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1-) In a room with a volume of 20000 m^3, 600 m^3 of fresh clean air is given per minute and exposed at the same rate. a) If y(0)=0 initial condition, find the expression that gives the amount of fresh air y(t) at any time. b) Find the time in minutes after which 90 % of the air in the room will be clean air.

a) $y' = 600 - \frac{600}{20000} y$

$y' + 0{,}03y = 600$

$P(x) = 0{,}03$

$Q(x) = 600$

TEY $e^{\int P dt} = e^{\int 0{,}03 dt} = e^{0{,}03t}$

$y \cdot e^{0{,}03t} = \int Q(x) \cdot e^{0{,}03t} \, dt$

$y \cdot e^{0{,}03t} = 600 \cdot \frac{1}{0{,}03} \cdot e^{0{,}03t} + C_1$

$y = 20000 + C_1 e^{-0{,}03t}$

$y(0) = 0 = 20000 + C_1 e^{-0{,}03 \cdot 0} \Rightarrow C_1 = -20000$

$$\boxed{y = 20000 - 20000 e^{-0{,}03t} = 20000\left(1 - e^{-0{,}03t}\right)}$$

b) $V = 2000 \ \%90 = 18000 \ m^3$

$18000 = 20000\left(1 - e^{-0{,}003t}\right)$

$0{,}9 = 1 - e^{-0{,}003t}$

$e^{-0{,}003t} = 0{,}1$

$$\boxed{t = 76{,}75 \text{ min}}$$

2-) A small copper ball whose temperature is Ti = 90 °C is put into ice water (0 °C) at time t = 0. It was observed that after t = 1 minute, the temperature of the ball dropped to 40 °C. Using Newton's law of cooling, find the temperature of the ball at t=5 minutes.

$$T(t) = T_a + (T_{ball} - T_a)e^{-kt}$$

INITIAL CON. $T(1) = 40°C$

$$T(1) = 40 = 0 + (90 - 0)e^{-k}$$

$$90e^{-k} = 40 \quad e^{-k} = \frac{40}{90}$$

$$-k = \ln\frac{40}{90}, \; k = -\ln\frac{40}{90}, \; k = 0{,}811$$

$$\Rightarrow T(t) = T_a + (T_{ball} - T_a)e^{-0{,}811t}$$

$$\boxed{\Rightarrow T(5) = 1{,}56°C}$$

3-) The water temperature in a tank in the room is 200 °C at 1:00 o'clock and the ambient temperature is 80 °C. At 1:30, the temperature of the water is 160 °C. a) What is the temperature of the water at 2:00 and b) when will the temperature be 100 °C?

$$\frac{dT}{dt} = -k(T - T_o)$$

$$\int \frac{1}{T - T_a} dT = \int -k\,dt$$

$$\ln(T - T_a) = -kt + \ln c$$

$$\ln(T - T_o) - \ln C = -kt$$

$$\frac{T - T_o}{C} = -kt$$

$$T(t) = T_a + Ce^{-kt}$$

$$T_o = 80°C$$

$$T(0) = 200°C,\ T(30) = 160°C$$

$$T(t) = 80 + Ce^{-kt}$$

$$T(0) = 200 = 80 + C$$

$$C = 200 - 80 = 120$$

$$T(t) = 80 + 120e^{-kt}$$

$$T(30) = 160 = 80 + 120e^{-30k}$$

$$k = 0{,}135$$

$$T(t) = 80 + 120e^{-0{,}0135}$$

$$T(60) = 80 + 120e^{-0{,}0135 \cdot 60}$$

$$\boxed{= 133{,}38°C} \quad \text{AT } 2:00$$

$$T(t) = 100 = 80 + 120e^{-0{,}0135t}$$

$$\boxed{t = 132{,}72 \text{ min}} = (1:00) + 132{,}72' = 3:00$$

4-) A person with a mass of 120 kg reaches a constant speed of 200 km/h after jumping from the plane. If he opens his parachute as soon as he reaches a constant speed, how long does it take for him to reach 10% of the speed v = 2.5 m/s in order to land safely on the ground?

