JERRY GINSBERG

ENGINEERING DYNAMICS

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ENGINEERING DYNAMICS

Engineering Dynamics is a new treatment of kinematics and classical and analytical dynamics based on Ginsberg's popular *Advanced Engineering Dynamics* Second Edition. Like its predecessor, this book conveys physical and analytical understanding of the basic principles of dynamics, but it is more comprehensive and better addresses real-world complexities. Every section has been rewritten, and many topics have been added or enhanced. Several derivations are new, and others have been reworked to make them more accessible, general, and elegant. Many new examples are provided, and those that were retained have been reworked. They all use a careful pedagogical structure that mirrors the text presentation. Instructors will appreciate the significant enhancement of the number and variety of homework exercises. All of the text illustrations have been redrawn to enhance their clarity.

Jerry Ginsberg began his academic career at Purdue University in 1969, and he was a Fulbright–Hays Advanced Research Fellow in 1975. He moved to Georgia Tech in 1980, where he became the first holder of the Woodruff Chair in Mechanical Systems in 1988. Professor Ginsberg has worked in a broad range of areas in mechanical vibrations and acoustics, for which he developed and applied specialized mathematical and computational solutions that provide greater insight in comparison with standard numerical techniques. Professor Ginsberg is the author of more than 150 technical and archival papers and the highly regarded textbooks: *Advanced Engineering Dynamics*, *Mechanical and Structural Vibrations*, *Statics*, and *Dynamics* (the last two with Joseph Genin), as well as two chapters in *Nonlinear Acoustics*. He is a Fellow of the Acoustical Society of America and of the American Society of Mechanical Engineers, and he has served as an associate editor of the *Journal of the Acoustical Society* and of the ASME *Journal of Vibration and Acoustics*. He received the Georgia Tech Distinguished Professor Award in 1994, the Archie Higdon Distinguished Educator Award from ASEE in 1998, the Trent–Crede Medal from ASA in 2005, and the Per Bruel Gold Medal from ASME in 2007. He has delivered a number of distinguished lectures, including the 2001 ASME Rayleigh Lecture and the 2003 Special Lecture for the Noise Control and Acoustics Division of ASME, as well as keynote speeches at several meetings, including the Second International Congress on Dynamics, Vibrations, and Control in Beijing in 2006.

Engineering Dynamics

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Preface

It has been more than a decade since the second edition of *Advanced Engineering Dynamics* was published. Although I was pleased with that effort, my experience teaching dynamics with that book as a companion has given me insights that I either did not have or did not fully appreciate. I tried to satisfy multiple objectives as I wrote the present book. I wished to convey both physical and analytical understanding of the fundamental principles, and to expose the beauty of the discipline as a tightly woven sequence of concepts. I wanted to address the complexities of real-world engineering problems and explore the implications of dynamics for other subjects, but to do so in a manner that is accessible to those who come to it from a wide range of experiences. I wanted to provide a self-contained resource from which the motivated reader could learn directly. At first, I thought this book would just be a third edition of *Advanced Engineering Dynamics*, but as I progressed, I realized that the expanded scope and the amount of material that is either new or redone necessitated treating it as a new work.

The subject of dynamics is an interdisciplinary blend of physics, applied mathematics, computational methods, and basic logic. The least difficult aspect of the subject is the basic physical laws, most of which are at least a century old. A primary element that has moved the study of dynamics from natural philosophy to engineering is the development of powerful tools for describing motion and for solving equations of motion. Throughout my career I have operated under the premise that the world is complicated, and that a good text should prepare the student to address these complications. One of the methods I use here to meet this imperative is to provide examples that carefully guide the reader from the inception of a solution to its conclusion. I have tried to select examples that have most of the elements one might encounter in practice but are not so intricate as to mask the tautological features of the solution. An important feature of these examples is that the question of why a solution is assembled in a certain manner is regarded to be as important as the actual steps. In many cases I have used the same system to illustrate alternative approaches or different topics, which tends to give the treatment of those systems some of the aspects of the case study approach.

