## 2.3 Canonical Forms and Eigen Values of Linear Differential Equation Systems

Systems of any order coupled differential equations can be expressed in term of a first order set of differential equations which can be expressed in matrix form. Consider such a set of homogeneous set of linear systems with constant coefficients.

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}\ x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}\ x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}\ x_2 + \cdots + a_{nn}x_n \end{aligned}$$

or in matrix form; $x' = Ax$

where

$$x' = \begin{bmatrix} x_1{}' \\ x_2' \\ . \\ x_n{}' \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix}$$

Let possible solutions be;

$$\begin{aligned} x_1 &= \alpha_1 e^{\lambda t} \\ x_2 &= \alpha_2 e^{\lambda t} \\ &\vdots \\ x_n &= \alpha_n e^{\lambda t} \end{aligned}$$

in matrix form; $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} e^{\lambda t}$, where $x$ and $\alpha$ are vectors.

Substitution of this solution proposal matrix expression $x = \alpha e^{\lambda t}$ into the given set in matrix form; $x' = Ax$, yields that

$$x = \alpha e^{\lambda t}$$

$$A\alpha - \lambda\alpha = 0$$

$$(A - \lambda I)\alpha = 0$$

As expected, for $\alpha$ to have a nontrivial solution, the condition $|A - \lambda I| = 0$ must be satisfied. This condition is known as "*characteristic equation*". Notice that $|\lambda I - A| = 0$, is also the same expression. The solutions of this expression are called *characteristic values* or more commonly, *eigenvalues* of the system, which are;

$$\lambda = \{\lambda_1 \,;\, \lambda_2 \,;\, \ldots \,;\, \lambda_n\} \text{ eigenvalues,}$$

Substitution of each $\lambda_i$ produces a corresponding $\alpha_i$ which is known as "*characteristic vector*" or "*eigenvector*". Each eigenvector corresponds to a solution proposal $x_i = \propto_i e^{\lambda_i t}$.

General solution is the linear combination of these solution proposals, provided that each of them is linearly independent. Otherwise some more manipulations are needed to be carried out to obtain corresponding independent solutions. As known, a popular measure for linear dependency of solution proposals (or any other functions) is that if the Wronskian determinant is not zero, $W(x_1, x_2, \ldots, x_n) \neq 0$, then the proposals are *linearly independent*, otherwise *dependent*. According to the eigenvalues, three cases can arise.

**<u>Case 1</u>**: All eigenvalues are distinct; that is

$$\lambda_1 \neq \lambda_2 \neq \ \ldots \ \neq \lambda_n$$

In this case, all of the proposals are linearly independent which can be verified using Wronskian determinant. Therefore, the general solution is obtained as the linear combination of the independent parameters;

$$x_{gen} \ = \ C_1 x_1 + C_2 x_2 + \cdots + C_n x_n$$

where $C_1, C_2, \ldots \ldots, C_n$ are arbitrary constants.

## <u>Example 2.6</u>

Consider the homogeneous linear system, and solve them using matrix method.

$$\begin{array}{llll} x_1' = & 7x_1 & -x_2 & +6x_3 \\ x_2' = & -10x_1 & +4x_2 & -12x_3 \\ x_3' = & -2x_1 & +x_2 & -x_3 \end{array}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \text{if} \qquad x = \alpha e^{\lambda t} \qquad \text{then} \qquad \begin{array}{l} x_1 = \alpha_1 e^{\lambda t} \\ x_2 = \alpha_2 e^{\lambda t} \\ x_3 = \alpha_3 e^{\lambda t} \end{array}$$

Characteristic equation;

$$|\lambda I - A| = \begin{vmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)\begin{vmatrix} 4-\lambda & -12 \\ 1 & -1-\lambda \end{vmatrix} + \begin{vmatrix} -10 & -12 \\ -2 & -1-\lambda \end{vmatrix} + 6\begin{vmatrix} -10 & 4-\lambda \\ -2 & 1 \end{vmatrix} = 0$$

$$(7-\lambda)[(4-\lambda)(-1-\lambda)+12] + [-10(-1-\lambda)-24] + [-10+2(4-\lambda)] = 0$$

after some manipulations;

$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$;or $(\lambda-2)(\lambda-3)(\lambda-5) = 0$ , so eigenvalues are

$\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 5$ ; are all distinct. Let's obtain the corresponding eigenvectors;

For $\lambda = \lambda_1 = 2$;

$$A\alpha = \lambda\alpha \xrightarrow{\textbf{yields}} \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 2\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\begin{array}{rrrl} 7\alpha_1 & -\alpha_2 & +6\alpha_3 & = 2\alpha_1 \\ -10\alpha_1 & +4\alpha_2 & -12\alpha_3 & = 2\alpha_2 \\ -2\alpha_1 & +\alpha_2 & -\alpha_3 & = 2\alpha_3 \end{array} \xrightarrow{\textbf{yields}} \begin{array}{rrrl} 5\alpha_1 & -\alpha_2 & +6\alpha_3 & = 0 \\ -10\alpha_1 & +2\alpha_2 & -12\alpha_3 & = 0 \\ -2\alpha_1 & +\alpha_2 & -3\alpha_3 & = 0 \end{array}$$

