lambda_n b) + A_n\lambda_n^2 b \sinh(\lambda_n b) + B_n\lambda_n^2 \cosh(\lambda_n b)\} \sin(\lambda_n x) = 0 \tag{6}$$

$$\sum_{n=1}^{\infty} \{A_n\lambda_n^2 b \cosh(\lambda_n b) + B_n\lambda_n^2 \sinh(\lambda_n b)\} \cos(\lambda_n x) = -\frac{p}{2} \ . \tag{7}$$

To invert the series, we multiply (7) by $\cos(\lambda_m x)$ and integrate from $-a$ to a, obtaining

$$\begin{aligned} a\{A_n\lambda_n^2 b \cosh(\lambda_n b) + B_n\lambda_n^2 \sinh(\lambda_n b)\} &= -\frac{p}{2}\int_{-a}^{a} \cos(\lambda_m x)dx \\ &= -\frac{p \sin(\lambda_m a)}{\lambda_m} \\ &= \frac{(-1)^m p}{\lambda_m} \ , \end{aligned} \tag{8}$$

using (2). The homogeneous equation (6) is satisfied for all x if

$$A_m \lambda_m \cosh(\lambda_m b) + A_m \lambda_m^2 b \sinh(\lambda_m b) + B_m \lambda_m^2 \cosh(\lambda_m b) = 0 \ . \tag{9}$$

and hence, solving (8,9) for A_m, B_m we have

$$\begin{aligned} A_m &= \frac{(-1)^m p \cosh(\lambda_m b)}{\lambda_m^2 a \{\lambda_m b - \sinh(\lambda_m b)\cosh(\lambda_m b)\}} \\ B_m &= -\frac{(-1)^m p (\cosh(\lambda_m b) + \lambda_m b \sinh(\lambda_m b))}{\lambda_m^3 a \{\lambda_m b - \sinh(\lambda_m b)\cosh(\lambda_m b)\}} \ , \end{aligned} \tag{10}$$

where λ_m is given by (2).

5.9. The large plate $y>0$ is loaded at its remote boundaries so as to produce a state of uniform tensile stress

$$\sigma_{xx} = S \ ; \ \sigma_{xy} = \sigma_{yy} = 0 \ ,$$

the boundary $y=0$ being traction-free. We now wish to determine the perturbation in this simple state of stress that will be produced if the traction-free boundary had a slight waviness, defined by the line

$$y = \epsilon \cos(\lambda x) \ ,$$

where $\lambda\epsilon \ll 1$. To solve this problem

(i) Start with the stress function

$$\phi = \frac{Sy^2}{2} + f(y)\cos(\lambda x)$$

and determine $f(y)$ if the function is to be biharmonic.

(ii) The perturbation will be localized near $y=0$, so select only those terms in $f(y)$ that decay as $y\to\infty$.

(iii) Find the stress components and use the stress transformation equations to determine the tractions on the wavy boundary. Notice that the inclination of the wavy surface to the plane $y=0$ will be everywhere small if $\lambda\epsilon\ll 1$ and hence the trigonometric functions involving this angle can be approximated using $\sin(x)\approx x,\ \cos(x)\approx 1, x\ll 1$.

(iv) Choose the free constants in $f(y)$ to satisfy the traction-free boundary condition on the wavy surface.

(v) Determine the maximum tensile stress and hence the stress concentration factor as a function of $\lambda\epsilon$.

The given stress function will be harmonic if

$$f(y) = Ae^{\lambda y} + Bye^{\lambda y} + Ce^{-\lambda y} + Dye^{-\lambda y} \ , \tag{1}$$

from equation (5.98). However, since the plate is large, we expect the disturbance to be localized near the wavy edge and hence we can restrict attention to the decaying terms, giving

$$\phi = \frac{Sy^2}{2} + (Ce^{-\lambda y} + Dye^{-\lambda y})\cos(\lambda x) \ . \tag{2}$$

The corresponding stress components are

$$\sigma_{xx} = S + \lambda(C\lambda e^{-\lambda y} - 2De^{-\lambda y} + D\lambda y e^{-\lambda y})\cos(\lambda x) \tag{3}$$
$$\sigma_{xy} = \lambda(C\lambda e^{-\lambda y} - De^{-\lambda y} + D\lambda y e^{-\lambda y})\sin(\lambda x)\ ; \tag{4}$$
$$\sigma_{yy} = -\lambda^2(Ce^{-\lambda y} + Dye^{-\lambda y})\cos(\lambda x) \tag{5}$$

Suppose we define a coordinate system x', y' such that y' is locally normal to the wavy surface. Since $\lambda\epsilon \ll 1$, the angle θ between x, y and x', y' is small, so the stress transformation equations (1.15–1.17) simplify to

$$\sigma_{x'x'} = \sigma_{xx} + 2\theta\sigma_{xy} \tag{6}$$
$$\sigma_{x'y'} = \sigma_{xy} + \theta(\sigma_{yy} - \sigma_{xx}) \tag{7}$$
$$\sigma_{y'y'} = \sigma_{yy} - 2\theta\sigma_{xy}\ . \tag{8}$$

We also note that θ is the slope of the surface $y = \epsilon\cos(\lambda x)$ and hence

$$\theta = -\lambda\epsilon\sin(\lambda x)\ . \tag{9}$$

The tractions on the wavy surface can now be obtained by substituting (3–5) into (6–8), using (9) and replacing y by $\epsilon\cos(\lambda x)$. However, before doing this, it is worth noting that many of the resulting terms will be second order in the small quantity $\lambda\epsilon$ and hence can be neglected. We first note that as we might anticipate, the expressions (6–8) consist of the unperturbed stress components $\sigma_{xx}, \sigma_{xy}, \sigma_{yy}$ modified by small multipliers of other combinations of stress components. The perturbation in the uniform stress field defined by the term $f(y)\cos(\lambda x)$ in ϕ is introduced to cancel these small perturbations and hence will also be small compared with the unperturbed field $\sigma_{xx} = S$. Thus, the perturbation terms can be dropped in those terms in (6–8) that are multiplied by θ. In addition, the substitution of $y = \epsilon\cos(\lambda x)$ into the exponential terms in (3–5) will yield terms of order $\exp(\lambda\epsilon) \approx 1 + \lambda\epsilon$. Since these terms are again multiplied by small terms, only the leading (unit) term needs to be retained. The tractions on the wavy surface therefore simplify to

$$\sigma_{x'y'} = \lambda(C\lambda - D)\sin(\lambda x) + S\lambda\epsilon\sin(\lambda x) \tag{10}$$
$$\sigma_{y'y'} = -\lambda^2 C\cos(\lambda x)\ . \tag{11}$$

They will be zero for all x if

$$C = 0\ \ ;\ \ D = S\epsilon \tag{12}$$

defining the stress function

$$\phi = \frac{Sy^2}{2} + S\epsilon y e^{-\lambda y}\cos(\lambda x)\ . \tag{13}$$

The complete stress field is

$$\sigma_{xx} = S + S\lambda\epsilon(-2e^{-\lambda y} + \lambda y e^{-\lambda y})\cos(\lambda x) \tag{14}$$
$$\sigma_{xy} = S\lambda\epsilon(-e^{-\lambda y} + \lambda y e^{-\lambda y})\sin(\lambda x)\ ; \tag{15}$$
$$\sigma_{yy} = -S\lambda^2\epsilon y e^{-\lambda y}\cos(\lambda x)\ . \tag{16}$$

The maximum stress is σ_{xx} at the troughs of the waves ($\lambda x = \pi$), where

$$\sigma_{xx} = S(1 + 2\lambda\epsilon) \ . \tag{17}$$

Thus, the stress concentration factor is $(1 + 2\lambda\epsilon)$.

CHAPTER 6

6.1. Show that if $\zeta = x + \imath y$ and $\sin(\zeta) - \zeta = 0$, where x, y are real variables, then

$$f(x) \equiv \cos x\sqrt{x^2 - \sin^2 x} + \sin x \ln(\sin x) - \sin x \ln\left(x + \sqrt{x^2 - \sin^2 x}\right) = 0 \ .$$

Using Maple or Mathematica to plot the function $f(x)$, find the first six roots of this equation and hence determine the first six values of $\lambda_R b$ for the antisymmetric mode.

If $\zeta = x + \imath y$,

$$\sin(\zeta) = \sin(x + \imath y) = \sin x \cosh y + \imath \cos x \sinh y \tag{1}$$

and the condition $\sin(\zeta) - \zeta = 0$ takes the form

$$\sin x \cosh y + \imath \cos x \sinh y - x - \imath y = 0 \ . \tag{2}$$

Equating real and imaginary parts, we have

$$\sin x \cosh y - x = 0 \ \ ; \ \ \cos x \sinh y - y = 0 \ . \tag{3}$$

From the first of these equations,

$$y = \cosh^{-1}\left(\frac{x}{\sin x}\right) = \ln\left(\frac{x}{\sin x} + \sqrt{\frac{x^2}{\sin^2 x} - 1}\right) = \ln(x + \sqrt{x^2 - \sin^2 x}) - \ln(\sin x) \ . \tag{4}$$

Substituting this result into $(3)_2$ and noting that

$$\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{\frac{x^2}{\sin^2 x} - 1} \ , \tag{5}$$

we obtain

$$\cos x\sqrt{\frac{x^2}{\sin^2 x} - 1} - \ln(x + \sqrt{x^2 - \sin^2 x}) + \ln(\sin x) = 0 \ , \tag{6}$$

or multiplying through by $\sin x$,

$$\cos x\sqrt{x^2 - \sin^2 x} + \sin x \ln(\sin x) - \sin x \ln\left(x + \sqrt{x^2 - \sin^2 x}\right) = 0 \ . \tag{7}$$

The function $\ln(\sin x)$ is real only in the ranges $2n\pi < x < (2n+1)\pi$ where n is an integer, so $f(x)$ defines a real function only in these segments. The function is

plotted in Figure 1, which shows a zero at $x = 0$ and additional zeros in each of the following segments.

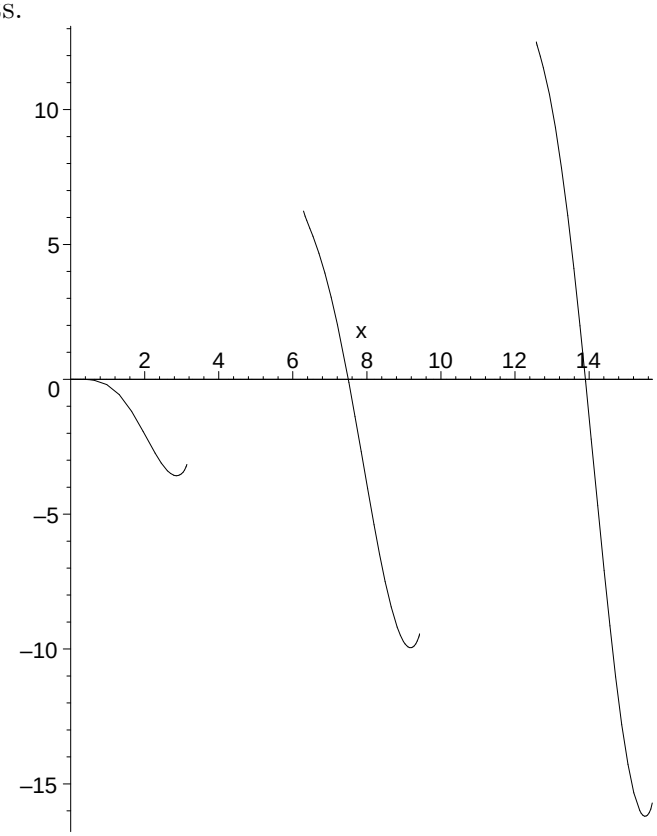


Figure 1

Use of Maple to solve for these roots gives the first 6 roots (for $2\lambda b$) as 7.498, 13.900, 20.239, 26.555, 32.860, 39.139...

6.2. Devise a method similar to that outlined in Problem 6.1 to determine the first six values of $\lambda_R b$ for the symmetric mode.

For the symmetric mode, the governing equation (6.21) is

$$\sin(2\lambda b) + 2\lambda b = 0 \ . \tag{1}$$

Writing $\zeta = 2\lambda b = x + \imath y$, we have

$$\sin(\zeta) = \sin(x + \imath y) = \sin x \cosh y + \imath \cos x \sinh y \tag{2}$$

and the condition $\sin(\zeta)+\zeta=0$ takes the form

$$\sin x \cosh y + \imath \cos x \sinh y + x + \imath y = 0 \tag{3}$$

Equating real and imaginary parts, we have

$$\sin x \cosh y + x = 0 \ \ ; \ \ \cos x \sinh y + y = 0 \ . \tag{4}$$

From the first of these equations,

$$y = \cosh^{-1}\left(-\frac{x}{\sin x}\right) \tag{5}$$

and we notice that this will have a real solution only if $\sin x < 0$. With this assumption, we have

$$y = \ln\left(-\frac{x}{\sin x} + \sqrt{\frac{x^2}{\sin^2 x} - 1}\right) = \ln(x + \sqrt{x^2 - \sin^2 x}) - \ln(-\sin x) \ . \tag{6}$$

Substituting this result into $(4)_2$ and noting that

$$\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{\frac{x^2}{\sin^2 x} - 1} \ , \tag{7}$$

we obtain

$$\cos x \sqrt{\frac{x^2}{\sin^2 x} - 1} + \ln(x + \sqrt{x^2 - \sin^2 x}) - \ln(-\sin x) = 0 \ , \tag{8}$$

or multiplying through by $-\sin x$,

$$\cos x \sqrt{x^2 - \sin^2 x} + \sin x \ln(-\sin x) - \sin x \ln\left(x + \sqrt{x^2 - \sin^2 x}\right) = 0 \ . \tag{9}$$

The function $\sin x$ is negative only in the ranges $(2n-1)\pi < x < 2n\pi$ where n is an integer and a plot of $f(x)$ will define a real function only in these segments. A Maple plot of $f(x)$ shows one zero in each segment. Use of Maple to solve for these roots gives the first 6 roots (for $2\lambda b$) as 4.212, 10.713, 17.073, 23.398, 29.708, 36.010...

6.3. A displacement function representation for plane strain problems can be developed[1] in terms of two harmonic functions ϕ, ω in the form

$$2\mu u_x = \frac{\partial \phi}{\partial x} + y\frac{\partial \omega}{\partial x} \;\; ; \;\; 2\mu u_y = \frac{\partial \phi}{\partial y} + y\frac{\partial \omega}{\partial y} - (3-4\nu)\omega$$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial x^2} + y\frac{\partial^2 \omega}{\partial x^2} - 2\nu\frac{\partial \omega}{\partial y} \;\; ; \;\; \sigma_{xy} = \frac{\partial^2 \phi}{\partial x \partial y} + y\frac{\partial^2 \omega}{\partial x \partial y} - (1-2\nu)\frac{\partial \omega}{\partial x}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial y^2} + y\frac{\partial^2 \omega}{\partial y^2} - 2(1-\nu)\frac{\partial \omega}{\partial y} \; .$$

Use this representation to formulate the eigenvalue problem of the long strip $x > 0, -b < y < b$ whose edges $y = \pm b$ are both bonded to a rigid body. Find the eigenvalue equation for symmetric and antisymmetric modes and comment on the expected decay rates for loading of the strip on the end $x = 0$.

We require solutions for the stresses and hence the stress functions that decay with $\exp(-\lambda x)$, so suitable harmonic displacement functions are

$$\phi = Ae^{-\lambda x}\cos(\lambda y) + Be^{-\lambda x}\sin(\lambda y) \;\; ; \;\; \omega = Ce^{-\lambda x}\cos(\lambda y) + De^{-\lambda x}\sin(\lambda y) \; . \tag{1}$$

The displacements are then obtained as

$$2\mu u_x = e^{-\lambda x}\{-A\lambda\cos(\lambda y) - B\lambda\sin(\lambda y) - C\lambda y\cos(\lambda y) - D\lambda y\sin(\lambda y)\} \tag{2}$$

$$\begin{aligned} 2\mu u_y = e^{-\lambda x}\{&-A\lambda\sin(\lambda y) + B\lambda\cos(\lambda y) - C\lambda y\sin(\lambda y) + D\lambda y\cos(\lambda y) \\ &-(3-4\nu)C\cos(\lambda y) - (3-4\nu)D\sin(\lambda y)\} \end{aligned} \tag{3}$$

Equating these to zero on the boundaries $y = \pm b$, we obtain

$$-A\lambda\cos(\lambda b) - B\lambda\sin(\lambda b) - C\lambda b\cos(\lambda b) - D\lambda b\sin(\lambda b) = 0 \tag{4}$$

$$-A\lambda\cos(\lambda b) + B\lambda\sin(\lambda b) + C\lambda b\cos(\lambda b) - D\lambda b\sin(\lambda b) = 0 \tag{5}$$

$$\begin{aligned} &-A\lambda\sin(\lambda b) + B\lambda\cos(\lambda b) - C\lambda b\sin(\lambda b) + D\lambda b\cos(\lambda b) \\ &\qquad -(3-4\nu)C\cos(\lambda b) - (3-4\nu)D\sin(\lambda b) = 0 \end{aligned} \tag{6}$$

$$\begin{aligned} &A\lambda\sin(\lambda b) + B\lambda\cos(\lambda b) - C\lambda b\sin(\lambda b) - D\lambda b\cos(\lambda b) \\ &\qquad -(3-4\nu)C\cos(\lambda b) + (3-4\nu)D\sin(\lambda b) = 0 \end{aligned} \tag{7}$$

and partitioning the matrix by separating symmetric and antisymmetric terms, we obtain

$$A\lambda\cos(\lambda b) + D\lambda b\sin(\lambda b) = 0 \tag{8}$$

$$-A\lambda\sin(\lambda b) + D\lambda b\cos(\lambda b) - (3-4\nu)D\sin(\lambda b) = 0 \tag{9}$$

$$B\lambda\sin(\lambda b) + C\lambda b\cos(\lambda b) = 0 \tag{10}$$

$$B\lambda\cos(\lambda b) - C\lambda b\sin(\lambda b) - (3-4\nu)C\cos(\lambda b) = 0 \tag{11}$$

[1] This solution is a two-dimensional version of a three-dimensional solution developed in Chapter 21 and tabulated in Table 21.1 as solutions A and B.

The symmetric terms (those involving A, D) have a non-trivial solution if

$$\lambda \cos(\lambda b)\{\lambda b \cos(\lambda b) - (3 - 4\nu)\sin(\lambda b)\} = -\lambda^2 b \sin^2(\lambda b) \tag{12}$$

and hence

$$(3 - 4\nu)\sin(2\lambda b) - 2\lambda b = 0 \ . \tag{13}$$

The antisymmetric terms have a non-trivial solution if

$$\lambda \sin(\lambda b)\{-\lambda b \sin(\lambda b) - (3 - 4\nu)\cos(\lambda b)\} = \lambda^2 b \cos^2(\lambda b) \tag{14}$$

and hence

$$(3 - 4\nu)\sin(2\lambda b) + 2\lambda b = 0 \ . \tag{15}$$

These equations reduce to those of the traction-free problem of §6.2.2 if $\nu = 0.5$ — i.e. if the material is incompressible. However, the symmetry of the eigenfunctions is reversed –i.e. the lowest eigenvalue for the bonded problem occurs for the antisymmetric mode. An investigation similar to that used in Problems 6.1, 6.2 shows that reducing Poisson's ratio causes a modest reduction in the decay rate. For example, at $\nu = 0$, the first eigenvalue for $2\lambda_R b$ is 3.844, compared with 4.212 for $\nu = 0.5$.

6.4. Use the displacement function representation of Problem 6.3 to formulate the eigenvalue problem for the long strip $x > 0, -b < y < b$ whose edges $y = \pm b$ are in frictionless contact with a rigid body (so that the normal displacement is zero, but the frictional (tangential) traction is zero). Find the eigenvalue equation for symmetric and antisymmetric modes and comment on the expected decay rates for loading of the strip on the end $x = 0$.

We require solutions for the stresses and hence the stress functions that decay with $\exp(-\lambda x)$, so suitable harmonic displacement functions are

$$\phi = Ae^{-\lambda x}\cos(\lambda y) + Be^{-\lambda x}\sin(\lambda y) \;\; ; \;\; \omega = Ce^{-\lambda x}\cos(\lambda y) + De^{-\lambda x}\sin(\lambda y) \; . \tag{1}$$

The displacement u_y and the shear stress σ_{xy} are then obtained as

$$\begin{aligned} 2\mu u_y &= e^{-\lambda x}\{-A\lambda\sin(\lambda y) + B\lambda\cos(\lambda y) - C\lambda y\sin(\lambda y) + D\lambda y\cos(\lambda y) \\ &\qquad -(3-4\nu)C\cos(\lambda y) - (3-4\nu)D\sin(\lambda y)\} \qquad (2) \\ \sigma_{xy} &= e^{-\lambda x}\{A\lambda^2\sin(\lambda y) - B\lambda^2\cos(\lambda y) + C\lambda^2 y\sin(\lambda y) + (1-2\nu)C\lambda\cos(\lambda y) \\ &\qquad -D\lambda^2 y\cos(\lambda y) + (1-2\nu)D\lambda\sin(\lambda y)\} \qquad (3) \end{aligned}$$

The frictionless contact boundary conditions require that $u_y = 0, \sigma_{xy} = 0$ on $y = \pm b$ and hence

$$\begin{aligned} -A\lambda\sin(\lambda b) + B\lambda\cos(\lambda b) - C\lambda b\sin(\lambda b) + D\lambda b\cos(\lambda b) & \\ -(3-4\nu)C\cos(\lambda b) - (3-4\nu)D\sin(\lambda b) &= 0 \qquad (4) \\ A\lambda\sin(\lambda b) + B\lambda\cos(\lambda b) - C\lambda b\sin(\lambda b) - D\lambda b\cos(\lambda b) & \\ -(3-4\nu)C\cos(\lambda b) + (3-4\nu)D\sin(\lambda b) &= 0 \qquad (5) \\ A\lambda^2\sin(\lambda b) - B\lambda^2\cos(\lambda b) + C\lambda^2 b\sin(\lambda b) & \\ +(1-2\nu)C\lambda\cos(\lambda b) - D\lambda^2 b\cos(\lambda b) + (1-2\nu)D\lambda\sin(\lambda b) &= 0 \qquad (6) \\ -A\lambda^2\sin(\lambda b) - B\lambda^2\cos(\lambda b) + C\lambda^2 b\sin(\lambda b) & \\ +(1-2\nu)C\lambda\cos(\lambda b) + D\lambda^2 b\cos(\lambda b) - (1-2\nu)D\lambda\sin(\lambda b) &= 0 \qquad (7) \end{aligned}$$

Partitioning the matrix by separating symmetric and antisymmetric terms, we obtain

$$\begin{aligned} -A\lambda\sin(\lambda b) + D\lambda b\cos(\lambda b) - (3-4\nu)D\sin(\lambda b) &= 0 \qquad (8) \\ A\lambda^2\sin(\lambda b) - D\lambda^2 b\cos(\lambda b) + (1-2\nu)D\lambda\sin(\lambda b) &= 0 \qquad (9) \\ B\lambda\cos(\lambda b) - C\lambda b\sin(\lambda b) - (3-4\nu)C\cos(\lambda b) &= 0 \qquad (10) \\ -B\lambda^2\cos(\lambda b) + C\lambda^2 b\sin(\lambda b) + (1-2\nu)C\lambda\cos(\lambda b) &= 0 \qquad (11) \end{aligned}$$

The symmetric terms (those involving A, D) have a non-trivial solution if

$$-\lambda\sin(\lambda b)\{-\lambda^2 b\cos(\lambda b) + (1-2\nu)\lambda\sin(\lambda b)\} = \{\lambda b\cos(\lambda b) - (3-4\nu)\sin(\lambda b)\}\lambda^2\sin(\lambda b) \tag{12}$$

from which

$$2(1-\nu)\sin^2(\lambda b) = 0\ . \tag{13}$$

The eigenvalues of this equation are $\lambda b = n\pi$ where n is an integer and the first non-zero eigenvalue is at $\lambda b = \pi$.

The antisymmetric terms have a non-trivial solution if

$$\lambda\cos(\lambda b)\{\lambda^2 b\sin(\lambda b)+(1-2\nu)\lambda\cos(\lambda b)\} = -\lambda^2\cos(\lambda b)\{-\lambda b\sin(\lambda b)-(3-4\nu)\cos(\lambda b)\} \tag{14}$$

and hence

$$2(1-\nu)\cos^2(\lambda b) = 0\ . \tag{15}$$

The eigenvalues of this equation are $\lambda b = (2n-1)\pi/2$ where n is an integer and the first non-zero eigenvalue is at $\lambda b = \pi/2$.

6.5. Use the displacement function representation of Problem 6.3 to formulate the eigenvalue problem for the long strip $x>0, -b<y<b$ which is bonded to a rigid surface at $y=-b$, the other long edge $y=b$ being traction-free. Notice that this problem is not symmetrical, so the problem will not partition into symmetric and antisymmetric modes.

We require solutions for the stresses and hence the stress functions that decay with $\exp(-\lambda x)$, so suitable harmonic displacement functions are

$$\phi = Ae^{-\lambda x}\cos(\lambda y) + Be^{-\lambda x}\sin(\lambda y) \;\; ; \;\; \omega = Ce^{-\lambda x}\cos(\lambda y) + De^{-\lambda x}\sin(\lambda y) \; . \tag{1}$$

The displacement u_x, u_y and the stress components σ_{xy}, σ_{yy} are then obtained as

$$\begin{aligned}
2\mu u_x &= e^{-\lambda x}\{-A\lambda\cos(\lambda y) - B\lambda\sin(\lambda y) - C\lambda y\cos(\lambda y) - D\lambda y\sin(\lambda y)\} && (2)\\
2\mu u_y &= e^{-\lambda x}\{-A\lambda\sin(\lambda y) + B\lambda\cos(\lambda y) - C\lambda y\sin(\lambda y) + D\lambda y\cos(\lambda y) \\
&\qquad -(3-4\nu)C\cos(\lambda y) - (3-4\nu)D\sin(\lambda y)\} && (3)\\
\sigma_{xy} &= e^{-\lambda x}\{A\lambda^2\sin(\lambda y) - B\lambda^2\cos(\lambda y) + C\lambda^2 y\sin(\lambda y) + (1-2\nu)C\lambda\cos(\lambda y) \\
&\qquad -D\lambda^2 y\cos(\lambda y) + (1-2\nu)D\lambda\sin(\lambda y)\} && (4)\\
\sigma_{yy} &= e^{-\lambda x}\{-A\lambda^2\cos(\lambda y) - B\lambda^2\sin(\lambda y) - C\lambda^2 y\cos(\lambda y) + 2(1-\nu)C\lambda\sin(\lambda y) \\
&\qquad -D\lambda^2 y\sin(\lambda y) - 2(1-\nu)D\lambda\cos(\lambda y)\} \; . && (5)
\end{aligned}$$

The boundary conditions

$$u_x = u_y = 0 \;\; ; \;\; y = -b \tag{6}$$

$$\sigma_{yx} = \sigma_{yy} = 0 \;\; ; \;\; y = b \tag{7}$$

then yield the equations

$$\begin{aligned}
-A\cos(\lambda b) + B\sin(\lambda b) + Cb\cos(\lambda b) - Db\sin(\lambda b) &= 0 && (8)\\
A\lambda\sin(\lambda b) + B\lambda\cos(\lambda b) - C\lambda b\sin(\lambda b) - (3-4\nu)C\cos(\lambda b) & \\
-D\lambda b\cos(\lambda b) + (3-4\nu)D\sin(\lambda b) &= 0 && (9)\\
A\lambda\sin(\lambda y) - B\lambda\cos(\lambda b) + C\lambda b\sin(\lambda b) + (1-2\nu)C\cos(\lambda b) & \\
-D\lambda b\cos(\lambda b) + (1-2\nu)D\sin(\lambda b) &= 0 && (10)\\
-A\lambda\cos(\lambda b) - B\lambda\sin(\lambda b) - C\lambda b\cos(\lambda b) + 2(1-\nu)C\sin(\lambda b) & \\
-D\lambda b\sin(\lambda b) - 2(1-\nu)D\cos(\lambda b) &= 0 \; . && (11)
\end{aligned}$$

Evaluating the determinant of the coefficient matrix, we find that non-trivial solutions occur when λb satisfies the equation

$$(3-4\nu)\sin^2(2\lambda b) + (2\lambda b)^2 - 4(1-\nu)^2 = 0 \; . \tag{12}$$

This equation has one real root in the range $0 < 2\lambda b < \pi/2$ in addition to an infinite series of complex roots. The real root varies from $\lambda b = 0.37$ for $\nu = 0.5$ to $\lambda b = 0.595$ for $\nu = 0$, so the decay of end loading in this system is quite slow.

6.6. Use Mathieu's method to approximate the stresses in the square $-a<x<a,\ -a<y<a$ subjected to the tractions

$$\sigma_{xx}(a,y) = \sigma_{xx}(-a,y) = \frac{Sy^2}{a^2}\ ,$$

all the remaining tractions being zero. Start by using (6.35, 6.36) to define a polynomial first approximation to the solution. Then use the series (6.31) to define a *corrective solution* **— i.e. the stress function which when added to ϕ_0 defines the complete solution. The constants X_m, Y_n in the corrective solution will then decay with increasing m, n and a good approximation can be found by truncating the series at $m=n=2$.**

The boundary conditions are

$$\sigma_{xx}(a,y) = \sigma_{xx}(-a,y) = \frac{Sy^2}{a^2}\ ; \quad \sigma_{xy}(a,y) = -\sigma_{xy}(-a,y) = 0$$

$$\sigma_{yx}(x,a) = -\sigma_{yx}(x,-a) = 0\ ; \quad \sigma_{yy}(x,a) = \sigma_{yy}(x,-a) = 0\ ,$$

which in the terminology of equations (6.29, 6.30) corresponds to

$$f(y) = \frac{Sy^2}{a^2}\ ; \quad g(y) = h(x) = \ell(x) = 0\ .$$

Substituting into (6.36), we have

$$\int_{-a}^{a} f(y)\left(y^2 - \frac{a^2}{3}\right)dy = \frac{S}{a^2}\int_{-a}^{a}\left(y^4 - \frac{a^2y^2}{3}\right)dy = \frac{2S}{a^2}\left(\frac{a^5}{5} - \frac{a^5}{9}\right) = \frac{8Sa^3}{45}\ .$$

Hence

$$G = \frac{45}{4(a^4+a^4)}\frac{8Sa^2}{45} = \frac{S}{a^2}$$

and

$$\phi_0 = \frac{S}{24a^2}\left[(y^2-a^2)^2 - (x^2-a^2)^2\right]\ .$$

The stresses corresponding to this stress function are

$$\sigma_{xx} = \frac{S(3y^2-a^2)}{6a^2}\ ; \quad \sigma_{yy} = -\frac{S(3x^2-a^2)}{6a^2}\ ; \quad \sigma_{xy} = 0\ .$$

and on the boundaries,

$$\sigma_{xx}(a,y) = \sigma_{xx}(-a,y) = \frac{S(3y^2-a^2)}{6a^2}\ ; \quad \sigma_{xy}(a,y) = -\sigma_{xy}(-a,y) = 0$$

$$\sigma_{yx}(x,a) = -\sigma_{yx}(x,-a) = 0\ ; \quad \sigma_{yy}(x,a) = \sigma_{yy}(x,-a) = -\frac{S(3x^2-a^2)}{6a^2}\ ,$$

It follows that the corrective solution must satisfy the boundary conditions

$$\sigma_{xx}(a,y) = \sigma_{xx}(-a,y) = \frac{Sy^2}{a^2} - \frac{S(3y^2-a^2)}{6a^2} \; ; \quad \sigma_{xy}(a,y) = -\sigma_{xy}(-a,y) = 0$$

$$\sigma_{yx}(x,a) = -\sigma_{yx}(x,-a) = 0 \; ; \quad \sigma_{yy}(x,a) = \sigma_{yy}(x,-a) = \frac{S(3x^2-a^2)}{6a^2} \; ,$$

or

$$f(y) = \frac{Sy^2}{a^2} - \frac{S(3y^2-a^2)}{6a^2} = \frac{S(3y^2+a^2)}{6a^2} \; ; \quad g(y) = h(x) = 0 \; ; \quad \ell(x) = \frac{S(3x^2-a^2)}{6a^2} \; .$$

Substituting into the equations

$$f_n = \int_{-a}^{a} f(y)\cos(\beta_n y)dy \; ; \quad \ell_m = \int_{-a}^{a} \ell(x)\cos(\alpha_m x)dx$$

and evaluating the integrals, we have

$$f_0 = \frac{2Sa}{3} \; ; \quad f_1 = -\frac{2Sa}{\pi^2} \; ; \quad f_2 = \frac{Sa}{2\pi^2}$$

$$\ell_0 = 0 \; ; \quad \ell_1 = -\frac{2Sa}{\pi^2} \; ; \quad \ell_2 = \frac{Sa}{2\pi^2}$$

and hence

$$A_0 = \frac{f_0}{4a} = \frac{S}{6} \; ; \quad B_0 = \frac{\ell_0}{4a} = 0 \; .$$

Also, the expressions for H_m, K_n reduce to

$$\begin{aligned} H_m &= \frac{(-1)^{m+1}\ell_m}{a} \\ K_n &= \frac{(-1)^n f_n}{a} \; , \end{aligned}$$

giving

$$H_1 = -\frac{2S}{\pi^2} \; ; \quad H_2 = -\frac{S}{2\pi^2} \; ; \quad K_1 = \frac{2S}{\pi^2} \; ; \quad K_2 = \frac{S}{2\pi^2} \; .$$

From here on, the calculations were performed in Maple. We substitute these values of H_m, K_n into the first two of each of the equations (6.34) and solve for X_1, X_2, Y_1, Y_2. The original constants A_1, A_2, B_1, B_2 are then recovered and substituted into the stress function (6.33), along with A_0, B_0 and the results are added to the term ϕ_0 above.

To check the accuracy of the solution, we compare the tractions on the surface $x = a$ from the approximate solution with the required tractions in Figure 1(a). In this figure, we have normalized the expressions by setting $S = 1, a = 1$. The two curves are actually indistinguishable, so we plot the difference between them in Figure 1(b), which shows that the maximum error is of the order of $4 \times 10^{-3} S$ — in other words, better than 0.5%.

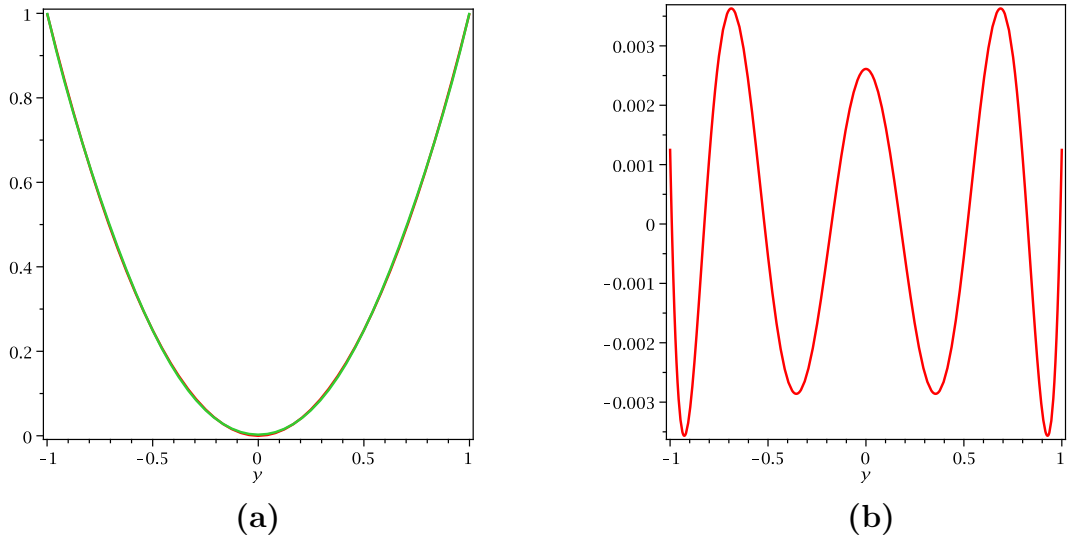


Figure 1: Tractions on $x = a$

Figure 2 shows the tractions, normalized by S, on the nominally traction-free surface $y = a$. The error here is comparable with that on $x = a$.

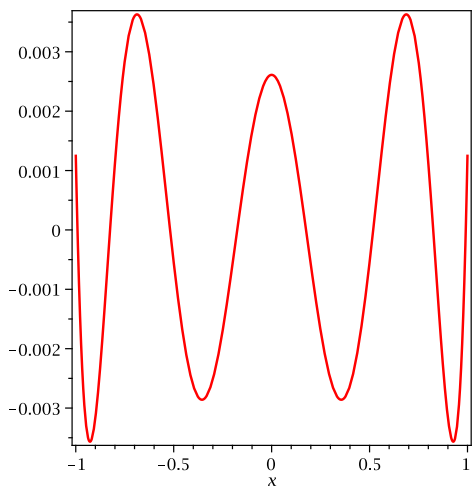


Figure 2 Tractions on $y = a$

CHAPTER 7

7.1. Every particle of an elastic body of density ρ experiences a force

$$F = \frac{C\delta m}{r^2}$$

directed towards the origin, where C is a constant, r is the distance from the origin and δm is the mass of the particle. Find a body force potential V that satisfies these conditions.

The force towards the origin can be described as a body force

$$p_r = -\frac{C\rho}{r^2}$$

per unit volume and this has components

$$p_x = p_r \cos\theta = -\frac{C\rho x}{r^3} = -\frac{C\rho x}{(x^2+y^2)^{3/2}}$$

$$p_y = p_r \sin\theta = -\frac{C\rho y}{r^3} = -\frac{C\rho y}{(x^2+y^2)^{3/2}} \ .$$

We therefore seek a potential V, such that

$$\frac{\partial V}{\partial x} = \frac{C\rho x}{(x^2+y^2)^{3/2}} \ ; \ \frac{\partial V}{\partial y} = \frac{C\rho y}{(x^2+y^2)^{3/2}} \ .$$

We can proceed by partial integration as in §7.2.4, but it is clear by inspection that the required potential is

$$V = -\frac{C\rho}{\sqrt{(x^2+y^2)}} = -\frac{C\rho}{r} \ .$$

7.2. Verify that the body force distribution

$$p_x = Cy \ ; \ p_y = -Cx$$

is non-conservative, by substituting into equation (7.6). Use the technique of §7.2.4 to attempt to construct a body force potential V for this case. Identify the step at which the procedure breaks down.

Substituting into equation (7.6), we have

$$\frac{\partial p_y}{\partial x} - \frac{\partial p_x}{\partial y} = -C - C = -2C \neq 0 \tag{1}$$

and hence (7.6) is not satisfied.

Following the procedure of §7.2.4, we start with

$$p_x = -\frac{\partial V}{\partial x} = Cy \tag{2}$$

and hence after partial integration,

$$V = -Cxy + f(y) \tag{3}$$

Substituting this value of V into

$$p_y = -\frac{\partial V}{\partial y} = -Cx \tag{4}$$

we obtain

$$Cx + f'(y) = -Cx \ , \tag{5}$$

or

$$\frac{df}{dy} = -2Cx \ . \tag{6}$$

There is no way we can find a function of y only whose derivative is a function of x, so equation (6) has no solution. This is the point at which the procedure breaks down.

7.3. To construct a particular solution for the stress components in plane strain due to a non-conservative body force distribution, it is proposed to start by representing the displacement components in the form

$$u_x = \frac{1}{2\mu}\frac{\partial \psi}{\partial y} \;\; ; \;\; u_y = -\frac{1}{2\mu}\frac{\partial \psi}{\partial x} \;\; ; \;\; u_z = 0 \; .$$

Use the strain-displacement equations (1.51) and Hooke's law (1.71) to find expressions for the stress components in terms of ψ. Substitute these results into the equilibrium equations (2.2, 2.3) to find the governing equations for the stress function ψ.

What is the condition that must be satisfied by the body force distribution $\boldsymbol{p}$ if these equations are to have a solution? Show that this condition is satisfied if the body force distribution can be written in terms of a potential function W as

$$p_x = \frac{\partial W}{\partial y} \;\; ; \;\; p_y = -\frac{\partial W}{\partial x} \; .$$

For the special case

$$p_x = Cy \;\; ; \;\; p_y = -Cx \; ,$$

find a particular solution for ψ in the axisymmetric polynomial form

$$\psi = A(x^2 + y^2)^n \; ,$$

where A is a constant and n is an appropriate integer power. Show that this solution can be used to obtain the stress components (7.46–7.48). Suggest ways in which this method might be adapted to give a particular solution for more generally non-conservative body force distributions.

Using the strain displacement equations, the only non-zero strains are

$$e_{xx} = \frac{1}{2\mu}\frac{\partial^2 \psi}{\partial x \partial y} \;\; ; \;\; e_{yy} = -\frac{1}{2\mu}\frac{\partial^2 \psi}{\partial x \partial y} \;\; ; \;\; e_{xy} = \frac{1}{4\mu}\left(\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2}\right) \; .$$

It follows that the dilatation

$$e = e_{xx} + e_{yy} + e_{zz} = 0$$

and Hooke's law (1.71) gives

$$\sigma_{xx} = \frac{\partial^2 \psi}{\partial x \partial y} \;\; ; \;\; \sigma_{yy} = -\frac{\partial^2 \psi}{\partial x \partial y} \;\; ; \;\; \sigma_{xy} = \frac{1}{2}\left(\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2}\right) \; ,$$

the other stress components being zero.

Substituting into the first two equilibrium equations, we then obtain

$$\frac{\partial^3\psi}{\partial x^2\partial y} + \frac{1}{2}\frac{\partial^3\psi}{\partial y^3} - \frac{1}{2}\frac{\partial^3\psi}{\partial x^2\partial y} + p_x = 0$$

$$\frac{1}{2}\frac{\partial^3\psi}{\partial y^2\partial x} - \frac{\partial^3\psi}{\partial x^3} - \frac{\partial^3\psi}{\partial y^2\partial x} + p_y = 0$$

In other words

$$\frac{\partial}{\partial y}\nabla^2\psi = -2p_x \ \ ; \ \ \frac{\partial}{\partial x}\nabla^2\psi = 2p_y \ .$$

If

$$p_x = \frac{\partial W}{\partial y} \ \ ; \ \ p_y = -\frac{\partial W}{\partial x} \ ,$$

both equilibrium equations will be satisfied by the choice

$$\nabla^2\psi = -2W \ .$$

For the special case

$$p_x = Cy \ \ ; \ \ p_y = -Cx \ ,$$

we have

$$W = \frac{C(x^2+y^2)}{2} \ .$$

This is a second degree polynomial in x, y, so we anticipate needing a fourth degree polynomial for ψ (since ∇^2 involves two differentiations). Using the trial function

$$\psi = A(x^2+y^2)^2 = A(x^4+2x^2y^2+y^4) \ ,$$

we have

$$\nabla^2\psi = A(12x^2+4y^2+4x^2+12y^2) = 16A(x^2+y^2) \ .$$

Thus, the governing equation is satisfied as long as we choose

$$A = -\frac{C}{16}$$

and hence

$$\psi = -\frac{C(x^4+2x^2y^2+y^4)}{16} \ .$$

The corresponding non-zero stress components are then obtained as

$$\sigma_{xx} = \frac{\partial^2\psi}{\partial x\partial y} = -\frac{Cxy}{2} \ \ ; \ \ \sigma_{yy} = -\frac{\partial^2\psi}{\partial x\partial y} = \frac{Cxy}{2} \ \ ; \ \ \sigma_{xy} = \frac{1}{2}\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right) = \frac{C(x^2-y^2)}{4} \ .$$

Comparing the body forces with equations (7.15, 7.16), we can describe the state of rotational acceleration by choosing

$$C = -\rho\dot{\Omega} \ .$$

Substituting for C in the above stress equations yields equations (7.46–7.48).

A more general body force distribution could be described by combining this representation with the conservative formulation of §7.1 — i.e. defining the body forces in terms of two potentials through

$$p_x = -\frac{\partial V}{\partial x} + \frac{\partial W}{\partial y} \ \ ; \ \ p_y = -\frac{\partial V}{\partial y} - \frac{\partial W}{\partial x} \ .$$

7.4. If the elastic displacement $\boldsymbol{u}$ varies in time, there will generally be accelerations $\boldsymbol{a} = \ddot{\boldsymbol{u}}$ and hence body forces $\boldsymbol{p} = -\rho\ddot{\boldsymbol{u}}$, from equation (7.11). Use this result and equation (2.17) to develop the general equation of linear Elastodynamics.

Show that this equation is satisfied by a displacement field of the form

$$u_x = f(x - c_1 t) \ ; \ u_y = g(x - c_2 t) \ ; \ u_z = 0 \ ,$$

where f, g are any functions and c_1, c_2 are two constants that depend on the material properties. Find the values of c_1, c_2 and comment on the physical significance of this solution.

Substituting $\boldsymbol{p} = -\rho\ddot{\boldsymbol{u}}$ into equation (2.17), we obtain

$$\boldsymbol{\nabla}\text{div } \boldsymbol{u} + (1 - 2\nu)\nabla^2\boldsymbol{u} - \frac{(1 - 2\nu)\rho}{\mu}\ddot{\boldsymbol{u}} = 0 \ ,$$

which is the general equation of Elastodynamics.

In the special case where

$$u_x = f(x - c_1 t) \ ; \ u_y = g(x - c_2 t) \ ; \ u_z = 0 \ ,$$

we have

$$\ddot{u_x} = c_1^2 f''(x - c_1 t) \ ; \ \ddot{u_y} = c_2^2 g''(x - c_1 t)$$

and

$$\text{div } \boldsymbol{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = f'(x - c_1 t) \ ,$$

since u_y is independent of y.

The component of the governing equation in the x-direction therefore reduces to

$$f''(x - c_1 t) + (1 - 2\nu)f''(x - c_1 t) - \frac{(1 - 2\nu)\rho c_1^2}{\mu}f''(x - c_1 t) = 0$$

and this is satisfied for all functions f provided

$$1 + (1 - 2\nu) - \frac{(1 - 2\nu)\rho c_1^2}{\mu} = 0 \ .$$

Solving for c_1, we obtain

$$c_1 = \pm\sqrt{\frac{2(1 - \nu)\mu}{(1 - 2\nu)\rho}} \ .$$

A similar procedure with the y-component of the governing equation yields

$$(1 - 2\nu)g''(x - c_1 t) - \frac{(1 - 2\nu)\rho c_2^2}{\mu}g''(x - c_1 t) = 0 \ ,$$

since div $\boldsymbol{u}$ is independent of y. This equation is satisfied for all g provided

$$c_2 = \pm\sqrt{\frac{\mu}{\rho}}\,.$$

These solutions describe stress fields which maintain the same form but which propagate through the body at speeds c_1, c_2 respectively. In other words, they are solutions for elastic wave propagation. *Longitudinal waves* (or *dilatational waves*) involving function f and speed c_1 correspond to the case where the direction of the displacement is parallel with the direction of propagation, whilst *transverse waves* (or *shear waves*) involve displacements perpendicular to the direction of propagation.

7.5. The beam $-b < y < b,\ 0 < x < L$ is built-in at the edge $x = L$ and is loaded merely by its own weight, ρg per unit volume.

Find a solution for the stress field, using the weak conditions on the end $x = 0$.

For gravitational loading, we can use the body force potential $V = \rho g y$ (see equation (7.10)), for which $\nabla^2 V = 0$ and hence $\nabla^4 \phi = 0$ from (7.8).

From Mechanics of Materials considerations, we note that the bending moment will vary with x^2 and hence the bending stress varies with $x^2 y$ suggesting a 5th degree polynomial.

The problem is clearly odd in y, since reversal of sign in g is equivalent to reversal of the y-axis direction. What is less obvious is that it is even in x. To see this, consider a beam of length $2L$ loaded by its own weight and simply supported at its ends. This is even in x. Now superpose a state of pure bending (which is also even in x) chosen so as to make the bending moment zero at $x = 0$. One half of the beam now looks like the problem under consideration, which is therefore even in x.

Using these arguments, we conclude that the most general stress function required is

$$\phi = Ax^4 y + Bx^2 y^3 + Cy^5 + Dx^2 y + Ey^3 \ . \tag{1}$$

The strong boundary conditions are

$$\sigma_{xy} = 0 \quad ; \quad y = \pm b \ ; \tag{2}$$

$$\sigma_{yy} = 0 \quad ; \quad y = \pm b \ . \tag{3}$$

Notice that it will only be necessary to enforce these conditions at one of the two boundaries, since the other will then follow from the symmetry condition in y.

We have

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -4Ax^3 - 6Bxy^2 - 2Dx \ , \tag{4}$$

which must be zero for all x on $y = \pm b$. We conclude that

$$A = 0 \quad ; \quad 3Bb^2 + D = 0 \ . \tag{5}$$

We also have

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + V = 12Ax^2 y + 2By^3 + 2Dy + \rho g y \ , \tag{6}$$

which with the second strong boundary condition yields the additional equation

$$2Bb^2 + 2D + \rho g = 0 \ . \tag{7}$$

Equations (5,7) can be solved for B, D with the result

$$B = \frac{\rho g}{4b^2} \quad ; \quad D = -\frac{3\rho g}{4} \ . \tag{8}$$

The condition $\nabla^4\phi = 0$ then gives

$$120Ay + 24By + 120Cy = 0 \tag{9}$$

and hence

$$C = -\frac{B}{5} = -\frac{\rho g}{20b^2} \ . \tag{10}$$

Only one constant, E, remains to be found from the weak conditions

$$\int_{-b}^{b} \sigma_{xy} dy = 0 \ \ ; \ \ \int_{-b}^{b} \sigma_{xx} dy = 0 \ \ ; \ \ \int_{-b}^{b} \sigma_{xx} y dy = 0 \tag{11}$$

on $x = 0$. However, the symmetry condition in x guarantees that the first of these will be satisfied identically and the antisymmetry condition in y guarantees that the second condition will be satisfied (Readers can satisfy themselves of this by substituting the original stress function into the two conditions in question.). Only the third (moment) condition remains to be satisfied. We have

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} + V = 6Bx^2y + 20Cy^3 + 6Ey + \rho g y \tag{12}$$

and hence we require

$$\int_{-b}^{b} (20Cy^3 + 6Ey + \rho g y) y dy = \frac{40Cb^5}{5} + \frac{12Eb^3}{3} + \frac{2\rho g b^3}{3} = 0 \ . \tag{13}$$

Substituting for C from (10), we have $E = -\rho g/15$ and the final solution is defined by

$$V = \rho g y \ ; \tag{14}$$

$$\phi = \rho g \left(-\frac{y^5}{20b^2} + \frac{y^3x^2}{4b^2} - \frac{y^3}{15} - \frac{3yx^2}{4} \right) \ ; \tag{15}$$

$$\sigma_{xx} = \rho g c \left(-\frac{y^3}{b^3} + \frac{3x^2y}{2b^3} + \frac{3y}{5b} \right) \ ; \tag{16}$$

$$\sigma_{xy} = \frac{3\rho g x}{2} \left(1 - \frac{y^2}{b^2} \right) \ ; \tag{17}$$

$$\sigma_{yy} = -\frac{\rho g y}{2} \left(1 - \frac{y^2}{b^2} \right) \ . \tag{18}$$

7.6. One wall of a multistory building of height H is approximated as a thin plate $-b < x < b$, $0 < y < H$. During an earthquake, ground motion causes the building to experience a uniform acceleration a in the x-direction. Find the resulting stresses in the wall if the material has density ρ and the edges $x = \pm b$, $y = H$ can be regarded as traction-free.

The acceleration is

$$a_x = a \quad ; \quad a_y = 0 \tag{1}$$

and hence we have

$$p_x = -\frac{\partial V}{\partial x} = -\rho a \quad ; \quad p_y = -\frac{\partial V}{\partial y} = 0 \tag{2}$$

A suitable particular integral is

$$V = \rho a x \ . \tag{3}$$

From Mechanics of Materials considerations, we note that the bending moment will vary with y^2 and hence the bending stress varies with xy^2 suggesting a 5th degree polynomial.

The problem is clearly odd in x, since reversal of sign in a is equivalent to reversal of the x-axis direction. We therefore propose the stress function

$$\phi = C_1 xy + C_2 x^3 + C_3 xy^2 + C_4 x^3 y + C_5 xy^3 + C_6 x^5 + C_7 x^3 y^2 + C_8 xy^4 \ . \tag{4}$$

The corresponding stress components are

$$\begin{aligned}
\sigma_{xx} &= 2C_3 x + 6C_5 xy + 2C_7 x^3 + 12C_8 xy^2 + \rho a x && (5)\\
\sigma_{xy} &= -C_1 - 2C_3 y - 3C_4 x^2 - 3C_5 y^2 - 6C_7 x^2 y - 4C_8 y^3 && (6)\\
\sigma_{yy} &= 6C_2 x + 6C_4 xy + 20C_6 x^3 + 6C_7 xy^2 + \rho a x \ . && (7)
\end{aligned}$$

The strong boundary conditions

$$\sigma_{xx} = \sigma_{xy} = 0 \ ; \quad x = \pm b \tag{8}$$

gives

$$\begin{aligned}
2C_3 b + 6C_5 by + 2C_7 b^3 + 12C_8 by^2 + \rho a b &= 0 && (9)\\
-C_1 - 2C_3 y - 3C_4 b^2 - 3C_5 y^2 - 6C_7 b^2 y - 4C_8 y^3 &= 0 && (10)
\end{aligned}$$

and these conditions must be satisfied for all x, giving

$$\begin{aligned}
C_8 &= 0 && (11)\\
C_5 &= 0 && (12)\\
2C_3 + 2C_7 b^2 + \rho a &= 0 && (13)\\
-2C_3 - 6C_7 b^2 &= 0 && (14)\\
-C_1 - 3C_4 b^2 &= 0 \ . && (15)
\end{aligned}$$

Equations (13, 14) can be solved to give

$$C_3 = -\frac{\rho a}{4} \; ; \quad C_7 = \frac{\rho a}{4b^2} \; . \tag{16}$$

The modified biharmonic equation (7.8) gives

$$120C_6 x + 24C_7 x + 24C_8 x = 0 \tag{17}$$

and hence, using (11, 16),

$$C_6 = -\frac{C_7}{5} = -\frac{\rho a}{20b^2} \; . \tag{18}$$

The symmetry condition ensures that there is no (vertical) axial force on the building, but the two other weak conditions on $y = H$ are

$$\int_{-b}^{b} \sigma_{yx}(x, H) dx = 0 \tag{19}$$

$$\int_{-b}^{b} \sigma_{yy}(x, H) x dx = 0 \; . \tag{20}$$

Substituting for the stresses from (6,7), but dropping the terms with zero coefficients (11, 12), we have

$$-2C_1 b - 4C_3 H b - 2C_4 b^3 - 4C_7 H b^3 = 0 \tag{21}$$

$$4C_2 b^3 + 4C_4 H b^3 + 8C_6 b^5 + 4C_7 H^2 b^3 + \frac{2\rho a b^3}{3} = 0 \; . \tag{22}$$

Solving for the remaining constants, we obtain

$$C_1 = \frac{3\rho a H}{2} \; ; \quad C_2 = \frac{\rho a H^2}{4b^2} - \frac{\rho a}{15} \; ; \quad C_3 = -\frac{\rho a}{4} \; ; \quad C_4 = -\frac{\rho a H}{2b^2} \; ; \quad C_5 = 0 \; ; \tag{23}$$

$$C_6 = -\frac{\rho a}{20b^2} \; ; \quad C_7 = \frac{\rho a}{4b^2} \quad C_8 = 0 \tag{24}$$

and the final stress function is

$$\phi = \frac{\rho a}{60b^2} \left(90Hb^2 xy + 15H^2 x^3 - 4b^2 x^3 - 45B^2 xy^2 - 30Hx^3 y - 3x^5 + 15x^3 y^2 \right) \; . \tag{25}$$

The stress components are

$$\sigma_{xx} = -\frac{\rho a x (b^2 - x^2)}{2b^2} \tag{26}$$

$$\sigma_{xy} = -\frac{3\rho a (H - y)(b^2 - x^2)}{2b^2} \tag{27}$$

$$\sigma_{yy} = \frac{\rho a}{10b^2} (15x(H - y)^2 + 6b^2 x - 10x^3) \; . \tag{28}$$

This problem is physically identical to the gravitational loading problem 7.5 and as in that case, some simplification could be achieved by moving the origin to the top of the building, in which case the problem would also be even in y.

7.7. A tall thin rectangular plate $-a < x < a$, $-b < y < b$ ($b \gg a$) is supported on the vertical edges $x = \pm a$ and loaded only by its own weight (density ρ). Find the stresses in the plate using the weak boundary conditions on the horizontal edges $y = \pm b$ and assuming that the support tractions consist only of uniform shear.

The body force potential for gravitational loading is given by equation (7.10) as

$$V = \rho g y \ . \tag{1}$$

The problem is even in x and odd in y, so we first try the simplest such function which is a third degree polynomial in x, y —

$$\phi = C_1 x^2 y + C_2 y^3 \ . \tag{2}$$

The corresponding stresses are

$$\begin{aligned} \sigma_{xx} &= 6C_2 y + \rho g y & (3) \\ \sigma_{xy} &= -2C_1 x & (4) \\ \sigma_{yy} &= 2C_1 y + \rho g y \ . & (5) \end{aligned}$$

The tractions on $x = \pm a$ are

$$\begin{aligned} \sigma_{xx} &= 6C_2 y + \rho g y & (6) \\ \sigma_{xy} &= \mp 2C_1 a & (7) \end{aligned}$$

and these will consist of uniform shear if

$$C_2 = -\frac{\rho g}{6} \ . \tag{8}$$

On the end $y = b$, two of the weak conditions are satisfied by symmetry considerations, leaving only the condition

$$\int_{-a}^{a} \sigma_{yy}(x, b) dx = 0 \ . \tag{9}$$

It is clear from equation (5) that this condition will be satisfied (also in the strong sense) if

$$C_1 = -\frac{\rho g}{2} \ . \tag{10}$$

The final stress field is then obtained as

$$\begin{aligned} \sigma_{xx} &= 0 & (11) \\ \sigma_{xy} &= \rho g x & (12) \\ \sigma_{yy} &= 0 \ . & (13) \end{aligned}$$

7.8. Figure 7.4(a) shows a triangular cantilever, defined by the boundaries $y = 0, \quad y = x \tan\alpha$ and built-in at $x = a$. It is loaded by its own weight, ρg per unit volume. Find a solution for the complete stress field and compare the maximum tensile bending stress with that predicted by the Mechanics of Materials theory.

Would the maximum tensile stress be lower if the alternative configuration of Figure 7.4(b) were used?

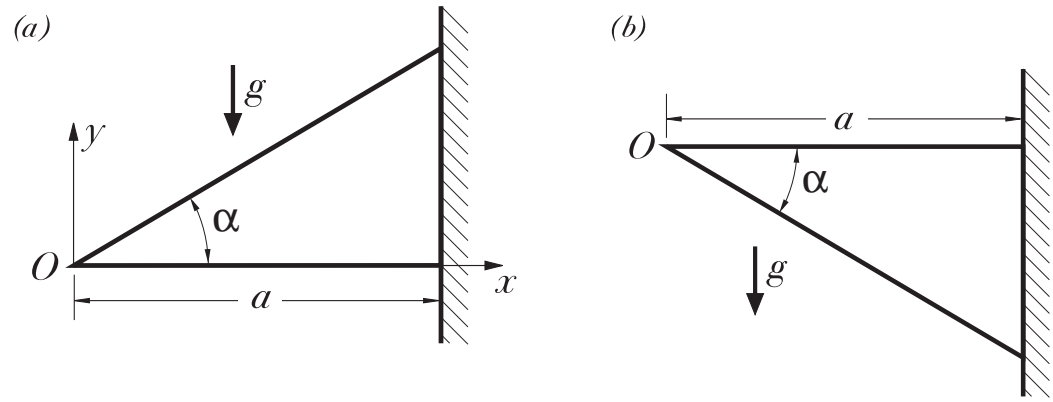


Figure 7.4

The self-weight can be described by a gravitational potential

$$V = \rho g y \tag{1}$$

as in equation (7.10).

Thinking of the gravitational loading as a distributed load depending upon x, the load per unit length varies with x, the shear force with x^2 and hence the bending moment with x^3. However, the second moment of area I of the beam also varies with x^3 and hence we anticipate a stress field in which the dominant term in σ_{xx} is

$$\sigma_{xx} = \frac{My}{I} \sim \frac{x^3 y}{x^3} = y \ . \tag{2}$$

Thus, we expect the Airy stress function ϕ to be a third degree polynomial in x, y.

The most general non-trivial function of this form is

$$\phi = C_1 x^3 + C_2 x^2 y + C_3 x y^2 + C_4 y^3 + C_5 x^2 + C_6 x y + C_7 y^2 \ , \tag{3}$$

defining the stress field

$$\sigma_{xx} \ = \ \frac{\partial^2 \phi}{\partial y^2} + V = 2C_3 x + 6C_4 y + 2C_7 + \rho g y \ ; \tag{4}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -2C_2 x - 2C_3 y - C_6 \; ; \tag{5}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} + V = 6C_1 x + 2C_2 y + 2C_5 + \rho g y \; . \tag{6}$$

All the terms in (2) satisfy the biharmonic equation identically.

The boundary conditions require that the surfaces $y = 0$ and $y = x \tan\alpha$ be traction-free.

For the surface $y = 0$, we have

$$\sigma_{yy} = 6C_1 x + 2C_5 = 0 \quad ; \quad \sigma_{xy} = -2C_2 x - C_6 = 0 \; . \tag{7}$$

Both of these equations have to be satisfied for all x and hence

$$C_1 = C_2 = C_5 = C_6 = 0 \; . \tag{8}$$

For the traction-free condition on $y = x \tan\alpha$, we consider the element of material shown in Figure 1 and conclude that

$$\sigma_{xx} \tan\alpha - \sigma_{xy} = 0 \quad ; \quad \sigma_{xy} \tan\alpha - \sigma_{yy} = 0 \tag{9}$$

on $y = x \tan\alpha$.

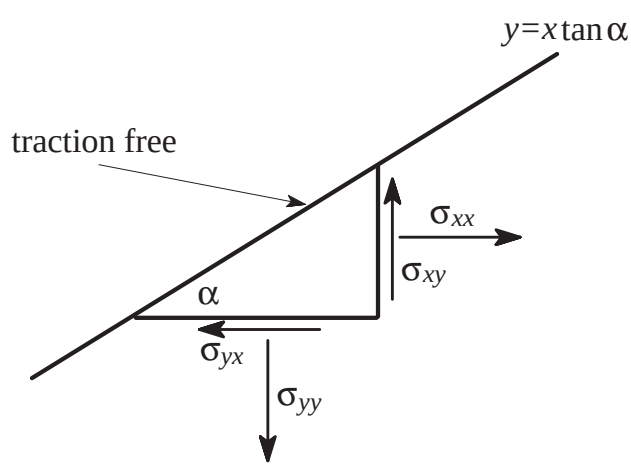


Figure 1

Substituting for the stress components, using (8) and replacing y by $x \tan\alpha$, we have

$$(2C_3 x + 6C_4 x \tan\alpha + 2C_7 + \rho g x \tan\alpha) \tan\alpha + 2C_3 x \tan\alpha = 0 \; ; \tag{10}$$

$$-2C_3 x \tan^2\alpha - 2C_5 - \rho g x \tan\alpha = 0 \; . \tag{11}$$

These equations must be satisfied for all x, so

$$C_5 = C_7 = 0 \tag{12}$$

and

$$
\begin{aligned}
4C_3 + 6C_4 \tan\alpha &= -\rho g \tan\alpha \ ; && (13)\\
2C_3 \tan\alpha &= -\rho g \ . && (14)
\end{aligned}
$$

The final solution for the constants is therefore

$$
C_3 = -\frac{\rho g \cot\alpha}{2} \quad ; \quad C_4 = \frac{\rho g (2\cot^2\alpha - 1)}{6} \ , \tag{15}
$$

all the other constants being zero.

The complete solution for the stress field is therefore defined by

$$
\begin{aligned}
\phi &= -\frac{\rho g x y^2 \cot\alpha}{2} + \frac{\rho g y^3 (2\cot^2\alpha - 1)}{6} \ ; && (16)\\
\sigma_{xx} &= -\rho g x \cot\alpha + 2\rho g y \cot^2\alpha \ ; && (17)\\
\sigma_{xy} &= \rho g y \cot\alpha \ ; && (18)\\
\sigma_{yy} &= \rho g y \ . && (19)
\end{aligned}
$$

The maximum tensile stress will occur at the point $x = a, y = a\tan\alpha$, where

$$
\begin{aligned}
\sigma_{xx} &= -\rho g a \cot\alpha + 2\rho g a \tan\alpha \cot^2\alpha = \rho g a \cot\alpha \ ; && (20)\\
\sigma_{xy} &= \rho g a \tan\alpha \cot\alpha = \rho g a \ ; && (21)\\
\sigma_{yy} &= \rho g a \tan\alpha \ . && (22)
\end{aligned}
$$

At this point, we know that the stress state must be one of uniaxial tension, since the inclined surface is traction-free.

We can obtain this stress by using the transformation equation (1.15), with the result

$$
\begin{aligned}
\sigma_{\max} &= \sigma_{xx}\cos^2\alpha + 2\sigma_{xy}\sin\alpha\cos\alpha + \sigma_{yy}\sin^2\alpha\\
&= \rho g a \cot\alpha \cos^2\alpha + 2\rho g a \sin\alpha\cos\alpha + \rho g a \tan\alpha \sin^2\alpha\\
&= \frac{\rho g a}{\sin\alpha\cos\alpha} \ . && (23)
\end{aligned}
$$

The elementary solution of this problem is

$$
\sigma_{\max} = \frac{My}{I} = \rho g \left(\frac{a^2\tan\alpha}{2} . \frac{a}{3} . \frac{a\tan\alpha}{2} \right) \Big/ \left(\frac{(a\tan\alpha)^3}{12} \right) = \rho g a \cot\alpha \ , \tag{24}
$$

which is also the stress σ_{xx} at $(a, a\tan\alpha)$ in the exact solution.

The exact solution predicts a larger stress than the elementary solution by the ratio

$$
\frac{\sigma_{\max}(\text{exact})}{\sigma_{\max}(\text{elementary})} = \frac{1}{\cos^2\alpha} \ . \tag{25}
$$

The above solution can also be used for the problem of Figure 7.4(b) if we set $y = -y$ and $g = -g$. The result is that corresponding points in the two Figures have equal and opposite normal stresses. In particular, the maximum tensile stress will now occur at $(a, 0)$ and is equal and opposite to the value of σ_{xx} at that point given by equation (17). Thus,

$$\sigma_{\text{max}} = \rho g a \cot \alpha \ , \tag{26}$$

which is equal to the prediction of the elementary theory. For the configuration of Figure 7.4(b), there will be a larger stress equal and opposite to that given by equation (23) at the point $(a, -a \tan \alpha)$, but it will be compressive.

7.9. The thin rectangular plate $-a < x < a, -b < y < b$ with $a \gg b$ rotates about the y-axis at constant angular velocity Ω. All surfaces of the plate are traction-free. Find a solution for the stress field, using strong boundary conditions on the long edges $y = \pm b$ and weak boundary conditions on the ends $x = \pm a$.

For rotation about the y-axis, the acceleration is

$$a_x = -\Omega^2 x \quad ; \quad a_y = 0 \tag{1}$$

and hence we have

$$p_x = -\frac{\partial V}{\partial x} = \rho\Omega^2 x \quad ; \quad p_y = -\frac{\partial V}{\partial y} = 0 \tag{2}$$

A suitable particular integral is

$$V = -\frac{\rho\Omega^2 x^2}{2} \ . \tag{3}$$

The problem is symmetrical about both axes and (3) suggests a fourth degree polynomial — i.e.

$$\phi = A_1 x^4 + A_2 x^2 y^2 + A_3 y^4 + A_4 x^2 + A_5 y^2 \ , \tag{4}$$

which contains all the terms of degree 4 and below with the required symmetry.

The stresses are then obtained as

$$\sigma_{xx} = 2A_2 x^2 + 12A_3 y^2 + 2A_5 - \frac{1}{2}\rho\Omega^2 x^2 \tag{5}$$

$$\sigma_{yy} = 12A_1 x^2 + 2A_2 y^2 + 2A_4 - \frac{1}{2}\rho\Omega^2 x^2 \tag{6}$$

$$\sigma_{yx} = -4A_2 xy \ . \tag{7}$$

The edges of the plate are traction-free and hence the strong boundary conditions are

$$\sigma_{yx} = 0 \ ; \ y = \pm b \tag{8}$$

$$\sigma_{yy} = 0 \ ; \ y = \pm b \ . \tag{9}$$

Substituting from equations (5–7), we require

$$-4A_2 xb = 0 \tag{10}$$

$$12A_1 x^2 + 2A_2 b^2 + 2A_4 - \frac{1}{2}\rho\Omega^2 x^2 = 0 \tag{11}$$

and these equations must be satisfied for all x, leading to

$$A_2 = 0 \; ; \quad A_1 = \frac{1}{24}\rho\Omega^2 \; ; \quad A_4 = 0 \; . \tag{12}$$

Substituting (4) into the modified biharmonic equation (7.8), we obtain

$$24A_1 + 8A_2 + 24A_3 = (1-\nu)\rho\Omega^2 \tag{13}$$

and hence

$$24A_3 = (1-\nu)\rho\Omega^2 - \rho\Omega^2 = -\nu\rho\Omega^2 \tag{14}$$

from (12), giving

$$A_3 = -\frac{\nu}{24}\rho\Omega^2 \; . \tag{15}$$

The remaining constant A_5 is obtained from the weak condition

$$\int_{-b}^{b} \sigma_{xx} dy = 0 \;\; ; \;\; x = \pm a \; , \tag{16}$$

the other two weak conditions being satisfied identically by virtue of the symmetry of the problem. Substituting (5) into (16), setting $x = a, A_2 = 0$ and performing the integration, we obtain

$$8A_3 b^3 + 4A_5 b - \rho\Omega^2 a^2 b = 0 \; . \tag{17}$$

Substituting for A_3, we then obtain

$$A_5 = \rho\Omega^2 \left(\frac{a^2}{4} + \frac{\nu b^2}{12} \right) \; . \tag{18}$$

The final stress field is then recovered from equations (5–7, 12, 15, 18) as

$$\sigma_{xx} = \rho\Omega^2 \left(\frac{(a^2 - x^2)}{2} + \frac{\nu(b^2 - 3y^2)}{6} \right) \tag{19}$$

$$\sigma_{yy} = 0 \tag{20}$$

$$\sigma_{yx} = 0 \; . \tag{21}$$

7.10. Solve Problem 7.9 for the case where $a \ll b$. In this case you should use strong boundary conditions on $x = \pm a$ and weak boundary conditions on $y = \pm b$.

For rotation about the y-axis, the acceleration is

$$a_x = -\Omega^2 x \quad ; \quad a_y = 0 \tag{1}$$

and hence we have

$$p_x = -\frac{\partial V}{\partial x} = \rho\Omega^2 x \quad ; \quad p_y = -\frac{\partial V}{\partial y} = 0 \tag{2}$$

A suitable particular integral is

$$V = -\frac{\rho\Omega^2 x^2}{2} \ . \tag{3}$$

The problem is symmetrical about both axes and (3) suggests a fourth degree polynomial — i.e.

$$\phi = A_1 x^4 + A_2 x^2 y^2 + A_3 y^4 + A_4 x^2 + A_5 y^2 \ , \tag{4}$$

which contains all the terms of degree 4 and below with the required symmetry.

The stresses are then obtained as

$$\sigma_{xx} = 2A_2 x^2 + 12A_3 y^2 + 2A_5 - \frac{1}{2}\rho\Omega^2 x^2 \tag{5}$$

$$\sigma_{yy} = 12A_1 x^2 + 2A_2 y^2 + 2A_4 - \frac{1}{2}\rho\Omega^2 x^2 \tag{6}$$

$$\sigma_{yx} = -4A_2 xy \ . \tag{7}$$

The edges of the plate are traction-free and hence the strong boundary conditions are

$$\sigma_{xx} = 0 \ ; \ x = \pm a \tag{8}$$

$$\sigma_{xy} = 0 \ ; \ x = \pm a \ . \tag{9}$$

Substituting from equations (5–7), we require

$$2A_2 a^2 + 12A_3 y^2 + 2A_5 - \frac{1}{2}\rho\Omega^2 a^2 = 0 \tag{10}$$

$$-4A_2 ay = 0 \ . \tag{11}$$

and these equations must be satisfied for all y, leading to

$$A_2 = 0 \ ; \quad A_3 = 0 \ ; \quad A_5 = \frac{1}{4}\rho\Omega^2 a^2 \ . \tag{12}$$

Substituting (4) into the modified biharmonic equation (7.8), we obtain

$$24A_1 + 8A_2 + 24A_3 = (1-\nu)\rho\Omega^2 \tag{13}$$

and hence

$$A_1 = -\frac{(1-\nu)}{24}\rho\Omega^2 \ , \tag{14}$$

using (12)

The remaining constant A_4 is obtained from the weak condition

$$\int_{-a}^{a} \sigma_{yy} dx = 0 \ \ ; \ \ y = \pm b \ , \tag{15}$$

the other two weak conditions being satisfied identically by virtue of the symmetry of the problem. From (6), with $y = b$ and $A_2 = 0$, we have

$$\sigma_{yy}(x,b) = -\frac{\nu\rho\Omega^2 x^2}{2} + 2A_4 \tag{16}$$

and hence, from (15) on performing the integration,

$$-\frac{\nu\rho\Omega^2 a^3}{3} + 4A_4 a \ , \tag{17}$$

or

$$A_4 = \frac{\nu\rho\Omega^2 a^2}{12} \ . \tag{18}$$

The final stress field is then recovered from equations (5–7, 12, 14, 18) as

$$\sigma_{xx} = \frac{1}{2}\rho\Omega^2(a^2 - x^2) \tag{19}$$

$$\sigma_{yy} = \frac{\nu}{6}\rho\Omega^2(a^2 - 3x^2) \tag{20}$$

$$\sigma_{yx} = 0 \ . \tag{21}$$

7.11. A thin triangular plate bounded by the lines $y = \pm x \tan\alpha$, $x = a$ rotates about the axis $x = a$ at constant angular velocity Ω. The inclined edges of the plate $y = \pm x \tan\alpha$ are traction-free. Find a solution for the stress field, using strong boundary conditions on the inclined edges (weak boundary conditions will then be implied on $x = a$).

For rotation about the axis $x = a$, the acceleration is

$$a_x = \Omega^2(a - x) \quad ; \quad a_y = 0 \tag{1}$$

and hence we have

$$p_x = -\frac{\partial V}{\partial x} = -\rho\Omega^2(a - x) \quad ; \quad p_y = -\frac{\partial V}{\partial y} = 0 \tag{2}$$

A suitable particular integral is

$$V = \frac{\rho\Omega^2(2ax - x^2)}{2} \ . \tag{3}$$

The problem is symmetrical about the x-axis and (3) suggests a fourth degree polynomial — i.e.

$$\phi = C_1x^2 + C_2y^2 + C_3x^3 + C_4xy^2 + C_5x^4 + C_6x^2y^2 + C_7y^4 \ , \tag{4}$$

which contains all the terms of degree 4 and below with the required symmetry.

The stresses are then obtained as

$$\sigma_{xx} = 2C_2 + 2C_4x + 2C_6x^2 + 12C_7y^2 + \frac{\rho\Omega^2(2ax - x^2)}{2} \tag{5}$$

$$\sigma_{yy} = 2C_1 + 6C_3x + 12C_5x^2 + 2C_6y^2 + \frac{\rho\Omega^2(2ax - x^2)}{2} \tag{6}$$

$$\sigma_{yx} = -2C_4y - 4C_6xy \ . \tag{7}$$

By considering the equilibrium of a small triangular element at the traction-free boundary, we conclude from equilibrium considerations that

$$\sigma_{xx}\sin\alpha - \sigma_{xy}\cos\alpha = 0 \tag{8}$$

$$\sigma_{xy}\sin\alpha - \sigma_{yy}\cos\alpha = 0 \tag{9}$$

on $y = x\tan\alpha$. Applying these conditions in the strong sense, we obtain

$$\left(2C_2 + 2C_4x + 2C_6x^2 + 12C_7x^2\tan^2\alpha + \frac{\rho\Omega^2(2ax - x^2)}{2}\right)\sin\alpha + (2C_4x + 4C_6x^2)\tan\alpha\cos\alpha = 0 \tag{10}$$

$$-(2C_4x + 4C_6x^2)\tan\alpha\sin\alpha - \left(2C_1 + 6C_3x + 12C_5x^2 + 2C_6x^2\tan^2\alpha + \frac{\rho\Omega^2(2ax - x^2)}{2}\right)\cos\alpha = 0 \ . \tag{11}$$

These equations must be satisfied for all x and hence

$$
\begin{aligned}
2C_2 \sin\alpha &= 0 && (12)\\
(2C_4 + \rho\Omega^2 a)\sin\alpha + 2C_4 \sin\alpha &= 0 && (13)\\
\left(2C_6 + 12C_7 \tan^2\alpha - \frac{\rho\Omega^2}{2}\right)\sin\alpha + 4C_6 \sin\alpha &= 0 && (14)\\
2C_1 \cos\alpha &= 0 && (15)\\
-2C_4 \tan\alpha \sin\alpha - (6C_3 + \rho\Omega^2 a)\cos\alpha &= 0 && (16)\\
-4C_6 \tan\alpha \sin\alpha - \left(12C_5 + 2C_6 \tan^2\alpha - \frac{\rho\Omega^2}{2}\right)\cos\alpha &= 0\ . && (17)
\end{aligned}
$$

One additional equation is obtained by substituting (4) into the modified biharmonic equation (7.8), giving

$$24C_5 + 8C_6 + 24C_7 = (1-\nu)\rho\Omega^2\ . \tag{18}$$

Solving the seven equations (12–18) for the constants $C_1 ... C_7$, we obtain

$$C_1 = C_2 = 0\ ; \quad C_3 = \frac{\rho\Omega^2 a(\tan^2\alpha - 2)}{12}\ ; \quad C_4 = -\frac{\rho\Omega^2 a}{4}\ ;$$

$$C_5 = \frac{\rho\Omega^2(3(1-\nu)\tan^4\alpha - 5\tan^2\alpha + 3)}{24(3\tan^4\alpha - 2\tan^2\alpha + 3)}\ ; \quad C_6 = \frac{\rho\Omega^2(1+\nu\tan^2\alpha)}{4(3\tan^4\alpha - 2\tan^2\alpha + 3)}\ ;$$

$$C_7 = \frac{\rho\Omega^2(3\tan^2\alpha - 2 - 3\nu)}{24(3\tan^4\alpha - 2\tan^2\alpha + 3)}\ . \tag{19}$$

The expressions for the stresses are complicated, but are readily obtained for particular values of ν, α by substituting (19) into (5–7).

7.12. A thin triangular plate bounded by the lines $y = \pm x \tan\alpha$, $x = a$ is initially at rest, but is subjected to tractions at the edge $x = a$ causing an angular acceleration $\dot{\Omega}$ about the perpendicular axis through the point $(a, 0)$. The inclined surfaces $y = \pm x \tan\alpha$ are traction-free. Find a solution for the instantaneous stress field, using the technique of §7.4.2.

As in §7.4.2, a suitable particular solution for the body force field due to rotational acceleration is given by equations (7.46–7.48). However, in this case, the axis of rotation passes through the point $(a, 0)$ and hence we replace x in these equations by $(x - a)$, obtaining

$$\sigma_{xx} = \frac{\rho\dot{\Omega}(x-a)y}{2} \tag{1}$$

$$\sigma_{yy} = -\frac{\rho\dot{\Omega}(x-a)y}{2} \tag{2}$$

$$\sigma_{xy} = -\frac{\rho\dot{\Omega}((x-a)^2 - y^2)}{4} \ . \tag{3}$$

For the homogeneous solution, we note that the geometry is symmetric about the x-axis and the particular solution is odd in y. The unwanted tractions due to the particular solution vary quadratically on the inclined surfaces, producing a bending moment with a leading term proportional to x^4. However, the second moment of area of the section is proportional to x^3, so we might anticipate quadratic stresses and hence a fourth order polynomial for ϕ. The most general fourth order polynomial with the required symmetry is

$$\phi = C_1 xy + C_2 x^2 y + C_3 y^3 + C_4 x^3 y + C_5 x y^3 \ . \tag{4}$$

The stress components obtained from this function, superposed on the particular solution (1–3) give

$$\sigma_{xx} = 6C_3 y + 6C_5 xy + \frac{\rho\dot{\Omega}(x-a)y}{2} \tag{5}$$

$$\sigma_{xy} = -C_1 - 2C_2 x - 3C_4 x^2 - 3C_5 y^2 - \frac{\rho\dot{\Omega}((x-a)^2 - y^2)}{4} \tag{6}$$

$$\sigma_{yy} = 2C_2 y + 6C_4 xy - \frac{\rho\dot{\Omega}(x-a)y}{2} \ . \tag{7}$$

By considering the equilibrium of a small triangular element at the traction-free boundary, we conclude from equilibrium considerations that

$$\sigma_{xx}\sin\alpha - \sigma_{xy}\cos\alpha = 0 \tag{8}$$

$$\sigma_{xy}\sin\alpha - \sigma_{yy}\cos\alpha = 0 \tag{9}$$

on $y = x\tan\alpha$. Applying these conditions in the strong sense, we obtain

$$\begin{aligned}
&\left(6C_3 x\tan\alpha + 6C_5 x^2\tan\alpha + \frac{\rho\dot{\Omega}(x-a)x\tan\alpha}{2}\right)\sin\alpha \\
&-\left(-C_1 - 2C_2 x - 3C_4 x^2 - 3C_5 x^2\tan^2\alpha - \frac{\rho\dot{\Omega}((x-a)^2 - x^2\tan^2\alpha)}{4}\right)\cos\alpha &= 0 \\
&\left(-C_1 - 2C_2 x - 3C_4 x^2 - 3C_5 x^2\tan^2\alpha - \frac{\rho\dot{\Omega}((x-a)^2 - x^2\tan^2\alpha)}{4}\right)\sin\alpha \\
&\qquad -\left(2C_2 x\tan\alpha + 6C_4 x^2\tan\alpha - \frac{\rho\dot{\Omega}(x-a)x\tan\alpha}{2}\right)\cos\alpha &= 0\,.
\end{aligned}$$

Notice that it is not necessary to consider the other boundary $y = -x\tan\alpha$, since this will be taken care of by the assumed symmetry properties of the solution.

These equations must be satisfied for all x, so for each equation, the coefficients of x^2, x^1, x^0 must be zero, yielding the six equations

$$\left(6C_5\tan\alpha + \frac{\rho\dot{\Omega}\tan\alpha}{2}\right)\sin\alpha - \left(-3C_4 - 3C_5\tan^2\alpha - \frac{\rho\dot{\Omega}(1-\tan^2\alpha)}{4}\right)\cos\alpha = 0 \tag{10}$$

$$\left(6C_3\tan\alpha + \frac{\rho\dot{\Omega}(-a)\tan\alpha}{2}\right)\sin\alpha - \left(-2C_2 - \frac{\rho\dot{\Omega}(-2a)}{4}\right)\cos\alpha = 0 \tag{11}$$

$$-\left(-C_1 - \frac{\rho\dot{\Omega}a^2}{4}\right)\cos\alpha = 0 \tag{12}$$

$$\left(-3C_4 - 3C_5\tan^2\alpha - \frac{\rho\dot{\Omega}(1-\tan^2\alpha)}{4}\right)\sin\alpha - \left(6C_4\tan\alpha - \frac{\rho\dot{\Omega}\tan\alpha}{2}\right)\cos\alpha = 0\,. \tag{13}$$

$$\left(-2C_2 - \frac{\rho\dot{\Omega}(-2a)}{4}\right)\sin\alpha - \left(2C_2\tan\alpha - \frac{\rho\dot{\Omega}(-a)\tan\alpha}{2}\right)\cos\alpha = 0\,. \tag{14}$$

$$\left(-C_1 - \frac{\rho\dot{\Omega}a^2}{4}\right)\sin\alpha = 0\,. \tag{15}$$

Equations (12,15) are clearly identical, so, solving (10–14) for the five constants, we obtain

$$C_1 = -\frac{\rho\dot{\Omega}a^2}{4}\ ;\ \ C_2 = 0\ ;\ \ C_3 = \frac{\rho\dot{\Omega}a}{12\sin^2\alpha}\ ;\ \ C_4 = \frac{\rho\dot{\Omega}}{24\cos^2\alpha}\ ;\ \ C_5 = -\frac{\rho\dot{\Omega}}{24\sin^2\alpha}\,. \tag{16}$$

Substituting these results into (5–7), we obtain the final stress components as

$$\sigma_{xx} = \frac{\rho\dot{\Omega}y(x + 2(a - x)\cos^2\alpha)}{4\sin^2\alpha} \tag{17}$$

$$\sigma_{xy} = -\frac{\rho\dot{\Omega}(x^2 + (x^2 - 3y^2 - 4ax)\cos^2\alpha + 2x(2a - x)\cos^4\alpha)}{2\sin^2(2\alpha)} \tag{18}$$

$$\sigma_{yy} = \frac{\rho\dot{\Omega}y(x + 2(a - x)\cos^2\alpha)}{4\cos^2\alpha} \ . \tag{19}$$

7.13. The thin square plate $-a < x < a, -a < y < a$ rotates about the z-axis at constant angular velocity Ω. Use Mathieu's method (§6.4 and Problem 6.6) to develop a series solution that satisfies all the boundary conditions in the strong sense. Make a contour plot of the maximum principal stress based on truncating the series at $m = n = 2$, and identify the location and magnitude of the maximum tensile stress. By what percentage does this value differ from the approximation obtained from §7.3.1?

Since the body is square and the loading is symmetrical about both axes, we anticipate a solution in which $\phi(x, y) = \phi(y, x)$. We shall therefore construct a particular solution of this form. The body force potential is

$$V = -\frac{1}{2}\rho\Omega^2(x^2 + y^2) \ ,$$

from (7.23). It is also helpful to choose the particular solution so as to give no shear tractions on the boundaries, since this agrees with the form of Mathieu's solution used in §6.4 and will therefore cause the corrective functions $g(y), h(x)$ to be zero and probably accelerate convergence. This can be achieved with the Airy function

$$\phi_P = A(x^4 + y^4) \ .$$

Substituting into (7.24), we have

$$\nabla^4\phi_P = 48A = 2\rho(1-\nu)\Omega^2$$

and hence

$$A = \frac{\rho(1-\nu)\Omega^2}{24} \quad \text{and} \quad \phi_P = \frac{\rho(1-\nu)\Omega^2(x^4+y^4)}{24} \ .$$

The stress components associated with this function are

$$\begin{aligned}
\sigma_{xx} &= \frac{\partial^2\phi_P}{\partial y^2} + V = \frac{\rho(1-\nu)\Omega^2 y^2}{2} - \frac{1}{2}\rho\Omega^2(x^2+y^2) \\
&= -\frac{\rho\Omega^2}{2}\left[x^2 + \nu y^2\right] \\
\sigma_{xy} &= -\frac{\partial^2\phi_P}{\partial x\partial y} = 0 \\
\sigma_{yy} &= -\frac{\rho\Omega^2}{2}\left[y^2 + \nu x^2\right] \ .
\end{aligned}$$

On the surface $x = a$, we have

$$\begin{aligned}
\sigma_{xx} &= -\frac{\rho\Omega^2}{2}\left[a^2 + \nu y^2\right] \\
\sigma_{xy} &= 0
\end{aligned}$$

and on $y = a$,

$$\begin{aligned}\sigma_{xy} &= 0 \\ \sigma_{yy} &= -\frac{\rho\Omega^2}{2}\left[a^2 + \nu x^2\right] .\end{aligned}$$

To cancel these unwanted tractions, we therefore need to superpose a corrective solution of the form of equations (6.29, 6.30) with

$$f(y) = \frac{\rho\Omega^2}{2}\left[a^2 + \nu y^2\right] \; ; \quad g(y) = 0$$

$$h(x) = 0 \; ; \quad \ell(x) = \frac{\rho\Omega^2}{2}\left[a^2 + \nu x^2\right] .$$

Substituting into (6.36), we find

$$G = 0 \; ,$$

which we could have predicted because of the symmetry $x \leftrightarrow y$.

Substituting into the equations

$$f_n = \int_{-a}^{a} f(y)\cos(\beta_n y)dy \; ; \quad \ell_m = \int_{-a}^{a} \ell(x)\cos(\alpha_m x)dx$$

and evaluating the integrals, we have

$$f_0 = \ell_0 = \frac{\rho\Omega^2 a^3(3+\nu)}{3} \; ; \quad f_1 = \ell_1 = -\frac{2\nu\rho\Omega^2 a^3}{\pi^2} \; ; \quad f_2 = \ell_2 = \frac{\nu\rho\Omega^2 a^3}{2\pi^2}$$

and hence

$$A_0 = \frac{f_0}{4a} = \frac{\rho\Omega^2 a^2(3+\nu)}{12} \; ; \quad B_0 = \frac{\ell_0}{4a} = \frac{\rho\Omega^2 a^2(3+\nu)}{12} .$$

Also, the expressions for H_m, K_n reduce to

$$\begin{aligned}H_m &= \frac{(-1)^{m+1}\ell_m}{a} \\ K_n &= \frac{(-1)^n f_n}{a} ,\end{aligned}$$

giving

$$H_1 = -\frac{2\nu\rho\Omega^2 a^2}{\pi^2} \; ; \quad H_2 = -\frac{\nu\rho\Omega^2 a^2}{2\pi^2} \; ; \quad K_1 = \frac{2\nu\rho\Omega^2 a^2}{\pi^2} \; ; \quad K_2 = \frac{\nu\rho\Omega^2 a^2}{2\pi^2} .$$

From here on, the calculations were performed in Maple. We substitute these values of H_m, K_n into the first two of each of the equations (6.34) and solve for X_1, X_2, Y_1, Y_2. The original constants A_1, A_2, B_1, B_2 are then recovered and substituted into the stress function (6.33), along with A_0, B_0 and the results are added to the term ϕ_0 above.

To check the accuracy of the solution, we compare the tractions on the surface $x = a$ for the case $a = 1$, normalized with the maximum tensile stress which occurs at the origin. No Mathieu correction is necessary if $\nu = 0$, since the solution of §7.3.1 then satisfies the boundary conditions exactly in the strong sense. The error is proportional to ν, so we present the worst case which occurs for $\nu = 0.5$. For comparison, we also present the tractions on this boundary implied by the solution of §7.3.1. The corrected solution (green curve) is almost indistinguishable from the correct zero traction condition on this scale.

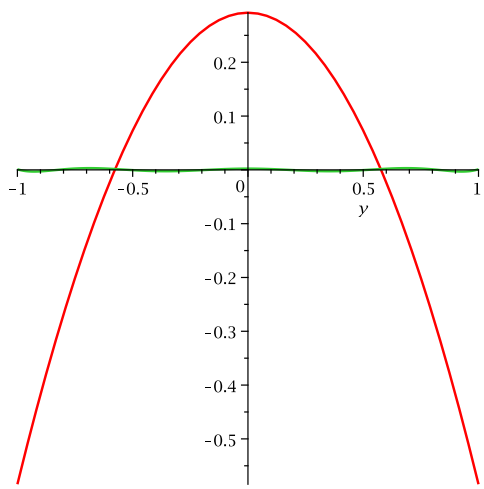


Figure 1: Tractions on $x = a$

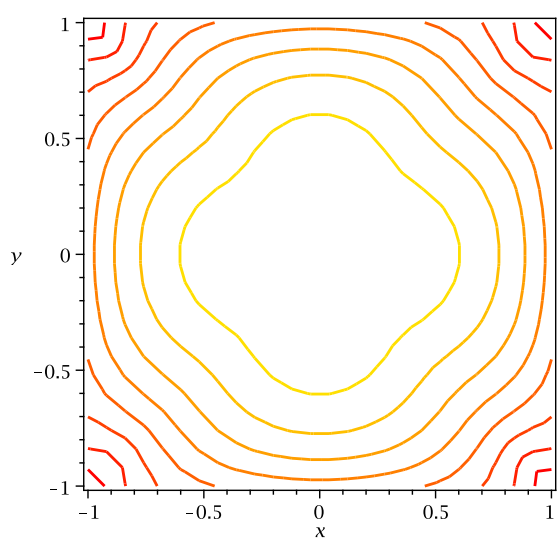


Figure 2: Contour plot of maximum principal stress

Figure 2 shows a contour plot of maximum tensile stress, showing that the maximum value occurs at the centre. This calculation was performed by reading the file principalstresses2D. The minimum value occurs at the corners and is zero.

Figure 3 shows the percentage error in the maximum principal stress predicted by the approximate solution of §7.3.1. There is no error for $\nu = 0$, but the approximate solution underestimates the maximum principal stress by 12% for $\nu = 0.5$.

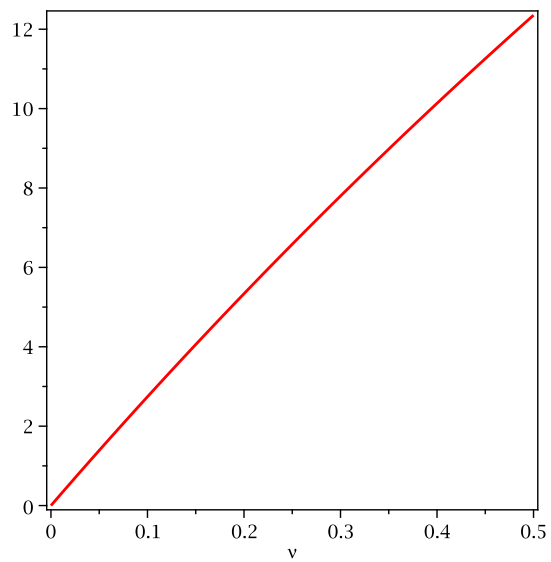


Figure 3: Percentage error in maximum principal stress in the approximate solution of §7.3.1

CHAPTER 8

8.1. A large plate with a small central hole of radius a is subjected to in-plane hydrostatic compression $\sigma_{xx} = \sigma_{yy} = -S$, $\sigma_{xy} = 0$ at the remote boundaries. Find the stress field in the plate if the surface of the hole is traction-free.

The unperturbed stress field can be described by the stress function

$$\phi = -\frac{Sx^2}{2} - \frac{Sy^2}{2} = -\frac{Sr^2}{2} \ . \tag{1}$$

This is axisymmetric ($n = 0$) and hence we add in perturbation terms from Table 8.1 that give axisymmetric stresses that decay as $r \to \infty$ — i.e.

$$\phi == -\frac{Sr^2}{2} + A\ln(r) + B\theta \ . \tag{2}$$

The stresses are then obtained as

$$\sigma_{rr} = -S + \frac{A}{r^2} \tag{3}$$

$$\sigma_{r\theta} = \frac{B}{r^2} \tag{4}$$

$$\sigma_{\theta\theta} = -S - \frac{A}{r^2} \tag{5}$$

For the hole to be traction-free, we require

$$\sigma_{rr} = \sigma_{r\theta} = 0 \ ; \quad r = a \tag{6}$$

and hence

$$-S + \frac{A}{a^2} = 0 \tag{7}$$

$$\frac{B}{a^2} = 0 \ , \tag{8}$$

with solution

$$A = Sa^2 \ ; \quad B = 0 \ . \tag{9}$$

The final stress function is therefore

$$\phi = -\frac{Sr^2}{2} + Sa^2\ln(r) \tag{10}$$

and the stress components are

$$\sigma_{rr} = -S\left(1 - \frac{a^2}{r^2}\right) \tag{11}$$

$$\sigma_{r\theta} = 0 \tag{12}$$

$$\sigma_{\theta\theta} = -S\left(1 + \frac{a^2}{r^2}\right) \ . \tag{13}$$

8.2. A large rectangular plate is loaded in such a way as to generate the unperturbed stress field

$$\sigma_{xx} = Cy^2 \ ; \ \sigma_{yy} = -Cx^2 \ ; \ \sigma_{xy} = 0 \ .$$

The plate contains a small traction-free circular hole of radius a centred on the origin. Find the perturbation in the stress field due to the hole.

For the unperturbed stress field, we have

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = Cy^2 \tag{1}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = 0 \tag{2}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = -Cx^2 \ . \tag{3}$$

From (2), we conclude that ϕ cannot contain polynomial terms with both x and y multipliers (such as x^2y^2) and hence the integral of (1) must be of the form

$$\phi = \frac{Cy^4}{12} + f(x) \tag{4}$$

where $f(x)$ is a function of x only. Substitution into (3) shows that

$$\frac{d^2 f}{dx^2} = -Cx^2 \tag{5}$$

and hence, discarding null constant and linear terms,

$$f(x) = -\frac{Cx^4}{12} \ . \tag{6}$$

In other words, the unperturbed stress function is

$$\phi = \frac{C(y^4 - x^4)}{12} \tag{7}$$

We need to check that this is biharmonic. We find that

$$\nabla^4 \phi = 2C - 2C = 0 \tag{8}$$

as required.

Casting ϕ in polar coordinates, we have

$$\phi = \frac{Cr^4(\sin^4\theta - \cos^4\theta)}{12} = \frac{Cr^4((1-\cos(2\theta))^2 - (1+\cos(2\theta))^2}{48} = -\frac{Cr^4\cos(2\theta)}{12} \ . \tag{9}$$

Notice incidentally that this has the form $r^{n+1}\cos(n\theta)$ and hence is biharmonic (as we just verified) by virtue of equations (8.23, 8.25).

To complete the solution, we add in terms from Table 8.1 that have $\cos(2\theta)$ dependence and for which the stresses decay as $r \to \infty$ — i.e.

$$\phi = -\frac{Cr^4\cos(2\theta)}{12} + Ar^{-2}\cos(2\theta) + B\cos(2\theta)\ . \tag{10}$$

The corresponding stress components (from the Table or by differentiation) are

$$\sigma_{rr} = -\frac{6A\cos(2\theta)}{r^4} - \frac{4B\cos(2\theta)}{r^2} \tag{11}$$

$$\sigma_{r\theta} = -\frac{Cr^2\sin(2\theta)}{2} - \frac{6A\sin(2\theta)}{r^4} - \frac{2B\sin(2\theta)}{r^2} \tag{12}$$

$$\sigma_{\theta\theta} = -\frac{Cr^2\cos(2\theta)}{2} + \frac{6A\cos(2\theta)}{r^4} \tag{13}$$

For the hole to be traction-free, we require

$$\sigma_{rr} = \sigma_{r\theta} = 0\ ; \quad r = a \tag{14}$$

and hence

$$-\frac{6A}{a^4} - \frac{4B}{a^2} = 0 \tag{15}$$

$$-\frac{Ca^2}{2} - \frac{6A}{a^4} - \frac{2B}{a^2} = 0\ , \tag{16}$$

with solution

$$A = -\frac{Ca^6}{6}\ ; \quad B = \frac{Ca^4}{4}\ . \tag{17}$$

The final stress function is therefore

$$\phi = -\frac{C}{12}\left(r^4 + \frac{2a^6}{r^2} - 3a^4\right)\cos(2\theta) \tag{18}$$

and the stress components are

$$\sigma_{rr} = -Ca^2\left(\frac{a^2}{r^2} - \frac{a^4}{r^4}\right)\cos(2\theta) \tag{19}$$

$$\sigma_{r\theta} = -\frac{C}{2}\left(r^2 - \frac{2a^6}{r^4} + \frac{a^4}{r^2}\right)\sin(2\theta) \tag{20}$$

$$\sigma_{\theta\theta} = -C\left(r^2 + \frac{a^6}{r^4}\right)\cos(2\theta)\ . \tag{21}$$

8.3. Figure 8.3 shows a thin uniform circular disk, which rotates at constant speed Ω about the diametral axis $y=0$, all the surfaces being traction-free. Determine the complete stress field in the disk.

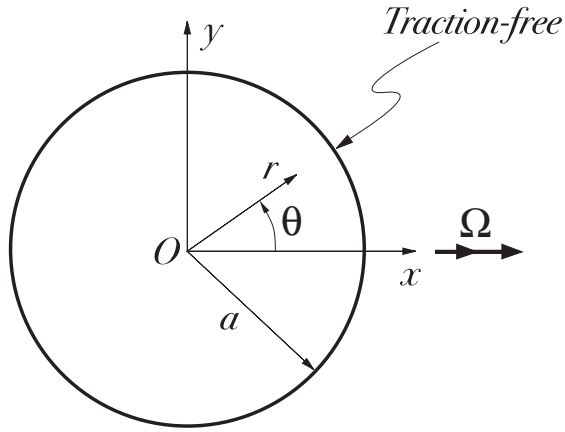


Figure 8.3: Thin disk rotating about a diametral axis.

The rotation about $y = 0$ implies accelerations

$$a_x = 0 \; ; \quad a_y = -\Omega^2 y \tag{1}$$

and hence 'D'Alembert body forces'

$$p_x = 0 \; ; \quad p_y = -\frac{\partial V}{\partial y} = \rho\Omega^2 y \; . \tag{2}$$

A suitable particular integral for the body force potential V is

$$V = -\frac{1}{2}\rho\Omega^2 y^2 = -\frac{1}{2}\rho\Omega^2 r^2 \sin^2\theta = -\frac{1}{4}\rho\Omega^2 r^2 (1 - \cos 2\theta) \; . \tag{3}$$

The stress function ϕ must satisfy

$$\nabla^4\phi = -(1-\nu)\nabla^2 V = (1-\nu)\rho\Omega^2 \; . \tag{4}$$

A particular solution is obtained most easily in x, y coordinates. Since the right hand side of (4) is a constant, any fourth degree polynomial form can be used and the simplest is the axisymmetric function $\phi = Kr^4 = K(x^2+y^2)^2$, where K is a constant to be determined. Substituting into equation (4), we find

$$\nabla^4\phi = K\nabla^4(x^4 + 2x^2y^2 + y^4) = (24 + 16 + 24)K = 64K \; . \tag{5}$$

Thus, equation (4) is satisfied if $K = (1-\nu)\rho\Omega^2/64$ — i.e.

$$\phi = \frac{(1-\nu)\rho\Omega^2 r^4}{64} \; . \tag{6}$$

To complete the solution, we must superpose a solution of the homogeneous equation — i.e. a biharmonic function. This function must vary with θ in the same way as the terms in the particular integral and/or the stress components implies by the body-force potential. Examining equations (3,6), we anticipate requiring terms that are independent of θ — i.e. axisymmetric terms — and terms that vary with $\cos 2\theta$. Furthermore, the stress components must be bounded at $r = 0$. We therefore select from Table 8.1, those terms that satisfy these requirements. They are to be found in the $n = 0$ section and in the section for general n, with n set equal to 2. The resulting stress function, including the particular solution from (6), is

$$\phi = \frac{(1-\nu)\rho\Omega^2 r^4}{64} + Ar^2 + Br^2\cos 2\theta + Cr^4\cos 2\theta \ . \tag{7}$$

The corresponding stress components are

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + V \\
&= \frac{(1-\nu)\rho\Omega^2 r^2}{16} + 2A - 2B\cos 2\theta - \frac{\rho\Omega^2 r^2}{4} + \frac{\rho\Omega^2 r^2\cos 2\theta}{4} \ ; && (8)\\
\sigma_{\theta\theta} &= \frac{\partial^2\phi}{\partial r^2} + V \\
&= \frac{3(1-\nu)\rho\Omega^2 r^2}{16} + 2A + 2B\cos 2\theta + 12Cr^2\cos 2\theta - \frac{\rho\Omega^2 r^2}{4} + \frac{\rho\Omega^2 r^2\cos 2\theta}{4} \ ; && (9)\\
\sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 2B\sin 2\theta + 6Cr^2\sin 2\theta \ . && (10)
\end{aligned}$$

We now wish to impose the boundary conditions

$$\sigma_{rr} = \sigma_{\theta r} = 0 \tag{11}$$

for all θ on $r = a$. We have

$$\begin{aligned}
\frac{(1-\nu)\rho\Omega^2 a^2}{16} + 2A - 2B\cos 2\theta - \frac{\rho\Omega^2 a^2}{4} + \frac{\rho\Omega^2 a^2\cos 2\theta}{4} &= 0 \ ; && (12)\\
2B\sin 2\theta + 6Ca^2\sin 2\theta &= 0 && (13)
\end{aligned}$$

and these equations are satisfied for all θ if and only if

$$A = \frac{(3+\nu)\rho\Omega^2 a^2}{32} \ ; \ B = \frac{\rho\Omega^2 a^2}{8} \ ; \ C = -\frac{\rho\Omega^2}{24} \ . \tag{14}$$

The final solution for the stresses is then

$$\begin{aligned}
\sigma_{rr} &= \frac{(3+\nu)\rho\Omega^2}{16}(a^2 - r^2) - \frac{\rho\Omega^2}{4}(a^2 - r^2)\cos 2\theta \ ; && (15)\\
\sigma_{r\theta} &= \frac{\rho\Omega^2}{4}(a^2 - r^2)\sin 2\theta \ ; && (16)\\
\sigma_{\theta\theta} &= -\frac{(1+3\nu)\rho\Omega^2 r^2}{16} + \frac{(3+\nu)\rho\Omega^2 a^2}{16} + \frac{\rho\Omega^2}{4}(a^2 - r^2)\cos 2\theta \ . && (17)
\end{aligned}$$

8.4. A series of experiments is conducted in which a thin plate is subjected to biaxial tension/compression, σ_1, σ_2, the plane surface of the plate being traction-free (i.e. $\sigma_3 = 0$).

Unbeknown to the experimenter, the material contains microscopic defects which can be idealized as a sparse distribution of small holes through the thickness of the plate. Show graphically the relation which will hold at yield between the tractions σ_1, σ_2 applied to the defective plate, if the Tresca (maximum shear stress) criterion applies for the undamaged material.

The stress state due to a hole in a body in biaxial tension or compression can be written down by superposition of (i) the effect of σ_1 and (ii) the effect of σ_2. Referring to Figure 1 and equation (8.76), we conclude that σ_1 produces a circumferential stress

$$\sigma_{\theta\theta} = 3\sigma_1 \quad \text{at A ;} \tag{1}$$

$$= -\sigma_1 \quad \text{at B .} \tag{2}$$

Similarly, σ_2 produces

$$\sigma_{\theta\theta} = 3\sigma_2 \quad \text{at B ;} \tag{3}$$

$$= -\sigma_2 \quad \text{at A .} \tag{4}$$

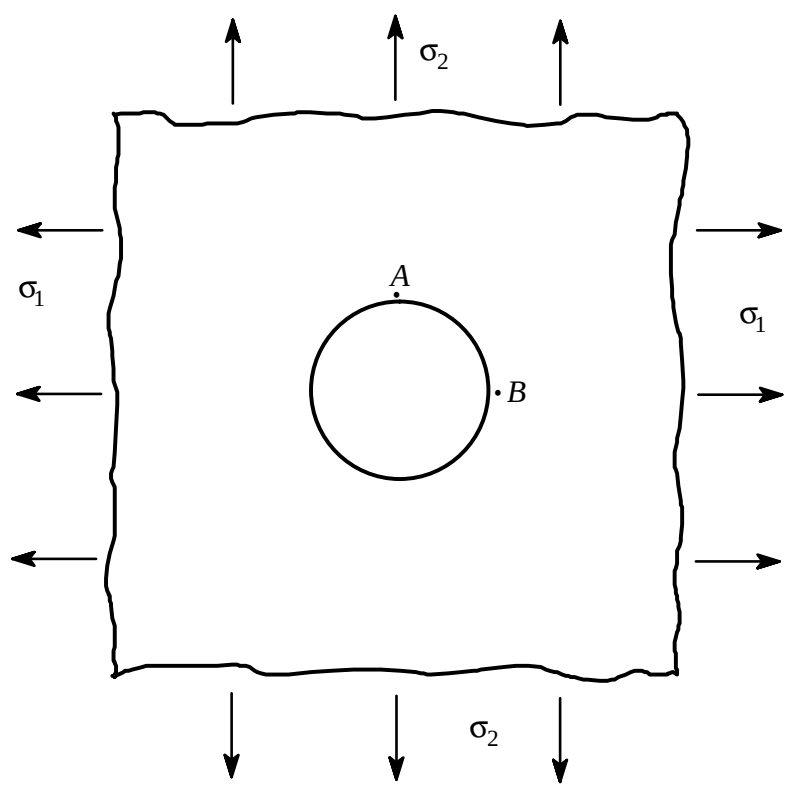


Figure 1

The maximum stress under combined loading will occur either at A or B, since (i) stresses increase as r is decreased, leading to a maximum at the edge of the hole and (ii) the stresses due to both σ_1 and σ_2 vary with $\cos(2\theta)$ and hence reach maximum values at $\theta = 0, \frac{\pi}{2}$ etc.

Summing the effects of σ_1 and σ_2 at A, B we have

$$\sigma_{\theta\theta}(A) = 3\sigma_1 - \sigma_2 \; ; \tag{5}$$

$$\sigma_{\theta\theta}(B) = 3\sigma_2 - \sigma_1 \; . \tag{6}$$

Both the other stress components at A, B are zero, since $r = a$ is a traction-free surface. Thus, A, B are both in uniaxial tension or compression. Failure will occur at A if

$$|3\sigma_1 - \sigma_2| = \sigma_0 \tag{7}$$

and at B if

$$|3\sigma_2 - \sigma_1| = \sigma_0 \; , \tag{8}$$

where σ_0 is the stress at which the unflawed material fails in uniaxial tension. This defines the four lines shown in Figure 2.

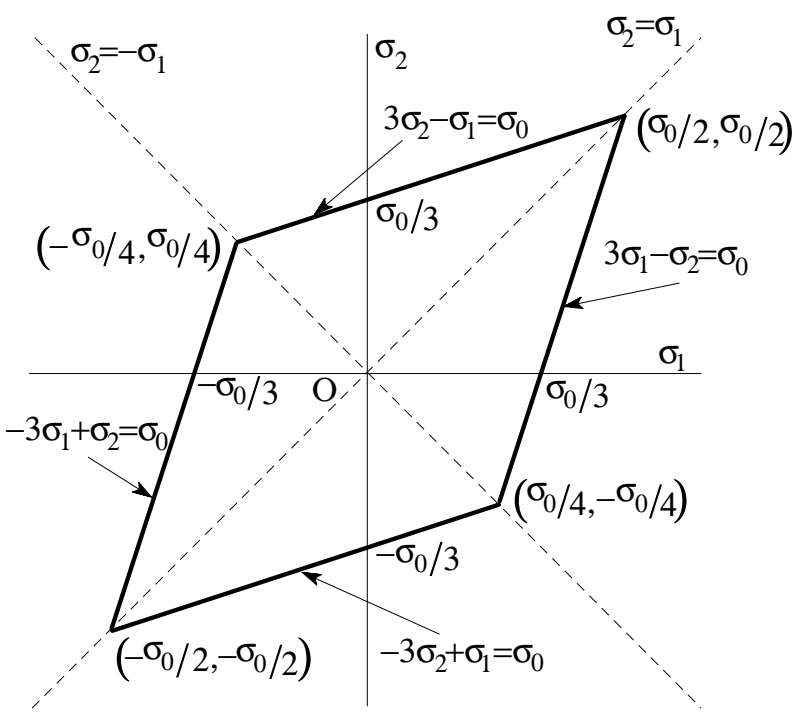


Figure 2

The experimentalist doesn't know that the material has holes and therefore would present a figure like Figure 3. In other words, he or she would plot the results in terms of the uniaxial tensile strength of the *flawed* material

$$\sigma_0^{'} = \frac{\sigma_0}{3} \ .$$

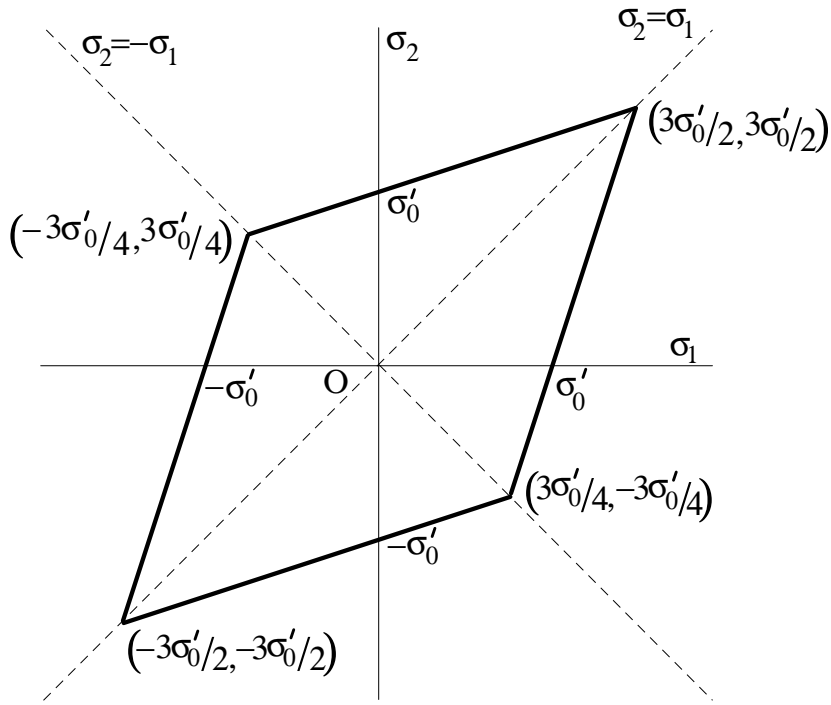


Figure 3

8.5. The circular disk $0 \le r < a$ is subjected to uniform compressive tractions $\sigma_{rr} = -S$ in the two arcs $-\pi/4 < \theta < \pi/4$ and $3\pi/4 < \theta < 5\pi/4$, the remainder of the surface $r = a$ being traction-free. Expand these tractions as a Fourier series in θ and hence develop a series solution for the stress field. Use Maple or Mathematica to produce a contour plot of the Von Mises stress σ_E, using a series truncated at 10 terms.

The loading is symmetrical about both x- and y-axes, showing that only cosine terms with even multipliers ($n = 2i$) are required. We also restrict attention to stress functions from Table 8.1 that are bounded at the origin, giving

$$\phi := A_0 r^2 + \sum_{i=1}^{\infty} (A_i r^2 i + B_i r^{(2i+1)}) \cos(2i\theta) \ . \tag{1}$$

The corresponding stress components are

$$\begin{aligned}
\sigma_{rr} &= 2A_0 - \sum_{i=1}^{\infty} (2i(2i-1)A_i r^{(2i-2)} + (2i+1)(2i-2)B_i r^{2i}) \cos(2i\theta) && (2)\\
\sigma_{r\theta} &= \sum_{i=1}^{\infty} (2i(2i-1)A_i r^{(2i-2)} + 2i(2i+1)B_i r^{2i}) \sin(2i\theta) && (3)\\
\sigma_{\theta\theta} &= 2A_0 + \sum_{i=1}^{\infty} (2i(2i-1)A_i r^{(2i-2)} + (2i+1)(2i+2)B_i r^{2i}) \cos(2i\theta) \ . && (4)
\end{aligned}$$

The traction boundary conditions on the surface $r = a$ in the range[1] $-\pi/2 < \theta < \pi/2$ can be written

$$\begin{aligned}
\sigma_{r\theta}(a,\theta) &= 0 && (5)\\
\sigma_{rr}(a,\theta) &= -S \ ; \quad -\pi/4 < \theta < \pi/4 && (6)\\
&= 0 \ ; \quad \pi/4 < |\theta| < \pi/2 \ . && (7)
\end{aligned}$$

The unmixed condition (5) requires that

$$2i(2i-1)A_i a^{(2i-2)} + 2i(2i+1)B_i a^{2i} = 0 \tag{8}$$

whilst the mixed condition leads to the Fourier equation

$$\begin{aligned}
2A_0 - \sum_{i=1}^{\infty} (2i(2i-1)A_i r^{(2i-2)} + (2i+1)(2i-2)B_i r^{2i}) \cos(2i\theta) &= -S \ ; \ -\pi/4 < \theta < \pi/4\\
&= 0 \ ; \ \pi/4 < |\theta| < \pi/2 \ .
\end{aligned}$$

[1] The boundary conditions in the other half of the disk will be taken care of by the assumed symmetry conditions.

To invert this equation, we multiply both sides by $\cos(2j\theta)$ and integrate in the range $-\pi/2 < \theta < \pi/2$ obtaining

$$\begin{aligned} 2A_0\pi &= -S\int_{-\pi/4}^{\pi/4} d\theta \\ &= -\frac{S\pi}{2} \qquad (9) \\ (2j(2j-1)A_j a^{(2j-2)} + (2j+1)(2j-2)B_j a^{2j})\frac{\pi}{2} &= -S\int_{-\pi/4}^{\pi/4}\cos(2j\theta)d\theta \\ &= -\frac{S}{j}\sin\left(\frac{\pi j}{2}\right) \;;\quad j \neq 0 \,. \qquad (10) \end{aligned}$$

Now if j is even, $\sin(\pi j/2) = 0$ and equations (8,10) will give $A_j = B_j = 0$. If j is odd, we can write

$$j = 2m - 1 \,, \qquad (11)$$

where m is an integer, in which case

$$\sin\left(\frac{\pi j}{2}\right) = \sin\left(\frac{(2m-1)\pi}{2}\right) = -(-1)^m \,. \qquad (12)$$

Equations (8,10) can then be rewritten as

$$\begin{aligned} 2(2m-1)(4m-3)A_m a^{(4m-4)} + 2(2m-1)(4m-1)B_m a^{(4m-2)} &= 0 \\ (2(2m-1)(4m-3)A_m a^{(4m-4)} + (4m-1)(4m-4)B_m a^{(4m-2)})\frac{\pi}{2} &= \frac{(-1)^m S}{(2m-1)} \,. \end{aligned}$$

Solving for A_m, B_m, we have

$$A_m = \frac{(-1)^m S}{\pi(2m-1)(4m-3)a^{(4m-4)}} \;;\quad B_m = -\frac{(-1)^m S}{\pi(2m-1)(4m-1)a^{(4m-2)}} \qquad (13)$$

and (9) gives

$$A_0 = -\frac{S}{4} \,. \qquad (14)$$

The complete solution for the stress components is therefore

$$\begin{aligned} \sigma_{rr} &= -\frac{S}{2} - \frac{2S}{\pi}\sum_{m=1}^{\infty}(-1)^m\left\{\left(\frac{r}{a}\right)^{(4m-4)} - \frac{(4m-3)}{(2m-1)}\left(\frac{r}{a}\right)^{(4m-2)}\right\}\cos((4m-2)\theta) \\ \sigma_{r\theta} &= \frac{2S}{\pi}\sum_{m=1}^{\infty}(-1)^m\left\{\left(\frac{r}{a}\right)^{(4m-4)} - \left(\frac{r}{a}\right)^{(4m-2)}\right\}\sin((4m-2)\theta) \\ \sigma_{\theta\theta} &= -\frac{S}{2} + \frac{2S}{\pi}\sum_{m=1}^{\infty}(-1)^m\left\{\left(\frac{r}{a}\right)^{(4m-4)} - \frac{(4m-1)}{(2m-1)}\left(\frac{r}{a}\right)^{(4m-2)}\right\}\cos((4m-2)\theta) \,. \end{aligned}$$

A contour plot of σ_E is shown in Figure 1.

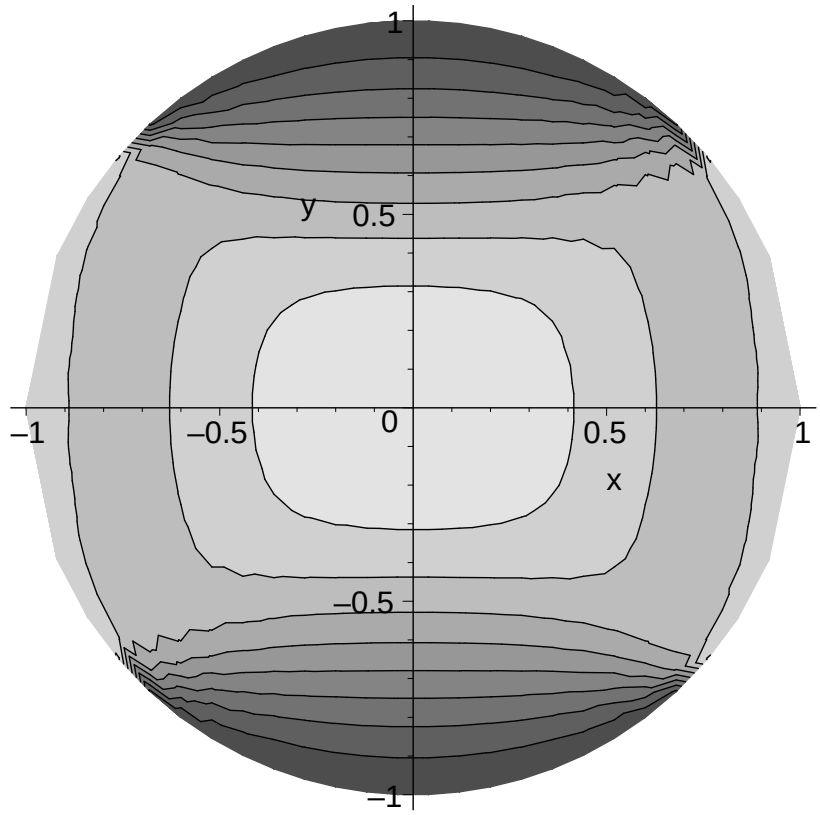


Figure 1

The variation of σ_E around the right hand half of the disk is shown in Figure 2. It is clear from these figures that the region between the two loaded segments is approximately in uniaxial compression, whilst there is a steep stress gradient to zero stress in the upper and lower regions of the disk.

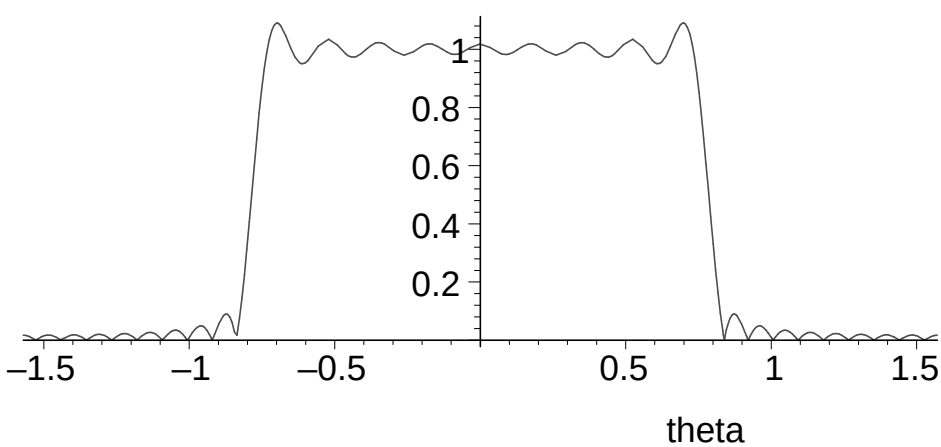


Figure 2

8.6. A hole of radius a in a large elastic plate is loaded only by a self-equilibrated distribution of normal pressure $p(\theta)$ that varies around the circumference of the hole. By expanding $p(\theta)$ as a Fourier series in θ and using Table 8.1, show that the hoop stress $\sigma_{\theta\theta}$ at the edge of the hole is given by

$$\sigma_{\theta\theta}(a,\theta) = 2\bar{p} - p(\theta) \ ,$$

where

$$\bar{p} = \frac{1}{2\pi}\int_0^{2\pi} p(\theta)d\theta$$

is the mean value of $p(\theta)$.

The boundary conditions at $r = a$ are

$$\begin{aligned} \sigma_{rr}(a,\theta) &= -p(\theta) \\ \sigma_{r\theta}(a,\theta) &= 0 \end{aligned}$$

We expand $p(\theta)$ as the Fourier series

$$p(\theta) = p_0 + \sum_{n=2}^{\infty}\left[p_n^c\cos(n\theta) + p_n^s\sin(n\theta)\right] \ .$$

Notice that we do not include any terms corresponding to $n = 1$ since these would correspond to non-zero force resultants and we are told that the pressure distribution is self-equilibrated. The coefficients in this series can be obtained using the classical Fourier series inversion. In particular, we have

$$\int_0^{2\pi} p_0 d\theta = \int_0^{2\pi} p(\theta)d\theta$$

and hence

$$p_0 = \frac{1}{2\pi}\int_0^{2\pi} p(\theta)d\theta = \bar{p} \ .$$

Consider the representative non-axisymmetric term $p_n^c\cos(n\theta)$. Corresponding terms from the Michell solution involve the stress function

$$\phi = Ar^{-n}\cos(n\theta) + Br^{-n+2}\cos(n\theta) \ ,$$

where A, B are arbitrary constants and we have included only those terms from Table 8.1 that decay as $r \to \infty$. From the Table, we then have

$$\sigma_{r\theta} = -An(n+1)r^{-n-2}\sin(n\theta) - Bn(n-1)r^{-n}\sin(n\theta)$$

and this will be zero on $r = a$ if

$$An(n+1)a^{-n-2} + Bn(n-1)a^{-n} = 0 \quad \text{or} \quad A(n+1) = -B(n-1)a^2 \ .$$

The corresponding value of $\sigma_{rr} - \sigma_{\theta\theta}$ at $r = a$ is

$$\begin{aligned}
\sigma_{rr} - \sigma_{\theta\theta} &= -An(n+1)a^{-n-2}\cos(n\theta) - B(n+2)(n-1)a^{-n}\cos(n\theta) \\
&\quad -An(n+1)a^{-n-2}\cos(n\theta) - B(n-1)(n-2)a^{-n}\cos(n\theta) \\
&= 2B(n-1)na^{-n}\cos(n\theta) - B(n+2)(n-1)a^{-n}\cos(n\theta) \\
&\quad -B(n-1)(n-2)a^{-n}\cos(n\theta) \ ,
\end{aligned}$$

using the above substitution for $A(n+1)$. However,

$$2B(n-1)n - B(n+2)(n-1) - B(n-1)(n-2) = B(n-1)(2n-(n+2)-(n-2)) = 0 \ .$$

Thus, this term causes $\sigma_{rr} - \sigma_{\theta\theta} = 0$ at $r = a$. An exactly similar result would be obtained for the term involving $\sin(2\theta)$. Thus, for that part of the Fourier series excluding the axisymmetric term $\bar{p}$ — i.e. for $p(\theta) - \bar{p}$,

$$\sigma_{\theta\theta}(a,\theta) = \sigma_{rr}(a,\theta) = -(p(\theta) - \bar{p}) \ .$$

For the axisymmetric term $\bar{p}$, the only stress function giving stresses that decay at infinity is

$$\phi = C\ln(r) \ ,$$

for which

$$\sigma_{rr} = -\sigma_{\theta\theta} = \frac{C}{r^2} \ .$$

Thus, the boundary condition gives

$$\sigma_{rr}(a,\theta) = -\bar{p} = \frac{C}{a^2} \quad \text{and} \quad \sigma_{\theta\theta}(a,\theta) = -\frac{C}{a^2} = \bar{p} \ .$$

Superposing these effects, we deduce that the entire pressure distribution defined by the Fourier series causes a hoop stress

$$\sigma_{\theta\theta}(a,\theta) = -(p(\theta) - \bar{p}) + \bar{p} = 2\bar{p} - p(\theta) \ .$$

CHAPTER 9

9.1. Find the displacement field corresponding to the stress field of equations (5.78–5.80). The beam is simply-supported at $x=\pm a$. You will need to impose this in the form of appropriate weak conditions on the displacements.

Find the vertical displacement at the points $(0,0)$, $(0,-b)$, $(0,b)$ and compare them with the predictions of the elementary bending theory.

To determine the displacement field, we have

$$\begin{aligned}
e_{xx} &= \frac{\partial u_x}{\partial x} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} \\
&= \frac{p}{20Eb^3}(15x^2y - 5(2+\nu)y^3 - 15a^2y + 3(2+5\nu)b^2y + 10\nu b^3) \qquad (1)\\
e_{yy} &= \frac{\partial u_y}{\partial y} = \frac{\sigma_{yy}}{E} - \frac{\nu\sigma_{xx}}{E} \\
&= \frac{p}{20Eb^3}(5(1+2\nu)y^3 - 15\nu x^2y + 15\nu a^2y - 3(5+2\nu)b^2y - 10b^3) \qquad (2)\\
2e_{xy} &= \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = \frac{2\sigma_{xy}(1+\nu)}{E} = \frac{3p(1+\nu)x}{2Eb^3}(b^2-y^2)\ . \qquad (3)
\end{aligned}$$

using (5.78–5.80).

Integrating the first two equations, we have

$$\begin{aligned}
u_x &= \frac{p}{20Eb^3}(5x^3y - 5(2+\nu)xy^3 - 15a^2xy + 3(2+5\nu)b^2xy + 10\nu b^3x) + f(y) \\
u_y &= \frac{p}{20Eb^3}\left(\frac{5(1+2\nu)y^4}{4} - \frac{15\nu x^2y^2}{2} + \frac{15\nu a^2y^2}{2} - \frac{3(5+2\nu)b^2y^2}{2} - 10b^3y\right) + g(x)
\end{aligned}$$

and substituting into (3), we have

$$-\frac{3p\nu xy^2}{4Eb^3} + g'(x) + \frac{p}{20Eb^3}(5x^3 - 15(2+\nu)xy^2 - 15a^2x + 3(2+5\nu)b^2x) + f'(y) = \frac{3p(1+\nu)}{2Eb^3}(xb^2 - xy^2)\ , \qquad (4)$$

which simplifies to

$$g'(x) + f'(y) + \frac{p}{20Eb^3}(5x^3 - 15a^2x - 3(8+5\nu)b^2x) = 0\ . \qquad (5)$$

Separating functions of x and y, we have

$$g'(x) + \frac{p}{20Eb^3}(5x^3 - 15a^2x - 3(8+5\nu)b^2x) = -f'(y) = C\ , \qquad (6)$$

with solution

$$g(x) = Cx + A - \frac{p}{20Eb^3}\left(\frac{5x^4}{4} - \frac{15a^2x^2}{2} - \frac{3(8+5\nu)b^2x^2}{2}\right) \; ; \quad f(y) = -Cy + B \; . \quad (7)$$

The displacements are therefore

$$\begin{aligned} u_x &= \frac{p}{20Eb^3}(5x^3y - 5(2+\nu)xy^3 - 15a^2xy + 3(2+5\nu)b^2xy + 10\nu b^3x) - Cy + B \\ u_y &= \frac{p}{20Eb^3}\left(\frac{5(1+2\nu)y^4}{4} - \frac{15\nu x^2y^2}{2} + \frac{15\nu a^2y^2}{2} - \frac{3(5+2\nu)b^2y^2}{2} - 10b^3y\right) \\ &\quad + Cx + A - \frac{p}{20Eb^3}\left(\frac{5x^4}{4} - \frac{15a^2x^2}{2} - \frac{3(8+5\nu)b^2x^2}{2}\right) \end{aligned}$$

The problem is symmetrical about $x = 0$, so we expect u_x to be odd and u_y to be even in x. Thus, $B = C = 0$. To determine the remaining rigid-body constant, we impose the weak condition

$$\int_{-b}^{b} u_y(a, y)dy = 0 \quad (8)$$

and hence

$$\begin{aligned} &\frac{p}{20Eb^3}\left(\frac{(1+2\nu)b^5}{2} - 5\nu a^2b^3 + 5\nu a^2b^3 - (5+2\nu)b^5\right) \\ &+2Ab - \frac{p}{20Eb^3}\left(\frac{5a^4b}{2} - 15a^4b - 3(8+5\nu)b^3a^2\right) = 0 \; . \end{aligned}$$

Solving for A, we have

$$A = \frac{p}{80Eb^3}((9+2\nu)b^4 - 6(8+5\nu)a^2b^2 - 25a^4) \; . \quad (9)$$

The vertical displacements at $(0, 0)$, $(0, -b)$, $(0, b)$ are

$$u_y(0,0) = -\frac{p}{80Eb^3}(25a^4 + 6(8+5\nu)a^2b^2 - (9+2\nu)b^4) \quad (10)$$

$$u_y(0,-b) = -\frac{p}{80Eb^3}(25a^4 + 48a^2b^2 - 24b^4) \quad (11)$$

$$u_y(0,b) = -\frac{p}{80Eb^3}(25a^4 + 48a^2b^2 + 56b^4) \; . \quad (12)$$

The elementary bending theory gives

$$u_y(0) = -\frac{5pa^4}{16Eb^3} \quad (13)$$

and corresponds to the first term in each of the above expressions. The remaining terms are smaller than the leading term in the ratio $48a^2/25b^2$, which will be small as long as $a/b \ll 1$.

9.2. The rectangular plate $-a<x<a$, $-b<y<b$ is bonded to rigid supports at $x=\pm a$, the edges $y=\pm b$ being traction-free. The plate is loaded only by its own weight in the negative y-direction.

(i) Find a solution for the stress field assuming uniform shear tractions on $x=\pm a$. Use strong boundary conditions on $x=\pm a$ and weak conditions on $y=\pm b$.

(ii) Find the displacements corresponding to this stress field. Do the resulting expressions permit the built-in boundary condition at $x=\pm a$ to be satisfied in the strong sense?

(iii) What are the restrictions on the ratio a/b for this solution to be a reasonable approximation to the physical problem?

The gravitational force can be described by the body force potential

$$V = \rho g y \ .$$

The problem is symmetric in x and antisymmetric in y, so the lowest order polynomial possible is

$$\phi = Ax^2 y + By^3 \ . \tag{1}$$

The corresponding stress components are

$$\begin{aligned} \sigma_{xx} &= 6By + \rho g y & (2) \\ \sigma_{xy} &= -2Ax & (3) \\ \sigma_{yy} &= 2Ay + \rho g y \ . & (4) \end{aligned}$$

If the plate is supported by shear tractions only on $x=\pm a$, we have

$$\sigma_{xx}(a,y) = 0 \tag{5}$$

and hence

$$B = -\frac{\rho g}{6} \ . \tag{6}$$

Equation (3) ensures that the shear traction will be constant on $x=\pm a$, so the strong boundary conditions for stress are satisfied.

We also have the weak condition

$$\int_{-a}^{a} \sigma_{yy}(x,b) = 0 \ , \tag{7}$$

from which

$$A = -\frac{\rho g}{2} \ . \tag{8}$$

The complete stress field is therefore

$$\sigma_{xx} = 0 \tag{9}$$
$$\sigma_{xy} = \rho g x \tag{10}$$
$$\sigma_{yy} = 0 \ , \tag{11}$$

from (2–4, 6, 8).

To determine the displacement field, we have

$$e_{xx} = \frac{\partial u_x}{\partial x} = \frac{\sigma_{xx}}{E} - \frac{\nu \sigma_{yy}}{E} = 0 \tag{12}$$
$$e_{yy} = \frac{\partial u_y}{\partial y} = \frac{\sigma_{yy}}{E} - \frac{\nu \sigma_{xx}}{E} = 0 \tag{13}$$
$$2e_{xy} = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} = \frac{2\sigma_{xy}(1+\nu)}{E} = \frac{2(1+\nu)\rho g x}{E} \ . \tag{14}$$

Integrating the first two equations, we have

$$u_x = F(y) \ ; \quad u_y = G(x) \tag{15}$$

and substituting into (14), we have

$$F'(y) + G'(x) = \frac{2(1+\nu)\rho g x}{E} \ . \tag{16}$$

Separating functions of x and y, we have

$$G'(x) = \frac{2(1+\nu)\rho g x}{E} + C \ ; \quad f'(y) = -C \ , \tag{17}$$

with solution

$$G(x) = \frac{(1+\nu)\rho g x^2}{E} + Cx + A \ ; \quad F(y) = -Cy + B \tag{18}$$

and hence

$$u_x = -Cy + B \ ; \quad u_y = \frac{(1+\nu)\rho g x^2}{E} + Cx + A \ . \tag{19}$$

We require

$$u_x = u_y = 0 \ ; \quad x = \pm a \tag{20}$$

and hence

$$C = 0 \ ; \quad B = 0 \ ; \quad A = -\frac{(1+\nu)\rho g a^2}{E} \ . \tag{21}$$

The final solution is therefore

$$u_x = 0 \ ; \quad u_y = -\frac{(1+\nu)\rho g (a^2 - x^2)}{E} \ . \tag{22}$$

Notice that the displacement conditions are satisfied in the strong sense on $x = \pm a$, as required.

9.3. A state of pure shear, $\sigma_{xy} = S$ in a large plate is perturbed by the presence of a rigid circular inclusion in the region $r < a$. The inclusion is perfectly bonded to the plate and is prevented from moving, so that $u_r = u_\theta = 0$ at $r = a$.

Find the complete stress field in the plate and hence determine the stress concentration factor due to the inclusion based on (i) the maximum shear stress criterion or (ii) the maximum tensile stress criterion.

Is it necessary to apply a force or a moment to the inclusion to prevent it from moving?

As in the case of the circular hole in shear (§8.3.2), the unperturbed solution is $\sigma_{xy} = S$; $\sigma_{xx} = \sigma_{yy} = 0$ and

$$\phi = -Sxy = -\frac{Sr^2 \sin 2\theta}{2} \tag{1}$$

(see equation (8.37)).

We select from Table 8.1, the stress function terms that involve stress components decaying with r and varying in the same way with θ — (i.e. $\phi \sim \sin 2\theta$). Thus,

$$\phi = -\frac{Sr^2 \sin 2\theta}{2} + A \sin 2\theta + \frac{B \sin 2\theta}{r^2} \ . \tag{2}$$

The *displacements* associated with this stress function can be written down from Table 9.1 using the last three lines with $n = 2$. We obtain

$$2\mu u_r = Sr \sin 2\theta + \frac{A(\kappa+1)\sin 2\theta}{r} + \frac{2B \sin 2\theta}{r^3} \ ; \tag{3}$$

$$2\mu u_\theta = Sr \cos 2\theta + \frac{A(\kappa-1)\cos 2\theta}{r} - \frac{2B \cos 2\theta}{r^3} \ . \tag{4}$$

We have to satisfy the boundary conditions $u_r = u_\theta = 0$ at $r = a$ and hence

$$\frac{A(\kappa+1)}{a} + \frac{2B}{a^3} = -Sa \ ; \tag{5}$$

$$\frac{A(\kappa-1)}{a} - \frac{2B}{a^3} = -Sa \ . \tag{6}$$

The solution of these two simultaneous equations is

$$A = -\frac{sa^2}{\kappa} \quad ; \quad B = \frac{Sa^4}{2\kappa} \tag{7}$$

and the complete stress field can now be written down from equations (8.39–8.41) by substituting for A, B, with the result

$$\sigma_{rr} = S\left(1 + \frac{4a^2}{\kappa r^2} - \frac{3a^4}{\kappa r^4}\right) \sin 2\theta \ ; \tag{8}$$

$$\sigma_{r\theta} = S\left(1 - \frac{2a^2}{\kappa r^2} + \frac{3a^4}{\kappa r^4}\right)\cos 2\theta \; ; \tag{9}$$

$$\sigma_{\theta\theta} = S\left(-1 + \frac{3a^4}{\kappa r^4}\right)\sin 2\theta \; . \tag{10}$$

Clearly the maximum stress will occur at $r = a$, since stresses increase as r is decreased. At this radius, we have

$$\sigma_{rr} = S\left(1 + \frac{1}{\kappa}\right)\sin 2\theta \; ; \tag{11}$$

$$\sigma_{r\theta} = S\left(1 + \frac{1}{\kappa}\right)\cos 2\theta \; ; \tag{12}$$

$$\sigma_{\theta\theta} = S\left(-1 + \frac{3}{\kappa}\right)\sin 2\theta \; . \tag{13}$$

The maximum in-plane shear stress is

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_{rr} - \sigma_{\theta\theta}}{2}\right)^2 + \sigma_{r\theta}^2} \tag{14}$$

$$= S\sqrt{\left(1 - \frac{1}{\kappa}\right)^2 \sin^2 2\theta + \left(1 + \frac{1}{\kappa}\right)^2 \cos^2 2\theta} \tag{15}$$

$$= S\sqrt{\left(1 + \frac{1}{\kappa}\right)^2 - \frac{4}{\kappa}\sin^2 2\theta} \tag{16}$$

and this is maximum when $\theta = 0$, being

$$\tau_{\max} = S\left(1 + \frac{1}{\kappa}\right) \; . \tag{17}$$

The maximum tensile stress is

$$\sigma_1 = \frac{\sigma_{rr} + \sigma_{\theta\theta}}{2} + \sqrt{\left(\frac{\sigma_{rr} - \sigma_{\theta\theta}}{2}\right)^2 + \sigma_{r\theta}^2} \tag{18}$$

$$= \frac{2S\sin 2\theta}{\kappa} + S\sqrt{\left(1 + \frac{1}{\kappa}\right)^2 - \frac{4}{\kappa}\sin^2 2\theta} \; . \tag{19}$$

To find the value of θ at which this is a maximum, write $\sin 2\theta = x$ in equation (19) and then differentiate with respect to x. The maximum will occur when

$$\frac{\partial \sigma_1}{\partial x} = 0 = \frac{2s}{\kappa} + \frac{\left(-\frac{4}{\kappa}\right) Sx}{\sqrt{\left(1 + \frac{1}{\kappa}\right)^2 - \frac{4x^2}{\kappa}}} \; . \tag{20}$$

In other words

$$\sqrt{\left(1 + \frac{1}{\kappa}\right)^2 - \frac{4x^2}{\kappa}} = 2x$$

$$\left(1+\frac{1}{\kappa}\right)^2 - \frac{4x^2}{\kappa} = 4x^2$$

$$\left(1+\frac{1}{\kappa}\right)^2 = 4x^2\left(1+\frac{1}{\kappa}\right)$$

and hence

$$x = \pm\frac{1}{2}\sqrt{1+\frac{1}{\kappa}}\ . \tag{21}$$

Thus, the maximum tensile stress occurs at an angle $\theta = \pm\theta_0$, where

$$\sin 2\theta_0 = \frac{1}{2}\sqrt{1+\frac{1}{\kappa}}\ . \tag{22}$$

Notice incidentally that $\kappa = (3-\nu)/(1+\nu)$ (plane stress) or $3-4\nu$ (plane strain) and in both cases $\kappa > 1$ for practical values of ν, so $x = \sin 2\theta_0 < 1/\sqrt{2}$ and $\theta_0 < \pi/8$.

The value of the maximum stress is obtained by substituting $\theta = \theta_0$ into equation (19), with the result

$$\sigma_1(\max) = \frac{S}{\kappa}\sqrt{1+\frac{1}{\kappa}} + S\sqrt{\left(1+\frac{1}{\kappa}\right)^2 - \frac{1}{\kappa}\left(1+\frac{1}{\kappa}\right)} \tag{23}$$

$$= S\left(1+\frac{1}{\kappa}\right)^{\frac{3}{2}}\ . \tag{24}$$

It is clear that the maximum tensile stress also has stationary points at $\theta = 0, \pi/4$, but elementary calculations show that at these points

$$\sigma_1 = S\left(1+\frac{1}{\kappa}\right)\ ,$$

which must therefore correspond to local minima.

Thus, in summary, the stress concentration factor is

$$\text{S.C.F.} = \left(1+\frac{1}{\kappa}\right) \quad ; \quad \text{Maximum shear stress criterion;} \tag{25}$$

$$= \left(1+\frac{1}{\kappa}\right)^{\frac{3}{2}} \quad ; \quad \text{Maximum tensile stress criterion.} \tag{26}$$

For Poisson's ratio in the range $0 < \nu < 0.5$, $3 > \kappa > 1$. We therefore have $2 < \text{S.C.F.(shear)} < \frac{4}{3}$ and $2.828 > \text{S.C.F.(tension)} > 1.540$. In all cases, the stress concentration factor is greater when using the maximum tensile stress criterion, as was also the case for the hole in a plate in shear.

Finally, no force or moment is needed to prevent the inclusion from moving. Because of the orthogonality of the terms in a Fourier series, the equilibrium equations (9.36–9.38) will only make non-zero contributions on terms with $n = 0, 1$.

9.4. A thin annular disk, inner radius b and outer radius a rotates at constant speed Ω about the axis of symmetry, all the surfaces being traction-free. Determine the stress field in the disk.

The body force for uniform rotation at speed Ω can be written in terms of a potential

$$V = -\frac{\rho\Omega^2 r^2}{2} \ , \tag{1}$$

from equation (7.22). The stress function must satisfy equation (7.8) which here takes the form

$$\nabla^4\phi = 2(1-\nu)\rho\Omega^2 \ .$$

Since the problem is axisymmetric, we seek a particular solution of this equation in the form $\phi = Cr^4$, and hence

$$64C = 2(1-\nu)\rho\Omega^2 \quad \text{or} \quad C = \frac{(1-\nu)\rho\Omega^2 r^4}{32} \ .$$

A particular solution for the stress field is therefore

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + V = -\frac{(3+\nu)\rho\Omega^2 r^2}{8} \tag{2}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0 \tag{3}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} + V = -\frac{(1+3\nu)\rho\Omega^2 r^2}{8} \ . \tag{4}$$

This solution is bounded at $r = 0$ and can therefore be used for the solid disk which is simply-connected. It follows that the displacement field associated with these stresses must be single-valued and it is not necessary to calculate them to ensure this.

To complete the solution, we must superpose functions from Tables 8.1, 9.1 so as to make the surfaces $r = a, b$ traction-free and the additional displacements single-valued. The solution is axisymmetric and involves no shear stresses $\sigma_{r\theta}$ so appropriate stress functions from Table 8.1 are r^2, $r^2\ln(r)$ and $\ln(r)$. However, examining these three terms in Table 9.1, we see that $r^2\ln(r)$ is the only one that gives multivalued displacements (with a θ-multiplier), so we exclude it, leaving

$$\phi = Ar^2 + B\ln(r) \tag{5}$$

for the corrective (homogeneous) solution. Adding the stresses from these terms in Table 8.1 to those from the particular solution, we have

$$\sigma_{rr} = -\frac{(3+\nu)\rho\Omega^2 r^2}{8} + 2A + \frac{B}{r^2} \tag{6}$$

$$\sigma_{r\theta} = 0 \tag{7}$$

$$\sigma_{\theta\theta} = -\frac{(1+3\nu)\rho\Omega^2 r^2}{8} + 2A - \frac{B}{r^2} \ . \tag{8}$$

We require $\sigma_{rr} = \sigma_{r\theta} = 0$ on $r = a, b$ and hence

$$-\frac{(3+\nu)\rho\Omega^2 a^2}{8} + 2A + \frac{B}{a^2} = 0 \tag{9}$$

$$-\frac{(3+\nu)\rho\Omega^2 b^2}{8} + 2A + \frac{B}{b^2} = 0 \ , \tag{10}$$

with solution

$$A = \frac{(3+\nu)\rho\Omega^2(a^2+b^2)}{16} \ ; \quad B = -\frac{(3+\nu)\rho\Omega^2 a^2 b^2}{8} \ . \tag{11}$$

The final stress field is therefore

$$\begin{aligned} \sigma_{rr} &= \frac{(3+\nu)\rho\Omega^2}{8}\left(a^2+b^2-r^2+\frac{a^2b^2}{r^2}\right) = \frac{(3+\nu)\rho\Omega^2(a^2-r^2)(r^2-b^2)}{8r^2} \\ \sigma_{r\theta} &= 0 \\ \sigma_{\theta\theta} &= \frac{\rho\Omega^2}{8}\left[(3+\nu)\left(a^2+b^2+\frac{a^2b^2}{r^2}\right)-(1+3\nu)r^2\right] \ . \end{aligned} \tag{12}$$

We note that the maximum tensile stress occurs at $r = b$ (the inner radius) and is

$$\sigma_{\theta\theta}(r=a) = \frac{\rho\Omega^2[(3+\nu)a^2+(1-\nu)b^2]}{4} \ . \tag{13}$$

9.5. A large rectangular plate is subjected to simple bending, such that the stress field is given by

$$\sigma_{xx} = Cy \ ; \ \sigma_{xy} = \sigma_{yy} = 0 \ .$$

The plate contains a small traction-free circular hole of radius a centred on the origin. Find the perturbation in the stress field due to the hole.

The unperturbed field is described by the stress function

$$\phi = \frac{Cy^3}{6} = \frac{Cr^3 \sin^3\theta}{6} = \frac{Cr^3}{24}(3\sin\theta - \sin(3\theta)) \ . \tag{1}$$

For the corrective solution, we therefore require terms giving stresses varying with $\sin\theta$ and $\sin(3\theta)$ that decay as $r \to \infty$. The most general stress function of this kind is

$$\phi = \frac{C_1 \sin(3\theta)}{r^3} + \frac{C_2 \sin(3\theta)}{r} + \frac{C_3 \sin\theta}{r} + C_4 r \ln(r) \sin\theta + C_5 r\theta\cos\theta \ . \tag{2}$$

Notice that there are 3 functions giving $\sin\theta$ terms in the stresses, but for this term we shall have an extra condition requiring that the displacements be single valued.

The corresponding stress components, including the unperterbed solution (1) are

$$\begin{aligned}
\sigma_{rr} &= \frac{Cr(\sin\theta + \sin(3\theta))}{4} - \frac{12C_1 \sin(3\theta)}{r^5} - \frac{10C_2 \sin(3\theta)}{r^3} - \frac{2C_3 \sin\theta}{r^3} \\
&\quad + \frac{C_4 \sin\theta}{r} - \frac{2C_5 \sin\theta}{r} && (3) \\
\sigma_{r\theta} &= \frac{Cr(-\cos\theta + \cos(3\theta))}{4} + \frac{12C_1 \cos(3\theta)}{r^5} + \frac{6C_2 \cos(3\theta)}{r^3} + \frac{2C_3 \cos\theta}{r^3} \\
&\quad - \frac{C_4 \cos\theta}{r} && (4) \\
\sigma_{\theta\theta} &= \frac{Cr(3\sin\theta - \sin(3\theta))}{4} + \frac{12C_1 \sin(3\theta)}{r^5} + \frac{2C_2 \sin(3\theta)}{r^3} + \frac{2C_3 \sin\theta}{r^3} \\
&\quad + \frac{C_4 \sin\theta}{r} && (5)
\end{aligned}$$

On the boundary $r = a$, we require $\sigma_{rr} = \sigma_{r\theta} = 0$ and hence

$$\begin{aligned}
\frac{Ca(\sin\theta + \sin(3\theta))}{4} - \frac{12C_1 \sin(3\theta)}{a^5} - \frac{10C_2 \sin(3\theta)}{a^3} - \frac{2C_3 \sin\theta}{a^3} \\
+ \frac{C_4 \sin\theta}{a} - \frac{2C_5 \sin\theta}{a} &= 0 && (6) \\
\frac{Ca(-\cos\theta + \cos(3\theta))}{4} + \frac{12C_1 \cos(3\theta)}{a^5} + \frac{6C_2 \cos(3\theta)}{a^3} + \frac{2C_3 \cos\theta}{a^3} \\
- \frac{C_4 \cos\theta}{a} &= 0 && (7)
\end{aligned}$$

and hence, equating coefficients of $\sin\theta, \cos\theta, \sin(3\theta), \cos(3\theta)$,

$$\frac{Ca}{4} - \frac{2C_3}{a^3} + \frac{C_4}{a} - \frac{2C_5}{a} = 0 \tag{8}$$

$$\frac{Ca}{4} - \frac{12C_1}{a^5} - \frac{10C_2}{a^3} = 0 \tag{9}$$

$$-\frac{Ca}{4} + \frac{2C_3}{a^3} - \frac{C_4}{a} = 0 \tag{10}$$

$$\frac{Ca}{4} + \frac{12C_1}{a^5} + \frac{6C_2}{a^3} = 0\ . \tag{11}$$

In addition, there will be a discontinuity in u_r at $\theta = 0$ unless

$$-C_4(\kappa + 1) + C_5(\kappa - 1) = 0\ . \tag{12}$$

Solving (8–12) for the unknown constants $C_1, ...C_5$, we have

$$C_1 = -\frac{Ca^6}{12}\ ;\ \ C_2 = C_3 = \frac{Ca^4}{8}\ ;\ \ C_4 = C_5 = 0\ . \tag{13}$$

The final stress field is therefore

$$\sigma_{rr} = \frac{Cr(\sin\theta + \sin(3\theta))}{4} + \frac{Ca^6\sin(3\theta)}{r^5} - \frac{5Ca^4\sin(3\theta)}{4r^3} - \frac{Ca^4\sin\theta}{4r^3} \tag{14}$$

$$\sigma_{r\theta} = \frac{Cr(-\cos\theta + \cos(3\theta))}{4} - \frac{Ca^6\cos(3\theta)}{r^5} + \frac{3Ca^4\cos(3\theta)}{4r^3} + \frac{Ca^4\cos\theta}{4r^3} \tag{15}$$

$$\sigma_{\theta\theta} = \frac{Cr(3\sin\theta - \sin(3\theta))}{4} - \frac{Ca^6\sin(3\theta)}{r^5} + \frac{Ca^4\sin(3\theta)}{4r^3} + \frac{Ca^4\sin\theta}{4r^3} \tag{16}$$

9.6. A heavy disk of density ρ and radius a is bonded to a rigid support at $r=a$. The gravitational force acts in the direction $\theta=-\pi/2$. Find the stresses and displacements in the disk.

Hint: The easiest method is to find the stresses and displacements for a simple particular solution for the body force and then superpose 'homogeneous' terms using Tables 8.1 and 9.1.

The gravitational body force can be described by the potential

$$V = \rho g y = \rho g r \sin\theta \ , \tag{1}$$

from equation (7.10). The corresponding stress components are

$$\begin{aligned} \sigma_{rr} &= \rho g r \sin\theta & (2)\\ \sigma_{r\theta} &= 0 & (3)\\ \sigma_{\theta\theta} &= \rho g r \sin\theta \ . & (4) \end{aligned}$$

To find the displacements associated with this particular solution, we write

$$\begin{aligned} e_{rr} &= e_{rr} = \frac{\sigma_{rr}}{E} - \frac{\nu\sigma_{\theta\theta}}{E} = \frac{\rho g(1-\nu) r \sin\theta}{E} & (5)\\ e_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} = \frac{\sigma_{\theta\theta}}{E} - \frac{\nu\sigma_{rr}}{E} = \frac{\rho g(1-\nu) r \sin\theta}{E} & (6)\\ 2e_{r\theta} &= \frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = \frac{2\sigma_{r\theta}(1+\nu)}{E} = 0 \ . & (7) \end{aligned}$$

Integrating equation (5), we have

$$u_r = \frac{\rho g(1-\nu) r^2 \sin\theta}{2E} + f(\theta) \ . \tag{8}$$

Substituting this value into (6) and multiplying though by r, we have

$$\frac{\partial u_\theta}{\partial\theta} = \frac{\rho g(1-\nu) r^2 \sin\theta}{E} - \frac{\rho g(1-\nu) r^2 \sin\theta}{2E} - f(\theta) \tag{9}$$

which can be integrated to give

$$u_\theta = -\frac{\rho g(1-\nu) r^2 \cos\theta}{2E} - \int f(\theta) d\theta + g(r) \ . \tag{10}$$

Substituting for u_r, u_θ into (7), we obtain the equation

$$\frac{f'(\theta)}{r} + g'(r) + \frac{1}{r}\int f(\theta) d\theta - \frac{g(r)}{r} = 0 \tag{11}$$

for the unknown functions f, g. However, since this equation is homogeneous, these functions describe only the rigid body terms in the displacement. A particular solution for the displacement field due to the stresses (2–4) is therefore

$$u_r = \frac{\rho g(1-\nu)r^2 \sin\theta}{2E} \tag{12}$$

$$u_\theta = -\frac{\rho g(1-\nu)r^2 \cos\theta}{2E} \ . \tag{13}$$

To complete the solution, we select terms from Tables 8.1, 9.1 so as to satisfy the boundary conditions

$$u_r = u_\theta = 0 \ ; \quad r = a \ . \tag{14}$$

Since the disk is solid, these terms must also be bounded at $r = 0$.

The only bounded term giving displacements varying with θ as in equations (12,13) is $\phi = Ar^3 \sin\theta$, for which

$$u_r = \frac{A(\kappa-2)r^2 \sin\theta}{2\mu} \tag{15}$$

$$u_\theta = -\frac{A(\kappa+2)r^2 \cos\theta}{2\mu} \ . \tag{16}$$

Noting that $2\mu(1+\nu) = E$ and, for plane stress

$$\kappa = \frac{(3-\nu)}{(1+\nu)} \ , \tag{17}$$

we have

$$\frac{(\kappa-2)}{2\mu} = \frac{(1-3\nu)}{E} \ ; \quad \frac{(\kappa+2)}{2\mu} = \frac{(5+\nu)}{E} \ . \tag{18}$$

The full displacement field can therefore be written

$$u_r = \frac{\rho g(1-\nu)r^2 \sin\theta}{2E} + \frac{A(1-3\nu)r^2 \sin\theta}{E} + B\cos\theta + C\sin\theta \tag{19}$$

$$u_\theta = -\frac{\rho g(1-\nu)r^2 \cos\theta}{2E} - \frac{A(5+\nu)r^2 \cos\theta}{E} - B\sin\theta + C\cos\theta + Dr \ , \tag{20}$$

where we have also added in an arbitrary rigid body displacement.

To satisfy the boundary conditions (14) for all θ, we therefore require $B = D = 0$ and

$$\frac{\rho g(1-\nu)a^2}{2E} + \frac{A(1-3\nu)a^2}{E} + C = 0 \tag{21}$$

$$-\frac{\rho g(1-\nu)a^2}{2E} - \frac{A(5+\nu)a^2}{E} + C = 0 \ , \tag{22}$$

with solution

$$A = -\frac{\rho g(1-\nu)}{2(3-\nu)} \; ; \;\; C = -\frac{\rho g(1-\nu^2)a^2}{E(3-\nu)} \; . \tag{23}$$

Substituting into (19,20), we obtain the final displacement field as

$$u_r = -\frac{\rho g(1-\nu^2)(a^2-r^2)\sin\theta}{E(3-\nu)} \tag{24}$$

$$u_\theta = -\frac{\rho g(1-\nu^2)(a^2-r^2)\cos\theta}{E(3-\nu)} \; . \tag{25}$$

The stress field is obtained from (2–4, 14) and Table 8.1 as

$$\sigma_{rr} = \rho g r \sin\theta + 2Ar\sin\theta = \frac{2\rho g r \sin\theta}{(3-\nu)} \tag{26}$$

$$\sigma_{r\theta} = -2Ar\cos\theta = \frac{\rho g(1-\nu) r \cos\theta}{(3-\nu)} \tag{27}$$

$$\sigma_{\theta\theta} = \rho g r \sin\theta + 6Ar\sin\theta = \frac{2\rho g \nu r \sin\theta}{(3-\nu)} \; . \tag{28}$$

9.7. A heavy disk of density ρ, elastic properties μ, ν and radius $a+\epsilon$ is pressed into a frictionless hole of radius a in a rigid body.

What is the minimum value of ϵ if the disk is to remain in contact with the hole at all points when the gravitational force acts in the direction $\theta = -\pi/2$.

If there was no body force, we would require to load the disk so as to produce a displacement

$$u_r(a) = -\epsilon \ . \tag{1}$$

This is an axisymmetric problem ($n = 0$) and it requires a bounded stress field at the origin. The only acceptable term in Table 8.1 is $\phi = Ar^2$, which gives

$$2\mu u_r = A(\kappa - 1)r \ . \tag{2}$$

Thus, to satisfy (1), we require

$$A(\kappa - 1)r = -2\mu\epsilon \tag{3}$$

and hence

$$A = -\frac{2\mu\epsilon}{(\kappa - 1)} = -\frac{\mu(1+\nu)\epsilon}{a(1-\nu)} \ , \tag{4}$$

using $\kappa = (3-\nu)/(1+\nu)$ for plane stress.

It follows from Table 8.1 that the stress components are

$$\sigma_{rr} = 2A = -\frac{2\mu(1+\nu)\epsilon}{a(1-\nu)} = -\frac{E\epsilon}{a(1-\nu)} \tag{5}$$

$$\sigma_{r\theta} = 0 \tag{6}$$

$$\sigma_{\theta\theta} = 2A = -\frac{2\mu(1+\nu)\epsilon}{a(1-\nu)} = -\frac{E\epsilon}{a(1-\nu)} \ . \tag{7}$$

In particular, this defines the compressive traction (the contact pressure) at $r = a$.

The gravitational body force can be described by a potential

$$V = \rho g y = \rho g r \sin\theta \ , \tag{8}$$

from equation (7.10). The stress components (5–7) are then modified to

$$\sigma_{rr} = -\frac{E\epsilon}{a(1-\nu)} + \rho g r \sin\theta \tag{9}$$

$$\sigma_{r\theta} = 0 \tag{10}$$

$$\sigma_{\theta\theta} = -\frac{E\epsilon}{a(1-\nu)} + \rho g r \sin\theta \ . \tag{11}$$

The boundary conditions for this problem are

$$\sigma_{r\theta} = 0 \tag{12}$$

$$u_r = -\epsilon \tag{13}$$

at $r = a$. The frictionless boundary condition (12) is already satisfied by the stress field (9–11), but we anticipate that the θ-varying terms in these stresses will generate displacements u_r varying with $\sin\theta$. At first sight, it is tempting to include the bounded stress function $\phi = Br^3 \sin\theta$ to remove these unwanted displacements at $r = a$, but this will also introduce a θ-varying component in $\sigma_{r\theta}$ which will force B to be zero. In fact, the θ-varying displacements at $r = a$ can be cancelled by superposing a rigid body displacement, which involves no additional stresses, and since we are not asked to calculate the displacement field, we do not need to find the magnitude of these terms. Instead, we note that (9–11) defines the final stress field and hence full contact will be maintained between the disk and the support as long as $\sigma_{rr}(a, \theta)$ remains compressive (negative) for all θ. The most likely separation point is the top of the disk, $\theta = \pi/2$, leading to the criterion

$$-\frac{E\epsilon}{a(1-\nu)} + \rho g a < 0 \tag{14}$$

and hence

$$\epsilon > \frac{\rho g a^2 (1-\nu)}{E} \ . \tag{15}$$

9.8. A rigid circular inclusion of radius a in a large elastic plate is subjected to a force F in the x-direction.

Find the stress field in the plate if the inclusion is perfectly bonded to the plate at $r=a$ and the stresses tend to zero as $r\to\infty$.

The system is isotropic, so the inclusion must move in the same direction as the force — i.e. $u_x = \delta, u_y = 0$, where δ is an unknown constant to be determined. Transforming this result into polar coordinates, we conclude that the boundary conditions to be imposed at $r = a$ are

$$u_r = \delta\cos\theta \quad ; \quad u_\theta = -\delta\sin\theta \; . \tag{1}$$

This in turn leads us to select those stress function terms that give the same Fourier dependence in the displacements. These are the terms that lie in the second block in Table 9.1. However, there are two restrictions that must be applied:-

1. The stresses are required to decay as $r \to \infty$, so the term $r^3\cos\theta$ in ϕ must be excluded (see Table 8.1).

2. The terms $\theta\sin\theta, \theta\cos\theta$ that occur in some of the displacements in Table 9.1 are multivalued (see §9.3.1) and cannot occur in a complete annulus. However, *a combination* of such terms can be admitted, provided the coefficients are chosen such that the multivalued displacement terms cancel.

With these considerations in mind, we start with the stress function

$$\phi = Ar\theta\sin\theta + Br\log r\cos\theta + \frac{C\cos\theta}{r} \; . \tag{2}$$

The stresses and displacements can then be written down from Tables 8.1, 9.1 respectively as

$$\sigma_{rr} = \left(\frac{2A}{r} + \frac{B}{r} - \frac{2C}{r^3}\right)\cos\theta \; ; \tag{3}$$

$$\sigma_{r\theta} = \left(\frac{B}{r} - \frac{2C}{r^3}\right)\sin\theta \; ; \tag{4}$$

$$\sigma_{\theta\theta} = \left(\frac{B}{r} + \frac{2C}{r^3}\right)\cos\theta \; . \tag{5}$$

$$\begin{aligned} 2\mu u_r &= \frac{A}{2}[(\kappa-1)\theta\sin\theta - \cos\theta + (\kappa+1)\log r\cos\theta] \\ &\quad + \frac{B}{2}[(\kappa+1)\theta\sin\theta - \cos\theta + (\kappa-1)\log r\cos\theta] + \frac{C\cos\theta}{r^2} \; ; \end{aligned} \tag{6}$$

$$\begin{aligned} 2\mu u_\theta &= \frac{A}{2}[(\kappa-1)\theta\cos\theta - \sin\theta - (\kappa+1)\log r\sin\theta] \\ &\quad + \frac{B}{2}[(\kappa+1)\theta\cos\theta - \sin\theta - (\kappa-1)\log r\sin\theta] + \frac{C\sin\theta}{r^2} \; . \end{aligned} \tag{7}$$

The multivalued term in (6) is

$$\frac{A(\kappa-1)\theta\sin\theta}{2}+\frac{B(\kappa+1)\theta\sin\theta}{2}$$

and it can be be made zero by choosing A, B such that

$$A(\kappa-1)+B(\kappa+1)=0\ . \tag{8}$$

The same condition cancels the multivalued term in (7).

Two additional equations for the constants A, B, C are obtained by enforcing the boundary conditions (1) at $r=a$. We obtain

$$\frac{A}{2}[-1+(\kappa+1)\log a]+\frac{B}{2}[-1+(\kappa-1)\log a]+\frac{C}{a^2} = 2\mu\delta\ ; \tag{9}$$

$$\frac{A}{2}[-1-(\kappa+1)\log a]+\frac{B}{2}[-1-(\kappa-1)\log a]+\frac{C}{a^2} = -2\mu\delta\ . \tag{10}$$

Finally, one equation can be obtained from the requirement that the total force on the inclusion be F —

$$F = \int_0^{2\pi}(-\sigma_{rr}\cos\theta+\sigma_{r\theta}\sin\theta)rd\theta \tag{11}$$

$$= \int_0^{2\pi}\left[-\left(2A+B-\frac{2C}{r^2}\right)\cos^2\theta+\left(B-\frac{2C}{r^2}\right)\sin^2\theta\right]d\theta \tag{12}$$

$$= -2A\pi\ . \tag{13}$$

Thus,

$$A=-\frac{F}{2\pi}\ , \tag{14}$$

from equation (14) and

$$B=-\frac{A(\kappa-1)}{\kappa+1}=\frac{F(\kappa-1)}{2\pi(\kappa+1)}\ , \tag{15}$$

from (8,15).

Finally, we determine C by adding (9,10) to obtain

$$-A-B+\frac{2C}{a^2}=0 \tag{16}$$

and hence

$$C=-\frac{Fa^2}{2\pi(\kappa+1)}\ , \tag{17}$$

using (14,15).

The complete stress field is now obtained by substituting these results into (3–5), with the result

$$\sigma_{rr} = \frac{F\cos\theta}{2\pi(\kappa+1)r}\left(\frac{2a^2}{r^2} - (\kappa+3)\right) ; \tag{18}$$

$$\sigma_{r\theta} = \frac{F\sin\theta}{2\pi(\kappa+1)r}\left(\frac{2a^2}{r^2} + (\kappa-1)\right) ; \tag{19}$$

$$\sigma_{\theta\theta} = \frac{F\cos\theta}{2\pi(\kappa+1)r}\left(-\frac{2a^2}{r^2} + (\kappa-1)\right) . \tag{20}$$

Notice that if the size of the inclusion becomes very small, $a \to 0$ the a^2/r^2 terms vanish and the solution reduces to Kelvin's solution for a point force in an infinite solid (see §13.1).

9.9. A rubber bushing comprises a hollow cylinder of rubber $a < r < b$, bonded to concentric thin-walled steel cylinders at $r = a$ and $r = b$, as shown in Figure 1. The steel cylinders may be considered as rigid compared with the rubber.

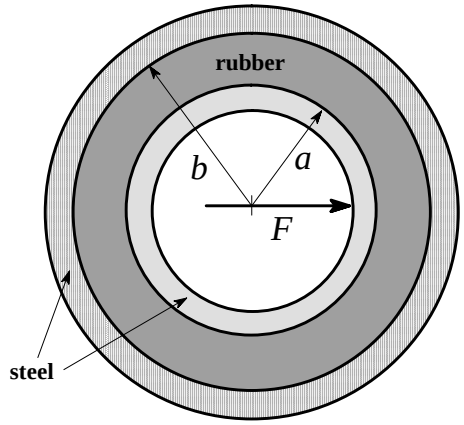


Figure 1

The outer cylinder is held fixed, whilst a force F per unit axial length is applied to the inner cylinder in the x-direction. Find the stiffness of the bushing under this loading — i.e. the relationship between F and the resulting displacement δ of the inner cylinder in the x-direction.

In particular, find the stiffness for a long cylinder (plane strain conditions) and a short cylinder (plane stress), assuming that rubber has a shear modulus μ and is incompressible ($\nu = 0.5$).

Plot a graph of the dimensionless stiffness $F/\mu\delta$ as a function of the ratio a/b in the range $F/\mu\delta < 100$. Can you provide a simple physical explanation of the difference between the plane stress and plane strain results?

Hint: The hollow cylinder is a multiply-connected body.

The system is isotropic, so the inner cylinder must move in the same direction as the force — i.e. $u_x = \delta, u_y = 0$, where δ is an unknown constant to be determined. Transforming this result into polar coordinates, we conclude that the boundary conditions to be imposed are

$$u_r = \delta \cos\theta \; ; \quad r = a \tag{1}$$

$$u_\theta = -\delta \sin\theta \; ; \quad r = a \tag{2}$$

$$u_r = u_\theta = 0 \; ; \quad r = b \tag{3}$$

This in turn leads us to select those stress function terms that give the same Fourier dependence in the displacements. These are the terms that lie in the second block in Table 9.1. Notice that the terms $\theta \sin\theta, \theta\cos\theta$ that occur in some of the displacements in Table 9.1 are multivalued (see §9.3.1) and cannot occur in a complete annulus, but *a combination* of such terms can be admitted, provided the coefficients are chosen such that the multivalued displacement terms cancel.