$$V_i = 200\,km/h = \frac{200 \cdot 1000}{3600} = 55{,}56\,m/s.$$

$$V_f = 2{,}5\ \% 10 = 0{,}25\,m/s.$$

$$c \cdot V_i = m \cdot g$$

$$c = \frac{120 \cdot 9{,}81}{55{,}56} = 21{,}187$$

$$mg - cV = -m\left(\frac{dV}{dt}\right)$$

$$\int dt = \int \frac{dV}{\frac{c}{m}V - g}$$

$$\int_{55{,}56}^{0{,}25} \frac{dV}{\frac{c}{m}V - g} = \int_0^t dt$$

$$\int_{55{,}56}^{0{,}25} \frac{1}{\frac{21{,}187}{120} \cdot V - 9{,}81}\, dV = t$$

$$\boxed{t = 56{,}96\,sec}$$

5-) Consider the homogenous linear system, and solve it using matrix method.

$$x_1' = 7x_1 - x_2 + 6x_3 + \cos 2t$$
$$x_2' = -10x_1 + 4x_2 - 12x_3$$
$$x_3' = -2x_1 + x_2 - x_3$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos 2t$$

$$|A - \lambda I| = \begin{bmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{bmatrix} = 0$$

$$= (7-\lambda)[(4-\lambda)(-1-\lambda) - (-12)(1)] - (-1)[(-10)(-1-\lambda) - (-12)(-2)] + 6[(-10)(1) - (4-\lambda)(-2)] = 0$$
$$(7-\lambda)(-4-4\lambda+\lambda+\lambda^2+12) \quad - \quad (-1)(10+10\lambda-24) \quad + \quad 6(-10+8-2\lambda) = 0$$
$$(7-\lambda)(\lambda^2-3\lambda+8) \quad +10\lambda-14 \quad -12-12\lambda = 0$$
$$7\lambda^2 - 21\lambda + 56 - \lambda^3 + 3\lambda^2 - 8\lambda + 10\lambda - 14 - 12 - 12\lambda = 0$$
$$-\lambda^3 + 10\lambda^2 - 31\lambda + 30 = 0$$

$$\Rightarrow \lambda_1 = 2, \ \lambda_2 = 3, \ \lambda_3 = 5$$

For $\lambda = 2$

$$A\alpha = \lambda\alpha \rightarrow \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

LET $\alpha_1 = 1, \ \alpha_3 = -1, \ \alpha_2 = -1$

$$\Rightarrow \text{For } \lambda = 2 \Rightarrow \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \xrightarrow{x_1 = \alpha e^{\lambda_1 t}} x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{bmatrix}$$

For $\lambda = 3$

$$A\alpha = \lambda\alpha \rightarrow \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 3 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\left. \begin{aligned} 7\alpha_1 - \alpha_2 + 6\alpha_3 &= 3\alpha_1 \\ -10\alpha_1 + 4\alpha_2 - 12\alpha_3 &= 3\alpha_2 \\ -2\alpha_1 + \alpha_2 - \alpha_3 &= 3\alpha_3 \end{aligned} \right\} \alpha_1 = -\alpha_3, \quad \alpha_2 = 2\alpha_3$$

LET $\alpha_1 = 1,\ \alpha_3 = -1,\ \alpha_2 = -2$

$$\Rightarrow \text{FOR } \lambda = 2 \Rightarrow \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \xrightarrow{x_2 = \alpha e^{\lambda_2 t}} x = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix}$$