First two are linearly dependent. So two of the independent expressions;

$$\begin{array}{rrrl} 5\alpha_1 & -\alpha_2 & +6\alpha_3 & = 0 \\ -2\alpha_1 & +\alpha_2 & -3\alpha_3 & = 0 \end{array} \xrightarrow{\textbf{yields}} \begin{array}{rrl} -\alpha_2 & +6\alpha_3 & = -5\alpha_1 \\ +\alpha_2 & -3\alpha_3 & = \ \ 2\alpha_1 \end{array} \xrightarrow{\textbf{yields}} \begin{array}{rl} \alpha_3 & = -\alpha_1 \\ \alpha_2 & = -\alpha_1 \end{array}$$

Let $\alpha_1 = 1\ then\ \ \alpha_2 = -1,\ \alpha_3 = -1$ so the corresponding eigenvector

For $\lambda_1 = 2 \Longrightarrow \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ then the corresponding solution;

$$x_1 = \alpha e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{bmatrix}$$

For $\lambda = \lambda_2 = 3$;

$$A\alpha = \lambda\alpha \xrightarrow{\textbf{yields}} \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 3\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\begin{array}{rrrl} 7\alpha_1 & -\alpha_2 & +6\alpha_3 & = 3\alpha_1 \\ -10\alpha_1 & +4\alpha_2 & -12\alpha_3 & = 3\alpha_2 \\ -2\alpha_1 & +\alpha_2 & -\alpha_3 & = 3\alpha_3 \end{array} \xrightarrow{\textbf{yields}} \begin{array}{rrrl} 4\alpha_1 & -\alpha_2 & +6\alpha_3 & = 0 \\ -10\alpha_1 & +\alpha_2 & -12\alpha_3 & = 0 \\ -2\alpha_1 & +\alpha_2 & -4\alpha_3 & = 0 \end{array}$$

Let $\alpha_1 = 1$ then

$$\begin{aligned} -\alpha_2 \quad +6\alpha_3 \quad &= -4 \\ +\alpha_2 \quad -12\alpha_3 \quad &= 10 \\ +\alpha_2 \quad -4\alpha_3 \quad &= 2 \end{aligned}$$ so the corresponding eigenvectors $\alpha_2 = -2,\ \alpha_3 = -1$

For $\lambda_2 = 3 \rightarrow \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ then the corresponding solution;

$$x_2 = \alpha e^{\lambda_2 t} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix}$$

For $\lambda = \lambda_3 = 5$;

$$A\alpha = \lambda\alpha \xrightarrow{\textbf{yields}} \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 5 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\begin{matrix} 7\alpha_1 & -\alpha_2 & +6\alpha_3 & = 5\alpha_1 \\ -10\alpha_1 & +4\alpha_2 & -12\alpha_3 & = 5\alpha_2 \\ -2\alpha_1 & +\alpha_2 & -\alpha_3 & = 5\alpha_3 \end{matrix} \xrightarrow{\textbf{yields}} \begin{matrix} 2\alpha_1 & -\alpha_2 & +6\alpha_3 & = 0 \\ -10\alpha_1 & -\alpha_2 & -12\alpha_3 & = 0 \\ -2\alpha_1 & +\alpha_2 & -6\alpha_3 & = 0 \end{matrix}$$

First and last are linearly dependent. Let $\alpha_1 = 3$ then two of the independent expressions;

$$\begin{matrix} 2\alpha_1 & -\alpha_2 & +6\alpha_3 & = 0 \\ -10\alpha_1 & -\alpha_2 & -12\alpha_3 & = 0 \end{matrix} \xrightarrow{\textbf{yields}} \begin{matrix} -\alpha_2 & +6\alpha_3 & = -2\alpha_1 \\ -\alpha_2 & -12\alpha_3 & = 10\alpha_1 \end{matrix} \xrightarrow{\textbf{yields}} \begin{matrix} \alpha_3 & = -2 \\ \alpha_2 & = -6 \end{matrix}$$

So the corresponding eigenvector

For $\lambda_3 = 5 \rightarrow \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix}$ then the corresponding solution;

$$x_3 = \alpha e^{\lambda_3 t} = \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix} e^{5t} = \begin{bmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{bmatrix}$$

Since general solution is expressed in terms of their linear combinations

$$x_{gen} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C_1 \begin{bmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{bmatrix} + C_2 \begin{bmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix} + C_3 \begin{bmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{bmatrix}$$ then;

$$x_1 = C_1 e^{2t} + C_2 e^{3t} + 3C_3 e^{5t}$$

$$x_2 = -C_1 e^{2t} - 2C_2 e^{3t} - 6C_3 e^{5t}$$

$$x_3 = -C_1e^{2t} - C_2e^{3t} - 2C_3e^{5t}$$