With these considerations in mind, we start with the stress function

$$\phi = Ar^3\cos\theta + Br\theta\sin\theta + Cr\log r\cos\theta + \frac{D\cos\theta}{r} \ . \tag{4}$$

The stresses and displacements can then be written down from Tables 8.1, 9.1 respectively as

$$\sigma_{rr} = \left(2Ar + \frac{2B}{r} + \frac{C}{r} - \frac{2D}{r^3}\right)\cos\theta \ ; \tag{5}$$

$$\sigma_{r\theta} = \left(2Ar + \frac{C}{r} - \frac{2D}{r^3}\right)\sin\theta \ ; \tag{6}$$

$$\sigma_{\theta\theta} = \left(6Ar + \frac{C}{r} + \frac{2D}{r^3}\right)\cos\theta \ . \tag{7}$$

$$\begin{aligned} 2\mu u_r &= A(\kappa-2)r^2\cos\theta + \frac{B}{2}[(\kappa-1)\theta\sin\theta - \cos\theta + (\kappa+1)\log r\cos\theta] \\ &\quad + \frac{C}{2}[(\kappa+1)\theta\sin\theta - \cos\theta + (\kappa-1)\log r\cos\theta] + \frac{D\cos\theta}{r^2} + 2\mu U\cos\theta \ ; \quad (8) \\ 2\mu u_\theta &= A(\kappa+2)r^2\cos\theta + \frac{B}{2}[(\kappa-1)\theta\cos\theta - \sin\theta - (\kappa+1)\log r\sin\theta] \\ &\quad + \frac{C}{2}[(\kappa+1)\theta\cos\theta - \sin\theta - (\kappa-1)\log r\sin\theta] + \frac{D\sin\theta}{r^2} - 2\mu U\sin\theta \ , \quad (9) \end{aligned}$$

where U represents an unknown rigid body displacement in the x-direction.

The multivalued term in (8) is

$$\frac{B(\kappa-1)\theta\sin\theta}{2} + \frac{C(\kappa+1)\theta\sin\theta}{2}$$

and it can be be made zero by choosing B, C such that

$$B(\kappa-1) + C(\kappa+1) = 0 \ . \tag{10}$$

The same condition cancels the multivalued term in (9).

Four additional equations for the constants A, B, C, D, U are obtained by enforcing the boundary conditions (1–4) at $r = a, b$. We obtain

$$A(\kappa-2)a^2 + \frac{B}{2}[-1 + (\kappa+1)\ln a] + \frac{C}{2}[-1 + (\kappa-1)\ln a] + \frac{D}{a^2} + 2\mu U = 2\mu\delta \ ;$$

$$
\begin{aligned}
A(\kappa+2)a^2 + \frac{B}{2}[-1-(\kappa+1)\ln a] + \frac{C}{2}[-1-(\kappa-1)\log a] + \frac{D}{a^2} - 2\mu U &= -2\mu\delta \\
A(\kappa-2)b^2 + \frac{B}{2}[-1+(\kappa+1)\ln b] + \frac{C}{2}[-1+(\kappa-1)\ln b] + \frac{D}{b^2} + 2\mu U &= 0 \\
A(\kappa+2)b^2 + \frac{B}{2}[-1-(\kappa+1)\ln b] + \frac{C}{2}[-1-(\kappa-1)\log b] + \frac{D}{b^2} - 2\mu U &= 0\ .
\end{aligned}
$$

Solving these equations, we obtain

$$
A = -\frac{\mu\delta}{\kappa^2(a^2+b^2)\ln(b/a) - (b^2-a^2)} \tag{11}
$$

$$
B = -\frac{\mu\delta\kappa(\kappa+1)(a^2+b^2)}{\kappa^2(a^2+b^2)\ln(b/a) - (b^2-a^2)} \tag{12}
$$

$$
C = -\frac{\mu\delta\kappa(\kappa-1)(a^2+b^2)}{\kappa^2(a^2+b^2)\ln(b/a) - (b^2-a^2)} \tag{13}
$$

$$
D = \frac{\mu\delta\kappa a^2 b^2}{\kappa^2(a^2+b^2)\ln(b/a) - (b^2-a^2)} \tag{14}
$$

Also, the applied force is given by

$$
F = \int_0^{2\pi} (-\sigma_{rr}\cos\theta + \sigma_{r\theta}\sin\theta) r d\theta \tag{15}
$$

$$
= \int_0^{2\pi} \left[-\left(2A + B - \frac{2C}{r^2}\right)\cos^2\theta + \left(B - \frac{2C}{r^2}\right)\sin^2\theta \right] d\theta \tag{16}
$$

$$
= -2B\pi \tag{17}
$$

$$
= \frac{2\pi\mu\delta\kappa(\kappa+1)(a^2+b^2)}{\kappa^2(a^2+b^2)\ln(b/a) - (b^2-a^2)} \tag{18}
$$

For plane stress, we have

$$
\kappa = \frac{3-\nu}{1+\nu} = \frac{5}{3} \tag{19}
$$

and

$$
k \equiv \frac{F}{\delta} = \frac{80\pi\mu(a^2+b^2)}{25(a^2+b^2)\ln(b/a) - 9(b^2-a^2)}\ , \tag{20}
$$

whilst for plane strain

$$
\kappa = 3 - 4\nu = 1 \tag{21}
$$

and

$$
k = \frac{4\pi\mu(a^2+b^2)}{(a^2+b^2)\ln(b/a) - (b^2-a^2)}\ . \tag{22}
$$

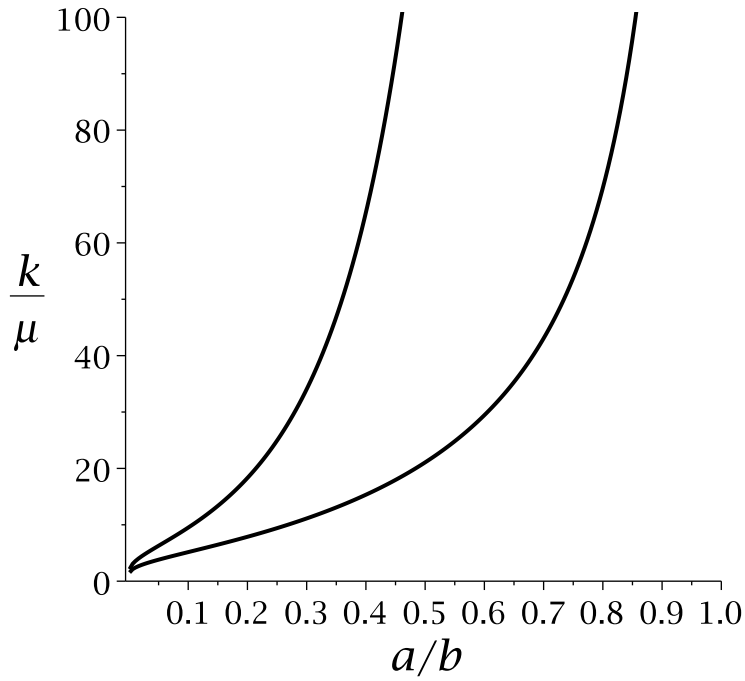


Figure 1

The plot is shown in Figure 1. Notice that the plane strain solution becomes very stiff at significantly lower values of a/b than does the plane stress solution. This is because the material is incompressible. Displacement of the inner cylinder to the right reduces the volume of rubber on the right and increases it on the left. In the case of plane strain, the only way this can happen is by rubber being displaced circumferentially around the annular space, but it is bonded at the two edges, which makes this difficult. In plane stress, the rubber is free to expand in the z-direction. In other words, the rubber will bulge out axially on the right and pull in on the left. This involves much less flow of material and hence is easier to do.

CHAPTER 10

10.1. A curved beam of inner radius a and outer radius $5a$ is subjected to a force F at its end as shown in Figure 10.5. The line of action of the force passes through the mid-point of the beam.

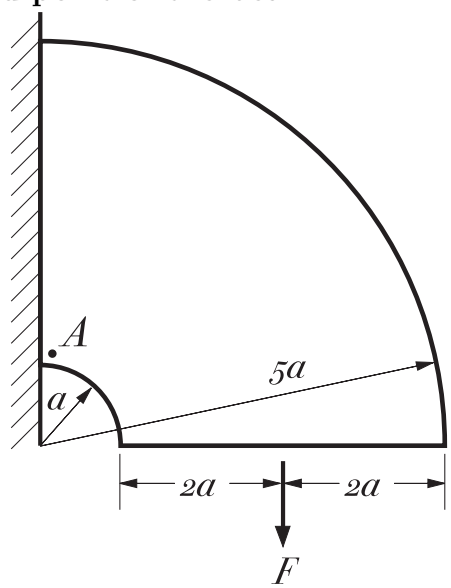


Figure 10.5: Curved beam loaded by a central force.

By superposing the solutions for the curved beam subjected to an end force and to pure bending respectively, find the hoop stress $\sigma_{\theta\theta}$ at the point $A(a, \pi/2)$ and compare it with the value predicted by the elementary Mechanics of Materials bending theory.

The solution of equations (10.20–10.29) describes the stress field due to a force whose line of action passes through the origin as in Figure 10.3. The same solution would apply to the problem of Figure 10.5 if (i) the force F were displaced to the left through a distance $3a$ and (ii) θ were measured from the support, rather than the free end. In this configuration, the tensile stress at A would correspond to $\sigma_{\theta\theta}$ at $\theta = 0$ and would therefore be zero from equation (10.23).

To obtain the solution of the problem in Figure 10.5, we need to superpose a bending moment $M_0 = 3Fa$, using the results of §10.1.1, with $b = 5a$.

The hoop stress at the point A ($r = a$) is

$$\sigma_{\theta\theta} = 2A + B(2\ln a + 3) - \frac{C}{a^2} = 2B - \frac{2C}{a^2}\ , \tag{1}$$

from (10.4) and using (10.10) to eliminate A.

We find

$$
\begin{aligned}
N_0 &= a^4((25-1)^2 - 100\ln^2(5)) = 317a^4 \ ; && (2)\\
B &= \frac{6Fa}{317a^4}(24a^2) = 0.454\frac{F}{a} \ ; && (3)\\
C &= \frac{12Fa}{317}25\ln(5) = 1.523Fa \ , && (4)
\end{aligned}
$$

from (10.15, 10.16).

Thus, the hoop stress at A is

$$\sigma = 0.908\frac{F}{a} - 3.046\frac{F}{a} = -2.138\frac{F}{a} \ . \qquad (5)$$

Elementary bending theory would predict a value

$$\sigma = \frac{My}{I} = \frac{12.(3Fa).(2a)}{(4a)^3} = -1.125\frac{F}{a} \ , \qquad (6)$$

so there is a stress concentration factor at A of 1.9 associated with the small inner radius of the curved bar.

10.2. The curved beam $a<r<b$, $0<\theta<\pi/2$ is built in at $\theta=\pi/2$ and loaded only by its own weight, which acts in the direction $\theta=-\pi/2$. Find the stress field in the beam.

For a particular solution, we use the body force potential

$$V = \rho g y = \rho g r \sin\theta \ , \tag{1}$$

the corresponding stress components being

$$\begin{aligned}
\sigma_{rr} &= \rho g r \sin\theta && (2)\\
\sigma_{r\theta} &= 0 && (3)\\
\sigma_{\theta\theta} &= \rho g r \sin\theta \ . && (4)
\end{aligned}$$

This solution implies the existence of sinusoidally varying normal tractions on the surfaces $r = a, b$, so by comparison with the solution of §10.3.1, we conclude that we require the special stress function of equation (10.35) in addition to the terms in (10.20) — i.e.

$$\begin{aligned}
\phi &= (Ar^3 + Br^{-1} + Cr\ln(r))\sin\theta + Dr\theta\cos\theta \\
&\quad + A'r^3(\ln(r)\sin\theta + \theta\cos\theta) + B'r^{-1}(\theta\cos\theta - \ln r\sin\theta) \\
&\quad C'r\ln(r)\theta\cos\theta + D'r(\ln^2 r\sin\theta - \theta^2\sin\theta) \ . && (5)
\end{aligned}$$

The corresponding stress field is obtained by superposing (2–4), (10.21–10.23) and (10.36–10.38), with the result

$$\begin{aligned}
\sigma_{rr} &= \rho g r\sin\theta + (2Ar - 2Br^{-3} + Cr^{-1} - 2Dr^{-1})\sin\theta \\
&\quad + (2A'r - 2B'r^{-3} + C'r^{-1} - 2D'r^{-1})\theta\cos\theta \\
&\quad + (2A'r\ln(r) - A'r + 2B'r^{-3}\ln(r) - 3B'r^{-3} \\
&\quad - 2C'r^{-1}\ln(r) + 2D'r^{-1}\ln(r) - 4D'r^{-1})\sin\theta && (6)\\
\sigma_{r\theta} &= (-2Ar + 2Br^{-3} - Cr^{-1})\cos\theta \\
&\quad (2A'r - 2B'r^{-3} + C'r^{-1})\theta\sin\theta \\
&\quad + (-2A'r\ln(r) - 3A'r - 2B'r^{-3}\ln(r) + 3B'r^{-3} \\
&\quad - C'r^{-1} - 2D'r^{-1}\ln(r))\cos\theta && (7)\\
\sigma_{\theta\theta} &= \rho g r\sin\theta + (6Ar + 2Br^{-3} + Cr^{-1})\sin\theta \\
&\quad + (6A'r + 2B'r^{-3} + C'r^{-1})\theta\cos\theta \\
&\quad + (6A'r\ln(r) + 5A'r - 2B'r^{-3}\ln(r) + 3B'r^{-3} \\
&\quad + 2D'r^{-1}\ln(r) + 2D'r^{-1})\sin\theta \ . && (8)
\end{aligned}$$

As in §10.3.1, we require that the tractions with θ-multipliers on $r = a, b$ be zero, leading to the equations (10.39–10.42).

We also require the $\sin\theta$ terms in σ_{rr} and the $\cos\theta$ terms in $\sigma_{r\theta}$ to go to zero on these surfaces, giving

$$\begin{aligned}
\rho g a + 2Aa - 2Ba^{-3} + Ca^{-1} - 2Da^{-1} + 2A'a\ln(a) - A'a \\
+2B'a^{-3}\ln(a) - 3B'a^{-3} - 2C'a^{-1}\ln(a) + 2D'a^{-1}\ln(a) - 4D'a^{-1} &= 0 \quad (9) \\
\rho g b + 2Ab - 2Bb^{-3} + Cb^{-1} - 2Db^{-1} + 2A'b\ln(b) - A'b \\
+2B'b^{-3}\ln(b) - 3B'b^{-3} - 2C'b^{-1}\ln(b) + 2D'b^{-1}\ln(b) - 4D'b^{-1} &= 0 \quad (10) \\
-2Aa + 2Ba^{-3} - Ca^{-1} - 2A'a\ln(a) - 3A'a \\
-2B'a^{-3}\ln(a) + 3B'a^{-3} - C'a^{-1} - 2D'a^{-1}\ln(a) &= 0 \quad (11) \\
-2Ab + 2Bb^{-3} - Cb^{-1} - 2A'b\ln(b) - 3A'b \\
-2B'b^{-3}\ln(b) + 3B'b^{-3} - C'b^{-1} - 2D'b^{-1}\ln(b) &= 0\,. \quad (12)
\end{aligned}$$

We also require the surface $\theta = 0$ to be traction-free. The normal traction is identically zero on this surface, but there remains a shear traction

$$\begin{aligned}
\sigma_{\theta r}(r,0) &= -2Ar + 2Br^{-3} - Cr^{-1} - 2A'r\ln(r) - 3A'r \\
&\quad -2B'r^{-3}\ln(r) + 3B'r^{-3} - C'r^{-1} - 2D'r^{-1}\ln(r)\,. \quad (13)
\end{aligned}$$

We therefore apply the weak condition

$$\int_a^b \sigma_{r\theta}(r,0)dr = 0\,, \quad (14)$$

obtaining

$$\begin{aligned}
&-A(b^2-a^2) + \frac{B(b^2-a^2)}{a^2b^2} - C\ln(b/a) - A'(b^2\ln(b) - a^2\ln(a)) - A'(b^2-a^2) \\
&+\frac{B'\ln(b)}{b^2} - \frac{B'\ln(a)}{a^2} - \frac{B'(b^2-a^2)}{a^2b^2} - C'\ln(b/a) - D'(\ln^2(b) - \ln^2(a)) = 0 \quad (15)
\end{aligned}$$

The constants $A, B, C, D, A_1, B_1, C_1, D_1$ are then obtained by solving the equations (9–12), (15) and (10.39-10.42). The solution is really only practicable in Maple or Mathematica. Some of the simpler expressions are

$$\begin{aligned}
A_1 &= -\frac{\rho g(b^2-a^2)}{4N_1} && (16) \\
B_1 &= -\frac{\rho g a^2b^2(b^2-a^2)}{4N_1} && (17) \\
C_1 &= \frac{\rho g(b^4-a^4)}{2N_1} && (18) \\
D_1 &= 0 && (19) \\
D &= \frac{\rho g(a^2+b^2)\{a^2-b^2+2a^2\ln(b)-2b^2\ln(a)\}}{4N_1} && (20)
\end{aligned}$$

where N_1 is defined by (10.29).

10.3. The curved beam $a < r < b$, $0 < \theta < \pi/2$ is built in at $\theta = \pi/2$ and loaded by a uniform normal pressure $\sigma_{rr} = -S$ at $r = b$, the other edges being traction-free. Find the stress field in the beam.

This is not a problem requiring a special stress function. There are two ways to see this.

(i) One is to consider first the problem of the complete annulus $a < r < b$ loaded by uniform normal tractions on the entire boundary $r = b$. This problem cannot require a special function, because any terms involving θ-multipliers would then be multivalued. In fact this is a quite elementary axisymmetric problem. Now suppose we cut through the annulus at $\theta = 0$. We shall require to cancel any force and/or moment on the end to produce the solution of the present problem, but the stress functions required for this process are included in §10.1.

(ii) An alternative argument is simply to examine the equations (10.10–10.13) for the $n = 0$ terms in the Michell solution. We notice that the redundancy of these equations arises from the fact that (10.12) and (10.13) are identical. These are the equations defining the shear tractions on the surfaces $r = a, b$ and in the present problem both are required to be zero. Thus, although we shall obtain four equations and four unknowns, equations (10.12, 10.13) will both be satisfied by the choice $D = 0$, exactly as in §10.1.1. Notice that this would not work if uniform shear tractions were imposed on these surfaces. In that case we *would* need a special stress function.

In a manual calculation, the easiest way to approach this problem is to follow the superposition method outlined in (i) above. We first solve the axisymmetric problem of the complete annulus, then calculate the force and moment transmitted across the surface $\theta = 0$ and finally solve two end loading problems to remove these unwanted loads. In a Maple or Mathematica solution, one could simply include a stress function comprised of (10.1) and (10.20) with sine and cosine interchanged. Imposition of boundary conditions on the curved surfaces and the end would then yield enough equations to solve for the constants.

For the complete annulus, the displacements must be single-valued, so from Table 9.1 we see that the only permissible terms are

$$\phi = C_1 r^2 + C_2 \ln(r) \ . \tag{1}$$

The stress field is

$$\sigma_{rr} = 2C_1 + \frac{C_2}{r^2} \tag{2}$$

$$\sigma_{r\theta} = 0 \tag{3}$$

$$\sigma_{\theta\theta} = 2C_1 - \frac{C_2}{r^2} \ . \tag{4}$$

The boundary conditions are

$$\begin{aligned}
\sigma_{rr}(a,\theta) &= 0 && (5)\\
\sigma_{rr}(b,\theta) &= -S && (6)\\
\sigma_{r\theta}(a,\theta) &= 0 && (7)\\
\sigma_{r\theta}(b,\theta) &= 0 && (8)\\
& && (9)
\end{aligned}$$

and hence

$$\begin{aligned}
2C_1 + \frac{C_2}{a^2} &= 0 && (10)\\
2C_1 + \frac{C_2}{b^2} &= -S\ , && (11)
\end{aligned}$$

with solution

$$C_1 = -\frac{Sb^2}{2(b^2-a^2)}\ ; \quad C_2 = \frac{Sa^2b^2}{(b^2-a^2)}\ . \tag{12}$$

The corresponding stress components are

$$\begin{aligned}
\sigma_{rr} &= -\frac{Sb^2}{(b^2-a^2)} + \frac{Sa^2b^2}{(b^2-a^2)r^2} && (13)\\
\sigma_{r\theta} &= 0 && (14)\\
\sigma_{\theta\theta} &= -\frac{Sb^2}{(b^2-a^2)} - \frac{Sa^2b^2}{(b^2-a^2)r^2} && (15)
\end{aligned}$$

and the force and moment transmitted across the surface $\theta = 0$ are

$$\begin{aligned}
F &= \int_a^b \sigma_{\theta\theta}(r,0)dr = -Sb && (16)\\
M &= \int_a^b \sigma_{\theta\theta}(r,0)rdr = -\frac{Sb^2}{2} - \frac{Sa^2b^2\ln(b/a)}{(b^2-a^2)}\ . && (17)
\end{aligned}$$

Notice that the simple expression for F could have been deduced from equilibrium and symmetry considerations, by cutting the complete annulus at $\theta = 0$ and $\theta = \pi$.

To cancel the unwanted moment, we simply superpose the solution of §10.1.1 with

$$M_0 = \frac{Sb^2}{2} + \frac{Sa^2b^2\ln(b/a)}{(b^2-a^2)} \tag{18}$$

The corrective stress function is given by

$$\phi = A_0r^2 + B_0r^2\ln(r) + C_0\ln(r)\ , \tag{19}$$

where

$$A_0 = -\frac{M_0}{N_0}(b^2 - a^2 + 2b^2\ln b - 2a^2\ln a)$$

$$B_0 = \frac{2M_0}{N_0}(b^2 - a^2) \;\; ; \;\; C_0 = \frac{4M_0}{N_0} a^2 b^2 \ln \frac{b}{a} \tag{20}$$

and

$$N_0 \equiv (b^2 - a^2)^2 - 4a^2 b^2 \ln^2(b/a) \; , \tag{21}$$

from (10.1, 10.15, 10.16).

To cancel the force F of equation (16), we need to modify the solution of §10.1.2 to describe a normal rather than a tangential force on the surface $\theta = 0$. For this purpose, we interchange sine and cosine in (10.20) obtaining the stress function

$$\phi = (A_1 r^3 + B_1 r^{-1} + C_1 r \ln(r)) \cos\theta + D_1 r\theta \sin\theta \; . \tag{22}$$

The corresponding stress components are

$$\sigma_{rr} = (2A_1 r - 2B_1 r^{-3} + C_1 r^{-1} + 2D_1 r^{-1}) \cos\theta \tag{23}$$

$$\sigma_{r\theta} = (2A_1 r - 2B_1 r^{-3} + C_1 r^{-1}) \sin\theta \tag{24}$$

$$\sigma_{\theta\theta} = (6A_1 r + 2B_1 r^{-3} + C_1 r^{-1}) \cos\theta \tag{25}$$

and the homogeneous boundary conditions (analogous to (10.24–10.27)) on $r = a, b$ require

$$2A_1 a - \frac{2B_1}{a^3} + \frac{C_1}{a} + \frac{2D_1}{a} = 0 \tag{26}$$

$$2A_1 b - \frac{2B_1}{b^3} + \frac{C_1}{b} - \frac{2D_1}{b} = 0 \tag{27}$$

$$2A_1 a - \frac{2B_1}{a^3} + \frac{C_1}{a} = 0 \tag{28}$$

$$2A_1 b - \frac{2B_1}{b^3} + \frac{C_1}{b} = 0 \; . \tag{29}$$

The tensile force on $\theta = 0$ is

$$F_1 = \int_a^b \sigma_{\theta\theta}(r, 0) dr = 3A_1(b^2 - a^2) + \frac{B_1(b^2 - a^2)}{a^2 b^2} + C_1 \ln(b/a) \; . \tag{30}$$

Solving (26–30) for A_1, B_1, C_1, D_1 as in §10.1.2, we obtain

$$A_1 = -\frac{F_1}{2N_1} \;\; ; \;\; B_1 = \frac{F_1 a^2 b^2}{2N_1} \;\; ; \;\; C_1 = \frac{F_1(a^2 + b^2)}{N_1} \;\; ; \;\; D = 0 \; , \tag{31}$$

where

$$N_1 = (a^2 - b^2) + (a^2 + b^2) \ln(b/a) \; . \tag{32}$$

The complete solution of the problem is now obtained by superposing the three stress functions (1, 19, 22) to obtain

$$\phi = C_1 r^2 + C_2 \ln(r) + A_0 r^2 + B_0 r^2 \ln(r) + C_0 \ln(r) + (A_1 r^3 + B_1 r^{-1} + C_1 r \ln(r)) \cos\theta \; ,$$

where the constants are given by (12), (20) and (31), with $F_1 = Sb$.

10.4. The curved beam $a < r < b$, $0 < \theta < \pi/2$ is built in at $\theta = \pi/2$ and loaded by a uniform shear traction $\sigma_{r\theta} = S$ at $r = b$, the other edges being traction-free. Find the stress field in the beam.

The loading is independent of θ, but a solution cannot be obtained using only the stress function (10.1), since no choice of the constants will permit the stress component (10.3) to represent a uniform traction on $r = b$ and a zero traction on $r = a$. Thus, this is one of the problems requiring a special stress function.

To find this function, we start with the general form

$$\begin{aligned}\phi_1 &= (A_1 r^{\lambda+2} + B_1 r^{\lambda} + C_1 r^{-\lambda+2} + D_1 r^{-\lambda})\cos(\lambda\theta) \\ &\quad +(A_2 r^{\lambda+2} + B_2 r^{\lambda} + C_2 r^{-\lambda+2} + D_2 r^{-\lambda})\sin(\lambda\theta) \end{aligned} \tag{1}$$

and differentiate with respect to λ, obtaining

$$\begin{aligned}\phi_2 &= A_1 r^{\lambda+2}(\ln(r)\cos(\lambda\theta) - \theta\sin(\lambda\theta)) + B_1 r^{\lambda}(\ln(r)\cos(\lambda\theta) - \theta\sin(\lambda\theta)) \\ &\quad -C_1 r^{-\lambda+2}(\ln(r)\cos(\lambda\theta) + \theta\sin(\lambda\theta)) - D_1 r^{-\lambda}(\ln(r)\cos(\lambda\theta) + \theta\sin(\lambda\theta)) \\ &\quad +A_2 r^{\lambda+2}(\ln(r)\sin(\lambda\theta) + \theta\cos(\lambda\theta)) + B_2 r^{\lambda}(\ln(r)\sin(\lambda\theta) + \theta\cos(\lambda\theta)) \\ &\quad -C_2 r^{-\lambda+2}(\ln(r)\sin(\lambda\theta) - \theta\cos(\lambda\theta)) - D_2 r^{-\lambda}(\ln(r)\sin(\lambda\theta) - \theta\cos(\lambda\theta)) \end{aligned} \tag{2}$$

Setting $\lambda = 0$ in this expression, we then recover the new function

$$\phi_2(0) = A_1 r^2\ln(r) + B_1\ln(r) - C_1 r^2\ln(r) - D_1\ln(r) + A_2 r^2\theta + B_2\theta + C_2 r^2\theta + D_2\theta\ . \tag{3}$$

However, we note that this function is a linear combination of the terms $r^2\ln(r), \ln(r)$, $r^2\theta, \theta$, three of which we already included in equation (10.1). We must therefore perform a second differentiation on ϕ_2, obtaining

$$\begin{aligned}\phi_3 &= A_1 r^{\lambda+2}(\ln^2(r)\cos(\lambda\theta) - 2\theta\ln(r)\sin(\lambda\theta) - \theta^2\cos(\lambda\theta)) \\ &\quad +B_1 r^{\lambda}(\ln^2(r)\cos(\lambda\theta) - 2\theta\ln(r)\sin(\lambda\theta) - \theta^2\cos(\lambda\theta)) \\ &\quad +C_1 r^{-\lambda+2}(\ln^2(r)\cos(\lambda\theta) + 2\theta\ln(r)\sin(\lambda\theta) - \theta^2\cos(\lambda\theta)) \\ &\quad -D_1 r^{-\lambda}(\ln^2(r)\cos(\lambda\theta) + 2\theta\ln(r)\sin(\lambda\theta) - \theta^2\cos(\lambda\theta)) \\ &\quad +A_2 r^{\lambda+2}(\ln^2(r)\sin(\lambda\theta) + 2\theta\ln(r)\cos(\lambda\theta) - \theta^2\sin(\lambda\theta)) \\ &\quad +B_2 r^{\lambda}(\ln^2(r)\sin(\lambda\theta) + 2\theta\ln(r)\cos(\lambda\theta) - \theta^2\sin(\lambda\theta)) \\ &\quad +C_2 r^{-\lambda+2}(\ln^2(r)\sin(\lambda\theta) - \theta\ln(r)\cos(\lambda\theta) - \theta^2\sin(\lambda\theta)) \\ &\quad -D_2 r^{-\lambda}(\ln^2(r)\sin(\lambda\theta) - \theta\ln(r)\cos(\lambda\theta) - \theta^2\sin(\lambda\theta))\ . \end{aligned} \tag{4}$$

Setting $\lambda = 0$, this reduces to

$$\begin{aligned}\phi_3 &= A_1 r^2(\ln^2(r) - \theta^2) + B_1(\ln^2(r) - \theta^2) + C_1 r^2(\ln^2(r) - \theta^2) - D_1(\ln^2(r) - \theta^2) \\ &\quad +2A_2 r^2\theta\ln(r) + 2B_2\theta\ln(r) - 2C_2 r^2\theta\ln(r) + 2D_2\theta\ln(r)\ , \end{aligned} \tag{5}$$

which is a linear combination of the new biharmonic functions $r^2(\ln^2(r)-\theta^2), (\ln^2(r)-\theta^2), r^2\theta\ln(r), \theta\ln(r)$.

To solve the problem, we therefore use the stress function

$$\begin{aligned}\phi &= C_1r^2 + C_2r^2\ln(r) + C_3\ln(r) + C_4\theta + C_5r^2\theta + C_6r^2(\ln^2(r)-\theta^2) \\ &\quad + C_7(\ln^2(r)-\theta^2) + C_8r^2\theta\ln(r) + C_9\theta\ln(r)\ . \end{aligned} \tag{6}$$

The corresponding stress components are

$$\begin{aligned}\sigma_{rr} &= 2C_1 + C_2(2\ln(r)+1) + \frac{C_3}{r^2} + 2C_5\theta + 2C_6(\ln^2(r)+\ln(r)-\theta^2-1) \\ &\quad + \frac{2C_7(\ln(r)-1)}{r^2} + 2C_8\theta(\ln(r)+1) + \frac{2C_9\theta}{r^2} \end{aligned} \tag{7}$$

$$\sigma_{r\theta} = \frac{C_4}{r^2} - C_5 + 2C_6\theta - \frac{2C_7\theta}{r^2} - C_8(\ln(r)+1) + \frac{C_9(\ln(r)-1)}{r^2} \tag{8}$$

$$\begin{aligned}\sigma_{\theta\theta} &= 2C_1 + C_2(2\ln(r)+3) - \frac{C_3}{r^2} + 2C_5\theta + 2C_6(\ln^2(r)+3\ln(r)-\theta^2+1) \\ &\quad - \frac{2C_7(\ln(r)-1)}{r^2} + C_8\theta(2\ln(r)+3) - \frac{2C_9\theta}{r^2}\ . \end{aligned} \tag{9}$$

To satisfy the boundary conditions

$$\sigma_{rr} = 0\ ; \quad r = a, b \tag{10}$$

$$\sigma_{r\theta} = S\ ; \quad r = b \tag{11}$$

$$= 0\ ; \quad r = 0 \tag{12}$$

for all θ, we require

$$2C_1 + C_2(2\ln(a)+1) + \frac{C_3}{a^2} + 2C_6(\ln^2(a)+\ln(a)-1) + \frac{2C_7(\ln(a)-1)}{a^2} = 0 \tag{13}$$

$$2C_1 + C_2(2\ln(b)+1) + \frac{C_3}{b^2} + 2C_6(\ln^2(b)+\ln(b)-1) + \frac{2C_7(\ln(b)-1)}{b^2} = 0 \tag{14}$$

$$2C_5 + 2C_8(\ln(a)+1) + \frac{2C_9}{a^2} = 0 \tag{15}$$

$$2C_5 + 2C_8(\ln(b)+1) + \frac{2C_9}{b^2} = 0 \tag{16}$$

$$C_6 = 0 \tag{17}$$

$$\frac{C_4}{b^2} - C_5 - C_8(\ln(b)+1) + \frac{C_9(\ln(b)-1)}{b^2} = S \tag{18}$$

$$\frac{C_4}{a^2} - C_5 - C_8(\ln(a)+1) + \frac{C_9(\ln(a)-1)}{a^2} = 0 \tag{19}$$

$$2C_6\theta - \frac{2C_7\theta}{a^2} = 0 \tag{20}$$

$$2C_6\theta - \frac{2C_7\theta}{b^2} = 0\ . \tag{21}$$

We also wish to satisfy the three weak conditions

$$F_x = \int_a^b \sigma_{\theta r}(r,0)dr = 0 \tag{22}$$

$$F_y = \int_a^b \sigma_{\theta\theta}(r,0)dr = 0 \tag{23}$$

$$M = \int_a^b \sigma_{\theta\theta}(r,0)rdr = 0\ . \tag{24}$$

However, we cannot necessarily expect to satisfy all these conditions with the present solution. It may be necessary to superpose the solutions of §§10.1.1, 10.1.2. However, all the stress functions involved in the solution of §10.1.1 are included in equation (6), so it must be possible to satisfy (24). We have

$$\sigma_{\theta\theta}(r,0) = 2C_1 + C_2(2\ln(r)+3) - \frac{C_3}{r^2} + 2C_6(\ln^2(r)+3\ln(r)+1) - \frac{2C_7(\ln(r)-1)}{r^2} \tag{25}$$

and substituting into (24) and evaluating the integral, we obtain

$$\begin{aligned} C_1(b^2-a^2) + C_2(b^2\ln(b) - a^2\ln(a) + b^2 - a^2) - C_3\ln(b/a) \\ +C_6(b^2\ln^2(b) - a^2\ln^2(a) + 2b^2\ln(b) - 2a^2\ln(a)) \\ +C_7(2\ln(b/a) - \ln^2(b) + \ln^2(a)) \quad = \quad 0\ . \end{aligned} \tag{26}$$

Solving (13–21) and (26) for the 9 constants, we obtain

$$\begin{gathered} C_1 = C_2 = C_3 = C_6 = C_7 = 0 \\ C_4 = \frac{Sa^2b^2(2b^2(2\ln(a)-1)\ln(b/a) - b^2 + a^2)}{N_2} \\ C_5 = \frac{Sb^2(b^2 - a^2 + 2b^2\ln(b) - 2a^2\ln(a))}{N_2} \\ C_8 = -\frac{2Sb^2(b^2-a^2)}{N_2}\ ;\ \ C_9 = -\frac{4Sa^2b^4\ln(b/a)}{N_2}\ , \end{gathered} \tag{27}$$

where

$$N_2 = (b^2-a^2)^2 - 4a^2b^2\ln^2(b/a)\ . \tag{28}$$

With this solution, we then check the values of F_x, F_y defined by equations (22, 23). Substituting the constants (27) into (25) and using this result to evaluate the integral in (23), we find that $F_y = 0$, so this condition is already satisfied.

We also have

$$\sigma_{\theta r}(r,0)\frac{C_4}{r^2} - C_5 - C_8(\ln(r)+1) + \frac{C_9(\ln(r)-1)}{r^2} \tag{29}$$

and substituting into (22) and using (27), we find

$$F_x = -Sb\ . \tag{30}$$

Thus, to complete the solution we must superpose the solution of §10.1.2 with F replaced by Sb.

10.5. The curved beam $a<r<b$, $0<\theta<\pi/2$ rotates about the axis $\theta=\pi/2$ at speed Ω, the edges $r=a,b$ and $\theta=0$ being traction-free. Find the stresses in the beam. Hint: The easiest method is probably to solve first the problem of the complete annular ring $a<r<b$ rotating about $\theta=\pi/2$ and then correct the boundary condition at $\theta=0$ (in the weak sense) by superposing a suitable homogeneous solution.

Following the 'hint', we first consider the problem of the complete annular ring. The inretia force due to rotation is

$$p_x = \rho\Omega^2 x \; ; \quad p_y = 0 \tag{1}$$

and can be described by a body force potential

$$V = -\frac{\rho\Omega^2 x^2}{2} = -\frac{\rho\Omega^2 r^2\cos^2\theta}{2} = -\frac{\rho\Omega^2 r^2}{4} - \frac{\rho\Omega^2 r^2\cos(2\theta)}{4} \; . \tag{2}$$

To complete the solution for the annular disk, we need to add those terms from Table 8.1 that are either axisymmetric or include a $\cos(2\theta)$ multiplier and that do not correspond to multivalued displacements in Table 9.1. Thus, we choose

$$\phi = C_1 r^2 + C_2\ln(r) + (C_3 r^4 + C_4 + C_5 r^2 + C_6 r^{-2})\cos(2\theta) \; . \tag{3}$$

The stress components, including the particular solution, are then

$$\sigma_{rr} = -\frac{\rho\Omega^2 r^2}{4} - \frac{\rho\Omega^2 r^2\cos(2\theta)}{4} + 2C_1 + \frac{C_2}{r^2} + \left\{-\frac{4C_4}{r^2} - 2C_5 - \frac{6C_6}{r^2}\right\}\cos(2\theta) \tag{4}$$

$$\sigma_{r\theta} = \left\{6C_3 r^2 - \frac{2C_4}{r^2} + 2C_5 - \frac{6C_6}{r^2}\right\}\sin(2\theta) \tag{5}$$

$$\sigma_{\theta\theta} = -\frac{\rho\Omega^2 r^2}{4} - \frac{\rho\Omega^2 r^2\cos(2\theta)}{4} + 2C_1 - \frac{C_2}{r^2} + \left\{12C_3 r^2 + 2C_5 + \frac{6C_6}{r^2}\right\}\cos(2\theta) \; . \tag{6}$$

The traction-free boundary conditions $\sigma_{rr} = \sigma_{r\theta} = 0$ on $r = a, b$ lead to the equations

$$-\frac{\rho\Omega^2 a^2}{4} + 2C_1 + \frac{C_2}{a^2} = 0 \tag{7}$$

$$-\frac{\rho\Omega^2 a^2}{4} - \frac{4C_4}{a^2} - 2C_5 - \frac{6C_6}{a^2} = 0 \tag{8}$$

$$6C_3 a^2 - \frac{2C_4}{a^2} + 2C_5 - \frac{6C_6}{a^2} = 0 \tag{9}$$

$$-\frac{\rho\Omega^2 b^2}{4} + 2C_1 + \frac{C_2}{b^2} = 0 \tag{10}$$

$$-\frac{\rho\Omega^2 b^2}{4} - \frac{4C_4}{b^2} - 2C_5 - \frac{6C_6}{b^2} = 0 \tag{11}$$

$$6C_3 b^2 - \frac{2C_4}{b^2} + 2C_5 - \frac{6C_6}{b^2} = 0 \; , \tag{12}$$

with solution

$$C_1 = \frac{\rho\Omega^2(a^2+b^2)}{8} \; ; \; C_2 = -\frac{\rho\Omega^2 a^2 b^2}{4} \; ; \; C_3 = \frac{\rho\Omega^2(a^4+4a^2b^2+b^4)}{24(b^2-a^2)^2}$$
$$C_4 = \frac{\rho\Omega^2 a^2 b^2(a^4+a^2b^2+b^4)}{4(b^2-a^2)^2} \; ; \; C_5 = -\frac{\rho\Omega^2(a^6+2a^4b^2+2a^2b^4+b^6)}{8(b^2-a^2)^2} \; ;$$
$$C_6 = -\frac{\rho\Omega^2 a^4 b^4(a^2+b^2)}{8(b^2-a^2)^2} \; . \tag{13}$$

When we make a cut at $\theta = 0$, the tractions on the cut will be

$$\sigma_{r\theta}(r,0) = 0 \tag{14}$$
$$\sigma_{\theta\theta}(r,0) = -\frac{\rho\Omega^2 r^2}{2} + 2C_1 - \frac{C_2}{r^2} + 12C_3 r^2 + 2C_5 + \frac{6C_6}{r^2} \tag{15}$$
$$= -\frac{\rho\Omega^2 a^2 b^2}{4(b^2-a^2)^2}\left(-12r^2 + 3(a^2+b^2) - \frac{(b^2-a^2)^2}{r^2} + \frac{3a^2b^2(a^2+b^2)}{r^4}\right) \; . \tag{16}$$

The normal tractions $\sigma_{\theta\theta}$ are equivalent to a force F and a moment M given by

$$F = \int_a^b \sigma_{\theta\theta} dr = 0 \tag{17}$$
$$M = \int_a^b \sigma_{\theta\theta} r dr = \frac{\rho\Omega^2 a^2 b^2 \ln(b/a)}{4} \; . \tag{18}$$

Notice that the fact that $F = 0$ could have been predicted from Newton's laws of motion, making a cut across the plane $\theta = 0, \pi$.

To complete the solution, we therefore must superpose the solution of §10.1.1 with

$$M_0 = -\frac{\rho\Omega^2 a^2 b^2 \ln(b/a)}{4} \; . \tag{19}$$

Thus, the complete stress function is

$$\phi = (C_1+A)r^2 + Br^2\ln(r) + (C_2+C)\ln(r) + (C_3 r^4 + C_4 + C_5 r^2 + C_6 r^{-2})\cos(2\theta) \; , \tag{20}$$

where $C_1, C_2, C_3, C_5, C_5, C_6$ are given by (13) and A, B, C are given by (10.15, 10.16) with M_0 replaced by (19).

A more direct solution

We notice at the end of this calculation that only 7 independent stress functions are required. We could have foreseen this by arguing from Newton's law that the $F = 0$ (see above). It then follows that the only terms required are those involved in the complete ring, supplemented by the term $r^2\ln(r)$ which is inadmissible for the complete ring because it gives multivalued displacements, but which features in the solution for the moment.

We can therefore start from the stress function

$$\phi = C_1 r^2 + C_2 r^2 \ln(r) + C_3 \ln(r) + (C_4 r^4 + C_5 + C_6 r^2 + C_7 r^{-2}) \cos(2\theta) \ , \tag{21}$$

proceed exactly as before, but add in the condition that the moment

$$M = \int_a^b \sigma_{\theta\theta} r dr = 0 \ . \tag{22}$$

10.6. Figure 10.6 shows a crane hook of thickness t that is loaded by a force F acting through the centre of the curved section. Find the stress field in the curved portion of the hook and compare the maximum tensile stress with that predicted by the elementary bending theory. Neglect the self-weight of the hook.

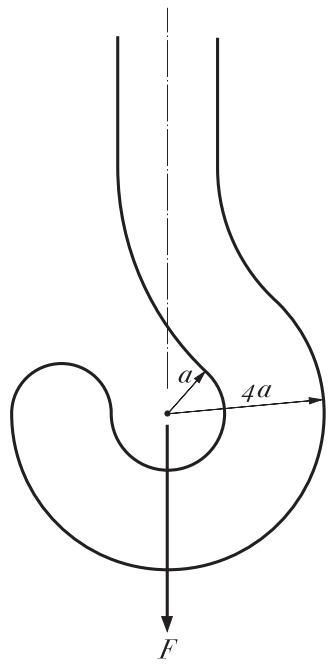


Figure 10.6: The crane hook.

This problem is identical with that of Figure 10.3 if (i) we measure θ from the line of action of the force, (ii) change the sign of the force (since the force in Figure 3 acts towards the center of curvature) and (iii) replace b by $4a$. It follows that the complete stress field in this coördinate system is defined by equations (10.21–10.23) and (10.28, 10.29) — i.e.

$$\sigma_{rr} = (2Ar - 2Br^{-3} + Cr^{-1})\sin\theta \tag{1}$$

$$\sigma_{r\theta} = (-2Ar + 2Br^{-3} - Cr^{-1})\cos\theta \tag{2}$$

$$\sigma_{\theta\theta} = (6Ar + 2Br^{-3} + Cr^{-1})\sin\theta\ , \tag{3}$$

with

$$A = -\frac{F}{2N_1}\ ;\ B = \frac{8Fa^4}{N_1}\ ;\ C = \frac{17Fa^2}{N_1} \tag{4}$$

and

$$N_1 = -15a^2 + 17a^2\ln(4) = 8.567a^2\ . \tag{5}$$

Substituting for the constants, we obtain

$$\sigma_{rr} = \frac{F}{8.567a^2}\left(-r - \frac{16a^4}{r^3} + \frac{17a^4}{r}\right)\sin\theta \tag{6}$$

$$\sigma_{r\theta} = \frac{F}{8.567a^2}\left(r + \frac{16a^4}{r^3} - \frac{17a^4}{r}\right)\cos\theta \tag{7}$$

$$\sigma_{\theta\theta} = \frac{F}{8.567a^2}\left(-3r + \frac{16a^4}{r^3} + \frac{17a^4}{r}\right)\sin\theta\ . \tag{8}$$

Elementary Mechanics of Materials considerations suggest that the maximum tensile stress will occur at the inner radius $r = a$ at $\theta = 0$. The elementary theory gives

$$I = \frac{(3a)^3}{12}$$

and

$$\sigma_{\max} = \frac{F}{3a} + \frac{2.5a \times F \times 1.5a}{I} = \frac{5F}{3a} = \frac{1.667F}{a}\ . \tag{9}$$

The Elasticity solution for the maximum tensile stress is

$$\frac{F}{8.567a}(-3 + 16 + 17) = \frac{3.502F}{a}\ , \tag{10}$$

implying a stress concentration factor of 2.1.

CHAPTER 11

11.1. Figure 11.9 shows a half plane, $y<0$, subjected to a uniform shear traction, $\sigma_{xy}=S$ on the half-line, $x>0,\ y=0$, the remaining tractions on $y=0$ being zero.

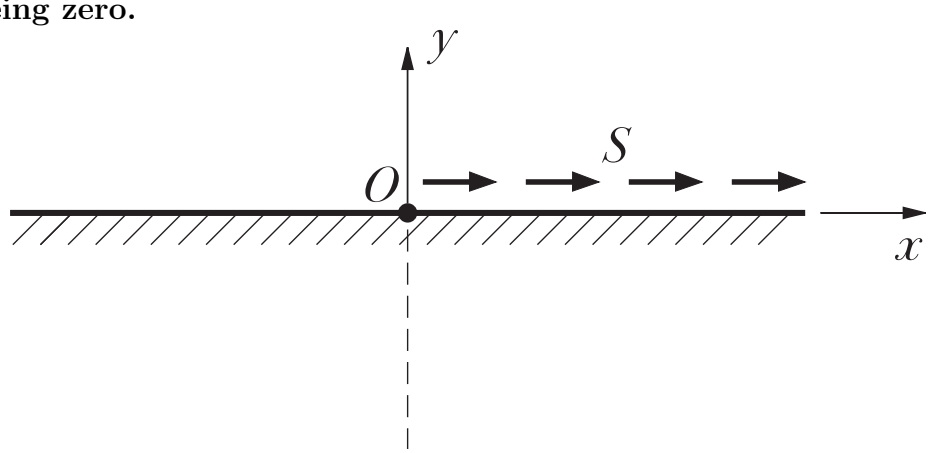


Figure 11.9: The half plane with shear tractions.

Find the complete stress field in the half plane. Note: This is a problem requiring a special stress function with a logarithmic multiplier (see §11.1.4 above).

The half-space is defined by $-\pi < \theta < 0$ in polar coordinates and the boundary conditions of the problem are

$$\sigma_{r\theta} = S \quad ; \quad \theta = 0\ ; \tag{1}$$
$$\sigma_{\theta\theta} = 0 \quad ; \quad \theta = 0;\ ; \tag{2}$$
$$\sigma_{r\theta} = 0 \quad ; \quad \theta = -\pi;\ ; \tag{3}$$
$$\sigma_{\theta\theta} = 0 \quad ; \quad \theta = -\pi;\ . \tag{4}$$

Since the tractions are independent of r, we anticipate needing a stress function proportional to r^2 — i.e.

$$\phi = r^2(A\cos(2\theta) + B\sin(2\theta) + C + D\theta)\ . \tag{5}$$

from equation (11.4). However, we know that the half-space ($2\alpha = \pi$) is an eigenvalue of the uniform loading problem (see §11.1.4) and hence we expect that the algebraic equations obtained from using (5) in the boundary conditions (1–4) would lead to an impossible (singular) problem. We therefore need to generate some additional special stress functions. The form of the functions required is given in §11.1.4, but for completeness we here develop them from the general form, using the technique outlined in §10.3.2.

We start with the general form

$$\phi = r^n(A\cos n\theta + B\sin n\theta + C\cos(n-2)\theta + D\sin(n-2)\theta) \tag{6}$$

and differentiate it with respect to n, obtaining a new stress function with four terms

$$\begin{aligned}\phi' &= r^n[A'(\ln r\cos n\theta - \theta\sin n\theta) + B'(\ln r\sin n\theta + \theta\cos n\theta)\\ &+C'(\ln r\cos(n-2)\theta - \theta\sin(n-2)\theta) + D'(\ln r\sin(n-2)\theta + \theta\cos(n-2)\theta)]\ . \end{aligned}\tag{7}$$

Setting $n = 2$ in this new function yields the function

$$\phi' = A'r^2(\ln r\cos 2\theta - \theta\sin 2\theta) + B'r^2(\ln r\sin 2\theta + \theta\cos 2\theta) + C'r^2\ln r + D'r^2\theta\ . \tag{8}$$

The last term in this new function already appears in (5) above — we had to develop it at an earlier stage to avoid a degeneracy in the function for $n = 0$ for any wedge angle. Thus, to replace this repeated function, we need to apply L'Hôpital's rule a second time on the function in question. Starting with

$$f(r,\theta) = r^n\sin(n-2)\theta$$

and differentiating *twice* with respect to n, we obtain

$$r^n[\ln^2 r\sin(n-2)\theta + 2\ln r\theta\cos(n-2)\theta - \theta^2\sin(n-2)\theta\ .$$

Now, setting $n = 2$, we obtain the new biharmonic function

$$r^2\theta\ln r\ .$$

The complete stress function appropriate for this problem is thus

$$\begin{aligned}\phi &= Ar^2\cos(2\theta) + Br^2\sin(2\theta) + Cr^2 + Dr^2\theta + A'r^2(\ln r\cos(2\theta) - \theta\sin(2\theta))\\ &\quad +B'r^2(\ln r\sin(2\theta) + \theta\cos 2\theta) + C'r^2\ln r + D'r^2\theta\ln r\ . \end{aligned}\tag{9}$$

We now substitute this function into the expressions (8.10, 8.11) to find the stress components, which are

$$\begin{aligned}\sigma_{rr} &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2}\\ &= -2A\cos(2\theta) - 2B\sin(2\theta) + 2C + 2D\theta + A'(-2\ln r\cos(2\theta) - 3\cos(2\theta) + 2\theta\sin(2\theta))\\ &+B'(-2\ln r\sin(2\theta) - 3\sin(2\theta) - 2\theta\cos(2\theta)) + C'(2\ln r + 1) + D'(2\theta\ln r + \theta)\ ; \end{aligned}\tag{10}$$

$$\begin{aligned}\sigma_{\theta\theta} &= \frac{\partial^2\phi}{\partial r^2}\\ &= 2A\cos(2\theta) + 2B\sin(2\theta) + 2C + 2D\theta + A'(2\ln r\cos(2\theta) + 3\cos(2\theta) - 2\theta\sin(2\theta))\\ &+B'(2\ln r\sin(2\theta) + 3\sin(2\theta) + 2\theta\cos(2\theta)) + C'(2\ln r + 3) + D'(2\theta\ln r + 3\theta)\ ; \end{aligned}\tag{11}$$

$$\begin{aligned}\sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)\\ &= 2A\sin(2\theta) - 2B\cos(2\theta) - D + A'(2\ln r\sin(2\theta) + 3\sin(2\theta) + 2\theta\cos(2\theta))\\ &\quad +B'(-2\ln r\cos(2\theta) - 3\cos(2\theta) + 2\theta\sin(2\theta)) + D'(-\ln r - 1)\ . \end{aligned}\tag{12}$$

Substituting these expressions into the boundary conditions (1–4), we obtain

$$
\begin{aligned}
-2B - D + B'(-2\ln r - 3) + D'(-\ln r - 1) &= S\,; && (13)\\
2A + +2C + A'(2\ln r + 3) + C'(2\ln r + 3) &= 0\,; && (14)\\
-2B - D + A'(-2\pi) + B'(-2\ln r - 3) + D'(-\ln r - 1) &= 0\,; && (15)\\
2A + 2C - 2D\pi + A'(2\ln r + 3) & && \\
+B'(-2\pi) + C'(2\ln r + 3) + D'(-2\pi\ln r - 3\pi) &= 0\,. && (16)
\end{aligned}
$$

Now these equations must be satisfied for all r and hence the logarithmic terms and the constant terms must *separately* satisfy them. Thus, we obtain two sets of four equations which are

$$
\begin{aligned}
-2B' - D' &= 0\,; && (17)\\
2A' + 2C' &= 0\,; && (18)\\
-2B' - D' &= 0\,; && (19)\\
2A' + 2 + C' - 2\pi D' &= 0\,, && (20)
\end{aligned}
$$

for the logarithmic terms and

$$
\begin{aligned}
-2B - D - 3B' - D' &= S\,; && (21)\\
2A + 2C + 3A' + 3C' &= 0\,; && (22)\\
-2B - D - 2\pi A' - 3B' - D' &= 0\,; && (23)\\
2A + 2C - 2\pi D + 3A' - 2\pi B' + 3C' - 3\pi D' &= 0\,, && (24)
\end{aligned}
$$

for the constant terms.

The first set of equations are not linearly independent (the first and third equations are identical). As in the problem in §10.3, the matrix of coefficients is identical with that obtained from the original stress function (5) alone and we know that this is singular for the 180° wedge. Equations (18–21) therefore permit the non-trivial solution

$$C' = -A' \quad ; \quad B' = D' = 0\,, \tag{25}$$

where A' can be chosen arbitrarily. Substituting into (22–25), we then have

$$
\begin{aligned}
-2B - D &= S\,; && (26)\\
2A + 2C &= 0\,; && (27)\\
-2B - D - 2\pi A' &= 0\,; && (28)\\
2A + 2C - 2\pi D &= 0\,. && (29)
\end{aligned}
$$

This is a set of four equations for five unknowns. One of the unknowns must therefore be arbitrary. Solving the equations, we obtain

$$B = -\frac{S}{2} \quad ; \quad C = -A \quad ; \quad D = 0 \quad ; \quad A' = \frac{S}{2\pi}\,, \tag{30}$$

where A can be chosen arbitrarily. It is easy to see what this corresponds to. If we pick out the terms involving A, C, with $C = -A$, we obtain the stress function

$$\phi = Ar^2(\cos 2\theta - 1) = -2Ar^2 \sin^2\theta = -2Ay^2 \tag{31}$$

and this corresponds to a state of uniform uniaxial tensile stress $\sigma_{xx} = -4A$; $\sigma_{xy} = \sigma_{yy} = 0$. It is clear that whatever solution we obtained to the problem of Figure 11.9, we could always superpose such a state of uniaxial tension without violating the boundary conditions and hence A could only be determined if we had additional information, such as the state of stress at $x \to \infty$. For the present, we shall simplify the solution by setting $A = 0$, recognizing that the additional term can always be included at the end if required.

With these values, we finally have

$$\phi = -\frac{Sr^2 \sin(2\theta)}{2} + \frac{Sr^2(\ln r \cos(2\theta) - \theta \sin(2\theta))}{2\pi} - \frac{Sr^2 \ln r}{2\pi} \tag{32}$$

and the complete stress field is

$$\begin{aligned}
\sigma_{rr} &= S\sin(2\theta) + \frac{S}{2\pi}(-2\ln r\cos(2\theta) - 3\cos(2\theta) + 2\theta\sin(2\theta)) - \frac{S}{2\pi}(2\ln r + 1) \\
\sigma_{\theta\theta} &= -S\sin(2\theta) + \frac{S}{2\pi}(2\ln r\cos(2\theta) + 3\cos(2\theta) - 2\theta\sin(2\theta)) - \frac{S}{2\pi}(2\ln r + 3) \\
\sigma_{r\theta} &= S\cos(2\theta) + \frac{S}{2\pi}(2\ln r\sin(2\theta) + 3\sin(2\theta) + 2\theta\cos(2\theta)) \ .
\end{aligned} \tag{33}$$

11.2. The half plane $y<0$ is subjected to a uniform normal pressure $\sigma_{yy}=-S$ on the half-line, $x>0,\ y=0$, the remaining tractions on $y=0$ being zero. Find the complete stress field in the half plane.

The boundary conditions for this problem in polar coördinates are

$$\begin{aligned}
\sigma_{\theta r} &= 0\,; \quad \theta = 0 && (1)\\
&= 0\,; \quad \theta = -\pi && (2)\\
\sigma_{\theta\theta} &= -S\,; \quad \theta = 0 && (3)\\
&= 0\,; \quad \theta = -\pi\,. && (4)\\
& && (5)
\end{aligned}$$

Since the tractions are uniform (varying with r^0), we start with the stress function (11.4), for which the stress components are

$$\begin{aligned}
\sigma_{rr} &= -2A_1\cos(2\theta) + 2A_2 - 2A_3\sin(2\theta) + 2A_4\theta && (6)\\
\sigma_{r\theta} &= 2A_1\sin(2\theta) - 2A_3\cos(2\theta) - A_4 && (7)\\
\sigma_{\theta\theta} &= 2A_1\cos(2\theta) + 2A_2 + 2A_3\sin(2\theta) + 2A_4\theta\,, && (8)
\end{aligned}$$

from (11.5–11.7). Substituting into the boundary conditions, we obtain

$$\begin{aligned}
-2A_3 - A_4 &= 0\,; \quad \theta = 0 && (9)\\
-2A_3 - A_4 &= 0\,; \quad \theta = -\pi && (10)\\
2A_1 + 2A_2 &= -S\,; \quad \theta = 0 && (11)\\
2A_1 + 2A_2 - 2\pi A_4 &= 0\,; \quad \theta = -\pi\,. && (12)\\
& && (13)
\end{aligned}$$

Notice that although these equations are not linearly-independent, the redundancy does not lead to an inconsistency, so we do not require a special stress function. Also, since two of the equations are identical, one of the four constants can be chosen arbitrarily. The solution is

$$A_2 = -A_1 - \frac{S}{2}\,; \quad A_3 = \frac{S}{4\pi}\,; \quad A_4 = -\frac{S}{2\pi}\,, \tag{14}$$

where A_1 may be chosen arbitrarily. It is easy to see what this corresponds to. If we pick out the terms involving A_1, A_2, with $A_2 = -A_1$, we obtain the stress function

$$\phi = -A_1 r^2(1-\cos(2\theta)) = -2A_1 r^2\sin^2\theta = -2A_1 y^2 \tag{15}$$

and this corresponds to a state of uniform uniaxial tensile stress $\sigma_{xx} = -4A_1$; $\sigma_{xy} = \sigma_{yy} = 0$. It is clear that whatever solution we obtained to the given problem, we could always superpose such a state of uniaxial tension without violating the boundary

conditions and hence A_1 could only be determined if we had additional information, such as the state of stress at $x \to \infty$. For the present, we shall simplify the solution by setting $A_1 = 0$, recognizing that the additional term can always be included at the end if required.

The stress field then reduces to

$$\sigma_{rr} = -\frac{S}{2\pi}(\sin(2\theta) + 2\theta + 2\pi) \tag{16}$$

$$\sigma_{r\theta} = \frac{S}{2\pi}(1 - \cos(2\theta)) \tag{17}$$

$$\sigma_{\theta\theta} = \frac{S}{2\pi}(\sin(2\theta) - 2\theta - 2\pi) \ . \tag{18}$$

11.3. The wedge $-\alpha < \theta < \alpha$ is loaded by a concentrated moment M_0 at the apex, the plane edges being traction free. Use dimensional arguments to show that the stress components must all have the separated-variable form

$$\sigma = \frac{f(\theta)}{r^2} \ .$$

Use this result to choose a suitable stress function and hence find the complete stress field in the wedge.

The moment transmitted across the surface $r = a$ is

$$M = \int_{-\alpha}^{\alpha} \sigma_{r\theta} r^2 d\theta \ , \tag{1}$$

since the elemental force $\sigma_{r\theta} r d\theta$ has a moment arm r. Notice that the line of action of the force $\sigma_{rr} r d\theta$ passes through the apex and hence makes no contribution to the moment.

If equation (1) is to define a constant value M_0, the stress component $\sigma_{r\theta}$ must have the form

$$\sigma_{r\theta} = \frac{f(\theta)}{r^2} \ . \tag{2}$$

The most general stress function giving components of this form is

$$\phi = C_1 \ln(r) + C_2\theta + C_3 \cos(2\theta) + C_4 \sin(2\theta) \ , \tag{3}$$

from Table 8.1, the corresponding stress components being

$$\sigma_{rr} = \frac{C_1}{r^2} - \frac{4C_3 \cos(2\theta)}{r^2} - \frac{4C_4 \sin(2\theta)}{r^2} \tag{4}$$

$$\sigma_{r\theta} = \frac{C_2}{r^2} - \frac{2C_3 \sin(2\theta)}{r^2} + \frac{2C_4 \cos(2\theta)}{r^2} \tag{5}$$

$$\sigma_{\theta\theta} = -\frac{C_1}{r^2} \ . \tag{6}$$

For the surfaces of the wedge to be traction free, we require

$$\sigma_{\theta r} = \sigma_{\theta\theta} = 0 \ ; \quad \theta = \pm\alpha \tag{7}$$

and hence

$$\frac{C_2}{r^2} - \frac{2C_3 \sin(2\alpha)}{r^2} + \frac{2C_4 \cos(2\alpha)}{r^2} = 0 \tag{8}$$

$$\frac{C_2}{r^2} + \frac{2C_3 \sin(2\alpha)}{r^2} + \frac{2C_4 \cos(2\alpha)}{r^2} = 0 \tag{9}$$

$$-\frac{C_1}{r^2} = 0 \tag{10}$$

$$-\frac{C_1}{r^2} = 0 \ . \tag{11}$$

The last two equations are clearly identical, so there is a non-trivial solution to this system of equations, which is

$$C_1 = C_3 = 0 \; ; \quad C_2 = -2C_4 \cos(2\alpha) \; . \tag{12}$$

Substituting these results into (4–6), we have

$$\sigma_{rr} = -\frac{4C_4 \sin(2\theta)}{r^2} \tag{13}$$

$$\sigma_{r\theta} = -\frac{2C_4}{r^2}(\cos(2\alpha) - \cos(2\theta) \tag{14}$$

$$\sigma_{\theta\theta} = 0 \; . \tag{15}$$

To determine the remaining constant, we substitute into equation (1) with $M = M_0$, obtaining

$$M_0 = -2C_4 \int_{-\alpha}^{\alpha} (\cos(2\alpha) - \cos(2\theta) d\theta = -2C_4(2\alpha \cos(2\alpha) - \sin(2\alpha)) \; . \tag{16}$$

Thus,

$$C_4 = -\frac{M_0}{2(2\alpha \cos(2\alpha) - \sin(2\alpha))} \tag{17}$$

and the final stress field is

$$\sigma_{rr} = \frac{2M_0 \sin(2\theta)}{(2\alpha \cos(2\alpha) - \sin(2\alpha))r^2} \tag{18}$$

$$\sigma_{r\theta} = \frac{M_0(\cos(2\alpha) - \cos(2\theta)}{(2\alpha \cos(2\alpha) - \sin(2\alpha))r^2} \tag{19}$$

$$\sigma_{\theta\theta} = 0 \; . \tag{20}$$

11.4. Show that $\phi = Ar^2\theta$ can be used as a stress function and determine the tractions which it implies on the boundaries of the region $-\pi/2 < \theta < \pi/2$. Hence show that the stress function appropriate to the loading of Figure 11.10 is

$$\phi = -\frac{F}{4\pi a}(r_1^2\theta_1 + r_2^2\theta_2)$$

where $r_1, r_2, \theta_1, \theta_2$ are defined in the Figure.

Determine the principal stresses at the point B.

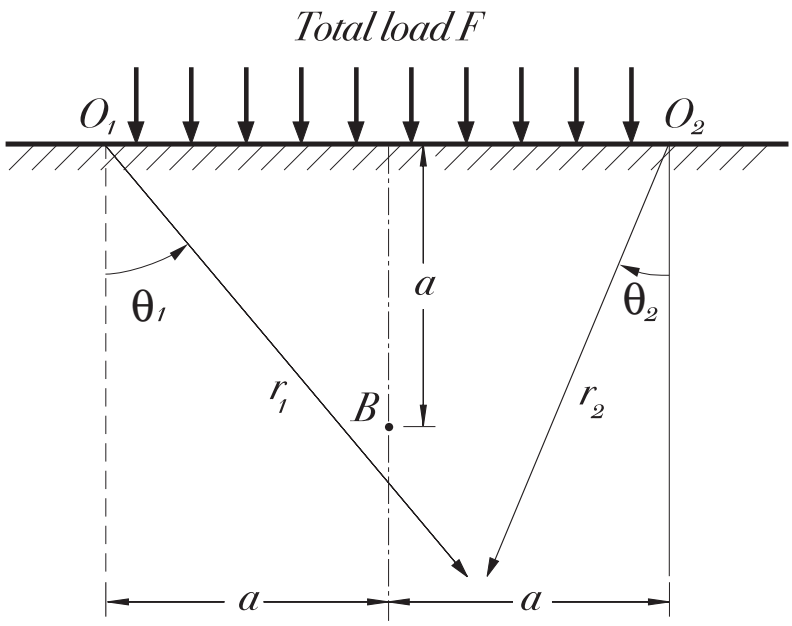


Figure 11.10: Uniform loading over a discrete region.

With $\phi = Ar^2\theta$, we have

$$\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} = 2A\theta + 2A\theta + 0 = 4\theta \tag{1}$$

and hence $\nabla^2\phi = 0$. Thus, the proposed function is biharmonic and can therefore be used as an Airy stress function.

The corresponding stress components are

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2} = 2A\theta\ ; \tag{2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = 2A\theta\ ; \tag{3}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = -A\ . \tag{4}$$

On the boundaries $\theta = \pm\pi/2$, these reduce to the tractions

$$\sigma_{r\theta} = -A \quad ; \quad \theta = \frac{\pi}{2} \; ; \tag{5}$$

$$\sigma_{\theta\theta} = \pi A \quad ; \quad \theta = \frac{\pi}{2} \; ; \tag{6}$$

$$\sigma_{r\theta} = -A \quad ; \quad \theta = -\frac{\pi}{2} \; ; \tag{7}$$

$$\sigma_{\theta\theta} = -\pi A \quad ; \quad \theta = -\frac{\pi}{2} \; , \tag{8}$$

which are illustrated in Figure 1. Notice that the shear traction retains the same direction through the origin, but the normal traction is uniform tension on the right and uniform compression on the left.

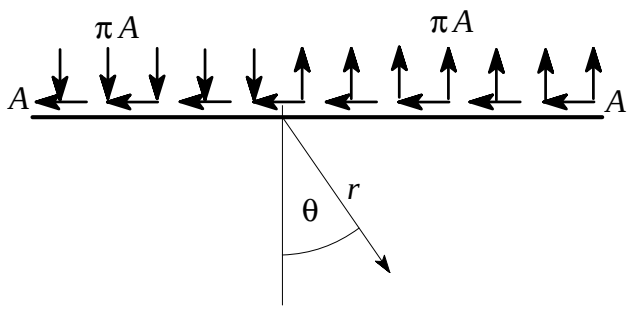


Figure 1

To solve the problem of Figure 11.10, we use the above stress function centered on the origin O_1 and the mirror image solution centered on the origin O_2. The mirror image solution will give a uniform shear traction with opposite direction, so the two shear tractions will cancel. The normal tractions will also cancel *except* in the region between O_1 and O_2 as shown in Figure 2.

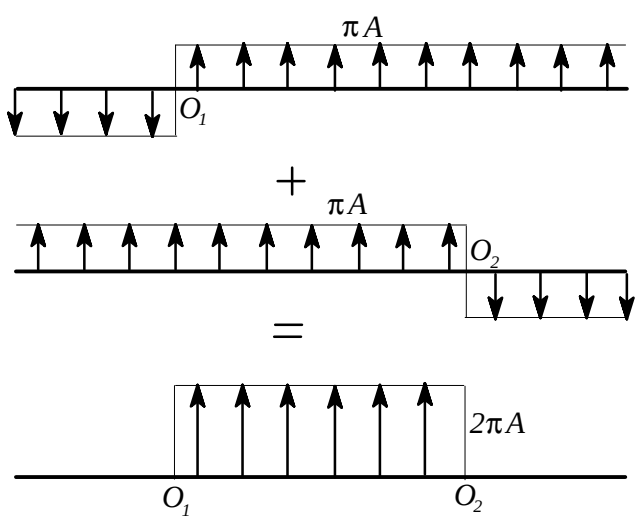


Figure 2

The traction in the central region will be $\pi A + \pi A = 2\pi A$, so if the total load is to be F and compressive, we require $2\pi A = -F/(2a)$ and hence

$$A = -\frac{F}{4\pi a} \ , \tag{9}$$

as specified in the problem statement. Notice that the direction of the angles θ_1, θ_2 in Figure 11.10 takes care of the requirement that the second superposed solution be a mirror image of the first.

At the point B, $r_1 = r_2 = a\sqrt{2}$ and $\theta_1 = \theta_2 = \frac{\pi}{4}$. Thus, the stress components resulting from the term centered on O_1 are

$$\sigma_{rr} = -\frac{F}{8a} \ ; \tag{10}$$

$$\sigma_{\theta\theta} = -\frac{F}{8a} \ ; \tag{11}$$

$$\sigma_{r\theta} = \frac{F}{4\pi a} \ . \tag{12}$$

These components are defined in the r_1, θ_1 coordinate system and hence appear on a small element at B as shown in Figure 3(a). The mirror image function gives an identical stress state at this point and therefore the complete solution is twice the values of equations (10–12). Finally, we use Mohr's circle to find the principal stresses, as shown in Figure 3(b), obtaining

$$\sigma_1 = -\frac{F}{2a}\left(\frac{1}{2} - \frac{1}{\pi}\right) \quad ; \quad \sigma_2 = -\frac{F}{2a}\left(\frac{1}{2} + \frac{1}{\pi}\right) \ . \tag{13}$$

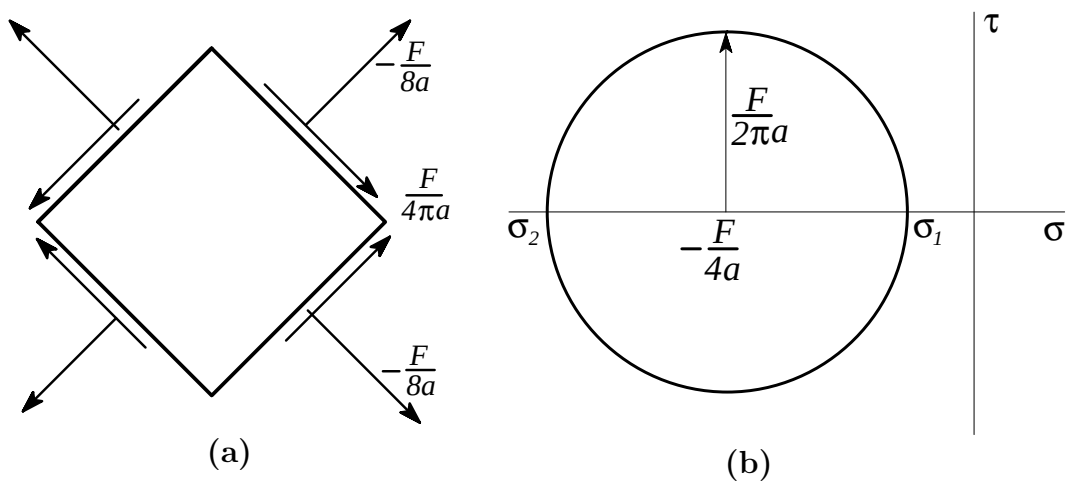


Figure 3

11.5. A wedge-shaped concrete dam is subjected to a hydrostatic pressure $\rho g h$ varying with depth h on the vertical face $\theta = 0$ as shown in Figure 11.11, the other face $\theta = \alpha$ being traction-free. The dam is also loaded by self-weight, the density of concrete being $\rho_c = 2.3\rho$.

Find the minimum wedge angle α if there is to be no tensile stress in the dam.

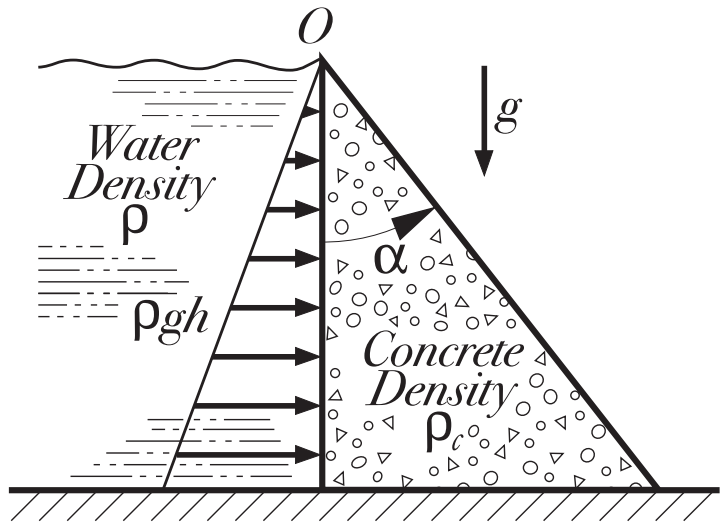


Figure 11.11: The wedge-shaped dam.

We define a polar coordinate system such that the vertical face of the dam is the line $\theta = 0$. The boundary conditions of the problem are then

$$\sigma_{\theta r} = 0 \quad ; \quad \theta = 0 \; ; \tag{1}$$

$$\sigma_{\theta\theta} = -\rho g r \quad ; \quad \theta = 0 \; ; \tag{2}$$

$$\sigma_{\theta r} = 0 \quad ; \quad \theta = \alpha \; ; \tag{3}$$

$$\sigma_{\theta\theta} = 0 \quad ; \quad \theta = \alpha \; . \tag{4}$$

We also have a body force due to gravity. In rectangular coordinates, the $x-$axis is the line $\theta = 0$ and hence points downwards. Hence, an appropriate body force potential is

$$V = -\rho_c g x = -\rho_c g r \cos\theta \; . \tag{5}$$

All these results point to a stress field which is of the form $r f(\theta)$. We therefore select a stress function of the form $r^3 f(\theta)$. Using Table 8.1, or equation (8.59), the most general stress function of this form is

$$\phi = A\cos 3\theta + B\sin 3\theta + C\cos\theta + D\sin\theta \; , \tag{6}$$

corresponding to the stress field

$$\sigma_{rr} = -6Ar\cos 3\theta - 6Br\sin 3\theta + 2Cr\cos\theta + 2Dr\sin\theta - \rho_c gr\cos\theta \; ; \tag{7}$$
$$\sigma_{r\theta} = 6Ar\sin 3\theta - 6Br\cos 3\theta + 2Cr\sin\theta - 2Dr\cos\theta \; ; \tag{8}$$
$$\sigma_{\theta\theta} = 6Ar\cos 3\theta + 6Br\sin 3\theta + 6Cr\cos\theta + 6Dr\sin\theta - \rho_c gr\cos\theta \; , \tag{9}$$

where we note that the additional terms $+V$ have been added into the expressions for the normal stresses only obtained from Table 8.1.

The boundary conditions (1–4) now define the following four equations for the four unknowns A, B, C, D:-

$$-6B - 2D = 0 \; ; \tag{10}$$
$$6A + 6C = (\rho_c - \rho)g \; ; \tag{11}$$
$$6A\sin 3\alpha - 6B\cos 3\alpha + 2C\sin\alpha - 2D\cos\alpha = 0 \; ; \tag{12}$$
$$6A\cos 3\alpha + 6B\sin 3\alpha + 6C\cos\alpha + 6D\sin\alpha = \rho_c g\cos\alpha \; . \tag{13}$$

To express everything in terms of $\sin\alpha, \cos\alpha$, we shall need the trigonometric results

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha \quad ; \quad \sin 3\alpha = -4\sin^3\alpha + 3\sin\alpha \; . \tag{14}$$

From (10,11), we have

$$D = -3B \quad ; \quad C = \frac{1}{6}(\rho_c - \rho)g - A \; . \tag{15}$$

Using these results, the remaining two equations (12,13) can be written

$$6A(-4\sin^3\alpha + 3\sin\alpha) - 6B(4\cos^3\alpha - 3\cos\alpha) + \frac{1}{3}(\rho_c - \rho)g\sin\alpha - 2A\sin\alpha + 6B\cos\alpha = 0 \; ; \tag{16}$$

$$6A(4\cos^3\alpha - 3\cos\alpha) + 6B(-4\sin^3\alpha + 3\sin\alpha) + (\rho_c - \rho)g\cos\alpha - 6A\cos\alpha - 18B\sin\alpha = \rho_c g\cos\alpha \tag{17}$$

and hence

$$A(-24\sin^3\alpha + 16\sin\alpha) + B(-24\cos^3\alpha + 24\cos\alpha) = -\frac{1}{3}(\rho_c - \rho)g\sin\alpha \; ; \tag{18}$$
$$A(24\cos^3\alpha - 24\cos\alpha) - 24B\sin^3\alpha = \rho g\cos\alpha \; , \tag{19}$$

which can be written

$$A(-3\sin^2\alpha + 2) + 3B\sin\alpha\cos\alpha = -\frac{1}{24}(\rho_c - \rho)g \; ; \tag{20}$$
$$A\cos\alpha + B\sin\alpha = -\frac{\rho g\cos\alpha}{24\sin^2\alpha} \; . \tag{21}$$

These equations can be solved to give

$$A = \frac{(\rho_c - \rho)g}{24} - \frac{\rho g\cot^2\alpha}{8} \quad ; \quad B = \frac{\rho g\cot^3\alpha}{12} - \frac{\rho_c g\cot\alpha}{24} \; . \tag{22}$$

Substitution in (15) then yields

$$C = \frac{(\rho_c - \rho)g}{8} + \frac{\rho g \cot^2 \alpha}{8} \quad ; \quad D = \frac{\rho_c g \cot \alpha}{8} - \frac{\rho g \cot^3 \alpha}{4} \tag{23}$$

and the full stress field is obtained by substituting these constants back into equations (7–9).

Strength of Materials arguments suggest that the maximum tensile stress will be the bending stress σ_{rr} at $\theta = 0$. This can be verified by plotting the maximum principal stress

$$\sigma_{\max} = \frac{\sigma_{rr} + \sigma_{\theta\theta}}{2} + \sqrt{\left(\frac{\sigma_{rr} + \sigma_{\theta\theta}}{2}\right)^2 + \sigma_{r\theta}^2} \tag{24}$$

as a function of θ in $0 < \theta < \alpha$ for a variety of cases.

The minimum wedge angle for the stresses in the dam to be everywhere compressive is therefore obtained by setting $\sigma_{rr}(r, 0) = 0$ and solving for α.

We then have

$$-6A + 2C = \rho_c g \ , \tag{25}$$

from equation (7). Substituting for A, C and simplifying, we obtain

$$\rho g \cot^2 \alpha = \rho_c g \quad ; \quad \cot^2 \alpha = \frac{\rho_c}{\rho} = 2.3 \tag{26}$$

and hence

$$\alpha = 33.4^o \ . \tag{27}$$

It is easily verified that for $\alpha > 33.4^o$, the stress σ_{rr} at $\theta = 0$ is compressive, whereas for $\alpha < 33.4^o$ it is tensile.

11.6. Figure 11.12 shows a 45° triangular plate ABC built in at BC and loaded by a uniform pressure p_0 on the upper edge AB. The inclined edge AC is traction free.

Find the stresses in the plate, using weak boundary conditions on the edge BC (which do not need to be explicitly enforced). Hence compare the maximum tensile stress with the prediction of the elementary bending theory.

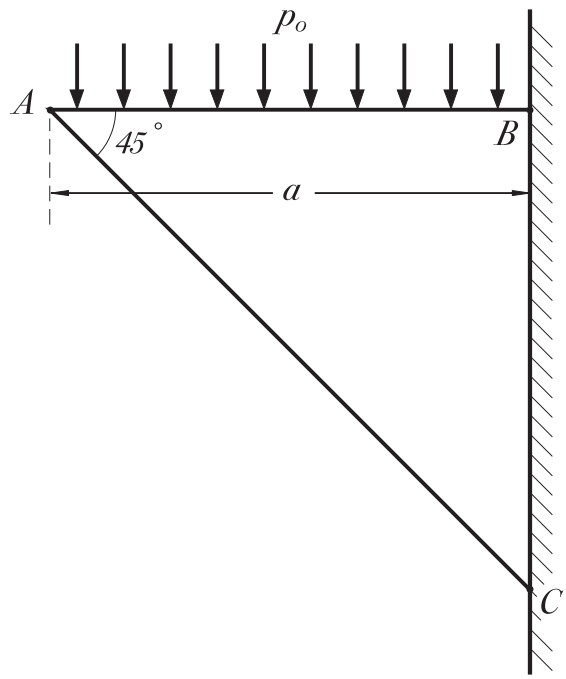


Figure 11.12

Following §11.1.1, we start with the stress function

$$\phi = r^2\{A_1\cos(2\theta) + A_2 + A_3\sin(2\theta) + A_4\theta\} \ , \tag{1}$$

for which the stress components are

$$\sigma_{rr} = -2A_1\cos(2\theta) + 2A_2 - 2A_3\sin(2\theta) + 2A_4\theta \tag{2}$$

$$\sigma_{r\theta} = 2A_1\sin(2\theta) - 2A_3\cos(2\theta) - A_4 \tag{3}$$

$$\sigma_{\theta\theta} = 2A_1\cos(2\theta) + 2A_2 + 2A_3\sin(2\theta) + 2A_4\theta \ . \tag{4}$$

The boundary conditions are

$$\sigma_{\theta\theta}(r,0) = -p_0 \tag{5}$$

$$\sigma_{\theta r}(r,0) = 0 \tag{6}$$

$$\sigma_{\theta\theta}(r,-\pi/4) = 0 \tag{7}$$

$$\sigma_{\theta r}(r,-\pi/4) = 0 \tag{8}$$

and hence

$$
\begin{aligned}
2A_1 + 2A_2 &= -p_0 && (9)\\
-2A_3 - A_4 &= 0 && (10)\\
2A_2 - 2A_3 - \frac{\pi A_4}{2} &= 0 && (11)\\
-2A_1 - A_4 &= 0\,. && (12)
\end{aligned}
$$

Solving these equations, we obtain

$$
A_1 = A_3 = -\frac{p_0}{(4-\pi)}\ ;\quad A_2 = \frac{p_0(\pi-2)}{2(4-\pi)}\ ;\quad A_4 = \frac{2p_0}{(4-\pi)}\ . \tag{13}
$$

The final stress field is therefore

$$
\begin{aligned}
\sigma_{rr} &= \frac{p_0\{2\cos(2\theta) + \pi - 2 + 2\sin(2\theta) + 4\theta\}}{(4-\pi)} && (14)\\
\sigma_{r\theta} &= \frac{2p_0\{-\sin(2\theta) + \cos(2\theta) - 1\}}{(4-\pi)} && (15)\\
\sigma_{\theta\theta} &= \frac{p_0\{-2\cos(2\theta) + \pi - 2 - 2\sin(2\theta) + 4\theta\}}{(4-\pi)}\ . && (16)
\end{aligned}
$$

The maximum tensile stress occurs at $\theta = 0$ and is

$$
\sigma_{rr}(r,0) = \frac{p_0\{2+\pi-2\}}{(4-\pi)} = \frac{\pi p_0}{(4-\pi)} = 3.66p_0\ .
$$

Notice that there is a bigger compressive stress at $\theta = -\pi/4$ given by

$$
\sigma_{rr}(r,-\pi/4) = \frac{p_0\{\pi-2-2-\pi\}}{(4-\pi)} = -\frac{4p_0}{(4-\pi)} = -4.66p_0\ .
$$

For the elementary theory,

$$
M = \frac{p_0 a^2}{2}\ ;\quad y_{\max} = \frac{a}{2}\ ;\quad I = \frac{a^3}{12}\ ,
$$

so

$$
\sigma_{\max} = \frac{p_0 a^2}{2}\cdot\frac{a}{2}\cdot\frac{12}{a^3} = 3p_0\ .
$$

Thus, the more exact maximum tensile stress is greater by 22%.

11.7. The wedge $-\alpha<\theta<\alpha$ is bonded to a rigid body on both edges $\theta=\pm\alpha$. Use the eigenfunction expansion of §11.2.2 to determine the characteristic equations that must be satisfied by the exponent λ in the stress function (11.38) for symmetric and antisymmetric stress fields. Show that these equations reduce to (11.53, 11.54) if $\nu=0.5$ and plane strain conditions are assumed.

The boundary conditions of the problem are

$$u_r = u_\theta = 0\ ; \quad \theta = \pm\alpha \tag{1}$$

and hence equations (11.75, 11.76) require that

$$\begin{aligned}
&-A_1(\lambda+1)\cos(\lambda+1)\alpha + A_2(\kappa-\lambda)\cos(\lambda-1)\alpha \\
&-A_3(\lambda+1)\sin(\lambda+1)\alpha + A_4(\kappa-\lambda)\sin(\lambda-1)\alpha &= 0 && (2)\\
&A_1(\lambda+1)\sin(\lambda+1)\alpha + A_2(\kappa+\lambda)\sin(\lambda-1)\alpha \\
&-A_3(\lambda+1)\cos(\lambda+1)\alpha - A_4(\kappa+\lambda)\cos(\lambda-1)\alpha &= 0 && (3)\\
&-A_1(\lambda+1)\cos(\lambda+1)\alpha + A_2(\kappa-\lambda)\cos(\lambda-1)\alpha \\
&+A_3(\lambda+1)\sin(\lambda+1)\alpha - A_4(\kappa-\lambda)\sin(\lambda-1)\alpha &= 0 && (4)\\
&-A_1(\lambda+1)\sin(\lambda+1)\alpha - A_2(\kappa+\lambda)\sin(\lambda-1)\alpha \\
&-A_3(\lambda+1)\cos(\lambda+1)\alpha - A_4(\kappa+\lambda)\cos(\lambda-1)\alpha &= 0\ . && (5)
\end{aligned}$$

To expose the symmetry, we take sums and differences of these equations in pairs, obtaining

$$\begin{aligned}
-A_1(\lambda+1)\cos(\lambda+1)\alpha + A_2(\kappa-\lambda)\cos(\lambda-1)\alpha &= 0 && (6)\\
A_1(\lambda+1)\sin(\lambda+1)\alpha + A_2(\kappa+\lambda)\sin(\lambda-1)\alpha &= 0 && (7)\\
-A_3(\lambda+1)\sin(\lambda+1)\alpha + A_4(\kappa-\lambda)\sin(\lambda-1)\alpha &= 0 && (8)\\
-A_3(\lambda+1)\cos(\lambda+1)\alpha - A_4(\kappa+\lambda)\cos(\lambda-1)\alpha &= 0\ . && (9)
\end{aligned}$$

Eigenvalues for the symmetric terms A_1, A_2 occur when

$$[-(\lambda+1)\cos(\lambda+1)\alpha][(\kappa+\lambda)\sin(\lambda-1)\alpha]-[(\kappa-\lambda)\cos(\lambda-1)\alpha][(\lambda+1)\sin(\lambda+1)\alpha] = 0\ , \tag{10}$$

which simplifies to

$$\lambda\sin(2\alpha) - \kappa sin(2\lambda\alpha) = 0 \tag{11}$$

after cancelling a factor of $(\lambda-1)$.

For the antisymmetric terms A_3, A_4, we have

$$[-(\lambda+1)\sin(\lambda+1)\alpha][-(\kappa+\lambda)\cos(\lambda-1)\alpha]-[(\kappa-\lambda)\sin(\lambda-1)\alpha][-(\lambda+1)\cos(\lambda+1)\alpha] = 0\ , \tag{12}$$

which simplifies to

$$\lambda\sin(2\alpha) + \kappa sin(2\lambda\alpha) = 0\ . \tag{13}$$

In plane strain, $\kappa = 3-4\nu$, so if $\nu = 0.5$, $\kappa = 1$. In this case, equations (11,13) reduce to (11.54, 11.53) respectively.

11.8. Find the equation that must be satisfied by λ if the stress function (11.38) is to define a non-trivial solution of the problem of the half plane $0<\theta<\pi$, traction-free on $\theta=0$ and in frictionless contact with a rigid plane surface at $\theta=\pi$. Find the lowest value of λ that satisfies this equation and obtain explicit expressions for the form of the corresponding singular stress field near the corner. This solution is of importance in connection with the frictionless indentation of a smooth elastic body by a rigid body with a sharp corner.

The boundary conditions of the problem are

$$\begin{aligned}
\sigma_{\theta r} &= 0\,; \quad \theta=0 && (1)\\
\sigma_{\theta\theta} &= 0\,; \quad \theta=0 && (2)\\
\sigma_{\theta r} &= 0\,; \quad \theta=\pi && (3)\\
u_\theta &= 0\,; \quad \theta=\pi && (4)
\end{aligned}$$

Using equations (11.40, 11.41, 11.75, 11.76), we obtain the conditions

$$\begin{aligned}
-A_3\lambda(\lambda+1) - A_4\lambda(\lambda-1) &= 0 && (5)\\
A_1\lambda(\lambda+1) + A_2\lambda(\lambda+1) &= 0 && (6)\\
A_1\lambda(\lambda+1)\sin(\lambda+1)\pi + A_2\lambda(\lambda-1)\sin(\lambda-1)\pi & \\
-A_3\lambda(\lambda+1)\cos(\lambda+1)\pi - A_4\lambda(\lambda-1)\cos(\lambda-1)\pi &= 0 && (7)\\
A_1(\lambda+1)\sin(\lambda+1)\pi + A_2(\kappa+\lambda)\sin(\lambda-1)\pi & \\
-A_3(\lambda+1)\cos(\lambda+1)\pi - A_4(\kappa+\lambda)\cos(\lambda-1)\pi &= 0\,. && (8)
\end{aligned}$$

We can use (5,6) to eliminate A_1, A_3 in (7,8), giving

$$\begin{aligned}
-A_2\lambda(\lambda+1)\sin(\lambda+1)\pi + A_2\lambda(\lambda-1)\sin(\lambda-1)\pi & \\
+A_4\lambda(\lambda-1)\cos(\lambda+1)\pi - A_4\lambda(\lambda-1)\cos(\lambda-1)\pi &= 0 && (9)\\
-A_2(\lambda+1)\sin(\lambda+1)\pi + A_2(\kappa+\lambda)\sin(\lambda-1)\pi & \\
+A_4(\lambda-1)\cos(\lambda+1)\pi - A_4(\kappa+\lambda)\cos(\lambda-1)\pi &= 0 && (10)
\end{aligned}$$

and these two equations must be identical, giving

$$\begin{aligned}
&[-\lambda(\lambda+1)\sin(\lambda+1)\pi + \lambda(\lambda-1)\sin(\lambda-1)\pi]\\
&\quad\times[(\lambda-1)\cos(\lambda+1)\pi - (\kappa+\lambda)\cos(\lambda-1)\pi]\\
&= [\lambda(\lambda-1)\cos(\lambda+1)\pi - \lambda(\lambda-1)\cos(\lambda-1)\pi]\\
&\quad\times[-(\lambda+1)\sin(\lambda+1)\pi + (\kappa+\lambda)\sin(\lambda-1)\pi] && (11)
\end{aligned}$$

Multiplying out this expression and using trigonometrical identities, we obtain the simpler expression

$$\lambda(\kappa+1)\sin(2\pi\lambda) = 0\,. \qquad (12)$$

The solutions of this equation are

$$\lambda = \frac{1}{2}, 1, \frac{3}{2}, \ldots \tag{13}$$

The singular stress field is associated with the eigenvalue $\lambda = 1/2$. Substituting this value into equations (5–8) we obtain

$$-\frac{3A_3}{4} + \frac{A_4}{4} = 0 \tag{14}$$

$$\frac{A_1}{4} + \frac{3A_2}{4} = 0 \tag{15}$$

$$-\frac{3A_1}{4} + \frac{A_2}{4} = 0 \tag{16}$$

$$-\frac{3A_1}{2} - \frac{(2\kappa+1)A_2}{2} = 0\,. \tag{17}$$

Of these equations, (15–17) are satisfied by the choice $A_1 = A_2 = 0$ and (18) can be satisfied by

$$A_4 = 3A_3 \equiv C \tag{18}$$

where we have defined a new constant C as the multiplier on the singular stress field. Substituting these values and $\lambda = 1/2$ into the stress components (11.39–11.41), we then obtain the singular stress field as

$$\sigma_{rr} = -\frac{C}{4\sqrt{r}}\left\{5\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{2}\right)\right\} \tag{19}$$

$$\sigma_{r\theta} = \frac{C}{4\sqrt{r}}\left\{\cos\left(\frac{\theta}{2}\right) - \cos\left(\frac{3\theta}{2}\right)\right\}\,, \tag{20}$$

$$\sigma_{\theta\theta} = -\frac{C}{4\sqrt{r}}\left\{\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{3\theta}{2}\right)\right\}\,. \tag{21}$$

11.9. Two large bodies of similar materials with smooth, continuously differentiable curved surfaces make frictionless contact. Use the asymptotic method of §11.2 to examine the asymptotic stress fields near the edge of the resulting contact area. The solution of this problem must also satisfy two *inequalities*: that the contact tractions in the contact region must be compressive and that the normal displacements in the non-contact region must cause a non-negative gap. Show that one or other of these conditions is violated, whatever sign is taken for the multiplier on the first eigenfunction. What conclusion do you draw from this result?

We suppose that the contact area occupies the region $\theta = 0$ and that the two half-planes are defined by $-\pi < \theta < 0$ and $0 < \theta < \pi$ respectively. Separation with a positive gap is required between the two surfaces $\theta = \pm\pi$.

The equality boundary conditions are

$$\begin{aligned}
\sigma_{\theta r} = \sigma_{\theta\theta} &= 0\,; \quad \theta = \pm\pi && (1)\\
\sigma_{\theta r} &= 0\,; \quad \theta = 0 && (2)\\
\sigma^1_{\theta\theta} &= \sigma^2_{\theta\theta}\,; \quad \theta = 0 && (3)\\
u^1_\theta &= u^2_\theta\,; \quad \theta = 0\,, && (4)
\end{aligned}$$

where superscripts $1, 2$ relate to the half planes $-\pi < \theta < 0$ and $0 < \theta < \pi$ respectively.

We represent the stresses and displacements in the two half-planes by the stress functions

$$\phi_i = r^{\lambda+1}\{A_{i1}\cos(\lambda+1)\theta + A_{i2}\cos(\lambda-1)\theta + A_{i3}\sin(\lambda+1)\theta + A_{i4}\sin(\lambda-1)\theta\}\,, \quad (5)$$

where $i = 1, 2$ respectively. Using equations (11.40, 11.41, 11.76), we can then write equations (1–4) as

$$\begin{aligned}
&-A_{11}\lambda(\lambda+1)\sin(\lambda+1)\pi - A_{12}\lambda(\lambda-1)\sin(\lambda-1)\pi\\
&-A_{13}\lambda(\lambda+1)\cos(\lambda+1)\pi - A_{14}\lambda(\lambda-1)\cos(\lambda-1)\pi = 0 && (6)\\
&A_{21}\lambda(\lambda+1)\sin(\lambda+1)\pi + A_{22}\lambda(\lambda-1)\sin(\lambda-1)\pi\\
&-A_{23}\lambda(\lambda+1)\cos(\lambda+1)\pi - A_{24}\lambda(\lambda-1)\cos(\lambda-1)\pi = 0 && (7)\\
&A_{11}\lambda(\lambda+1)\cos(\lambda+1)\pi + A_{12}\lambda(\lambda+1)\cos(\lambda-1)\pi\\
&-A_{13}\lambda(\lambda+1)\sin(\lambda+1)\pi - A_{14}\lambda(\lambda+1)\sin(\lambda-1)\pi = 0 && (8)\\
&A_{21}\lambda(\lambda+1)\cos(\lambda+1)\pi + A_{22}\lambda(\lambda+1)\cos(\lambda-1)\pi\\
&+A_{23}\lambda(\lambda+1)\sin(\lambda+1)\pi + A_{24}\lambda(\lambda+1)\sin(\lambda-1)\pi = 0 && (9)\\
&-A_{13}\lambda(\lambda+1) - A_{14}\lambda(\lambda-1) = 0 && (10)\\
&-A_{23}\lambda(\lambda+1) - A_{24}\lambda(\lambda-1) = 0 && (11)\\
&A_{11}\lambda(\lambda+1) + A_{12}\lambda(\lambda+1) = A_{21}\lambda(\lambda+1) + A_{22}\lambda(\lambda+1) && (12)\\
&-A_{13}(\lambda+1) - A_{14}(\kappa+\lambda) = -A_{23}(\lambda+1) - A_{24}(\kappa+\lambda)\,. && (13)
\end{aligned}$$

We first note that $\lambda = 0, -1$ are not acceptable eigenvalues, so factors $\lambda, (\lambda+1)$ can be cancelled where they occur. Also, $(\lambda+1)\pi$ and $(\lambda-1)\pi$ differ by 2π, so

$$\sin(\lambda+1)\pi = \sin(\lambda-1)\pi \; ; \quad \cos(\lambda+1)\pi = \cos(\lambda-1)\pi \; . \tag{14}$$

Using (10,11) to eliminate A_{13}, A_{23} in the remaining equations and using the above results to simplify the equations where possible, we obtain

$$-A_{11}(\lambda+1)\sin(\lambda+1)\pi - A_{12}(\lambda-1)\sin(\lambda+1)\pi = 0 \tag{15}$$
$$A_{21}(\lambda+1)\sin(\lambda+1)\pi + A_{22}(\lambda-1)\sin(\lambda+1)\pi = 0 \tag{16}$$
$$A_{11}(\lambda+1)\cos(\lambda+1)\pi + A_{12}(\lambda+1)\cos(\lambda+1)\pi - 2A_{14}\sin(\lambda+1)\pi = 0 \tag{17}$$
$$A_{21}(\lambda+1)\cos(\lambda+1)\pi + A_{22}(\lambda+1)\cos(\lambda+1)\pi + 2A_{24}\sin(\lambda+1)\pi = 0 \tag{18}$$

$$A_{11} + A_{12} = A_{21} + A_{22} \tag{19}$$
$$A_{14} = A_{24} \; . \tag{20}$$

Using (15,16) to eliminate A_{11}, A_{21} in (19), we obtain

$$A_{12}\sin(\lambda+1)\pi = A_{22}\sin(\lambda+1)\pi \tag{21}$$

and hence either

$$\sin(\lambda+1)\pi = 0 \tag{22}$$

or

$$A_{12} = A_{22} \; . \tag{23}$$

In the latter case, (19) then also gives $A_{11} = A_{21}$ and the eigenfunction is symmetric in the two half-planes — i.e.

$$A_{i1} = A_{i2} \; ; \quad i = 1, 4 \; . \tag{24}$$

For the symmetric case, adding and subtracting equations (17,18) and using (24) yields

$$A_{11}\cos(\lambda+1)\pi + A_{12}\cos(\lambda+1)\pi = 0 \tag{25}$$
$$A_{14} = 0 \; , \tag{26}$$

where we have cancelled a factor of $(\lambda+1)$ in (25). It follows from (15,25) that non-trivial solutions of the equations will occur if

$$\cos(\lambda+1)\pi = 0 \text{ i.e. } \lambda = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ... \tag{27}$$

or

$$\sin(\lambda+1)\pi = 0 \text{ i.e. } \lambda = 1, 2, 3, ... \tag{28}$$

The first (singular) term is $\lambda = 1/2$ for which

$$A_{11}(\lambda+1) = -A_{12}(\lambda-1)\ ; \quad A_{13} = A_{14} = 0\ . \tag{29}$$

The contact pressure at the interface $\theta = 0$ is

$$\begin{aligned} p(r) &= -\sigma_{\theta\theta}(r,0) = -r^{\lambda-1}\{A_{11}\lambda(\lambda+1) + A_{12}\lambda(\lambda+1)\} && (30)\\ &= -2A_{12}\lambda r^{\lambda-1} && (31)\\ &= -A_{12}r^{-1/2}\ , && (32) \end{aligned}$$

from (11.41, 29). Thus, the inequality $p(r) > 0$ requires that $A_{12} < 0$. The gap opened at $\theta = \pm\pi$ is

$$\begin{aligned} g(r) &= u_\theta(r,-\pi) - u_\theta(r,\pi) && (33)\\ &= r^\lambda\{-A_{11}(\lambda+1)\sin(\lambda+1)\pi - A_{12}(\kappa+\lambda)\sin(\lambda-1)\pi \\ &\quad -A_{13}(\lambda+1)\cos(\lambda+1)\pi - A_{14}(\kappa+\lambda)\cos(\lambda-1)\pi \\ &\quad -A_{21}(\lambda+1)\sin(\lambda+1)\pi - A_{22}(\kappa+\lambda)\sin(\lambda-1)\pi \\ &\quad +A_{23}(\lambda+1)\cos(\lambda+1)\pi + A_{24}(\kappa+\lambda)\cos(\lambda-1)\pi\}/(2\mu) && (34)\\ &= -\frac{(\kappa+1)A_{12}r^\lambda\sin(\lambda+1)\pi}{\mu} && (35)\\ &= \frac{(\kappa+1)A_{12}r^{1/2}}{\mu}\ , && (36) \end{aligned}$$

using (24, 29). Thus, the inequality $g(r) > 0$ requires that $A_{12} > 0$. The two inequalities therefore lead to inconsistencies and the only acceptable value for A_{12} is zero.

The eigenvalue $\lambda = 3/2$ yields

$$\begin{aligned} p(r) &= -3A_{12}r^{1/2} && (37)\\ g(r) &= -\frac{(\kappa+1)A_{12}r^{3/2}}{\mu} && (38) \end{aligned}$$

and this time both inequalities lead to the same conclusion, $A_{12} < 0$.

The integer eigenvalues result from condition (22) and in this case equations (15–20) reduce to

$$A_{11} + A_{12} = 0 \tag{39}$$

$$A_{21} + A_{22} = 0 \tag{40}$$

$$A_{14} = A_{24}\ . \tag{41}$$

It follows from (30,39) that the contact pressure $p(r) = 0$ in the corresponding eigenfunction. Thus, the integer eigenvalues correspond to cases where the two half-planes

do not interact in any way. The first non-trivial solution $\lambda = 1$ corresponds to uniform tensile stress $\sigma_{xx} = S_i$ in one or both bodies.

We conclude that the first non-zero term in the contact pressure comes from the eigenvalue $\lambda = 3/2$ and goes to zero at the edge of the contact with $r^{1/2}$. The corresponding gap goes to zero at the edge with $r^{3/2}$.

11.10. Find the equation that must be satisfied by λ if the stress function (11.38) is to define a non-trivial solution of the problem of the wedge $0<\theta<\pi/2$, traction-free on $\theta=0$ and bonded to a rigid plane surface at $\theta=\pi/2$. Do not attempt to solve the equation.

The boundary conditions of the problem are

$$\begin{aligned}
\sigma_{\theta r} &= 0\,; \quad \theta=0 && (1)\\
\sigma_{\theta\theta} &= 0\,; \quad \theta=0 && (2)\\
u_r &= 0\,; \quad \theta=\pi/2 && (3)\\
u_\theta &= 0\,; \quad \theta=\pi/2 && (4)
\end{aligned}$$

Using equations (11.40, 11.41, 11.75, 11.76), we then obtain the conditions

$$\begin{aligned}
-A_3\lambda(\lambda+1) - A_4\lambda(\lambda-1) &= 0 && (5)\\
A_1\lambda(\lambda+1) + A_2\lambda(\lambda+1) &= 0 && (6)\\
-A_1(\lambda+1)\cos\left(\frac{(\lambda+1)\pi}{2}\right) + A_2(\kappa-\lambda)\cos\left(\frac{(\lambda-1)\pi}{2}\right) & \\
-A_3(\lambda+1)\sin\left(\frac{(\lambda+1)\pi}{2}\right) + A_4(\kappa-\lambda)\sin\left(\frac{(\lambda-1)\pi}{2}\right) &= 0 && (7)\\
A_1(\lambda+1)\sin\left(\frac{(\lambda+1)\pi}{2}\right) + A_2(\kappa+\lambda)\sin\left(\frac{(\lambda-1)\pi}{2}\right) & \\
-A_3(\lambda+1)\cos\left(\frac{(\lambda+1)\pi}{2}\right) - A_4(\kappa+\lambda)\cos\left(\frac{(\lambda-1)\pi}{2}\right) &= 0\,. && (8)
\end{aligned}$$

We can use (5,6) to eliminate A_1, A_3 in (7,8), giving

$$\begin{aligned}
A_2(\lambda+1)\cos\left(\frac{(\lambda+1)\pi}{2}\right) + A_2(\kappa-\lambda)\cos\left(\frac{(\lambda-1)\pi}{2}\right) & \\
+A_4(\lambda-1)\sin\left(\frac{(\lambda+1)\pi}{2}\right) + A_4(\kappa-\lambda)\sin\left(\frac{(\lambda-1)\pi}{2}\right) &= 0 && (9)\\
-A_2(\lambda+1)\sin\left(\frac{(\lambda+1)\pi}{2}\right) + A_2(\kappa+\lambda)\sin\left(\frac{(\lambda-1)\pi}{2}\right) & \\
+A_4(\lambda-1)\cos\left(\frac{(\lambda+1)\pi}{2}\right) - A_4(\kappa+\lambda)\cos\left(\frac{(\lambda-1)\pi}{2}\right) &= 0\,. && (10)
\end{aligned}$$

and these two equations must be identical, giving

$$
\begin{aligned}
&\left[(\lambda+1)\cos\left(\frac{(\lambda+1)\pi}{2}\right)+(\kappa-\lambda)\cos\left(\frac{(\lambda-1)\pi}{2}\right)\right] \\
&\times\left[(\lambda-1)\cos\left(\frac{(\lambda+1)\pi}{2}\right)-(\kappa+\lambda)\cos\left(\frac{(\lambda-1)\pi}{2}\right)\right] \\
&=\left[-(\lambda+1)\sin\left(\frac{(\lambda+1)\pi}{2}\right)+(\kappa+\lambda)\sin\left(\frac{(\lambda-1)\pi}{2}\right)\right] \\
&\times\left[(\lambda-1)\sin\left(\frac{(\lambda+1)\pi}{2}\right)+(\kappa-\lambda)\sin\left(\frac{(\lambda-1)\pi}{2}\right)\right] \ . \qquad (11)
\end{aligned}
$$

Multiplying out this expression and using trigonometrical identities, we obtain the simpler expression

$$
\kappa^2+1-4\lambda^2+2\kappa\cos(\pi\lambda)=0 \ . \qquad (12)
$$

CHAPTER 12

12.1. Figure 12.16 shows a disk of radius a subjected to two equal and opposite forces F at the points A, B, the rest of the boundary $r = a$ being traction-free.

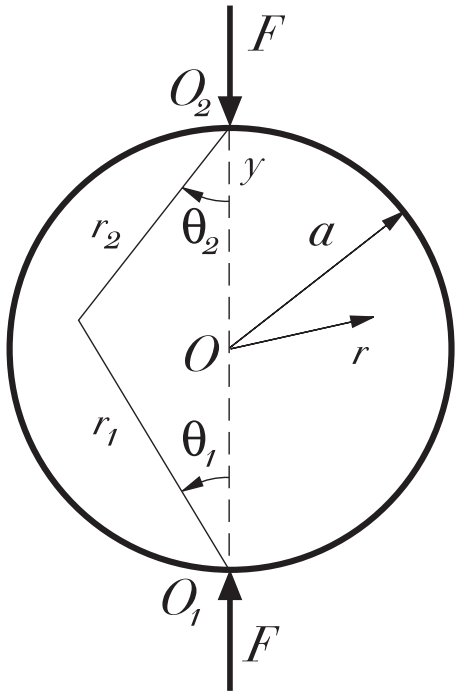


Figure 12.16: Disk loaded by concentrated forces.

The stress function

$$\phi = -\frac{F}{\pi}(r_1\theta_1 \sin\theta_1 + r_2\theta_2 \sin\theta_2)$$

is proposed to account for the localized effect of the forces. Find the stress field due to this function and, in particular, find the tractions implied upon the boundary $r = a$. Then complete the solution by superposing appropriate stress functions from Table 8.1, so as to satisfy the traction-free boundary condition.

Consider first the stress function

$$\phi_1 = -\frac{F r_1 \theta_1 \sin\theta_1}{\pi} \tag{1}$$

in the coordinate system r_1, θ_1.

The corresponding stresses can be written down from Table 8.1 as

$$\sigma_{r_1 r_1} = -\frac{2F\cos\theta_1}{\pi r_1} \quad ; \quad \sigma_{r_1\theta_1} = \sigma_{\theta_1\theta_1} = 0 \tag{2}$$

and the analysis of §12.3 shows that it involves the application of a concentrated compressive force F at the origin O_1.

Similar results are obtained for the second term in the stress function, with r_1, θ_1 replaced by r_2, θ_2.

We next superpose these stresses and transform them into the r, θ coordinate system, based on the centre O of the disk. We first note from Figure 1 that for points on the surface of the disk, the subtended angle is a right angle and hence

$$\cos\theta_1 = \frac{r_1}{2a} \quad ; \quad \cos\theta_2 = \frac{r_2}{2a} \ . \tag{3}$$

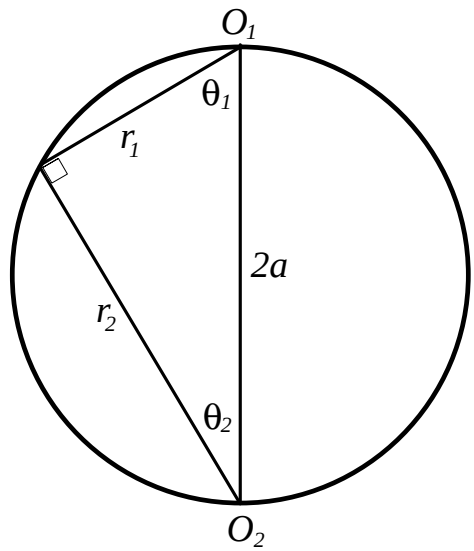


Figure 1

We conclude that, for points on the surface,

$$\sigma_{r_1 r_1} = -\frac{F}{\pi a} \quad ; \quad \sigma_{r_1\theta_1} = \sigma_{\theta_1\theta_1} = 0 \tag{4}$$

and the stresses are therefore independent of position around the surface[1].

The same geometric result from Figure 1 also implies that, for points on the surface of the disk, the r_1 direction coincides with the θ_2 direction and *vice versa*. The two sets of stress components can therefore be superposed as in Figure 2, without a preliminary application of the transformation equations.

[1]Noting that the state of stress in the Flamant solution is everywhere one of uniaxial compression or tension, we can deduce also from this result that lines of constant stress (e.g. constant tensile stress or constant maximum shear stress) are circles all with their centres on the line of action of the force and passing through its point of application. As a result, a photoelastic picture of the stress field due to a point force will show such a circular fringe pattern.

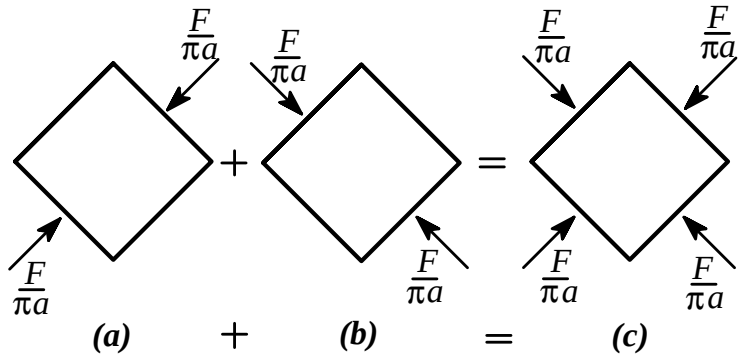


Figure 2

Now the combined stress state (Figure 2(c)) at the surface of the disk is a simple state of biaxial hydrostatic compression and hence will transform into the same state in any coordinate system. Thus, the traction on the surface of the disk implied by the stated stress function is one of uniform compression, with no shear tractions — i.e.

$$\sigma_{rr} = -\frac{F}{\pi a} \quad ; \quad \sigma_{r\theta} = 0 \ . \tag{5}$$

To complete the solution, we need to superpose a stress state involving uniform *tensile* tractions on the disk surface and this is obtained from the function

$$\phi_3 = \frac{Fr^2}{2\pi a} \tag{6}$$

from Table 8.1, yielding

$$\sigma_{rr} = \sigma_{\theta\theta} = \frac{F}{\pi a} \quad ; \quad \sigma_{r\theta} = 0 \ . \tag{7}$$

The complete stress function for the problem of Figure 12.16 is therefore

$$\phi = -\frac{F}{\pi}(r_1\theta_1 \sin\theta_1 + r_2\theta_2 \sin\theta_2) + \frac{Fr^2}{2\pi a} \ . \tag{8}$$

12.2. The disk $0 \leq r < a$ is accelerated from rest by two concentrated forces F acting in the positive θ-direction at the points $(a, \pm\pi/2)$, as shown in Figure 12.17. Use the Flamant solution in appropriate local coördinates to describe the concentrated forces and the solution of §7.4.1 to describe the inertia forces. Complete the solution by superposing appropriate stress functions from Table 8.1, so as to satisfy the traction-free boundary condition.

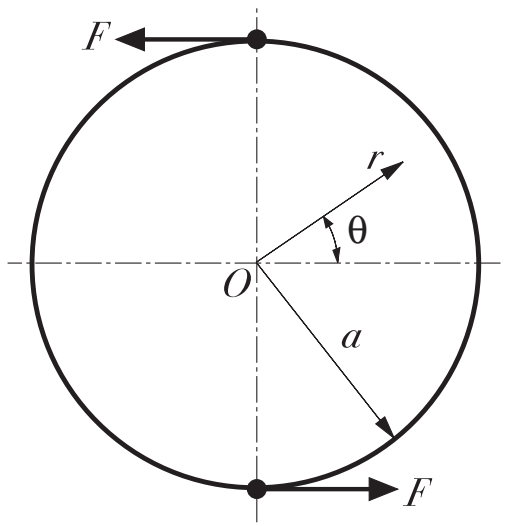


Figure 12.17: Disk accelerated by concentrated tangential forces.

We use the local coordinates shown in Figure 1(a). In these terms, the Flamant solution for the two concentrated forces is defined by the stress function

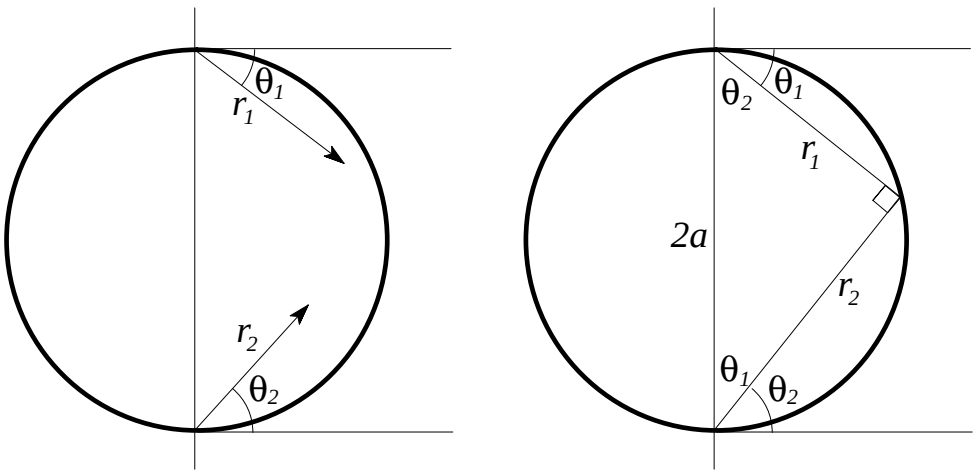
$$\phi = \frac{F}{\pi}(r_2\theta_2 \sin\theta_2 - r_1\theta_1 \sin\theta_1) \tag{1}$$

and the stress components due to the two separate terms can be described by the equations

$$\sigma_{r_1r_1} = -\frac{2F\cos\theta_1}{\pi r_1} \quad ; \quad \sigma_{r_1\theta_1} = \sigma_{\theta_1\theta_1} = 0 \tag{2}$$

and

$$\sigma_{r_2r_2} = \frac{2F\cos\theta_2}{\pi r_2} \quad ; \quad \sigma_{r_2\theta_2} = \sigma_{\theta_2\theta_2} = 0\ . \tag{3}$$



(a) **(b)**

Figure 1

We also note from Figure 1(b) that for points on the surface of the disk, the subtended angle is a right angle and hence

$$\cos\theta_2 = \sin\theta_1 \ ; \quad r_1 = 2a\sin\theta_1 \ ; \quad r_2 = 2a\cos\theta_1 \ . \tag{4}$$

Thus,

$$\sigma_{r_1r_1} = -\frac{F\cos\theta_1}{\pi a\sin\theta_1} \quad ; \tag{5}$$

$$\sigma_{r_2r_2} = \frac{F\cos\theta_2}{\pi a\sin\theta_1} \ . \tag{6}$$

These two stresses act on orthogonal surfaces of a small element at the surface of the disk and we can transform them to find the implied tractions on the disk surface using equations (1.15–1.17). The disk surface is an angle θ_1 counterclockwise from the surface r_2 and hence the normal and shear tractions are given by

$$\sigma_{rr} = \sigma_{r_2r_2}\cos^2\theta_1 + \sigma_{r_1r_1}\sin^2\theta_1 \tag{7}$$

$$\sigma_{r\theta} = (\sigma_{r_1r_1} - \sigma_{r_2r_2})\sin\theta_1\cos\theta_1 \ . \tag{8}$$

Substituting from (5,6) and simplifying, we find

$$\sigma_{rr} = 0 \ ; \quad \sigma_{r\theta} = -\frac{F}{\pi a} \ . \tag{9}$$

To complete the solution, we need to cancel these uniform shear tractions on $r = a$. They will clearly cause an angular acceleration $\dot{\Omega}$ and the solution is given in §7.4.1. Setting

$$\frac{F}{\pi a} = \sigma_{r\theta}(a,\theta) = \frac{\rho\dot{\Omega}a^2}{4} \ , \tag{10}$$

we have

$$\dot{\Omega} = \frac{4F}{\rho \pi a^3} \ , \tag{11}$$

where we have changed the sign from equation (7.43) so as to define the angular acceleration as counter-clockwise positive. The complete stress field associated with this solution is

$$\sigma_{r\theta} = \frac{F r^2}{\pi a^3} \ ; \quad \sigma_{rr} = \sigma_{\theta\theta} = 0 \tag{12}$$

If we then superpose the stress fields (2, 3, 12), the boundary conditions of the problem will be satisfied (since the traction (10) will cancel the resultant traction (9)) and the angular acceleration due to the original point forces will be given by (11). This superposition therefore defines the solution of the problem.

12.3. Figure 12.18 shows a heavy disk of radius a and density ρ supported by a concentrated force $\pi a^2 \rho g$. Find a solution for the stress field in the disk by combining the Flamant solution with appropriate terms from Table 8.1 in a coordinate system centred on the disk.

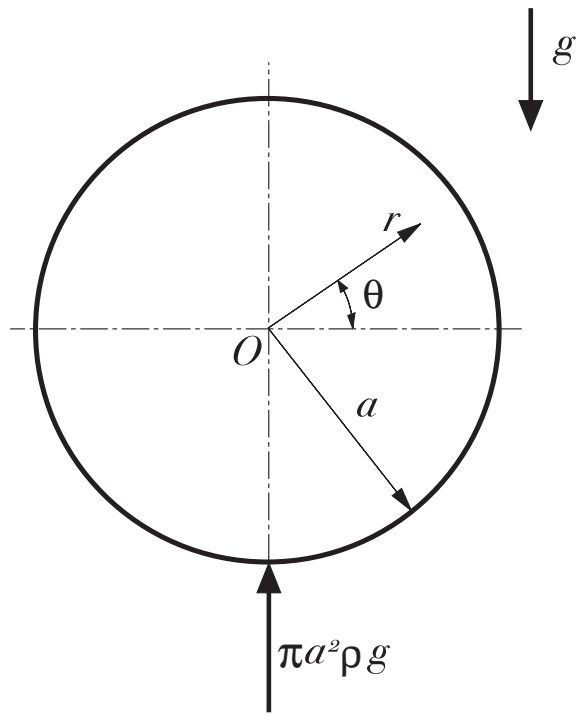


Figure 12.18: Heavy disk supported by a concentrated force.

Our strategy in solving this problem will be to describe the body force by the potential

$$V = \rho g y \tag{1}$$

and the concentrated force $F = \pi a^2 \rho g$ by the Flamant solution stress function

$$\phi_1 = -\frac{FR\psi \sin\psi}{\pi} = -a^2 \rho g R \psi \sin\psi \ , \tag{2}$$

from §12.3, where R is measured from the point of application of the force and ψ is measured from the positive y-axis, representing the upward vertical, as shown in Figure 1.

To complete the solution, we then need to superpose functions chosen so as to make the dotted circle traction-free.

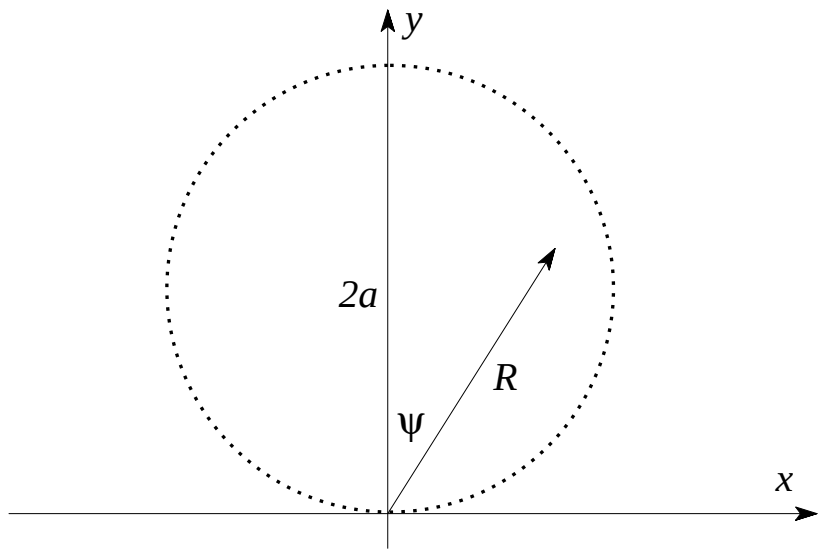


Figure 1

The particular solution of equations (1,2) defines the stress field

$$\sigma_{RR} = -\frac{2a^2\rho g\cos\psi}{R} + \rho g R\cos\psi \tag{3}$$

$$\sigma_{R\psi} = 0 \tag{4}$$

$$\sigma_{\psi\psi} = \rho g R\cos\psi\ . \tag{5}$$

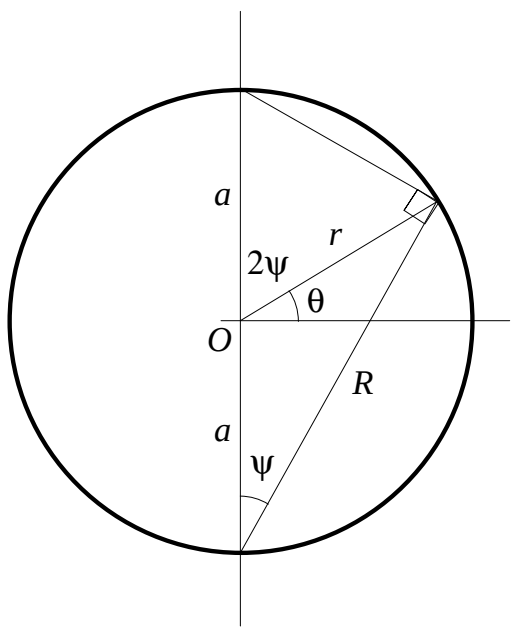


Figure 2

On the circle in Figure 2, we can use the result that the angle in a semi-circle is a right angle to obtain $R = 2a\cos\psi$ and hence

$$\begin{aligned}
\sigma_{RR} &= -a\rho g + 2a\rho g\cos^2\psi && (6)\\
\sigma_{R\psi} &= 0 && (7)\\
\sigma_{\psi\psi} &= 2a\rho g\cos^2\psi\ . && (8)
\end{aligned}$$

To obtain the tractions on the dotted circle, we must use the transformation equations (1.15–1.17). The local orthogonal directions R, ψ must be rotated clockwise through an angle ψ to reach the set r, θ. We therefore replace "x, y" by R, ψ, "x', y'" by r, θ and "θ" by $-\psi$ in (1.15–1.17) obtaining

$$\begin{aligned}
\sigma_{rr} &= \sigma_{RR}\cos^2\psi + \sigma_{\psi\psi}\sin^2\psi - 2\sigma_{R\psi}\sin\psi\cos\psi = a\rho g\cos^2\psi = \frac{a\rho g(1+\cos(2\psi))}{2}\\
\sigma_{r\theta} &= \sigma_{R\psi}(\cos^2\psi - \sin^2\psi) - (\sigma_{\psi\psi} - \sigma_{RR})\sin\psi\cos\psi = -a\rho g\sin\psi\cos\psi\\
&= -\frac{a\rho g\sin(2\psi)}{2}
\end{aligned}$$

Furthermore, noting that the angle subtended at the centre O is 2ψ, we have

$$2\psi = \theta - \frac{\pi}{2}\ ;\quad \cos(2\psi) = \sin\theta\ ;\quad \sin(2\psi) = \cos\theta \qquad (9)$$

and

$$\begin{aligned}
\sigma_{rr} &= \frac{a\rho g(1+\sin\theta)}{2} && (10)\\
\sigma_{r\theta} &= -\frac{a\rho g\cos\theta}{2}\ . && (11)
\end{aligned}$$

To cancel these tractions, we need axisymmetric ($n = 0$) and $\sin\theta$ terms from Table 8.1 that are bounded at $r = 0$ in the r, θ coördinate system — i.e.

$$\phi_2 = Ar^2 + Br^3\sin\theta\ . \qquad (12)$$

The corresponding stress components are

$$\begin{aligned}
\sigma_{rr} &= 2A + 2Br\sin\theta && (13)\\
\sigma_{r\theta} &= -2Br\cos\theta && (14)
\end{aligned}$$

and these will cancel the tractions due to ϕ_1 and V on $r = a$ if we choose

$$\begin{aligned}
\frac{a\rho g}{2} + 2A &= 0 && (15)\\
\frac{a\rho g}{2} + 2Ba &= 0 && (16)\\
-\frac{a\rho g}{2} - 2Ba &= 0\ . && (17)
\end{aligned}$$

The last two of these conditions are identical[2], so the three equations are all satisfied by the choice

$$A = -\frac{a\rho g}{4} \; ; \quad B = -\frac{\rho g}{4} \; . \tag{18}$$

The final solution can therefore be written in terms of the body force potential (1) and the stress function

$$\phi = \phi_1 + \phi_2 = -a^2 \rho g R \psi \sin\psi - \frac{a\rho g r^2}{4} - \frac{\rho g r^3 \sin\theta}{4} \; . \tag{19}$$

If it is desired to express the solution in a single coördinate system, the easiest choice is the Cartesian system, in terms of which

$$r = \sqrt{x^2 + (y-a)^2} \; ; \quad r\sin\theta = y - a \; ; \quad R = \sqrt{x^2 + y^2} \; ; \quad \psi = -\tan^{-1}(y/x) \; ; \quad R\sin\psi = -x \; .$$

We then have $V = \rho g y$ and

$$\phi = -a^2 \rho g x \tan^{-1}(y/x) - \frac{a\rho g(x^2 + (y-a)^2)}{4} - \frac{\rho g (y-a)(x^2 + (y-a)^2)}{4} \; . \tag{20}$$

This expression can be simplified slightly by expanding the last two terms and dropping null linear and constant terms, giving

$$\phi = -a^2 \rho g x \tan^{-1}(y/x) - \frac{\rho g (x^2 y + y^3 + a x^2 - 2a y^2)}{4} \; . \tag{21}$$

[2]We should have expected this redundancy, since the concentrated force was carefully chosen so as to balance the weight of the disk. An alternative approach would be to leave the force F as an unknown, in which case we would have had three linearly independent equations at this stage for the three unknowns A, B, F. Of course, the solution for F would then be found to correspond to the weight of the disk.

12.4. A rigid flat punch of width $2a$ is pressed into an elastic half-plane by a force F whose line of action is displaced a distance b from the centreline, as shown in Figure 12.19.

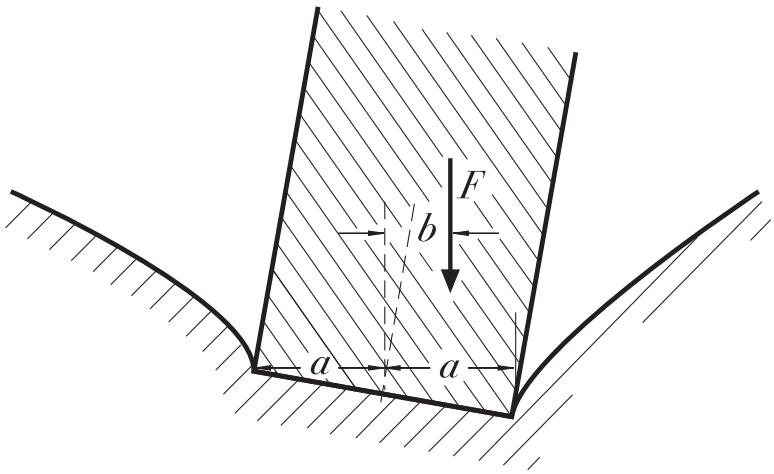


Figure 12.19: Punch with an eccentric load.

(i) Assuming that the punch makes contact over the entire face $-a<x<a$, find the pressure distribution $p(x)$ and the angle of tilt ot the punch. Assume the half-plane is prevented from rotating at infinity.

(ii) Hence find the maximum value of b for which there is contact throughout $-a<x<a$.

(iii) Re-solve the problem, assuming that b is larger than the critical value found in (ii). Contact will now occur in the range $c<x<a$ and the unknown left hand end of the contact region $(x=c)$ must be found from a smoothness condition on $p(x)$. Express c and $p(x)$ as functions of F, x and b.

(i). The punch is flat, corresponding to $u_0(x) = 0$ in equation (12.29). Thus, $u_n = 0$ for all n.

From (12.43), we have

$$p_0 = -\frac{F}{\pi a}\ ; \tag{1}$$

$$p_1 = -\frac{2Fb}{\pi a^2}\ . \tag{2}$$

From (12.40), with $u_1 = 0$, the angle of tilt is

$$C_1 = \frac{p_1(\kappa+1)}{4\mu} = -\frac{Fb(\kappa+1)}{2\pi a^2\mu}\ . \tag{3}$$

The contact pressure distribution is obtained by substituting for p_0, p_1 into (12.36) — i.e.

$$p(\theta) = -\frac{F}{\pi a \sin\theta} - \frac{2Fb\cos\theta}{\pi a^2 \sin\theta} = -\frac{F(a + 2b\cos\theta)}{\pi a^2 \sin\theta} \ . \tag{4}$$

Recalling that $a\cos\theta = x$, this implies that

$$p(x) = -\frac{F(a^2 + 2bx)}{\pi a^2 \sqrt{a^2 - x^2}} \ . \tag{5}$$

(ii). The contact traction $p(x)$ is required to be compressive (negative) everywhere in $-a < x < a$. This condition will first be violated at $x = -a$ when $a^2 - 2ba = 0$ — i.e.

$$b = \frac{a}{2} \ . \tag{6}$$

(iii). If $b > a/2$, the left edge of the punch will separate from the half-plane and contact will extend only from c to a, where c remains to be found as a function of b.

We can apply the same analysis provided we redefine the coordinate system such that (i) the origin is midway between c and a and (ii) 'a' is replaced by the half-width of the new contact area $a - c$. Thus we define new quantities:-

$$x^* = x - \frac{c + a}{2} \ ; \tag{7}$$

$$a^* = \frac{a - c}{2} \ ; \tag{8}$$

$$b^* = b - \frac{c + a}{2} \ . \tag{9}$$

The analysis of (i) above applies to the new problem provided we replace x, a, b by x^*, a^*, b^* respectively. Furthermore, since the contact pressure is required to be smooth at $x = c$ which corresponds to $x^* = -a^*$, we must have

$$b^* = \frac{a^*}{2} \ . \tag{10}$$

Notice that the problem is now the same as that of a punch extending from c to a *with b chosen so that contact is just about to be lost at $x = c$.* In fact that part of the punch face in $-a < x < c$ could be removed without changing the problem.

Using (8,9), the condition (10) implies that

$$b - \frac{c + a}{2} = \frac{a - c}{4} \tag{11}$$

and hence

$$c = 4b - 3a \ . \tag{12}$$

Adding *s to (5) and using (10), we obtain

$$p(x^*) = -\frac{F(a^{*2} + a^* x^*)}{\pi a^{*2}\sqrt{a^{*2} - x^{*2}}} = -\frac{F}{\pi a^*}\sqrt{\frac{a^* + x^*}{a^* - x^*}} \tag{13}$$

and substituting for x^*, a^* we then obtain

$$p(x) = -\frac{2F}{\pi(a-c)}\sqrt{\frac{x-c}{a-x}} = -\frac{F}{2\pi(a-b)}\sqrt{\frac{x-4b+3a}{a-x}} \ . \tag{14}$$

Finally, the angle of tilt is

$$C_1 = -\frac{Fb^*(\kappa+1)}{2\pi a^{*2}\mu} = -\frac{F(\kappa+1)}{4\pi a^*\mu} \ , \tag{15}$$

using (10) and hence

$$C_1 = -\frac{F(\kappa+1)}{8\pi\mu(a-b)} \ , \tag{16}$$

from (8,12).

Notice that as $b \to a$, $c \to a$ and $C_1 \to \infty$. In other words, the angle of tilt increases without limit as the line of action of the force approaches the corner.

12.5. Two half-planes $y > 0$ and $y < 0$ of the same material are welded together along the section $-a < x < a$ of their common interface $y = 0$. Equal and opposite forces F are now applied at infinity tending to load the weld in tension. Use a symmetry argument to deduce conditions that must be satisfied on the symmetry line $y = 0$ and hence determine the tensile stresses transmitted by the weld as a function of x. Find also the *stress intensity factor* K_I**, defined as**

$$K_I \equiv \lim_{x \to a^-} \sigma_{yy}(x,0)\sqrt{2\pi(a-x)} \; .$$

By symmetry, there can be no shear stresses on the weld line $-a < x < a$ $y = 0$ and this line must remain plane. In other words

$$\sigma_{yx}(x,0) = 0 \; ; \quad -a < x < a \tag{1}$$

$$\frac{\partial u_y}{\partial x}(x,0) = 0 \; ; \quad -a < x < a \; . \tag{2}$$

The rest of the surface $y = 0$ is traction-free.

These conditions are identical to those for the indentation of a half-plane by a frictionless flat rigid punch of width $2a$ (see equation (12.46). The only difference is that the transmitted force is now tensile rather than compressive. We therefore have

$$\sigma_{yy}(x,0) = \frac{F}{\pi\sqrt{a^2 - x^2}} \; , \tag{3}$$

from (12.47). The stress intensity factor is

$$\begin{aligned} K_I &\equiv \lim_{x \to a^-} \sigma_{yy}(x,0)\sqrt{2\pi(a-x)} \\ &= \lim_{x \to a^-} \frac{F\sqrt{2\pi(a-x)}}{\pi\sqrt{a^2 - x^2}} \\ &= \lim_{x \to a^-} F\sqrt{\frac{2}{\pi(a+x)}} \\ &= \frac{F}{\sqrt{\pi a}} \; . \end{aligned} \tag{4}$$

12.6. A flat rigid punch is pressed into the surface $y=0$ of the elastic half-plane $y>0$ by a force F. A tangential force T is then applied to the punch. If Coulomb friction conditions apply at the interface with coefficient f and Dundurs constant $\beta = 0$, show that no microslip will occur until T reaches the value fF at which point gross slip starts.

As in §12.8, we first assume that stick occurs everywhere — i.e. that there are no regions of microslip. If the punch moves a distance h to the right, we then have

$$u_x(x,0) = h \; ; \quad -a < x < a \; , \tag{1}$$

and hence

$$\frac{\partial u_x}{\partial x}(x,0) = 0 \; ; \quad -a < x < a \; , \tag{2}$$

where the punch is assumed to contact in the region $-a < x < a$ as in §12.5.1.

Equation (12.66) then requires that

$$-\frac{(\kappa+1)}{4\pi\mu}\int_{-a}^{a}\frac{p_x(\xi)d\xi}{(x-\xi)} + \frac{(\kappa-1)}{4\mu}p_y(x) = 0 \; ; \quad -a < x < a \; . \tag{3}$$

The behaviour of the system then depends qualitatively on the parameter $\kappa - 1$. The case $\beta = 0$ corresponds to $\kappa = 1$, and equation (3) reduces to

$$\int_{-a}^{a}\frac{p_x(\xi)d\xi}{(x-\xi)} = 0 \; . \tag{4}$$

This is identical with the homogeneous version of (12.32) and has the solution

$$p_x(x) = \frac{C}{\sqrt{a^2 - x^2}} \; , \tag{5}$$

where C is an arbitrary constant (see §12.5.1). The constant is determined from the condition that the total tangential force is T, giving

$$T = \int_{-a}^{a} p_x(\xi)d\xi = \int_{-a}^{a}\frac{Cd\xi}{\sqrt{a^2-\xi^2}} = \pi C \; . \tag{6}$$

Thus, $C = T/\pi$ and

$$p_x(x) = \frac{T}{\pi\sqrt{a^2 - x^2}} \; . \tag{7}$$

The contact pressure for this case is given by (12.47) as

$$p(x) = \frac{F}{\pi\sqrt{a^2 - x^2}} \tag{8}$$

where we note that the implied sign convention for F is opposite to that used in §12.5.2. It follows that $|p_x(x)| < fp(x)$ for all x as long as $|T| < fF$ and hence that

no microslip occurs anywhere until the tangential force is sufficient to cause gross slip (slip everywhere).

This solution holds as long as $\kappa = 1$. This is equivalent to the statement that Dundurs constant $\beta = 0$ (see §12.7). Since one body is rigid, it requires that $\nu = 0.5$ in plane strain. The situation is very different if $\kappa \neq 1$. The system of equations is then coupled (see §12.7) and we must determine both p_x and p_y from the coupled integral equations

$$-\frac{(\kappa+1)}{\pi}\int_{-a}^{a}\frac{p_y(\xi)d\xi}{(x-\xi)} - (\kappa-1)p_x(x) = 0 \tag{9}$$

$$-\frac{(\kappa+1)}{\pi}\int_{-a}^{a}\frac{p_x(\xi)d\xi}{(x-\xi)} + (\kappa-1)p_y(x) = 0\ . \tag{10}$$

This problem is discussed by K.L.Johnson *Contact Mechanics* §2.8, where it is shown that no coefficient of friction is sufficient to prevent some microslip for $\kappa \neq 1$.

12.7. Express the stress function

$$\phi = \frac{F_y r\theta\cos\theta}{\pi}$$

in Cartesian coördinates and hence find the stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{yy}$ due to the point force F_y in Figure 12.2.

Use this result and the integration procedure of §12.4 (Figure 12.4) to determine the stress field in the half-plane $y < 0$ due to the Hertzian traction distribution of equation (12.60). Make a contour plot of the Von Mises stress σ_E of equation (1.31) and identify the maximum value and its location.

In Cartesian coördinates we have

$$\phi = \frac{Fx\tan^{-1}(y/x)}{\pi} \ . \tag{1}$$

The stresses can then be obtained as

$$\sigma_{xx} = -\frac{2Fx^2y}{\pi(x^2+y^2)^2} \tag{2}$$

$$\sigma_{xy} = -\frac{2Fxy^2}{\pi(x^2+y^2)^2} \tag{3}$$

$$\sigma_{yy} = -\frac{2Fy^3}{\pi(x^2+y^2)^2} \tag{4}$$

The stress components due to a traction distribution $p(x)$ are therefore

$$\sigma_{xx} = -\frac{2y}{\pi}\int_{-a}^{a}\frac{(x-\xi)^2p(\xi)d\xi}{((x-\xi)^2+y^2)^2} \tag{5}$$

$$\sigma_{xy} = -\frac{2y^2}{\pi}\int_{-a}^{a}\frac{(x-\xi)p(\xi)d\xi}{((x-\xi)^2+y^2)^2} \tag{6}$$

$$\sigma_{yy} = -\frac{2y^3}{\pi}\int_{-a}^{a}\frac{p(\xi)d\xi}{((x-\xi)^2+y^2)^2} \ . \tag{7}$$

Using the traction distribution $p(\xi)$ of equation (12.60), we have

$$\sigma_{xx} = \frac{4Fy}{\pi^2a^2}\int_{-a}^{a}\frac{(x-\xi)^2\sqrt{a^2-\xi^2}d\xi}{((x-\xi)^2+y^2)^2} \tag{8}$$

$$\sigma_{xy} = \frac{4Fy^2}{\pi^2a^2}\int_{-a}^{a}\frac{(x-\xi)\sqrt{a^2-\xi^2}d\xi}{((x-\xi)^2+y^2)^2} \tag{9}$$

$$\sigma_{yy} = \frac{4Fy^3}{\pi^2a^2}\int_{-a}^{a}\frac{\sqrt{a^2-\xi^2}d\xi}{((x-\xi)^2+y^2)^2} \ . \tag{10}$$

These integrals can be obtained in closed form, but Maple and Mathematica will require some cajoling to generate the results. The substitution

$$\frac{\xi}{a} = \frac{(1-t^2)}{(1+t^2)} \tag{11}$$

gives

$$d\xi = -\frac{4atdt}{(1+t^2)^2} \; ; \quad \sqrt{a^2-\xi^2} = \frac{2at}{(1+t^2)} \tag{12}$$

and converts the integrals into ratios of polynomials in t^2 that can be split into simpler terms as partial fractions. We also note that the range $-a < \xi < a$ translates into $\infty > t > 0$. Thus, for example,

$$I_{yy} \equiv \int_{-a}^{a} \frac{\sqrt{a^2-\xi^2}d\xi}{((x-\xi)^2+y^2)^2} = \int_0^\infty \frac{8a^2t^2(1+t^2)dt}{(A+Bt^2+Ct^4)^2} \; , \tag{13}$$

where

$$A = (x-a)^2+y^2 \; ; \quad B = 2(x^2+y^2-a^2) \; ; \quad C = (x+a)^2+y^2 \; . \tag{14}$$

Equation (13) can then be split into the simpler integrals

$$\begin{aligned} I_{yy} &= \frac{8a^2}{C}\left(\int_0^\infty \frac{dt}{(A+Bt^2+Ct^4)} + (C-B)\int_0^\infty \frac{t^2dt}{(A+Bt^2+Ct^4)^2}\right. \\ &\quad \left. -A\int_0^\infty \frac{dt}{(A+Bt^2+Ct^4)^2}\right) \; . \end{aligned} \tag{15}$$

To evaluate these terms, we record the results

$$I_1 \equiv \int_0^\infty \frac{dt}{(A+Bt^2+Ct^4)} = \frac{\pi}{2\sqrt{A}\sqrt{B+2\sqrt{AC}}} \tag{16}$$

$$I_2 \equiv \int_0^\infty \frac{t^2dt}{(A+Bt^2+Ct^4)} = \frac{\pi}{2\sqrt{C}\sqrt{B+2\sqrt{AC}}} \tag{17}$$

$$I_3 \equiv \int_0^\infty \frac{dt}{(A+Bt^2+Ct^4)^2} = \frac{\pi(B+3\sqrt{AC})}{4A^{3/2}(B+2\sqrt{AC})^{3/2}} \tag{18}$$

$$I_3 \equiv \int_0^\infty \frac{t^2dt}{(A+Bt^2+Ct^4)^2} = \frac{\pi}{4\sqrt{A}(B+2\sqrt{AC})^{3/2}} \; , \tag{19}$$

see for example Gradshteyn and Ryzhik, §2.161. Thus,

$$I_{yy} = \frac{8a^2}{C}(I_1 + (C-B)I_4 - AI_3) = \frac{2\pi a^2(C+\sqrt{AC})}{CA^{1/2}(B+2\sqrt{AC})^{3/2}} \tag{20}$$

and

$$\sigma_{yy} = \frac{4Fy^3 I_{yy}}{\pi^2 a^2} = \frac{8Fy^3(C+\sqrt{AC})}{\pi C A^{1/2}(B+2\sqrt{AC})^{3/2}} \ . \tag{21}$$

Applying a similar procedure to the integral in equation (9), we have

$$\begin{aligned} I_{xy} &\equiv \int_{-a}^{a} \frac{(x-\xi)\sqrt{a^2-\xi^2}d\xi}{((x-\xi)^2+y^2)^2} \\ &= \int_0^\infty \frac{8a^2t^2(x-a+(x+a)t^2)dt}{(A+Bt^2+Ct^4)^2} \\ &= \frac{8a^2}{C}((x+a)I_1 + [(x-a)C-(x+a)B]I_4 - (x+a)AI_3) \ . \end{aligned} \tag{22}$$

For the integral in equation (8), we have

$$\begin{aligned} I_{xx} &\equiv \int_{-a}^{a} \frac{(x-\xi)^2\sqrt{a^2-\xi^2}d\xi}{((x-\xi)^2+y^2)^2} \\ &= \int_0^\infty \frac{8a^2t^2(x-a+(x+a)t^2)^2dt}{(A+Bt^2+Ct^4)^2(1+t^2)} \\ &= C_1I_1 + C_2I_2 + C_3I_3 + C_4I_4 + C_5 \ . \int_0^\infty \frac{dt}{(1+t^2)} \ , \end{aligned} \tag{23}$$

where $C_1, ...C_5$ are constants that can be determined by multiplying the integrands on each side of the equation by the factor $(A+Bt^2+Ct^4)^2(1+t^2)$ and equating coefficients of powers of t^2. A Maple file for these operations is available on request from the author. After evaluating the stresses and substituting for the expressions A, B, C from (14), it is prudent to check some of the results for accuracy. Figure 1 shows a plot of the stress components just below the surface (at $y = -0.01$, with $a = 1$ and $F = 1$). As expected, the stress component σ_{yy} reproduces the Hertzian distribution of equation (12.60) and the shear stress $\sigma_{xy} \approx 0$. Notice that the orthogonal normal stress σ_{xx} has a similar form to σ_{yy}. It can be shown that $\sigma_{xx} = \sigma_{yy}$ at the surface $y = 0$, but this is a singular point for some of the expressions being calculated.

The Von Mises stress depends on whether plane stress or plane stress conditions are assumed. Under plane strain conditions, $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$, whereas for plane stress $\sigma_{zz} = 0$. In Figures 2 and 3, we show contour plots for both cases.

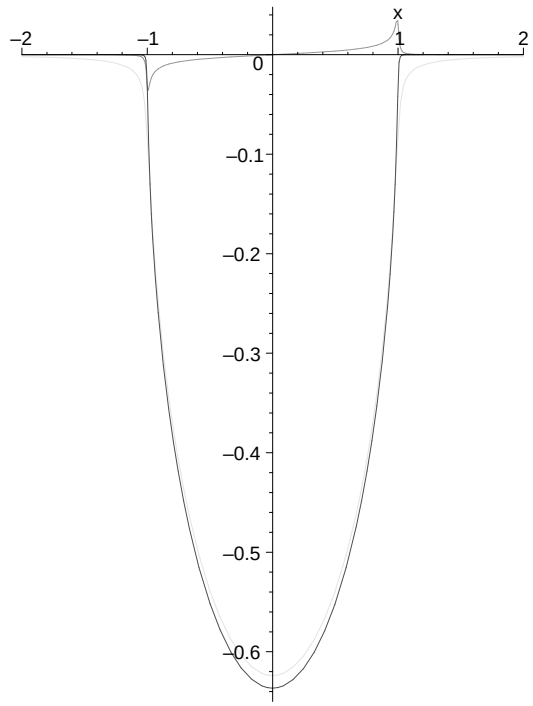


Figure 1: Stress components near the surface.

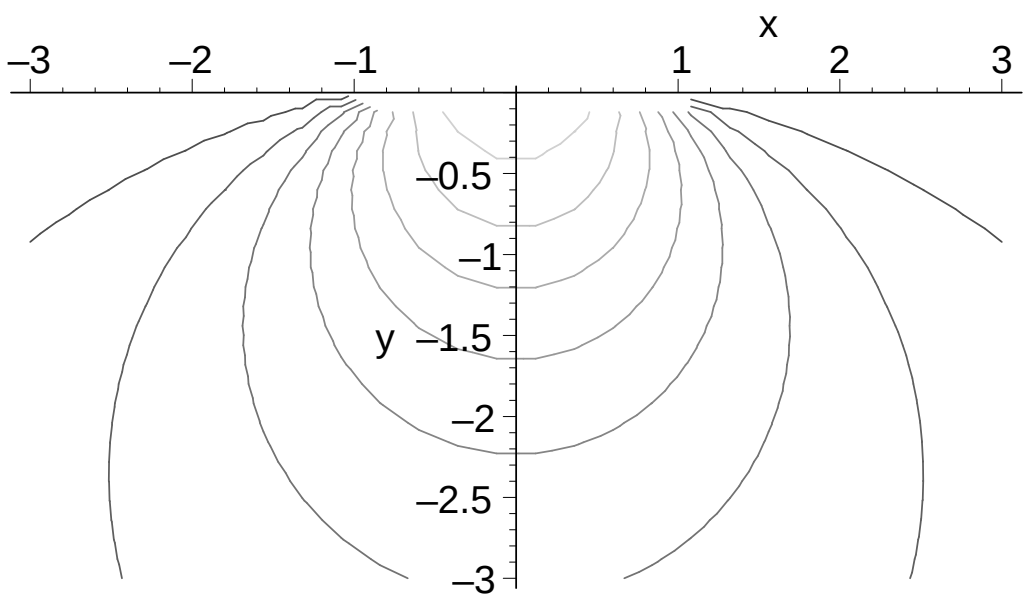


Figure 2: Von Mises stress contours for plane stress.

For plane stress, the maximum occurs at the origin — i.e. at the mid point of the loaded region, where

$$\sigma_{xx} = \sigma_{yy} = -\frac{2F}{\pi a} \; ; \quad \sigma_{xy} = 0 \tag{24}$$

and hence

$$\sigma_E = \frac{2F}{\pi a} = \frac{0.637F}{a} \; , \tag{25}$$

from (1.19).

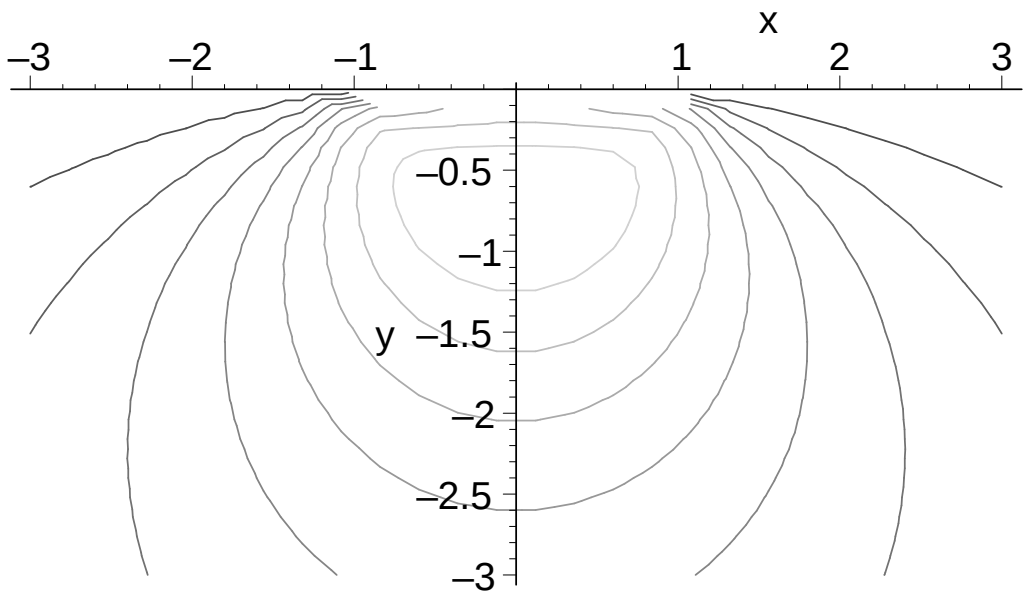


Figure 3: Von Mises stress contours for plane strain with $\nu = 0.3$.

For plane strain, the maximum occurs below the midpoint. For $\nu = 0.3$, numerical iteration shows that the maximum occurs at $y \approx -0.704a$, at which point

$$\sigma_E = \frac{0.355F}{a} \; . \tag{26}$$

This value is significantly lower than the plane stress value (25) because of the constraint offered by the stress σ_{zz}. This constraint increases with ν. In Figure 4, we show plots of σ_E along the axis for various values of ν. Notice that below about $\nu = 0.1$, the maximum occurs at the surface $y = 0$, as in the plane stress case. Of course, if $\nu = 0$, the plane strain and plane stress solutions are identical.

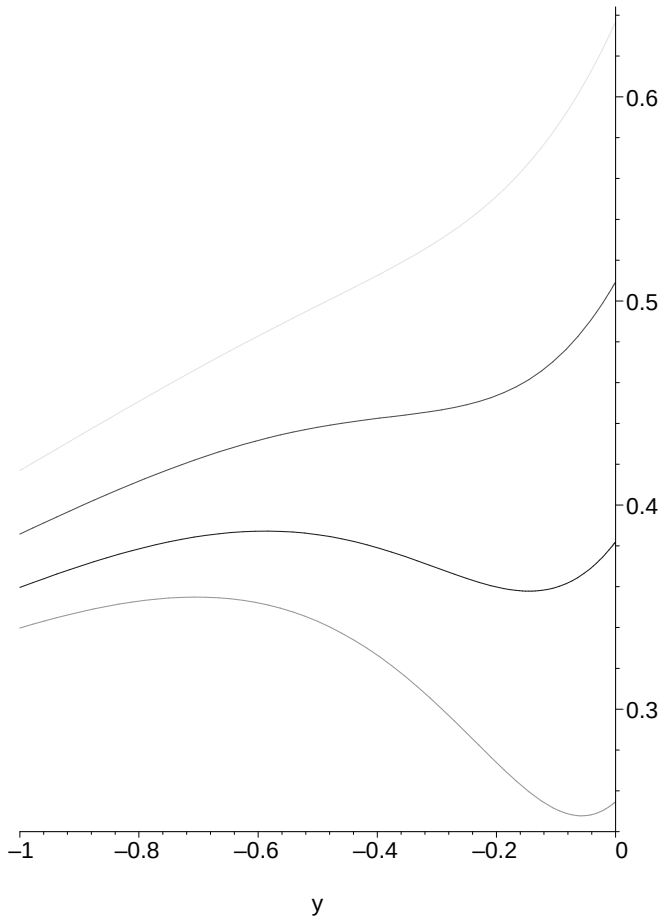


Figure 4: Variation of Von Mises stress along $x = 0$ for plane strain with $\nu = 0, 0.1, 0.2, 0.3$.

12.8. From §12.5.2 and equation (12.47), it follows that the contact traction distribution

$$\begin{aligned} p(x,a) &= \frac{1}{\sqrt{a^2-x^2}}\ ; \quad |x|<a \\ &= 0\ ; \quad |x|>a \end{aligned}$$

produces zero surface slope du_y/dx in $|x|<a$. Use equation (12.27) to determine the corresponding value of slope *outside* the contact area ($|x|>a$) and hence construct the discontinuous function

$$u(x,a) \equiv \frac{du_y}{dx}\ ,$$

such that $p(x,a)$ produces $u(x,a)$.

Linear superposition then shows that the more general traction distribution

$$p(x) = \int_0^b g(a)p(x,a)da$$

produces the surface slope

$$\frac{du_y}{dx} = \int_0^b g(a)u(x,a)da\ ,$$

where $g(a)$ is any function of a. In effect this is a superposition of a range of 'flat punch' traction distributions over different width strips up to a maximum semi-width of b, so the pressure will still be zero for $|x|>b$.

Use this representation to solve the problem of the indentation by a wedge of semi-angle $\pi/2-\alpha$ ($\alpha \ll 1$), for which

$$\frac{du_0}{dx} = -|\alpha|\ ; \quad |x|<b\ ,$$

where b is the semi-width of the contact area. In particular, find the contact traction distribution and the relation between b and the applied force F.

Hint: You will find that the boundary condition leads to an Abel integral equation, whose solution is given in Table 27.2 in Chapter 27.

The surface slope due to a distribution of contact traction in $-a<\xi<a$

$$\frac{du_y}{dx} = -\frac{(\kappa+1)}{4\pi\mu}\int_{-a}^{a}\frac{p(\xi)d\xi}{(x-\xi)}\ ,$$

from (12.27)

For the unit flat punch distribution,

$$p(\xi) = \frac{1}{\sqrt{a^2-\xi^2}}\ , \tag{1}$$

we have

$$\begin{aligned}
\frac{du_y}{dx} \equiv u(x,a) &= -\frac{(\kappa+1)}{4\pi\mu}\int_{-a}^{a}\frac{d\xi}{(x-\xi)\sqrt{a^2-\xi^2}} \\
&= 0\ ; \qquad -a<x<a \qquad (2)\\
&= -\frac{(\kappa+1)\mathrm{sgn}(x)}{4\mu\sqrt{x^2-a^2}}\ ; \quad |x|>a\ . \qquad (3)
\end{aligned}$$

This last integral can be obtained from the integral in (13.43) by replacing the factor ξ in the numerator by $x-(x-\xi)$, and then splitting (13.43) into two integrals, one of which can be evaluated by elementary means.

To evaluate the integral

$$\frac{du_y}{dx} = \int_0^b g(a)u(x,a)da\ ,$$

for the more general distribution, we need to split the range of integration at the point x if $0<x<b$. Thus

$$\frac{du_y}{dx} = \int_0^x g(a)u(x,a)da + \int_x^b g(a)u(x,a)da\ ; \quad 0<x<b\ .$$

Now in the second integral, $x<a<b$ and hence $x<a$, so the integrand is zero from (2). Thus, we have

$$\frac{du_y}{dx} = -\frac{(\kappa+1)}{4\mu}\int_0^x -\frac{g(a)da}{\sqrt{x^2-a^2}};\ ; \quad 0<x<b\ .$$

To solve the wedge indentation problem, we need to choose $g(a)$ so as to satisfy the equation

$$-\frac{(\kappa+1)}{4\mu}\int_0^x \frac{g(a)da}{\sqrt{x^2-a^2}} = \alpha;\ ; \quad 0<x<b\ .$$

or

$$\int_0^x \frac{g(a)da}{\sqrt{x^2-a^2}} = -\frac{4\mu\alpha}{(\kappa+1)}\ .$$

This equation is of the form of the first equation in Table 29.2 with $t\to a$ and the solution is

$$g(a) = -\frac{2}{\pi}\frac{d}{da}\int_0^a \frac{4\mu\alpha t dt}{(\kappa+1)\sqrt{a^2-t^2}} = -\frac{8\mu\alpha}{\pi(\kappa+1)}$$

The contact traction distribution is then obtained as

$$p(x) = -\frac{8\mu\alpha}{\pi(\kappa+1)}\int_0^b p(x,a)da\ .$$

Once again, we split the range of integration at $x=a$ obtaining

$$p(x) = -\frac{8\mu\alpha}{\pi(\kappa+1)}\int_0^x p(x,a)da - \frac{8\mu\alpha}{\pi(\kappa+1)}\int_x^b p(x,a)da\ .$$

In this case, the *first* integral is zero, since $a < x$ implies $x > a$. We therefore have

$$\begin{aligned} p(x) &= -\frac{8\mu\alpha}{\pi(\kappa+1)}\int_x^b \frac{da}{\sqrt{a^2-x^2}} \\ &= -\frac{8\mu\alpha\cosh^{-1}(b/x)}{\pi(\kappa+1)} \end{aligned} \tag{4}$$

By symmetry, this solution also applies in the region $-b < x < 0$ with x replaced by $|x|$. The total contact force is

$$\begin{aligned} F &= -2\int_0^b p(x)dx \\ &= \frac{16\mu\alpha}{\pi(\kappa+1)}\int_0^b \cosh^{-1}(b/x)dx \\ &= \frac{8\mu\alpha b}{(\kappa+1)} \ . \end{aligned} \tag{5}$$

12.9. Modify the method outlined in Problem 12.8 to solve the problem of the indentation of a half plane by an unsymmetrical wedge, for which

$$\begin{aligned}\frac{du_0}{dx} &= -\alpha\ ; \quad 0 < x < c \\ &= \beta\ ; \quad -d < x < 0\ ,\end{aligned}$$

where the unsymmetrical contact area extends from $-d$ to c. In particular, find the contact pressure distribution and both c and d as functions of the applied force F.

The modification involves moving the origin to destroy the symmetry once the function $u(x,a)$ has been determined. In other words, instead of superposing a set of symmetrically disposed 'flat punch' distributions, superpose a similar set arranged so that the common point is displaced from the origin.

The surface slope due to a distribution of contact traction in $-a < \xi < a$

$$\frac{du_y}{dx} = -\frac{(\kappa+1)}{4\pi\mu}\int_{-a}^{a}\frac{p(\xi)d\xi}{(x-\xi)}\ ,$$

from (12.27)

For the unit flat punch distribution,

$$p(\xi) = \frac{1}{\sqrt{a^2-\xi^2}}\ , \tag{1}$$

we have

$$\begin{aligned}\frac{du_y}{dx} \equiv u(x,a) &= -\frac{(\kappa+1)}{4\pi\mu}\int_{-a}^{a}\frac{d\xi}{(x-\xi)\sqrt{a^2-\xi^2}} \\ &= 0\ ; \qquad -a < x < a \qquad (2) \\ &= -\frac{(\kappa+1)\mathrm{sgn}(x)}{4\mu\sqrt{x^2-a^2}}\ ; \quad |x| > a\ . \qquad (3)\end{aligned}$$

This last integral can be obtained from the integral in (13.43) by replacing the factor ξ in the numerator by $x-(x-\xi)$, and then splitting (13.43) into two integrals, one of which can be evaluated by elementary means.

Let's now move the origin to a different (non-symmetric) point in the contact area by writing $x \to (x-b)$,; $\xi \to (\xi - b)$, where $|b| < a$. It follows that the pressure distribution

$$p(\xi) = \frac{1}{\sqrt{a^2-[\xi-b]^2}} \tag{4}$$

produces surface slope

$$\begin{aligned}\frac{du_y}{dx} &= 0\ ; \qquad -a+b<x<a+b \\ &= -\frac{(\kappa+1)\mathrm{sgn}(x)}{4\mu\sqrt{(x-b)^2-a^2}}\ ; \quad |x-b|>a\ . \end{aligned} \tag{5}$$

Using the notation

$$s \equiv a+b\ ; \quad \gamma \equiv \frac{(a-b)}{(a+b)} \tag{6}$$

we can write these results as

$$p(\xi,s) = \frac{1}{\sqrt{(s-\xi)(\xi+\gamma s)}}\ ; \quad -\gamma s<\xi<s \tag{7}$$

$$= 0\ ; \quad \xi>s \text{ or } \xi<-\gamma s\ . \tag{8}$$

producing

$$\begin{aligned}\frac{du_y}{dx} \equiv u(x,s) &= 0\ ; \qquad -\gamma s<x<s \\ &= -\frac{(\kappa+1)\mathrm{sgn}(x)}{4\mu\sqrt{(x-s)(\gamma s+x)}}\ ; \quad x>s \text{ or } x<-\gamma s\ . \end{aligned} \tag{9}$$

Now construct a more general solution by superposition in the form

$$p(\xi) = \int_0^c p(\xi,s)g(s)ds$$

$$= \int_\xi^c \frac{g(s)ds}{\sqrt{(s-\xi)(\xi+\gamma s)}}\ ; \quad 0<\xi<c \tag{10}$$

$$= \int_{-\xi/\gamma}^c \frac{g(s)ds}{\sqrt{(s-\xi)(\xi+\gamma s)}}\ ; \quad -\gamma c<\xi<0 \tag{11}$$

$$= 0\ ; \quad \xi>s \text{ or } \xi<-\gamma s\ . \tag{12}$$

The corresponding expression for the surface slope is

$$\frac{du_y}{dx} = \int_0^c u(x,s)g(s)ds$$

$$= -\frac{(\kappa+1)}{4\mu}\int_0^x \frac{g(s)ds}{\sqrt{(x-s)(\gamma s+x)}}\ ; \quad 0<x<c \tag{13}$$

$$= \frac{(\kappa+1)}{4\mu}\int_0^{-x/\gamma} \frac{g(s)ds}{\sqrt{(x-s)(\gamma s+x)}}\ ; \quad -\gamma c<x<0\ . \tag{14}$$

For the wedge problem, we want both the last two expressions to be independent of x, since the slope of the wedge is constant in these regions. If we make the change of variable $s = xy$ in the first integral, we obtain

$$\int_0^x \frac{g(s)ds}{\sqrt{(x-s)(\gamma s + x)}} = \int_0^1 \frac{g(xy)dy}{\sqrt{(1-y)(\gamma y + 1)}} \tag{15}$$

and this will be independent of x if and only if $g(s)$ is a constant. A similar argument applies to the second integral. We therefore set $g(s) = C$ and evaluate the integrals, obtaining

$$\begin{aligned}
\frac{du_y}{dx} &= -\frac{C(\kappa+1)}{4\mu} \int_0^x \frac{dc}{\sqrt{(x-c)(\gamma c + x)}} \\
&= -\frac{C(\kappa+1)}{4\mu\sqrt{\gamma}} \arccos\left(\frac{\gamma-1}{\gamma+1}\right) \; ; \quad 0 < x < d \qquad (16) \\
&= \frac{C(\kappa+1)}{4\mu\sqrt{\gamma}} \int_0^{-x/\gamma} \frac{dc}{\sqrt{(x-c)(\gamma c + x)}} \\
&= \frac{C(\kappa+1)}{4\mu\sqrt{\gamma}} \arccos\left(\frac{1-\gamma}{\gamma+1}\right) \; ; \quad -\gamma d < x < 0 \; . \qquad (17)
\end{aligned}$$

The parameters C and γ can now be chosen so as to make the slope expressions equal to α and $-\beta$ respectively on the two wedge faces. If we define θ such that

$$\arccos\left(\frac{1-\gamma}{\gamma+1}\right) = \frac{\pi}{2} - \theta \; , \tag{18}$$

then

$$\arccos\left(\frac{\gamma-1}{\gamma+1}\right) = \frac{\pi}{2} + \theta \tag{19}$$

and the boundary conditions require that

$$\begin{aligned}
-\frac{C(\kappa+1)}{4\mu\sqrt{\gamma}} \left(\frac{\pi}{2} + \theta\right) &= \alpha \qquad (20) \\
\frac{C(\kappa+1)}{4\mu\sqrt{\gamma}} \left(\frac{\pi}{2} - \theta\right) &= -\beta \; . \qquad (21)
\end{aligned}$$

Taking the ratio of these equations, we have

$$\alpha\left(\frac{\pi}{2} - \theta\right) = \beta\left(\frac{\pi}{2} + \theta\right) \tag{22}$$

with solution

$$\theta = \frac{\pi(\alpha - \beta)}{2(\beta + \alpha)} \; , \tag{23}$$

after which

$$\gamma = \left(\frac{1-\sin\theta}{1+\sin\theta}\right) \; ; \quad C = -\frac{4\mu\sqrt{\gamma}(\alpha+\beta)}{\pi(\kappa+1)} \; . \tag{24}$$

The contact traction distribution is then obtained from equations (10, 11) as

$$\begin{aligned} p(\xi) &= \int_{\xi}^{c} \frac{Cds}{\sqrt{(s-\xi)(\xi+\gamma s)}} \\ &= \frac{C}{\sqrt{\gamma}} \cosh^{-1}\left(\frac{2\gamma c + (1-\gamma)\xi}{(1+\gamma)\xi}\right) \; ; \quad 0 < \xi < c \qquad (25) \\ &= \int_{\xi/\gamma}^{c} \frac{Cds}{\sqrt{(s-\xi)(\xi+\gamma s)}} \\ &= \frac{C}{\sqrt{\gamma}} \cosh^{-1}\left(\frac{2\gamma c + (1-\gamma)\xi}{(1+\gamma)\xi}\right) \; ; \quad -\gamma c < \xi < 0 \; , \qquad (26) \end{aligned}$$

so the same expression holds on both sides of the vertex.

To obtain the total force F, we write

$$F = \int_A \int_0^c p(\xi, s) C ds d\xi \; , \tag{27}$$

where the domain of integration A comprises the complete surface of the half plane. Changing the order of integration, we then get

$$\begin{aligned} F &= C \int_0^c \int_A p(\xi, s) d\xi ds \\ &= C \int_0^c \int_{-\gamma s}^{s} \frac{d\xi}{\sqrt{(s-\xi)(\xi+\gamma s)}} \\ &= C \int_0^c \pi ds = \pi C c \\ &= \frac{4\mu\sqrt{\gamma}(\alpha+\beta)c}{(\kappa+1)} \; , \end{aligned} \tag{28}$$

using (24). In general, α, β and F will be given, so we can use (24) to determine γ, C and (28) to determine c. The contact traction distribution is then given by (26).