Almost every section of the text has been rewritten relative to *Advanced Engineering Dynamics*, and yet it should be clearly recognizable as being a descendant of its predecessor. New explanations for fundamental concepts have been introduced. Derivations have been reworked, sometimes to increase their generality and sometimes to enhance their elegance, but always to make them more accessible. Like the previous text, Chapter 1 develops the fundamental physical laws for a particle, but a section has been added to help the reader use mathematical software as an analytical aid. The early pioneers, who provided us with most of the basic laws and concepts, were correctly considered to be natural philosophers because they provided a framework for understanding how our world works. Partially as a recognition of their importance, I have expanded the biographical section at the conclusion of Chapter 1. I have also tried to bring out this philosophical perspective in the technical development.

Kinematics is the framework supporting the laws of dynamics. Being comfortable with the former vastly aids one to address the various kinetics concepts. For this reason a thorough treatment of the kinematics of particle and rigid-body motion is the focus of the initial development. The development is broad without going into specialized concepts that are primarily used in a confined topical area. The treatment in Chapter 2 of the kinematics of particle motion now derives the basic formulas for cylindrical and spherical coordinates prior to the tensor-oriented derivation of the comparable formulas for arbitrary orthogonal coordinate systems. This enables one to omit the more mathematical derivation without sacrificing fundamental concepts. An item of particular note is the expanded exploration in Chapter 3 of displacement of points relative to various reference frames, which should clarify many of the problematic aspects of the description of relative motion. I have found that it significantly assists students who are not practiced in visualizing spatial motion, and students in computer-aided design have told me that it aided them greatly in that subject. More important, the treatment leads to a derivation of the kinematics equations for relative motion that is simultaneously elegant and intuitive—there should be no misunderstanding of the significance of the various terms. Chapter 4 addresses the kinematics of systems that are subject to kinematical constraints, with an emphasis on linkages and rolling. The modifications here relative to the previous text are mostly incremental, but greater emphasis is now placed on the parallelism of the analysis of displacement, velocity, and acceleration. The example of a cardan joint should be enlightening in this regard.

The treatment of momentum-based concepts for rigid-body motion has now been split into two chapters. Chapter 5 focuses on the fundamental concept of angular momentum and the implications of its variability. The treatment of inertia properties has been expanded. The emphasis in this chapter is on making the angular momentum of a rigid body a quantity that the reader understands on a fundamental physical level, rather than merely being a quantity to be evaluated. These concepts are employed in Chapter 6 to implement the Newton–Euler equations for a single rigid body. The extension of such a formulation to a system of rigid bodies has been expanded. The chapter closes with a treatment of impulsive forces and their role in collisions, which is a topic that was omitted from the previous text.

Prior to delving into the mathematical concepts associated with analytical mechanics, the treatment in Chapter 7 begins by developing the principle of virtual work and applying it to a rigid frame and the analogous dynamic linkage. Seeing how the principle of dynamic virtual work, which has been incorrectly attributed to d'Alembert, can be used directly to formulate equations of motion enables one to view the concepts of analytical mechanics from the Newton–Euler perspective. This approach provides a strong motivation for the theoretical developments that follow. A noteworthy aspect of Chapter 7 is the expanded usage of the configuration space in conjunction with generalized coordinates, so that all of the basic aspects of Lagrangian mechanics now receive a parallel treatment between the configuration and physical spaces. This is the area in which my awareness has expanded most. It is also an area in which misconception is occasionally encountered in the technical literature. For example, a recent journal article stated that virtual displacement and virtual work are confusing concepts! A primary objective of Chapter 7 is to make it unlikely that such statements will continue to be made. The reader will find graphical descriptions of scleronomic, rheonomic, and nonholonomic constraints that should enhance understanding of their unifying features, as well as their differences. One of the desirable outcomes of the presentation should be a greater understanding of the fundamental philosophy underlying analytical mechanics, and of the associated concepts pertaining to generalized coordinates, generalized velocities, and virtual displacement.