FOR $\lambda = 5$

$$A\alpha = \lambda\alpha \rightarrow \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 5 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\left.\begin{aligned} 7\alpha_1 - \alpha_2 + 6\alpha_3 &= 5\alpha_1 \\ -10\alpha_1 + 4\alpha_2 - 12\alpha_3 &= 5\alpha_2 \\ -2\alpha_1 + \alpha_2 - \alpha_3 &= 5\alpha_3 \end{aligned}\right\} \quad \alpha_1 = -1{,}5\alpha_3,\quad \alpha_2 = 3\alpha_3$$

LET $\alpha_1 = 3,\ \alpha_3 = -2,\ \alpha_2 = -6$

$$\Rightarrow \text{FOR } \lambda = 2 \Rightarrow \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix} \xrightarrow{x_3 = \alpha e^{\lambda_3 t}} x_3 = \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix} e^{5t} = \begin{bmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{bmatrix}$$

$$X_{hom} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C_1 \begin{bmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{bmatrix} + C_2 \begin{bmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix} + C_3 \begin{bmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{bmatrix}$$

$\Rightarrow H = \{e^{2t}, e^{3t}, e^{5t}\}$ AND $D = \{\cos 2t, \sin 2t\}$

$\Rightarrow H \cap D = \emptyset$

$x_p = C_1 \cos 2t + C_2 \sin 2t$

$x_p' = -2C_1 \sin 2t + 2C_2 \cos 2t$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos 2t$$

$$-2C_1 \sin 2t + 2C_2 \cos 2t = A\, C_1 \cos 2t + A\, C_2 \sin 2t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos 2t$$

$$\sin 2t \to -2C_1 = AC_2 \qquad \cos 2t \to 2C_2 = AC_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\cos 2t \to 2C_2 = AC_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to 2AC_2 = A^2 C_1 + A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\to -4C_1 = A^2 C_1 + A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(-4I - A^2) C_1 = A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \frac{1}{-4I - A^2} \cdot A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -0{,}094 \\ -0{,}311 \\ -0{,}078 \end{bmatrix}$$

$$2C_2 = AC_1 \Rightarrow C_2 = \frac{AC_1}{2}$$

$$C_2 = \begin{bmatrix} -0{,}408 \\ 0{,}3169 \\ -0{,}022 \end{bmatrix}$$

$$x_{part} = \cos 2t \begin{bmatrix} -0{,}094 \\ -0{,}311 \\ -0{,}078 \end{bmatrix} + \sin 2t \begin{bmatrix} -0{,}408 \\ 0{,}3169 \\ -0{,}022 \end{bmatrix}$$

$$x_{hom} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C_1 \begin{bmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{bmatrix} + C_2 \begin{bmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix} + C_3 \begin{bmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{bmatrix}$$

$$x_{gen} = x_{hom} + x_{part}$$

$$\boxed{\begin{aligned} x_1 &= C_1 e^{2t} + C_2 e^{3t} + 3C_3 e^{5t} - 0{,}094 \cos 2t - 0{,}408 \sin 2t \\ x_2 &= -C_1 e^{3t} - 2C_2 e^{3t} - 6C_3 e^{5t} - 0{,}311 \cos 2t + 0{,}3169 \sin 2t \\ x_3 &= -C_1 e^{2t} - C_2 e^{3t} - 2C_3 e^{5t} - 0{,}078 \cos 2t - 0{,}022 \sin 2t \end{aligned}}$$

6-) Find series solution of the differential equation

$$6x^2 y'' + 7xy' - \left(1+x^2\right) y = 0 \; about \; x_0 = 0$$

$$y'' + \frac{7x}{6x^2} y' - \frac{1+x^2}{6x^2} y = 0$$

* COEFFICIENT FUNCTIONS ARE NOT ANALYTIC ONLY AT $x=0$

$$x P(x) \to \frac{7}{6} \;, \quad x^2 Q(x) \to \frac{-(1+x^2)}{6} \Rightarrow \text{REGULAR SINGULAR POINT}$$

