Method of Reducing Eqns. to Bessel Eqns.

There are many differential eqns. that can be reduced to Bessel or modified Bessel eqns. by transformation of variables. Then the reduced eqn. can be solved in terms of Bessel functions. If a differential eqn. can be put into the form

$$\frac{d}{dx}\left(x^a \frac{dy}{dx}\right) + bx^c y = 0$$

where a, b, c are real constants. It can be expressed into a Bessel eqn. by transforming both independent and dependent variables. The transformation can be obtained by defining independent and dependent variables as follows:

$$t = \alpha\sqrt{b}\, x^{1/\alpha} \text{ and } u = x^{-v/\alpha} y$$

Moreover, α and v are calculated as

$$\alpha = \frac{2}{c-a+2} \quad \text{and} \quad v = \frac{1-a}{c-a+2}$$

Thus, using the transformation parameters the differential eqn. will be transformed to

$$t^2 \frac{d^2u}{dt^2} + t\frac{du}{dt} + (t^2 - v^2)u = 0$$

Then the soln. becomes $u(t) = Z_v$ where Z_v is the Bessel function soln of the transformed eqn. If $b>0$, Z_v denotes Bessel functions of first kind; J_v and Y_v; and if $b<0$ Z_{iv} denotes the Bessel functions of second kind, I_v and K_v

Finally, one can write the soln. for y as:

$$y(x) = x^{v/\alpha} Z_v(\alpha\sqrt{b}\, x^{1/\alpha})$$

Example 4.3

Solve the differential eqn.

$$y'' + \left(e^{2x} - \frac{1}{9}\right)y = 0$$

$z = e^x \Rightarrow dz = e^x dx$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}e^x = \frac{dy}{dz}z$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dz}\frac{dz}{dx}\left(z\frac{dy}{dz}\right) = z\frac{dy}{dz} + z^2\frac{d^2y}{dz^2}$$

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + \left(z^2 - \frac{1}{9}\right)y = 0$$

$$y = C_1 J_{1/3}(z) + C_2 J_{-1/3}(z)$$

$$y = C_1 J_{1/3}(e^x) + C_2 J_{-1/3}(e^x)$$

Example 4.4

Solve the differential eqn. $xy'' + 2y' + xy = 0$

$$y = zx^{-1/2} \Rightarrow y' = -\frac{zx^{-3/2}}{2} + \frac{dz}{dx}x^{-1/2}$$

$$y'' = \frac{3zx^{-5/2}}{4} - \frac{x^{-3/2}}{2}\frac{dz}{dx} - \frac{x^{-3/2}dz}{2\,dx} + x^{-1/2}\frac{d^2z}{dx^2} = \frac{3zx^{-5/2}}{4} - x^{-3/2}\frac{dz}{dx} + x^{-1/2}\frac{d^2z}{dx^2}$$

Substituting into the differential eqn., yields that

$$\frac{3}{4}zx^{-3/2} - x^{-1/2}\frac{dz}{dx} + x^{-1/2}\frac{d^2z}{dx^2} - zx^{-3/2} + 2x^{-1/2}\frac{dz}{dx} + zx^{1/2} = 0$$

$$x^{1/2}\frac{d^2z}{dx^2} + x^{-1/2}\frac{dz}{dx} - \frac{1}{4}zx^{-3/2} + zx^{1/2} = 0$$

Scale by $x^{3/2}$

$$x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + \left(x^2 - \frac{1}{4}\right)z = 0$$