Chapter 8 explores reasons why one might need to employ Lagrange's equations with constrained generalized coordinates, and then goes on to present solution methods for the differential equations of such systems. The treatment has been expanded considerably. Several numerical algorithms for solving such equations of motion consistent with constraint equations are examined. The algorithms are worked through carefully, and their relative merits are discussed. The objective here is to prepare the reader to handle situations involving nonholonomic constraints, friction, and geometrical complexity, all of which are at the forefront of contemporary research and practice. The computed results for the example of unsteady rolling of a disk should be of interest to all. I am not sure that the nature of this solution has been recognized previously.

Chapter 9 is a treatment of alternative formulations of equations of motion. Discussion of alternative analytical approaches to formulating equations of motion was reserved for this chapter because I believe it is best to begin by providing a set of tools that can be employed reliably, even though they may not be optimal for any one situation. The reader who has reached the later chapters will have the level of capability required to appreciate the availability of alternatives. Much of the material in this chapter did not appear in *Advanced Engineering Dynamics*. It begins with an introduction to calculus of variations in conjunction with Hamilton's principle to derive equations of motion for continua. Usage of variational principles to formulate approximate solutions of field equations was pioneered by Ritz, whom I hold in high esteem. Although the topic is tangential to a course in rigid-body dynamics, everyone should recognize that the study of vibratory systems is intimately dependent on classical mechanics. A large part of this chapter is devoted to explorations of the Gibbs–Appell equations and Kane's equations. I have endeavored here to clarify the relationship between these formulations and to give a balanced discussion of their relative merits. Writing Chapters 8 and 9 has increased my esteem for Lagrange's contribution; the discussion explains why.

Chapter 10, which treats gyroscopic effects, has been updated. The discussion of inertial guidance systems has been clarified and modernized. Although the latest guidance technology is less reliant on these concepts, understanding them serves to enhance mastery of dynamics. One of the developments in this chapter that anyone should find interesting is the analysis of the precession of the equinoxes, which appears as an example in Chapter 10 based on the analysis of gravitational torque in Chapter 5.

This project has required more effort than any of my prior books. In part this comes from treating this book as a new work, while simultaneously making sure it retains what was good in the previous version. However, the most time-consuming aspect entailed the selection of additional examples, as well as the modification of examples I used previously. Also, as an instructor, I realized that keeping a course and textbook fresh requires a large number of homework exercises. The instructor who uses this text will find that both the number and variety of homework exercises have been greatly increased.

As I look over the finished manuscript, I am quite satisfied. I believe that I have met the goals that guided me throughout this project. I hope you agree.

Acknowledgments

From a personal standpoint I have found that the most vital prerequisite to writing a technical book is enthusiasm for the project. For this, I thank all the students I have worked with at Georgia Tech, both students in my graduate dynamics courses and the research assistants with whom I have worked. Seeing them progress as they studied with *Advanced Engineering Dynamics* as their guide was rewarding and made me desirous to build on that success. Being able to try out new ideas on them was enormously helpful, and their tolerance as I arranged and rearranged topics is very much appreciated. I thank all of the faculty in the G. W. Woodruff School of Mechanical Engineering at Georgia Tech for tolerating my focus on this project, which often caused me to neglect my usual academic chores. Two of my colleagues warrant special mention. Dr. Aldo Ferri is exceptionally knowledgeable and insightful. If I had some idea or approach that I was not certain would be suitable, he was the first person whose counsel I would seek. I especially valued his willingness to bring me back to reality when it was warranted to do so. I also very much appreciate John Papastavridis. He is a remarkable resource for anyone who is concerned with the history of mechanics or who needs to find a good entry to some of the less-explored areas of dynamics.