In the special case where $\beta = \alpha$, equation (24) gives $\gamma = 1$,

$$F = \frac{8\mu\alpha c}{(\kappa+1)} \; , \tag{29}$$

from (28) and

$$p(\xi) = -\frac{8\mu\alpha \cosh^{-1}(c/\xi)}{\pi(\kappa+1)} \; , \tag{30}$$

from (26, 24).

Notice that the unsymmetrical solution can also be used to solve the problem in which a symmetrical wedge is loaded by a force that acts away from the vertex and that is not restrained from rigid-body rotation.

CHAPTER 13

13.1. Figure 13.6 shows a large body with a circular hole of radius a, subjected to a concentrated force F, tangential to the hole. The remainder of the hole surface is traction-free and the stress field is assumed to tend to zero as $r \to 0$.

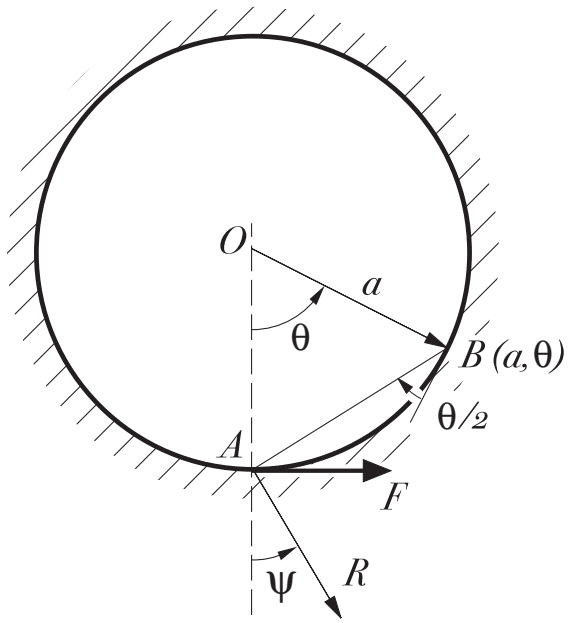


Figure 13.6

(i) **Find the stress components at the point $B(a,\theta)$ on the surface of the hole, due to the candidate stress function**

$$\phi = \frac{F}{\pi} R\psi \cos\psi$$

in polar coördinates R, ψ centred on the point A as shown.

(ii) **Transform these stress components into the polar coördinate system r, θ centred on the centre of the hole, O.**

Note: *For points on the surface of the hole* $(r = a)$, **we have** $R = 2a\sin(\theta/2)$, $\sin\psi = \cos(\theta/2)$, $\cos\psi = -\sin(\theta/2)$.

(iii) **Complete the solution by superposing stress functions (in r, θ) from Tables 8.1, 9.1 with appropriate Fourier components in the tractions and determining the multiplying constants from the conditions that (a) the surface of the hole be traction-free (except at A) and (b) the displacements be everywhere single-valued.**

The stresses due to

$$\phi = \frac{F}{\pi} R\psi \cos(\psi)$$

are most naturally expressed in the R, ψ coordinate system as

$$\sigma_{RR} = -\frac{2F \sin\psi}{\pi R} \quad ; \quad \sigma_{R\psi} = \sigma_{\psi\psi} = 0 \ , \tag{1}$$

from Table 8.1.

At the point $B(a, \theta)$, we have

$$R = 2a \sin\left(\frac{\theta}{2}\right) \quad ; \quad \sin\psi = \cos\left(\frac{\theta}{2}\right) \tag{2}$$

and hence

$$\sigma_{RR} = -2F \cos\left(\frac{\theta}{2}\right) \Big/ \left[2\pi a \sin\left(\frac{\theta}{2}\right)\right] = -\frac{F}{\pi a} \cot\left(\frac{\theta}{2}\right) \ . \tag{3}$$

The stress transformation equations then permit us to write

$$\sigma_{\theta\theta} = \sigma_{RR} \cos^2\left(\frac{\theta}{2}\right) = -\frac{F}{\pi a} \cos^2\left(\frac{\theta}{2}\right) \cot\left(\frac{\theta}{2}\right) \ ; \tag{4}$$

$$\sigma_{rr} = \sigma_{RR} \sin^2\left(\frac{\theta}{2}\right) = -\frac{F}{\pi a} \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) = -\frac{F \sin\theta}{2\pi a} \ ; \tag{5}$$

$$\sigma_{r\theta} = \sigma_{RR} \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) = -\frac{F}{\pi a} \cos^2\left(\frac{\theta}{2}\right) = -\frac{F(1+\cos\theta)}{2\pi a} \ , \tag{6}$$

where we have used the fact that $\sigma_{R\psi} = \sigma_{\psi\psi} = 0$.

For the surface of the hole to be traction-free, we require

$$\sigma_{r\theta} = \sigma_{rr} = 0 \quad ; \quad r = a, \text{ all } \theta \tag{7}$$

Equations (5,6) define terms containing $\sin\theta, \cos\theta$ in these tractions and also a constant term in $\sigma_{r\theta}$. To cancel them, we need to superpose the stress function[1]

$$\phi = A\theta + Br\theta\cos\theta + Cr\ln(r)\sin\theta + \frac{D\sin\theta}{r} \tag{8}$$

from Table 8.1, leading to the following stress components at the point $B(a, \theta)$:-

$$\sigma_{rr} = -\frac{2B\sin\theta}{a} + \frac{C\sin\theta}{a} - \frac{2D\sin\theta}{a^3} - \frac{F\sin\theta}{2\pi a} \ ; \tag{9}$$

$$\sigma_{r\theta} = \frac{A}{a^2} - \frac{C\cos\theta}{a} + \frac{2D\cos\theta}{a^3} - \frac{F(1+\cos\theta)}{2\pi a} \ ; \tag{10}$$

$$\sigma_{\theta\theta} = \frac{C\sin\theta}{a} + \frac{2D\sin\theta}{a^3} - \frac{F}{\pi a}\cos^2\left(\frac{\theta}{2}\right)\cot\left(\frac{\theta}{2}\right) \ . \tag{11}$$

[1]Notice that we only include terms that decay as $r \to \infty$.

To satisfy the traction-free boundary conditions (7) for all θ, the constants A, B, C, D must satisfy the conditions

$$-2B + C - \frac{2D}{a^2} = \frac{F}{2\pi} \; ; \tag{12}$$

$$A = \frac{Fa}{2\pi} \; ; \tag{13}$$

$$-C + \frac{2D}{a^2} = \frac{F}{2\pi} \; . \tag{14}$$

One additional equation is obtained from the condition that the displacements be single-valued. The possible discontinuity can be concentrated on the line $\theta = \psi = 0$ and arises from the terms $\theta \cos\theta, \psi \cos\psi$ in u_r, which is the same as u_R on this line.

Taking corresponding terms from Table 9.1, we find that the magnitude of the possible discontinuity is

$$\frac{F(\kappa - 1)}{2\pi} + \frac{B(\kappa - 1)}{2} - \frac{C(\kappa + 1)}{2}$$

and hence continuity of displacements demands that

$$B - \frac{C(\kappa + 1)}{(\kappa - 1)} = -\frac{F}{\pi} \; . \tag{15}$$

The four equations (12–15) can now be solved to give

$$A = \frac{Fa}{2\pi} \; ; \; B = -\frac{F}{2\pi} \; ; \; C = \frac{F(\kappa - 1)}{2\pi(\kappa + 1)} \; ; \; D = \frac{F\kappa a^2}{2\pi(\kappa + 1)} \; . \tag{16}$$

The complete stress function follows on substituting these results into equation (8) and superposing the term

$$\frac{F}{\pi} R\psi \cos(\psi) \; .$$

13.2. Use a method similar to that outlined in Problem 13.1 to find the stress field if the force F acts in the direction *normal* to the surface of the otherwise traction-free hole $r=a$.

If a normal (compressive) force is applied at the point A, the appropriate stress function is

$$\phi = -\frac{F}{\pi} R\psi \sin(\psi) \ ,$$

from §12.3. The corresponding stress components are most naturally expressed in the R, ψ coordinate system as

$$\sigma_{RR} = -\frac{2F\cos\psi}{\pi R} \quad ; \quad \sigma_{R\psi} = \sigma_{\psi\psi} = 0 \ , \tag{1}$$

from Table 8.1.

At the point $B(a, \theta)$, we have

$$R = 2a\sin\left(\frac{\theta}{2}\right) \quad ; \quad \cos\psi = -\sin\left(\frac{\theta}{2}\right) \tag{2}$$

and hence

$$\sigma_{RR} = 2F\sin\left(\frac{\theta}{2}\right) \Big/ \left[2\pi a \sin\left(\frac{\theta}{2}\right)\right] = \frac{F}{\pi a} \ . \tag{3}$$

The stress transformation equations then permit us to write

$$\sigma_{\theta\theta} = \sigma_{RR}\cos^2\left(\frac{\theta}{2}\right) = \frac{F}{\pi a}\cos^2\left(\frac{\theta}{2}\right) = \frac{F}{2\pi a}(1+\cos\theta) \tag{4}$$

$$\sigma_{rr} = \sigma_{RR}\sin^2\left(\frac{\theta}{2}\right) = \frac{F}{\pi a}\sin^2\left(\frac{\theta}{2}\right) = \frac{F}{2\pi a}(1-\cos\theta) \tag{5}$$

$$\sigma_{r\theta} = \sigma_{RR}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{F}{\pi a}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{F\sin\theta}{2\pi a} \ , \tag{6}$$

where we have used the fact that $\sigma_{R\psi} = \sigma_{\psi\psi} = 0$.

For the surface of the hole to be traction-free, we require

$$\sigma_{r\theta} = \sigma_{rr} = 0 \quad ; \quad r = a, \text{ all } \theta \tag{7}$$

Equations (5,6) define terms containing $\sin\theta, \cos\theta$ in these tractions and also a constant term in σ_{rr}. To cancel them, we need to superpose the stress function[2]

$$\phi = A\ln(r) + Br\theta\sin\theta + Cr\ln(r)\cos\theta + \frac{D\cos\theta}{r} \tag{8}$$

[2]Notice that we only include terms that decay as $r \to \infty$.

from Table 8.1, leading to the following traction components at the point $B(a,\theta)$:-

$$\sigma_{rr} = \frac{F}{2\pi a}(1-\cos\theta) + \frac{A}{a^2} + \frac{2B\cos\theta}{a} + \frac{C\cos\theta}{a} - \frac{2D\cos\theta}{a^3} \tag{9}$$

$$\sigma_{r\theta} = \frac{F\sin\theta}{2\pi a} + \frac{C\sin\theta}{a} - \frac{2D\sin\theta}{a^3} \ . \tag{10}$$

To satisfy the traction-free boundary conditions (7) for all θ, the constants A, B, C, D must satisfy the conditions

$$2B + C - \frac{2D}{a^2} = \frac{F}{2\pi} \ ; \tag{11}$$

$$A = -\frac{Fa}{2\pi} \ ; \tag{12}$$

$$C - \frac{2D}{a^2} = -\frac{F}{2\pi} \ . \tag{13}$$

One additional equation is obtained from the condition that the displacements be single-valued. The possible discontinuity can be concentrated on the line $\theta = \psi = 0$ and arises from the terms $\theta\sin\theta, \psi\sin\psi$ in u_θ, which is the same as u_ψ on this line.

Taking corresponding terms from Table 9.1, we find that the magnitude of the possible discontinuity is

$$-\frac{F(\kappa-1)}{2\pi} + \frac{B(\kappa-1)}{2} + \frac{C(\kappa+1)}{2}$$

and hence continuity of displacements demands that

$$B + \frac{C(\kappa+1)}{(\kappa-1)} = \frac{F}{\pi} \ . \tag{14}$$

The four equations (12–15) can now be solved to give

$$A = -\frac{Fa}{2\pi} \ ; \ B = -\frac{F}{2\pi} \ ; \ C = -\frac{F(\kappa-1)}{2\pi(\kappa+1)} \ ; \ D = -\frac{F\kappa a^2}{2\pi(\kappa+1)} \ . \tag{15}$$

The complete stress function follows on substituting these results into equation (8) and superposing the term

$$-\frac{F}{\pi}R\psi\sin(\psi) \ .$$

13.3. A large plate is in a state of pure bending such that $\sigma_{xx} = \sigma_{xy} = 0$, $\sigma_{yy} = Cx$, where C is a constant.

We now introduce a crack in the range $-a < x < a,\ y = 0$.

(i) Find a suitable corrective solution which, when superposed on the simple bending field, will make the surfaces of the crack free of tractions. (Hint: represent the corrective solution by a distribution of climb dislocations in $-a < x < a$. Don't forget the closure condition (13.41)).

(ii) Find the corresponding crack opening displacement as a function of x and show that it has an unacceptable negative value in $-a < x < 0$.

(iii) Re-solve the problem assuming that there is frictionless contact in some range $-a < x < b$, where b is a constant to be determined. The dislocations will now have to be distributed only in $b < x < a$ and b is found from a continuity condition at $x = b$. (Move the origin to the mid-point of the range $b < x < a$.)

(iv) For the case with contact, find expressions for (a) the crack opening displacement, (b) the stress intensity factor at $x = a$, (c) the dimension b and (d) the contact traction in $-a < x < b$ and hence verify that the contact inequalities are satisfied.

The unperturbed solution is $\sigma_{yy} = Cx$ and hence the corrective solution must satisfy the boundary conditions

$$\sigma_{yy} = -Cx \quad ; \quad \sigma_{xy} = 0 \quad ; \quad y = 0,\ -a < x < a\ . \tag{1}$$

Using equation (13.34), we therefore require a distribution of climb dislocations $B_y(\xi)$ such that

$$\int_{-a}^{a} \frac{B_y(\xi)d\xi}{(x-\xi)} = \frac{\pi C(\kappa+1)x}{2\mu} \quad ; \quad -a < x < a\ . \tag{2}$$

Making the change of variable $x = a\cos\phi, \xi = a\cos\theta$, we obtain

$$\int_0^{\pi} \frac{B_y(\theta)\sin\theta d\theta}{(\cos\phi - \cos\theta)} = \frac{\pi C(\kappa+1)a\cos\phi}{2\mu} \quad ; \quad 0 < \phi < \pi\ . \tag{3}$$

Now (12.35) with $n = 2$ gives

$$\begin{aligned} \int_0^{\pi} \frac{\cos 2\theta d\theta}{(\cos\phi - \cos\theta)} &= -\frac{\pi\sin 2\phi}{\sin\phi} \quad ; \quad 0 < \phi < \pi \\ &= -\frac{2\pi\sin\phi\cos\phi}{\sin\phi} \quad ; \quad 0 < \phi < \pi \\ &= -2\pi\cos\phi \quad ; \quad 0 < \phi < \pi \end{aligned} \tag{4}$$

and hence a solution of Equation (3) is

$$B_y(\theta)\sin\theta = -\frac{C(\kappa+1)a\cos 2\theta}{4\mu} \ . \tag{5}$$

This is not the most general solution, since we can add in an arbitrary multiple of the homogeneous solution as in equation (13.39) obtaining

$$B_y(\theta) = -\frac{C(\kappa+1)a\cos 2\theta}{4\mu\sin\theta} + \frac{A}{\sin\theta} \ , \tag{6}$$

where A is an arbitrary constant.

The closure condition (13.41) demands that

$$\int_0^\pi aB_y(\theta)\sin\theta d\theta = 0 \tag{7}$$

and hence

$$\int_0^\pi a\left(A - \frac{C(\kappa+1)a\cos 2\theta}{4\mu}\right)d\theta = \pi Aa = 0 \tag{8}$$

— i.e. $A = 0$, since the second term in the integral is zero[3].

The dislocation distribution is now completely determined. In Cartesian coordinates, it is

$$B_y(x) = -\frac{C(\kappa+1)(2x^2-a^2)}{4\mu\sqrt{a^2-x^2}} \ . \tag{9}$$

The crack opening displacement (C.O.D.) is determined from equation (13.47) as

$$\begin{aligned} \text{C.O.D.} &= \int_{-a}^{x} B_y(\xi)d\xi \\ &= -\frac{C(\kappa+1)}{4\mu}\int_{-a}^{x}\frac{(2x^2-a^2)dx}{\sqrt{a^2-x^2}} \\ &= \frac{C(\kappa+1)x\sqrt{a^2-x^2}}{4\mu} \ . \end{aligned} \tag{10}$$

This expression is plotted in Figure 1 and clearly indicates that interpenetration is predicted in the region $-a < x < 0$, as we might have anticipated from the loading conditions.

[3]This will also be true for the most general case. If the dislocation distribution is written as a complete Fourier series

$$B_y(\theta)\sin\theta = A + \sum_{n=1}^{\infty} c_n\cos(n\theta) \ ,$$

all the terms except A will make zero contribution to the closure condition integral and hence we can still conclude that the multiplier A on the homogeneous solution of the equation for B_y is zero.

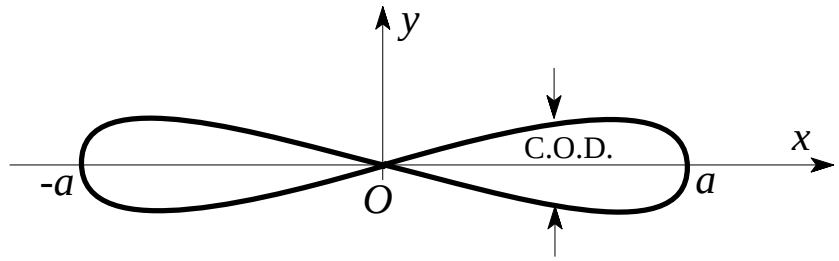


Figure 1: Crack opening displacement

Even though this solution is physically unreasonable, we proceed to determine the stresses on the crack plane $y = 0$ in the interests of completeness. Following equation (13.42), we have

$$\sigma_{yy}(x,0) = -\frac{2\mu}{\pi(\kappa_1)}\int_{-a}^{a}\frac{B_y(\xi)d\xi}{(x-\xi)} = \frac{C}{2\pi}\int_{-a}^{a}\frac{(2\xi^2-a^2)d\xi}{(x-\xi)\sqrt{a^2-\xi^2}} \ . \tag{11}$$

To evaluate this integral, we first develop a general recurrence relation between integrals of the class

$$I_n \equiv \int_{-a}^{a}\frac{\xi^n d\xi}{(x-\xi)\sqrt{a^2-\xi^2}} \ . \tag{12}$$

Noting that $\xi = x-(x-\xi)$, we can write $\xi^n = x\xi^{n-1}-(x-\xi)\xi^{n-1}$ and hence

$$I_n = x\int_{-a}^{a}\frac{\xi^{n-1}d\xi}{(x-\xi)\sqrt{a^2-\xi^2}} - \int_{-a}^{a}\frac{\xi^{n-1}d\xi}{\sqrt{a^2-\xi^2}} = xI_{n-1} - \int_{-a}^{a}\frac{\xi^{n-1}d\xi}{\sqrt{a^2-\xi^2}} \ . \tag{13}$$

The remaining integral is by symmetry zero if n is even and is easily evaluated using the substitution $\xi = a\cos\theta$ if n is odd. We therefore obtain the recurrence relation

$$I_n = xI_{n-1} \qquad n \text{ even} \tag{14}$$

$$= xI_{n-1} - \frac{1.3.5....(n-2)\pi}{2.4.6....(n-1)} \ ; \quad n = 3,5,7,... \tag{15}$$

$$= xI_0 - \pi \qquad n = 1 \tag{16}$$

For the present problem, we have

$$I_1 = xI_0 - \pi = -\pi + \frac{|x|}{\sqrt{x^2-a^2}} \quad ; \quad |x| > a \ , \tag{17}$$

from (13.43) and hence

$$I_0 = \frac{\pi \mathrm{sgn}(x)}{\sqrt{x^2-a^2}} \quad ; \quad |x| > a \ . \tag{18}$$

Also,

$$I_2 = xI_1 = -\pi x + \frac{\pi x^2 \mathrm{sgn}(x)}{\sqrt{x^2 - a^2}} \quad ; \quad |x| > a \ . \tag{19}$$

Using these results in equation (11), we obtain

$$\begin{aligned} \sigma_{yy}(x,0) &= \frac{CI_2}{\pi} - \frac{Ca^2 I_0}{2\pi} \\ &= -Cx + \frac{Cx^2 \mathrm{sgn}(x)}{\sqrt{x^2 - a^2}} - \frac{Ca^2 \mathrm{sgn}(x)}{2\sqrt{x^2 - a^2}} \quad ; \quad |x| > a \ . \end{aligned} \tag{20}$$

Adding in the unperturbed solution $\sigma_{yy} = Cx$, we then have

$$\sigma_{yy}(x,0) = \frac{C(2x^2 - a^2)\mathrm{sgn}(x)}{2\sqrt{x^2 - a^2}} \ . \tag{21}$$

The partially-closed crack

We now consider the case where the crack is permitted to close in the region $-a < x < b$. Dislocations will now be required only in the open region $b < x < a$ and it is convenient to move the origin to the mid-point of this range by the change of variable

$$x^* = x - \frac{(a+b)}{2} \quad ; \quad a^* = \frac{(a-b)}{2} \ . \tag{22}$$

The open region $b < x < a$ then transforms to $-a^* < x^* < a^*$ and the analysis of §13.3.2 can be used in this new coordinate system.

The unperturbed solution can be written

$$\sigma_{yy} = Cx = Cx^* + \frac{C(a+b)}{2} \tag{23}$$

and hence, for the corrective solution, we seek a distribution of dislocations $B_y(\xi^*)$ such that

$$\int_{-a^*}^{a^*} \frac{B_y(\xi^*)d\xi^*}{(x^* - \xi^*)} = \frac{\pi C(\kappa+1)(2x^* + (a+b))}{4\mu)} \quad ; \quad -a^* < x^* < a^* \ . \tag{24}$$

The first term $\pi C(\kappa+1)x^*/2\mu$ in the right hand side leads to the same solution as before, except that a, x are replaced by a^*, x^*. This gives a contribution

$$B_y(x^*) = -\frac{C(\kappa+1)(2x^{*2} - a^{*2})}{4\mu\sqrt{a^{*2} - x^{*2}}} \ . \tag{25}$$

from (9). The second term in the right hand side of (24) is a constant $\pi C(\kappa+1)(a+b)/4\mu$. Its contribution to $B_y(x^*)$ can therefore be written down from §13.3.2 by replacing S by $C(a+b)/2$ — i.e.

$$B_y(x^*) = -\frac{C(\kappa+1)(a+b)x^*}{4\mu\sqrt{a^{*2} - x^{*2}}} \ . \tag{26}$$

from (13.40) with $A = 0$.

Assembling these two terms, we have the corrective solution

$$B_y(x^*) = -\frac{C(\kappa+1)[2x^{*2} - a^{*2} + (a+b)x^*]}{4\mu\sqrt{a^{*2} - x^{*2}}} \ . \tag{27}$$

We can now determine the extent of the contact area by noting that the crack must open smoothly at $x = b$ ($x^* = -a^*$) and hence B_y at this point must be zero. Equation (27) then requires that $2a^{*2} - a^{*2} - (a+b)a^* = 0$ — i.e. $a^* = a + b$. Substituting for a^* from (22), we obtain

$$\frac{(a-b)}{2} = a + b \tag{28}$$

and hence

$$b = -\frac{a}{3} \ . \tag{29}$$

Finally, we assemble the solutions for crack opening displacement and crack plane tractions. This is most conveniently done in the x^* coordinate system, since the equations are then similar to those in the preceding problems. We have

$$\begin{aligned} \text{C.O.D.} &= \int_{-a^*}^{x^*} B_y(\xi^*)d\xi^* \\ &= -\frac{C(\kappa+1)}{4\mu}\int_{-a^*}^{x^*} \frac{(2x^{*2} - a^{*2} + (a+b)x^*)dx}{\sqrt{a^{*2} - x^{*2}}} \\ &= \frac{C(\kappa+1)(x^* + a + b)\sqrt{a^{*2} - x^{*2}}}{4\mu} \ . \end{aligned} \tag{30}$$

Substituting for x^*, a^*, this can be written in the form

$$\text{C.O.D.} = \frac{C(\kappa+1)}{4\mu}\left(\frac{a}{3} + x\right)^{\frac{3}{2}}\sqrt{a - x} \ . \tag{31}$$

Notice that the crack opening displacement goes to zero with distance from the crack tip to the power $\frac{1}{2}$, but at the smooth contact point it goes to zero with distance to the power $\frac{3}{2}$. This latter behaviour is also observed in other smooth contact problems, such as the Hertz problem of §12.5.3.

The tractions on the crack plane can be written

$$\begin{aligned} \sigma_{yy}(x,0) &= -\frac{2\mu}{\pi(\kappa+1)}\int_{-a^*}^{a^*} \frac{B_y(\xi^*)d\xi^*}{(x^* - \xi^*)} \\ &= \frac{C}{2\pi}\int_{-a^*}^{a^*} \frac{(2\xi^{*2} - a^{*2} + (a+b))d\xi^*}{(x^* - \xi^*)\sqrt{a^{*2} - \xi^{*2}}} \\ &= \frac{C[(2x^{*2} - a^{*2}) + (a+b)x^*]\text{sgn}(x^*)}{2\sqrt{x^{*2} - a^{*2}}} \quad ; \quad |x^*| > a^* \ , \end{aligned} \tag{32}$$

using the integrals I_0, I_1, I_2 defined above and noting that $|x| = x\,\mathrm{sgn}(x)$.

Transforming this into the original coordinate system and simplifying, we obtain

$$C\left(x - \frac{2a}{3}\right)\sqrt{\frac{x+\frac{a}{3}}{x-a}}\mathrm{sgn}(x) \quad ; \quad x > a \text{ and } x < -\frac{a}{3}\ . \tag{33}$$

Notice that this traction is bounded at $x \to -a/3$, but singular at $x \to a$ as we would expect. Also, σ_{yy} is compressive (negative) in $-a < x < -a/3$, as it must be to satisfy the unilateral contact condition.

The complete traction distribution on the plane and the crack opening displacement are illustrated in Figure 2.

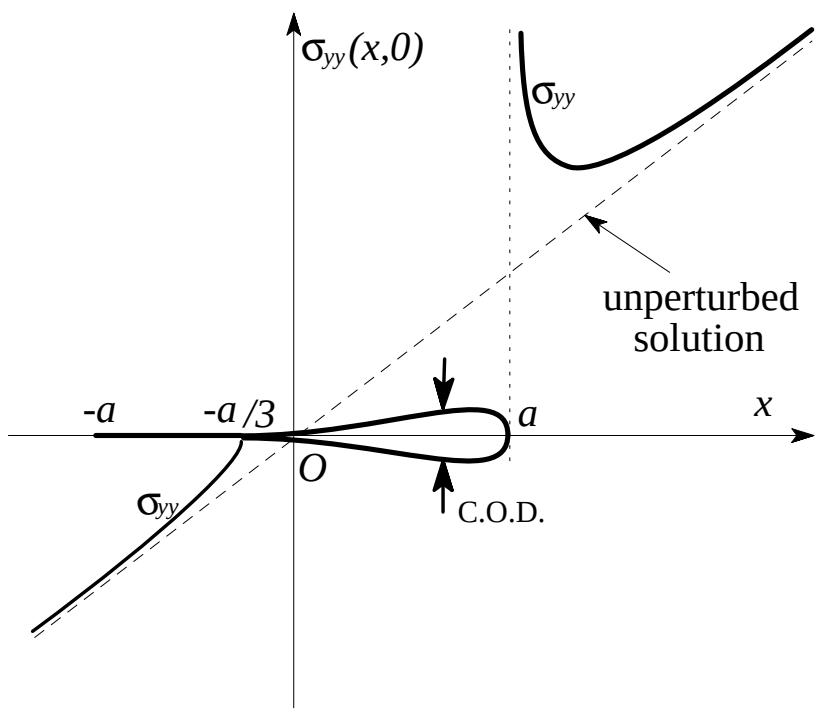


Figure 2: Crack opening displacement and tractions on the crack plane

Finally, we determine the stress intensity factor K_I at $x = a$ as

$$\begin{aligned} K_I &= \lim_{x\to a^+} \sigma_{yy}(x,0)\sqrt{2\pi(x-a)} \\ &= \frac{2\sqrt{2\pi}Ca^{\frac{3}{2}}}{3\sqrt{3}}\ . \end{aligned} \tag{34}$$

13.4. A state of uniform shear $\sigma_{xy} = S, \sigma_{xx} = \sigma_{yy} = 0$ in a large block of material is perturbed by the presence of the plane crack $-a < x < a, y = 0$. Representing the perturbation due to the crack by a distribution of glide dislocations along the crack line, find the shear stress distribution on the line $x > a, y = 0$, the relative motion between the crack faces in $-a < x < a, y = 0$ and the mode II stress intensity factor K_{II} defined as

$$K_{II} \equiv \lim_{x \to a^+} \sigma_{yx}(x, 0)\sqrt{2\pi(x-a)} \ .$$

If the crack surfaces are to be traction-free, the corrective solution must satisfy the boundary conditions

$$\sigma_{yx} = -S \ ; \ \ \sigma_{yy} = 0 \ ; \quad -a < x < a, \ y = 0 \tag{1}$$

$$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \to 0 \ ; \quad r \to \infty \ . \tag{2}$$

We represent the corrective solution by a distribution $B_x(\xi)$ of glide dislocations on the line of the crack. As in §13.3.2, but using equation (13.27) for the glide dislocation, we then have

$$\sigma_{yx} = -\frac{2\mu}{\pi(\kappa+1)} \int_{-a}^{a} \frac{B_x(\xi)d\xi}{(x-\xi)} = -S \ . \tag{3}$$

This is identical to Equation (13.35) and hence has solution

$$B_x(\xi) = -\frac{S(\kappa+1)\xi}{2\mu\sqrt{a^2-\xi^2}} \ , \tag{4}$$

by analogy with (13.40) with $A = 0$ from the closure condition.

The stress distribution on the crack line is then found as

$$\begin{aligned} \sigma_{yx} &= -\frac{2\mu}{\pi(\kappa+1)} \int_{-a}^{a} \frac{B_x(\xi)d\xi}{(x-\xi)} && (5) \\ &= \frac{S}{\pi} \int_{-a}^{a} \frac{\xi d\xi}{(x-\xi)\sqrt{a^2-\xi^2}} \\ &= S\left(-1 + \frac{|x|}{\sqrt{x^2-a^2}}\right) \ ; \ \ |x| > a, \ y = 0 && (6) \end{aligned}$$

and adding in the unperturbed solution, we have

$$\sigma_{yx}(x, 0) = \frac{S|x|}{\sqrt{x^2-a^2}} \ ; \ \ |x| > a \ . \tag{7}$$

The mode II stress intensity factor is

$$K_{II} \equiv \lim_{x \to a^+} \frac{S|x|\sqrt{2\pi(x-a)}}{\sqrt{x^2-a^2}} = S\sqrt{\pi a} \ . \tag{8}$$

The relative motion between the crack faces is a tangential shift

$$h(x) = u_x(x, 0^+) - u_x(x, 0^-) \ . \tag{9}$$

A unit positive glide dislocation causes a unit positive shift in $x > \xi$ (to the right of the dislocation) and hence the shift is

$$h(x) = \int_{-a}^{x} B_x(\xi) d\xi = \frac{S(\kappa+1)\sqrt{a^2-x^2}}{2\mu} \ . \tag{10}$$

13.5. A state of uniform general bi-axial stress $\sigma_{xx} = S_{xx}, \sigma_{yy} = S_{yy}, \sigma_{xy} = S_{xy}$ in a large block of material is perturbed by the presence of the plane crack $-a < x < a, y = 0$. Note that the crack will close completely if $S_{yy} < 0$ and will be fully open if $S_{yy} > 0$. Use the method proposed in Problem 13.4 to find the mode II stress intensity factor K_{II} for both cases, assuming that Coulomb friction conditions hold in the closed crack with coefficient f.

If the block contains a large number of widely separated similar cracks of all possible orientations and if the block fails when at any one crack

$$\sqrt{K_I^2 + K_{II}^2} = K_{Ic} \ ,$$

where the fracture toughness K_{Ic} is a material constant, sketch the biaxial failure surface for the material — i.e. the locus of all failure points in principal biaxial stress space (σ_1, σ_2).

If $S_{yy} > 0$, the crack will open. In this case, the surfaces of the crack are traction-free and there will be a stress intensity factor

$$K_I = S_{yy}\sqrt{\pi a} \ , \tag{1}$$

from (13.46). The shear loading S_{xy} will also lead to mode II loading. The solution of this problem is the same as that given in Problem 13.4, leading to a mode II stress intensity factor

$$K_{II} = S_{xy}\sqrt{\pi a} \ . \tag{2}$$

The situation changes if $S_{yy} < 0$, since then the crack does not open. Instead, the unperturbed normal traction is transmitted across the crack faces as a contact load and there is no mode I stress intensity factor. The contact traction will be equal to S_{yy} — i.e. the contact *pressure* will be $p(x) = -S_{yy}$ — and friction will be sufficient to prevent motion as long as $|S_{xy}| < -fS_{yy}$ (remember that S_{yy} is negative). In this case there will be no mode II stress intensity factor either and failure cannot occur.

If $S_{yy} < 0$ and $|S_{xy}| > -fS_{yy}$, slip will occur at the interface and the Coulomb law requires that the traction there will be $\pm fS_{yy}$, the sign depending on the direction of slip. If $S_{xy} > 0$, we require the negative sign and the *corrective* solution is the difference between the final state (with $\sigma_{yx} = -fS_{yy}$) and the unperturbed state with $\sigma_{yx} = S_{xy}$. Thus, the corrective solution must satisfy the condition

$$\sigma_{yx}(x, 0) = -fS_{yy} - S_{xy} \ ; \quad -a < x < a \ . \tag{3}$$

This problem is exactly similar to a crack loaded by a modified traction $fS_{yy} + S_{xy}$ and leads therefore to a mode II stress intensity factor

$$K_{II} = (S_{xy} + fS_{yy})\sqrt{\pi a} \ . \tag{4}$$

If there is no friction, $f = 0$ and this reduces to the case of the open crack. Also, note that the friction *reduces* the stress intensity factor, because $S_{yy} < 0$ (compressive).

Summarizing these results and using the given failure criterion, we conclude that failure will occur if either

$$S_{yy} > 0 \text{ and } \sqrt{S_{xy}^2 + S_{yy}^2} > \frac{K_{Ic}}{\sqrt{\pi a}} \tag{5}$$

or

$$S_{yy} < 0 \text{ and } S_{xy} + f S_{yy} > \frac{K_{Ic}}{\sqrt{\pi a}} \ . \tag{6}$$

Notice that failure depends only on the tractions S_{yy}, S_{xy} that act on the crack surface. It is independent of S_{xx}. If we have a random array of cracks of all orientations in a biaxial stress field, failure will be determined by the orientation that gives the maximum value of the parameters defined in equations (5,6).

To explore the consequences for the biaxial failure surface, we shall represent the stress state by a Mohr's circle and write σ, τ for S_{yy}, S_{xy} respectively. The quantity $\sqrt{S_{xy}^2 + S_{yy}^2} = \sqrt{\tau^2 + \sigma^2}$ then defines the distance from the origin to a point on the circle representing the orientation of the crack. Failure is most likely to occur when this distance is a maximum, subject to the condition that $\sigma > 0$. Figure 1(a) illustrates the case where $\sigma_1 + \sigma_2 > 0$, in which case it is clear that the maximum distance OA is σ_1, leading to the criterion

$$\sigma_1 + \sigma_2 > 0 \text{ and } \sigma_1 > \frac{K_{Ic}}{\sqrt{\pi a}} \ . \tag{7}$$

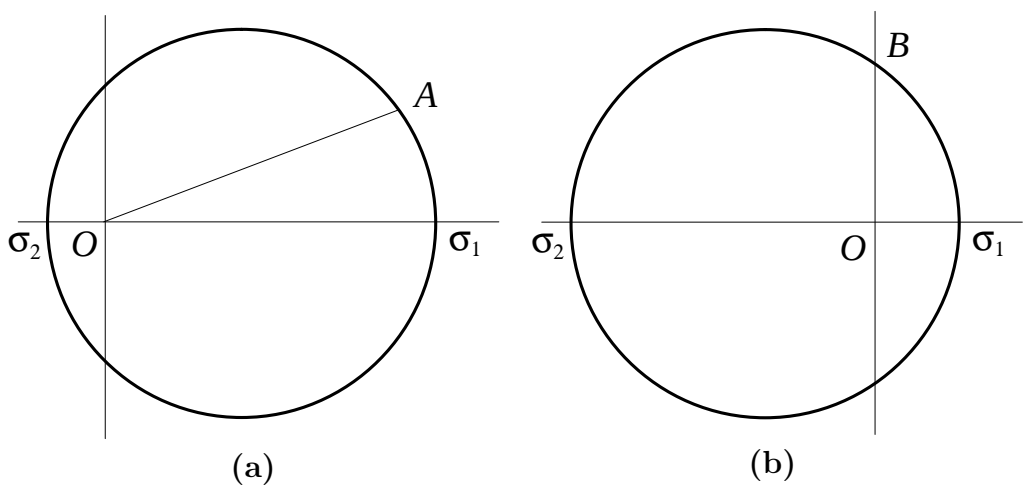


Figure 1

The case $\sigma_1 + \sigma_2 < 0$ is illustrated in Figure 1(b). In this case, the centre of the circle lies to the left of the origin and the maximum distance corresponding to a point satisfying the condition $\sigma > 0$ is the line OB. Using the Mohr's circle equations (1.15, 1.16), we have

$$\sigma = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta \tag{8}$$

$$\tau = (\sigma_2 - \sigma_1) \sin\theta \cos\theta \ , \tag{9}$$

which can be written in the double angle notation

$$\sigma = \frac{(\sigma_1 + \sigma_2)}{2} + \frac{(\sigma_1 - \sigma_2)}{2}\cos(2\theta) \tag{10}$$

$$\tau = \frac{(\sigma_2 - \sigma_1)}{2}\sin(2\theta) \ , \tag{11}$$

The point B is defined by the condition $\sigma = 0$ and hence

$$\cos(2\theta) = -\frac{(\sigma_1 + \sigma_2)}{(\sigma_1 - \sigma_2)} \ . \tag{12}$$

We therefore have

$$OB = \tau = \frac{(\sigma_2 - \sigma_1)}{2}\sqrt{1 - \frac{(\sigma_1 + \sigma_2)^2}{(\sigma_1 - \sigma_2)^2}} = \sqrt{-\sigma_1\sigma_2} \ , \tag{13}$$

where we have assumed that $\sigma_1 > 0 > \sigma_2$. In this range, failure is predicted if

$$\sqrt{-\sigma_1\sigma_2} > \frac{K_{Ic}}{\sqrt{\pi a}} \ . \tag{14}$$

In the region $\sigma < 0$, the criterion (6) predicts failure if

$$\tau + f\sigma > \frac{K_{Ic}}{\sqrt{\pi a}} \ . \tag{15}$$

Using equations (10,11) we have

$$\tau + f\sigma = \frac{(\sigma_2 - \sigma_1)}{2}\sin(2\theta) + \frac{f(\sigma_1 + \sigma_2)}{2} + \frac{f(\sigma_1 - \sigma_2)}{2}\cos(2\theta) \tag{16}$$

and the maximum value of this expression occurs when θ is given by

$$(\sigma_2 - \sigma_1)\cos(2\theta) - f(\sigma_1 - \sigma_2)\sin(2\theta) = 0 \tag{17}$$

— i.e.

$$\tan(2\theta) = -\frac{1}{f} \ . \tag{18}$$

Substitutiting this value into (16), we then obtain

$$(\tau + f\sigma)_{\max} = \frac{1}{2}(\sigma_1(\sqrt{1 + f^2} + f) - \sigma_2(\sqrt{1 + f^2} - f)) \ , \tag{19}$$

implying that failure will occur when

$$\sigma_1(\sqrt{1 + f^2} + f) - \sigma_2(\sqrt{1 + f^2} - f) > \frac{2K_{Ic}}{\sqrt{\pi a}} \ . \tag{20}$$

However, this result applies only as long as the predicted value of $\sigma < 0$ and hence

$$\frac{(\sigma_1 + \sigma_2)}{2} + \frac{(\sigma_1 - \sigma_2)f}{2\sqrt{1+f^2}} < 0 \ , \tag{21}$$

from (10,18). Multiplying through by the positive quantity $2\sqrt{1+f^2}$, we have

$$\sigma_1(\sqrt{1+f^2} + f) + \sigma_2(\sqrt{1+f^2} - f) < 0 \ . \tag{22}$$

Collecting results, we conclude that for $\sigma_1 > \sigma_2$, failure occurs for

$$\begin{aligned}
\sigma_1 &> \frac{K_{Ic}}{\sqrt{\pi a}} \ ; \qquad \sigma_1 > -\sigma_2 \\
\sqrt{-\sigma_1\sigma_2} &> \frac{K_{Ic}}{\sqrt{\pi a}} \ ; \qquad -\sigma_2 > \sigma_1 > -\left(\frac{\sqrt{1+f^2} - f}{\sqrt{1+f^2} + f}\right)\sigma_2 \\
\sigma_1(\sqrt{1+f^2} + f) - \sigma_2(\sqrt{1+f^2} - f) &> \frac{2K_{Ic}}{\sqrt{\pi a}} \ ; \qquad \sigma_1 < -\left(\frac{\sqrt{1+f^2} - f}{\sqrt{1+f^2} + f}\right)\sigma_2 \ .
\end{aligned}$$

The corresponding failure envelope is sketched in Figure 3, where we have used symmetry arguments to extend into the region $\sigma_2 > \sigma_1$. Notice that failure cannot occur for arbitrarily large hydrostatic compression ($\sigma_1 = \sigma_2 < 0$), since in that case the crack remains closed and has no tendency to slip.

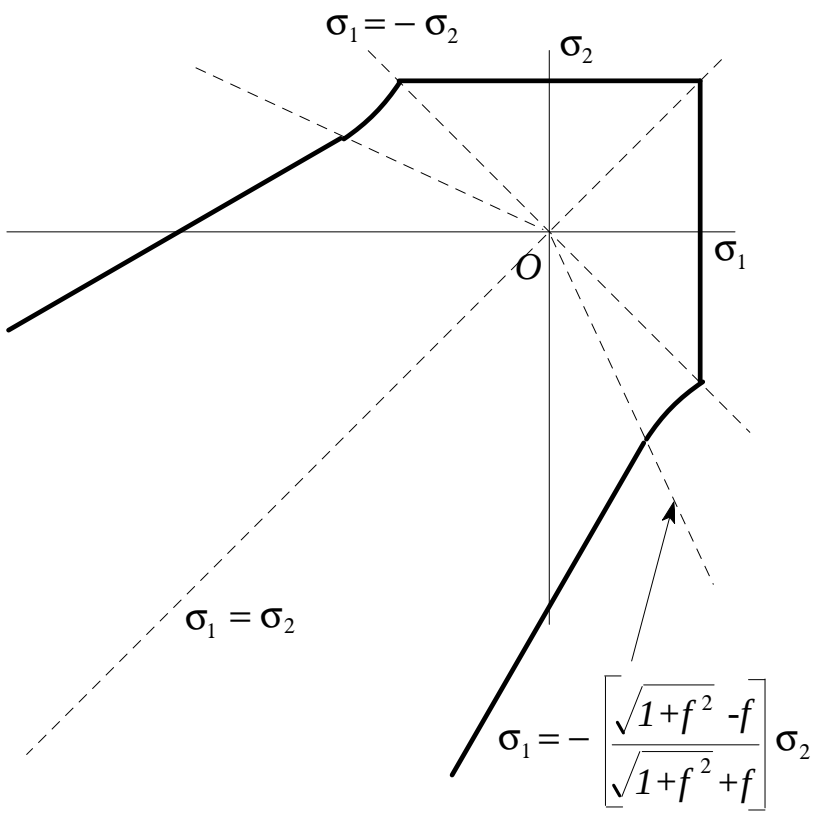


Figure 3

13.6. Find the stress field due to a climb dislocation of unit strength located at the point $(0, a)$ in the traction-free half-plane $y > 0$, using the following method:-

(i) Write the stress function (13.15) in Cartesian coördinates.

(ii) Find the solution for a dislocation at $(0, a)$ in the full plane $-\infty < y < \infty$ by making a change of origin.

(iii) Superpose additional singularities centred on the image point $(0, -a)$ to make the surface $y = 0$ traction-free. Notice that these singularities are outside the actual body and are therefore admissible. Appropriate functions (in polar coördinates centred on $(0, -a)$) are

$$C_1 r \ln(r) \cos\theta \ ; \quad \frac{C_2 \cos\theta}{r} \ ; \quad C_3 \sin(2\theta) \ .$$

The stress function for a dislocation of strength B_y in an infinite body is given by equations (13.15, 13.17) as

$$\phi = -\frac{2\mu B_y}{\pi(\kappa+1)} r \ln(r) \cos\theta \ . \tag{1}$$

Noting that $x = r\cos\theta$ and $r = \sqrt{x^2 + y^2}$, we can rewrite this expression in Cartesian coördinates as

$$\phi = -\frac{\mu B_y}{\pi(\kappa+1)} x \ln(x^2 + y^2) \ . \tag{2}$$

We can move the dislocation to the point $(0, a)$ by replacing y by $y - a$, obtaining

$$\phi = -\frac{\mu B_y}{\pi(\kappa+1)} x \ln(x^2 + (y-a)^2) \ . \tag{3}$$

The singular functions

$$C_1 r \ln(r) \cos\theta \ ; \quad \frac{C_2 \cos\theta}{r} \ ; \quad C_3 \sin(2\theta)$$

can be written in Cartesian coördinates as

$$\frac{C_1}{2} x \ln(x^2 + y^2) \ ; \quad \frac{C_2 x}{x^2 + y^2} \ ; \quad \frac{2C_3 xy}{x^2 + y^2} \ .$$

To move these singularities to the image point $(0, -a)$ we replace y by $y + a$ obtaining the candidate stress function

$$\phi = -\frac{\mu B_y}{\pi(\kappa+1)} x \ln(x^2 + (y-a)^2) + \frac{C_1}{2} x \ln(x^2 + (y+a)^2) + \frac{C_2 x}{x^2 + (y+a)^2} + \frac{2C_3 x(y+a)}{x^2 + (y+a)^2} \ .$$

The corresponding stress components are

$$\begin{aligned}
\sigma_{xx} &= -\frac{2\mu B_y x(x^2-(y-a)^2)}{\pi(\kappa+1)(x^2+(y-a)^2)^2}+\frac{C_1 x(x^2-(y+a)^2)}{(x^2+(y+a)^2)^2} \\
&\quad -\frac{2C_2 x(x^2-3(y+a)^2)}{(x^2+(y+a)^2)^3}-\frac{4C_3 x(y+a)(3x^2-(y+a)^2)}{(x^2+(y+a)^2)^3} \qquad (4)\\
\sigma_{xy} &= -\frac{2\mu B_y (y-a)(x^2-(y-a)^2)}{\pi(\kappa+1)(x^2+(y-a)^2)^2}+\frac{C_1 (y+a)(x^2-(y+a)^2)}{(x^2+(y+a)^2)^2} \\
&\quad -\frac{2C_2 (y+a)(3x^2-(y+a)^2)}{(x^2+(y+a)^2)^3}+\frac{2C_3(x^4-6x^2(y+a)^2+(y+a)^4)}{(x^2+(y+a)^2)^3} \qquad (5)\\
\sigma_{yy} &= -\frac{2\mu B_y x(x^2+3(y-a)^2)}{\pi(\kappa+1)(x^2+(y-a)^2)^2}+\frac{C_1 x(x^2+3(y+a)^2)}{(x^2+(y+a)^2)^2} \\
&\quad +\frac{2C_2 x(x^2-3(y+a)^2)}{(x^2+(y+a)^2)^3}+\frac{4C_3 x(y+a)(x^2-3(y+a)^2)}{(x^2+(y+a)^2)^3}\ . \qquad (6)
\end{aligned}$$

On the surface $y=0$, the tractions are

$$\begin{aligned}
\sigma_{xy}(x,0) &= \frac{2\mu B_y a(x^2-a^2)}{\pi(\kappa+1)(x^2+a^2)^2}+\frac{C_1 a(x^2-a^2)}{(x^2+a^2)^2} \\
&\quad -\frac{2C_2 a(3x^2-a^2)}{(x^2+a^2)^3}+\frac{2C_3(x^4-6x^2a^2+a^4)}{(x^2+a^2)^3} \qquad (7)\\
\sigma_{yy}(x,0) &= -\frac{2\mu B_y x(x^2+3a^2)}{\pi(\kappa+1)(x^2+a^2)^2}+\frac{C_1 x(x^2+3a^2)}{(x^2+a^2)^2} \\
&\quad +\frac{2C_2 x(x^2-3a^2)}{(x^2+a^2)^3}+\frac{4C_3 xa(x^2-3a^2)}{(x^2+a^2)^3}\ , \qquad (8)
\end{aligned}$$

which we require to be zero for all x.

Multiplying these expressions by $(x^2+a^2)^3$, we obtain the functions

$$\begin{aligned}
f_1(x) &= \frac{2\mu B_y a(x^2-a^2)(x^2+a^2)}{\pi(\kappa+1)}+C_1 a(x^2-a^2)(x^2+a^2) \\
&\quad -2C_2 a(3x^2-a^2)+2C_3(x^4-6x^2a^2+a^4) \qquad (9)\\
f_2(x) &= -\frac{2\mu B_y x(x^2+3a^2)(x^2+a^2)}{\pi(\kappa+1)}+C_1 x(x^2+3a^2)(x^2+a^2) \\
&\quad +2C_2 x(x^2-3a^2)+4C_3 ax(x^2-3a^2) \qquad (10)
\end{aligned}$$

respectively. These can be expanded as polynomials in x as

$$\begin{aligned}
f_1(x) &= \frac{2\mu B_y a(x^4-a^4)}{\pi(\kappa+1)}+C_1 a(x^4-a^4) \\
&\quad -2C_2 a(3x^2-a^2)+2C_3(x^4-6x^2a^2+a^4) \qquad (11)
\end{aligned}$$

$$
\begin{aligned}
f_2(x) &= -\frac{2\mu B_y(x^5+4a^2x^3+3a^4x)}{\pi(\kappa+1)} + C_1(x^5+4a^2x^3+3a^4x) \\
&\quad +2C_2(x^3-3a^2x) + 4C_3a(x^3-3a^2x)\ . \qquad (12)
\end{aligned}
$$

The coefficients of each power of x must be separately zero, giving the six equations

$$
\begin{aligned}
\frac{2\mu B_y a}{\pi(\kappa+1)} + C_1 a + 2C_3 &= 0 \qquad (13)\\
-6C_2 a - 12C_3 a^2 &= 0 \qquad (14)\\
-\frac{2\mu B_y a^5}{\pi(\kappa+1)} - C_1 a^5 + 2C_2 a^3 + 2C_3 a^4 &= 0 \qquad (15)\\
-\frac{2\mu B_y}{\pi(\kappa+1)} + C_1 &= 0 \qquad (16)\\
-\frac{6\mu B_y a^2}{\pi(\kappa+1)} + 4C_1 a^2 + 2C_2 + 4C_3 a &= 0 \qquad (17)\\
-\frac{6\mu B_y a^4}{\pi(\kappa+1)} + 3C_1 a^4 - 6C_2 a^2 - 12C_3 a^3 &= 0\ . \qquad (18)
\end{aligned}
$$

These equations are not linearly independent. We can solve them for the three constants C_1, C_2, C_3, obtaining

$$
C_1 = \frac{2\mu B_y}{\pi(\kappa+1)}\ ; \quad C_2 = \frac{4\mu B_y a^2}{\pi(\kappa+1)}\ ; \quad C_3 = -\frac{2\mu B_y a}{\pi(\kappa+1)}\ , \qquad (19)
$$

and hence

$$
\begin{aligned}
\phi &= -\frac{\mu B_y}{\pi(\kappa+1)} x\ln(x^2+(y-a)^2) + \frac{\mu B_y}{\pi(\kappa+1)} x\ln(x^2+(y+a)^2) \\
&\quad + \frac{4\mu B_y a^2 x}{\pi(\kappa+1)(x^2+(y+a)^2)} - \frac{4\mu B_y a x(y+a)}{\pi(\kappa+1)(x^2+(y+a)^2)} \\
&= \frac{\mu B_y}{\pi(\kappa+1)}\left[x\ln\left(\frac{x^2+(y+a)^2}{x^2+(y-a)^2}\right) - \frac{4axy}{(x^2+(y+a)^2)}\right]\ .
\end{aligned}
$$

The stress components are then recovered by substituting these expressions into equations (5–7).

13.7. Solve Problem 6 using Aderogba's result (13.59, 13.57) for step (iii).

The stress function for a dislocation of strength B_y in an infinite body is given by equations (13.15, 13.17) as

$$\phi = -\frac{2\mu B_y}{\pi(\kappa+1)} r \ln(r) \cos\theta \ . \tag{1}$$

Noting that $x = r\cos\theta$ and $r = \sqrt{x^2+y^2}$, we can rewrite this expression in Cartesian coördinates as

$$\phi = -\frac{\mu B_y}{\pi(\kappa+1)} x \ln(x^2+y^2) \ . \tag{2}$$

We can move the dislocation to the point $(0, a)$ by replacing y by $y-a$, obtaining

$$\phi_0 = -\frac{\mu B_y}{\pi(\kappa+1)} x \ln(x^2+(y-a)^2) \ . \tag{3}$$

Substituting into equation (13.59), we have

$$\mathcal{L}\left\{\phi_0(x,-y)\right\} = -\frac{\mu B_y}{\pi(\kappa+1)}\left[-x\ln\left(x^2+(y+a)^2\right) + \frac{4xy(y+a)}{x^2+(y+a)^2} - \frac{4xy^2}{x^2+(y+a)^2}\right]$$

and hence the complete stress function is

$$\phi = \frac{\mu B_y}{\pi(\kappa+1)}\left[x\ln\left(\frac{x^2+(y+a)^2}{x^2+(y-a)^2}\right) - \frac{4axy}{x^2+(y+a)^2}\right] \ ,$$

from (13.57).

13.8. Find the stress field due to a concentrated force F applied in the x-direction at the point $(0, a)$ in the half-plane $y > 0$, if the surface of the half-plane is traction-free. Verify that the results reduce to those of §12.3 in the limit as $a \to 0$.

The starting point for this problem is clearly the Kelvin solution for the force F in the x-direction in an *infinite* solid, for which the stress function (in plane stress) is

$$\phi = -\frac{F}{2\pi} r\theta \sin\theta + \frac{F(1-\nu)}{4\pi} r \ln(r) \cos\theta \ , \tag{1}$$

from equations (13.1, 13.10, 13.11).

Using the relations $x = r\cos\theta$, $y = r\sin\theta$, $\theta = \arctan(y/x)$ and $r = \sqrt{x^2+y^2}$, we can rewrite this expression in Cartesian coördinates as

$$\phi = -\frac{F}{2\pi} y \arctan(y/x) + \frac{F(1-\nu)}{8\pi} x \ln(x^2+y^2) \ . \tag{2}$$

We can then move the force to the point $(0, a)$ by replacing y by $y - a$, obtaining

$$\phi_0 = -\frac{F}{2\pi}(y-a) \arctan\left(\frac{y-a}{x}\right) + \frac{F(1-\nu)}{8\pi} x \ln(x^2 + (y-a)^2) \ . \tag{3}$$

In preparation for the use of equation (13.59), we have

$$\phi_0(x, -y) = -\frac{F}{2\pi}(y+a) \arctan\left(\frac{y+a}{x}\right) + \frac{F(1-\nu)}{8\pi} x \ln(x^2 + (y+a)^2) \ .$$

We then have

$$\nabla^2 \phi_0(x, -y) = -\frac{(1+\nu)Fx}{2\pi(x^2 + (y+a)^2)}$$

and

$$\mathcal{L}\phi_0(x, -y) = -\frac{F(y-a)}{2\pi} \arctan\left(\frac{y+a}{x}\right) - \frac{F(1-\nu)x}{8\pi} \ln(x^2 + (y+a)^2) - \frac{(1+\nu)Faxy}{2\pi(x^2+(y+a)^2)} \ .$$

Equation (13.57) then gives

$$\begin{aligned}
\phi &= \phi_0 + \mathcal{L}\phi_0(x, -y) \\
&= -\frac{F(y-a)}{2\pi} \arctan\left(\frac{y-a}{x}\right) + \frac{F(1-\nu)x}{8\pi} \ln(x^2 + (y-a)^2) \\
&\quad -\frac{F(y-a)}{2\pi} \arctan\left(\frac{y+a}{x}\right) - \frac{F(1-\nu)x}{8\pi} \ln(x^2 + (y+a)^2) - \frac{(1+\nu)Faxy}{2\pi(x^2+(y+a)^2)} \\
&= -\frac{F(y-a)}{2\pi} \left[\arctan\left(\frac{y+a}{x}\right) + \arctan\left(\frac{y-a}{x}\right)\right] \\
&\quad + \frac{F(1-\nu)x}{8\pi} \ln\left(\frac{x^2 + (y-a)^2}{x^2 + (y+a)^2}\right) - \frac{(1+\nu)Faxy}{2\pi(x^2+(y+a)^2)}
\end{aligned}$$

The stress components are then recovered by substituting these expressions into equations (4.6).

In the limiting case where $a \to 0$, we have

$$\phi \to -\frac{Fy}{\pi} \arctan\left(\frac{y}{x}\right)$$

which agrees with the results of §12.3.

13.9. Two dissimilar elastic half planes are bonded on the interface $y = 0$ and have material properties as defined in §13.4. Use Aderogba's formula (13.57) to determine the stress field due to a climb dislocation at the point $(0, a)$ in the half-plane $y > 0$. In particular, find the location and magnitude of the maximum shear traction at the interface.

The solution for the climb dislocation at the origin in a homogeneous half plane is defined by the stress function

$$\phi_0 = -\frac{\mu_1 B_y x \ln(x^2 + y^2)}{\pi(\kappa_1 + 1)} ,$$

from equation (13.15). Moving this to the point $(0, a)$, we obtain

$$\phi_0 = -\frac{\mu_1 B_y) x \ln(x^2 + (y - a)^2)}{\pi(\kappa_1 + 1)} ,$$

so

$$\phi_0(x, -y) = -\frac{\mu_1 B_y x \ln(x^2 + (y + a)^2)}{\pi(\kappa_1 + 1)} .$$

The procedure is now to substitute these expressions into (13.57) and then determine the stress components from

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} ; \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} ; \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} .$$

The algebraic expressions are lengthy, but the operations are routine in Maple or Mathematica. One point to watch however is that the integral functions

$$f_1 = \iint \nabla^2 \{\} dy dy$$

and

$$f_2 = \iint \nabla^2 \{\} dy dy - 2y \int \nabla^2 \{\} dy$$

will generally not be biharmonic. We therefore have to evaluate the expressions

$$\nabla^4 f_1 ; \quad \nabla^4 f_2 ,$$

both of which will be found to be functions of x only. It is then a simple matter to restore biharmonicity by subtracting the functions

$$\iiiint \nabla^4 f_1 dx dx dx dx ; \quad \iiiint \nabla^4 f_2 dx dx dx dx$$

from f_1, f_2 respectively.

Here we simply report the values of the traction components on the interface $x = 0$, which are

$$\sigma_{xy}(x,0) = \frac{2\mu_1 B_y(1+\alpha)a(x^2+2\beta a^2-a^2)}{\pi(\kappa_1+1)(1-\beta^2)(x^2+a^2)^2}$$

$$\sigma_{yy}(x,0) = -\frac{2\mu_1 B_y(1+\alpha)x(x^2+3a^2-2\beta a^2)}{\pi(\kappa_1+1)(1-\beta^2)(x^2+a^2)^2} \ .$$

The maximum shear traction transmitted across the interface occurs when

$$\frac{d}{dx}\left(\frac{x^2+(2\beta-1)a^2)}{(x^2+a^2)^2}\right) = 0$$

which occurs at $x = \pm a\sqrt{3-4\beta}$. The value of this maximum traction is

$$\tau_{\max} = \frac{2\mu_1 B_y(1+\alpha)(2-\beta)}{\pi(\kappa_1+1)(1-\beta^2)a} \ .$$

13.10. Use equation (13.60) to solve Problem 8.2.

The unperturbed field is defined by

$$\sigma_{xx} = Cy^2 \; ; \quad \sigma_{yy} = -Cx^2 \; ; \quad \sigma_{xy} = 0$$

so

$$\frac{\partial^2 \phi}{\partial x^2} = -Cx^2 \; ; \quad \frac{\partial^2 \phi}{\partial y^2} = Cy^2 \; ; \quad \frac{\partial^2 \phi}{\partial x \partial y} = 0$$

and

$$\nabla^2 \phi = Cy^2 - Cx^2 \; ; \quad \nabla^4 \phi = 2C - 2C = 0$$

as required. An appropriate function ϕ satisfying these conditions can be found by any convenient partial integration. For example

$$\phi = \int \int Cy^2 dy dy = \frac{Cy^4}{12} + yf(x) + g(x) \; .$$

Substitution in the last equation then gives

$$\frac{\partial^2 \phi}{\partial x \partial y} = y \frac{df}{dx} = 0 \; ,$$

whose only solution is a constant which leads to a null field, so we can take $f = 0$. Substitution in the first equation then gives

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{d^2 g}{dx^2} = -Cx^2 \; ,$$

with solution

$$g = -\frac{Cx^4}{12}$$

again dropping null terms. Thus, the unperturbed stress function is

$$\phi_0 = \frac{C(y^4 - x^4)}{12} = \frac{Cr^4(\sin^4\theta - \cos^4\theta)}{12} \; .$$

We then have

$$\frac{r^2}{a^2}\phi_0\left(\frac{a^2}{r}, \theta\right) = \frac{Ca^6(\sin^4\theta - \cos^4\theta)}{12r^2}$$

$$\nabla^2 \left\{ \frac{r^2}{a^2}\phi_0\left(\frac{a^2}{r}, \theta\right) \right\} = 0$$

and

$$\mathcal{L}\left\{ \frac{r^2}{a^2}\phi_0\left(\frac{a^2}{r}, \theta\right) \right\} = Ca^4(1 - 2\cos^2\theta)\left(\frac{a^2}{6r^2} - \frac{1}{4}\right) \; .$$

Also, since $\nabla^2\phi = Cy^2 - Cx^2$, we have

$$A = 0 \ ,$$

so

$$\phi = \frac{Cr^4(\sin^4\theta - \cos^4\theta)}{12} - Ca^4(1 - 2\cos^2\theta)\left(\frac{a^2}{6r^2} - \frac{1}{4}\right) \ ,$$

which can be restated in multiple angle terms as

$$\phi = \frac{C}{12}\left(3a^4 - r^4 - \frac{2a^6}{r^2}\right)\cos(2\theta) \ .$$

Finally, the perturbed stress field is obtained as

$$\begin{aligned}
\sigma_{rr} &= \frac{Ca^4(a^2 - r^2)\cos(2\theta)}{r^4} \\
\sigma_{r\theta} &= \frac{C(2a^6 - a^4r^2 - r^6)\sin(2\theta)}{2r^4} \\
\sigma_{\theta\theta} &= -\frac{C(a^6 + r^6)\cos(2\theta)}{r^4} \ .
\end{aligned}$$

CHAPTER 14

14.1. A direct electric current I flows along a conductor of rectangular cross-section $-4a < x < 4a, \quad -a < y < a$, all the surfaces of which are traction free. The conductor is made of copper of electrical resistivity ρ, thermal conductivity K, Young's modulus E, Poisson's ratio ν and coefficient of thermal expansion α. Assuming the current density to be uniform and neglecting electromagnetic effects, estimate the thermal stresses in the conductor when the temperature has reached a steady state.

The definition of resistivity is such that a unit cube will have resistance ρ for conduction of electricity between opposite faces. It follows that the power loss in such a cube is — and hence the heat generated per unit time per unit volume — is

$$Q = i^2 \rho \ , \tag{1}$$

where i is the *current density* — i.e. the current flux per a unit area.

For our problem, a current I flows through a cross-section of area $16a^2$ and hence $i = I/16a^2$. We therefore have

$$Q = \rho \left(\frac{I}{16a^2} \right)^2 = \frac{\rho I^2}{256a^4} \ . \tag{2}$$

For steady-state heat conduction in plane strain, we can use (14.16) and (14.7) to obtain

$$\nabla^4 \phi = \frac{E\alpha Q}{K(1-\nu)} \ . \tag{3}$$

Substituting for Q from (2), we then find

$$\nabla^4 \phi = \frac{E\alpha\rho I^2}{256K(1-\nu)a^4} \ , \tag{4}$$

which is constant, suggesting a 4th order polynomial as the particular solution.

The problem is symmetrical in x and y and the boundaries are traction free, so we try the most general 4th degree polynomial with these symmetries — i.e.

$$\phi = Ax^4 + Bx^2y^2 + Cy^4 + Dx^2 + Fy^2 \ . \tag{5}$$

This will function both as particular and homogeneous solution.

Equation (3) imposes the constraint

$$24A + 8B + 24C = \frac{E\alpha\rho I^2}{256K(1-\nu)a^4} \ . \tag{6}$$

The stress components are

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = 2Bx^2 + 12Cy^2 + 2F \; ; \tag{7}$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -4Bxy \; ; \tag{8}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = 12Ax^2 + 2By^2 + 2D \; . \tag{9}$$

We want to impose strong traction-free boundary conditions on the long edges $y = \pm a$ — i.e.

$$\sigma_{xy} = 0 \quad ; \quad y = a \; ; \tag{10}$$

$$\sigma_{yy} = 0 \quad ; \quad y = a \; ; \tag{11}$$

$$\sigma_{xy} = 0 \quad ; \quad y = -a \; ; \tag{12}$$

$$\sigma_{yy} = 0 \quad ; \quad y = -a \; . \tag{13}$$

The use of symmetry means that we only need to enforce the first two if these, with the result

$$12Ax^2 + 2Ba^2 + 2D = 0 \quad ; \quad -4Bax = 0 \tag{14}$$

and the only way both these equations can be satisfied for all x is if A, B, D are all zero. Using this result in (6), we obtain

$$C = \frac{E\alpha\rho I^2}{6144K(1-\nu)a^4} \; . \tag{15}$$

Two of the three weak conditions on the ends $x = \pm 4a$ are satisfied identically by virtue of the symmetry of the problem. The remaining condition is

$$\int_{-a}^{a} \sigma_{xx} dy = 0, \tag{16}$$

from which we obtain

$$\int_{-a}^{a} (12Cy^2 + 2F) dy = \frac{24Ca^3}{3} + 4Fa = 0 \tag{17}$$

and hence

$$F = -2Ca^2 = -\frac{2E\alpha\rho I^2}{6144K(1-\nu)a^4} \; . \tag{18}$$

The complete stress field is therefore defined by

$$\phi = \frac{E\alpha\rho I^2 (y^4 - 2a^2 y^2)}{6144K(1-\nu)a^4} \tag{19}$$

and

$$\sigma_{xx} = \frac{E\alpha\rho I^2(3y^2 - a^2)}{1536K(1-\nu)a^4} \quad ; \quad \sigma_{xy} = \sigma_{yy} = 0 \ . \tag{20}$$

Notice that the stress field is not affected by the thermal boundary conditions at the edges, even though these affect the temperature distribution. This would remain true even if we used the arguments of Chapter 6 to impose the strong rather than weak boundary conditions at the ends.

14.2. A fuel element in a nuclear reactor can be regarded as a solid cylinder of radius a. During operation, heat is generated at a rate $Q_0(1+Ar^2/a^2)$ per unit volume, where r is the distance from the axis of the cylinder and A is a constant.

Assuming that the element is immersed in a fluid at pressure p and that *axial* expansion is prevented, find the radial and circumferential thermal stresses produced in the steady state.

For steady-state heat conduction in plane strain, we can use (14.16) and (14.7) to obtain

$$\nabla^4\phi = \frac{E\alpha Q}{K(1-\nu)} = \frac{E\alpha Q_0}{K(1-\nu)}\left(1+\frac{Ar^2}{a^2}\right) . \tag{1}$$

The problem is axisymmetric and the origin is included, so only bounded stress functions are permitted. This and equation (1) suggests a stress function of the form

$$\phi = \frac{E\alpha Q_0}{K(1-\nu)}(Br^6 + Cr^4 + Dr^2) . \tag{2}$$

Notice that odd powers of r would imply discontinuities in derivatives at $r = 0$ and are therefore not permissible.

Substituting (2) into (1), we obtain

$$\frac{E\alpha Q_0}{K(1-\nu)}(576Br^2 + 64C) = \frac{E\alpha Q_0}{K(1-\nu)}\left(1+\frac{Ar^2}{a^2}\right) \tag{3}$$

and hence, equating coefficients of r,

$$B = \frac{A}{576a^2} \quad ; \quad C = \frac{1}{64} . \tag{4}$$

The stress components are

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{E\alpha Q_0}{K(1-\nu)}(6Br^4 + 4Cr^2 + 2D) ; \tag{5}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0 ; \tag{6}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = \frac{E\alpha Q_0}{K(1-\nu)}(30Br^4 + 12Cr^2 + 2D) . \tag{7}$$

We have one strong boundary condition, which is $\sigma_{rr} = -p$ at $r = a$ and hence

$$\frac{E\alpha Q_0}{K(1-\nu)}(6Ba^4 + 4Ca^2 + 2D) = -p , \tag{8}$$

giving

$$D = -3Ba^4 - 2Ca^2 - \frac{pK(1-\nu)}{2E\alpha Q_0} \ . \tag{9}$$

With these results, the final solution is defined by

$$\phi = \frac{E\alpha Q_0}{K(1-\nu)} \left(\frac{A(r^6 - 3r^2)}{576a^2} + \frac{r^4 - 2a^2}{64} \right) - \frac{pr^2}{2} \tag{10}$$

and

$$\sigma_{rr} = \frac{E\alpha Q_0}{K(1-\nu)} \left(\frac{A(r^4 - a^4)}{96a^2} + \frac{(r^2 - a^2)}{16} \right) - p \ ; \tag{11}$$

$$\sigma_{r\theta} = 0 \ ; \tag{12}$$

$$\sigma_{\theta\theta} = \frac{E\alpha Q_0}{K(1-\nu)} \left(\frac{A(5r^4 - a^4)}{96a^2} + \frac{(3r^2 - a^2)}{16} \right) - p \ . \tag{13}$$

14.3. The instantaneous temperature distribution in the thin plate $-a < x < a, -b < y < b$ is defined by

$$T(x,y) = T_0 \left(\frac{x^2}{a^2} - 1 \right) ,$$

where T_0 is a positive constant. Find the magnitude and location of (i) the maximum tensile stress and (ii) the maximum shear stress in the plate if the edges $x = \pm a, y = \pm b$ are traction-free and $a \gg b$.

From

$$T(x,y) = T_0 \left(\frac{x^2}{a^2} - 1 \right) \tag{1}$$

we have

$$\nabla^2 T = \frac{2T_0}{a^2} \tag{2}$$

and hence

$$\nabla^4 \phi = -\frac{2E\alpha T_0}{a^2} , \tag{3}$$

from (14.6).

The problem is symmetric in x, y and (3) suggests a fourth order polynomial solution. We therefore try

$$\phi = C_1 x^4 + C_2 x^2 y^2 + C_3 y^4 + C_4 x^2 + C_5 y^2 , \tag{4}$$

for which the stress components are

$$\sigma_{xx} = 2C_2 x^2 + 12C_3 y^2 + 2C_5 \tag{5}$$

$$\sigma_{xy} = -4C_2 xy \tag{6}$$

$$\sigma_{yy} = 12C_1 x^2 + 2C_2 y^2 + 2C_4 . \tag{7}$$

We impose the strong boundary conditions $\sigma_{yx} = \sigma_{yy} = 0$ on $y = \pm b$, giving

$$C_2 = 0 \tag{8}$$

$$12C_1 x^2 + 2C_2 b^2 + 2C_4 = 0 \tag{9}$$

and hence

$$C_1 = C_2 = C_4 = 0 . \tag{10}$$

Using these results in the modified biharmonic equation (3) we obtain

$$24C_3 = -\frac{2E\alpha T_0}{a^2} \tag{11}$$

and hence

$$C_3 = -\frac{E\alpha T_0}{12a^2} . \tag{12}$$

On the ends $x = \pm a$ we impose the weak traction-free condition[1]

$$\int_{-b}^{b} \sigma_{xx}(a, y) dy = 0 \tag{13}$$

giving

$$\int_{-b}^{b} (12C_3 y^2 + 2C_5) dy = 8C_3 b^3 + 4C_5 b = 0 \tag{14}$$

and hence

$$C_5 = -2C_3 b^2 = \frac{E\alpha T_0 b^2}{6a^2} \ . \tag{15}$$

The final solution is therefore defined by the stress function

$$\phi = \frac{E\alpha T_0}{12a^2}(2b^2 y^2 - y^4) \tag{16}$$

and the stress components

$$\sigma_{xx} = \frac{E\alpha T_0}{3a^2}(b^2 - 3y^2) \tag{17}$$

$$\sigma_{xy} = 0 \tag{18}$$

$$\sigma_{yy} = 0 \ . \tag{19}$$

The stress state is everywhere one of uniaxial tension. The maximum tensile stress is

$$\sigma_{\max} = \frac{E\alpha T_0 b^2}{3a^2} \tag{20}$$

and it occurs at all points on the axis $y = 0$. Notice however that end effects associated with the weak boundary conditions at the ends will modify the stress locally and since this edge is traction-free, we can conclude that the maximum stress is on that part of the axis $y = 0$ distant from the ends.

There is a maximum *compressive* stress

$$-\frac{2E\alpha T_0 b^2}{3a^2}$$

at $y = \pm b$, so the maximum shear stress also occurs on these lines and is

$$\tau_{\max} = \frac{E\alpha T_0 b^2}{3a^2} \ . \tag{21}$$

[1]The other two weak conditions on this edge are satisfied identically by virtue of the assumed symmetry.

14.4. The half plane $y>0$ is subject to periodic heating at the surface $y=0$, such that the surface temperature is

$$T(0,t) = T_0 \cos(\omega t) \ .$$

Show that the temperature field

$$T(y,t) = T_0 e^{-\lambda y} \cos(\omega t - \lambda y)$$

satisfies the heat conduction equation (14.16) with no internal heat generation, provided that

$$\lambda = \sqrt{\frac{\omega}{2k}} \ .$$

Find the corresponding thermal stress field as a function of y, t if the surface of the half plane is traction-free.

Using appropriate material properties, estimate the maximum tensile stress generated in a large rock due to diurnal temperature variation, with a maximum daytime temperature of 30°C and minimum nighttime temperature of 10°C.

The first part of the problem can be completed merely by substituting the given temperature field into (14.16) and showing that the latter is satisfied. However, we shall here give a more deductive proof of this result.

Since the heating is periodic, it is reasonable to seek a solution of the heat conduction equation of the form

$$T(y,t) = \Re\{f(y)\exp(\imath\omega t)\} \ . \tag{1}$$

Substituting this expression into (14.16) and cancelling the factor $\exp(\imath\omega t)$, we obtain the ordinary differential equation

$$\frac{d^2 f}{dy^2} - \frac{\imath\omega f}{k} = 0 \ , \tag{2}$$

which is satisfied by

$$f = \exp(by) \tag{3}$$

if

$$b = \pm\sqrt{\frac{\imath\omega}{k}} \ . \tag{4}$$

Writing

$$\imath = \exp\left(\frac{\imath\pi}{2}\right) \tag{5}$$

and hence

$$\sqrt{\imath} = \exp\left(\frac{\imath\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{\imath}{\sqrt{2}} \ . \tag{6}$$

We can therefore write (4) in the form

$$b = \pm\lambda(1+\imath) \ , \tag{7}$$

where

$$\lambda = \sqrt{\frac{\omega}{2k}} \tag{8}$$

and the general solution of (2) is

$$f = A\exp(\lambda(1+\imath)y) + B\exp(-\lambda(1+\imath)y) \ . \tag{9}$$

We seek a solution that decays as $y \to \infty$, so we set $A = 0$. Substituting for f into (1), we then obtain

$$T(y,t) = \Re\{B\exp(-\lambda(1+\imath)y + \imath\omega t)\} \ . \tag{10}$$

At $y = 0$, we have

$$T(0,t) = \Re\{B\exp(\imath\omega t)\} = T_0\cos(\omega t) \tag{11}$$

and hence $B = T_0$. Equation (10) then yields

$$T(y,t) = T_0\exp(-\lambda y)\cos(\omega t - \lambda y)\} \ . \tag{12}$$

For plane strain, we require

$$\nabla^4\phi = -\frac{E\alpha}{(1-\nu)}\nabla^2 T = \frac{E\alpha\omega T_0}{(1-\nu)k}\exp(-\lambda y)\sin(\omega t - \lambda y) \ , \tag{13}$$

from (14.7, 12). We can find a particular solution by writing

$$\phi_P = C\exp(-\lambda y)\sin(\omega t - \lambda y) \ . \tag{14}$$

It follows by differentiation that

$$\nabla^4\phi_P = -4\lambda^4 C\exp(-\lambda y)\sin(\omega t - \lambda y) \tag{15}$$

and hence (13) is satisfied if we choose

$$C = -\frac{E\alpha\omega T_0}{4\lambda^4(1-\nu)k} = \frac{E\alpha T_0}{2\lambda^2(1-\nu)} \ , \tag{16}$$

using (8).

The stresses associated with the particular solution are

$$\sigma_{xx} = -\frac{E\alpha T_0}{(1-\nu)}\exp(-\lambda y)\cos(\omega t - \lambda y) \tag{17}$$

$$\sigma_{xy} = 0 \tag{18}$$

$$\sigma_{yy} = 0 \tag{19}$$

and we notice that they satisfy the traction-free boundary condition identically. We therefore do not need to superpose a homogeneous solution.

The stress is everywhere one of uniaxial tension and the maximum tensile stress occurs at $y = 0$ and is

$$\sigma_{\max} = \frac{E\alpha T_0}{(1-\nu)} \ . \tag{20}$$

For granite, appropriate properties are around $E = 50$ GPa, $\nu = 0.28$ and $\alpha = 6.5 \times 10^{-6}$ per degree C. If the temperature cycles between 10 and 30°C, the amplitude is 10°C and the maximum tensile stress is

$$\sigma_{\max} = \frac{50 \times 10^3 \times 6.5 \times 10^{-6} \times 10}{0.72} = 4.5 \text{ MPa} \ . \tag{21}$$

14.5. The layer $0<y<h$ rests on a frictionless rigid foundation at $y=0$ and the surface $y=h$ is traction-free. The foundation is a thermal insulator and the free surface is subjected to the steady state heat input

$$q_y = q_0 \cos(mx) \ .$$

Use Dundurs' theorem to show that the layer will not separate from the foundation and find the amplitude of the sinusoidal perturbation in the free surface due to thermal distortion.

The thermal boundary conditions are

$$\begin{aligned} q_y &= q_0 \cos(mx) \ ; \quad y = h && (1) \\ &= 0 \ ; \qquad y = 0 && (2) \end{aligned}$$

Dundurs theorem then states that if the surfaces are traction-free, the surface $y = 0$ will remain plane, permitting it to remain in contact with the rigid foundation.

At $y = h$, we have

$$\frac{d^2 u_y}{dx^2} = \frac{\alpha(1+\nu) q_0 \cos(mx)}{K} \ , \tag{3}$$

so

$$u_y = -\frac{\alpha(1+\nu) q_0 \cos(mx)}{K m^2} + Ax + B \ , \tag{4}$$

where A, B are two arbitrary constants describing a possible rigid-body displacement. However, these constants must be zero, since the heat input averaged over the surface $y = h$ is zero. We therefore have

$$u_y = -\frac{\alpha(1+\nu) q_0 \cos(mx)}{K m^2} \ . \tag{5}$$

CHAPTER 15

15.1. The long rectangular bar $0 < x < a, 0 < y < b$, $a \gg b$ is built in to a rigid support at $x = a$ and loaded by a uniform shear traction $\sigma_{yz} = S$ at $y = b$. The remaining surfaces are free of traction. Find a solution for the displacement and stress fields, using strong boundary conditions on the edges $y = 0, b$.

The boundary conditions for this problem are

$$\begin{aligned} \sigma_{yz} &= S\ ; \quad y = b && (1)\\ &= 0\ ; \quad y = 0 && (2)\\ \sigma_{xz} &= 0\ ; \quad x = 0 && (3)\\ u_z &= 0\ ; \quad x = a\ . && (4) \end{aligned}$$

We shall try a general second degree polynomial for f — i.e.

$$f = C_1 x^2 + C_2 xy + C_3 y^2 + C_4 x + C_5 y + C_6\ , \tag{5}$$

giving

$$\begin{aligned} \sigma_{xz} &= \mu(2C_1 x + C_2 y + C_4) && (6)\\ \sigma_{yz} &= \mu(C_2 x + 2C_3 y + C_5)\ . && (7) \end{aligned}$$

Applying the boundary conditions (1,2) on the long edges in the strong sense, we have

$$\begin{aligned} \mu(C_2 x + 2C_3 b + C_5) &= S && (8)\\ \mu(C_2 x + C_5) &= 0 && (9) \end{aligned}$$

and hence

$$C_2 = 0\ ; \quad C_5 = 0\ ; \quad C_3 = \frac{S}{2\mu b}\ . \tag{10}$$

There is no body force, so

$$\nabla^2 f = 2C_1 + 2C_3 = 0\ , \tag{11}$$

from (15.7), giving

$$C_1 = -\frac{S}{2\mu b}\ . \tag{12}$$

The traction on $x = 0$ is

$$\sigma_{xz}(0, y) = \mu(C_2 y + C_4) = C_4 \mu \tag{13}$$

and hence (3) can be satisfied in the strong sense by choosing $C_4 = 0$. The displacement condition (4) can be satisfied only in the weak sense

$$\int_0^b u_z(a, y) = 0 \ . \tag{14}$$

Noting that

$$u_z = \frac{S}{2\mu b}(y^2 - x^2) + C_6 \ , \tag{15}$$

from (15.1, 5, 10, 12), we have

$$u_z(a, y) = \frac{S}{2\mu b}(y^2 - a^2) + C_6 \tag{16}$$

and the condition (14) gives

$$\frac{S}{2\mu b}\left(\frac{b^3}{3} - a^2 b\right) + C_6 b = 0 \tag{17}$$

and hence

$$C_6 = \frac{S}{2\mu b}\left(a^2 - \frac{b^2}{3}\right) \ . \tag{18}$$

The final solution is

$$u_z = f = \frac{S}{2\mu b}\left(y^2 - x^2 + a^2 - \frac{b^2}{3}\right) \tag{19}$$

$$\sigma_{xz} = -\frac{Sx}{b} \tag{20}$$

$$\sigma_{yz} = \frac{Sy}{b} \ . \tag{21}$$

15.2. Figure 15.3 shows the cross section of a long bar of equilateral triangular cross section of side a. The three faces of the bar are built in to rigid supports and the axis of the bar is vertical, resulting in gravitational loading $p_z = -\rho g$.

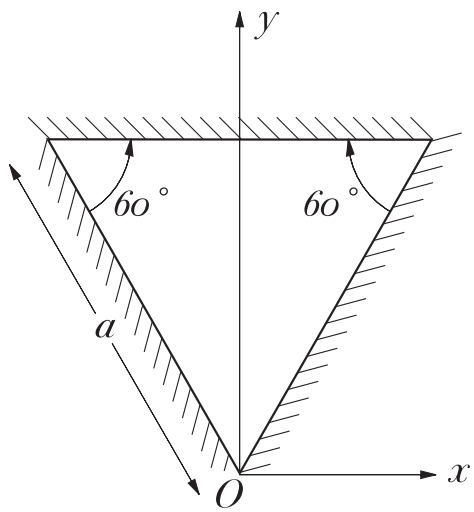


Figure 15.3: Cross-section of the triangular bar.

Show that an exact solution for the stress and displacement fields can be obtained using a stress function of the form

$$f = C(y - \sqrt{3}x)(y + \sqrt{3}x)\left(y - \frac{\sqrt{3}a}{2}\right) .$$

Find the value of the constant C and the location and magnitude of the maximum shear stress.

The boundaries of the triangle are

$$y = \pm x \tan 60^o \text{ and } y = a \cos 60^o \tag{1}$$

— i.e.

$$y = \pm x\sqrt{3} \text{ and } y = a\frac{\sqrt{3}}{2} . \tag{2}$$

Thus, the cubic polynomial

$$f = C(y - \sqrt{3}x)(y + \sqrt{3}x)\left(y - \frac{\sqrt{3}a}{2}\right) \tag{3}$$

is zero around all three edges. Expanding the product, we have

$$f = C\left(y^3 - 3x^2y - \frac{\sqrt{3}ay^2}{2} + \frac{3\sqrt{3}ax^2}{2}\right) \tag{4}$$

and hence

$$\nabla^2 f = C\left(6y - 6y - \sqrt{3}a + 3\sqrt{3}a\right) = 2\sqrt{3}Ca\ . \tag{5}$$

The body force is

$$p_z = -\rho g \tag{6}$$

and hence equation (15.6) can be satisfied by choosing

$$C = \frac{\rho g}{2\sqrt{3}\mu a}\ . \tag{7}$$

The final solution is therefore

$$u_z = f = \frac{\rho g}{2\sqrt{3}\mu a}\left(y^3 - 3x^2 y - \frac{\sqrt{3}ay^2}{2} + \frac{3\sqrt{3}ax^2}{2}\right) \tag{8}$$

$$\sigma_{xz} = \mu\frac{\partial f}{\partial x} = \frac{\rho g}{2a}(-2\sqrt{3}xy + 3ax) \tag{9}$$

$$\sigma_{yz} = \mu\frac{\partial f}{\partial y} = \frac{\rho g}{2a}(\sqrt{3}y^2 - \sqrt{3}x^2 - ay)\ . \tag{10}$$

Using these expressions and (15.12) to draw a contour plot of the maximum shear stress, we find that the maximum value occurs at the mid-points of the three faces (e.g. at $(0, \sqrt{3}a/2)$ and is given by

$$|\boldsymbol{\tau}|_{\max} = \frac{\sqrt{3}\rho g a}{8}\ . \tag{11}$$

15.3. A state of uniform shear $\sigma_{xz} = S, \sigma_{yz} = 0$ in a large block of material is perturbed by the presence of a small hole whose boundary is defined by the equation $r = a$ in cylindrical polar coordinates r, θ, z. Find the complete stress field in the block and hence determine the appropriate stress concentration factor.

For the unperturbed state of uniform shear, we require

$$\frac{\partial f}{\partial y} = 0 \; ; \quad \frac{\partial f}{\partial x} = \frac{S}{\mu} \; , \tag{1}$$

from (15.5). Integrating the second equation, we have

$$f = \frac{Sx}{\mu} + g(y) \tag{2}$$

and a solution satisfying (15.7) is obtained by setting the arbitrary function $g(y)$ to zero.

Writing equation (2) in polar coördinates, we then have

$$f = \frac{Sr\cos\theta}{\mu} \; . \tag{3}$$

To complete the solution, we must superpose a solution of (15.7) that has the same θ-dependence as (3) and that decays to zero as $r \to \infty$. The appropriate function[1] is $C\cos\theta/r$, giving

$$f = \frac{Sr\cos\theta}{\mu} + \frac{C\cos\theta}{r} \tag{4}$$

and

$$\sigma_{zr} = S\cos\theta - \frac{C\mu\cos\theta}{r^2} \; ; \quad \sigma_{z\theta} = -S\sin\theta - \frac{C\mu\sin\theta}{r^2} \; , \tag{5}$$

from equation (15.11).

For the hole to be traction-free, we require

$$\sigma_{zr} = 0 \; ; \quad r = a \tag{6}$$

and hence

$$S - \frac{C\mu}{a^2} = 0 \tag{7}$$

or

$$C = \frac{Sa^2}{\mu} \; . \tag{8}$$

[1]Notice that functions of the form $r^p\cos(p\theta), r^p\sin(p\theta), r^{-p}\cos(p\theta)$ and $r^{-p}\sin(p\theta)$ are harmonic for all values of p.

The final stress field is therefore

$$\sigma_{zr} = S\left(1 - \frac{a^2}{r^2}\right)\cos\theta \; ; \quad \sigma_{z\theta} = -S\left(1 + \frac{a^2}{r^2}\right)\sin\theta \; . \tag{9}$$

The maximum shear stress occurs at the point $(a, -\pi/2)$ and is

$$\sigma_{z\theta}(a, \pi/2) = 2S \; . \tag{10}$$

The stress concentration factor is therefore equal to 2.

15.4. A state of uniform shear $\sigma_{xz} = S, \sigma_{yz} = 0$ in a large block of material is perturbed by the presence of a rigid circular inclusion whose boundary is defined by the equation $r = a$ in cylindrical polar coordinates r, θ, z. The inclusion is perfectly bonded to the elastic material and is prevented from moving, so that $u_z = 0$ at $r = a$. Find the complete stress field in the block and hence determine the appropriate stress concentration factor.

For the unperturbed state of uniform shear, we require

$$\frac{\partial f}{\partial y} = 0 \; ; \quad \frac{\partial f}{\partial x} = \frac{S}{\mu} \; , \tag{1}$$

from (15.5). Integrating the second equation, we have

$$f = \frac{Sx}{\mu} + g(y) \tag{2}$$

and a solution satisfying (15.7) is obtained by setting the arbitrary function $g(y)$ to zero.

Writing equation (2) in polar coördinates, we then have

$$f = \frac{Sr\cos\theta}{\mu} \; . \tag{3}$$

To complete the solution, we must superpose a solution of (15.7) that has the same θ-dependence as (3) and that decays to zero as $r \to \infty$. The appropriate function[2] is $C\cos\theta/r$, giving

$$f = \frac{Sr\cos\theta}{\mu} + \frac{C\cos\theta}{r} \; . \tag{4}$$

The inclusion is prevented from moving, so we require

$$u_z = f = 0 \; ; \quad r = a \tag{5}$$

and hence

$$\frac{Sa}{\mu} + \frac{C}{a} = 0 \tag{6}$$

or

$$C = -\frac{Sa^2}{\mu} \; . \tag{7}$$

We therefore have

$$f = \frac{S}{\mu}\left(r - \frac{a^2}{r}\right)\cos\theta \tag{8}$$

[2]Notice that functions of the form $r^p\cos(p\theta), r^p\sin(p\theta), r^{-p}\cos(p\theta)$ and $r^{-p}\sin(p\theta)$ are harmonic for all values of p.

and hence

$$\sigma_{zr} = S\left(1 + \frac{a^2}{r^2}\right)\cos\theta \; ; \quad \sigma_{z\theta} = -S\left(1 - \frac{a^2}{r^2}\right)\sin\theta \; , \tag{9}$$

from (15.11).

The maximum shear stress occurs at the point $(a, 0)$ and is

$$\sigma_{zr}(a, 0) = 2S \; . \tag{10}$$

The stress concentration factor is therefore equal to 2.

15.5. The body defined by $a < r < b, 0 < \theta < \pi/2, -\infty < z < \infty$ is built in at $\theta = \pi/2$ and loaded by a shear force F per unit length in the z-direction on the surface $\theta = 0$. The curved surfaces $r = a, b$ are traction free. Find the stress field in the body, using the strong boundary conditions on the curved surfaces and weak conditions elsewhere.

The boundary conditions of the problem are

$$\sigma_{rz} = 0 \; ; \quad r = a, b \tag{1}$$

$$\int_a^b \sigma_{\theta z} dr = F(b-a) \; ; \quad \theta = 0 \; . \tag{2}$$

It is clear from equilibrium considerations that the weak condition (2) must actually be satisfied for *all* values of θ, since the force on the end has to be transmitted across every θ-surface. This suggests that the stress field will be independent of θ.

The function

$$f = C\theta \tag{3}$$

satisfies (15.7) and defines the stress components

$$\sigma_{rz} = 0 \; ; \quad \sigma_{\theta z} = \frac{C\mu}{r} \; , \tag{4}$$

from (15.11). This solution satisfies the strong boundary conditions (1) and can be made to satisfy (2) by choosing C such that

$$\int_a^b \frac{C\mu dr}{r} = C\mu \ln(b/a) = F(b-a) \tag{5}$$

— i.e.

$$C = \frac{F(b-a)}{\mu \ln(b/a)} \; . \tag{6}$$

The complete stress field is therefore

$$\sigma_{rz} = 0 \; ; \quad \sigma_{\theta z} = \frac{F(b-a)}{r \ln(b/a)} \; . \tag{7}$$

15.6. The body defined by $a < r < b, 0 < \theta < \pi/2, -\infty < z < \infty$ is built in at $\theta = \pi/2$, such that the z-axis is vertical. It is loaded only by its own weight with density ρ, the curved surfaces $r = a, b$ being traction free. Find the stress field in the body, using the strong boundary conditions on the curved surfaces and weak conditions elsewhere.

The only loading is the gravitational body force

$$p_z = \rho g \tag{1}$$

and hence

$$\mu \nabla^2 f = -\rho g \ , \tag{2}$$

from (15.6). A simple particular solution of this equation is obtained by writing

$$f = Cr^2 \ , \tag{3}$$

from which

$$4C\mu = -\rho g \tag{4}$$

and

$$C = -\frac{\rho g}{4\mu} \ ; \quad f = -\frac{\rho g r^2}{4\mu} \ . \tag{5}$$

Substituting into equations (15.11), we obtain

$$\sigma_{zr} = -\frac{\rho g r}{2} \ ; \quad \sigma_{z\theta} = 0 \ . \tag{6}$$

To complete the solution, we need to superpose a solution of the homogeneous equation (15.7) so as to satisfy the strong boundary conditions

$$\sigma_{zr} = 0 \ ; \quad r = a, b \ . \tag{7}$$

The axisymmetric form of the tractions (6) suggests the use of an axisymmetric function, but equilibrium considerations show that the shear force and hence the shear stress $\sigma_{z\theta}$ must increase linearly with θ, implying a term proportional to θ^2 in f. It is therefore clear that this is a degenerate problem requiring a special stress function.

To find the required function, we notice that the first two successive derivatives with respect to n of the harmonic function $r^n \cos(n\theta)$ are

$$r^n \ln(r) \cos(n\theta) - \theta r^n \sin(n\theta) \quad \text{and} \quad r^n \ln^2(r) \cos(n\theta) - 2\theta r^n \ln(r) \sin(n\theta) - \theta^2 r^n \cos(n\theta) \ .$$

Setting $n = 0$ (which would generate axisymmetric solutions for the undifferentiated function) then yields the harmonic functions

$$\ln(r) \quad \text{and} \quad (\ln^2(r) - \theta^2) \ .$$

We therefore propose the solution

$$f = -\frac{\rho g r^2}{4\mu} + A\ln(r) + B(\ln^2(r) - \theta^2) \ , \tag{8}$$

which defines the stress components

$$\sigma_{zr} = -\frac{\rho g r}{2} + \frac{A\mu}{r} + \frac{2B\mu\ln(r)}{r} \ ; \quad \sigma_{z\theta} = -\frac{2B\mu\theta}{r} \ . \tag{9}$$

The boundary conditions (7) then provide the two conditions

$$-\frac{\rho g a}{2} + \frac{A\mu}{a} + \frac{2B\mu\ln(a)}{a} = 0 \tag{10}$$

$$-\frac{\rho g b}{2} + \frac{A\mu}{b} + \frac{2B\mu\ln(b)}{b} = 0 \tag{11}$$

for the unknown constants A, B, with solution

$$A = \frac{\rho g[a^2\ln(b) - b^2\ln(a)]}{2\mu\ln(b/a)} \ ; \quad B = \frac{\rho g(b^2 - a^2)}{4\mu\ln(b/a)} \ . \tag{12}$$

We therefore have

$$\sigma_{zr} = \frac{\rho g}{2}\left(-r + \frac{a^2\ln(b/r) + b^2\ln(r/a)}{r\ln(b/a)}\right) \tag{13}$$

$$\sigma_{z\theta} = -\frac{\rho g(b^2 - a^2)\theta}{2r\ln(b/a)} \ . \tag{14}$$

Equation (14) also satisfies the traction-free condition on the end $\theta = 0$ in the strong sense, so equations (13, 14) define the complete stress field in the body.

15.7. The body defined by $a<r<b, 0<\theta<\pi/2, -\infty<z<\infty$ is built in at $\theta=\pi/2$ and loaded by the uniform shear traction $\sigma_{rz}=S$ on $r=a$, the other surfaces being traction free. Find the stress field in the body, using the strong boundary conditions on the curved surfaces and weak conditions elsewhere. Hint: This is a degenerate problem in the sense of §10.3.1 and requires a special stress function. Equilibrium arguments suggest that the stress component $\sigma_{\theta z}$ must increase with θ, implying a function f varying with θ^2. You can construct a suitable function by starting with the harmonic function $r^n\cos(n\theta)$, differentiating twice with respect to n and then setting $n=0$.

The first two successive derivatives with respect to n of the harmonic function $r^n\cos(n\theta)$ are

$$r^n\ln(r)\cos(n\theta)-\theta r^n\sin(n\theta) \quad\text{and}\quad r^n\ln^2(r)\cos(n\theta)-2\theta r^n\ln(r)\sin(n\theta)-\theta^2 r^n\cos(n\theta)\ .$$

Setting $n=0$ (which would generate axisymmetric solutions for the undifferentiated function) then yields the functions

$$\ln(r) \quad\text{and}\quad (\ln^2(r)-\theta^2)\ ,$$

which are both harmonic and hence satisfy the governing equation (15.7). We therefore propose the solution

$$f = A\ln(r) + B(\ln^2(r)-\theta^2)\ , \tag{1}$$

which defines the stress components

$$\sigma_{zr} = \frac{A\mu}{r} + \frac{2B\mu\ln(r)}{r}\ ; \quad \sigma_{z\theta} = -\frac{2B\mu\theta}{r}\ . \tag{2}$$

The boundary conditions

$$\sigma_{zr} = S\ ; \quad r=a \tag{3}$$
$$= 0\ ; \quad r=b \tag{4}$$

then provide the two conditions

$$\frac{A\mu}{a} + \frac{2B\mu\ln(a)}{a} = S \tag{5}$$
$$\frac{A\mu}{b} + \frac{2B\mu\ln(b)}{b} = 0 \tag{6}$$

for the unknown constants A, B, with solution

$$A = \frac{Sa\ln(b)}{\mu\ln(b/a)}\ ; \quad B = -\frac{Sa}{2\mu\ln(b/a)}\ . \tag{7}$$

We therefore have

$$\sigma_{zr} = \frac{Sa\ln(b/r)}{r\ln(b/a)}\ ; \quad \sigma_{z\theta} = \frac{Sa\theta}{r\ln(b/a)}\ . \tag{8}$$

Equations (8) also satisfies the traction-free condition on the end $\theta = 0$ in the strong sense and hence define the complete stress field in the body.

15.8. The body $-\alpha<\theta<\alpha, 0\leq r<a$ is supported at $r=a$ and loaded only by a uniform antiplane shear traction $\sigma_{\theta z}=S$ on the surface $\theta=\alpha$, the other surface being traction-free. Find the complete stress field in the body, using strong boundary conditions on $\theta=\pm\alpha$ and weak conditions on $r=a$.

We require the traction $\sigma_{\theta z}$ to be uniform — i.e. to vary with r^0 on the surface $\theta=\alpha$. Referring to equation (15.11), we see that this suggests that the displacement function f should vary with r^1.

The most general solution of (15.6) with this behaviour is

$$f = Ar\cos\theta + Br\sin\theta \ , \tag{1}$$

where A, B are two arbitrary constants. The corresponding stress components are

$$\sigma_{zr} = A\mu\cos\theta + B\mu\sin\theta \tag{2}$$

$$\sigma_{z\theta} = -A\mu\sin\theta + B\mu\cos\theta \tag{3}$$

The boundary conditions

$$\sigma_{z\theta} = S \ ; \quad \theta=\alpha \tag{4}$$

$$= 0 \ ; \quad \theta=-\alpha \tag{5}$$

then yield the equations

$$-A\mu\sin\alpha + B\mu\cos\alpha = S \tag{6}$$

$$A\mu\sin\alpha + B\mu\cos\alpha = 0 \ , \tag{7}$$

with solution

$$A = -\frac{S}{2\mu\sin\alpha} \ ; \quad B = \frac{S}{2\mu\cos\alpha} \ . \tag{8}$$

The complete stress field is therefore

$$\sigma_{zr} = -\frac{S\cos\theta}{2\sin\alpha} + \frac{S\sin\theta}{2\cos\alpha} \tag{9}$$

$$\sigma_{z\theta} = \frac{S\sin\theta}{2\sin\alpha} + \frac{S\cos\theta}{2\cos\alpha} \ . \tag{10}$$

The weak condition

$$\int_{-\alpha}^{\alpha} u_z(a,\theta)d\theta = 0 \tag{11}$$

has no effect on the stress field. It simply enables us to satisfy a rigid-body displacement condition. With the present solution, the displacement at $r=a$ is

$$u_z = -\frac{Sr\cos\theta}{2\mu\sin\alpha} + \frac{Sr\sin\theta}{2\mu\cos\alpha} \ , \tag{12}$$

from (1,8). We can superpose a rigid-body displacement $u_z = C$ without affecting the stress field and the constant C can then be found by substituting into (11) and performing the integral.

15.9. The half-space $-\pi < \theta < 0$ is bonded to a rigid body in the region $\theta = 0$, the remaining surface $\theta = -\pi$ being traction-free. If the rigid body is loaded in the z-direction (out-of-plane), use an asymptotic argument analogous to that in §11.2 to determine the exponent of the most singular term in the stress field near the origin.

Following the convention in Chapter 11, we define the parameter λ such that admissible singularities (stresses less singular than r^{-1}) correspond to positive values of λ. From (15.11), it is clear that the function f must therefore contain the factor r^{λ}.

The most general solution of (15.6) containing this factor is

$$f = Ar^{\lambda}\cos(\lambda\theta) + Br^{\lambda}\sin(\lambda\theta) \ . \tag{1}$$

The corresponding stresses are

$$\sigma_{zr} = A\mu\lambda r^{(\lambda-1)}\cos(\lambda\theta) + B\mu\lambda r^{(\lambda-1)}\sin(\lambda\theta) \tag{2}$$

$$\sigma_{z\theta} = -A\mu\lambda r^{(\lambda-1)}\sin(\lambda\theta) + B\mu\lambda r^{(\lambda-1)}\cos(\lambda\theta) \ . \tag{3}$$

The boundary conditions for the present problem are

$$u_z = f = 0 \ ; \quad \theta = 0 \tag{4}$$

$$\sigma_{\theta z} = 0 \ ; \quad \theta = -\pi \ . \tag{5}$$

Substuting from equations (1,3), we obtain

$$A = 0 \tag{6}$$

$$A\lambda\sin(\lambda\pi) + B\lambda\cos(\lambda\pi) = 0 \tag{7}$$

and there is a non-trivial solution if and only if

$$\cos(\lambda\pi) = 0 \tag{8}$$

— i.e.

$$\lambda = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ... \tag{9}$$

The most singular term corresponds to $\lambda = \frac{1}{2}$ and yields a stress field that is singular with $r^{-1/2}$.

15.10. The wedge $-\alpha < \theta < \alpha$ has traction-free surfaces $\theta = \pm\alpha$ and is loaded at large r in antiplane shear. Develop an asymptotic (eigenfunction) expansion for the most general stress field near the apex $r=0$ and plot the exponent of the dominant term for (i) symmetric and (ii) antisymmetric stress fields as functions of the complete wedge angle 2α.

Following the convention in Chapter 11, we define the parameter λ such that admissible singularities (stresses less singular than r^{-1}) correspond to positive values of λ. From (15.11), it is clear that the function f must therefore contain the factor r^λ.

The most general solution of (15.6) containing this factor is

$$f = Ar^\lambda \cos(\lambda\theta) + Br^\lambda \sin(\lambda\theta) \ . \tag{1}$$

The corresponding stresses are

$$\begin{aligned} \sigma_{zr} &= A\mu\lambda r^{(\lambda-1)} \cos(\lambda\theta) + B\mu\lambda r^{(\lambda-1)} \sin(\lambda\theta) && (2) \\ \sigma_{z\theta} &= -A\mu\lambda r^{(\lambda-1)} \sin(\lambda\theta) + B\mu\lambda r^{(\lambda-1)} \cos(\lambda\theta) \ . && (3) \end{aligned}$$

The boundary conditions for the present problem are

$$\sigma_{\theta z} = 0 \ ; \quad \theta = \pm\alpha \ . \tag{4}$$

Substuting from equation (3), we obtain

$$\begin{aligned} -A\lambda \sin(\lambda\alpha) + B\lambda \cos(\lambda\alpha) &= 0 && (5) \\ A\lambda \sin(\lambda\alpha) + B\lambda \cos(\lambda\alpha) &= 0 \ . && (6) \end{aligned}$$

As in Chapter 11, we can separate the symmetric and antisymmetric solutions (partition the matrix) by taking respectively the sum and difference of these two equations, obtaining

$$\begin{aligned} A \sin(\lambda\alpha) &= 0 && (7) \\ B \cos(\lambda\alpha) &= 0 \ , && (8) \end{aligned}$$

where we have also cancelled the factor λ.

It follows that symmetric solutions ($B = 0$) occur when

$$\sin(\lambda\alpha) = 0 \tag{9}$$

and hence

$$\lambda = \frac{n\pi}{\alpha} \ , \tag{10}$$

where n is an integer. The dominant term corresponds to $n = 1$ and is

$$\lambda_S = \frac{\pi}{\alpha} \ . \tag{11}$$

However, $\lambda_S > 1$ (and hence the stresses are non-singular) for all wedge angles $0 < 2\alpha < 2\pi$, so there are no singular symmetric solutions.

Antisymmetric solutions ($A = 0$) occur when

$$\cos(\lambda\alpha) = 0 \tag{12}$$

and hence

$$\lambda = \frac{(2n-1)\pi}{2\alpha} \ , \tag{13}$$

where n is an integer. The dominant term corresponds to $n = 1$ and is

$$\lambda_A = \frac{\pi}{2\alpha} \ . \tag{14}$$

Singular solutions ($\lambda_A < 1$) occur for wedge angles $\pi < 2\alpha < 2\pi$. The most singular solution corresponds to the case of the crack $2\alpha = 2\pi$, for which $\lambda_A = \frac{1}{2}$ and the stress field is square-root singular. This permits us to define a mode III stress intensity factor for cracks in out-of-plane loading.

15.11. Two large parallel cylinders each of radius R are pressed together by a normal force P per unit length. An axial tangential force Q per unit length is then applied, tending to make the cylinders slide relative to each other along the axis as shown in Figure 15.4. If the coefficient of friction is f and $Q < fP$, find the extent of the contact region, the extent of the slip and stick regions and the distribution of normal and tangential traction in the contact area[3]

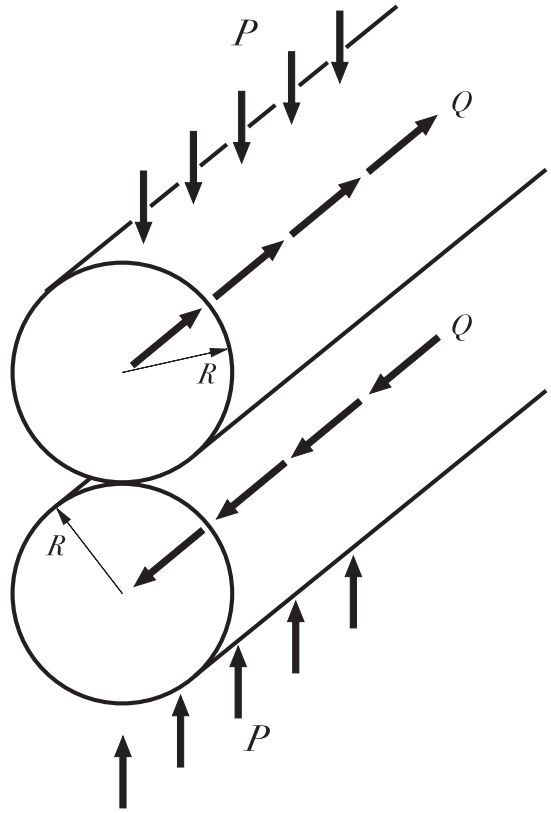


Figure 15.4: Contacting cylinders loaded out of plane.

The normal force produces a (compressive) contact traction distribution

$$p_y(x) = -\frac{2P\sqrt{a^2 - x^2}}{\pi a^2}\,, \tag{1}$$

from equation (12.75), where

$$a = \sqrt{\frac{PR}{4\pi}\left(\frac{(\kappa_1 + 1)}{\mu_1} + \frac{(\kappa_2 + 1)}{\mu_2}\right)}\,, \tag{2}$$

[3]This problem is analogous to Mindlin's problem discussed in §12.8.1.

since $R_1 = R_2 = R$. Notice that in this problem it is not necessary to assume that the materials of the two cylinders are identical, since there is no coupling between in-plane and out-of-plane loading. For example, the force F in Figure 15.2 produces no displacement in the y-direction.

Denoting the contact traction in the z-direction by p_z and the corresponding surface displacement by u_z, we can write

$$u_z = -\frac{1}{\pi\mu}\int_{-a}^{a} p_z(\xi)\ln|x-\xi|d\xi + C\ , \tag{3}$$

from (15.34) and hence

$$u_z^{'} = -\frac{1}{\pi\mu}\int_{-a}^{a}\frac{p_z(\xi)d\xi}{x-\xi}\ . \tag{4}$$

The relative shift $h = u_{z1} - u_{z2}$ is then given by

$$h'(x) = -\frac{1}{\pi M}\int_{-a}^{a}\frac{p_z(\xi)d\xi}{x-\xi}\ , \tag{5}$$

where

$$\frac{1}{M} = \frac{1}{\mu_1} + \frac{1}{\mu_2}\ . \tag{6}$$

Following the technique in §12.8.1, we now write the traction

$$p_z(x) = \frac{2fP\sqrt{a^2-x^2}}{\pi a^2} + p_z^*(x)\ . \tag{7}$$

Assuming that slip occurs in the regions $-a < x < -c$ and $c < x < a$, the corrective traction p_z^* will be zero in these regions. The combined tractions must be sufficient to give $h'(x) = 0$ in the stuck region $-c < x < c$ and hence

$$-\frac{2fP}{\pi^2 Ma^2}\int_{-a}^{a}\frac{\sqrt{a^2-\xi^2}d\xi}{x-\xi} - \frac{1}{\pi\mu}\int_{-c}^{c}\frac{p_z^*(\xi)d\xi}{x-\xi} = 0\ ; \quad -c < x < c\ . \tag{8}$$

The first term in this equation is easily evaluated (as in §12.8.1) as

$$-\frac{2fP}{\pi^2 Ma^2}\int_{-a}^{a}\frac{\sqrt{a^2-\xi^2}d\xi}{x-\xi} = \frac{2fPx}{M\pi a^2}\ . \tag{9}$$

The corrective traction p_z^* is therefore determined by the integral equation

$$\frac{1}{\pi\mu}\int_{-c}^{c}\frac{p_z^*(\xi)d\xi}{x-\xi} = \frac{2fPx}{M\pi a^2}\ ; \quad -c < x < c\ , \tag{10}$$

with solution

$$p_z^*(x) = -\frac{2fP\sqrt{c^2-x^2}}{\pi a^2}\ ; \quad -c < x < c\ . \tag{11}$$

The complete tangential traction distribution can therefore be written

$$p_z(x) = \frac{2fP}{\pi a^2}\{\sqrt{a^2 - x^2} - H(c^2 - x^2)\sqrt{c^2 - x^2}\} \;\; ; \;\; -a < x < a \; . \tag{12}$$

Finally, we determine the extent of the stick zone from the condition

$$Q = \int_{-a}^{a} p_z(x)dx = fP - fP\left(\frac{c}{a}\right)^2 \; . \tag{13}$$

Thus,

$$c = a\sqrt{1 - \frac{Q}{fP}} \; . \tag{14}$$

15.12. Solve Problem 15.11 for the case where the cylinders are rolling against each other, as in Carter's problem (§12.8.2). If the rolling speed is Ω, find the relative (axial) creep velocity.

Now suppose that the axes of the cylinders are very slightly misaligned through some angle ϵ which is insufficient to make the contact conditions vary significantly along the axis. If the cylinders run in bearings which prevent relative axial motion, find the axial bearing force and the maximum misalignment which can occur without there being gross slip.

The normal force produces a (compressive) contact traction distribution

$$p_y(x) = -\frac{2P\sqrt{a^2 - x^2}}{\pi a^2} , \tag{1}$$

from equation (12.75), where

$$a = \sqrt{\frac{PR}{4\pi}\left(\frac{(\kappa_1 + 1)}{\mu_1} + \frac{(\kappa_2 + 1)}{\mu_2}\right)} , \tag{2}$$

since $R_1 = R_2 = R$. Notice that in this problem it is not necessary to assume that the materials of the two cylinders are identical, since there is no coupling between in-plane and out-of-plane loading. For example, the force F in Figure 15.2 produces no displacement in the y-direction.

Denoting the contact traction in the z-direction by p_z and the corresponding surface displacement by u_z, we can write

$$u_z = -\frac{1}{\pi\mu}\int_{-a}^{a} p_z(\xi)\ln|x - \xi| d\xi + C , \tag{3}$$

from (15.34) and hence

$$u_z^{'} = -\frac{1}{\pi\mu}\int_{-a}^{a} \frac{p_z(\xi)d\xi}{x - \xi} . \tag{4}$$

As in §12.8.2, the stick condition in the moving frame of reference translates to the condition that

$$V(u_{z1}^{'} - u_{z2}^{'}) \equiv Vh' = -\dot{C} , \tag{5}$$

where $\dot{C}$ is an as yet unknown axial creep velocity and

$$V = R\Omega \tag{6}$$

is the rolling velocity.

Writing the tangential traction in the form

$$p_z(x) = \frac{2fP\sqrt{a^2 - x^2}}{\pi a^2} + p_z^*(x) \tag{7}$$

and assuming a corrective traction

$$p_z^*(x) = -\frac{2fP}{\pi a^2}\sqrt{(a-x)(x-c)}\ ; \quad c < x < a\ , \tag{8}$$

we can evaluate (4) to obtain

$$u_{z1}^{'} - u_{z2}^{'} = \frac{fP(a+c)}{M\pi a^2}\ , \tag{9}$$

which is independent of x as required. The complete tangential traction distribution can therefore be written as

$$p_z(x) = \frac{2fP}{\pi a^2}\{\sqrt{a^2-x^2} - H(x-c)\sqrt{(a-x)(x-c)}\}\ \ ;\ \ -a < x < a\ . \tag{10}$$

We determine the extent of the stick zone from the condition

$$Q = \int_{-a}^{a} p_z(x)dx = fP - fP\left(\frac{a-c}{2a}\right)^2\ . \tag{11}$$

Thus,

$$c = a\left(1 - 2\sqrt{1-\frac{Q}{fP}}\right)\ . \tag{12}$$

We also determine the creep velocity $\dot{C}$ from (5, 9, 12), with the result

$$\dot{C} = -\frac{2fR\Omega P}{M\pi a}\left(1-\sqrt{1-\frac{Q}{fP}}\right)\ . \tag{13}$$

Alternatively, we could eliminate the unknown contact semi-width a, using (2) to obtain an expression involving only the given quantities P, Q, R, Ω, f and the material properties — i.e.

$$\dot{C} = -\frac{4f\Omega}{M}\left(1-\sqrt{1-\frac{Q}{fP}}\right)\sqrt{\frac{PR}{\pi}}\Bigg/\sqrt{\frac{(\kappa_1+1)}{\mu_1}+\frac{(\kappa_2+1)}{\mu_2}}\ . \tag{14}$$

If two rigid cylinders are misaligned by a small angle ϵ, the local rolling velocity vectors $\boldsymbol{V}$ at the interface will be inclined by the same angle and there will therefore be a relative velocity along the axis of

$$\dot{C} = -V\epsilon = -\Omega R\epsilon\ . \tag{15}$$

If the cylinders are deformable and this relative velocity is accommodated by elastic deformation, we can calculate the tangential force Q by equating the expressions (14, 15) — i.e.

$$-\Omega R\epsilon = -\frac{4f\Omega}{M}\left(1-\sqrt{1-\frac{Q}{fP}}\right)\sqrt{\frac{PR}{\pi}}\Bigg/\sqrt{\frac{(\kappa_1+1)}{\mu_1}+\frac{(\kappa_2+1)}{\mu_2}}\ . \tag{16}$$

Solving for Q, we obtain

$$Q = fP - fP\left(1 - \frac{M\epsilon}{4f}\sqrt{\frac{\pi R}{P}\left(\frac{(\kappa_1+1)}{\mu_1} + \frac{(\kappa_2+1)}{\mu_2}\right)}\right)^2 . \tag{17}$$

The limiting condition for gross slip is $Q = fP$ and hence

$$\epsilon = \frac{4f}{M}\sqrt{\frac{P}{\pi R}} \Bigg/ \sqrt{\frac{(\kappa_1+1)}{\mu_1} + \frac{(\kappa_2+1)}{\mu_2}} . \tag{18}$$

For similar materials with Poisson's ratio ν, this reduces to

$$\epsilon = 2f\sqrt{\frac{2P}{\pi\mu(1-\nu)R}} , \tag{19}$$

which can also be written in terms of the contact semi-width a as

$$\epsilon = \frac{2fa}{R(1-\nu)} . \tag{20}$$

Thus, for there to be no gross slip, the misalignment must be small of the order of the ratio between the contact area size and the cylinder radii.

15.13. A state of uniform shear $\sigma_{xz}=0, \sigma_{yz}=S$ in a large block of material is perturbed by the presence of the plane crack $-a<x<a, y=0$. Representing the perturbation due to the crack by a distribution of screw dislocations along the crack line, find the shear stress distribution on the line $x>a, y=0$, the relative motion between the crack faces in $-a<x<a, y=0$ and the mode III stress intensity factor K_{III} defined as

$$K_{III} \equiv \lim_{x\to a^+} \sigma_{yz}(x,0)\sqrt{2\pi(x-a)}\ .$$

The stress components in polar coördinates due to a screw dislocation are given in equation (15.36) as

$$\sigma_{zr} = 0\ ; \quad \sigma_{z\theta} = \frac{\mu B_z}{2\pi r}\ . \tag{1}$$

Converting to Cartesian coördinates and restricting attention to the line $y = 0$, we have

$$\sigma_{zx}(x,0) = 0\ ; \quad \sigma_{zy}(x,0) = \frac{\mu B_z}{2\pi x}\ . \tag{2}$$

Notice that in $x < 0$, $r = -x$, but $\sigma_{z\theta} = -\sigma_{zy}$, so the second of equations (2) holds on the entire line $y = 0$.

If we now distribute dislocations along the segment $-a < \xi < a$, the resulting tractions on the line $y = 0$ will be

$$\sigma_{zy}(x,0) = \frac{\mu}{2\pi}\int_{-a}^{a} \frac{B_z(\xi)d\xi}{x-\xi}\ , \tag{3}$$

where $B_z(\xi)$ is the dislocation density.

For the unperturbed problem, the stress field is $\sigma_{zy} = S$ and we require the crack segment to be traction-free. It follows that we must choose $B_z(\xi)$ to satisfy the condition

$$\int_{-a}^{a} \frac{B_z(\xi)d\xi}{x-\xi} = -\frac{2\pi S}{\mu}\ ; \quad -a < x < a \tag{4}$$

and by analogy with equations (13.33, 13.38), we obtain

$$B_z(\xi) = -\frac{2S\xi}{\mu\sqrt{a^2-\xi^2}}\ . \tag{5}$$

The stress distribution on the uncracked portion of the crack plane is then given by

$$\sigma_{zy} = S - \frac{2S}{\mu}\int_{-a}^{a} \frac{\xi d\xi}{(x-\xi)\sqrt{a^2-\xi^2}} = \frac{S|x|}{\sqrt{x^2-a^2}}\ ; \quad |x| > a \tag{6}$$

and the mode III stress intensity factor is

$$K_{III} \equiv \lim_{x\to a^+} \sigma_{yz}(x,0)\sqrt{2\pi(x-a)} = S\sqrt{\pi a}\ . \tag{7}$$

The relative motion between the crack faces is

$$\Delta = \int_{-a}^{x} B_z(\xi) d\xi = \frac{2S}{\mu} \int_{-a}^{x} \frac{\xi d\xi}{\sqrt{a^2 - \xi^2}} = \frac{2S\sqrt{a^2 - x^2}}{\mu} \ . \tag{8}$$

CHAPTER 16

16.1. Find the solution for torsion of a solid circular shaft of radius a by setting $b=a$ in the solution for the elliptical bar. Express the stress function and hence the stress components in polar coördinates and verify that these results and the relation between T and β agree with those given by the elementary Mechanics of Materials theory of torsion. Also, verify that there is no warping of the cross-section $f=0$ in this case.

Setting $b=a$ in equations (16.26), we obtain

$$C = -\frac{T}{\pi a^2}\ ; \quad \beta = \frac{2T}{\pi \mu a^4}\ . \tag{1}$$

Substituting for C into (16.22), we then obtain

$$\varphi = \frac{T(a^2 - r^2)}{\pi a^4}\ . \tag{2}$$

The stress components in polar coördinates can be obtained from (16.7) as

$$\sigma_{zr} = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} = 0 \tag{3}$$

$$\sigma_{z\theta} = -\frac{\partial \varphi}{\partial r} = \frac{2Tr}{\pi a^4}\ . \tag{4}$$

The Mechanics of Materials theory gives

$$\beta = \frac{T}{\mu J}\ ; \quad \sigma_{z\theta} = \frac{Tr}{J}\ , \tag{5}$$

where

$$J = \frac{\pi a^4}{2}\ . \tag{6}$$

Substituting this value for J, we recover the expressions from the exact solution.

To determine the warping function f, we can substitute for φ, β into equations (16.11), obtaining (after some simplification)

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0\ . \tag{7}$$

It follows that f is independent of x, y and hence that there is no warping of the cross-section in this case.

16.2. A bar transmitting a torque T has the cross-section of a thin sector $0 < r < a, -\alpha < \theta < \alpha$, where $\alpha \ll 1$. Develop an approximate solution for the shear stress distribution and the twist per unit length, using strong boundary conditions on the edges $\theta = \pm\alpha$ only.

Suppose we choose

$$\varphi_P = -\frac{\mu\beta r^2}{2} , \tag{1}$$

from equation (16.17). The complete solution is required to satisfy the strong condition

$$\varphi = 0 ; \quad \theta = \pm\alpha \tag{2}$$

for all r and this will be possible only if the homogeneous solution φ_H also contains terms varying with r^2. The most general such solution is

$$\varphi_H = Ar^2 \cos(2\theta) + Br^2 \sin(2\theta) \tag{3}$$

(see equation (16.19)). We therefore take

$$\varphi = \varphi_P + \varphi_H = -\frac{\mu\beta r^2}{2} + Ar^2 \cos(2\theta) + Br^2 \sin(2\theta) , \tag{4}$$

in which case the condition (2) requires that

$$A = 0 ; \quad B = \frac{\mu\beta}{2\cos(2\alpha)} . \tag{5}$$

Thus,

$$\varphi = \frac{\mu\beta r^2(\cos(2\theta) - \cos(2\alpha))}{2\cos(2\alpha)} . \tag{6}$$

The transmitted torque is

$$T = 2\int_0^a \int_{-\alpha}^{\alpha} \varphi r d\theta dr = \frac{\mu\beta a^4(\tan(2\alpha) - 2\alpha)}{4} \tag{7}$$

and hence

$$\beta = \frac{4T}{\mu a^4(\tan(2\alpha) - 2\alpha)} . \tag{8}$$

The stress components in polar coördinates can be obtained from (16.7) as

$$\sigma_{zr} = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = -\frac{\mu\beta r \sin(2\theta)}{\cos(2\alpha)} \tag{9}$$

$$\sigma_{z\theta} = -\frac{\partial\varphi}{\partial r} = -\frac{\mu\beta r(\cos(2\theta) - \cos(2\alpha))}{\cos(2\alpha)} . \tag{10}$$

Alternatively, we can use (8) to express the stresses in terms of the torque T, obtaining

$$\sigma_{zr} = -\frac{4Tr\sin(2\theta)}{a^4(\sin(2\alpha) - 2\alpha\cos(2\alpha))} \tag{11}$$

$$\sigma_{z\theta} = -\frac{4Tr(\cos(2\theta) - \cos(2\alpha))}{a^4(\sin(2\alpha) - 2\alpha\cos(2\alpha))} \,. \tag{12}$$

16.3. A bar transmitting a torque T has the cross-section of an equilateral triangle of side a. Find the equations of the three sides of the triangle in Cartesian coördinates, taking the origin at one corner and the x-axis to bisect the opposite side. Express these equations in the form $f_i(x,y)=0$, $i=1,2,3$.

Show that the function

$$\varphi = Cf_1(x,y)f_2(x,y)f_3(x,y)$$

satisfies the boundary condition (16.10) and can be made to satisfy (16.12) with a suitable choice of the constant C. Hence find the stress field in the bar and make a contour plot of the maximum shear stress. Why can this method not be used for a more general triangular cross-section?

The two sides of the triangle that meet at the origin make angles $\pm\pi/6$ with the x-axis and hence have equations

$$y = \pm x \tan\frac{\pi}{6} = \pm\frac{x}{\sqrt{3}} \ . \tag{1}$$

The third side has the equation

$$x = a\cos\frac{\pi}{6} = \frac{\sqrt{3}a}{2} \ . \tag{2}$$

These equations can be cast in the form $f_i(x,y)=0$ as

$$\begin{aligned} f_1(x,y) &= \sqrt{3}y - x = 0 && (3)\\ f_2(x,y) &= \sqrt{3}y + x = 0 && (4)\\ f_3(x,y) &= \sqrt{3}a - 2x = 0 \ . && (5) \end{aligned}$$

The function

$$\begin{aligned} \varphi &= Cf_1(x,y)f_2(x,y)f_3(x,y) = C(\sqrt{3}y - x)(\sqrt{3}y + x)(\sqrt{3}a - 2x)\\ &= C(2x^3 - 6xy^2 - \sqrt{3}ax^2 + 3\sqrt{3}ay^2) && (6) \end{aligned}$$

clearly goes to zero on all three boundaries and hence satisfies (16.10). It follows that

$$\nabla^2\varphi = C(12x - 12x - 2\sqrt{3}a + 6\sqrt{3}a) = 4\sqrt{3}Ca \tag{7}$$

and this will satisfy (16.12) if we choose

$$4\sqrt{3}Ca = -2\mu\beta \ . \tag{8}$$

The torque

$$T = 2\int_0^{\sqrt{3}a/2}\int_{-x/\sqrt{3}}^{x/\sqrt{3}} \varphi dydx = -\frac{3Ca^5}{40} \tag{9}$$

and hence

$$C = -\frac{40T}{3a^5} \; ; \quad \beta = \frac{80T}{\sqrt{3}\mu a^4} \tag{10}$$

$$\varphi = -\frac{40T(2x^3 - 6xy^2 - \sqrt{3}ax^2 + 3\sqrt{3}ay^2)}{3a^5} \; . \tag{11}$$

The stress components are

$$\sigma_{zx} = \frac{\partial \varphi}{\partial y} = -\frac{80T(-2xy + \sqrt{3}ay)}{a^5} \tag{12}$$

$$\sigma_{zy} = -\frac{\partial \varphi}{\partial x} = \frac{80T(3x^2 - 3y^2 - \sqrt{3}ax)}{3a^5} \; . \tag{13}$$

The magnitude of the shear stress

$$\tau = \sqrt{\sigma_{zx}^2 + \sigma_{zy}^2} \tag{14}$$

is plotted in Figure 1. Notice that the maximum value occurs at the mid-points of the sides — for example, at $(\sqrt{3}a/2, 0)$. At this point, we have

$$\tau_{\max} = \frac{20T}{a^3} \; . \tag{15}$$

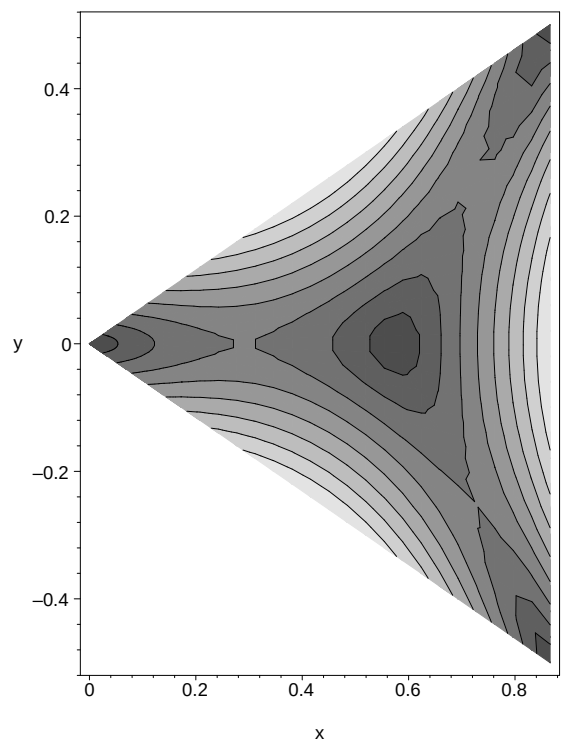


Figure 1: Contour plot of resultant shear stress τ.

We can choose a coordinate system for a more general triangle with the origin at one vertex and the opposite side parallel with the y-axis. The equations of the three sides will then be of the form $f_i(x,y)=0$, $i=1,2,3$, where

$$f_1 = y - C_1 x \; ; \quad f_2 = y - C_2 x \; ; \quad f_3 = x - C_3 \; . \tag{16}$$

Choosing the trial function

$$\begin{aligned} \varphi &= C(y - C_1 x)(y - C_2 x)(x - C_3) \\ &= C(y^2 x - (C_1 + C_2)x^2 y + C_1 C_2 x^3 - C_3 y^2 + (C_1 + C_2)C_3 xy - C_1 C_2 C_3 x^2) \end{aligned} \tag{17}$$

we obtain

$$\nabla^2 \varphi = C(2x - 2(C_1 + C_2)y + 6C_1 C_2 x - 2C_3 - 2C_1 C_2 C_3) \; . \tag{18}$$

This expression will be constant (independent of x and y) if an only if

$$C_1 + C_2 = 0 \quad \text{and} \quad 6C_1 C_2 + 2 = 0 \; . \tag{19}$$

The only solution of these equations is

$$C_1 = -C_2 = \frac{1}{\sqrt{3}} \; , \tag{20}$$

which defines an equilateral triangle, as in the above example.

16.4. A bar transmitting a torque T has the cross-section of a sector $0<r<a, -\alpha<\theta<\alpha$, where α is not small compared with unity.

(i) Find a combination of the functions r^2 and $r^2\cos(2\theta)$ that satisfies the governing equation (16.12) and the boundary condition (16.10) *on the straight edges $\theta=\pm\alpha$ only.*

(ii) Superpose a series of harmonic functions of the form $r^\gamma\cos(\gamma\theta)$ where γ is chosen to satisfy (16.10) on $\theta=\pm\alpha$.

(iii) Choose the coefficients on these terms so as to satisfy the boundary condition on the curved edge $r=a$ and hence obtain expressions for the torsional rigidity K and the maximum shear stress.

(iv) Evaluate your results for the special case $\alpha=\pi/6$ and compare them with those obtained using the thin-walled approximation of §16.3.

We first choose a particular solution

$$\varphi_P = -\frac{\mu\beta r^2}{2} \ , \tag{1}$$

from equation (16.17). The complete solution is required to satisfy the strong condition

$$\varphi = 0 \ ; \quad \theta = \pm\alpha \tag{2}$$

for all r and this will be possible only if the homogeneous solution φ_H also contains terms varying with r^2. The most general such solution is

$$\varphi_H = Ar^2\cos(2\theta) + Br^2\sin(2\theta) \tag{3}$$

(see equation (16.19)). We therefore take

$$\varphi = \varphi_P + \varphi_H = -\frac{\mu\beta r^2}{2} + Ar^2\cos(2\theta) + Br^2\sin(2\theta) \ , \tag{4}$$

in which case the condition (2) requires that

$$A = \frac{\mu\beta}{2\cos(2\alpha)} \ ; \quad B = 0 \ . \tag{5}$$

Thus,

$$\varphi = \frac{\mu\beta r^2(\cos(2\theta) - \cos(2\alpha))}{2\cos(2\alpha)} \tag{6}$$

satisfies the strong boundary conditions on $\theta = \pm\alpha$.

We next seek solutions of the form $\varphi = r^{\gamma_n}\cos(\gamma_n\theta)$ that also satisfy equation (2). This will be the case if

$$\cos(\gamma_n\alpha) = 0 \tag{7}$$

and hence

$$\gamma_n = \frac{(2n-1)\pi}{2\alpha}\ , \tag{8}$$

where n is an integer. The stress components are first derivatives φ and hence those associated with the function $\varphi = r^{\gamma_n}\cos(\gamma_n\theta)$ will vary with $r^{\gamma_n - 1}$. The arguments of §11.2.1 then show that we must restrict attention to values of γ_n in the range $\gamma_n > 0$, which implies that only positive integer values of n should be allowed[1]. Superposing a series of terms of this form on the stress function of equation (6), we obtain

$$\varphi = \frac{\mu\beta r^2(\cos(2\theta) - \cos(2\alpha))}{2\cos(2\alpha)} + \sum_{n=1}^{\infty} C_n r^{\gamma_n}\cos(\gamma_n\theta)\ , \tag{9}$$

where C_n are a set of arbitrary constants that can be chosen so as to satisfy the boundary condition $\varphi = 0$ on $r = a$ in the strong sense. This implies

$$\frac{\mu\beta a^2(\cos(2\theta) - \cos(2\alpha))}{2\cos(2\alpha)} + \sum_{n=1}^{\infty} C_n a^{\gamma_n}\cos(\gamma_n\theta) = 0 \tag{10}$$

and hence

$$\sum_{n=1}^{\infty} C_n a^{\gamma_n}\cos(\gamma_n\theta) = -\frac{\mu\beta a^2(\cos(2\theta) - \cos(2\alpha))}{2\cos(2\alpha)} \tag{11}$$

in $-\alpha < \theta < \alpha$.

Multiplying both sides of this equation by $\cos(\gamma_m\theta)$, where m is an integer, and integrating in the range $-\alpha < \theta < \alpha$, we obtain

$$\begin{aligned} C_m a^{\gamma_m}\alpha &= -\frac{\mu\beta a^2}{2\cos(2\alpha)}\int_{-\alpha}^{\alpha}(\cos(2\theta) - \cos(2\alpha))\cos(\gamma_m\theta)d\theta \\ &= \frac{4(-1)^{(m+1)}\mu\beta a^2}{\gamma_m(4-\gamma_m^2)}\ , \end{aligned} \tag{12}$$

where we have used the results

$$\cos(\gamma_m\alpha) = 0\ ; \quad \sin(\gamma_m\alpha) = (-1)^{(m+1)}\ . \tag{13}$$

It follows that

$$C_m = \frac{4(-1)^{(m+1)}\mu\beta a^{(2-\gamma_m)}}{\alpha\gamma_m(4-\gamma_m^2)}\ . \tag{14}$$

[1]Notice that the most singular term then corresponds to $n = 1$ and this will result in bounded stresses in all terms if $2\alpha \le \pi$ — i.e. if the angle of the sector is not re-entrant. For re-entrant corners, $2\alpha > \pi$ there well be one eigenvalue corresponding to a singular stress field and the remaining terms will be bounded

and hence

$$\varphi = \frac{\mu\beta r^2(\cos(2\theta) - \cos(2\alpha))}{2\cos(2\alpha)} - \frac{4\mu\beta a^2}{\alpha}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^{\gamma_n}\frac{(-1)^n\cos(\gamma_n\theta)}{\gamma_n(4-\gamma_n^2)} \ . \tag{15}$$

The transmitted torque is

$$T = 2\int_{-\alpha}^{\alpha}\int_0^a \varphi r dr d\theta = \mu\beta a^4\left(\frac{(\sin(2\alpha) - 2\alpha)}{8\cos(2\alpha)} + \frac{8}{\alpha}\sum_{n=1}^{\infty}\frac{1}{\gamma_n^2(4-\gamma_n^2)(2+\gamma_n)}\right) \ , \tag{16}$$

from equations (15) and (16.14). Numerical evaluation of this expression for particular values of α shows that it is essentially converged after two terms of the series. For a sector of 60° ($\alpha = \pi/6$), we obtain

$$K = \frac{T}{\mu\beta} = 0.0349a^4 \ . \tag{17}$$

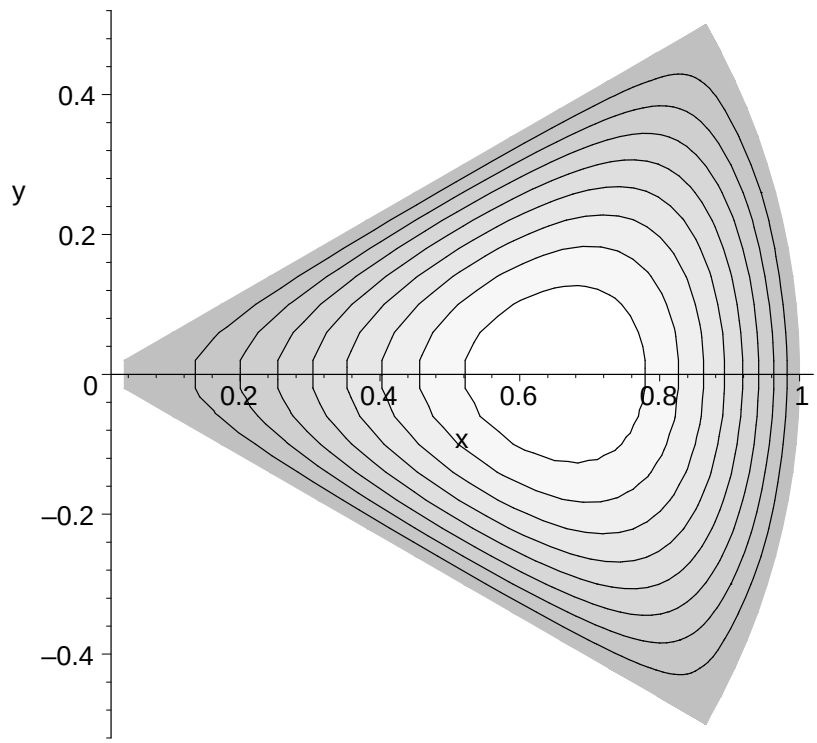


Figure 1: Contour plot of φ for a sector of 60°.

Figure 1 shows a contour plot of φ for a sector of 60°, using 4 terms in the series. Notice that the contours are already almost parallel with the curved surface, showing that the boundary condition at this surface is quite well satisfied. This figure also

shows that the maximum stress (the location of the closest contours) occurs either at $(a, 0)$ or near the point $(a/2. \pm \alpha)$. On $\theta = 0$, we have

$$\sigma_{z\theta} = \frac{\partial \varphi}{\partial r} = \frac{\mu\beta r(\cos(2\theta) - \cos(2\alpha))}{\cos(2\alpha)} - \frac{4\mu\beta a^2}{\alpha} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^{\gamma_n} \frac{(-1)^n \cos(\gamma_n \theta)}{r(4 - \gamma_n^2)} \tag{18}$$

and hence the resultant shear stress at $(a, 0)$ is

$$\tau = \frac{\mu\beta a(\cos(2\theta) - \cos(2\alpha))}{\cos(2\alpha)} - \frac{4\mu\beta a}{\alpha} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\gamma_n \theta)}{(4 - \gamma_n^2)} \ . \tag{19}$$

The resultant shear stress on the line $\theta = \alpha$ is

$$\sigma_{zr} = -\frac{1}{r}\frac{\partial \varphi}{\partial \theta} = \mu\beta r \tan(2\alpha) - \frac{4\mu\beta a}{\alpha} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^{(\gamma_n - 1)} \frac{1}{(4 - \gamma_n^2)} \ . \tag{20}$$

For $\alpha = \pi/6$, numerical calculations show that $\tau(a, 0) = 0.446\mu\beta a$ and the maximum value of σ_{zr} on $\theta = \alpha$ occurs at $r = 0.562a$ and is

$$\tau_{\max} = 0.490\mu\beta a = \frac{14.0T}{a^3} \ . \tag{21}$$

The approximate theory predicts

$$K = \frac{1}{3}\int_0^{a\cos\alpha} (2x\tan\alpha)^3 dx = \frac{2a^4 \sin^3\alpha \cos\alpha}{3} \ , \tag{22}$$

where we have neglected the contribution of the segment beyond the vertical line $x = a\cos\alpha$. For $\alpha = \pi/6$, this evaluates to $K = 0.072a^4$, which is more than twice the correct value of $0.0349a^4$. Even for a sector of 2° ($\alpha = \pi/180$), the approximate theory overestimates the stiffness by 4.3%. For the maximum shear stress, the approximate theory gives

$$\tau_{\max} = 2\mu\beta a \cos\alpha \tag{23}$$

and for $\alpha = \pi/6$ this gives

$$\tau_{\max} = 1.732\mu\beta a = \frac{24.0T}{a^3} \ . \tag{24}$$

16.5. Figure 16.4 shows the cross-section of a circular shaft of radius a with a circular groove of radius b whose centre lies on the surface of the ungrooved shaft. Construct a suitable stress function for this problem using the particular solution

$$\varphi_P = C(x^2 + y^2 - b^2)$$

and a homogeneous solution φ_H comprising appropriate multipliers of the harmonic functions $r\cos\theta$ and $r^{-1}\cos\theta$ centred on the point B, which in Cartesian coördinates can be written

$$x \text{ and } \frac{x}{x^2 + y^2} .$$

Remember that the resulting function must be zero at all points on both the circles $x^2+y^2=b^2$ and $(x-a)^2+y^2=a^2$. Hence find the stress field in the bar and make a contour plot of the maximum shear stress for the case where $b=0.1a$.

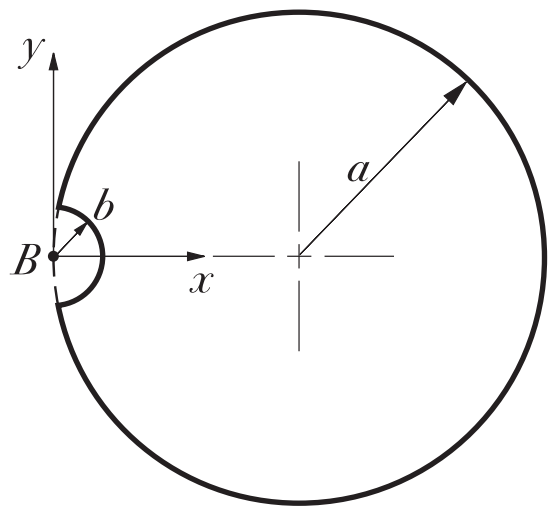


Figure 16.4: Circular shaft with a semi-circular groove.

We start with the trial function

$$\varphi = C(x^2 + y^2 - b^2) + Ax + \frac{Bx}{x^2 + y^2} . \tag{1}$$

Substituting into (16.12), we obtain

$$\nabla^2\varphi = 4C = -2\mu\beta , \tag{2}$$

which can be satisfied by choosing

$$C = -\frac{\mu\beta}{2} \ . \tag{3}$$

On the circle $x^2+y^2=b^2$, we have

$$\varphi = Ax + \frac{Bx}{b^2} \tag{4}$$

and this will be zero everywhere on the circle provided

$$A + \frac{B}{b^2} = 0 \ . \tag{5}$$

On the circle $(x-a)^2+y^2=a^2$, we have

$$y^2 = 2ax - x^2 \tag{6}$$

and hence

$$\begin{aligned} \varphi &= C(x^2 + 2ax - x^2 - b^2) + Ax + \frac{Bx}{2ax} \\ &= C(2ax - b^2) + Ax + \frac{B}{2a} \ . \end{aligned} \tag{7}$$

This will be zero everywhere on the circle if

$$2Ca + A = 0 \quad \text{and} \quad -Cb^2 + \frac{B}{2a} = 0 \ . \tag{8}$$

Although there are four equations (3, 5, 8) for three unknowns A, B, C, it is easily verified that the expressions

$$A = \mu\beta a \ ; \quad B = -\mu\beta a b^2 \tag{9}$$

and (3) satisfy them all and hence the solution is defined by the stress function

$$\varphi = -\frac{\mu\beta}{2}\left(x^2 + y^2 - b^2 - 2ax + \frac{2ab^2x}{(x^2+y^2)}\right) \ . \tag{10}$$

The stress components are then given by

$$\sigma_{zx} = \frac{\partial\varphi}{\partial y} = -\mu\beta\left(y - \frac{2ab^2xy}{(x^2+y^2)^2}\right) \tag{11}$$

$$\sigma_{zy} = -\frac{\partial\varphi}{\partial x} = \mu\beta\left(x - a + \frac{ab^2}{(x^2+y^2)} - \frac{2ab^2x^2}{(x^2+y^2)^2}\right) \tag{12}$$

Figure 1 shows the required contour plot of

$$\tau = \sqrt{\sigma_{zx}^2 + \sigma_{zy}^2} \tag{13}$$

for the case $b = 0.1a$.

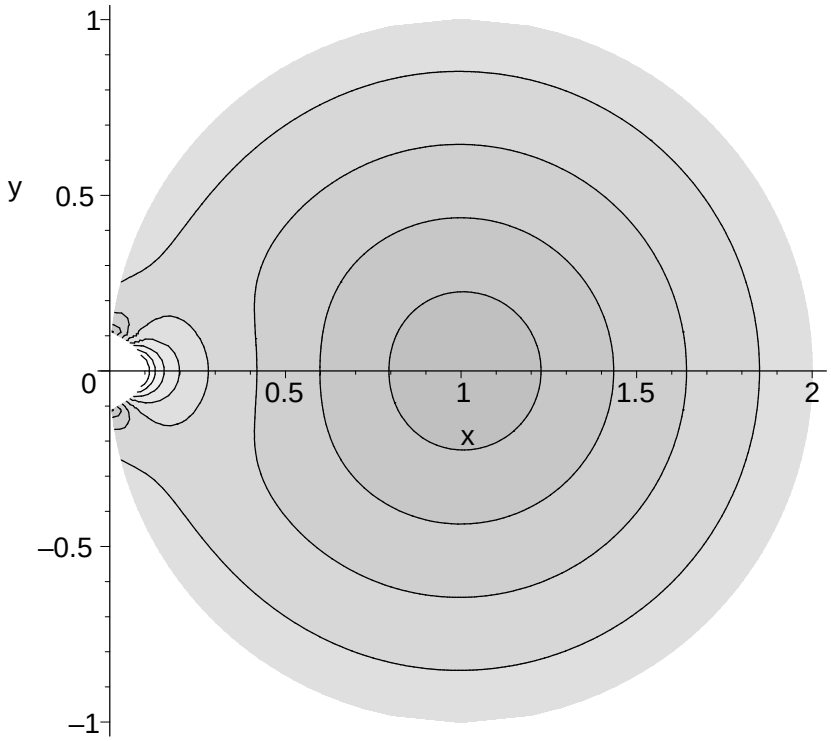


Figure 1: Contour plot of τ for $b = 0.1a$.

16.6. Figure 16.5 shows part of a thin-walled open section that is loaded in torsion. Find the stress function φ in the curved section, assuming that it depends upon r only and hence estimate[2] the maximum shear stress in the corner if the twist per unit length is β and the shear modulus is μ.

Find the corresponding maximum shear stress in the straight segment, distant from the corner and hence find the stress concentration factor due to the corner. Plot the stress concentration factor as a function of the ratio of the inner radius to the section thickness (b/c).

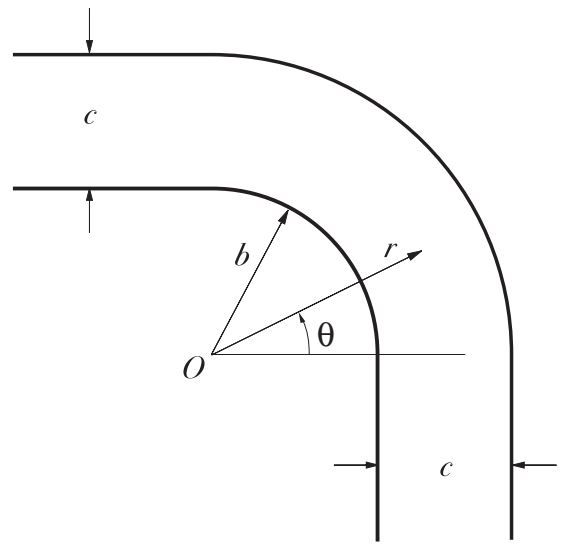


Figure 16.5: Stress concentration due to a bend.

If the stress function φ is independent of θ, equation (16.12) reduces to the ordinary differential equation

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} \equiv \frac{1}{r}\frac{d}{dr}r\frac{d\varphi}{dr} = -2\mu\beta \; . \tag{1}$$

Integrating this expression, we obtain the general solution as

$$\varphi = -\frac{\mu\beta r^2}{2} + A\ln(r) + B \; , \tag{2}$$

where A, B are arbitrary constants of integration. We choose these constants to satisfy the boundary conditions

$$\varphi = 0 \; ; \quad r = b, (b+c) \tag{3}$$

[2]This solution is approximate because the stress function is not independent of θ in the curved region. Otherwise there would be a discontinuity at the curved/straight transition.

and hence

$$A\ln(b) + B = \frac{\mu\beta b^2}{2}\ ; \quad A\ln(b+c) + B = \frac{\mu\beta(b+c)^2}{2} \tag{4}$$

with solution

$$A = \frac{\mu\beta c(2b+c)}{2\ln(1+c/b)}\ ; \quad B = \frac{\mu\beta\{b^2\ln(b+c) - (b+c)^2\ln(b)\}}{2\ln(1+c/b)}\ . \tag{5}$$

The only non-zero stress component is

$$\sigma_{z\theta} = -\frac{\partial\varphi}{\partial r} = \mu\beta r - \frac{A}{r} \tag{6}$$

and the maximum occurs at $r = b$ and is

$$\tau_{\max} = \left|\mu\beta b - \frac{A}{b}\right| = -\mu\beta b + \frac{\mu\beta c(2b+c)}{2b\ln(1+c/b)}\ . \tag{7}$$

Distant from the corner, we shall have $\tau_{\max} = \mu\beta c$, from the second of equations (16.39), so the stress concentration factor is

$$S.C.F. = \frac{(2+c/b)}{2\ln(1+c/b)} - \frac{b}{c}\ . \tag{8}$$

This expression is plotted in Figure 1 as a function of the ratio b/c. It is unbounded as $b/c \to 0$ and of course tends to unity at large values of b/c.

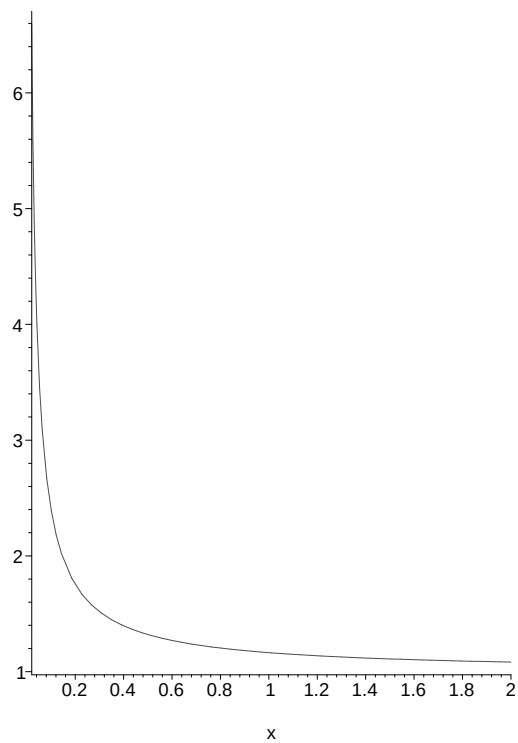


Figure 1: Stress Concentration Factor as a function of the ratio b/c.

16.7. Approximations to the stresses in a square bar in torsion can be obtained by adding harmonic functions of the form $r^n \cos(n\theta)$ to the axisymmetric particular solution (16.17).

Express the function

$$\varphi = r^2 + r^4 \cos(4\theta)$$

in Cartesian coördinates and make a contour plot. Select the contour which is the best approximation to a square with rounded corners and scale the resulting function to obtain an approximate solution for the maximum shear stress in a square bar of side a when the twist per unit length is β. Compare the result with that obtained from Figure 16.2 when $b/a=1$.

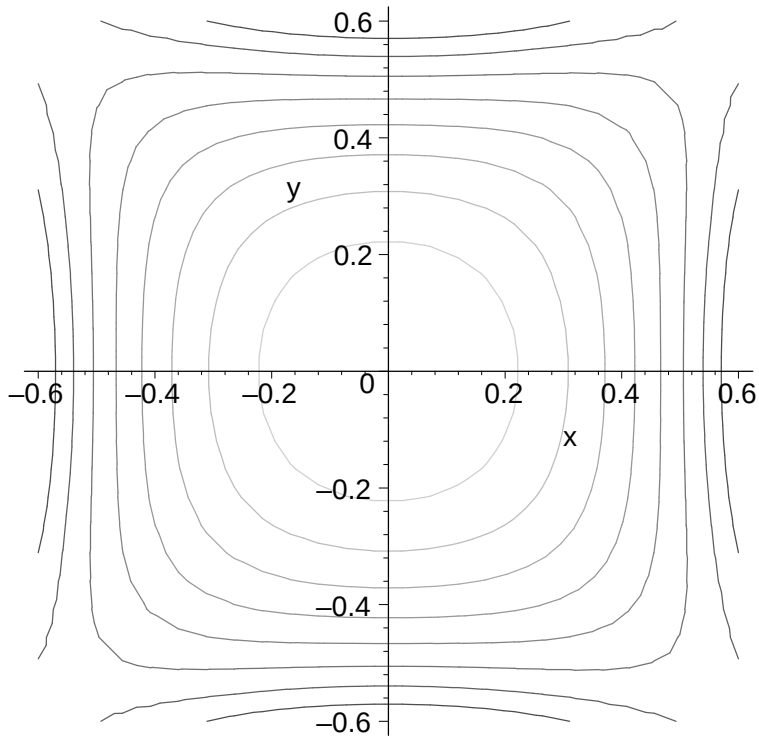


Figure 1: Contour plot of $\varphi = r^2 + r^4 \cos(4\theta)$.

The contour plot is shown in Figure 1. The best fit to a square with rounded corners is given by the contour that passes through the point (0.455,0) and hence corresponds to a value

$$\varphi = 0.455^2 + 0.455^4 = 0.2499 \ . \tag{1}$$

Thus, we can construct a function satisfying the condition $\varphi = 0$ on this boundary as

$$\varphi = C(r^2 + r^4 \cos(4\theta) - 0.2499) \ , \tag{2}$$

where C is an arbitrary constant. This would be the required function for the special case $a = 0.91$. To generalize to a square of side a, we can replace 'r' by $0.91r/a$, obtaining the function

$$\varphi = C\left\{\left(\frac{0.91r}{a}\right)^2 + \left(\frac{0.91r}{a}\right)^4 \cos(4\theta) - 0.2499\right\} . \tag{3}$$

We then have

$$\nabla^2\varphi = 4C\left(\frac{0.91}{a}\right)^2 = -2\mu\beta \tag{4}$$

and hence

$$C = -\frac{\mu\beta a^2}{2 \times 0.91^2} . \tag{5}$$

The stress function is therefore

$$\varphi = \frac{\mu\beta}{2}\left\{0.3018a^2 - r^2 - \frac{0.91^2 r^4 \cos(4\theta)}{a^2}\right\} . \tag{6}$$

The applied torque is

$$T = 2\int\int_A \varphi dA . \tag{7}$$

This integral could be evaluated in either polar or Cartesian coördinates, but the latter choice avoids the resolution of a factor 0/0 at $\theta = \pi/4$. In Cartesian coördinates,

$$\varphi = \frac{\mu\beta}{2}\left\{0.3018a^2 - x^2 - y^2 - \frac{0.91^2(x^4 - 6x^2y^2 + y^4)}{a^2}\right\} \tag{8}$$

and the boundary curve in the sector $0 < \theta < \pi/4$ is defined by

$$0.3018a^2 - x^2 - y^2 - \frac{0.91^2(x^4 - 6x^2y^2 + y^4)}{a^2} = 0 . \tag{9}$$

This is a quadratic equation for x^2 and it has one positive root, giving a real root for x as a function of y, which we denote by $x_1(y)$. The maximum value of y in this sector occurs when $x = y$ on the boundary and hence

$$0.3018a^2 - 2y^2 + \frac{4 \times 0.91^2 y^4}{a^2} = 0 . \tag{10}$$

We denote the smaller real root of this equation by y_1. The integral (7) can then be expressed as

$$T = 16\int_0^{y_1}\int_y^{x_1(y)} \varphi(x,y)dxdy , \tag{11}$$

since the domain of integration can be divided into 8 identical sectors such as the one bounded by the lines $x = y$ and $y = 0$. Equation (11) can be evaluated numerically as

$$T = 0.1493\mu\beta a^4 . \tag{12}$$

The maximum shear stress occurs at $(a/2, 0)$ and is

$$\tau_{\max} = \left| \frac{\partial \varphi}{\partial x} \left(\frac{a}{2}, 0 \right) \right| = 0.707 \mu \beta a \ . \tag{13}$$

For comparison, the exact expressions from Figure 16.2 are

$$\frac{3K}{16} \left(\frac{2}{a} \right)^4 = 0.4218 \ ; \quad \frac{8\tau_{\max}}{3T} \left(\frac{a}{2} \right)^3 = 1.6010 \ , \tag{14}$$

where we note that the square of side a in this Figure is achieved by replacing both a and b by $a/2$, since the rectangle in §16.4 is of side $2a \times 2b$. Simplifying these expressions, we obtain

$$K = \frac{T}{\mu \beta} = 0.1406 a^4 \tag{15}$$

and

$$\tau_{\max} = 0.675 \mu \beta a \ . \tag{16}$$

Thus the approximate method gives a quite good accuracy for both the stiffness and the maximum shear stress.

16.8. We know from Mechanics of Materials that a bar cross-section posesses a *shear centre* which is the point through which a shear force must act if there is to be no twist of the section. Maxwell's reciprocal theorem[3] implies that the shear centre must also be the only point in the section that has no in-plane displacement when the bar is loaded in torsion and hence that equations (16.1) are correct if and only if the shear centre is taken as origin.

Rederive the torsion theory using a more general origin and hence show that the formulation in terms of Prandtl's stress function remains true for all choices of origin.

Suppose that the shear centre is located at the point (a, b). In that case, equations (16.1) should be replaced by

$$u_x = -\beta z(y-b) \;\; ; \;\; u_y = \beta z(x-a) \;\; ; \;\; u_z = \beta f(x,y) \tag{1}$$

and the shear strains are

$$e_{xy} = 0 \;\; ; \;\; e_{zx} = \frac{\beta}{2}\left(\frac{\partial f}{\partial x} - y + b\right) \;\; ; \;\; e_{zy} = \frac{\beta}{2}\left(\frac{\partial f}{\partial y} + x - a\right) . \tag{2}$$

We then obtain

$$\sigma_{xy} = 0 \;\; ; \;\; \sigma_{zx} = \mu\beta\left(\frac{\partial f}{\partial x} - y + b\right) \;\; ; \;\; \sigma_{zy} = \mu\beta\left(\frac{\partial f}{\partial y} + x - a\right) \tag{3}$$

from Hooke's law and substituting into the third equilibrium equation, we have

$$\nabla^2 f = 0 \tag{4}$$

as before.

Defining Prandtl's stress function through the relations

$$\sigma_{zx} = \frac{\partial \varphi}{\partial y} \;\; ; \;\; \sigma_{zy} = -\frac{\partial \varphi}{\partial x} , \tag{5}$$

we have

$$\frac{\partial \varphi}{\partial y} = \mu\beta\left(\frac{\partial f}{\partial x} - y + b\right) \;\; ; \;\; \frac{\partial \varphi}{\partial x} = -\mu\beta\left(\frac{\partial f}{\partial y} + x - a\right) \tag{6}$$

and eliminating f between these two equations yields

$$\nabla^2 \varphi = -2\mu\beta , \tag{7}$$

as before. Since the stress definitions are the same in the new and old coördinate systems, the expression for the torque (16.14) and the boundary condition (16.10) remain unchanged. Thus, the problem statement in terms of Prandtl's stress function remains unchanged by the change of origin and hence any convenient coöordinate system can be chosen.

[3] See Chapter 30.

16.9. A titanium alloy turbine blade is idealized by the section of Figure 16.6. The leading edge is a semicircle of radius 2 mm and the inner and outer edges of the trailing section are cylindrical surfaces of radius 43 mm and 35 mm respectively, with centres of curvature on the line Oy. Find the twist per unit length β and the maximum shear stress when the section is loaded by a torque of 5 Nm. The shear modulus for the alloy is 43 GPa.

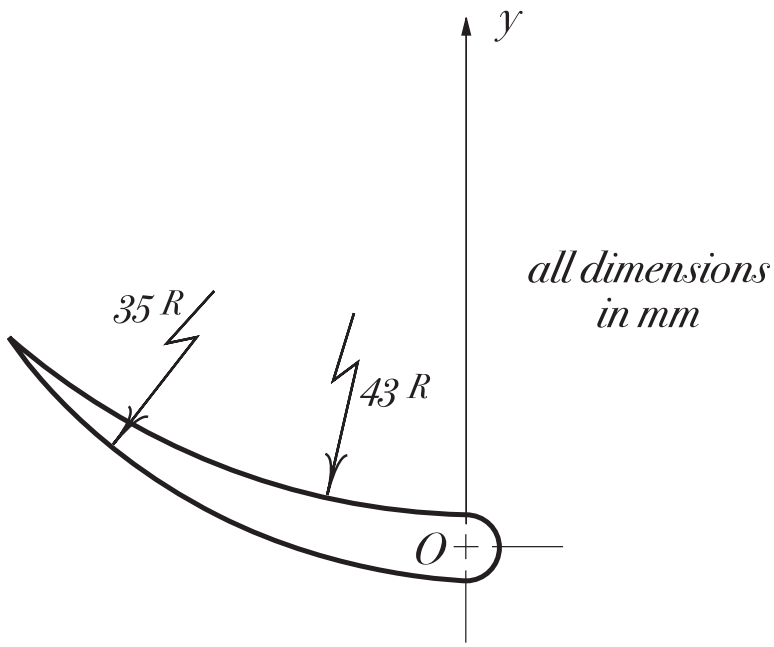


Figure 16.6: Turbine blade cross-section.

The thickness of the section at O is 4 mm, so the two centres of curvature O_1, O_2 must be separated by

$$43 - 35 + 4 = 12 \text{ mm} .$$

The mean radius of the section is

$$a = \frac{43 + 35}{2} = 39 \text{ mm}$$

and the thickness can be approximated as in Figure 1 as

$$t \approx 35 + 12\cos\theta - 43 = 12\cos\theta - 8 .$$

The end of the section is defined by $t = 0$ and hence

$$\cos\theta = \frac{8}{12} \quad ; \quad \theta = 0.841 \text{ rad.}$$

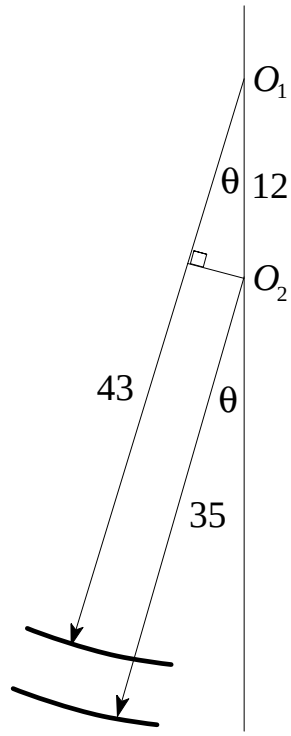


Figure 1

Using these results, we can calculate K from equation (16.38) as

$$K_1 = \frac{1}{3}\int_0^{0.841} (12\cos\theta - 8)^3 39 d\theta = 314 \text{ mm}^4 \ .$$

The rounded end of the section (radius 2mm) will add slightly to this value. For this correction we have

$$K_2 = \frac{1}{3}\int_0^a t^3 dx = \frac{1}{3}\int_0^a (2\sqrt{a^2 - x^2})^3 dx \ ,$$

where $a = 2$ mm. Evaluating the integral, we find

$$K_2 = \frac{\pi a^4}{2} = 8\pi = 25 \text{ mm}^4 \ .$$

The total stiffness is therefore

$$K = K_1 + K_2 = 339 \text{ mm}^4 \ .$$

If we apply a torque of 5Nm, the twist per unit length will be

$$\beta = \frac{T}{K\mu} = \frac{5}{339 \times 10{-}12 \times 43 \times 10^9} = 0.343 \text{ radians per m.}$$

The maximum shear stress is

$$\tau_{\max} = \mu\beta t_{\max} = 43 \times 10^9 \times 0.343 \times 4 \times 10^{-3} = 59 \times 10^6 \text{ Pa (59 MPa).}$$

16.10. The W200×22 I-beam of Figure 16.7 is made of steel $G = 80$ GPa) and is 15 m long. One end is built in and the other is loaded by two forces F constituting a torque, as shown. Find the torsional rigidity K for the beam section and hence determine the value of F needed to cause the end to twist through an angle of 20°. Does the result surprise you? How large an angle of twist[4] could a person of average strength produce in such a beam using his or her hands only?

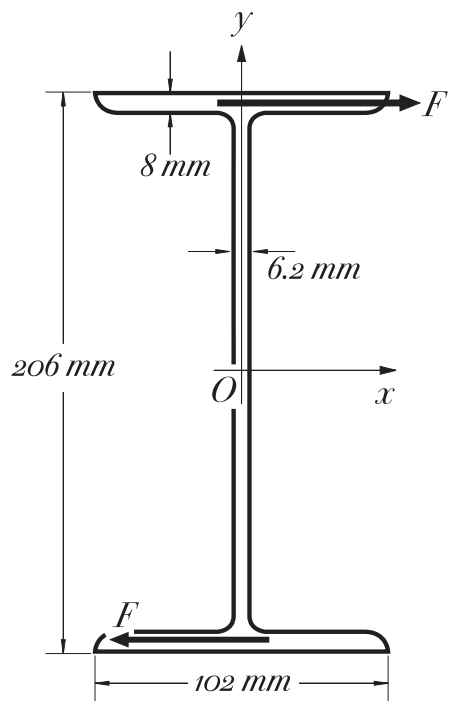


Figure 16.7: I-beam twisted by two forces.

The torsional stiffness of the beam section is

$$K = \frac{1}{3} \times (206 - 16) \times 6.2^3 + \frac{2}{3} \times 102 \times 8^3 = 49910 \text{ mm}^4 .$$

For a twist per unit length of 20° over a length of 15 m, we require a torque T given by

$$\frac{20 \times \pi}{180 \times 15 \times 10^3} = \frac{T}{80 \times 10^3 \times 49910}$$

and hence

$$T = \frac{20\pi \times 80 \times 10^3 \times 49910}{180 \times 15 \times 10^3} = 92900 \text{ Nmm}.$$

[4]If you want to impress your unsuspecting colleagues with your engineering expertise (or your immense strength), ask them to guess the answer to this question and then perform the experiment.

This corresponds to a pair of forces F, where $T = F \times 198$ and hence

$$F = \frac{92900}{198} = 469 \text{ N}.$$

This is a relatively small force. Most people could quite easily apply a force of 100 N (≈ 25 lb) on such a beam and hence cause a rotation of

$$\theta = \frac{100 \times 20}{469} \approx 4.3^o \ .$$

16.11. Show that the torsional rigidity of a closed thin-walled steel tube of uniform thickness is reduced by the factor

$$\frac{12A^2}{A_s^2}$$

when a longitudinal slit is made down the whole length of the tube to form an open section. In this expression, A is the area enclosed by the mean line of the closed section and A_s is the cross-sectional area of the steel.

Find also by what ratio the maximum shear stress will increase for a given applied torque.

For the closed section, the torsional rigidity is obtained from equations (16.57, 16.58) as

$$K_C = \frac{T}{\mu\beta} = 4A^2 \Big/ \oint \frac{dS}{t} = \frac{4A^2 t}{S} ,$$

since t is uniform. For the open section it is

$$K_O = \frac{St^3}{3} ,$$

from equation (16.39). The cross-sectional area of the steel is

$$A_s = St .$$

The ratio

$$\frac{K_C}{K_O} = \frac{4A^2 t}{S}\frac{3}{St^3} = \frac{12A^2}{S^2t^2} = \frac{12A^2}{A_s^2} .$$

The maximum shear stress for the closed section is

$$\tau_C = \frac{\varphi_1}{t} = \frac{2\mu\beta A}{S} ,$$

from equation (16.57). For the open section,

$$\tau_O = \mu\beta t ,$$

from (16.39). Thus

$$\frac{\tau_O}{\tau_C} = \frac{\mu\beta t S}{2\mu\beta A} = \frac{A_s}{2A} .$$

In other words, the maximum stress is increased in the ratio $A_s/2A$.

16.12. The outer and inner circular walls of a closed thin-walled circular tube have slightly different centres, as shown in Figure 16.8, so that the thickness of the section varies according to

$$t = t_0(1 - \epsilon \cos\theta) \ ,$$

where $\epsilon < 1$. The mean radius of the tube is a. Find the torsional rigidity K and the maximum shear stress when the bar is loaded by a torque T. Plot your results as functions of ϵ in the range $0 \leq \epsilon < 1$.

