Legendre Diff Eqn & Legendre Polynomials

$(1-x^2)y'' + 2xy' + p(p+1)y = 0$, $p \in \mathbb{R}$

The normalized eqn is

$$y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0$$

Since $\frac{2x}{1-x^2}$ and $\frac{p(p+1)}{1-x^2}$ are all analytic at $x=0$, in fact for $|x|<1$, $x=0$ is an ordinary point for the differential equation

Let $x = p(p+1)$, then an approximate soln. proposed is

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} \; , \; y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$$

$$(1-x^2)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + 2x\sum_{n=1}^{\infty} c_n n x^{n-1} - x\sum_{n=0}^{\infty} c_n x^n = 0$$

Shifting the first term $n \to n+2$ one has

$$\sum_{n=0}^{\infty} c_{n+2}(n+2)(n+1)x^n - \sum_{n=2}^{\infty} c_n n(n-1)x^n$$
$$-2\sum_{n=1}^{\infty} c_n n x^n + x\sum_{n=0}^{\infty} c_n x^n = 0$$

The common range is $2 \to \infty$, we cut extra terms to

$$2c_2 + 6c_3 x + \sum_{n=2}^{\infty} c_{n+2}(n+2)(n+1)x^n$$
$$- \sum_{n=2}^{\infty} c_n(n)(n-1)x^n - 2c_1 x - 2\sum_{n=2}^{\infty} c_n n x^n + x c_0 + x\sum_{n=0}^{\infty} c_n x^n = 0$$

$$(2c_2 + 2c_0) + (x c_1 - 2c_1 + 6c_3)x$$
$$+ \sum_{n=2}^{\infty}\{c_n[x - n(n-1) - 2n] + c_{n+2}(n+2)(n+1)x^n\}$$

No restriction on c_0. Equate the coeffs. of $x^i = 0$ to zero one has

$$c_2 = -\frac{\lambda}{2}c_0 = -\frac{p(p+1)}{2}c_0$$

$$c_3 = \frac{-1}{6}(\lambda - 2)c_1 = \frac{2 - p(p+1)}{6}c_1 = \frac{(-p-1)(p+2)}{3!}c_1$$

The recurrence relation is

$$c_{n+2} = \frac{p(p+1) - n(n-1) - 2n}{(n+2)(n+1)}c_n$$

$$c_{n+2} = \frac{-(p-n)(p+n+1)}{(n+2)(n+1)}c_n \; , \; n \geq 2$$

Therefore

$n=2 \Rightarrow c_4 = \frac{-(p-2)(p+3)}{4\cdot 3}c_2 = \frac{(p-2)(p+1)(p+3)}{4!}c_0$

$n=3 \Rightarrow c_5 = -\frac{(p-3)(p+4)}{5\cdot 4}c_3 = \frac{(p-3)(p-1)(p+2)(p+4)}{5!}c_1$

Substitution of these into the soln. proposed

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$

$$y_{gen} = c_0 + c_1 x - \frac{p(p+1)}{2!}c_0 x^2 - \frac{(p-1)(p+2)}{3!}c_1 x^3$$
$$+ \frac{(p-2)p(p+1)(p+3)}{4!}c_0 x^4 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!}c_1 x^5$$
$$+ \dots$$

Grouping c_0, c_1 terms

$$y_{gen} = c_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{(p-2)p(p+1)(p+3)}{4!}x^4 - \dots\right] + c_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!}x^5 + \dots\right]$$

Simplifying by summation symbols

$$y_{gen} = c_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{\prod_{k=0}^{p-1}(p-2k)(p+2k+1)}{2!}x^2\right]$$
$$+ c_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{\prod_{k=0}^{p-1}(p-2k-1)(p+2k+2)}{(2+1)!}x^3\right]$$

With more simplifications

$$y_{gen} = c_0\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{\prod_{k=0}^{n-1}(p-2k)(p+2k+1)}{(2n)!}x^{2n}\right]$$
$$+ c_1\left[x + \sum_{n=1}^{\infty}(-1)^n \frac{\prod_{k=0}^{n-1}(p-2k-1)(p+2k+2)}{(2n+1)!}x^{2n+1}\right]$$