Many faculty at other institutions have approached me in a variety of venues. There is no way to replace the gratification that came from their compliments for *Advanced Engineering Dynamics.* I am sure that my desire to sustain such gratification was responsible in some measure for my decision to redo that work. One person who stands out in particular in this regard is Weidong Zhu at the University of Maryland Baltimore Campus. His suggestions were very useful.

Peter Gordon, my editor at Cambridge University Press, deserves special recognition. If anyone needs a definition of "enthusiasm," they should have a conversation with Peter. I am especially impressed by his overall perspective for the fundamental issues in the field of dynamics, as well as his general understanding of the publishing world. I found our conversations to be quite informative.

Although my colleagues and students were vital to this effort, I could not have pursued it without the support of my family. My grandchildren, Leah Morgan Ginsberg, Elizabeth Rachel Ginsberg, and Abigail Rose Ginsberg, are wondrous in individual ways, and the careful reader will find them mentioned somewhere herein. One of my hopes is that seeing their grandfather pursue a self-guided path to learning will stimulate them to do the same. My son, Mitchell Robert Ginsberg, and his wife, Tracie Sears Ginsberg, amaze me with the diversity of their talents, especially in their parenting skills.

My son, Daniel Brian Ginsberg, and his wife, Jessica Rose, are equally awe-inspiring, especially in their dedication to improving our society. My children and grandchildren show tremendous tolerance for my preoccupation with research, teaching, and writing and gave me much encouragement when I was ill.

Three who are no longer with us were very important to this effort. My mother, Raechel Bass Ginsberg, pushed me to excel. Although I did not always appreciate how important her emphasis was, I now hope that I am equally successful in this aspect relative to my grandchildren. My father, David Ginsberg, set an intellectual example for me that is unmatched. Despite never going beyond the eighth grade, he showed me the joy of intellectual achievement. He learned for the sake of doing so. He voraciously read all subjects, and did so with me in his lap when I was young. His example started me off on this long path. The last of the departed whom I very much miss is my dog Tundra, a magnificent and imposing Alaskan Malamute, who despite her great size was the most gentle animal I have ever encountered. She loved everyone, especially children, and taught me that there is another plane of existence. For her, a long brisk walk was a great reward, and those walks often provided the mechanism by which I changed my perspective when I could not resolve thorny issues in the preparation of this book.

More than anyone else, I am indebted to my wife, Rona Axelrod Ginsberg, who is the foundation of our family. Even after thirty-nine years of marriage, she still surprises me with her talents and expertise. Her love, understanding, support, and dedication sustained me. I sometimes think that I do not adequately convey my love for her, and I fear that my effort here is equally inadequate.

CHAPTER 1

Basic Considerations

Since ancient times many researchers have devoted themselves to predicting and explaining how bodies move under the action of forces. This is the scope of the subject of dynamics, which consists of two phases: kinematics and kinetics. A kinematical analysis entails a quantitative description of the motion of bodies without concern for what is causing the motion. Sometimes that is all that is required, as would be the case if we needed to ascertain the output motion of a gear system or linkage. More significantly, a kinematical analysis will always be a key component of a kinetics study, which analyzes the interplay between forces and motion. Indeed, we will see that the kinematical description provides the skeleton on which the laws of kinetics are applied.

A primary objective will be the development of procedures for applying kinematics and kinetics principles in a logical and consistent manner, so that one may successfully analyze systems that have novel features. Particular emphasis will be placed on three-dimensional systems, some of which feature phenomena that are counterintuitive for those whose experience is limited to systems that move in a plane. A related objective is development of the capability to address realistic situations encountered in current engineering practice.