$$z = C_1 J_{1/2}(x) + C_2 Y_{1/2}(x)$$

$$yx^{1/2} = C_1 J_{1/2}(x) + C_2 Y_{1/2}(x)$$

$$y = x^{-1/2}\left[C_1 J_{1/2}(x) + C_2 Y_{1/2}(x)\right]$$

Example 4.5

Evaluate $\int x^{-1} J_4$

$$J_4(x) = \frac{2(3)}{x}J_3 - J_2$$

$$\int x^{-1}J_4 = \int x^{-1}\left(\frac{6}{x}J_3 - J_2\right)$$

$$\int x^{-1}J_4 = \int 6x^{-2}J_3 - \int x^{-1}J_2$$

$$= 6x^{-2}J_2 + x^{-1}J_1$$

Example: Solve the D.E. in terms of Bessel functions $y'' + xy = 0$

$a = 0,\ b = 1,\ c = 1$

$$\alpha = \frac{2}{c-a+2} = \frac{2}{3}, \quad n = \frac{1-a}{c-a+2} = \frac{1}{3}$$

$$t = \alpha\sqrt{b}\, x^{1/\alpha} = \frac{2}{3}x^{3/2}, \quad u = x^{-n/\alpha}y = x^{-1/2}y$$

$$t^2\frac{d^2u}{dt^2} + t\frac{du}{dt} + (t^2 + n^2)u = 0$$

$$y(x) = x^{\frac{1}{2}}J_n\left(\alpha\sqrt{b}\, x^{1/\alpha}\right)$$

$$y(x) = C_1 x^{1/2}J_{1/3}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + C_2 x^{1/2}J_{-1/3}\left(\frac{2}{3}x^{3/2}\right)$$

The Gamma function

Definition for Gamma function is

$$\Gamma(x) = \int_0^\infty x^{p-1}e^{-x}dx \quad p \in R \ \& \ p > 0$$

Let's verify some of its features

1) $p = 1 \Rightarrow \Gamma(1) = \int_0^\infty x^0 e^{-x}dx = -e^{-x}\Big|_0^\infty = -(0-1) = 1$ so; $\Gamma(1) = 1$

2) $\Gamma(p+1) = p\Gamma(p)$; $p > 0$

$\Gamma(n+1) = n!$

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}dx \qquad u = x^p \Rightarrow du = e^{-x}dx$$

$$du = px^{p-1}dx \Rightarrow v = -e^{-x}$$

$$\Gamma(p+1) = -x^p e^{-x}\Big|_{x=0}^\infty + \int_0^\infty e^{-x}px^{p-1}dx$$

$$= p\int_0^\infty x^{p-1}e^{-x}dx$$

so $\Gamma(p+1) = p\Gamma(p)$

$p = 1 \Rightarrow \Gamma(2) = 1\Gamma(1) = 1 = 1!$

$p = 2 \Rightarrow \Gamma(3) = 2\Gamma(2) = 2.1 = 2!$

$p = 3 \Rightarrow \Gamma(4) = 3\Gamma(3) = 3.2.1 = 3!$

$p = n \Rightarrow \Gamma(n) = (n-1)!$

Verify that;

$$\Gamma\left(\frac{1}{2}\right) = \left(-\frac{1}{2}\right)! = \sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{\frac{1}{2}-1}e^{-x}dx = \int_0^\infty x^{-1/2}e^{-x}dx$$

Substitute

$$x = u^2 \Rightarrow dx = 2u\,du \Rightarrow \frac{dx}{\sqrt{x}} = 2\,du$$

Insert in

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}du$$

Taking square of both sides

$$\Gamma^2\left(\frac{1}{2}\right) = 4\int_0^\infty e^{-u^2}du\int_0^\infty e^{-v^2}dv = 4\iint_0^\infty e^{-(u^2+v^2)}du\,dv$$

the double integral is evaluated by transforming to polar coordinates

$$u = r\cos\theta,\ v = r\sin\theta \Rightarrow dA = r\,dr\,d\theta = du\,dv$$

$$\Gamma^2\left(\frac{1}{2}\right) = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\left(-\frac{e^{-r^2}}{2}\right)\Big|_0^\infty d\theta = 4\int_0^{\pi/2}\left(0 + \frac{1}{2}\right)d\theta$$

$$\Gamma^2\left(\frac{1}{2}\right) = 2\frac{\pi}{2} = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \left(\frac{1}{2}\right)! = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}! = \frac{3\sqrt{\pi}}{2^2}$$