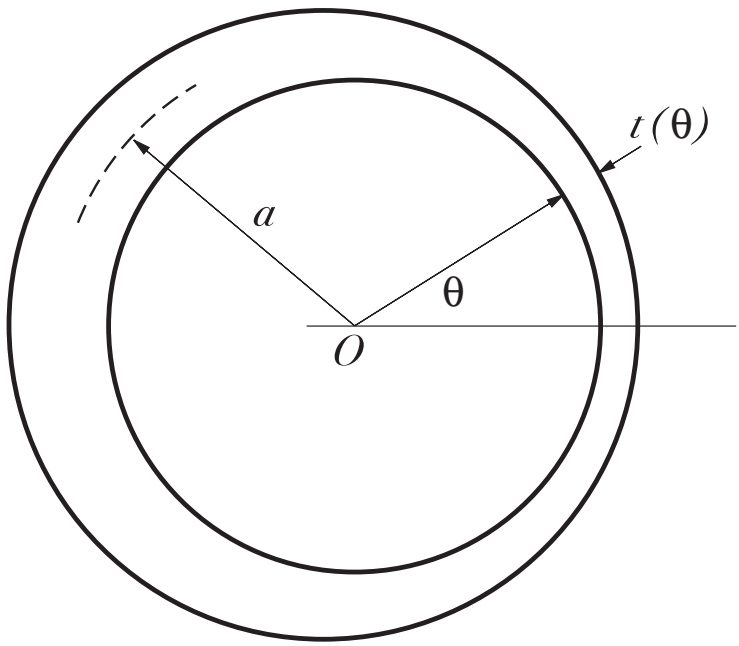


Figure 16.8: Cylindrical tube with variable wall thickness.

The enclosed area is $A = \pi a^2$ and

$$\oint_S \frac{ds}{t} = \int_0^{2\pi} \frac{a d\theta}{t_0(1 - \epsilon\cos\theta)} = \frac{2\pi a}{t_0\sqrt{1-\epsilon^2}} \ .$$

It follows that

$$K = 4A^2 \Big/ \oint_S \frac{ds}{t} = \frac{4\pi^2 a^4}{2\pi a / t_0\sqrt{1-\epsilon^2}} = 2\pi a^3 t_0 \sqrt{1-\epsilon^2} \ .$$

The shear stress is

$$\tau(\theta) = \frac{T}{2At(\theta)} = \frac{T}{2\pi a^2 t_0(1 - \epsilon\cos\theta)}$$

and the maximum value occurs at $\theta = 0$, being

$$\tau_{\max} = \frac{T}{2\pi a^2 t_0 (1-\epsilon)} \ .$$

The twist per unit length is

$$\frac{\theta}{L} = \frac{T}{GK} = \frac{T}{2\pi G a^3 t_0 \sqrt{1-\epsilon^2}} \ .$$

16.13. Show that the stress function

$$\varphi = -\frac{\mu\beta}{2}\left(x^2 + y^2 - 2a^2 - 2ax + \frac{4a^3 x}{(x^2+y^2)}\right)$$

defines the exact solution for the cross-section defined by the intersection of the circles

$$x^2 + y^2 = 2a^2 \quad \textbf{and} \quad (x-a)^2 + y^2 = a^2 \ .$$

Find the corresponding torque T and the maximum shear stress $\tau_{\max}$ and express the latter as a function of T, a only.

Now find an approximate solution to this problem by assuming that it is a 'thin-walled section' — i.e. using equations (16.36, 16.38). What is the percentage error in the approximate solution (i) in the stiffness K and (ii) in $\tau_{\max}$? Are these quantities overestimated or underestimated by the approximate solution?

The given stress function can be written in polar coordinates as

$$\varphi = -\frac{\mu\beta}{2}\left(r^2 - 2a^2 - 2ar\cos\theta + \frac{4a^3\cos\theta}{r}\right)$$

and the only non-harmonic term is the first one. Thus,

$$\nabla^2\varphi = -\frac{\mu\beta}{2}(4) = -2\mu\beta \ ,$$

as required. We also notice that on $r^2 = 2a^2$,

$$\varphi = -\frac{\mu\beta}{2}\left(-2\sqrt{2}a^2\cos\theta + 2\sqrt{2}a^2\cos\theta\right) = 0$$

and on $(x-a)^2 + y^2 = a^2$ (or $x^2 + y^2 = 2ax$),

$$\varphi = -\frac{\mu\beta}{2}\left(-2a^2 + \frac{4a^3 x}{2ax}\right) = 0 \ .$$

To determine the torque T and hence to perform the integration of φ over the area of the cross section, we first need to find the location of the intersection between the two circles. Equating

$$x^2 + y^2 = 2a^2 \quad \text{and} \quad x^2 + y^2 = 2ax \ ,$$

we have

$$x = a \quad \text{and} \quad a^2 + y^2 = 2a^2 \quad \text{or} \quad y = \pm a \ .$$

in other words, the intersection points are (a, a) and $(a, -a)$ and we can write the torque as

$$\begin{aligned} T = 2\int\int \varphi d\Omega &= \int_{-a}^{a}\int_{\sqrt{2a^2-y^2}}^{a+\sqrt{a^2-y^2}} \varphi(x,y)dxdy \\ &= \frac{\mu\beta a^4\left(16-5\pi\right)}{4} = 0.073\mu\beta a^4 \ , \end{aligned}$$

using Maple. In other words,

$$K = \frac{T}{\mu\beta} = 0.0073a^4 \ .$$

The maximum shear stress will be σ_{zy} at the inner edge $x = \sqrt{2}a$, $y = 0$ and is

$$\sigma_{zy} = -\frac{\partial\varphi}{\partial x} \ .$$

Since the derivative is with respect to x, we can simplify by first finding φ on $y = 0$ as

$$\varphi(x,0) = -\frac{\mu\beta}{2}\left(x^2 - 2a^2 - 2ax + \frac{4a^3}{x}\right) \ ,$$

so that

$$\sigma_{zy}(x,0) = \frac{\mu\beta}{2}\left(2x - 2a - \frac{4a^3}{x^2}\right)$$

and at $x = \sqrt{2}a$,

$$\sigma_{zy}(\sqrt{2}a,0) = \frac{\mu\beta}{2}\left(2\sqrt{2}a - 2a - \frac{4a^3}{2a^2}\right) = \mu\beta a\left(\sqrt{2}-2\right) \ .$$

Eliminating $\mu\beta$ to express the maximum shear stress in terms of T, we have

$$\tau_{\max} = \frac{4T\left(2-\sqrt{2}\right)}{a^3\left(16-5\pi\right)} = \frac{8.02T}{a^3} \ .$$

The approximate solution

There is some ambiguity in defining the mean line for a section that is not thin, so the solution given here is not unique. However, the simplest approach is to define a circle that passes through the points $(a, \pm a)$ and the mid-point at $y = 0$ which is at

$$x = \frac{2a + \sqrt{2}a}{2} \ .$$

The radius R of this circle must satisfy the equation

$$R^2 = a^2 + \left(R - \left(\frac{2a + \sqrt{2}a}{2} - a\right)\right)^2 ,$$

or

$$R^2 = a^2 - \left(R - \frac{a}{\sqrt{2}}\right)^2 = \frac{3a^2}{2} - \sqrt{2}aR + R^2 ,$$

with solution

$$R = \frac{3a}{2\sqrt{2}} .$$

The subtended semi-angle of the end points would then be

$$\alpha = \arcsin\left(\frac{a}{R}\right) = \arcsin\left(\frac{2\sqrt{2}}{3}\right) = \arccos\left(\frac{1}{3}\right) .$$

The maximum thickness of the section is

$$t_{\max} = 2a - \sqrt{2}a = 0.586a ,$$

and a reasonable approximation to the thickness is therefore

$$t(\theta) = t_{\max}\frac{(\cos\theta - \cos\alpha)}{(1 - \cos\alpha)} ,$$

Using this expression, we have

$$K = \frac{1}{3}\int_{-\alpha}^{\alpha} t^3 R d\theta = \frac{t_{\max}^3 R}{3(1 - \cos\alpha)^3}\int_{-\alpha}^{\alpha}(\cos\theta - \cos\alpha)^3 d\theta = 0.0765a^4 ,$$

using Maple. This overestimates the stiffness by about 5% relative to the exact solution.

The maximum stress from the approximate solution is

$$\tau_{\max} = \mu\beta t_{\max} = \frac{T t_{\max}}{K} = \frac{0.586T}{0.0765a^3} = \frac{7.655T}{a^3} .$$

This underestimates the exact value, also by about 5%.

These results are quite good in view of the approximation involved. However, other approximations to the thickness give different results. For example, using

$$t(\theta) = t_{\max}\frac{\alpha^2 - \theta^2}{\alpha^2} ,$$

we find

$$K = 0.080a^4 \quad \text{and} \quad \tau_{\max} = \frac{7.324T}{a^3} ,$$

which are both in error by about 10%.

16.14. Figure 16.9 shows the cross-section of a solid circular bar of radius a which is slit to the centre along the line $\theta = \pm\pi$. The bar is subjected to a prescribed twist β per unit length.

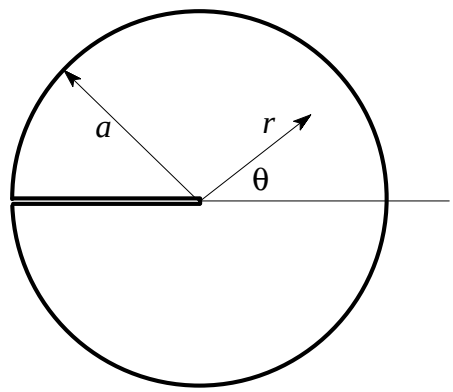


Figure 16.9

(i) **Show that the particular solution**

$$\varphi_P = -\mu\beta y^2 = -\mu\beta r^2 \sin^2(\theta)$$

satisfies the governing equation and also satisfies the condition that $\varphi = 0$ on the slit $\theta = \pm\pi$.

(ii) **To complete the solution, we need to superpose a series of stress functions each of which satisfies the homogeneous equation**

$$\nabla^2 \varphi_H = 0$$

and $\varphi = 0$ on $\theta = \pm\pi$. Show that the functions $r^\gamma \cos(\gamma\theta)$ meet these conditions with a suitable choice of γ.

(iii) **Use these solutions to construct a series solution of the problem and use the boundary condition on the curved edge $r = a$ to define equations that the coefficients of the series must satisfy.**

(iv) **Solve the equations and hence determine the mode III stress-intensity factor K_{III} at $r = 0$, defined as**

$$K_{III} = \lim_{r \to 0} \sigma_{z\theta}(r, 0)\sqrt{2\pi r}\ .$$

(i) Substituting φ_P into the governing equation, we have

$$\nabla^2 \varphi_P = -2\mu\beta$$

as required (equation (16.12)).

On $\theta = \pi$, we have

$$\varphi_P(r, \pi) = -\mu\beta r^2 \sin^2(\pi) = 0$$

and the same result is obtained for $\theta = -\pi$.

(ii) The function $r^\gamma \cos(\gamma\theta)$ is harmonic for all values of γ, since

$$\nabla^2 (r^\gamma \cos(\gamma\theta)) = r^{\gamma-2} \cos(\gamma\theta) \left(\gamma(\gamma - 1) + \gamma - \gamma^2 \right) = 0 \, .$$

On $\theta = \pm\pi$,

$$r^\gamma \cos(\gamma\theta) = r^\gamma \cos(\gamma\pi)$$

and this is zero for all r if

$$\cos(\gamma\pi) = 0 \quad \text{or} \quad \gamma = n + \frac{1}{2} \, ,$$

where n is any integer. We can therefore define a more general function satisfying the governing equation and the boundary conditions on $\theta = \pm\pi$ as

$$\varphi = \varphi_P + \sum_{n=0}^{\infty} C_n r^{n+1/2} \cos\left(\frac{(2n+1)\theta}{2} \right) \, .$$

(iii) On $r = a$, we have

$$\varphi = -\mu\beta a^2 \sin^2(\theta) + \sum_{n=0}^{\infty} C_n a^{n+1/2} \cos\left(\frac{(2n+1)\theta}{2} \right)$$

and to make this zero, we require

$$\sum_{n=0}^{\infty} C_n a^{n+1/2} \cos\left(\frac{(2n+1)\theta}{2} \right) = \mu\beta a^2 \sin^2(\theta) \, .$$

(iv) Multiplying both sides of this equation by

$$\cos\left(\frac{(2m+1)\theta}{2} \right) \, ,$$

where m is an integer and noting that

$$\begin{aligned} \int_{-\pi}^{\pi} \cos\left(\frac{(2m+1)\theta}{2} \right) \cos\left(\frac{(2n+1)\theta}{2} \right) d\theta &= 0 \quad m \neq n \\ &= \pi \quad m = n \, , \end{aligned}$$

we obtain

$$\pi C_m a^{m+1/2} = \mu\beta a^2 \int_{-\pi}^{\pi} \cos\left(\frac{(2m+1)\theta}{2}\right) \sin^2(\theta) d\theta = -\frac{32(-1)^m \mu\beta a^2}{(8m^3 + 12m^2 - 26m - 15)}$$

(using Maple) and hence

$$C_m = -\frac{32(-1)^m \mu\beta a^{3/2-m}}{\pi(8m^3 + 12m^2 - 26m - 15)} \ .$$

The stress intensity factor can be defined as

$$K_{III} = \lim_{r\to 0} \sigma_{z\theta}(r,0)\sqrt{2\pi r} \ ,$$

by analogy with equation (13.43). In polar coordinates, we can write

$$\sigma_{z\theta} = -\frac{\partial\varphi}{\partial r}$$

and hence the square-root singular term will arise from a term in φ proportional to $r^{1/2}$, which in turn arises from the term of the series with $n = 0$ — i.e.

$$C_0 r^{1/2} \cos\left(\frac{\theta}{2}\right) = \frac{32\mu\beta a^{3/2} r^{1/2}}{15\pi} \cos\left(\frac{\theta}{2}\right) \ .$$

From this term, we obtain

$$\sigma_{z\theta} = -\frac{16\mu\beta a^{3/2} r^{-1/2}}{15\pi} \cos\left(\frac{\theta}{2}\right)$$

and

$$\sigma_{z\theta}(r,0) = -\frac{16\mu\beta a^{3/2} r^{-1/2}}{15\pi} \ .$$

Thus,

$$K_{III} = -\frac{16\mu\beta a^{3/2}\sqrt{2\pi}}{15\pi} = -\frac{16\sqrt{2}\mu\beta a^{3/2}}{15\sqrt{\pi}} \ .$$

CHAPTER 17

17.1. Find an expression for the *local* twist per unit length of the bar, defined as

$$\beta(x,y) = \frac{\partial \omega_z}{\partial z} \ ,$$

where ω_z is the rotation defined in equation (1.47). Hence show that the *average* twist per unit length for a general unsymmetrical section will be zero if $C=0$ in equation (17.10).

From equation (1.47), we have

$$\omega_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \tag{1}$$

and hence

$$\beta(x,y) = \frac{\partial \omega_z}{\partial z} = \frac{1}{2}\left(\frac{\partial^2 u_y}{\partial x \partial z} - \frac{\partial^2 u_x}{\partial y \partial z}\right) \tag{2}$$

Writing

$$\frac{\partial u_y}{\partial z} = 2e_{yz} - \frac{\partial u_z}{\partial y} \ ; \quad \frac{\partial u_x}{\partial z} = 2e_{xz} - \frac{\partial u_z}{\partial x} \ , \tag{3}$$

we have

$$\begin{aligned}
\beta(x,y) &= \frac{1}{2}\left(2\frac{\partial e_{yz}}{\partial x} - \frac{\partial^2 u_z}{\partial x \partial y} - 2\frac{\partial e_{xz}}{\partial y} + \frac{\partial^2 u_z}{\partial x \partial y}\right) && (4)\\
&= \frac{\partial e_{yz}}{\partial x} - \frac{\partial e_{xz}}{\partial y} && (5)\\
&= \frac{1}{2\mu}\left(\frac{\partial \sigma_{yz}}{\partial x} - \frac{\partial \sigma_{xz}}{\partial y}\right) && (6)\\
&= -\frac{1}{2\mu}\nabla^2 \phi && (7)\\
&= -\frac{\nu}{2\mu(1+\nu)}(Ay - Bx) \ , && (8)
\end{aligned}$$

using (17.8, 17.10), with $C = 0$.

The average value of β is therefore

$$\frac{1}{A_\Omega}\int\int_\Omega \beta dxdy = -\frac{\nu}{2\mu(1+\nu)A_\Omega}\int\int_\Omega (Ay - Bx)dxdy \ , \tag{9}$$

where A_Ω is the area of the cross-section. The integral in this expression is zero, since it is a linear combination of the first moments of area of the section, which are both zero because the origin has been taken at the centroid. It follows that the average value of β is zero.

17.2. A bar of elliptical cross-section defined by the boundary

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

is loaded by a shear force $F_x = F$ acting along the negative x-axis. Find expressions for the shear stresses σ_{zx}, σ_{zy} on the cross-section.

For the ellipse,

$$I_x = \frac{\pi a b^3}{4} \; ; \;\; I_y = \frac{\pi a^3 b}{4} \; ; \;\; I_{xy} = 0 \tag{1}$$

and hence

$$A = \frac{4F}{\pi a^3 b} \; ; \;\; B = 0 \; , \tag{2}$$

from (17.6).

We therefore have

$$\frac{\partial \phi}{\partial t} = \frac{2Fx^2}{\pi a^3 b} \frac{\partial y}{\partial t} \tag{3}$$

on the boundary, from (17.13). Substituting

$$x^2 = a^2 - \frac{a^2 y^2}{b^2} \tag{4}$$

and integrating, we have

$$\phi = \int \frac{2F}{\pi a b} \left(1 - \frac{y^2}{b^2} \right) dy = \frac{2F}{\pi a b} \left(y - \frac{y^3}{3b^2} \right) \tag{5}$$

on the boundary defined by (4).

As in the circular example of §17.4.1, equations (5) and (17.10) and the symmetry of the problem suggest the third order polynomial

$$\phi = C_1 y^3 + C_2 x^2 y + C_3 y \; . \tag{6}$$

Equations (5) and (17.10) then require that

$$C_1 y^3 + C_2 \left(a^2 - \frac{a^2 y^2}{b^2} \right) y + C_3 y = \frac{2F}{\pi a b} \left(y - \frac{y^3}{3b^2} \right) \tag{7}$$

$$6C_1 y + 2C_2 y = \frac{4\nu F y}{(1+\nu)\pi a^3 b} \tag{8}$$

and these equations must be satisfied for all y, giving

$$C_1 - \frac{C_2 a^2}{b^2} = -\frac{2F}{3\pi a b^3} \tag{9}$$

$$C_2 a^2 + C_3 = \frac{2F}{\pi a b} \tag{10}$$

$$6C_1 + 2C_2 = \frac{4\nu F}{(1+\nu)\pi a^3 b} \; . \tag{11}$$

Solving for the constants, we obtain

$$C_1 = -\frac{2F(1-2\nu)}{3\pi(1+\nu)ab(3a^2+b^2)} \tag{12}$$

$$C_2 = \frac{2F\{(1+\nu)a^2+\nu b^2\}}{\pi(1+\nu)a^3b(3a^2+b^2)} \tag{13}$$

$$C_3 = \frac{2F\{2(1+\nu)a^2+b^2\}}{\pi(1+\nu)ab(3a^2+b^2)} \,. \tag{14}$$

The stress components are then obtained from (6, 12–14) and (17.8) as

$$\sigma_{zx} = \frac{\partial\phi}{\partial y} - \frac{2Fx^2}{\pi a^3 b} = \frac{2F\{[2(1+\nu)a^2+b^2](a^2-x^2)-(1-2\nu)y^2\}}{\pi(1+\nu)a^3b(3a^2+b^2)} \tag{15}$$

$$\sigma_{zy} = -\frac{\partial\phi}{\partial x} = -\frac{4F\{(1+\nu)a^2+\nu b^2\}xy}{\pi(1+\nu)a^3b(3a^2+b^2)} \,. \tag{16}$$

As a check on these results, we note that they reduce to (17.29) if we interchange x and y and write $b = a$.

17.3. A bar of equilateral triangular cross-section defined by the boundaries

$$y = \sqrt{3}x \ ; \ y = -\sqrt{3}x \ ; \ y = \frac{\sqrt{3}a}{2}$$

is loaded by a shear force $F_y = F$ acting along the negative y-axis. Find expressions for the shear stresses σ_{zx}, σ_{zy} on the cross-section for the special case where $\nu = 0.5$. Note: You will need to move the origin to the centroid of the section $(0, a/\sqrt{3})$, since equation (17.3) is based on this choice of origin.

With the change in origin, the equations of the boundaries of the triangle are

$$y = \sqrt{3}\left(x - \frac{a}{3}\right) \ ; \ y = -\sqrt{3}\left(x + \frac{a}{3}\right) \ ; \ y = \frac{a}{2\sqrt{3}} \ . \tag{1}$$

The centroidal second moment of area of the triangle (about any axis) is

$$I = \frac{1}{36} \times a \times \left(\frac{\sqrt{3}a}{2}\right)^3 = \frac{a^4}{32\sqrt{3}} \ , \tag{2}$$

so

$$A = 0 \ ; \quad B = \frac{32\sqrt{3}F}{a^4} \ , \tag{3}$$

from equation (17.6). It follows that the stress function ϕ must satisfy the equation

$$\nabla^2\phi = -\frac{32Fx}{\sqrt{3}a^4} \ , \tag{4}$$

from (17.10), where we have set $\nu = 0.5$.

For the boundary value of ϕ, we note that

$$d\phi = -\frac{16\sqrt{3}Fy^2dx}{a^4} \ , \tag{5}$$

from (17.13). On the edge $y = \sqrt{3}(x - a/3)$, we have

$$\phi = -\frac{16\sqrt{3}F}{a^4}\int\left(3x^2 - 2ax + \frac{a^2}{3}\right)dx = -\frac{16\sqrt{3}F}{a^4}\left(x^3 - ax^2 + \frac{a^2x}{3}\right) \ , \tag{6}$$

where we have dropped the constant of integration, since ϕ is not determined to within an arbitrary constant. Also, the symmetry of the problem suggests that ϕ will be an odd function of x, so it is reasonable to assume that $\phi = 0$ on $x = 0$.

On the edge $y = -\sqrt{3}(x + a/3)$, we have

$$\phi = -\frac{16\sqrt{3}F}{a^4}\int\left(3x^2 + 2ax + \frac{a^2}{3}\right)dx = -\frac{16\sqrt{3}F}{a^4}\left(x^3 + ax^2 + \frac{a^2x}{3}\right) \ . \tag{7}$$

Finally, on the edge $y = a/2\sqrt{3}$, we have

$$\phi = -\frac{4F}{\sqrt{3}a^2}\int dx = -\frac{4Fx}{\sqrt{3}a^2} \ . \tag{8}$$

It is easily verified that (for example) equations (6, 8) both give

$$\phi = -\frac{2F}{\sqrt{3}a} \ , \tag{9}$$

when $x = a/2$, where the corresponding lines intersect, thus ensuring continuity of ϕ around the boundary. This continuity is of course a consequence of the choice of origin, as proved in §17.3.1.

In view of equations (4, 6–8), we use as trial function a polynomial of third degree and below that is odd in x, the most general such form being

$$\phi = C_1x^3 + C_2xy^2 + C_3xy + C_4x \ . \tag{10}$$

The governing equation (4) then requires that

$$6C_1 + 2C_2 = -\frac{32F}{\sqrt{3}a^4} \ , \tag{11}$$

where we have cancelled the common factor x, and the boundary conditions require

$$\begin{aligned}
C_1x^3+C_2xy^2+C_3xy+C_4x &= -\frac{16\sqrt{3}F}{a^4}\left(x^3-ax^2+\frac{a^2x}{3}\right) \ ; \quad y=\sqrt{3}\left(x-\frac{a}{3}\right) \\
C_1x^3+C_2xy^2+C_3xy+C_4x &= -\frac{16\sqrt{3}F}{a^4}\left(x^3+ax^2+\frac{a^2x}{3}\right) \ ; \quad y=-\sqrt{3}\left(x+\frac{a}{3}\right) \\
C_1x^3+C_2xy^2+C_3xy+C_4x &= -\frac{4Fx}{\sqrt{3}a^2} \ ; \quad y = \frac{a}{2\sqrt{3}} \ .
\end{aligned} \tag{12}$$

Using the equations of the boundary lines to eliminate y in the boundary conditions, we obtain

$$\begin{aligned}
C_1x^3 + 3C_2x\left(x-\frac{a}{3}\right)^2 + \sqrt{3}C_3x\left(x-\frac{a}{3}\right) + C_4x &= -\frac{48\sqrt{3}F}{a^4}\left(\frac{x^3}{3}-\frac{ax^2}{3}+\frac{a^2x}{9}\right) \\
C_1x^3 + 3C_2x\left(x+\frac{a}{3}\right)^2 - \sqrt{3}C_3x\left(x+\frac{a}{3}\right) + C_4x &= -\frac{48\sqrt{3}F}{a^4}\left(\frac{x^3}{3}+\frac{ax^2}{3}+\frac{a^2x}{9}\right) \\
C_1x^3 + \frac{C_2xa^2}{12} + \frac{C_3xa}{2\sqrt{3}} + C_4x &= -\frac{4Fx}{\sqrt{3}a^2} \ .
\end{aligned} \tag{13}$$

Finally, expanding these expressions and equating coefficients of powers of x, we obtain

$$C_1 + 3C_2 = -\frac{16\sqrt{3}F}{a^4} \tag{14}$$

$$
\begin{aligned}
-2C_2a + \sqrt{3}C_3 &= \frac{16\sqrt{3}F}{a^3} && (15)\\
\frac{C_2a^2}{3} - \frac{C_3a}{\sqrt{3}} + C_4 &= -\frac{16F}{\sqrt{3}a^2} && (16)\\
C_1 &= 0 && (17)\\
\frac{C_2a^2}{12} + \frac{C_3a}{2\sqrt{3}} + C_4 &= -\frac{4F}{\sqrt{3}a^2}\ . && (18)
\end{aligned}
$$

These equations and (11) have the solution

$$
C_1 = 0\ ;\ \ C_2 = -\frac{16F}{\sqrt{3}a^4}\ ;\ \ C_3 = \frac{16F}{3a^3}\ ;\ \ C_4 = -\frac{16F}{3\sqrt{3}a^2}\ . \tag{19}
$$

The stress components are then recovered from equations (17.8) as

$$
\begin{aligned}
\sigma_{zx} &= 2C_2xy + C_3x\\
&= \frac{16Fx}{\sqrt{3}a^4}\left(\frac{a}{\sqrt{3}} - 2y\right) && (20)\\
\sigma_{zy} &= C_2y^2 - C_3y - C_4 - \frac{16\sqrt{3}Fy^2}{a^4}\\
&= \frac{16F}{\sqrt{3}a^4}\left(\frac{a}{\sqrt{3}} - 2y\right)\left(\frac{a}{\sqrt{3}} + y\right)\ . && (21)
\end{aligned}
$$

We note that σ_{zy} is independent of x and σ_{zx} is a linear function of x. It follows that for a given value of y, the maximum resultant shear stress must occur at the maximum value of x and hence on the edges $y = -a/\sqrt{3} \pm \sqrt{3}x$. On these edge, we have

$$
\tau = \sqrt{\sigma_{zx}^2 + \sigma_{zy}^2} = \frac{16F}{a^4}\left(\frac{a}{\sqrt{3}} - 2y\right)\left(\frac{a}{\sqrt{3}} + y\right)\ . \tag{22}
$$

This equation can be obtained more easily by remarking that at the edge the shear stress must be parallel to the edge. Thus for example,

$$
\tau\cos\left(\frac{\pi}{6}\right) = \sigma_{zy}\ , \tag{23}
$$

leading to the same result. The maximum shear stress occurs at $y = -a/4\sqrt{3}$, which is the mid-point of the two vertical edges, and is

$$
\tau_{\max} = \frac{6F}{a^2}\ . \tag{24}
$$

Notice that equations (11, 14–18) define six equations for four unknowns. The fact that these equations have a solution depends on the triangle being equilateral and also on the assumption $\nu = 0.5$. Timoshenko has shown that a closed form polynomial

solution can be obtained for a more general isosceles triangle if the semi-angle α in the triangle satisfies the relation

$$\tan^2 \alpha = \frac{\nu}{(1+\nu)} \ .$$

The equilateral case treated here is a special case of this result.

17.4. A bar of equilateral triangular cross-section defined by the boundaries

$$y = \sqrt{3}x \ ; \ y = -\sqrt{3}x \ ; \ y = \frac{\sqrt{3}a}{2}$$

is loaded by a shear force $F_x = F$. Find expressions for the shear stresses σ_{zx}, σ_{zy} on the cross-section for the special case where $\nu = 0.5$, assuming the constant $C = 0$ in equation (17.10). You will need to move the origin to the centroid of the section $(0, a/\sqrt{3})$, since equation (17.3) is based on this choice of origin. Show that the resultant force $\boldsymbol{F}$ acts through the centroid.

With the change in origin, the equations of the boundaries of the triangle are

$$y = \sqrt{3}\left(x - \frac{a}{3}\right) \ ; \ y = -\sqrt{3}\left(x + \frac{a}{3}\right) \ ; \ y = \frac{a}{2\sqrt{3}} \ . \tag{1}$$

The centroidal second moment of area of the triangle (about any axis) is

$$I = \frac{1}{36} \times a \times \left(\frac{\sqrt{3}a}{2}\right)^3 = \frac{a^4}{32\sqrt{3}} \ , \tag{2}$$

so

$$A = \frac{32\sqrt{3}F}{a^4} \ ; \quad B = 0 \ , \tag{3}$$

from equation (17.6). It follows that the stress function ϕ must satisfy the equation

$$\nabla^2\phi = \frac{32Fy}{\sqrt{3}a^4} \ , \tag{4}$$

from (17.10), where we have set $\nu = 0.5$.

For the boundary value of ϕ, we note that

$$d\phi = \frac{16\sqrt{3}Fx^2dy}{a^4} \ , \tag{5}$$

from (17.13). On the edge $y = \sqrt{3}(x - a/3)$, we have

$$x = \frac{a + \sqrt{3}y}{3} \tag{6}$$

and

$$\phi = \frac{16F}{3\sqrt{3}a^4}\int(a^2 + 2\sqrt{3}ay + 3y^2)dy = \frac{16F}{3\sqrt{3}a^4}(a^2y + \sqrt{3}ay^2 + y^3) \ , \tag{7}$$

where we have set the constant of integration to zero, since ϕ is not determined to within an arbitrary constant.

On the edge $y = -\sqrt{3}(x + a/3)$,

$$x = -\frac{a + \sqrt{3}y}{3} \tag{8}$$

and the same result is obtained for ϕ.

Finally, on the edge $y = a/2\sqrt{3}$, we have $d\phi = 0$ from (5), so ϕ is constant along this line. Continuity at the corners, shows that this constant value must be

$$\phi = \frac{38F}{27a} \ . \tag{9}$$

In view of equations (4, 7), we use as trial function a polynomial of third degree and below that is even in x, the most general such form being

$$\phi = C_1 y^3 + C_2 x^2 y + C_3 y^2 + C_4 x^2 + C_5 y + C_6 \ . \tag{10}$$

The governing equation (4) then requires that

$$6C_1 y + 2C_2 y + 2C_3 + 2C_4 = \frac{32Fy}{\sqrt{3}a^4} \ , \tag{11}$$

and hence

$$C_3 = -C_4 \tag{12}$$

and

$$6C_1 + 2C_2 = \frac{32F}{\sqrt{3}a^4} \tag{13}$$

The boundary conditions require

$$\begin{aligned} C_1 y^3 + C_2 x^2 y + C_4(x^2 - y^2) + C_5 y + C_6 &= \frac{16F}{3\sqrt{3}a^4}(a^2 y + \sqrt{3}ay^2 + y^3) \ ; \quad x = \pm\frac{a + \sqrt{3}y}{3} \\ C_1 y^3 + C_2 x^2 y + C_4(x^2 - y^2) + C_5 y + C_6 &= \frac{38F}{27a} \ ; \quad y = \frac{a}{2\sqrt{3}} \ . \end{aligned} \tag{14}$$

Using the equations of the boundary lines to eliminate x in the the first equation and y in the second, we obtain

$$\begin{aligned} C_1 y^3 + C_2 \left(\frac{(a + \sqrt{3}y)^2}{9}\right) y + C_4 \left(\frac{(a + \sqrt{3}y)^2}{9}\right) - C_4 y^2 + C_5 y + C_6 \\ = \frac{16F}{3\sqrt{3}a^4}(a^2 y + \sqrt{3}ay^2 + y^3) \end{aligned}$$

$$\frac{C_1 a^3}{24\sqrt{3}} + \frac{C_2 a x^2}{2\sqrt{3}} + C_4 x^2 - \frac{C_4 a^2}{12} + \frac{C_5 a}{2\sqrt{3}} + C_6 = \frac{38F}{27a} \ . \tag{15}$$

Equating coefficients of y, x respectively in these two equations, we then obtain

$$
\begin{aligned}
C_1 + \frac{C_2}{3} &= \frac{16F}{3\sqrt{3}a^4} && (16)\\
\frac{2C_2 a}{3\sqrt{3}} + \frac{C_4}{3} - C_4 &= \frac{16F}{3a^3} && (17)\\
\frac{C_2 a^2}{9} + \frac{2C_4 a}{3\sqrt{3}} + C_5 &= \frac{16F}{3\sqrt{3}a^2} && (18)\\
\frac{C_4 a^2}{9} + C_6 &= 0 && (19)\\
\frac{C_2 a}{2\sqrt{3}} + C_4 &= 0 && (20)\\
\frac{C_1 a^3}{24\sqrt{3}} - \frac{C_4 a^2}{12} + \frac{C_5 a}{2\sqrt{3}} + C_6 &= \frac{38F}{27a} \ . && (21)
\end{aligned}
$$

The solution of these equations and (12,13) is

$$
C_1 = 0 \ ; \ \ C_2 = \frac{16F}{\sqrt{3}a^4} \ ; \ \ C_3 = -C_4 = \frac{8F}{3a^3} \ ; \ \ C_5 = \frac{16F}{3\sqrt{3}a^2} \ ; \ \ C_6 = \frac{8F}{27a} \ . \qquad (22)
$$

The stress components are then recovered from equations (17.8) and (10) as

$$
\begin{aligned}
\sigma_{zx} &= 6C_1 y^2 + C_2 x^2 + 2C_3 y + C_5 - \frac{16\sqrt{3}Fx^2}{a^4} \\
&= \frac{16F}{3a^3}\left(y + \frac{a}{\sqrt{3}}\right) - \frac{32Fx^2}{\sqrt{3}a^4} && (23)\\
\sigma_{zy} &= -2C_2 xy - 2C_4 x \\
&= \frac{16Fx}{\sqrt{3}a^4}\left(\frac{a}{\sqrt{3}} - 2y\right) \ . && (24)
\end{aligned}
$$

The maximum stress occurs at the mid-point of the top edge $y = a/2\sqrt{3}$ and is

$$
\tau_{\max} = \frac{8F}{\sqrt{3}a^2} \ . \qquad (25)
$$

The resultant force F will act along a line a distance d from the x-axis, where

$$
Fd = \int_{-a/\sqrt{3}}^{a/2\sqrt{3}} \int_{-(a+\sqrt{3}y)/3}^{(a+\sqrt{3}y)/3} (y\sigma_{zx} - x\sigma_{zy}) dx dy \ . \qquad (26)
$$

Substituting for the stresses from equations (23,24) and evaluating the integral, we find that $d = 0$ and hence that the force acts through the origin, which is also the centroid of the section.

17.5. Solve the problem of the rectangular bar of §17.4.2 for the case where $a > b$. You will need to satisfy the boundary conditions in the strong sense on $y = \pm b$ and in the weak sense on $x = \pm a$. Compare the solution you get at this stage with the predictions of the elementary Mechanics of Materials theory. Then superpose an appropriate series of harmonic functions (sinusoidal in y and hyperbolic in x) to obtain an exact solution. In what range of the ratio a/b is it necessary to include the series to obtain 1% accuracy in the maximum shear stress?

As in §17.4.2, we start with the stress function

$$\phi = C_1 x^3 + C_2 x y^2 + C_3 x \ , \tag{1}$$

but we choose to satisfy the first of conditions (17.34) in the strong sense, giving

$$3C_1 x^2 + C_2 b^2 + C_3 = -\frac{Bb^2}{2} = -\frac{3F}{8ab} \ , \tag{2}$$

from (17.33). This equation will be satisfied for all x provided that

$$C_1 = 0 \ ; \quad C_2 b^2 + C_3 = -\frac{3F}{8ab} \ . \tag{3}$$

The governing equation (17.36) then gives

$$C_2 = -\frac{3F\nu}{8ab^3(1+\nu)} \tag{4}$$

and hence

$$C_3 = -\frac{3F}{8ab} - C_2 b^2 = -\frac{3F}{8ab(1+\nu)} \ . \tag{5}$$

We conclude that

$$\phi = -\frac{3F(\nu x y^2 + b^2 x)}{8ab^3(1+\nu)} \ . \tag{6}$$

Since ϕ is an even function of y, the weak form of the second of (17.34) is satisfied by symmetry.

The stresses implied by this approximate solution are

$$\sigma_{zx} = \frac{\partial \phi}{\partial y} = -\frac{3F\nu x y}{4ab^3(1+\nu)} \tag{7}$$

$$\sigma_{zy} = -\frac{\partial \phi}{\partial x} - \frac{By^2}{2} = \frac{3F(\nu y^2 + b^2)}{8ab^3(1+\nu)} - \frac{3Fy^2}{8ab^3} \tag{8}$$

$$= \frac{3F(b^2 - y^2)}{8ab^3(1+\nu)} \ . \tag{9}$$

This solution satisfies the traction-free condition on $y = \pm b$ exactly as we should expect, but it is rather surprising to see that it differs from the Mechanics of Materials solution by a factor of $(1+\nu)$ in the denominator, implying that the resultant of the tractions is a force $F/(1+\nu)$ rather than F. The explanation of this paradox is that the approximation on the short edges $x = \pm a$ constitutes a moment per unit length, which is statically equivalent to a shear force at the ends of the edges, as we know from Kirchhoff's theory of plates. These unwanted tractions will of course be removed by the series terms.

To complete the solution, we superpose a series of harmonic functions ϕ_n chosen to satisfy the condition $\phi_n = 0$ on $y = \pm b$. In this way, the governing equation and the boundary conditions on $y = \pm b$ are unaffected and the remaining constants can be chosen to satisfy the boundary conditions on $x = \pm a$ in the strong sense. Here we need even functions of y and odd functions of x satisfying $\phi_n(\pm b, x) = 0$, leading to the solution

$$\phi = -\frac{3F(\nu x y^2 + b^2 x)}{8ab^3(1+\nu)} + \sum_{n=1}^{\infty} D_n \cos\left(\frac{(2n-1)\pi y}{2b}\right) \sinh\left(\frac{(2n-1)\pi x}{2b}\right) , \tag{10}$$

where D_n is a set of arbitrary constants. This function satisfies the second of (17.34) in the strong sense if

$$\sum_{n=1}^{\infty} \frac{(2n-1)\pi D_n}{2b} \sin\left(\frac{(2n-1)\pi y}{2b}\right) \sinh\left(\frac{(2n-1)\pi a}{2b}\right) = -\frac{3F\nu y}{4b^3(1+\nu)} , \tag{11}$$

Multiplying both sides of this equation by $\sin((2m-1)\pi x/2b)$ and integrating over the range $-b < x < b$, we then obtain

$$\begin{aligned} \frac{(2m-1)\pi D_m}{2} \sinh\left(\frac{(2m-1)\pi a}{2b}\right) &= -\frac{3F\nu}{4b^3(1+\nu)} \int_{-b}^{b} y \sin\left(\frac{(2m-1)\pi y}{2b}\right) dy \\ &= \frac{6F\nu(-1)^m}{(2m-1)^2\pi^2 b(1+\nu)} \end{aligned} \tag{12}$$

and hence

$$D_m = \frac{12F\nu(-1)^m}{(2m-1)^3\pi^3 b(1+\nu)} \Big/ \sinh\left(\frac{(2m-1)\pi a}{2b}\right) . \tag{13}$$

The stress components are then given by

$$\begin{aligned} \sigma_{zx} &= \frac{\partial \phi}{\partial y} \\ &= -\frac{3F\nu x y}{4ab^3(1+\nu)} - \sum_{n=1}^{\infty} \frac{(2n-1)\pi D_n}{2b} \sin\left(\frac{(2n-1)\pi y}{2b}\right) \sinh\left(\frac{(2n-1)\pi x}{2b}\right) \end{aligned} \tag{14}$$

$$\begin{aligned} \sigma_{zy} &= -\frac{\partial \phi}{\partial x} - \frac{3Fy^2}{8ab^3} \\ &= \frac{3F(b^2-y^2)}{8ab^3(1+\nu)} + \sum_{n=1}^{\infty} \frac{(2n-1)\pi D_n}{2b} \cos\left(\frac{(2n-1)\pi y}{2b}\right) \cosh\left(\frac{(2n-1)\pi x}{2b}\right) . \end{aligned} \tag{15}$$

From §17.4.2, we anticipate that the maximum stress will occur at $(0, \pm a)$, where

$$\begin{aligned}\tau &= \frac{3F}{8ab(1+\nu)} + \frac{6F\nu}{\pi^2 b^2(1+\nu)} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \coth\left(\frac{(2n-1)\pi a}{2b}\right) \\ &= \frac{3F}{8ab}\left\{\frac{1}{(1+\nu)} + \frac{16F\nu a}{\pi^2 b(1+\nu)} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \coth\left(\frac{(2n-1)\pi a}{2b}\right)\right\} , \qquad (16)\end{aligned}$$

where we have taken out a factor of $3F/ab$ to expose the comparison with the Mechanics of Materials solution. It is easily shown that in contrast to the case $a < b$, it is always necessary to include some terms of the series to achieve 1% accuracy. In fact, for $a > b$, the stress distribution is significantly non-uniform along the line $y = 0$ with a maximum at $x = \pm a$. For example, for $a/b = 10$ and $\nu = 0.3$, the elementary theory underestimates the maximum stress by a factor of 4, whereas for $a/b = 30$ and $\nu = 0.3$, the underestimate is by a factor of 11. In fact there is a localized region of very non-uniform stress near the edges of the plate.

For $a/b \gg 1$ and $\nu \neq 0$, a larger maximum is obtained at points on the upper and lower surfaces near the corner of the rectangle. An upper limit to this maximum is provided by the value of σ_{zx} in the approximate solution (7) at the corner (a, b) — i.e.

$$\tau_{\max} < \frac{3F\nu}{4b^2(1+\nu)} . \qquad (17)$$

This problem can also be solved using the solution of equation (17.48), but the series converges more slowly than (16) for $a/b > 4$.

17.6. Find the distribution of shear stress in a *hollow* cylindrical bar of inner radius b and outer radius a loaded by a shear force $F_y = F$ acting through the origin. You will need to supplement the polynomial solution by the singular harmonic functions $x/(x^2+y^2)$ and $(x^3-3xy^2)/(x^2+y^2)^3$ with appropriate multipliers[1].

For the hollow cylindrical bar, we have

$$I_x = I_y = \frac{\pi(a^4 - b^4)}{4} \tag{1}$$

and hence

$$A = 0\ ; \quad B = \frac{4F}{\pi(a^4 - b^4)}\ . \tag{2}$$

On each of the circular boundaries, we require

$$\phi = \int \frac{2Fy^2 dx}{\pi(a^4 - b^4)}\ , \tag{3}$$

from (17.13). On the outer boundary, $y^2 = a^2 - x^2$ and hence

$$\phi = \int \frac{2F(a^2 - x^2)dx}{\pi(a^4 - b^4)} = -\frac{2F(3a^2x - x^3)}{3\pi(a^4 - b^4)} \quad \text{on } x^2 + y^2 = a^2\ , \tag{4}$$

where we have set the constant of integration to zero to preserve antisymmetry about $x=0$. On the inner boundary, $y^2 = b^2 - x^2$ and a similar procedure yields

$$\phi = \int \frac{2F(b^2 - x^2)dx}{\pi(a^4 - b^4)} = -\frac{2F(3b^2x - x^3)}{3\pi(a^4 - b^4)} \quad \text{on } x^2 + y^2 = b^2\ . \tag{5}$$

We assume a stress function of the form

$$\phi = C_1x^3 + C_2xy^2 + C_3x + \frac{C_4x}{(x^2 + y^2)} + \frac{C_5(x^3 - 3xy^2)}{(x^2 + y^2)^3}\ . \tag{6}$$

The governing equation (17.10) then requires that

$$6C_1x + 2C_2x = -\frac{4F\nu x}{\pi(a^4 - b^4)(1 + \nu)} \tag{7}$$

and the boundary conditions (4,5) require

$$\begin{aligned} C_1x^3+C_2xy^2+C_3x+\frac{C_4x}{(x^2+y^2)}+\frac{C_5(x^3-3xy^2)}{(x^2+y^2)^3} &= -\frac{2F(3a^2x-x^3)}{3\pi(a^4-b^4)}\ ;\ x^2+y^2=a^2 \\ C_1x^3+C_2xy^2+C_3x+\frac{C_4x}{(x^2+y^2)}+\frac{C_5(x^3-3xy^2)}{(x^2+y^2)^3} &= -\frac{2F(3b^2x-x^3)}{3\pi(a^4-b^4)}\ ;\ x^2+y^2=b^2\ . \end{aligned}$$

[1]These functions are Cartesian versions of the functions $r^{-1}\cos\theta$ and $r^{-3}\cos 3\theta$ respectively.

In other words,

$$C_1x^3+C_2x(a^2-x^2)+C_3x+\frac{C_4x}{a^2}+\frac{C_5(x^3-3x(a^2-x^2))}{a^6} = -\frac{2F(3a^2x-x^3)}{3\pi(a^4-b^4)} \quad (8)$$

$$C_1x^3+C_2x(b^2-x^2)+C_3x+\frac{C_4x}{b^2}+\frac{C_5(x^3-3x(b^2-x^2))}{b^6} = -\frac{2F(3b^2x-x^3)}{3\pi(a^4-b^4)} \ . \quad (9)$$

Equating coefficients of x, x^3 in equations (7,8,9), we obtain the five equations

$$6C_1 + 2C_2 = -\frac{4F\nu}{\pi(a^4-b^4)(1+\nu)} \quad (10)$$

$$C_1 - C_2 - \frac{2C_5}{a^6} = \frac{2F}{3\pi(a^4-b^4)} \quad (11)$$

$$C_2a^2 + C_3 + \frac{C_4}{a^2} - \frac{3C_5}{a^4} = -\frac{6Fa^2}{3\pi(a^4-b^4)} \quad (12)$$

$$C_1 - C_2 - \frac{2C_5}{b^6} = \frac{2F}{3\pi(a^4-b^4)} \quad (13)$$

$$C_2b^2 + C_3 + \frac{C_4}{b^2} - \frac{3C_5}{b^4} = -\frac{6Fb^2}{3\pi(a^4-b^4)} \quad (14)$$

for the constants $C_1, ...C_5$, with solution

$$C_1 = \frac{F(2-\nu)}{6\pi(a^4-b^4)(1+\nu)} \ ; \ \ C_2 = -\frac{F(2+3\nu)}{2\pi(a^4-b^4)(1+\nu)}$$

$$C_3 = -\frac{F(6+5\nu)}{2\pi(a^2-b^2)(1+\nu)} \ ; \ \ C_4 = \frac{Fa^2b^2(6+5\nu)}{2\pi(a^4-b^4)(1+\nu)} \ ; \ \ C_5 = 0 \ . \quad (15)$$

The stress components are then obtained from (17.8) and (2, 6, 15) as

$$\sigma_{zx} = -\frac{Fxy}{\pi(a^4-b^4)(1+\nu)}\left((2+3\nu) + \frac{a^2b^2(6+5\nu)}{(x^2+y^2)^2}\right) \quad (16)$$

$$\sigma_{zy} = \frac{F}{2\pi(a^4-b^4)(1+\nu)}\left((6+5\nu)(a^2+b^2) - (2-\nu)x^2 - (2+\nu)y^2 + \frac{(6+5\nu)a^2b^2(x^2-y^2)}{(x^2+y^2)^2}\right) \ . \quad (17)$$

17.7. A bar has the cross-section of a sector $0<r<a, -\alpha<\theta<\alpha$, where α is not small compared with unity. The bar is loaded by a shear force F in the direction $\theta=0$.

(i) Find a closed form solution for the shear stresses in the bar, using the strong boundary conditions on the straight edges $\theta=\pm\alpha$ and a weak boundary condition on the curved edge $r=a$.

(ii) Superpose a series of harmonic functions ϕ_n of the form $r^\gamma \sin(\gamma\theta)$ where γ is chosen to satisfy $\phi_n=0$ on $\theta=\pm\alpha$.

(iii) Choose the coefficients on these terms so as to satisfy the boundary condition on the curved edge $r=a$ and hence obtain an expression for the maximum shear stress.

We first need to find the location of the centroid of the section and the second moment of area I_y. The area of the section A is

$$A = \int_0^a \int_{-\alpha}^{\alpha} r d\theta dr = \alpha a^2 \tag{1}$$

and the centroid is located at the point $(\bar{x}, 0)$, where

$$A\bar{x} = \int_0^a \int_{-\alpha}^{\alpha} r x d\theta dr = \int_0^a \int_{-\alpha}^{\alpha} r^2 \cos\theta d\theta dr = \frac{2a^3 \sin\alpha}{3} . \tag{2}$$

Thus

$$\bar{x} = \frac{2a \sin\alpha}{3\alpha} . \tag{3}$$

The second moment of area about the y-axis through the origin is

$$I = \int_0^a \int_{-\alpha}^{\alpha} r x^2 d\theta = \int_0^a \int_{-\alpha}^{\alpha} r^3 \cos^2\theta d\theta = \frac{a^4(\alpha + \sin\alpha\cos\alpha)}{4} \tag{4}$$

and hence the centroidal second moment is

$$I_y = I - A\bar{x}^2 = \frac{a^4(\alpha + \sin\alpha\cos\alpha)}{4} - \frac{4a^4 \sin^2\alpha}{9\alpha} . \tag{5}$$

The boundary conditions in this problem are most naturally stated in terms of polar coordinates. Transforming coordinates as in §8.1, we obtain

$$\sigma_{zr} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} - \frac{A(r\cos\theta - \bar{x})^2 \cos\theta}{2} - \frac{Br^2 \sin^3\theta}{2} \tag{6}$$

$$\sigma_{z\theta} = -\frac{\partial\phi}{\partial r} + \frac{A(r\cos\theta - \bar{x})^2 \sin\theta}{2} - \frac{Br^2 \sin^2\theta \cos\theta}{2} , \tag{7}$$

where we have replaced x by $r\cos\theta - \bar{x}$, since equations (17.8) depend on the choice of the centroid of the section as the origin of coordinates. The same change of origin requires us to replace (17.10) by

$$\nabla^2\phi = \frac{\nu}{(1+\nu)}\{Ar\sin\theta - B(r\cos\theta - \bar{x})\} \ . \tag{8}$$

In the present problem,

$$A = \frac{F}{I_y} \ ; \quad B = 0 \tag{9}$$

and hence

$$\nabla^2\phi = \frac{F\nu r\sin\theta}{I_y(1+\nu)} \tag{10}$$

$$\sigma_{zr} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} - \frac{F(r\cos\theta - \bar{x})^2\cos\theta}{2I_y} \ ; \quad \sigma_{z\theta} = -\frac{\partial\phi}{\partial r} + \frac{F(r\cos\theta - \bar{x})^2\sin\theta}{2I_y} \ . \tag{11}$$

To satisfy the strong boundary conditions on $\theta = \pm\alpha$, we require

$$\sigma_{z\theta} = 0 \ ; \quad \theta = \pm\alpha \tag{12}$$

and hence

$$\frac{\partial\phi}{\partial r} = \frac{F(r\cos\alpha - \bar{x})^2\sin\alpha}{2I_y} \ ; \quad \theta = \pm\alpha \ . \tag{13}$$

In view of (10,13), we require a solution for ϕ that is odd in θ and that contains powers of r up to r^3. A particular solution of (10) satisfying these conditions is

$$\phi_P = Cr^3\sin\theta \ , \tag{14}$$

where C is a constant to be determined. Substituting into (10), we have

$$8Cr\sin\theta = \frac{F\nu r\sin\theta}{I_y(1+\nu)} \tag{15}$$

and hence

$$C = \frac{F\nu}{8I_y(1+\nu)} \ . \tag{16}$$

A more general solution can then be obtained by superposing appropriate harmonic functions as

$$\phi = \frac{F\nu r^3\sin\theta}{8I_y(1+\nu)} + C_1 r\sin\theta + C_2 r^2\sin\theta\cos\theta + C_3 r^3(4\sin^3\theta - 3\sin\theta) \ , \tag{17}$$

where the additional terms are all of the form $r^n\sin(n\theta)$ with $n = 1, 2, 3$. The boundary condition (13) then requires that

$$\frac{3F\nu r^2\sin\alpha}{8I_y(1+\nu)} + C_1\sin\alpha + 2C_2 r\sin\alpha\cos\alpha + 3C_3 r^2(4\sin^3\alpha - 3\sin\alpha) = \frac{F(r\cos\alpha - \bar{x})^2\sin\alpha}{2I_y} \tag{18}$$

and hence, equating powers of r,

$$\frac{3F\nu\sin\alpha}{8I_y(1+\nu)} + 3C_3(4\sin^3\alpha - 3\sin\alpha) = \frac{F\cos^2\alpha\sin\alpha}{2I_y} \tag{19}$$

$$2C_2\sin\alpha\cos\alpha = -\frac{F\bar{x}\cos\alpha\sin\alpha}{I_y} \tag{20}$$

$$C_1\sin\alpha = \frac{F\bar{x}^2\sin\alpha}{2I_y} \ . \tag{21}$$

Thus

$$C_1 = \frac{F\bar{x}^2}{2I_y} \ ; \ \ C_2 = -\frac{F\bar{x}}{2I_y} \ ; \ \ C_3 = \frac{F\{4(1+\nu)\cos^2\alpha - 3\nu\}}{24I_y(1+\nu)(4\sin^2\alpha - 3)} \ . \tag{22}$$

The weak condition on $r = a$ is satisfied by equilibrium considerations, since it is the only remaining boundary.

For an exact solution, we now superpose the series terms

$$\phi_S = \sum_{n=1}^{\infty} D_n r^{\gamma_n}\sin(\gamma_n\theta) \ . \tag{23}$$

These additional terms will not contribute additional stresses on the boundaries $\theta = \pm\alpha$ as long as

$$\sin(\gamma_n\alpha) = 0 \tag{24}$$

and hence

$$\gamma_n = \frac{n\pi}{\alpha} \tag{25}$$

The complete stress function is then

$$\begin{aligned}\phi \ = \ & \frac{F\nu r^3\sin\theta}{8I_y(1+\nu)} + \frac{F\bar{x}^2 r\sin\theta}{2I_y} - \frac{F\bar{x}r^2\sin\theta\cos\theta}{2I_y} \\ & + \frac{F\{4(1+\nu)\cos^2\alpha - 3\nu\}r^3(4\sin^3\theta - 3\sin\theta)}{24I_y(1+\nu)(4\sin^2\alpha - 3)} + \sum_{n=1}^{\infty} D_n r^{n\pi/\alpha}\sin\left(\frac{n\pi\theta}{\alpha}\right)\end{aligned} \tag{26}$$

and the constants D_n must be chosen so as to satisfy the strong boundary condition

$$\sigma_{zr} = 0 \ ; \quad r = a \ . \tag{27}$$

Equations (11, 27, 28) then give

$$\begin{aligned}& \frac{F\nu a^2\cos\theta}{8I_y(1+\nu)} + \frac{F\bar{x}^2\cos\theta}{2I_y} - \frac{F\bar{x}a(\cos^2\theta - \sin^2\theta)}{2I_y} \\ & + \frac{F\{4(1+\nu)\cos^2\alpha - 3\nu\}a^2(4\sin^2\theta\cos\theta - \cos\theta)}{8I_y(1+\nu)(4\sin^2\alpha - 3)} \\ & + \sum_{n=1}^{\infty}\frac{n\pi D_n}{\alpha}a^{(n\pi/\alpha - 1)}\cos\left(\frac{n\pi\theta}{\alpha}\right) - \frac{F(a\cos\theta - \bar{x})^2\cos\theta}{2I_y} = 0\end{aligned} \tag{28}$$

and hence

$$\sum_{n=1}^{\infty} \frac{n\pi D_n}{\alpha} a^{(n\pi/\alpha-1)} \cos\left(\frac{n\pi\theta}{\alpha}\right) = \frac{Fa(a\cos^3\theta - \bar{x})}{2I_y} - \frac{F\nu a^2\cos\theta}{8I_y(1+\nu)} - \frac{F\{4(1+\nu)\cos^2\alpha - 3\nu\}a^2(3\cos\theta - 4\cos^3\theta)}{8I_y(1+\nu)(4\sin^2\alpha - 3)} . \tag{29}$$

Multiplying both sides of this equation by $\cos(m\pi\theta/\alpha)$ and integrating in the range $-\alpha < \theta < \alpha$, we obtain

$$\begin{aligned} m\pi D_m a^{(m\pi/\alpha-1)} &= \frac{6(-1)^m F a^2\alpha^2 \sin\alpha\{(m^2\pi^2 - \alpha^2)\cos^2\alpha - 2\alpha^2\}}{2I_y(m^2\pi^2 - \alpha^2)(m^2\pi^2 - 9\alpha^2)} \\ &\quad - \frac{F(-1)^m \nu a^2\alpha^2 \sin\alpha}{2I_y(1+\nu)(m^2\pi^2 - \alpha^2)} \\ &\quad - \frac{3(-1)^m F a^2\alpha^2 \sin\alpha\{4(1+\nu)\cos^2\alpha - 3\nu\}}{4I_y(1+\nu)(m^2\pi^2 - 9\alpha^2)} \\ &= \frac{(-1)^m F a^2\alpha^2 \sin\alpha\{7\nu m^2\pi^2 - 3(8+5\nu)\alpha^2\}}{4I_y(1+\nu)(m^2\pi^2 - \alpha^2)(m^2\pi^2 - 9\alpha^2)} \end{aligned} \tag{30}$$

and hence

$$D_m = \frac{(-1)^m F a^{(3-m\pi/\alpha)}\alpha^2 \sin\alpha\{7\nu m^2\pi^2 - 3(8+5\nu)\alpha^2\}}{4I_y(1+\nu)m\pi(m^2\pi^2 - \alpha^2)(m^2\pi^2 - 9\alpha^2)} \tag{31}$$

The final solution for the stresses is obtained by substituting for D_n in equation (26) and then substituting the resulting expression into (11).

17.8. Solve Problem 17.7 for the case where the force F acts in the direction $\theta = \pi/2$. Assume that the line of action of F is such as to make the constant C in (17.10) be zero.

We first need to find the location of the centroid of the section and the second moment of area I_y. The area of the section A is

$$A = \int_0^a \int_{-\alpha}^{\alpha} r d\theta dr = \alpha a^2 \tag{1}$$

and the centroid is located at the point $(\bar{x}, 0)$, where

$$A\bar{x} = \int_0^a \int_{-\alpha}^{\alpha} r x d\theta dr = \int_0^a \int_{-\alpha}^{\alpha} r^2 \cos\theta d\theta dr = \frac{2a^3 \sin\alpha}{3} \ . \tag{2}$$

Thus

$$\bar{x} = \frac{2a \sin\alpha}{3\alpha} \ . \tag{3}$$

We also have

$$I_x = \int_0^a \int_{-\alpha}^{\alpha} r^3 \sin^2\theta d\theta dr = \frac{a^4(\alpha - \sin\alpha\cos\alpha)}{4} \ . \tag{4}$$

The boundary conditions in this problem are most naturally stated in terms of polar coordinates. Transforming coordinates as in §8.1, we obtain

$$\sigma_{zr} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} - \frac{A(r\cos\theta - \bar{x})^2 \cos\theta}{2} - \frac{Br^2 \sin^3\theta}{2} \tag{5}$$

$$\sigma_{z\theta} = -\frac{\partial\phi}{\partial r} + \frac{A(r\cos\theta - \bar{x})^2 \sin\theta}{2} - \frac{Br^2 \sin^2\theta \cos\theta}{2} \ , \tag{6}$$

where we have replaced x by $r\cos\theta - \bar{x}$, since equations (17.8) depend on the choice of the centroid of the section as the origin of coordinates. The same change of origin requires us to replace (17.10) by

$$\nabla^2\phi = \frac{\nu}{(1+\nu)}\{Ar\sin\theta - B(r\cos\theta - \bar{x})\} \ . \tag{7}$$

In the present problem,

$$A = 0 \ ; \quad B = \frac{F}{I_x} \tag{8}$$

and hence

$$\nabla^2\phi = -\frac{F\nu(r\cos\theta - \bar{x})}{(1+\nu)I_x} \tag{9}$$

$$\sigma_{zr} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} - \frac{Fr^2 \sin^3\theta}{2I_x} \ ; \quad \sigma_{z\theta} = -\frac{\partial\phi}{\partial r} - \frac{Fr^2 \sin^2\theta \cos\theta}{2I_x} \ . \tag{10}$$

To satisfy the strong boundary conditions on $\theta = \pm\alpha$, we require

$$\sigma_{z\theta} = 0 \; ; \quad \theta = \pm\alpha \tag{11}$$

and hence

$$\frac{\partial\phi}{\partial r} = -\frac{Fr^2 \sin^2\theta \cos\theta}{2I_x} \; ; \quad \theta = \pm\alpha \; . \tag{12}$$

In view of (9, 12), we require a solution for ϕ that is even in θ and that contains powers of r up to r^3. A particular solution of (9) satisfying these conditions is

$$\phi_P = A_1 r^3 \cos\theta + A_2 r^2 \; , \tag{13}$$

where A_1, A_2 are constants to be determined. Substituting into (9), we have

$$8A_1 r\cos\theta + 4A_2 = -\frac{F\nu(r\cos\theta - \bar{x})}{(1+\nu)I_x} \tag{14}$$

and hence

$$A_1 = -\frac{F\nu}{8I_x(1+\nu)} \; ; \quad A_2 = \frac{F\nu\bar{x}}{4I_x(1+\nu)} \; . \tag{15}$$

A more general solution can then be obtained by superposing appropriate harmonic functions as

$$\phi = \frac{F\nu(2\bar{x}r^2 - r^3\cos\theta)}{8I_x(1+\nu)} + C_1 r\cos\theta + C_2 r^2(2\cos^2\theta - 1) + C_3 r^3(4\cos^3\theta - 3\cos\theta) \; , \tag{16}$$

where the additional terms are all of the form $r^n\cos(n\theta)$ with $n = 1, 2, 3$. The boundary condition (12) then requires that

$$\frac{F\nu(4\bar{x}r - 3r^2\cos\alpha)}{8I_x(1+\nu)} + C_1\cos\alpha + 2C_2 r(2\cos^2\alpha - 1) + 3C_3 r^2(4\cos^3\alpha - 3\cos\alpha) = -\frac{Fr^2\sin^2\alpha\cos\alpha}{2I_x} \tag{17}$$

and hence, equating powers of r,

$$-\frac{3F\nu\cos\alpha}{8I_x(1+\nu)} + 3C_3(4\cos^3\alpha - 3\cos\alpha) = -\frac{F\sin^2\alpha\cos\alpha}{2I_x} \tag{18}$$

$$\frac{F\nu\bar{x}}{2I_x(1+\nu)} + 2C_2(2\cos^2\alpha - 1) = 0 \tag{19}$$

$$C_1\cos\alpha = 0 \tag{20}$$

Thus

$$C_1 = 0 \; ; \quad C_2 = -\frac{F\nu\bar{x}}{4I_x(1+\nu)(2\cos^2\alpha - 1)} \; ; \quad C_3 = \frac{F\{3\nu - 4(1+\nu)\sin^2\alpha\}}{24I_x(1+\nu)(4\cos^2\alpha - 3)} \; . \tag{21}$$

The weak condition on $r = a$ is satisfied by equilibrium considerations, since it is the only remaining boundary.

For an exact solution, we now superpose the series terms

$$\phi_S = \sum_{n=1}^{\infty} D_n r^{\gamma_n} \cos(\gamma_n \theta) \ . \tag{22}$$

These additional terms will not contribute additional stresses on the boundaries $\theta = \pm\alpha$ as long as

$$\cos(\gamma_n \alpha) = 0 \tag{23}$$

and hence

$$\gamma_n = \frac{(2n-1)\pi}{2\alpha} \tag{24}$$

The complete stress function is then

$$\begin{aligned} \phi \ = \ & \frac{F\nu(2\bar{x}r^2 - r^3\cos\theta)}{8I_x(1+\nu)} - \frac{F\nu\bar{x}r^2(2\cos^2\theta - 1)}{4I_x(1+\nu)(2\cos^2\alpha - 1)} \\ & + \frac{F\{3\nu - 4(1+\nu)\sin^2\alpha\}r^3(4\cos^3\theta - 3\cos\theta)}{24I_x(1+\nu)(4\cos^2\alpha - 3)} \\ & + \sum_{n=1}^{\infty} D_n r^{(2n-1)\pi/2\alpha} \cos\left(\frac{(2n-1)\pi\theta}{2\alpha}\right) \end{aligned} \tag{25}$$

and the constants D_n must be chosen so as to satisfy the strong boundary condition

$$\sigma_{zr} = 0 \ ; \quad r = a \ . \tag{26}$$

Equations (10, 25, 26) then give

$$\begin{aligned} & \frac{F\nu a^2 \sin\theta}{8I_x(1+\nu)} + \frac{F\nu\bar{x}a\sin\theta\cos\theta}{I_x(1+\nu)(2\cos^2\alpha - 1)} \\ & - \frac{F\{3\nu - 4(1+\nu)\sin^2\alpha\}a^2(4\sin\theta\cos^2\theta - \sin\theta)}{8I_x(1+\nu)(4\cos^2\alpha - 3)} \\ & - \sum_{n=1}^{\infty} \frac{(2n-1)\pi D_n}{2\alpha} D_n a^{\{(2n-1)\pi/2\alpha - 1\}} \sin\left(\frac{(2n-1)\pi\theta}{2\alpha}\right) - \frac{Fa^2\sin^3\theta}{2I_x} = 0 \end{aligned} \tag{27}$$

and hence, after some simplification,

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(2n-1)\pi D_n a^{\{(2n-1)\pi/2\alpha - 1\}}}{2\alpha} \sin\left(\frac{(2n-1)\pi\theta}{2\alpha}\right) \\ & = \frac{Fa^2\{(2\sin^2\alpha - \nu)\sin\theta + 2(1-2\nu)\sin^3\theta\}}{4I_x(1+\nu)(4\cos^2\alpha - 3)} + \frac{F\nu\bar{x}a\sin\theta\cos\theta}{I_x(1+\nu)(2\cos^2\alpha - 1)} \ . \end{aligned} \tag{28}$$

Multiplying both sides of this equation by $\sin((2m-1)\pi\theta/2\alpha)$ and integrating in the range $-\alpha < \theta < \alpha$, we obtain

$$\frac{(2m-1)\pi D_m a^{\{(2m-1)\pi/2\alpha-1\}}}{2} = \frac{2(-1)^m F a^2 \alpha^2 \cos\alpha(2\sin^2\alpha - \nu)}{I_x(1+\nu)(4\cos^2\alpha - 3)((2m-1)^2\pi^2 - 4\alpha^2)}$$
$$+\frac{12(-1)^m F a^2 \alpha^2 (1-2\nu)\cos\alpha\{4\alpha^2(\cos^2\alpha - 3) - (2m-1)^2\pi^2(\cos^2\alpha - 1)}{I_x(1+\nu)(4\cos^2\alpha - 3)((2m-1)^2\pi^2 - 4\alpha^2)((2m-1)^2\pi^2 - 36\alpha^2)}$$
$$+\frac{8(-1)^m F \nu \bar{x} a \alpha^2}{I_x(1+\nu)((2m-1)^2\pi^2 - 16\alpha^2)} \,. \qquad (29)$$

The final solution for the stresses is obtained by solving for D_m, substituting into equation (25) and then substituting the resulting expression into (10).

CHAPTER 18

18.1. By writing

$$\zeta^n = e^{n\ln(\zeta)} \ ,$$

show that

$$\frac{\partial^m}{\partial n^m}\left(\zeta^n\right) = \zeta^n \ln^m(\zeta) \ .$$

Using equations (18.3), find the real and imaginary parts of the function $\zeta^n \ln^2(\zeta)$ and verify that they are both harmonic.

Using

$$\zeta^n = e^{n\ln(\zeta)} \ ,$$

we have

$$\frac{\partial \zeta^n \ln^m(\zeta)}{\partial n} = \frac{\partial}{\partial n} e^{n\ln(\zeta)} \ln^m(\zeta) = \ln(\zeta) e^{n\ln(\zeta)} \ln^m(\zeta) = \zeta^n \ln^{m+1}(\zeta) \ .$$

It follows immediately by repeated use of this result, starting from $m = 0$ that

$$\frac{\partial^m}{\partial n^m}\left(\zeta^n\right) = \zeta^n \ln^m(\zeta) \ .$$

Using

$$\zeta = re^{\imath\theta} \ ,$$

we have

$$\zeta^n = r^n[\cos(n\theta) + \imath\sin(n\theta)] \ ; \quad \ln(\zeta) = \ln(r) + \imath\theta$$

and

$$\zeta^n \ln^2(\zeta) = r^n[\cos(n\theta) + \imath\sin(n\theta)](\ln^2(r) + 2\imath\ln(r)\theta - \theta^2) \ .$$

The real part of this function is

$$\Re(\zeta^n \ln^2(\zeta)) = r^n(\ln^2(r) - \theta^2)\cos(n\theta) - 2r^n\ln(r)\theta\sin(n\theta)$$

and the imaginary part is

$$\Im(\zeta^n \ln^2(\zeta)) = r^n(\ln^2(r) - \theta^2)\sin(n\theta) + 2r^n\ln(r)\theta\cos(n\theta) \ .$$

It is easily verified that each of these real functions satisfies the Laplace equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

using Mathematica or Maple.

18.2. Verify that the real stress function $\phi = r^2\theta$ satisfies the biharmonic equation (8.16). Then express it in the complex form (18.27).

$$\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 4\theta \ ,$$

so

$$\nabla^4\phi = \nabla^2(4\theta) = 0 \ .$$

Using (18.4), we have

$$r^2\theta = \frac{\imath\zeta\bar{\zeta}}{2}\ln\left(\frac{\bar{\zeta}}{\zeta}\right) = \frac{\imath\zeta\bar{\zeta}}{2}\ln(\bar{\zeta}) - \frac{\imath\zeta\bar{\zeta}}{2}\ln(\zeta) \ .$$

By inspection, this is of the form

$$\phi = \bar{\zeta}g_2 + \zeta\bar{g}_2 \ ,$$

where

$$g_2 = -\frac{\imath\zeta}{2}\ln(\zeta) \ ,$$

since we would then have

$$\bar{g}_2 = \frac{\imath\bar{\zeta}}{2}\ln(\bar{\zeta})$$

(recall that in forming the conjugate we change ζ to $\bar{\zeta}$ but also change $\imath$ to $-\imath$).

18.3. Express the real biharmonic function $\phi = r\theta\sin\theta$ (from equation (8.59) or Table 8.1) in the complex form (18.27).

We first note that

$$r\sin\theta = y = -\frac{\imath}{2}(\zeta - \bar{\zeta}) \; ; \quad \theta = \imath \ln\left(\frac{\bar{\zeta}}{\zeta}\right) \; ,$$

from (18.1, 18.4) respectively, so

$$r\theta\sin\theta = \frac{(\zeta - \bar{\zeta})}{2}\ln\left(\frac{\bar{\zeta}}{\zeta}\right) = \frac{1}{2}\left[\zeta\ln(\bar{\zeta}) - \zeta\ln(\zeta) - \bar{\zeta}\ln(\bar{\zeta}) + \bar{\zeta}\ln(z)\right] \; .$$

By inspection, this is of the form (18.27) with

$$g_1 = -\frac{\zeta\ln(\zeta)}{2} \; ; \quad g_2 = \frac{\ln(\zeta)}{2} \; .$$

18.4. Find the imaginary part of the holomorphic function whose real part is the harmonic function $\ln(r)$.

Suppose that $\ln(r)$ is the real part of the holomorphic function f, so

$$f + \bar{f} = 2\ln(r) = \ln(r^2) = \ln(\zeta\bar{\zeta}) = \ln(\zeta) + \ln(\bar{\zeta}) \ ,$$

using (18.4). Clearly

$$f = \ln(\zeta) = \ln(r) + \imath\theta \ ,$$

using (18.3). It follows that the imaginary part of f is

$$\Im(f) = \theta \ .$$

18.5. Use the residue theorem to evaluate the contour integral

$$\oint_S \frac{\zeta^n d\zeta}{(\zeta^2 - a^2)} ,$$

where n is a positive integer and the contour S is the circle $|\zeta| = 2a$.

The integrand has two poles at $\zeta = \pm a$, both of which lie inside the contour. At $\zeta = a$, we have

$$\lim_{z \to a} \frac{\zeta^n(\zeta - a)}{(\zeta^2 - a^2)} = \lim_{z \to a} \frac{\zeta^n}{(\zeta + a)} = \frac{a^{n-1}}{2}$$

and at $\zeta = -a$,

$$\lim_{z \to -a} \frac{\zeta^n(\zeta + a)}{(\zeta^2 - a^2)} = \lim_{z \to -a} \frac{\zeta^n}{(\zeta - a)} = \frac{(-a)^n}{(-2a)} = \frac{(-a)^{n-1}}{2}$$

Thus, using the residue theorem,

$$\oint_S \frac{\zeta^n d\zeta}{(\zeta^2 - a^2)} = 2\pi\imath \left(\frac{a^{n-1}}{2} + \frac{(-a)^{n-1}}{2} \right) .$$

This expression sums to zero if n is even and to

$$2\pi\imath a^{n-1}$$

if n is odd.

18.6. Find the real and imaginary parts of the function $\sin(\zeta)$ and hence find the points in the x, y-plane at which $\sin(\zeta) = 0$.

Use the residue theorem to evaluate the integral

$$\oint_S \frac{d\zeta}{\sin(\zeta)}$$

for a contour S that encloses only the pole at $\zeta = 0$. By choosing S as a rectangle two of whose sides are the lines $x = \pm\pi/2,\ -\infty < y < \infty$, use your result to evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{dy}{\cosh y} \ .$$

$$\sin(\zeta) = \sin(x + \imath y) = \sin(x)\cosh(y) + \imath\cos(x)\sinh(y) \ ,$$

since

$$\cos(\imath y) = \cosh(y) \ ; \quad \sin(\imath y) = \imath\sinh(y) \ .$$

For $\sin(\zeta) = 0$, we require both the real and imaginary parts to be zero. The real function $\cosh(y)$ is never zero, so for the real part $\sin(x)\cosh(y)$ to be zero, we require

$$\sin(x) = 0 \quad \text{implying} \quad x = n\pi \ ,$$

where n is any positive or negative integer, including zero. $\sin(x)$ and $\cos(x)$ are never zero at the same values of x, so for the imaginary part $\cos(x)\sinh(y)$ to be simultaneously zero at a point where $x = n\pi$, we require $\sinh(y) = 0$ and this occurs only at $y = 0$. We conclude that the zeros of the complex function $\sin(\zeta)$ occur at the points

$$\zeta = (n\pi, 0)$$

on the real axis.

For the integral

$$\oint_S \frac{d\zeta}{\sin(\zeta)} \ ,$$

the residue at the origin is

$$\text{Res}(0) = \lim_{\zeta\to 0} \frac{\zeta}{\sin(\zeta)} \ .$$

Using l'Hôpital's rule (or expanding $\sin(\zeta)$ for small values of ζ), we obtain

$$\text{Res}(0) = \frac{1}{\cos(0)} = 1$$

and hence

$$\oint_S \frac{d\zeta}{\sin(\zeta)} = 2\pi\imath \ .$$

With the specified contour, the contribution to the contour integral from the line $x = \pi/2,\ -\infty < y < \infty$ is

$$\int_{-\infty}^{\infty} \frac{\imath dy}{\cosh y}$$

since on this line $\cos(x) = 0, \sin(x) = 1$ and $\sin(\zeta) = \cosh y$ and $d\zeta = \imath dy$. The contribution from the other line, $x = -\pi/2,\ -\infty < y < \infty$ is the same, since there is only a sign change in $\sin(x)$, and this corresponds to the requirement that this integral be evaluated from $y = \infty$ to $y = -\infty$ to preserve the anticlockwise direction around the contour. Thus, we conclude that

$$\oint_S \frac{d\zeta}{\sin(\zeta)} = 2\imath \int_{-\infty}^{\infty} \frac{dy}{\cosh y} = 2\pi\imath$$

and hence the real integral

$$\int_{-\infty}^{\infty} \frac{dy}{\cosh y} = \pi \ .$$

18.7. Show that on the unit circle,

$$\zeta + \frac{1}{\zeta} = \cos\theta \; ; \quad \zeta^n + \frac{1}{\zeta^n} = \cos(n\theta) \; .$$

Use this result and the notation $\omega = e^{\imath\phi}$ to express the real integral (12.35) in terms of an integral around the contour shown in Figure 18.4. Use the residue theorem to evaluate this integral and hence prove equation (12.35). Explain in particular how you determine the contribution from the small semi-circular paths around the poles $\zeta = \omega$, $\zeta = 1/\omega$.

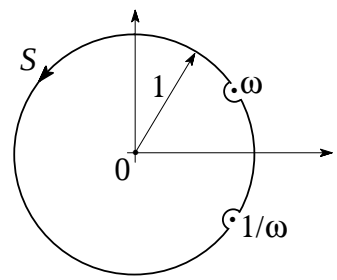


Figure 18.4:

We have $\zeta = re^{\imath\theta}$ and hence on the unit circle, where $r = 1$,

$$\zeta = e^{\imath\theta} = \cos\theta + \imath\sin\theta \; ; \quad \zeta^n = e^{\imath n\theta} = \cos(n\theta) + \imath\sin(n\theta) \; .$$

It follows that

$$\cos\theta = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right) \; ; \quad \cos\phi = \frac{1}{2}\left(\omega + \frac{1}{\omega}\right) \; ; \quad \cos(n\theta) = \frac{1}{2}\left(\zeta^n + \frac{1}{\zeta^n}\right) \; ; \quad \cos\phi = \frac{1}{2}\left(\omega + \frac{1}{\omega}\right)$$

and

$$d\theta = -\frac{\imath d\zeta}{\zeta} \; .$$

Thus, the integral in (12.35) is

$$\begin{aligned}
\int_0^\pi \frac{\cos(n\theta)d\theta}{(\cos\phi - \cos\theta)} &= \frac{1}{2}\int_0^{2\pi} \frac{\cos(n\theta)d\theta}{(\cos\phi - \cos\theta)} \quad \text{(by symmetry)} \\
&= -\frac{\imath}{2}\oint \frac{\left(\zeta^n + \frac{1}{\zeta^n}\right)d\zeta}{\left(\omega + \frac{1}{\omega} - \zeta - \frac{1}{\zeta}\right)\zeta} \\
&= -\frac{\imath}{2}\oint \frac{(\zeta^{2n}+1)\omega d\zeta}{\zeta^n(\omega\zeta - 1)(\omega - \zeta)} \; ,
\end{aligned}$$

where the contour integral has to be performed around the unit circle.

However, the points $\zeta = \omega$, $\zeta = 1/\omega$ lie on the unit circle, so we need to modify the path so as to go around these poles. In Figure 18.4, we choose a path going inside the poles, so that the only remaining residue is that at $z = 0$. We shall calculate this residue first and then address the question about the effect of the small semi-circles around the poles.

The contour integral

For the pole of order n at the origin, we need to evaluate

$$f^{[n-1]}(\zeta) \equiv \frac{d^{n-1}f(\zeta)}{d\zeta^{n-1}} \quad \text{where} \quad f(\zeta) = \frac{(\zeta^{2n}+1)\omega}{(\omega\zeta - 1)(\omega - \zeta)}$$

and in fact the term ζ^{2n} in the numerator is sufficiently strong to cancel the singularity, so this term makes no contribution to the residue. We can therefore drop it and write

$$f(\zeta) = \frac{\omega}{(\omega\zeta - 1)(\omega - \zeta)} \ .$$

Using partial fractions, we write

$$f(\zeta) = f_1(\zeta) + f_2(\zeta) \quad \text{where} \quad f_1(z) = \frac{C_1}{(\omega\zeta - 1)} \ ; \quad f_2(\zeta) = \frac{C_2}{(\omega - \zeta)} \ .$$

Expanding, we then have

$$\frac{\omega}{(\omega\zeta - 1)(\omega - \zeta)} = \frac{C_1}{(\omega\zeta - 1)} + \frac{C_2}{(\omega - \zeta)} = \frac{C_1(\omega - \zeta) + C_2(\omega\zeta - 1)}{(\omega\zeta - 1)(\omega - \zeta)} \ .$$

Equating coefficients of ζ and constants in the numerator, we have

$$-C_1 + C_2\omega = 0 \ ; \quad C_1\omega - C_2 = \omega$$

with solution

$$C_1 = \frac{\omega^2}{(\omega^2 - 1)} \ ; \quad C_2 = \frac{\omega}{(\omega^2 - 1)} \ .$$

It follows that

$$f_1^{[n-1]}(\zeta) = \frac{(-1)^{n-1}(n-1)!C_1\omega^{n-1}}{(\omega\zeta - 1)^n}$$

and at $\zeta = 0$ this evaluates to

$$f_1^{[n-1]}(0) = \frac{(-1)^{n-1}(n-1)!C_1\omega^{n-1}}{(-1)^n} = -(n-1)!C_1\omega^{n-1} \ .$$

Similarly,

$$f_2^{[n-1]}(\zeta) = \frac{(n-1)!C_2}{(\omega - \zeta)^n}$$

and at $\zeta = 0$ this evaluates to

$$f_2^{[n-1]}(0) = \frac{(n-1)!C_2}{\omega^n} \ .$$

Adding the two contributions and substituting for C_1, C_2, we have

$$f^{[n-1]}(0) = \frac{(n-1)!}{(\omega^2-1)\omega^{n-1}} - \frac{(n-1)!\omega^{n+1}}{(\omega^2-1)} = \left(\frac{1}{\omega^n} - \omega^n\right)\frac{(n-1)!\omega}{(\omega^2-1)} \ .$$

and hence

$$\mathrm{Res}(0) = \frac{f^{[n-1]}(0)}{(n-1)!} = \left(\frac{1}{\omega^n} - \omega^n\right)\frac{\omega}{(\omega^2-1)} \ .$$

However

$$\frac{1}{\omega^n} - \omega^n = e^{-\imath n\phi} - e^{\imath n\phi} = -2\imath\sin(n\phi) \ ,$$

and

$$\frac{(\omega^2-1)}{\omega} = \omega - \frac{1}{\omega} = 2\imath\sin\phi \ ,$$

so

$$\mathrm{Res}(0) = -\frac{\sin(n\phi)}{\sin\phi}$$

and

$$\oint \frac{(\zeta^{2n}+1)\omega d\zeta}{\zeta^n(\omega\zeta-1)(\omega-\zeta)} = 2\pi\imath\mathrm{Res}(0) = -\frac{2\pi\imath\sin(n\phi)}{\sin\phi} \ .$$

The semi-circular paths

The semicircular clockwise paths around the poles at $\omega, 1/\omega$ will contribute to this contour integral. The contribution will be $-\pi\imath$ times the sum of the residues at these two poles, since the residues correspond to full circular paths traversed anticlockwise. To evaluate these residues, we write the integrand in the form

$$-\frac{(\zeta^{2n}+1)}{\zeta^n(\zeta-1/\omega)(\zeta-\omega)}$$

and find

$$\mathrm{Res}(\omega) = -\frac{(\omega^{2n}+1)}{\omega^n(\omega-1/\omega)} = -\frac{(\omega^{2n}+1)}{\omega^{n-1}(\omega^2-1)}$$

and

$$\mathrm{Res}(1/\omega) = -\frac{(\omega^{-2n}+1)}{\omega^{-n}(1/\omega-\omega)} = -\frac{(1+\omega^{2n})}{\omega^n(1/\omega-\omega)} = \frac{(\omega^{2n}+1)}{\omega^{n-1}(\omega^2-1)} \ .$$

Since these two contributions are equal and opposite, they are self-cancelling and hence the two semi-circular paths also make self-cancelling contributions.

We conclude that the contour integral is equal to the integral around just those parts of the contour that lie on the unit circle. If we now allow the radii of the semi-circular detours to approach zero, we shall approach the principal value of the integral (12.35), since the principal value is obtained by excluding a small symmetric region on each side of the singular point and then approaching the limit as the size of this excluded region goes to zero. Thus,

$$\int_0^{2\pi} \frac{\cos(n\theta)d\theta}{(\cos\phi - \cos\theta)} = -\imath\left(-\frac{2\pi\imath\sin(n\phi)}{\sin\phi}\right) = -\frac{2\pi\sin(n\phi)}{\sin\phi}$$

and by symmetry

$$\int_0^{\pi} \frac{\cos(n\theta)d\theta}{(\cos\phi - \cos\theta)} = -\frac{\pi\sin(n\phi)}{\sin\phi} \ ,$$

confirming the result (12.35).

18.8. Find the real harmonic function $\phi(r,\theta)$ defined inside the circle $0 \leq r < a$ whose boundary values are

$$\begin{aligned} \phi(a,\theta) &= 1\,; \quad -\alpha < \theta < \alpha \\ &= 0\,; \quad \alpha < |\theta| < \pi\,. \end{aligned}$$

From equation (18.55), we have

$$f(\zeta) = -\bar{C}_0 + \frac{1}{2\pi\imath}\oint \frac{\phi(s)ds}{(s-\zeta)}\,.$$

Here the integral becomes

$$\frac{1}{2\pi\imath}\oint \frac{\phi(s)ds}{(s-\zeta)} = \int_{\theta=-\alpha}^{\theta=\alpha} \frac{ds}{(s-\zeta)} = \ln(s-\zeta)|_{\theta=-\alpha}^{\theta=\alpha} = \ln\left(\frac{a\exp(\imath\alpha)-\zeta}{a\exp(-\imath\alpha)-\zeta}\right)\,.$$

Separating real and imaginary parts, we have

$$a\exp(\imath\alpha)-\zeta = (a\cos\alpha - r\cos\theta) + \imath(a\sin\alpha - r\sin\theta)$$

$$a\exp(-\imath\alpha)-\zeta = (a\cos\alpha - r\cos\theta) - \imath(a\sin\alpha + r\sin\theta)$$

and

$$\begin{aligned} (a\cos(\alpha)-r\cos\theta)^2 + (a\sin\alpha - r\sin\theta)^2 &= a^2\cos^2\alpha + r^2\cos^2\theta - 2ar\cos\theta\cos\alpha \\ &\quad + a^2\sin^2\alpha + r^2\sin^2\theta - 2ar\sin\alpha\sin\theta \\ &= a^2 + r^2 - 2ar\cos(\theta-\alpha)\,. \end{aligned}$$

Thus,

$$\ln(a\exp(\imath\alpha)-\zeta) = \frac{1}{2}\ln\left(a^2+r^2-2ar\cos(\theta-\alpha)\right) + \imath\arctan\left(\frac{a\sin\alpha - r\sin\theta}{a\cos\alpha - r\cos\theta}\right)$$

Similarly

$$\begin{aligned} (a\cos(\alpha)-r\cos\theta)^2 + (a\sin\alpha + r\sin\theta)^2 &= a^2\cos^2\alpha + r^2\cos^2\theta - 2ar\cos\theta\cos\alpha \\ &\quad + a^2\sin^2\alpha + r^2\sin^2\theta + 2ar\sin\alpha\sin\theta \\ &= a^2 + r^2 - 2ar\cos(\theta+\alpha)\,. \end{aligned}$$

and

$$\ln(a\exp(-\imath\alpha)-\zeta) = \frac{1}{2}\ln\left(a^2+r^2-2ar\cos(\theta+\alpha)\right) - \imath\arctan\left(\frac{a\sin\alpha + r\sin\theta}{a\cos\alpha - r\cos\theta}\right)$$

and

$$\phi(r,\theta) = f + \bar{f} = \frac{1}{\pi}\arctan\left(\frac{a\sin\alpha - r\sin\theta}{a\cos\alpha - r\cos\theta}\right) + \frac{1}{\pi}\arctan\left(\frac{a\sin\alpha + r\sin\theta}{a\cos\alpha - r\cos\theta}\right) - C_0 - \bar{C}_0 \ .$$

To evaluate the constant $C_0 + \bar{C}_0$, we use (18.56), obtaining

$$C_0 + \bar{C}_0 = \frac{1}{2\pi\imath}\ln(\exp(2\imath\alpha)) = \frac{\alpha}{\pi} \ .$$

Thus,

$$\phi(r,\theta) = \frac{1}{\pi}\left(\arctan\left(\frac{a\sin\alpha - r\sin\theta}{a\cos\alpha - r\cos\theta}\right) + \arctan\left(\frac{a\sin\alpha + r\sin\theta}{a\cos\alpha - r\cos\theta}\right) - \alpha\right) \ .$$

18.9. Find the real harmonic function $\phi(r,\theta)$ defined in the *external* region $r > a$ satisfying the boundary conditions

$$\phi(a,\theta) = \phi_3 \sin(3\theta) \; ; \quad \phi(r,\theta) \to 0 \; ; \quad r \to \infty \; ,$$

where ϕ_3 is a real constant.

We can write

$$\phi(a,\theta) = \phi_3 \cos(3\theta) = \frac{\phi_3(e^{3\imath\theta} + e^{-3\imath\theta})}{2} = \frac{\phi_3}{2}\left(\frac{s^3}{a^3} + \frac{a^3}{s^3}\right)$$

and it follows that

$$\frac{1}{2\pi\imath}\oint \frac{\phi(s)ds}{(s-\zeta)} = \frac{\phi_3}{4\pi a^3\imath}\oint \frac{s^3 ds}{(s-\zeta)} + \frac{\phi_3 a^3}{4\pi\imath}\oint \frac{ds}{s^3(s-\zeta)} \; .$$

Now recall that this is an exterior problem, so the point ζ is outside the contour S. It follows that the first integral on the right-hand side is zero and the second is not zero (as it would have been if ζ had been inside S). To evaluate it, we use the residue theorem, noting that the only pole is at $s = 0$ and is of order 3. Thus, in the terminology of §18.5.1,

$$f_1(s) = \frac{\phi_3 a^3}{4\pi\imath(s-\zeta)}$$

and $n = 3$. We therefore have

$$f_1^{[n-1]} = \frac{d^2}{d\zeta^2}\left(\frac{\phi_3 a^3}{4\pi\imath(s-\zeta)}\right) = \frac{\phi_3 a^3}{2\pi\imath(s-\zeta)^3}$$

and

$$\mathrm{Res}(0) = -\frac{\phi_3 a^3}{4\pi\imath\zeta^3}$$

from (18.39). Thus,

$$f = -\frac{1}{2\pi\imath}\oint \frac{\phi(s)ds}{(s-\zeta)} = -2\pi\imath\mathrm{Res}(0) = \frac{\phi_3 a^3}{2\zeta^3}$$

and

$$\begin{aligned}
\phi = f + \bar{f} &= \frac{\phi_3 a^3}{2\zeta^3} + \frac{\phi_3 a^3}{2\bar{\zeta}^3} \\
&= \frac{\phi_3 a^3}{2r^3}\left(e^{-3\imath\theta} + e^{3\imath\theta}\right) \\
&= \frac{\phi_3 a^3 \cos(3\theta)}{r^3} \; .
\end{aligned}$$

18.10. Find the real harmonic function $\phi(x,y)$ in the half plane $y \geq 0$ satisfying the boundary conditions

$$\phi(x,0) = \frac{1}{(1+x^2)^2} \; ; \quad \phi(x,y) \to 0 \; ; \quad \sqrt{x^2+y^2} \to \infty \; .$$

As in Example 2 in §18.6.3, we can write

$$\phi(x,y) = f(\zeta) + \bar{f}(\bar{\zeta}) \; ,$$

where

$$f(\zeta) = -\bar{C}_0 + \frac{1}{2\pi\imath} \oint_S \frac{ds}{(1+s^2)^2(s-\zeta)}$$

and the contour S includes the edge $y = 0$ and encircles the half plane $y > 0$. The integrand

$$\frac{1}{(1+s^2)^2(s-\zeta)} = \frac{1}{(s+\imath)^2(s-\imath)^2(s-\zeta)}$$

has a simple pole at $s = \zeta$ and double poles ($n = 2$) at $s = \imath$ and $s = -\imath$, the former of which lies within the contour. We therefore need to compute the residues at $s = \zeta$ and $s = \imath$.

At $s = \imath$, the function f_1 in (18.31) is

$$f_1(s) = \frac{1}{(s+\imath)^2(s-\zeta)} \; .$$

Since $n = 2$, we need to evaluate

$$f_1^{[1]}(s) = \frac{df_1(s)}{ds} = -\frac{2}{(s+\imath)^3(s-\zeta)} - \frac{1}{(s+\imath)^2(s-\zeta)^2}$$

and at $s = \imath$, this takes the value

$$f_1^{[1]}(\imath) = -\frac{2}{(2\imath)^3(\imath-\zeta)} - \frac{1}{(2\imath)^2(\imath-\zeta)^2} = \frac{1}{4\imath(\imath-\zeta)} + \frac{1}{4(\imath-\zeta)^2}$$

or

$$f_1^{[1]}(\imath) = \frac{1+1+\imath z}{4(\imath-\zeta)^2} = \frac{2+\imath z}{4(\imath-\zeta)^2} \; .$$

Thus, the residue at $s = \imath$ is

$$\mathrm{Res}(\imath) = \frac{2+\imath z}{4(\imath-\zeta)^2} \; .$$

At the simple pole, we have

$$\mathrm{Res}(\zeta) = \frac{1}{(\zeta+\imath)^2(\zeta-\imath)^2} \; .$$

Adding the two residues, we obtain

$$\frac{2+\imath z}{4(\imath-\zeta)^2}+\frac{1}{(\zeta+\imath)^2(\zeta-\imath)^2}=\frac{4+(2+\imath z)((\zeta+\imath)^2}{4(\zeta+\imath)^2(\zeta-\imath)^2}\ ,$$

the numerator of which is

$$4+(2+\imath\zeta)(\zeta^2+2\imath\zeta-1)=4+2\zeta^2+4\imath\zeta-2+\imath\zeta^3-2\zeta^2-\imath\zeta=-\imath\zeta^3-\imath\zeta+2\ .$$

We thereore obtain

$$f(\zeta)=\frac{2-\imath\zeta^3-\imath\zeta}{4(\zeta+\imath)^2(\zeta-\imath)^2}-\bar{C}_0=\frac{2-\imath\zeta^3-\imath\zeta}{4(1+\zeta^2)^2}-\bar{C}_0$$

and

$$\phi=f+\bar{f}=\frac{2-\imath\zeta^3-\imath\zeta}{4(1+\zeta^2)^2}+\frac{2+\imath\bar{\zeta}^3+\imath\bar{\zeta}}{4(1+\bar{\zeta}^2)^2}-\bar{C}_0-C_0\ .$$

To determine the constant C_0, we evaluate this expression at $\zeta=\bar{\zeta}=0$ giving

$$\phi(0,0)=1=\frac{2}{4}+\frac{2}{4}-\bar{C}_0-C_0\ .$$

and hence $\bar{C}_0+C_0=0$.

Hence

$$\begin{aligned}\phi &= \frac{2-\imath\zeta^3-\imath\zeta}{4(1+\zeta^2)^2}+\frac{2+\imath\bar{\zeta}^3+\imath\bar{\zeta}}{4(1+\bar{\zeta}^2)^2}\\ &= \frac{(2-\imath\zeta^3-\imath\zeta)(1+\bar{\zeta}^2)^2+(2+\imath\bar{\zeta}^3+\imath\bar{\zeta})(1+\zeta^2)^2}{4(1+\zeta^2)^2(1+\bar{\zeta}^2)^2}\end{aligned}$$

This can be expressed as a function of x,y by substituting for $\zeta,\bar{\zeta}$. This is most easily done using Maple or Mathematica. We obtain

$$\phi(x,y)=\frac{(1-y^2)^2(2+y-y^3)+(4+y-12y^2+2y^3-3y^5)x^2+(2-y-3y^3)x^4-yx^6}{2[(1+x^2-y^2)^2-4x^2y^2]^2}\ .$$

It is easily verified that this reduces to the required expression on $y=0$.

18.11. Use the mapping function (18.62) to find the holomorphic function $f(\omega)$ such that the real harmonic function

$$\phi = f + \bar{f}$$

satisfies the boundary condition $\phi = A\xi$ on the ellipse

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1$$

and $f \to 0$ as $|\omega| \to \infty$, where A is a real constant. Do not attempt to express ϕ as an explicit real function of ξ, η.

We first note that (18.62) maps the circle $r = 1$ into the ellipse and that on this boundary

$$\xi = c(1+m)\cos\theta = a\cos\theta \ ,$$

from (18.62, 18.63). Thus, the boundary conditions on the circular boundary $r = 1$ are

$$\phi(1,\theta) = A\xi = Aa\cos\theta \ .$$

We can write

$$\phi(1,\theta) = Aa\cos\theta = \frac{Aa(e^{\imath\theta} + e^{-\imath\theta})}{2} = \frac{Aa}{2}\left(s + \frac{1}{s}\right) \ ,$$

where $s = \exp(\imath\theta)$ is a complex-variable representation of the boundary of the unit circle. It follows that

$$\frac{1}{2\pi\imath}\oint \frac{\phi(s)ds}{(s-\zeta)} = \frac{Aa}{4\pi\imath}\oint \frac{sds}{(s-\zeta)} + \frac{Aa}{4\pi\imath}\oint \frac{ds}{s(s-\zeta)} \ .$$

Now recall that this is an exterior problem, so the point ζ is outside the contour S. It follows that the first integral on the right-hand side is zero and the second is not zero (as it would have been if ζ had been inside S). To evaluate it, we use the residue theorem, obtaining

$$\mathrm{Res}(0) = \frac{Aa}{4\pi\imath}\frac{1}{(-\zeta)}$$

from (18.37). Thus, equation (18.59) yields

$$f = -\frac{1}{2\pi\imath}\oint \frac{\phi(s)ds}{(s-\zeta)} = -2\pi\imath\mathrm{Res}(0) = \frac{Aa}{2\zeta} \ .$$

To check that this does indeed satisfy the boundary conditions on $r = 1$, we calculate

$$\begin{aligned} \phi = f + \bar{f} &= \frac{Aa}{2\zeta} + \frac{Aa}{2\bar{\zeta}} \\ &= \frac{Aae^{-\imath\theta}}{2r} + \frac{Aae^{\imath\theta}}{2r} \\ &= \frac{Aa\cos\theta}{r} \ , \end{aligned}$$

as required. We could of course have obtained this result semi-intuitively as in Chapter 8, but we give the direct solution here as a rehearsal of the complex-variable procedure.

It remains only to map the function f back into the ω-domain. From (18.68), we have

$$\zeta = \frac{\omega + \sqrt{\omega^2 - 4mc^2}}{2c} \ .$$

We also note from (18.66) that

$$2c = a + b \ ; \quad 4mc^2 = a^2 - b^2 \ ,$$

so

$$\zeta = \frac{\omega + \sqrt{\omega^2 - a^2 + b^2}}{(a+b)}$$

and the required holomorphic function of ω is

$$f = \frac{Aa(a+b)}{2\left(\omega + \sqrt{\omega^2 - a^2 + b^2}\right)} \ .$$

18.12. Show that the function

$$\omega(\zeta) = \zeta^{1/2}$$

maps the quarter plane $\xi \geq 0, \eta \geq 0$ into the half plane $y > 0$. Use this result and appropriate results from the example in §18.6.3 to find the real harmonic function $\phi(\xi, \eta)$ in the quarter plane with boundary values

$$\phi(\xi, 0) = \frac{1}{1+\xi^4} \; ; \quad \xi > 0 \; , \qquad \phi(0, \eta) = \frac{1}{1+\eta^4} \; ; \quad \eta > 0 \; .$$

The given function ω maps

$$\rho \exp(\imath\vartheta) \to r^{1/2} \exp\left(\frac{\imath\theta}{2}\right)$$

and in particular

$$0 \leq \vartheta \leq \frac{\pi}{2} \to 0 \leq \theta \leq \pi \; .$$

On the ξ-axis, $\xi = x^{1/2}, \; x > 0$ and hence

$$\phi(x, 0) = \frac{1}{1+x^2} \; ; \quad x > 0 \; .$$

On the η-axis, $\imath\eta = (x)^{1/2} = \imath(-x)^{1/2}, \; x < 0$ and hence

$$\phi(x, 0) = \frac{1}{1+x^2} \; ; \quad x < 0 \; .$$

Thus, in the ζ-plane (the half plane), the boundary condition is

$$\phi(x, 0) = \frac{1}{1+x^2} \; ; \quad -\infty < x < \infty$$

and the solution is given in Example 2 in §18.6.3 as $\phi = f + \bar{f}$, where

$$f(\zeta) = \frac{\imath}{2(\zeta + \imath)} \; .$$

The relation $\omega(\zeta) = \zeta^{1/2}$ implies $\zeta = \omega^2$ and hence

$$f(\omega) = \frac{\imath}{2(\omega^2 + \imath)} \; ; \quad \bar{f}(\bar{\omega}) = -\frac{\imath}{2(\bar{\omega}^2 - \imath)} \; .$$

Thus,

$$\phi = f + \bar{f} = \frac{\imath(\bar{\omega}^2 - \imath - \omega^2 - \imath)}{2((\omega^2 + \imath)(\bar{\omega}^2 - \imath)} \; .$$

$$\omega^2 = \xi^2 + 2\imath\xi\eta - \eta^2 \ ; \quad \bar{\omega}^2 = \xi^2 - 2\imath\xi\eta - \eta^2 \ ,$$

so

$$(\omega^2+\imath)(\bar{\omega}^2-\imath) = (\xi^2-\eta^2+\imath(2\xi\eta+1))(\xi^2-\eta^2-\imath(2\xi\eta+1)) = (\xi^2-\eta^2)^2+(2\xi\eta+1)^2 \ .$$

Using these results, we obtain

$$\phi = \frac{(2\xi\eta+1)}{(\xi^2-\eta^2)^2+(2\xi\eta+1)^2} \ .$$

As a check, we note that

$$\phi(\xi,0) = \frac{1}{1+\xi^4} \ ; \quad \xi > 0 \ , \qquad \phi(0,\eta) = \frac{1}{1+\eta^4} \ ; \quad \eta > 0 \ ,$$

as required. Notice that in combination with the method of §18.6.3, this mapping can be used for any boundary-value problem for the quarter plane.

CHAPTER 19

19.1. A state of uniform shear $\sigma_{xz} = S, \sigma_{yz} = 0$ in a large block of material is perturbed by the presence of a rigid circular inclusion whose boundary is defined by the equation $r = a$ in cylindrical polar coordinates r, θ, z. The inclusion is perfectly bonded to the elastic material and is prevented from moving, so that $u_z = 0$ at $r = a$. Use the complex-variable formulation to find the complete stress field in the block and hence determine the appropriate stress concentration factor.

In the unperturbed solution, $\sigma_{zx} = S$ everywhere. From (19.17, 19.18),

$$\Psi = \overline{h'} = S$$

and hence a particular solution (omitting a constant of integration which corresponds merely to a rigid-body displacement) is

$$\overline{h} = S\overline{\zeta}\ ; \quad h = S\zeta\ ,$$

where we note that S is a real constant, so $\overline{S} = S$.

We can assume that the rigid inclusion has zero axial displacement without loss of generality, which corresponds to $h + \overline{h} = 0$ at $r = a$ in view of equation (19.16). The corrective solution must therefore satisfy the conditions

$$h + \overline{h} = -S(\zeta + \overline{\zeta})\ ; \quad \zeta = ae^{\imath\theta}$$

and $h \to 0$ as $\zeta \to \infty$. This an exterior problem for which we use equation (18.59). We obtain

$$h = -\frac{1}{2\pi\imath}\oint \frac{-S(s+\overline{s})ds}{(s-\zeta)} = \frac{S}{2\pi\imath}\oint \frac{(s+\overline{s})ds}{(s-\zeta)}\ .$$

On $r = a$, we have $\overline{s} = a^2/s$ and hence

$$h = \frac{S}{2\pi\imath}\oint \left(s + \frac{a^2}{s}\right)\frac{ds}{(s-\zeta)}\ .$$

We can use the residue theorem (18.34) to compute the contour integral. Since ζ is outside the contour, only the pole a^2/s at $s = 0$ makes a contribution and we obtain

$$\mathrm{Res}(0) = -\frac{Sa^2}{2\pi\imath\zeta} \quad \text{giving} \quad h = -\frac{Sa^2}{\zeta}\ . \tag{1}$$

The complete stress function (unperturbed + corrective solution) is then

$$h = S\zeta - \frac{Sa^2}{\zeta}$$

and the stress field is obtained from (19.18) as

$$\Psi = \overline{h'} = S + \frac{Sa^2}{\overline{\zeta}^2} \ .$$

On $\zeta = s$, we have

$$\Psi = S + S\exp(2\imath\theta) = S[1 + \cos(2\theta) + \imath\sin(2\theta)] \ .$$

Thus, the resultant shear stress at a general point on the interface is

$$\tau = S\sqrt{(1+\cos(2\theta))^2 + \sin^2(2\theta)} = S\sqrt{2 + 2\cos(2\theta)} = S\cos\theta \ .$$

The maximum occurs at $\theta = 0, \pi$ and is $\tau = 2S$ corresponding to a stress concentration factor of 2.

19.2. Show that the function (18.62) maps the surfaces of the crack $-a < \xi < a, \eta = 0$ onto the unit circle if $m = 1, c = a/2$. Use this result and the complex-variable formulation to solve the antiplane problem of a uniform stress field $\sigma_{\eta z} = S$ perturbed by a traction-free crack. In particular, find the mode III stress intensity factor K_{III}.

The ellipse corresponding to the unit circle has semi-axes

$$a = c(1+m) \; ; \quad b = c(1-m) \; ,$$

from equation (18.66), so with $m = 1$, we have $a = 2c, b = 0$. This is an ellipse with zero minor axis, which therefore degenerates to a plane crack occupying the line $-a < \xi < a, \eta = 0$.

In the unperturbed solution, $\sigma_{\eta z} = S$ everywhere. From (19.17, 19.18),

$$\Psi = \overline{h'} = \imath S$$

and hence a particular solution (omitting a constant of integration which corresponds merely to a rigid-body displacement) is

$$\overline{h} = \imath S \overline{\omega} \; ; \quad h = -\imath S \omega \; ,$$

where we note that S is a real constant, so $\overline{S} = S$. In particular, we have

$$h_x + \imath h_y = -\imath S(\xi + \imath\eta) \quad \text{so} \quad h_y = -S\xi$$

on $< \xi < a, \eta = 0$. For the crack problem, we must superpose a corrective solution so as to reduce h_y to zero on this surface.

From (18.63) with $m = 1, c = a/2$ and $r = 1$, we have

$$\xi = a\cos\theta \; .$$

Thus, in the ζ-plane, the required corrective function must satisfy the boundary conditions

$$h_y = Sa\cos\theta \; ; \quad r = 1$$

and $h_y \to 0$ as $\zeta \to \infty$. This problem can be solved using the direct method[1] of §18.6.2, but it is clear that the real harmonic function satisfying these conditions is

$$h_y = \frac{Sa\cos\theta}{r} = \frac{Sa(e^{\imath\theta} + e^{-\imath\theta})}{2r} = \frac{Sa}{2}\left(\frac{1}{\zeta} + \frac{1}{\overline{\zeta}}\right) \; .$$

Using (19.28), we then have

$$-\frac{\imath h}{2} = \frac{Sa}{2\zeta} \quad \text{and} \quad h = \frac{\imath Sa}{\zeta} \; .$$

[1]For an illustration of the direct method with a very similar function, see the example in §19.2.1.

To move back to the ω-plane, we must express ζ as a function of ω. We obtain

$$\zeta = \frac{\omega \pm \sqrt{\omega^2 - a^2}}{a}$$

from (18.68). The corrective solution in the ω-plane is therefore

$$h = \frac{\imath S a^2}{\omega + \sqrt{\omega^2 - a^2}} = \imath S \left(\omega - \sqrt{\omega^2 - a^2}\right)$$

and the full solution is defined by

$$h = -\imath S \omega + \imath S \left(\omega - \sqrt{\omega^2 - a^2}\right) = -\imath S \sqrt{\omega^2 - a^2} \ .$$

The corresponding stress field is given by

$$\Psi = \overline{h'} = \frac{\imath S \overline{\omega}}{\sqrt{\overline{\omega}^2 - a^2}} \ .$$

In particular, on the crack face, where $\omega = \xi$ and $|\xi| < a$, we have

$$\Psi = \frac{S\xi}{\sqrt{a^2 - \xi^2}} \ .$$

This is real, showing that the stress component $\sigma_{\eta z} = 0$ on the crack face as required. On the crack plane on the right of the crack, we have $\xi > a, \eta = 0$ and

$$\Psi = \frac{\imath S \xi}{\sqrt{\xi^2 - a^2}}$$

and hence

$$\sigma_{\eta z} = \frac{S\xi}{\sqrt{\xi^2 - a^2}} \ .$$

On the left of the crack, in $\xi < -a, \eta = 0$, we need to take care with the sign of the square root. We choose to define $\sqrt{\omega}$ such that

$$\sqrt{\omega} = r^{1/2} e^{\imath\theta/2} \ ; \quad \sqrt{\overline{\omega}} = r^{1/2} e^{-\imath\theta/2} \ ,$$

with $-\pi < \theta < \pi$. With this interpretation, $\sqrt{\overline{\omega}}$ is discontinuous in the imaginary part at $\theta = \pm\pi$. It jumps from $\imath\sqrt{-\xi}$ to $-\imath\sqrt{-\xi}$ as we cross the negative real axis from below.

The term $\sqrt{\overline{\omega}^2 - a^2}$ can be factorized as

$$\sqrt{\overline{\omega}^2 - a^2} = \sqrt{\overline{\omega} - a}\sqrt{\overline{\omega} + a}$$

and with the above interpretation of the square roots, we have

$$\sqrt{\overline{\omega}^2 - a^2} = \sqrt{\xi - a}\sqrt{\xi + a} e^{-\imath\pi} = -\sqrt{\xi^2 - a^2}$$

in $\xi < -a, \eta = 0$, where for both roots the local value of $\theta = \pm\pi$. In effect, the two discontinuities (sign changes) cancel in the product, giving a continuous function. Of course, this must happen, since the technique we have used guarantees that the resulting function of ω is holomorphic (and hence continuous) throughout the region surrounding the crack. In the intermediate range $-a < \xi < a$, only one of the two square roots has a discontinuity and we therefore obtain different values on the two sides of the crack, but this is acceptable, since there is no physical necessity for continuity across the crack faces. Incidentally, this discussion illustrates how we have to be extremely careful in choosing the correct branch of multi-valued complex functions.

The two expressions for $\sigma_{\eta z}$ on the plane $\eta = 0$ can be combined as

$$\sigma_{\eta z} = \frac{S|\xi|}{\sqrt{\xi^2 - a^2}} \; ; \quad |\xi| > a \; .$$

Also, the mode III stress intensity factor is

$$K_{III} = \lim_{\xi \to a^+} \sigma_{\eta z}(\xi, 0)\sqrt{2\pi(\xi - a)} = S\sqrt{\pi a} \; .$$

19.3. Show that the function $\omega(\zeta) = \zeta^\lambda$ maps the wedge-shaped region $0 < \vartheta < \alpha$ in polar coördinates (ρ, ϑ) into the half-plane $y > 0$ if $\lambda = \alpha/\pi$.

Use this mapping to solve the antiplane problem of the wedge of subtended angle α loaded only by a concentrated out-of-plane force F at the point $\rho = a, \vartheta = 0$.

The given function ω maps

$$\rho \exp(\imath\vartheta) \to r^\lambda \exp^{\imath\lambda\theta}$$

and in particular

$$0 \le \vartheta \le \alpha \to 0 \le \theta \le \frac{\alpha}{\lambda} \ .$$

Thus, it will map the wedge into the half plane if

$$\frac{\alpha}{\lambda} = \pi \quad \text{or} \quad \lambda = \frac{\alpha}{\pi} \ .$$

The boundaries of the wedge are traction-free everywhere except at the point $\rho = a, \vartheta = 0$, so h_y is constant everywhere on the boundary, but there is a jump in value of magnitude F at this point. Thus, the boundary-value problem for h_y can be written

$$\begin{aligned} h_y &= F \ ; \quad \vartheta = \alpha \\ &= F \ ; \quad \vartheta = 0, 0 < \rho < a \\ &= 0 \ ; \quad \vartheta = 0, \rho > a \end{aligned}$$

The point $\rho = a, \vartheta = 0$ maps into $r = b, \theta = 0$, where

$$b = a^{1/\lambda} \ .$$

Thus, the boundary-value problem in the ζ-plane is defined by

$$\begin{aligned} h_y &= F \ ; \quad \theta = \pi \\ &= F \ ; \quad \theta = 0, 0 < r < b \\ &= 0 \ ; \quad \theta = 0, r > b \end{aligned}$$

or

$$\begin{aligned} h_y &= F \ ; \quad y = 0, x < b \\ &= 0 \ ; \quad y = 0, x > b \ . \end{aligned}$$

The real harmonic function

$$h_y = \frac{F\theta}{\pi} = \frac{\imath F}{2\pi}\left(\ln(\overline{\zeta}) - \ln(\zeta)\right) = \frac{F}{2\pi\imath}\left(\ln(\zeta) - \ln(\overline{\zeta})\right) \ ,$$

(using (18.4)) satisfies the condition

$$\begin{aligned} h_y &= F\ ; \quad y = 0, x < 0 \\ &= 0\ ; \quad y = 0, x > 0\ . \end{aligned}$$

and is the imaginary part of the function

$$h = \frac{F}{\pi} \ln(\zeta)\ .$$

It follows that the solution of our problem (achieved by moving the origin to the point $(b, 0)$) is

$$h = \frac{F}{\pi} \ln(\zeta - b)\ .$$

Mapping this back into the ω-plane, we have

$$\zeta - b = \omega^{1/\lambda} - a^{1/\lambda} = \rho^{1/\lambda} \cos(\vartheta/\lambda) - a^{1/\lambda} + \imath \rho^{1/\lambda} \sin(\vartheta/\lambda)\ .$$

Thus, the solution of the antiplane problem is defined by

$$h = \frac{F}{\pi} \ln\left(\omega^{1/\lambda} - a^{1/\lambda}\right)\ .$$

and the stresses are

$$\Psi = \overline{h'} = \frac{d}{d\overline{\omega}} \frac{F}{\pi} \ln\left(\overline{\omega}^{1/\lambda} - a^{1/\lambda}\right) = \frac{F}{\pi \lambda \overline{\omega}(1 - (a/\overline{\omega})^{1/\lambda})}\ ,$$

or

$$\Psi = \frac{F}{\alpha \overline{\omega}(1 - (a/\overline{\omega})^{\pi/\alpha})}\ .$$

19.4. Use the transformation equations (1.15–1.17) to prove the relation (19.48).

Using the stress transformation equations

$$\begin{aligned}
\sigma_{x'x'} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta \\
\sigma_{x'y'} &= \sigma_{xy}(\cos^2\theta - \sin^2\theta) + (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta \\
\sigma_{y'y'} &= \sigma_{yy}\cos^2\theta + \sigma_{xx}\sin^2\theta - 2\sigma_{xy}\sin\theta\cos\theta \ ,
\end{aligned}$$

we have

$$\begin{aligned}
\Phi_\alpha &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta + 2\imath\Big(\sigma_{xy}(\cos^2\theta - \sin^2\theta) \\
&\quad +(\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta) - \sigma_{yy}\cos^2\theta - \sigma_{xx}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta \qquad (1) \\
&= \Big(\cos^2\theta - 2\imath\sin\theta\cos\theta - \sin^2\theta\Big)\sigma_{xx} + \Big(4\sin\theta\cos\theta + 2\imath(\cos^2\theta - \sin^2\theta))\Big)\sigma_{xy} \\
&\quad + \Big(\sin^2\theta + 2\imath\sin\theta\cos\theta - \cos^2\theta\Big)\sigma_{yy} \\
&= \Big(\cos^2\theta - 2\imath\sin\theta\cos\theta - \sin^2\theta\Big)\sigma_{xx} + 2\imath\Big(-2\imath\sin\theta\cos\theta + \cos^2\theta - \sin^2\theta)\Big)\sigma_{xy} \\
&\quad - \Big(-\sin^2\theta - 2\imath\sin\theta\cos\theta + \cos^2\theta\Big)\sigma_{yy} \\
&= e^{-2\imath\theta}\left(\sigma_{xx} + 2\imath\sigma_{xy} - \sigma_{yy}\right) = e^{-2\imath\theta}\Phi \ ,
\end{aligned}$$

since

$$e^{-2\imath\theta} = \cos^2\theta - 2\imath\sin\theta\cos\theta - \sin^2\theta \ .$$

19.5. Express the plane strain equilibrium equations (2.2, 2.3 with $\sigma_{xz} = \sigma_{yz} = 0$) in terms of Θ, Φ, p and $\zeta, \overline{\zeta}$.

The two-dimensional in-plane equilibrium equations are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + p_x = 0 \; ; \quad \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + p_y = 0 \; .$$

To convert to complex variable form, we note that

$$\Theta + \Phi = 2\sigma_{xx} + 2\imath\sigma_{xy} \; ; \quad \Theta - \Phi = 2\sigma_{yy} - 2\imath\sigma_{xy} \; ; \quad p = p_x + \imath p_y \; ,$$

so

$$\sigma_{xx} = \frac{1}{2}\Re\left(\Theta + \Phi\right) = \frac{(2\Theta + \Phi + \overline{\Phi})}{4} \; ; \quad \sigma_{xy} = \frac{1}{2}\Im\left(\Phi\right) = \frac{(\Phi - \overline{\Phi})}{4\imath}$$

$$\sigma_{yy} = \frac{1}{2}\Re\left(\Theta - \Phi\right) = \frac{(2\Theta - \Phi - \overline{\Phi})}{4} \; ; \quad p_x = \frac{p + \overline{p}}{2} \; ; \quad p_y = \frac{p - \overline{p}}{2\imath} \; .$$

Also, from (18.5, 18.6),

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \overline{\zeta}} \; ; \quad \imath\frac{\partial}{\partial y} = \frac{\partial}{\partial \overline{\zeta}} - \frac{\partial}{\partial \zeta} \; ,$$

so

$$\frac{\partial}{\partial y} = \imath\frac{\partial}{\partial \zeta} - \imath\frac{\partial}{\partial \overline{\zeta}} \; .$$

Using these results, the first equilibrium equation becomes

$$\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \overline{\zeta}}\right)\frac{(2\Theta + \Phi + \overline{\Phi})}{4} + \imath\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \overline{\zeta}}\right)\frac{(\Phi - \overline{\Phi})}{4\imath} + \frac{p + \overline{p}}{2} = 0 \; ,$$

or, after multiplying by 2 and simplifying,

$$\frac{\partial \Theta}{\partial \zeta} + \frac{\partial \Theta}{\partial \overline{\zeta}} + \frac{\partial \Phi}{\partial \zeta} + \frac{\partial \overline{\Phi}}{\partial \overline{\zeta}} + p + \overline{p} = 0 \; .$$

The second equation becomes

$$\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \overline{\zeta}}\right)\frac{(\Phi - \overline{\Phi})}{4\imath} + \imath\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \overline{\zeta}}\right)\frac{(2\Theta - \Phi - \overline{\Phi})}{4} + \frac{p - \overline{p}}{2\imath} = 0 \; ,$$

or, after multiplying by $2\imath$ and simplifying,

$$\frac{\partial \Phi}{\partial \zeta} - \frac{\partial \overline{\Phi}}{\partial \overline{\zeta}} - \frac{\partial \Theta}{\partial \zeta} + \frac{\partial \Theta}{\partial \overline{\zeta}} + p - \overline{p} = 0 \; .$$

Adding or subtracting the two equations, we can show that both results can be combined in the single complex equation

$$\frac{\partial \Phi}{\partial \zeta} + \frac{\partial \Theta}{\partial \overline{\zeta}} + p = 0 \ .$$

To verify this, we expand the left-hand side as

$$\begin{aligned}
\frac{\partial \Phi}{\partial \zeta} + \frac{\partial \Theta}{\partial \overline{\zeta}} + p &= \frac{1}{2}\left(\frac{\partial}{\partial x} - \imath \frac{\partial}{\partial y}\right)(\sigma_{xx} + 2\imath\sigma_{xy} - \sigma_{yy}) + \frac{1}{2}\left(\frac{\partial}{\partial x} + \imath \frac{\partial}{\partial y}\right)(\sigma_{xx} + \sigma_{yy}) + p_x + \imath p_y \\
&= \frac{1}{2}\frac{\partial \sigma_{xx}}{\partial x} - \frac{\imath}{2}\frac{\partial \sigma_{xx}}{\partial y} + \imath \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} - \frac{1}{2}\frac{\partial \sigma_{yy}}{\partial x} + \frac{\imath}{2}\frac{\partial \sigma_{yy}}{\partial y} + \frac{1}{2}\frac{\partial \sigma_{xx}}{\partial x} \\
&\quad + \frac{\imath}{2}\frac{\partial \sigma_{xx}}{\partial y} + \frac{1}{2}\frac{\partial \sigma_{yy}}{\partial x} + \frac{\imath}{2}\frac{\partial \sigma_{yy}}{\partial y} + p_x + \imath p_y \\
&= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + p_x + \imath\left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + p_y\right)
\end{aligned}$$

and we observe that the real and imaginary parts are respectively the left-hand sides of the two equilibrium equations.

19.6. By analogy with equations (19.46, 19.47), we can define the complex strains e, ε as

$$e = e_{xx} + e_{yy} \; ; \quad \varepsilon = e_{xx} + 2\imath e_{xy} - e_{yy} \; ,$$

where we note that $e = \operatorname{div} \boldsymbol{u}$ is the dilatation in plane strain. Express the elastic constitutive law (1.71) as a relation between e, ε and Θ, Φ.

From (1.71), we have

$$\sigma_{xx} = \lambda e + 2\mu e_{xx} \; ; \quad \sigma_{xy} = 2\mu e_{xy} \; ; \quad \sigma_{yy} = \lambda e + 2\mu e_{yy} \; .$$

It follows that

$$\Theta = \lambda e + 2\mu e_{xx} + \lambda e + 2\mu e_{yy} = 2\lambda e + 2\mu(e_{xx} + e_{yy}) = 2(\lambda + \mu)e$$

$$\Phi = \lambda e + 2\mu e_{xx} + 4\imath\mu e_{xy} - \lambda e - 2\mu e_{yy} = 2\mu(e_{xx} + 2\imath e_{xy} - e_{yy}) = 2\mu\varepsilon \; .$$

19.7. Express the compatibility equation (2.8) in terms of the complex strains e, ε defined in Problem 6 and $\zeta, \overline{\zeta}$.

The compatibility equation is

$$\frac{\partial^2 e_{xx}}{\partial y^2} - 2\frac{\partial^2 e_{xy}}{\partial x \partial y} + \frac{\partial^2 e_{yy}}{\partial x^2} = 0 \ .$$

To convert to complex variable form, we note that

$$e + \varepsilon = 2e_{xx} + 2\imath e_{xy} \ ; \quad e - \varepsilon = 2e_{yy} - 2\imath e_{xy} \ ,$$

so

$$e_{xx} = \frac{1}{2}\Re\left(e + \varepsilon\right) = \frac{(2e + \varepsilon + \overline{\varepsilon})}{4} \ ; \quad e_{xy} = \frac{1}{2}\Im\left(\varepsilon\right) = \frac{(\varepsilon - \overline{\varepsilon})}{4\imath}$$

$$e_{yy} = \frac{1}{2}\Re\left(e - \varepsilon\right) = \frac{(2e - \varepsilon - \overline{\varepsilon})}{4} \ .$$

Also, from (18.5, 18.6),

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \overline{\zeta}} \ ; \quad \imath\frac{\partial}{\partial y} = \frac{\partial}{\partial \overline{\zeta}} - \frac{\partial}{\partial \zeta} \ ,$$

so

$$\frac{\partial}{\partial y} = \imath\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \overline{\zeta}}\right) \ .$$

Using these results, the compatibility equation becomes

$$\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \overline{\zeta}}\right)^2 \frac{(2e + \varepsilon + \overline{\varepsilon})}{4} - 2\imath\left(\frac{\partial}{\partial \zeta} - \frac{\partial}{\partial \overline{\zeta}}\right)\left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \overline{\zeta}}\right)\frac{(\varepsilon - \overline{\varepsilon})}{4\imath} + \left(\frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \overline{\zeta}}\right)^2 \frac{(2e - \varepsilon - \overline{\varepsilon})}{4} = 0$$

Expanding, we then obtain

$$\frac{\partial^2 e}{\partial \zeta^2} + \frac{\partial^2 e}{\partial \overline{\zeta}^2} - \frac{1}{2}\frac{\partial^2 \varepsilon}{\partial \zeta^2} - \frac{\partial^2 \varepsilon}{\partial \zeta \partial \overline{\zeta}} + \frac{1}{2}\frac{\partial^2 \varepsilon}{\partial \overline{\zeta}^2} + \frac{1}{2}\frac{\partial^2 \overline{\varepsilon}}{\partial \zeta^2} - \frac{\partial^2 \overline{\varepsilon}}{\partial \zeta \partial \overline{\zeta}} - \frac{1}{2}\frac{\partial^2 \overline{\varepsilon}}{\partial \overline{\zeta}^2} = 0 \ .$$

However, the left-hand side of this equation is the sum of a function and its conjugate (as it must be since it is real), so a sufficient condition is

$$\frac{\partial^2 e}{\partial \zeta^2} - \frac{1}{2}\frac{\partial^2 \varepsilon}{\partial \zeta^2} - \frac{\partial^2 \varepsilon}{\partial \zeta \partial \overline{\zeta}} + \frac{1}{2}\frac{\partial^2 \varepsilon}{\partial \overline{\zeta}^2} = 0 \ .$$

19.8. Use (19.9) and (1.46) to obtain a complex-variable expression for the in-plane rotation ω_z. Then apply your result to equation (19.57) to verify that the term involving C_1 is consistent with a rigid-body rotation.

Applying (19.9) to (19.33), we have

$$2\,\mathrm{curl}\,\boldsymbol{u} = \imath\left(\frac{\partial\overline{u}}{\partial\overline{\zeta}} - \frac{\partial u}{\partial\zeta}\right)\boldsymbol{k}\ ,$$

so

$$\omega_z = \frac{\imath}{2}\left(\frac{\partial\overline{u}}{\partial\overline{\zeta}} - \frac{\partial u}{\partial\zeta}\right)\ ,$$

from (1.46).

Applying this expression to (19.55), we have

$$2\mu\frac{\partial u}{\partial\zeta} = -4(1-\nu)\imath C_1\ ;\quad 2\mu\frac{\partial\overline{u}}{\partial\overline{\zeta}} = 4(1-\nu)\imath C_1$$

(since C_1 is a real constant) and

$$2\mu\omega_z = \frac{\imath}{2}\left(4(1-\nu)\imath C_1 + 4(1-\nu)\imath C_1\right) = -4(1-\nu)C_1\ .$$

Thus, the rotation is the same at all points in the body, which is consistent with a rigid-body rotation.

19.9. Use the method of §19.4 to determine the displacements due to the Airy stress function $\phi = r^2\theta$. Express the results in terms of the components u_r, u_θ, using the transformation equation (19.13).

Using (18.4) we have

$$r^2 = \zeta\overline{\zeta}\ ; \quad \theta = \frac{\imath}{2}\ln\left(\frac{\overline{\zeta}}{\zeta}\right)$$

and hence

$$\phi = \frac{\imath\zeta\overline{\zeta}}{2}\ln\left(\frac{\overline{\zeta}}{\zeta}\right) = \frac{\imath\zeta\overline{\zeta}\ln(\overline{\zeta})}{2} - \frac{\imath\zeta\overline{\zeta}\ln(\zeta)}{2}\ .$$

By inspection, we see that this can be written in the form (18.27), with

$$g_1 = 0\ ; \quad g_2 = -\frac{\imath\zeta\ln(\zeta)}{2}\ .$$

From (19.62, 19.64), we then have

$$\chi = \imath\zeta\ln(\zeta)\ ; \quad \theta = 0$$

and substitution into (19.44) yields

$$\begin{aligned} 2\mu(u_x + \imath u_y) &= -\kappa\imath\zeta\ln(\zeta) - \imath\zeta\ln(\overline{\zeta}) - \imath\zeta \\ &= -\kappa\imath re^{\imath\theta}(\ln(r) + \imath\theta) - \imath re^{\imath\theta}(\ln(r) - \imath\theta) - \imath re^{\imath\theta} \end{aligned}$$

Using (19.13) and rotating through θ anticlockwise, we then have

$$\begin{aligned} 2\mu(u_r + \imath u_\theta) &= -\kappa\imath r(\ln(r) + \imath\theta) - \imath r(\ln(r) - \imath\theta) - \imath r \\ &= -(\kappa+1)\imath r\ln(r) + (\kappa-1)r\theta - \imath r \end{aligned}$$

and hence

$$2\mu u_r = (\kappa-1)r\theta\ ; \quad 2\mu u_\theta = -(\kappa+1)r\ln(r) - r\ .$$

19.10. Substitute the Laurent expansion (19.94) into the contour integral

$$\oint_S \frac{\overline{s}\chi'(s)ds}{(s-\zeta)} ,$$

where ζ is a point outside the circular contour S of radius a. Use the residue theorem to evaluate each term in the resulting series and hence verify that

$$\frac{1}{2\pi\imath}\oint \frac{\overline{s}\chi'(s)ds}{(s-\zeta)} = -\frac{a^2\chi'(\zeta)}{\zeta} .$$

From (19.94),

$$\chi(\zeta) = \frac{A_1}{\zeta} + \frac{A_2}{\zeta^2} + \frac{A_3}{\zeta^3} + \ldots ; \quad \chi'(\zeta) = -\frac{A_1}{\zeta^2} - \frac{2A_2}{\zeta^3} - \frac{3A_3}{\zeta^4} - \ldots$$

and hence

$$\overline{s}\chi'(s) = -\frac{A_1\overline{s}}{s^2} - \frac{2A_2\overline{s}}{s^3} - \frac{3A_3\overline{s}}{s^4} - \ldots = -\frac{A_1a^2}{s^3} - \frac{2A_2a^2}{s^4} - \frac{3A_3a^2}{s^5} - \ldots .$$

Consider a representative term of the form

$$\oint \frac{ds}{s^n(s-\zeta)} .$$

The point ζ is outside the contour, so the only poles are those at $s = 0$. In the notation of §18.5.1, we have

$$f_1 = \frac{1}{(s-\zeta)} ; \quad f_1^{[n-1]}(s) = \frac{(-1)^{(n-1)}(n-1)!}{(s-\zeta)^n} ; \quad f_1^{[n-1]}(0) = -\frac{(n-1)!}{\zeta^n} .$$

and

$$\mathrm{Res}(0) = \frac{f_1^{[n-1]}(0)}{(n-1)!} = -\frac{1}{\zeta^n} .$$

Using this result, we have

$$\frac{1}{2\pi\imath}\oint \frac{\overline{s}\chi'(s)ds}{(s-\zeta)} = \frac{A_1a^2}{\zeta^3} + \frac{2A_2a^2}{\zeta^4} + \frac{3A_3a^2}{\zeta^5} - \ldots = -\frac{a^2\chi'(\zeta)}{\zeta} .$$

19.11. Use the method of §19.6.1 to find the complex stresses Θ, Φ for Problem 12.1, where a disk of radius a is compressed by equal and opposite forces F acting at the points $\zeta = \pm a$.

In this case, we have $T = -F\delta(\theta) + F\delta(\theta - \pi)$ and hence

$$\psi(s) = \frac{\imath F}{2}\mathrm{sgn}(\theta) \quad \text{and} \quad \overline{\psi}(\overline{s}) = -\frac{\imath F}{2}\mathrm{sgn}(\theta) \ .$$

The integral in (19.84) with $f(s) = \psi(s)$ takes the form

$$\begin{aligned} \frac{1}{2\pi\imath}\oint \frac{\psi(s)ds}{(s-\zeta)} &= \frac{F}{4\pi}\left(-\ \ln(s-\zeta)\Big|_{s=-a}^{s=a} + \ \ln(s-\zeta)\Big|_{s=a}^{s=-a}\right) \\ &= \frac{F}{2\pi}\ln\left(\frac{\zeta+a}{\zeta-a}\right) \ , \end{aligned}$$

so

$$\chi(\zeta) = \frac{1}{2\pi\imath}\oint \frac{f(s)ds}{(s-\zeta)} - \overline{A}_1\zeta - 2\overline{A}_2 a^2 = \frac{F}{2\pi}\ln\left(\frac{\zeta+a}{\zeta-a}\right) - \overline{A}_1\zeta - 2\overline{A}_2 a^2 \ ,$$

from (19.83). We can find the constants A_1, A_2 from (19.87), but to avoid doing more integration, it is simpler to repeat the process of differentation and evaluation for this specific function. We have

$$\begin{aligned} \chi' &= -\frac{Fa}{\pi(\zeta^2 - a^2)} - \overline{A}_1 \\ \chi'' &= \frac{2Fa\zeta}{\pi(\zeta^2 - a^2)^2} \ . \end{aligned}$$

Evaluating these expressions at $\zeta = 0$ and equating them to the corresponding Taylor series expansions, we find

$$\frac{F}{\pi a} - \overline{A}_1 = A_1 \ ; \quad 0 = 2A_2 \ ,$$

so

$$A_1 + \overline{A}_1 = \frac{F}{\pi a} \ ; \quad A_2 = 0 \ .$$

The imaginary part of A_1 corresponds to a rigid-body rotation and can be chosen arbitrarily to be zero, giving

$$A_1 = \overline{A}_1 = \frac{F}{2\pi a} \quad \text{and} \quad \chi(\zeta) = \frac{1}{2\pi\imath}\oint \frac{f(s)ds}{(s-\zeta)} - \overline{A}_1\zeta - 2\overline{A}_2 a^2 = \frac{F}{2\pi}\ln\left(\frac{\zeta+a}{\zeta-a}\right) - \frac{F\zeta}{2\pi a} \ .$$

Next, the integral in (19.91) takes the form

$$\begin{aligned} \frac{1}{2\pi\imath}\oint \frac{\overline{\psi(s)}ds}{(s-\zeta)} &= \frac{F}{4\pi}\left(\ \ln(s-\zeta)\Big|_{s=-a}^{s=a} - \ \ln(s-\zeta)\Big|_{s=a}^{s=-a}\right) \\ &= -\frac{F}{2\pi}\ln\left(\frac{\zeta+a}{\zeta-a}\right) \ . \end{aligned}$$

so

$$\theta(\zeta) = \frac{1}{2\pi\imath}\oint \frac{\overline{f(s)}ds}{(s-\zeta)} - \gamma\overline{A}_0 - \frac{a^2\left(\chi'(z)-\chi'(0)\right)}{\zeta} = -\frac{F}{2\pi}\ln\left(\frac{\zeta+a}{\zeta-a}\right) - \overline{A}_0 - \frac{a^2\left(\chi'(z)-\chi'(0)\right)}{\zeta} \ .$$

$$\chi'(\zeta) - \chi'(0) = -\frac{Fa}{\pi(\zeta^2-a^2)} - \frac{F}{\pi a} = -\frac{F(a^2+\zeta^2-a^2}{\pi a(\zeta^2-a^2)} = -\frac{F\zeta^2}{\pi a(\zeta^2-a^2)} \ ,$$

so

$$\theta(\zeta) = -\frac{F}{2\pi}\ln\left(\frac{\zeta+a}{\zeta-a}\right) - \overline{A}_0 + \frac{Fa\zeta}{\pi(\zeta^2-a^2)} \ .$$

There is no necessity to determine the constant $\overline{A}_0$ since it corresponds to a rigid-body displacement and will not contribute to the stresses.

Using these results, we have

$$\chi'(\zeta) = -\frac{Fa}{\pi(\zeta^2-a^2)} - \frac{F}{2\pi a} = -\frac{F}{2\pi a}\left(\frac{\zeta^2+a^2}{\zeta^2-a^2}\right)$$

$$\chi''(\zeta) = \frac{2Fa\zeta}{\pi(\zeta^2-a^2)^2}$$

$$\theta'(\zeta) = \frac{Fa}{\pi(\zeta^2-a^2)} + \frac{Fa}{\pi(\zeta^2-a^2)} - \frac{2Fa\zeta^2}{\pi(\zeta^2-a^2)^2} = -\frac{2Fa^3}{\pi(\zeta^2-a^2)^2} \ .$$

The complex stresses can now be calculated as

$$\Theta = -2(\chi' + \overline{\chi'}) = \frac{F}{\pi a}\left[\left(\frac{\zeta^2+a^2}{\zeta^2-a^2}\right) + \left(\frac{\overline{\zeta}^2+a^2}{\overline{\zeta}^2-a^2}\right)\right]$$

$$\Phi = 2(\zeta\,\overline{\chi''} + \overline{\theta}') = 2\left(\frac{2Fa\zeta\overline{\zeta}}{\pi(\overline{\zeta}^2-a^2)^2} - \frac{2Fa^3}{\pi(\overline{\zeta}^2-a^2)^2}\right) = \frac{4Fa(\zeta\overline{\zeta}-a^2)}{\pi(\overline{\zeta}^2-a^2)^2} \ .$$

19.12. A state of uniform uniaxial stress, $\sigma_{xx} = S$ is perturbed by the presence of a traction-free circular hole of radius a. Use the method of §19.6.2 to find the complex stresses Θ, Φ. [Note: the Airy stress function solution of this problem is given in §8.4.1.]

As usual, we start by describing the unperturbed state of stress which is

$$\sigma_{xx} = S \; ; \quad \sigma_{xy} = \sigma_{yy} = 0 \quad \text{or} \quad \Theta = \Phi = S \; .$$

From equations (19.50, 19.51), it is clear that a particular solution can be obtained from the linear functions

$$\chi = A\zeta \; ; \quad \theta = B\zeta \; ,$$

where A, B are constants. Substituting into (19.50, 19.51), we then have

$$\Theta = -4A \; ; \quad \Phi = 2B$$

and hence

$$A = -\frac{S}{4} \; ; \quad B = \frac{S}{2} \; ; \quad \chi = -\frac{S\zeta}{4} \; ; \quad \theta = \frac{S\zeta}{2} \; .$$

It then follows from (19.52) that

$$\psi = -\frac{S\zeta}{4} - \frac{S\zeta}{4} + \frac{S\overline{\zeta}}{2} = -\frac{S\zeta}{2} + \frac{S\overline{\zeta}}{2} \; .$$

Notice as a check that (19.54) then returns the correct complex stresses.

For the complete (perturbed) solution, we want the hole to be traction-free and hence $\psi(s) = 0$. In the unperturbed case, we have

$$\psi_0(s) = -\frac{Ss}{2} + \frac{S\overline{s}}{2} = -\frac{Ss}{2} + \frac{Sa^2}{2s} \; ,$$

so the perturbation ψ_1 must be chosen so that $\psi(s) = \psi_0(s) + \psi_1(s) = 0$ and hence

$$\psi_1(s) = \frac{Ss}{2} - \frac{Sa^2}{2s} \; .$$

This is an exterior problem, so we use (19.95) with $\gamma = 1, f(s) = \psi_1(s)$, giving

$$\chi(\zeta) = -\frac{1}{2\pi\imath} \oint \left(\frac{Ss}{2} - \frac{Sa^2}{2s} \right) \frac{ds}{(s-\zeta)} \; .$$

Since the point ζ is outside the contour, the only pole is at $s = 0$ giving a residue

$$\text{Res}(0) = -\frac{Sa^2}{2(-\zeta)} = \frac{Sa^2}{2\zeta} \; .$$

Thus,

$$\oint \left(\frac{Ss}{2} - \frac{Sa^2}{2s} \right) \frac{ds}{(s-\zeta)} = 2\pi\imath \mathrm{Res}(0) = 2\pi\imath \left(\frac{Sa^2}{2\zeta} \right)$$

and

$$\chi(\zeta) = -\frac{Sa^2}{2\zeta} \ .$$

It follows that

$$\chi'(\zeta) = \frac{Sa^2}{2\zeta^2} \quad \text{and} \quad -\frac{a^2\chi'(\zeta)}{\zeta} = -\frac{Sa^4}{2\zeta^3} \ .$$

Next, using (19.96), we have

$$\theta(\zeta) = -\frac{1}{2\pi\imath} \oint \frac{\overline{f(s)}ds}{(s-\zeta)} - \frac{a^2\chi'(\zeta)}{\zeta}$$

and

$$\overline{f(s)} = \frac{Sa^2}{2s} - \frac{Ss}{2} \ .$$

Thus

$$\oint \frac{\overline{f(s)}ds}{(s-\zeta)} = \oint \left(\frac{Sa^2}{2s} - \frac{Ss}{2} \right) \frac{ds}{(s-\zeta)}$$

and again the only pole is at $s = 0$, giving

$$\mathrm{Res}(0) = \frac{Sa^2}{2(-\zeta)} = -\frac{Sa^2}{2\zeta} \quad \text{and} \quad \frac{1}{2\pi\imath} \oint \frac{\overline{f(s)}ds}{(s-\zeta)} = -\frac{Sa^2}{2\zeta} \ .$$

Thus,

$$\theta(\zeta) = \frac{Sa^2}{2\zeta} - \frac{Sa^4}{2\zeta^3} \ .$$

This is not asked for in the question, but as a check, we compare these complex potentials with the perturbation terms in the real stress function ϕ from §8.4.1, which is

$$\phi = -\frac{Sa^2}{2}\ln(r) + \frac{Sa^2}{2}\cos(2\theta) - \frac{Sa^4}{4r^2}\cos(2\theta) \ ,$$

from (8.65, 8.73). We can express this in complex form as in §18.4.1, obtaining

$$\phi = -\frac{Sa^2}{4}\ln(\zeta) - \frac{Sa^2}{4}\ln(\overline{\zeta}) + \frac{Sa^2\zeta}{4\overline{\zeta}} + \frac{Sa^2\overline{\zeta}}{4\zeta} - \frac{Sa^4}{8\overline{\zeta}^2} - \frac{Sa^4}{8\zeta^2}$$

and this is of the form (18.27) with

$$g_1 = -\frac{Sa^2}{4}\ln(\zeta) - \frac{Sa^4}{8\zeta^2} \ ; \quad g_2 = \frac{Sa^2}{4\zeta} \quad \text{so} \quad g_1' = -\frac{Sa^2}{4\zeta} + \frac{Sa^4}{4\zeta^3} \ .$$

It follows from (19.62, 19.64) that

$$\chi = -2g_2 = -\frac{Sa^2}{2\zeta} \; ; \quad \theta = -2g_1' = \frac{Sa^2}{2\zeta} - \frac{Sa^4}{2\zeta^3} \; ,$$

agreeing with the previous results.

The complete solution is then obtained by summing the unperturbed solution and the perturbation, giving

$$\chi = -\frac{S\zeta}{4} - \frac{Sa^2}{2\zeta} \; ; \quad \theta = \frac{S\zeta}{2} + \frac{Sa^2}{2\zeta} - \frac{Sa^4}{2\zeta^3} \; ,$$

so

$$\chi' = -\frac{S}{4} + \frac{Sa^2}{2\zeta^2} \; ; \quad \chi'' = -\frac{Sa^2}{\zeta^3} \; ; \quad \theta' = \frac{S}{2} - \frac{Sa^2}{2\zeta^2} + \frac{3Sa^4}{2\zeta^4} \; .$$

The complex stresses are now obtained as

$$\Theta = -2(\chi' + \overline{\chi'}) = -2\left(-\frac{S}{4} + \frac{Sa^2}{2\zeta^2} - \frac{S}{4} + \frac{Sa^2}{2\overline{\zeta}^2}\right)$$

$$\Phi = 2(\zeta\,\overline{\chi''} + \overline{\theta'}) = 2\left(-\frac{Sa^2\zeta}{\overline{\zeta}^3} + \frac{S}{2} - \frac{Sa^2}{2\overline{\zeta}^2} + \frac{3Sa^4}{2\overline{\zeta}^4}\right)$$

or

$$\Theta = S - \frac{Sa^2}{\zeta^2} - \frac{Sa^2}{\overline{\zeta}^2}$$

$$\Phi = S - \frac{2Sa^2\zeta}{\overline{\zeta}^3} - \frac{Sa^2}{\overline{\zeta}^2} + \frac{3Sa^4}{\overline{\zeta}^4} \; .$$

Notice that

$$\Phi_\theta = \sigma_{rr} + 2\imath\sigma_{r\theta} - \sigma_{\theta\theta} = e^{-2\imath\theta}\Phi = \frac{\overline{\zeta}\Phi}{\zeta} \; .$$

Thus

$$\Phi_\theta = \frac{S\overline{\zeta}}{\zeta} - \frac{2Sa^2}{\overline{\zeta}^2} - \frac{Sa^2}{\zeta\overline{\zeta}} + \frac{3Sa^4}{\zeta\overline{\zeta}^3}$$

which expands as

$$\Phi_\theta = Se^{-2\imath\theta} - \frac{2Sa^2e^{2\imath\theta}}{r^2} - \frac{Sa^2}{r^2} + \frac{3Sa^4e^{2\imath\theta}}{r^4}$$

This corresponds to

$$\sigma_{rr} - \sigma_{\theta\theta} = -\frac{Sa^2}{r^2} + \left(S - \frac{2Sa^2}{r^2} + \frac{3Sa^4}{r^4}\right)\cos(2\theta)$$

and

$$\sigma_{r\theta} = \frac{S}{2}\left(-1 - \frac{2a^2}{r^2} + \frac{3Sa^4}{r^4}\right)\sin(2\theta) \ .$$

Also

$$\Theta = \sigma_{rr} + \sigma_{\theta\theta} = S - \frac{Sa^2 e^{-2\imath\theta}}{r^2} - \frac{Sa^2 e^{2\imath\theta}}{r^2} = S - \frac{2Sa^2}{r^2}\cos(2\theta) \ ,$$

agreeing with (8.74–8.76).

19.13. A state of uniform shear stress, $\sigma_{xy} = S$ is perturbed by the presence of a bonded rigid circular inclusion of radius a. Use the method of §19.6.2 to find the complex stresses Θ, Φ.

The unperturbed field is defined by

$$\Theta = 0 \; ; \quad \Phi = 2\imath S \; .$$

The first condition can be satisfied by choosing $\chi = 0$ and since

$$\Phi = -2(\zeta\overline{\chi}'' + \overline{\theta}') \; ,$$

we then have

$$-2\overline{\theta}' = 2\imath S \quad \text{or} \quad \overline{\theta}' = -\imath S$$

and hence

$$\overline{\theta} = -\imath S\overline{\zeta} \; ; \quad \theta = \imath S\zeta \; .$$

Thus a particular solution for the complex potentials is

$$\chi_0 = 0 \; ; \quad \theta_0 = \imath S\zeta \; ,$$

giving

$$f_0 = -\kappa\chi + \zeta\,\overline{\chi'} + \overline{\theta} = -\imath S\overline{\zeta} \; ,$$

from (19.81), which also implies

$$\overline{f_0} = \imath S\zeta \; .$$

We require f to be zero on the circle $|\zeta| = a$, so the corrective solution f_1 must satisfy

$$f_1(s) = \frac{\imath S a^2}{s} \; ; \quad \overline{f_1(s)} = -\imath S s \; ,$$

since on the circle, if $\zeta = s$, then $\overline{\zeta} = a^2/s$. Using this expression in (19.95) with $\gamma = -\kappa$, we have

$$-\kappa\chi_1 = -\frac{1}{2\pi\imath}\oint \frac{\imath S a^2 ds}{s(s-\zeta)} \; .$$

Since this is an exterior problem, the point ζ lies outside the contour and the only residue inside the contour is that at $s = 0$. Evaluating the integral using the residue theorem, we therefore obtain

$$\kappa\chi_1 = \frac{\imath S a^2}{(-\zeta)} = -\frac{\imath S a^2}{\zeta} \; ,$$

so

$$\chi_1 = -\frac{\imath S a^2}{\kappa\zeta} \; ; \quad \chi_1' = \frac{\imath S a^2}{\kappa\zeta^2} \; .$$

Equation (19.96) then gives

$$\theta_1 = \frac{1}{2\pi\imath}\oint \frac{\imath S s ds}{(s-\zeta)} - \frac{\imath S a^4}{\kappa\zeta^3}$$

and the integral is zero since there are no poles inside the contour, leaving

$$\theta_1 = -\frac{\imath S a^4}{\kappa\zeta^3} \ .$$

Adding in the unperturbed solution, the complete stress field is defined by the potentials

$$\chi = -\frac{\imath S a^2}{\kappa\zeta} \ ; \quad \theta = \imath S\zeta - \frac{\imath S a^4}{\kappa\zeta^3} \ .$$

We then have

$$\chi' = \frac{\imath S a^2}{\kappa\zeta^2} \ ; \quad \chi'' = -\frac{2\imath S a^2}{\kappa\zeta^3} \ ; \quad \theta' = \imath S + \frac{3\imath S a^4}{\kappa\zeta^4}$$

and the complex stresses are

$$\Theta = 2\left(\frac{\imath S a^2}{\kappa\zeta^2} - \frac{\imath S a^2}{\kappa\overline{\zeta}^2}\right) = \frac{2\imath S a^2}{\kappa}\left(\frac{1}{\zeta^2} - \frac{1}{\overline{\zeta}^2}\right)$$

$$\Phi = -2\left(\frac{2\imath S a^2\zeta}{\kappa\overline{\zeta}^3} - \imath S - \frac{3\imath S a^4}{\kappa\overline{\zeta}^4}\right) = 2\imath S - \frac{2\imath S}{\kappa}\left(\frac{2a^2\zeta}{\overline{\zeta}^3} - \frac{3a^4}{\overline{\zeta}^4}\right) \ .$$

This is not asked for in the question, but as a check, we compare these complex potentials with the real stress function solution of Problem 9.3.

In polar coordinates, we can write

$$\Theta = \frac{2\imath S a^2}{\kappa r^2}\left(e^{-2\imath\theta} - e^{2\imath\theta}\right) = \frac{2Sa^2}{\imath\kappa r^2}\left(e^{2\imath\theta} - e^{-2\imath\theta}\right) = \frac{4Sa^2}{\kappa r^2}\sin(2\theta)$$

$$\Phi = 2\imath S - \frac{2\imath S}{\kappa}\left(\frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)e^{4\imath\theta} \ ,$$

so

$$\begin{aligned}\Phi_\theta &= \Phi e^{-2\imath\theta} = 2\imath S e^{-2\imath\theta} - \frac{2\imath S}{\kappa}\left(\frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)e^{2\imath\theta} \\ &= 2S\left(\imath\cos(2\theta) + \sin(2\theta)\right) - \frac{2S}{\kappa}\left(\frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\left(\imath\cos(2\theta) - \sin(2\theta)\right) \ .\end{aligned}$$

Extracting real and imaginary parts, we then have (for example)

$$2\sigma_{rr} = \frac{4Sa^2}{\kappa r^2}\sin(2\theta) + 2S\sin(2\theta) + \frac{2S}{\kappa}\left(\frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\sin(2\theta) = 2S\left(1 + \frac{4a^2}{\kappa r^2} - \frac{3a^4}{r^4}\right)\sin(2\theta)$$

agreeing with the solution of Problem 9.3.

19.14. A state of uniform uniaxial stress, $\sigma_{yy} = S$ is perturbed by the presence of a traction-free elliptical hole of semi-axes a, b. Find the complex potentials χ, θ as functions of ω and hence determine the stress concentration factor as a function of the ratio b/a. Verify that it tends to the value 3 when $b = a$.

As in the example in §19.7.1, the unperturbed solution is defined by

$$\Theta = S \; ; \quad \Phi = -S \; ,$$

$$\chi_0 = \frac{S\omega}{4} \; ; \quad \theta_0 = \frac{S\omega}{2} \quad \text{and hence} \quad \psi_0 = \frac{S\omega}{2} + \frac{S\overline{\omega}}{2} \; ,$$

The mapping function (19.105) is

$$\omega = c\left(\zeta + \frac{m}{\zeta}\right) \; ; \quad \overline{\omega} = c\left(\overline{\zeta} + \frac{m}{\overline{\zeta}}\right) \; .$$

Using this to map ψ_0 into the ζ-plane and then setting $\zeta = s$, $\overline{\zeta} = 1/s$, corresponding to the unit circle $|\zeta| = 1$, we obtain

$$\psi_0(s) = \frac{Sc}{2}\left(\zeta + \frac{m}{\zeta} + \overline{\zeta} + \frac{m}{\overline{\zeta}}\right) = \frac{Sc(1+m)}{2}\left(s + \frac{1}{s}\right) \; ,$$

or

$$\psi_0(s) = \frac{Sa}{2}\left(s + \frac{1}{s}\right) \; ,$$

using (18.66). This is identical with the expression obtained for the special case of the crack in the example in §19.7.1, so similar operations to those reported there lead to

$$\chi_\zeta(\zeta) = -\frac{Sa}{2\zeta}$$

We also have

$$\overline{\psi_0} = \frac{Sc}{2}\left(\overline{\zeta} + \frac{m}{\overline{\zeta}} + \zeta + \frac{m}{\zeta}\right) = \psi_0 \; .$$

Also,

$$\chi_1' = \frac{Sa}{2\zeta^2} \; ; \quad \overline{\chi'} = \frac{Sa}{2\overline{\zeta}^2}$$

and

$$\frac{\zeta(1+m\zeta^2)\chi_1'(\zeta)}{(\zeta^2 - m)} = \frac{Sa(1+m\zeta^2)}{2\zeta(\zeta^2 - m)} \; ,$$

so, as in the text example, (19.107) gives

$$\theta_1 = -\frac{Sa}{2\zeta} - \frac{Sa(1+m\zeta^2)\chi_1'(\zeta)}{2\zeta(\zeta^2 - m)} = -\frac{Sa[(1-m) + (1+m)\zeta^2]}{2\zeta(\zeta^2 - m)} \; .$$

The inverse of the mapping function (choosing the positive square root) is

$$\zeta = \frac{\omega + \sqrt{\omega^2 - 4mc^2}}{2c} ,$$

so

$$\frac{1}{\zeta} = \frac{2c}{\omega + \sqrt{\omega^2 - 4mc^2}} = \frac{2c(\omega - \sqrt{\omega^2 - 4mc^2})}{\omega^2 - \omega^2 + 4mc^2} = \frac{\omega - \sqrt{\omega^2 - 4mc^2}}{2mc}$$

and

$$\begin{aligned} \zeta^2 - m &= \frac{\omega^2 + 2\omega\sqrt{\omega^2 - 4mc^2} + \omega^2 - 4mc^2 - 4mc^2}{4c^2} \\ &= \frac{\sqrt{\omega^2 - 4mc^2}(\omega + \sqrt{\omega^2 - 4mc^2})}{2c^2} \\ &= \frac{\zeta\sqrt{\omega^2 - 4mc^2}}{c} . \end{aligned}$$

so

$$\frac{\zeta}{(\zeta^2 - m)} = \frac{c}{\sqrt{\omega^2 - 4mc^2}} .$$

and

$$\frac{1}{\zeta(\zeta^2 - m)} = \frac{(\omega - \sqrt{\omega^2 - 4mc^2})^2}{4m^2c\sqrt{\omega^2 - 4mc^2}} .$$

Mapping the perturbed terms to the ω-plane and adding the unperturbed term, we obtain

$$\begin{aligned} \chi &= \frac{S\omega}{4} - \frac{Sa(\omega - \sqrt{\omega^2 - 4mc^2})}{4mc} = \frac{S\omega}{4} - \frac{S(m+1)(\omega - \sqrt{\omega^2 - 4mc^2})}{4m} \\ &= -\frac{S\omega}{4m} + \frac{S(m+1)\sqrt{\omega^2 - 4mc^2}}{4m} \end{aligned}$$

$$\begin{aligned} \theta &= \frac{S\omega}{2} - \frac{Sc(1+m)}{2}\left[\frac{(1-m)(\omega - \sqrt{\omega^2 - 4mc^2})^2}{4m^2c\sqrt{\omega^2 - 4mc^2}} + \frac{c(1+m)}{\sqrt{\omega^2 - 4mc^2}}\right] \\ &= \frac{S\omega}{2} - \frac{Sc(1+m)}{2\sqrt{\omega^2 - 4mc^2}}\left[\frac{(1-m)(\omega - \sqrt{\omega^2 - 4mc^2})^2}{4m^2c} + c(1+m)\right] \\ &= \frac{S\omega}{2} - \frac{Sc(1+m)}{2\sqrt{\omega^2 - 4mc^2}}\left[\frac{\omega^2(1-m)}{2m^2c} + \frac{c(m^2+2m-1)}{m} - \frac{(1-m)\omega\sqrt{\omega^2 - 4mc^2}}{2m^2c}\right] \\ &= \frac{S\omega}{2}\left(1 + \frac{(1+m)(1-m)}{2m^2}\right) - \frac{Sc(1+m)}{2\sqrt{\omega^2 - 4mc^2}}\left[\frac{\omega^2(1-m)}{2m^2c} + \frac{c(m^2+2m-1)}{m}\right] \end{aligned} \quad (1)$$

The maximum stress will occur at the point $(a, 0)$, where $\sigma_{\xi\xi} = \sigma_{\xi\eta} = 0$, since this is a traction-free boundary. It follows that

$$\sigma_{\eta\eta}(a, 0) = \Theta(a, 0) .$$

We have

$$\chi' = -\frac{S}{4m} + \frac{S(m+1)\omega}{4m\sqrt{\omega^2 - 4mc^2}}$$

and hence

$$\Theta = 2\left(-\frac{S}{4m} + \frac{S(m+1)\omega}{4m\sqrt{\omega^2 - 4mc^2}} - \frac{S}{4m} + \frac{S(m+1)\overline{\omega}}{4m\sqrt{\overline{\omega}^2 - 4mc^2}}\right) .$$

At $(a, 0)$,

$$\omega^2 - 4mc^2 \to a^2 - \frac{4ma^2}{(1+m)^2} = \frac{a^2(1-m)^2}{(1+m)^2}$$

so

$$\begin{aligned}
\Theta &= 2\left(-\frac{S}{4m} + \frac{S(m+1)^2}{4m(1-m)} - \frac{S}{4m} + \frac{S(m+1)^2}{4m(1-m)}\right) \\
&= \frac{S}{m}\left(-1 + \frac{(m+1)^2}{(1-m)}\right) \\
&= \frac{S(m+3)}{(1-m)}
\end{aligned}$$

and the stress concentration factor is

$$\frac{(m+3)}{(1-m)} = \frac{2a}{b} + 1 ,$$

using (18.66). This expression clearly tends to 3 when $a = b$ and increases without limit as $b/a \to 0$.

19.15. Use the method of §19.7.1 to find the complete stress field for Problem 13.4 in which a state of uniform shear stress $\sigma_{xy} = S$ is perturbed by the presence of a crack occupying the region $-a < x < a, y = 0$.

The unperturbed field is defined by

$$\Theta = 0 \; ; \quad \Phi = 2\imath S \; ,$$

and a particular solution for the complex potentials is

$$\chi_0 = 0 \; ; \quad \theta_0 = \imath S \omega \quad \text{giving} \quad \psi_0 = -\imath S \overline{\omega} \; .$$

Mapping this to the ζ-plane, we have

$$\psi_0(\zeta, \overline{\zeta}) = -\frac{\imath S a}{2} \left(\overline{\zeta} + \frac{1}{\overline{\zeta}} \right)$$

and on the unit circle $\zeta = s, \overline{\zeta} = 1/s$,

$$\psi_0(s) = -\frac{\imath S a}{2} \left(\frac{1}{s} + s \right) \; .$$

The corrective solution must therefore satisfy

$$\psi_1(s) = \frac{\imath S a}{2} \left(\frac{1}{s} + s \right) \; ,$$

so from (19.106),

$$\chi_\zeta(\zeta) = -\frac{1}{2\pi\imath} \oint \frac{\imath S a}{2} \left(\frac{1}{s} + s \right) \frac{ds}{(s - \zeta)}$$

This is the same integral as (19.108) except for a multiplier of $-\imath$, so we have

$$\chi_\zeta(\zeta) = \frac{\imath S a}{2\zeta} \; .$$

The conjugate function

$$\overline{\psi_1(s)} = -\frac{\imath S a}{2} \left(\frac{1}{\overline{s}} + \overline{s} \right) = -\frac{\imath S a}{2} \left(s + \frac{1}{s} \right) \; ,$$

so

$$\theta_\zeta(\zeta) = \frac{1}{2\pi\imath} \oint \frac{\imath S a}{2} \left(s + \frac{1}{s} \right) \frac{ds}{(s - \zeta)} + \frac{\zeta(1 + \zeta^2)}{(\zeta^2 - 1)} \frac{\imath S a}{2\zeta^2} = -\frac{\imath S a}{2\zeta} + \frac{\imath S a (1 + \zeta^2)}{2\zeta(\zeta^2 - 1)}$$

i.e.

$$\chi_\zeta(\zeta) = \frac{\imath S a}{2\zeta} \; ; \quad \theta_\zeta(\zeta) = \frac{\imath S a}{\zeta(\zeta^2 - 1)} \; .$$

We observe that χ_ζ is equal to $-\imath$ times the corresponding function in the example in §19.7.1, so it follows that

$$\chi(\omega) = \frac{-\imath S\sqrt{\omega^2 - a^2}}{2} + \frac{\imath S\omega}{2} \ .$$

For θ, we have

$$\zeta = \frac{\omega + \sqrt{\omega^2 - a^2}}{a}$$

$$\begin{aligned} \zeta^2 - 1 &= \frac{2\omega^2 + 2\omega\sqrt{\omega^2 - a^2} + \omega^2 - a^2 - a^2}{a^2} = \frac{2(\omega^2 - a^2 + \omega\sqrt{\omega^2 - a^2})}{a^2} \\ &= \frac{2\sqrt{\omega^2 - a^2}(\sqrt{\omega^2 - a^2} + \omega)}{a^2} \end{aligned}$$

and hence

$$\frac{\imath Sa}{\zeta(\zeta^2 - 1)} = \frac{\imath Sa^4}{2\sqrt{\omega^2 - a^2}(\sqrt{\omega^2 - a^2} + \omega)(\omega + \sqrt{\omega^2 - a^2})} = \frac{\imath S(\sqrt{\omega^2 - a^2} - \omega)^2}{2\sqrt{\omega^2 - a^2}} \ ,$$

giving the perturbation in $\theta(\omega)$ as

$$\theta_1(\omega) = -\imath S\omega + \frac{\imath S(2\omega^2 - a^2)}{2\sqrt{\omega^2 - a^2}} \ .$$

The complete solution (perturbed+unperturbed) is then

$$\chi = \frac{\imath S\omega}{2} - \frac{\imath S\sqrt{\omega^2 - a^2}}{2} \ ; \quad \theta = \frac{\imath S(2\omega^2 - a^2)}{2\sqrt{\omega^2 - a^2}} \ .$$

We then have

$$\chi' = \frac{\imath S}{2} - \frac{\imath S\omega}{2\sqrt{\omega^2 - a^2}}$$

$$\chi'' = -\frac{\imath S}{2\sqrt{\omega^2 - a^2}} + \frac{\imath S\omega^2}{2(\omega^2 - a^2)^{3/2}} = \frac{\imath Sa^2}{2(\omega^2 - a^2)^{3/2}}$$

$$\theta' = \frac{2\imath S\omega}{\sqrt{\omega^2 - a^2}} - \frac{\imath S\omega(2\omega^2 - a^2)}{2(\omega^2 - a^2)^{3/2}} = \frac{\imath S(4\omega^3 - 4a^2\omega - 2\omega^3 + a^2\omega)}{2(\omega^2 - a^2)^{3/2}} = \frac{\imath S(2\omega^3 - 3a^2\omega)}{2(\omega^2 - a^2)^{3/2}} \ .$$

and

$$\Theta = 2(\chi' + \overline{\chi}') = -\frac{\imath S\omega}{\sqrt{\omega^2 - a^2}} + \frac{\imath S\overline{\omega}}{\sqrt{\overline{\omega}^2 - a^2}}$$

$$\begin{aligned} \Phi &= -2(\omega\overline{\chi}'' + \overline{\theta}') \\ &= -2\left(-\frac{\imath Sa^2\omega}{2(\overline{\omega}^2 - a^2)^{3/2}} - \frac{\imath S(2\overline{\omega}^3 - 3a^2\overline{\omega})}{2(\overline{\omega}^2 - a^2)^{3/2}}\right) \\ &= \imath S\left(\frac{a^2\omega + 2\overline{\omega}^3 - 3a^2\overline{\omega}}{(\overline{\omega}^2 - a^2)^{3/2}}\right) \ . \end{aligned}$$

On the plane $\eta = 0$ to the right of the crack, $\omega = \overline{\omega} = \xi$ and

$$\Theta = 0 \quad \text{and} \quad \Phi = \imath S \left(\frac{a^2\xi + 2\xi^3 - 3a^2\xi}{(\xi^2 - a^2)^{3/2}} \right) = \frac{2\imath S\xi}{\sqrt{\xi^2 - a^2}} \ ,$$

corresponding to

$$\sigma_{xx} = \sigma_{yy} = 0 \ ; \quad \sigma_{xy} = \frac{S\xi}{\sqrt{\xi^2 - a^2}} \ .$$

which is the mode II equivalent of equation (13.44). Now check inside the crack, where $\omega = \overline{\omega} = \xi$, but

$$\sqrt{\zeta^2 - a^2} = \imath\sqrt{a^2 - \xi^2} \ ; \quad \sqrt{\overline{\zeta}^2 - a^2} = -\imath\sqrt{a^2 - \xi^2} \ ,$$

since it is the conjugate expression. We then obtain

$$\Theta = -\frac{\imath S\xi}{\imath\sqrt{a^2 - \xi^2}} + \frac{\imath S\xi}{-\imath\sqrt{a^2 - \xi^2}} = -\frac{2S\xi}{\sqrt{a^2 - \xi^2}}$$

and

$$\Phi = \imath S \left(\frac{a^2\xi + 2\xi^3 - 3a^2\xi}{\imath(a^2 - \xi^2)^{3/2}} \right) = -\frac{2S\xi}{\sqrt{a^2 - \xi^2}}$$

Thus,

$$\sigma_{xx} = \Re\left(\frac{\Phi + \Theta}{2}\right) = 0 \ ; \quad \sigma_{yy} = \Re\left(\frac{\Theta - \Phi}{2}\right) = -\frac{2S\xi}{\sqrt{a^2 - \xi^2}} \ ; \quad \sigma_{xy} = \frac{1}{2\imath}\Im(\Phi) = 0 \ ,$$

which satisfies the traction-free boundary condition.

CHAPTER 20

20.1. Show that the function

$$\psi = y\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial y}$$

will be harmonic ($\nabla^2\psi = 0$) if ϕ is harmonic. Show also that ψ will be biharmonic if ϕ is biharmonic.

Differentiating by parts, we have

$$\frac{\partial\psi}{\partial x} = y\frac{\partial^2\phi}{\partial x^2} - \frac{\partial\phi}{\partial y} - x\frac{\partial^2\phi}{\partial x\partial y} \tag{1}$$

and

$$\frac{\partial^2\psi}{\partial x^2} = y\frac{\partial^3\phi}{\partial x^3} - 2\frac{\partial^2\phi}{\partial x\partial y} - x\frac{\partial^3\phi}{\partial x^2\partial y}\ . \tag{2}$$

By a similar process, we also have

$$\frac{\partial^2\psi}{\partial y^2} = -x\frac{\partial^3\phi}{\partial y^3} + 2\frac{\partial^2\phi}{\partial x\partial y} + y\frac{\partial^3\phi}{\partial x\partial y^2} \tag{3}$$

$$\frac{\partial^2\psi}{\partial z^2} = y\frac{\partial^3\phi}{\partial x\partial z^2} - x\frac{\partial^3\phi}{\partial z^2\partial y}\ . \tag{4}$$

Summing the last three expressions, we obtain

$$\nabla^2\psi = y\frac{\partial}{\partial x}\nabla^2\phi - x\frac{\partial}{\partial y}\nabla^2\phi \tag{5}$$

In other words,

$$\nabla^2\left(y\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial y}\right) = \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\nabla^2\phi \tag{6}$$

and the differential operators

$$\nabla^2 \quad \text{and} \quad \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

commute.

Applying the same operation twice, we have

$$\begin{aligned}\nabla^4\left(y\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial y}\right) &= \nabla^2\left\{\nabla^2\left(y\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial y}\right)\right\} \\ &= \nabla^2\left\{\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\nabla^2\phi\right\} \\ &= \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\nabla^4\phi\ . \end{aligned} \tag{7}$$

It follows immediately from (6) that if ϕ is harmonic, ψ will be harmonic and from (7) that if ϕ is biharmonic, ψ will be biharmonic.

20.2. Find the most general solution of equation (20.4) that is spherically symmetric — i.e. depends only on the distance $R=\sqrt{x^2+y^2+z^2}$ from the origin. Use your solution to find the stress and displacement field in a hollow spherical container of inner radius b and outer radius a, loaded only by internal pressure p_0 at $R=b$. Note: Strain-displacement relations and expressions for the gradient and Laplacian operator in spherical polar coördinates are given in §21.4.2 below.