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

$$r(r-1) + p_0 r + q_0 = 0$$

$$\lim_{x\to 0} xP(x) = \lim_{x\to 0} \frac{7}{6} = \frac{7}{6} = p_0$$

$$\lim_{x\to 0} x^2 Q(x) = \lim_{x\to 0} \frac{-(1+x^2)}{6} = -\frac{1}{6} = q_0$$

FROBENIUS THEOREM

$$y(x) = c_0 x^r + c_1 x^{r+1} + c_2 x^{r+2} + \dots = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y'(x) = c_0 r x^{r-1} + c_1 (r+1) x^r + c_2 (r+2) x^{r+1} + \dots = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$

$$y''(x) = c_0 r(r-1) x^{r-2} + c_1 (r+1) r x^{r-1} + c_2 (r+1)(r+2) x^r + \dots$$

$$= \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

$$6x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} + 7x \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$

$$-(1+x^2) \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$6 \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r} + 7 \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$- \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0$$

$$6\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + 7\sum_{n=0}^{\infty}(n+r)c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$-\sum_{n=2}^{\infty} c_{n-2}x^{n+r} = 0$$

$$6r(r-1)C_0 x^r + 6(r+1)(r)C_1 x^{r+1} + 7rC_0 x^r + 7(r+1)C_1 x^{r+1}$$

$$-C_0 x^r - C_1 x^{r+1} + \sum_{n=2}^{\infty}\left[6(n+r)(n+r-1)C_n + 7(n+r)C_n - C_n - C_{n-2}\right]x^{n+r} = 0$$

$$\left[6r(r-1)+7r-1\right]C_0 x^r + \left[6(r+1)(r)+7(r+1)-1\right]C_1 x^{r+1}$$

$$+\sum_{n=2}^{\infty}\left[6(n+r)(n+r-1)C_n + 7(n+r)C_n - C_n - C_{n-2}\right]x^{n+r} = 0$$

$$(6r^2+r-1)C_0 x^r = 0 \Rightarrow r_1 = \frac{1}{3},\ r_2 = -\frac{1}{2},\ r_1 - r_2 \neq N$$

$$(6r^2+13r+6)C_1 x^{r+1} = 0 \Rightarrow r_3 = -\frac{2}{3},\ r_4 = -\frac{3}{2},\ r_3 - r_4 \neq N$$

$$\left[6(n+r)(n+r-1)+7(n+r)-1\right]C_n - C_{n-2} = 0$$

$$C_n = \frac{C_{n-2}}{6(n+r)(n+r-1)+7(n+r)-1}$$

For $r_1 = 1/3$

$$n=2 \Rightarrow C_2 = \frac{C_0}{6(2+1/3)(2+1/3-1)+7(2+1/3)-1} \Rightarrow C_0 = 34C_2$$

$$n=4 \Rightarrow C_4 = \frac{C_2}{116},\quad C_4 = \frac{C_0}{3944}$$

$$y_1 = C_0 x^{1/3}\left(1 + \frac{x^2}{34} + \frac{x^4}{3944} + \cdots\right)$$

For $r_2 = -1/2$

$$n=2 \Rightarrow C_2 = \frac{C_0}{6(2-1/2)(2-1/2-1)+7(2-1/2)-1} \Rightarrow C_0 = 14C_2$$

$n=4 \Rightarrow C_4 = \frac{C_2}{76}$, $C_4 = \frac{C_0}{1064}$

$$y_2 = C_0 x^{-1/2}\left(1 + \frac{x^2}{14} + \frac{x^4}{1064} + \cdots\right)$$

LET $C_0 = 1$

$$y_{gen} = C_1 y_1 + C_2 y_2$$

$$\boxed{y_{gen} = C_1 x^{1/3}\left(1 + \frac{x^2}{34} + \frac{x^4}{3944} + \cdots\right) + C_2 x^{-1/2}\left(1 + \frac{x^2}{14} + \frac{x^4}{1064}\right)}$$