Therefore the general soln. is

$$y_{gen} = c_0 y_1 + c_1 y_2$$

where

$$y_1 = 1 + \sum_{n=1}^{\infty}(-1)^n \frac{\prod_{k=0}^{n-1}(p-2k)(p+2k+1)}{(2n)!}x^{2n}$$

$$y_2 = x + \sum_{n=1}^{\infty}(-1)^n \frac{\prod_{k=0}^{n-1}(p-2k-1)(p+2k+2)}{(2n+1)!}x^{2n+1}$$

$$y_1 = 1 - \frac{p(p+1)}{2!}x^2 + \frac{(p-2)p(p+1)(p+3)}{4!}x^4 - \sum_{p=3}^{\infty}\frac{\alpha\cdots(\alpha-2p+2)(\alpha+1)\cdots(\alpha+2p-1)}{2p!}x^{2p}$$

$$y_2 = x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!}x^5 - \sum_{p=3}^{\infty}(-1)^p\frac{(\alpha-1)\cdots(\alpha-2p+1)(\alpha+2)\cdots(\alpha+2p)}{(2p+1)!}x^{p+1}$$

$$y_{gen} = y_1 + y_2 \quad \text{where } y_1 = c_0 y_1 \quad y_2 = c_1 y_2$$

It can be observed that for odd values of p's the solution y_1 (and y_1) reduces to a polynomial with order p ($n=p$). Similarly for even values of p's, the solution y_2 (y_2) reduces to a polynomial with order p ($n=p$). Let's obtain the polynomials using y_1 and y_2

$p=0 \Rightarrow P_0(x) = y_1|_{p=0} = c_0 1$

$p=2 \Rightarrow P_2(x) = y_1|_{p=2} = c_0\left[1 - 3x^2\right]$

$p=4 \Rightarrow P_4(x) = y_1|_{p=4} = c_0\left[1 - 10x^2 + \frac{35}{3}x^4\right]$

$\vdots$

$p=1 \Rightarrow P_1(x) = y_2|_{p=1} = c_1 x$

$p=3 \Rightarrow P_3(x) = y_2|_{p=3} = c_1\left[x - \frac{5}{3}x^3\right]$

$p=5 \Rightarrow P_5(x) = y_2|_{p=5} = c_1\left[x - \frac{14}{3}x^3 + \frac{21}{5}x^5\right]$

$\vdots$

If c_0 and c_1 are chosen in a such way that $P_i(x) = 1$ for $x=1 \Rightarrow P_i(1) = 1$ then the resulting polynomials will be simplified and still be the solns of the Legendre eqns. as follows

$c_0 = 1 \Rightarrow P_0(x) = 1$

$c_0 = -\frac{1}{2} \Rightarrow P_2(x) = \frac{1}{2}(3x^2 - 1)$

$c_0 = \frac{1}{8} \Rightarrow P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$c_1 = 1 \Rightarrow P_1(x) = x$

$c_1 = -\frac{3}{2} \Rightarrow P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$c_1 = \frac{15}{8} \Rightarrow P_3(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

Note that these polynomials are the solns. of Legendre differential eqns. only for the interval $-1 < x < +1$

A generalized expression for Legendre polynomials is also given as follows

$$P_p(x) = \frac{1}{2^p p!}\frac{d^p}{dx^p}(x^2 - 1)^p \; ; \; p \geq 0, \text{integer}$$

for example, for $p=3$;

$$P_3(x) = \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1)$$

$$= \frac{1}{48}(120x^3 - 72x)$$

then

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{2^4 \cdot 4!}\frac{d^4}{dx^4}(x^2-1)^4$$

$$P_4(x) = \frac{1}{16\cdot 4\cdot 3\cdot 2\cdot 1}\frac{d^4}{dx^4}\left[1(x^2)^4 1^0 - 4(x^2)^3 1^1 + 6(x^2)^2 1^2 - 4(x^2)^1 1^3 + 1(x^2)^0 1^4\right]$$

$$\frac{1}{16\cdot 4\cdot 3\cdot 2}\frac{d^4}{dx^4}\left[x^8 - 4x^6 + 6x^4 - 4x^2 + 1\right] =$$

$$\frac{1}{16\cdot 4\cdot 3\cdot 2}\frac{d^3}{dx^3}\left[8x^7 - 24x^5 + 24x^3 - 8x\right]$$

$$\frac{1}{16\cdot 4\cdot 3\cdot 2}\frac{d^2}{dx^2}\left[56x^6 - 120x^4 + 72x^2 - 8\right]$$

$$= \frac{1}{16\cdot 4\cdot 3\cdot 2}\frac{d}{dx}\left[336x^5 - 480x^3 + 144x\right]$$

$$\frac{1}{16\cdot 4\cdot 3\cdot 2}\left[1680x^4 - 1440x^2 + 144\right] = \frac{1}{16}\left[70x^4 - 60x^2 + 3\right]$$

$$P_4(x) = \frac{1}{16}\left[70x^4 - 60x^2 + 3\right]$$