If a function ϕ is a function only of the distance R from the origin, the Laplace operator (21.35) reduces to

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial R^2} + \frac{2}{R}\frac{\partial\phi}{\partial R} = C \tag{1}$$

from (20.4). The general solution of this ordinary differential equation is

$$\phi = A + \frac{B}{R} + \frac{CR^2}{6} \ . \tag{2}$$

The only non-zero displacement component will be u_R where

$$2\mu u_R = \frac{\partial\phi}{\partial R} = -\frac{B}{R^2} + \frac{CR}{3} \tag{3}$$

and the non-zero strain components are

$$e_{RR} = \frac{\partial u_R}{\partial R} = \frac{B}{\mu R^3} + \frac{C}{6\mu} \tag{4}$$

$$e_{\theta\theta} = e_{\beta\beta} = \frac{u_R}{R} = -\frac{B}{2\mu R^3} + \frac{C}{6\mu} \ . \tag{5}$$

The dilatation is

$$e = e_{RR} + e_{\theta\theta} + e_{\beta\beta} = \frac{C}{2\mu} \tag{6}$$

and the non-zero stress components are

$$\sigma_{RR} = \lambda e + 2\mu e_{RR} = \frac{C\lambda}{2\mu} + \frac{2B}{R^3} + \frac{C}{3} \tag{7}$$

$$\sigma_{\theta\theta} = \sigma_{\beta\beta} = \lambda e + 2\mu e_{\theta\theta} = \frac{C\lambda}{2\mu} - \frac{B}{R^3} + \frac{C}{3} \ . \tag{8}$$

These equations can be simplified using (1.69), giving

$$\sigma_{RR} = \frac{C(1+\nu)}{3(1-2\nu)} + \frac{2B}{R^3} \ ; \quad \sigma_{\theta\theta} = \sigma_{\beta\beta} = \frac{C(1+\nu)}{3(1-2\nu)} - \frac{B}{R^3} \ . \tag{9}$$

For the hollow spherical container, the boundary conditions are

$$\sigma_{RR} = -p_0 \; ; \quad r = b \tag{10}$$

$$= 0 \; ; \quad r = a \; . \tag{11}$$

These imply

$$\frac{C(1+\nu)}{3(1-2\nu)} + \frac{2B}{b^3} = -p_0 \; ; \quad \frac{C(1+\nu)}{3(1-2\nu)} + \frac{2B}{a^3} = 0 \tag{12}$$

with solution

$$B = -\frac{p_0 b^3 a^3}{2(a^3 - b^3)} \; ; \quad C = \frac{3p_0(1-2\nu)b^3}{(1+\nu)(a^3-b^3)} \; . \tag{13}$$

The stresses and displacements are therefore

$$2\mu u_R = \frac{p_0 b^3 a^3}{2(a^3-b^3)R^2} + \frac{p_0(1-2\nu)b^3 R}{(1+\nu)(a^3-b^3)} \tag{14}$$

$$\sigma_{RR} = -\frac{p_0 b^3}{(a^3-b^3)}\left(\frac{a^3}{R^3} - 1\right) \tag{15}$$

$$\sigma_{\theta\theta} = \sigma_{\beta\beta} = \frac{p_0 b^3}{(a^3-b^3)}\left(1 + \frac{a^3}{2R^3}\right) \; . \tag{16}$$

20.3. By expressing the vector operators in index notation, show that

$$\boldsymbol{\nabla}\mathbf{div}\ \boldsymbol{\nabla}\mathbf{div}\ \boldsymbol{V} \equiv \nabla^2\boldsymbol{\nabla}\mathbf{div}\ \boldsymbol{V} \equiv \boldsymbol{\nabla}\mathbf{div}\ \nabla^2\boldsymbol{V}\ ,$$

where $\boldsymbol{V}$ is any vector field.

From (1.10, 1.11, 1.13), we have

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial x_i}\ ; \quad \text{div}\ \boldsymbol{V} = \frac{\partial V_j}{\partial x_j}\ ; \quad \nabla^2 f = \frac{\partial^2 f}{\partial x_k \partial x_k}\ .$$

It follows that

$$\boldsymbol{\nabla}\text{div}\ \boldsymbol{V} = \frac{\partial^2 V_j}{\partial x_j \partial x_i}$$

and

$$\boldsymbol{\nabla}\text{div}\ \boldsymbol{\nabla}\text{div}\ \boldsymbol{V} = \frac{\partial^4 V_j}{\partial x_j \partial x_i \partial x_i \partial x_k}$$

and

$$\nabla^2\boldsymbol{\nabla}\text{div}\ \boldsymbol{V} = \frac{\partial^4 V_j}{\partial x_j \partial x_i \partial x_k \partial x_k}\ .$$

Also

$$\boldsymbol{\nabla}\text{div}\ \nabla^2\boldsymbol{V} = \frac{\partial^4 V_j}{\partial x_k \partial x_k \partial x_j \partial x_j}\ .$$

Since the symbols used for dummy indices (i.e. those that are repeated and hence summed) are arbitrary, we see that all three expressions are the same. In fact, these three vector operations correspond to the three similar terms in equation (20.9).

20.4. Starting from equation (20.10), use (1.51, 1.71, 1.69) to develop general expressions for the stress components σ_{ij} in the Galerkin formulation, in terms of F_i and the elastic constants μ, ν.

Equation (20.10) is

$$2\mu u_i = 2(1-\nu)\frac{\partial^2 F_i}{\partial x_j \partial x_j} - \frac{\partial^2 F_j}{\partial x_i \partial x_j} \ . \tag{1}$$

We then have

$$2\mu e = 2\mu \frac{\partial u_i}{\partial x_i} = 2(1-\nu)\frac{\partial^3 F_i}{\partial x_j \partial x_j \partial x_i} - \frac{\partial^3 F_j}{\partial x_j \partial x_i \partial x_i} = (1-2\nu)\frac{\partial^3 F_i}{\partial x_j \partial x_j \partial x_i} \tag{2}$$

$$\begin{aligned}
2\mu e_{kl} &= \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \\
&= \frac{1}{2}\left(2(1-\nu)\frac{\partial^3 F_k}{\partial x_j \partial x_j \partial x_l} - \frac{\partial^3 F_j}{\partial x_j \partial x_k \partial x_l} + 2(1-\nu)\frac{\partial^3 F_l}{\partial x_j \partial x_j \partial x_k} - \frac{\partial^3 F_j}{\partial x_j \partial x_l \partial x_k}\right) \\
&= (1-\nu)\frac{\partial^2}{\partial x_j \partial x_j}\left(\frac{\partial F_k}{\partial x_l} + \frac{\partial F_l}{\partial x_k}\right) - \frac{\partial^3 F_j}{\partial x_j \partial x_k \partial x_l} \ .
\end{aligned} \tag{3}$$

The stress components are

$$\begin{aligned}
\sigma_{kl} &= \lambda \delta_{kl} e + 2\mu e_{kl} \\
&= \frac{\lambda(1-2\nu)\delta_{kl}}{2\mu}\frac{\partial^3 F_i}{\partial x_j \partial x_j \partial x_i} + (1-\nu)\frac{\partial^2}{\partial x_j \partial x_j}\left(\frac{\partial F_k}{\partial x_l} + \frac{\partial F_l}{\partial x_k}\right) - \frac{\partial^3 F_j}{\partial x_j \partial x_k \partial x_l} \\
&= \nu \delta_{kl}\frac{\partial^3 F_i}{\partial x_j \partial x_j \partial x_i} + (1-\nu)\frac{\partial^2}{\partial x_j \partial x_j}\left(\frac{\partial F_k}{\partial x_l} + \frac{\partial F_l}{\partial x_k}\right) - \frac{\partial^3 F_j}{\partial x_j \partial x_k \partial x_l} \ .
\end{aligned} \tag{4}$$

20.5. Verify that the function $\psi = 1/\sqrt{x^2+y^2+z^2}$ is harmonic everywhere except at the origin, where it is singular. This function can therefore be used as a Papkovich-Neuber displacement function for a body that does not contain the origin, such as the infinite body with a spherical hole centred on the origin. Show by partial integration that it is impossible to find a function H such that

$$\frac{\partial H}{\partial z} = \psi$$

and $\nabla^2 H = 0$, except possibly at the origin.

Differentiating, we have

$$\frac{\partial \psi}{\partial x} = -\frac{x}{(x^2+y^2+z^2)^{3/2}} \; ; \quad \frac{\partial^2 \psi}{\partial x^2} = -\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} \tag{1}$$

and hence

$$\begin{aligned} \nabla^2 \psi &\equiv \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial y^2} \\ &= -\frac{3}{(x^2+y^2+z^2)^{3/2}} + \frac{3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} \\ &= 0 \; . \end{aligned} \tag{2}$$

Thus, the function ψ is indeed harmonic.

Integrating the relation

$$\frac{\partial H}{\partial z} = \psi \; , \tag{3}$$

we obtain

$$H = \int \psi dz = \ln(\sqrt{x^2+y^2+z^2} + z) + f(x,y) \; , \tag{4}$$

where $f(x,y)$ is an arbitrary function of x and y only. If H is to be harmonic, we require

$$\nabla^2 H = 0 \tag{5}$$

and substituting from (4) and performing the differentiation, this reduces to

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \; . \tag{6}$$

Writing $r = \sqrt{x^2+y^2}$, we have

$$\sqrt{x^2+y^2+z^2} + z = \sqrt{r^2+z^2} + z \; . \tag{7}$$

If $r/|z| \to 0$,

$$\sqrt{r^2+z^2} = |z|\sqrt{1+\frac{r^2}{z^2}} \to |z|\left(1+\frac{r^2}{2z^2}\right) \tag{8}$$

and if $z < 0$, $|z| = -z$ and

$$\sqrt{x^2 + y^2 + z^2} + z \to \frac{r^2}{-2z} \to 0 \ . \tag{9}$$

Thus, the function H will have a logarithmic singularity at all points on the negative z-axis unless there is a cancelling singularity in the function $f(x, y)$. This singularity leads to a discontinuity in derivatives through the origin. For example, the derivative

$$\frac{\partial}{\partial x} \ln(\sqrt{x^2 + y^2 + z^2} + z) = \frac{x}{(\sqrt{x^2 + y^2 + z^2} + z)\sqrt{x^2 + y^2 + z^2}} \tag{10}$$

and at the point $(x, 0, -a)$ $(x \ll a)$ this can be approximated by

$$\frac{2a}{x\sqrt{x^2 + a^2}}$$

which changes sign from $+\infty$ to $-\infty$ as x passes through zero. Thus, the function will not be harmonic at any such singular point.

However, if $f(x, y)$ has a cancelling singularity, H will then be singular on the *positive* z-axis, since $f(x, y)$ is independent of z and must have the same behaviour on both positive and negative axes, whereas the logarithmic function in (4) is bounded on the positive z-axis, where

$$\sqrt{x^2 + y^2 + z^2} + z \to 2z \ . \tag{11}$$

Thus, no choice of $f(x, y)$ will make the function H harmonic on both positive and negative z-axes.

20.6. Repeat the derivation of §20.3 using equation (20.44) in place of (20.12) in order to generalize the Papkovich-Neuber solution to include a body force field $\boldsymbol{p}$. In particular, find the governing equations for the functions $\boldsymbol{\psi}, \phi$.

The Galerkin solution for a general body force $\boldsymbol{p}$ can be written in the form of equation (20.11) — i.e.

$$2\mu\mathbf{u} = 2(1-\nu)\nabla^2\boldsymbol{F} - \boldsymbol{\nabla}\text{div } \boldsymbol{F} \ , \tag{1}$$

where

$$\nabla^4\boldsymbol{F} = -\frac{\boldsymbol{p}}{(1-\nu)} \tag{2}$$

from (20.44).

As in §20.3, we define the function $\boldsymbol{\psi}$ such that

$$\boldsymbol{\psi} = -\frac{1}{2}\nabla^2\boldsymbol{F} \ , \tag{3}$$

from which we have

$$\nabla^2\boldsymbol{\psi} = -\frac{1}{2}\nabla^4\boldsymbol{F} = \frac{\boldsymbol{p}}{2(1-\nu)} \ , \tag{4}$$

using (2).

Since $\boldsymbol{\psi}$ is not harmonic, we shall get additional terms in equation (20.14). For example, we have

$$\nabla^2(x\psi_x) = 2\frac{\partial\psi_x}{\partial x} + x\nabla^2(\psi_x) \tag{5}$$

and the second term will not be zero if $\boldsymbol{\psi}$ is not harmonic. It follows that

$$\begin{aligned}
\nabla^2(\boldsymbol{r}.\boldsymbol{\psi}) &= \nabla^2(x\psi_x + y\psi_y + z\psi_z) \\
&= 2 \text{ div } \boldsymbol{\psi} + \boldsymbol{r}.\nabla^2\boldsymbol{\psi} \\
&= 2 \text{ div } \boldsymbol{\psi} + \frac{\boldsymbol{r}.\boldsymbol{p}}{2(1-\nu)} \\
&= -\nabla^2 \text{ div } \boldsymbol{F} + \frac{\boldsymbol{r}.\boldsymbol{p}}{2(1-\nu)} \ ,
\end{aligned} \tag{6}$$

using (3) and (4).

If we define the function

$$\phi \equiv -\boldsymbol{r}.\boldsymbol{\psi} - \text{div } \boldsymbol{F} \tag{7}$$

so that

$$-\text{div } \boldsymbol{F} = \boldsymbol{r}.\boldsymbol{\psi} + \phi \ , \tag{8}$$

we have

$$\nabla^2\phi = -\frac{\boldsymbol{r}.\boldsymbol{p}}{2(1-\nu)} \ , \tag{9}$$

from (6).

We can then eliminate div $\boldsymbol{F}$ and $\nabla^2\boldsymbol{F}$ from (1), using (3) and (8), with the result

$$2\mu\boldsymbol{u} = -4(1-\nu)\boldsymbol{\psi} + \boldsymbol{\nabla}(\boldsymbol{r}.\boldsymbol{\psi} + \phi) \ , \tag{10}$$

where

$$\nabla^2\boldsymbol{\psi} = \frac{\boldsymbol{p}}{2(1-\nu)} \ ; \quad \nabla^2\phi = -\frac{\boldsymbol{r}.\boldsymbol{p}}{2(1-\nu)} \ . \tag{11}$$

20.7. Find a particular solution for the Galerkin vector $\boldsymbol{F}$ for the non-conservative body force field

$$p_x = -\rho\dot{\Omega}y \;\; ; \;\; p_y = \rho\dot{\Omega}x \;\; ; \;\; p_z = 0 \; .$$

The Galerkin vector has to satisfy equation (20.44), which in the present case can be written

$$\nabla^4 F_x = \frac{\rho\dot{\Omega}y}{(1-\nu)} \tag{1}$$

$$\nabla^4 F_y = -\frac{\rho\dot{\Omega}x}{(1-\nu)} \tag{2}$$

$$\nabla^4 F_z = 0 \; . \tag{3}$$

There is no unique solution for this problem. We note that ∇^4 is a 4th order operator, so to obtain linear functions of the right hand side, we need F_x, F_y to be 5th order in x or y. A simple choice would be

$$F_x = C_1 y^5 \; ; \quad F_y = C_2 x^5 \; ; \quad F_z = 0 \; . \tag{4}$$

Substitution into (1,2) then gives

$$C_1 = \frac{\rho\dot{\Omega}}{120(1-\nu)} \; ; \quad C_2 = -\frac{\rho\dot{\Omega}}{120(1-\nu)} \; . \tag{5}$$

A more elegant or 'symmetrical' solution would recognize that

$$\boldsymbol{p} = -\rho\dot{\Omega}\boldsymbol{r} \times \boldsymbol{k} \tag{6}$$

or in cylindrical polar coördinates,

$$p_r = 0 \; ; \quad p_\theta = \rho\dot{\Omega}r \; ; \quad p_z = 0 \tag{7}$$

and seek a similar form for $\boldsymbol{F}$. We can write

$$\boldsymbol{F} = Cr^5 \boldsymbol{e_\theta} \; , \tag{8}$$

where $\boldsymbol{e_\theta}$ is the unit vector in the θ-direction. We then have

$$F_x = -Cr^5 \sin\theta = -C(x^2+y^2)y \; ; \quad F_y = Cr^5 \cos\theta = C(x^2+y^2)x \tag{9}$$

and substitution into (1,2) yields

$$C = -\frac{\rho\dot{\Omega}}{168(1-\nu)} \; . \tag{10}$$

Thus, in this particular solution

$$F_x = \frac{\rho\dot{\Omega}(x^2+y^2)y}{168(1-\nu)} \; ; \quad F_y = -\frac{\rho\dot{\Omega}(x^2+y^2)x}{168(1-\nu)} \; ; \quad F_z = 0 \; , \tag{11}$$

or

$$F_r = 0 \; ; \quad F_\theta = -\frac{\rho\dot{\Omega}r^5}{168(1-\nu)} \; ; \quad F_z = 0 \; . \tag{12}$$

20.8 Prove the result (used in deriving (20.24)) that

$$\textbf{curl } \boldsymbol{\nabla}\phi = 0 \ ,$$

for any scalar function ϕ.

For a vector $\boldsymbol{V}$, we have

$$\text{curl } \boldsymbol{V} = \boldsymbol{\nabla}\times\boldsymbol{V} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} \ ,$$

so, for example, the z-component of curl $\boldsymbol{V}$ is

$$\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \ .$$

However, if $\boldsymbol{V} = \boldsymbol{\nabla}\phi$,

$$V_y = \frac{\partial \phi}{\partial y} \ ; \quad V_x = \frac{\partial \phi}{\partial x}$$

and

$$\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} = 0 \ .$$

Thus, the z-component of curl $\boldsymbol{\nabla}\phi$ is zero and a similar proof can be used for the remaining components.

20.9. (i) Verify that the Galerkin vector

$$\boldsymbol{F} = Cx^3y\boldsymbol{k}$$

satisfies the governing equation (20.12) and determine the corresponding displacement vector $\boldsymbol{u}$.

(ii) Show that

$$\boldsymbol{F}_0 = \Big((1-2\nu)\nabla^2 + \boldsymbol{\nabla}\mathbf{div}\ \Big)\boldsymbol{f}$$

defines a null Galerkin vector (i.e. the corresponding displacements are everywhere zero) if $\nabla^4\boldsymbol{f} = 0$.

(iii) Using this result or otherwise, find an alternative Galerkin vector for the displacement field of part (i) that is confined to the xy-plane — i.e. such that $F_z = 0$.

(i)

$$\nabla^2\boldsymbol{F} = \frac{\partial^2\boldsymbol{F}}{\partial x^2} + \frac{\partial^2\boldsymbol{F}}{\partial y^2} + \frac{\partial^2\boldsymbol{F}}{\partial z^2} = 6Cxy\boldsymbol{k}$$

so

$$\nabla^4\boldsymbol{F} = \nabla^2\left(\nabla^2\boldsymbol{F}\right) = \nabla^2\left(6Cxy\boldsymbol{k}\right) = 0\ .$$

We also have

$$\text{div}\ \boldsymbol{F} = \frac{\partial}{\partial z}\left(Cx^3y\right) = 0\ ,$$

so from (20.11)

$$2\mu\boldsymbol{u} = 2(1-\nu)\nabla^2\boldsymbol{F} - \boldsymbol{\nabla}\text{div}\ \boldsymbol{F} = 12C(1-\nu)xy\boldsymbol{k}\ ,$$

or

$$u_x = u_y = 0\ ; \quad u_z = \frac{6C(1-\nu)xy}{\mu}\ .$$

(ii) If

$$\boldsymbol{F}_0 = \Big((1-2\nu)\nabla^2 + \boldsymbol{\nabla}\text{div}\ \Big)\boldsymbol{f}$$

is used as a Galerkin vector, then

$$\begin{aligned} 2\mu\boldsymbol{u} &= 2(1-\nu)\nabla^2\boldsymbol{F_0} - \boldsymbol{\nabla}\text{div}\ \boldsymbol{F}_0 \\ &= 2(1-\nu)(1-2\nu)\nabla^4\boldsymbol{f} + 2(1-\nu)\nabla^2\boldsymbol{\nabla}\text{div}\ \boldsymbol{f} \\ &\quad -(1-2\nu)\boldsymbol{\nabla}\text{div}\ \nabla^2\boldsymbol{f} - \boldsymbol{\nabla}\text{div}\ \boldsymbol{\nabla}\text{div}\ \boldsymbol{f}\ . \end{aligned}$$

The last three operators are identical (see Problem 20.3) and

$$2(1-\nu) - (1-2\nu) - 1 = 0\ ,$$

so

$$2\mu \boldsymbol{u} = 2(1-\nu)(1-2\nu)\nabla^4 \boldsymbol{f} = 0$$

if $\boldsymbol{f}$ is biharmonic.

(iii) To find an alternative Galerkin vector confined to the xy-plane, we need to subtract a null vector with the same z-component as $\boldsymbol{F}$.

Suppose we use the form of Part (ii) and choose a biharmonic vector $\boldsymbol{f}$ that is confined to the xy-plane. The z-component of $\boldsymbol{F}_0$ will then be

$$\frac{\partial}{\partial z}\text{div } \boldsymbol{f} = \frac{\partial^2 f_x}{\partial x \partial z} + \frac{\partial^2 f_y}{\partial y \partial z} = Cx^3 y \ .$$

We can obtain a partial integral of this equation (for example) by writing

$$f_x = 0 \ ; \quad \frac{\partial f_y}{\partial z} = \int Cx^3 y dy = \frac{Cx^3 y^2}{2} + g(x,z) \ ; \quad f_z = 0 \ .$$

Substituting into the biharmonic equation, we find

$$\nabla^4 f_y = 12Cx + \nabla^4 g = 0 \quad \text{so} \quad \nabla^4 g = -12Cx \ .$$

A particular integral of this equation is

$$g = -\frac{Cx^5}{10} \quad \text{so} \quad \frac{\partial f_y}{\partial z} = \frac{Cx^3 y^2}{2} - \frac{Cx^5}{10} \ .$$

Repeating the partial integration procedure, we have

$$f_y = \frac{Cx^3 y^2 z}{2} - \frac{Cx^5 z}{10} + h(x,y)$$

and

$$\nabla^4 f_y = 12Cxz - 12Cxz + \nabla^4 h = 0 \quad \text{so} \quad \nabla^4 h = 0 \ .$$

Thus, a suitable null field is defined by

$$f_x = 0 \ ; \quad f_y = \frac{Cx^3 y^2 z}{2} - \frac{Cx^5 z}{10} \ ; \quad f_z = 0$$

and hence

$$\text{div } \boldsymbol{f} = \frac{\partial f_y}{\partial y} = Cx^3 yz \ ; \quad \nabla^2 f_y = 3Cxy^2 z + Cx^3 z - 2Cx^3 z = 3Cxy^2 z - Cx^3 z$$

and

$$\begin{aligned}
\boldsymbol{F}_0 &= \Big((1-2\nu)\nabla^2 + \boldsymbol{\nabla}\text{div}\Big)\boldsymbol{f} \\
&= 3\boldsymbol{j}C(1-2\nu)xy^2 z - \boldsymbol{j}C(1-2\nu)x^3 z + 3C\boldsymbol{i}x^2 yz + C\boldsymbol{j}x^3 z + C\boldsymbol{k}x^3 y \\
&= 3\boldsymbol{j}C(1-2\nu)xy^2 z + 2\boldsymbol{j}C\nu x^3 z + 3C\boldsymbol{i}x^2 yz + C\boldsymbol{k}x^3 y \ .
\end{aligned}$$

We notice that this has the required z-component, so subtracting it from the original Galerkin vector we obtain

$$\boldsymbol{F} = -3C\boldsymbol{i}x^2yz - 3\boldsymbol{j}C(1-2\nu)xy^2z - 2\boldsymbol{j}C\nu x^3z \ .$$

As a check on this result, let's calulate the displacement field cooresponding to this new Galerkin vector. We have

$$\begin{aligned}
\nabla^2\boldsymbol{F} &= \frac{\partial^2\boldsymbol{F}}{\partial x^2} + \frac{\partial^2\boldsymbol{F}}{\partial y^2} + \frac{\partial^2\boldsymbol{F}}{\partial z^2} \\
&= -6C\boldsymbol{i}yz - 6\boldsymbol{j}C(1-2\nu)xz - 12\boldsymbol{j}C\nu xz - 6C\boldsymbol{j}xz \\
&= -6C\boldsymbol{i}yz - 6\boldsymbol{j}Cxz \ ,
\end{aligned}$$

so

$$\nabla^4\boldsymbol{F} = \nabla^2\left(\nabla^2\boldsymbol{F}\right) = \nabla^2\left(-6C\boldsymbol{i}yz - 6\boldsymbol{j}Cxz\right) = 0 \ .$$

We also have

$$\text{div } \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = -6Cxyz - 6C(1-2\nu)xyz = -12C(1-\nu)xyz \ ,$$

so from (20.11)

$$\begin{aligned}
2\mu\boldsymbol{u} &= 2(1-\nu)\nabla^2\boldsymbol{F} - \boldsymbol{\nabla}\text{div } \boldsymbol{F} \\
&= -12C(1-\nu)\boldsymbol{i}yz - 12\boldsymbol{j}C(1-\nu)xz + 12\boldsymbol{i}C(1-\nu)yz \\
&\quad +12\boldsymbol{j}C(1-\nu)xz + 12\boldsymbol{k}C(1-\nu)xy \\
&= 12\boldsymbol{k}C(1-\nu)xy \ ,
\end{aligned}$$

or

$$u_x = u_y = 0 \ ; \quad u_z = \frac{6C(1-\nu)xy}{\mu}$$

as before.

CHAPTER 21

21.1. Use solution E to solve the elementary torsion problem for a cylindrical bar of radius b transmitting a torque T, the surface $r = b$ being traction-free. You will need to find a suitable harmonic potential function — it will be an axisymmetric polynomial.

From the elementary Mechanics of Materials theory, we anticipate $\sigma_{\theta z} \sim r$ and hence $\psi \sim r^2 z$ — i.e. a 3rd order polynomial in r, z.

Only even powers of r are permissible to avoid discontinuities in derivatives at the axis. Thus, we include the terms

$$\psi = Ar^2 z + Bz^3 \ . \tag{1}$$

The function ψ is required to be harmonic, so

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} = 2Az + 2Az + 6Bz = 0 \ , \tag{2}$$

from which $A = -3B/2$ and

$$\psi = -\frac{B}{2}(3r^2 z - 2z^3) \ . \tag{3}$$

Substituting this expression into Solution E, Table 21.1, we have

$$\sigma_{\theta z} = -\frac{\partial^2 \psi}{\partial r \partial z} = 3Br \ ; \tag{4}$$

$$\sigma_{\theta r} = \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{\partial^2 \psi}{\partial r^2} = -\frac{3B}{2}(2z - 2z) = 0 \ . \tag{5}$$

All the other stress components can be seen to be zero by inspection, since they involve derivatives with respect to θ.

This solution clearly satisfies the boundary condition that the surface $r = b$ be traction-free (i.e. $\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0$). To determine the constant B, we impose the equilibrium condition

$$T = 2\pi \int_0^b r\sigma_{\theta z} r dr = 6\pi B \int_0^b r^3 dr = \frac{3\pi B b^4}{2} \ . \tag{6}$$

Thus, $B = 2T/3\pi b^4$,

$$\psi = -\frac{T(3r^2 z - 2z^3)}{3\pi b^4} \ , \tag{7}$$

$$2\mu u_\theta = -2\frac{\partial \psi}{\partial r} = \frac{4Trz}{\pi b^4} \tag{8}$$

— i.e.

$$u_\theta = \frac{2Trz}{\mu\pi b^4} \ , \tag{9}$$

$$\sigma_{\theta z} = \frac{2Tr}{\pi b^4} \tag{10}$$

and all other stress and displacement components are zero.

All of these results agree with the those of the elementary torsion theory, confirming that the latter is an exact theory for the case of uniform torsion of a circular cylinder.

21.2. Find a way of taking a combination of solutions A,B,E so as to satisfy identically the global conditions

$$\sigma_{zy} = \sigma_{zz} = 0 \ ; \ \textbf{all } x, y, \ \ z = 0$$

Thus, σ_{zx} should be the only non-zero traction component on the surface.

On the surface $z = 0$, the combination of Solutions A,B,E gives

$$\sigma_{zz} = \frac{\partial^2 \phi}{\partial z^2} - 2(1-\nu)\frac{\partial \omega}{\partial z} \ ; \tag{1}$$

$$\sigma_{yz} = \frac{\partial^2 \phi}{\partial y \partial z} - (1-2\nu)\frac{\partial \omega}{\partial y} - \frac{\partial^2 \Psi}{\partial x \partial z} \ . \tag{2}$$

To make these expressions go to zero, we introduce a new harmonic function χ and define ϕ, ψ, ω in terms of it through

$$\phi = 2(1-\nu)\frac{\partial \chi}{\partial x} \quad ; \quad \omega = \frac{\partial^2 \chi}{\partial x \partial z} \quad ; \quad \Psi = \frac{\partial \chi}{\partial y} \ . \tag{3}$$

Then, on $z = 0$,

$$\sigma_{zz} = 2(1-\nu)\frac{\partial^3 \chi}{\partial z^2 \partial x} - 2(1-\nu)\frac{\partial^3 \chi}{\partial z^2 \partial x} = 0 \ ; \tag{4}$$

$$\sigma_{yz} = 2(1-\nu)\frac{\partial^3 \chi}{\partial x \partial y \partial z} - (1-2\nu)\frac{\partial^3 \chi}{\partial x \partial y \partial z} - \frac{\partial^3 \chi}{\partial x \partial y \partial z} = 0 \ . \tag{5}$$

We now use this substitution to generate expressions for the complete stress and displacement components throughout the half-space. We have

$$2\mu u_x = \frac{\partial \phi}{\partial x} + z\frac{\partial \omega}{\partial x} + 2\frac{\partial \Psi}{\partial y} = 2(1-\nu)\frac{\partial^2 \chi}{\partial x^2} + z\frac{\partial^3 \chi}{\partial x^2 \partial z} + 2\frac{\partial^2 \chi}{\partial y^2} \ ; \tag{6}$$

$$\begin{aligned} 2\mu u_y &= \frac{\partial \phi}{\partial y} + z\frac{\partial \omega}{\partial y} - 2\frac{\partial \Psi}{\partial x} = 2(1-\nu)\frac{\partial^2 \chi}{\partial x \partial y} + z\frac{\partial^3 \chi}{\partial x \partial y \partial z} - 2\frac{\partial^2 \chi}{\partial x \partial y} \\ &= -2\nu\frac{\partial^2 \chi}{\partial x \partial y} + z\frac{\partial^3 \chi}{\partial x \partial y \partial z} \ ; \end{aligned} \tag{7}$$

$$\begin{aligned} 2\mu u_z &= \frac{\partial \phi}{\partial z} + z\frac{\partial \omega}{\partial z} - (3-4\nu)\omega = 2(1-\nu)\frac{\partial^2 \chi}{\partial x \partial z} + z\frac{\partial^3 \chi}{\partial z^2 \partial x} - (3-4\nu)\frac{\partial^2 \chi}{\partial x \partial z} \\ &= -(1-2\nu)\frac{\partial^2 \chi}{\partial x \partial z} + z\frac{\partial^3 \chi}{\partial z^2 \partial x} \ ; \end{aligned} \tag{8}$$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial x^2} + z\frac{\partial^2 \omega}{\partial x^2} - 2\nu\frac{\partial \omega}{\partial z} + 2\frac{\partial^2 \Psi}{\partial x \partial y}$$

$$
\begin{aligned}
&= 2(1-\nu)\frac{\partial^3\chi}{\partial x^3} + z\frac{\partial^4\chi}{\partial x^3\partial z} - 2\nu\frac{\partial^3\chi}{\partial z^2\partial x} + 2\frac{\partial^3\chi}{\partial y^2\partial x}\ ; && (9)\\
\sigma_{xy} &= \frac{\partial^2\phi}{\partial x\partial y} + z\frac{\partial^2\omega}{\partial x\partial y} + \frac{\partial^2\Psi}{\partial y^2} - \frac{\partial^2\Psi}{\partial x^2}\\
&= 2(1-\nu)\frac{\partial^3\chi}{\partial x^2\partial y} + z\frac{\partial^4\chi}{\partial x^2\partial y\partial z} + \frac{\partial^3\chi}{\partial y^3} - \frac{\partial^3\chi}{\partial x^2\partial y}\\
&= (1-2\nu)\frac{\partial^3\chi}{\partial x^2\partial y} + \frac{\partial^3\chi}{\partial y^3} + z\frac{\partial^4\chi}{\partial x^2\partial y\partial z}\ ; && (10)\\
\sigma_{yy} &= \frac{\partial^2\phi}{\partial y^2} + z\frac{\partial^2\omega}{\partial y^2} - 2\nu\frac{\partial\omega}{\partial z} - 2\frac{\partial^2\Psi}{\partial x\partial y}\\
&= 2(1-\nu)\frac{\partial^3\chi}{\partial y^2\partial x} + z\frac{\partial^4\chi}{\partial y^2\partial x\partial z} - 2\nu\frac{\partial^3\chi}{\partial z^2\partial x} - 2\frac{\partial^3\chi}{\partial y^2\partial x}\\
&= -2\nu\left(\frac{\partial^3\chi}{\partial y^2\partial x} + \frac{\partial^3\chi}{\partial z^2\partial x}\right) + z\frac{\partial^4\chi}{\partial y^2\partial x\partial z}\\
&= 2\nu\frac{\partial^3\chi}{\partial x^3} + z\frac{\partial^4\chi}{\partial y^2\partial x\partial z}\ ; && (11)\\
\sigma_{xz} &= \frac{\partial^2\phi}{\partial x\partial z} + z\frac{\partial^2\omega}{\partial x\partial z} - (1-2\nu)\frac{\partial\omega}{\partial x} + \frac{\partial^2\Psi}{\partial y\partial z}\\
&= 2(1-\nu)\frac{\partial^3\chi}{\partial x^2\partial z} + z\frac{\partial^4\chi}{\partial x^2\partial z^2} - (1-2\nu)\frac{\partial^3\chi}{\partial x^2\partial z} + \frac{\partial^3\chi}{\partial y^2\partial z}\\
&= \frac{\partial^3\chi}{\partial x^2\partial z} + \frac{\partial^3\chi}{\partial y^2\partial z} + z\frac{\partial^4\chi}{\partial x^2\partial z^2}\\
&= -\frac{\partial^3\chi}{\partial z^3} + z\frac{\partial^4\chi}{\partial x^2\partial z^2}\ ; && (12)\\
\sigma_{yz} &= \frac{\partial^2\phi}{\partial y\partial z} + z\frac{\partial^2\omega}{\partial y\partial z} - (1-2\nu)\frac{\partial\omega}{\partial y} - \frac{\partial^2\Psi}{\partial x\partial z}\\
&= 2(1-\nu)\frac{\partial^3\chi}{\partial x\partial y\partial z} + z\frac{\partial^4\chi}{\partial x\partial y\partial z^2} - (1-2\nu)\frac{\partial^3\chi}{\partial x\partial y\partial z} - \frac{\partial^3\chi}{\partial x\partial y\partial z}\\
&= +z\frac{\partial^4\chi}{\partial x\partial y\partial z^2}\ ; && (13)\\
\sigma_{zz} &= \frac{\partial^2\phi}{\partial z^2} + z\frac{\partial^2\omega}{\partial z^2} - 2(1-\nu)\frac{\partial\omega}{\partial z}\\
&= 2(1-\nu)\frac{\partial^3\chi}{\partial z^2\partial x} + z\frac{\partial^4\chi}{\partial x\partial z^3} - 2(1-\nu)\frac{\partial^3\chi}{\partial z^2\partial x}\\
&= z\frac{\partial^4\chi}{\partial x\partial z^3}\ . && (14)
\end{aligned}
$$

Notice that the last two stress components σ_{yz}, σ_{zz} are identically zero on $z = 0$ as required, because of the z multiplier.

21.3. An important class of frictional contact problems involves the steady sliding of an indenter in the x-direction across the surface of the half-space $z>0$. Assuming Coulomb's friction law to apply with coefficient of friction f, the appropriate boundary conditions are

$$\sigma_{xz} = -f\sigma_{zz} \;\; ; \;\; \sigma_{zy} = 0 \; , \qquad (21.52) \tag{1}$$

in the contact area, the rest of the surface $z=0$ being traction-free.

Show that (21.52) are actually *global* conditions for this problem and find a way of combining Solutions A,B,E so as to satisfy them identically.

We first note that in the traction-free region, all three tractions $\sigma_{xz}, \sigma_{zy}, \sigma_{zz} = 0$ and hence $\sigma_{xz} = -f\sigma_{zz}$ *a fortiori*.

Taking a combination of solutions A,B,E, we have the following tractions on $z = 0$:-

$$\sigma_{zx} = \frac{\partial^2\phi}{\partial x\partial z} - (1-2\nu)\frac{\partial\omega}{\partial x} + \frac{\partial^2\Psi}{\partial y\partial z} \; ; \tag{2}$$

$$\sigma_{yz} = \frac{\partial^2\phi}{\partial y\partial z} - (1-2\nu)\frac{\partial\omega}{\partial y} - \frac{\partial^2\Psi}{\partial x\partial z} \; ; \tag{3}$$

$$\sigma_{zz} = \frac{\partial^2\phi}{\partial z^2} - 2(1-\nu)\frac{\partial\omega}{\partial z} \; . \tag{4}$$

The two global conditions therefore require that

$$\frac{\partial^2\phi}{\partial x\partial z} - (1-2\nu)\frac{\partial\omega}{\partial x} + \frac{\partial^2\Psi}{\partial y\partial z} = -f\left(\frac{\partial^2\phi}{\partial z^2} - 2(1-\nu)\frac{\partial\omega}{\partial z}\right) \; ; \tag{5}$$

$$\frac{\partial^2\phi}{\partial y\partial z} - (1-2\nu)\frac{\partial\omega}{\partial y} - \frac{\partial^2\Psi}{\partial x\partial z} = 0 \; ; \tag{6}$$

Equation (6) can be satisfied by defining a new function θ through the relation

$$\frac{\partial^2\Psi}{\partial x\partial z} = \frac{\partial^2\phi}{\partial y\partial z} - (1-2\nu)\frac{\partial\omega}{\partial y} \equiv \frac{\partial^3\theta}{\partial x\partial y\partial z} \; , \tag{7}$$

which is satisfied identically if

$$\Psi = \frac{\partial\theta}{\partial y} \quad ; \quad (1-2\nu)\omega = \frac{\partial\phi}{\partial z} - \frac{\partial^2\theta}{\partial x\partial z} \; . \tag{8}$$

Substituting these results for Ψ, ω into (5), we have

$$\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z} + \frac{\partial^3\theta}{\partial x^2\partial z} + \frac{\partial^3\theta}{\partial y^2\partial z} = -f\left(\frac{\partial^2\phi}{\partial z^2} - \frac{2(1-\nu)}{(1-2\nu)}\left[\frac{\partial^2\phi}{\partial z^2} - \frac{\partial^3\theta}{\partial x\partial z^2}\right]\right) \; . \tag{9}$$

Noting that θ and its derivatives are harmonic, we have

$$\frac{\partial^3\theta}{\partial x^2\partial z}+\frac{\partial^3\theta}{\partial y^2\partial z}=-\frac{\partial^3\theta}{\partial z^3} \tag{10}$$

and hence, (9) reduces to

$$-\frac{\partial^3\theta}{\partial z^3}=-f\left(-\frac{1}{(1-2\nu)}\frac{\partial^2\phi}{\partial z^2}+\frac{2(1-\nu)}{(1-2\nu)}\frac{\partial^3\theta}{\partial x\partial z^2}\right) \tag{11}$$

— i.e.

$$\frac{f}{(1-2\nu)}\frac{\partial^2\phi}{\partial z^2}=-\frac{\partial^3\theta}{\partial z^3}+\frac{2f(1-\nu)}{(1-2\nu)}\frac{\partial^3\theta}{\partial x\partial z^2}\ , \tag{12}$$

which is satisfied if

$$\phi=-\frac{(1-2\nu)}{f}\frac{\partial\theta}{\partial z}+2(1-\nu)\frac{\partial\theta}{\partial x}\ . \tag{13}$$

We then have

$$\begin{aligned}(1-2\nu)\omega &= -\frac{(1-2\nu)}{f}\frac{\partial^2\theta}{\partial z^2}+2(1-\nu)\frac{\partial^2\theta}{\partial x\partial z}-\frac{\partial^2\theta}{\partial x\partial z}\\ &= -\frac{(1-2\nu)}{f}\frac{\partial^2\theta}{\partial z^2}+(1-2\nu)\frac{\partial^2\theta}{\partial x\partial z}\ . \end{aligned}\tag{14}$$

To avoid division by f, it is convenient to define a new harmonic function through $\theta=f\chi$, in terms of which we have

$$\phi = (1-2\nu)\frac{\partial\chi}{\partial z}+2f(1-\nu)\frac{\partial\chi}{\partial x}\ ; \tag{15}$$

$$\Psi = f\frac{\partial\chi}{\partial y}\ ; \tag{16}$$

$$\omega = -\frac{\partial^2\chi}{\partial z^2}+f\frac{\partial^2\chi}{\partial x\partial z}\ . \tag{17}$$

Finally, we substitute for ϕ,ω,Ψ in the results from Table 19.1, obtaining

$$\begin{aligned}2\mu u_x &= \frac{\partial\phi}{\partial x}+z\frac{\partial\omega}{\partial x}+2\frac{\partial\Psi}{\partial y}\\ &= (1-2\nu)\frac{\partial^2\chi}{\partial z\partial x}+2f(1-\nu)\frac{\partial^2\chi}{\partial x^2}-z\frac{\partial^3\chi}{\partial z^2\partial x}+fz\frac{\partial^3\chi}{\partial z\partial x^2}+2f\frac{\partial^2\chi}{\partial y^2}\ ; \end{aligned}\tag{18}$$

$$\begin{aligned}2\mu u_y &= \frac{\partial\phi}{\partial y}+z\frac{\partial\omega}{\partial y}-2\frac{\partial\Psi}{\partial x}\\ &= -(1-2\nu)\frac{\partial^2\chi}{\partial y\partial z}-2f\nu\frac{\partial^2\chi}{\partial x\partial y}-z\frac{\partial^3\chi}{\partial z^2\partial y}+zf\frac{\partial^3\chi}{\partial x\partial y\partial z}\ ; \end{aligned}\tag{19}$$

$$\begin{aligned}2\mu u_z &= \frac{\partial\phi}{\partial z}+z\frac{\partial\omega}{\partial z}-(3-4\nu)\omega\\ &= 2(1-\nu)\frac{\partial^2\chi}{\partial z^2}-(1-2\nu)f\frac{\partial^2\chi}{\partial x\partial z}-z\frac{\partial^3\chi}{\partial z^3}+fz\frac{\partial^3\chi}{\partial z^2\partial x}\ . \end{aligned}\tag{20}$$

$$
\begin{aligned}
\sigma_{xx} &= \frac{\partial^2\phi}{\partial x^2} + z\frac{\partial^2\omega}{\partial x^2} - 2\nu\frac{\partial\omega}{\partial z} + 2\frac{\partial^2\Psi}{\partial x\partial y} \\
&= -(1-2\nu)\frac{\partial^3\chi}{\partial x^2\partial z} + 2\nu\frac{\partial^3\chi}{\partial z^3} + 2\nu f\frac{\partial^3\chi}{\partial x\partial y^2} - 2f\frac{\partial^3\chi}{\partial x\partial z^2} - z\frac{\partial^4\chi}{\partial x^2\partial z^2} + fz\frac{\partial^4\chi}{\partial x^3\partial z} ; \qquad (21) \\
\sigma_{xy} &= \frac{\partial^2\phi}{\partial x\partial y} + z\frac{\partial^2\omega}{\partial x\partial y} + \frac{\partial^2\Psi}{\partial y^2} - \frac{\partial^2\Psi}{\partial x^2} \\
&= -(1-2\nu)\frac{\partial^3\chi}{\partial x\partial y\partial z} + f(1-2\nu)\frac{\partial^3\chi}{\partial x^2\partial y} + f\frac{\partial^3\chi}{\partial y^3} - z\frac{\partial^4\chi}{\partial x\partial y\partial z^2} + fz\frac{\partial^4\chi}{\partial x^2\partial y\partial z} ; \qquad (22) \\
\sigma_{yy} &= \frac{\partial^2\phi}{\partial y^2} + z\frac{\partial^2\omega}{\partial y^2} - 2\nu\frac{\partial\omega}{\partial z} - 2\frac{\partial^2\Psi}{\partial x\partial y} \\
&= -(1-2\nu)\frac{\partial^3\chi}{\partial y^2\partial z} + 2\nu f\frac{\partial^3\chi}{\partial x^3} + 2\nu\frac{\partial^3\chi}{\partial z^3} - z\frac{\partial^4\chi}{\partial y^2\partial z^2} + fz\frac{\partial^4\chi}{\partial x\partial y^2\partial z} ; \qquad (23) \\
\sigma_{xz} &= \frac{\partial^2\phi}{\partial x\partial z} + z\frac{\partial^2\omega}{\partial x\partial z} - (1-2\nu)\frac{\partial\omega}{\partial x} + \frac{\partial^2\Psi}{\partial y\partial z} \\
&= -f\frac{\partial^3\chi}{\partial z^3} - z\frac{\partial^4\chi}{\partial x\partial z^3} + fz\frac{\partial^4\chi}{\partial x^2\partial z^2} ; \qquad (24) \\
\sigma_{yz} &= \frac{\partial^2\phi}{\partial y\partial z} + z\frac{\partial^2\omega}{\partial y\partial z} - (1-2\nu)\frac{\partial\omega}{\partial y} - \frac{\partial^2\Psi}{\partial x\partial z} \\
&= -z\frac{\partial^4\chi}{\partial y\partial z^3} + fz\frac{\partial^4\chi}{\partial x\partial y\partial z^2} ; \qquad (25) \\
\sigma_{zz} &= \frac{\partial^2\phi}{\partial z^2} + z\frac{\partial^2\omega}{\partial z^2} - 2(1-\nu)\frac{\partial\omega}{\partial z} \\
&= \frac{\partial^3\chi}{\partial z^3} - z\frac{\partial^4\chi}{\partial z^4} + fz\frac{\partial^4\chi}{\partial x\partial z^3} . \qquad (26)
\end{aligned}
$$

21.4. A large number of thin fibres are bonded to the surface $z=0$ of the elastic half space $z>0$, with orientation in the x-direction. The fibres are sufficiently rigid to prevent axial displacement u_x, but sufficiently thin to offer no restraint to lateral motion u_y, u_z.

This composite block is now indented by a frictionless rigid punch. Find a linear combination of Solutions A, B and E that satisfies the global conditions of the problem and in particular find expressions for the surface values of the normal displacement u_z and the normal traction σ_{zz} in terms of the one remaining harmonic potential function.

At the surface $z = 0$, we have

$$2\mu u_x = \frac{\partial \phi}{\partial x} + 2\frac{\partial \Psi}{\partial y} \tag{1}$$

$$\sigma_{yz} = \frac{\partial^2 \phi}{\partial y \partial z} - (1-2\nu)\frac{\partial \omega}{\partial y} - \frac{\partial^2 \Psi}{\partial x \partial z} \,. \tag{2}$$

The boundary conditions of the problem force these quantities to zero[1] and hence we need to generate a solution in which

$$\frac{\partial \phi}{\partial x} + 2\frac{\partial \Psi}{\partial y} = 0 \tag{3}$$

$$\frac{\partial^2 \phi}{\partial y \partial z} - (1-2\nu)\frac{\partial \omega}{\partial y} - \frac{\partial^2 \Psi}{\partial x \partial z} = 0 \,. \tag{4}$$

Equation (3) can be satisfied by defining both ϕ and Ψ in terms of a new harmonic function Θ, such that

$$\frac{\partial \phi}{\partial x} = -2\frac{\partial \Psi}{\partial y} = 2\frac{\partial^2 \Theta}{\partial x \partial y} \tag{5}$$

and hence

$$\phi = 2\frac{\partial \Theta}{\partial y} \,; \quad \Psi = -\frac{\partial \Theta}{\partial x} \,. \tag{6}$$

Substituting these results into (4), we have

$$2\frac{\partial^3 \Theta}{\partial y^2 \partial z} - (1-2\nu)\frac{\partial \omega}{\partial y} + \frac{\partial^3 \Theta}{\partial x^2 \partial z} = 0 \tag{7}$$

and this can be satisfied by defining a new harmonic function χ through

$$2\frac{\partial^3 \Theta}{\partial y^2 \partial z} + \frac{\partial^3 \Theta}{\partial x^2 \partial z} = (1-2\nu)\frac{\partial \omega}{\partial y} = (1-2\nu)\frac{\partial}{\partial y}\left(2\frac{\partial^3 \chi}{\partial y^2 \partial z} + \frac{\partial^3 \chi}{\partial x^2 \partial z}\right) \tag{8}$$

[1]Notice that though the contact is frictionless, we do not have $\sigma_{zx} = 0$, since the fibres can transmit loads in the x-direction.

or

$$\Theta = (1-2\nu)\frac{\partial \chi}{\partial y}\ ; \quad \omega = 2\frac{\partial^3 \chi}{\partial y^2 \partial z} + \frac{\partial^3 \chi}{\partial x^2 \partial z}\ . \tag{9}$$

Using (6), we then have

$$\phi = 2(1-2\nu)\frac{\partial^2 \chi}{\partial y^2} \tag{10}$$

$$\omega = 2\frac{\partial^3 \chi}{\partial y^2 \partial z} + \frac{\partial^3 \chi}{\partial x^2 \partial z} \tag{11}$$

$$\Psi = -(1-2\nu)\frac{\partial^2 \chi}{\partial x \partial y} \tag{12}$$

and the complete solution is obtained by substituting these expressions into Table 21.1. We obtain

$$2\mu u_x = 2z\frac{\partial^4 \chi}{\partial x \partial y^2 \partial z} + z\frac{\partial^4 \chi}{\partial x^3 \partial z} \tag{13}$$

$$2\mu u_y = -2(1-2\nu)\frac{\partial^3 \chi}{\partial y \partial z^2} + 2z\frac{\partial^4 \chi}{\partial y^3 \partial z} + z\frac{\partial^4 \chi}{\partial x^2 \partial y \partial z} \tag{14}$$

$$2\mu u_z = -4(1-\nu)\frac{\partial^3 \chi}{\partial y^2 \partial z} - (3-4\nu)\frac{\partial^3 \chi}{\partial x^2 \partial z} + 2z\frac{\partial^4 \chi}{\partial y^2 \partial z^2} + z\frac{\partial^4 \chi}{\partial x^2 \partial z^2} \tag{15}$$

$$\sigma_{xx} = -2\nu\frac{\partial^4 \chi}{\partial x^2 \partial z^2} - 4\nu\frac{\partial^4 \chi}{\partial y^2 \partial z^2} + 2z\frac{\partial^5 \chi}{\partial x^2 \partial y^2 \partial z} + z\frac{\partial^5 \chi}{\partial x^4 \partial z} \tag{16}$$

$$\sigma_{xy} = -(1-2\nu)\frac{\partial^4 \chi}{\partial x \partial y \partial z^2} + 2z\frac{\partial^5 \chi}{\partial x \partial y^3 \partial z} + z\frac{\partial^5 \chi}{\partial x^3 \partial y \partial z} \tag{17}$$

$$\sigma_{yy} = -2\frac{\partial^4 \chi}{\partial y^2 \partial z^2} - 2\nu\frac{\partial^4 \chi}{\partial x^2 \partial z^2} + 2z\frac{\partial^5 \chi}{\partial y^4 \partial z} + z\frac{\partial^5 \chi}{\partial x^2 \partial y^2 \partial z} \tag{18}$$

$$\sigma_{xz} = (1-2\nu)\frac{\partial^4 \chi}{\partial x \partial z^3} + 2z\frac{\partial^5 \chi}{\partial \partial x y^2 \partial z y^2} + z\frac{\partial^5 \chi}{\partial x^3 \partial z^2} \tag{19}$$

$$\sigma_{yz} = 2z\frac{\partial^5 \chi}{\partial y^3 \partial z^3} + z\frac{\partial^5 \chi}{\partial x^2 \partial y \partial z^2} \tag{20}$$

$$\sigma_{zz} = 2\frac{\partial^4 \chi}{\partial z^4} + 2\nu\frac{\partial^4 \chi}{\partial x^2 \partial z^2} + 2z\frac{\partial^5 \chi}{\partial y^2 \partial z^3} + z\frac{\partial^5 \chi}{\partial x^2 \partial z^3}\ , \tag{21}$$

where we have used the result $\nabla^2 \chi = 0$ to simplify expressions where appropriate.

In particular, the surface values of the normal displacement u_z and the normal traction σ_{zz} are

$$2\mu u_z(x,y,0) = -2(1-\nu)\frac{\partial^3 \chi}{\partial y^2 \partial z} - (3-4\nu)\frac{\partial^3 \chi}{\partial x^2 \partial z} \tag{22}$$

$$\sigma_{zz}(x,y,0) = 2\frac{\partial^4 \chi}{\partial z^4} + 2\nu\frac{\partial^4 \chi}{\partial x^2 \partial z^2}\ . \tag{23}$$

21.5. Show that if ϕ, ω are taken to be independent of x, a linear combination of Solutions A and B defines a state of plane strain in the yz-plane (i.e. $u_x = 0, \sigma_{xy} = \sigma_{xz} = 0$ and all the remaining stress and displacement components are independent of x).

An infinite layer $0<z<h$ rests on a frictionless rigid foundation at $z=0$, whilst the surface $z=h$ is loaded by the normal traction

$$\sigma_{zz} = -p_0 + p_1 \cos(my) \ ,$$

where $p_1<p_0$. Find the distribution of contact pressure at $z=0$.

If the functions ϕ, ω are independent of x, we can eliminate the x-derivatives in Table 21.1, obtaining the solution

$$2\mu u_x = 0 \tag{1}$$

$$2\mu u_y = \frac{\partial \phi}{\partial y} + z\frac{\partial \omega}{\partial y} \tag{2}$$

$$2\mu u_z = \frac{\partial \phi}{\partial z} + z\frac{\partial \omega}{\partial z} - (3-4\nu)\omega \tag{3}$$

$$\sigma_{xx} = -2\nu\frac{\partial \omega}{\partial z} \tag{4}$$

$$\sigma_{xy} = 0 \tag{5}$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial y^2} + z\frac{\partial^2 \omega}{\partial y^2} - 2\nu\frac{\partial \omega}{\partial z} \tag{6}$$

$$\sigma_{xz} = 0 \tag{7}$$

$$\sigma_{yz} = \frac{\partial^2 \phi}{\partial y \partial z} + z\frac{\partial^2 \omega}{\partial y \partial z} - (1-2\nu)\frac{\partial \omega}{\partial y} \tag{8}$$

$$\sigma_{zz} = \frac{\partial^2 \phi}{\partial z^2} + z\frac{\partial^2 \omega}{\partial z^2} - 2(1-\nu)\frac{\partial \omega}{\partial z} \ . \tag{9}$$

Since $u_x, \sigma_{xy}, \sigma_{xz}$, this solution defines a state of plane strain in the yz-plane.

The boundary conditions of the problem are

$$\sigma_{zy} = 0 \ ; \quad z = 0 \tag{10}$$

$$u_z = 0 \ ; \quad z = 0 \tag{11}$$

$$\sigma_{zy} = 0 \ ; \quad z = h \tag{12}$$

$$\sigma_{zz} = -p_0 + p_1 \cos(my) \ ; \quad z = h \ . \tag{13}$$

The term $-p_0$ will simply cause a trivial state of uniaxial compression $\sigma_{zz} = -p_0$ throughout the layer. For the remaining term in (13), we need harmonic functions of the form $\exp(\pm mz)\cos(my)$ in ϕ and ω. Writing

$$\phi = C_1 e^{mz}\cos(my) + C_2 e^{-mz}\cos(my) \ ; \quad \omega = C_3 e^{mz}\cos(my) + C_4 e^{-mz}\cos(my) \ , \tag{14}$$

we obtain

$$
\begin{aligned}
2\mu u_z &= C_1 m e^{mz}\cos(my) - C_2 m e^{-mz}\cos(my) + C_3 m z e^{mz}\cos(my) \\
&\quad - C_4 m z e^{-mz}\cos(my) - (3-4\nu)C_3 e^{mz}\cos(my) - (3-4\nu)C_4 e^{-mz}\cos(my) \qquad (15)\\
\sigma_{yz} &= -\Big(C_1 m e^{mz} - C_2 m e^{-mz} + C_3 m z e^{mz} - C_4 m z e^{-mz} \\
&\quad -(1-2\nu)(C_3 e^{mz} + C_4 e^{-mz})\Big)\, m\sin(my) \qquad (16)\\
\sigma_{zz} &= \Big(C_1 m e^{mz} + C_2 m e^{-mz} + C_3 m z e^{mz} + C_4 m z e^{-mz} \\
&\quad -2(1-\nu)(C_3 e^{mz} - C_4 e^{-mz})\Big)\, m\cos(my)\ . \qquad (17)
\end{aligned}
$$

The boundary conditions therefore require that

$$
\begin{aligned}
C_1 m - C_2 m - (1-2\nu)(C_3 + C_4) &= 0 \qquad (18)\\
C_1 m - C_2 m - (3-4\nu)(C_3 + C_4) &= 0 \qquad (19)\\
C_1 m e^{mh} - C_2 m e^{-mh} + C_3 m h e^{mh} - C_4 m h e^{-mh} & \\
-(1-2\nu)(C_3 e^{mh} + C_4 e^{-mh}) &= 0 \qquad (20)\\
C_1 m^2 e^{mh} + C_2 m^2 e^{-mh} + C_3 m^2 h e^{mh} + C_4 m^2 h e^{-mh} & \\
-2(1-\nu)(C_3 m e^{mh} - C_4 m e^{-mh}) &= p_1\ . \qquad (21)
\end{aligned}
$$

Solving these four equations for C_1, C_2, C_3, C_4, we obtain

$$
\begin{aligned}
C_1 &= C_2 = \frac{p_1 e^{mh}\Big((1-2\nu)(1-e^{2mh}) + mh(1+e^{2mh})\Big)}{m^2(e^{4mh} + 4mhe^{2mh} - 1)} \qquad (22)\\
C_3 &= -C_4 = -\frac{p_1 e^{mh}(e^{2mh}-1)}{m(e^{4mh} + 4mhe^{2mh} - 1)}\ . \qquad (23)
\end{aligned}
$$

The contact pressure at $z = 0$ is

$$
p(x) = -\sigma_{zz}(x,y,h) = p_0 - (C_1 m + C_2 m - 2(1-\nu)(C_3 - C_4))\, m\cos(my) \qquad (24)
$$

using (17), where we have re-introduced the uniform compressive stress $-p_0$.

Substituting for the constants from (22,23) and simplifying, we obtain

$$
p(x) = p_0 - \frac{2p_1 e^{mh}\Big(mhe^{2mh} + mh + e^{2mh} - 1\Big)\cos(my)}{(e^{4mh} + 4mhe^{2mh} - 1)}; \qquad (25)
$$

21.6 A massive asteroid passes close to the surface of the Earth causing gravitational forces that can be described by a known harmonic body force potential $V(x, y, z)$. Assuming that the affected region of the Earth can be approximated as the traction-free half space $z > 0$, find a representation of the resulting stress and displacement fields in terms of a single harmonic potential function. An appropriate strategy is:-

(i) Using the particular solution of §20.5.1, show that the potential ϕ must be biharmonic.

(ii) Show that this condition can be satisfied by writing $\phi = z\chi$, where χ is a *harmonic* function, and find the relation between χ and V.

(iii) Find the tractions on the surface of the half space in terms of derivatives of χ.

(iv) Superpose appropriate combinations of Solutions A and/or B so as to render this surface traction free for all functions χ.

Find the resulting stress field for the special case of a spherical asteroid of radius a, for which

$$V = -\frac{4\pi\gamma\rho\rho_0 a^3}{3\sqrt{x^2 + y^2 + (z+h)^2}} ,$$

where γ is the universal gravitational constant, ρ, ρ_0 are the densities of the asteroid and the Earth respectively, and the centre of the sphere is at the point $(0, 0, -h)$, a height $h > a$ above the Earth's surface.

If the Earth and the asteroid have the same density ($\rho_0 = \rho$) and the self-weight of the Earth is assumed to cause hydrostatic compression of magnitude $\rho g h$, where g is the acceleration due to gravity, is it possible for the passage of the asteroid to cause tensile stresses anywhere in the Earth's crust. If so, where?

From equation (20.40), we have

$$\nabla^2\phi = \frac{(1-2\nu)V}{(1-\nu)}$$

and since V is harmonic, we have

$$\nabla^4\phi = \frac{(1-2\nu)}{(1-\nu)}\nabla^2 V = 0 ,$$

so ϕ is biharmonic.

Writing

$$\phi = z\chi \ ,$$

and differentiating by parts, we have

$$\nabla^2\phi = 2\frac{\partial\chi}{\partial z} + z\nabla^2\chi$$

and if χ is harmonic,

$$\nabla^2\phi = 2\frac{\partial\chi}{\partial z} \ .$$

Thus,

$$\nabla^4\phi = 2\nabla^2\frac{\partial\chi}{\partial z} = 2\frac{\partial}{\partial z}\nabla^2\chi = 0 \ ,$$

which satisfies the condition that ϕ be biharmonic. Thus, we can write the particular solution in terms of χ provided that

$$2\frac{\partial\chi}{\partial z} = \frac{(1-2\nu)V}{(1-\nu)} \ .$$

The stress components

We can find the stress components corresponding to the particular solution by substituting $\phi = z\chi$ into equations (20.41, 20.42), noting that

$$\frac{\nu V}{(1-\nu)} = \frac{2\nu}{(1-2\nu)V}\frac{\partial\chi}{\partial z} \ .$$

We obtain

$$\begin{aligned}
\sigma_{xx} &= \frac{\nu V}{(1-\nu)} + \frac{\partial^2\phi}{\partial x^2} = \frac{2\nu}{(1-2\nu)}\frac{\partial\chi}{\partial z} + z\frac{\partial^2\chi}{\partial x^2} \\
\sigma_{yy} &= \frac{2\nu}{(1-2\nu)}\frac{\partial\chi}{\partial z} + z\frac{\partial^2\chi}{\partial y^2} \\
\sigma_{zz} &= \frac{\nu V}{(1-\nu)} + \frac{\partial^2\phi}{\partial z^2} = \frac{2\nu}{(1-2\nu)}\frac{\partial\chi}{\partial z} + 2\frac{\partial\chi}{\partial z} + z\frac{\partial^2\chi}{\partial z^2} \\
&= \frac{2(1-\nu)}{(1-2\nu)}\frac{\partial\chi}{\partial z} + z\frac{\partial^2\chi}{\partial z^2} \\
\sigma_{xy} &= \frac{\partial^2\phi}{\partial x\partial y} = z\frac{\partial^2\chi}{\partial x\partial y} \\
\sigma_{yz} &= \frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial\chi}{\partial y} + z\frac{\partial^2\chi}{\partial y\partial z} \\
\sigma_{zx} &= \frac{\partial\chi}{\partial x} + z\frac{\partial^2\chi}{\partial z\partial x} \ .
\end{aligned}$$

In particular the tractions on $z = 0$ are

$$\sigma_{zx} = \frac{\partial \chi}{\partial x} \; ; \quad \sigma_{zy} = \frac{\partial \chi}{\partial y} \; ; \quad \sigma_{zz} = \frac{2(1-\nu)}{(1-2\nu)} \frac{\partial \chi}{\partial z} \; .$$

Superposing Solutions A,B, we would have (on $z = 0$)

$$\sigma_{zx} = \frac{\partial \chi}{\partial x} + \frac{\partial^2 \varphi}{\partial z \partial x} - (-2\nu) \frac{\partial \omega}{\partial x} \; ; \quad \sigma_{zy} = \frac{\partial \chi}{\partial y} + \frac{\partial^2 \varphi}{\partial z \partial y} - (1-2\nu) \frac{\partial \omega}{\partial y}$$

$$\sigma_{zz} = \frac{2(1-\nu)}{(1-2\nu)} \frac{\partial \chi}{\partial z} + \frac{\partial^2 \varphi}{\partial z^2} - 2(1-\nu) \frac{\partial \omega}{\partial z} \; .$$

and these will all be zero if

$$\varphi = 0 \; ; \quad \omega = \frac{\chi}{(1-2\nu)} \; .$$

It is convenient to define a new function ψ such that

$$\psi = \frac{2(1-\nu)\chi}{(1-2\nu)} = 2(1-\nu)\omega$$

so that

$$\chi = \frac{(1-2\nu)\psi}{2(1-\nu)} \; ; \quad \omega = \frac{\psi}{2(1-\nu)} \; ; \quad \phi = \frac{(1-2\nu)z\psi}{2(1-\nu)} \; ,$$

after which the stress components are obtained as

$$\begin{aligned}
\sigma_{xx} &= \frac{2\nu}{(1-2\nu)} \frac{\partial \chi}{\partial z} + z \frac{\partial^2 \chi}{\partial x^2} + z \frac{\partial^2 \omega}{\partial x^2} - 2\nu \frac{\partial \omega}{\partial z} \\
&= \frac{\nu}{(1-\nu)} \frac{\partial \psi}{\partial z} + \frac{(1-2\nu)}{2(1-\nu)} z \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2(1-\nu)} z \frac{\partial^2 \psi}{\partial x^2} - \frac{\nu}{(1-\nu)} \frac{\partial \psi}{\partial z} \\
&= z \frac{\partial^2 \psi}{\partial x^2} \\
\sigma_{yy} &= z \frac{\partial^2 \psi}{\partial y^2} \\
\sigma_{zz} &= \frac{2(1-\nu)}{(1-2\nu)} \frac{\partial \chi}{\partial z} + z \frac{\partial^2 \chi}{\partial z^2} + z \frac{\partial^2 \omega}{\partial z^2} - 2(1-\nu) \frac{\partial \omega}{\partial z} \\
&= \frac{\partial \psi}{\partial z} + \frac{(1-2\nu)}{2(1-\nu)} z \frac{\partial^2 \psi}{\partial z^2} + \frac{1}{2(1-\nu)} z \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial \psi}{\partial z} \\
&= z \frac{\partial^2 \psi}{\partial z^2} \\
\sigma_{xy} &= z \frac{\partial^2 \chi}{\partial x \partial y} + z \frac{\partial^2 \omega}{\partial x \partial y} = z \frac{\partial^2 \psi}{\partial x \partial y}
\end{aligned}$$

$$
\begin{aligned}
\sigma_{yz} &= \frac{\partial \chi}{\partial y} + z\frac{\partial^2 \chi}{\partial y \partial z} + z\frac{\partial^2 \omega}{\partial y \partial z} - (1-2\nu)\frac{\partial \omega}{\partial y} \\
&= \frac{(1-2\nu)}{2(1-\nu)}\frac{\partial \psi}{\partial y} + \frac{(1-2\nu)}{2(1-\nu)} z\frac{\partial^2 \psi}{\partial y \partial z} + \frac{1}{2(1-\nu)} z\frac{\partial^2 \psi}{\partial y \partial z} - \frac{(1-2\nu)}{2(1-\nu)}\frac{\partial \psi}{\partial y} \\
&= z\frac{\partial^2 \psi}{\partial y \partial z} \\
\sigma_{zx} &= z\frac{\partial^2 \psi}{\partial z \partial x} \ .
\end{aligned}
$$

For the displacements

$$
2\mu u_x = \frac{\partial \phi}{\partial x} + z\frac{\partial \omega}{\partial x} = \frac{(1-2\nu)}{2(1-\nu)} z\frac{\partial \psi}{\partial x} + \frac{1}{2(1-\nu)} z\frac{\partial \psi}{\partial x} = z\frac{\partial \psi}{\partial x} \ .
$$

$$
2\mu u_z = \frac{\partial \phi}{\partial z} + z\frac{\partial \omega}{\partial x} - (3-4\nu)\omega = \frac{(1-2\nu)\psi}{2(1-\nu)} + \frac{(1-2\nu)}{2(1-\nu)} z\frac{\partial \psi}{\partial z} + \frac{1}{2(1-\nu)} z\frac{\partial \psi}{\partial z} - \frac{(3-4\nu)\psi}{2(1-\nu)}
$$

and hence

$$
\sigma_{xx} = z\frac{\partial^2 \psi}{\partial x^2} \ ; \quad \sigma_{yy} = z\frac{\partial^2 \psi}{\partial y^2} \ ; \quad \sigma_{zz} = z\frac{\partial^2 \psi}{\partial z^2}
$$

$$
\sigma_{xy} = z\frac{\partial^2 \psi}{\partial x \partial y} \ ; \quad \sigma_{yz} = z\frac{\partial^2 \psi}{\partial y \partial z} \ ; \quad \sigma_{zx} = z\frac{\partial^2 \psi}{\partial z \partial x}
$$

$$
2\mu u_x = z\frac{\partial \psi}{\partial x} \ ; \quad 2\mu u_y = z\frac{\partial \psi}{\partial y} \ ; \quad 2\mu u_z = z\frac{\partial \psi}{\partial z} - \psi \ ,
$$

where

$$
\frac{\partial \psi}{\partial z} = V \ .
$$

For the special case

$$
V = -\frac{4\pi\gamma\rho\rho_0 a^3}{3\sqrt{x^2+y^2+(z+h)^2}} \ ,
$$

we can find a harmonic partial integral as

$$
\psi = -\frac{4\pi\gamma\rho\rho_0 a^3}{3} \ln\left(\sqrt{x^2+y^2+(z+h)^2} + z + h\right)
$$

The stress components are then obtained as

$$
\begin{aligned}
\sigma_{xx} &= -\frac{4\pi\gamma\rho\rho_0 a^3 z}{3}\left(\frac{1}{R^*(R^*+z^*)} - \frac{x^2}{R^{*2}(R^*+z^*)^2} - \frac{x^2}{R^{*3}(R^*+z^*)}\right) \\
\sigma_{yy} &= -\frac{4\pi\gamma\rho\rho_0 a^3}{3}\left(\frac{1}{R^*(R^*+z^*)} - \frac{y^2}{R^{*2}(R^*+z^*)^2} - \frac{x^2}{R^{*3}(R^*+z^*)}\right) \\
\sigma_{zz} &= \frac{4\pi\gamma\rho\rho_0 a^3 z z^*}{3R^{*3}}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{xy} &= \frac{4\pi\gamma\rho\rho_0 a^3 xyz}{3}\left(\frac{1}{R^{*2}(R^*+z^*)^2}+\frac{1}{R^{*3}(R^*+z^*)}\right) \\
\sigma_{yz} &= \frac{4\pi\gamma\rho\rho_0 a^3 zy}{3R^{*3}} \\
\sigma_{zx} &= \frac{4\pi\gamma\rho\rho_0 a^3 zx}{3R^{*3}} ,
\end{aligned}
$$

where

$$z^* = z + h \; ; \quad R^* = \sqrt{x^2+y^2+(z+h)^2} \; .$$

The maximum tensile stress is σ_{zz} on the axis $x = y = 0$, where

$$\sigma_{zz} = \frac{4\pi\gamma\rho\rho_0 a^3 z}{3(z+h)^2} \; .$$

The maximum value of

$$\frac{z}{(z+h)^2}$$

occurs when

$$(z+h)^2 = 2z(z+h) \quad \text{or} \quad z = h \; ,$$

so the maximum tensile stress is

$$\sigma_{\max} = \frac{\pi\gamma\rho\rho_0 a^3}{3h} \; .$$

If the radius of the Earth is b, its mass is

$$M = \frac{4\pi\rho b^3}{3}$$

and the acceleration due to gravity at the surface is equal to the force exerted on a unit mass, which is

$$g = \frac{M\gamma}{b^2} = \frac{4\pi\rho\gamma b}{3} \; .$$

It follows that the hydrostatic pressure is

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -\rho g z = -\frac{4\pi\rho^2\gamma b z}{3}$$

and hence along the z-axis, the combined stress due to the asteroid and self-weight of the Earth is

$$\sigma_{zz} = \frac{4\pi\gamma\rho^2 a^3 z}{3(z+h)^2} - \frac{4\pi\rho^2\gamma b z}{3} = \frac{4\pi\gamma\rho^2 a z}{3}\left(\frac{a^2}{(z+h)^2} - \frac{b}{a}\right) \; .$$

Since $h > a$ and $z > 0$ and we assume $b \gg a$, this must be negative (compressive) for all z. Notice that if we relax the assumption $b \gg a$, the half space approximation would be inappropriate. Of course, previous tectonic motion may have left the Earth's crust in a state of stress different from hydrostatic compression, so this does not imply that a passing asteroid can have no effect on earthquake behaviour.

CHAPTER 22

22.1. The temperature $T(R)$ in a hollow sphere $b < R < a$ depends only on the radius R and the surfaces of the sphere are traction-free. Use a method similar to that in §22.1.1 to obtain expressions for the stress and displacement components for any function $T(R)$.

Note: The most general spherically symmetrical isothermal state of stress (i.e. the general homogeneous solution) can be written by superposing (i) the function A/R in solution A of Table 21.2 and (ii) a uniform state of hydrostatic stress. The identity

$$\frac{d^2}{dR^2} + \frac{2}{R}\frac{d}{dR} \equiv \frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{d}{dR}\right)$$

may also prove useful.

If the temperature and ϕ depend only on R, equations (22.8, 21.35) give

$$\frac{d^2\phi}{dR^2} + \frac{2}{R}\frac{d\phi}{dR} = \frac{2\mu(1+\nu)\alpha T(R)}{(1-\nu)} \tag{1}$$

and using the given identity, we have If the temperature and ϕ depend only on R, equations (22.8, 21.35) give

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{d\phi}{dR}\right) = \frac{2\mu(1+\nu)\alpha T}{(1-\nu)} \ . \tag{2}$$

Multiplying through by R^2 and integrating, we then obtain

$$R^2\frac{d\phi}{dR} = \frac{2\mu(1+\nu)\alpha}{(1-\nu)}\int R^2 T(R)dR \ . \tag{3}$$

Substituting into equations (22.14), we then obtain

$$u_R = \frac{1}{2\mu}\frac{d\phi}{dR} = \frac{(1+\nu)\alpha}{(1-\nu)R^2}\int R^2 T(R)dR \tag{4}$$

$$\sigma_{RR} = -\frac{2}{R}\frac{d\phi}{dR} = -\frac{4\mu(1+\nu)\alpha}{(1-\nu)R^3}\int R^2 T(R)dR \tag{5}$$

$$\begin{aligned}\sigma_{\theta\theta} &= \sigma_{\beta\beta} = -\frac{1}{R}\frac{d\phi}{dR} - \frac{d^2\phi}{dR^2} \\ &= \frac{2\mu(1+\nu)\alpha}{(1-\nu)}\left(\frac{1}{R^3}\int R^2 T(R)dR - T(R)\right) \ .\end{aligned} \tag{6}$$

A uniform state of isothermal hydrostatic stress

$$\sigma_{RR} = \sigma_{\theta\theta} = \sigma_{\beta\beta} = B \tag{7}$$

would produce a radial strain

$$e_{RR} = \frac{\partial u_R}{\partial R} = \frac{B(1-2\nu)}{2\mu(1+\nu)} \tag{8}$$

and hence

$$u_R = \frac{B(1-2\nu)R}{2\mu(1+\nu)} \ . \tag{9}$$

Superposing the thermoelastic terms, the hydrostatic terms and the results of substituting $\phi = A/R$ into Solution A in Table 21.2, we obtain

$$u_R = \frac{(1+\nu)\alpha}{(1-\nu)R^2}\int_b^R R^2 T(R)dR - \frac{A}{2\mu R^2} + \frac{B(1-2\nu)}{2\mu(1+\nu)} \tag{10}$$

$$\sigma_{RR} = -\frac{4\mu(1+\nu)\alpha}{(1-\nu)R^3}\int_b^R R^2 T(R)dR + \frac{2A}{R^3} + B \tag{11}$$

$$\sigma_{\theta\theta} = \frac{2\mu(1+\nu)\alpha}{(1-\nu)}\left(\frac{1}{R^3}\int_b^R R^2 T(R)dR - T(R)\right) - \frac{A}{R^3} + B \ . \tag{12}$$

Notice that we have set the lower limit in the integrals to b. This is essentially an arbitrary choice. Changing it would merely change the value of the constant A.

If the surfaces of the sphere $R = a, b$ are traction-free, we have

$$\sigma_{RR} = 0 \ ; \quad R = a, b \tag{13}$$

and hence

$$-\frac{4\mu(1+\nu)\alpha}{(1-\nu)a^3}\int_b^a R^2 T(R)dR + \frac{2A}{a^3} + B = 0 \tag{14}$$

$$-\frac{2A}{b^3} + B = 0 \ . \tag{15}$$

Solving for A, B, we obtain

$$A = -\frac{2\mu(1+\nu)\alpha b^3}{(1-\nu)(a^3-b^3)}\int_b^a R^2 T(R)dR \ ; \quad B = \frac{4\mu(1+\nu)\alpha}{(1-\nu)(a^3-b^3)}\int_b^a R^2 T(R)dR \ . \tag{16}$$

The final stress field is then obtained by substituting these expressions into equations (10-12).

22.2. The temperature field in the elastic half-space $z>0$ is defined by

$$T(x,y,z) = f(z)\cos(mx) \ ,$$

where $f(z)$ is a known function of z only. The surface $z=0$ of the half-space is traction-free. Show that the normal surface displacement $u_z(x,y,0)$ can be written

$$u_z(x,y,0) = u_0 \cos(mx) \ ,$$

where

$$u_0 = 2\alpha(1+\nu)\int_0^\infty e^{-ms} f(s)ds \ .$$

Important Note: This problem *cannot* be solved using Solutions T or P, since the temperature field will only be harmonic for certain special functions $f(z)$. To solve the problem, assume a strain potential ϕ of the form

$$\phi = g(z)\cos(mx) \ ,$$

substitute into (22.10) and solve the resulting ordinary differential equation for $g(z)$. You can then superpose appropriate potentials from solutions A and B of Table 21.1 to satisfy the traction-free boundary condition.

Assuming that

$$\phi = g(z)\cos(mx) \ , \tag{1}$$

substituting into (22.10) and cancelling a factor of $\cos(mx)$, we have

$$\frac{d^2 g(z)}{dz^2} - m^2 g(z) = \frac{2\mu(1+\nu)\alpha f(z)}{(1-\nu)} \ . \tag{2}$$

The left hand side of this equation can be factorized into two linear operators — i.e.

$$\left(\frac{d}{dz} - m\right)\left(\frac{d}{dz} + m\right) g = \frac{2\mu(1+\nu)\alpha f(z)}{(1-\nu)} \ . \tag{3}$$

Appropriate integrating factors then permit the equation to be written

$$e^{mz}\frac{d}{dz}\left(e^{-2mz}\frac{d}{dz}(e^{mz}g)\right) = \frac{2\mu(1+\nu)\alpha f(z)}{(1-\nu)} \ , \tag{4}$$

the solution of which can be written

$$g(z) = \frac{2\mu(1+\nu)\alpha e^{-mz}}{(1-\nu)}\int_0^z\int_0^\zeta e^{(2m\zeta - ms)} f(s)dsd\zeta + Ae^{mz} + Be^{-mz} \ , \tag{5}$$

where A, B are arbitrary constants of integration.

The integral term in (5) can be simplified by changing the order of integration and performing the inner integral, with the result

$$g(z) = \frac{\mu(1+\nu)\alpha e^{mz}}{m(1-\nu)} \int_0^z e^{-m\zeta} f(\zeta) d\zeta - \frac{\mu(1+\nu)\alpha e^{-mz}}{m(1-\nu)} \int_0^z e^{m\zeta} f(\zeta) d\zeta + A e^{mz} + B e^{-mz} \ . \tag{6}$$

We also note that

$$\frac{dg}{dz} = \frac{\mu(1+\nu)\alpha e^{mz}}{(1-\nu)} \int_0^z e^{-m\zeta} f(\zeta) d\zeta + \frac{\mu(1+\nu)\alpha e^{-mz}}{(1-\nu)} \int_0^z e^{m\zeta} f(\zeta) d\zeta + A m e^{mz} - B m e^{-mz} \ . \tag{7}$$

We shall require that the stresses and hence $g(z)$ be bounded as $z \to \infty$. At large values of z, the terms with positive exponential multipliers will be unbounded unless

$$\frac{\mu(1+\nu)\alpha}{m(1-\nu)} \int_0^\infty e^{-m\zeta} f(\zeta) d\zeta + A = 0 \tag{8}$$

and hence

$$A = -\frac{\mu(1+\nu)\alpha}{m(1-\nu)} \int_0^\infty e^{-m\zeta} f(\zeta) d\zeta \ . \tag{9}$$

The stress and displacement components are

$$2\mu u_z = \frac{\partial \phi}{\partial z} \ ; \quad \sigma_{zx} = \frac{\partial^2 \phi}{\partial z \partial x} \ ; \quad \sigma_{zz} = -\frac{\partial^2 \phi}{\partial x^2} \ , \tag{10}$$

from equations (22.11, 22.12). Substituting for ϕ from (1,6), we obtain

$$\begin{aligned} 2\mu u_z &= \frac{\mu(1+\nu)\alpha e^{mz} \cos(mx)}{(1-\nu)} \int_0^z e^{-m\zeta} f(\zeta) d\zeta \\ &\quad + \frac{\mu(1+\nu)\alpha e^{-mz} \cos(mx)}{(1-\nu)} \int_0^z e^{m\zeta} f(\zeta) d\zeta + A m e^{mz} \cos(mx) \\ &\quad - B m e^{-mz} \cos(mx) \end{aligned} \tag{11}$$

$$\begin{aligned} \sigma_{zx} &= -\frac{m\mu(1+\nu)\alpha e^{mz} \sin(mx)}{(1-\nu)} \int_0^z e^{-m\zeta} f(\zeta) d\zeta \\ &\quad - \frac{m\mu(1+\nu)\alpha e^{-mz} \sin(mx)}{(1-\nu)} \int_0^z e^{m\zeta} f(\zeta) d\zeta - A m^2 e^{mz} \sin(mx) \\ &\quad + B m^2 e^{-mz} \sin(mx) \end{aligned} \tag{12}$$

$$\begin{aligned} \sigma_{zz} &= \frac{m\mu(1+\nu)\alpha e^{mz} \cos(mx)}{(1-\nu)} \int_0^z e^{-m\zeta} f(\zeta) d\zeta \\ &\quad - \frac{m\mu(1+\nu)\alpha e^{-mz} \cos(mx)}{(1-\nu)} \int_0^z e^{m\zeta} f(\zeta) d\zeta + A m^2 e^{mz} \cos(mx) \\ &\quad + B m^2 e^{-mz} \cos(mx) \ . \end{aligned} \tag{13}$$

The terms multiplied by the constants A, B are harmonic and hence correspond to an arbitrary harmonic function in Solution A, since both Solution A and the thermoelastic displacement potential start with the definition $2\mu\boldsymbol{u} = \boldsymbol{\nabla}\phi$. Thus, there is nothing to be gained from adding in additional terms from Solution A. We therefore add just the term

$$\omega = Ce^{-mz}\cos(mx) \tag{14}$$

in Solution B, giving the additional stress and displacement components

$$2\mu u_z = -Cmze^{-mz}\cos(mx) - C(3-4\nu)e^{-mz}\cos(mx) \tag{15}$$

$$\sigma_{zx} = Cm^2ze^{-mz}\sin(mx) + C(1-2\nu)me^{-mz}\sin(mx) \tag{16}$$

$$\sigma_{zz} = Cm^2ze^{-mz}\cos(mx) + 2C(1-\nu)me^{-mz}\cos(mx)\ . \tag{17}$$

The remaining constants B, C must be chosen so as to satisfy the traction-free condition

$$\sigma_{zx} = \sigma_{zz} = 0\ ; \quad z = 0 \tag{18}$$

Adding corresponding terms in equations (12, 13) and (16, 17), imposing this condition and cancelling factors of $\sin(mx), \cos(mx)$, we obtain

$$-Am^2 + Bm^2 + C(1-2\nu)m = 0 \tag{19}$$

$$Am^2 + Bm^2 + 2C(1-\nu)m = 0\ . \tag{20}$$

The solution of these equations is

$$B = A(3-4\nu) = -\frac{\mu(1+\nu)(3-4\nu)\alpha}{m(1-\nu)}\int_0^\infty e^{-m\zeta}f(\zeta)d\zeta \tag{21}$$

$$C = -2mA = \frac{2\mu(1+\nu)\alpha}{(1-\nu)}\int_0^\infty e^{-m\zeta}f(\zeta)d\zeta\ . \tag{22}$$

We can now determine the surface displacement by adding equations (11, 15) and setting $z = 0$. We obtain

$$2\mu u_z(x,y,0) = Am\cos(mx) - Bm\cos(mx) - C(3-4\nu)\cos(mx) \tag{23}$$

$$= 4\mu(1+\nu)\alpha\cos(mx)\int_0^\infty e^{-m\zeta}f(\zeta)d\zeta \tag{24}$$

and hence

$$u_z(x,y,0) = 2(1+\nu)\alpha\cos(mx)\int_0^\infty e^{-m\zeta}f(\zeta)d\zeta\ . \tag{25}$$

22.3. During a test on an automotive disk brake, it was found that the temperature T of the disk could be approximated by the expression

$$T = \frac{5}{4}(1 - e^{-\tau}) - \frac{r^2}{2a^2}(1 - e^{-\tau}) + \frac{r^4}{8a^4}\tau e^{-\tau} \ ,$$

where $\tau = t/t_0$ is a dimensionless time, a is the radius of the disk, r is the distance from the axis, t is time and t_0 is a constant. The disk is continuous at $r=0$ and all its surfaces can be assumed to be traction-free.

Derive expressions for the radial and circumferential thermal stresses as functions of radius and time, and hence show that the maximum value of the stress difference $(\sigma_{\theta\theta} - \sigma_{rr})$ at any given time always occurs at the outer edge of the disk.

At any given time, the temperature distribution can be written

$$T = C_1 + C_2 r^2 + C_3 r^4 \ , \tag{1}$$

where

$$C_1 = \frac{5}{4}\left(1 - e^{-t/t_0}\right) \quad ; \quad C_2 = -\frac{\left(1 - e^{-t/t_0}\right)}{2a^2} \quad ; \quad C_3 = \frac{(t/t_0)e^{-t/t_0}}{8a^4} \ . \tag{2}$$

With this notation,

$$\int_0^r rT(r)dr = \int_0^r (C_1 r + C_2 r^3 + C_3 r^5)dr = \frac{C_1 r^2}{2} + \frac{C_2 r^4}{4} + \frac{C_3 r^6}{6} \tag{3}$$

and the stress components can be written down from the plane stress version of (22.24-22.26) as

$$\sigma_{rr} = -2\mu(1+\nu)\alpha\left(\frac{C_1}{2} + \frac{C_2 r^2}{4} + \frac{C_3 r^4}{6}\right) + A \ ; \tag{4}$$

$$\sigma_{\theta\theta} = 2\mu(1+\nu)\alpha\left(-\frac{C_1}{2} - \frac{3C_2 r^2}{4} - \frac{5C_3 r^4}{6}\right) + A \ ; \tag{5}$$

$$\sigma_{r\theta} = 0 \ . \tag{6}$$

Notice that the term B/r^2 has been omitted from these equations, since the stresses must be bounded at the origin.

The constant A is determined from the condition that the outer edge of the disk be traction-free — i.e. $\sigma_{rr} = 0$ at $r = a$ and

$$-2\mu(1+\nu)\alpha\left(\frac{C_1}{2} + \frac{C_2 a^2}{4} + \frac{C_3 a^4}{6}\right) + A \ , \tag{7}$$

giving

$$A = 2\mu(1+\nu)\alpha\left(\frac{C_1}{2} + \frac{C_2 a^2}{4} + \frac{C_3 a^4}{6}\right) . \tag{8}$$

Substituting in the stress equations (4–6), we obtain

$$\sigma_{rr} = 2\mu(1+\nu)\alpha\left(\frac{C_2(a^2-r^2)}{4} + \frac{C_3(a^4-r^4)}{6}\right) ; \tag{9}$$

$$\sigma_{\theta\theta} = 2\mu(1+\nu)\alpha\left(\frac{C_2(a^2-3r^2)}{4} + \frac{C_3(a^4-5r^4)}{6}\right) ; \tag{10}$$

$$\sigma_{r\theta} = 0 . \tag{11}$$

The stress difference $\sigma_{\theta\theta} - \sigma_{rr}$ is

$$\sigma_{\theta\theta} - \sigma_{rr} = -2\mu(1+\nu)\alpha\left(\frac{C_2 r^2}{2} + \frac{2C_3 r^4}{3}\right) \tag{12}$$

and is zero at $r = 0$.

At any given time t, the maximum value must therefore occur *either* at $r = a$ *or* where

$$\frac{\partial(\sigma_{\theta\theta} - \sigma_{rr})}{\partial r} = 0 .$$

Now

$$\frac{\partial(\sigma_{\theta\theta} - \sigma_{rr})}{\partial r} = -2\mu(1+\nu)\alpha\left(C_2 r + \frac{8C_3 r^3}{3}\right) \tag{13}$$

and this is equal to zero when

$$r^2 = -\frac{3C_2}{8C_3} = \frac{3a^2\left(1-e^{-t/t_0}\right)}{2(t/t_0)e^{-t/t_0}} = \frac{3a^2\left(e^{-t/t_0}-1\right)}{2(t/t_0)} . \tag{14}$$

But

$$3\left(e^{-t/t_0}-1\right) = 3\left(\frac{t}{t_0} + \frac{1}{2!}\left(\frac{t}{t_0}\right)^2 + \frac{1}{3!}\left(\frac{t}{t_0}\right)^3 + ...\right) , \tag{15}$$

which is greater that $2t/t_0$ for all $t > 0$. Thus, the supposed maximum point is outside the disk ($r > a$) for all t and we conclude that the stress difference increases monotonically with r in $0 < r < a$ and that the maximum therefore occurs at $r = a$.

22.4. A heat exchanger tube consists of a long hollow cylinder of inner radius a and outer radius b, maintained at temperatures T_a, T_b respectively. Find the steady-state stress distribution in the tube if the curved surfaces are traction-free and plane strain conditions can be assumed.

In the steady state, the temperature distribution is harmonic and is therefore of the form

$$T = C + D\ln(r) \ , \tag{1}$$

where C, D are arbitrary constants. These constants must satisfy the conditions

$$T(a) = C + D\ln(a) = T_a \ ; \quad T(b) = C + D\ln(b) = T_b \tag{2}$$

and hence

$$C = \frac{T_a\ln(b) - T_b\ln(a)}{\ln(b/a)} \ ; \quad D = \frac{(T_b - T_a)}{\ln(b/a)} \tag{3}$$

The temperature distribution is therefore

$$T(r) = \frac{T_a\ln(b/r) + T_b\ln(r/a)}{\ln(b/a)} \tag{4}$$

The stresses are then given by equations (22.24, 22.25). We note that

$$\int_a^r rT(r)dr = \int_a^r (Cr + Dr\ln(r))dr = \frac{C(r^2-a^2)}{2} + \frac{D(r^2\ln(r) - a^2\ln(a))}{2} - \frac{D(r^2-a^2)}{4} \tag{5}$$

and hence

$$\sigma_{rr} = \frac{\mu(1+\nu)\alpha}{(1+\nu)r^2}\left(C(r^2-a^2) + D(r^2\ln(r) - a^2\ln(a)) - \frac{D(r^2-a^2)}{2}\right) + A + \frac{B}{r^2} \ , \tag{6}$$

where we have replaced $2\mu A, 2\mu B$ by A, B respectively.

The constants A, B are determined from the traction-free conditions

$$\sigma_{rr} = 0 \ ; \quad r = a, b \tag{7}$$

and hence

$$\begin{aligned} A + \frac{B}{a^2} &= 0 \\ \frac{2\mu(1+\nu)\alpha}{(1+\nu)b^2}\left(\frac{C(b^2-a^2)}{2} + \frac{D(b^2\ln(b) - a^2\ln(a))}{2} - \frac{D(b^2-a^2)}{4}\right) + A + \frac{B}{b^2} &= 0 \ . \end{aligned}$$

Solving for A, B and substituting for C, D, we have

$$A = -\frac{\mu(1+\nu)\alpha}{(1+\nu)(b^2-a^2)}\left(b^2T_b - a^2T_a - \frac{(T_b-T_a)(b^2-a^2)}{2\ln(b/a)}\right) \tag{8}$$

$$B = \frac{\mu(1+\nu)\alpha a^2}{(1+\nu)(b^2-a^2)}\left(b^2T_b - a^2T_a - \frac{(T_b-T_a)(b^2-a^2)}{2\ln(b/a)}\right) \ . \tag{9}$$

Substituting for A, B, C, D into equation (6), we then obtain

$$\sigma_{rr} = \frac{\mu(1+\nu)\alpha(T_b - T_a)(a^2r^2\ln(r/a) + b^2r^2\ln(b/r) - a^2b^2\ln(b/a))}{(1+\nu)(b^2-a^2)\ln(b/a)r^2} \ . \tag{10}$$

The circumferential stress is obtained from (22.25) as

$$\sigma_{\theta\theta} = \frac{\mu(1+\nu)\alpha(T_b - T_a)(a^2(b^2-a^2) - r^2b^2(\ln(b/a) + \ln(r/a)) + r^2a^2\ln(r/a) - a^2b^2\ln(b/a))}{(1+\nu)(b^2-a^2)\ln(b/a)r^2} \ . \tag{11}$$

22.5. The steady-state temperature $T(\xi,\eta)$ in the quarter plane $\xi \geq 0, \eta \geq 0$ is determined by the boundary conditions

$$\begin{aligned} T(\xi,0) &= 0 \\ T(0,\eta) &= T_0\ ; \quad 0<\eta<a \\ &= 0\ \ ; \quad \eta > a\ . \end{aligned}$$

Use the function

$$\omega(\zeta) = \zeta^{1/2}$$

to map the problem into the half plane $y > 0$ and solve for T using the method of §18.6.3.

Then determine the complex displacement u throughout the quarter plane, using equation (22.32).

The given function ω maps

$$\rho \exp(\imath\vartheta) \to r^{1/2} \exp\left(\frac{\imath\theta}{2}\right)$$

and in particular

$$0 \leq \vartheta \leq \frac{\pi}{2} \to 0 \leq \theta \leq \pi\ .$$

On the ξ-axis, $\xi = x^{1/2},\ x > 0$ and hence

$$T(x,0) = 0\ ; \quad x > 0\ .$$

On the η-axis, $\imath\eta = (x)^{1/2} = \imath(-x)^{1/2},\ x < 0$ and hence

$$\begin{aligned} T(x,0) &= T_0\ ; \quad -a^2 < x < 0 \\ &= 0\ ; \quad x < -a^2 \end{aligned} \tag{1}$$

Thus, in the ζ-plane (the half plane), the boundary condition is

$$\begin{aligned} T(x,0) &= T_0\ ; \quad -a^2 < x < 0 \\ &= 0\ ; \quad x < -a^2 \text{ and } x > 0\ . \end{aligned}$$

This is essentially the same boundary-value problem as is treated in Example 1 in §18.6.3, except for a change of scale and origin. We multiply by T_0, move the origin to the point $-a^2/2$ and then replace the quantities in the example according to the scheme

$$a \to \frac{a^2}{2}\ ; \quad \zeta \to \zeta + \frac{a^2}{2}\ ,$$

obtaining

$$g(\zeta) = \frac{T_0}{2\pi\imath} \ln\left(\frac{\zeta}{\zeta + a^2}\right)\ .$$

Moving back to the ω-plane, using the inverse mapping $\zeta = \omega^2$, we have

$$g(\omega) = \frac{T_0}{2\pi\imath} \ln\left(\frac{\omega^2}{\omega^2 + a^2}\right)$$

and hence

$$T(\zeta) = g + \overline{g} = \frac{T_0}{2\pi\imath} \ln\left(\frac{\omega^2(\overline{\omega}^2 + a^2)}{\overline{\omega}^2(\omega^2 + a^2)}\right) \ .$$

We now substitute into (22.32), obtaining

$$\frac{\partial u}{\partial \omega} = \frac{\alpha(1+\nu)T_0}{\pi\imath} \ln\left(\frac{\omega^2}{\omega^2 + a^2}\right)$$

and hence

$$u = \frac{\alpha(1+\nu)T_0}{\pi\imath} \left[\omega \ln\left(\frac{\omega^2}{\omega^2 + a^2}\right) - 2a \arctan\left(\frac{\omega}{a}\right)\right] \ .$$

22.6. Use dimensional arguments and continuity of heat flux to show that the two-dimensional steady-state temperature field in the half plane $y > 0$ due to a line heat source Q per unit length at the origin is

$$T(x, y) = \frac{Q}{\pi K} \ln(r) + C \ ,$$

where $r = \sqrt{x^2 + y^2}$ is the distance from the origin, K is the thermal conductivity and C is a constant.

Use this result and equation (22.32) to determine the displacement throughout the half plane. Assume that $C = 0$ and neglect rigid-body displacements. In particular, determine the displacements at the surface $y = 0$ and verify that they satisfy Dundurs' theorem (14.21).

The conditions of the heat flow problem are satisfied if the heat flux is purely radial, since this will give no heat flux across the boundary except at the origin where there is a singularity. Since the same total heat flux Q has to pass over every semicircular area at radius r, the heat flux must be

$$q_r = -K \frac{dT}{dr} = \frac{Q}{\pi r}$$

and integrating, we obtain

$$T = \frac{Q}{\pi K} \ln(r) + C \ .$$

To use equation (22.32), we must first express T in the form of (22.31) which we do using the result

$$r^2 = \zeta \bar{\zeta} \ ; \quad \ln(r) = \frac{\ln(r^2)}{2} = \frac{(\ln(\zeta) + \ln(\bar{\zeta}))}{2} \ .$$

It follows that

$$T = \frac{Q(\ln(\zeta) + \ln(\bar{\zeta}))}{2\pi K} \quad \text{or} \quad g(\zeta) = \frac{Q \ln(\zeta)}{2\pi K} \ .$$

Substituting into (22.32), integrating with respect to ζ and dropping the rigid-body terms, we obtain the complete displacement field as

$$u = u_x + \imath u_y = \int \frac{\alpha(1+\nu) Q \ln(\zeta) d\zeta}{\pi K} = \frac{\alpha(1+\nu) Q(\zeta \ln(\zeta) - \zeta)}{\pi K} \ .$$

On the surface $y = 0$, we have $\zeta = x$. If $x > 0$, we have

$$u_x + \imath u_y = \frac{\alpha(1+\nu) Q(x \ln(x) - x)}{\pi K}$$

and this is real, so $u_y = 0$. If $x < 0$,

$$\ln(x) = \ln|x| + \imath\pi$$

and

$$u_x + \imath u_y = \frac{\alpha(1+\nu)Q(x\ln|x| + \imath\pi x - x)}{\pi K} \ ,$$

giving

$$u_y = \frac{\alpha(1+\nu)Qx}{K} \ .$$

It follows that the negative real axis rotates clockwise through an angle

$$\frac{\alpha(1+\nu)Q}{K}$$

Dundurs theorem predicts that the (convex) curvature at the surface is given by

$$\frac{\alpha(1+\nu)q}{K} \ ,$$

where q is the heat input per unit length. If a total heat flux Q is input over some length a, the flux will be $q = Q/a$ and the total change in slope across the heated region will be

$$\alpha(1+\nu)qaK = \frac{\alpha(1+\nu)Q}{K} \ .$$

Thus, the present result is consistent with Dundurs' theorem.

22.7. A uniform heat flux $q_x = q_0$ in a large body in a state of plane strain is perturbed by the presence of a rigid non-conducting circular inclusion of radius a. The inclusion is bonded to the matrix at $r = a$. Find the complete stress field in the matrix and identify the location and magnitude of the maximum tensile stress. Assume plane strain conditions.

Since the inclusion is non-conducting, the temperature field in the matrix is given by equation (22.34) and a particular solution for the stress and displacement field is (22.37) — i.e.

$$\begin{aligned} u_r &= -\frac{(1+\nu)\alpha q_0}{2K}\left(r^2 + a^2(\ln(r)+1)\right)\cos\theta \\ u_\theta &= \frac{(1+\nu)\alpha q_0}{2K(1-\nu)}\left(r^2 + a^2\ln(r)\right)\sin\theta \end{aligned} \tag{1}$$

$$\sigma_{rr} = \frac{\mu(1+\nu)\alpha q_0 a^2\cos\theta}{2K(1-\nu)r}\ ; \quad \sigma_{r\theta} = \frac{\mu(1+\nu)\alpha q_0 a^2\sin\theta}{2K(1-\nu)r}$$

$$\sigma_{\theta\theta} = \frac{\mu(1+\nu)\alpha q_0 a^2\cos\theta}{2K(1-\nu)r}\ . \tag{2}$$

Since the inclusion is rigid, the appropriate boundary conditions are

$$u_r = \delta\cos\theta\ ; \quad u_\theta = -\delta\sin\theta\ ; \quad r = a\ , \tag{3}$$

where δ represents an unknown rigid-body displacement of the inclusion in the x-direction (we know by symmetry that it cannot move in the y-direction or rotate). As in §22.1.3, we can satisfy these conditions by superposing the Airy stress function

$$\varphi = C_2 r\theta\sin\theta + C_3 r\ln(r)\cos\theta + \frac{C_4\cos\theta}{r}\ , \tag{4}$$

from Table 8.1. The corresponding stresses (including the particular solution) are

$$\begin{aligned} \sigma_{rr} &= \frac{\mu(1+\nu)\alpha q_0 a^2\cos\theta}{2K(1-\nu)r} + \frac{2C_2\cos\theta}{r} + \frac{C_3\cos\theta}{r} - \frac{2C_4\cos\theta}{r^3} \\ \sigma_{r\theta} &= \frac{\mu(1+\nu)\alpha q_0 a^2\sin\theta}{2K(1-\nu)r} + \frac{C_3\sin\theta}{r} - \frac{2C_4\sin\theta}{r^3} \\ \sigma_{\theta\theta} &= \frac{\mu(1+\nu)\alpha q_0 a^2\cos\theta}{2K(1-\nu)r} + \frac{C_3\cos\theta}{r} + \frac{2C_4\cos\theta}{r^3} \end{aligned} \tag{5}$$

Also, the displacements, from Table 9.1 and (1) are

$$u_r = -\frac{(1+\nu)\alpha q_0}{2K}\left(r^2 + a^2(\ln(r)+1)\right)\cos\theta$$

$$
\begin{aligned}
&+\frac{C_2}{4\mu}\{2(1-2\nu)\theta\sin\theta-\cos\theta+4(1-\nu)\ln(r)\cos\theta\} \\
&+\frac{C_3}{4\mu}\{4(1-\nu)\theta\sin\theta-\cos\theta+2(1-2\nu)\ln(r)\cos\theta\}+\frac{C_4\cos\theta}{2\mu r^2} \\
u_\theta \;=\;& \frac{(1+\nu)\alpha q_0}{2K(1-\nu)}\left(r^2+a^2\ln(r)\right)\sin\theta \\
&+\frac{C_2}{4\mu}\{2(1-2\nu)\theta\cos\theta-\sin\theta-4(1-\nu)\ln(r)\sin\theta\} \\
&+\frac{C_3}{4\mu}\{4(1-\nu)\theta\cos\theta-\sin\theta-2(1-2\nu)\ln(r)\sin\theta\}+\frac{C_4\sin\theta}{2\mu r^2}\;, \qquad (6)
\end{aligned}
$$

where we have used the plane strain assumption $\kappa = 3-4\nu$.

For the displacements to be single-valued, the terms in $\theta\sin\theta, \theta\cos\theta$ must cancel and hence

$$
2(1-2\nu)C_2+4(1-\nu)C_3 = 0\;. \qquad (7)
$$

The boundary conditions (3) give the two equations

$$
\begin{aligned}
-\frac{(1+\nu)\alpha q_0}{2K}\left(a^2+a^2(\ln(a)+1)\right)+\frac{C_2}{4\mu}\{-1+4(1-\nu)\ln(a)\& \\
+\frac{C_3}{4\mu}\{-1+2(1-2\nu)\ln(a)\}+\frac{C_4}{2\mu a^2} \;=\; \delta \qquad (8) \\
\frac{(1+\nu)\alpha q_0}{2K(1-\nu)}\left(a^2+a^2\ln(a)\right)+\frac{C_2}{4\mu}\{-1-4(1-\nu)\ln(a)\& \\
+\frac{C_3}{4\mu}\{-1-2(1-2\nu)\ln(a)\}+\frac{C_4}{2\mu a^2} \;=\; -\delta \qquad (9)
\end{aligned}
$$

Finally, we require that there be no net force on the dislocation, so that

$$
F = \int_0^{2\pi}(\sigma_{rr}\cos\theta-\sigma_{r\theta}\sin\theta)r\,d\theta = 0 \qquad (10)
$$

at $r=a$. As in §13.1, the only term contributing to F is C_3 and hence $C_3=0$. It then follows that $C_2=0$ from (7) and hence the remaining constants are determined from the simplified equations

$$
\begin{aligned}
-\frac{(1+\nu)\alpha q_0}{2K}\left(a^2+a^2(\ln(a)+1)\right)+\frac{C_4}{2\mu a^2} \;&=\; \delta \qquad (11) \\
\frac{(1+\nu)\alpha q_0}{2K(1-\nu)}\left(a^2+a^2\ln(a)\right)+\frac{C_4}{2\mu a^2} \;&=\; -\delta\;, \qquad (12)
\end{aligned}
$$

with solution

$$
\delta = -\frac{\alpha(1+\nu)q_0a^2(1+2\ln(a))}{4K(1-\nu)}\;;\quad C_4 = \frac{\alpha(1+\nu)\mu q_0a^4}{2K(1-\nu)}\;. \qquad (13)
$$

The stress field is then recovered as

$$\sigma_{rr} = \frac{\alpha(1+\nu)\mu q_0 a^2(r^2-a^2)\cos\theta}{K(1-\nu)r^3} \; ; \quad \sigma_{r\theta} = \frac{\alpha(1+\nu)\mu q_0 a^2(r^2-a^2)\sin\theta}{K(1-\nu)r^3}$$

$$\sigma_{\theta\theta} = \frac{\alpha(1+\nu)\mu q_0 a^2(r^2+a^2)\cos\theta}{K(1-\nu)r^3} \; . \tag{14}$$

The maximum tensile stress is $\sigma_{\theta\theta}$ at $(a, 0)$ and is

$$\sigma_{\max} = \frac{2\alpha(1+\nu)\mu q_0 a}{K(1-\nu)} \; . \tag{15}$$

Notice that the stress field is actually identical with that obtained when the inclusion is replaced by a hole. Another way to have predicted this would have been to determine the displacements for the solution of §22.1.3, in which case we would have found that they take the form of a rigid body displacement at $r = a$.

22.8. A uniform heat flux $q_x = q_0$ in a large body in a state of plane strain with elastic and thermal properties $\mu_1, \nu_1, \alpha_1, K_1$ is perturbed by the presence of a circular inclusion of radius a with properties $\mu_2, \nu_2, \alpha_2, K_2$. The inclusion is bonded to the matrix at $r = a$. Find the complete stress field in the matrix and the inclusion and identify the location and magnitude of the maximum tensile stress in each. Assume plane strain conditions. Note: You will find the algebra for this problem formidable if you do not use Maple or Mathematica.

The unperturbed solution in an infinite body of the matrix material can be taken from §22.1.3 as

$$T_1 = -\frac{q_0 r \cos\theta}{K_1} \; ; \quad \phi = \varphi = -\frac{\mu_1(1+\nu_1)\alpha_1 q_0 r^3 \cos\theta}{4K_1(1-\nu_1)} \; , \tag{1}$$

$$u_r = -\frac{(1+\nu_1)\alpha_1 q_0 r^2 \cos\theta}{2K_1} \; ; \quad u_\theta = -\frac{(1+\nu_1)\alpha_1 q_0 r^2 \sin\theta}{2K_1} \; ; \quad \sigma_{rr} = \sigma_{r\theta} = \sigma_{\theta\theta} = 0 \; . \tag{2}$$

The temperature perturbation must vary with $\cos\theta$, tend to the unperturbed solution as $r \to \infty$, be bounded at the origin and be harmonic. The only solution satisfying these conditions is

$$T_1 = -\frac{q_0 r \cos\theta}{K_1} + \frac{C_1 \cos\theta}{r} \; ; \quad T_2 = C_2 r \cos\theta \; , \tag{3}$$

for which the corresponding radial heat flux component is

$$q_r^{(1)} = q_0 \cos\theta + \frac{K_1 C_1 \cos\theta}{r^2} \; ; \quad q_r^{(2)} = -K_2 C_2 \cos\theta \; . \tag{4}$$

At the interface $r = a$, we must have

$$T_1(a,\theta) = T_2(a,\theta) \; ; \quad q_r^{(1)}(a,\theta) = q_r^{(2)}(a,\theta) \; , \tag{5}$$

giving

$$\frac{C_1}{a} - C_2 a = \frac{q_0 a}{K_1} \; ; \quad \frac{K_1 C_1}{a^2} + K_2 C_2 = -q_0 \; . \tag{6}$$

The solution of these equations is

$$C_1 = -\frac{q_0 a^2 (K_1 - K_2)}{K_1(K_1 + K_2)} \; ; \quad C_2 = -\frac{2q_0}{(K_1 + K_2)} \tag{7}$$

and hence

$$T_1 = -\frac{q_0}{K_1}\left(r + \frac{a^2(K_1 - K_2)}{(K_1 + K_2)r}\right)\cos\theta \; ; \quad T_2 = -\frac{2q_0 r}{(K_1 + K_2)}\cos\theta \; . \tag{8}$$

A particular solution for the thermoelastic stress field can be obtained by writing

$$\phi = -\frac{\mu_1(1+\nu_1)\alpha_1 q_0 r^3 \cos\theta}{4K_1(1-\nu_1)} + C_3 r \ln(r)\cos\theta \ . \tag{9}$$

Substituting into (22.10) then yields the result

$$C_3 = -\frac{\mu_1(1+\nu_1)\alpha_1 q_0 (K_1-K_2)a^2}{4K_1(K_1+K_2)(1-\nu_1)} \tag{10}$$

and this additional term in ϕ yields additional stress and displacement components

$$\begin{aligned} u_r^{(1)} &= -\frac{(1+\nu_1)\alpha_1 q_0 (K_1-K_2)a^2(1+\ln(r))\cos\theta}{2K_1(K_1+K_2)(1-\nu_1)} \\ u_\theta^{(1)} &= \frac{(1+\nu_1)\alpha_1 q_0 (K_1-K_2)a^2 \ln(r)\sin\theta}{2K_1(K_1+K_2)(1-\nu_1)} \end{aligned} \tag{11}$$

$$\begin{aligned} \sigma_{rr}^{(1)} &= \frac{\mu_1(1+\nu_1)\alpha_1 q_0 (K_1-K_2)a^2 \cos\theta}{K_1(K_1+K_2)(1-\nu_1)r} \\ \sigma_{r\theta}^{(1)} &= \frac{\mu_1(1+\nu_1)\alpha_1 q_0 (K_1-K_2)a^2 \sin\theta}{K_1(K_1+K_2)(1-\nu_1)r} \\ \sigma_{\theta\theta}^{(1)} &= \frac{\mu_1(1+\nu_1)\alpha_1 q_0 (K_1-K_2)a^2 \cos\theta}{K_1(K_1+K_2)(1-\nu_1)r} \ . \end{aligned} \tag{12}$$

In the inclusion, we write

$$\phi = C_4 r^3 \cos\theta \tag{13}$$

by analogy with (1) and routine calculations show that

$$C_4 = -\frac{\mu_2(\alpha_2(1+\nu_2)q_0}{2(K_1+K_2)(1-\nu_2)} \tag{14}$$

giving displacement and stress components

$$u_r^{(2)} = -\frac{3(\alpha_2(1+\nu_2)q_0 r^2\cos\theta}{4(K_1+K_2)(1-\nu_2)} \ ; \quad u_\theta^{(2)} = \frac{\alpha_2(1+\nu_2)q_0 r^2 \sin\theta}{4(K_1+K_2)(1-\nu_2)} \tag{15}$$

$$\sigma_{rr}^{(2)} = \frac{\mu_2\alpha_2(1+\nu_2)q_0 r\cos\theta}{(K_1+K_2)(1-\nu_2)} \ ; \quad \sigma_{r\theta}^{(2)} = \frac{\mu_2\alpha_2(1+\nu_2)q_0 r\sin\theta}{(K_1+K_2)(1-\nu_2)}$$

$$\sigma_\theta^{(2)} = \frac{3\mu_2\alpha_2(1+\nu_2)q_0 r\cos\theta}{(K_1+K_2)(1-\nu_2)} \tag{16}$$

To complete the solution, we add in Airy stress function terms from Tables 8.1, 9.1 to satisfy the boundary conditions at $r = a$. In the interests of brevity, we note first that the term $r\theta\sin\theta$ is unique in corresponding to a force acting at the origin and this must be excluded from the stress function in the matrix if the inclusion is

not restrained from moving. With this term excluded, considerations of single valued displacements then eliminate the term $r\ln(r)\cos\theta$ as in Chapter 13. We therefore add merely the terms

$$\varphi_1 = \frac{C_5\cos\theta}{r}\ ; \quad \varphi_2 = C_6 r\cos\theta \tag{17}$$

in the matrix and inclusion respectively. The complete stress and displacement fields in the matrix are then obtained by assembling (2, 11, 12) and the appropriate expressions from (17) and Tables 8.1, 9.1 as

$$\begin{aligned}
u_r^{(1)} &= -\frac{(1+\nu_1)\alpha_1 q_0 r^2\cos\theta}{2K_1} - \frac{(1+\nu_1)\alpha_1 q_0(K_1-K_2)a^2(1+\ln(r))\cos\theta}{2K_1(K_1+K_2)(1-\nu_1)} + \frac{C_5\cos\theta}{2\mu_1 r^2} \\
u_\theta^{(1)} &= -\frac{(1+\nu_1)\alpha_1 q_0 r^2\sin\theta}{2K_1} + \frac{(1+\nu_1)\alpha_1 q_0(K_1-K_2)a^2\ln(r)\sin\theta}{2K_1(K_1+K_2)(1-\nu_1)} + \frac{C_5\sin\theta}{2\mu_1 r^2}
\end{aligned} \tag{18}$$

$$\begin{aligned}
\sigma_{rr}^{(1)} &= \frac{\mu_1(1+\nu_1)\alpha_1 q_0(K_1-K_2)a^2\cos\theta}{K_1(K_1+K_2)(1-\nu_1)r} - \frac{2C_5\cos\theta}{r^3} \\
\sigma_{r\theta}^{(1)} &= \frac{\mu_1(1+\nu_1)\alpha_1 q_0(K_1-K_2)a^2\sin\theta}{K_1(K_1+K_2)(1-\nu_1)r} - \frac{2C_5\sin\theta}{r^3} \\
\sigma_{\theta\theta}^{(1)} &= \frac{\mu_1(1+\nu_1)\alpha_1 q_0(K_1-K_2)a^2\cos\theta}{K_1(K_1+K_2)(1-\nu_1)r} + \frac{2C_5\cos\theta}{r^3}\ .
\end{aligned} \tag{19}$$

A similar assembly for the inclusion yields

$$\begin{aligned}
u_r^{(2)} &= -\frac{3\alpha_2(1+\nu_2)q_0 r^2\cos\theta}{4(K_1+K_2)(1-\nu_2)} + \frac{(1-4\nu)C_6 r^2\cos\theta}{2\mu_2} \\
u_\theta^{(2)} &= \frac{\alpha_2(1+\nu_2)q_0 r^2\sin\theta}{4(K_1+K_2)(1-\nu_2)} + \frac{(5-4\nu)C_6 r^2\sin\theta}{2\mu_2} \\
\sigma_{rr}^{(2)} &= \frac{\mu_2\alpha_2(1+\nu_2)q_0 r\cos\theta}{(K_1+K_2)(1-\nu_2)} + 2C_6 r\cos\theta \\
\sigma_{r\theta}^{(2)} &= \frac{\mu_2\alpha_2(1+\nu_2)q_0 r\sin\theta}{(K_1+K_2)(1-\nu_2)} + 2C_6 r\sin\theta \\
\sigma_{\theta\theta}^{(2)} &= \frac{3\mu_2\alpha_2(1+\nu_2)q_0 r\cos\theta}{(K_1+K_2)(1-\nu_2)} + 6C_6 r\cos\theta\ .
\end{aligned}$$

The interface conditions at $r = a$ are

$$u_r^{(1)} - u_r^{(2)} = \delta\cos\theta\ ; \quad u_\theta^{(1)} - u_\theta^{(2)} = -\delta\sin\theta\ ; \quad \sigma_{rr}^{(1)} = \sigma_{rr}^{(2)}\ ; \quad \sigma_{r\theta}^{(1)} = \sigma_{r\theta}^{(2)} \tag{20}$$

from which

$$\begin{aligned}
\frac{C_5}{2\mu_1 a^2} - \frac{(1-4\nu)C_6 a^2}{2\mu_2} - \delta &= \frac{(1+\nu_1)\alpha_1 q_0 a^2}{2K_1} \\
&\quad + \frac{(1+\nu_1)\alpha_1 q_0(K_1-K_2)a^2(1+\ln(a))}{2K_1(K_1+K_2)(1-\nu_1)} - \frac{3\alpha_2(1+\nu_2)q_0 a^2}{4(K_1+K_2)(1-\nu_2)}
\end{aligned} \tag{21}$$

$$\frac{C_5}{2\mu_1 a^2} - \frac{(5-4\nu)C_6 a^2}{2\mu_2} + \delta = \frac{(1+\nu_1)\alpha_1 q_0 a^2}{2K_1} - \frac{(1+\nu_1)\alpha_1 q_0 (K_1-K_2)a^2 \ln(a)}{2K_1(K_1+K_2)(1-\nu_1)} + \frac{\alpha_2(1+\nu_2)q_0 r^2}{4(K_1+K_2)(1-\nu_2)} \tag{22}$$

$$-\frac{2C_5}{a^3} - 2C_6 a = -\frac{\mu_1(1+\nu_1)\alpha_1 q_0 (K_1-K_2)a}{K_1(K_1+K_2)(1-\nu_1)} + \frac{\mu_2\alpha_2(1+\nu_2)q_0 a}{(K_1+K_2)(1-\nu_2)} \tag{23}$$

$$-\frac{2C_5}{a^3} - 2C_6 a = -\frac{\mu_1(1+\nu_1)\alpha_1 q_0 (K_1-K_2)a}{K_1(K_1+K_2)(1-\nu_1)} + \frac{\mu_2\alpha_2(1+\nu_2)q_0 a}{(K_1+K_2)(1-\nu_2)} \ . \tag{24}$$

Notice that the last two equations are identical, so the remaining three independent equations can be solved for C_5, C_6, δ giving

$$\begin{aligned} C_5 &= \frac{q_0\mu_1 a^4 C}{2[\mu_1(3-4\nu_1)+\mu_2](1-\nu_2)(1-\nu_1)K_1(K_1+K_2)} \\ C_6 &= \frac{q_0\mu_2[(\mu_1-\mu_2)K_1\alpha_2(1+\nu_2) - 2\mu_1\alpha_1(1+\nu_1)(K_1+K_2)]}{2[\mu_1(3-4\nu_1)+\mu_2](1-\nu_2)K_1(K_1+K_2)} \ , \end{aligned} \tag{25}$$

where

$$\begin{aligned} C &= (1+\nu_1)\alpha_1(1-\nu_2)[\mu_2(K_2(1-2\nu_1)+K_1(3-2\nu_1)) + \mu_1(K_1-K_2)(3-4\nu_1)] \\ &\quad -4\mu_2\alpha_2 K_1(1-\nu_1)^2(1+\nu_2) \ . \end{aligned} \tag{26}$$

The expressions for stresses are too complicated to present here, but it is clear that the maximum values of $\sigma_{rr}, \sigma_{\theta\theta}$ must occur at the poles where $r = a, \theta = 0, \pi$. Some simplification can be achieved by defining the ratios

$$\hat{\mu} = \frac{\mu_2}{\mu_1} \ ; \quad \hat{\alpha} = \frac{\alpha_2}{\alpha_1} \ ; \quad \hat{K} = \frac{K_2}{K_1}$$

and restricting attention to the case $\nu_1 = \nu_2 = 1/3$. We then obtain

$$\sigma_{rr}^{(1)}(a,0) = -\frac{8\alpha_1\mu_1 q_0 a\hat{\mu}(1+\hat{K}-2\hat{\alpha})}{K_1(5+3\hat{\mu})(1+\hat{K})} \tag{27}$$

$$\sigma_{\theta\theta}^{(1)}(a,0) = \frac{4\alpha_1\mu_1 q_0 a(5+5\hat{\mu}-5\hat{K}-\hat{\mu}\hat{K}-4\hat{\mu}\hat{\alpha})}{K_1(5+3\hat{\mu})(1+\hat{K})} \tag{28}$$

$$\sigma_{rr}^{(2)}(a,0) = -\frac{8\alpha_1\mu_1 q_0 a\hat{\mu}(1+\hat{K}-2\hat{\alpha})}{K_1(5+3\hat{\mu})(1+\hat{K})} \tag{29}$$

$$\sigma_{\theta\theta}^{(1)}(a,0) = -\frac{24\alpha_1\mu_1 q_0 a\hat{\mu}(1+\hat{K}-2\hat{\alpha})}{K_1(5+3\hat{\mu})(1+\hat{K})} \ . \tag{30}$$

The solution is antisymmetric in z, so the stresses at the other pole (a, π) are equal and opposite to those given above.

22.9. A rigid non-conducting body slides at velocity V in the positive x-direction over the surface of the thermoelastic half-space $z>0$, causing frictional tractions σ_{zx} and frictional heating q_z given by

$$\sigma_{zx} = -f\sigma_{zz} \ ; \ \sigma_{zy} = 0 \ ; \ q_z = V\sigma_{zx} \ ,$$

where f is the coefficient of friction. Find a solution in terms of one potential function that satisfies these global boundary conditions in the steady state.

If the half-space remains in contact with the sliding body throughout the plane $z=0$, we have the additional global boundary condition $u_z = 0$. Show that a non-trivial steady-state solution to this problem can be obtained using the harmonic function

$$Ce^{-mz}\cos(my) \ ,$$

where C is an arbitrary constant and m is a parameter which must be chosen to satisfy the displacement boundary condition. Comment on the physical significance of this solution.

On the surface $z=0$ a combination of Solutions A, B, E and T gives

$$\begin{aligned}
2\mu u_z &= \frac{\partial\phi}{\partial z} - (3-4\nu)\omega + \chi && (1)\\
\sigma_{zx} &= \frac{\partial^2\phi}{\partial x\partial z} - (1-2\nu)\frac{\partial\omega}{\partial x} + \frac{\partial^2\Psi}{\partial y\partial z} + \frac{\partial\chi}{\partial x} && (2)\\
\sigma_{zy} &= \frac{\partial^2\phi}{\partial y\partial z} - (1-2\nu)\frac{\partial\omega}{\partial y} - \frac{\partial^2\Psi}{\partial x\partial z} + \frac{\partial\chi}{\partial y} && (3)\\
\sigma_{zz} &= \frac{\partial^2\phi}{\partial z^2} - 2(1-\nu)\frac{\partial\omega}{\partial z} && (4)\\
q_z &= -\frac{(1-\nu)K}{\mu\alpha(1+\nu)}\frac{\partial^2\chi}{\partial z^2} \ . && (5)
\end{aligned}$$

Thus, to satisfy the stated conditions, we require

$$\begin{aligned}
\frac{\partial^2\phi}{\partial x\partial z} - (1-2\nu)\frac{\partial\omega}{\partial x} + \frac{\partial^2\Psi}{\partial y\partial z} + \frac{\partial\chi}{\partial x} - f\frac{\partial^2\phi}{\partial z^2} + 2(1-\nu)f\frac{\partial\omega}{\partial z} &= 0 && (6)\\
\frac{\partial^2\phi}{\partial y\partial z} - (1-2\nu)\frac{\partial\omega}{\partial y} - \frac{\partial^2\Psi}{\partial x\partial z} + \frac{\partial\chi}{\partial y} &= 0 && (7)\\
\frac{(1-\nu)K}{\mu\alpha(1+\nu)}\frac{\partial^2\chi}{\partial z^2} - fV\frac{\partial^2\phi}{\partial z^2} + 2(1-\nu)fV\frac{\partial\omega}{\partial z} &= 0 \ , && (8)
\end{aligned}$$

where we have used (6) to simplify (8).

We can eliminate Ψ by differentiating (6) with respect to x and (7) with respect to y and adding the results. We obtain

$$-\frac{\partial^3\phi}{\partial z^3} - f\frac{\partial^3\phi}{\partial x\partial z^2} + 2(1-\nu)f\frac{\partial^2\omega}{\partial x\partial z} + (1-2\nu)\frac{\partial^2\omega}{\partial z^2} - \frac{\partial^2\chi}{\partial z^2} = 0 \ , \tag{9}$$

where we have used the fact that the functions are all harmonic to replace (for example)

$$\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2} = -\frac{\partial^2\omega}{\partial z^2} \ .$$

Eliminating χ between (7) and (8), we then obtain

$$\frac{\partial^2}{\partial z^2}\left(\kappa\frac{\partial\phi}{\partial z} + \kappa f\frac{\partial\phi}{\partial x} + \phi\right) = \frac{\partial}{\partial z}\left(2(1-\nu)\kappa f\frac{\partial\omega}{\partial x} + \kappa(1-2\nu)\frac{\partial\omega}{\partial z} + 2(1-\nu)\omega\right) \ , \tag{10}$$

where

$$\kappa = \frac{(1-\nu)K}{fV\mu\alpha(1+\nu)} \ . \tag{11}$$

This equation can be satisfied by defining a new function Θ such that

$$\phi = 2(1-\nu)\kappa f\frac{\partial\Theta}{\partial x} + \kappa(1-2\nu)\frac{\partial\Theta}{\partial z} + 2(1-\nu)\Theta \tag{12}$$

$$\omega = \kappa\frac{\partial^2\Theta}{\partial z^2} + \kappa f\frac{\partial^2\Theta}{\partial x\partial z} + \frac{\partial\Theta}{\partial z} \ . \tag{13}$$

Substituting back into (8), we then have

$$\frac{\partial^3\Theta}{\partial z^3} + \frac{\partial^2\chi}{\partial z^2} = 0 \ , \tag{14}$$

which is satisfied if

$$\chi = -\frac{\partial\Theta}{\partial z} \ . \tag{15}$$

Finally, we substitute (12, 13, 15) into (7), obtaining

$$\kappa f\frac{\partial^3\Theta}{\partial x\partial y\partial z} - \frac{\partial^2\Psi}{\partial x\partial z} = 0 \ , \tag{16}$$

which is satisfied if

$$\Psi = \kappa f\frac{\partial\Theta}{\partial y} \ . \tag{17}$$

The required solution is then obtained by substituting (12, 13, 15, 17) into Solutions A, B, E and T.

In particular, we obtain

$$2\mu u_z(x,y,0) = -(1-2\nu)\kappa f \frac{\partial^2\Theta}{\partial x \partial z} - 2(1-\nu)\kappa \frac{\partial^2\Theta}{\partial z^2} - 2(1-\nu)\frac{\partial\Theta}{\partial z} \ . \tag{18}$$

If we choose

$$\Theta = Ce^{-mz}\cos(my) \ , \tag{19}$$

the condition $u_z(x,y,0) = 0$ gives

$$-2(1-\nu)\kappa m^2 C\cos(my) + 2(1-\nu)mC\cos(my) = 0 \tag{20}$$

and a non-trivial solution is obtained if

$$m = \frac{1}{\kappa} = \frac{fV\mu\alpha(1+\nu)}{(1-\nu)K} \ . \tag{21}$$

Under these conditions, a self-sustaining non-uniform pressure distribution can occur. Essentially, the thermoelastic distortion due to frictional heating exactly balances the elastic displacement due to the pressure. For smaller values of m (or for example for larger values of the sliding speed V), the thermoelastic distortion exceeds the elastic deformation and a non-uniform disturbance of this form will grow with time. In other words, the system is unstable for perturbations of wavelength greater than

$$\frac{2\pi(1-\nu)K}{fV\mu\alpha(1+\nu)} \ .$$

22.10. The half-space $z>0$ is bonded to a rigid body at the surface $z=0$. Show that in the steady state, the surface temperature $T(x,y,0)$ and the normal stress $\sigma_{zz}(x,y,0)$ at the bond are related by the equation

$$\sigma_{zz}(x,y,0) = -\frac{2\mu\alpha(1+\nu)T(x,y,0)}{(3-4\nu)} \ .$$

On the surface $z=0$ a combination of Solutions A, B and T gives

$$\begin{aligned}
2\mu u_x &= \frac{\partial\phi}{\partial x} && (1)\\
2\mu u_y &= \frac{\partial\phi}{\partial y} && (2)\\
2\mu u_z &= \frac{\partial\phi}{\partial z} - (3-4\nu)\omega + \chi && (3)\\
\sigma_{zz} &= \frac{\partial^2\phi}{\partial z^2} - 2(1-\nu)\frac{\partial\omega}{\partial z} && (4)\\
T &= \frac{(1-\nu)}{\mu\alpha(1+\nu)}\frac{\partial\chi}{\partial z} \ . && (5)
\end{aligned}$$

If the body is bonded to a rigid body, we require

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= 0 && (6)\\
\frac{\partial\phi}{\partial y} &= 0 && (7)\\
\frac{\partial\phi}{\partial z} - (3-4\nu)\omega + \chi &= 0 \ . && (8)
\end{aligned}$$

Thes equations can clearly be satisfied by the choice

$$\phi = 0 \ ; \quad \chi = (3-4\nu)\omega \ , \qquad (9)$$

leaving one free function to define the temperature field.

The temperature and normal traction at the surface then take the form

$$\begin{aligned}
\sigma_{zz} &= -2(1-\nu)\frac{\partial\omega}{\partial z} && (10)\\
T &= \frac{(1-\nu)(3-4\nu)}{\mu\alpha(1+\nu)}\frac{\partial\omega}{\partial z} && (11)
\end{aligned}$$

and it follows immediately that

$$\sigma_{zz}(x,y,0) = -\frac{2\mu\alpha(1+\nu)T(x,y,0)}{(3-4\nu)} \ .$$

22.11. The traction-free surface of the half space $z>0$ is uniformly heated in the annular region $a<r<b$ and unheated elsewhere, so that

$$\begin{aligned} q_z(r,\theta,0) &= \frac{Q}{\pi(b^2-a^2)} \;;\quad a<r<b \\ &= 0 \qquad 0\le r<a \quad \textbf{and} \quad r>b\,. \end{aligned}$$

Use equation (22.58) to determine the slope of the surface

$$\frac{\partial u_z}{\partial r}(r,\theta,0)$$

in the steady state. By taking the limit of this solution as $a\to b$ or otherwise, find the surface displacement due to a heat input Q uniformly distributed along a circle of radius b. Comment on the implications for steady-state thermoelastic contact problems.

The heating is axisymmetric, so u_z is a function of r only and equation (22.36) takes the form

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr}\right) = 0\;;\quad 0\le r<a \tag{1}$$

$$= \frac{Q\alpha(1+\nu)}{\pi K(b^2-a^2)}\;;\quad a<r<b \tag{2}$$

$$= 0\;;\quad r>b\,. \tag{3}$$

Integrating this expression, we find that

$$\frac{du_z}{dr} = \frac{C_1}{r}\;;\quad 0\le r<a \tag{4}$$

$$= \frac{Q\alpha(1+\nu)r}{\pi K(b^2-a^2)}+\frac{C_2}{r}\;;\quad a<r<b \tag{5}$$

$$= \frac{C_3}{r}\;;\quad r>b\,. \tag{6}$$

The constants C_1, C_2, C_3 must be chosen so as to ensure continuity of slope at $r = 0, a, b$ and hence

$$C_1 = 0 \tag{7}$$

$$\frac{C_1}{a} = \frac{Q\alpha(1+\nu)a}{2\pi K(b^2-a^2)}+\frac{C_2}{a} \tag{8}$$

$$\frac{Q\alpha(1+\nu)b}{2\pi K(b^2-a^2)}+\frac{C_2}{b} = \frac{C_3}{b}\,. \tag{9}$$

Solving these equations, we obtain

$$C_1 = 0\;;\quad C_2 = -\frac{Q\alpha(1+\nu)a^2}{2\pi K(b^2-a^2)}\;;\quad C_3 = \frac{Q\alpha(1+\nu)}{2\pi K}\,. \tag{10}$$

Thus, the surface is flat and horizontal inside the heated annulus.

In the limit $a \rightarrow b$ the heating occurs only along the line $r = b$ and the slope is

$$\frac{du_z}{dr} = 0 \; ; \quad 0 \leq r < b \tag{11}$$

$$= \frac{Q\alpha(1+\nu)}{2\pi K r} \; ; \quad r > b \; . \tag{12}$$

This result shows that axisymmetric heating *outside* some radius b does not distort the surface inside $r = b$. Thus, for example, if an axisymmetric punch is pressed into the surface by a force F, establishing a circular contact area of radius a, the contact pressure distribution and the size of the contact area would be unaffected by changes in the (axisymmetric) heat input or output outside the contact circle, as long as these changes did not affect the heat flow through the contact area.

CHAPTER 23

23.1. Starting from the potential function solution of Problem 21.2, use dimensional arguments to determine a suitable *axisymmetric* harmonic singular potential to solve the problem of a concentrated *tangential* force F in the x-direction, applied to the surface of the half-space at the origin.

We require the stresses to decay with R^{-2}. Noting that the solution of Problem 21.2 defines stresses as third derivatives of the stress function χ, we conclude that we need a potential of order R. From equation (23.23), we therefore select ϕ_{-2} — i.e.

$$\chi = A(z\ln(R+z) - R)\ . \tag{1}$$

We first collect some information about the derivatives of this function, temporarily dropping the multiplying constant A. We have

$$\begin{aligned}
\frac{\partial\chi}{\partial x} &= \frac{xz}{(R+z)R} - \frac{x}{R} = -\frac{x}{(R+z)}\ ;\\
\frac{\partial\chi}{\partial y} &= -\frac{y}{(R+z)}\ ;\\
\frac{\partial\chi}{\partial z} &= \ln(R+z)\ ;\\
\frac{\partial^2\chi}{\partial x^2} &= -\frac{1}{(R+z)} + \frac{x^2}{R(R+z)^2}\ ;\\
\frac{\partial^2\chi}{\partial y^2} &= -\frac{1}{(R+z)} + \frac{y^2}{R(R+z)^2}\ ;\\
\frac{\partial^2\chi}{\partial z^2} &= \frac{1}{R}\ ;\\
\frac{\partial^2\chi}{\partial x\partial y} &= \frac{xy}{R(R+z)^2}\ ;\\
\frac{\partial^2\chi}{\partial y\partial z} &= \frac{y}{R(R+z)}\ ;\\
\frac{\partial^2\chi}{\partial z\partial x} &= \frac{x}{R(R+z)}\ ;\\
\frac{\partial^3\chi}{\partial x^3} &= \frac{3x}{R(R+z)^2} - \frac{x^3}{R^3(R+z)^2} - \frac{2x^3}{R^2(R+z)^3}\ ;\\
\frac{\partial^3\chi}{\partial y^3} &= \frac{3y}{R(R+z)^2} - \frac{y^3}{R^3(R+z)^2} - \frac{2y^3}{R^2(R+z)^3}\ ;\\
\frac{\partial^3\chi}{\partial z^3} &= -\frac{z}{R^3}\ ;
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^3\chi}{\partial x^2\partial z} &= \frac{1}{R(R+z)} - \frac{x^2}{R^3(R+z)} - \frac{x^2}{R^2(R+z)^2}\ ; \\
\frac{\partial^3\chi}{\partial y^2\partial z} &= \frac{1}{R(R+z)} - \frac{y^2}{R^3(R+z)} - \frac{y^2}{R^2(R+z)^2}\ ; \\
\frac{\partial^3\chi}{\partial z^2\partial x} &= -\frac{x}{R^3}\ ; \\
\frac{\partial^3\chi}{\partial z^2\partial y} &= -\frac{y}{R^3}\ ; \\
\frac{\partial^3\chi}{\partial x^2\partial y} &= \frac{y}{R(R+z)^2} - \frac{x^2y}{R^3(R+z)^2} - \frac{2x^2y}{R^2(R+z)^3}\ ; \\
\frac{\partial^3\chi}{\partial y^2\partial x} &= \frac{x}{R(R+z)^2} - \frac{xy^2}{R^3(R+z)^2} - \frac{2xy^2}{R^2(R+z)^3}\ ; \\
\frac{\partial^3\chi}{\partial x\partial y\partial z} &= -\frac{xy}{R^3(R+z)} - \frac{xy}{R^2(R+z)^2}\ ; \\
\frac{\partial^4\chi}{\partial x^2\partial z^2} &= -\frac{1}{R^3} + \frac{3x^2}{R^5}\ ;
\end{aligned}
$$

etc.

Symmetry conditions show that there can be no net force in the y or z-directions when this function is substituted into the solution of Problem 21.2. To see this, note that an axisymmetric function is even in x, so the stress σ_{zz} will be odd in x and will therefore sum to zero over any infinite z-plane. Also, both solution and function are even in y, so there can be no net force in the y-direction. To determine the multipying constant A, we first find the stress component σ_{xz} and then impose equilibrium conditions on a plane $z = h$. We have

$$
\begin{aligned}
\sigma_{xz} &= -\frac{\partial^3\chi}{\partial z^3} + z\frac{\partial^4\chi}{\partial x^2\partial z^2} \\
&= \frac{Az}{R^3} - \frac{Az}{R^3} + \frac{3Azx^2}{R^5} = \frac{3Azx^2}{R^5} && (2) \\
&= \frac{3Azr^2\cos^2\theta}{R^5} && (3)
\end{aligned}
$$

The total force on the plane $z = h$ can then be written

$$
\begin{aligned}
F &= -\int_0^\infty \int_0^{2\pi} \frac{3Ahr^3\cos^2\theta}{R^5} d\theta dr && (4) \\
&= -\int_0^\infty \frac{3A\pi hr^3 dr}{R^5}\ , && (5)
\end{aligned}
$$

where $R^2 = z^2 + h^2$.

We have

$$
\int_0^\infty \frac{r^3 dr}{R^5} = \left[-\frac{r^2}{3R^3}\right]_0^\infty + \frac{1}{3}\int_0^\infty \frac{2rdr}{R^3} = 0 + \left[-\frac{2}{3R}\right]_0^\infty = \frac{2}{3h} \qquad (6)
$$

and hence

$$F = -3A\pi h\left(\frac{2}{3h}\right) = -2A\pi \tag{7}$$

and

$$A = -\frac{F}{2\pi}\ . \tag{8}$$

We can now calculate the displacement and stress components, using the results of Problem 21.2. For example,

$$\begin{aligned}
2\mu u_x &= 2(1-\nu)\frac{\partial^2\chi}{\partial x^2} + z\frac{\partial^3\chi}{\partial x^2\partial z} + 2\frac{\partial^2\chi}{\partial y^2} \\
&= \frac{F}{2\pi}\frac{[(2(1-\nu)R^2 + (3-2\nu)Rz + z^2)R^2 + x^2(2\nu R^2 + 2Rz + z^2)]}{R^3(R+z)^2}\ ; && (9)\\
2\mu u_y &= -2\nu\frac{\partial^2\chi}{\partial x\partial y} + z\frac{\partial^3\chi}{\partial x\partial y\partial z} \\
&= \frac{Fxy}{2\pi R(R+z)}\left(\frac{2\nu}{(R+z)} + \frac{z}{R^2} + \frac{z}{R(R+z)}\right)\ ; && (10)\\
2\mu u_z &= -(1-2\nu)\frac{\partial^2\chi}{\partial x\partial z} + z\frac{\partial^3\chi}{\partial z^2\partial x} = \frac{Fx}{2\pi R}\left(\frac{(1-2\nu)}{(R+z)} + \frac{z}{R^2}\right)\ ; && (11)\\
\sigma_{xx} &= 2(1-\nu)\frac{\partial^3\chi}{\partial x^3} + z\frac{\partial^4\chi}{\partial x^3\partial z} - 2\nu\frac{\partial^3\chi}{\partial z^2\partial x} + 2\frac{\partial^3\chi}{\partial y^2\partial x} \\
&= \frac{Fx}{2\pi R^3}\left[-\frac{3x^2}{R^2} + \frac{(1-2\nu)}{(R+z)^2}\left(R^2 - y^2 - \frac{2Ry^2}{R+z}\right)\right]\ ; && (12)\\
\sigma_{xy} &= (1-2\nu)\frac{\partial^3\chi}{\partial x^2\partial y} + \frac{\partial^3\chi}{\partial y^3} + z\frac{\partial^4\chi}{\partial x^2\partial y\partial z} \\
&= \frac{Fy}{2\pi R^3}\left[-\frac{3x^2}{R^2} + \frac{(1-2\nu)}{(R+z)^2}\left(-R^2 + x^2 + \frac{2Rx^2}{R+z}\right)\right]\ ; && (13)\\
\sigma_{yy} &= 2\nu\frac{\partial^3\chi}{\partial x^3} + z\frac{\partial^4\chi}{\partial y^2\partial x\partial z} \\
&= \frac{Fx}{2\pi R^3}\left[-\frac{3y^2}{R^2} + \frac{(1-2\nu)}{(R+z)^2}\left(3R^2 - x^2 - \frac{2Rx^2}{R+z}\right)\right]\ ; && (14)\\
\sigma_{xz} &= -\frac{\partial^3\chi}{\partial z^3} + z\frac{\partial^4\chi}{\partial x^2\partial z^2} = -\frac{3Fzx^2}{2\pi R^5}\ ; && (15)\\
\sigma_{yz} &= z\frac{\partial^4\chi}{\partial x\partial y\partial z^2} = -\frac{3Fxyz}{2\pi R^5}\ ; && (16)\\
\sigma_{zz} &= z\frac{\partial^4\chi}{\partial x\partial z^3} = -\frac{3Fz^2x}{2\pi R^5}\ .
\end{aligned}$$

Notice incidentally that the expressions for the three stress components σ_{zx}, σ_{zy}, σ_{zz} on the z-plane are identical except for a multiplier of x, y, z respectively. It follows

that the resultant traction on any differential element of area in any z-plane has a direction such that its line of action passes through the origin and a magnitude of

$$-\frac{3Fxz}{2\pi R^4} \ .$$

23.2. Find the stresses and displacements[1] corresponding to the use of $\Psi = 1/R$ in Solution E. Show that the surfaces of the cone, $r = z\tan\beta$ are free of traction and hence use the solution to determine the stresses in a conical shaft of semi-angle β_0, transmitting a torque, T.

Find the maximum shear stress at the section $z = c$, and compare it with the maximum shear stress in a cylindrical bar of radius $a = c\tan\beta_0$ transmitting the same torque. Note that (i) the *maximum* shear stress will not generally be either $\sigma_{z\theta}$ or $\sigma_{r\theta}$, but must be found by appropriate coördinate transformation, and (ii) it may not *necessarily* occur at the outer radius of the cone, $r = c\tan\beta_0$.

The function $1/R$ is axisymmetric and hence the only non-zero stress components are those for which there are terms without θ derivatives — i.e.

$$\sigma_{\theta r} = \frac{1}{r}\frac{\partial\Psi}{\partial r} - \frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2} = -\frac{3r^2}{R^5}\ ; \tag{1}$$

$$\sigma_{\theta z} = -\frac{\partial^2\Psi}{\partial r\partial z} = -\frac{3rz}{R^5}\ . \tag{2}$$

Consider the shear stresses on the $\theta-$plane in a coordinate system normal and tangential to a conical surface as shown in Figure 1.

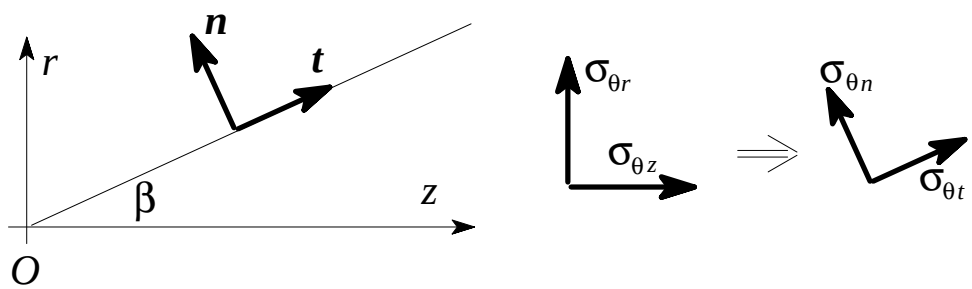


Figure 1: Stress components normal and tangential to a conical surface

We have

$$\sigma_{\theta n} = \sigma_{\theta r}\cos\beta - \sigma_{\theta z}\sin\beta = -\frac{3r}{R^5}(r\cos\beta - z\sin\beta)\ ; \tag{3}$$

$$\sigma_{\theta t} = \sigma_{\theta z}\cos\beta + \sigma_{\theta r}\sin\beta = -\frac{3r}{R^5}(z\cos\beta + r\sin\beta)\ . \tag{4}$$

However, $r = z\tan\beta$ and hence $\sigma_{\theta n} = 0$ — i.e. all conical surfaces radiating from the origin are traction-free[2]. Thus, this stress function can be used to describe the

[1] This solution is known as the *centre of rotation*.

[2] Notice that the two shear stress components can be combined and transformed as a vector on the θ-plane. The fact that $\sigma_{\theta n} = 0$ implies that the complementary shear stress $\sigma_{n\theta} = 0$ — i.e. that there is no θ-direction traction on the n-surface, which is the surface of the cone.

state of stress in a conical shaft transmitting a torque. We take $\Psi = A/R$ and choose the arbitrary constant A so as to give the required value of transmitted torque T.

From equilibrium considerations we have

$$\begin{aligned} T &= 2\pi \int_0^a \sigma_{\theta z} r^2 dr = -6\pi z A \int_0^a \frac{r^3 dr}{(r^2+z^2)^{\frac{5}{2}}} \\ &= 2\pi z A \left[\frac{r^2}{(r^2+z^2)^{\frac{3}{2}}} + \frac{2}{(r^2+z^2)^{\frac{1}{2}}} \right]_{r=0}^{r=a} \\ &= 2\pi A \left(\frac{za^2}{(a^2+z^2)^{\frac{3}{2}}} + \frac{2z}{(a^2+z^2)^{\frac{1}{2}}} - 2 \right) . \end{aligned} \tag{5}$$

Using the trigonometric relations

$$\sin\beta = \frac{a}{(a^2+z^2)^{\frac{1}{2}}} \quad ; \quad \cos\beta = \frac{z}{(a^2+z^2)^{\frac{1}{2}}} , \tag{6}$$

we have

$$T = 2\pi A(\sin^2\beta\cos\beta + 2\cos\beta - 2) = -2\pi A(2 + \cos^3\beta - 3\cos\beta) \tag{7}$$

and hence

$$A = \frac{-T}{2\pi(2+\cos^3\beta - 3\cos\beta)} . \tag{8}$$

It is worth noting here that the expression for T depends on β only and is independent of z, as indeed it must be if we are to have the solution for a conical shaft transmitting a constant torque. This result alone is sufficient to establish that the surface of the cone is traction-free, since the only possible tractions ($\sigma_{n\theta}$)would be axisymmetric and would therefore lead to a change in torque with z.

Maximum stress

Since the only non-zero stress components can be summed as a vector on the θ-plane (see equations (3,4)) above, the maximum shear stress at any given point must be $\sigma_{\theta t}$, which is

$$\sigma_{\theta t} = -\frac{3Ar}{R^4}(\cos^2\beta + \sin^2\beta) = -\frac{3Ar}{R^4} , \tag{9}$$

using $r = R\sin\beta, z = R\cos\beta$.

For a given value of z, the maximum value of this stress occurs when

$$\frac{\partial\sigma_{\theta t}}{\partial r} = 0 = \frac{\partial}{\partial r}\left(\frac{3Ar}{R^4}\right) = -3A\left(\frac{1}{R^4} - \frac{4r^2}{R^6}\right) \tag{10}$$

— i.e.

$$4r^2 = R^2 \quad ; \quad \frac{r}{R} = \sin\beta = \frac{1}{2} \quad ; \quad \beta = 30^o . \tag{11}$$

If the cone angle $\beta < 30^o$, the maximum shear stress is at the outer edge and is of magnitude

$$\tau_{\max} = \frac{3Aa}{R_a^4} = -\frac{3A\sin^4\beta}{a^3} = \frac{3T\sin^4\beta}{2\pi a^3(2+\cos^3\beta - 3\cos\beta)} \ , \tag{12}$$

for $\beta < 30^o$, where a is the local radius of the cone cross-section and $R_a = a/\sin\beta$ is the distance from the origin to the point on the surface of the cone.

If $\beta > 30^o$, the maximum shear stress occurs within the body at $\beta = 30^o$ and is of magnitude

$$\tau_{\max} = \frac{3Ab}{R_b^4} = -\frac{3A\sin^4 30^o}{2b^3} \ , \tag{13}$$

where $a = z\tan\beta, b = z\tan 30^o$ and hence $b = a\tan 30^o/\tan\beta$.

Thus,

$$\tau_{\max} = \frac{3T\sin^4 30^o\tan^3\beta}{2\pi(2+\cos^3\beta - 3\cos\beta)a^3\tan^3 30^o} = \frac{9\sqrt{3}T\tan^3\beta}{32\pi a^3(2+\cos^3\beta - 3\cos\beta)} \ , \tag{14}$$

for $\beta > 30^o$.

The variation of $\sigma_{\theta t}$ with radius at constant z for this case is illustrated in Figure 2.

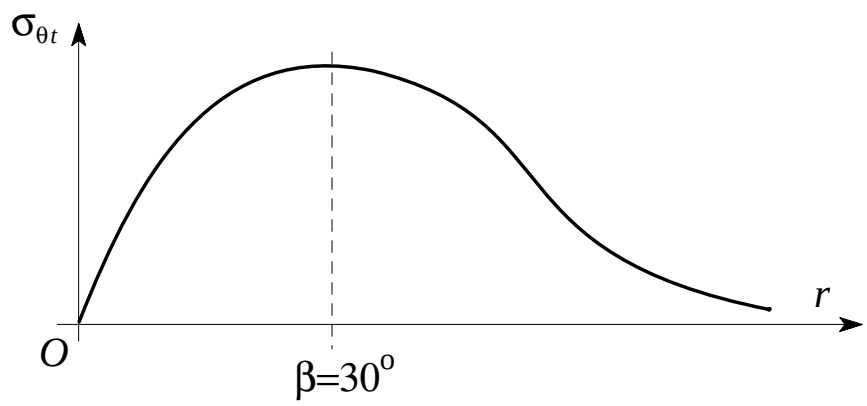


Figure 2: Variation of $\sigma_{\theta t}$ with radius at constant z, for $\beta > 30^o$.

The elementary torsion theory gives

$$\tau_{\max} = \frac{2T}{\pi a^3} \ . \tag{15}$$

This is *higher* than the value given by equations (12,14) for $\beta < 43^o$. In other words, the elementary theory is pessimistic. However, for $\beta > 43^o$, the elementary theory increasingly underestimates the maximum stress.

23.3. An otherwise uniform cylindrical bar of radius b has a small spherical hole of radius a on the axis.

Combine the function $\Psi = Az/R^3$ in solution E with the elementary torsion solution for the bar without a hole (see, for example, Problem 21.1) and show that, with a suitable choice of the arbitrary constant A, the surface of the hole, $R=a$ can be made traction-free. Hence deduce the stress field near the hole for this problem.

Assume that $b \gg a$, so that the perturbation due to the hole has a negligible influence on the stresses near the outer surface, $r=b$.

The stress function $\Psi = Az/R^3$ produces the stress components

$$\sigma_{\theta r} = \frac{1}{r}\frac{\partial \Psi}{\partial r} - \frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} = -\frac{15Azr^2}{R^7} \; ; \tag{1}$$

$$\sigma_{\theta z} = -\frac{\partial^2 \Psi}{\partial r \partial z} = \frac{3Ar}{R^5} - \frac{15Az^2 r}{R^7} \; . \tag{2}$$

In addition, the unperturbed solution for elementary torsion is

$$\sigma_{\theta z} = \frac{2Tr}{\pi b^4} \; , \tag{3}$$

where T is the transmitted torque.

Using Figure 1, we can deduce that the surface of the hole will be traction free if and only if

$$\sigma_{\theta z} \cos\beta + \sigma_{\theta r} \sin\beta = 0 \tag{4}$$

on $R = a$.

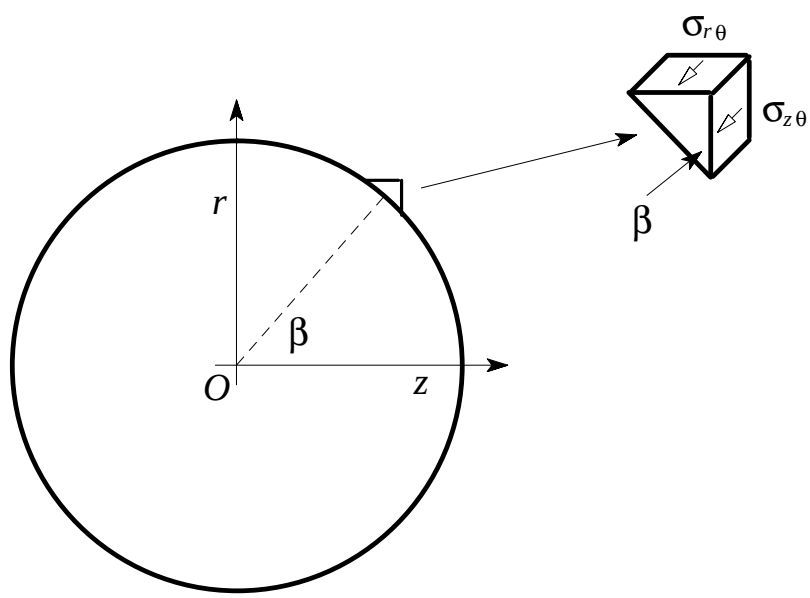


Figure 1: Traction-free boundary condition at the hole surface

Substituting for the stress components from equations (1–3), we require

$$\left(\frac{3Ar}{R^5} - \frac{15Az^2r}{R^7} + \frac{2Tr}{\pi b^4}\right)\frac{z}{R} - \left(\frac{15Azr^2}{R^7}\right)\frac{r}{R} = 0 \quad ; \quad R = a \ . \tag{5}$$

Noting that $r^2 + z^2 = R^2$, this equation simplifies to

$$-\frac{12Azr}{R^6} + \frac{2Trz}{\pi b^4 R} = 0 \quad ; \quad R = a \ , \tag{6}$$

which is satisfied if

$$A = \frac{Ta^5}{6\pi b^4} \ . \tag{7}$$

With this value, the complete stress field is

$$\sigma_{\theta r} = -\frac{5Ta^5 zr^2}{2\pi b^4 R^7} \ ; \tag{8}$$

$$\sigma_{\theta z} = \frac{Ta^5 r}{2\pi b^4 R^5} - \frac{5Ta^5 z^2 r}{2\pi b^4 R^7} + \frac{2Tr}{\pi b^4} \ . \tag{9}$$

Taking out the elementary solution (3) as a factor, we have

$$\sigma_{\theta r} = -\frac{2Tr}{\pi b^4}\left(\frac{5a^5 zr}{4R^7}\right) \ ; \tag{10}$$

$$\sigma_{\theta z} = \frac{2Tr}{\pi b^4}\left(\frac{a^5}{4R^5} - \frac{5a^5 z^2}{4R^7} + 1\right) \ . \tag{11}$$

In particular, the stress at the edge of the hole $z = 0, r = a$ is

$$\sigma_{\theta z}(r, 0) = \frac{2Tr}{\pi b^4}\left(\frac{1}{4} + 1\right) \ , \tag{12}$$

and corresponds to a local stress concentration factor of 1.25. Notice however that the hole is not in a location of high stress, so there will be regions of the shaft (at the outer surface) where the stresses exceed this value.

23.4. The area A on the surface of an elastic half-space is subjected to a purely normal pressure $p(x, y)$ corresponding to the total force, F. The loaded region is now expanded in a self-similar manner by multiplying all its linear dimensions by the same ratio, λ. Each point in the new contact area is subjected to a self-similar loading such that the pressure at the point $(\lambda x, \lambda y)$ is $Cp(x, y)$, where the constant C is chosen to ensure that the total force F is independent of λ.

Show that the deflection at corresponding points $(\lambda x, \lambda y, \lambda z)$ will then be proportional to F/λ.

If C were equal to unity — i.e. if the pressure at corresponding points in the two fields were equal — the total force

$$F = \int\int_A p(x, y)dxdy \tag{1}$$

would be proportional to λ^2, since all elements of area over which the pressure acts expand according to this ratio. It follows that the total force will remain constant if and only if we choose

$$C = \frac{1}{\lambda^2} \tag{2}$$

Suppose the function $f(\xi, \eta, \zeta)$ is harmonic ($\nabla^2 f = 0$). It follows that the functions

$$\phi_1 = f(x, y, z) \quad ; \quad \phi_2 = Bf(bx, by, bz) \tag{3}$$

are also harmonic, if B, b are constants.

Consider the stress states defined by the use of ϕ_1, ϕ_2 respectively in Solution F. In each case, there will be no shear tractions on the surface. The normal tractions at the point $(x, y, 0)$ in each case will be

$$\sigma_{zz1}(x, y, 0) \equiv -p_1(x, y) = \frac{\partial^2 f}{\partial \zeta^2}(x, y, 0) \; ; \tag{4}$$

$$\sigma_{zz2}(x, y, 0) \equiv -p_2(x, y) = Bb^2 \frac{\partial^2 f}{\partial \zeta^2}(bx, by, 0) \; . \tag{5}$$

Since x, y are arbitrary in these equations, the traction at $\lambda x, \lambda y$ due to ϕ_2 can be found by replacing x, y by $\lambda x, \lambda y$, respectively, with the result

$$\sigma_{zz2}(\lambda x, \lambda y, 0) \equiv -p_2(\lambda x, \lambda y) = Bb^2 \frac{\partial^2 f}{\partial \zeta^2}(b\lambda x, b\lambda y, 0) \; . \tag{6}$$

We can make $p_2(\lambda x, \lambda y) = Cp_1(x, y)$ by choosing $b\lambda = 1$ and $Bb^2 = C$ — i.e.

$$b = \frac{1}{\lambda} \quad ; \quad B = \frac{C}{b^2} = C\lambda^2 = 1 \; , \tag{7}$$

using equation (2).

The deflection component u_z is given by

$$2\mu u_z = z\frac{\partial^2\phi}{\partial z^2} - 2(1-\nu)\frac{\partial\phi}{\partial z} \ , \tag{8}$$

from Table 16.2. Substituting for ϕ_2 from (3), we have

$$2\mu u_{z2} = zBb^2\frac{\partial^2 f}{\partial z^2}(bx, by, bz) - 2(1-\nu)Bb\frac{\partial f}{\partial z}(bx, by, bz) \tag{9}$$

and the value at the point $(\lambda x, \lambda y, \lambda z)$ is

$$\begin{aligned} 2\mu u_{z2}(\lambda x, \lambda y, \lambda z) &= \lambda zBb^2\frac{\partial^2 f}{\partial z^2}(\lambda bx, \lambda by, \lambda bz) - 2(1-\nu)Bb\frac{\partial f}{\partial z}(\lambda bx, \lambda by, \lambda bz) \\ &= \frac{1}{\lambda}\left[z\frac{\partial^2 f}{\partial z^2}(x, y, z) - 2(1-\nu)\frac{\partial f}{\partial z}(x, y, z)\right] \ , \end{aligned} \tag{10}$$

using (7). We also have

$$2\mu u_{z1}(x, y, z) = z\frac{\partial^2 f}{\partial z^2}(x, y, z) - 2(1-\nu)\frac{\partial f}{\partial z}(x, y, z) \tag{11}$$

and hence

$$u_{z2}(\lambda x, \lambda y, \lambda z) = \frac{1}{\lambda}u_{z1}(x, y, z) \ . \tag{12}$$

Similar results can be proved in the same way for the displacement components u_x, u_y.

We have assumed in this proof that the force F was the same in the two solutions. If the load also varies whilst the fields remain in other respects similar, the displacements will also be linear with F, giving displacements at corresponding points proportional to F/λ.

23.5. The surface of the half space $z > 0$ is traction-free and a concentrated heat source Q is applied at the origin. Determine the stress and displacement field in the half space in the steady state.

The traction-free condition can be satisfied by using solution P of Table 22.1. You can then use dimensional arguments to determine the dependence of the heat flux on R and hence to choose the appropriate singular potential.

The total heat flux away from the source is constant and hence the heat flux per unit area must vary with R^{-2}. In Solution P, heat flux is described by a third derivative of ψ, so we need to start with a singular solution of order +1. We therefore select the function

$$\psi = C(z\ln(R+z) - R) \tag{1}$$

from equation (23.23).

Substituting into Solution P, we have

$$q_z = \frac{(1-\nu)K}{\mu\alpha(1+\nu)}\frac{\partial^3\psi}{\partial z^3} = -\frac{(1-\nu)KCz}{\mu\alpha(1+\nu)R^3}\,. \tag{2}$$

The total heat input to the half space can be found as

$$Q = -\frac{2\pi(1-\nu)K}{\mu\alpha(1+\nu)Ch}\int_0^\infty \frac{rdr}{(r^2+h^2)^{3/2}} = -\frac{2\pi(1-\nu)KC}{\mu\alpha(1+\nu)} \tag{3}$$

and hence we must choose the arbitrary constant

$$C = -\frac{Q\mu\alpha(1+\nu)}{2\pi(1-\nu)K}\,. \tag{4}$$

The stress and displacement field then follows immediately on substituting (1,4) into Solution P. We obtain

$$2\mu u_r = \frac{Q\mu\alpha(1+\nu)r}{\pi K(R+z)} \tag{5}$$

$$2\mu u_z = \frac{Q\mu\alpha(1+\nu)\ln(R+z)}{\pi K} \tag{6}$$

$$\sigma_{rr} = \frac{Q\mu\alpha(1+\nu)}{\pi K(R+z)} \tag{7}$$

$$\sigma_{\theta\theta} = -\frac{Q\mu\alpha(1+\nu)z}{\pi KR(R+z)}\,. \tag{8}$$

The remaining stress components $\sigma_{zz}, \sigma_{r\theta}, \sigma_{\theta z}, \sigma_{zr}$ are all zero.

23.6. Use Aderogba's formula (23.33) to determine the potentials ϕ, ψ, ω defining the stress and displacement fields in the traction-free half space $z > 0$, subjected to a concentrated force in the x-direction acting at the point $(0, 0, a)$.

From equation (23.10), we know that

$$\omega = \psi_z = -\frac{F}{8\pi(1-\nu)R} \; ; \quad \phi = \psi_x = \psi_y = 0$$

defines the solution for a force F acting at the origin in the z-direction. Since the Papkovich-Neuber solution is isotropic, it follows that

$$\psi_x = -\frac{F}{8\pi(1-\nu)R} \; ; \quad \phi = \psi_y = \psi_z = 0$$

defines the solution for the same force in the x-direction. Writing this in complex notation, we have

$$\psi = \overline{\psi} = -\frac{F}{8\pi(1-\nu)\sqrt{\zeta\bar{\zeta} + z^2}} \; ; \quad \phi = \omega = 0 \; .$$

To move the force to the point $(0, 0, a)$, we use equation (20.21) with

$$\boldsymbol{s} = a\boldsymbol{k} \quad \text{so} \quad \boldsymbol{s} \cdot \boldsymbol{\psi} = a\psi_z = 0 \; ,$$

since $\boldsymbol{\psi}$ has no z-component. Thus,

$$\psi^{(0)} = \overline{\psi}^{(0)} = -\frac{F}{8\pi(1-\nu)\sqrt{\zeta\bar{\zeta} + (z-a)^2}} \; ; \quad \phi^{(0)} = \omega^{(0)} = 0 \; .$$

Suitable harmonic partial integrals of this function can be obtained using the pattern of equations (23.25) as

$$\int \psi^{(0)} dz = \frac{F}{8\pi(1-\nu)} \ln\left(\sqrt{\zeta\bar{\zeta} + (z-a)^2} - (z-a)\right)$$

$$\iint \psi^{(0)} dz dz = \frac{F(z-a)}{8\pi(1-\nu)} \ln\left(\sqrt{\zeta\bar{\zeta} + (z-a)^2} - (z-a)\right) + \frac{F\sqrt{\zeta\bar{\zeta} + (z-a)^2}}{8\pi(1-\nu)} \; .$$

Notice that we use the form (23.25) instead of (23.23), since after replacing z by $-z$, this will ensure that the singularities occur only on the negative z-axis.

We then have

$$
\begin{aligned}
\mathcal{J}\left\{\psi^{(0)}\right\} &= \frac{\partial\psi^{(0)}}{\partial\zeta} + \frac{\partial\overline{\psi}^{(0)}}{\partial\bar{\zeta}} \\
&= \frac{F(\zeta+\bar{\zeta})}{16\pi(1-\nu)(\zeta\bar{\zeta}+(z-a)^2)^{3/2}}\ , \\
\mathcal{J}\left\{\int\psi^{(0)}dz\right\} &= \frac{F(\zeta+\bar{\zeta})}{16\pi(1-\nu)\left(\sqrt{\zeta\bar{\zeta}+(z-a)^2}-(z-a)\right)\sqrt{\zeta\bar{\zeta}+(z-a)^2}} \\
\mathcal{J}\left\{\iint\psi^{(0)}dzdz\right\} &= \frac{F(\zeta+\bar{\zeta})}{16\pi(1-\nu)\left(\sqrt{\zeta\bar{\zeta}+(z-a)^2}-(z-a)\right)}
\end{aligned}
$$

$$
\zeta\overline{\psi}^{(0)} + \bar{\zeta}\psi^{(0)} = -\frac{F(\zeta+\bar{\zeta})}{8\pi(1-\nu)\sqrt{\zeta\bar{\zeta}+(z-a)^2}}\ ,
$$

$$
\frac{\partial}{\partial z}\left(\zeta\overline{\psi}^{(0)} + \bar{\zeta}\psi^{(0)}\right) = \frac{F(z-a)(\zeta+\bar{\zeta})}{8\pi(1-\nu)(\zeta\bar{\zeta}+(z-a)^2)^{3/2}}
$$

and

$$
\begin{aligned}
\mathcal{L}\left\{\psi^{(0)}\right\} &= \frac{1}{2}\frac{\partial}{\partial z}\left(\zeta\overline{\psi}^{(0)} + \bar{\zeta}\psi^{(0)}\right) - z\mathcal{J}\left\{\psi^{(0)}\right\} \\
&= \frac{F(z-a)(\zeta+\bar{\zeta})}{16\pi(1-\nu)(\zeta\bar{\zeta}+(z-a)^2)^{3/2}} - \frac{Fz(\zeta+\bar{\zeta})}{16\pi(1-\nu)(\zeta\bar{\zeta}+(z-a)^2)^{3/2}} \\
&= -\frac{Fa(\zeta+\bar{\zeta})}{16\pi(1-\nu)(\zeta\bar{\zeta}+(z-a)^2)^{3/2}} \\
\mathcal{L}\left\{\int\psi^{(0)}dz\right\} &= -\frac{F(\zeta+\bar{\zeta})\left(\sqrt{\zeta\bar{\zeta}+(z-a)^2}+a\right)}{16\pi(1-\nu)\sqrt{\zeta\bar{\zeta}+(z-a)^2}\left(\sqrt{\zeta\bar{\zeta}+(z-a)^2}-(z-a)\right)}\ ,
\end{aligned}
$$

using Maple.

We now substitute $z \to -z$, obtaining

$$
\begin{aligned}
\left[\mathcal{L}\left\{\int\psi^{(0)}dz\right\}\right]_{z\to-z} &= -\frac{F(\zeta+\bar{\zeta})\left(\sqrt{\zeta\bar{\zeta}+(z+a)^2}+a\right)}{16\pi(1-\nu)\sqrt{\zeta\bar{\zeta}+(z+a)^2}\left(\sqrt{\zeta\bar{\zeta}+(z+a)^2}+(z+a)\right)} \\
\left[\mathcal{J}\left\{\iint\psi^{(0)}dzdz\right\}\right]_{z\to-z} &= \frac{F(\zeta+\bar{\zeta})}{16\pi(1-\nu)\left(\sqrt{\zeta\bar{\zeta}+(z+a)^2}+(z+a)\right)} \\
\left[\psi^{(0)}\right]_{z\to-z} &= -\frac{F}{8\pi(1-\nu)\sqrt{\zeta\bar{\zeta}+(z+a)^2}}
\end{aligned}
$$

$$
\begin{aligned}
\left[\mathcal{J}\left\{\int \psi^{(0)} dz\right\}\right]_{z\to -z} &= \frac{F(\zeta+\bar{\zeta})}{16\pi(1-\nu)\left(\sqrt{\zeta\bar{\zeta}+(z+a)^2}+(z+a)\right)\sqrt{\zeta\bar{\zeta}+(z+a)^2}} \\
\left[\mathcal{L}\left\{\psi^{(0)}\right\}\right]_{z\to -z} &= -\frac{Fa(\zeta+\bar{\zeta})}{16\pi(1-\nu)(\zeta\bar{\zeta}+(z+a)^2)^{3/2}}
\end{aligned}
$$

Assembling the potentials from equation (23.33), we then have

$$
\begin{aligned}
\phi &= -\frac{F(\kappa-1)(\zeta+\bar{\zeta})\left(\sqrt{\zeta\bar{\zeta}+(z+a)^2}+a\right)}{16\pi(1-\nu)\sqrt{\zeta\bar{\zeta}+(z+a)^2}\left(\sqrt{\zeta\bar{\zeta}+(z+a)^2}+(z+a)\right)} \\
&\quad +\frac{F(\kappa^2-1)(\zeta+\bar{\zeta})}{32\pi(1-\nu)\left(\sqrt{\zeta\bar{\zeta}+(z+a)^2}+(z+a)\right)} \\
\psi &= -\frac{F}{8\pi(1-\nu)\sqrt{\zeta\bar{\zeta}+(z-a)^2}}-\frac{F}{8\pi(1-\nu)\sqrt{\zeta\bar{\zeta}+(z+a)^2}} \\
\omega &= -\frac{F(\kappa-1)(\zeta+\bar{\zeta})}{16\pi(1-\nu)\left(\sqrt{\zeta\bar{\zeta}+(z+a)^2}+(z+a)\right)\sqrt{\zeta\bar{\zeta}+(z+a)^2}}+\frac{Fa(\zeta+\bar{\zeta})}{8\pi(1-\nu)(\zeta\bar{\zeta}+(z+a)^2)^{3/2}}
\end{aligned}
$$

23.7. Determine the potentials ϕ, ψ, ω defining the stress and displacement field in the traction-free half space $z > 0$ due to a centre of dilatation of strength ΔV located at the point $(0, 0, a)$.

The potential defining a centre of dilatation at the origin is

$$\phi = -\frac{\mu \Delta V}{2\pi R} ,$$

from equation (23.5) and $\psi = \omega = 0$ which is equivalent to $\boldsymbol{\psi} = 0$. To move to the point $(0, 0, a)$, we use equation (20.21), but since $\boldsymbol{\psi} = 0$ this does not change ϕ, leaving

$$\phi = -\frac{\mu \Delta V}{2\pi \sqrt{r^2 + (z-a)^2}} .$$

Thus, in the notation of equation (23.33), we have

$$\phi^{(0)} = -\frac{\mu \Delta V}{2\pi \sqrt{r^2 + (z-a)^2}} ; \quad \psi^{(0)} = \omega^{(0)} = 0 .$$

It then follows that

$$\frac{\partial \phi^{(0)}}{\partial z} = \frac{\mu \Delta V (z-a)}{2\pi (r^2 + (z-a)^2)^{3/2}}$$

$$\left[\phi^{(0)}\right]_{z \to -z} = -\frac{\mu \Delta V}{2\pi \sqrt{r^2 + (z+a)^2}}$$

$$\left[\frac{\partial \phi^{(0)}}{\partial z}\right]_{z \to -z} = -\frac{\mu \Delta V (z+a)}{2\pi (r^2 + (z+a)^2)^{3/2}}$$

and using (23.33),

$$\phi = -\frac{\mu \Delta V}{2\pi \sqrt{r^2 + (z-a)^2}} - \frac{\mu \kappa \Delta V}{2\pi \sqrt{r^2 + (z+a)^2}} .$$

$$\psi = 0 ; \quad \omega = \frac{\mu \Delta V (z+a)}{\pi (r^2 + (z+a)^2)^{3/2}} .$$

CHAPTER 24

24.1. Use equation (24.22) to evaluate the function $P_1^2(x)$ and verify that it satisfies Legendre's equation (24.9) with $m=2$ and $n=1$. Construct the appropriate harmonic potential function from equation (24.24) and verify that it is singular on the z-axis.

Starting from

$$P_1(x) = x \ , \tag{1}$$

we have

$$\int_x^1 P_1(x)dx = \frac{1-x^2}{2} \tag{2}$$

and

$$\int_x^1 \left(\int_x^1 P_1(x)dx \right) dx = \frac{(1-x)}{2} - \frac{(1-x^3)}{6} \ . \tag{3}$$

It follows from (24.22) that

$$P_1^2(x) = \frac{(1-x)}{2(1-x^2)} - \frac{(1-x^3)}{6(1-x^2)} = \frac{2-x-x^2}{6(1+x)} \ . \tag{4}$$

Legendre's equation (24.9) with $m=2$ and $n=1$ takes the form

$$(1-x^2)\frac{d^2g}{dx^2} - 2x\frac{dg}{dx} + 2g - \frac{4g}{(1-x^2)} \ . \tag{5}$$

Differentiating equation (4), we obtain

$$\frac{d}{dx}P_1^2(x) = \frac{-3-2x-x^2}{(1+x)^2} \ ; \quad \frac{d^2}{dx^2}P_1^2(x) = \frac{4}{(1+x)^3} \ . \tag{6}$$

It is easily verified by substitution that these expressions and (4) satisfy equation (5).

The corresponding spherical harmonics are

$$R^1 P_1^2(z/R)\cos(2\theta) \quad \text{and} \quad R^1 P_1^2(z/R)\cos(2\theta)$$

Using equation (4), we have (for example)

$$R^1 P_1^2(z/R)\cos(2\theta) = \frac{(2R^2 - zR - z^2)\cos(2\theta)}{6(R+z)} \tag{7}$$

and this is clearly singular on the negative z-axis, where $R = -z$. On the positive z-axis, the function tends to zero and is also continuous in all derivatives.

24.2. The functions (23.23) are singular only on the negative z-axis ($r = 0, z < 0$). Similar functions singular only on the positive z-axis are given in equations (23.25). Develop the first three of (24.42) by superposition, noting that $\ln(R+z)+\ln(R-z)=2\ln(r)$.

The first three functions from equation (23.23) are

$$\ln(R+z) \tag{1}$$

$$z\ln(R+z) - R \tag{2}$$

$$\frac{1}{4}\{(2z^2 - r^2)\ln(R+z) - 3Rz + r^2\} \ . \tag{3}$$

Corresponding functions singular only on the positive z-axis are therefore

$$\ln(R-z) \tag{4}$$

$$-z\ln(R-z) - R \tag{5}$$

$$\frac{1}{4}\{(2z^2 - r^2)\ln(R-z) + 3Rz + r^2\} \ , \tag{6}$$

where we have simply replaced z by $-z$ throughout.

Adding (1) and (4), we obtain

$$\ln(R+z) + \ln(R-z) = \ln(R^2 - z^2) = \ln(r^2) = 2\ln(r) \ . \tag{7}$$

Subtracting (5) from (2), we obtain

$$z\ln(R+z) - R + z\ln(R-z) + R = z\ln(R^2 - z^2) = 2z\ln(r) \ . \tag{8}$$

Adding (3) and (6), we obtain

$$\begin{aligned}\frac{1}{4}&\{(2z^2 - r^2)\ln(R+z) - 3Rz + r^2 + (2z^2 - r^2)\ln(R-z) + 3Rz + r^2\} \\ &= \frac{1}{4}\{(2z^2 - r^2)\ln(R^2 - z^2) + 2r^2\} = \frac{1}{2}\{(2z^2 - r^2)\ln(r) + r^2\} \ . \end{aligned} \tag{9}$$

The expressions (7,8,9) are identical to the first three of (24.42) except for arbitrary numerical multipliers.

24.3. Use the methodology of §5.1 to construct the most general harmonic polynomial function of degree 3 in Cartesian coördinates x, y, z. Decompose the resulting polynomial into a set of spherical harmonics.

The most general polynomial of degree 3 is

$$\phi = C_1x^3+C_2x^2y+C_3x^2z+C_4xy^2+C_5xyz+C_6xz^2+C_7y^3+C_8y^2z+C_9yz^2+C_{10}z^3 \ . \quad (1)$$

This function will be harmonic ($\nabla^2\phi = 0$) if

$$6C_1x + 2C_2y + 2C_3z + 2C_4x + 2C_6x + 6C_7y + 2C_8z + 2C_9y + 6C_{10}z = 0 \quad (2)$$

for all x, y, z and hence

$$\begin{aligned} 6C_1 + 2C_4 + 2C_6 &= 0 & (3)\\ 2C_2 + 6C_7 + 2C_9 &= 0 & (4)\\ 2C_3 + 2C_8 + 6C_{10} &= 0 \ . & (5) \end{aligned}$$

We can use these constraint equations to eliminate C_1, C_7, C_{10}, obtaining

$$\begin{aligned} \phi &= A_1(3y^2x - x^3) + A_2(3z^2x - x^3) + A_3(3z^2y - y^3) + A_4(3x^2y - y^3) \\ &\quad +A_5(3x^2z - z^3) + A_6(3y^2z - z^3) + A_7xyz \ , \end{aligned} \quad (6)$$

where we have defined new arbitrary constants through the relations

$$C_2 = 3A_4 \ ; \ \ C_3 = 3A_5 \ ; \ \ C_4 = 3A_1 \ ; \ \ C_5 = A_7 \ ; \ \ C_6 = 3A_2 \ ; \ \ C_8 = 3A_6 \ ; \ \ C_9 = 3A_3 \ . \quad (7)$$

To express these functions in terms of spherical harmonics, we note that

$$x = r\cos\theta \ ; \quad y = r\sin\theta \ ; \quad r = R(1 - z^2/R^2)^{1/2} \ . \quad (8)$$

Thus,

$$\begin{aligned} 3y^2x - x^3 &= r^3(3\sin^2\theta\cos\theta - \cos^3\theta) = r^3(3\cos\theta - 4\cos^3\theta) = -r^3\cos(3\theta) \\ &= -R^3(1 - z^2/R^2)^{3/2}\cos(3\theta) = \frac{R^3P_3^3(z/R)\cos(3\theta)}{15} \qquad (9) \\ 3z^2x - x^3 &= 3z^2r\cos\theta - r^3\cos^3\theta = 3z^2r\cos\theta - \frac{r^3(cos(3\theta) + 3\cos\theta)}{4} \\ &= \frac{3}{4}(4z^2r - r^3)\cos\theta - \frac{r^3\cos(3\theta)}{4} \\ &= \frac{3R^3}{4}\left(\frac{4z^2}{R^2} - \left(1 - \frac{z^2}{R^2}\right)\right)(1 - z^2/R^2)^{1/2}\cos\theta - \frac{R^3(1 - z^2/R^2)^{3/2}\cos(3\theta)}{4} \\ &= \frac{3R^3}{4}\left(\frac{5z^2}{R^2} - 1\right)(1 - z^2/R^2)^{1/2}\cos\theta - \frac{R^3(1 - z^2/R^2)^{3/2}\cos(3\theta)}{4} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{R^3 P_3^1(z/R)\cos\theta}{2} + \frac{R^3 P_3^3(z/R)\cos(3\theta)}{60} && (10)\\
3z^2y - y^3 &= 3z^2 r\sin\theta - r^3\sin^3\theta = 3z^2 r\sin\theta - \frac{r^3(sin(3\theta) - 3\sin\theta)}{4}\\
&= \frac{3}{4}(4z^2 r - r^3)\sin\theta + \frac{r^3\sin(3\theta)}{4}\\
&= -\frac{R^3 P_3^1(z/R)\sin\theta}{2} - \frac{R^3 P_3^3(z/R)\sin(3\theta)}{60} && (11)\\
3x^2y - y^3 &= r^3(3\cos^2\theta\sin\theta - \sin^3\theta) = r^3(3\sin\theta - 4\sin^3\theta) = r^3\sin(3\theta)\\
&= -\frac{R^3 P_3^3(z/R)\sin(3\theta)}{15} && (12)\\
3x^2z - z^3 &= 3r^2 z\cos^2\theta - z^3 = \frac{3r^2 z(1+\cos(2\theta))}{2} - z^3\\
&= \frac{(3r^2z - 2z^3)}{2} + \frac{3r^2 z\cos(2\theta)}{2} = \frac{(3R^2z - 5z^3)}{2} + \frac{3R^2 z}{2}\left(1 - \frac{z^2}{R^2}\right)\cos(2\theta)\\
&= -R^3 P_3(z/R) + \frac{R^3 P_3^2\cos(2\theta)}{10} && (13)\\
3y^2z - z^3 &= 3r^2 z\sin^2\theta - z^3 = \frac{3r^2 z(1-\cos(2\theta))}{2} - z^3\\
&= -R^3 P_3(z/R) - \frac{R^3 P_3^2\cos(2\theta)}{10} && (14)\\
xyz &= r^2 z\sin\theta\cos\theta = \frac{1}{2}r^2 z\sin(2\theta) = \frac{1}{2}R^3\left(1 - \frac{z^2}{R^2}\right)\frac{z}{R}\sin(2\theta)\\
&= \frac{R^3 P_3^2(z/R)\sin(2\theta)}{30}\ . && (15)
\end{aligned}
$$

Thus, the most general third degree polynomial has seven degrees of freedom corresponding to the seven constants A_1, A_7, but it can also be written as a linear combination of the seven independent spherical harmonics

$$R^3 P_3(z/R)\ ;\quad R^3 P_3^1(z/R)\cos\theta\ ;\quad R^3 P_3^1(z/R)\sin\theta\ ;\quad R^3 P_3^2(z/R)\cos(2\theta)\ ;$$

$$R^3 P_3^2(z/R)\sin(2\theta)\ ;\quad R^3 P_3^3(z/R)\cos(3\theta)\ ;\quad R^3 P_3^3(z/R)\sin(3\theta)\ .$$

24.4. Express the recurrence relation of equations (24.43–24.46) in spherical polar coördinates R, θ, β. Check your result by deriving the first three functions starting from the source solution $\phi_0 = 1/R$ and compare the results with the expressions derived in the Maple or Mathematica file 'spn'.

If we write

$$f_m(r, z) = F_m(R, \beta) \ , \tag{1}$$

where

$$R = \sqrt{r^2 + z^2} \ ; \quad \beta = \tan^{-1}(z/R) \ ; \quad r = R \sin\beta \ ; \quad z = R\cos\beta \ , \tag{2}$$

we have

$$\frac{\partial f_m}{\partial r} = \frac{\partial F_m}{\partial R}\frac{\partial R}{\partial r} + \frac{\partial F_m}{\partial \beta}\frac{\partial \beta}{\partial r} = \sin\beta\frac{\partial F_m}{\partial R} + \frac{\cos\beta}{R}\frac{\partial F_m}{\partial \beta} \ . \tag{3}$$

It then follows from equation (24.46) that

$$F_{m+1} = \sin\beta\frac{\partial F_m}{\partial R} + \frac{\cos\beta}{R}\frac{\partial F_m}{\partial \beta} - \frac{mF_m}{R\sin\beta} \ . \tag{4}$$

Notice that the change of variable here from r, z to R, β leaves θ unchanged, so the proof of §24.7 is sufficient to show that

$$\phi_{m+1} = F_{m+1}\cos((m+1)\theta) \tag{5}$$

is harmonic.

Starting with

$$F_0 = \frac{1}{R} \ , \tag{6}$$

we then obtain

$$F_1 = \sin\beta\frac{\partial F_0}{\partial R} + \frac{\cos\beta}{R}\frac{\partial F_0}{\partial \beta} = -\frac{\sin\beta}{R^2} \tag{7}$$

$$F_2 = \sin\beta\frac{\partial F_1}{\partial R} + \frac{\cos\beta}{R}\frac{\partial F_1}{\partial \beta} - \frac{F_1}{R\sin\beta} = \frac{3\sin^2\beta}{R^3} \tag{8}$$

$$F_3 = \sin\beta\frac{\partial F_2}{\partial R} + \frac{\cos\beta}{R}\frac{\partial F_2}{\partial \beta} - \frac{2F_2}{R\sin\beta} = -\frac{15\sin^3\beta}{R^4} \ . \tag{9}$$

These expressions agree with the functions $S11, S22, S33$ from the Maple and Mathematica files 'spn'.

CHAPTER 25

25.1. The solid cylinder $0 \le r < a,\ 0 < z < L$ is supported at the end $z = L$ and subjected to a uniform shear traction $\sigma_{rz} = S$ on the curved boundary $r = a$. The end $z = 0$ is traction-free.

Find the complete stress field in the cylinder using strong boundary conditions on the curved boundary and weak conditions at the ends.

In the absence of torsion, Solutions A,B provide a sufficiently general representation for the problem and since the loading is axisymmetric, we anticipate using Legendre polynomial solutions similar to those used in §25.1.1

To determine the order of the polynomials required, we first consider the elementary Mechanics of Materials solution of the same problem. The only force resultant that can occur is an axial force, which the elementary theory predicts will lead to an axial stress σ_{zz} that is uniformly distributed over the cross-section. The uniform traction $\sigma_{rz} = S$ will cause the axial force to increase linearly with z and hence the Mechanics of Materials solution of this problem therefore involves an axial stress σ_{zz} that increases linearly with z, but is uniform over the cross section.

To accommodate this in Solutions A,B, we need at least a 3rd order polynomial in ϕ and a 2nd order polynomial in ω — i.e.

$$\phi = AR^3 P_3(z/R) = A\left(z^3 - \frac{3}{2}zr^2\right) ; \tag{1}$$

$$\omega = BR^2 P_2(z/R) = \left(z^2 - \frac{1}{2}r^2\right) , \tag{2}$$

from equations (24.27).

We would normally have to include multiples of all lower order polynomials as well. However, the problem is odd in z, so only odd polynomials would appear in ϕ and even polynomials in ω. The only lower order polynomials satisfying this condition involve displacements but no stresses — they correspond therefore to rigid-body displacements and need not be included here, since the question does not ask for displacements.

Substituting (1,2) into solutions A,B, we obtain

$$\sigma_{rr} = \frac{\partial^2\phi}{\partial r^2} + z\frac{\partial^2\omega}{\partial r^2} - 2\nu\frac{\partial\omega}{\partial z} = -3Az - B(1+4\nu)z ; \tag{3}$$

$$\sigma_{rz} = \frac{\partial^2\phi}{\partial r\partial z} + z\frac{\partial^2\omega}{\partial r\partial z} - (1-2\nu)\frac{\partial\omega}{\partial r} = -3Ar + B(1-2\nu)r ; \tag{4}$$

$$\sigma_{zz} = \frac{\partial^2\phi}{\partial z^2} + z\frac{\partial^2\omega}{\partial z^2} - 2(1-\nu)\frac{\partial\omega}{\partial z} = 6Az - 2(1-2\nu)Bz ; \tag{5}$$

$$\sigma_{\theta\theta} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{z}{r}\frac{\partial\omega}{\partial r} - 2\nu\frac{\partial\omega}{\partial z} = -3Az - B(1+4\nu)z , \tag{6}$$

where we have dropped derivatives with respect to θ since the functions are axisymmetric. The remaining stress components $\sigma_{\theta r} = \sigma_{\theta z} = 0$ everywhere.

The strong boundary conditions on the curved edge $r = a$ state that

$$\sigma_{rz} = S \quad ; \quad \sigma_{rr} = \sigma_{r\theta} = 0 \quad ; \quad \text{all } z,\, r = a \ , \tag{7}$$

leading to the equations

$$-3Aa + B(1-2\nu)a = T \quad ; \quad -3A - B(1+4\nu) = 0 \ . \tag{8}$$

The solution is

$$A = -\frac{S(1+4\nu)}{6a(1+\nu)} \quad ; \quad B = \frac{S}{2a(1+\nu)} \tag{9}$$

and the stress components, obtained by substituting these results back into (3–5), are

$$\sigma_{rz} = \frac{Sr}{a} \quad ; \quad \sigma_{zz} = -\frac{Sz}{a} \quad ; \quad \sigma_{rr} = \sigma_{\theta\theta} = 0 \ . \tag{10}$$

The end of the bar $z = 0$ should ideally be traction-free, but the solution of equations (10) involves a non-zero value of σ_{rz} on this surface. However, the weak conditions are already satisfied by the solution, since an axisymmetric distribution of σ_{rz} does not sum to a force or a moment. The axial force is identically zero because $\sigma_{zz} = 0$ on the end in the strong sense, as can be seen from equations (10).

Notice that the elementary theory correctly predicts the expression for σ_{zz} in this problem, since the axial force is $F = -2\pi a T z$ (compressive) and hence

$$\sigma_{zz} = \frac{F}{A} = -\frac{2\pi a S z}{\pi a^2} = -\frac{Sz}{a} \ , \tag{11}$$

where $A = \pi a^2$ is the cross-sectional area of the bar.

25.2. The hollow cylinder $b<r<a,\ 0<z<L$ is supported at the end $z=L$ and subjected to a uniform shear traction $\sigma_{rz}=S$ on the outer boundary $r=a$, the inner boundary $r=b$ and the end $z=0$ being traction-free.

Find the complete stress field in the cylinder using strong boundary conditions on the curved boundary and weak conditions at the ends.

In the absence of torsion, Solutions A,B provide a sufficiently general representation for the problem and since the loading is axisymmetric, we anticipate using Legendre polynomial solutions similar to those used in §25.1.1 However, these will need to be supplemented by solutions singular on the line $r=0$ as in §25.1.2 to permit boundary conditions to be satisfied on the surface of the hole.

To determine the order of the polynomials required, we first consider the elementary Mechanics of Materials solution of the same problem. The only force resultant that can occur is an axial force, which the elementary theory predicts will lead to an axial stress σ_{zz} that is uniformly distributed over the cross-section. The axial force uniform traction $\sigma_{rz}=S$ will cause the axial force to increase linearly with z and hence the Mechanics of Materials solution of this problem therefore involves an axial stress σ_{zz} that increases linearly with z, but is uniform over the cross section.

To accommodate this in Solutions A,B, we need at least a 3rd order polynomial in ϕ and a 2nd order polynomial in ω — i.e.

$$\phi = AR^3P_3(z/R) = A\left(z^3-\frac{3}{2}zr^2\right)\ ; \tag{1}$$

$$\omega = BR^2P_2(z/R) = \left(z^2-\frac{1}{2}r^2\right)\ , \tag{2}$$

from equations (24.27).

The corresponding singular potentials will be of the form

$$\varphi_n(r,z) = R^nP_n(z/R)\ln(r) + S_n(r,z) \tag{3}$$

and are listed in equation (24.42)..

The dominant z-variation of the resulting stress field will generally be of order n in the singular potential, since there will be terms in which the differential operators have been applied exclusively to the $\ln(r)$ multiplier. By contrast, in the bounded polynomials, the resulting stress field will be polynomial of order $(n-2)$ (in ϕ) and hence the dominant $z-$varying term will be of order n. For this reason, it is sufficient to carry the singular potentials to two orders lower than the bounded polynomials.

In the present problem, this involves the introduction of the potential φ_1 of equation (24.42) in ϕ and φ_0 in ω, leading to the representation

$$\phi = A\left(z^3-\frac{3}{2}zr^2\right) + Cz\ln(r)\ ; \tag{4}$$

$$\omega = \left(z^2-\frac{1}{2}r^2\right) + D\ln(r)\ . \tag{5}$$

The stress components are now obtained as

$$\sigma_{rr} = \frac{\partial^2\phi}{\partial r^2} + z\frac{\partial^2\omega}{\partial r^2} - 2\nu\frac{\partial\omega}{\partial z} = -3Az - \frac{Cz}{r^2} - B(1+4\nu)z - \frac{Dz}{r^2} \; ; \tag{6}$$

$$\sigma_{rz} = \frac{\partial^2\phi}{\partial r\partial z} + z\frac{\partial^2\omega}{\partial r\partial z} - (1-2\nu)\frac{\partial\omega}{\partial r} = -3Ar + \frac{C}{r} + B(1-2\nu)r - \frac{(1-2\nu)D}{r} \; ; \tag{7}$$

$$\sigma_{zz} = \frac{\partial^2\phi}{\partial z^2} + z\frac{\partial^2\omega}{\partial z^2} - 2(1-\nu)\frac{\partial\omega}{\partial z} = 6Az - 2(1-2\nu)Bz \; ; \tag{8}$$

$$\sigma_{\theta\theta} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{z}{r}\frac{\partial\omega}{\partial r} - 2\nu\frac{\partial\omega}{\partial z} = -3Az + \frac{Cz}{r^2} - B(1+4\nu)z + \frac{Dz}{r^2} \; . \tag{9}$$

The strong boundary conditions are

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0 \quad ; \quad r = b, \text{ all } z \; ; \tag{10}$$

$$\sigma_{rr} = \sigma_{r\theta} = 0 \quad ; \quad r = a, \text{ all } z \; ; \tag{11}$$

$$\sigma_{rz} = S \quad ; \quad r = a, \text{ all } z \tag{12}$$

and with (6,7) they yield the equations

$$-3A - \frac{C}{b^2} - B(1+4\nu) - \frac{D}{b^2} = 0 \; ; \tag{13}$$

$$-3Aa + \frac{C}{b} + B(1-2\nu)b - \frac{(1-2\nu)D}{b} = 0 \; ; \tag{14}$$

$$-3A - \frac{C}{a^2} - B(1+4\nu) - \frac{D}{a^2} = 0 \; ; \tag{15}$$

$$-3Ab + \frac{C}{a} + B(1-2\nu)a - \frac{(1-2\nu)D}{a} = S \; . \tag{16}$$

The solution of these equations is

$$\begin{aligned} A &= -\frac{(1+4\nu)Sa}{6(a^2-b^2)(1+\nu)} \quad ; \quad B = \frac{Sa}{2(a^2-b^2)(1+\nu)} \quad ; \\ C &= -\frac{Sb^2a}{2(a^2-b^2)(1-\nu)} \quad ; \quad D = \frac{Sb^2a}{2(a^2-b^2)(1-\nu)} \end{aligned} \tag{17}$$

and substituting back into the stress equations (6–9), we find

$$\sigma_{rz} = \frac{Sa(r^2-b^2)}{(a^2-b^2)r} \quad ; \quad \sigma_{zz} = -\frac{2Saz}{(a^2-b^2)} \; , \tag{18}$$

all the other stress components being zero.

The end of the bar $z = 0$ should ideally be traction-free, but the solution of equations (18) involves a non-zero value of σ_{rz} on this surface. However, the weak conditions are already satisfied by the solution, since an axisymmetric distribution of

σ_{rz} does not sum to a force or a moment. The axial force is identically zero because $\sigma_{zz} = 0$ on the end in the strong sense, as can be seen from equation (18).

Notice that the elementary theory correctly predicts the expression for σ_{zz} in this problem, since the axial force is $F = -2\pi a T z$ (compressive) and hence

$$\sigma_{zz} = \frac{F}{A} = -\frac{2\pi a S z}{\pi(a^2 - b^2)} = -\frac{2Saz}{(a^2 - b^2)} \ , \tag{19}$$

where $A = \pi(a^2 - b^2)$ is the cross-sectional area of the bar.

25.3. The solid cylinder $(0 \le r < a, -L < z < L)$ is subjected to the traction distribution

$$\begin{aligned} \sigma_{rr} &= -p_0(L^2 - z^2) \ ; \quad \sigma_{zr} = 0 \ ; \quad r = a \ ; \\ \sigma_{zr} &= \sigma_{zz} = 0 \ ; \quad z = \pm L \ . \end{aligned}$$

Find the complete stress field in the cylinder using strong boundary conditions on the curved boundary and weak conditions at the ends.

The loading in this case is axisymmetric and it does not produce any axial force. It should therefore be sufficient to include just those potentials that generate stress components σ_{rr} proportional to z^2 and below and that preserve symmetru about $z = 0$. Since Solution A involves two differentiations and Solution B one differentiation, this implies order 4 in ϕ and order 3 in ω — i.e.

$$\phi = A_1 R^4 P_4(z/R) + A_2 R^2 P_2(z/R) = A_1 \left(z^4 - 3z^2 r^2 + \frac{3}{8} r^4 \right) + A_2 \left(z^2 - \frac{1}{2} r^2 \right) \quad (1)$$

$$\omega = B_1 R^3 P_3(z/R) + B_2 R^1 P_1(z/R) = B_1 \left(z^3 - \frac{3}{2} z r^2 \right) + B_2 z \ , \quad (2)$$

from equations (24.27).

The stress components are now obtained as

$$\begin{aligned} \sigma_{rr} &= \frac{\partial^2 \phi}{\partial r^2} + z \frac{\partial^2 \omega}{\partial r^2} - 2\nu \frac{\partial \omega}{\partial z} \\ &= \frac{3A_1}{2}(3r^2 - 4z^2) - A_2 - 3B_1\{(1+2\nu)z^2 + \nu r^2\} - 2\nu B_2 \quad (3) \\ \sigma_{rz} &= \frac{\partial^2 \phi}{\partial r \partial z} + z \frac{\partial^2 \omega}{\partial r \partial z} - (1-2\nu) \frac{\partial \omega}{\partial r} \\ &= -6(2A_1 + B_1 \nu) z r \quad (4) \\ \sigma_{zz} &= \frac{\partial^2 \phi}{\partial z^2} + z \frac{\partial^2 \omega}{\partial z^2} - 2(1-\nu) \frac{\partial \omega}{\partial z} \\ &= 6A_1(2z^2 - r^2) + 2A_2 + 3B_1\{(1-\nu)r^2 + 2\nu z^2\} - 2(1-\nu)B_2 \quad (5) \\ \sigma_{\theta\theta} &= \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{z}{r}\frac{\partial \omega}{\partial r} - 2\nu \frac{\partial \omega}{\partial z} \\ &= \frac{3A_1}{2}(r^2 - 4z^2) - A_2 - 3B_1\{(1+2\nu)z^2 + \nu r^2\} - 2\nu B_2 \ . \quad (6) \end{aligned}$$

The strong boundary conditions are

$$\sigma_{rz} = \sigma_{rr} = 0 \ ; \quad r = a \quad (7)$$

and hence

$$\begin{aligned} \frac{3A_1}{2}(3a^2 - 4z^2) - A_2 - 3B_1\{(1+2\nu)z^2 + \nu a^2\} - 2\nu B_2 &= 0 \quad (8) \\ -6(2A_1 + B_1\nu) z a &= 0 \ . \quad (9) \end{aligned}$$

These equations must be satisfied for all z and hence

$$
\begin{aligned}
6A_1 - 3B_1(1+2\nu) &= p_0 && (10)\\
\frac{9A_1a^2}{2} - A_2 - 3B_1\nu a^2 - 2\nu B_2 &= -p_0L^2 && (11)\\
2A_1 + B_1\nu &= 0\ . && (12)
\end{aligned}
$$

The axial force on $z = L$ is obtained from (25.1) as

$$
F = 2\pi\int_0^a \sigma_{zz}(r,L)rdr = -3\pi A_1a^4 + \frac{3\pi(1-\nu)B_1a^4}{2} + 2\pi A_2a^2 - 2(1-\nu)\pi B_2a^2\ . \quad (13)
$$

and this must be zero providing a fourth equation for the constants A_1, A_2, B_1, B_2. Solving these equations, we obtain

$$
A_1 = \frac{\nu p_0}{6(1+\nu)}\ ;\quad A_2 = \frac{p_0(4(1-\nu)L^2+\nu a^2)}{4(1+\nu)}\ ;\quad B_1 = -\frac{p_0}{3(1+\nu)}\ ;\quad B_2 = \frac{p_0(4L^2-a^2)}{4(1+\nu)} \quad (14)
$$

and substituting into equations (3–6) we obtain the final stress components as

$$
\begin{aligned}
\sigma_{rr} &= p_0(z^2-L^2) + \frac{\nu p_0(a^2-r^2)}{4(1+\nu)} && (15)\\
\sigma_{rz} &= 0 && (16)\\
\sigma_{zz} &= \frac{p_0(a^2-2r^2)}{2(1+\nu)} && (17)\\
\sigma_{\theta\theta} &= p_0(z^2-L^2) + \frac{\nu p_0(a^2-3r^2)}{4(1+\nu)}\ . && (18)
\end{aligned}
$$

25.4. The hollow cylinder $b < r < a, 0 < z < L$ is subjected to the traction distribution

$$\begin{aligned}
\sigma_{rr} &= \sigma_{r\theta} = \sigma_{rz} = 0 \ ; \ r = a \ ; \\
\sigma_{rz} &= S \ ; \ \sigma_{rr} = \sigma_{r\theta} = 0 \ ; \ r = b \ ; \\
\sigma_{zr} &= \sigma_{zz} = 0 \ ; \ z = 0 \ .
\end{aligned}$$

Find the complete stress field in the cylinder using strong boundary conditions on the curved boundary and weak conditions at the ends.

In the absence of torsion, Solutions A,B provide a sufficiently general representation for the problem and since the loading is axisymmetric, we anticipate using Legendre polynomial solutions similar to those used in §25.1.1. However, these will need to be supplemented by solutions singular on the line $r = 0$ as in §25.1.2 to permit boundary conditions to be satisfied on the surface of the hole.

To determine the order of the polynomials required, we first consider the elementary Mechanics of Materials solution of the same problem. The only force resultant that can occur is an axial force, which the elementary theory predicts will lead to an axial stress σ_{zz} that is uniformly distributed over the cross-section. The axial force uniform traction $\sigma_{rz} = S$ will cause the axial force to increase linearly with z and hence the Mechanics of Materials solution of this problem therefore involves an axial stress σ_{zz} that increases linearly with z, but is uniform over the cross section.

To accommodate this in Solutions A,B, we need at least a 3rd order polynomial in ϕ and a 2nd order polynomial in ω — i.e.

$$\phi = AR^3 P_3(z/R) = A\left(z^3 - \frac{3}{2}zr^2\right) \ ; \tag{1}$$

$$\omega = BR^2 P_2(z/R) = \left(z^2 - \frac{1}{2}r^2\right) \ , \tag{2}$$

from equations (24.27).

The corresponding singular potentials will be of the form

$$\varphi_n(r, z) = R^n P_n(z/R)\ln(r) + S_n(r, z) \tag{3}$$

and are listed in equation (24.42)..

The dominant z-variation of the resulting stress field will generally be of order n in the singular potential, since there will be terms in which the differential operators have been applied exclusively to the $\ln(r)$ multiplier. By contrast, in the bounded polynomials, the resulting stress field will be polynomial of order $(n-2)$ (in ϕ) and hence the dominant $z-$varying term will be of order n. For this reason, it is sufficient to carry the singular potentials to two orders lower than the bounded polynomials.

In the present problem, this involves the introduction of the potential φ_1 of equation (24.42) in ϕ and φ_0 in ω, leading to the representation

$$\phi = A\left(z^3 - \frac{3}{2}zr^2\right) + Cz\ln(r) \; ; \tag{4}$$

$$\omega = \left(z^2 - \frac{1}{2}r^2\right) + D\ln(r) \; . \tag{5}$$

The stress components are now obtained as

$$\sigma_{rr} = \frac{\partial^2\phi}{\partial r^2} + z\frac{\partial^2\omega}{\partial r^2} - 2\nu\frac{\partial\omega}{\partial z} = -3Az - \frac{Cz}{r^2} - B(1+4\nu)z - \frac{Dz}{r^2} \; ; \tag{6}$$

$$\sigma_{rz} = \frac{\partial^2\phi}{\partial r\partial z} + z\frac{\partial^2\omega}{\partial r\partial z} - (1-2\nu)\frac{\partial\omega}{\partial r} = -3Ar + \frac{C}{r} + B(1-2\nu)r - \frac{(1-2\nu)D}{r} \; ; \tag{7}$$

$$\sigma_{zz} = \frac{\partial^2\phi}{\partial z^2} + z\frac{\partial^2\omega}{\partial z^2} - 2(1-\nu)\frac{\partial\omega}{\partial z} = 6Az - 2(1-2\nu)Bz \; ; \tag{8}$$

$$\sigma_{\theta\theta} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{z}{r}\frac{\partial\omega}{\partial r} - 2\nu\frac{\partial\omega}{\partial z} = -3Az + \frac{Cz}{r^2} - B(1+4\nu)z + \frac{Dz}{r^2} \; . \tag{9}$$

The strong boundary conditions are

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0 \quad ; \quad r = a, \text{ all } z \; ; \tag{10}$$

$$\sigma_{rr} = \sigma_{r\theta} = 0 \quad ; \quad r = b, \text{ all } z \; ; \tag{11}$$

$$\sigma_{rz} = S \quad ; \quad r = b, \text{ all } z \tag{12}$$

and with (6,7) they yield the equations

$$-3A - \frac{C}{a^2} - B(1+4\nu) - \frac{D}{a^2} = 0 \; ; \tag{13}$$

$$-3Aa + \frac{C}{a} + B(1-2\nu)a - \frac{(1-2\nu)D}{a} = 0 \; ; \tag{14}$$

$$-3A - \frac{C}{b^2} - B(1+4\nu) - \frac{D}{b^2} = 0 \; ; \tag{15}$$

$$-3Ab + \frac{C}{b} + B(1-2\nu)b - \frac{(1-2\nu)D}{b} = S \; . \tag{16}$$

The solution of these equations is

$$A = \frac{(1+4\nu)Sb}{6(a^2-b^2)(1+\nu)} \quad ; \quad B = -\frac{Sb}{2(a^2-b^2)(1+\nu)} \quad ;$$

$$C = \frac{Sa^2b}{2(a^2-b^2)(1-\nu)} \quad ; \quad D = -\frac{Sa^2b}{2(a^2-b^2)(1-\nu)} \tag{17}$$

and substituting back into the stress equations (6–9), we find

$$\sigma_{rz} = \frac{Sb(a^2-r^2)}{(a^2-b^2)r} \quad ; \quad \sigma_{zz} = \frac{2Sbz}{(a^2-b^2)} \; , \tag{18}$$

all the other stress components being zero.

The end of the bar $z = 0$ should ideally be traction-free, but the solution of equations (18) involves a non-zero value of σ_{rz} on this surface. However, the weak conditions are already satisfied by the solution, since an axisymmetric distribution of σ_{rz} does not sum to a force or a moment. The axial force is identically zero because $\sigma_{zz} = 0$ on the end in the strong sense, as can be seen from equation (18).

Notice that the elementary theory correctly predicts the expression for σ_{zz} in this problem, since the axial force is $F = 2\pi a S z$ (compressive) and hence

$$\sigma_{zz} = \frac{F}{A} = \frac{2\pi b S z}{\pi(a^2 - b^2)} = \frac{2Sbz}{(a^2 - b^2)} \,, \tag{19}$$

where $A = \pi(a^2 - b^2)$ is the cross-sectional area of the bar.

25.5. A heat exchanger tube $b < r < a, -L < z < L$ is subjected to the steady-state temperature distribution

$$\begin{aligned} T &= T_0\left(1+\frac{z}{10a}\right) \quad ; \quad r=b \\ &= -\frac{T_0 z}{10a} \quad ; \quad r=a\,. \end{aligned}$$

Find the complete stress field in the tube if all the surfaces are traction-free. Find the location and a general expression for the magnitude of (i) the maximum tensile stress and (ii) the maximum Von Mises equivalent stress for the case where $b=0.8a$ and $L=20a$. Find the magnitude of these quantities when $T_0 = 50^o$C and the material is copper for which $E = 121$ GPa, $\nu=0.33$ and $\alpha=17\times10^{-6}$ per ^{o}C.

This problem is axisymmetric, so we need to use Solutions A, B and T only. However, there is no symmetry in z.

We first solve the thermal problem using only Solution T. The temperature varies linearly with z and temperature is a first derivative in Solution T, so we start with second order polynomials and below in χ from equations (24.27) and the corresponding singular potentials from equations (24.42) — i.e.

$$\chi = C_1 z + \frac{C_2(2z^2-r^2)}{2} + C_3 z\ln(r) + \frac{C_4\{(2z^2-r^2)\ln(r)+r^2}{2}\,, \tag{1}$$

where we have excluded the function $\ln(r)$, since this gives a null temperature field. The corresponding temperature field is then

$$\begin{aligned} T &= \frac{(1-\nu)}{\mu\alpha(1+\nu)}\frac{\partial\chi}{\partial z} \\ &= \frac{(1-\nu)}{\mu\alpha(1+\nu)}\{C_1 + C_2 z + C_3\ln(r) + C_4 z\ln(r)\}\,. \end{aligned} \tag{2}$$

The thermal boundary conditions therefore require that

$$\frac{(1-\nu)}{\mu\alpha(1+\nu)}\{C_1 + C_2 z + C_3\ln(b) + C_4 z\ln(b)\} = T_0\left(1+\frac{z}{10a}\right) \tag{3}$$

$$\frac{(1-\nu)}{\mu\alpha(1+\nu)}\{C_1 + C_2 z + C_3\ln(a) + C_4 z\ln(a)\} = -\frac{T_0 z}{10a} \tag{4}$$

for all z, and hence

$$\frac{(1-\nu)}{\mu\alpha(1+\nu)}\{C_1 + C_3\ln(b)\} = T_0 \tag{5}$$

$$\frac{(1-\nu)}{\mu\alpha(1+\nu)}\{C_2 + C_4\ln(b)\} = \frac{T_0 z}{10a} \tag{6}$$

$$\frac{(1-\nu)}{\mu\alpha(1+\nu)}\{C_1 + C_3\ln(a)\} = 0 \tag{7}$$

$$\frac{(1-\nu)}{\mu\alpha(1+\nu)}\{C_2 + C_4\ln(a)\} = -\frac{T_0}{10a}\ . \tag{8}$$

Solving these equations for the constants C_1, C_2, C_3, C_4, we obtain

$$C_1 = \frac{\mu\alpha(1+\nu)T_0\ln(a)}{(1-\nu)\ln(a/b)} \tag{9}$$

$$C_2 = \frac{\mu\alpha(1+\nu)T_0\ln(ab)}{20a(1-\nu)\ln(a/b)} \tag{10}$$

$$C_3 = -\frac{\mu\alpha(1+\nu)T_0}{(1-\nu)\ln(a/b)} \tag{11}$$

$$C_4 = -\frac{\mu\alpha(1+\nu)T_0}{10a(1-\nu)\ln(a/b)}\ . \tag{12}$$

This partial solution will imply the existence of tractions on the curved surfaces $r = a, b$ that must be nullified by the superposition of appropriate potentials in Solutions A,B. To determine the required order of these potentials, we first calculate the stress components σ_{rr}, σ_{rz} associated with χ alone. We obtain

$$\begin{aligned}\sigma_{rr} &= z\frac{\partial^2\chi}{\partial r^2} - 2\frac{\partial\chi}{\partial z} \\ &= -\frac{\mu\alpha(1+\nu)T_0}{20a(1-\nu)\ln(a/b)}\left(5z\ln\left(\frac{ab}{r^2}\right) + 40a\ln\left(\frac{a}{r}\right) - \frac{20a^2z}{r^2} - \frac{2z^3}{r^2} - z\right)\end{aligned} \tag{13}$$

$$\begin{aligned}\sigma_{rz} &= \frac{\partial\chi}{\partial r} + z\frac{\partial^2\chi}{\partial r\partial z} \\ &= -\frac{\mu\alpha(1+\nu)T_0}{20a(1-\nu)\ln(a/b)}\left(r\ln\left(\frac{ab}{r^2}\right) + \frac{40az}{r} + \frac{6z^2}{r} + r\right)\ .\end{aligned} \tag{14}$$

The z^3 terms in σ_{rr} show that we need to use polynomials of degree 5 and below in ϕ and degree 4 and below in ω. The singular terms can be terminated at 2 degrees lower than this, leaving the functions

$$\begin{aligned}\phi &= \frac{C_5}{2}(2z^2 - r^2) + \frac{C_6}{2}(2z^3 - 3zr^2) + \frac{C_7}{8}(8z^4 - 24z^2r^2 + 3r^4) \\ &\quad + \frac{C_8}{8}(8z^5 - 40z^3r^2 + 15zr^4) + C_9\ln(r) + C_{10}z\ln(r) \\ &\quad + \frac{C_{11}}{2}\{(2z^2 - r^2)\ln(r) + r^2\} + \frac{C_{12}}{2}\{(2z^3 - 3zr^2)\ln(r) + 3zr^2\}\end{aligned} \tag{15}$$

$$\begin{aligned}\omega &= C_{13}z + \frac{C_{14}}{2}(2z^2 - r^2) + \frac{C_{15}}{2}(2z^3 - 3zr^2) + \frac{C_{16}}{8}(8z^4 - 24z^2r^2 + 3r^4) \\ &\quad + C_{17}\ln(r) + C_{18}z\ln(r) + \frac{C_{19}}{2}\{(2z^2 - r^2)\ln(r) + r^2\}\ .\end{aligned} \tag{16}$$

The number of arbitrary constants in these expressions makes a Maple or Mathematica solution the only viable option. The procedure is to substitute the given expressions into Solutions A, B and T to find the corresponding stress components. The strong boundary conditions are

$$\sigma_{rr} = \sigma_{rz} = 0 \; ; \quad r = a, b \tag{17}$$

and these conditions must be satisfied for all z, leading to separate conditions for each power of z in the resulting tractions. One additional condition is obtained from the weak condition that the force F on either end given by equation (25.1) be zero.

After solving for the unknown constants, the maximum tensile stress and the Von Mises equivalent stress can be found as functions of r, z, using equations (1.12–1.19) or the Maple and Mathematica files 'principalstresses'. Contour plots of these quantities for the given case are shown in Figures 1,2.

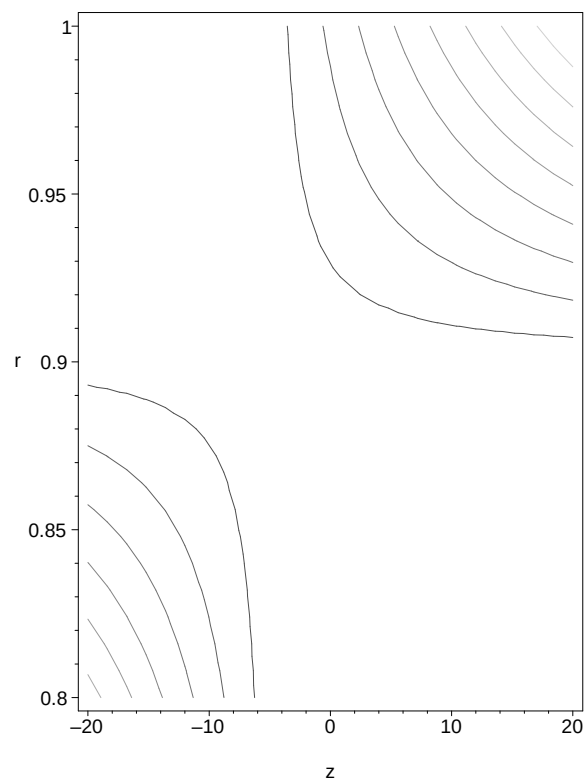


Figure 1: Contour plot of maximum tensile stress.

The maximum tensile stress occurs at $r = a, z = L$ and is

$$\sigma_{\max} = 355 \text{ MPa} \; . \tag{18}$$

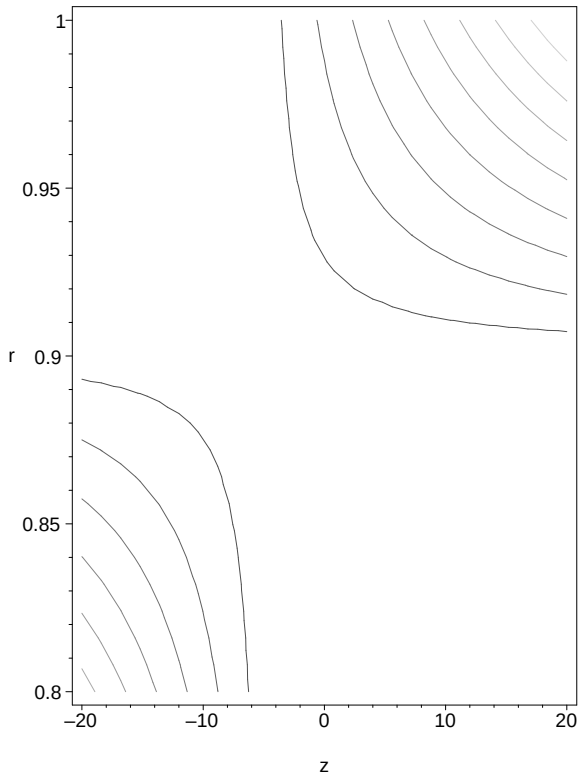


Figure 2: Contour plot of Von Mises equivalent stress.

The maximum Von Mises stress occurs at $r = b, z = L$ and is

$$\sigma_{\max} = 412 \text{ MPa} . \tag{19}$$

25.6. The circular plate $0 \le r < a, -h/2 < z < h/2$ is simply supported at $r = a$ and loaded by a shear traction $\sigma_{zr} = Sr$ on the top surface $z = h/2$. All the other tractions on the plane surfaces $z = \pm h/2$ are zero. Find the stresses in the plate, using strong boundary conditions on $z = \pm h/2$ and weak boundary conditions on $r = a$.

Following the rule in §5.2.2, we anticipate requiring a polynomial of order $(m+4) = 5$ (for shear loading) in Solution A and one order lower in Solution B. We therefore start with the trial function

$$\begin{aligned}
\phi &= \frac{A_5}{8}(8z^5 - 40z^3r^2 + 15zr^4) + \frac{A_4}{8}(8z^4 - 24z^2r^2 + 3r^4) + \frac{A_3}{2}(2z^3 - 3zr^2) \\
&\quad + \frac{A_2}{2}(2z^2 - r^2) \qquad (1) \\
\omega &= \frac{B_4}{8}(8z^4 - 24z^2r^2 + 3r^4) + \frac{B_3}{2}(2z^3 - 3zr^2) + \frac{B_2}{2}(2z^2 - r^2) + B_1 z \ , \qquad (2)
\end{aligned}$$

where we have omitted those terms leading only to rigid-body displacements. Substituting into Table 19.1, the stresses are obtained as

$$\begin{aligned}
\sigma_{rr} &= \frac{5A_5}{2}(9r^2z - 4z^3) + \frac{3A_4}{2}(3r^2 - 4z^2) - 3A_3z - A_2 - 2B_4(3+4\nu)z^3 \\
&\quad + \frac{3B_4}{2}(3+8\nu)r^2z - 3B_3(1+2\nu)z^2 + 3\nu B_3 r^2 - B_2(1+4\nu)z - 2B_1\nu \\
\sigma_{\theta\theta} &= \frac{5A_5}{2}(3r^2z - 4z^3) + \frac{3A_4}{2}(r^2 - 4z^2) - 3A_3z - A_2 - 2B_4(3+4\nu)z^3 \\
&\quad + \frac{3B_4}{2}(1+8\nu)r^2z - 3B_3(1+2\nu)z^2 + 3\nu B_3 r^2 - B_2(1-4\nu)z - 2B_1\nu \\
\sigma_{rz} &= \frac{15A_5}{2}(r^3 - 4rz^2) - 12A_4rz - 3A_3r - 6B_4(1+2\nu)z^2r - \frac{3B_4}{2}(1-2\nu)r^3 \\
&\quad - 6B_3\nu zr + B_2(1-2\nu)r \qquad (3) \\
\sigma_{zz} &= 10A_5(2z^3 - 3r^2z) + 6A_4(2z^2 - r^2) + 6A_3z + 2A_2 + 4B_4(1+2\nu)z^3 \\
&\quad + 6B_4(1-2\nu)r^2z + 6B_3\nu z^2 + 3B_3(1-\nu)r^2 - 2B_2(1-2\nu)z - 2B_1(1-\nu) \ .
\end{aligned}$$

The strong boundary conditions

$$\begin{aligned}
\sigma_{rz} &= Sr \ ; \quad z = \frac{h}{2} \qquad (4) \\
&= 0 \ ; \quad z = -\frac{h}{2} \qquad (5) \\
\sigma_{zz} &= 0 \ ; \quad z = \pm\frac{h}{2} \qquad (6)
\end{aligned}$$

then require that

$$\frac{15A_5}{2}(r^3 - rh^2) - 6A_4rh - 3A_3r - \frac{3B_4(1+2\nu)h^2r}{2} - \frac{3B_4}{2}(1-2\nu)r^3$$

$$
\begin{aligned}
-3B_3\nu hr + B_2(1-2\nu)r &= Sr \\
\frac{15A_5}{2}(r^3 - rh^2) + 6A_4 rh - 3A_3 r - \frac{3B_4(1+2\nu)h^2 r}{2} - \frac{3B_4}{2}(1-2\nu)r^3 & \\
+3B_3\nu hr + B_2(1-2\nu)r &= 0 \\
\frac{5A_5}{2}(h^3 - 6r^2 h) + 3A_4(h^2 - 2r^2) + 3A_3 h + 2A_2 + \frac{3B_4}{2}(1+2\nu)h^3 & \\
+3B_4(1-2\nu)r^2 h + \frac{3B_3}{2}\nu h^2 + 3B_3(1-\nu)r^2 - B_2(1-2\nu)h - 2B_1(1-\nu) &= 0 \\
-\frac{5A_5}{2}(h^3 - 6r^2 h) + 3A_4(h^2 - 2r^2) - 3A_3 h + 2A_2 - \frac{3B_4}{2}(1+2\nu)h^3 & \\
-3B_4(1-2\nu)r^2 h + \frac{3B_3}{2}\nu h^2 + 3B_3(1-\nu)r^2 + B_2(1-2\nu)h - 2B_1(1-\nu) &= 0
\end{aligned}
$$

Separating powers of r in these equations yields

$$
\begin{aligned}
\frac{15A_5}{2} - \frac{3B_4}{2}(1-2\nu) &= 0 && (7) \\
-\frac{15A_5 h^2}{2} - 6A_4 h - 3A_3 - \frac{3B_4(1+2\nu)h^2}{2} - 3B_3\nu h + B_2(1-2\nu) &= S && (8) \\
-\frac{15A_5 h^2}{2} + 6A_4 h - 3A_3 - \frac{3B_4(1+2\nu)h^2}{2} + 3B_3\nu h + B_2(1-2\nu) &= 0 && (9) \\
-15A_5 h - 6A_4 + 3B_4(1-2\nu)h + 3B_3(1-\nu) &= 0 && (10) \\
\frac{5A_5 h^3}{2} + 3A_4 h^2 + 3A_3 h + 2A_2 + \frac{3B_4}{2}(1+2\nu)h^3 + \frac{3B_3}{2}\nu h^2 & \\
-B_2(1-2\nu)h - 2B_1(1-\nu) &= 0 && (11) \\
+15A_5 h - 6A_4 - 3B_4(1-2\nu)h + 3B_3(1-\nu) &= 0 && (12) \\
-\frac{5A_5 h^3}{2} + 3A_4 h^2 - 3A_3 h + 2A_2 - \frac{3B_4}{2}(1+2\nu)h^3 + \frac{3B_3}{2}\nu h^2 & \\
+B_2(1-2\nu)h - 2B_1(1-\nu) &= 0 && (13)
\end{aligned}
$$

Two additional equations are obtained by substituting (3) into the weak conditions

$$
\int_{-h/2}^{h/2} \sigma_{rr} dz = 0 \;\; ; \;\; \int_{-h/2}^{h/2} z\sigma_{rr} dz = 0 \;\; ; \;\; r = a \qquad (14)
$$

and evaluating the integrals, with the result

$$
\begin{aligned}
\frac{A_4(9a^2 h - h^3)}{2} - A_2 h - \frac{B_3(1+2\nu)h^3}{4} + 3B_3\nu a^2 h - 2B_1\nu h &= 0 \\
\frac{A_5(15a^2 h^3 - h^5)}{8} - \frac{A_3 h^3}{4} - \frac{B_4(3+4\nu)h^5}{40} + \frac{B_4(3+8\nu)a^2 h^3}{8} - \frac{B_2(1+4\nu)h^3}{12} &= 0\,.
\end{aligned}
$$

Solving these equations for the unknown constants, we obtain

$$
A_2 = -\frac{S\{3(3+\nu)(1-\nu)a^2 - (2+5\nu-\nu^2)h^2\}}{24(1+\nu)h}
$$

$$
\begin{aligned}
A_3 &= -\frac{S\{15(3+\nu)(1-2\nu)a^2-(11+11\nu-6\nu^2)h^2\}}{120(1+\nu)h^2} \\
A_4 &= -\frac{S(1-\nu)}{12h}\ ; \quad A_5 = -\frac{S(1-2\nu)}{20h^2} \\
B_1 &= -\frac{S\{3(3+\nu)a^2+(1-\nu)h^2\}}{24(1+\nu)h} \\
B_2 &= -\frac{S\{15(3+\nu)a^2-(1+3\nu)h^2\}}{40(1+\nu)h^2} \\
B_3 &= -\frac{S}{6h}\ ; \quad B_4 = -\frac{S}{4h^2}\ .
\end{aligned}
$$

The stresses are then obtained by substituting into (3) as

$$
\begin{aligned}
\sigma_{rr} &= \frac{S\{(2+\nu)(120z^3+60z^2h-18zh^2-5h^3)+15(3+\nu)(6z+h)(a^2-r^2)\}}{120h^2} \\
\sigma_{\theta\theta} &= \frac{S\{(2+\nu)(120z^3+60z^2h-18zh^2-5h^3)+15(6z+h)[(3+\nu)a^2-(1+3\nu)r^2)]\}}{120h^2} \\
\sigma_{rz} &= \frac{Sr(6z-h)(2z+h)}{4h^2} \\
\sigma_{zz} &= \frac{S(2z+h)(h^2-4z^2)}{4h^2}\ .
\end{aligned}
$$

25.7. The circular plate $0 \le r < a, -h/2 < z < h/2$ rotates at constant speed Ω about the z-axis, all the surfaces being traction-free. Find the stresses in the plate, using strong boundary conditions on $z = \pm h/2$ and weak boundary conditions on $r = a$.

For rotation at constant speed about the z-axis, the accelerations are

$$a_r = -\Omega^2 r \; ; \quad a_\theta = 0 \; ; \quad a_z = 0 \tag{1}$$

corrsponding to inertial body forces

$$p_r = \rho\Omega^2 r \; ; \quad p_\theta = 0 \; ; \quad p_z = 0 \; . \tag{2}$$

These forces can be described in terms of an axisymmetric body-force potential

$$V = -\frac{\rho\Omega^2 r^2}{2} \tag{3}$$

(see equation (20.38)). Equation (20.40) then takes the form

$$\nabla^2 \phi = -\frac{(1-2\nu)\rho\Omega^2 r^2}{2(1-\nu)} \tag{4}$$

and a particular solution is

$$\phi_P = -\frac{(1-2\nu)\rho\Omega^2 r^4}{32(1-\nu)} \; . \tag{5}$$

Substituting into the polar coordinate versions[1] of equations (20.41, 20.42), we obtain the particular solution as

$$\sigma_{rr} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2 \phi_P}{\partial r^2} = -\frac{(3-2\nu)\rho\Omega^2 r^2}{8(1-\nu)} \tag{6}$$

$$\sigma_{\theta\theta} = \frac{\nu V}{(1-\nu)} + \frac{1}{r}\frac{\partial \phi_P}{\partial r} = -\frac{(1+2\nu)\rho\Omega^2 r^2}{8(1-\nu)} \tag{7}$$

$$\sigma_{zz} = \frac{\nu V}{(1-\nu)} = -\frac{\nu\rho\Omega^2 r^2}{2(1-\nu)} \; , \tag{8}$$

where we have omitted derivatives with respect to θ and z, since ϕ_P is axisymmetric. The three shear stresses $\sigma_{r\theta} = \sigma_{\theta z} = \sigma_{zr} = 0$ by symmetry.

To complete the solution, we superpose appropriate axisymmetric polynomials in Solutions A and B. These additional terms must remove the unwanted tractions on the surfaces $z = \pm h/2$ in the strong sense. These tractions vary with r^2, suggesting

[1] These are identical to the expressions for Solution A of Table 21.1, except for an additional term $\nu V/(1-\nu)$ in the normal stresses.

4th order polynomials in ϕ and 3rd order in ω. The problem is also symmetrical about the plane $z = 0$, so we select the polynomials

$$\begin{aligned}
\phi &= A_4 R^4 P_4(z/R) + A_2 R^2 P_2(z/R) = \frac{A_4}{8}(35z^4 - 30z^2R^2 + 3R^4) + \frac{A_2}{2}(2z^2 - r^2) \\
\omega &= B_3 R^3 P_3(z/R) + B_1 R P_1(z/R) = \frac{B_3}{2}(2z^3 - 3zr^2) + B_1 z \ ,
\end{aligned}$$

from (24.27).

The non-zero stress components are then obtained from Table 9.1 and (6–8) as

$$\begin{aligned}
\sigma_{rr} &= \frac{3A_4}{2}(3r^2 - 4z^2) - A_2 - 3B_3(1+2\nu)z^2 + 3\nu B_3 r^2 - 2B_1\nu - \frac{(3-2\nu)\rho\Omega^2 r^2}{8(1-\nu)} \\
\sigma_{\theta\theta} &= +\frac{3A_4}{2}(r^2 - 4z^2) - A_2 - 3B_3(1+2\nu)z^2 + 3\nu B_3 r^2 - 2B_1\nu - \frac{(1+2\nu)\rho\Omega^2 r^2}{8(1-\nu)} \\
\sigma_{rz} &= -12A_4 rz - 6B_3\nu zr \\
\sigma_{zz} &= 6A_4(2z^2 - r^2) + 2A_2 + 6B_3\nu z^2 + 3B_3(1-\nu)r^2 - 2B_1(1-\nu) - \frac{\nu\rho\Omega^2 r^2}{2(1-\nu)} \ .
\end{aligned}$$

The strong boundary conditions

$$\sigma_{zr} = \sigma_{zz} = 0 \ ; \quad z = \pm\frac{h}{2} \tag{9}$$

then require that

$$-12A_4 - 6B_3\nu = 0 \tag{10}$$

$$-6A_4 + 3B_3(1-\nu) - \frac{\nu\rho\Omega^2}{2(1-\nu)} = 0 \tag{11}$$

$$3A_4h^2 + 2A_2 + \frac{3B_3\nu h^2}{2} - 2B_1(1-\nu) = 0 \tag{12}$$

One additional equation is obtained by substituting for σ_{rr} into the weak condition

$$\int_{-h/2}^{h/2} \sigma_{rr} dz = 0 \ ; \quad r = a \tag{13}$$

and evaluating the integral, with the result

$$\frac{A_4(9a^2h - h^3)}{2} - A_2 h - \frac{B_3(1+2\nu)h^3}{4} + 3B_3\nu a^2 h - 2B_1\nu h - \frac{(3-2\nu)\rho\Omega^2 h^3}{96(1-\nu)} = 0 \ . \tag{14}$$

Solving (10–13) for the unknown constants, we obtain

$$A_2 = -\frac{\rho\Omega^2\{3(3+\nu)(1-\nu)a^2 - \nu(1+\nu)h^2\}}{24(1+\nu)} \ ; \quad A_4 = -\frac{\nu^2\rho\Omega^2}{12(1-\nu)}$$

$$B_1 = -\frac{\rho\Omega^2\{3(3+\nu)(1-\nu)a^2 - \nu(1+\nu)h^2\}}{24(1-\nu^2)} \; ; \quad B_3 = \frac{\nu\rho\Omega^2}{6(1-\nu)} \; .$$

Substituting these results into the stress components, we then obtain

$$\begin{aligned}
\sigma_{rr} &= \frac{\rho\Omega^2\{3(3+\nu)(1-\nu)(a^2-r^2) + \nu(1+\nu)(h^2-12z^2)\}}{24(1-\nu)} \\
\sigma_{\theta\theta} &= \frac{\rho\Omega^2\{3(1-\nu)[(3+\nu)a^2 - (1+3\nu)r^2] + \nu(1+\nu)(h^2-12z^2)\}}{24(1-\nu)} \; .
\end{aligned}$$

and

$$\sigma_{zz} = \sigma_{r\theta} = \sigma_{\theta z} = \sigma_{zr} = 0 \; . \tag{15}$$

25.8. The solid cylinder $0 \le r < a, 0 < z < L$ with its axis horizontal is built in at $z = L$ and loaded only by its own weight (density ρ). Find the complete stress field in the cylinder.

We suppose that gravity acts in the negative y-direction, so that

$$p_x = 0 \; ; \quad p_y = -\rho g \; ; \quad p_z = 0 \; . \tag{1}$$

These forces can be described in terms of the body-force potential

$$V = \rho g y = \rho g r \sin\theta \tag{2}$$

(see equation (20.38)). Equation (20.40) then takes the form

$$\nabla^2 \phi = \frac{(1-2\nu)\rho g r \sin\theta}{(1-\nu)} \tag{3}$$

and a particular solution is

$$\phi_P = \frac{(1-2\nu)\rho g r^3 \sin\theta}{8(1-\nu)} \; . \tag{4}$$

Substituting into the polar coordinate versions[2] of equations (20.41, 20.42), we obtain the particular solution as

$$\sigma_{rr} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2 \phi_P}{\partial r^2} = \frac{(3-2\nu)\rho g r \sin\theta}{4(1-\nu)} \tag{5}$$

$$\sigma_{\theta\theta} = \frac{\nu V}{(1-\nu)} + \frac{1}{r}\frac{\partial \phi_P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_P}{\partial \theta^2} = \frac{(1+2\nu)\rho g r \sin\theta}{4(1-\nu)} \tag{6}$$

$$\sigma_{zz} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2 \phi_P}{\partial z^2} = \frac{\nu \rho g r \sin\theta}{(1-\nu)} \tag{7}$$

$$\sigma_{r\theta} = \frac{1}{r}\frac{\partial^2 \phi_P}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \phi_P}{\partial \theta} = \frac{(1-2\nu)\rho g r \cos\theta}{4(1-\nu)} \tag{8}$$

$$\sigma_{\theta z} = \frac{1}{r}\frac{\partial^2 \phi_P}{\partial \theta \partial z} = 0 \tag{9}$$

$$\sigma_{zr} = \frac{\partial^2 \phi_P}{\partial z \partial r} = 0 \; . \tag{10}$$

For the homogeneous solution, we shall require terms in Solutions A,B that vary with $\sin\theta$ and terms in Solution E that vary with $\cos\theta$. The stresses in the particular solution vary with z^0, but the gravitational load causes bending moments which

[2] These are identical to the expressions for Solution A of Table 21.1, except for an additional term $\nu V/(1-\nu)$ in the normal stresses.

increases the order of the required polynomial by 3. We therefore use 5th order and below in Solutions A,B and 4th order and below in Solution B. The problem is also even in z, so we define

$$\begin{aligned}
\phi &= (A_3F_{13} + A_5F_{15})\sin\theta = \left(\frac{3A_3}{2}(4z^2r - r^3) + \frac{15A_5}{8}(8z^4r - 12z^2r^3 + r^5)\right)\sin\theta \\
\omega &= (B_2F_{12} + B_4F_{14})\sin\theta = \left(3B_2zr + \frac{5B_4}{2}(4z^3r - 3zr^3)\right)\sin\theta \qquad (11) \\
\Psi &= (E_3F_{13} + E_5F_{15})\cos\theta = \left(\frac{3E_3}{2}(4z^2r - r^3) + \frac{15E_5}{8}(8z^4r - 12z^2r^3 + r^5)\right)\cos\theta\ ,
\end{aligned}$$

where the functions F_{1i} are given in equation (24.55) and we have omitted terms that generate only rigid-body displacements. The stress components from Table 21.1, including the body force terms, are then

$$\begin{aligned}
\sigma_{rr} &= -\frac{3}{2}(6A_3 + 5A_5(18z^2 - 5r^2) + 4\nu B_2 + 10B_4((3+4\nu)z^2 - \nu r^2) \\
&\quad -4E_3 + 10E_5(r^2 - 6z^2))r\sin\theta + \frac{(3-2\nu)\rho g r\sin\theta}{4(1-\nu)} \qquad (12) \\
\sigma_{\theta\theta} &= -\frac{3}{2}(2A_3 + 5A_5(6z^2 - r^2) + 4\nu B_2 + 10B_4((1+4\nu)z^2 - \nu r^2) \\
&\quad +4E_3 - 10E_5(r^2 - 6z^2))r\sin\theta + \frac{(1+2\nu)\rho g r\sin\theta}{4(1-\nu)} \qquad (13) \\
\sigma_{zz} &= 3(4A_3 + 15A_5(4z^2 - r^2) - 2B_2 \\
&\quad +5B_4(4\nu z^2 + (1-\nu)r^2))r\sin\theta + \frac{\nu\rho g r\sin\theta}{(1-\nu)} \qquad (14) \\
\sigma_{r\theta} &= -\frac{3}{2}(2A_3 + 5A_5(6z^2 - r^2) + 10B_4z^2 - 4E_3 \\
&\quad +20E_5(r^2 - 3z^2))r\cos\theta + \frac{(1-2\nu)\rho g r\cos\theta}{4(1-\nu)} \qquad (15) \\
\sigma_{\theta z} &= (12A_3 + 15A_5(4z^2 - 3r^2) + 6\nu B_2 + 5B_4(4(1+\nu)z^2 - 3\nu r^2) \\
&\quad -12E_3 + 15E_5(9r^2 - 4z^2))z\cos\theta \qquad (16) \\
\sigma_{zr} &= (12A_3 + 15A_5(4z^2 - 9r^2) + 6\nu B_2 + 5B_4(4(1+\nu)z^2 - 9\nu r^2) \\
&\quad -12E_3 + 15E_5(3r^2 - 4z^2))z\sin\theta\ . \qquad (17)
\end{aligned}$$

The tractions $\sigma_{rr}, \sigma_{r\theta}, \sigma_{rz}$ must be zero on $r = a$, giving

$$\begin{aligned}
-\frac{3}{2}(6A_3 + 5A_5(18z^2 - 5a^2) + 4\nu B_2 + 10B_4((3+4\nu)z^2 - \nu a^2) - 4E_3 \\
+10E_5(a^2 - 6z^2))a + \frac{(3-2\nu)\rho g a}{4(1-\nu)} &= 0 \quad (18) \\
-\frac{3}{2}(2A_3 + 5A_5(6z^2 - a^2) + 10B_4z^2
\end{aligned}$$

$$
\begin{aligned}
-4E_3 + 20E_5(a^2 - 3z^2))a + \frac{(1-2\nu)\rho g a}{4(1-\nu)} &= 0 \quad (19)\\
12A_3 + 15A_5(4z^2 - 9a^2) + 6\nu B_2 + 5B_4(4(1+\nu)z^2 - 9\nu a^2) & \\
-12E_3 + 15E_5(3a^2 - 4z^2) &= 0 \quad (20)
\end{aligned}
$$

and these equations must be satisfied for all z. Equating coefficients of powers of z, we obtain the six equations

$$
\begin{aligned}
90A_5 + 10B_4(3+4\nu) - 60E_5 &= 0 \quad (21)\\
-\frac{3}{2}(6A_3 - 25A_5a^2 + 4\nu B_2 - 10B_4\nu a^2 - 4E_3 + 10E_5a^2)a + \frac{(3-2\nu)\rho g a}{4(1-\nu)} &= 0 \quad (22)\\
30A_5 + 10B_4 - 60E_5 &= 0 \quad (23)\\
-\frac{3}{2}(2A_3 - 5A_5 - a^2 - 4E_3 + 20E_5a^2)a + \frac{(1-2\nu)\rho g a}{4(1-\nu)} &= 0 \quad (24)\\
60A_5 + 20B_4(1+\nu) + 15E_5 &= 0 \quad (25)\\
12A_3 - 135A_5 9a^2 + 6\nu B_2 - 45B_4\nu a^2 - 12E_3 + 45E_5a^2 &= 0 \quad (26)
\end{aligned}
$$

On the free end $z = 0$, the tractions are

$$
\begin{aligned}
\sigma_{zr} &= 0 \quad (27)\\
\sigma_{z\theta} &= 0 \quad (28)\\
\sigma_{zz} &= 3(4A_3 - 15A_4r^2 - 2B_2 + 5B_4(1-\nu)r^2)r\sin\theta + \frac{\nu\rho g r\sin\theta}{(1-\nu)} \quad (29)\\
& \quad (30)
\end{aligned}
$$

The only non-trivial weak condition to satisfy is the moment condition

$$
M = \int_0^a \int_0^{2\pi} \sigma_{zz} r^2 \sin\theta d\theta dr = 0 \ , \quad (31)
$$

from which we obtain

$$
\frac{5\pi(B_4(1-\nu) - 3A_5)a^6}{2} + \frac{3\pi(2A_3 - B_2(1-\nu))a^4}{2} + \frac{\pi\nu\rho g a^4}{4(1-\nu)} = 0 \ . \quad (32)
$$

Solving equations (21–26, 32) for the constants $A_3, A_5, B_2, B_4, E_3, E_5$, we obtain

$$
A_3 = \frac{\rho g(5 + 2\nu - 13\nu^2 - 6\nu^3)}{36(1-\nu)(1+\nu)^2} \ ; \quad A_5 = \frac{\rho g(1+2\nu)}{90a^2(1+\nu)} \ ; \quad B_2 = \frac{\rho g(1+2\nu)(3+\nu)}{18(1-\nu)(1+\nu)^2}
$$

$$
B_4 = -\frac{\rho g}{30a^2(1+\nu)} \ ; \quad E_3 = \frac{\rho g(1+5\nu+2\nu^2)}{72(1+\nu)^2} \ ; \quad E_5 = \frac{\nu\rho g}{90a^2(1+\nu)} \ . \quad (33)
$$

Substituting these values into (12–17) yields the final expressions for the stress components, which are

$$\sigma_{rr} = -\frac{\rho g(5+2\nu)(a^2-r^2)r\sin\theta}{12(1+\nu)a^2} \tag{34}$$

$$\sigma_{\theta\theta} = \frac{\rho g\{(1-2\nu)r^2-3(1+2\nu)a^2\}r\sin\theta}{12(1+\nu)a^2} \tag{35}$$

$$\sigma_{zz} = \frac{\rho g\{12(1+\nu)z^2+(2+\nu)(2a^2-3r^2)\}r\sin\theta}{6(1+\nu)a^2} \tag{36}$$

$$\sigma_{r\theta} = -\frac{\rho g(1-2\nu)(a^2-r^2)r\cos\theta}{12(1+\nu)a^2} \tag{37}$$

$$\sigma_{\theta z} = \frac{\rho g\{(3+2\nu)a^2-(1-2\nu)r^2\}z\cos\theta}{2(1+\nu)a^2} \tag{38}$$

$$\sigma_{zr} = \frac{\rho g(3+2\nu)(a^2-r^2)z\sin\theta}{2(1+\nu)a^2}\ . \tag{39}$$

25.9. The hollow cylinder $b < r < a, 0 < z < L$ with its axis horizontal is built in at $z = L$ and loaded only by its own weight (density ρ). Find the complete stress field in the cylinder.

We suppose that gravity acts in the negative y-direction, so that

$$p_x = 0 \; ; \quad p_y = -\rho g \; ; \quad p_z = 0 \; . \tag{1}$$

These forces can be described in terms of the body-force potential

$$V = \rho g y = \rho g r \sin\theta \tag{2}$$

(see equation (20.38)). Equation (20.40) then takes the form

$$\nabla^2 \phi = \frac{(1-2\nu)\rho g r \sin\theta}{(1-\nu)} \tag{3}$$

and a particular solution is

$$\phi_P = \frac{(1-2\nu)\rho g r^3 \sin\theta}{8(1-\nu)} \; . \tag{4}$$

Substituting into the polar coordinate versions[3] of equations (20.41, 20.42), we obtain the particular solution as

$$\sigma_{rr} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2 \phi_P}{\partial r^2} = \frac{(3-2\nu)\rho g r \sin\theta}{4(1-\nu)} \tag{5}$$

$$\sigma_{\theta\theta} = \frac{\nu V}{(1-\nu)} + \frac{1}{r}\frac{\partial \phi_P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_P}{\partial \theta^2} = \frac{(1+2\nu)\rho g r \sin\theta}{4(1-\nu)} \tag{6}$$

$$\sigma_{zz} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2 \phi_P}{\partial z^2} = \frac{\nu \rho g r \sin\theta}{(1-\nu)} \tag{7}$$

$$\sigma_{r\theta} = \frac{1}{r}\frac{\partial^2 \phi_P}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \phi_P}{\partial \theta} = \frac{(1-2\nu)\rho g r \cos\theta}{4(1-\nu)} \tag{8}$$

$$\sigma_{\theta z} = \frac{1}{r}\frac{\partial^2 \phi_P}{\partial \theta \partial z} = 0 \tag{9}$$

$$\sigma_{zr} = \frac{\partial^2 \phi_P}{\partial z \partial r} = 0 \; . \tag{10}$$

For the homogeneous solution, we shall require terms in Solutions A,B that vary with $\sin\theta$ and terms in Solution E that vary with $\cos\theta$. The stresses in the particular solution vary with z^0, but the gravitational load causes bending moments which

[3] These are identical to the expressions for Solution A of Table 21.1, except for an additional term $\nu V/(1-\nu)$ in the normal stresses.

increases the order of the required polynomial by 3. We therefore use 5th order and below in Solutions A,B and 4th order and below in Solution B. Because the cylinder is hollow, we shall also require the logarithmically singular potentials derived from equations (24.54), but we anticipate that these can be terminated at two orders lower than the bounded terms. The problem is also even in z, so we start with the functions

$$
\begin{aligned}
\phi &= (A_1F_{13} + A_2F_{15} + A_3S_{10} + A_4S_{12})\sin\theta \\
&= \left(\frac{3A_1}{2}(4z^2r-r^3)+\frac{15A_2}{8}(8z^4r-12z^2r^3+r^5)+\frac{A_3}{r}+A_4\left(-r\ln(r)+\frac{z^2}{r}+\frac{r}{2}\right)\right)\sin\theta \\
\omega &= (B_1F_{12} + B_2F_{14} + B_3S_{11})\sin\theta \\
&= \left(3B_1zr + \frac{5B_2}{2}(4z^3r - 3zr^3) + \frac{B_3z}{r}\right)\sin\theta \qquad (11) \\
\Psi &= (E_1F_{13} + E_2F_{15} + E_3S_{10} + E_4S_{12})\cos\theta \\
&= \left(\frac{3E_1}{2}(4z^2r-r^3)+\frac{15E_2}{8}(8z^4r-12z^2r^3+r^5)+\frac{E_3}{r}+E_4\left(-r\ln(r)+\frac{z^2}{r}+\frac{r}{2}\right)\right)\cos\theta\ ,
\end{aligned}
$$

where the functions F_{1i}, S_{1i} are given in equations (24.55, 24.56) and we have omitted terms that generate only rigid-body displacements. The stress components from Table 19.1, including the body force terms, are then

$$
\begin{aligned}
\sigma_{rr} &= \Bigg(-9A_1r - \frac{15A_2r(18z^2-5r^2)}{2} - 6\nu B_1r - 15B_2r((3+4\nu)z^2-\nu r^2) \\
&\quad +6E_1r - 15E_2r(r^2-6z^2) + \frac{2A_3}{r} + \frac{A_4(2z^2-r^2)}{r^3} + \frac{2B_3(z^2-\nu r^2)}{r^3} \\
&\quad +\frac{4E_3}{r^3} + \frac{2E_4(r^2+2z^2)}{r^3}\Bigg)\sin\theta + \frac{(3-2\nu)\rho g r\sin\theta}{4(1-\nu)} \qquad (12) \\
\sigma_{\theta\theta} &= \Bigg(-3A_1r - \frac{15A_2r(6z^2-r^2)}{2} - 6\nu B_1r - 15B_2r((1+4\nu)z^2-\nu r^2) \\
&\quad -6E_1r + 15E_2r(r^2-6z^2) - \frac{2A_3}{r} - \frac{A_4(2z^2+r^2)}{r^3} - \frac{2B_3(z^2+\nu r^2)}{r^3} \\
&\quad -\frac{4E_3}{r^3} - \frac{2E_4(r^2+2z^2)}{r^3}\Bigg)\sin\theta + \frac{(1+2\nu)\rho g r\sin\theta}{4(1-\nu)} \qquad (13) \\
\sigma_{zz} &= \Big(12A_1r + 45A_2r(4z^2-r^2) - 6B_1r \\
&\quad +15B_2r(4\nu z^2+(1-\nu)r^2) + \frac{2A_4}{r} - \frac{2B_3(1-\nu)}{r}\Big)\sin\theta + \frac{\nu\rho g r\sin\theta}{(1-\nu)} \qquad (14) \\
\sigma_{r\theta} &= \Bigg(-3A_1r - \frac{15A_2r(6z^2-r^2)}{2} - 15B_2rz^2 + 6E_1r \\
&\quad -30E_2r(r^2-3z^2) - \frac{2A_3}{r^3} - \frac{A_4(r^2+2z^2)}{r^3} - \frac{2B_3z^2}{r^3} - \frac{4E_3}{r^3} - \frac{4E_4z^2}{r^3}\Bigg)\cos\theta
\end{aligned}
$$

$$+\frac{(1-2\nu)\rho g r\cos\theta}{4(1-\nu)} \tag{15}$$

$$\begin{aligned}\sigma_{\theta z} &= \Big(12A_1 + 15A_2(4z^2-3r^2) + 6\nu B_1 + 5B_2(4(1+\nu)z^2 - 3\nu r^2) \\ &\quad -12E_1 + 15E_2(9r^2-4z^2) + \frac{2A_4}{r^2} + \frac{2B_3\nu}{r^2} + \frac{2E_4}{r^2}\Big) z\cos\theta \end{aligned} \tag{16}$$

$$\begin{aligned}\sigma_{zr} &= \Big(12A_1 + 15A_2(4z^2-9r^2) + 6\nu B_1 + 5B_2(4(1+\nu)z^2 - 9\nu r^2) \\ &\quad -12E_1 + 15E_2(3r^2-4z^2) - \frac{2A_4}{r^2} - \frac{2B_3\nu}{r^2} - \frac{2E_4}{r^2}\Big) z\sin\theta\ . \end{aligned} \tag{17}$$

The tractions $\sigma_{rr}, \sigma_{r\theta}, \sigma_{rz}$ must be zero on $r = a$, giving

$$\begin{aligned} -9A_1a - \frac{15A_2a(18z^2-5a^2)}{2} - 6\nu B_1 a - 15B_2a((3+4\nu)z^2 - \nu a^2) \\ +6E_1a - 15E_2a(a^2-6z^2) + \frac{2A_3}{a} + \frac{A_4(2z^2-a^2)}{a^3} + \frac{2B_3(z^2-\nu a^2)}{a^3} \\ +\frac{4E_3}{a^3} + \frac{2E_4(a^2+2z^2)}{a^3} + \frac{(3-2\nu)\rho g a}{4(1-\nu)} &= 0 \end{aligned} \tag{18}$$

$$\begin{aligned} -3A_1a - \frac{15A_2a(6z^2-a^2)}{2} - 15B_2az^2 + 6E_1a \\ -30E_2a(a^2-3z^2) - \frac{2A_3}{a^3} - \frac{A_4(a^2+2z^2)}{a^3} - \frac{2B_3z^2}{a^3} - \frac{4E_3}{a^3} - \frac{4E_4z^2}{a^3} \\ +\frac{(1-2\nu)\rho g a}{4(1-\nu)} &= 0 \end{aligned} \tag{19}$$

$$\begin{aligned} 12A_1 + 15A_2(4z^2-9a^2) + 6\nu B_1 + 5B_2(4(1+\nu)z^2 - 9\nu a^2) \\ -12E_1 + 15E_2(3a^2-4z^2) - \frac{2A_4}{a^2} - \frac{2B_3\nu}{a^2} - \frac{2E_4}{a^2} &= 0 \end{aligned} \tag{20}$$

and these equations must be satisfied for all z. Equating coefficients of powers of z, we obtain the six equations

$$-135A_2a - 15B_2a(3+4\nu) + 90E_2a + \frac{2A_4}{a^3} + \frac{2B_3}{a^3} + \frac{4E_4}{a^3} = 0 \tag{21}$$

$$\begin{aligned} -9A_1a + \frac{75A_2a^3}{2} - 6\nu B_1a + 15B_2\nu a^3 + 6E_1a - 15E_2a^3 + \frac{2A_3}{a} \\ -\frac{A_4}{a} - \frac{2B_3\nu}{a} + \frac{4E_3}{a^3} + \frac{2E_4}{a} + \frac{(3-2\nu)\rho g a}{4(1-\nu)} &= 0 \end{aligned} \tag{22}$$

$$-45A_2a - 15B_2a + 90E_2a - \frac{2A_4}{a^3} - \frac{2B_3}{a^3} - \frac{4E_4}{a^3} = 0 \tag{23}$$

$$-3A_1a + \frac{15A_2a^3}{2} + 6E_1a - 30E_2a^3 - \frac{2A_3}{a^3} - \frac{A_4}{a} - \frac{4E_3}{a^3} + \frac{(1-2\nu)\rho g a}{4(1-\nu)} = 0 \tag{24}$$

$$60A_2 + 20B_2(1+\nu) - 60E_2 = 0 \tag{25}$$

$$12A_1 - 135A_2a^2 + 6\nu B_1 - 45B_2\nu a^2 - 12E_1 + 45E_2a^2 - \frac{2A_4}{a^2} - \frac{2B_3\nu}{a^2} - \frac{2E_4}{a^2} = 0 \tag{26}$$

On the free end $z = 0$, the tractions are

$$\sigma_{zr} = 0 \tag{27}$$

$$\sigma_{z\theta} = 0 \tag{28}$$

$$\sigma_{zz} = \Big(12A_1 r + 45A_2 r(-r^2) - 6B_1 r + 15B_2 r(4\nu(1-\nu)r^2) + \frac{2A_4}{r} - \frac{2B_3(1-\nu)}{r}\Big)\sin\theta + \frac{\nu\rho g r\sin\theta}{(1-\nu)} \tag{29}$$

$$\tag{30}$$

The only non-trivial weak condition to satisfy is the moment condition

$$M = \int_b^a \int_0^{2\pi} \sigma_{zz} r^2 \sin\theta d\theta dr = 0 \ , \tag{31}$$

from which we obtain

$$\frac{5\pi(B_2(1-\nu) - 3A_2)(a^6 - b^6)}{2} + \frac{3\pi(2A_1 - B_1(1-\nu))(a^4 - b^4)}{2} + \pi(A_4 - B_3(1-\nu))(a^2 - b^2) + \frac{\pi\nu\rho g(a^4 - b^4)}{4(1-\nu)} = 0 \ . \tag{32}$$

Solving equations (21–26, 31) for the constants $A_1, A_2, A_3, A_4, B_1, B_2, B_3, E_1, E_2, E_3, E_4$ and substituting into (12–17), we obtain the final stress field as

$$\sigma_{rr} = \frac{\rho g(a^2-r^2)(r^2-b^2)\{4(1+\nu)(1+2\nu)a^2b^2-(1-\nu)(5+2\nu)r^2(a^2+b^2)\}\sin\theta}{12(1-\nu^2)(a^2+b^2)^2r^3}$$

$$\sigma_{\theta\theta} = \frac{\rho g\sin\theta}{12(1-\nu^2)(a^2+b^2)^2r^3}\{(1-\nu)(1+2\nu)r^4[(a^2+b^2)r^2 - 3(a^4+b^4)] + 2(1+2\nu)a^2b^2[(1+\nu)a^2b^2 - 3(3+\nu)r^4] - 3(1-2\nu)(5+3\nu)(a^2+b^2)a^2b^2r^2\}$$

$$\sigma_{zz} = -\frac{\rho g\sin\theta}{6(1-\nu^2)(a^2+b^2)^2r}\{(1-\nu)(2+\nu)r^2[3(a^2+b^2)r^2 - 2(a^4+b^4)] - 3(6+3\nu-\nu^2)(a^2+b^2)a^2b^2 + 4(8+5\nu-\nu^2)a^2b^2r^2 - 12(1-\nu^2)(a^2+b^2)r^2z^2\}$$

$$\sigma_{r\theta} = -\frac{\rho g(a^2-r^2)(r^2-b^2)\{4(1+\nu)(1+2\nu)a^2b^2+(1-\nu)(1-2\nu)r^2(a^2+b^2)\}\cos\theta}{12(1-\nu^2)(a^2+b^2)^2r^3}$$

$$\sigma_{\theta z} = \frac{\rho g\{(3+2\nu)(a^2b^2 + a^2r^2 + r^2b^2) - (1-2\nu)r^4\}z\cos\theta}{2(1+\nu)(a^2+b^2)r^2}$$

$$\sigma_{zr} = \frac{\rho g(3+2\nu)(a^2-r^2)(r^2-b^2)z\sin\theta}{2(1+\nu)(a^2+b^2)r^2} \ .$$

25.10. The hollow cylinder $b \le r < a, -L < z < L$ rotates at constant speed Ω about the x-axis, all the surfaces being traction-free. Find the complete stress field in the cylinder.

For rotation at constant speed about the x-axis, the accelerations are

$$a_x = 0 \; ; \quad a_y = -\Omega^2 y \; ; \quad a_z = -\Omega^2 z \tag{1}$$

corrsponding to inertial body forces

$$p_x = 0 \; ; \quad p_y = \rho\Omega^2 y \; ; \quad p_z = \rho\Omega^2 z \; . \tag{2}$$

These forces can be described in terms of a body-force potential

$$V = -\frac{\rho\Omega^2(y^2+z^2)}{2} = \frac{\rho\Omega^2(r^2(\cos(2\theta)-1)-2z^2)}{4} \tag{3}$$

(see equation (20.38)). Equation (20.40) then takes the form

$$\nabla^2\phi = -\frac{(1-2\nu)\rho\Omega^2(y^2+z^2)}{2(1-\nu)} \tag{4}$$

and a particular solution is

$$\phi_P = -\frac{(1-2\nu)\rho\Omega^2 y^2 z^2}{4(1-\nu)} = \frac{(1-2\nu)\rho\Omega^2 r^2 z^2(\cos(2\theta)-1)}{8(1-\nu)} \; . \tag{5}$$

Substituting into the polar coordinate versions[4] of equations (20.41, 20.42), we obtain the particular solution as

$$\sigma_{rr} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2\phi_P}{\partial r^2} = -\frac{\rho\Omega^2\{(z^2+\nu r^2)(1-\cos(2\theta)+2\nu z^2\cos(2\theta)\}}{4(1-\nu)} \tag{6}$$

$$\sigma_{\theta\theta} = \frac{\nu V}{(1-\nu)} + \frac{1}{r}\frac{\partial\phi_P}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi_P}{\partial\theta^2} = \frac{\rho\Omega^2\{(\nu r^2-(1-2\nu)z^2)\cos(2\theta)-z^2-\nu r^2\}}{4(1-\nu)} \tag{7}$$

$$\sigma_{zz} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2\phi_P}{\partial z^2} = -\frac{\rho\Omega^2\{(1-\nu)r^2(1-\cos(2\theta))+2\nu z^2\}}{4(1-\nu)} \tag{8}$$

$$\sigma_{r\theta} = \frac{1}{r}\frac{\partial^2\phi_P}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\phi_P}{\partial\theta} = -\frac{\rho\Omega^2(1-2\nu)z^2\sin(2\theta)}{4(1-\nu)} \tag{9}$$

$$\sigma_{\theta z} = \frac{1}{r}\frac{\partial^2\phi_P}{\partial\theta\partial z} = -\frac{\rho\Omega^2(1-2\nu)rz\sin(2\theta)}{2(1-\nu)} \tag{10}$$

$$\sigma_{zr} = \frac{\partial^2\phi_P}{\partial z\partial r} = \frac{\rho\Omega^2(1-2\nu)rz(\cos(2\theta)-1)}{2(1-\nu)} \; . \tag{11}$$

[4] These are identical to the expressions for Solution A of Table 21.1, except for an additional term $\nu V/(1-\nu)$ in the normal stresses.

These expressions involve axisymmetric terms and $\cos(2\theta)$ terms up to second order in z. For the $\cos(2\theta)$ terms, we need to generate functions F_{mn}, S_{mn} of §24.7 with $m = 2$, using the relation

$$f_2 = \frac{\partial f1}{\partial r} - \frac{f_1}{r} \tag{12}$$

and the functions from (24.55, 24.56). These functions are generated by the Maple and Mathematica files 'cyln' and 'holn'. For example, we define

$$F_{24} = \frac{\partial F_{15}}{\partial r} - \frac{F_{15}}{r} = -\frac{15}{2}(6z^2r^2 - r^4) \tag{13}$$

$$S_{22} = \frac{\partial S_{12}}{\partial r} - \frac{S_{12}}{r} = -1 - \frac{2z^2}{r^2} \ . \tag{14}$$

Trial and error with the Maple and Mathematica file 'ABErtz' shows that stress components of the appropriate order and symmetry are obtained with the functions

$$\phi = (A_1F_{22}+A_2F_{24}+A_3S_{20}+A_4S_{22})\cos(2\theta)+A_5F_2+A_6F_4+A_7S_0+A_8S_2 \tag{15}$$

$$\omega = (B_1F_{23} + B_2S_{21})\cos(2\theta) + B_3F_1 + B_4F_3 + B_5S_1 \tag{16}$$

$$\Psi = (E_1F_{24} + E_2S_{22})\sin(2\theta) \ , \tag{17}$$

where $F_n = R^nP_n(z/R)$ from (24.27) and $S_n = \varphi_n$ from (24.42). The additional terms F_{22}, S_{20} might be included in Ψ, but it will be found that these define identical stress and displacement fields to those due to the corresponding terms in ϕ, so they can be excluded without loss of generality.

Since there are 15 separate constants, it is not really practical to perform the resulting algebra manually, but the Maple or Mathematica solution is straightforward. We substitute the functions (15–17) into 'ABErtz' to obtain the stress components, add in the particular solution of (6–11) and then determine the traction components $\sigma_{rr}, \sigma_{r\theta}, \sigma_{rz}$ on the surfaces $r = a, b$. Each of these tractions must be zero for all z, leading to separate equations for each power of z.

One additional equation is obtained from the weak boundary condition

$$F = \int_b^a \int_0^{2\pi} \sigma_{zz}(r, \theta, L) r d\theta dr = 0 \ . \tag{18}$$

Notice that this condition need only be imposed on one end in view of the symmetry of the problem. Only the axisymmetric terms make a contribution to this condition.

The final stress field is then obtained by solving the resulting equations for the constants $A_1 - A_8, B_1 - B_5, E_1, E_2$ and substituting into the expressions for the stress components. Some of the less complex expressions are

$$\sigma_{rr} = \frac{\rho\Omega^2(a^2 - r^2)(r^2 - b^2)}{4r^2}\left\{\left(1 + \frac{3(a^2 + b^2)a^2b^2}{(a^2 - b^2)^2r^2}\right)\cos(2\theta) + \frac{(3 + \nu)}{4(1 - \nu^2)}\right\} \tag{19}$$

$$\sigma_{r\theta} = \frac{\rho\Omega^2(a^2 - r^2)(r^2 - b^2)\{3b^4a^2 + 3a^4b^2 + (b^4 + 4a^2b^2 + a^4)r^2\}}{4(a^2 - b^2)^2r^4} \tag{20}$$

$$\sigma_{rz} = \sigma_{z\theta} = 0 \ . \tag{21}$$

25.11. The solid cylinder $0 \le r < a, -L < z < L$ is accelerated from rest by two equal and opposite forces F applied in the positive and negative y-directions respectively at the ends $z = \pm L$. All the other surface tractions are zero. Find the complete stress field in the cylinder at the instant when the forces are applied[5] (when the angular velocity is zero).

The forces in the y-direction will cause angular accelerations about the x-axis which can be written

$$a_x = 0 \; ; \quad a_y = \dot{\Omega} z \; ; \quad a_z = -\dot{\Omega} y \; , \tag{1}$$

where $\dot{\Omega}$ is a constant to be determined. The corresponding inertia body forces are

$$p_x = 0 \; ; \quad p_y = -\rho \dot{\Omega} z \; ; \quad p_z = \rho \dot{\Omega} y \; . \tag{2}$$

These forces are non-conservative and hence cannot be described by a body force potential. However, we can use the same particular solution as in §7.4.2. In the present coordinate system, this leads to the stress components

$$\sigma_{xx} = 0 \tag{3}$$

$$\sigma_{yy} = \frac{\rho \dot{\Omega} y z}{2} \tag{4}$$

$$\sigma_{zz} = -\frac{\rho \dot{\Omega} y z}{2} \tag{5}$$

$$\sigma_{xy} = 0 \tag{6}$$

$$\sigma_{yz} = -\frac{\rho \dot{\Omega} (y^2 - z^2)}{4} \tag{7}$$

$$\sigma_{zx} = 0 \; . \tag{8}$$

To convert these results to cylindrical polar coordinates r, θ, z, we have

$$\begin{aligned} \sigma_{rr} &= \sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + 2\sigma_{xy} \sin\theta \cos\theta = \frac{\rho \dot{\Omega} r z \sin^3\theta}{2} \\ &= \frac{\rho \dot{\Omega} r z}{8} \{3 \sin\theta - \sin(3\theta)\} \end{aligned} \tag{9}$$

$$\begin{aligned} \sigma_{\theta\theta} &= \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - 2\sigma_{xy} \sin\theta \cos\theta = \frac{\rho \dot{\Omega} r z \sin\theta \cos^2\theta}{2} \\ &= \frac{\rho \dot{\Omega} r z}{8} \{3 \sin\theta + \sin(3\theta)\} \end{aligned} \tag{10}$$

$$\sigma_{zz} = -\frac{\rho \dot{\Omega} r z \sin\theta}{2} \tag{11}$$

$$\sigma_{r\theta} = (\sigma_{yy} - \sigma_{xx}) \sin\theta \cos\theta + \sigma_{xy} (\cos^2\theta - \sin^2\theta) = \frac{\rho \dot{\Omega} r z \sin^2\theta \cos\theta}{2}$$

[5] This is a three-dimensional version of the problem considered in §7.4.2.

$$
\begin{aligned}
&= \frac{\rho\dot{\Omega} r z}{8}\{\cos\theta - \cos(3\theta)\} && (12)\\
\sigma_{\theta z} &= \sigma_{zy}\cos\theta - \sigma_{zx}\sin\theta = \frac{\rho\dot{\Omega}(z^2 - r^2\sin^2\theta)\cos\theta}{4} \\
&= \frac{\rho\dot{\Omega}}{16}\{(4z^2 - r^2)\cos\theta + r^2\cos(3\theta)\} && (13)\\
\sigma_{zr} &= \sigma_{zx}\cos\theta + \sigma_{zy}\sin\theta = -\frac{\rho\dot{\Omega}(z^2 - r^2\sin^2\theta)\sin\theta}{4} \\
&= \frac{\rho\dot{\Omega}}{16}\{(4z^2 - 3r^2)\sin\theta + r^2\sin(3\theta)\}\ . && (14)
\end{aligned}
$$

These results imply tractions varying with $\sin\theta$ and $\sin(3\theta)$ on $r = a$. The tractions also vary with z^2. The $z^2\sin\theta$ terms will generate bending moments requiring higher order terms in the homogeneous solution. Following the arguments in §7.4.2, we anticipate needing even polynomial terms up to 6th order in z. The $\sin(3\theta)$ terms do not generate bending moments and therefore need only be carried up to 4th order polynomials. With these considerations in mind, we therefore superpose the stresses resulting from the functions

$$
\begin{aligned}
\phi &= (A_1F_{12} + A_2F_{14} + A_3F_{16})\sin\theta + A_4\sin(3\theta) && (15)\\
\omega &= (B_1F_{11} + B_2F_{13} + B_3F_{15})\sin\theta + B_4\sin(3\theta) && (16)\\
\Psi &= (E_1F_{12} + E_2F_{14} + E_3F_{16})\cos\theta + E_4\cos(3\theta)\ , && (17)
\end{aligned}
$$

where the functions F_{1i} are given in equation (24.55) and the remaining functions can be developed in the same way using (24.46). These functions are calculated in the Maple and Mathematica file 'cyln'.

Since there are 12 separate constants, it is not really practical to perform the resulting algebra manually, but the Maple or Mathematica solution is straightforward. We substitute the functions (15–17) into 'ABErtz' to obtain the stress components, add in the particular solution of (9–14) and then determine the traction components $\sigma_{rr}, \sigma_{r\theta}, \sigma_{rz}$ on the surface $r = a, b$ Each of these tractions must be zero for all z, leading to separate equations for each power of z and each Fourier term $\sin\theta$ or $\sin(3\theta)$. One additional equation is obtained from the weak boundary condition

$$
M = \int_0^a \int_0^{2\pi} \sigma_{zz}(r,\theta,L) r^2 \sin\theta d\theta dr = 0\ . \qquad (18)
$$

Notice that this condition need only be imposed on one end in view of the symmetry of the problem.

The final stress field is then obtained by solving the resulting equations for the constants $A_1 - A_4, B_1 - B_4, E_1 - E_4$ and substituting into the expressions for the stress components. We obtain

$$
\sigma_{rr} = \frac{\rho\dot{\Omega}(5 + 2\nu)(a^2 - r^2) r z \sin\theta}{12(1+\nu)a^2}
$$

$$
\begin{aligned}
\sigma_{\theta\theta} &= \frac{\rho\dot{\Omega}\{3(1+2\nu)a^2-(1-2\nu)r^2\}rz\sin\theta}{12(1+\nu)a^2} \\
\sigma_{zz} &= -\frac{\rho\dot{\Omega}\{(2+\nu)(3r^2-2a^2)+4(1+\nu)(L^2-z^2)\}rz\sin\theta}{6(1+\nu)a^2} \\
\sigma_{r\theta} &= \frac{\rho\dot{\Omega}(1-2\nu)(a^2-r^2)rz\cos\theta}{12(1+\nu)a^2} \\
\sigma_{rz} &= \frac{\rho\dot{\Omega}(a^2-r^2)\sin\theta}{48(1+\nu)a^2}\left(4(3+2\nu)(L^2-3z^2)+2(5+2\nu)r^2-\frac{(19+35\nu+18\nu^2)a^2}{(1+\nu)}\right) \\
\sigma_{\theta z} &= \frac{\rho\dot{\Omega}(a^2-r^2)\sin\theta}{48(1+\nu)a^2}\left(2(1-2\nu)r^2(r^2+2L^2-6z^2)+(3+2\nu)a^2(3z^2-L^2)\right. \\
&\quad \left.+\frac{a^2\{(19+35\nu+18\nu^2)a^2-(7+3\nu+2\nu^2)r^2\}}{(1+\nu)}\right) . \qquad (19)
\end{aligned}
$$

The rotational acceleration $\dot{\Omega}$ can be expressed in terms of the force F through

$$
F=\int_0^a\int_0^{2\pi}(\sigma_{rz}(r,\theta,L)\sin\theta+\sigma_{\theta z}(r,\theta,L)\cos\theta)r^2d\theta dr=\frac{\pi\rho\dot{\Omega}a^2(3a^2+4L^2)}{12} \qquad (20)
$$

and hence

$$
\rho\dot{\Omega}=\frac{12F}{\pi a^2(3a^2+4L^2)} \qquad (21)
$$

25.12. The circular plate $0 \le r < a, -h/2 < z < h/2$ rotates at constant speed Ω about the x-axis, all the surfaces being traction-free. Find the stresses in the plate, using strong boundary conditions on $z = \pm h/2$ and weak boundary conditions on $r = a$. Note: This problem is not axisymmetric. It is the three-dimensional version of Problem 8.3.

For rotation at constant speed about the x-axis, the accelerations are

$$a_x = 0 \; ; \quad a_y = -\Omega^2 y \; ; \quad a_z = -\Omega^2 z \tag{1}$$

corrsponding to inertial body forces

$$p_x = 0 \; ; \quad p_y = \rho\Omega^2 y \; ; \quad p_z = \rho\Omega^2 z \; . \tag{2}$$

These forces can be described in terms of a body-force potential

$$V = -\frac{\rho\Omega^2(y^2+z^2)}{2} = \frac{\rho\Omega^2(r^2(\cos(2\theta)-1)-2z^2)}{4} \tag{3}$$

(see equation (20.38)). Equation (20.40) then takes the form

$$\nabla^2\phi = -\frac{(1-2\nu)\rho\Omega^2(y^2+z^2)}{2(1-\nu)} \tag{4}$$

and a particular solution is

$$\phi_P = -\frac{(1-2\nu)\rho\Omega^2 y^2 z^2}{4(1-\nu)} = \frac{(1-2\nu)\rho\Omega^2 r^2 z^2(\cos(2\theta)-1)}{8(1-\nu)} \; . \tag{5}$$

Substituting into the polar coordinate versions[6] of equations (20.41, 20.42), we obtain the particular solution as

$$\sigma_{rr} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2\phi_P}{\partial r^2} = -\frac{\rho\Omega^2\{(z^2+\nu r^2)(1-\cos(2\theta)+2\nu z^2\cos(2\theta)\}}{4(1-\nu)} \tag{6}$$

$$\sigma_{\theta\theta} = \frac{\nu V}{(1-\nu)} + \frac{1}{r}\frac{\partial\phi_P}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi_P}{\partial\theta^2} = \frac{\rho\Omega^2\{(\nu r^2-(1-2\nu)z^2)\cos(2\theta)-z^2-\nu r^2\}}{4(1-\nu)} \tag{7}$$

$$\sigma_{zz} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2\phi_P}{\partial z^2} = -\frac{\rho\Omega^2\{(1-\nu)r^2(1-\cos(2\theta))+2\nu z^2\}}{4(1-\nu)} \tag{8}$$

$$\sigma_{r\theta} = \frac{1}{r}\frac{\partial^2\phi_P}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\phi_P}{\partial\theta} = -\frac{\rho\Omega^2(1-2\nu)z^2\sin(2\theta)}{4(1-\nu)} \tag{9}$$

$$\sigma_{\theta z} = \frac{1}{r}\frac{\partial^2\phi_P}{\partial\theta\partial z} = -\frac{\rho\Omega^2(1-2\nu)rz\sin(2\theta)}{2(1-\nu)} \tag{10}$$

$$\sigma_{zr} = \frac{\partial^2\phi_P}{\partial z\partial r} = \frac{\rho\Omega^2(1-2\nu)rz(\cos(2\theta)-1)}{2(1-\nu)} \; . \tag{11}$$

[6] These are identical to the expressions for Solution A of Table 21.1, except for an additional term $\nu V/(1-\nu)$ in the normal stresses.

These expressions involve axisymmetric terms and $\cos(2\theta)$ terms up to second order in r. The problem is also symmetric (even) in z. For the $\cos(2\theta)$ terms, we need to generate functions F_{mn} of §24.7 with $m = 2$, using the relation

$$f_2 = \frac{\partial f_1}{\partial r} - \frac{f_1}{r} \tag{12}$$

and the functions from (24.55). These functions are generated by the Maple and Mathematica files 'cyln'. For example, we define

$$F_{24} = \frac{\partial F_{15}}{\partial r} - \frac{F_{15}}{r} = -\frac{15}{2}(6z^2r^2 - r^4) \ . \tag{13}$$

Trial and error with the Maple and Mathematica file 'ABErtz' shows that stress components of the appropriate order and symmetry are obtained with the functions

$$\phi = (A_1F_{22}+A_2F_{24})\cos(2\theta)+A_3F_2+A_4F_4 \tag{14}$$

$$\omega = B_1F_{23}\cos(2\theta) + B_2F_1 + B_3F_3 \tag{15}$$

$$\Psi = E_1F_{24}\sin(2\theta) \ , \tag{16}$$

where $F_n = R^nP_n(z/R)$ from (24.27). The additional term F_{22} might be included in Ψ, but it will be found that it defines identical stress and displacement fields to those due to the corresponding terms in ϕ, so it can be excluded without loss of generality.

The stress components are then obtained from Table 9.1 and (6–11) as

$$\begin{aligned}
\sigma_{rr} &= 3\{2A_1 + 30A_2(z^2 - r^2) + 10B_1(z^2 - \nu r^2) + 30E_1(2z^2 - r^2)\}\cos(2\theta) \\
&\quad -A_3 + \frac{3A_4}{2}(3r^2 - 4z^2) - 2\nu B_2 - 3B_3((1+2\nu)z^2 - \nu r^2) \\
&\quad -\frac{\rho\Omega^2\{(z^2 + \nu r^2)(1 - \cos(2\theta) + 2\nu z^2\cos(2\theta)\}}{4(1-\nu)} \\
\sigma_{\theta\theta} &= 3\{-2A_1 - 30A_2z^2 - 10B_1(z^2 + \nu r^2) - 30E_1(2z^2 - r^2)\}\cos(2\theta) \\
&\quad -A_3 + \frac{3A_4}{2}(r^2 - 4z^2) - 2\nu B_2 - 3B_3((1+2\nu)z^2 - \nu r^2) \\
&\quad +\frac{\rho\Omega^2\{(\nu r^2 - (1-2\nu)z^2)\cos(2\theta) - z^2 - \nu r^2\}}{4(1-\nu)} \\
\sigma_{zz} &= 30(3A_2 - B_1(1-\nu))r^2\cos(2\theta) + 2A_3 + 6A_4(2z^2 - r^2) - 2B_2(1-\nu) \\
&\quad +3B_3(2\nu z^2 + (1-\nu)r^2) - \frac{\rho\Omega^2\{(1-\nu)r^2(1-\cos(2\theta)) + 2\nu z^2\}}{4(1-\nu)} \\
\sigma_{rz} &= 60(3A_2 + \nu B_1 + 3E_1)rz\cos(2\theta) - 6(2A_4 + \nu B_3)rz \\
&\quad +\frac{\rho\Omega^2(1-2\nu)rz(\cos(2\theta) - 1)}{2(1-\nu)} \\
\sigma_{\theta z} &= -60(3A_2 + \nu B_1 + 3E_1)rz\sin(2\theta) - \frac{\rho\Omega^2(1-2\nu)rz\sin(2\theta)}{2(1-\nu)}
\end{aligned} \tag{17}$$

$$\sigma_{r\theta} = -3\{2A_1 + 15A_2(2z^2 - r^2) + 10B_1 z^2 + 30E_1(2z^2 - r^2)\}\sin(2\theta) - \frac{\rho\Omega^2(1-2\nu)z^2\sin(2\theta)}{4(1-\nu)} .$$

The tractions on the surface $z = h/2$ are therefore

$$\begin{aligned}
\sigma_{\theta z} &= -30(3A_2 + \nu B_1 + 3E_1)rh\sin(2\theta) - \frac{\rho\Omega^2(1-2\nu)rh\sin(2\theta)}{4(1-\nu)} \\
\sigma_{rz} &= 30(3A_2 + \nu B_1 + 3E_1)rh\cos(2\theta) - 3(2A_4 + \nu B_3)rh \\
&\quad + \frac{\rho\Omega^2(1-2\nu)rh(\cos(2\theta) - 1)}{4(1-\nu)} \\
\sigma_{zz} &= 30(3A_2 - B_1(1-\nu))r^2\cos(2\theta) + 2A_3 + 3A_4(h^2 - 2r^2) - 2B_2(1-\nu) \\
&\quad + \frac{3B_3}{2}(\nu h^2 + 2(1-\nu)r^2) - \frac{\rho\Omega^2\{2(1-\nu)r^2(1-\cos(2\theta)) + \nu h^2\}}{8(1-\nu)} .
\end{aligned}$$

These have to be zero for all r, θ and hence the coefficients of all powers of r must be zero for both axisymmetric and $\cos(2\theta)$ terms, giving

$$\begin{aligned}
30(3A_2 + \nu B_1 + 3E_1) + \frac{\rho\Omega^2(1-2\nu)}{4(1-\nu)} &= 0 \\
-3(2A_4 + \nu B_3) - \frac{\rho\Omega^2(1-2\nu)}{4(1-\nu)} &= 0 \\
30(3A_2 - B_1(1-\nu)) + \frac{\rho\Omega^2}{4} &= 0 \\
2A_3 + 3A_4 h^2 - 2B_2(1-\nu) + \frac{3B_3\nu h^2}{2} - \frac{\rho\Omega^2\nu h^2}{8(1-\nu)} &= 0 \\
-6A_4 + 3B_3(1-\nu) - \frac{\rho\Omega^2}{4} &= 0 .
\end{aligned}$$

For the weak conditions, we require that the radial and tangential forces F_r, F_θ be zero for all θ at $r = a$, where

$$F_r = \int_{-h/2}^{h/2} \sigma_{rr} dz \; ; \quad F_\theta = \int_{-h/2}^{h/2} \sigma_{r\theta} dz \; . \tag{18}$$

The remaining weak conditions

$$F_z = \int_{-h/2}^{h/2} \sigma_{rz} dz = 0 \; ; \quad M = \int_{-h/2}^{h/2} \sigma_{rr} z dz = 0 \tag{19}$$

are satisfied identically by symmetry.

This yields three additional equations for the constants $A_1 - A_4, B_1 - B_3, E_1$. Solving these equations and substituting into the stress equations, we obtain the final

stress field as

$$
\begin{aligned}
\sigma_{rr} &= \frac{\rho\Omega^2(3+\nu)(a^2-r^2)}{16} - \frac{\rho\Omega^2(a^2-r^2)\cos(2\theta)}{4} - \frac{\rho\Omega^2\nu(3+\nu)(12z^2-h^2)}{48(1-\nu)} \\
\sigma_{\theta\theta} &= \frac{\rho\Omega^2(3+\nu)a^2}{16} - \frac{\rho\Omega^2(1+3\nu)r^2}{16} + \frac{\rho\Omega^2(a^2-r^2)\cos(2\theta)}{4} \\
&\quad - \frac{\rho\Omega^2\nu(3+\nu)(12z^2-h^2)}{48(1-\nu)} \\
\sigma_{zz} &= \frac{\rho\Omega^2(h^2-4z^2)}{8} \\
\sigma_{r\theta} &= \frac{\rho\Omega^2(a^2-r^2)\sin(2\theta)}{4} \\
\sigma_{rz} &= 0 \\
\sigma_{\theta z} &= 0\ .
\end{aligned}
$$

These expressions agree with those obtained for Problem 8.3, except for the final terms in $\sigma_{rr}, \sigma_{\theta\theta}$ and the non-zero expression for σ_{zz}.

The maximum tensile stress occurs at the origin and is

$$
\sigma_{\max} = \frac{\rho\Omega^2}{16}\left((7+\nu)a^2 + \frac{\nu(3+\nu)h^2}{3(1-\nu)}\right)\ . \tag{20}
$$

The maximum Von Mises stress σ_E occurs at the points $(0, \theta, \pm h/2)$.

CHAPTER 26

26.1. A traction-free solid sphere of radius a and density ρ rotates at constant angular velocity Ω about the z-axis[1]. Find the complete stress field in the sphere. Find the location and magnitude of (i) the maximum tensile stress and (ii) the maximum Von Mises stress and also (iii) the increase in the equatorial diameter and (iv) the decrease in the polar diameter due to the rotation.

For rotation at constant speed about the z-axis, the accelerations are

$$a_r = -\Omega^2 r \; ; \quad a_\theta = 0 \; ; \quad a_z = 0 \tag{1}$$

corrsponding to inertial body forces

$$p_r = \rho\Omega^2 r \; ; \quad p_\theta = 0 \; ; \quad p_z = 0 \; . \tag{2}$$

These forces can be described in terms of an axisymmetric body-force potential

$$V = -\frac{\rho\Omega^2 r^2}{2} = -\frac{\rho\Omega^2 R^2 \sin^2\beta}{2} \tag{3}$$

(see equation (20.38)). Equation (20.40) then takes the form

$$\nabla^2\phi = -\frac{(1-2\nu)\rho\Omega^2 r^2}{2(1-\nu)} \tag{4}$$

and a particular solution is

$$\phi_P = -\frac{(1-2\nu)\rho\Omega^2 r^4}{32(1-\nu)} = -\frac{(1-2\nu)\rho\Omega^2 R^4 \sin^4\beta}{32(1-\nu)} \; . \tag{5}$$

Substituting into the spherical polar coordinate versions[2] of equations (20.41, 20.42), we obtain the particular solution as

$$\sigma_{RR} = \frac{\rho\Omega^2 R^2[3(1-2\nu)\cos^2\beta - 3 + 2\nu]\sin^2\beta}{8(1-\nu)} \tag{6}$$

$$\sigma_{\theta\theta} = -\frac{\rho\Omega^2 R^2(1-2\nu)\sin^2\beta}{8(1-\nu)} \tag{7}$$

$$\sigma_{\beta\beta} = \frac{\rho\Omega^2 R^2[3(1-2\nu)\cos^2\beta - 4\nu]\sin^2\beta}{8(1-\nu)} \tag{8}$$

$$\sigma_{\beta R} = -\frac{3\rho\Omega^2 R^2(1-2\nu)\sin^3\beta\cos\beta}{8(1-\nu)} \tag{9}$$

[1] This problem is of some interest in connection with the stresses and displacements in the earth's crust due to diurnal rotation.

[2] These are identical to the expressions for Solution A of Table 21.2, except for an additional term $\nu V/(1-\nu)$ in the normal stresses.

$$\sigma_{R\theta} = \sigma_{\theta\beta} = 0 \ . \tag{10}$$

To satisfy the traction-free boundary conditions

$$\sigma_{RR} = \sigma_{R\beta} = 0 \ ; \quad R = a \tag{11}$$

we need to superpose bounded polynomials in Solutions A and B. The particular solution suggests the terms $R^4 P_4(\cos\beta), R^2 P_2(\cos\beta)$ in ϕ and $R^3 P_3(\cos\beta), R P_1(\cos\beta)$ — i.e.

$$\phi = \phi_P + \frac{A_1 R^4}{8}(35\cos^4\beta - 30\cos^2\beta + 3) + \frac{A_2 R^2}{2}(3\cos^2\beta - 1) \tag{12}$$

$$\omega = \frac{B_1 R^3}{2}(5\cos^3\beta - 3\cos\beta) + B_2 R\cos\beta \ . \tag{13}$$

The stress components are then obtained from Table 21.2 as

$$\begin{aligned}\sigma_{RR} = {} & \frac{3A_1R^2(35\cos(4\beta) + 20\cos(2\beta) + 9)}{16} + \frac{A_2(3\cos(2\beta) + 1)}{2} \\ & + \frac{3\nu B_1 R^2(5\cos(4\beta) + 6\cos(2\beta) + 3)}{4} - B_2[(1-2\nu)\cos(2\beta) + 1] \\ & + \frac{\rho\Omega^2 R^2[3(1-2\nu)\cos(4\beta) - 4(3-2\nu)\cos(2\beta) + 9 - 2\nu]}{64(1-\nu)}\end{aligned} \tag{14}$$

$$\begin{aligned}\sigma_{\theta\theta} = {} & -\frac{3A_1R^2(5\cos(2\beta) + 3)}{4} - A_2 - \frac{3B_1R^2((1+3\nu)\cos(2\beta) + 1 + \nu)}{2} - 2B_2\nu \\ & + \frac{\rho\Omega^2R^2(1-2\nu)(\cos(2\beta) - 1)}{16(1-\nu)}\end{aligned} \tag{15}$$

$$\begin{aligned}\sigma_{\beta\beta} = {} & -\frac{3A_1R^2(35\cos(4\beta) - 3)}{16} - \frac{A_2(3\cos(2\beta) - 1)}{2} \\ & - \frac{3B_1R^2(5\nu\cos(4\beta) + 2(2+\nu)\cos(2\beta) + \nu)}{4} + B_2[(1-2\nu)\cos(2\beta) - 1] \\ & + \frac{\rho\Omega^2R^2[3(1-2\nu)\cos(4\beta) + 16\nu\cos(2\beta) - 3 - 10\nu]}{16(1-\nu)}\end{aligned} \tag{16}$$

$$\begin{aligned}\sigma_{\beta R} = {} & -\frac{15A_1R^2(7\sin(4\beta) + 2\sin(2\beta))}{16} - \frac{3A_2\sin(2\beta)}{2} \\ & - \frac{3B_1R^2(5\nu\sin(4\beta) + 2(1+\nu)\sin(2\beta))}{4} - (1-2\nu)B_2\sin(2\beta) \\ & + \frac{3\rho\Omega^2R^2(1-2\nu)[\sin(4\beta) - 2\sin(2\beta)]}{64(1-\nu)}\end{aligned} \tag{17}$$

$$\sigma_{R\theta} = \sigma_{\theta\beta} = 0 \ . \tag{18}$$

Notice that we have converted these to multiple angle form, but it would be equally possible to group terms with powers of $\sin\beta$ or $\cos\beta$.

The boundary conditions (11) then require that

$$\frac{105A_1}{16} + \frac{15\nu B_1}{4} + \frac{3(1-2\nu)\rho\Omega^2}{64(1-\nu)} = 0 \quad (19)$$

$$\frac{15A_1a^2}{4} + \frac{3A_2}{2} + \frac{9\nu B_1a^2}{2} - (1-2\nu)B_2 - \frac{(3-2\nu)\rho\Omega^2a^2}{16(1-\nu)} = 0 \quad (20)$$

$$\frac{27A_1a^2}{16} + \frac{A_2}{2} + \frac{9\nu B_1a^2}{4} - B_2 + \frac{(9-2\nu)\rho\Omega^2a^2}{64(1-\nu)} = 0 \quad (21)$$

$$-\frac{105A_1R^2}{16} - \frac{15\nu B_1R^2}{4} + \frac{3(1-2\nu)\rho\Omega^2R^2}{64(1-\nu)} = 0 \quad (22)$$

$$-\frac{15A_1a^2}{8} - \frac{3A_2}{2} - \frac{3(1+\nu)B_1a^2}{2} - (1-2\nu)B_2 - \frac{3(1-2\nu)\rho\Omega^2a^2}{32(1-\nu)} = 0\,, \quad (23)$$

with solution

$$A_1 = \frac{\rho\Omega^2(3-21\nu+10\nu^2)}{60(1-\nu)(7+5\nu)}\,; \quad A_2 = -\frac{4\rho\Omega^2a^2(3-2\nu-3\nu^2)}{5(1-\nu^2)(7+5\nu)}$$

$$B_1 = \frac{\rho\Omega^2(3+\nu)}{6(1-\nu)(7+5\nu)}\,; \quad B_2 = -\frac{\rho\Omega^2a^2(3-\nu)}{10(1-\nu^2)}\,. \quad (24)$$

Substituting into (14–18), we then obtain the complete stress field as

$$\sigma_{RR} = \frac{\rho\Omega^2(a^2-R^2)[9+7\nu-5(3-\nu-2\nu^2)\cos(2\beta)]}{10(7+5\nu)(1-\nu)} \quad (25)$$

$$\sigma_{\theta\theta} = \frac{\rho\Omega^2[2(12+\nu-5\nu^2)a^2-(13+14\nu+5\nu^2)R^2-5(1-\nu^2)R^2\cos(2\beta)]}{10(7+5\nu)(1-\nu)} \quad (26)$$

$$\sigma_{\beta\beta} = \frac{\rho\Omega^2[5(3-\nu-2\nu^2)(a^2-R^2)\cos(2\beta)+(9+7\nu)a^2-(3+19\nu+10\nu^2)R^2]}{10(7+5\nu)(1-\nu)} \quad (27)$$

$$\sigma_{\beta R} = \frac{(3+2\nu)\rho\Omega^2(a^2-R^2)\sin(2\beta)}{2(7+5\nu)} \quad (28)$$

$$\sigma_{R\theta} = \sigma_{\theta\beta} = 0\,. \quad (29)$$

We also obtain the radial displacement at the surface of the sphere from equtions (12, 13, 24) and Table 21.2, which is

$$u_R(a,\beta) = \frac{\rho\Omega^2a^3[6-17\nu-15\nu^2-5(2+3\nu+\nu^2)\cos(2\beta)]}{20(7+5\nu)(1+\nu)}\,. \quad (30)$$

The stresses are quadratic functions of the coordinates and the maximum values must occur either on the surface or at the center. At the center, the principal stresses are

$$\sigma_1 = \sigma_2 = \sigma_{RR}(0,\pi/2) = \frac{\rho\Omega^2a^2(12+\nu-5\nu^2)}{5(7+5\nu)(1-\nu)} \quad (31)$$

$$\sigma_3 = \sigma_{RR}(0,0) = -\frac{\rho\Omega^2a^2(3-6\nu-5\nu^2)}{5(7+5\nu)(1-\nu)} \quad (32)$$

and the maximum shear stress is

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\rho\Omega^2 a^2(3+2\nu)}{2(7+5\nu)} \ . \tag{33}$$

At $R = a$, the principal stresses are

$$\sigma_{RR} = 0 \tag{34}$$

$$\sigma_{\theta\theta} = \frac{\rho\Omega^2 a^2[11 - 12\nu - 15\nu^2 - 5(1-\nu^2)\cos(2\beta)]}{10(7+5\nu)(1-\nu)} \tag{35}$$

$$\sigma_{\beta\beta} = \frac{\rho\Omega^2 a^2(3 - 6\nu - 5\nu^2)}{5(7+5\nu)(1-\nu)} \ . \tag{36}$$

The largest of these stresses is $\sigma_{\theta\theta}$ at the equator ($\beta = \pi/2$), but this is still lower than σ_1 of equation (31) for all values of Poisson's ratio in the admissible range $-1 < \nu < 0.5$. The maximum shear stress also always occurs at the origin and hence we have

$$\sigma_{\max} = \frac{\rho\Omega^2 a^2(12 + \nu - 5\nu^2)}{5(7+5\nu)(1-\nu)} \ ; \quad \tau_{\max} = \frac{\rho\Omega^2 a^2(3+2\nu)}{2(7+5\nu)} \ . \tag{37}$$

The equatorial diameter increases by

$$\Delta_e = 2u_R(a, \pi/2) = \frac{\rho\Omega^2 a^3(8 - \nu - 5\nu^2)}{5(7+5\nu)(1+\nu)} \tag{38}$$

whilst the polar diameter *decreases* by

$$\Delta_p = -2u_R(a, 0) = \frac{2\rho\Omega^2 a^3(1 + 8\nu + 5\nu^2)}{5(7+5\nu)(1+\nu)} \ . \tag{39}$$

26.2. A traction-free hollow sphere of inner radius b, outer radius a and density ρ rotates at constant angular velocity Ω about the z-axis. Find the complete stress field in the sphere. Find also the increase in the outer equatorial diameter and the decrease in the outer polar diameter due to the rotation.

For rotation at constant speed about the z-axis, the accelerations are

$$a_r = -\Omega^2 r \; ; \quad a_\theta = 0 \; ; \quad a_z = 0 \tag{1}$$

corrsponding to inertial body forces

$$p_r = \rho\Omega^2 r \; ; \quad p_\theta = 0 \; ; \quad p_z = 0 \; . \tag{2}$$

These forces can be described in terms of an axisymmetric body-force potential

$$V = -\frac{\rho\Omega^2 r^2}{2} = -\frac{\rho\Omega^2 R^2 \sin^2\beta}{2} \tag{3}$$

(see equation (20.38)). Equation (20.40) then takes the form

$$\nabla^2\phi = -\frac{(1-2\nu)\rho\Omega^2 r^2}{2(1-\nu)} \tag{4}$$

and a particular solution is

$$\phi_P = -\frac{(1-2\nu)\rho\Omega^2 r^4}{32(1-\nu)} = -\frac{(1-2\nu)\rho\Omega^2 R^4 \sin^4\beta}{32(1-\nu)} \; . \tag{5}$$

Substituting into the spherical polar coordinate versions[3] of equations (20.41, 20.42), we obtain the particular solution as

$$\sigma_{RR} = \frac{\rho\Omega^2 R^2[3(1-2\nu)\cos^2\beta - 3 + 2\nu]\sin^2\beta}{8(1-\nu)} \tag{6}$$

$$\sigma_{\theta\theta} = -\frac{\rho\Omega^2 R^2(1-2\nu)\sin^2\beta}{8(1-\nu)} \tag{7}$$

$$\sigma_{\beta\beta} = \frac{\rho\Omega^2 R^2[3(1-2\nu)\cos^2\beta - 4\nu]\sin^2\beta}{8(1-\nu)} \tag{8}$$

$$\sigma_{\beta R} = -\frac{3\rho\Omega^2 R^2(1-2\nu)\sin^3\beta\cos\beta}{8(1-\nu)} \tag{9}$$

$$\sigma_{R\theta} = \sigma_{\theta\beta} = 0 \; . \tag{10}$$

[3]These are identical to the expressions for Solution A of Table 21.2, except for an additional term $\nu V/(1-\nu)$ in the normal stresses.

To satisfy the traction-free boundary conditions

$$\sigma_{RR} = \sigma_{R\beta} = 0 \; ; \quad R = a, b \tag{11}$$

we need to superpose bounded polynomials and corresponding harmonics of the form (24.18) singular only at the origin in Solutions A and B. The particular solution suggests the bounded terms $R^4 P_4(\cos\beta), R^2 P_2(\cos\beta)$ in ϕ and $R^3 P_3(\cos\beta), R P_1(\cos\beta)$ and hence we choose

$$\begin{aligned} \phi &= \phi_P + \frac{A_1 R^4}{8}(35\cos^4\beta - 30\cos^2\beta + 3) + \frac{A_2 R^2}{2}(3\cos^2\beta - 1) \\ &\quad + \frac{A_3}{2R^3}(3\cos^2\beta - 1) + \frac{A_4}{R} \qquad (12) \\ \omega &= \frac{B_1 R^3}{2}(5\cos^3\beta - 3\cos\beta) + B_2 R\cos\beta + \frac{B_3\cos\beta}{R^3} \; . \qquad (13) \end{aligned}$$

Notice that the singular terms are chosen to have the same Legendre polynomial form as the bounded terms except that (i) the highest order terms are excluded[4] and (ii) the term $1/R$ is included though there is no corresponding bounded terms since this would correspond merely to an additive constant in ϕ.

The stress components are then obtained as

$$\begin{aligned} \sigma_{RR} &= \frac{3A_1R^2(35\cos(4\beta) + 20\cos(2\beta) + 9)}{16} + \frac{A_2(3\cos(2\beta) + 1)}{2} \\ &\quad + \frac{3A_3(1 + 3\cos(2\beta))}{R^5} + \frac{2A_4}{R^3} + \frac{3\nu B_1 R^2(5\cos(4\beta) + 6\cos(2\beta) + 3)}{4} \\ &\quad - B_2[(1-2\nu)\cos(2\beta) + 1] + \frac{B_3[(5-\nu)\cos(2\beta) + 5 - 3\nu]}{R^3} \\ &\quad + \frac{\rho\Omega^2 R^2[3(1-2\nu)\cos(4\beta) - 4(3-2\nu)\cos(2\beta) + 9 - 2\nu]}{64(1-\nu)} \qquad (14) \\ \sigma_{\theta\theta} &= -\frac{3A_1R^2(5\cos(2\beta) + 3)}{4} - A_2 - \frac{3A_3(3 + 5\cos(2\beta))}{4R^5} - \frac{A_4}{R^3} \\ &\quad - \frac{3B_1R^2((1+3\nu)\cos(2\beta) + 1 + \nu)}{2} - 2B_2\nu - \frac{B_3[3(1-2\nu)\cos(2\beta) + 3 - 2\nu]}{2R^3} \\ &\quad + \frac{\rho\Omega^2R^2(1-2\nu)(\cos(2\beta) - 1)}{16(1-\nu)} \qquad (15) \\ \sigma_{\beta\beta} &= -\frac{3A_1R^2(35\cos(4\beta) - 3)}{16} - \frac{A_2(3\cos(2\beta) - 1)}{2} - \frac{3A_3(1 + 7\cos(2\beta)}{4R^5} - \frac{A_4}{R^3} \\ &\quad - \frac{3B_1R^2(5\nu\cos(4\beta) + 2(2+\nu)\cos(2\beta) + \nu)}{4} + B_2[(1-2\nu)\cos(2\beta) - 1] \end{aligned}$$

[4] If in doubt about whether it is legitimate to exclude the highest order singular terms, the simpler procedure is just to include them and let the solution procedure show whether the corresponding coefficients are zero.

$$
\begin{aligned}
& -\frac{B_3[(1-2\nu)\cos(2\beta)+5-6\nu]}{2R^3} \\
& +\frac{\rho\Omega^2 R^2[3(1-2\nu)\cos(4\beta)+16\nu\cos(2\beta)-3-10\nu]}{16(1-\nu)}
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\sigma_{\beta R} \;=\; & -\frac{15A_1R^2(7\sin(4\beta)+2\sin(2\beta))}{16}-\frac{3A_2\sin(2\beta)}{2}+\frac{6A_3\sin(2\beta)}{R^5} \\
& -\frac{3B_1R^2(5\nu\sin(4\beta)+2(1+\nu)\sin(2\beta))}{4}-(1-2\nu)B_2\sin(2\beta) \\
& +\frac{B_3(1+\nu)\sin(2\beta)}{R^3}+\frac{3\rho\Omega^2R^2(1-2\nu)[\sin(4\beta)-2\sin(2\beta)]}{64(1-\nu)}
\end{aligned} \tag{17}
$$

The boundary conditions (11) then require that

$$
\begin{aligned}
\frac{105A_1}{16}+\frac{15\nu B_1}{4}+\frac{3(1-2\nu)\rho\Omega^2}{64(1-\nu)} &= 0 \\
\frac{15A_1a^2}{4}+\frac{3A_2}{2}+\frac{9A_3}{a^5}+\frac{9\nu B_1a^2}{2}-B_2(1-2\nu)+\frac{B_3(5-\nu)}{a^3}-\frac{(3-2\nu)\rho\Omega^2a^2}{16(1-\nu)} &= 0 \\
\frac{15A_1b^2}{4}+\frac{3A_2}{2}+\frac{9A_3}{b^5}+\frac{9\nu B_1b^2}{2}-B_2(1-2\nu)+\frac{B_3(5-\nu)}{b^3}-\frac{(3-2\nu)\rho\Omega^2b^2}{16(1-\nu)} &= 0 \\
\frac{27A_1a^2}{16}+\frac{A_2}{2}+\frac{3A_3}{a^5}+\frac{2A_4}{a^3}+\frac{9\nu B_1a^2}{4}-B_2+\frac{B_3(5-3\nu)}{a^3}+\frac{(9-2\nu)\rho\Omega^2a^2}{64(1-\nu)} &= 0 \\
\frac{27A_1b^2}{16}+\frac{A_2}{2}+\frac{3A_3}{b^5}+\frac{2A_4}{b^3}+\frac{9\nu B_1b^2}{4}-B_2+\frac{B_3(5-3\nu)}{b^3}+\frac{(9-2\nu)\rho\Omega^2b^2}{64(1-\nu)} &= 0 \\
-\frac{105A_1}{16}-\frac{15\nu B_1}{4}+\frac{3(1-2\nu)\rho\Omega^2}{64(1-\nu)} &= 0 \\
-\frac{15A_1a^2}{8}-\frac{3A_2}{2}+\frac{6A_3}{a^5}-\frac{3B_1(1+\nu)a^2}{2}-(1-2\nu)B_2+\frac{B_3(1+\nu)}{a^3}-\frac{3(1-2\nu)\rho\Omega^2a^2}{64(1-\nu)} &= 0 \\
-\frac{15A_1b^2}{8}-\frac{3A_2}{2}+\frac{6A_3}{b^5}-\frac{3B_1(1+\nu)b^2}{2}-(1-2\nu)B_2+\frac{B_3(1+\nu)}{b^3}-\frac{3(1-2\nu)\rho\Omega^2b^2}{64(1-\nu)} &= 0\,.
\end{aligned}
$$

Beyond this point, the solution can only conveniently be performed in Maple or Mathematica. We solve the above equations for the constants $A_1, A_2, A_3, A_4, B_1, B_2, B_3$ and substitute into (14-17) to obtain the complete stress field.

The increase in the outer equatorial diameter is

$$
\Delta_1 = 2u_R(a, \pi/2)\ , \tag{18}
$$

whilst the decrease in the outer polar diameter is

$$
\Delta_2 = -2u_R(a, 0)\ . \tag{19}
$$

These quantities are plotted in Figure 1 for the case where $\nu = 0.3$.

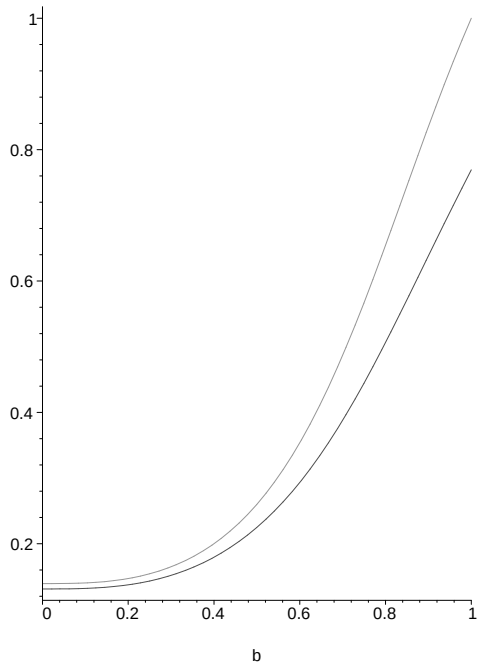


Figure 1: Change in equatorial and polar diameters Δ_1, Δ_2 as a function of the ratio b/a. The upper curve corresponds to the dimensionless quantity $\mu\Delta_2/\rho\Omega^2 a^3$ and the lower curve to $\mu\Delta_1/\rho\Omega^2 a^3$.

26.3. A hollow sphere of inner radius b and outer radius a is filled with a gas at pressure p. Find the complete stress field in the sphere and the increase in the contained volume due to the internal pressure.

This problem is spherically symmetrical. In other words, the stress field is independent depends only on R. We first explore the effect of using Solution A only with a function ϕ that depends only on R. In spherical polar coordinates, the Laplace equation (24.1) then degenerates to

$$\frac{d^2\phi}{dR^2} + \frac{2}{R}\frac{d\phi}{dR} = 0 \ , \tag{1}$$

with general solution

$$\phi = A + \frac{B}{R} \ , \tag{2}$$

where A, B are arbitrary constants. The corresponding non-zero stresses and displacements are obtained from Table 21.2 as

$$2\mu u_R = \frac{\partial\phi}{\partial R} = -\frac{B}{R^2} \tag{3}$$

$$\sigma_{RR} = \frac{\partial^2\phi}{\partial R^2} = \frac{2B}{R^3} \tag{4}$$

$$\sigma_{\theta\theta} = \sigma_{\beta\beta} = \frac{1}{R}\frac{\partial\phi}{\partial R} = -\frac{B}{R^3} \ . \tag{5}$$

This solution provides us with one arbitrary constant which is not sufficient to satisfy the two boundary conditions

$$\sigma_{RR} = -p \ ; \quad R = b \tag{6}$$

$$= 0 \ ; \quad R = a \ . \tag{7}$$

To add extra generality, we superpose a state of uniform hydrostatic tension

$$\sigma_{RR} = \sigma_{\theta\theta} = \sigma_{\beta\beta} = C \tag{8}$$

$$\sigma_{R\theta} = \sigma_{\theta\beta} = \sigma_{\beta R} = 0 \ , \tag{9}$$

for which the displacements are easily shown to be

$$2\mu u_R = \frac{C(1-2\nu)R}{(1+\nu)} \ ; \quad u_\theta = u_\beta = 0 \ . \tag{10}$$

Adding (4) and (8), we have

$$\sigma_{RR} = C + \frac{2B}{R^3} \tag{11}$$

and hence the boundary conditions (6,7) require that

$$C + \frac{2B}{b^3} = -p \; ; \quad C + \frac{2B}{a^3} = 0 \; , \tag{12}$$

with solution

$$B = -\frac{pa^3b^3}{2(a^3-b^3)} \; ; \quad C = \frac{pb^3}{(a^3-b^3)} \; . \tag{13}$$

The final stress and displacement field is therefore

$$2\mu u_R = -\frac{B}{R^2} + \frac{C(1-2\nu)R}{(1+\nu)} = \frac{pa^3b^3}{2(a^3-b^3)R^2} + \frac{p(1-2\nu)b^3R}{(1+\nu)(a^3-b^3)} \tag{14}$$

$$\sigma_{RR} = \frac{2B}{R^3} + C = -\frac{pb^3}{(a^3-b^3)}\left(\frac{a^3}{R^3} - 1\right) \tag{15}$$

$$\sigma_{\theta\theta} = \sigma_{\beta\beta} = -\frac{B}{R^3} + C = \frac{pb^3}{(a^3-b^3)R^3}\left(1 + \frac{a^3}{2b^3}\right) \; . \tag{16}$$

At the inner surface $R = b$, the radial displacement is

$$u_R(b,\theta,\beta) = \frac{pa^3b}{4\mu(a^3-b^3)} + \frac{p(1-2\nu)b^4}{2\mu(1+\nu)(a^3-b^3)} = \frac{pab[(1+\nu)a^2 + 2(1-2\nu)b^2]}{4\mu(1+\nu)(a^3-b^3)} \; . \tag{17}$$

The increase in the contained volume is therefore

$$\Delta V = 4\pi b^2 u_R(b,\theta,\beta) = \frac{\pi pab^3[(1+\nu)a^2 + 2(1-2\nu)b^2]}{\mu(1+\nu)(a^3-b^3)} \; . \tag{18}$$

26.4. A state of uniaxial tension, $\sigma_{zz}=S$ in a large body is perturbed by the presence of a rigid spherical inclusion in the region $0\leq R<a$. The inclusion is perfectly bonded to the surrounding material, so that the boundary condition at the interface in a suitable frame of reference is one of zero displacement ($u_R=u_\theta=u_\beta=0$; $R=a$).

Develop a solution for the stress field and find the stress concentration factor.

This problem is defined by the conditions

$$\begin{aligned} \sigma_{zz} &\rightarrow S \;;\; R\rightarrow\infty && (1)\\ \sigma_{rr},\sigma_{\theta\theta},\sigma_{r\theta},\sigma_{\theta z},\sigma_{zr} &\rightarrow 0 \;;\; R\rightarrow\infty && (2)\\ u_R=u_\theta=u_\beta &= 0 \;;\; R=a\,. && (3)\end{aligned}$$

The unperturbed problem is clearly identical to that obtained in §26.1.2 and we anticipate the complete solution being obtained from the potential functions (26.10, 26.11), with of course different values for the coefficients A_2, A_3, B_2 on the singular terms.

Thus,

$$\begin{aligned} \phi &= A_1R^2P_2(\cos\beta)+A_2R^{-3}P_2(\cos\beta)+A_3R^{-1}P_0(\cos\beta) && (4)\\ \omega &= B_1R^1P_1(\cos\beta)+B_2R^{-2}P_1(\cos\beta)\,, && (5)\end{aligned}$$

with

$$A_1=\frac{S\nu}{(1+\nu)}\;;\;B_1=-\frac{S}{2(1+\nu)}\,. \qquad (6)$$

Substituting in Table 21.2, we obtain the non-zero displacement components as

$$\begin{aligned} 2\mu u_R &= \frac{A_1R[1+3\cos(2\beta)]}{2}-\frac{3A_2[1+3\cos(2\beta)]}{4R^4}-\frac{A_3}{R^2}\\ &\quad -B_1R(1-2\nu)[1+\cos(2\beta)]-\frac{B_2(5-4\nu)[1+\cos(2\beta)]}{2R^2} && (7)\\ 2\mu u_\beta &= -\frac{3A_1R\sin(2\beta)}{2}-\frac{3A_2\sin(2\beta)}{2R^4}+B_1R(1-2\nu)\sin(2\beta)\\ &\quad +\frac{B_2(1-2\nu)\sin(2\beta)}{R^2}\,. && (8)\end{aligned}$$

The boundary conditions (3) therefore require that

$$\begin{aligned} \frac{3A_1a}{2}-\frac{9A_2}{4a^4}-B_1a(1-2\nu)-\frac{B_2(5-4\nu)}{2a^2} &= 0 && (9)\\ \frac{A_1a}{2}-\frac{3A_2}{4a^4}-\frac{A_3}{a^2}-B_1a(1-2\nu)-\frac{B_2(5-4\nu)}{2a^2} &= 0 && (10)\\ -\frac{3A_1a}{2}-\frac{3A_2}{2a^4}+B_1a(1-2\nu)+\frac{B_2(1-2\nu)}{a^2} &= 0\,. && (11)\end{aligned}$$

Solving these equations and using (6), we obtain

$$A_2 = -\frac{Sa^5}{2(4-5\nu)}\ ; \quad A_3 = -\frac{Sa^3(3+19\nu-20\nu^2)}{2(1+\nu)(4-5\nu)}\ ; \quad B_2 = \frac{5Sa^3}{4(4-5\nu)}\ . \tag{12}$$

Substituting (6, 12) into (4,5) and the resulting potential functions into Table 21.2, we obtain the final stress field as

$$\begin{aligned}
\sigma_{RR} &= \frac{S[1+\cos(2\beta)]}{2} - \frac{3Sa^5[1+3\cos(2\beta)]}{2(4-5\nu)R^5} \\
&\quad + \frac{Sa^3}{4(4-5\nu)R^3}\left(\frac{19-28\nu+25}{(1+\nu)} + 5(5-\nu)\cos(2\beta)\right) && (13)\\
\sigma_{\theta\theta} &= \frac{3S}{8(4-5\nu)}\left[\frac{a^5[3+\cos(2\beta)]}{R^5} - \frac{a^3}{R^3}\left(\frac{(3-5\nu)(1-2\nu)}{(1+\nu)} + 5(1-2\nu)\cos(2\beta)\right)\right] && (14)\\
\sigma_{\beta\beta} &= \frac{S[1-\cos(2\beta)]}{2} + \frac{3Sa^5[1+7\cos(2\beta)]}{8(4-5\nu)R^5} \\
&\quad - \frac{Sa^3}{4(4-5\nu)R^3}\left(\frac{(19-5\nu)(1-2\nu)}{(1+\nu)} + 5(1-2\nu)\cos(2\beta)\right) && (15)\\
\sigma_{\beta R} &= \frac{S\sin(2\beta)}{2}\left(1 + \frac{6a^5}{(4-5\nu)R^5} - \frac{5(1+\nu)a^3}{2(4-5\nu)R^3}\right)\ . && (16)
\end{aligned}$$

Graphical exploration of this stress field shows that the maximum tensile stress always occurs at the poles $R = a, \beta = 0, \pi$ and is

$$\sigma_{\max} = \sigma_{RR}(a,0) = \frac{9S(1-\nu)}{(4-5\nu)(1+\nu)}\ , \tag{17}$$

giving a stress concentration factor

$$\mathrm{SCF} = \frac{9(1-\nu)}{(4-5\nu)(1+\nu)}\ . \tag{18}$$

26.5. A rigid spherical inclusion of radius a in a large elastic body is subjected to a force F in the z-direction.

Find the stress field if the inclusion is perfectly bonded to the elastic body at $R{=}a$ and the stresses tend to zero as $R{\to}\infty$.

When $R \gg a$, the stress field must approach that due to a concentrated force at the origin (the Kelvin solution), given in §23.1.2. This field is described by the potential

$$\omega = -\frac{F}{8\pi(1-\nu)R} \ , \tag{1}$$

from equation (23.10).

However, there will some modification in this stress field near the inclusion and we therefore require an additional singular term in solution A. This must be an antisymmetric term in z, corresponding to an odd numbered Legendre polynomial, so we provisionally try the simplest such term, which is

$$\phi = CR^{-2}P_1(cos(\beta)) = \frac{C\cos\beta}{R^2} \ . \tag{2}$$

Suppose that the inclusion moves a distance δ as a rigid body in the direction of the force. Since it is bonded to the elastic body, we must then have

$$u_z = \delta \ ; \quad u_r = u_\theta = 0 \tag{3}$$

on $R = a$ and hence

$$u_R = \delta\cos\beta \ ; \quad u_\beta = -\delta\sin\beta \ ; \quad u_\theta = 0 \tag{4}$$

on $R = a$.

Substituting (1, 2) into Table 21.2, we have

$$2\mu u_R = \frac{F\cos\beta}{2\pi R} - \frac{2C\cos\beta}{R^3} \tag{5}$$

$$2\mu u_\beta = -\frac{F(3-4\nu)\sin\beta}{8\pi(1-\nu)R} - \frac{C\sin\beta}{R^3} \ . \tag{6}$$

The boundary conditions (4) therefore require

$$\frac{F}{2\pi a} - \frac{2C}{a^3} = 2\mu\delta \tag{7}$$

$$-\frac{F(3-4\nu)}{8\pi(1-\nu)a} - \frac{C}{a^3} = -2\mu\delta \ , \tag{8}$$

with solution

$$C = \frac{Fa^2}{24\pi(1-\nu)} \ ; \quad \delta = \frac{F(5-6\nu)}{24\pi\mu(1-\nu)a} \ . \tag{9}$$

The complete stress field is then obtained from (1, 2, 9) and Table 21.1 as

$$\sigma_{RR} = \frac{F\cos\beta}{4\pi(1-\nu)}\left(-\frac{(2-\nu)}{R^2}+\frac{a^2}{R^4}\right) \tag{10}$$

$$\sigma_{\theta\theta} = \frac{F\cos\beta}{8\pi(1-\nu)}\left(\frac{(1-2\nu)}{R^2}-\frac{a^2}{R^4}\right) \tag{11}$$

$$\sigma_{\beta\beta} = \frac{F\cos\beta}{8\pi(1-\nu)}\left(\frac{(1-2\nu)}{R^2}-\frac{a^2}{R^4}\right) \tag{12}$$

$$\sigma_{\beta R} = \frac{F\sin\beta}{8\pi(1-\nu)}\left(\frac{(1-2\nu)}{R^2}+\frac{a^2}{R^4}\right) . \tag{13}$$

Also, the displacements are

$$2\mu u_R = \frac{F\cos\beta}{2\pi R} - \frac{Fa^2\cos\beta}{12\pi(1-\nu)R^3} \tag{14}$$

$$2\mu u_\beta = -\frac{F(3-4\nu)\sin\beta}{8\pi(1-\nu)R} - \frac{Fa^2\sin\beta}{24\pi(1-\nu)R^3} , \tag{15}$$

from (5,6,9).

26.6. Investigate the steady-state thermal stress field due to the disturbance of an otherwise uniform heat flux, $q_z = Q$, by an insulated spherical hole of radius a.

You must first determine the perturbed temperature field, noting (i) that the temperature T is harmonic and hence can be represented in terms of spherical harmonics and (ii) that the insulated boundary condition implies that $\partial T/\partial R = 0$ at $R = a$.

The thermal stress field can now be obtained using Solution T supplemented by appropriate functions in Solutions A,B, chosen so as to satisfy the traction-free condition at the hole.

The unperturbed field corresponds to the temperature field

$$T = -\frac{Qz}{K} \ . \tag{1}$$

This will not generate any stresses unless the body is constrained. We start with Solution T, for which

$$T = \frac{(1-\nu)}{\mu\alpha(1+\nu)}\frac{\partial\chi}{\partial z} \ . \tag{2}$$

showing that χ must be a second order axisymmetric harmonic polynomial — i.e.

$$\chi = C_0 R^2 P_2(z/R) = \frac{C_0(2z^2 - r^2)}{2} \ , \tag{3}$$

from (24.27). Substitution into (2) and (1) then gives

$$C_0 = -\frac{\mu\alpha(1+\nu)Q}{2(1-\nu)K} \ . \tag{4}$$

Table 22.1 shows that this will generate non-zero stress components that are first order polynomials in r, z and hence infinite stresses will be implied at $R \to \infty$. To restore the stress-free condition, we therefore superpose solutions A and B with third and second order polynomials respectively — i.e.

$$\phi = A_0 R^3 P_3(z/R) = \frac{A_0(2z^3 - 3zr^2)}{2} \ ; \quad \omega = B_0 R^2 P_2(z/R) = \frac{B_0(2z^2 - r^2)}{2} \ . \tag{5}$$

The non-zero stress components in cylindrical coordinates are now obtained from Tables 21.1, 22.1 as

$$\sigma_{rr} = -3A_0 z - (1+4\nu)B_0 z - 5C_0 z \tag{6}$$

$$\sigma_{\theta\theta} = -3A_0 z - (1+4\nu)B_0 z - 5C_0 z \tag{7}$$

$$\sigma_{zz} = 6A_0 z - 2(1-2\nu)B_0 z + 2C_0 z \tag{8}$$

$$\sigma_{rz} = -3A_0 r + (1-2\nu)B_0 r - C_0 r \ . \tag{9}$$

For these to be zero, we require

$$
\begin{aligned}
-3A_0 - (1+4\nu)B_0 - 5C_0 &= 0 && (10)\\
6A_0 - 2(1-2\nu)B_0 + 2C_0 &= 0 && (11)\\
-3A_0 + (1-2\nu)B_0 - C_0 &= 0\,, && (12)
\end{aligned}
$$

with solution

$$
A_0 = -\frac{(1-\nu)C_0}{(1+\nu)} = \frac{\mu\alpha Q}{2K}\,; \quad B_0 = -\frac{2C_0}{(1+\nu)} = \frac{\mu\alpha Q}{2K(1-\nu)}\,. \tag{13}
$$

This is a rather tedious way of expressing the stress-free state, but the advantage of doing this is that when we add the perturbation terms, they will be identical in form to the unperturbed terms.

The unperturbed solution can be rewritten in spherical polar coordinates as

$$
\begin{aligned}
\chi &= \frac{C_0 R^2(3\cos(2\beta)+1)}{4} && (14)\\
\phi &= \frac{A_0 R^3(5\cos(3\beta)+3\cos\beta)}{8} && (15)\\
\omega &= \frac{B_0 R^2(3\cos(2\beta)+1)}{4}\,, && (16)
\end{aligned}
$$

from (24.11). We therefore superpose singular spherical harmonics with the same β variation and even orders lower, obtaining the solution

$$
\begin{aligned}
\chi &= \frac{C_0 R^2(3\cos(2\beta)+1)}{4} + \frac{C_1(3\cos(2\beta)+1)}{4R^3} + \frac{C_2}{R} && (17)\\
\phi &= \frac{A_0 R^3(5\cos(3\beta)+3\cos\beta)}{8} + \frac{A_1(5\cos(3\beta)+3\cos\beta)}{8R^4} + \frac{A_2\cos\beta}{R^2} && (18)\\
\omega &= \frac{B_0 R^2(3\cos(2\beta)+1)}{4} + \frac{B_1(3\cos(2\beta)+1)}{4R^3} + \frac{B_2}{R}\,. && (19)
\end{aligned}
$$

The radial heat flux is then obtained from Table 22.2 as

$$
q_R = Q\cos\beta - \frac{(1-\nu)KC_1(15\cos(3\beta)+9\cos\beta)}{2\mu\alpha(1+\nu)R^5} - \frac{2(1-\nu)KC_2\cos\beta}{\mu\alpha(1+\nu)R^3} \tag{20}
$$

and this will be zero for all β at $R=a$ if

$$
C_1 = 0\,; \quad C_2 = \frac{\mu\alpha(1+\nu)Qa^3}{2(1-\nu)K}\,. \tag{21}
$$

For the mechanical boundary conditions, we require

$$
\sigma_{RR} = \sigma_{R\beta} = 0 \tag{22}
$$

for all β on $R = a$. Substituting into Tables 21.2, 22.2, we obtain

$$\begin{aligned}
\sigma_{RR} &= \frac{\mu\alpha(1+\nu)Qa^3\cos\beta}{(1-\nu)KR^2} + \frac{5A_1(5\cos(3\beta)+3\cos\beta)}{2R^6} + \frac{6A_2\cos\beta}{R^4} \\
&\quad + \frac{3B_1\{(9-\nu)\cos(3\beta)+(15-7\nu)\cos\beta\}}{4R^4} + \frac{(2-\nu)B_2\cos\beta}{R^2} \qquad (23) \\
\sigma_{R\beta} &= \frac{\mu\alpha(1+\nu)Qa^3\sin\beta}{2(1-\nu)KR^2} + \frac{15A_1(5\sin(3\beta)+\sin\beta)}{8R^6} + \frac{3A_2\sin\beta}{R^4} \\
&\quad + \frac{3B_1\{(7+2\nu)\sin(3\beta)+(11-6\nu)\sin\beta\}}{8R^4} - \frac{(1-2\nu)B_2\sin\beta}{R^2} \qquad (24)
\end{aligned}$$

and these components will satisfy the boundary conditions (22) if

$$\begin{aligned}
\frac{25A_1}{2a^6} + \frac{3B_1(9-\nu)}{4a^4} &= 0 \qquad (25) \\
\frac{\mu\alpha(1+\nu)Qa^3}{(1-\nu)Ka^2} + \frac{15A_1}{2a^6} + \frac{6A_2}{a^4} + \frac{3(15-7\nu)B_1}{4a^4} + \frac{(2-\nu)B_2}{a^2} &= 0 \qquad (26) \\
\frac{75A_1}{8a^6} + \frac{3(7+2\nu)B_1}{8a^4} &= 0 \qquad (27) \\
\frac{\mu\alpha(1+\nu)Qa^3\sin\beta}{2(1-\nu)KR^2} + \frac{15A_1}{8a^6} + \frac{3A_2}{a^4} + \frac{3(11-6\nu)B_1}{8a^4} - \frac{(1-2\nu)B_2}{a^2} &= 0\,. \qquad (28)
\end{aligned}$$

The solution of these equations is

$$A_1 = B_1 = B_2 = 0\,; \quad A_2 = -\frac{\mu\alpha(1+\nu)Qa^5}{6(1-\nu)K}\,. \qquad (29)$$

Substituting these values and (21) into (17–19) and re-evaluating the stress components from Tables 21.2, 22.2, we obtain

$$\begin{aligned}
\sigma_{RR} &= \frac{\mu\alpha(1+\nu)Qa\cos\beta}{(1-\nu)K}\left(\frac{a^2}{R^2} - \frac{a^4}{R^4}\right) \qquad (30) \\
\sigma_{\theta\theta} &= \frac{\mu\alpha(1+\nu)Qa\cos\beta}{2(1-\nu)K}\left(\frac{a^2}{R^2} + \frac{a^4}{R^4}\right) \qquad (31) \\
\sigma_{\beta\beta} &= \frac{\mu\alpha(1+\nu)Qa\cos\beta}{2(1-\nu)K}\left(\frac{a^2}{R^2} + \frac{a^4}{R^4}\right) \qquad (32) \\
\sigma_{R\beta} &= \frac{\mu\alpha(1+\nu)Qa\sin\beta}{2(1-\nu)K}\left(\frac{a^2}{R^2} - \frac{a^4}{R^4}\right) \qquad (33)
\end{aligned}$$

$$\sigma_{R\theta} = \sigma_{\theta\beta} = 0\,. \qquad (34)$$

We also record the temperature and the heat flux components which are

$$T = -\frac{Q\cos\beta}{K}\left(R + \frac{a^3}{2R^2}\right)\ ;\ \ q_R = Q\cos\beta\left(1 - \frac{a^3}{R^3}\right)\ ;\ \ q_\beta = -Q\sin\beta\left(1 + \frac{a^3}{2R^3}\right)\,. \qquad (35)$$

26.7. A state of uniaxial tension, $\sigma_{zz} = S$ in a large body (the matrix) of elastic properties μ_1, ν_1 is perturbed by the presence of a spherical inclusion in the region $0 \le R < a$ with elastic properties μ_2, ν_2. The inclusion is perfectly bonded to the matrix.

Develop a solution for the stress field and find the two stress concentration factors

$$K_1 = \frac{\sigma_1^{\max}}{S} \quad ; \quad K_2 = \frac{\sigma_2^{\max}}{S} \ ,$$

where $\sigma_1^{\max}, \sigma_2^{\max}$ are respectively the maximum tensile stresses in the matrix and in the inclusion.

Plot figures showing how K_1, K_2 vary with the modulus ratio μ_1/μ_2 and with the two Poisson's ratios.

Distant from the inclusion ($R \to \infty$), we must have

$$\sigma_{zz}^{(1)} \ \to \ S \tag{1}$$

$$\sigma_{rr}^{(1)}, \sigma_{\theta\theta}^{(1)}, \sigma_{r\theta}^{(1)}, \sigma_{\theta z}^{(1)}, \sigma_{zr}^{(1)} \ \to \ 0 \tag{2}$$

where the superscript $^{(1)}$ refers to the stress field in the matrix.

At the interface $R = a$ we require continuity of all displacement components

$$u_R^{(1)} = u_R^{(2)} \tag{3}$$

$$u_\theta^{(1)} = u_\theta^{(2)} \tag{4}$$

$$u_\beta^{(1)} = u_\beta^{(2)} \ , \tag{5}$$

where $^{(2)}$ refers to the inclusion.

Also, equilibrium considerations demand continuity in those traction components that act on the surface $R = a$

$$\sigma_{RR}^{(1)} = \sigma_{RR}^{(2)} \tag{6}$$

$$\sigma_{R\theta}^{(1)} = \sigma_{R\theta}^{(2)} \tag{7}$$

$$\sigma_{R\beta}^{(1)} = \sigma_{R\beta}^{(2)} \tag{8}$$

The unperturbed problem is clearly identical to that obtained in §26.1.2 and we anticipate the complete solution in the matrix being obtained from the potential functions (26.10, 26.11), with of course different values for the coefficients A_2, A_3, B_2 on the singular terms.

Thus,

$$\phi^{(1)} = A_1 R^2 P_2(\cos\beta) + A_2 R^{-3} P_2(\cos\beta) + A_3 R^{-1} P_0(\cos\beta) \tag{9}$$

$$\omega^{(1)} = B_1 R^1 P_1(\cos\beta) + B_2 R^{-2} P_1(\cos\beta) \ , \tag{10}$$

with

$$A_1 = \frac{S\nu}{(1+\nu)} \;\; ; \;\; B_1 = -\frac{S}{2(1+\nu)} \; . \tag{11}$$

Substituting in Table 21.2, we obtain the non-zero displacement and traction components as

$$\begin{aligned}
2\mu_1 u_R^{(1)} &= \frac{A_1 R[1+3\cos(2\beta)]}{2} - \frac{3A_2[1+3\cos(2\beta)]}{4R^4} - \frac{A_3}{R^2} \\
&\quad -B_1 R(1-2\nu_1)[1+\cos(2\beta)] - \frac{B_2(5-4\nu_1)[1+\cos(2\beta)]}{2R^2} && (12)\\
2\mu_1 u_\beta^{(1)} &= -\frac{3A_1 R\sin(2\beta)}{2} - \frac{3A_2\sin(2\beta)}{2R^4} + B_1 R(1-2\nu_1)\sin(2\beta) \\
&\quad + \frac{B_2(1-2\nu_1)\sin(2\beta)}{R^2} \; , && (13)\\
\sigma_{RR}^{(1)} &= \left(3A_1 + \frac{18A_2}{R^5} - 2(1-2\nu_1)B_1 + \frac{2(5-\nu_1)B_2}{R^3}\right)\cos^2\beta \\
&\quad -A_1 - \frac{6A_2}{R^5} + \frac{2A_3}{R^3} - 2\nu_1 B_1 - \frac{2\nu_1 B_2}{R^3} && (14)\\
\sigma_{R\beta}^{(1)} &= \left(-3A_1 + \frac{12A_2}{R^5} - 2(1-2\nu_1)B_1 - \frac{2(1+\nu_1)B_2}{R^3}\right)\sin\beta\cos\beta && (15)
\end{aligned}$$

where μ_1, ν_1 are the elastic properties of the matrix.

The stress field in the inclusion cannot be singular at the origin and hence can only include the bounded terms of the same form as (9, 10) — i.e.

$$\begin{aligned}
\phi^{(2)} &= A_4 R^2 P_2(\cos\beta) && (16)\\
\omega^{(2)} &= B_3 R^1 P_1(\cos\beta) \; , && (17)
\end{aligned}$$

giving

$$\begin{aligned}
2\mu_2 u_R^{(2)} &= \frac{A_4 R[1+3\cos(2\beta)]}{2} - B_3 R(1-2\nu_2)[1+\cos(2\beta)] && (18)\\
2\mu_2 u_\beta^{(2)} &= -\frac{3A_4 R\sin(2\beta)}{2} + B_3 R(1-2\nu_2)\sin(2\beta) && (19)\\
\sigma_{RR}^{(2)} &= (3A_4 - 2(1-2\nu_2)B_3)\cos^2\beta - A_4 - 2\nu_2 B_3 && (20)\\
\sigma_{R\beta}^{(2)} &= (-3A_4 - 2(1-2\nu_2)B_3)\sin\beta\cos\beta && (21)
\end{aligned}$$

where μ_2, ν_2 are the elastic properties of the matrix.

Substituting (12–15, 18–21) into the conditions (3–8) and separating coefficients of similar functions of β, we obtain the algebraic equations

$$\frac{3A_1 a}{2\mu_1} - \frac{9A_2}{4\mu_1 a^4} - \frac{B_1 a(1-2\nu_1)}{\mu_1} - \frac{B_2(5-4\nu_1)}{2\mu_1 a^2} - \frac{3A_4 a}{2\mu_2} + \frac{B_3 a(1-2\nu_2)}{\mu_2} = 0 \tag{22}$$

$$\frac{A_1 a}{2\mu_1} - \frac{3A_2}{4\mu_1 a^4} - \frac{A_3}{\mu_1 a^2} - \frac{B_1 a(1-2\nu_1)}{\mu_1} - \frac{B_2(5-4\nu_1)}{2\mu_1 a^2} - \frac{A_4 a}{2\mu_2} + \frac{B_3 a(1-2\nu_2)}{\mu_2} = 0 \quad (23)$$

$$-\frac{3A_1 a}{2\mu_1} - \frac{3A_2}{2\mu_1 a^4} + \frac{B_1 a(1-2\nu_1)}{\mu_1} + \frac{B_2(1-2\nu_1)}{\mu_1 a^2} + \frac{3A_4 a}{2\mu_2} - \frac{B_3 a(1-2\nu_2)}{\mu_2} = 0 \quad (24)$$

$$3A_1 + \frac{18A_2}{a^5} - 2(1-2\nu_1)B_1 + \frac{2(5-\nu_1)B_2}{a^3} - 3A_4 + 2(1-2\nu_2)B_3 = 0 \quad (25)$$

$$-A_1 - \frac{6A_2}{a^5} + \frac{2A_3}{a^3} - 2\nu_1 B_1 - \frac{2\nu_1 B_2}{a^3} + A_4 + 2\nu_2 B_3 = 0 \quad (26)$$

$$-3A_1 + \frac{12A_2}{R^5} - 2(1-2\nu_1)B_1 - \frac{2(1+\nu_1)B_2}{R^3} + 3A_4 + 2(1-2\nu_2)B_3 = 0 \quad (27)$$

From this point, the solution is routine but algebraically complicated and is only really practicable in Maple or Mathematica. We solve equations (22–27) for A_2, A_3, A_4, B_2, B_3 and use the results and (11) to determine the stress field throughout the body.

The stress in the inclusion is uniform and is also equal to σ_{RR} at the pole $(a, 0)$ in the matrix in view of condition (6).The maximum stress is either this common value or $\sigma_{\beta\beta}$ at the equator $(a, \pi/2)$in the matrix.

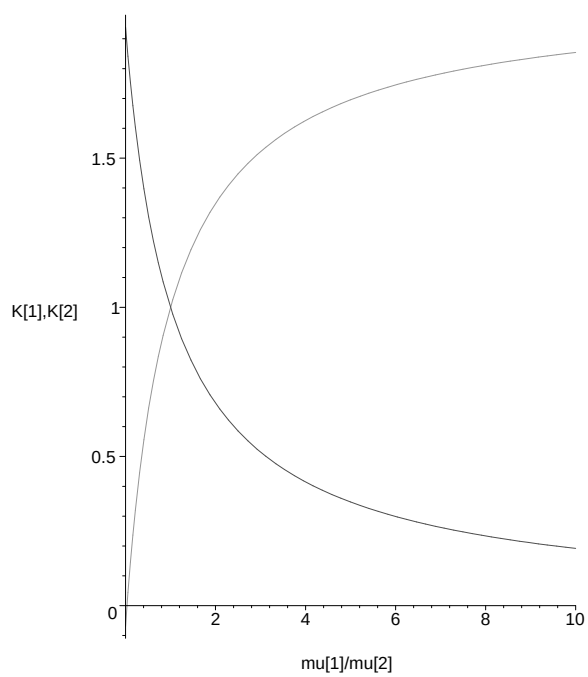


Figure 1

Figure 1 shows the variation of these two stresses with the modulus ratio μ_1/μ_2 for the special case $\nu_1 = \nu_2 = 0.3$. The two limits in this plot correspond to the case of a rigid inclusion ($\mu_1/\mu_2 \to 0$) and a spherical hole $\mu_1/\mu_2 \to \infty$. In the former case, the maximum stress occurs in the inclusion and at the poles in the matrix, whereas

in the latter it occurs of course in the matrix. Maximum stresses for intermediate values of modulus ratio interpolate between these extremes.

For this case, both stress concentration factors are equal to unity when the moduli are the same, since $\nu_1 = \nu_2$ and hence the material is homogeneous. Figure 2 shows the stress concentratio factors for a case in which the moduli are still equal ($\mu_1 = \mu_2$), but the Poisson's ratios are different. We keep $\nu_1 = 0$ and vary ν_2. The maximum stress occurs in the inclusion when $\nu_2 = 0.5$ and is equal to $5S/3$.

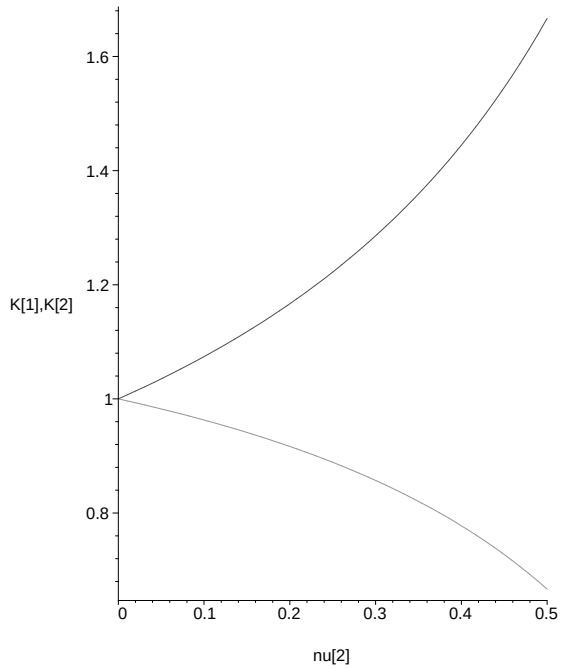


Figure 2

26.8. A sphere of radius a and density ρ rests on a rigid plane horizontal surface. Assuming that the support reaction consists of a concentrated vertical force equal to the weight of the sphere, find the complete stress field in the sphere.

Hint: Start with the solution for a point force acting on the surface of a half-space (the Boussinesq solution of §23.2.1) and determine the tractions on an imaginary spherical surface passing through the point of application of the force. Comparison with the corresponding cylindrical problems 12.1, 12.2, 12.3, would lead one to expect that these tractions could be removed by a finite series of spherical harmonics, but this is not the case. In fact, the tractions are still weakly (logarithmically) singular, though the dominant $(1/R)$ singularity associated with the point force has been removed. A truncated series of spherical harmonics will give an approximate solution, but the series will be rather slowly convergent. A more rapidly convergent solution can be obtained by first removing the logarithmic singularity.

The Boussinesq solution is defined by the function

$$\varphi = -\frac{F}{2\pi}\ln(R+z) \tag{1}$$

of equation (23.16), in Solution F of Table 21.3. This in turn can be expressed in terms of Solutions A and B, using equations (21.37). We obtain

$$\phi_0 = (1-2\nu)\varphi = -\frac{F(1-2\nu)}{2\pi}\ln(R+z) \tag{2}$$

$$\omega_0 = \frac{\partial\varphi}{\partial z} = -\frac{F}{2\pi R}\,. \tag{3}$$

These can be written in spherical polar coordinates *centred on the point of application of the force* as

$$\phi_0 = -\frac{F(1-2\nu)}{2\pi}\ln(R+R\cos\beta) \tag{4}$$

$$\omega_0 = -\frac{F}{2\pi R} \tag{5}$$

and the corresponding non-zero stress components are obtained by substitution into Table 21.2 as

$$\sigma_{RR} = \frac{F[1-2\nu-2(2-\nu)\cos\beta]}{2\pi R^2} \tag{6}$$

$$\sigma_{\theta\theta} = -\frac{F(1-2\nu)(1-\cos\beta-\cos^2\beta)}{2\pi R^2(1+\cos\beta)} \tag{7}$$

$$\sigma_{\beta\beta} = \frac{F(1-2\nu)\cos^2\beta}{2\pi R^2(1+\cos\beta)} \tag{8}$$

$$\sigma_{\beta R} = \frac{F(1-2\nu)\sin\beta\cos\beta}{2\pi R^2(1+\cos\beta)} \ . \tag{9}$$

Points on the surface of the sphere are defined by the equation

$$R = 2a\cos\beta \ , \tag{10}$$

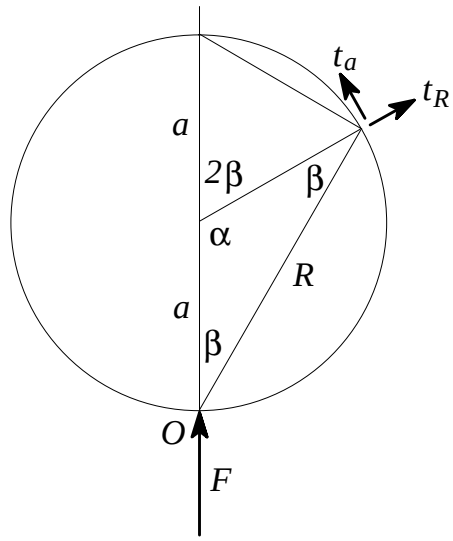


Figure 1

as shown in Figure 1. At this point we have

$$\sigma_{RR} = \frac{F[1-2\nu-2(2-\nu)\cos\beta]}{8\pi a^2\cos^2\beta} \tag{11}$$

$$\sigma_{\theta\theta} = -\frac{F(1-2\nu)(1-\cos\beta-\cos^2\beta)}{8\pi a^2\cos^2\beta(1+\cos\beta)} \tag{12}$$

$$\sigma_{\beta\beta} = \frac{F(1-2\nu)}{8\pi a^2(1+\cos\beta)} \tag{13}$$

$$\sigma_{\beta R} = \frac{F(1-2\nu)\sin\beta}{8\pi a^2\cos\beta(1+\cos\beta)} \ . \tag{14}$$

However, to find the implied tractions on the spherical surface, we need to rotate the local coordinate system *clockwise* through an angle β, obtaining the radial and tangential tractions

$$t_R = \sigma_{RR}\cos^2\beta + \sigma_{\beta\beta}\sin^2\beta + 2(-\sigma_{\beta R})(-\sin\beta)\cos\beta \tag{15}$$

$$t_\alpha = (\sigma_{\beta\beta}-\sigma_{RR})(-\sin\beta)\cos\beta + (-\sigma_{\beta R})(\cos^2\beta-\sin^2\beta) \ , \tag{16}$$

from (1.16, 1.17). In these equations, notice that the sign convention for a Cartesian coordinate system (x, y) aligned with (R, β) at the surface would involve a negative shear stress ($\sigma_{xy} = -\sigma_{\beta R}$) and the clockwise rotation introduces a negative sign into the terms involving $\sin\beta$. The direction of the shear traction t_α is chosen so as to be consistent with the angle α subtended at the centre of the sphere, defined such that $\alpha = 0$ corresponds to the point of application of the force. From Figure 1, we then have

$$2\beta = \pi - \alpha \ . \tag{17}$$

Substituting from (11–14) into (15, 16) and simplifying, we obtain

$$t_R = \frac{F[4(1-2\nu) - (7-8\nu)\cos\beta]}{8\pi a^2} \tag{18}$$

$$t_\alpha = \frac{F[2(1-2\nu) - 3\cos\beta - (7-8\nu)\cos^2\beta]\sin\beta}{8\pi a^2(1+\cos\beta)\cos\beta} \ , \tag{19}$$

or in terms of α,

$$t_R = \frac{F[4(1-2\nu) - (7-8\nu)\sin(\alpha/2)]}{8\pi a^2} \tag{20}$$

$$t_\alpha = -\frac{F[3 + 6\sin(\alpha/2) - (7-8\nu)\cos\alpha]\cos(\alpha/2)}{16\pi a^2(1+\sin(\alpha/2))\sin(\alpha/2)} \ , \tag{21}$$

These expressions are not of Fourier form in the angle α and hence the process of adding terms to satisfy the traction boundary conditions on the sphere will be more complex than in the two-dimensional case. More seriously, the traction t_α is singular as $\alpha \to 0$ — i.e. at the point of application of the force. This result was first remarked by Sternberg and Rosenthal[5]. It might still be possible to satisfy the boundary conditions using an infinite series of spherical harmonics, but the series would probably be only very slowly convergent because of the singularity.

To prevent this problem and improve the convergence of the series, we need to superpose additional potentials chosen so as to cancel the singularity. As a preliminary to this process, we can expand (21) near $\alpha = 0$ by writing

$$\sin\alpha \approx \alpha \ ; \quad \sin(\alpha/2) \approx \frac{\alpha}{2} \ ; \quad \cos\alpha \approx 1 - \frac{\alpha^2}{2} \ ; \quad \cos(\alpha/2) \approx \frac{1-\alpha^2}{8} \ , \tag{22}$$

obtaining

$$t_\alpha \approx \frac{F(1-2\nu)}{2\pi a^2 \alpha} - \frac{F(5-4\nu)}{8\pi a^2} + O(\alpha) \tag{23}$$

as $\alpha \to 0$.

[5]E.Sternberg and F.Rosenthal, The elastic sphere under concentrated loads, *ASME J.Applied Mechanics,* Vol. 19, (1952), pp. 413-423. See also I.Guerrero and M.J.Turtletaub, The elastic sphere under arbitrary concentrated surface loads, *J.Elasticity,* Vol. 2 (1972), pp. 21-33.

To choose a suitable potential to cancel the first term, notice that the surface is very nearly plane when $\alpha \to 0$, so we can look for potentials giving shear tractions on the surface of the half plane with this singular form. This may seem curious, since the original Boussinesq solution gave identically zero tractions on this plane, but we note that the singular tractions are one order lower than the singularity associated with the point force, which is order R^{-2}. We therefore choose additional potentials from (23.22) that are one order less singular than those in (4, 5). Notice that these will introduce singular tractions in both t_R and t_α and the former is undesirable, so we use the same combination as implied in equation (21.41) to make the dominant singular term in t_R be zero.

We therefore choose

$$\phi_1 = 2(1-\nu)CR[\cos\beta \ln(R + R\cos\beta) - 1] \tag{24}$$

$$\omega_1 = C\ln(R + R\cos\beta) \tag{25}$$

Substituting into Table 21.2 and following the same procedure as for the Boussinesq solution above, we find the tractions due to this potential as $\alpha \to 0$ to be

$$t_R \approx -\frac{C}{2a} + O(\alpha) \tag{26}$$

$$t_\alpha \approx -\frac{C}{a\alpha} + \frac{C(1-2\nu)}{2a} + O(\alpha)\ . \tag{27}$$

Superposing this on the original stress function, it is clear that we can cancel the unwanted singularity by choosing

$$\frac{F(1-2\nu)}{2\pi a^2} - \frac{C}{a} = 0\ , \tag{28}$$

or

$$C = \frac{F(1-2\nu)}{2\pi a}\ . \tag{29}$$

With this choice, the tractions everywhere on the surface of the sphere are bounded, but we notice from equation (27) that the traction t_α will be non-zero at $\alpha = 0$. Now it is easily verified that all the axisymmetric spherical harmonics give zero values of shear traction $\sigma_{R\beta}$ on the axis $\beta = 0$ and this is because a non zero value would involve a local loading looking like the 'cowlick' of Figure 2. In other words, although the magnitude of the traction is continuous, it's direction changes discontinuously at the origin. This is itself a kind of singularity. In fact, by analogy with Problem 11.1, we might expect that some of the stress components associated with this loading would have logarithmic singularities at the origin.

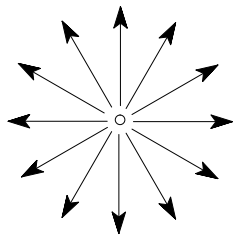


Figure 2: Cowlick of non-zero shear tractions at the pole.

This additional (weaker) singularity can also be removed by superposing the next higher order potentials, once again chosen from (23.22) so as to satisfy (21.41) — i.e.

$$\phi_2 = \frac{(1-\nu)AR^2}{2}[(2\cos^2\beta - \sin^2\beta)\cos\beta\ln(R + R\cos\beta) - 3\cos\beta + \sin^2\beta] \quad (30)$$

$$\omega_2 = AR[\cos\beta\ln(R + R\cos\beta) - 1] \ . \quad (31)$$

Substituting into Table 21.2 and following the same asymptotic procedure as before, we find the tractions due to the superposition of all the above potentials at $\alpha \to 0$ to be

$$t_R \approx -\frac{F(1-2\nu)}{2\pi a^2} + O(\alpha) \quad (32)$$

$$t_\alpha \approx -\frac{F(1+12\nu-16\nu^2)}{8\pi a^2} + A + O(\alpha) \ . \quad (33)$$

and hence we can eliminate the cowlick by choosing

$$A = \frac{F(1+12\nu-16\nu^2)}{8\pi a^2} \ . \quad (34)$$

Thus, a suitably smooth form of the point force solution for the sphere is provided by the potentials

$$\begin{aligned}\phi &= \phi_0 + \phi_1 + \phi_2 \\ &= -\frac{F(1-2\nu)}{2\pi}\ln(R + R\cos\beta) + \frac{2F(1-2\nu)(1-\nu)R[\cos\beta\ln(R + R\cos\beta) - 1]}{2\pi a} \\ &\quad + \frac{F(1+12\nu-16\nu^2)(1-\nu)R^2}{16\pi a^2}[(2\cos^2\beta - \sin^2\beta)\cos\beta\ln(R + R\cos\beta) \\ &\quad -3\cos\beta + \sin^2\beta] \end{aligned} \quad (35)$$

$$\begin{aligned}\omega &= \omega_0 + \omega_1 + \omega_2 \\ &= -\frac{F}{2\pi R} + \frac{F(1-2\nu)\ln(R + R\cos\beta)}{2\pi a} \\ &\quad + \frac{F(1+12\nu-16\nu^2)R[\cos\beta\ln(R + R\cos\beta) - 1]}{8\pi a^2} \end{aligned} \quad (36)$$

To confirm that the tractions now remaining to be removed are smooth, we plot them in Figure 3 as functions of α.

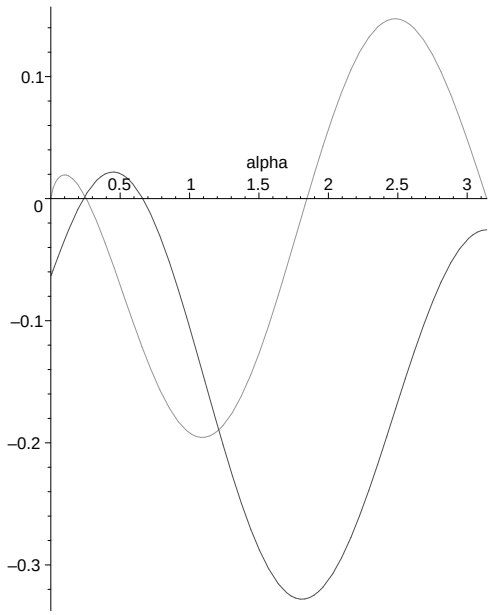


Figure 3: Tractions on the spherical surface associated with the stress functions of equations (35, 36). The curve passing through the origin is the shear traction t_α.

Gravitational loading

The body force due to self-weight is conveniently introduced as a hydrostatic stress

$$\sigma_{RR} = \sigma_{\theta\theta} = \sigma_{\alpha\alpha} = -\rho g \tilde{R} \cos\alpha \ , \tag{37}$$

where $\tilde{R}$ is here measured from the centre of the sphere. This adds an additional term

$$-\rho g a \cos\alpha$$

into the traction component t_R, whilst leaving t_α unchanged. Also, we note that the force F must support the weight of the sphere, so

$$F = \frac{4\pi\rho g a^3}{3} \ . \tag{38}$$

At this point in the Maple or Mathematica solution, it is a good idea to check to make sure that the remaining tractions on the surface of the sphere add up to a zero force[6]. This is done by evaluating the integral

$$2\pi a^2 \int_0^\pi (t_R \cos\alpha - t_\alpha \sin\alpha) \sin\alpha d\alpha \ .$$

If everything is correct so far, this should sum to zero.

[6]This would probably reveal for example a sign error in the direction of the tractions or of the gravitational component.

Spherical harmonics

To complete the solution, we superpose a series of spherical harmonics in Solutions A and B centred on the centre of the sphere. Thus, we add the new potentials

$$\phi_3 = \sum_{i=1}^{n} A_i \tilde{R}^{i+1} P_{i+1}(\cos\alpha)\omega = \sum_{i=1}^{n} B_i \tilde{R}^i P_i(\cos\alpha) \ , \tag{39}$$

substitute into Table 21.2 and then evaluate the additional tractions on the surface $R = a$.

Since we can only use a finite number of terms in the series, we can satisfy the traction-free boundary condition only in a weak sense. There are many ways that such weak conditions might be chosen, but the most convergent is to use a Galerkin or 'weighted residual' method. For example, if we write the approximate tractions in the form

$$\tilde{t}_R = t_R^P + \sum_{i=1}^{n} A_i t_{Ri}^A(\alpha) + \sum_{i=1}^{n} B_i t_{Ri}^B(\alpha) \tag{40}$$

$$\tilde{t}_\alpha = t_\alpha^P + \sum_{i=1}^{n} A_i t_{\alpha i}^A(\alpha) + \sum_{i=1}^{n} B_i t_{\alpha i}^B(\alpha) \ , \tag{41}$$

we can define an error measure

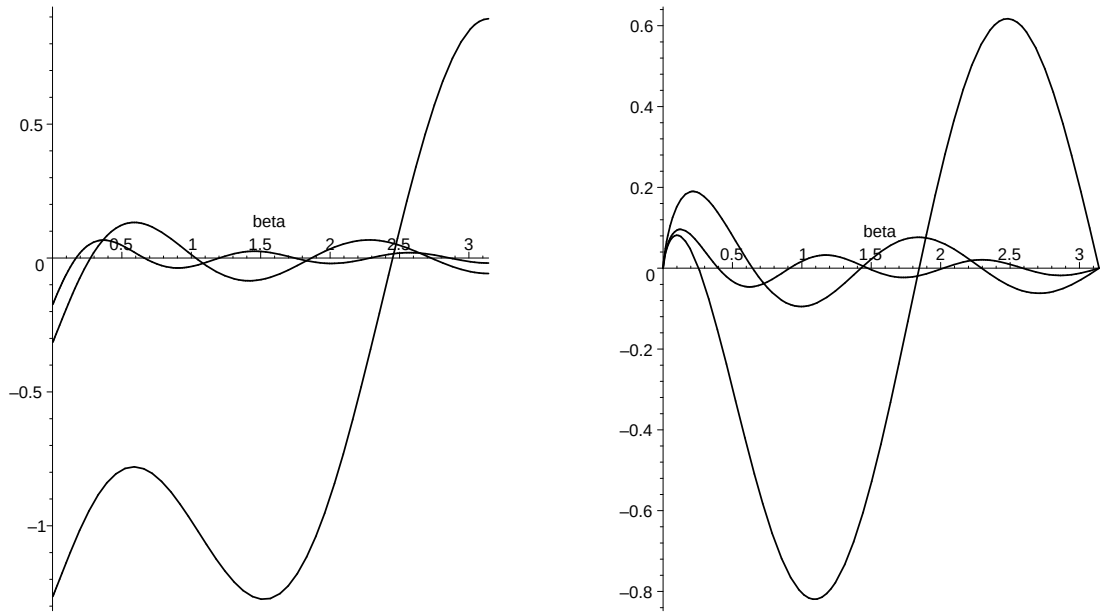
$$E = \int_0^\alpha (\tilde{t}_R^2 + \tilde{t}_\alpha^2) d\alpha \ . \tag{42}$$

An optimal choice of the constants A_i, B_i can then be made by requiring

$$\frac{\partial E}{\partial A_i} = 0 \ ; \quad \frac{\partial E}{\partial B_i} = 0 \ , \tag{43}$$

for $i = 1, n$. Notice that the error measure has been weighted uniformly in $0 < \alpha < \pi$. An alternative choice here would be to weight according to the volume of surface of the sphere, which would introduce a factor of $sin\alpha$ into the integral. This would probably give better accuracy near the equator (where there is more surface area) and less near the poles.

The effect of this process is to weight the traction-free condition according to the shape of the individual spherical harmonic traction fields. The process is only really practicable in Maple or Mathematica. In Figure 4 we present the self-equilibrated tractions t_R due to self weight and the concentrated force alone and with the approximations obtained using $n = 2$ and $n = 4$ respectively. The process is clearly convergent and as with Fourier series approximations, the error exhibits more zero crossings as the number of terms increases. Oscillations near $\alpha = 0$ (Gibb's phenomenon) are to be expected with large numbers of terms, but this effect has been to some extent mitigated by the removal of the stronger discontinuous effects in the above analysis.



(a) (b)

Figure 4: Tractions in the approximate solution (a) $\tilde{t}_R$ and (b) $\tilde{t}_\alpha$, normalized by $\rho g a$ for the case $\nu = 0.3$.

26.9. A sphere of radius a is loaded by two equal and opposite point forces F applied at the poles $\beta = 0, 2\pi$). Find the stress field in the sphere. (Read the 'hint' for Problem 26.8).

The Boussinesq solution is defined by the function

$$\varphi = -\frac{F}{2\pi}\ln(R+z) \tag{1}$$

of equation (23.16), in Solution F of Table 21.3. This in turn can be expressed in terms of Solutions A and B, using equations (21.37). We obtain

$$\phi_0 = (1-2\nu)\varphi = -\frac{F(1-2\nu)}{2\pi}\ln(R+z) \tag{2}$$

$$\omega_0 = \frac{\partial\varphi}{\partial z} = -\frac{F}{2\pi R}\ . \tag{3}$$

These can be written in spherical polar coordinates *centred on the point of application of the force* as

$$\phi_0 = -\frac{F(1-2\nu)}{2\pi}\ln(R+R\cos\beta) \tag{4}$$

$$\omega_0 = -\frac{F}{2\pi R} \tag{5}$$

and the corresponding non-zero stress components are obtained by substitution into Table 21.2 as

$$\sigma_{RR} = \frac{F[1-2\nu-2(2-\nu)\cos\beta]}{2\pi R^2} \tag{6}$$

$$\sigma_{\theta\theta} = -\frac{F(1-2\nu)(1-\cos\beta-\cos^2\beta)}{2\pi R^2(1+\cos\beta)} \tag{7}$$

$$\sigma_{\beta\beta} = \frac{F(1-2\nu)\cos^2\beta}{2\pi R^2(1+\cos\beta)} \tag{8}$$

$$\sigma_{\beta R} = \frac{F(1-2\nu)\sin\beta\cos\beta}{2\pi R^2(1+\cos\beta)}\ . \tag{9}$$

Points on the surface of the sphere are defined by the equation

$$R = 2a\cos\beta\ , \tag{10}$$

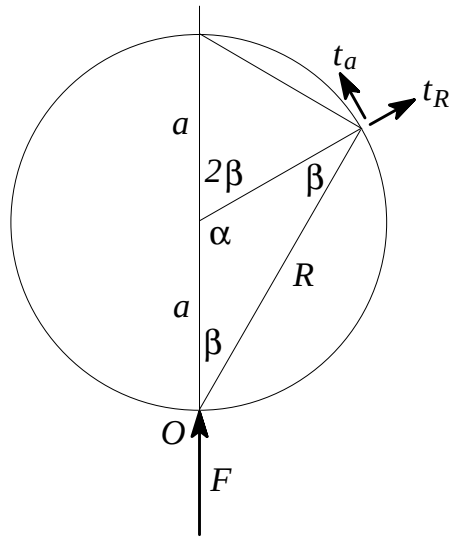


Figure 1

as shown in Figure 1. At this point we have

$$\sigma_{RR} = \frac{F[1-2\nu-2(2-\nu)\cos\beta]}{8\pi a^2\cos^2\beta} \tag{11}$$

$$\sigma_{\theta\theta} = -\frac{F(1-2\nu)(1-\cos\beta-\cos^2\beta)}{8\pi a^2\cos^2\beta(1+\cos\beta)} \tag{12}$$

$$\sigma_{\beta\beta} = \frac{F(1-2\nu)}{8\pi a^2(1+\cos\beta)} \tag{13}$$

$$\sigma_{\beta R} = \frac{F(1-2\nu)\sin\beta}{8\pi a^2\cos\beta(1+\cos\beta)} \ . \tag{14}$$

However, to find the implied tractions on the spherical surface, we need to rotate the local coordinate system *clockwise* through an angle β, obtaining the radial and tangential tractions

$$t_R = \sigma_{RR}\cos^2\beta + \sigma_{\beta\beta}\sin^2\beta + 2(-\sigma_{\beta R})(-\sin\beta)\cos\beta \tag{15}$$

$$t_\alpha = (\sigma_{\beta\beta}-\sigma_{RR})(-\sin\beta)\cos\beta + (-\sigma_{\beta R})(\cos^2\beta-\sin^2\beta) \ , \tag{16}$$

from (1.16, 1.17). In these equations, notice that the sign convention for a Cartesian coordinate system (x, y) aligned with (R, β) at the surface would involve a negative shear stress ($\sigma_{xy} = -\sigma_{\beta R}$) and the clockwise rotation introduces a negative sign into the terms involving $\sin\beta$. The direction of the shear traction t_α is chosen so as to be consistent with the angle α subtended at the centre of the sphere, defined such that $\alpha = 0$ corresponds to the point of application of the force. From Figure 1, we then have

$$2\beta = \pi - \alpha \ . \tag{17}$$

Substituting from (11–14) into (15, 16) and simplifying, we obtain

$$t_R = \frac{F[4(1-2\nu)-(7-8\nu)\cos\beta]}{8\pi a^2} \tag{18}$$

$$t_\alpha = \frac{F[2(1-2\nu)-3\cos\beta-(7-8\nu)\cos^2\beta]\sin\beta}{8\pi a^2(1+\cos\beta)\cos\beta}\ , \tag{19}$$

or in terms of α,

$$t_R = \frac{F[4(1-2\nu)-(7-8\nu)\sin(\alpha/2)]}{8\pi a^2} \tag{20}$$

$$t_\alpha = -\frac{F[3+6\sin(\alpha/2)-(7-8\nu)\cos\alpha]\cos(\alpha/2)}{16\pi a^2(1+\sin(\alpha/2))\sin(\alpha/2)}\ , \tag{21}$$

These expressions are not of Fourier form in the angle α and hence the process of adding terms to satisfy the traction boundary conditions on the sphere will be more complex than in the two-dimensional case. More seriously, the traction t_α is singular as $\alpha \to 0$ — i.e. at the point of application of the force. This result was first remarked by Sternberg and Rosenthal[7]. It might still be possible to satisfy the boundary conditions using an infinite series of spherical harmonics, but the series would probably be only very slowly convergent because of the singularity.

To prevent this problem and improve the convergence of the series, we need to superpose additional potentials chosen so as to cancel the singularity. As a preliminary to this process, we can expand (21) near $\alpha = 0$ by writing

$$\sin\alpha \approx \alpha\ ;\quad \sin(\alpha/2) \approx \frac{\alpha}{2}\ ;\quad \cos\alpha \approx 1-\frac{\alpha^2}{2}\ ;\quad \cos(\alpha/2) \approx \frac{1-\alpha^2}{8}\ , \tag{22}$$

obtaining

$$t_\alpha \approx \frac{F(1-2\nu)}{2\pi a^2\alpha} - \frac{F(5-4\nu)}{8\pi a^2} + O(\alpha) \tag{23}$$

as $\alpha \to 0$.

To choose a suitable potential to cancel the first term, notice that the surface is very nearly plane when $\alpha \to 0$, so we can look for potentials giving shear tractions on the surface of the half plane with this singular form. This may seem curious, since the original Boussinesq solution gave identically zero tractions on this plane, but we note that the singular tractions are one order lower than the singularity associated with the point force, which is order R^{-2}. We therefore choose additional potentials from (23.22) that are one order less singular than those in (4, 5). Notice that these will introduce singular tractions in both t_R and t_α and the former is undesirable, so we use the same combination as implied in equation (21.41) to make the dominant singular term in t_R be zero.

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We therefore choose

$$\phi_1 = 2(1-\nu)CR[\cos\beta\ln(R+R\cos\beta)-1] \tag{24}$$
$$\omega_1 = C\ln(R+R\cos\beta) \tag{25}$$

Substituting into Table 21.2 and following the same procedure as for the Boussinesq solution above, we find the tractions due to this potential as $\alpha \to 0$ to be

$$t_R \approx -\frac{C}{2a} + O(\alpha) \tag{26}$$
$$t_\alpha \approx -\frac{C}{a\alpha} + \frac{C(1-2\nu)}{2a} + O(\alpha)\ . \tag{27}$$

Superposing this on the original stress function, it is clear that we can cancel the unwanted singularity by choosing

$$\frac{F(1-2\nu)}{2\pi a^2} - \frac{C}{a} = 0\ , \tag{28}$$

or

$$C = \frac{F(1-2\nu)}{2\pi a}\ . \tag{29}$$

With this choice, the tractions everywhere on the surface of the sphere are bounded, but we notice from equation (27) that the traction t_α will be non-zero at $\alpha = 0$. Now it is easily verified that all the axisymmetric spherical harmonics give zero values of shear traction $\sigma_{R\beta}$ on the axis $\beta = 0$ and this is because a non zero value would involve a local loading looking like the 'cowlick' of Figure 2. In other words, although the magnitude of the traction is continuous, it's direction changes discontinuously at the origin. This is itself a kind of singularity. In fact, by analogy with Problem 11.1, we might expect that some of the stress components associated with this loading would have logarithmic singularities at the origin.

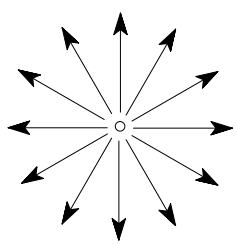


Figure 2: Cowlick of non-zero shear tractions at the pole.

This additional (weaker) singularity can also be removed by superposing the next higher order potentials, once again chosen from (23.22) so as to satisfy (21.41) — i.e.

$$\phi_2 = \frac{(1-\nu)AR^2}{2}[(2\cos^2\beta - \sin^2\beta)\cos\beta\ln(R+R\cos\beta) - 3\cos\beta + \sin^2\beta] \tag{30}$$
$$\omega_2 = AR[\cos\beta\ln(R+R\cos\beta)-1]\ . \tag{31}$$

Substituting into Table 21.2 and following the same asymptotic procedure as before, we find the tractions due to the superposition of all the above potentials at $\alpha \to 0$ to be

$$t_R \approx -\frac{F(1-2\nu)}{2\pi a^2} + O(\alpha) \tag{32}$$

$$t_\alpha \approx -\frac{F(1+12\nu-16\nu^2)}{8\pi a^2} + A + O(\alpha)\ . \tag{33}$$

and hence we can eliminate the cowlick by choosing

$$A = \frac{F(1+12\nu-16\nu^2)}{8\pi a^2}\ . \tag{34}$$

Thus, a suitably smooth form of the point force solution for the sphere is provided by the potentials

$$\begin{aligned}\phi &= \phi_0 + \phi_1 + \phi_2 \\ &= -\frac{F(1-2\nu)}{2\pi}\ln(R + R\cos\beta) + \frac{2F(1-2\nu)(1-\nu)R[\cos\beta\ln(R+R\cos\beta)-1]}{2\pi a} \\ &\quad + \frac{F(1+12\nu-16\nu^2)(1-\nu)R^2}{16\pi a^2}[(2\cos^2\beta - \sin^2\beta)\cos\beta\ln(R+R\cos\beta) \\ &\quad -3\cos\beta + \sin^2\beta]\end{aligned} \tag{35}$$

$$\begin{aligned}\omega &= \omega_0 + \omega_1 + \omega_2 \\ &= -\frac{F}{2\pi R} + \frac{F(1-2\nu)\ln(R+R\cos\beta)}{2\pi a} \\ &\quad + \frac{F(1+12\nu-16\nu^2)R[\cos\beta\ln(R+R\cos\beta)-1]}{8\pi a^2}\end{aligned} \tag{36}$$

Superposition of two forces

The solution so far describes the effect of a single concentrated force at $\alpha = 0$, as shown in Figure 1. If we denote the tractions from the potentials (35, 36) by $\bar{t}_R(\alpha), \bar{t}_\alpha(\alpha)$, the effect of the additional force at $\alpha = \pi$ can be obtained by replacing α by $\pi - \alpha$ and changing the sign of t_α. Thus, the superposition of two forces leads to the tractions

$$t_R = \bar{t}_R(\alpha) + \bar{t}_R(\pi-\alpha)\ ; \quad t_\alpha = \bar{t}_\alpha(\alpha) - \bar{t}_\alpha(\pi-\alpha)\ . \tag{37}$$

This traction field is symmetric with respect to $\alpha = \pi/2$, as we should expect from the symmetry of the problem.

Spherical harmonics

To complete the solution, we superpose a series of spherical harmonics in Solutions A and B centred on the centre of the sphere. Thus, we add the new potentials

$$\phi_3 = \sum_{i=1}^{n} A_i \tilde{R}^{2i} P_{2i}(\cos\alpha)\omega = \sum_{i=1}^{n} B_i \tilde{R}^{2i-1} P_{2i-1}(\cos\alpha) \ , \tag{38}$$

substitute into Table 21.2 and then evaluate the additional tractions on the surface $R = a$. Notice that we have selected only even terms in ϕ and odd terms in ω in view of the symmetry of the problem.

Since we can only use a finite number of terms in the series, we can satisfy the traction-free boundary condition only in a weak sense. There are many ways that such weak conditions might be chosen, but the most convergent is to use a Galerkin or 'weighted residual' method. For example, if we write the approximate tractions in the form

$$\tilde{t}_R = t_R^P + \sum_{i=1}^{n} A_i t_{Ri}^A(\alpha) + \sum_{i=1}^{n} B_i t_{Ri}^B(\alpha) \tag{39}$$

$$\tilde{t}_\alpha = t_\alpha^P + \sum_{i=1}^{n} A_i t_{\alpha i}^A(\alpha) + \sum_{i=1}^{n} B_i t_{\alpha i}^B(\alpha) \ , \tag{40}$$

we can define an error measure

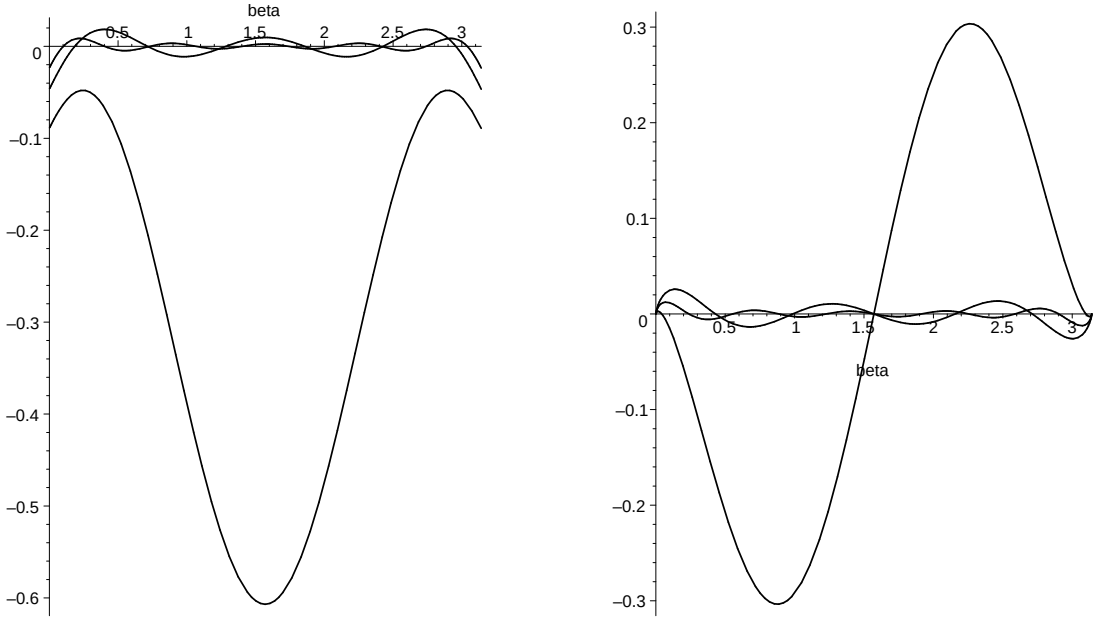
$$E = \int_0^\alpha (\tilde{t}_R^2 + \tilde{t}_\alpha^2) d\alpha \ . \tag{41}$$

An optimal choice of the constants A_i, B_i can then be made by requiring

$$\frac{\partial E}{\partial A_i} = 0 \ ; \quad \frac{\partial E}{\partial B_i} = 0 \ , \tag{42}$$

for $i = 1, n$. Notice that the error measure has been weighted uniformly in $0 < \alpha < \pi$. An alternative choice here would be to weight according to the volume of surface of the sphere, which would introduce a factor of $\sin\alpha$ into the integral. This would probably give better accuracy near the equator (where there is more surface area) and less near the poles.

The effect of this process is to weight the traction-free condition according to the shape of the individual spherical harmonic traction fields. The process is only really practicable in Maple or Mathematica. In Figure 3 we present the self-equilibrated tractions t_R, t_α due to the concentrated forces alone and with the approximations obtained using $n = 2$ and $n = 4$ respectively. The process is clearly convergent and as with Fourier series approximations, the error exhibits more zero crossings as the number of terms increases. Oscillations near $\alpha = 0$ (Gibb's phenomenon) are to be expected with large numbers of terms, but this effect has been to some extent mitigated by the removal of the stronger discontinuous effects in the above analysis.



(a) (b)

Figure 3: Tractions in the approximate solution (a) $\tilde{t}_R$ and (b) $\tilde{t}_\alpha$, normalized by F/a^2 for the case $\nu = 0.3$.

26.10. A cylindrical bar of radius b transmits a torque T. It also contains a small spherical hole of radius a ($\ll b$) on the axis. Find the perturbation in the elementary torsional stress field due to the hole.

The elementary torsion theory predicts that

$$\sigma_{r\theta} = 0 \; ; \quad \sigma_{z\theta} = \frac{Tr}{J} \; , \tag{1}$$

where J is the polar second moment of area. In order to get stresses varying with r in Solution E, we shall need a third order polynomial, so we take

$$\Psi = \frac{C_1}{2}(2z^3 - 3zr^2) + \frac{C_2}{2}(2z^2 - r^2) + C_3 z \; , \tag{2}$$

from equations (24.27). Susbtituting into Solution E (Table 21.1), we obtain

$$\sigma_{z\theta} = 3C_1 r \; , \tag{3}$$

all the remaining stress components being zero. The constant C_1 can be determined by imposing the condition

$$\int_0^{2\pi} \int_0^b \sigma_{z\theta} r^2 dr = T \; , \tag{4}$$

giving

$$C_1 = \frac{2T}{3\pi b^4} \; . \tag{5}$$

The remaining potentials in equation (2) give no stresses and can be omitted. Thus, the elementary torsion problem is solved by the potential

$$\Psi = \frac{T}{3\pi b^4}(2z^3 - 3zr^2) \tag{6}$$

in Solution E.

We now convert to spherical polar coordinates by using the equivalent form from equations (24.11)

$$\Psi = \frac{C_1 R^3}{8}(5\cos(3\beta) + 3\cos\beta) = \frac{TR^3}{12\pi b^4}(5\cos(3\beta) + 3\cos\beta) \; . \tag{7}$$

The corresponding stress components are then obtained from Table 21.2 as

$$\sigma_{R\theta} = \frac{TR\sin(2\beta)}{\pi b^4} \tag{8}$$

$$\sigma_{\beta\theta} = \frac{TR[\cos(2\beta) - 1]}{\pi b^4} \; . \tag{9}$$

The singular potential of similar form in β is $R^n P_n(\cos\beta)$ with order $n = -3-1 = -4$, but substitution of this potential into Solution E yields stress components that

vary with $\sin(4\beta), \cos(4\beta)$. We therefore take singular terms that start two orders lower[8] ($n = -2$), writing

$$\Psi = \frac{TR^3}{12\pi b^4} + B_1 R^{-2} \cos\beta + B_2 R^{-1} \ . \tag{10}$$

The corresponding stress components are

$$\sigma_{R\theta} = \frac{TR\sin(2\beta)}{\pi b^4} - \frac{6B_1 \sin(2\beta)}{R^4} - \frac{3B_2 \sin\beta}{R^3} \tag{11}$$

$$\sigma_{\beta\theta} = \frac{TR[\cos(2\beta) - 1]}{\pi b^4} + \frac{3B_1[\cos(2\beta) - 1]}{2R^4} \ . \tag{12}$$

The non-zero traction on the surface of the hole $R = a$ is therefore

$$\sigma_{R\theta} = \frac{Ta\sin(2\beta)}{\pi b^4} - \frac{6B_1 \sin(2\beta)}{a^4} - \frac{3B_2 \sin\beta}{a^3} \tag{13}$$

and this will be zero for all β if and only if

$$\frac{Ta}{\pi b^4} - \frac{6B_1}{a^4} = 0 \ ; \quad \frac{3B_2}{a^3} = 0 \ . \tag{14}$$

In other words,

$$B_1 = \frac{Ta^5}{6\pi b^4} \ ; \quad B_2 = 0 \ . \tag{15}$$

The complete stress function is therefore

$$\Psi = \frac{TR^3}{12\pi b^4}(5\cos(3\beta) + 3\cos\beta) + \frac{Ta^5}{6\pi b^4 R^2}\cos\beta \tag{16}$$

and the complete stress field is

$$\sigma_{R\theta} = \frac{Ta}{\pi b^4}\left(\frac{R}{a} - \frac{a^4}{R^4}\right)\sin(2\beta) \tag{17}$$

$$\sigma_{\beta\theta} = -\frac{Ta}{\pi b^4}\left(\frac{R}{a} + \frac{a^4}{4R^4}\right)[1 - \cos(2\beta)] \ . \tag{18}$$

[8]This is a consequence of the same degeneracy that causes the low order polynomials in Ψ to give no stresses. It would be reasonable to use the full sequence of singular potentials in a Maple or Mathematica solution, starting with $S3$ and then satisfy traction-free conditions for each of the Fourier terms obtained in the tractions on $R = a$. This would lead to the conclusion that the multipliers on the two highest order terms are zero and hence to the same final solution. This approach is not used here, principally in the interests of brevity.

26.11. A vertical conical tower, $0 < \beta < \beta_0$ of height h and density ρ is loaded only by its own weight. Find the stress field in the tower.

Taking the axis of the tower to be the positive z-axis, the gravitational loading defines a body force

$$p_x = p_y = 0 \; ; \quad p_z = \rho g \; , \tag{1}$$

which can be defined in terms of the body-force potential[9]

$$V = -\rho g z = \rho g R \cos\beta \; . \tag{2}$$

The function ϕ of §20.5.1 must therefore satisfy the equation

$$\nabla^2 \phi = -\frac{(1-2\nu)\rho g z}{(1-\nu)} \; , \tag{3}$$

which has the particular solution

$$\phi = -\frac{(1-2\nu)\rho g z^3}{6(1-\nu)} = -\frac{(1-2\nu)\rho g R^3 \cos^3\beta}{6(1-\nu)} \; . \tag{4}$$

We notice that equations (20.41, 20.42) are identical to those of Solution A except for the additional term $\nu V/(1-\nu)$ in the normal stresses. Thus, we can include the particular solution by (i) adding the non-harmonic term (4) into ϕ of Solution A and (ii) adding $\nu V/(1-\nu)$ into each of the normal stresses.

The particular solution leads us to expect terms in R^3 in Solution A and hence terms in R^2 in Solution B. We therefore define

$$\phi = \frac{AR^3}{8}(5\cos(3\beta) + 3\cos\beta) - \frac{(1-2\nu)\rho g R^3 \cos^3\beta}{6(1-\nu)} \tag{5}$$

$$\omega = \frac{BR^2}{2}(3\cos(2\beta) + 1) \; . \tag{6}$$

The corresponding stress components (after adding the term $\nu V/(1-\nu)$ to the normal stresses) are

$$\sigma_{RR} = R\cos\beta \Big\{ 3A(5\cos^2\beta - 3) + B[1 - 8\nu - 3(1-4\nu)\cos^2\beta]$$

[9] A simpler particular solution for the body force is the state of hydrostatic compression

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -\rho g z \; ; \quad \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \; .$$

This clearly satisfies the equilibrium equations and the resulting strains are linear functions of z only and hence satisfy all the compatibility equations. In spherical polar coordinates, this solution takes the form

$$\sigma_{RR} = \sigma_{\theta\theta} = \sigma_{\beta\beta} = -\rho g R \cos\beta \; ; \quad \sigma_{R\theta} = \sigma_{\theta\beta} = \sigma_{\beta R} = 0 \; .$$

Using this particular solution, we would use the stress functions of equations (5,6), but without the additional term in ϕ.

$$-\frac{\nu+(1-2\nu)\rho g\cos^2\beta}{(1-\nu)}\Bigg\} \tag{7}$$

$$\sigma_{\theta\theta} = R\cos\beta\left\{-3A-B(1+4\nu)-\frac{\nu\rho g}{(1-\nu)}\right\} \tag{8}$$

$$\sigma_{\beta\beta} = R\cos\beta\Bigg\{3A(4-5\cos^2\beta)+B[3(1-4\nu)\cos^2\beta-4(1-2\nu)] + \frac{(1-2\nu)\rho g\cos^2\beta}{(1-\nu)}-\rho g\Bigg\} \tag{9}$$

$$\sigma_{\beta R} = R\sin\beta\Bigg\{-3A(5\cos^2\beta-1)+B[3(1-4\nu)\cos^2\beta-(1-2\nu)] + \frac{(1-2\nu)\rho g\cos^2\beta}{(1-\nu)}\Bigg\} \tag{10}$$

$$\sigma_{R\theta} = \sigma_{\theta\beta} = 0\ . \tag{11}$$

The surface of the cone is traction-free, so

$$\sigma_{\beta R} = \sigma_{\beta\beta} = 0\ ; \quad \beta = \beta_0\ , \tag{12}$$

leading to the equations

$$\begin{aligned} -3A(5\cos^2\beta_0-1)+B(3(1-4\nu)\cos^2\beta_0-(1-2\nu))+\frac{(1-2\nu)\rho g\cos^2\beta_0}{(1-\nu)} &= 0 \\ 3A(4-5\cos^2\beta_0)+B(3(1-4\nu)\cos^2\beta_0-4(1-2\nu))+\frac{(1-2\nu)\rho g\cos^2\beta_0}{(1-\nu)}-\rho g &= 0\ . \end{aligned}$$

The solution of these equations is

$$A = \frac{(1-3\nu+2\nu^2+3\nu\cos^2\beta_0)\rho g)}{18(1-\nu^2)\cos^2\beta_0}\ ; \quad B = \frac{(1-\nu-(2+\nu)\cos^2\beta_0)\rho g)}{6(1-\nu^2)\cos^2\beta_0}\ . \tag{13}$$

The final stress field is obtained by substituting these results into equations (7–10), giving

$$\sigma_{RR} = -\frac{\rho g R\cos\beta(1+\cos^2\beta_0-\cos^2\beta)}{3\cos^2\beta_0} \tag{14}$$

$$\sigma_{\theta\theta} = -\frac{\rho g R\sin^2\beta_0\cos\beta}{3\cos^2\beta_0} \tag{15}$$

$$\sigma_{\beta\beta} = \frac{\rho g R\cos\beta(\cos^2\beta_0-\cos^2\beta)}{3\cos^2\beta_0} \tag{16}$$

$$\sigma_{\beta R} = \frac{\rho g R\sin\beta(\cos^2\beta_0-\cos^2\beta)}{3\cos^2\beta_0} \tag{17}$$

$$\sigma_{R\theta} = \sigma_{\theta\beta} = 0\ . \tag{18}$$

26.12. A vertical hollow conical tower, $\beta_1 < \beta < \beta_2$ of height h and density ρ is loaded only by its own weight. Find the stress field in the tower. Are there any conditions (values of β_1, β_2) for which tensile stresses are developed in the tower? If so, when and where?

Taking the axis of the tower to be the positive z-axis, the gravitational loading defines a body force

$$p_x = p_y = 0 \; ; \quad p_z = \rho g \; , \tag{1}$$

which can be defined in terms of the body-force potential[10]

$$V = -\rho g z = -\rho g R \cos\beta \; . \tag{2}$$

The function ϕ of §20.5.1 must therefore satisfy the equation

$$\nabla^2 \phi = -\frac{(1-2\nu)\rho g z}{(1-\nu)} \; , \tag{3}$$

which has the particular solution

$$\phi = -\frac{(1-2\nu)\rho g z^3}{6(1-\nu)} = -\frac{(1-2\nu)\rho g R^3 \cos^3\beta}{6(1-\nu)} \; . \tag{4}$$

We notice that equations (20.41, 20.42) are identical to those of Solution A except for the additional term $\nu V/(1-\nu)$ in the normal stresses. Thus, we can include the particular solution by (i) adding the non-harmonic term (4) into ϕ of Solution A and (ii) adding $\nu V/(1-\nu)$ into each of the normal stresses.

The particular solution leads us to expect terms in R^3 in Solution A and hence terms in R^2 in Solution B. Since the cone is hollow, we require both bounded harmonics $R^n P_n(\cos\beta)$ derived from equations (24.11) and the logarithmic Q-series harmonics $R^n Q_n(\cos\beta)$ derived from equations (24.32). We therefore define

$$\begin{aligned} \phi &= -\frac{(1-2\nu)\rho g R^3 \cos^3\beta}{6(1-\nu)} + \frac{A_1 R^3}{2}(5\cos^3\beta - 3\cos\beta) \\ &\quad + A_2 R^3 \left[\frac{1}{4}(5\cos^3\beta - 3\cos\beta)\ln\left(\frac{1+\cos\beta}{1-\cos\beta}\right) - \frac{5\cos^2\beta}{2} + \frac{2}{3}\right] \qquad (5) \\ \omega &= \frac{B_1 R^2}{2}(3\cos^2\beta - 1) + B_2 R^2 \left[\frac{1}{4}(3\cos^2\beta - 1)\ln\left(\frac{1+\cos\beta}{1-\cos\beta}\right) - \frac{3\cos\beta}{2}\right] . \qquad (6) \end{aligned}$$

[10] A simpler particular solution for the body force is the state of hydrostatic compression

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -\rho g z \; ; \quad \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \; .$$

This clearly satisfies the equilibrium equations and the resulting strains are linear functions of z only and hence satisfy all the compatibility equations. In spherical polar coordinates, this solution takes the form

$$\sigma_{RR} = \sigma_{\theta\theta} = \sigma_{\beta\beta} = -\rho g R \cos\beta \; ; \quad \sigma_{R\theta} = \sigma_{\theta\beta} = \sigma_{\beta R} = 0 \; .$$

Using this particular solution, we would use the stress functions of equations (5,6), but without the additional term in ϕ.

From this point onwards, it is only really practicable to perform the calculations in Maple or Mathematica. We substitute the expressions (5,6) into Table 21.2 and add the term $\nu V/(1-\nu)$ to the normal stresses.

The surface of the cone is traction-free, so

$$\sigma_{\beta R} = \sigma_{\beta\beta} = 0 \; ; \quad \beta = \beta_1, \beta_2 \; . \tag{7}$$

All of these traction components will be linearly proportional to R, so the boundary conditions provide four simultaneous equations for the unknown constants A_1, A_2, B_1, B_2.

Plots of the resulting stress maximum tensile stress, using the file 'principal-stresses2D' show that the stresses are compressive everywhere for all values of β_1 if $\beta_2 < 65^o$, when $\nu = 0.3$. For $\beta_2 > 65^o$, there is a range of values of β_1 giving tensile stresses ($\sigma_{RR} > 0$) at the boundary $\beta = \beta_2$. For example, tensile stresses occur for $\beta_2 = 75^o$ and $25^o < \beta_1 < 69^o$.

Membrane theory solution

When the shell is very thin ($\beta_2 - \beta_1 \ll 1$), we might expect the membrane theory of shells to apply. We define the mean angle

$$\beta_0 = \frac{(\beta_1 + \beta_2)}{2} \; . \tag{8}$$

The cross-sectional radius of the cone at spherical radius R is

$$r = R \sin \beta_0 \tag{9}$$

and the volume of a thin hollow cone out to radius R is

$$V = 2\pi \left(\frac{R^2 \Delta\beta}{2} \right) \times \frac{2R \sin \beta_0}{3} = \frac{2\pi R^3 \Delta\beta \sin \beta_0}{3} \tag{10}$$

corresponding to a weight

$$W = \frac{2\pi \rho g R^3 \Delta\beta \sin \beta_0}{3} \; , \tag{11}$$

where

$$\Delta\beta = \beta_1 - \beta_2 \; . \tag{12}$$

The thickness of the shell at this section is

$$t = R\Delta\beta \tag{13}$$

and the cross-sectional area of a cut along a line at constant R is

$$2\pi r t = 2\pi R^2 \Delta\beta \sin \beta_0 \; . \tag{14}$$

We therefore obtain the meridional membrane stress σ_{RR} from the equation

$$2\pi R^2 \Delta\beta \sin\beta_0 \sigma_{RR} \cos\beta_0 = -\frac{2\pi\rho g R^3 \Delta\beta \sin\beta_0}{3} \tag{15}$$

and hence

$$\sigma_{RR} = -\frac{\rho g R}{3\cos\beta_0} \ . \tag{16}$$

For the circumferential stress $\sigma_{\theta\theta}$, we use the equation

$$\frac{\sigma_{RR}}{R_1} + \frac{\sigma_{\theta\theta}}{R_2} = -\rho g \sin\beta_0 \ , \tag{17}$$

where for the cone, $R_1 = \infty$ and

$$R_2 = R\tan\beta_0 \ . \tag{18}$$

Substituting these values, we obtain

$$\sigma_{\theta\theta} = -\frac{\rho g R \sin^2\beta_0}{\cos\beta_0} \ . \tag{19}$$

Thus, the membrance theory predicts compressive stresses throughout the shell as long as $\beta_0 < \pi/2$.

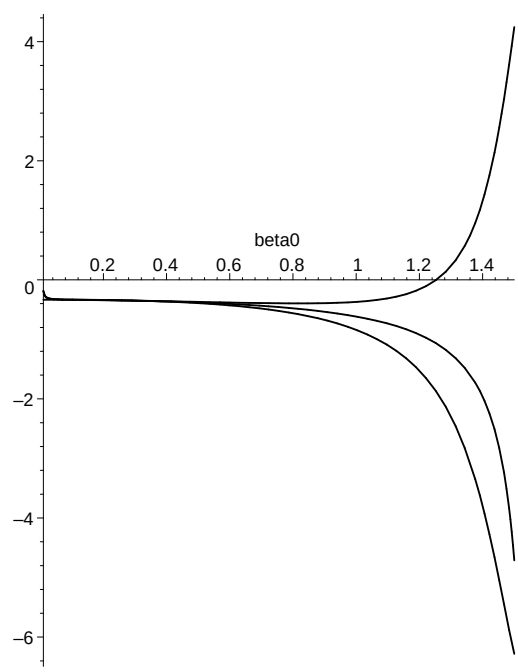


Figure 1

In Figure 1, we plot the meridional stress σ_{RR} at the inner and outer surfaces $\beta = \beta_1, \beta_2$ respectively as a function of β_0 for the case where $\Delta\beta = 0.1$. We also plot the membrane solution (16). All three curves are almost coincident for $\beta_0 < 0.6$, but for higher values, the stress at the outer surface eventually becomes tensile. The membrane estimate is approximately the mean value between the extremes of the exact solution. Somewhat similar results are obtained for the circumferential stress $\sigma_{\theta\theta}$.

26.13. The hollow conical bar $\beta_1 < \beta < \beta_2$ is loaded by a tensile axial force F, the curved surfaces $\beta = \beta_1, \beta_2$ being traction-free. Use the stress functions

$$\phi = A_1\{\ln(R) + \ln(1+\cos(\beta))\} + A_2\{\ln(R) + \ln(1-\cos(\beta))\} \;\; ; \;\; \omega = \frac{B}{R}$$

to obtain a solution for the stress field.

As in §26.2.1, we expect the stresses to vary with R^{-2}, which implies that ω varies with R^{-1} and ϕ with R^0. The choice of ϕ in equation (26.20) was motivated by the fact that the cone was solid and hence that singularities could not be permitted on the positive z-axis ($\beta = 0$). This axis is not part of the hollow cone, so this restriction is removed, permitting us to include the additional term $A_2\{\ln(R) + \ln(1-\cos(\beta))\}$, which provides the extra degree of freedom needed to satisfy the boundary conditions on the inner surface.

The traction-free boundary conditions are

$$\sigma_{\beta R} = \sigma_{\beta\beta} = 0 \; ; \quad \beta = \beta_1, \beta_2 \; . \tag{1}$$

Substituting the given stress function into Table 21.2, we obtain

$$\sigma_{RR} = -\frac{A_1}{R^2} - \frac{A_2}{R^2} + \frac{2(2-\nu)B\cos\beta}{R^2} \tag{2}$$

$$\sigma_{\theta\theta} = \frac{A_1}{R^2(1+\cos\beta)} + \frac{A_2}{R^2(1-\cos\beta)} - \frac{(1-2\nu)B\cos\beta}{R^2} \tag{3}$$

$$\sigma_{\beta\beta} = \frac{A_1\cos\beta}{R^2(1+\cos\beta)} - \frac{A_2\cos\beta}{R^2(1-\cos\beta)} - \frac{(1-2\nu)B\cos\beta}{R^2} \tag{4}$$

$$\sigma_{\beta R} = \frac{A_1\sin\beta}{R^2(1+\cos\beta)} - \frac{A_2\sin\beta}{R^2(1-\cos\beta)} - \frac{(1-2\nu)B\sin\beta}{R^2} \tag{5}$$

and hence the strong boundary conditions (1) require

$$\frac{A_1\sin\beta_1}{(1+\cos\beta_1)} - \frac{A_2\sin\beta_1}{(1-\cos\beta_1)} - (1-2\nu)B\sin\beta_1 = 0 \tag{6}$$

$$\frac{A_1\sin\beta_2}{(1+\cos\beta_2)} - \frac{A_2\sin\beta_2}{(1-\cos\beta_2)} - (1-2\nu)B\sin\beta_2 = 0 \tag{7}$$

$$\frac{A_1\cos\beta_1}{(1+\cos\beta_1)} - \frac{A_2\cos\beta_1}{(1-\cos\beta_1)} - (1-2\nu)B\cos\beta_1 \tag{8}$$

$$\frac{A_1\cos\beta_2}{(1+\cos\beta_2)} - \frac{A_2\cos\beta_2}{(1-\cos\beta_2)} - (1-2\nu)B\cos\beta_2 \; . \tag{9}$$

These equations are all satisfied if we choose

$$A_1 = \frac{(1-2\nu)B(1+\cos\beta_1)(1+\cos\beta_2)}{2} \; ; \quad A_2 = -\frac{(1-2\nu)B(1-\cos\beta_1)(1-\cos\beta_2)}{2} \; . \tag{10}$$

One further equation to determine the remaining constant B is obtained from the equilibrium condition

$$\int_0^{2\pi}\int_{\beta_1}^{\beta_2}\{\sigma_{RR}(R_b,\theta,\beta)\cos\beta-\sigma_{R\beta}(R_b,\theta,\beta)\sin\beta\}R_b^2\sin\beta d\beta d\theta=F\ , \tag{11}$$

modified from equation (26.19). Substituting for the stress components and using (10), we obtain

$$-2\pi B(\cos\beta_2-\cos\beta_1)(\cos^2\beta_2+2\nu\cos\beta_1\cos\beta_2+\cos^2\beta_1)=F \tag{12}$$

and hence

$$B=-\frac{F}{2\pi(\cos\beta_2-\cos\beta_1)(\cos^2\beta_2+2\nu\cos\beta_1\cos\beta_2+\cos^2\beta_1)}\ . \tag{13}$$

Substituting this result into (10) and (2–5), we obtain the non-zero stress components as

$$\sigma_{RR} = \frac{F\{(1-2\nu)(\cos\beta_1+\cos\beta_2)-2(2-\nu)\cos\beta\}}{2\pi R^2(\cos\beta_2-\cos\beta_1)(\cos^2\beta_2+2\nu\cos\beta_1\cos\beta_2+\cos^2\beta_1)} \tag{14}$$

$$\sigma_{\theta\theta} = -\frac{F(1-2\nu)(\cos^3\beta-\cos\beta_1\cos\beta_2\cos\beta+\cos\beta_1+\cos\beta_2-2\cos\beta)}{2\pi R^2(\cos\beta_2-\cos\beta_1)(\cos^2\beta_2+2\nu\cos\beta_1\cos\beta_2+\cos^2\beta_1)\sin^2\beta} \tag{15}$$

$$\sigma_{\beta\beta} = \frac{F(1-2\nu)(\cos\beta_2-\cos\beta)(\cos\beta-\cos\beta_1)\cos\beta}{2\pi R^2(\cos\beta_2-\cos\beta_1)(\cos^2\beta_2+2\nu\cos\beta_1\cos\beta_2+\cos^2\beta_1)\sin^2\beta} \tag{16}$$

$$\sigma_{\beta R} = \frac{F(1-2\nu)(\cos\beta_2-\cos\beta)(\cos\beta-\cos\beta_1)}{2\pi R^2(\cos\beta_2-\cos\beta_1)(\cos^2\beta_2+2\nu\cos\beta_1\cos\beta_2+\cos^2\beta_1)\sin\beta}\ . \tag{17}$$

26.14. A *horizontal* solid conical bar, $0<\beta<\beta_0$ of length L and density ρ is built in at $z=L$ and loaded only by its own weight. Find the stress field in the bar and the vertical displacement of the vertex.

An appropriate gravitational body-force potential is

$$V = \rho g y = \rho g R \sin\beta \sin\theta \ . \tag{1}$$

To find a particular solution ϕ_P we write

$$\phi_P = D R^3 \sin\beta \sin\theta \tag{2}$$

and substitute into (20.40), obtaining

$$\nabla^2 \phi_P = 10 D R \sin\beta \sin\theta = \frac{(1-2\nu)\rho g R \sin\beta \sin\theta}{(1-\nu)} \ , \tag{3}$$

which is satisfied if

$$D = \frac{\rho g (1-2\nu)}{10(1-\nu)} \ . \tag{4}$$

The problem has no length scale, showing that the potentials must be self-similar and capable of expression in the separated-variable form

$$R^n f(\theta, \beta) \ .$$

In view (2) we must have $n=3$ in ϕ, ψ and $n=2$ in ω. Thus we write

$$\phi_H = A R^3 P_3^1(\cos\beta) \sin\theta \tag{5}$$

$$\omega = B R^2 P_2^1(\cos\beta) \sin\theta \tag{6}$$

$$\psi = C R^3 P_3^1(\cos\beta) \cos\theta \tag{7}$$

and hence

$$\phi = \phi_H + \phi_P = \left(-\frac{3A}{8}(\sin\beta + 5\sin(3\beta)) + \frac{\rho g (1-2\nu)\sin\beta}{10(1-\nu)} \right) R^3 \sin\theta \tag{8}$$

$$\omega = -\frac{3BR^2}{2} \sin(2\beta) \sin\theta \tag{9}$$

$$\psi = -\frac{3CR^3(\sin\beta + 5\sin(3\beta))\cos\theta}{8} \ . \tag{10}$$

Using Maple or Mathematica, we now substitute these expressions into Table 21.2 to obtain the stress components and set up three algebraic equations for A, B, C from the traction-free boundary conditions

$$\sigma_{\beta R} = \sigma_{\beta\theta} = \sigma_{\beta\beta} = 0 \ . \tag{11}$$

Notice that the tractions on these surfaces all have the same power of R and the same trigonometric function of θ, so one equation only is obtained from each condition. Solving for A, B, C, we obtain

$$A = -\frac{\rho g(6 - 6\nu - 12\nu^2 - (6 - 11\nu - 7\nu^2)\cos^2\beta_0)}{90(1-\nu^2)\sin^2\beta_0} \tag{12}$$

$$B = -\frac{\rho g(3 - 3\nu - (4 + 2\nu)\cos^2\beta_0)}{18(1-\nu^2)\sin^2\beta_0} \tag{13}$$

$$C = -\frac{\rho g(2 + 2\nu - (2 + \nu)\cos^2\beta_0)}{36(1+\nu)\sin^2\beta_0} . \tag{14}$$

The final stress field is then obtained as

$$\sigma_{RR} = \frac{\rho g R(\sin^2\beta_0 + \cos^2\beta)\sin\beta\sin\theta}{3\sin^2\beta_0} \tag{15}$$

$$\sigma_{\theta\theta} = -\frac{\rho g R\sin\beta\sin\theta}{3} \tag{16}$$

$$\sigma_{\beta\beta} = -\frac{\rho g R(\cos^2\beta - \cos^2\beta_0)\sin\beta\sin\theta}{3\sin^2\beta_0} \tag{17}$$

$$\sigma_{\beta R} = \frac{\rho g R(\cos^2\beta - \cos^2\beta_0)\cos\beta\sin\theta}{3\sin^2\beta_0} \tag{18}$$

$$\sigma_{R\theta} = \frac{\rho g R\cos\theta}{3} \tag{19}$$

$$\sigma_{\theta\beta} = 0 \tag{20}$$

Displacement of the vertex

In order to determine the displacement of the vertex, we must find the displacement components, superpose an arbitrary rigid-body displacement and evaluate the arbitrary constants by applying suitable weak statements of the built-in condition on the disk $z = L, 0 < r < L\tan\beta_0$. This is most easily done in Cartesian coordinates, in which

$$V = \rho g y \tag{21}$$

$$\phi_P = Dy(x^2 + y^2 + z^2) \tag{22}$$

$$\phi_H = \frac{3Ay(x^2 + y^2 - 4z^2)}{2} \tag{23}$$

$$\omega = -3Bzy \tag{24}$$

$$\psi = \frac{3Cx(x^2 + y^2 - 4z^2)}{2} . \tag{25}$$

The displacement components are then

$$2\mu u_x = \frac{\partial\phi}{\partial x} + z\frac{\partial\omega}{\partial x} + 2\frac{\partial\psi}{\partial y}$$

$$
\begin{aligned}
&= (3A + 6C + 2D)xy && (26)\\
2\mu u_y &= \frac{\partial \phi}{\partial y} + z\frac{\partial \omega}{\partial y} - 2\frac{\partial \psi}{\partial x} \\
&= \frac{1}{2}[(3A - 18C + 2D)x^2 + 3(3A - 2C + 2D)y^2 - 2(6A + 3B - 12C - D)z^2] && (27)\\
2\mu u_z &= \frac{\partial \phi}{\partial z} + z\frac{\partial \omega}{\partial z} - (3 - 4\nu)\omega \\
&= -2(6A + 3(1 - 2\nu)B - D)yz\ . && (28)
\end{aligned}
$$

The problem is even in x and odd in y, so a sufficienly general rigid-body displacement is

$$
u_x = 0\ ; \quad u_y = C_1 - C_2 z\ ; \quad u_z = C_2 y\ , \tag{29}
$$

where C_1, C_2 are two arbitrary constants representing vertical translation and rotation about the x-axis respectively. We have not included a rigid-body translation in the z-direction, since this would not affect the vertical motion of the vertex. Appropriate weak boundary conditions at $z = L, 0 < r < L\tan\beta_0$ are

$$
\int_{-L\tan\beta_0}^{L\tan\beta_0} \int_{-\sqrt{L^2\tan^2\beta_0 - y^2}}^{\sqrt{L^2\tan^2\beta_0 - y^2}} u_y(x, y, L)\,dx\,dy = 0 \tag{30}
$$

$$
\int_{-L\tan\beta_0}^{L\tan\beta_0} \int_{-\sqrt{L^2\tan^2\beta_0 - y^2}}^{\sqrt{L^2\tan^2\beta_0 - y^2}} u_z(x, y, L)\,y\,dx\,dy = 0\ . \tag{31}
$$

The constants are obtained by adding (29) into (26–28) and substituting the results into (30, 31). Solving the resulting equations for C_1, C_2, we obtain

$$
C_1 = -\frac{\rho g L^2[4(1+\nu) - 2(1+\nu)\cos^2\beta_0 + (1-2\nu)\cos^4\beta_0]}{36\mu(1+\nu)\sin^2\beta_0\cos^2\beta_0} \tag{32}
$$

$$
C_2 = -\frac{\rho g L\cos^2\beta_0}{6\mu(1+\nu)\sin^2\beta_0}\ . \tag{33}
$$

The vertical displacement of the vertex is then

$$
u_y(0,0,0) = C_1 = -\frac{\rho g L^2[4(1+\nu) - 2(1+\nu)\cos^2\beta_0 + (1-2\nu)\cos^4\beta_0]}{36\mu(1+\nu)\sin^2\beta_0\cos^2\beta_0}\ . \tag{34}
$$

26.15. Figure 26.2 shows the cross-section of a concrete dam which may be approximated as a sector of a truncated hollow cone. The dam is loaded by hydrostatic pressure (water density ρ) and by self weight (concrete density $\rho_c = 2.3\rho$). Assuming the stress field is axisymmetric — i.e. that the tractions at the side support are the same as those that would occur on the equivalent plane in a complete truncated hollow cone loaded all around the circumference — determine the stress field in the dam using strong conditions on the curved surfaces and weak (traction-free) conditions on the truncated end.

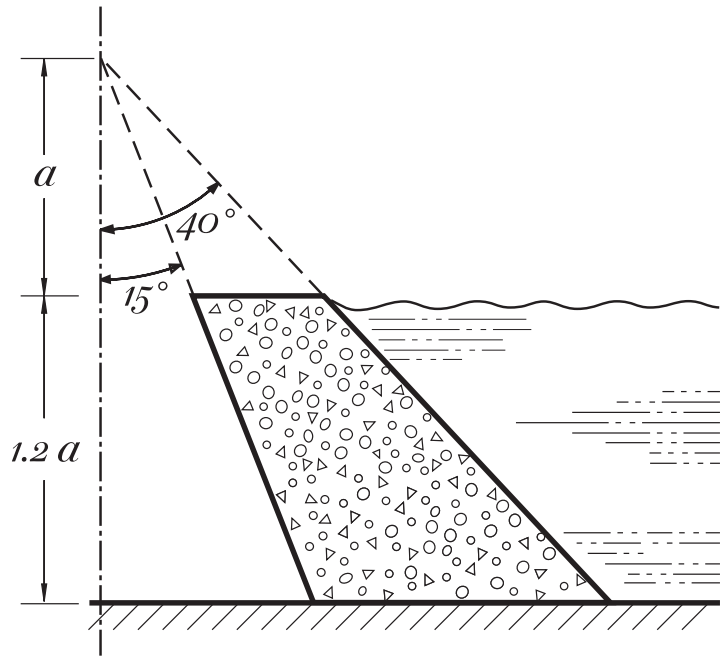


Figure 26.2: Truncated hollow conical dam.

The boundary conditions on the curved surfaces of the dam are

$$\sigma_{\beta\beta} = \sigma_{\beta R} = 0 \; ; \quad \beta = 15^o \tag{1}$$

$$\sigma_{\beta R} = 0 \; ; \quad \beta = 40^o \tag{2}$$

$$\sigma_{\beta\beta} = -\rho g(z-a) = \rho g(a - R\cos 40^o) \; ; \quad \beta = 40^o \; . \tag{3}$$

In addition, we have self-weight loading which corresponds to a body force

$$p_x = p_y = 0 \; ; \quad p_z = \rho_c g \; . \tag{4}$$

The solution strategy will be to solve the problem for the untruncated cone, extending the boundary conditions to the vertex. This can be expected to yield a stress

field that does not satisfy the weak boundary condition on the truncation surface, but by symmetry, the only resultant on this surface is an axial force. We can therefore superpose the solution of Problem 26.13 with a multiplier chosen so as to make this resultant force vanish.

The simplest way to describe the body force is to use the hydrostatic particular solution

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -\rho_c g z \; ; \quad \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \; . \tag{5}$$

This clearly satisfies the equilibrium equations and the resulting strains are linear functions of z only and hence satisfy all the compatibility equations. In spherical polar coordinates, this solution takes the form

$$\sigma_{RR} = \sigma_{\theta\theta} = \sigma_{\beta\beta} = -\rho_c g R \cos\beta \; ; \quad \sigma_{R\theta} = \sigma_{\theta\beta} = \sigma_{\beta R} = 0 \; . \tag{6}$$

The only non-zero traction boundary condition (3) involves terms of power R^0 and R^1 and the gravitational stresses are R^1, so we need potentials of order R^2, R^3 in ϕ and R^1, R^2 in ω. Since the cone is hollow, we shall also require the Q-series potentials of the same order from equations (24.34). Thus, we define

$$\begin{aligned}
\phi &= \frac{A_1 R^2}{2}(3\cos^2\beta - 1) + \frac{A_2 R^3}{2}(5\cos^3\beta - 3\cos\beta) \\
&\quad + A_3 R^2 \left[\frac{1}{4}(3\cos^2\beta - 1)\ln\left(\frac{1+\cos\beta}{1-\cos\beta}\right) - \frac{3\cos\beta}{2}\right] \\
&\quad + A_4 R^3 \left[\frac{1}{4}(5\cos^3\beta - 3\cos\beta)\ln\left(\frac{1+\cos\beta}{1-\cos\beta}\right) - \frac{5\cos^2\beta}{2} + \frac{2}{3}\right] \qquad (7) \\
\omega &= B_1 R\cos\beta + \frac{B_2 R^2}{2}(3\cos^2\beta - 1) + B_3 R\left[\frac{\cos\beta}{2}\ln\left(\frac{1+\cos\beta}{1-\cos\beta}\right) - 1\right] \\
&\quad + B_4 R^2 \left[\frac{1}{4}(3\cos^2\beta - 1)\ln\left(\frac{1+\cos\beta}{1-\cos\beta}\right) - \frac{3\cos\beta}{2}\right] \; . \qquad (8)
\end{aligned}$$

From this point on, it is only really practical to solve the problem in Maple or Mathematica. We calculate the stresses by substituting (7, 8) into Table 21.2 and add in the particular solution (6). On each boundary we have two tractions, each of which has terms that are linear in R and constant terms, giving eight boundary conditions in all, when the stress field is substituted into (1–3). These equations can be solved for the eight constants $A_1, ...B_4$. For the particular case $\nu = 0.3$, we obtain

$$\begin{aligned}
&A_1 = -0.349\rho g a \; ; \quad A_2 = 0.020\rho g \; ; \quad A_3 = -0.023\rho g a \; ; \quad A_4 = -0.031\rho g \; ; \\
&B_1 = -1.29\rho g a \; ; \quad B_2 = -0.490\rho g \; ; \quad B_3 = -0.062\rho g a \; ; \quad B_4 = 0.137\rho g \; . \qquad (9)
\end{aligned}$$

Since the truncation surface $z = a$ is a surface in the cylindrical coordinate system, it is convenient to transform to this system in order to determine the resultant axial force that needs to be removed. We substitute

$$R = \sqrt{r^2 + z^2} \; ; \quad \cos\beta = \frac{z}{\sqrt{r^2 + z^2}} \tag{10}$$

into (7, 8) and the resulting expressions into Table 21.1. We then add in the gravitational term (5) into σ_{zz} and evaluate the tensile force

$$F = 2\pi \int_{a\tan 15^o}^{a\tan 40^o} \sigma_{zz}(r,\theta,a) r dr \ , \tag{11}$$

obtaining the result

$$F = -0.250\rho g a^3 \tag{12}$$

for $\nu = 0.3$.

To remove this unwanted force, we start with the potential function suggested in Problem 26.13

$$\phi = A_5\{\ln(R) + \ln(1+\cos(\beta))\} + A_6\{\ln(R) + \ln(1-\cos(\beta))\} \ ; \ \omega = \frac{B5}{R} \ . \tag{13}$$

Substituting these results into Table 21.2, we obtain

$$\sigma_{RR} = -\frac{A_5}{R^2} - \frac{A_6}{R^2} + \frac{2(2-\nu)B_5\cos\beta}{R^2} \tag{14}$$

$$\sigma_{\theta\theta} = \frac{A_5}{R^2(1+\cos\beta)} + \frac{A_6}{R^2(1-\cos\beta)} - \frac{(1-2\nu)B_5\cos\beta}{R^2} \tag{15}$$

$$\sigma_{\beta\beta} = \frac{A_5\cos\beta}{R^2(1+\cos\beta)} - \frac{A_6\cos\beta}{R^2(1-\cos\beta)} - \frac{(1-2\nu)B_5\cos\beta}{R^2} \tag{16}$$

$$\sigma_{\beta R} = \frac{A_5\sin\beta}{R^2(1+\cos\beta)} - \frac{A_6\sin\beta}{R^2(1-\cos\beta)} - \frac{(1-2\nu)B_5\sin\beta}{R^2} \tag{17}$$

These additional terms should not introduce any additional tractions on $\beta = 15^o, 40^o$ and hence we require

$$\frac{A_5\sin 15^o}{(1+\cos 15^o)} - \frac{A_6\sin 15^o}{(1-\cos 15^o)} - (1-2\nu)B_5\sin 15^o = 0 \tag{18}$$

$$\frac{A_5\sin 40^o}{(1+\cos 40^o)} - \frac{A_6\sin 40^o}{(1-\cos 40^o)} - (1-2\nu)B_5\sin 40^o = 0 \tag{19}$$

$$\frac{A_5\cos 15^o}{(1+\cos 15^o)} - \frac{A_6\cos 15^o}{(1-\cos 15^o)} - (1-2\nu)B_5\cos 15^o \tag{20}$$

$$\frac{A_5\cos 40^o}{(1+\cos 40^o)} - \frac{A_6\cos 40^o}{(1-\cos 40^o)} - (1-2\nu)B_5\cos 40^o \ . \tag{21}$$

These equations are all satisfied if we choose

$$A_5 = \frac{(1-2\nu)B_5(1+\cos 15^o)(1+\cos 40^o)}{2}$$

$$A_6 = -\frac{(1-2\nu)B_5(1-\cos 15^o)(1-\cos 40^o)}{2} \ . \tag{22}$$

One further equation to determine the remaining constant B_5 is obtained from the equilibrium condition

$$\int_0^{2\pi} \int_{15^o}^{40^o} \{\sigma_{RR}(R_b, \theta, \beta) \cos\beta - \sigma_{R\beta}(R_b, \theta, \beta) \sin\beta\} R_b^2 \sin\beta d\beta d\theta = -F \ , \tag{23}$$

modified from equation (26.19). Notice that we don't need to transform to cylindrical coordinates to satisfy this condition, since there are no tractions on the curved surfaces and hence the transmitted force must be the same at every cross section of any shape.

Substituting for the stress components and using (22), we obtain

$$-2\pi B_5(\cos 40^o - \cos 15^o)(\cos^2 40^o + 2\nu \cos 15^o \cos 40^o + \cos^2 15^o) = -F \tag{24}$$

and hence

$$B_5 = 0.318\rho g a^3 \ ; \quad A_5 = 0.221\rho g a^3 \ ; \quad A_6 = -0.000508\rho g a^3 \tag{25}$$

for $\nu = 0.3$, using (22).

The final stress field is obtained by superposing the stress functions (7, 8, 13), substituting into Table 21.2 and then superposing the gravitational stress field (6). Numerical exploration of the stress field shows that the principal stresses are everywhere compressive.

26.16. Find the stress field in the solid conical shaft $0 < \beta < \beta_0$, loaded only by a torque T at the vertex.

The torque will be given by

$$T = 2\pi \int_0^{\beta_0} \sigma_{R\theta} R^3 \sin^2 \beta d\beta \tag{1}$$

and this will be independent of R if and only if $\sigma_{R\theta}$ varies with R^{-3}. This implies the use of a function Ψ in Solution E that varies with R^{-1} — i.e. the source solution of §23.1 — and hence

$$\Psi = \frac{C}{R} \ . \tag{2}$$

Substituting this expression into Table 21.2, we obtain

$$\sigma_{R\theta} = -\frac{3C \sin \beta}{R^3} \ . \tag{3}$$

$$\sigma_{RR} = \sigma_{\theta\theta} = \sigma_{\beta\beta} = \sigma_{\beta\theta} = \sigma_{\beta R} = 0 \ . \tag{4}$$

Notice that this solution satisfies the strong boundary conditions that the conical surfaces be traction-free — i.e.

$$\sigma_{\beta R} = \sigma_{\beta\theta} = \sigma_{\beta\beta} = 0 \ ; \quad \beta = \beta_0 \ . \tag{5}$$

To determine the constant C, we substitute (3) into (1) and evaluate the integral, obtaining

$$T = -2\pi C(2 + cos^3\beta_0 - 3\cos\beta_0) \tag{6}$$

and hence

$$C = -\frac{T}{2\pi(2 + cos^3\beta_0 - 3\cos\beta_0)} \tag{7}$$

The final stress field is therefore

$$\sigma_{R\theta} = \frac{3T \sin \beta}{2\pi R^3(2 + cos^3\beta_0 - 3\cos\beta_0)} \ . \tag{8}$$

$$\sigma_{RR} = \sigma_{\theta\theta} = \sigma_{\beta\beta} = \sigma_{\beta\theta} = \sigma_{\beta R} = 0 \ . \tag{9}$$

26.17. The solid conical bar $0<\beta<\beta_0$ is loaded by uniform torsional shear tractions $\sigma_{\beta\theta}=S$, all other traction components being zero. Find the stress field in the bar. Note: The 'obvious' stress function for this problem is $R^2P_2(\cos\beta)$, but it corresponds only to rigid-body rotation and gives zero stress components. As in Chapter 8, we need a special stress function for this case, which is in fact ϕ_{-3} from equation (23.25). You will need to write $z=R\cos\beta$ to express this function in spherical polar coördinates. It will contain the term $R^2\ln(R)$, but as in §10.3, the same degeneracy that causes the elementary potential to give zero stresses will cause there to be no logarithmic terms in the stress components.

The problem has no length scale, showing that the stress field must be self-similar and capable of expression in the separated-variable form

$$\sigma = R^n f(\beta) \ . \tag{1}$$

In view of the uniform surface tractions, it is clear that n must be zero and hence the potential in Solution E must be of the same form but with $n = 2$.

As stated in the question, the 'obvious' function $R^2P_2(\cos\beta)$ degenerates here, so we use the form

$$\begin{aligned} \psi = C\phi_{-3} &= \frac{C}{4}\{(2z^2-r^2)\ln(R+z)-3Rz+r^2\} \\ &= = \frac{CR^2}{4}\{(3\cos^2\beta-1)\ln(R(1+\cos\beta))-3\cos\beta+\sin^2\beta\} \ . \end{aligned} \tag{2}$$

Substituting into Solution E (Table 21.2), we obtain the non-zero stress components as

$$\sigma_{\beta\theta} = -\frac{C(1-\cos\beta)(2+\cos\beta)}{2(1+\cos\beta)} \ ; \quad \sigma_{R\theta} = \frac{C\sin\beta}{2} \ . \tag{3}$$

The boundary condition is

$$\sigma_{\beta\theta} = S \ ; \quad \beta = \beta_0 \tag{4}$$

and hence

$$-\frac{C(1-\cos\beta_0)(2+\cos\beta_0)}{2(1+\cos\beta_0)} = S \ , \tag{5}$$

or

$$C = -\frac{2S(1+\cos\beta_0)}{(1-\cos\beta_0)(2+\cos\beta_0)} \ . \tag{6}$$

Finally, we substitute for C in equation (2) obtaining the stress field as

$$\sigma_{\beta\theta} = \frac{S(1+\cos\beta_0)(1-\cos\beta)(2+\cos\beta)}{(1+\cos\beta)(1-\cos\beta_0)(2+\cos\beta_0)} \ ; \quad \sigma_{R\theta} = -\frac{S(1+\cos\beta_0)\sin\beta}{(1-\cos\beta_0)(2+\cos\beta_0)} \ . \tag{7}$$

26.18. Use spherical polar coördinates to find the stresses in the half-space $z > 0$ due to a concentrated tangential force F in the x-direction, applied at the origin. Note: The half-space is equivalent to the cone $0 \le \beta < \pi/2$. This problem is the spherical polar equivalent of Problem 23.1.

Since there is a concentrated force at the origin, we anticipate stresses decaying with R^{-2}. Also, the problem is antisymmetric about the y-axis and symmetric about the x-axis, suggesting potentials involving $\cos\theta$ ($\sin\theta$ in Solution E).

We therefore select the potentials

$$\phi = AR^0 P_0^1(\cos\beta)\cos\theta = \frac{A\sin\beta\cos\theta}{(1+\cos\beta)} \tag{1}$$

$$\omega = BR^{-1} P_0^1(\cos\beta)\cos\theta = \frac{A\sin\beta\cos\theta}{R(1+\cos\beta)} \tag{2}$$

$$\psi = CR^0 P_0^1(\cos\beta)\sin\theta = \frac{A\sin\beta\cos\theta}{(1+\cos\beta)} \tag{3}$$

Notice that the same Legendre function P_0^1 appears in all three functions, even though they have different powers of R. This is because $n = 0$ and $n = -1$ are related through the equation $n = -p - 1$ (see §24.3.1).

Using Maple or Mathematica, we substitute these expressions into Table 21.2 and in particular find the tractions on the surface $\beta = \pi/2$, which are

$$\sigma_{\beta R} = -\frac{[A + B(1-\nu) - C]\cos\theta}{R^2} \tag{4}$$

$$\sigma_{\beta\theta} = -\frac{[A + B(1-\nu) - C]\sin\theta}{R^2} \tag{5}$$

$$\sigma_{\beta\beta} = \frac{[A - 2B(1-\nu)]\cos\theta}{R^2} \ . \tag{6}$$

Setting these tractions to zero yields just two independent conditions

$$A + B(1-\nu) - C = 0 \ ; \quad A - 2B(1-\nu) = 0 \ . \tag{7}$$

The remaining equation is obtained from the equilibrium condition

$$\int_0^{\pi/2}\int_0^{2\pi} [(\sigma_{RR}\sin\beta + \sigma_{R\beta}\cos\beta]\cos\theta - \sigma_{R\theta}\sin\theta]R^2\sin\beta d\theta d\beta + F = 0 \ . \tag{8}$$

Substituting for the stress components and evaluating the integral, we obtain

$$-\pi[A + B(1-2\nu) + C] + F = 0 \ . \tag{9}$$

Solving (7, 9) for A, B, C, we obtain

$$A = \frac{F(1-\nu)}{\pi} \ ; \quad B = -\frac{F}{2\pi} \ ; \quad C = \frac{F}{2\pi} \ . \tag{10}$$

The full stress field is then recovered by substituting these values into the stress expressions, giving

$$\sigma_{RR} = -\frac{F(1+\nu+(2-\nu)\cos\beta)\sin\beta\cos\theta}{\pi R^2(1+\cos\beta)} \tag{11}$$

$$\sigma_{\theta\theta} = \frac{F(1-2\nu)(2+\cos\beta)\sin\beta\cos\theta}{2\pi R^2(1+\cos\beta)^2} \tag{12}$$

$$\sigma_{\beta\beta} = \frac{F(1-2\nu)\cos^2\beta\sin\beta\cos\theta}{2\pi R^2(1+\cos\beta)^2} \tag{13}$$

$$\sigma_{\beta R} = \frac{F(1-2\nu)(1-\cos\beta)\cos\theta}{2\pi R^2(1+\cos\beta)} \tag{14}$$

$$\sigma_{R\theta} = \frac{F(1-2\nu)(1-\cos\beta)\sin\theta}{2\pi R^2(1+\cos\beta)} \tag{15}$$

$$\sigma_{\theta\beta} = \frac{F(1-2\nu)\sin(2\beta)\sin\theta}{4\pi R^2(1+\cos\beta)^2} \tag{16}$$

26.19. A solid sphere of radius a is loaded only by self-gravitation — i.e. each particle of the sphere is attracted to each other particle by a force $\gamma m_1 m_2/R^2$, where m_1, m_2 are the masses of the two particles, R is the distance betwen them and γ is the universal gravitational constant. Find the stress field in the sphere and in particular the location and magnitude of the maximum compressive stress and the maximum shear stress.

Assuming that the solution is to be applied to the earth, express the results in terms of the gravitational acceleration g at the surface.

The body force due to a point mass M at the origin in spherical polar coordinates is

$$p_R = -\frac{\partial V}{\partial R} = -\frac{\rho M \gamma}{R^2} \ , \tag{1}$$

the other two components being zero. This force field can be defined in terms of the potential

$$V = -\frac{\rho M \gamma}{R} + A \ , \tag{2}$$

where A is an arbitrary constant and this expression is harmonic ($\nabla^2 V = 0$) except at the origin.

A more general distribution of mass can be described by superposition of this solution, so we conclude that the potential V due to a general distribution of mass will be harmonic in any location where the mass density is zero.

Now consider the special case of a hollow sphere. If the density is uniform, the potential will be a function of R only and it must be harmonic both inside the hole and outside the sphere. The most general solution of this form is

$$V = C_1 + \frac{C_2}{R} \ . \tag{3}$$

Inside the hole, the second term must be zero for continuity at the origin[11]. It follows that V is constant inside the hole and hence that the body force there is zero. In other words, a particle located inside a hollow sphere would experience no gravitational force. Outside the sphere, the potential has the same form as that due to a single particle at the origin (2) whose mass can be shown to be equal to that of the hollow sphere[12].

For the complete solid sphere, the body force at radius R ($< a$) is therefore due only to the attraction of the mass M of the material of radius less than R, given by

$$M = \frac{4\pi\rho R^3}{3} \tag{4}$$

[11] In fact we already know that a non-zero value of C_2 would correspond to the location of a massive particle at the origin from equation (2).

[12] This clearly must be the case, since at very large values of R, the sphere will look more and more like a particle.

and we have

$$p_R = -\frac{\partial V}{\partial R} = -\left(\frac{4\pi\rho R^3}{3}\right)\frac{\rho\gamma}{R^2} = -\frac{4\pi\rho^2\gamma R}{3} \ , \tag{5}$$

from (2). Integrating this result we have

$$V = \frac{2\pi\rho^2\gamma R^2}{3} + A \ ; \quad 0 < R < a \ . \tag{6}$$

We next seek a potential function ϕ satisfying

$$\nabla^2\phi = \frac{(1-2\nu)V}{(1-\nu)} = \frac{2\pi\rho^2(1-2\nu)\gamma R^2}{3(1-\nu)} + \frac{(1-2\nu)A}{(1-\nu)} \ , \tag{7}$$

from (20.40). Since the problem is spherically symmetrical, we restrict attention to functions of R only, giving

$$\frac{d^2\phi}{dR^2} + \frac{2}{R}\frac{d\phi}{dR} = \frac{2\pi\rho^2(1-2\nu)\gamma R^2}{3(1-\nu)} + \frac{(1-2\nu)A}{(1-\nu)} \ , \tag{8}$$

from (21.35). The general solution of this equation is

$$\phi = \frac{\pi\rho^2(1-2\nu)\gamma R^4}{30(1-\nu)} + \frac{(1-2\nu)A}{4(1-\nu)} + \frac{B}{R} + C \ , \tag{9}$$

where B, C are two arbitrary constants which can be set to zero since the origin is not a singular point ($B = 0$) and the constant term C makes no contributions to the stress or displacement components.

The non-zero stress components are then obtained by analogy with equation (20.41) as[13]

$$\sigma_{RR} = \frac{\partial^2\phi}{\partial R^2} + \frac{\nu V}{(1-\nu)} = \frac{2\pi\rho^2(3-\nu)\gamma R^2}{15(1-\nu)} + \frac{A}{2(1-\nu)} \tag{10}$$

$$\sigma_{\theta\theta} = \sigma_{\beta\beta} = \frac{1}{R}\frac{\partial\phi}{\partial R} + \frac{\nu V}{(1-\nu)} = \frac{2\pi\rho^2(1+3\nu)\gamma R^2}{15(1-\nu)} + \frac{A}{2(1-\nu)} \tag{11}$$

The constant A must be chosen so as to satisfy the traction-free boundary condition

$$\sigma_{RR} = 0 \ ; \quad R = a \ , \tag{12}$$

giving

$$\frac{2\pi\rho^2(3-\nu)\gamma a^2}{15(1-\nu)} + \frac{A}{2(1-\nu)} = 0 \ , \tag{13}$$

[13]Notice that equations (20.41, 20.42) are equivalent to those in Solution A of Table 21.1, supplemented by a hydrostatic stress $\nu V/(1-\nu)$. The corresponding expressions in spherical polar coordinates are therefore conveniently obtained by adding the same hydrostatic stress to the components in Solution A in Table 21.2.

or

$$A = -\frac{4\pi\rho^2(3-\nu)\gamma a^2}{15} \ . \tag{14}$$

The final expressions for the stress components are therefore

$$\sigma_{RR} = -\frac{2\pi\rho^2(3-\nu)\gamma(a^2-R^2)}{15(1-\nu)} \tag{15}$$

$$\sigma_{\theta\theta} = \sigma_{\beta\beta} = -\frac{2\pi\rho^2\gamma[(3-\nu)a^2-(1+3\nu)R^2]}{15(1-\nu)} \ . \tag{16}$$

The maximum shear stress at any radius R is

$$\tau_{\max} = \frac{\sigma_{RR}-\sigma_{\theta\theta}}{2} = \frac{2\pi\rho^2(1-2\nu)\gamma R^2}{15(1-\nu)} \tag{17}$$

and the maximum value is $2\pi\rho^2(1-2\nu)\gamma a^2/15(1-\nu)$ at the surface $R=a$.

The maximum compressive stress occurs at the origin and is of magnitude

$$|\sigma|_{\max} = \frac{2\pi\rho^2(3-\nu)\gamma a^2}{15(1-\nu)} \ . \tag{18}$$

To apply this result to the earth, notice that the gravitational force acting on a particle of mass m at the surface is

$$F = \left(\frac{4\pi\rho a^3}{3}\right)\frac{m\gamma}{a^2} = \frac{4\pi\rho m\gamma a}{3} = mg \tag{19}$$

and hence

$$g = \frac{4\pi\rho\gamma a}{3} \ . \tag{20}$$

Using this result to eliminate γ in equations (15–18), we obtain

$$\sigma_{RR} = -\frac{\rho g(3-\nu)\gamma(a^2-R^2)}{10(1-\nu)a} \tag{21}$$

$$\sigma_{\theta\theta} = \sigma_{\beta\beta} = -\frac{\rho g[(3-\nu)a^2-(1+3\nu)R^2]}{10(1-\nu)a} \tag{22}$$

$$\tau_{\max} = \frac{\rho g(1-2\nu)\gamma R^2}{10(1-\nu)a} \tag{23}$$

$$|\sigma|_{\max} = \frac{\rho g(3-\nu)a}{10(1-\nu)} \ . \tag{24}$$

Notice that these results define the stresses which would be generated in a solid sphere if it were manufactured in a stress-free state, *after which* the gravitational loads were applied. In practice, astronomical bodies most likely developed by accretion

from systems of particles and the gravitational forces were present throughout this process. There is therefore every reason to expect the final stress field to differ from that obtained here. The problem of a self-gravitating sphere grown by accretion was considered by Brown, C.B. and Goodman, L.E., 1963. Gravitational stresses in accreted bodies, *Proceedings of the Royal Society of London, Series A* **276**, 571-576.

26.20. The solid cone $0 < \beta < \beta_0$ is loaded by a force in the x-direction applied at the vertex. The surface of the cone is otherwise traction-free. Find the stress field in the cone. Notice that Problem 26.18 is a special case of this solution for $\beta_0 = \pi/2$.

Since there is a concentrated force at the origin, we anticipate stresses decaying with R^{-2}. Also, the problem is antisymmetric about the y-axis and symmetric about the x-axis, suggesting potentials involving $\cos\theta$ ($\sin\theta$ in Solution E).

We therefore select the potentials

$$\phi = AR^0 P_0^1(\cos\beta)\cos\theta = \frac{A\sin\beta\cos\theta}{(1+\cos\beta)} \tag{1}$$

$$\omega = BR^{-1} P_0^1(\cos\beta)\cos\theta = \frac{A\sin\beta\cos\theta}{R(1+\cos\beta)} \tag{2}$$

$$\psi = CR^0 P_0^1(\cos\beta)\sin\theta = \frac{A\sin\beta\cos\theta}{(1+\cos\beta)} \tag{3}$$

Notice that the same Legendre function P_0^1 appears in all three functions, even though they have different powers of R. This is because $n = 0$ and $n = -1$ are related through the equation $n = -p - 1$ (see §24.3.1).

Using Maple or Mathematica, we substitute these expressions into Table 21.2 and in particular find the tractions on the surface $\beta = \beta_0$, which are

$$\sigma_{\beta R} = \tag{4}$$

$$\sigma_{\beta\theta} = \tag{5}$$

$$\sigma_{\beta\beta} = \quad . \tag{6}$$

Setting these tractions to zero yields just two independent conditions

$$A + B(1-\nu) - C = 0\,; \quad A - 2B(1-\nu) = 0\,. \tag{7}$$

The remaining equation is obtained from the equilibrium condition

$$\int_0^{\beta_0}\int_0^{2\pi} [(\sigma_{RR}\sin\beta + \sigma_{R\beta}\cos\beta]\cos\theta - \sigma_{R\theta}\sin\theta]R^2\sin\beta d\theta d\beta + F = 0\,. \tag{8}$$

Substituting for the stress components and evaluating the integral (in Maple), we obtain an additional condition for A, B, C which can then be solved in combination with (7). Substituting the resulting values of the constants into the stress equations then yields the complete stress field.

26.21. The surface of a solid sphere of radius a is traction-free, but is subjected to the steady-state heat flux

$$q_R = q_0(3\cos(2\beta) + 1) \ .$$

In other words, it is heated near the equator and cooled near the poles.

Find the complete stress and displacement field in the sphere. Also, find the difference between the largest and smallest external diameter of the thermally distorted sphere.

The temperature must be a harmonic function and the form of heat flux distribution suggests the spherical harmonic distribution

$$T(R,\theta,\beta) = C_1 R^2 P_2(\cos\beta) = \frac{C_1 R^2(3\cos(2\beta)+1)}{4} \ ,$$

from equation (24.11). The radial heat flux is then

$$q_R = -K\frac{\partial T}{\partial R} = -\frac{C_1 K R(3\cos(2\beta)+1)}{2}$$

and the condition $q_R = q_0(3\cos(2\beta)+1)$ at $R = a$ yields

$$C_1 = -\frac{2q_0}{Ka}$$

and

$$T(R,\theta,\beta) = -\frac{q_0 R^2(3\cos(2\beta)+1)}{2Ka} \ .$$

To obtain a particular thermoelastic solution, we need to find a function χ such that

$$\frac{\partial\chi}{\partial z} = \frac{\alpha\mu(1+\nu)T}{(1-\nu)} = -\frac{\alpha\mu q_0(1+\nu)R^2(3\cos(2\beta)+1)}{2K(1-\nu)a} \ .$$

This is most easily done using the cylindrical coordinate forms of equation (24.27). We have

$$R^3 P_3(z/R) = \frac{1}{2}(2z^3 - 3zr^2) \ ; \quad \frac{\partial}{\partial z}R^3 P_3(z/R) = \frac{3}{2}(2z^2 - r^2) = 3R^2 P_2(z/R)$$

and hence the required potential is

$$\chi = \frac{\alpha\mu(1+\nu)C_1 R^3 P_3(\cos\beta)}{3(1-\nu)} = -\frac{\alpha\mu q_0(1+\nu)R^3(5\cos(3\beta)+3\cos\beta)}{12K(1-\nu)a} \ .$$

The non-zero stress components from this particular solution are then obtained from Table 22.2 as

$$\begin{aligned}
\sigma_{RR} &= -\frac{\alpha\mu q_0(1+\nu)R^2(1+2\cos(2\beta)+5\cos(4\beta))}{2K(1-\nu)a} \\
\sigma_{\theta\theta} &= \frac{2\alpha\mu q_0(1+\nu)R^2(1+2\cos(2\beta))}{K(1-\nu)a} \\
\sigma_{\beta\beta} &= -\frac{\alpha\mu q_0(1+\nu)R^2(9-6\cos(2\beta)-5\cos(4\beta))}{2K(1-\nu)a} \\
\sigma_{R\beta} &= \frac{\alpha\mu q_0(1+\nu)R^2(4\sin(2\beta)+5\sin(4\beta))}{2K(1-\nu)a}
\end{aligned}$$

To complete the solution, we require spherical harmonics in Solutions A and B that give symmetric stress fields with variation in σ_{RR} up to $\cos(4\beta)$. We therefore select

$$\phi = C_2R^2P_2(\cos\beta)+C_3R^4P_4(\cos\beta) = \frac{C_2R^2(3\cos(2\beta)+1)}{4}+\frac{C_3R^4(35\cos(4\beta)+20\cos(2\beta)+9)}{64}$$

$$\omega = C_4RP_1(\cos\beta)+C_5R^3P_3(\cos\beta) = C_4R\cos\beta+\frac{C_5R^3(5\cos(3\beta)+3\cos\beta)}{8}$$

with solution

$$C_2 = -\frac{2\alpha\mu q_0 a(1+\nu)}{K(7+5\nu)}\ ;\quad C_3 = \frac{8\alpha\mu q_0(1+\nu)}{3Ka(7+5\nu)}\ ;\quad C_4 = 0\ ;\quad C_5 = \frac{8\alpha\mu q_0(1+\nu)}{Ka(7+5\nu)(1-\nu)}\ .$$

The final stress field is

$$\begin{aligned}
\sigma_{RR} &= -\frac{\alpha\mu q_0(1+\nu)(a^2-R^2)(1+3\cos(2\beta))}{Ka(7+5\nu)} \\
\sigma_{\theta\theta} &= \frac{2\alpha\mu q_0(1+\nu)(a^2-2R^2+3R^2\cos(2\beta))}{Ka(7+5\nu)} \\
\sigma_{\beta\beta} &= \frac{\alpha\mu q_0(1+\nu)(-a^2+5R^2+3(a^2-R^2)\cos(2\beta))}{Ka(7+5\nu)} \\
\sigma_{R\beta} &= \frac{3\alpha\mu q_0(1+\nu)(a^2-R^2)\sin(2\beta)}{Ka(7+5\nu)}
\end{aligned}$$

and the displacements are

$$u_R = -\frac{\alpha\mu q_0(1+\nu)R(a^2+2R^2)(1+3\cos(2\beta))}{2Ka(7+5\nu)}\ ;\quad u_\beta = \frac{\alpha\mu q_0(1+\nu)R(a^2-3R^2)\sin(2\beta)}{2Ka(7+5\nu)}\ .$$

CHAPTER 27

27.1. Verify that equation (27.17) defines a function satisfying equation (27.11) by direct substitution, using the result

$$\frac{\partial^2 g}{\partial z^2} = -\frac{\partial^2 g}{\partial r^2} - \frac{1}{r}\frac{\partial g}{\partial r}$$

(which is a restatement of the condition $\nabla^2 g = 0$) to eliminate the derivatives with respect to z.

Starting from

$$\phi = r^2 \frac{\partial^2 g}{\partial r^2} - r\frac{\partial g}{\partial r} \ , \tag{1}$$

we have

$$\begin{aligned}
\frac{\partial \phi}{\partial r} &= r\frac{\partial^2 g}{\partial r^2} + r^2\frac{\partial^3 g}{\partial r^3} - \frac{\partial g}{\partial r} && (2)\\
\frac{\partial^2 \phi}{\partial r^2} &= 3r\frac{\partial^3 g}{\partial r^3} + r^2\frac{\partial^4 g}{\partial r^4} && (3)\\
\frac{\partial^2 \phi}{\partial z^2} &= r^2\frac{\partial^4 g}{\partial r^2 \partial z^2} - r\frac{\partial^3 g}{\partial r \partial z^2} \ . && (4)
\end{aligned}$$

Also, using

$$\frac{\partial^2 g}{\partial z^2} = -\frac{\partial^2 g}{\partial r^2} - \frac{1}{r}\frac{\partial g}{\partial r} \tag{5}$$

we have

$$\begin{aligned}
\frac{\partial^3 g}{\partial r \partial z^2} &= \frac{1}{r^2}\frac{\partial g}{\partial r} - \frac{1}{r}\frac{\partial^2 g}{\partial r^2} - \frac{\partial^3 g}{\partial r^3} && (6)\\
\frac{\partial^4 g}{\partial r^2 \partial z^2} &= -\frac{2}{r^3}\frac{\partial g}{\partial r} + \frac{2}{r^2}\frac{\partial^2 g}{\partial r^2} - \frac{1}{r}\frac{\partial^3 g}{\partial r^3} - \frac{\partial^4 g}{\partial r^4} \ , && (7)
\end{aligned}$$

so

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial z^2} &= -\frac{2}{r}\frac{\partial g}{\partial r} + 2\frac{\partial^2 g}{\partial r^2} - r\frac{\partial^3 g}{\partial r^3} - r^2\frac{\partial^4 g}{\partial r^4} - \frac{1}{r}\frac{\partial g}{\partial r} + \frac{\partial^2 g}{\partial r^2} + r\frac{\partial^3 g}{\partial r^3} && (8)\\
&= -\frac{3}{r}\frac{\partial g}{\partial r} + 3\frac{\partial^2 g}{\partial r^2} - r^2\frac{\partial^4 g}{\partial r^4} \ . && (9)
\end{aligned}$$

Finally, substituting (2, 3, 9) into (27.11), we obtain

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial r^2} - \frac{3}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} &= 3r\frac{\partial^3 g}{\partial r^3} + r^2\frac{\partial^4 g}{\partial r^4} - 3\frac{\partial^2 g}{\partial r^2} - 3r\frac{\partial^3 g}{\partial r^3} + \frac{3}{r}\frac{\partial g}{\partial r}\\
&\quad -\frac{3}{r}\frac{\partial g}{\partial r} + 3\frac{\partial^2 g}{\partial r^2} - r^2\frac{\partial^4 g}{\partial r^4} && (10)\\
&= 0 \ . && (11)
\end{aligned}$$

27.2. The solid cylindrical bar $0 \le r < a, 0 < z < L$ is loaded by a uniform torsional shear traction $\sigma_{r\theta} = S$ on the surface $r = a$, the end $z = 0$ being traction-free. Find the stress field in the bar, using bounded polynomial solutions to equation (27.11).

Equilibrium arguments show that the torque in the bar must increase linearly with z. We know from §27.5 that the function ϕ_4 defines a constant torque in the cylindrical bar, so it is reasonable to start with the function

$$\phi = C\phi_5 = Czr^4 \ , \tag{1}$$

where C is an arbitrary constant. Substitution into (27.5, 27.7) then yields

$$\sigma_{z\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial r} = 4Czr \ ; \quad \sigma_{r\theta} = -\frac{1}{r^2}\frac{\partial \phi}{\partial z} = -Cr^2 \ . \tag{2}$$

The boundary condition

$$\sigma_{r\theta} = S \tag{3}$$

on $r = a$ then gives $-Ca^2 = S$ and hence

$$C = -\frac{S}{a^2} \ . \tag{4}$$

On the end $z = 0$, we have $\sigma_{z\theta} = 0$ and hence the end condition is satisfied in the strong sense. The solution is therefore given by

$$\phi = -\frac{Szr^4}{a^2} \ ; \quad \sigma_{z\theta} = -\frac{4Szr}{a^2} \ ; \quad \sigma_{r\theta} = \frac{Sr^2}{a^2} \ . \tag{5}$$

27.3. Show that the function

$$\varphi_1 = z$$

satisfies equation (27.11) and defines a state of stress that is independent of z. Combine this solution with a bounded polynomial solution of (27.11) to determine the stress field in the hollow cylindrical bar $a<r<b, 0<z<L$ built in at $z=L$ and loaded by a uniform torsional shear traction $\sigma_{r\theta}=S$ on the inner surface $r=a$, the outer surface $r=b$ and the end $z=0$ being traction-free.

The proposed function is independent of r and only linear in z so it is clear by inspection that it will satisfy (27.11).

The boundary condition on $r=b$ requires that $\phi(b,z)$ be independent of z, so the proposed function must be supplemented by a bounded term that also varies linearly with z. From equations (27.19), we therefore choose ϕ_5 giving a trial function

$$\phi = Az + Br^4 z \ . \tag{1}$$

The corresponding stress components are

$$\sigma_{r\theta} = -\frac{1}{r^2}\frac{\partial\phi}{\partial z} = -\frac{A}{r^2} - Br^2 \tag{2}$$

$$\sigma_{z\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial r} = 4Brz \ . \tag{3}$$

The strong boundary conditions are

$$\sigma_{r\theta} = S \ ; \quad r=a \tag{4}$$

$$= 0 \ ; \quad r=b \ , \tag{5}$$

giving the equations

$$-\frac{A}{a^2} - Ba^2 = S \ ; \quad -\frac{A}{b^2} - Bb^2 = 0 \ . \tag{6}$$

Solving for A, B, we have

$$A = -\frac{Sa^2b^4}{(b^4-a^4)} \ ; \quad B = \frac{Sa^2}{(b^4-a^4)} \tag{7}$$

and the final stress field is

$$\sigma_{r\theta} = \frac{Sa^2}{(b^4-a^4)}\left(\frac{b^4-r^4}{r^2}\right) \tag{8}$$

$$\sigma_{z\theta} = \frac{4Sa^2rz}{(b^4-a^4)} \ . \tag{9}$$

27.4. A solid cylindrical bar of radius b transmitting a torque T, contains a small spherical hole of radius a ($\ll b$) centred on the axis. Find a bounded polynomial solution to equation (27.11) to describe the stress field in the bar in the absence of the hole and hence determine the perturbation in this field due to hole by superposing a suitable singular solution.

In the absence of the hole, the stress field in the bar is given by the stress function (27.32). In particular,

$$\phi = \frac{Tr^4}{2\pi b^4} \; ; \quad \sigma_{z\theta} = \frac{2Tr}{\pi b^4} \; ; \quad \sigma_{r\theta} = 0 \; , \tag{1}$$

from (27.33–27.35).

To complete the solution we need to superpose a stress function that is singular at the origin and that has the same spherical polar form as the unperturbed solution. This is clearly ϕ_{-1} of equations (27.20). We therefore write

$$\phi = \frac{TR^4 \sin^4\beta}{2\pi b^4} + \frac{C \sin^4\beta}{R} \; , \tag{2}$$

where C is an arbitrary constant. On the surface of the hole, we have

$$\phi(R=a) = \frac{Ta^4 \sin^4\beta}{2\pi b^4} + \frac{C \sin^4\beta}{a} \; . \tag{3}$$

The hole is traction free and hence ϕ must be constant around it. This can be achieved by choosing C such that

$$\frac{Ta^4}{2\pi b^4} + \frac{C}{a} = 0 \; , \tag{4}$$

in which case $\phi = 0$ on $R = a$. Solving for C, we have

$$C = -\frac{Ta^5}{2\pi b^4} \tag{5}$$

and hence

$$\phi = \frac{TR^4 \sin^4\beta}{2\pi b^4} - \frac{Ta^5 \sin^4\beta}{2\pi b^4 R} = \frac{Tr^4}{2\pi b^4} - \frac{Ta^5 r^4}{2\pi b^4 (r^2+z^2)^{5/2}} \tag{6}$$

in cylindrical polar coordinates. The stress components can then be obtained as

$$\sigma_{\theta r} = -\frac{1}{r^2}\frac{\partial \phi}{\partial z} = -\frac{5Ta^5 z r^2}{2\pi b^4 (r^2+z^2)^{7/2}} \; ; \tag{7}$$

$$\sigma_{\theta z} = \frac{1}{r^2}\frac{\partial \phi}{\partial r} = \frac{2Tr}{\pi b^4} - \frac{2Ta^5 r}{\pi b^4 (r^2+z^2)^{5/2}} \left(1 - \frac{5r^2}{4(r^2+z^2)}\right) \; . \tag{8}$$

At the 'equator' $(a, 0)$, we have

$$\sigma_{\theta z} = \frac{2.5Ta}{\pi b^4} \; , \tag{9}$$

which exceeds the unperturbed stress at this point by 25%.

27.5. Develop a contour plot in the r, z plane of a function comprising

(i) the function ϕ_4 of equation (27.19),

(ii) a centre of rotation, centred on the point $(0, a)$,

(iii) an equal negative centre of rotation, centred on the point $(0, -a)$.

By experimenting with different multiplying constants on these functions, show that the solution can be obtained for a cylindrical bar with an approximately elliptical central hole, transmitting a torque T. If the bar has outer radius b and the hole has radius $0.1b$ at the mid-plane and is of total axial length $0.4b$, estimate the stress at the point $0.1b, 0$.

The stress function can be assembled as

$$\phi = r^4 + C(g(z+a) - g(z-a)) \ , \tag{1}$$

where

$$g(\zeta) = \frac{(\sqrt{r^2+\zeta^2} - \zeta)^2 (2\sqrt{r^2+\zeta^2} + \zeta)}{(r^2+\zeta^2)^{3/2}} \ . \tag{2}$$

The size of the hole and its ellipticity can be adjusted by changing the parameters C, a. In general, increasing C makes a larger hole and increasing a causes the hole to be longer along the z-axis. Figure 1 shows the contours obtained using the parameters $C = 0.0002, a = 0.3$. The hole has an aspect ratio of about two to one as required, though it is closer to a rounded rectangle than to an ellipse. A more elliptical shape could be obtained using a distribution of centres of rotation.

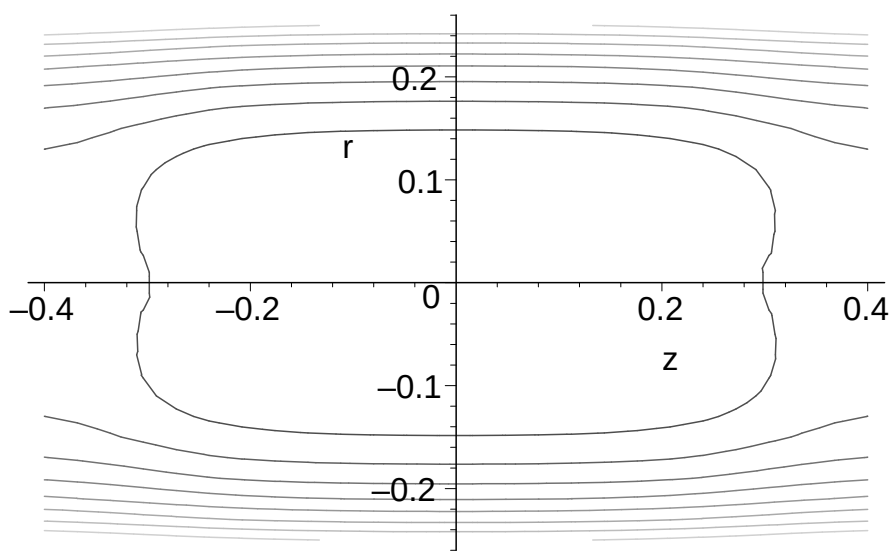


Figure 1

The stress function can be scaled so as to make the hole surface pass through the points $(0, 0.2b)$ and $(0.1b, 0)$. Noting that the present curve passes through the point $(0.148, 0)$ We obtain

$$\phi = \left(\frac{r}{b}\right)^4 + \frac{0.002}{0.148^4}\left(g\left(\frac{z}{b} + \frac{0.3}{0.148}\right) - g\left(\frac{z}{b} - \frac{0.3}{0.148}\right)\right) . \tag{3}$$

A plot of this function is given in Figure 2.

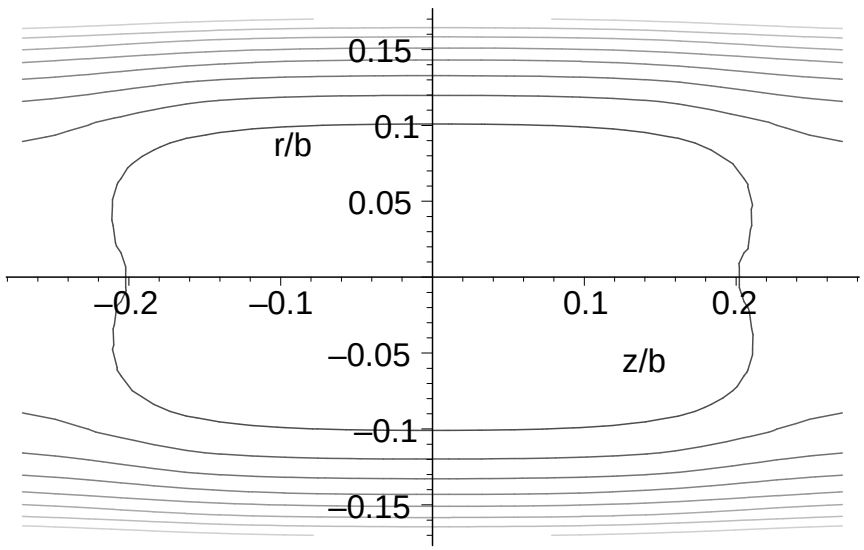


Figure 2

Finally, to ensure that the torque transmitted is equal to T, we ensure that the leading (unperturbed) term is $Tr^4/2\pi b^4$ by multiplying by $T/2\pi$, obtaining

$$\phi = \frac{T}{2\pi}\left[\left(\frac{r}{b}\right)^4 + \frac{0.002}{0.148^4}\left(g\left(\frac{z}{b} + \frac{0.3}{0.148}\right) - g\left(\frac{z}{b} - \frac{0.3}{0.148}\right)\right)\right] . \tag{4}$$

The general expression for the stress components can now be found from equations (27.5, 27.7). In particular, at $(0.1b, 0)$ we have

$$\sigma_{\theta z} = \frac{0.204T}{\pi b^3} . \tag{5}$$

This should be compared with the value $0.2T/\pi b^3$ at this point in the unperturbed solution. The increase in stress (about 2%) is negligible and is considerably less than that due to a circular hole (25%, see Problem 27.4). If the hole was very long, the stress at this point would approach that in a hollow bar which exceeds that in a solid bar only in the ratio $1/(1 - 0.1^4)$ — i.e. a 0.01% increase. As expected, the approximate elliptical solution is intermediate between these extremes.

27.6. Show that the lines $R-z=C$ define a set of parabolas, where C is a constant. Combine suitable functions from equations (27.18–27.24) to obtain a solution of (27.11) that is constant on these lines and hence solve the problem of a paraboloidal bar transmitting a torque T. Comment on the nature of the solution in the region $z \leq 0$ and in particular at the origin. Do your results place any restrictions on the applicability of the solution?

The lines $R-z=C$ correspond to

$$R = \sqrt{r^2 + z^2} = C + z \tag{1}$$

and hence

$$r^2 + z^2 = C^2 + 2Cz + z^2 \ , \quad r^2 = C^2 + 2Cz \tag{2}$$

or

$$z + \frac{C}{2} = \frac{r^2}{2C} \ . \tag{3}$$

This defines a parabola passing through the point $(0, -C/2)$.

The function

$$\phi = \phi_2 = A(R - z)^2 \tag{4}$$

from equation (27.22) satisfies (27.11) and is constant along the line and hence corresponds to the case in which the paraboloid of revolution is traction-free. In the range $z > 0$, $R - z = 0$ on the axis $r = 0$ and hence the transmitted torque from equation (27.4) is

$$T = 2\pi A C^2 \ . \tag{5}$$

Solving for the arbitrary constant A, we have

$$A = \frac{T}{2\pi C^2} \tag{6}$$

and hence

$$\phi = \frac{T(R - z)^2}{2\pi C^2} \ . \tag{7}$$

The stress components are

$$\sigma_{r\theta} = -\frac{1}{r^2}\frac{\partial \phi}{\partial z} = \frac{T(R - z)^2}{\pi C^2 r^2 R} = \frac{T(R - z)}{\pi C^2 R(R + z)} \tag{8}$$

$$\sigma_{z\theta} = \frac{1}{r^2}\frac{\partial \phi}{\partial r} = \frac{T(R - z)}{\pi C^2 r R} = \frac{Tr}{\pi C^2 R(R + z)} \ , \tag{9}$$

where we have used the fact that $r^2 = (R^2 - z^2) = (R - z)(R + z)$.

The factor $R(R+z)$ in the denominator of equations (8,9) imply that the stresses will be singular at the origin and also throughout the negative z-axis. Thus, the solution cannot be applied to that part of the paraboloidal shaft in the region $z < 0$.

We can apply the solution to the truncated paraboloid $z > h$, but we should note that for small values of h, the implied tractions on the circular surface $z = h$ will become more and more concentrated near the origin. Only weak boundary conditions can be imposed on this edge — in fact we have already imposed these conditions by specifying the transmitted torque — so the solution will only be correct near the end if the actual tractions follow those implied by equation (9). In this context, note that the relevant length scale is defined by the length C, so we anticipate problems if the truncation point lies in the range $h/C \ll 1$.

27.7. The solid cylindrical bar $0 \le r < a, z > 0$ is bonded to a rigid body at the curved surface $r = a$ and loaded by torsional tractions on the plane end $z = 0$. Using a displacement function

$$\psi = e^{-\lambda z} f(r) \ ,$$

find suitable non-trivial solutions for the stress field and hence determine the slowest rate at which the stress field will decay along the bar.

The displacement function ψ must satisfy the condition

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{3}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0 \ , \tag{1}$$

from equation (27.31). Substituting

$$\psi = e^{-\lambda z} f(r) \tag{2}$$

and cancelling the exponential factor, we obtain the ordinary differential equation

$$\frac{d^2 f}{dr^2} + \frac{3}{r}\frac{df}{dr} + \lambda^2 f = 0 \ . \tag{3}$$

The general solution of this equation is

$$f = \frac{C_1 J_1(\lambda r)}{r} + \frac{C_2 Y_1(\lambda r)}{r} \ , \tag{4}$$

where J_1, Y_1 are Bessel functions of the first and second kind respectively of order 1 and C_1, C_2 are arbitrary constants.

The second term is singular at $r = 0$ and must be dropped for the solid bar. The first term is bounded because $J_1(r) \to O(r)$ as $r \to 0$. The displacement $u_\theta = 0$ at $r = a$, giving the boundary condition

$$\psi(a, z) = \frac{C_1 e^{-\lambda z} J_1(\lambda a)}{a} = 0 \tag{5}$$

and this condition is satisfied for all z if

$$J_1(\lambda a) = 0 \ . \tag{6}$$

The first zero of this equation occurs at $\lambda a \approx 3.83$, implying a slower decay than that due to self-equilibrated tractions on the end of a traction-free bar (see §27.6).

CHAPTER 28

28.1. A circular cylindrical bar of radius a is loaded by the tangential tractions $\sigma_{r\theta} = S\cos\theta$ on $r = a$. The end of the bar $z = 0$ is traction-free. Find the complete stress field in the bar.

The tractions are uniform along the bar, so this is a problem in class $\mathcal{P}_1$, for which the solution of $\mathcal{P}_0$ is

$$\phi_0 = \mathcal{Q}_0^0 \; ; \quad \psi_0 = \mathcal{V}_0^0 \; ,$$

from equations (28.41). As in §28.9.1, we substitute into (28.53) with $j = 0$ and integrate with respect to z, obtaining

$$\phi_1 = \mathcal{Q}_0^1 + \mathcal{G}_1^1 \; ; \quad \psi_1 = \mathcal{V}_0^1 + \mathcal{H}_1^1 \; .$$

The given external loading produces no torsion of the bar, so we set $C_0 = 0$.

The corresponding in-plane stresses are

$$\begin{aligned} \Theta &= \frac{\left[2(1-\nu)\zeta\bar{\zeta} - (3+2\nu)(3-4\nu)a^2\right](A_0\zeta + \bar{A}_0\bar{\zeta})}{16(1-\nu^2)} \\ \Phi &= \frac{(1+4\nu)A_0\zeta^3}{24(1+\nu)} + \frac{\left[2(1-\nu)\zeta\bar{\zeta} - (3+2\nu)a^2\,\right]\bar{A}_0\zeta}{16(1-\nu^2)} \; , \end{aligned} \tag{1}$$

as in (28.56).

In-plane correction

On the boundary $r = a$, we require

$$T_r + \imath T_\theta = \imath S\cos\theta = \frac{\imath S(e^{\imath\theta} + e^{-\imath\theta})}{2}$$

and in view of the vector transformation equation (19.13), we have

$$T_r + \imath T_\theta = Te^{-\imath\theta}$$

and hence

$$T \equiv T_x + \imath T_y = (T_r + \imath T_\theta)e^{\imath\theta} = \frac{\imath S(e^{2\imath\theta} + 1)}{2} \; .$$

Equation (28.64) requires that

$$\int T ds = a\int T d\theta = \frac{a}{2}\int(\Phi e^{-\imath\theta} + \Theta e^{\imath\theta})d\theta \tag{2}$$

and hence

$$T = \frac{1}{2}(\Phi e^{-\imath\theta} + \Theta e^{\imath\theta}) \; .$$

On $r = a$, we have

$$\zeta = ae^{\imath\theta} \; ; \quad \bar{\zeta} = ae^{-\imath\theta} \; ; \quad \zeta\bar{\zeta} = a^2 \; ,$$

$$2(1-\nu)\zeta\bar{\zeta}-(3+2\nu)(3-4\nu)a^2 = -(7-4\nu-8\nu^2)a^2 \; ; \quad 2(1-\nu)\zeta\bar{\zeta}-(3+2\nu)a^2 = -(1+4\nu)a^2$$

and hence

$$\begin{aligned} \Theta &= -\frac{(7-4\nu-8\nu^2)a^3(A_0e^{\imath\theta}+\bar{A}_0e^{-\imath\theta})}{16(1-\nu^2)} \\ \Phi &= \frac{(1+4\nu)A_0a^3e^{3\imath\theta}}{24(1+\nu)} - \frac{(1+4\nu)a^3\bar{A}_0e^{\imath\theta}}{16(1-\nu^2)} \; . \end{aligned} \tag{3}$$

Then

$$\begin{aligned} \Phi e^{-\imath\theta} + \Theta e^{\imath\theta} &= \frac{(1+4\nu)A_0a^3e^{2\imath\theta}}{24(1+\nu)} - \frac{(1+4\nu)a^3\bar{A}_0}{16(1-\nu^2)} - \frac{(7-4\nu-8\nu^2)a^3(A_0e^{2\imath\theta}+\bar{A}_0)}{16(1-\nu^2)} \\ &= -\frac{\bar{A}_0a^3}{2} - \frac{(19-18\nu-16\nu^2)A_0a^3e^{2\imath\theta}}{48(1-\nu^2)} \end{aligned} \tag{4}$$

and hence

$$T - \frac{1}{2}(\Phi e^{-\imath\theta} + \Theta e^{\imath\theta}) = \frac{\imath S(e^{2\imath\theta}+1)}{2} + \frac{\bar{A}_0a^3}{4} + \frac{(19-18\nu-16\nu^2)A_0a^3e^{2\imath\theta}}{96(1-\nu^2)} \; .$$

The constant A_0 must be chosen so as to make the constant term in this expression vanish, since otherwise the integration (2) would give a multi-valued error that could not be corrected with the addition of a holomorphic function. We therefore choose

$$\bar{A}_0 = -\frac{2\imath S}{a^3} \; ; \quad A_0 = \frac{2\imath S}{a^3}$$

and with this value,

$$T - \frac{1}{2}(\Phi e^{-\imath\theta} + \Theta e^{\imath\theta}) = \frac{(43-18\nu-40\nu^2)\imath Se^{2\imath\theta}}{48(1-\nu^2)} \; .$$

These unwanted tractions have the same form as those in equation (28.67) and we therefore anticipate that they can be removed by adding the additional terms (28.68) with an appropriate choice of the constant C. We therefore write

$$\phi_1 = \mathcal{Q}_0^1 + \mathcal{G}_1^1 \; ; \quad \psi_1 = \mathcal{V}_0^1 + \mathcal{H}_1^1 + C\chi_0^2 \; .$$

Repeating the above calculations with these revised stress functions, we obtain

$$T - \frac{1}{2}(\Phi e^{-\imath\theta} + \Theta e^{\imath\theta}) = \frac{(43-18\nu-40\nu^2)\imath Se^{2\imath\theta}}{48(1-\nu^2)} + 2Cae^{2\imath\theta}$$

and this will be zero if

$$C = -\frac{(43-18\nu-40\nu^2)\imath S}{96(1-\nu^2)} \; .$$

Antiplane correction

Using the new stress function with this value for the constant C, we next evaluate the stress component Ψ and hence the traction T_z on $r = a$. However, this expression involves only terms with $A_1, \bar{A}_1, B_1$ and since we require $T_z = 0$, this is achieved by setting these constants to zero and hence eliminating the terms $\mathcal{G}_1^1, \mathcal{H}_1^1$ in ϕ, ψ respectively. With a more thoughtful initial analysis of the problem, we could have achieved some simplification by omitting these terms from the beginning, since it is clear that no antiplane tractions would be introduced by the in-plane correction and there are no applied antiplane tractions. However, we emphasise here the hierarchical and formal characteristic of the solution procedure.

End conditions

It remains to satisfy the weak traction-free conditions on the end $z=0$. We add the pure bending solution (28.26) with new constants A_2, B_2 into the stress field, which is achieved by adding the terms $\mathcal{G}_2^0, \mathcal{H}_2^0$ into ϕ, ψ respectively, giving

$$\phi = \mathcal{Q}_0^1 + \mathcal{G}_2^0 \; ; \quad \psi = \mathcal{V}_0^1 + \mathcal{H}_2^0 + C\chi_0^2 \; .$$

We then substitute these potentials into (28.18, 28.19) to obtain the complex stress components σ_{zz}, Ψ. In particular, the tractions on the end of the bar $z = 0$ are obtained as

$$\begin{aligned} \sigma_{zz}(0,\zeta,\bar{\zeta}) &= \frac{\imath S(2(9+19\nu+10\nu^2)a^2 - (2+\nu)\zeta\bar{\zeta})(\zeta-\bar{\zeta})}{12a^3(1+\nu)} + A_2\zeta + \bar{A}_2\bar{\zeta} + B_2 \\ \Psi(0,\zeta,\bar{\zeta}) &= 0 \end{aligned} \tag{5}$$

and the force and moment resultants are

$$F_z = \int_0^a \int_0^{2\pi} \sigma_{zz}(0,r,\theta) r d\theta dr = \pi B_2 a^2$$

$$M_x + \imath M_y = \int_0^a \int_0^{2\pi} \sigma_{zz}(0,r,\theta) r^2 e^{\imath\theta} dr d\theta = -\frac{\pi S a^3(7+18\nu+10\nu^2)}{12(1+\nu)} + \frac{\pi \bar{A}_2 a^4}{2} \; .$$

Thus, the weak conditions are satisfied by the choice

$$B_2 = 0 \; ; \quad \bar{A}_2 = \frac{\imath S(7+18\nu+10\nu^2)}{6a(1+\nu)} \; ; \quad A_2 = -\frac{\imath S(7+18\nu+10\nu^2)}{6a(1+\nu)}$$

and the final stress field is obtained from the potentials

$$\phi = \frac{(1-2\nu)\imath S}{(1-\nu^2)a^3}\left(\chi_4^1 - \overline{\chi}_4^1\right) + \frac{(3-4\nu)(3+2\nu)\imath S(\chi_2^1 - \overline{\chi}_2^1)}{8a(1-\nu^2)}$$

$$
\begin{aligned}
& -\frac{\imath S(7+18\nu+10\nu^2)(1-2\nu)}{12a(1+\nu)(1-\nu^2)}\left(\chi_2^1-\overline{\chi}_2^1\right) \\
\psi \quad = \quad & -\frac{2\imath S\chi_4^0}{(1-\nu^2)a^3}-\frac{(1-2\nu)\imath S\chi_2^2}{4(1-\nu^2)a^3}-\frac{(3+2\nu)\imath S\chi_2^0}{4(1-\nu^2)a}+\frac{\imath S(7+18\nu+10\nu^2)\chi_2^0}{6a(1+\nu)(1-\nu^2)} \\
& +\frac{\imath S(7+18\nu+10\nu^2)(1-2\nu)\chi_0^2}{48a(1+\nu)(1-\nu^2)}-\frac{\imath S(43-18\nu-40\nu^2)\chi_0^2}{96(1-\nu^2)}
\end{aligned}
$$

28.2. A circular cylindrical bar of radius a is loaded by a uniform radially-inward load F_0 per unit length along the line $\theta=0$. The end of the bar $z=0$ is traction-free. Find the complete stress field in the bar.

The tractions are uniform along the bar, so this is a problem in class $\mathcal{P}_2$, for which the solution of $\mathcal{P}_1$ is

$$\phi_0 = \mathcal{Q}_0^0 \; ; \quad \psi_0 = \mathcal{V}_0^0 \; ,$$

from equations (28.41).

We next integrate with respect to z, obtaining the particular solution

$$\phi_P = \mathcal{Q}_0^1 \; ; \quad \psi_P = \mathcal{V}_0^1 \; .$$

Notice that there is no need to add in the bending solution before performing the integration, since there are no zeroth order antiplane tractions in problem $\mathcal{P}_2$, so at the final stage, the process of cancelling these tractions would simply force all the coefficients in the added solution to zero. Notice also that the symmetry of the loading ensures that there will be no torsion, so we can set the constant C_0 to zero.

In-plane correction

The in-plane stresses corresponding to this particular solution are

$$\begin{aligned} \Theta &= \frac{\left[2(1-\nu)\zeta\bar{\zeta} - (3+2\nu)(3-4\nu)a^2\right](A_0\zeta + \bar{A}_0\bar{\zeta})}{16(1-\nu^2)} \\ \Phi &= \frac{(1+4\nu)A_0\zeta^3}{24(1+\nu)} + \frac{\left[2(1-\nu)\zeta\bar{\zeta} - (3+2\nu)a^2\right]\bar{A}_0\zeta}{16(1-\nu^2)} \; , \end{aligned} \tag{1}$$

from (28.16, 28.17).

On the boundary $r = a$, we require

$$T \equiv T_x + \imath T_y = \frac{1}{2}(\Phi e^{-\imath\theta} + \Theta e^{\imath\theta}) = -\frac{F_0\delta(\theta)}{a} \; .$$

On $r = a$, we have

$$\zeta = ae^{\imath\theta} \; ; \quad \bar{\zeta} = ae^{-\imath\theta} \; ; \quad \zeta\bar{\zeta} = a^2 \; ,$$

$$2(1-\nu)\zeta\bar{\zeta} - (3+2\nu)(3-4\nu)a^2 = -(7-4\nu-8\nu^2)a^2 \; ; \quad 2(1-\nu)\zeta\bar{\zeta} - (3+2\nu)a^2 = -(1+4\nu)a^2$$

and hence

$$\begin{aligned} \Theta &= -\frac{(7-4\nu-8\nu^2)a^3(A_0e^{\imath\theta} + \bar{A}_0e^{-\imath\theta})}{16(1-\nu^2)} \\ \Phi &= \frac{(1+4\nu)A_0a^3e^{3\imath\theta}}{24(1+\nu)} - \frac{(1+4\nu)a^3\bar{A}_0e^{\imath\theta}}{16(1-\nu^2)} \; . \end{aligned} \tag{2}$$

Then

$$\begin{aligned}\Phi e^{-\imath\theta} + \Theta e^{\imath\theta} &= \frac{(1+4\nu)A_0 a^3 e^{2\imath\theta}}{24(1+\nu)} - \frac{(1+4\nu)a^3\bar{A}_0}{16(1-\nu^2)} \\ &\quad - \frac{(7-4\nu-8\nu^2)a^3(A_0 e^{2\imath\theta} + \bar{A}_0)}{16(1-\nu^2)} \\ &= -\frac{\bar{A}_0 a^3}{2} - \frac{(19-18\nu-16\nu^2)A_0 a^3 e^{2\imath\theta}}{48(1-\nu^2)} \ . \end{aligned} \tag{3}$$

In order to satisfy the traction condition, we need to superpose the solution of an in-plane problem corresponding to the loading of a circular disc by a concentrated inward normal force F_0 per unit length, equilibrated by tractions of the form (3). The Airy stress function

$$\varphi = \frac{F_0 r\theta\sin\theta}{\pi} = -\frac{F_0(\zeta-\bar{\zeta})(\ln(\zeta)-\ln(\bar{\zeta}))}{4\pi} \ ,$$

represents a force in the negative x-direction at the origin, so a concentrated force at the point $x = a, y = 0$ can be described by the function

$$\varphi = -\frac{F_0(\zeta-\bar{\zeta})(\ln(\zeta-a)-\ln(\bar{\zeta}-a))}{4\pi} \ ,$$

Comparing this with (28.48), we see that

$$g_1(\zeta) = -\frac{F_0\zeta\ln(\zeta-a)}{4\pi} \ ; \quad g_2(\zeta) = \frac{F_0\ln(\zeta-a)}{4\pi}$$

and hence the corrective terms in ϕ, ψ are

$$\phi_H = \frac{F_0[\zeta\ln(\zeta-a)+\bar{\zeta}\ln(\bar{\zeta}-a)}{4\pi} \ ; \quad \psi_H = -\frac{F_0\ln(\zeta-a)}{2\pi} + D_1\zeta^2 + D_2\zeta \ ,$$

where D_1, D_2 are real arbitrary constants and the additional terms correspond to the Airy function $r^3\cos\theta, r^2$, which are the only non-trivial degrees of freedom in the Michell solution with the appropriate Fourier dependence (see for example the solutions to Problem 12.3).

We then repeat the calculation of tractions, using the modified stress functions

$$\phi = \phi_P + \phi_H \ ; \quad \psi = \psi_P + \psi_H \ .$$

The resulting expressions contain factors of $(1-e^{\imath\theta})$ and $(1-e^{-\imath\theta})$ in the denominator which can be cancelled into the numerator with sufficient cunning in Maple or Mathematica. However, a simple alternative is to expand the numerator in powers of $e^{\imath\theta}$ and force each coefficient to be zero. This yields six equations for three unknowns, but they are not independent and permit the solution

$$A_0 = \frac{2F_0}{\pi a^4} \ ; \quad D_1 = -\frac{F_0(19-18\nu-16\nu^2)}{96\pi(1-\nu^2)a^2} \ ; \quad D_2 = -\frac{F_0}{4\pi a} \ .$$

The interdependence comes from the fact that we have raised the power of the polynomial in $e^{\imath\theta}$ by avoiding the simplification of the complete expression.

End conditions

It is easily verified that the solution so far involves no zeroth order antiplane tractions on the lateral surfaces ($T_z = 0$) or on the end $\Psi(\zeta, \bar{\zeta}, 0) = 0$. Both results follow from the fact that the problem is symmetric in z.

It remains to satisfy the weak traction-free conditions on σ_{zz} on the end $z = 0$. The problem is antisymmetric in x, so the moment about the y-axis is identically zero. Thus, the non-zero force resultants are

$$\begin{aligned} F_z(0) &= \int_0^a \int_0^{2\pi} \sigma_{zz}(r, \theta, 0) r d\theta dr \\ M_x &= \int_0^a \int_0^{2\pi} \sigma_{zz}(r, \theta, 0) r^2 \cos\theta d\theta dr \ . \end{aligned}$$

To eliminate these terms, we add the potentials

$$\phi = \mathcal{G}_2^0 \ ; \quad \phi = \mathcal{H}_2^0$$

and imposing the condition that the end moments and force be zero, we obtain

$$A_2 = -\frac{F_0(7 + 6\nu + 10\nu^2)}{6\pi a^2(1 + \nu)} \ ; \quad B_2 = \frac{\nu F_0}{\pi a} \ .$$

The final solution is therefore obtained from the potentials

$$\begin{aligned} \phi &= \mathcal{Q}_0^1 + \mathcal{G}_2^0 + \frac{F_0[\zeta \ln(\zeta - a) + \bar{\zeta} \ln(\bar{\zeta} - a)}{4\pi} \\ \psi &= \mathcal{V}_0^1 + \mathcal{H}_2^0 - \frac{F_0 \ln(\zeta - a)}{2\pi} - \frac{F_0(19 - 18\nu - 16\nu^2)\zeta^2}{96\pi(1 - \nu^2)a^2} - \frac{F_0 \zeta}{4\pi a} \ , \end{aligned}$$

where the constants implied in the functions $\mathcal{Q}, \mathcal{V}, \mathcal{G}, \mathcal{H}$ are

$$A_0 = \frac{2F_0}{\pi a^4} \ ; \quad C_0 = 0 \ ; \quad A_2 = -\frac{F_0(7 + 6\nu + 10\nu^2)}{6\pi a^2(1 + \nu)} \ ; \quad B_2 = \frac{\nu F_0}{\pi a} \ .$$

28.3. A circular cylindrical bar of radius a is loaded by a uniform tangential load F_0 per unit length along the line $\theta=0$. The load acts in the positive θ-direction and the end of the bar $z=0$ is traction-free. Find the complete stress field in the bar.

The tractions are uniform along the bar, so this is a problem in class $\mathcal{P}_2$, for which the solution of $\mathcal{P}_1$ is

$$\phi_0 = \mathcal{Q}_0^0 \; ; \quad \psi_0 = \mathcal{V}_0^0 \; ,$$

from equations (28.41).

We next integrate with respect to z, obtaining the particular solution

$$\phi_P = \mathcal{Q}_0^1 \; ; \quad \psi_P = \mathcal{V}_0^1 \; .$$

Notice that there is no need to add in the bending solution before performing the integration, since there are no zeroth order antiplane tractions in problem $\mathcal{P}_2$, so at the final stage, the process of cancelling these tractions would simply force all the coefficients in the added solution to zero.

In-plane correction

The in-plane stresses corresponding to this particular solution are

$$\begin{aligned} \Theta &= \frac{\left[2(1-\nu)\zeta\bar{\zeta} - (3+2\nu)(3-4\nu)a^2\right](A_0\zeta + \bar{A}_0\bar{\zeta})}{16(1-\nu^2)} \\ \Phi &= \frac{(1+4\nu)A_0\zeta^3}{24(1+\nu)} + \frac{\left[2(1-\nu)\zeta\bar{\zeta} - (3+2\nu)a^2\right]\bar{A}_0\zeta}{16(1-\nu^2)} - \frac{\imath C_0\zeta^2}{2} \; , \end{aligned} \tag{1}$$

from (28.16, 28.17).

On the boundary $r = a$, we require

$$T \equiv T_x + \imath T_y = \frac{1}{2}(\Phi e^{-\imath\theta} + \Theta e^{\imath\theta}) = \frac{\imath F_0 \delta(\theta)}{a} \; .$$

On $r = a$, we have

$$\zeta = ae^{\imath\theta} \; ; \quad \bar{\zeta} = ae^{-\imath\theta} \; ; \quad \zeta\bar{\zeta} = a^2 \; ,$$

$$2(1-\nu)\zeta\bar{\zeta} - (3+2\nu)(3-4\nu)a^2 = -(7-4\nu-8\nu^2)a^2 \; ; \quad 2(1-\nu)\zeta\bar{\zeta} - (3+2\nu)a^2 = -(1+4\nu)a^2$$

and hence

$$\begin{aligned} \Theta &= -\frac{(7-4\nu-8\nu^2)a^3(A_0e^{\imath\theta} + \bar{A}_0e^{-\imath\theta})}{16(1-\nu^2)} \\ \Phi &= \frac{(1+4\nu)A_0a^3e^{3\imath\theta}}{24(1+\nu)} - \frac{(1+4\nu)a^3\bar{A}_0e^{\imath\theta}}{16(1-\nu^2)} - \frac{\imath C_0a^2e^{2\imath\theta}}{2} \; . \end{aligned} \tag{2}$$

Then

$$\begin{aligned}\Phi e^{-\imath\theta} + \Theta e^{\imath\theta} &= \frac{(1+4\nu)A_0 a^3 e^{2\imath\theta}}{24(1+\nu)} - \frac{(1+4\nu)a^3\bar{A}_0}{16(1-\nu^2)} - \frac{\imath C_0 a^2 e^{\imath\theta}}{2} \\ &\quad - \frac{(7-4\nu-8\nu^2)a^3(A_0 e^{2\imath\theta} + \bar{A}_0)}{16(1-\nu^2)} \\ &= -\frac{\bar{A}_0 a^3}{2} - \frac{(19-18\nu-16\nu^2)A_0 a^3 e^{2\imath\theta}}{48(1-\nu^2)} - \frac{\imath C_0 a^2 e^{\imath\theta}}{2} \ . \end{aligned} \tag{3}$$

In order to satisfy the traction condition, we need to superpose the solution of an in-plane problem corresponding to the loading of a circular disc by a concentrated tangential force F_0 per unit length, equilibrated by tractions of the form (3). The Airy stress function

$$\varphi = \frac{F_0 r\theta\cos\theta}{\pi} = -\frac{\imath F_0(\zeta+\bar{\zeta})(\ln(\zeta)-\ln(\bar{\zeta}))}{4\pi} \ ,$$

represents a force in the y-direction at the origin, so a concentrated force at the point $x = a, y = 0$ can be described by the function

$$\varphi = -\frac{\imath F_0(\zeta+\bar{\zeta}-2a)(\ln(\zeta-a)-\ln(\bar{\zeta}-a))}{4\pi} \ .$$

Comparing this with (28.48), we see that

$$g_1(\zeta) = -\frac{\imath F_0(\zeta-2a)\ln(\zeta-a)}{4\pi} \ ; \quad g_2(\zeta) = -\frac{\imath F_0\ln(\zeta-a)}{4\pi}$$

and hence the corrective terms in ϕ, ψ are

$$\phi_H = \frac{\imath F_0[(\zeta-2a)\ln(\zeta-a) - (\bar{\zeta}-2a)\ln(\bar{\zeta}-a)}{4\pi} \ ; \quad \psi_H = \frac{\imath F_0\ln(\zeta-a)}{2\pi} + \imath D_0\zeta^2 \ ,$$

where D_0 is an real arbitrary constant and the additional term corresponds to the Airy function $D_0 r^3\sin\theta$, which is the only non-trivial degree of freedom in the Michell solution with the appropriate Fourier dependence.

We then repeat the calculation of tractions, using the modified stress functions

$$\phi = \phi_P + \phi_H \ ; \quad \psi = \psi_P + \psi_H \ .$$

The resulting expressions contain factors of $(1-e^{\imath\theta})$ and $(1-e^{-\imath\theta})$ in the denominator which can be cancelled into the numerator with sufficient cunning in Maple or Mathematica. However, a simple alternative is to expand the numerator in powers of $e^{\imath\theta}$ and force each coefficient to be zero. This yields six equations for three unknowns, but they are not independent and permit the solution

$$A_0 = \frac{2\imath F_0}{\pi a^4} \ ; \quad C_0 = -\frac{2F_0}{\pi a^3} \ ; \quad D_0 = -\frac{F_0(19-18\nu-16\nu^2)}{96\pi(1-\nu^2)a^2} \ .$$

The interdependence comes from the fact that we have raised the power of the polynomial in $e^{\imath\theta}$ by avoiding the simplification of the complete expression.

End conditions

It is easily verified that the solution so far involves no zeroth order antiplane tractions on the lateral surfaces ($T_z = 0$) or on the end $\Psi(\zeta, \bar{\zeta}, 0) = 0$. Both results follow from the fact that the problem is symmetric in z.

It remains to satisfy the weak traction-free conditions on σ_{zz} on the end $z = 0$. The problem is antisymmetric in y, so the axial force F_z and the moment about the x-axis are identically zero. Thus, the only non-zero force resultant is

$$M_y = -\int_0^a \int_0^{2\pi} \sigma_{zz}(r, \theta, 0) r^2 \sin\theta d\theta dr \ .$$

To eliminate this term, we add the potentials

$$\phi = \mathcal{G}_2^0 \ ; \quad \phi = \mathcal{H}_2^0$$

and imposing the condition that the end moments and force be zero, we obtain

$$A_2 = -\frac{\imath F_0(7 + 18\nu + 10\nu^2)}{6\pi(1+\nu)a^2} \ ; \quad B_2 = 0 \ .$$

The final solution is therefore obtained from the potentials

$$\begin{aligned} \phi &= \mathcal{Q}_0^1 + \mathcal{G}_2^0 + \frac{\imath F_0[(\zeta - 2a)\ln(\zeta - a) - (\bar{\zeta} - 2a)\ln(\bar{\zeta} - a)}{4\pi} \\ \psi &= \mathcal{V}_0^1 + \mathcal{H}_2^0 + \frac{\imath F_0 \ln(\zeta - a)}{2\pi} - \frac{\imath F_0(19 - 18\nu - 16\nu^2)\zeta^2}{96\pi(1-\nu^2)a^2} \ , \end{aligned}$$

where the constants implied in the functions $\mathcal{Q}, \mathcal{V}, \mathcal{G}, \mathcal{H}$ are

$$A_0 = \frac{2\imath F_0}{\pi a^4} \ ; \quad C_0 = -\frac{2F_0}{\pi a^3} \ ; \quad A_2 = -\frac{\imath F_0(7 + 18\nu + 10\nu^2)}{6\pi(1+\nu)a^2} \ ; \quad B_2 = 0 \ .$$

28.4. The lateral surfaces Γ of a bar of equilateral trianglular cross-section are defined by the equations $x = \pm\sqrt{3}y, x = a$. The bar is twisted by in-plane tractions T on Γ that are independent of z, the end $z = 0$ being traction-free. The tractions on the surface $x = a$ are defined by

$$\sigma_{xx} = 0 \; ; \quad \sigma_{xy} = S$$

and those on the other two surfaces are similar under a rotation of $\pm 120°$.

This problem lies in class $\mathcal{P}_2$. Use the strategy suggested in Problem 16.3 to find a Prandtl stress function solution of the corresponding problem $\mathcal{P}_1$, convert it to P-N form as in §28.5.4 and then integrate with respect to z to obtain a particular solution of $\mathcal{P}_2$. Determine the tractions on the surface $x = a$ implied by this solution and discuss possible strategies for completing the in-plane correction, but do not attempt it.

The appropriate solution of $\mathcal{P}_1$ corresponds to the triangular bar loaded only by a torque at the end $z = 0$. The Prandtl function φ can be taken to be zero on the boundary Γ and a simple solution satisfying this condition is obtained as

$$\varphi = C(x - \sqrt{3}y)(x + \sqrt{3}y)(x - a) = C(x^3 - 3x^2y - ax^2 + 3ay^2) \; ,$$

where C is a constant. In the torsion problem, C would be determined from the given torque applied at the end, but here we must leave it free, since it will be determined as an integrability condition on the in-plane correction.

We then have

$$\nabla^2\varphi = 4Ca$$

and comparing with (28.50), we see that $A_0 = B_0 = 0, C_0 = -2Ca$ and

$$\varphi_P = Ca(x^2 + y^2) \; .$$

The equivalent P-N potentials are

$$\phi_P = 0 \; ; \quad \psi_P = \frac{\imath Ca\chi_1^1}{(1-\nu)} \; ,$$

from (28.31).

The harmonic homogeneous function

$$\varphi_H = \varphi - \varphi_P = C(x^3 - 3x^2y - 2ax^2 + 2ay^2)$$

and we can convert this to complex form by writing

$$x = \frac{1}{2}\left(\zeta + \bar{\zeta}\right) \; ; \quad y = -\frac{\imath}{2}\left(\zeta - \bar{\zeta}\right) \; ,$$

obtaining

$$\varphi = C\left(\frac{\zeta^3+\bar{\zeta}^3}{2} - a(\zeta^2+\bar{\zeta}^2)\right) .$$

This can clearly be written in the form

$$\varphi = g + \overline{g} ,$$

where

$$g = C\left(\frac{\zeta^3}{2} - a\zeta^2\right)$$

and hence

$$h = \imath C(\zeta^3 - 2a\zeta^2) ,$$

from (28.51). Substituting into (28.35) and adding in the particular solution, we obtain the P-N solution of $\mathcal{P}_1$ as

$$\begin{aligned}
\phi &= \frac{\imath Cz[(1-4\nu)(\zeta^3-\bar{\zeta}^3) - 4(1-2\nu)a(\zeta^2-\bar{\zeta}^2)]}{4(1-\nu)} \\
\psi &= \frac{\imath Cz(-3\bar{\zeta}^2 + 4a\bar{\zeta} + 2a\zeta)}{2(1-\nu)} .
\end{aligned}$$

A particular integral of ψ with respect to z is

$$h_1 = \frac{\imath Cz^2(-3\bar{\zeta}^2 + 4a\bar{\zeta} + 2a\zeta)}{4(1-\nu)} .$$

We then have

$$\nabla^2 h_1 = \frac{\partial^2 h_1}{\partial z^2} + 4\frac{\partial^2 h_1}{\partial\zeta\partial\bar{\zeta}} = \frac{\imath C(-3\bar{\zeta}^2 + 4a\bar{\zeta} + 2a\zeta)}{2(1-\nu)} .$$

Substituting into (28.21) and integrating, we then have

$$h_2 = -\iint \frac{\imath C(-3\bar{\zeta}^2 + 4a\bar{\zeta} + 2a\zeta)}{8(1-\nu)} d\zeta d\bar{\zeta} = -\frac{\imath C(-\zeta\bar{\zeta}^3 + 2a\zeta\bar{\zeta}^2 + a\zeta^2\bar{\zeta})}{8(1-\nu)}$$

and hence

$$\psi = h_1 + h_2 = \frac{\imath Cz^2(-3\bar{\zeta}^2 + 4a\bar{\zeta} + 2a\zeta)}{4(1-\nu)} - \frac{\imath C(-\zeta\bar{\zeta}^3 + 2a\zeta\bar{\zeta}^2 + a\zeta^2\bar{\zeta})}{8(1-\nu)} .$$

A similar procedure with ϕ yields

$$\begin{aligned}
\phi &= \frac{\imath Cz^2[(1-4\nu)(\zeta^3-\bar{\zeta}^3) - 4(1-2\nu)a(\zeta^2-\bar{\zeta}^2)]}{8(1-\nu)} \\
&\quad - \frac{\imath C}{4(1-\nu)}\left[\frac{(1-4\nu)(\zeta^4\bar{\zeta} - \zeta\bar{\zeta}^4)}{16} - \frac{(1-2\nu)a(\zeta^3\bar{\zeta} - \zeta\bar{\zeta}^3)}{3}\right]
\end{aligned}$$

Substituting into the P-N solution (28.16, 28.17), we obtain the in-plane stress components as

$$\begin{aligned}\Theta &= \frac{\imath C(3-4\nu)[2a(\zeta^2-\bar{\zeta}^2)-(\zeta^3-\bar{\zeta}^3)]}{4(1-\nu)}\\ \Phi &= -\frac{\imath C[-3\bar{\zeta}^2+4a\bar{\zeta}-4(1-\nu)a\zeta]}{4(1-\nu)}\end{aligned}$$

In particular, the tractions on $x=a$ are obtained as

$$\sigma_{xx}(a,y) = -\frac{C[(3-4\nu)a^2y+(3-8\nu)y^3]}{8(1-\nu)}\ ;\quad \sigma_{xy}(a,y) = \frac{C[(3-4\nu)a^3-(5-4\nu)ay^2]}{8(1-\nu)}\ .$$

The constant C can be obtained by demanding that the moment of these tractions about the centroid $(2a/3,0)$ be equal to that of the applied tractions. We obtain

$$\int_{-a/\sqrt{3}}^{a/\sqrt{3}}\left(\frac{\sigma_{xy}(a,0)a}{3}-\sigma_{xx}(a,0)y\right)dy = \int_{-a/\sqrt{3}}^{a/\sqrt{3}}\frac{Sady}{3}\ .$$

Evaluating the integrals, we obtain

$$\frac{(71-106\nu)\sqrt{3}Ca^5}{405(1-\nu)} = \frac{2Sa^2}{3\sqrt{3}}\quad\text{and hence}\quad C=\frac{90(1-\nu)S}{(71-106\nu)a^3}\ .$$

The corrective in-plane solution could be performed either using the real Airy stress function or the complex potentials of Chapter 19. Since the problem has threefold symmetry in polar coordinates, it is clear that real potentials would only involve terms of the form

$$f(r)\sin(3n\theta)$$

where n is an integer. In complex P-N form, a sufficiently general series of such terms can be written

$$\phi = \sum_{n=1}^{\infty}\imath C_n(\zeta^{3n}-\bar{\zeta}^{3n})\ ;\quad \psi = \sum_{n=1}^{\infty}\imath D_n\zeta^{3n+1}\ ,$$

where C_n, D_n are real constants.

The state of stress in the corners involves infinite stress gradients, since the shear stresses on two non-orthogonal planes with zero normal stresses are equal. We should therefore anticipate problems of convergence of the series in the corners. Improved convergence could be obtained by first subtracting out an appropriate corner solution, as suggested for the right-angle corner in §11.1.2.

28.5. The rectangular bar $-a<x<a, -b<y<b, 0<z<L$ is built in at $z=L$ and loaded only by the uniform shear tractions

$$\sigma_{xy}(a,y,z) = S \; ; \quad \sigma_{xy}(-a,y,z) = -S$$

on the surfaces $x=\pm a$. Find the complete stress field in the bar using weak conditions on the end $z=0$. Assume that $b \gg a$ and use weak boundary conditions on the edges $y=\pm b$. Hint: The problem is even in x and odd in y, which implies that $\Re(A_0)=C_0=0$. The easiest way to impose these symmetries on the rest of the solution is to use an Airy function for the in-plane corrective solution.

The applied in-plane tractions are independent of z, so this is a problem in class $\mathcal{P}_2$.

Problem $\mathcal{P}_1$

For the corresponding problem $\mathcal{P}_1$, we start with the functions

$$\phi = \mathcal{G}_0^1 \; ; \quad \psi = \mathcal{H}_0^1 - \frac{\imath C_0 \chi_1^1}{2(1-\nu)} \; ,$$

from (28.31). However, since the problem is even in x and odd in y, there is no torsion ($C_0 = 0$) and no bending about the y-axis ($A = 0$). We shall therefore omit these constants to reduce the following algebraic complexity. We note from (28.33) that the above particular solution yields antiplane stresses Ψ that are quadratic in $\zeta, \bar{\zeta}$. In view of equation $(28.35)_2$, we conclude that the polynomial antiplane correction $h(\zeta)$ must be cubic. We therefore take

$$h = \sum_{n=1}^{3} D_n \zeta^n \; ,$$

where the D_n are complex constants. The following steps are performed in the Maple file P285 and are merely described in words here. We substitute h into (28.35) to obtain the corrections in ϕ, ψ and compute the total antiplane stress Ψ. Next, we substitute $\zeta = x + \imath y, \bar{\zeta} = x - \imath y$ and separate real and imaginary parts to obtain σ_{zx}, σ_{zy} and hence the tractions on $x = \pm a$, $y = \pm b$. The tractions on $x = \pm a$ are quadratic in y and equating all the coefficients to zero yields six equations for the real and imaginary parts of the three constants D_n. These equations are not linearly independent, so one free constant remains, which is determined from the weak condition

$$\int_{-a}^{a} \sigma_{zy}(x,b)dx = 0 \; .$$

The corresponding condition on $y = -b$ is automatically satisfied by equilibrium considerations, but we verify this by including it in the solve command for the remaining

free constant. The result of this procedure is to give the constants

$$D_1 = \imath B\left(b^2 - \frac{\nu a^2}{3(1+\nu)}\right) \; ; \quad D_2 = 0 \; ; \quad D_3 = \frac{\imath B(1+2\nu)}{12(1+\nu)} \; ,$$

where $A_0 = \imath B, \overline{A}_0 = -\imath B, B_0 = 0$.

Problem $\mathcal{P}_2$

To move to Problem $\mathcal{P}_2$, we first integrate with respect to z, using the algorithm

$$g_1 = \int f(\zeta, \bar{\zeta}, z) dz \; ,$$

$$\nabla^2 g_1 = \frac{\partial^2 g_1}{\partial z^2} + 4\frac{\partial^2 g_1}{\partial \zeta \partial \bar{\zeta}} \; ,$$

$$g_2 = -\frac{1}{4}\iint \nabla^2 g_1 d\zeta d\bar{\zeta} \; ,$$

$$g = g_1 + g_2 \; .$$

The basis of this procedure is that if the function f is harmonic, then $\nabla^2 g_1$ must be independent of z. If we then integrate with respect to ζ and $\bar{\zeta}$, we can construct a function whose Laplacian just cancels $\nabla^2 g_1$, so that the final function $g_1 + g_2$ is a harmonic partial integral of f.

In-plane correction

We next calculate the in-plane stress components Θ, Φ and convert them to real form, obtaining

$$\begin{aligned}
\sigma_{xx} &= \frac{B(-(7-14\nu-8\nu^2)y^3 + 3(3+2\nu-8\nu^2)x^2y + 4(15-9\nu-24\nu^2)b^2y - 4\nu(5-8\nu)a^2y)}{48(1-\nu^2)} \\
\sigma_{xy} &= -\frac{B[(1-18\nu+8\nu^2)x^3 + 3(3+2\nu-8\nu^2)xy^2 + 4\nu a^2 x - 12(1+\nu)b^2 x]}{96(1-\nu^2)} \\
\sigma_{yy} &= \frac{B(-(29-2\nu-24\nu^2)y^3 + 3(1+14\nu-24\nu^2)x^2y + 12(7-\nu-8\nu^2)b^2y - 4\nu(7-8\nu)a^2y)}{96(1-\nu^2)}
\end{aligned}$$

This leads to tractions σ_{xx} on $x = \pm a$ that are cubic in y and tractions σ_{xy} that are quadratic in y. The symmetry ensures that no bending moments are developed, but the shear tractions will develop an 'axial' force in the y-direction varying with y^3.

The simplest way of performing the correction is to use an Airy stress function which we here denote by φ to avoid confusion with the complex potential. With the appropriate symmetries, we have

$$\varphi = C_1 y^5 + C_2 y^3 x^2 + C_3 y x^4 + C_4 y^3 + C_5 y x^2 \; . \tag{1}$$

We next calculate the stress components, add them into the above expressions, and determine the constants from the boundary conditions

$$\sigma_{xx}(a,y)=0\ ;\quad \sigma_{xy}(a,y)=0\ ;\quad \int_{-a}^{a}\sigma_{yy}(x,b)dx=0$$

and the biharmonic condition. Notice that the symmetries in the problem ensure that the corresponding conditions on $x=-a, y=-b$ are then satisfied and that the two remaining weak conditions on $y=b$ are identically satisfied. Equating coefficients in the two strong boundary conditions, we obtain a total of six equations for the six constants $C_1, C_2, C_3, C_4, C_5, B$, with solution

$$B=\frac{3S}{2ab^2}\ ;\ C_1=\frac{S(7-14\nu-8\nu^2)}{1280ab^2(1-\nu^2)}\ ;\ C_2=-\frac{S(3+2\nu-8\nu^2)}{128ab^2(1-\nu^2)}\ ;\ C_3=-\frac{S(1-18\nu+8\nu^2)}{256ab^2(1-\nu^2)}$$

$$C_4=-\frac{S[3(5-3\nu-8\nu^2)b^2-\nu(5-8\nu)a^2]}{96ab^2(1-\nu^2)}\ ;\ C_5=-\frac{S[(13-3\nu-16\nu^2)b^2+\nu a^2]}{32ab^2(1-\nu^2)}\ .$$

substituting these results into (1) and using the procedure 'rtocb', we can write φ in the form (28.48) with

$$g_1=\frac{\imath S[40\{3(1+4\nu)b^2+\nu a^2\}\zeta^3-(3+12\nu)\zeta^5]}{3840ab^2(1+\nu)}$$

$$g_2=\frac{\imath S[\{8(7-3\nu-10\nu^2)b^2-8\nu(1-2\nu)a^2\}\zeta^2+(1-4\nu)\zeta^4]}{512ab^2(1-\nu^2)}\ .$$

and hence determine the additional terms in ϕ,ψ from (28.49). For safety, we verify that the new complex P-N solution satisfies the in-plane boundary conditions. This completes the solution of problem $\mathcal{P}_2$

End conditions

The solution to $\mathcal{P}_2$ satisfies the condition $\Psi=0$ on $z=0$, as it must, since the problem is even in z. However, the normal traction on the end is non-zero and after substituting $\zeta=x+\imath y, \bar{\zeta}=x-\imath y$ we obtain

$$\sigma_{zz}(x,y,0)=\frac{S[2(1+\nu)y^3-3\nu x^2y+\nu(2+\nu)a^2y-(6+11\nu-5\nu^2)b^2y]}{2(1+\nu)b^2a}\ .$$

This function is odd in y and even in x, so we conclude that the axial force and the moment about the y-axis are each identically zero. However, there is a non-zero moment about the x-axis given by

$$M_x=\int_{-a}^{a}\int_{-b}^{b}\sigma_{zz}(x,y,0)ydydx=-\frac{2S[(24+25\nu)b^3-5\nu a^2b]}{15}\ .$$

This can clearly be removed by superposing a simple bending solution, but to be consistent with the previous development, we add the additional potentials $\mathcal{G}_2^0, \mathcal{H}_2^0$ into ϕ, ψ respectively, with $A[2] = \imath D, B_2 = 0$, where D is a real constant. Notice that this choice (of an imaginary constant for A_2 is dictated by symmetry considerations. Resolving for M_x and setting $M_x = 0$, we obtain

$$D = -\frac{S[(24 + 25\nu)b^2 - 5\nu a^2]}{20a^2 b}$$

after which the final stress field is recovered as

$$\begin{aligned}
\sigma_{xx} &= 0 \\
\sigma_{xy} &= \frac{Sx}{a} \\
\sigma_{yy} &= -\frac{S[(1+\nu)y(y^2 - b^2) - 3\nu x^2 y + \nu a^2 y]}{2(1+\nu)ab^2} \\
\sigma_{zx} &= 0 \\
\sigma_{zy} &= \frac{S[3(1+\nu)y(y^2 - b^2) - 3\nu x^2 y + \nu a^2 y]}{2(1+\nu)ab^2} \\
\sigma_{zz} &= \frac{S[10(1+\nu)y^3 - 15(1+\nu)yz^2 - 15\nu x^2 y - 6(1+\nu)b^2 y + 5\nu a^2 y]}{10(1+\nu)a^2 b}
\end{aligned}$$

28.6. A bar of square cross-section $-a<x<a, = -a<y<a, 0<z<L$ is built in at $z=L$ and loaded only by the uniform tractions

$$\sigma_{yy}(x,a,z) = -p_0$$

on the surface $y=a$. Find the complete stress field in the bar using only the closed-form terms from the solution of §17.4.2 for the problem $\mathcal{P}_1$ and an appropriate finite polynomial as an Airy stress function for the in-plane correction in $\mathcal{P}_2$. Plot appropriate stress components to estimate the errors in the elementary bending solution.

The tractions are uniform along the bar, so this is a problem in class $\mathcal{P}_2$.

Problem $\mathcal{P}_1$

The solution of $\mathcal{P}_1$ for the square bar with an end load F can be written in terms of the Prandtl stress function

$$\varphi = \frac{F}{8a^4(1+\nu)}\left(-\nu x^3 + \nu a^2 x - 3(1+\nu)a^2 x\right) = \frac{F}{8a^4(1+\nu)}\left(-\nu x^3 - (3+2\nu)a^2 x\right) , \tag{1}$$

from (17.42), where we have omitted the corrective series. At this stage, F is an unknown constant, so we use the expression

$$\varphi = D((3+2\nu)a^2 x + \nu x^3) ,$$

where D is a real constant. It follows that

$$\nabla^2\varphi = 6D\nu x$$

and comparison with (17.10) shows that

$$-\frac{B\nu}{(1+\nu)} = 6D\nu \quad \text{or} \quad D = -\frac{B}{6(1+\nu)}$$

and $A = C = 0$. This result could of course also have been obtained by eliminating F between $(17.33)_2$ and (1).

We next construct the particular solution

$$\varphi_P = -\frac{B[3(1+2\nu)y^2 x - (1-2\nu)x^3]}{24(1+\nu)}$$

and hence

$$\begin{aligned}\varphi_H &= \varphi - \varphi_P = -\frac{B}{6(1+\nu)}((3+2\nu)a^2 x + \nu x^3) + \frac{B[3(1+2\nu)y^2 x - (1-2\nu)x^3]}{24(1+\nu)} \\ &= \frac{B[4(3+2\nu)a^2 x + (1+2\nu)(3y^2 x - x^3)]}{24(1+\nu)} ,\end{aligned}$$

which is harmonic, as required.

Using the procedure 'rtoch' or otherwise, we can express this in the form

$$\varphi_H = g + \overline{g} \ ,$$

where

$$g = -\frac{B[4(3+2\nu)a^2\zeta + (1+2\nu)\zeta^3]}{48(1+\nu)} \ .$$

We then construct the function $h = 2\imath g$ and substitute it into equations (28.35) to obtain the solution of problem $\mathcal{P}_1$. These operations can also all be performed by applying the procedure 'PrandtltoPN' to the function (1).

Problem $\mathcal{P}_2$

To move to Problem $\mathcal{P}_2$, we first integrate with respect to z, using the algorithm

$$g_1 = \int f(\zeta, \bar{\zeta}, z) dz \ ,$$

$$\nabla^2 g_1 = \frac{\partial^2 g_1}{\partial z^2} + 4\frac{\partial^2 g_1}{\partial \zeta \partial \bar{\zeta}} \ ,$$

$$g_2 = -\frac{1}{4}\iint \nabla^2 g_1 d\zeta d\bar{\zeta} \ ,$$

$$g = g_1 + g_2 \ .$$

The basis of this procedure is that if the function f is harmonic, then $\nabla^2 g_1$ must be independent of z. If we then integrate with respect to ζ and $\bar{\zeta}$, we can construct a function whose Laplacian just cancels $\nabla^2 g_1$, so that the final function $g_1 + g_2$ is a harmonic partial integral of f.

In-plane correction

We next calculate the in-plane stress components Θ, Φ and convert them to real form, obtaining

$$\begin{aligned}
\sigma_{xx} &= \frac{By[64a^2\nu^2 + 24\nu^2x^2 - 6x^2\nu + 56\nu a^2 + 7y^2 - 9x^2 - 8\nu^2y^2 - 14y^2\nu - 60a^2]}{192(1-\nu^2)} \\
\sigma_{xy} &= -\frac{Bx[-8\nu^2x^2 + 24\nu^2y^2 - 6y^2\nu - 9y^2 - x^2 + 12a^2 + 8\nu a^2 + 18x^2\nu]}{192(1-\nu^2)} \\
\sigma_{yy} &= \frac{By[64a^2\nu^2 + 72\nu^2x^2 - 42x^2\nu + 40\nu a^2 + 29y^2 - 3x^2 - 24\nu^2y^2 - 2y^2\nu - 84a^2]}{192(1-\nu^2)}
\end{aligned}$$

The strong boundary conditions for the in-plane problem are

$$\sigma_{yy}(a,x) = -p_0 \ ; \quad \sigma_{yx}(x,a) = \sigma_{yy}(x,-a) = \sigma_{yx}(x,-a) = 0 \ ,$$

$$\sigma_{xx}(a,y) = \sigma_{xy}(a,y) = \sigma_{xx}(-a,y) = \sigma_{xy}(-a,y) = 0 \ ,$$

To correct the in-plane tractions we shall use a polynomial Airy stress function. The stress field is even in x and odd in y, so an appropriate Airy function is

$$\varphi = C_1 y^5 + C_2 y^3 x^2 + C_3 y x^4 + C_4 y^3 + C_5 y x^2 \ . \tag{2}$$

We calculate the stress components, add them into the above expressions, and determine the constants from the boundary conditions

$$\sigma_{xy}(a,y) = \sigma_{yx}(x,a) = 0$$

and the biharmonic condition. Notice that the symmetries in the problem ensure that the corresponding conditions on $x = -a, y = -a$ are then satisfied. Equating coefficients in these two strong boundary conditions, we obtain a total of five equations for the five constants C_1, C_2, C_3, C_4, C_5. However, these equations are not linearly independent and the constant C_4 is indeterminate. Substituting the solution into the combined stress field, we obtain

$$\begin{aligned}
\sigma_{xx} &= \frac{y[16Ba^2\nu^2 + 14\nu a^2 B - 15Ba^2 + 576C_4 - 576C_4\nu^2]}{48(1-\nu^2)} \\
\sigma_{xy} &= 0 \\
\sigma_{yy} &= -\frac{By[2\nu a^2 - y^2\nu + 3x^2\nu + 3a^2 - y^2]}{6(1+\nu)} \ .
\end{aligned}$$

Clearly the constant C_4 could be chosen to satisfy the additional boundary conditions

$$\sigma_{xx}(a,y) = \sigma_{xx}(-a,y) = 0 \ .$$

The solution is then defined by equation (2) with the constants

$$C_1 = \frac{B(14\nu - 7 + 8\nu^2)}{3840(1-\nu^2)} \ ; \quad C_2 = -\frac{B(-2\nu - 3 + 8\nu^2)}{384(1-\nu^2)} \ ;$$

$$C_3 = \frac{B(-18\nu + 1 + 8\nu^2)}{768(1-\nu^2)} \ ; \quad C_4 = -\frac{Ba^2(16\nu^2 + 14\nu - 15)}{288(1-\nu^2)} \ ; \quad C_5 = -\frac{Ba^2(2\nu + 3)}{96(1-\nu^2)} \ ,$$

and the stress components are

$$\sigma_{xx} = \sigma_{xy} = 0 \ ; \quad \sigma_{yy} = -\frac{By(2\nu a^2 + 3x^2\nu - y^2\nu + 3a^2 - y^2)}{6(1+\nu)} \ ,$$

giving tractions

$$\sigma_{yy}(x,a) = -\frac{By((2+\nu)a^2 + 3x^2\nu)}{6(1+\nu)} \ ; \quad \sigma_{yy}(x,-a) = \frac{By((2+\nu)a^2 + 3x^2\nu)}{6(1+\nu)} \ .$$

At this stage, the solution is still antisymmetric in y, but the original problem does not have this symmetry, since only the top edge of the bar is loaded. To generate a corrective problem that is still antisymmetric, we need to superpose an appropriate uniform tensile stress (which is a *symmetric* state). It is easy to see what magnitude this must have, since if we add $\sigma_{yy} = p_0/2$ to the original physical problem, we would end up with boundary conditions $\sigma_{yy}(x,a) = -p_0/2$, $\sigma_{yy}(x,-a) = +p_0/2$, which is antisymmetric as required. However, to avoid the necessity for this kind of thinking, we shall simply add an additional stress function term $C_6 x^2$ (which generates the required stress component) and choose its magnitude and the as yet unknown constant B so as to satisfy the weak boundary conditions on the two edges — i.e.

$$\int_{-a}^{a} \sigma_{yy}(x,a)dx = -2p_0 a \; ; \quad \int_{-a}^{a} \sigma_{yy}(x,-a)dx = 0 \; .$$

The constants are obtained as

$$B = \frac{3p_0}{2a^3} \; ; \quad C_6 = -\frac{p_0}{4} \; ,$$

the only non-zero in-plane stress component is

$$\sigma_{yy} = -\frac{p_0(3yx^2\nu - y^3\nu + 2y\nu a^2 + 2a^3\nu + 3ya^2 - y^3 + 2a^3)}{4a^3(1+\nu)} \; , \tag{3}$$

and the *error* in the tractions on $y = \pm a$ — i.e. the difference between the required boundary tractions and the tractions defined by the stress function — is

$$\hat{\sigma}_{yy}(x, \pm a) = \pm \frac{\nu p_0(3x^2 - a^2)}{4a^2(1+\nu)} \; .$$

This could be removed using an appropriate pair of Mathieu series, as in \$6.4, but we shall not pursue this here.

End conditions

The solution to $\mathcal{P}_2$ satisfies the condition $\Psi = 0$ on $z = 0$, as it must, since the problem is even in z. However, the normal traction on the end is non-zero and will comprise (i) the contribution from the three-dimensional potentials ϕ, ψ and (ii) the plane strain components from the Airy function φ, given by

$$\sigma_{zz} = \nu \nabla^2 \varphi \; ,$$

from (3.3) and (4.1). Adding these two components and substituting $\zeta = x + \imath y, \bar{\zeta} = x - \imath y$ we obtain

$$\sigma_{zz} = \frac{3p_0 z^2 y}{4a^3} - \frac{\nu p_0}{2} + \frac{p_0 y(2\nu^2 a^2 + 3\nu x^2 - 2\nu y^2 + 7\nu a^2 + 6a^2 - 2y^2)}{4(1+\nu)a^3} \; .$$

To complete the solution, we need to superpose a state of pure bending and axial force

$$\sigma_{zz} = C_1 + C_2 y \ ,$$

with the constants C_1, C_2 chosen to satisfy the weak conditions

$$\int_{-a}^{a}\int_{-a}^{a} \sigma_{zz}(x,y,0)dxdy = 0 \ ; \quad \int_{-a}^{a}\int_{-a}^{a} \sigma_{zz}(x,y,0)ydxdy = 0 \ .$$

This does not affect the other stress components and leaves the final expression

$$\sigma_{zz} = \frac{3p_0 z^2 y}{4a^3} + \frac{p_0 y(15\nu x^2 - 10\nu y^2 + \nu a^2 + 6a^2 - 10y^2)}{20(1+\nu)a^3}$$

The complete solution therefore is defined by the non-zero stress components

$$\begin{aligned}
\sigma_{yy} &= -\frac{p_0(3yx^2\nu - y^3\nu + 2y\nu a^2 + 2a^3\nu + 3ya^2 - y^3 + 2a^3)}{4a^3(1+\nu)} \\
\sigma_{zz} &= \frac{3p_0 z^2 y}{4a^3} + \frac{p_0 y(15\nu x^2 - 10\nu y^2 + \nu a^2 + 6a^2 - 10y^2)}{20(1+\nu)a^3} \\
\sigma_{zy} &= \frac{3p_0 z(a^2 - y^2)}{4a^3} + \frac{\nu p_0 z(3x^2 - a^2)}{4a^3(1+\nu)}
\end{aligned}$$

The elementary theory predicts only the bending stress

$$\sigma_{zz} = \frac{My}{I} = \frac{3p_0 z^2 y}{4a^3}$$

and the transverse shear stress

$$\sigma_{zy} = \frac{3p_0 z(a^2 - y^2)}{4a^3} \ .$$

The maximum tensile stress will occur at the upper edge of the bar. Clearly the elementary bending stress term will dominate if $z \gg a$. The additional terms in the three-dimensional solution reduce this maximum. As an estimate of this reduction, at $z = 2a$ and with $\nu = 0$, the maximum bending stress is reduced from

$$3p_0 \quad \text{to} \quad 2.8p_0 \ .$$

The Mises stress σ_E at this edge will be increased, since it will contain a significant contribution from the compressive traction $\sigma_{yy} = -p_0$.

CHAPTER 29

29.1. Using the result of Problem 23.4 or otherwise, show that if the punch profile $u_0(x,y)$ has the form

$$u_0 = r^p f(\theta)$$

where $p>0$ and $f(\theta)$ is any function of θ, the resulting frictionless contact problems at different indentation forces will be self-similar[1].

Show also that if l is a representative dimension of the contact area, the indentation force, F, and the rigid-body indentation, d, will vary according to

$$F \sim l^{p+1} \;\; ; \;\; d \sim l^p$$

and hence that the indentation has a stiffening load-displacement relation

$$F \sim d^{1+\frac{1}{p}} \; .$$

Verify that the Hertzian contact relations (§30.2.5 below) agree with this result.

Following the solutions of Problem 23.4, we use Solution F and suppose that the solution can be written in terms of a potential

$$\phi = Bf(bx, by, bz) \; , \tag{1}$$

where $f(\xi,\eta,\zeta)$ is a harmonic function and B, b are scaling parameters that will now take values that depend on the applied load. In other words, we anticipate that the solution for all possible loads will be describable by the same function, but with different values of B, b.

The normal surface displacement is

$$2\mu u_z(x,y,0) = -2(1-\nu)Bb\frac{\partial f}{\partial z}(bx,by,0) = -2(1-\nu)Bb\frac{\partial f}{\partial \zeta}(br\cos\theta, br\sin\theta, 0) \tag{2}$$

and in the contact region this is given by $u_0 = r^p f(\theta)$.

This condition can only be satisfied if, in the contact region,

$$\frac{\partial f}{\partial \zeta}(\xi,\eta,0) = (x^2+y^2)^{p/2} f(\theta) \; . \tag{3}$$

The contact condition then requires that

$$-2(1-\nu)Bb(br)^p f(\theta) = u_0 r^p f(\theta) \; , \tag{4}$$

[1] D.A.Spence, An eigenvalue problem for elastic contact with finite friction, *Proc. Cambridge Phil. Soc.,* Vol. 73 (1973), 249–268, has shown that this argument also extends to problems with Coulomb friction at the interface, in which case, the zones of stick and slip also remain self-similar with monotonically increasing indentation force.

which will be satisfied for all r, θ if

$$-2(1-\nu)Bb^{p+1} = u_0 \tag{5}$$

— i.e. if

$$B = -\frac{u_0 b^{-(p+1)}}{2(1-\nu)} \ . \tag{6}$$

The contact pressure distribution is

$$p(x,y) = -\sigma_{zz}(x,y,0) = -Bb^2 \frac{\partial^2 f}{\partial \zeta^2}(bx, by, 0) = \frac{u_0 b^{1-p}}{2(1-\nu)} \frac{\partial^2 f}{\partial \zeta^2}(bx, by, 0) \ . \tag{7}$$

In other words, the pressure at corresponding points in the self-similar solution varies with b^{1-p}.

The problem is invariant (except for possible scalar multipliers) in (ξ, η, ζ) space. In particular, the boundary of the contact region has the form $F(\xi, \eta) = 0$ and hence $F(bx, by) = 0$ on $\zeta = bz = 0$. It follows that linear dimensions of the contact area scale with b^{-1} and hence the total force

$$F = \int\int_A p(x,y) dx dy \tag{8}$$

scales with $b^{1-p} b^{-2} = b^{-(p+1)}$, since each of dx, dy scales with b^{-1}. These results can be summarized as

$$F \sim b^{-(p+1)} \quad ; \quad l \sim b^{-1} \quad ; \quad p(bx, by) \sim b^{1-p} \ . \tag{9}$$

The corresponding result for displacement is obtained from equations (2,6) as

$$u_z \sim Bb \sim b^{-p} \ , \tag{10}$$

a special case of which is $d \sim b^{-p}$.

We can now eliminate b between these relations. For example, from $l \sim b^{-1}$, we have

$$b \sim l^{-1} \tag{11}$$

and hence

$$d \sim b^{-p} \sim l^p \ . \tag{12}$$

In the same way,

$$F \sim b^{-(p+1)} \sim l^{p+1} \ . \tag{13}$$

Alternatively, starting from $d \sim b^{-p}$, we have

$$b \sim d^{-\frac{1}{p}} \tag{14}$$

and hence

$$F \sim b^{-(p+1)} \sim d^{\frac{p+1}{p}} \ , \tag{15}$$

which is a stiffening relation, since dF/dd increases with d if $p > 0$.

For the Hertzian problem,

$$u_0 = r^2 f(\theta) \tag{16}$$

and hence $p = 2$. We conclude that

$$d \sim l^2 \quad ; \quad F \sim l^3 \sim d^{\frac{3}{2}} \ . \tag{17}$$

The first of these relations is confirmed by (30.56) and the second by (30.59). The relation between F and d is confirmed by (30.57).

Notice that the traction-free condition outside A and the inequality conditions are also satisfied by the self-similar solution, since (i) quantities that are zero, remain zero during the scaling and (ii) the *sign* of physical quantities remains the same during scaling.

29.2. A frictionless rigid body is pressed into an elastic body of shear modulus μ and Poisson's ratio ν by a normal force F, establishing a contact area A and causing the rigid body to move a distance δ. Use the results of §29.1 to define the boundary-value problem for the potential function ϕ corresponding to the *incremental problem*, in which an infinitesimal additional force increment ΔF produces an incremental rigid-body displacement $\Delta\delta$.

Suppose that the rigid body is a perfect electrical conductor at potential V_0 and the 'potential at infinity' in the elastic body is maintained at zero. Define the boundary-value problem for the potential V in the elastic body, noting that the current density vector $\boldsymbol{i}$ is given by Ohm's law

$$\boldsymbol{i} = -\frac{1}{\rho}\boldsymbol{\nabla} V \ ,$$

where ρ is the electrical resistivity of the material. Include in your statement an expression for the total current I transmitted through the contact interface.

By comparing the two potential problems or otherwise, show that the electrical contact resistance $R = V_0/I$ is related to the incremental contact stiffness $dF/d\delta$ by the equation

$$\frac{1}{R} = \frac{(1-\nu)}{\rho\mu}\frac{dF}{d\delta} \ .$$

For the incremental problem, we have

$$u_z = \Delta\delta \ ; \quad \text{in } A \tag{1}$$

$$\sigma_{zz} = 0 \ ; \quad \text{in } \bar{A} \tag{2}$$

and hence

$$\frac{\partial\varphi}{\partial z} = -\frac{\mu\Delta\delta}{(1-\nu)} \ ; \quad \text{in } A \tag{3}$$

$$\frac{\partial^2\varphi}{\partial z^2} = 0 \ ; \quad \text{in } \bar{A} \ , \tag{4}$$

from (29.1, 29.2).

For the electrical conduction problem, we note that continuity of electric current demands that

$$\text{div } \boldsymbol{i} = 0 \tag{5}$$

and hence

$$-\frac{1}{\rho}\text{div } \boldsymbol{\nabla} V = -\frac{1}{\rho}\nabla^2 V = 0 \ . \tag{6}$$

In other words, the electrical potential V is a harmonic function. The boundary conditions on V are

$$V = V_0 \; ; \quad \text{in } A \tag{7}$$

$$\boldsymbol{i}.\boldsymbol{n} = -\frac{1}{\rho}\frac{\partial V}{\partial z} = 0 \; ; \quad \text{in } \bar{A} \; , \tag{8}$$

where the second equation states that there is no electric current flowing into the surface of the half space outside the contact area.

The two problems (3, 4) and (7, 8) are clearly identical except for a constant multiplier. Thus, if the potential $V(x, y, z)$ satisfies (7, 8), it follows that a potential

$$\frac{\partial \varphi}{\partial z}(x, y, z) = -\frac{\mu \Delta\delta}{(1-\nu)V_0} V(x, y, z) \tag{9}$$

will satisfy (3, 4), where we recall that $\partial\varphi/\partial z$ will be harmonic if φ is harmonic.

The total force is

$$\Delta F = -\int\int_A \sigma_{zz} dA = \int\int_A \frac{\partial^2 \varphi}{\partial z^2} dA \tag{10}$$

from (29.1). Using (9), we therefore have

$$\Delta F = -\frac{\mu \Delta\delta}{(1-\nu)V_0} \int\int_A \frac{\partial V}{\partial z} dA \; . \tag{11}$$

However, the current density in the contact area is

$$i_z = -\frac{1}{\rho}\frac{\partial V}{\partial z} \tag{12}$$

and hence the total current flow through the contact is

$$I = -\frac{1}{\rho}\int\int_A \frac{\partial V}{\partial z} dA = \frac{(1-\nu)V_0 \Delta F}{\rho\mu\Delta\delta} \; , \tag{13}$$

from (10). In other words,

$$\frac{1}{R} \equiv \frac{I}{V_0} = \frac{(1-\nu)}{\rho\mu}\frac{dF}{d\delta} \tag{14}$$

— i.e., the conductance through the contact area is proportional to the incremental stiffness of the elastic contact.

29.3. Use equations (18.9, 18.11) to show that the function

$$f(x,z) = \Im\left\{\arcsin\left(\frac{x+\imath z}{a}\right)\right\}$$

satisfies Laplace's equation. Show that the function

$$\frac{\partial\varphi}{\partial z} = A + Bf(x,z)$$

can be used to solve the plane strain equivalent of the flat punch problem of §12.5.2 with suitable choices for the constants A, B. In particular, obtain expressions for the contact pressure distribution and for the normal surface displacement u_z outside the contact area.

Note: For complex arguments, we can write[2]

$$\arcsin(\zeta) = \frac{1}{\imath}\ln(\imath\zeta + \sqrt{1-\zeta^2})\ .$$

Treating $\zeta = x + \imath z$ as the complex variable, it is clear that $f(x,z)$ is a function of ζ only and hence that it is harmonic.

With this definition, we have

$$\frac{\partial^2\varphi}{\partial z^2} = B\frac{\partial f}{\partial z}(x,z) = B\Im\left\{\frac{\imath}{\sqrt{a^2-(x+\imath z)^2}}\right\}\ . \tag{1}$$

On the surface $z = 0$,

$$\frac{\partial\varphi}{\partial z} = A + B\Im\left\{\arcsin\left(\frac{x}{a}\right)\right\} \tag{2}$$

$$\frac{\partial^2\varphi}{\partial z^2} = B\Im\left\{\frac{\imath}{\sqrt{a^2-x^2}}\right\}\ . \tag{3}$$

For $|x| < a$, $arcsin(x/a)$ is real and hence

$$\frac{\partial\varphi}{\partial z} = A\ ; \quad \frac{\partial^2\varphi}{\partial z^2} = \frac{B}{\sqrt{a^2-x^2}}\ , \tag{4}$$

whilst for $|x| > a$,

$$\begin{aligned}\frac{\partial\varphi}{\partial z} &= A + B\Im\left\{\frac{1}{\imath}\ln\left(\frac{\imath x}{a} + \sqrt{1-\frac{x^2}{a^2}}\right)\right\} \\ &= A - B\ln\left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2}-1}\right) \qquad (5)\\ \frac{\partial^2\varphi}{\partial z^2} &= 0\ . \qquad (6)\end{aligned}$$

[2]See I.S.Gradshteyn and I.M.Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, New York (1980), §1.622.1.

The problem of §12.5.2 can be defined by the boundary conditions

$$u_z = C \; ; \quad |x| < a \tag{7}$$

$$\sigma_{zz} = 0 \; ; \quad |x| > a \tag{8}$$

and hence

$$\frac{\partial \varphi}{\partial z} = -\frac{\mu C}{(1-\nu)} \; ; \quad |x| < a \tag{9}$$

$$\frac{\partial^2 \varphi}{\partial z^2} = 0 \; ; \quad |x| > a \; . \tag{10}$$

Condition (10) is satisfied by (6) for all values of A, B and (4, 9) give

$$A = -\frac{\mu C}{(1-\nu)} \; . \tag{11}$$

The contact pressure distribution is

$$p(x) = -\sigma_{zz}(x,0) = \frac{\partial^2 \varphi}{\partial z^2}(x,0) = \frac{B}{\sqrt{a^2 - x^2}} \; . \tag{12}$$

The constant B is determined from the total force F as

$$F = \int_{-a}^{a} \frac{B dx}{\sqrt{a^2 - x^2}} = \pi B \tag{13}$$

and hence

$$B = \frac{F}{\pi} \; . \tag{14}$$

Thus,

$$p(x) = \frac{F}{\pi \sqrt{a^2 - x^2}} \tag{15}$$

and the surface displacement outside the contact area is

$$u_z = -\frac{(1-\nu)}{\mu} \frac{\partial \varphi}{\partial z}(x,0) = C - \frac{(1-\nu)F}{\pi \mu} \ln \left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right) \; . \tag{16}$$

from (5, 14).

29.4. Show that for frictionless contact problems for the half-space $z > 0$, the normal stress components near the surface $z = 0$ satisfy the relation

$$\sigma_{xx} + \sigma_{yy} = (1 + 2\nu)\sigma_{zz} \ ,$$

where σ_{zz} is the applied traction.

For frictionless contact problems, we can use Solution F. At the surface $z = 0$, this gives

$$\sigma_{xx} = \frac{\partial^2 \varphi}{\partial x^2} + 2\nu \frac{\partial^2 \varphi}{\partial y^2} \ ; \quad \sigma_{xx} = \frac{\partial^2 \varphi}{\partial y^2} + 2\nu \frac{\partial^2 \varphi}{\partial x^2} \ ; \quad \sigma_{zz} = -\frac{\partial^2 \varphi}{\partial z^2} \ . \tag{1}$$

It follows that

$$\sigma_{xx} + \sigma_{xx} = (1 + 2\nu) \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \right) = -(1 + 2\nu) \frac{\partial^2 \varphi}{\partial z^2} \ , \tag{2}$$

since φ is harmonic. Thus using (3) we have

$$\sigma_{xx} + \sigma_{xx} = (1 + 2\nu)\sigma_{zz} \ . \tag{3}$$

CHAPTER 30

30.1. A harmonic potential function $\omega(r,z)$ in the region $z>0$ is defined by the surface values $\omega(r,0)=A(a^2-r^2)$; $0\le r<a$; $\omega(r,0)=0$; $r>a$. Select a suitable form for the function using Table 30.1 and then use the boundary conditions to determine the corresponding unknown function $g_i(t)$.

The given boundary values of ω are continuous at $r=a$. Recalling that the functions of equations (30.21) provide continuity in displacements and hence in the first derivative of ϕ_i if $g_i(t)$ is a bounded function, it is appropriate[1] to use a representation in which $\omega=\dfrac{\partial\phi_i}{\partial z}$.

Since

$$\omega(r,0)=0 \quad ; \quad r>a \ , \tag{1}$$

we see from Table 30.1 that the appropriate form is ϕ_3 and hence we write

$$\omega=\frac{\partial\phi_3}{\partial z}=\frac{\partial}{\partial z}\Im\int_0^a g_3(t)F(r,z,t)dt=\Im\int_0^a\frac{g_3(t)dt}{\sqrt{r^2+(z+it)^2}} \tag{2}$$

Table 30.1 then shows that the external boundary condition (1) is identically satisfied and the internal condition leads to the equation

$$\omega(r,0)=\frac{\partial\phi_3}{\partial z}(r,0)=-\int_r^a\frac{g_3(t)dt}{\sqrt{t^2-r^2}}=A(a^2-r^2) \quad ; \quad 0\le r<a \ . \tag{3}$$

This is an Abel equation for $g_3(t)$ whose solution is

$$g_3(x)=\frac{2}{\pi}\frac{d}{dx}\int_x^a\frac{A(a^2-r^2)rdr}{\sqrt{r^2-x^2}} \ . \tag{4}$$

Noting that

$$\int_x^a\frac{rdr}{\sqrt{r^2-x^2}}=\sqrt{a^2-x^2} \quad ; \quad \int_x^a\frac{r^3dr}{\sqrt{r^2-x^2}}=a^2\sqrt{a^2-x^2}-\frac{2}{3}(a^2-x^2)^{\frac{3}{2}} \ , \tag{5}$$

we have

$$\begin{aligned} g_3(x) &= \frac{2}{\pi}\frac{d}{dx}\left(a^2\sqrt{a^2-x^2}-a^2\sqrt{a^2-x^2}+\frac{2}{3}(a^2-x^2)^{\frac{3}{2}}\right) \\ &= -\frac{4Ax\sqrt{a^2-x^2}}{\pi} \ . \end{aligned} \tag{6}$$

This result and the definition (2) completely define the potential function ω throughout the half-space.

[1] It is not essential to use this argument. Suppose we looked at Table 30.1 and identified the zero in $\frac{\partial^2\phi_1}{\partial z^2}$ in $r>a$ with the corresponding zero in ω. The resulting respresentation would not necessarily be continuous at $r=a$. However, since the given boundary conditions have this continuity, it must appear in the final solution. Notice also that if this representation is used, the inner boundary condition will require an integration before it yields an Abel equation for g_1. This integration will introduce an arbitrary constant that must be determined from continuity conditions at the origin.

30.2. Use equation (30.27) to derive the expressions for $\partial^2\phi_1/\partial z^2$ in Table 30.1 from those for $\partial\phi_1/\partial r$.

From Table 30.1, we have

$$\frac{\partial\phi_1}{\partial r}(r,0) = \frac{A}{r} \quad ; \quad r > a \ , \tag{1}$$

where we have written

$$\int_0^a g_1(t)dt = A \ , \tag{2}$$

since this integral is not a function of r and hence is a constant.

It follows that

$$r\frac{\partial\phi_1}{\partial r}(r,0) = A \tag{3}$$

and hence (using (30.27)) that

$$\frac{\partial^2\phi_1}{\partial z^2}(r,0) = -\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi_1}{\partial r}(r,0) = 0 \ , \tag{4}$$

confirming the entry in Table 30.1.

For the region $0 \leq r < a$, we have

$$\frac{\partial\phi_1}{\partial r}(r,0) = \frac{A}{r} - \frac{1}{r}\int_r^a \frac{tg_1(t)dt}{}.\sqrt{t^2-r^2} \quad ; \quad 0 < r < a \ . \tag{5}$$

It follows that

$$r\frac{\partial\phi_1}{\partial r}(r,0) = A - \int_r^a \frac{tg_1(t)dt}{\sqrt{t^2-r^2}} \tag{6}$$

and hence

$$\frac{\partial}{\partial r}r\frac{\partial\phi_1}{\partial r}(r,0) = -\frac{d}{dr}\int_r^a \frac{tg_1(t)dt}{\sqrt{t^2-r^2}} \ . \tag{7}$$

Equation (30.27) then gives

$$\frac{\partial^2\phi_1}{\partial z^2}(r,0) = -\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi_1}{\partial r}(r,0) = \frac{1}{r}\frac{d}{dr}\int_r^a \frac{tg_1(t)dt}{\sqrt{t^2-r^2}} \ , \tag{8}$$

confirming the results of Table 30.1.

Notice that in each case, the partial derivatives $\frac{\partial}{\partial r}$ revert to ordinary derivatives when we restrict attention to the plane $z = 0$.

30.3 An elastic half space is indented by a rigid cylindrical punch of radius a with a *concave* spherical end of radius $R \gg a$. Find the contact pressure distribution $p(r)$ and hence determine the minimum force F_0 required to maintain contact over the entire punch surface. Do not attempt to solve the problem for $F < F_0$.

The profile is described by the function

$$u_0 = d + \frac{r^2}{2R} ,$$

which is the same as (30.36) except that the sign of the second term is changed to reflect the concave curvature. We then have

$$g_1(s) = -\frac{2\mu}{\pi(1-\nu)} \frac{d}{ds} \int_0^s \frac{r u_0(r) dr}{\sqrt{s^2 - r^2}} = -\frac{2\mu}{\pi(1-\nu)} \frac{d}{ds} \int_0^s \left(dr + \frac{r^3}{2R} \right) \frac{dr}{\sqrt{s^2 - r^2}} . \quad (1)$$

The integrals

$$\int_0^s \frac{r dr}{\sqrt{s^2 - r^2}} = -\sqrt{s^2 - r^2}\Big|_0^s = s$$

and

$$\begin{aligned} \int_0^s \frac{r^3 dr}{\sqrt{s^2 - r^2}} &= -r^2 \sqrt{s^2 - r^2}\Big|_0^s + 2 \int_0^s r \sqrt{s^2 - r^2} ds \\ &= 0 - \frac{2(s^2 - r^2)^{3/2}}{3}\Bigg|_0^s \\ &= \frac{2s^3}{3} , \end{aligned} \quad (2)$$

so

$$g_1(s) = -\frac{2\mu}{\pi(1-\nu)} \frac{d}{ds} \left(ds + \frac{s^3}{3R} \right) = -\frac{2\mu}{\pi(1-\nu)} \left(d + \frac{s^2}{R} \right) .$$

The contact pressure is then given by

$$p(r) = -\sigma_{zz} = \frac{\partial^2 \varphi}{\partial z^2} = \frac{1}{r} \frac{d}{dr} \int_r^a \frac{t g_1(t) dt}{\sqrt{t^2 - r^2}} ,$$

from Table 30.1.

Substituting for g_1 and using the results

$$\int_r^a \frac{t dt}{\sqrt{t^2 - r^2}} = \sqrt{t^2 - r^2}\Big|_r^a = \sqrt{a^2 - r^2}$$

$$\begin{aligned} \int_r^a \frac{t^3 dt}{\sqrt{t^2 - r^2}} &= t^2 \sqrt{t^2 - r^2}\Big|_r^a - 2 \int_r^a t \sqrt{t^2 - r^2} dt \\ &= a^2 \sqrt{a^2 - r^2} - \frac{2(t^2 - r^2)^{3/2}}{3}\Bigg|_r^a \\ &= a^2 \sqrt{a^2 - r^2} - \frac{2(a^2 - r^2)^{3/2}}{3} \end{aligned}$$

we obtain

$$
\begin{aligned}
p(r) &= -\frac{2\mu}{\pi(1-\nu)r}\frac{d}{dr}\left(d\sqrt{a^2-r^2}+\frac{a^2\sqrt{a^2-r^2}}{R}-\frac{2(a^2-r^2)^{3/2}}{3R}\right) \\
&= -\frac{2\mu}{\pi(1-\nu)r}\left(-\frac{dr}{\sqrt{a^2-r^2}}-\frac{a^2r}{R\sqrt{a^2-r^2}}+\frac{2r\sqrt{a^2-r^2}}{R}\right) \\
&= \frac{2\mu}{\pi(1-\nu)R}\left(\frac{Rd+a^2-2(a^2-r^2)}{\sqrt{a^2-r^2}}\right) \\
&= \frac{2\mu}{\pi(1-\nu)R}\left(\frac{Rd-a^2+2r^2}{\sqrt{a^2-r^2}}\right) . \qquad (3)
\end{aligned}
$$

The contact pressure must be positive and this condition will fail first at the origin $r = 0$ when

$$d = \frac{a^2}{R} .$$

In other words, equation (3) will define an unacceptable tensile contact traction at $r = 0$ unless $d > a^2/R$.

The total force applied to the punch is given by equation (30.35) as

$$F = -2\pi\int_0^a g_1(t)dt = \frac{4\mu}{(1-\nu)}\left(da+\frac{a^3}{3R}\right) ,$$

so

$$d = \frac{F(1-\nu)}{4\mu a}-\frac{a^2}{3R}$$

and hence to achieve the required minimum value of d, we require

$$\frac{F(1-\nu)}{4\mu a}-\frac{a^2}{3R} > \frac{a^2}{R}$$

or

$$\frac{F(1-\nu)}{4\mu a} > \frac{a^2}{R}+\frac{a^2}{3R} = \frac{4a^2}{3R}$$

or

$$F > \frac{16\mu a^3}{3R(1-\nu)} .$$

30.4. A rigid flat punch of radius a is bonded to the surface of the elastic half space $z > 0$. A torque T is then applied to the punch. Use Solution E to describe the stress field in the half space and Collins' method to solve the resulting boundary-value problem for Ψ. In particular, find the distribution of traction between the punch and the half space and the rotation of the punch due to the torque.

Suppose that the punch rotates through an angle α, in which case we will have

$$u_\theta = \alpha r \; ; \quad 0 < r < a, z = 0 \; . \tag{1}$$

We also require that the surface $r > a, z = 0$ be traction-free and hence in particular

$$\sigma_{z\theta} = 0 \; ; \quad r > a, z = 0 \; . \tag{2}$$

Using Solution E, these equations translate into the boundary conditions

$$-2\frac{\partial \Psi}{\partial r} = 2\mu\alpha r \; ; \quad 0 < r < a, z = 0 \tag{3}$$

$$-\frac{\partial^2 \Psi}{\partial r \partial z} = 0 \; ; \quad r > a, z = 0 \; . \tag{4}$$

Equation (4) shows that $\partial\Psi/\partial z$ is constant outside the contact area and this constant can be set to zero. Thus condition (4) can be satisfied by using ϕ_3 of Table 30.1. Condition (3) then requires that

$$\frac{2}{r}\int_0^r \frac{tg(t)dt}{\sqrt{r^2 - t^2}} = 2\mu\alpha r \; ; \quad 0 < r < a \; . \tag{5}$$

The solution of this Abel equation is obtained from Table 30.2 as

$$xg(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^x \frac{\mu\alpha r^3 dr}{\sqrt{x^2 - r^2}} = \frac{2}{\pi}\frac{d}{dx}\left(\frac{2\mu\alpha x^3}{3}\right) = \frac{4\mu\alpha x^2}{\pi} \tag{6}$$

and hence

$$g(x) = \frac{4\mu\alpha x}{\pi} \; . \tag{7}$$

The traction in the contact area is then found as

$$\begin{aligned} \sigma_{z\theta} &= -\frac{\partial^2 \Psi}{\partial r \partial z} = \frac{d}{dr}\int_r^a \frac{g(t)dt}{\sqrt{t^2 - r^2}} = \frac{4\mu\alpha}{\pi}\frac{d}{dr}\int_r^a \frac{tdt}{\sqrt{t^2 - r^2}} = \frac{4\mu\alpha}{\pi}\frac{d}{dr}\sqrt{a^2 - r^2} \\ &= -\frac{4\mu\alpha r}{\pi\sqrt{a^2 - r^2}} \; . \end{aligned} \tag{8}$$

The applied torque is

$$T = -\int_0^{2\pi}\int_0^a \sigma_{z\theta} r^2 dr d\theta = 8\mu\alpha \int_0^a \frac{r^3 dr}{\sqrt{a^2 - r^2}} = \frac{16\mu\alpha a^3}{3} \; . \tag{9}$$

Thus, the rotation α is

$$\alpha = \frac{3T}{16\mu a^3} \tag{10}$$

and the traction distribution is

$$\sigma_{z\theta} = -\frac{3Tr}{4\pi a^3\sqrt{a^2 - r^2}} \ . \tag{11}$$

30.5. A rigid sphere of radius R is pressed into an elastic half space of shear modulus μ, and Poisson's ratio ν. Assuming that the interface energy for this material combination is $\Delta\gamma$, find a relation between the indenting force F and the radius of the contact area a. You should use the conditions (29.17, 29.18) to determine the unknown constant d in (30.36) and then (30.35) to determine F.

Hence determine the contact radius when the sphere is unloaded ($F = 0$) and the tensile force required to pull the sphere out of contact[2].

As in the example in §30.2, we have

$$u_0 = d - \frac{r^2}{2R}$$

and hence

$$g_1(s) = -\frac{2\mu}{\pi(1-\nu)}\left(d - \frac{s^2}{R}\right) .$$

$$p(r) = \frac{2\mu}{\pi(1-\nu)R}\left(\frac{Rd + a^2 - 2r^2}{\sqrt{a^2 - r^2}}\right)$$

$$F = \frac{4\mu a}{(1-\nu)}\left(d - \frac{a^2}{3R}\right) , \tag{1}$$

from equations (30.37–30.39) with $b = a$.

Using (29.17), we then have

$$\begin{aligned} K_I &= -\lim_{r\to a} p(r)\sqrt{2\pi(a-r)} = -\frac{2\mu}{\pi(1-\nu)R}\lim_{r\to a}\sqrt{2\pi}\left(\frac{Rd + a^2 - 2r^2}{\sqrt{a+r}}\right) \\ &= -\frac{2\mu(Rd - a^2)}{\sqrt{\pi a}(1-\nu)R} \end{aligned}$$

Equation (29.18) with $\mu_1 = \mu, \nu_1 = \nu, \mu_2 = \infty$ gives

$$K_I = 2\sqrt{\frac{\mu\Delta\gamma}{(1-\nu)}}$$

and hence here we have

$$-\frac{2\mu(Rd - a^2)}{\sqrt{\pi a}(1-\nu)R} = 2\sqrt{\frac{\mu\Delta\gamma}{(1-\nu)}}$$

[2]This solution was first given by K.L.Johnson, K.Kendall and A.D.Roberts, Surface energy and the contact of elastic solids, *Proceedings of the Royal Society of London*, Vol.A324 (1971), pp.301–313.

or

$$(Rd - a^2) = -\frac{(1-\nu)R}{\mu}\sqrt{\frac{\pi a \mu \Delta\gamma}{(1-\nu)}}$$

Solving this equation for d, we have

$$d = \frac{a^2}{R} - \frac{(1-\nu)}{\mu}\sqrt{\frac{\pi a \mu \Delta\gamma}{(1-\nu)}}$$

and substituting into (1),

$$\begin{aligned} F &= \frac{4\mu a}{(1-\nu)}\left(\frac{2a^2}{3R} - \frac{(1-\nu)}{\mu}\sqrt{\frac{\pi a \mu \Delta\gamma}{(1-\nu)}}\right) \\ &= \frac{8\mu a^3}{3R(1-\nu)} - 4\sqrt{\frac{\pi a^3 \mu \Delta\gamma}{(1-\nu)}} \end{aligned} \tag{2}$$

If the sphere is unloaded, $F = 0$ and

$$\frac{2\mu a^{3/2}}{3R(1-\nu)} = \sqrt{\frac{\pi \mu \Delta\gamma}{(1-\nu)}}$$

or

$$\frac{4\mu^2 a^3}{9R^2(1-\nu)^2} = \frac{\pi \mu \Delta\gamma}{(1-\nu)}$$

giving

$$a = \left(\frac{9R^2 \pi \Delta\gamma (1-\nu)}{4\mu}\right)^{1/3} .$$

To find the maximum pull-off force, we set

$$\frac{\partial F}{\partial a} = 0$$

and hence

$$\frac{8\mu a^2}{R(1-\nu)} = 6\sqrt{\frac{\pi a \mu \Delta\gamma}{(1-\nu)}} ,$$

or

$$\frac{8\mu a^2}{R(1-\nu)} = 6\sqrt{\frac{\pi a \mu \Delta\gamma}{(1-\nu)}} ,$$

$$a^3 = \frac{9R^2 \pi \Delta\gamma (1-\nu)}{16\mu} .$$

Substituting this into (2) and simplifying, we obtain

$$F = \frac{3R\pi\Delta\gamma}{2} - 3R\pi\Delta\gamma = -\frac{3R\pi\Delta\gamma}{2} .$$

This result is negative because F is defined to be positive when compressive. Notice that the result is independent of the elastic modulus of the material. It depends only on the surface energy and the radius of the sphere.

30.6. (i) An elastic half-space is indented by a conical frictionless rigid punch as shown in Figure 30.3, the penetration of the apex being d. Find the contact pressure distribution, the total load F and the contact radius a. The semi-angle of the cone is $\frac{\pi}{2}-\alpha$, where $\alpha \ll 1$.

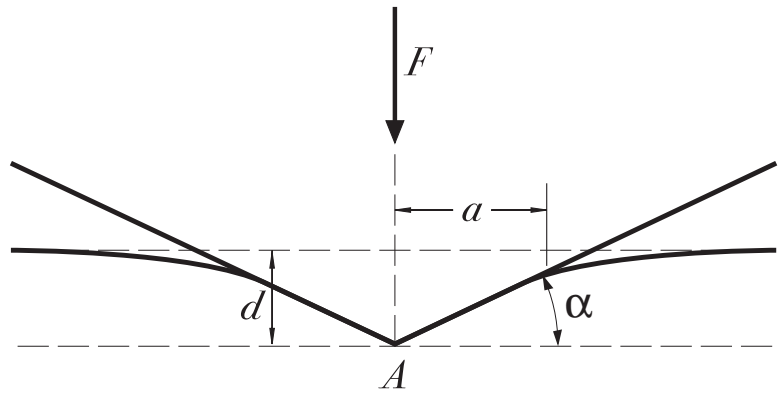


Figure 30.3: The conical punch.

(ii) What is the nature of the singularity in pressure at the vertex of the cone (point A).

(iii) Now suppose that the cone is truncated as shown in Figure 30.4. *Without finding the new contact pressure distribution*, **find the new relationship between F and a. Note: Take the radius of the truncated end of the cone as b and refer the profile to the point A where the vertex would have been before truncation.**

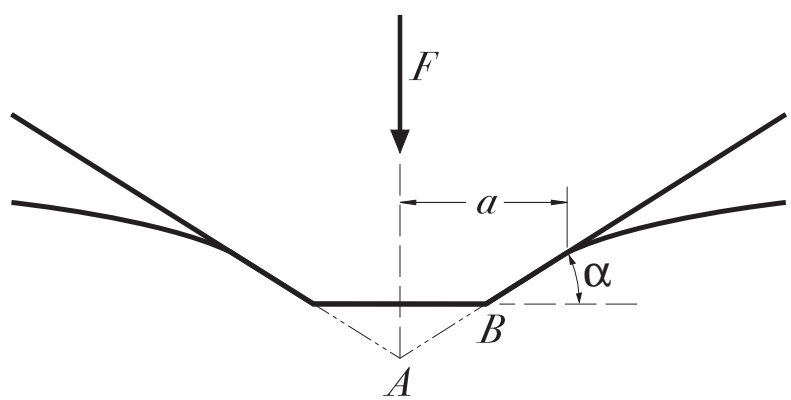


Figure 30.4: The truncated conical punch.

(iv) Would you expect a singularity in contact pressure at B and if so of what form. (The process of determining $p(r)$ is algebraically tedious. Try to find a simpler way to answer the question.)

For this problem, the normal surface displacement is

$$u_0 = d - \alpha r \ , \tag{3}$$

since $\alpha \ll 1$.

Using equation (30.33), we have

$$g_1(s) = -\frac{2\mu}{\pi(1-\nu)}\frac{d}{ds}\int_0^s \frac{(d-\alpha r)rdr}{\sqrt{s^2-r^2}} = -\frac{2\mu}{\pi(1-\nu)}\left(d - \frac{\alpha\pi s}{2}\right) \ . \tag{4}$$

The total force F is given by equation (30.35) as

$$F = -2\pi\int_0^a g_1(s)ds = \frac{4\mu}{(1-\nu)}\left(da - \frac{\alpha\pi a^2}{4}\right) \tag{5}$$

and we can therefore determine the contact radius a by writing

$$\frac{dF(a)}{da} = 0 \ . \tag{6}$$

Substituting for F from (3) and performing the differentiation, we obtain

$$d = \frac{\alpha\pi a}{2} \tag{7}$$

and hence

$$F(a) = \frac{\alpha\pi a^2\mu}{(1-\nu)} \tag{8}$$

from (3,5) — i.e.

$$a = \sqrt{\frac{F(1-\nu)}{\alpha\pi\mu}} \ . \tag{9}$$

The contact pressure distribution is

$$p(r) = \frac{1}{r}\frac{d}{dr}\int_r^a \frac{tg_1(t)dt}{\sqrt{t^2-r^2}} \ . \tag{10}$$

Substituting for g_1 from (2) and using the integrals

$$\int_r^a \frac{tdt}{\sqrt{t^2-r^2}} = \sqrt{a^2-r^2} \ ; \tag{11}$$

$$\int_r^a \frac{t^2dt}{\sqrt{t^2-r^2}} = \frac{1}{2}r^2\cosh^{-1}\left(\frac{a}{r}\right) + \frac{1}{2}a\sqrt{a^2-r^2} \ , \tag{12}$$

we obtain

$$p(r) = -\frac{2\mu}{\pi(1-\nu)r}\frac{d}{dr}\left[d\sqrt{a^2-r^2} - \frac{\alpha\pi}{4}r^2\cosh^{-1}\left(\frac{a}{r}\right) - a\frac{\alpha\pi}{4}\sqrt{a^2-r^2}\right] \ . \tag{13}$$

After substituting for d from equation (5) and performing the differentiation, this expression simplifies to

$$p(r) = \frac{\mu\alpha}{(1-\nu)} \cosh^{-1}\left(\frac{a}{r}\right) \ . \tag{14}$$

Notice that the contact pressure tends to zero as $r \to a$ as expected (this is equivalent to the condition $dF(a)/da = 0$), but is singular at the vertex of the cone $r = 0$. The inverse hyperbolic cosine can be written

$$\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1}) \tag{15}$$

and hence, as $x \to \infty$ it tends to $\log(2x)$.

It follows that for $r \ll a$,

$$p(r) \to \frac{\mu\alpha}{(1-\nu)} \log\left(\frac{2a}{r}\right) \tag{16}$$

and hence that the singularity is logarithmic.

The truncated cone

If the cone is truncated, the indentation is described by the function

$$u_0 = d - \alpha b \quad ; \quad 0 \le r \le b \ ; \tag{17}$$

$$= d - \alpha r \quad ; \quad b < r < a \ . \tag{18}$$

As before, we have

$$\begin{aligned} g_1(s) &= -\frac{2\mu}{\pi(1-\nu)} \frac{d}{ds} \int_0^s \frac{u_0 r dr}{\sqrt{s^2 - r^2}} \\ &= -\frac{2\mu}{\pi(1-\nu)} \frac{d}{ds} \left(\int_0^b \frac{(d-\alpha b) r dr}{\sqrt{s^2 - r^2}} + \int_b^s \frac{(d - \alpha r) r dr}{\sqrt{s^2 - r^2}} \right) \ . \end{aligned} \tag{19}$$

After tedious but routine algebraic manipulations, we obtain

$$g_1(s) = -\frac{2\mu}{\pi(1-\nu)}(d - \alpha b) \qquad ; \quad 0 \le s \le b \ ; \tag{20}$$

$$= -\frac{2\mu}{\pi(1-\nu)} \left[d - \alpha b - \alpha s \cos^{-1}\left(\frac{b}{s}\right) \right] \quad ; \quad b < s < a \tag{21}$$

and the total force is now given by

$$F(a) = -2\pi \int_0^a g_1(s) ds \ . \tag{22}$$

Substituting for g_1 and performing the integration, we obtain

$$F(a) = \frac{4\mu}{(1-\nu)} \left[da - \alpha b a - \frac{1}{2}\alpha a^2 \cos^{-1}\left(\frac{b}{a}\right) + \frac{1}{2}\alpha b \sqrt{a^2 - b^2} \right] \ . \tag{23}$$

Imposing the condition $dF(a)/da = 0$, which is in fact equivalent to setting $g_1(a) = 0$ (see §30.2.5), we find

$$d = \alpha b + \alpha a \cos^{-1}\left(\frac{b}{a}\right) \tag{24}$$

and hence

$$F(a) = \frac{2\mu}{(1-\nu)}\left[\alpha a^2 \cos^{-1}\left(\frac{b}{a}\right) + \alpha b\sqrt{a^2 - b^2}\right] . \tag{25}$$

Notice that $F(b) = 0$, as we should expect, since the flat portion of the truncated cone will make contact with the half-space even when a negligible force is applied.

Finally, there will be a logarithmic singularity at the point B. Locally, the problem is similar to the indentation of a half-plane by a wedge and this problem has the same local behaviour as the cone at its vertex. Using the arguments of §11.2, we note that the displacements vary linearly with distance from the edge and hence we expect stresses varying with distance to power zero. Of course, uniform stresses have this behaviour as well, but the sharp corner leads us to anticipate logarithmic terms as well (which also vary with distance to power zero in the sense of §11.2).

30.7. A rigid flat punch has rounded edges, as shown in Figure 30.5. The punch is pressed into an elastic half space by a force F. Assuming that the contact is frictionless, find the relation between F, the indentation u_0 and the radius a of the contact area.

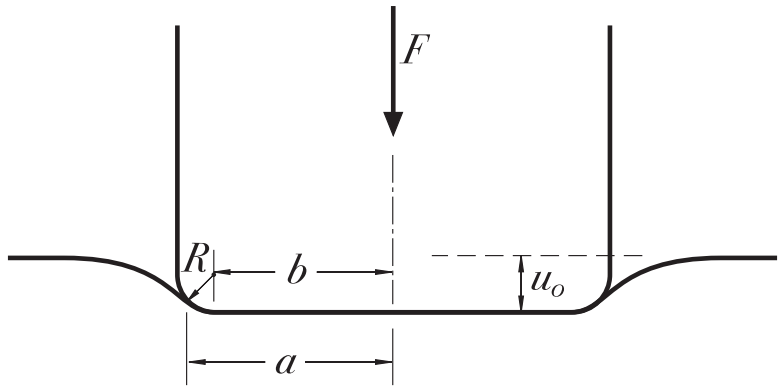


Figure 30.5: Flat punch with rounded corners.

The indentation function $u_0(r)$ of equation (30.1) is given by

$$u_0(r) = u_0 \; ; \quad 0 \le r < b \tag{1}$$

$$= u_0 - \frac{(r-b)^2}{2R} \; ; \quad b < r < a \; . \tag{2}$$

The function $g_1(s)$ of equation (30.33) is then given by

$$g_1(s) = -\frac{2\mu u_0}{\pi(1-\nu)} \frac{d}{ds} \int_0^s \frac{rdr}{\sqrt{s^2 - r^2}} \; ; \quad 0 \le s < b \tag{3}$$

and

$$g_1(s) = -\frac{2\mu}{\pi(1-\nu)} \frac{d}{ds} \left(\int_0^b \frac{u_0 rdr}{\sqrt{s^2 - r^2}} + \int_b^s \left(u_0 - \frac{(r-b)^2}{2R} \right) \frac{rdr}{\sqrt{s^2 - r^2}} \right) \tag{4}$$

$$= -\frac{2\mu}{\pi(1-\nu)} \frac{d}{ds} \left(\int_0^s \frac{u_0 rdr}{\sqrt{s^2 - r^2}} - \int_b^s \frac{(r-b)^2 rdr}{2R\sqrt{s^2 - r^2}} \right) \; ; \quad b < s < a \; , \tag{5}$$

where a is the outer radius of the contact area. Performing the integrals and differentiations, we obtain

$$g_1(s) = -\frac{2\mu u_0}{\pi(1-\nu)} \; ; \quad 0 \le r < b \tag{6}$$

$$= -\frac{2\mu u_0}{\pi(1-\nu)} + \frac{2\mu s}{\pi(1-\nu)R} \left(\sqrt{s^2 - b^2} - b \arccos\left(\frac{b}{s}\right) \right) \; ; \quad b < s < a \; . \tag{7}$$

The outer contact radius a is determined from the condition $g_1(a) = 0$ and hence

$$u_0 - \frac{a}{R}\left(\sqrt{a^2 - b^2} - b \arccos\left(\frac{b}{a}\right)\right) = 0 \ . \tag{8}$$

The total force is obtained from equation (30.35) as

$$\begin{aligned} F &= -2\pi \int_0^a g_1(s) ds \\ &= \frac{4\mu u_0}{(1-\nu)} \int_0^a ds - \frac{4\mu s}{(1-\nu)R} \int_b^a \left(\sqrt{s^2 - b^2} - b \arccos\left(\frac{b}{s}\right)\right) ds \\ &= \frac{2\mu}{3(1-\nu)R}\left((4a^2 - b^2)\sqrt{a^2 - b^2} - 3a^2 b \arccos\left(\frac{b}{a}\right)\right) \ , \end{aligned} \tag{9}$$

after evaluating the integrals and substituting for u_0 from equation (8).

Alternative proof using §30.2.5

We have

$$u_0'(r) = 0 \ ; \quad 0 \le r < b \tag{10}$$

$$= 0 - \frac{(r-b)}{R} \ ; \quad b < r < a \ . \tag{11}$$

Substituting into equation (30.51), we have

$$\begin{aligned} F &= \frac{4\mu}{(1-\nu)R} \int_b^a \frac{r^2 (r-b) dr}{\sqrt{a^2 - r^2}} \\ &= \frac{2\mu}{3(1-\nu)R}\left((4a^2 - b^2)\sqrt{a^2 - b^2} - 3a^2 b \arccos\left(\frac{b}{a}\right)\right) \ , \end{aligned}$$

using Maple. Also from (30.48),

$$d = \frac{a}{R} \int_0^a \frac{(r-b) dr}{\sqrt{a^2 - r^2}} = \frac{a}{R}\left(\sqrt{a^2 - b^2} - b \arccos\left(\frac{b}{a}\right)\right) \ . \tag{12}$$

30.8. An elastic half space is indented by an axisymmetric rigid punch with the power law profile Cr^s, so the displacement in the contact area is

$$u_0(r) = d - Cr^s \ ,$$

where C, s are constants. Show that the indentation force F, the contact radius a and the indentation depth d are related by the equation

$$F = \frac{4\mu a d}{(1-\nu)} \left(\frac{s}{s+1} \right) \ .$$

Substituting

$$u_0(r) = d - Cr^s \tag{1}$$

into the integral in (30.33), we obtain

$$\int_0^t \frac{r u_0(r) dr}{\sqrt{t^2 - r^2}} = \int_0^t \frac{(rd - Cr^{(s+1)}) dr}{\sqrt{t^2 - r^2}} \ . \tag{2}$$

Writing

$$x = \frac{r}{t} \ ; \quad r = tx \ , \tag{3}$$

we have

$$\int_0^t \frac{r u_0(r) dr}{\sqrt{t^2 - r^2}} = \int_0^1 \frac{(txd - Cx^{(s+1)} t^{(s+1)} dx}{\sqrt{1 - x^2}} = td - Cf(s) t^{(s+1)} \ , \tag{4}$$

where

$$f(s) = \int_0^1 \frac{x^{(s+1)} dx}{\sqrt{1 - x^2}} \tag{5}$$

is a function of s only.

It follows from (30.33) that

$$g_1(t) = -\frac{2\mu}{\pi(1-\nu)} \frac{d}{dt} \left(td - Cf(s) t^{(s+1)} \right) = -\frac{2\mu}{\pi(1-\nu)} \left(d - C(s+1) f(s) t^s \right) \ . \tag{6}$$

The contact area is determined from the condition

$$g_1(a) = 0 \tag{7}$$

and hence

$$C(s+1) f(s) a^s = d \ . \tag{8}$$

The total load is

$$\begin{aligned} F &= -2\pi \int_0^a g_1(t) dt = \frac{4\mu}{(1-\nu)} \int_0^a \left(d - C(s+1) f(s) t^s \right) dt \\ &= \frac{4\mu}{(1-\nu)} \left(ad - Cf(s) a^{(s+1)} \right) \ . \end{aligned} \tag{9}$$

Using (8) to substitute for $Cf(s)a^s$ in the second term, we then obtain

$$F = \frac{4\mu}{(1-\nu)} \left(ad - \frac{ad}{(s+1)} \right) = \frac{4\mu a d}{(1-\nu)} \frac{s}{(s+1)} \ . \tag{10}$$

Alternative proof

We have

$$u_0' = -Csr^{s-1} \ ,$$

so from (30.48),

$$d = Cas \int_0^a \frac{r^{s-1}dr}{\sqrt{a^2 - r^2}}$$

and from (30.53),

$$F = \frac{4C\mu s}{(1-\nu)} \int_0^a \frac{r^{s+1}(r)dr}{\sqrt{a^2 - r^2}} \ .$$

Integrating by parts,

$$\begin{aligned}
\int_0^a \frac{r^{s+1}dr}{\sqrt{a^2 - r^2}} &= -r^s\sqrt{a^2 - r^2} + s\int_0^a r^{s-1}\sqrt{a^2 - r^2}dr \\
&= 0 + s\int_0^a \frac{r^{s-1}(a^2 - r^2)dr}{\sqrt{a^2 - r^2}} \\
&= a^2 s\int_0^a \frac{r^{s-1}dr}{\sqrt{a^2 - r^2}} - s\int_0^a \frac{r^{s+1}dr}{\sqrt{a^2 - r^2}} \ .
\end{aligned}$$

Carrying the second term on the right-hand side over to the left, we then have

$$(s+1)\int_0^a \frac{r^{s+1}dr}{\sqrt{a^2 - r^2}} = a^2 s\int_0^a \frac{r^{s-1}dr}{\sqrt{a^2 - r^2}} \ .$$

Using these results, we then have

$$F = \frac{4C\mu a^2 s^2}{(s+1)(1-\nu)} \int_0^a \frac{r^{s-1}dr}{\sqrt{a^2 - r^2}} = \frac{4C\mu a^2 s^2}{(s+1)(1-\nu)} \frac{d}{Cas} = \frac{4\mu ads}{(s+1)(1-\nu)} \ .$$

30.9. The profile of a smooth axisymmetric frictionless rigid punch is described by the power law

$$u_0(r) = A_n r^{2n} \ ,$$

where n is an integer. The punch is pressed into an elastic half space by a force F. Find the indentation d, the radius of the contact area a and the contact pressure distribution $p(r)$. Check your results by comparison with the Hertz problem of §30.2.5 and give simplified expressions for the case of the fourth order punch

$$u_0(r) = A_2 r^4 \ .$$

For the power law punch, we have

$$u_0(r) = d - A_n r^{2n} \tag{1}$$

and hence substituting into (30.33) we have

$$g(s) = -\frac{2\mu}{\pi(1-\nu)} \frac{d}{ds} \int_0^s \frac{\left[d - A_n r^{2n}\right] r dr}{\sqrt{s^2 - r^2}} \ . \tag{2}$$

Writing

$$s = a \cos\theta \ ,$$

we have

$$\int_0^s \frac{r^{2n+1} dr}{\sqrt{s^2 - r^2}} = s^{2n+1} \int_0^{\pi/2} \cos^{2n+1}\theta d\theta = \frac{(2n)!! s^{2n+1}}{(2n+1)!!} \ , \tag{3}$$

where $(2n)!! = 2.4...(2n)$, $(2n+1)!! = 1.3...(2n+1)$ and we adopt the convention that $(0)!! = (-1)!! = (-2)!! = (-3)!! = 1$. We then have

$$\int_0^s \frac{\left[d - A_n r^{2n}\right] r dr}{\sqrt{s^2 - r^2}} = ds - \frac{A_n (2n)!! s^{2n+1}}{(2n+1)!!} \ , \tag{4}$$

and

$$g(s) = -\frac{2\mu}{\pi(1-\nu)} \left[d - \frac{A_n (2n)!! s^{2n}}{(2n-1)!!} \right] \ . \tag{5}$$

The load F can then be determined from equation (30.35) as

$$F = -2\pi \int_0^a g(t) dt = \frac{4\mu}{(1-\nu)} \int_0^a \left[d - \frac{A_n (2n)!! t^{2n}}{(2n-1)!!} \right] dt = \frac{4\mu}{(1-\nu)} \left[da - \frac{A_n (2n)!! a^{2n+1}}{(2n+1)!!} \right] \ . \tag{6}$$

Since the contact is smooth, the condition $dF/da = 0$ or $g(a) = 0$ gives

$$d = \frac{A_n (2n)!! a^{2n}}{(2n-1)!!} \tag{7}$$

and substitution into (6) gives

$$F(a) = \frac{4\mu}{(1-\nu)}\left[\frac{A_n(2n)!!a^{2n+1}}{(2n-1)!!} - \frac{A_n(2n)!!a^{2n+1}}{(2n+1)!!}\right] = \frac{8\mu A_n n(2n)!!a^{2n+1}}{(1-\nu)(2n+1)!!} \ . \tag{8}$$

For the contact pressure, we have

$$p(r) = \frac{1}{r}\frac{d}{dr}\int_r^a \frac{tg(t)dt}{\sqrt{t^2-r^2}} \tag{9}$$

and for smooth contact problems there is some advantage of integrating by parts to obtain

$$\int_r^a \frac{tg(t)dt}{\sqrt{t^2-r^2}} = \sqrt{a^2-r^2}g(a) - \int_r^a \sqrt{t^2-r^2}g'(t)dt \ . \tag{10}$$

The first term is zero since $g(a) = 0$ (see above) and hence

$$p(r) = -\frac{1}{r}\frac{d}{dr}\int_r^a \sqrt{t^2-r^2}g'(t)dt = \int_r^a \frac{g'(t)dt}{\sqrt{t^2-r^2}} \ . \tag{11}$$

This is actually equation (30.45) which we can use directly. Here we have

$$p(r) = \frac{2\mu A_n(2n)!!(2n)}{(2n-1)!!\pi(1-\nu)}\int_r^a \frac{t^{2n-1}dt}{\sqrt{t^2-r^2}} \ , \tag{12}$$

Writing $\cosh x = t/r$, we have

$$\int_r^a \frac{t^{2n-1}dt}{\sqrt{t^2-r^2}} = r^{2n-1}\int_0^{\cosh^{-1}(a/r)} \cosh^{2n-1} x dx$$

$$\begin{aligned} &= \frac{r^{2n-1}\sinh x}{(2n-1)}\left[\cosh^{2n-2} x + \sum_{k=0}^{n-2}\frac{(2n-2)!!(2n-2k-5)!!}{(2n-3)!!(2n-2k-4)!!}\cosh^{2n-2k-4} x\right]_0^{\cosh^{-1}(a/r)} \\ &= \frac{a^{2n-2}\sqrt{a^2-r^2}}{(2n-1)}\left[1 + \sum_{k=0}^{n-2}\frac{(2n-2)!!(2n-2k-5)!!}{(2n-3)!!(2n-2k-4)!!}\left(\frac{r}{a}\right)^{2k+2}\right] \end{aligned} \tag{13}$$

and hence

$$p(r) = \frac{2\mu A_n(2n)!!(2n)!!a^{2n-2}\sqrt{a^2-r^2}}{(2n-1)!!(2n-1)!!\pi(1-\nu)}\left[\frac{(2n-3)!!}{(2n-2)!!} + \sum_{k=0}^{n-2}\frac{(2n-2k-5)!!}{(2n-2k-4)!!}\left(\frac{r}{a}\right)^{2k+2}\right] \ . \tag{14}$$

For the special case of the Hertz problem, $n = 1$ and $A_1 = 1/2R$, giving

$$d = \frac{a^2}{R} \ ; \quad F = \frac{8\mu a^3}{3(1-\nu)R} \tag{15}$$

and

$$p(r) = \frac{4\mu\sqrt{a^2 - r^2}}{\pi(1-\nu)R} \tag{16}$$

agreeing with (30.56, 30.55, 30.60) respectively.

For a fourth order punch, $n = 2$,

$$d = \frac{8A_2a^4}{3} \; ; \quad F = \frac{128\mu A_2 a^5}{15(1-\nu)} \tag{17}$$

and

$$p(r) = \frac{64\mu A_2(a^2 + 2r^2)\sqrt{a^2 - r^2}}{9\pi(1-\nu)} \; . \tag{18}$$

Alternative solution using §30.2.5

Differentiating (1), we have

$$u_0'(r) = -2nA_n r^{2n-1} \tag{19}$$

and hence

$$d = -a\int_0^a \frac{u_0'(r)dr}{\sqrt{a^2 - r^2}} = 2nA_n a\int_0^a \frac{r^{2n-1}dr}{\sqrt{a^2 - r^2}} \; ,$$

from (30.48). Evaluating the integral using (3) with $s \to a, n \to (n-1)$, we have

$$d = \frac{A_n(2n)!!a^{2n}}{(2n-1)!!} \tag{20}$$

as before. The total force F is obtained from (30.53) as

$$F = -\frac{4\mu}{(1-\nu)}\int_0^a \frac{u_0'(r)r^2dr}{\sqrt{a^2 - r^2}} = \frac{8\mu n A_n}{(1-\nu)}\int_0^a \frac{r^{2n+1}dr}{\sqrt{a^2 - r^2}} = \frac{8\mu A_n n(2n)!!a^{2n+1}}{(1-\nu)(2n+1)!!} \; ,$$

using (3). If F is given, this equation allows us to find a, after which d is found from (20).

For the contact pressure distribution, we first note that

$$\int_0^s \frac{u_0'(r)dr}{\sqrt{s^2 - r^2}} = -2nA_n\int_0^s \frac{r^{2n-1}dr}{\sqrt{s^2 - r^2}} = -\frac{A_n(2n)!!s^{2n-1}}{(2n-1)!!}$$

by analogy with (20) with a replaced by s. Substituting this result into (30.49) and differentiating with respect to s, we have

$$g_1'(s) = -\frac{2\mu A_n(2n)!!s^{2n-1}}{\pi(1-\nu)(2n-1)!!}$$

after which the pressure is obtained from equation (30.45) as before.

30.10. For a smooth axisymmetric contact problem, the contact pressure must be continuous at $r = a$. Usually this condition is imposed explicitly and it serves to determine the radius a of the contact region. However, it can also be guaranteed by 'choice of form' as suggested in §30.3. Use this strategy to solve the Hertzian contact problem, for which

$$u_0'(r) = -\frac{r}{R} \ ,$$

where R is the radius of the indenter.

You will need to find a suitable representation of the form

$$\frac{\partial \varphi}{\partial z} = \phi_i \ ,$$

where ϕ_i is one of the four functions in equation (30.21), chosen so as to satisfy the homogeneous boundary condition (zero contact pressure in $r > a$). Then use Tables 30.1, 30.2 to set up and solve an Abel equation for the unknown function $g_i(t)$. In particular, find the contact pressure distribution $p(r)$ and the total load F needed to establish a circular contact area of radius a.

With the proposed representation, equations (29.1, 29.2) imply

$$p(r) = -\sigma_{zz}(r,0) = \frac{\partial^2 \varphi}{\partial z^2}(r,0) = \frac{\partial \phi_i}{\partial z}(r,0) \tag{1}$$

$$u_z(r,0) = -\frac{(1-\nu)}{\mu}\frac{\partial \varphi}{\partial z}(r,0) = -\frac{(1-\nu)}{\mu}\phi_i(r,0) \ . \tag{2}$$

The boundary conditions for the Hertz problem are

$$\frac{\partial u_z}{\partial r}(r,0) = -\frac{r}{R} \ ; \quad 0 \le r < a \tag{3}$$

$$p(r) = 0 \ ; \quad r > a \tag{4}$$

and hence

$$\frac{\partial \phi_i}{\partial r}(r,0) = \frac{\mu r}{(1-\nu)R} \ ; \quad 0 \le r < a \tag{5}$$

$$\frac{\partial \phi_i}{\partial z} = 0 \ ; \quad r > a \ . \tag{6}$$

Examination of Table 30.1 shows that the homogeneous condition (6)is satisfied identically by the choice $i = 3$. The inhomogeneous boundary condition (5) then requires

$$-\frac{1}{r}\int_0^r \frac{t g_3(t) dt}{\sqrt{r^2 - t^2}} = \frac{\mu r}{(1-\nu)R} \ ; \quad 0 \le r < a \ , \tag{7}$$

from Table 30.1, and hence

$$\int_0^r \frac{tg_3(t)dt}{\sqrt{r^2-t^2}} = -\frac{\mu r^2}{(1-\nu)R} \; ; \quad 0 \le r < a \; . \tag{8}$$

This Abel equation can be solved using Table 30.2, giving

$$xg_3(x) = -\frac{2\mu}{\pi(1-\nu)R}\int_0^x \frac{t^3dt}{\sqrt{x^2-t^2}} = -\frac{4\mu x^2}{\pi(1-\nu)R} \tag{9}$$

and hence

$$g_3(x) = -\frac{4\mu x}{\pi(1-\nu)R} \; . \tag{10}$$

The contact pressure can then be recovered from (1) and Table 30.1 as

$$p(r) = \frac{\partial\phi_3}{\partial z}(r,0) = -\int_r^a \frac{g_3(t)dt}{\sqrt{t^2-r^2}} = \frac{4\mu}{\pi(1-\nu)R}\int_r^a \frac{tdt}{\sqrt{t^2-r^2}} = \frac{4\mu\sqrt{a^2-r^2}}{\pi(1-\nu)R} \; . \tag{11}$$

The total load F is then

$$F = 2\pi\int_0^a rp(r)dr = \frac{8\mu}{(1-\nu)R}\int_0^a r\sqrt{a^2-r^2}dr = \frac{8\mu a^3}{3(1-\nu)R} \; . \tag{12}$$

These results agree of course with those obtained by the conventional method in §30.2.5.

30.11. The flat end of a rigid cylindrical punch of radius a is pressed into the curved surface of an elastic cylinder of radius $R \gg a$ by a centric force F sufficient to ensure contact throughout the punch face. Use Copson's formula (30.61–30.63) to determine the contact pressure distribution and hence find the minimum value of F required for full contact.

If we choose to define the y-axis to be parallel with the axis of the elastic cylinder, the function $u_0(x, y)$ will be

$$u_0(x, y) = d - \frac{x^2}{2R} = d - \frac{r^2 \cos^2 \theta}{2R} = d - \frac{r^2(1 + \cos(2\theta))}{4R} \ .$$

This is conveniently separated into an axisymmetric and a non-axisymmetric term as

$$u_0(r, \theta) = \left(d - \frac{r^2}{4R}\right) - \frac{r^2 \cos(2\theta)}{4R} \ .$$

The axisymmetric term is of the same form as (30.36) except that R is replaced by $2R$, so we can write down the pressure distribution from (30.38) as

$$p(r) = \frac{\mu}{\pi(1-\nu)R} \left(\frac{2Rd + a^2 - 2r^2}{\sqrt{a^2 - r^2}} \right) \ .$$

The non-axisymmetric term requires a potential function φ satisfying

$$\begin{aligned} \frac{\partial \varphi}{\partial z}(r, \theta, 0) &= \frac{\mu r^2 \cos(2\theta)}{4(1-\nu)R} \ ; \quad 0 \le r < a \\ \frac{\partial^2 \varphi}{\partial z^2}(r, \theta, 0) &= 0 \ ; \quad r > a \ , \end{aligned} \tag{1}$$

from (30.1, 30.2). This problem is of the form of equations (30.64, 30.60) with $m = 2$ and

$$f(r) = \frac{\mu r^2}{4(1-\nu)R} \ .$$

Equation (30.65) then gives

$$g(t) = \frac{2}{\pi t^4} \frac{d}{dt} \int_0^t \frac{s^3 f(s) ds}{\sqrt{t^2 - s^2}} = \frac{\mu}{2\pi(1-\nu)Rt^4} \frac{d}{dt} \int_0^t \frac{s^5 ds}{\sqrt{t^2 - s^2}} \ .$$

Using the result

$$\int_0^t \frac{s^5 ds}{\sqrt{t^2 - s^2}} = \frac{8t^5}{15} \ ,$$

we have

$$g(t) = \frac{\mu}{2\pi(1-\nu)Rt^4} \frac{8t^4}{3} = \frac{4\mu}{3\pi(1-\nu)R} \ .$$

Next, substituting in (30.62),

$$\begin{aligned}\frac{\partial^2\varphi}{\partial z^2}(r,\theta,0) &= r\cos(2\theta)\frac{d}{dr}\int_r^a \frac{tg(t)dt}{\sqrt{r^2-t^2}} = \frac{4\mu r\cos(2\theta)}{3\pi(1-\nu)R}\frac{d}{dr}\int_r^a \frac{tdt}{\sqrt{r^2-t^2}} \\ &= \frac{4\mu r\cos(2\theta)}{3\pi(1-\nu)R}\frac{d}{dr}\sqrt{a^2-r^2} = -\frac{4\mu r^2\cos(2\theta)}{3\pi(1-\nu)R\sqrt{a^2-r^2}} \ . \qquad (2)\end{aligned}$$

The contact pressure under the punch is therefore

$$\begin{aligned}p(r,\theta) &= \frac{\mu}{\pi(1-\nu)R}\left(\frac{2Rd+a^2-2r^2}{\sqrt{a^2-r^2}}\right) + \frac{4\mu r^2\cos(2\theta)}{3\pi(1-\nu)R\sqrt{a^2-r^2}} \\ &= \frac{\mu}{\pi(1-\nu)R\sqrt{a^2-r^2}}\left(2Rd+a^2-2r^2+\frac{4r^2\cos(2\theta)}{3}\right) \ .\end{aligned}$$

The total force is

$$F = \int_0^a\int_0^{2\pi} p(r,\theta)rd\theta dr$$

and using the results

$$\int_0^a \frac{rdr}{\sqrt{a^2-r^2}} = a \ ; \quad \int_0^a \frac{r^3dr}{\sqrt{a^2-r^2}} = \frac{2a^3}{3} \ ,$$

we have

$$F = \frac{2\mu}{(1-\nu)R}\left(2Rda+a^3-\frac{4a^3}{3}\right) = \frac{2\mu a}{(1-\nu)R}\left(2Rd-\frac{a^2}{3}\right) \ .$$

Separation will occur near $(a, \pm\pi/2)$ if

$$2Rd+a^2-2a^2-\frac{4a^2}{3} < 0 \quad \text{or} \quad d < \frac{7a^2}{6R} \ .$$

and hence

$$F < \frac{4\mu a^3}{(1-\nu)R} \ .$$

30.12. Solve Problem 11 by relating the non-axisymmetric component of $\varphi(r,\theta,z)$ to a corresponding axisymmetric harmonic function $f_0(r,z)$, as in §30.3.1. Notice that the required function varies with $\cos(2\theta)$, so you will need to apply equation (24.46) twice recursively to relate $f_2(r,z)$ to $f_0(r,z)$. This will introduce two arbitrary constants in the boundary conditions for f_0 and these must be determined from the continuity conditions $g_1(a) = g_1'(a) = 0$.

If we choose to define the y-axis to be parallel with the axis of the elastic cylinder, the function $u_0(x,y)$ will be

$$u_0(x,y) = d - \frac{x^2}{2R} = d - \frac{r^2\cos^2\theta}{2R} = d - \frac{r^2(1+\cos(2\theta))}{4R} \ .$$

This is conveniently separated into an axisymmetric and a non-axisymmetric term as

$$u_0(r,\theta) = \left(d - \frac{r^2}{4R}\right) - \frac{r^2\cos(2\theta)}{4R} \ .$$

The axisymmetric term is of the same form as (30.36) except that R is replaced by $2R$, so we can write down the pressure distribution from (30.38) as

$$p(r) = \frac{\mu}{\pi(1-\nu)R}\left(\frac{2Rd + a^2 - 2r^2}{\sqrt{a^2-r^2}}\right) \ .$$

The non-axisymmetric term requires a potential function φ satisfying

$$\begin{aligned} \frac{\partial\varphi}{\partial z}(r,\theta,0) &= \frac{\mu r^2\cos(2\theta)}{4(1-\nu)R} \ ; \quad 0 \le r < a \\ \frac{\partial^2\varphi}{\partial z^2}(r,\theta,0) &= 0 \ ; \quad r > a \ , \end{aligned} \tag{1}$$

from (30.1, 30.2). Writing

$$\varphi(r,\theta,z) = f_2(r,z)\cos(2\theta) \ ,$$

we conclude that $f_2(r,z)$ must satisfy the boundary conditions

$$\begin{aligned} \frac{\partial f_2}{\partial z}(r,0) &= \frac{\mu r^2}{4(1-\nu)R} \ ; \quad 0 \le r < a \\ \frac{\partial^2 f_2}{\partial z^2}(r,0) &= 0 \ ; \quad r > a \ . \end{aligned} \tag{2}$$

from (30.1, 30.2).

From the recurrence relation (24.46) with $m = 0, 1$ we have

$$f_2 = r\frac{\partial}{\partial r}\left(\frac{f_1}{r}\right) \ ; \quad f_1 = \frac{\partial f_0}{\partial r} \ .$$

implying

$$f_1(r) = r\int \frac{f_2(r)dr}{r} \ ; \quad f_0(r) = \int f_1(r)dr \ .$$

Applying these relations to the boundary conditions, we find

$$\frac{\partial f_1}{\partial z}(r,0) = \frac{\mu r}{4(1-\nu)R}\int r dr = \frac{\mu r^3}{8(1-\nu)R} + C_1 r$$

$$\frac{\partial f_0}{\partial z}(r,0) = \frac{\mu r^4}{32(1-\nu)R} + \frac{C_1 r^2}{2} + C_2 \tag{3}$$

in $0 \le r < a$. As in the example problem, we can integrate the external homogeneous boundary condition to

$$\frac{\partial^2 f_0}{\partial z^2}(r,0) = 0 \tag{4}$$

without loss of generality.

The boundary value problem for the axisymmetric function f_0 can be solved using the function ϕ_1 from Table 30.1. Condition (4) is then satisfied identically and (3) requires that

$$\int_0^r \frac{g_1(t)dt}{\sqrt{r^2-t^2}} = \frac{\mu r^4}{32(1-\nu)R} + \frac{C_1 r^2}{2} + C_2 \ .$$

The solution of this Abel equation is given by Table 30.2 as

$$\begin{aligned} g_1(x) &= \frac{2}{\pi}\frac{d}{dx}\int_0^x \left(\frac{\mu t^5}{32(1-\nu)R} + \frac{C_1 t^3}{2} + C_2 t\right)\frac{dt}{\sqrt{x^2-t^2}} \\ &= \frac{2}{\pi}\frac{d}{dx}\left(\frac{\mu x^5}{60(1-\nu)R} + \frac{C_1 x^3}{3} + C_2 x\right) \\ &= \frac{2}{\pi}\left(\frac{\mu x^4}{12(1-\nu)R} + C_1 x^2 + C_2\right) \ . \end{aligned}$$

We then have

$$g_1'(x) = \frac{2}{\pi}\left(\frac{\mu x^3}{3(1-\nu)R} + 2C_1 x\right)$$

and the two continuity conditions $g_1(a) = 0, g_1'(a) = 0$ give

$$C_1 = -\frac{\mu a^2}{6(1-\nu)R} \ ; \quad C_2 = \frac{\mu a^4}{6(1-\nu)R} - \frac{\mu a^4}{12(1-\nu)R} = \frac{\mu a^4}{12(1-\nu)R} \ .$$

Substituting these values into $g_1(x)$, we have

$$g_1(x) = \frac{\mu(x^2 - a^2)^2}{6\pi(1-\nu)R} \ .$$

Using this value in

$$\frac{\partial^2 f_0}{\partial z^2}(r,0) = \frac{1}{r}\frac{d}{dr}\int_r^a \frac{t_1 g(t)dt}{\sqrt{t^2 - r^2}} \ ; \quad r < a$$

from Table 30.1, we have

$$\frac{\partial^2 f_0}{\partial z^2}(r,0) = \frac{\mu}{6\pi(1-\nu)Rr}\frac{d}{dr}\int_r^a \frac{(t^5 - 2a^2t^3 + a^4 t)dt}{\sqrt{t^2 - r^2}} \ ; \quad r < a$$

Using the integrals

$$\int_r^a \frac{tdt}{\sqrt{t^2 - r^2}} = \sqrt{a^2 - r^2} \ ; \quad \int_r^a \frac{t^3 dt}{\sqrt{t^2 - r^2}} = a^2\sqrt{a^2 - r^2} - \frac{2(a^2 - r^2)^{3/2}}{3}$$

and

$$\int_r^a \frac{t^5 dt}{\sqrt{t^2 - r^2}} = a^4\sqrt{a^2 - r^2} - \frac{4a^2(a^2 - r^2)^{3/2}}{3} + \frac{8(a^2 - r^2)^{5/2}}{15} \ ,$$

we have

$$\begin{aligned}
\int_r^a \frac{(t^5 - 2a^2t^3 + a^4 t)dt}{\sqrt{t^2 - r^2}} &= a^4\sqrt{a^2 - r^2} - \frac{4a^2(a^2 - r^2)^{3/2}}{3} + \frac{8(a^2 - r^2)^{5/2}}{15} \\
&\quad -2a^4\sqrt{a^2 - r^2} + \frac{4a^2(a^2 - r^2)^{3/2}}{3} + a^4\sqrt{a^2 - r^2} \\
&= \frac{8(a^2 - r^2)^{5/2}}{15}
\end{aligned}$$

and

$$\frac{\partial^2 f_0}{\partial z^2}(r,0) = \frac{\mu}{6\pi(1-\nu)Rr}\frac{d}{dr}\frac{8(a^2 - r^2)^{5/2}}{15} = -\frac{4\mu(a^2 - r^2)^{3/2}}{9\pi(1-\nu)R} \ ; \quad r < a$$

Using the recurrence relation

$$\frac{\partial^2 f_1}{\partial z^2}(r,0) = \frac{d}{dr}\frac{\partial^2 f_0}{\partial z^2}(r,0) = \frac{4\mu r(a^2 - r^2)^{1/2}}{3\pi(1-\nu)R} \ ; \quad r < a$$

and

$$\frac{\partial^2 f_2}{\partial z^2}(r,0) = r\frac{d}{dr}\frac{1}{r}\frac{\partial^2 f_1}{\partial z^2}(r,0) = -\frac{4\mu r^2}{3\pi(1-\nu)R\sqrt{a^2 - r^2}}$$

Thus, the non-axisymmetric term contributes

$$p(r,\theta) = -\frac{\partial^2 \varphi}{\partial z^2}(r,\theta,0) = \frac{4\mu r^2 \cos(2\theta)}{3\pi(1-\nu)R\sqrt{a^2-r^2}}$$

and the complete pressure distribution is

$$p(r,\theta) = \frac{\mu}{\pi(1-\nu)R\sqrt{a^2-r^2}}\left(2Rd + a^2 - 2r^2 + \frac{4r^2\cos(2\theta)}{3}\right) .$$

The total force is

$$F = \int_0^a \int_0^{2\pi} p(r,\theta) r d\theta dr$$

and using the results

$$\int_0^a \frac{r dr}{\sqrt{a^2-r^2}} = a \; ; \quad \int_0^a \frac{r^3 dr}{\sqrt{a^2-r^2}} = \frac{2a^3}{3} ,$$

we have

$$F = \frac{2\mu}{(1-\nu)R}\left(2Rda + a^3 - \frac{4a^3}{3}\right) = \frac{2\mu a}{(1-\nu)R}\left(2Rd - \frac{a^2}{3}\right) .$$

Separation will occur near $(a, \pm\pi/2)$ if

$$2Rd + a^2 - 2a^2 - \frac{4a^2}{3} < 0 \quad \text{or} \quad d < \frac{7a^2}{6R} .$$

and hence

$$F < \frac{4\mu a^3}{(1-\nu)R} .$$

CHAPTER 31

31.1. An infinite homogeneous body contains an axisymmetric *external* **crack — i.e. the crack extends over the region $r > a,\ z = 0$. Alternatively, the external crack can be considered as two half-spaces bonded together over the region $0 \le r < a,\ z = 0$.**

The body is loaded in tension, such that the total tensile force transmitted is F. Find an expression for the stress field and in particular determine the mode I stress intensity factor K_I.

The problem is symmetrical about the crack plane $z = 0$ and hence we can conclude that (i) there will be no shear stress on this plane and (ii) the displacement component u_z will be zero in the bonded region. Solution F is an appropriate potential function representation, but if we were to set up the boundary-value problem from these conditions, we would find that the solution was a trivial state of zero stress.

In fact, what happens is that the force F causes the two half-spaces to move apart by some as yet undetermined distance d and the way to formulate the problem is to impose this as a rigid-body displacement and then determine a corrective solution that will reimpose continuity in displacement at the bonded region. The corrective solution for the half-space $z > 0$ is then defined by the boundary conditions:-

$$u_z = -\frac{d}{2} \quad ; \quad 0 \le r < a \; ; \tag{1}$$

$$\sigma_{zz} = 0 \quad ; \quad r > a \tag{2}$$

and $\sigma_{zx} = \sigma_{zy} = 0$ for all r.

This problem is mathematically[1] the same as the indentation of the half-space by a frictionless flat punch (§30.2.1 and Figure 30.1) with

$$u_0 = -\frac{d}{2} \; . \tag{3}$$

Equation (30.17) then gives the *tensile* stress in the bonded region as

$$\sigma_{zz}(r, 0) = \frac{\mu d}{\pi(1-\nu)\sqrt{a^2 - r^2}} \tag{4}$$

and the total force, from (30.19) is

$$F = \frac{2\mu d a}{(1-\nu)} \tag{5}$$

[1] To see this physically, simply reverse the tractions (and hence the displacements) for the problem in Figure 30.1 and join the central circle on the surface to a similar half-space loaded as a mirror image. All the conditions of the external crack problem are then met.

(remember that both the force and the displacement have opposite direction in this application, so the negative signs cancel).

Eliminating d between these equations, we have

$$\sigma_{zz}(r,0) = \frac{F}{2\pi a\sqrt{a^2 - r^2}} \ . \tag{6}$$

Finally, using the definition (31.18) for stress intensity factor, we have

$$K_I = \lim_{r \to a^-} \sigma_{zz}\sqrt{2\pi(a-r)} = \frac{F}{2\sqrt{\pi a^3}} \ . \tag{7}$$

31.2. An infinite homogeneous body containing a penny-shaped crack is subjected to torsional loading $\sigma_{z\theta}=Cr,\ R\to\infty$, where C is a constant[2]. Use Solution E (Table 21.1) to formulate the problem and solve the resulting boundary-value problem using the methods of §30.2.

The unperturbed solution is

$$\sigma_{\theta z} = Cr \tag{1}$$

and the corrective solution must therefore satisfy the conditions

$$\sigma_{\theta z} = -Cr \quad ; \quad \sigma_{rz} = \sigma_{zz} = 0 \tag{2}$$

on $z = 0, 0 \le r < a$ and, by symmetry,

$$\sigma_{zz} = 0 \quad ; \quad u_\theta = u_r = 0 \tag{3}$$

on $z = 0, r > a$.

All of these conditions are satisfied if we use Solution E with a function satisfying the boundary value problem

$$\frac{\partial^2 \Psi}{\partial r \partial z} = Cr \quad ; \quad 0 \le r < a \ , \ z = 0 \ ; \tag{4}$$

$$\frac{\partial \Psi}{\partial r} = 0 \quad ; \quad r > a \quad , \ z = 0 \ . \tag{5}$$

Referring to Table 30.1, we see that equation (5) can be satisfied identically by using ϕ_1 of equation (30.21), provided the function $g_1(t)$ satisfies the equation

$$\int_0^a g_1(t)dt = 0 \ . \tag{6}$$

The remaining boundary condition (4), then requires that

$$\frac{d}{dr}\int_0^r \frac{g_1(t)dt}{\sqrt{r^2 - t^2}} = 2Cr \ , \tag{7}$$

from Table 30.1, and hence

$$\int_0^r \frac{g_1(t)dt}{\sqrt{r^2 - t^2}} = Cr^2 + D \ , \tag{8}$$

where D is an arbitrary constant.

Equation (8) is an Abel integral equation for $g_1(t)$ and can be solved using Table 30.2 as

$$g_1(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^x \frac{t(Ct^2 + D)dt}{\sqrt{x^2 - t^2}} \ . \tag{9}$$

[2]This is known as the Reissner-Sagoci problem.

Noting that

$$\int_0^x \frac{tdt}{\sqrt{x^2-t^2}} = x \quad ; \quad \int_0^x \frac{t^3dt}{\sqrt{x^2-t^2}} = \frac{2x^3}{3} \ , \tag{10}$$

we have

$$\begin{aligned} g_1(x) &= \frac{2}{\pi}\frac{d}{dx}\left(\frac{2Cx^3}{3} + Dx\right) \\ &= \frac{4Cx^2}{\pi} + \frac{2D}{\pi} \ . \end{aligned} \tag{11}$$

We can now use equation (6) to determine the unknown constant D, giving

$$\int_0^a (2Cx^2 + D)dx = \frac{2Ca^3}{3} + Da = 0 \tag{12}$$

and hence

$$D = -\frac{2Ca^2}{3} \quad ; \quad g_1(x) = \frac{4C(3x^2-a^2)}{3\pi} \ . \tag{13}$$

We now calculate the stress $\sigma_{\theta z}$ on the plane $z = 0$ in $r > a$. Using Table 30.1, we have

$$\begin{aligned} \sigma_{\theta z} &= -\frac{1}{2}\frac{\partial^2\Psi}{\partial r\partial z} = -\frac{1}{2}\frac{d}{dr}\int_0^a \frac{g_1(t)dt}{\sqrt{r^2-t^2}} \\ &= -\frac{2C}{3\pi}\frac{d}{dr}\int_0^a \frac{(3t^2-a^2)dt}{\sqrt{r^2-t^2}} \ . \end{aligned} \tag{14}$$

Using the substitution $t = r\sin\theta$, we have

$$\int_0^a \frac{dt}{\sqrt{r^2-t^2}} = \int_{\cos^{-1}\left(\frac{a}{r}\right)}^{\frac{\pi}{2}} d\theta = \sin^{-1}\left(\frac{a}{r}\right) \ . \tag{15}$$

Also,

$$\begin{aligned} \int_0^a \frac{t^2dt}{\sqrt{r^2-t^2}} &= r^2\int_{\cos^{-1}\left(\frac{a}{r}\right)}^{\frac{\pi}{2}} \cos^2\theta d\theta \\ &= r^2\int_{\cos^{-1}\left(\frac{a}{r}\right)}^{\frac{\pi}{2}} \frac{(\cos(2\theta)+1)d\theta}{2} = r^2\left[\frac{\sin(2\theta)}{4} + \frac{\theta}{2}\right]_{\cos^{-1}\left(\frac{a}{r}\right)}^{\frac{\pi}{2}} \\ &= r^2\left[\frac{\sin\theta\cos\theta}{2} + \frac{\theta}{2}\right]_{\cos^{-1}\left(\frac{a}{r}\right)}^{\frac{\pi}{2}} = \frac{r^2}{2}\left(\frac{\pi}{2} - \frac{a}{r}\sqrt{1-\frac{a^2}{r^2}} - \cos^{-1}\left(\frac{a}{r}\right)\right) \\ &= \frac{r^2}{2}\sin^{-1}\left(\frac{a}{r}\right) - \frac{a}{2}\sqrt{r^2-a^2} \ . \end{aligned} \tag{16}$$

$$\tag{17}$$

Using these results in equation (14), we obtain

$$\begin{aligned}
\sigma_{\theta z} &= -\frac{2C}{3\pi}\frac{d}{dr}\left(\frac{3r^2}{2}\sin^{-1}\left(\frac{a}{r}\right) - \frac{3a}{2}\sqrt{r^2-a^2} - a^2\sin^{-1}\left(\frac{a}{r}\right)\right) \\
&= -\frac{2C}{3\pi}\left[3r\sin^{-}1\left(\frac{a}{r}\right) - \frac{3ar}{2\sqrt{r^2-a^2}} - \left(\frac{3r^2}{2}-a^2\right)\frac{a}{r\sqrt{r^2-a^2}}\right] \\
&= -\frac{2C}{3\pi}\left[3r\sin^{-1}\left(\frac{a}{r}\right) - (3r^2-a^2)\frac{a}{r\sqrt{r^2-a^2}}\right] . \qquad (18)
\end{aligned}$$

Notice that this is the corrective solution. If we add in the unperturbed solution (1), we obtain

$$\sigma_{\theta z} = \frac{2C}{3\pi}\left[3r\cos^{-1}\left(\frac{a}{r}\right) + (3r^2-a^2)\frac{a}{r\sqrt{r^2-a^2}}\right] . \qquad (19)$$

The stress intensity factor K_{III} is proportional to the singular term in this expression as $r \to a^+$ — i.e.

$$K_{III} = \lim_{r\to a^+}\sigma_{\theta z}\sqrt{2\pi(r-a)} = \lim_{r\to a^+}\frac{2C}{3\pi}\left[3r\cos^{-1}\left(\frac{a}{r}\right) + (3r^2-a^2)\frac{a}{r\sqrt{r^2-a^2}}\right]\sqrt{2\pi(r-a)} . \qquad (20)$$

Noting that

$$\lim_{r\to a^+}\frac{\sqrt{r-a}}{\sqrt{r^2-a^2}} = \lim_{r\to a^+}\frac{1}{\sqrt{r+a}} = \frac{1}{\sqrt{2a}} ,$$

we have

$$K_{III} = \frac{2C\sqrt{2\pi}}{3\pi}(3a^2-a^2)\frac{1}{\sqrt{2a}} = \frac{4Ca^{\frac{3}{2}}}{3\sqrt{\pi}} . \qquad (21)$$

31.3. The axisymmetric external crack of radius a (see Problem 31.1) is unloaded, but is subjected to the steady-state thermal conditions

$$\begin{aligned} T(r,z) &\rightarrow T_1 \ ; \ z \rightarrow \infty \\ &\rightarrow T_2 \ ; \ z \rightarrow -\infty \\ q_z(r,0) &= 0 \ ; \ r > a \ . \end{aligned}$$

In other words the extremities of the body are maintained at different temperatures T_1, T_2 causing heat to flow through the ligament $z=0, 0 \leq r < a$ and the crack faces are insulated.

Find the mode II stress intensity factor K_{II}.

The geometry is symmetric about the crack plane and the problem can be made antisymmetric about this plane by subtracting the uniform solution

$$T(r,z) = \frac{T_1 + T_2}{2} \ . \tag{1}$$

The corrective solution must then satisfy the antisymmetric conditions

$$T(r,z) \rightarrow \frac{T_1 - T_2}{2} \ ; \ z \rightarrow \infty \tag{2}$$

$$\rightarrow -\frac{T_1 - T_2}{2} \ ; \ z \rightarrow -\infty \tag{3}$$

$$q_z(r,0) = 0 \ ; \ r > a \tag{4}$$

and we conclude that the corrective temperature in the ligament $z=0, 0 \leq r < a$ must be zero. We can therefore write the corrective solution as a problem for the half space $z > 0$ with boundary conditions

$$T(r,z) \rightarrow \frac{T_1 - T_2}{2} \ ; \ z \rightarrow \infty \tag{5}$$

$$T(r,0) = 0 \ ; \ 0 \leq r < a \tag{6}$$

$$\frac{\partial T}{\partial z}(r,0) = 0 \ ; \ r > a \ . \tag{7}$$

Finally, we subtract a further uniform temperature

$$T(r,z) = \frac{T_1 - T_2}{2} \tag{8}$$

to define a second corrective problem for the half space through the conditions

$$T(r,z) \rightarrow 0 \ ; \ z \rightarrow \infty \tag{9}$$

$$T(r,0) = -\frac{T_1 - T_2}{2} \ ; \ 0 \leq r < a \tag{10}$$

$$\frac{\partial T}{\partial z}(r,0) = 0 \ ; \ r > a \ . \tag{11}$$

This last step is necessary to ensure that the potential solution tends to zero at large z.

Using Solution P, we have

$$\frac{\partial^2\psi}{\partial z^2}(r,0) = \frac{\mu\alpha(1+\nu)(T_1-T_2)}{2(1-\nu)} \;\; ; \;\; 0 \le r < a \tag{12}$$

$$\frac{\partial^3\psi}{\partial z^3}(r,0) = 0 \;\; ; \;\; r > a \tag{13}$$

and (13) can be satisfied identically along with appropriate continuity conditions by writing

$$\frac{\partial\psi}{\partial z} = \Re \int_0^a g_1(t)F(r,z,t)dt \; . \tag{14}$$

The other boundary condition (12) then requires that

$$\int_0^r \frac{g_1(t)dt}{\sqrt{r^2-t^2}} = \frac{\mu\alpha(1+\nu)(T_1-T_2)}{2(1-\nu)} \; , \tag{15}$$

from Table 30.1.

Using Table 30.2 to solve this Abel equation, we have

$$g_1(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^x \frac{\mu\alpha(1+\nu)(T_1-T_2)xdx}{2(1-\nu)\sqrt{x^2-t^2}} = \frac{\mu\alpha(1+\nu)(T_1-T_2)}{\pi(1-\nu)} \; . \tag{16}$$

This and Solution P completely defines the thermal problem, but we need to superpose an isothermal solution to satisfy the mechanical boundary conditions

$$\sigma_{zz} = 0 \; ; \;\; u_r = 0 \; ; \quad 0 \le r < a, z = 0 \tag{17}$$

$$\sigma_{zz} = \sigma_{zr} = 0 \; ; \quad r > a, z = 0 \; . \tag{18}$$

Solution P already satisfies the global condition $\sigma_{zz}(r,0) = 0$, and this can be preserved by using Solution G for the isothermal solution. The mixed conditions then require

$$\frac{\partial\psi}{\partial r} + \frac{\partial\chi}{\partial r} = 0 \; ; \quad 0 \le r < a, z = 0 \tag{19}$$

$$\frac{\partial^2\chi}{\partial r\partial z} = 0 \; ; \quad r > a, z = 0 \; . \tag{20}$$

Notice that these conditions are similar to those in (31.36, 31.37) except that they apply in complementary regions of the surface.

We can satisfy (20) by writing

$$\chi = \Im \int_0^a g_3(t)F(r,z,t)dt \; , \tag{21}$$

in which case (19) requires that

$$\frac{1}{r}\int_0^r \frac{tg_3(t)dt}{\sqrt{r^2-t^2}} = \frac{\partial\psi}{\partial r}\ ; \quad 0 \le r < a, z = 0\ . \tag{22}$$

To evaluate the right hand side of this equation, note that on the plane $z = 0$,

$$\frac{1}{r}\frac{d}{dr}r\frac{d\psi}{dr} = -\frac{\partial^2\psi}{\partial z^2} = -\frac{\mu\alpha(1+\nu)(T_1-T_2)}{2(1-\nu)}\ , \tag{23}$$

from (12). Integrating this expression, we have

$$r\frac{d\psi}{dr} = -\frac{\mu\alpha(1+\nu)(T_1-T_2)r^2}{4(1-\nu)} + C \tag{24}$$

and the constant C must be zero to preserve continuity of $d\psi/dr$ at the origin.

Using this result in (22), we have

$$\int_0^r \frac{tg_3(t)dt}{\sqrt{r^2-t^2}} = -\frac{\mu\alpha(1+\nu)(T_1-T_2)r^2}{4(1-\nu)}\ ; \quad 0 \le r < a\ , \tag{25}$$

and the solution of this Abel equation (using Table 30.2) is

$$g_3(x) = -\frac{\mu\alpha(1+\nu)(T_1-T_2)x}{\pi(1-\nu)}\ . \tag{26}$$

To determine the stress intensity factor, we must first evaluate the shear traction in the ligament which is

$$\sigma_{zr} = \frac{\partial^2\chi}{\partial z\partial r}\ . \tag{27}$$

Using Table 30.1, we have

$$\frac{\partial\chi}{\partial z} = -\int_r^a \frac{g_3(t)dt}{\sqrt{t^2-r^2}} = \frac{\mu\alpha(1+\nu)(T_1-T_2)}{\pi(1-\nu)}\int_r^a \frac{tdt}{\sqrt{t^2-r^2}} \tag{28}$$

and hence

$$\begin{aligned}\sigma_{zr} &= \frac{\mu\alpha(1+\nu)(T_1-T_2)}{\pi(1-\nu)}\frac{d}{dr}\int_r^a \frac{tdt}{\sqrt{t^2-r^2}} \\ &= -\frac{\mu\alpha(1+\nu)(T_1-T_2)r}{\pi(1-\nu)\sqrt{a^2-r^2}}\ .\end{aligned} \tag{29}$$

The stress intensity factor is

$$K_{II} = \lim_{r\to a}\sqrt{2\pi(a-r)}\sigma_{zr} = -\frac{\mu\alpha(1+\nu)(T_1-T_2)}{(1-\nu)}\sqrt{\frac{a}{\pi}}\ . \tag{30}$$

31.4. The antisymmetry of the thermoelastic penny-shaped crack problem of §31.2 implies that there is no crack-opening displacement and hence the assumption of insulation (31.21) is arguably rather unrealistic. A more realistic assumption might be that the heat flux across the crack faces is proportional to the local temperature difference — i.e.

$$q_z(r,0) = h(T(r,0^-) - T(r,0^+)) \ ; \ \ 0 \le r < a \ .$$

Use the methods of Chapter 30 to formulate the heat conduction problem. In particular, express the temperature in terms of one of the forms (30.21) and find the Abel integral equation which must be satisfied by the function $g(t)$. Do not attempt to solve this equation.

For this problem, the boundary conditions (31.21, 31.22) are modified to

$$q_z = -K\frac{\partial T}{\partial z} = h(T(r,0^-) - T(r,0^+)) \ ; \ \ 0 \le r < a \tag{1}$$

$$\to \ Q \ ; \quad R \to \infty \ . \tag{2}$$

The unperturbed solution corresponds to $q_z = Q$ everywhere and satisfies continity of temperature at the crack, so the perturbation must satisfy the conditions

$$q_z - h(T(r,0^-) - T(r,0^+)) = -Q \ ; \ \ 0 \le r < a \tag{3}$$

$$q_z \to 0 \ ; \quad R \to \infty \ , \tag{4}$$

or

$$-K\frac{\partial T}{\partial z} - h(T(r,0^-) - T(r,0^+)) = -Q \ ; \ \ 0 \le r < a \tag{5}$$

$$-K\frac{\partial T}{\partial z} \to 0 \ ; \quad R \to \infty \ . \tag{6}$$

As in the simpler problem of §31.2, the temperature field is antisymmetric about $z = 0$ and hence we can define a problem for the half-space $z > 0$ through the boundary conditions

$$-K\frac{\partial T}{\partial z} + 2hT(r,0) = -Q \ ; \ \ 0 \le r < a \tag{7}$$

$$T(r,0) = 0 \ ; \quad r > a \ . \tag{8}$$

Notice that the factor of 2 in equation (7) arises because the temperature at $(r,0^-)$ is equal and opposite to that at $(r,0^+)$, so the total temperature difference across the crack is twice $(r,0^+)$.

In terms of Solution P, these conditions can be written

$$\frac{\partial^3\psi}{\partial z^3} - \frac{2h}{K}\frac{\partial^2\psi}{\partial z^2} = -\frac{\mu\alpha(1+\nu)Q}{(1-\nu)K} \ ; \ \ 0 \le r < a \tag{9}$$

$$\frac{\partial^2\psi}{\partial z^2} = 0 \ ; \quad r > a \ . \tag{10}$$

As in §31.2, equation (10) and appropriate continuity conditions are satisfied by writing

$$\frac{\partial \psi}{\partial z} = \Im \int_0^a g(t) F(r, z, t) dt \ , \tag{11}$$

in which case (9) leads to the integral equation

$$\frac{1}{r}\frac{d}{dr}\int_0^r \frac{tg(t)dt}{\sqrt{r^2 - t^2}} + \frac{2h}{K}\int_r^a \frac{g(t)dt}{\sqrt{t^2 - r^2}} = -\frac{\mu\alpha(1+\nu)Q}{(1-\nu)K} \ \ ; \ \ 0 \leq r < a \ . \tag{12}$$

31.5. A long rectangular beam defined by the cross-section $-c < x < c$, $-d < y < d$ transmits a bending moment M about the positive x-axis. The beam contains a small penny-shaped crack of radius a in the cross-sectional plane $z = 0$ with its centre at the point $(0, b, 0)$ — i.e. the crack surface is defined by $0 \leq \sqrt{x^2 + (y-b)^2} < a$; $z = 0$. Assuming that the crack opens completely and that $c, d, d-b$ are all large compared with a, find the variation of K_I around the crack edge and hence determine the minimum value of b for the open-crack assumption to be valid. Find also the crack-opening displacement as a function of position within the crack.

Notice that the corrective problem involves both axisymmetric and non-axisymmetric terms. For the latter, you will need to apply the methods introduced in §30.3.

The unperturbed problem is one of simple bending about the x-axis. The appropriate second moment of area is

$$I = \frac{(2d)^3(2c)}{12} = \frac{4d^3c}{3}$$

and the stress field is

$$\sigma_{zz} = \frac{My}{I} = \frac{3My}{4d^3c} \ .$$

Moving the origin to the point $(0, b, 0)$ is equivalent to replacing y by $y + b$ giving

$$\sigma_{zz} = \frac{3M(y+b)}{4d^3c} = \frac{3Mb}{4d^3c} + \frac{3Mr\sin\theta}{4d^3c} \ .$$

The corrective solution must cancel these tractions inside the crack $0 \leq r < a$. The first (axisymmetric) term is identical to the problem treated in §31.1 with

$$S = \frac{3Mb}{4d^3c} \ ,$$

so we conclude that it contributes a stress intensity factor

$$K_I = 2S\sqrt{\frac{a}{\pi}} = \frac{3Mb}{2d^3c}\sqrt{\frac{a}{\pi}} \ ,$$

from equation (31.15) and a crack opening displacement

$$\delta(r) = u_z(r, 0^+) - u_z(r, 0^-) = \frac{3Mb(1-\nu)\sqrt{a^2 - r^2}}{\pi\mu d^3c} \ , \tag{1}$$

from (31.20).

For the non-axisymmetric term, we require a function $\varphi(r,\theta,z)$ satisfying the boundary conditions

$$\begin{aligned}\frac{\partial^2\varphi}{\partial z^2}(r,\theta,0) &= \frac{3Mr\sin\theta}{4d^3c}\ ; \quad 0\le r<a\\ \frac{\partial\varphi}{\partial z}(r,\theta,0) &= 0\ ; \quad r>a\end{aligned}$$

Writing

$$\varphi(r,\theta,z)=f_1(r,z)\sin\theta\ ,$$

this implies that

$$\begin{aligned}\frac{\partial^2 f_1}{\partial z^2}(r,\theta,0) &= \frac{3Mr}{4d^3c}\ ; \quad 0\le r<a\\ \frac{\partial f_1}{\partial z}(r,\theta,0) &= 0\ ; \quad r>a\end{aligned}$$

From the recurrence relation (24.46) with $m=0$ we have

$$f_1=\frac{\partial f_0}{\partial r}\ .$$

implying

$$f_0(r)=\int f_1(r)dr\ .$$

Applying this relation to the boundary conditions, we find

$$\frac{\partial^2 f_0}{\partial z^2}(r,0)=\frac{3Mr^2}{8d^3c}+C_1 \tag{2}$$

in $0\le r<a$. As in the example problem in §30.3.1, we can integrate the external homogeneous boundary condition to

$$\frac{\partial f_0}{\partial z}(r,0)=0 \tag{3}$$

without loss of generality.

The boundary value problem for the axisymmetric function f_0 can be solved using the function ϕ_3 from Table 30.1. Condition (3) is then satisfied identically and (2) requires that

$$\frac{1}{r}\frac{d}{dr}\int_0^r\frac{tg_3(t)dt}{\sqrt{r^2-t^2}}=\frac{3Mr^2}{8d^3c}+C_1\ ,$$

or

$$\int_0^r\frac{tg_3(t)dt}{\sqrt{r^2-t^2}}=\frac{3Mr^4}{32d^3c}+\frac{C_1r^2}{2}+C_2\ .$$

The solution of this Abel equation is given by Table 30.2 as

$$xg_3(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^x \left(\frac{3Mt^5}{32d^3c} + \frac{C_1t^3}{2} + C_2t\right)\frac{dt}{\sqrt{x^2-t^2}} \ .$$

Evaluating the integrals, we have

$$xg_3(x) = \frac{2}{\pi}\frac{d}{dx}\left(\frac{Mx^5}{20d^3c} + \frac{C_1x^3}{3} + C_2x\right) = \frac{2}{\pi}\left(\frac{Mx^4}{4d^3c} + C_1x^2 + C_2\right)$$

$$g_3(x) = \frac{2}{\pi}\left(\frac{Mx^3}{4d^3c} + C_1x + \frac{C_2}{x}\right) \ .$$

As in §31.1, we must set $C_2 = 0$ to maintain continuity at the origin. Also, since the application of the recurrence relation involves differentiation with respect to r, we also enforce $g_3(a) = 0$, giving

$$C_1 = -\frac{Ma^2}{4d^3c}$$

and

$$g_3(x) = \frac{Mx(x^2-a^2)}{2\pi d^3c} \ . \tag{4}$$

Table 30.1 then gives then have

$$\frac{\partial^2 f_0}{\partial z^2}(r,0) = \frac{1}{r}\frac{d}{dr}\int_0^a \frac{tg_3(t)dt}{\sqrt{r^2-t^2}} = \frac{M}{2\pi d^3cr}\frac{d}{dr}\int_0^a \frac{(t^4-a^2t^2)dt}{\sqrt{r^2-t^2}} \ ; \quad r > a \ .$$

The integral

$$\int_0^a \frac{(t^4-a^2t^2)dt}{\sqrt{r^2-t^2}} = \frac{3r^4}{8}\arcsin\left(\frac{a}{r}\right) - \frac{3ar^2\sqrt{r^2-a^2}}{8} + \frac{a^3\sqrt{r^2-a^2}}{4} - \frac{a^2r^2}{2}\arcsin\left(\frac{a}{r}\right)$$

so

$$\frac{d}{dr}\int_0^a \frac{(t^4-a^2t^2)dt}{\sqrt{r^2-t^2}} = \frac{(3r^3-2a^2r)}{2}\arcsin\left(\frac{a}{r}\right) - \frac{3ar\sqrt{r^2-a^2}}{2}$$

and

$$\frac{\partial^2 f_0}{\partial z^2}(r,0) = \frac{M}{4\pi d^3c}\left((3r^2-2a^2)\arcsin\left(\frac{a}{r}\right) - 3a\sqrt{r^2-a^2}\right) \ .$$

The recurrence relation then gives

$$\begin{aligned}
\frac{\partial^2 f_1}{\partial z^2}(r,0) &= \frac{\partial f_0}{\partial r} \\
&= \frac{M}{4\pi d^3c}\left(6r\arcsin\left(\frac{a}{r}\right) - \frac{(3r^2-2a^2)a}{r\sqrt{r^2-a^2}} - \frac{3ar}{\sqrt{r^2-a^2}}\right) \\
&= \frac{M}{4\pi d^3c}\left(6r\arcsin\left(\frac{a}{r}\right) - \frac{(6r^2-2a^2)a}{r\sqrt{r^2-a^2}}\right)
\end{aligned}$$

and

$$\sigma_{zz} = -\frac{\partial^2 \varphi}{\partial z^2}(r,0) = -\frac{M \sin\theta}{2\pi d^3 c}\left(3r \arcsin\left(\frac{a}{r}\right) - \frac{(3r^2 - a^2)a}{r\sqrt{r^2 - a^2}}\right) .$$

The corresponding stress-intensity factor is

$$K_I = \lim_{r \to a^+} \sigma_{zz}\sqrt{2\pi(r-a)} = \frac{Ma \sin\theta}{d^3 c}\sqrt{\frac{a}{\pi}}$$

and adding in the axisymmetric term we have

$$K_I = \frac{3Mb}{2d^3 c}\sqrt{\frac{a}{\pi}} + \frac{Ma \sin\theta}{d^3 c}\sqrt{\frac{a}{\pi}} .$$

The minimum occurs at $\theta = -\pi/2$ and is

$$K_I = \frac{3Mb}{2d^3 c}\sqrt{\frac{a}{\pi}} - \frac{Ma}{d^3 c}\sqrt{\frac{a}{\pi}}$$

and for this to be positive, we require

$$\frac{3b}{2} > a \quad \text{or} \quad b > \frac{2a}{3} .$$

For the crack opening displacement, we need

$$\frac{\partial f_0}{\partial z}(r,0) = -\int_r^a \frac{g_3(t)dt}{\sqrt{t^2 - r^2}} ; \quad r < a$$

and using (4),

$$\frac{\partial f_0}{\partial z}(r,0) = -\frac{M}{2\pi d^3 c}\int_r^a \frac{(t^3 - a^2 t)dt}{\sqrt{t^2 - r^2}} = -\frac{M}{2\pi d^3 c}\left(a^2\sqrt{a^2 - r^2} - \frac{2(a^2 - r^2)^{3/2}}{3} - a^2\sqrt{a^2 - r^2}\right)$$

In other words,

$$\frac{\partial f_0}{\partial z}(r,0) = \frac{M(a^2 - r^2)^{3/2}}{3\pi d^3 c} ,$$

$$\frac{\partial f_1}{\partial z}(r,0) = \frac{Mr\sqrt{a^2 - r^2}}{\pi d^3 c}$$

and

$$\frac{\partial \varphi}{\partial z}(r,\theta,0) = \frac{Mr\sqrt{a^2 - r^2}\sin\theta}{\pi d^3 c}$$

The crack-opening displacement is then

$$\delta(r,\theta) = \frac{2(1-\nu)}{\mu}\frac{\partial \varphi}{\partial z}(r,\theta,0) = \frac{2M(1-\nu)r\sqrt{a^2 - r^2}\sin\theta}{\pi\mu d^3 c} .$$

Adding in the axisymmetric term from equation (1), we have

$$\delta(r,\theta) = \frac{3Mb(1-\nu)\sqrt{a^2-r^2}}{\pi\mu d^3 c} + \frac{2M(1-\nu)r\sqrt{a^2-r^2}\sin\theta}{\pi\mu d^3 c} = \frac{3Mb(1-\nu)\sqrt{a^2-r^2}}{\pi\mu d^3 c}\left(1+\frac{2r\sin\theta}{3b}\right) .$$

For this to be positive for all r, θ, we require

$$\frac{2a}{3b} < 1 ,$$

as before.

CHAPTER 33

33.1. For an isotropic material, find a way to express the strain energy density U_0 of equation (33.4) in terms of the stress invariants of equations (1.28–1.30).

From equation (33.4), we have

$$U_0 = \frac{\left(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2\right)}{2E} - \frac{\nu(\sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} + \sigma_{xx}\sigma_{yy})}{E} + \frac{\left(\sigma_{yz}^2 + \sigma_{zx}^2 + \sigma_{xy}^2\right)}{2\mu} \ .$$

From (1.29), we have

$$\sigma_{yz}^2 + \sigma_{zx}^2 + \sigma_{xy}^2 = \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} + \sigma_{xx}\sigma_{yy} - I_2 \ ,$$

so

$$U_0 = \frac{\left(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2\right)}{2E} - \frac{\nu(\sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} + \sigma_{xx}\sigma_{yy})}{E} + \frac{(\sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} + \sigma_{xx}\sigma_{yy})}{2\mu} - \frac{I_2}{2\mu} \ .$$

The second and third terms can be combined, giving

$$U_0 = \frac{\left(\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2\right)}{2E} + \frac{(\sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} + \sigma_{xx}\sigma_{yy})}{E} - \frac{I_2}{2\mu} \ ,$$

since

$$-\frac{\nu}{E} + \frac{1}{2\mu} = -\frac{\nu}{E} + \frac{1+\nu}{E} = \frac{1}{E} \ .$$

Finally, we have

$$I_1^2 = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})^2 = \sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 + 2(\sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx} + \sigma_{xx}\sigma_{yy}) \ ,$$

so

$$U_0 = \frac{I_1^2}{2E} - \frac{I_2}{2\mu} = \frac{(I_1^2 - I_2(1+\nu))}{2E} \ .$$

33.2. An annular ring $a < r < b$ is loaded by uniform (axisymmetric) shear tractions $\sigma_{r\theta}$ at $r = a$ that are statically equivalent to a torque T per unit thickness. By considering the equilibrium of the annular region $a < r < s$ where $s < b$, find the shear stress $\sigma_{r\theta}$ as a function of radius and hence find the total strain energy stored in the annulus. Use this result to determine the rotation of the outer boundary relative to the inner boundary, assuming the material has a shear modulus μ.

At radius s, we have

$$\sigma_{r\theta}(2\pi s)(s) = T \quad \text{so} \quad \sigma_{r\theta} = \frac{T}{2\pi s^2} \ .$$

In this equation, $(2\pi s)$ is the area over which the stress acts and (s) is the moment arm. All the other stress components are zero by symmetry, so

$$U_0(s) = \frac{\sigma_{r\theta}^2}{2\mu} = \frac{T^2}{8\pi^2 \mu s^4} \ .$$

Integrating over the annulus, the total strain energy per unit thickness is

$$U = \int_a^b U_0(s) 2\pi s ds = \frac{T^2}{4\pi\mu} \int_a^b \frac{ds}{s^3} = -\left. \frac{T^2}{8\pi\mu s^2} \right|_a^b = \frac{T^2(b^2 - a^2)}{8\pi\mu a^2 b^2} \ .$$

If the relative rotation is through an angle β, the work done is $W = T\beta/2$. Equating this to U, we obtain

$$\beta = \frac{2U}{T} = \frac{T(b^2 - a^2)}{4\pi\mu a^2 b^2} \ .$$

33.3. The isotropic semi-infinite strip $-a < y < a, 0 < x < \infty$ is attached to a rigid support at $y = \pm a$, whilst the end $x = 0$ is subject to the antiplane displacement

$$u_z(0, y) = u_0 \left(1 - \frac{y^2}{a^2}\right) .$$

Since the resulting deformation will decay with x, assume a displacement of the form

$$u_z(x, y) = u_0 \left(1 - \frac{y^2}{a^2}\right) e^{-\lambda x}$$

and use the minimum total potential energy theorem to determine the optimal value of the decay rate λ.

For antiplane problems, the strain energy density reduces to

$$\begin{aligned} U_0 &= 2\mu(e_{yz}^2 + e_{zx}^2) = \frac{\mu}{2}\left[\left(\frac{\partial u_z}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial y}\right)^2\right] \\ &= \frac{\mu u_0^2}{2}\left[\lambda^2\left(1 - \frac{y^2}{a^2}\right)^2 + \frac{4y^2}{a^4}\right] e^{-2\lambda x} \\ &= \frac{\mu u_0^2}{2}\left[\lambda^2\left(1 - \frac{2y^2}{a^2} + \frac{y^4}{a^4}\right) + \frac{4y^2}{a^4}\right] e^{-2\lambda x} \end{aligned}$$

It follows that

$$\begin{aligned} U &= \int_0^\infty \int_{-a}^a U_0 dy dx = -\frac{\mu u_0^2}{2\lambda}\left[\lambda^2\left(a - \frac{2a}{3} + \frac{a}{5}\right) + \frac{4}{3a}\right] \\ &= -\mu u_0^2 \left[\frac{4\lambda a}{15} + \frac{2}{3\lambda a}\right] \end{aligned}$$

Thus, to minimize U, we take

$$\frac{\partial U}{\partial \lambda} = 0 \quad \text{or} \quad \frac{4a}{15} - \frac{2}{3\lambda^2 a} = 0 \quad \text{or} \quad \lambda a = \sqrt{\frac{5}{2}} .$$

33.4. The rectangular plate $-a < x < a, -b < y < b$ is subject to the tensile tractions

$$\sigma_{xx} = S\left(1 - \frac{y^2}{b^2}\right)$$

on the edges $x = \pm a$, all the other tractions being zero. Using appropriate third-order polynomials for the displacement components u_x, u_y, find an approximation for the displacement field and hence estimate the distribution of the stress component σ_{xx} on the plane $x = 0$. Comment on the way this result is affected by the aspect ratio b/a.

The problem is symmetric with respect to both axes, so the appropriate polynomial terms are

$$u_x = A_1x^3 + A_2xy^2 + A_3x \; ; \quad u_y = A_4y^3 + A_5yx^2 + A_6y$$

giving

$$e_{xx} = 3A_1x^2 + A_2y^2 + A_3 \; ; \quad e_{yy} = 3A_4y^2 + A_5x^2 + A_6$$

$$e_{xy} = A_2 + A_5 \; ; \quad e = e_{xx} + e_{yy} = 3A_1x^2 + A_2y^2 + A_3 + 3A_4y^2 + A_5x^2 + A_6 \; .$$

Then, from (33.4),

$$U_0 = \mu\left[\frac{\nu(3A_1x^2 + A_2y^2 + A_3 + 3A_4y^2 + A_5x^2 + A_6)^2}{(1-2\nu)} + (3A_1x^2 + A_2y^2 + A_3)^2 \right.$$
$$\left. + (3A_4y^2 + A_5x^2 + A_6)^2 + 2(A_2 + A_5)^2\right]$$

and we can calculate U from the equation

$$U = \int_{-a}^{a}\int_{-b}^{b} U_0 dydx$$

For V, we have (using symmetry)

$$V = -2\int_{-b}^{b} u_x(a,y)S\left(1 - \frac{y^2}{b^2}\right)dy = -\frac{8Sab}{3}\left(A_1a^2 + \frac{A_2b^2}{5} + A_3\right)$$

We then write $\Pi = U + V$ and impose the six conditions

$$\frac{\partial \Pi}{\partial A_i} \; ; \quad i = 1, 6 \; ,$$

obtaining six linear algebraic equations which we solve for A_i. The algebra is too lengthy to present here and is best performed in Mathematica or Maple, but for the special case where $\nu = 0$, we find

$$A_1 = 0 \; ; \quad A_2 = -\frac{S(2a^2 + 5b^2)}{2\mu(5a^4 + 2a^2b^2 + 5b^4)} \; ; \quad A_3 = \frac{S(10a^4 + 6a^2b^2 + 15b^4)}{6\mu(5a^4 + 2a^2b^2 + 5b^4)}$$

$$A_4 = 0 \; ; \quad A_5 = \frac{5Sb^2}{2\mu(5a^4 + 2a^2b^2 + 5b^4)} \; ; \quad A_6 = -\frac{5Sa^2b^2}{6\mu(5a^4 + 2a^2b^2 + 5b^4)} \; .$$

The stress σ_{xx} can be found by substituting back into e_{xx}, e_{yy} and using

$$\sigma_{xx} = \frac{2\mu\nu e}{(1-2\nu)} + 2\mu e_{xx} \; .$$

For the special case $\nu = 0$, we find

$$\sigma_{xx}(0,y) = \frac{S(10a^4 + (6a^2 + 15b^2)(b^2 - y^2)}{3(5a^4 + 2a^2b^2 + 5b^4)} \; .$$

This is a quadratic function of y like the applied tractions, but the distribution is more uniform than the tractions on $x = \pm a$. For example, the ratio

$$\frac{\sigma_{xx}(0,b)}{\sigma_{xx}(0,0)} = \frac{(10a^4 + 6a^2b^2 + 15b4)}{(10a^4)} = 1 + \frac{3b^2}{5a^2} + \frac{3b^4}{2a^4} \; .$$

If $b \ll a$, this tends to unity, showing that the stress across the cross-section is uniform as we should expect. At the other extreme, if $a \ll b$,

$$\sigma_{xx}(0,y) \to Sy^2 \; ,$$

again as we should expect. For general values of ν, we obtain

$$\frac{\sigma_{xx}(0,b)}{\sigma_{xx}(0,0)} = 1 + \frac{3b^2}{5a^2(1-\nu)} + \frac{3b^4}{2a^4} \; .$$

33.5. Use the Maple or Mathematica file 'uxy' or the method of §9.1 to find the displacement field associated with the Airy stress function

$$\phi = \frac{G}{24}\left((y^2-b^2)^2-(x^2-a^2)^2\right) \ .$$

Using this displacement field, find the value of the single degree of freedom G that best approximates the stress field in the rectangular bar $-a<x<a, -b<y<b$ subjected only to the normal tractions

$$\sigma_{xx} = f(y) \ ; \quad x = \pm a \ .$$

The stress field associated with the given stress function is

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = \frac{G(3y^2-b^2)}{6} \ ; \quad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = -\frac{G(3x^2-a^2)}{6} \ ; \quad \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = 0 \ .$$

In plane stress[1], we then have

$$e_{xx} = \frac{\partial u_x}{\partial x} = \frac{G}{6E}\left[(3y^2-b^2)+\nu(3x^2-a^2)\right]$$

$$e_{yy} = \frac{\partial u_y}{\partial y} = -\frac{G}{6E}\left[\nu(3y^2-b^2)+(3x^2-a^2)\right]$$

and integrating,

$$u_x = \frac{G}{6E}\left[3xy^2-b^2x+\nu(x^3-a^2x)\right]+f(y)$$

$$u_y = -\frac{G}{6E}\left[\nu(y^3-b^2y)+3x^2y-a^2y\right]+g(x) \ ,$$

where $f(y), g(x)$ are arbitrary functions of a single variable.

The shear strain condition requires that

$$\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x} = 0 \ ,$$

since the shear stress is everywhere zero. Imposing this condition, we obtain

$$\frac{G}{6E}\left[6xy\right]+f'(y)-\frac{G}{6E}\left[6xy\right]+g'(x) = 0 \ ,$$

or

$$f'(y) = -g'(x) \ .$$

[1]We shall find that the final result contains no elastic constants and hence is independent of the plane strain/plane stress assumption, as we should expect from §4.4.3.

This condition clearly leads only to rigid-body displacements which we can set to zero since the work done by any self-equilibrated traction distribution on a set of rigid-body displacements must necessarily be zero. We then have

$$u_x = \frac{G}{6E}\left[3xy^2 - b^2x + \nu(x^3 - a^2x)\right] \ ; \quad u_y = -\frac{G}{6E}\left[\nu(y^3 - b^2y) + 3x^2y - a^2y\right] \ .$$

To apply the Rayleigh-Ritz method to the specified tractions, we need to find the displacement u_x on the loaded surfaces $x = \pm a$. We have

$$u_x(\pm a, y) = \pm\frac{Ga}{6E}\left[3y^2 - b^2\right] \ .$$

The strain energy density is

$$U_0 = \frac{(\sigma_{xx}^2 + \sigma_{yy}^2 - 2\nu\sigma_{xx}\sigma_{yy})}{2E} \ ,$$

from (33.4) and integrating this over the domain $-a < x < a, -b < y < b$ (using the above expressions for the stresses and Maple for the integration), we obtain

$$U = \frac{2G^2ab(a^4 + b^4)}{45E} \ .$$

The potential energy of the applied tractions is

$$V = -2\int_{-b}^{b} u_x(a, y)f(y)dy \ ,$$

where the factor of 2 arises because of the equal contribution from the other edge $x = -a$. Evaluating the integral, we have

$$V = -\frac{Ga}{3E}\int_{-b}^{b}\left[3y^2 - b^2\right]f(y)dy$$

and hence

$$\Pi = U + V = \frac{2G^2ab(a^4 + b^4)}{45E} - \frac{Ga}{3E}\int_{-b}^{b}\left[3y^2 - b^2\right]f(y)dy \ .$$

The minimum total potential energy theorem then gives

$$\frac{\partial\Pi}{\partial G} = \frac{4Gab(a^4 + b^4)}{45E} - \frac{a}{3E}\int_{-b}^{b}\left[3y^2 - b^2\right]f(y)dy = 0$$

and hence

$$G = \frac{15}{4b(a^4 + b^4)}\int_{-b}^{b}\left[3y^2 - b^2\right]f(y)dy = 0 \ .$$

Notice that this expression agrees with the first term in equation (6.36).

33.6. A simple two-dimensional finite element mesh can be obtained by joining a series of nodes (x_i, y_i) $i = 1, N$ by straight lines, forming a set of triangular elements. If the antiplane displacement u_z takes the values u_i at the three nodes $i = 1, 2, 3$, develop an expression for a linear function of x, y that interpolates these nodal values. In other words determine the three constants A, B, C in

$$u_z(x, y) = Ax + By + C$$

such that $u_z(x_i, y_i) = u_i$ for $i = 1, 2, 3$. Hence determine the strain energy U stored in the element as a function of u_i and use the principle of stationary potential energy to determine the element stiffness matrix k relating the nodal forces F_i at the nodes to the nodal displacements through the equation

$$F_i = k_{ij} u_j \ .$$

We have

$$\begin{aligned} Ax_1 + By_1 + C &= u_1 \\ Ax_2 + By_2 + C &= u_2 \\ Ax_3 + By_3 + C &= u_3 \end{aligned}$$

with solution

$$\begin{aligned} A &= \frac{u_1(y_3 - y_2) + u_2(y_1 - y_3) + u_3(y_2 - y_1)}{y_1x_2 + y_2x_3 + y_3x_1 - x_1y_2 - x_2y_3 - x_3y_1} \\ B &= -\frac{u_1(x_3 - x_2) + u_2(x_1 - x_3) + u_3(x_2 - x_1)}{y_1x_2 + y_2x_3 + y_3x_1 - x_1y_2 - x_2y_3 - x_3y_1} \\ C &= \frac{u_1(x_3y_2 - x_2y_3) + u_2(x_1y_3 - x_3y_1) + u_3(x_2y_1 - x_1y_2)}{y_1x_2 + y_2x_3 + y_3x_1 - x_1y_2 - x_2y_3 - x_3y_1} \end{aligned}$$

The area of the triangle can be found by drawing a horizontal line through x_2, y_2 so as to meet the opposite side at the point X as shown in Figure 1, where

$$x = x_1 + \frac{(x_3 - x_1)(y_2 - y_1)}{(y_3 - y_1)} \ .$$

The length of this line is therefore

$$L = (x_2 - x_1) - \frac{(x_3 - x_1)(y_2 - y_1)}{(y_3 - y_1)}$$

and it forms the base of the two triangle into which it divides the original area. Thus, the total area of the triangle (the sum of the two separate areas) is

$$\begin{aligned} \mathcal{A} &= \frac{(y_3 - y_1)L}{2} = \frac{(x_2 - x_1)(y_3 - y_1)}{2} - \frac{(x_3 - x_1)(y_2 - y_1)}{2} \\ &= \frac{1}{2}\left[x_1y_2 + x_2y_3 + x_3y_1 - y_1x_2 - y_2x_3 - y_3x_1\right] \ . \end{aligned} \tag{1}$$

It follows that we can write the expressions for A, B, C in the form

$$\begin{aligned}
A &= -\frac{u_1(y_3-y_2)+u_2(y_1-y_3)+u_3(y_2-y_1)}{2\mathcal{A}} \\
B &= \frac{u_1(x_3-x_2)+u_2(x_1-x_3)+u_3(x_2-x_1)}{2\mathcal{A}} \\
C &= -\frac{u_1(x_3y_2-x_2y_3)+u_2(x_1y_3-x_3y_1)+u_3(x_2y_1-x_1y_2)}{2\mathcal{A}} \ .
\end{aligned}$$

The non-zero stress components are

$$\sigma_{zx} = \mu\frac{\partial u_z}{\partial x} = \mu A \ ; \quad \sigma_{zy} = \mu\frac{\partial u_z}{\partial y} = \mu B$$

and hence

$$U_0 = \frac{\mu^2(A^2+B^2)}{2\mu} = \frac{\mu(A^2+B^2)}{2} \ .$$

This is independent of x, y, so

$$U = \int_\Omega U_0 d\Omega = U_0\mathcal{A} = \frac{\mu\mathcal{A}(A^2+B^2)}{2} \ .$$

The potential energy due to the nodal forces (from (33.10)) is

$$V = -F_i u_i \quad \text{so} \quad \Pi = U - F_i u_i \ .$$

Applying the principle of stationary total potential energy, we then have (for example)

$$\frac{\partial U}{\partial u_1} - F_1 = 0$$

and hence

$$\begin{aligned}
F_1 &= \frac{\partial U}{\partial u_1} = \mu\mathcal{A}\left(A\frac{\partial A}{\partial u_1} + B\frac{\partial B}{\partial u_1}\right) \\
&= \mu\mathcal{A}\left(\frac{[u_1(y_3-y_2)+u_2(y_1-y_3)+u_3(y_2-y_1)](y_3-y_2)}{4\mathcal{A}^2}\right. \\
&\qquad \left. + \frac{[u_1(x_3-x_2)+u_2(x_1-x_3)+u_3(x_2-x_1)](x_3-x_2)}{4\mathcal{A}^2}\right)
\end{aligned}$$

from which

$$\begin{aligned}
k_{11} &= \frac{\mu}{4\mathcal{A}}\left((y_3-y_2)^2+(x_3-x_2)^2\right) \\
k_{12} &= \frac{\mu}{4\mathcal{A}}\left((y_1-y_3)(y_3-y_2)+(x_1-x_3)(x_3-x_2)\right) \\
k_{13} &= \frac{\mu}{4\mathcal{A}}\left((y_2-y_1)(y_1-y_3)+(x_2-x_1)(x_1-x_3)\right) \ .
\end{aligned}$$

The remaining coefficients can be obtained by permuting suffices.

33.7. Use a one degree-of-freedom approximation and Castigliano's second theorem to estimate the twist per unit length β of a bar whose triangular cross-section is defined by the three corners $(0,0),(a,0),(0,a)$ and which is loaded by a torque T. Also, estimate the location and magnitude of the maximum shear stress.

The equations of the three sides of the triangle are

$$x = 0 \; ; \quad y = 0 \; ; \quad y - x = a \; ,$$

so a suitable one-degree-of-freedom function can be constructed as

$$\phi = Axy(x - y + a) = A(x^2y - xy^2 + axy) \; .$$

We then have

$$\sigma_{zx} = \frac{\partial \phi}{\partial y} = A(x^2 - 2xy + ax) \; ; \quad \sigma_{zy} = -\frac{\partial \phi}{\partial x} = -A(2xy - y^2 + ay)$$

and

$$2\mu U_0 = A^2((x^2 - 2xy + ax)^2 + (2xy - y^2 + ay)^2)$$

Integrating over the trianglular area, we have

$$2\mu U = A^2 \int_0^a \int_0^{a-y} ((x^2 - 2xy + ax)^2 + (2xy - y^2 + ay)^2) dxdy = \frac{19A^2a^6}{90} \; .$$

(using Mathematica). Also,

$$\int\int_\Omega \phi d\Omega = \int_0^a \int_0^{a-y} A(x^2y - xy^2 + axy) = \frac{Aa^5}{24} \; .$$

It follows from equations (33.31, 33.32) (eliminating T) that

$$2\mu C = \frac{19A^2a^6}{90} - \frac{4\mu\beta Aa^5}{24}$$

and this will be stationary when

$$A = \frac{15\mu\beta}{38a} \; .$$

The torque is then given by

$$T = 2\int\int_\Omega \phi d\Omega = \frac{Aa^5}{12} = \frac{5\mu\beta a^4}{152} \; .$$

The maximum shear stress is likely to occur on one of the edges. On $x = 0$, we have

$$\sigma_{zy} = -\frac{\partial \phi}{\partial x} = -A(ay - y^2) \; ,$$

which is maximum when $y = a/2$, giving

$$\tau_{\max} = Aa^2 4 = \frac{15\mu\beta a}{152} \ .$$

On the inclined edge, the maximum must occur at the mid-point $(a/2, a/2)$, where the two components have equal magnitude equal to

$$A\left(\frac{a^2}{4} - \frac{a^2}{2} + \frac{a^2}{2}\right) = \frac{Aa^2}{4} \ .$$

The maximum shear stress at this point is

$$\tau_{\max} = \sqrt{\sigma_{zx}^2 + \sigma_{zy}^2} = \frac{Aa^2\sqrt{2}}{4} = \frac{15\sqrt{2}\mu\beta a}{152} \ .$$

This is the bigger of the two values and represents the approxmate value of the maximum shear stress. In terms of the torque, the maximum shear stress is

$$\tau_{\max} = \frac{15\sqrt{2}T}{5a^3} \ .$$

33.8. Using the single degree of freedom stress function

$$\phi = Ax(a^2 - x^2 - y^2) \ ,$$

estimate the torsional stiffness of the semicircular bar of radius a.

Differentiating ϕ, we have

$$\frac{\partial \phi}{\partial x} = A(a^2 - 3x^2 - y^2) \ ; \quad \frac{\partial \phi}{\partial y} = -2Axy$$

and hence

$$\begin{aligned}
\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 &= A^2(9x^4 + 10x^2y^2 + y^4 - 6a^2x^2 - 2a^2y^2 + a^4) \\
&= A^2r^4(9\cos^4\theta + 10\sin^2\theta\cos^2\theta + \sin^4\theta) \\
&\quad -A^2a^2r^2(6\cos^2\theta + 2\sin^2\theta) + A^2a^4 \ .
\end{aligned}$$

Integrating over the semi-circle, we have

$$U = \frac{1}{2\mu}\int_0^a \int_{-\pi/2}^{\pi/2} \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] r d\theta dr$$

and noting that

$$\int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta = \int_{-\pi/2}^{\pi/2} \sin^4\theta d\theta = \frac{3\pi}{8} \ ; \quad \int_{-\pi/2}^{\pi/2} \sin^2\theta\cos^2\theta d\theta = \frac{\pi}{8}$$

$$\int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta = \int_{-\pi/2}^{\pi/2} \sin^2\theta d\theta = \frac{\pi}{2} \ ,$$

we have

$$\begin{aligned}
U &= \int_0^a \frac{\pi A^2}{2\mu}\left[r^4\left(\frac{27}{8} + \frac{10}{8} + \frac{3}{8}\right) - a^2r^2(3+1) + a^4\right] r dr \\
&= \frac{\pi A^2}{2\mu}\int_0^a (5r^5 - 4a^2r^3 + a^4r) dr = \frac{\pi A^2 a^6}{2\mu}\left(\frac{5}{6} - 1 + \frac{1}{2}\right) = \frac{\pi A^2 a^6}{6\mu} \ .
\end{aligned}$$

Also

$$\int_0^a \int_{-\pi/2}^{\pi/2} \phi r d\theta dr = A\int_0^a \int_{-\pi/2}^{\pi/2} \left(a^2r^2\cos\theta - r^4\cos^3\theta - r^4\sin^2\theta\cos\theta\right) d\theta dr$$

and noting that

$$\int_{-\pi/2}^{\pi/2} \cos^3\theta d\theta = \frac{4}{3} \ ; \quad \int_{-\pi/2}^{\pi/2} \sin^2\theta\cos\theta d\theta = \frac{2}{3} \ ; \quad \int_{-\pi/2}^{\pi/2} \cos\theta d\theta = 2 \ ,$$

we have

$$\int_0^a \int_{-\pi/2}^{\pi/2} \phi r d\theta dr = A \int_0^a \left(2a^2 r^2 - \frac{4r^4}{3} - \frac{2r^4}{3} \right) dr = A \left(\frac{2a^5}{3} - \frac{4a^5}{15} - \frac{2a^5}{15} \right) = \frac{4Aa^5}{15} \ ,$$

so that the torque is

$$T = \frac{8Aa^5}{15} \ .$$

Equation (33.32) then gives

$$C = \frac{\pi A^2 a^6}{6\mu} - T\beta = \frac{\pi A^2 a^6}{6\mu} - \frac{8Aa^5\beta}{15}$$

and hence

$$\frac{dC}{dA} = \frac{\pi A a^6}{3\mu} - \frac{8a^5\beta}{15} = 0 \quad \text{giving} \quad A = \frac{8\mu\beta}{5\pi a} \ .$$

It follows that the torsional stiffness is

$$\frac{T}{\beta} = \frac{8a^5}{15} \frac{8\mu}{5\pi a} = \frac{64\mu a^4}{75\pi} \ .$$

33.9. The rectangular plate $-a < x < a, -b < y < b$ is subject to the tractions

$$\sigma_{xx} = \frac{Sy^2}{b^2}$$

on the edges $x = \pm a$, all the other tractions being zero. Find an approximate solution for the stress field using Castigliano's second theorem. An appropriate particular solution is

$$\phi_P = \frac{Sy^4}{12b^2} \ .$$

Use a single degree of freedom approximation of the form (33.41) for the homogeneous solution.

The boundaries are $x = \pm a, y = \pm b$, so the homogeneous function can be written as

$$\phi_H = A(x^2 - a^2)^2(y^2 - b^2)^2 \ ,$$

giving

$$\phi = \frac{Sy^4}{12b^2} + A(x^2 - a^2)^2(y^2 - b^2)^2$$

and

$$\sigma_{xx} = \frac{Sy^2}{b^2} + 4A(x^2 - a^2)^2(3y^2 - b^2)$$

$$\sigma_{yy} = 4A(3x^2 - a^2)(y^2 - b^2)^2$$

$$\sigma_{xy} = -16Axy(x^2 - a^2)(y^2 - b^2) \ .$$

As in §33.7.2, we can arbitrarily set $\nu = 0$, since the exact solution is independent of ν. We then have

$$U_0 = \frac{1}{4\mu}\left[\left(\frac{Sy^2}{b^2} + 4A(x^2 - a^2)^2(3y^2 - b^2)\right)^2 + \left(4A(3x^2 - a^2)(y^2 - b^2)^2\right)^2 \right.$$
$$\left. +2\left(16Axy(x^2 - a^2)(y^2 - b^2)\right)^2\right]$$

and

$$U = \int_{-a}^{a}\int_{-b}^{b} U_0 dydx = \frac{8192A^2a^5b^5(7a^4 + 4b^4 + 7b^4)}{11025\mu} + \frac{256SAa^5b^3}{225\mu} + \frac{S^2ab}{5\mu} \ .$$

This is a traction boundary-value problem, so there are no perturbations in the tractions and $C = U$. Imposing the condition

$$\frac{\partial C}{\partial A} = 0$$

and solving for A, we obtain

$$A = -\frac{49S}{64b^2(7a^4 + 4b^4 + 7b^4)} \ .$$

The complete solution is then obtained by substituting into the stress components $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ above.

33.10. A plate defined by the region $a < r < 2a,\ -\alpha < \theta < \alpha$ is loaded only by the uniform tensile tractions

$$\sigma_{rr}(a,\theta) = \frac{F}{2a\alpha}\ ; \quad \sigma_{rr}(2a,\theta) = \frac{F}{4a\alpha}$$

on the curved edges $r = a, 2a$. Find an approximate solution for the stress field using Castigliano's second theorem. Use a single degree of freedom approximation of the form (33.41) for the homogeneous solution ϕ_H.

The stress field

$$\sigma_{rr}(r,\theta) = \frac{F}{2r\alpha}$$

satisfies the boundary conditions and the equilibrium equations and corresponds to the stress function

$$\phi_P = \frac{Fr}{2\alpha}\ .$$

To check this, we calculate

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{F}{2r\alpha} \\
\sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = 0 \\
\sigma_{\theta\theta} &= \frac{\partial^2\phi}{\partial r^2} = 0\ .
\end{aligned}$$

For the homogeneous solution, we use the form (33.41), which here gives

$$\phi_H = A(r-a)^2(r-2a)^2(\theta^2-\alpha^2)^2\ ,$$

$$so\phi = \frac{Fr}{2\alpha} + A(r-a)^2(r-2a)^2(\theta^2-\alpha^2)^2\ ,$$

where A is the constant to be determined from the theorem.

The stress components are obtained as

$$\begin{aligned}
\sigma_{rr} &= \frac{F}{2r\alpha} + \frac{2A(2a-r)(a-r)}{r^2}\left(2(a-r)(2a-r)(3\theta^2-\alpha^2) - r(3a-2r)(\alpha^2-\theta^2)^2\right) \\
\sigma_{r\theta} &= \frac{(4A(4a^4-13a^2r^2+12ar^3-3r^4)\theta(\theta^2-\alpha^2))}{r^2} \\
\sigma_{\theta\theta} &= 2A(13a^2-18ar+6r^2)(\alpha^2-\theta^2)^2\ .
\end{aligned}$$

We next evaluate the strain energy density as

$$U_0 = \frac{1}{4\mu}\left[\sigma_{rr}^2 + \sigma_{\theta\theta}^2 + 2\sigma_{r\theta}^2\right]$$

and

$$U = C = \int_a^{2a} \int_{-\alpha}^{\alpha} U_0 r d\theta dr = \frac{1}{12600\alpha\mu} \Big[1575 \ln(2) F^2 - 1120(36 \ln(2) - 25) a^3 \alpha^5 AF$$
$$+128 a^6 \alpha^6 A^2 (-480\alpha^2 (88 \ln(2) - 61) + 4\alpha^4 (720 \ln(2) - 493) + 63(2480 \ln(2) - 1719))\Big]$$

Enforcing the condition

$$\frac{\partial C}{\partial A} = 0 \ ,$$

we obtain

$$A = \frac{35(36 \ln(2) - 25)F}{8a^3 \alpha (156240 \ln(2) - 108297 + (29280 - 42240 \ln(2))\alpha^2 + (2880 \ln(2) - 1972)\alpha^4)} \ .$$

After which the complete stress field is recovered.

CHAPTER 34

34.1. Figure 34.4 shows an elastic cube of side a which just fits between two parallel frictionless rigid walls. The block is now loaded by equal and opposite concentrated forces, F, at the mid-point of two opposite faces, as shown. 'Poisson's ratio' strains will cause the block to exert normal forces P on the walls. Find the magnitude of these forces.

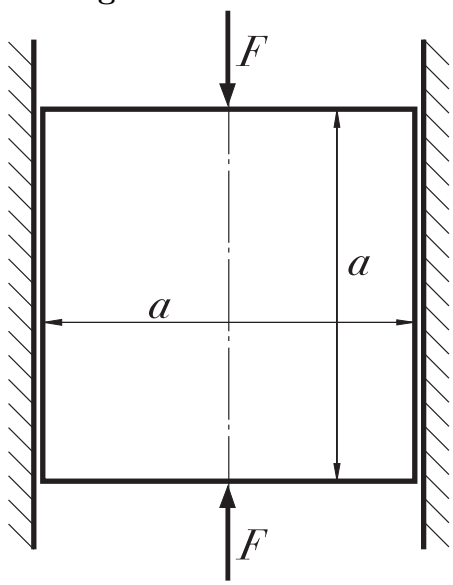


Figure 34.4: Loading of the constrained cube.

Would your result be changed if (i) the forces F were applied away from the mid-point or (ii) if their common line of action was inclined to the vertical?

The required total force is the sum of the normal compressive tractions on either face — i.e.

$$P = -\int\int_{\Gamma_1} \boldsymbol{t_1}.\boldsymbol{n} d\Gamma \ , \tag{1}$$

where Γ_1 is one of the two faces contacting the walls and $\boldsymbol{n}$ is the outward normal to Γ. Notice also that the wall is frictionless, so $\boldsymbol{t_1}$ does not contain any component that is not parallel to $\boldsymbol{n}$.

The integral form of equation (1) suggests an auxiliary solution in which a uniform pressure is exerted over the surface Γ_1. For convenience, we suppose that this pressure has unit magnitude, leading to strains $1/E, -\nu/E$ respectively in the directions between the walls and the two orthogonal directions.

If we apply the pressure against the surface S_1 and place the opposing face in contact with a frictionless rigid wall (so that there is no displacement at the other

face) we shall have normal displacements equal to $-a/E$ at Γ_1 and $\nu a/2E$ at the unloaded faces, assuming that the rigid body displacements are chosen to give a symmetrical solution.

We now apply equation (34.8) — i.e.

$$\int\int_{\Gamma} \boldsymbol{t_1}.\boldsymbol{u_2} d\Gamma = \int\int_{\Gamma} \boldsymbol{t_2}.\boldsymbol{u_1} d\Gamma \ . \tag{2}$$

The right hand side of this equation is zero, since the normal component of $\boldsymbol{u_1}$ is zero on the two faces in contact with the walls and $\boldsymbol{t_2}$ is a normal traction on these surfaces and zero everwhere else.

It follows that the left hand side of (2) must be zero. This integral has two components, one associated with the two concentrated forces and one with an integral over the contacting end faces. We obtain

$$-\int\int_{\Gamma_1} \boldsymbol{T_1}.\frac{a\boldsymbol{n}}{E} d\Gamma - 2F\frac{\nu a}{2E} = 0 \ . \tag{3}$$

It follows immediately that

$$P = -\int\int_{\Gamma_1} \boldsymbol{t_1}.\boldsymbol{n} d\Gamma = F\nu \ . \tag{4}$$

The above proof does not depend upon the line of action of the forces F passing through the mid-point and hence this result applies for vertical forces with any line of action.

If the forces were inclined to the vertical through some angle α, the work done by F against the displacements of the auxiliary solution could be obtained by summing the contributions from the vertical and horizontal components. From the vertical components, we would obtain

$$-2F\cos\alpha\frac{\nu a}{2E}$$

— i.e. the same result as in (3), except that F is replaced by its vertical component $F\cos\alpha$. In addition, the horizontal components $F\sin\alpha$ would contribute the term

$$2F\sin\alpha\frac{a\tan\alpha}{E} \ ,$$

since the relative horizontal displacement in the auxiliary solution between the two points of application is $a\tan\alpha/E$.

It follows that in this case

$$P = F(\nu\cos\alpha - \sin\alpha\tan\alpha) = F\cos\alpha(\nu - \tan^2\alpha) \ . \tag{5}$$

Notice that P would become negative if α were sufficiently large, indicating in practice that the block would fail to contact the walls. This is physically reasonable, since the horizontal component of F is compressing the block and tending to reduce its length. In practice, it is likely that some localised regions of separation would develop before this limit was reached, in which case the above analysis would be invalid.

34.2. Use the method of §§29.2, 34.3.2 to find the relations between the indentation force, F, the contact radius, a and the maximum indentation depth, d, for a rigid frictionless punch with a fourth order profile $u_0 = -Cr^4$ indenting an elastic half-space.

If the punch is pushed into the half-space by a distance δ, we have

$$u_1(r) = d - Cr^4 \ . \tag{1}$$

The auxiliary problem is that in which a punch of radius a is pushed a unit distance into the half-space and hence the contact pressure is

$$p_2(r) = \frac{2\mu}{\pi(1-\nu)\sqrt{a^2 - r^2}} \ , \tag{2}$$

from (30.17) with $u_0 = 1$.

Substituting (1,2) into (34.21), we obtain

$$F(a) = 2\pi \int_0^a \frac{2\mu(d - Cr^4) r dr}{\pi(1-\nu)\sqrt{a^2 - r^2}} = \frac{2\mu}{(1-\nu)} \left(da - \frac{8}{15} C a^5 \right) \ . \tag{3}$$

We then use the condition $dF(a)/da$ to determine the contact radius, with the result

$$d = \frac{8Ca^4}{3} \ . \tag{4}$$

It follows that

$$F(a) = \frac{64\mu C a^5}{15(1-\nu)} \ , \tag{5}$$

from (3,4), and hence

$$a = \left(\frac{15F(1-\nu)}{64\mu C} \right)^{\frac{1}{5}} \ , \tag{6}$$

after which d is given by (4).

Notice that these results agree with the general theorems proved in Problem 29.1.

34.3. The penny-shaped crack $0 \leq r < a, z = 0$ is opened by equal and opposite normal forces F applied to the two crack faces at a radius r_1. Find the volume of the opened crack.

The volume of the opened crack is

$$V = \int_0^a \int_0^{2\pi} \{u_z(r,\theta,0^+) - u_z(r,\theta,0^-)\} r d\theta dr \ . \tag{1}$$

Now the displacements $u_z(r,\theta,0^+)u_z(r,\theta,0^-)$, are in the direction of the inward normal to the crack surfaces, so the volume of the opened crack will be

$$V = \int\int_\Gamma \boldsymbol{u}_1.\boldsymbol{t}_2 d\Gamma = \int\int_\Gamma \boldsymbol{u}_2.\boldsymbol{t}_1 d\Gamma \tag{2}$$

if we use as auxiliary solution the problem of a crack opened by a unit pressure — i.e.

$$\sigma_{zz}(r,\theta,0) = -1 \ ; \quad \sigma_{zr}(r,\theta,0) = \sigma_{z\theta}(r,\theta,0) = 0 \ . \tag{3}$$

The solution of this problem is given in §31.1. In particular, we have

$$u_z(r,\theta,0^+) = \frac{2(1-\nu)\sqrt{a^2-r^2}}{\pi\mu} \tag{4}$$

from (31.19). With this choice, V of equation (2) is the work done by the two concentrated forces F moving through the displacements $(\boldsymbol{u}_2)$ of the auxiliary solution. The force on the surface $z = 0^+$ does work

$$W = \frac{2(1-\nu)F\sqrt{a^2-r_1^2}}{\pi\mu} \tag{5}$$

and an equal amount of work is done by the force on the surface $z = 0^-$. The volume of the opened crack is therefore

$$V = \frac{4(1-\nu)F\sqrt{a^2-r_1^2}}{\pi\mu} \ . \tag{6}$$

34.4. A long elastic cylinder of constant but arbitrary cross-sectional area and length L rests horizontally on a plane rigid surface. Show that gravitational loading will cause the cylinder to increase in length by an amount

$$\delta = \frac{\nu \rho g L h}{E} ,$$

where ρ, E, ν are the density, Young's modulus and Poisson's ratio respectively of the material and h is the height of the centre of gravity of the cylinder above the plane.

Suppose that the cylinder rests on the plane $y = 0$, its axis is parallel with the z-direction and the cross-section of the cylinder in the xy-plane is denoted by A. In this case, the gravitational body force is

$$p_y = -\rho g \; ; \quad p_x = p_z = 0 \quad \text{or} \quad \boldsymbol{p}_1 = -\boldsymbol{j}\rho g \; . \tag{1}$$

The average change in length of the cylinder is

$$\delta L = \frac{1}{A} \int \int_A \{u_z(x, y, L) - u_z(x, y, 0)\} dx dy \tag{2}$$

and we can make the integral in (2) look like the right hand side of (34.8) if we choose as auxiliary solution a state of unit uniaxial tension — i.e.

$$\sigma_{zz}^{(2)} = 1 \; , \tag{3}$$

all other stress components being zero. For this state of stress, we have

$$e_{xx} = e_{yy} = -\frac{\nu}{E} \; ; \quad e_{xy} = 0 \tag{4}$$

and hence the cross-section retains the same shape but gets smaller in the ratio $(1 - \nu/E)$. It follows that the vertical displacement will be

$$u_y^{(2)} = -\frac{\nu y}{E} \; . \tag{5}$$

Using this result in the left hand side of (34.8), we obtain

$$\begin{aligned} A\delta &= \int \int \int_\Omega \left(-\rho g\right)(-\frac{\nu y}{E}\right) dx dy dz \\ &= \frac{L \rho g \nu}{E} \int \int_A y dx dy \\ &= \frac{L \rho g \nu A h}{E} \end{aligned} \tag{6}$$

and hence

$$\delta = \frac{L \rho g \nu h}{E} \; . \tag{7}$$

34.5. A pressure vessel of arbitrary shape occupies a space of volume V_1 and encloses a hole of volume V_2, so that the volume of solid material is $V_M = V_1 - V_2$. If the vessel is now subjected to an external pressure p_1 and an internal pressure p_2, show that the total change in V_M is

$$\delta V_M = \frac{p_2 V_2 - p_1 V_1}{K_b} \ ,$$

where K_b is the bulk modulus of equation (1.76).

As in §34.3, we can obtain the change in volume δV_M using as auxiliary solution the state of unit hydrostatic tension

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 1 \ ; \ \ \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0 \ . \tag{1}$$

The stress-strain relations then give

$$e_{xx} = e_{yy} = e_{zz} = \frac{(1-2\nu)}{E} \ ; \ \ e_{xy} = e_{yz} = e_{zx} = 0 \tag{2}$$

and hence, in state 2, the body will simply deform into a larger, geometrically similar shape, any linear dimension L increasing to $L(1+(1-2\nu)/E)$. Suppose we choose a rigid-body reference frame such that the origin O remains fixed during this deformation. It follows that the displacement $\boldsymbol{u}_2^P$ at some point P defined by the position vector $\boldsymbol{r}_P$ will be

$$\boldsymbol{u}_2^P = \frac{(1-2\nu)}{E}\boldsymbol{r}_P \ . \tag{3}$$

Betti's theorem now gives

$$\begin{aligned} \delta V_M &= \int\int_{\Gamma_1+\Gamma_2} \boldsymbol{u}_1.\boldsymbol{n} d\Gamma = \int\int_{\Gamma_1+\Gamma_2} \boldsymbol{u}_1.\boldsymbol{t}_2 d\Gamma = \int\int_{\Gamma_1+\Gamma_2} \boldsymbol{u}_2.\boldsymbol{t}_1 d\Gamma \\ &= -\frac{(1-2\nu)}{E}\int\int_{\Gamma_1} \boldsymbol{n}.\boldsymbol{r} d\Gamma_1 - \frac{(1-2\nu)}{E}\int\int_{\Gamma_2} \boldsymbol{n}.\boldsymbol{r} d\Gamma_2 \ , \end{aligned} \tag{4}$$

where Γ_1 is the outer surface and Γ_2 is the inner surface of the body and the position vector $\boldsymbol{r}$ is measured to points on the respective boundaries. Now the volume of the pyramid constructed on the base $d\Gamma_1$ with apex at the origin is

$$\delta V = \frac{1}{3}\boldsymbol{n}.\boldsymbol{r} d\Gamma_1 \ . \tag{5}$$

It follows that

$$\int\int_{\Gamma_1} \boldsymbol{n}.\boldsymbol{r} d\Gamma_1 = 3V_1 \ ; \quad \int\int_{\Gamma_2} \boldsymbol{n}.\boldsymbol{r} d\Gamma_2 = -3V_2 \ , \tag{6}$$

where the negative sign in the second equation arises because the outward normal to the body at Γ_2 is actually the *inward* normal to the volume enclosed by Γ_2 (the hole).

Using these results in (4), we obtain

$$\delta V_M = \frac{(1-2\nu)(p_2 V_2 - p_1 V_1)}{3E} = \frac{(p_2 V_2 - p_1 V_1)}{K_b}, \tag{7}$$

using (1.76).

34.6. The long rectangular bar $0 < x < a, -b < y < b$, $a \gg b$ is believed to contain residual stresses σ_{xx} that are functions of y only, except near the ends, the other residual stress components being zero. To estimate the residual stress distribution, it is proposed to clamp the bar at $x = 0$, cut a groove as shown in Figure 34.5, using laser machining and measure the vertical displacement u (if any) of the end $x = a$ due to the resulting stress relief.

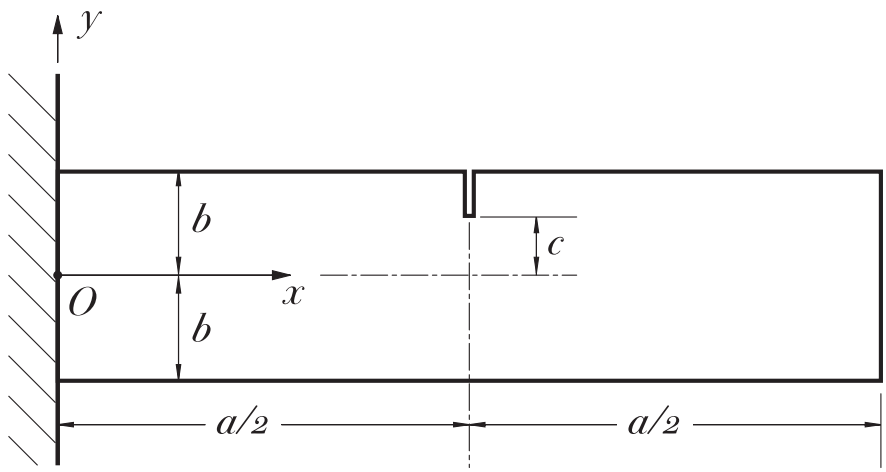


Figure 34.5: Groove cut in a bar with residual stresses.

What would be the auxiliary problem we would need to solve to establish an integral relation between the residual stress distribution and the displacement u, using the reciprocal theorem. (Do not attempt to solve the auxiliary problem!)

The average downward displacement at the free end is

$$u = -\frac{1}{2b}\int_{-b}^{b} u_y(a, y)dy \tag{1}$$

and we can make this look like the right hand side of (34.8) if we choose the auxiliary solution such that $\boldsymbol{t}_2$ is a uniform shear traction

$$\sigma_{yx}^{(2)}(a, y) = -\frac{1}{2b} \tag{2}$$

on the end $x = a$, the other surfaces (except the support $x = 0$) being traction free.

In the original state (1), the *release* of the residual stress $\sigma_{xx}^R(y)$ is equivalent to a *compressive* traction[1] $\sigma_{xx}^R(y)$ on the surfaces of the cut, so equation (34.8) would

[1]This is the same argument that makes the corrective solution for a crack loaded in tension be that for the same crack loaded on the faces by a compressive traction, as in §28.1.

then take the form

$$u = -\frac{1}{2b}\int_{-b}^{b} u_y(a,y)dy = \int_{c}^{b}\{u_x^{(2)}(a^+/2,y) - u_x^{(2)}(a^-/2,y)\}\sigma_{xx}^R(y)dy \ , \qquad (3)$$

since the compressive traction does positive work on $u_x^{(2)}(a^+/2,y)$ and negative work on $u_x^{(2)}(a^-/2,y)$.

Thus, if we could solve the auxiliary problem defined by (2), equation (3) defines an integral relation between $\sigma_{xx}^R(y)$ and u. By measuring u for various values of cut length c, we could use this relation to obtain the complete residual stress distribution $\sigma_{xx}^R(y)$.

34.7. By differentiating the solution of the Hertz problem of §30.2.5 with respect to x, show that the contact pressure distribution

$$p(r,\theta) = \frac{Cr\cos\theta}{\sqrt{a^2 - r^2}} \; ; \quad 0 \le r < a$$

produces the normal surface displacement

$$u_z = \frac{\pi C(1-\nu) r\cos\theta}{4\mu} \; ; \quad 0 \le r < a \; .$$

Use this result and the reciprocal theorem to determine the angle through which the punch in §34.3.2 will tilt about the y-axis if the force F is applied along a line passing through the origin.

From §30.2.5 and equation (30.60), we know that the pressure distribution

$$p(r) = \frac{4\mu\sqrt{a^2 - r^2}}{\pi(1-\nu)R}$$

in $0 < r < a$ produces a displacement

$$u_z(r) = d - \frac{r^2}{2R} \; .$$

Writing this in Cartesian coordinates

$$r^2 = x^2 + y^2 \; ,$$

we have that

$$p(x,y) = \frac{4\mu\sqrt{a^2 - x^2 - y^2}}{\pi(1-\nu)R}$$

produces

$$u_z(x,y) = d - \frac{(x^2 + y^2)}{2R} \; .$$

Differentiating with respect to x, we conclude that

$$p(x,y) = -\frac{4\mu x}{\pi(1-\nu)R\sqrt{a^2 - x^2 - y^2}} = -\frac{4\mu r\cos\theta}{\pi(1-\nu)R\sqrt{a^2 - r^2}}$$

will produce

$$u_z(x,y) = -\frac{x}{R} = -\frac{r\cos\theta}{R} \; ,$$

or, writing

$$C = -\frac{4\mu}{\pi(1-\nu)R} \; ,$$

$$p(r,\theta) = \frac{Cx}{\sqrt{a^2 - x^2 - y^2}} = \frac{Cr\cos\theta}{\sqrt{a^2 - r^2}} \; ; \quad 0 \le r < a \tag{1}$$

produces

$$u_z = \frac{\pi C(1-\nu)x}{4\mu} = \frac{\pi C(1-\nu)r\cos\theta}{4\mu} \; ; \quad 0 \le r < a \; . \tag{2}$$

We shall use (1, 2) as the auxiliary solution p_2, u_2 in an application of the reciprocal theorem to the problem where p_1, u_1 represents the problem of §34.3.2, modified so that the line of action of the force passes through the origin. We then have

$$u_1 = u_0(x,y) + C_1 + C_2 x + C_3 y = u_0(r,\theta) + C_1 + C_2 r\cos\theta + C_3 r\sin\theta \; ,$$

where C_2 is the angle through which the punch tilts about the y-axis and C_3 is the angle of tilt about the x-axis. Also, since the force acts through the origin, the resultant moment about each of these axes must be zero, giving

$$\int\int_A p_1(x,y)xdxdy = \int\int_A p_1(x,y)ydxdy = 0 \; . \tag{3}$$

Applying the reciprocal theorem, we have

$$\int\int p_1 u_2 dxdy = \int\int p_2 u_1 dxdy$$

and hence

$$\begin{aligned}
\frac{\pi(1-\nu)}{4\mu}\int\int p_1(x,y)xdxdy &= \int\int \frac{x}{\sqrt{a^2-x^2-y^2}}\left(u_0(x,y) + C_1 + C_2 x + C_3 y\right)dxdy \\
&= \int\int \frac{u_0(x,y)xdxdy}{\sqrt{a^2-x^2-y^2}} + \int\int \frac{C_1 xdxdy}{\sqrt{a^2-x^2-y^2}} \\
&\quad + \int\int \frac{C_2 x^2 dxdy}{\sqrt{a^2-x^2-y^2}} + \int\int \frac{C_3 yxdxdy}{\sqrt{a^2-x^2-y^2}} \; ,
\end{aligned} \tag{4}$$

where we have cancelled a common factor of C.

We know from (3) that the integral on the left-hand side of this equation is zero, so the right-hand side must also be zero. Also, the second and fourth integrals on the right-hand side are each zero because the integrands are odd in x, y respectively and the domain of integration is symmetric. We therefore conclude that

$$\int\int \frac{u_0(x,y)xdxdy}{\sqrt{a^2-x^2-y^2}} + \int\int \frac{C_2 x^2 dxdy}{\sqrt{a^2-x^2-y^2}} = 0 \; .$$

The second term can be evaluated in polar coordinates as

$$\int\int \frac{C_2 x^2 dxdy}{\sqrt{a^2-x^2-y^2}} = C_2\int_0^a\int_0^{2\pi} \frac{r^3\cos^2\theta d\theta dr}{\sqrt{a^2-r^2}} = \pi C_2\int_0^a \frac{r^3 dr}{\sqrt{a^2-r^2}} = \frac{2\pi C_2 a^3}{3} \tag{5}$$

and it follows that

$$C_2 = -\frac{3}{2\pi a^3}\int\int \frac{u_0(x,y)xdxdy}{\sqrt{a^2-x^2-y^2}} = -\frac{3}{2\pi a^3}\int_0^a\int_0^{2\pi} \frac{u_0(r,\theta)r^2\cos\theta d\theta dr}{\sqrt{a^2-r^2}} \; ,$$

which is the required result.

34.8. A long hollow cylinder of inner radius a and outer radius b has traction free surfaces, but it is subjected to a non-axisymmetric temperature increase $T(r,\theta)$. Find the change in the cross-sectional area of the hole due to thermoelastic distortion.

The change in area of the hole can be written

$$\delta A = \int_0^{2\pi} u_r(a,\theta) a d\theta \tag{1}$$

and we can make this equal to the right hand side of (34.27) by choosing an auxiliary solution such that $\boldsymbol{t}_2$ is a unit pressure on the inner surface. This is the solution of §9.3.2, with $p_0 = 1$ and hence

$$\begin{aligned}
\sigma_{rr}^{(2)} &= \frac{a^2}{(b^2-a^2)}\left(1-\frac{b^2}{r^2}\right) && (2)\\
\sigma_{r\theta}^{(2)} &= 0 && (3)\\
\sigma_{\theta\theta}^{(2)} &= \frac{a^2}{(b^2-a^2)}\left(1+\frac{b^2}{r^2}\right) . && (4)
\end{aligned}$$

To evaluate the left hand side of (34.27), we first note that in plane strain,

$$e_{rr} = \frac{(1-\nu)\sigma_{rr}}{2\mu} - \frac{\nu\sigma_{\theta\theta}}{2\mu} \; ; \quad e_{\theta\theta} = \frac{(1-\nu)\sigma_{\theta\theta}}{2\mu} - \frac{\nu\sigma_{rr}}{2\mu} , \tag{5}$$

from (3.4) and hence

$$\text{div } \boldsymbol{u} = e = e_{rr} + e_{\theta\theta} = \frac{(1-2\nu)(\sigma_{rr}+\sigma_{\theta\theta})}{2\mu} . \tag{6}$$

Substituting for the stresses from (2,4), we obtain

$$\text{div } \boldsymbol{u}^{(2)} = \frac{(1-2\nu)a^2}{\mu(b^2-a^2)} . \tag{7}$$

It then follows from (34.27) that

$$\begin{aligned}
\delta A &= \int_a^b \int_0^{2\pi} \left(\frac{(1-2\nu)a^2}{\mu(b^2-a^2)}\right)\left(\frac{2\mu(1+\nu)\alpha}{(1-2\nu)}\right) T(r,\theta) r d\theta dr \\
&= \frac{2\alpha(1+\nu)a^2}{(b^2-a^2)} \int_a^b \int_0^{2\pi} T(r,\theta) r d\theta dr . && (8)
\end{aligned}$$

34.9. The long solid cylinder $0 \le r < a, 0 < z < L$, $L \gg a$ is built in at $z = L$ and all the remaining surfaces are traction free. If the cylinder is now heated to an arbitrary non-uniform temperature T, find an integral expression for the displacement u_y at the free end.

The right hand side of (34.27) can be made to represent an average displacement u_y at $z = 0$ if we choose an auxiliary solution in which the same cylinder is supported at $z = L$ and loaded by a unit force in the y-direction. The solution of this problem is given by two different methods in §17.4.1 and §25.3.1. Writing $F = 1$ and replacing $\cos\theta, \sin\theta$ by $\sin\theta, -\cos\theta$ respectively in equations (25.58) (because the solution in §25.3.1 is for a force in the x-direction), we have

$$\sigma_{rr}^{(2)} = \sigma_{r\theta}^{(2)} = \sigma_{\theta\theta}^{(2)} = 0$$

$$\begin{aligned} \sigma_{zr}^{(2)} &= -\frac{(3+2\nu)(a^2-r^2)\sin\theta}{2\pi a^4(1+\nu)} \\ \sigma_{z\theta}^{(2)} &= -\frac{[(3+2\nu)a^2-(1-2\nu)r^2]\cos\theta}{2\pi a^4(1+\nu)} \\ \sigma_{zz}^{(2)} &= -\frac{4rz\sin\theta}{\pi a^4} \ . \end{aligned} \tag{1}$$

We then have

$$\text{div } \boldsymbol{u}_2 = e = \frac{(1-2\nu)(\sigma_{rr}+\sigma_{\theta\theta}+\sigma_{zz})}{E} = -\frac{2(1-2\nu)rz\sin\theta}{\mu(1+\nu)\pi a^4} \ . \tag{2}$$

Substituting into equation (34.27), we obtain

$$\begin{aligned} u_y &= -\frac{2\mu(1+\nu)\alpha}{(1-2\nu)}\int_0^L\int_0^a\int_0^{2\pi}\frac{2(1-2\nu)rz\sin\theta T(r,\theta,z)rd\theta drdz}{\mu(1+\nu)\pi a^4} \\ &= -\frac{4\alpha}{\pi a^4}\int_0^L\int_0^a\int_0^{2\pi}T(r,\theta,z)r^2z\sin\theta d\theta drdz \ . \end{aligned} \tag{3}$$

Notice that u_y will vary with position over the free end. What is actually calculated here is an average weighted with the shear traction distribution of equation (1) — i.e.

$$u_y = -\int_0^a\int_0^{2\pi}\{\sigma_{zr}^{(2)}u_r^{(1)}(r,\theta,0)+\sigma_{z\theta}^{(2)}u_\theta^{(1)}(r,\theta,0)\}rd\theta dr \ . \tag{4}$$

The negative sign in this equation arises because the free end is a negative z-plane, so (for example) the traction $\sigma_{zr}^{(2)}$ acts in the negative r-direction. Equation (4) and hence (3) is the best approximation to a rigid-body displacement in the sense defined by Renton[2].

[2] J.D.Renton, Generalized beam theory applied to shear stiffness, *Int. J. Solids Structures,* Vol. 27 (1991), 1955–1967.

34.10. A body Ω is raised to an arbitrary temperature field $T(x, y, z)$, but the boundaries are all free of traction. Show that the average value of the bulk stress $\bar{\sigma}$ of equation (1.75) is zero — i.e.

$$\iiint_\Omega \bar{\sigma} d\Omega = 0 \ .$$

Summing the three normal stress-strain relations, we obtain

$$e = e_{xx} + e_{yy} + e_{zz} = \frac{3(1-2\nu)\bar{\sigma}}{E} + 3\alpha T \tag{1}$$

and hence

$$\bar{\sigma} = \frac{Ee}{3(1-2\nu)} - \frac{E\alpha T}{(1-2\nu)} \ . \tag{2}$$

Thus

$$\iiint_\Omega \bar{\sigma} d\Omega = \frac{E}{3(1-2\nu)} \iiint_\Omega e d\Omega - \frac{E\alpha}{(1-2\nu)} \iiint_\Omega \bar{\sigma} T d\Omega \ . \tag{3}$$

In this expression, the integral in the first term represents the change ΔV in the volume of the body

$$\iiint_\Omega e d\Omega = \iiint_\Omega \text{div } \boldsymbol{u} d\Omega = \iint_S \boldsymbol{n}.\boldsymbol{u} dS = \Delta V \ , \tag{4}$$

using the divergence theorem. Substituting for this from equation (34.29), we find that the two terms on the right hand side of (3) cancel, giving

$$\iiint_\Omega \bar{\sigma} d\Omega = 0 \ . \tag{5}$$