The scope of this text is limited to situations that are accurately described by the classical laws of physics. The only kinetics laws we will take to be axiomatic are those of Newton, which are accurate whenever the object of interest is moving much more slowly than the speed of light. Newton's Laws pertain only to a particle. The derivation of a variety of principles that extend these laws to bodies having significant dimensions will be treated in depth. We will limit our attention to systems in which all bodies may be considered to be particles or rigid bodies. The dynamics of flexible bodies, which is the subject of vibrations, is founded on the kinematics and kinetics concepts we will establish. We shall begin by reviewing the fundamental aspects of Newton's Laws. Although the reader is likely to have already studied these concepts, the intent is to provide a consistent foundation for later developments.

1.1 VECTOR OPERATIONS

1.1.1 Algebra and Computations

Almost every quantity of importance in dynamics is vectorial in nature. Such quantities have a direction in which they are oriented, as well as a magnitude. The kinematical vectors of primary importance for our initial studies are position, velocity, and acceleration, and the kinetics quantities are force and moment. Some quantities have magnitude and direction, but are not vectors. One example, which will play a major role in Chapter 3, is a finite rotation about an axis. An additional requirement for vector quantities is that they add according to the parallelogram law. This entails a graphical representation of vectors in which an arrow indicates the direction of the vector and the length of the arrow is proportional to the magnitude of the vector. A graphical representation of the summation operation is shown in Fig. 1.1(a), which shows that the addition of two vectors $\bar{A}$ and $\bar{B}$ may be constructed in either of two ways. Vectors $\bar{A}$ and $\bar{B}$ may be placed tail to tail, and then considered to form two sides of a parallelogram. Then $\bar{A}+\bar{B}$ is the main diagonal, with the sense defined to be from the common tail to the opposite corner. An alternative picture places the tail of $\bar{B}$ at the head of $\bar{A}$. The sum then extends from the tail of $\bar{A}$ to the head of $\bar{B}$.

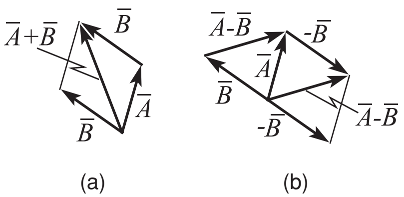


Figure 1.1. Diagrammatical construction of the sum and difference of two vectors.

An important aspect of these constructions is that a sum is independent of the sequence in which the vectors are added. This is the commutative property, which is stated as

$$\bar{A}+\bar{B}=\bar{B}+\bar{A}. \tag{1.1.1}$$

A diagram showing the sum of three vectors leads to the associative property,

$$\left(\bar{A}+\bar{B}\right)+\bar{C}=\bar{A}+\left(\bar{B}+\bar{C}\right). \tag{1.1.2}$$

Another important property comes from the observation that multiplying a vector by a scalar number does not affect its direction, but the magnitude is multiplied by that factor's absolute value, that is,

$$\left|\gamma\bar{A}\right|=|\gamma|\left|\bar{A}\right|. \tag{1.1.3}$$

A corollary of this property is that multiplying $\bar{A}+\bar{B}$ in Fig. 1.1(a) by a scalar changes the length of the diagonal, which requires that the individual sides be scaled by the same factor. Thus,

$$\gamma\left(\bar{A}+\bar{B}\right)=\gamma\bar{A}+\gamma\bar{B}, \tag{1.1.4}$$

which is the distributive property for vector addition.

If the γ factor in Eq. (1.1.3) is negative, $\gamma\bar{A}$ will be parallel to $\bar{A}$, but in the opposite sense. This observation leads to graphical rules for subtracting vectors. Multiplying a vector by -1 only reverses the sense of the vector. Because $\bar{A}-\bar{B}\equiv A+\left(-\bar{B}\right)$, the difference of two vectors may be constructed in one of three ways, as depicted graphically in Fig. 1.1(b). The difference may be formed by placing $\bar{A}$ and $-\bar{B}$ tail to tail, which forms a parallelogram. Then $\bar{A} - \bar{B}$ extends from the common tail to the opposite corner. A different rule leading to the same result comes from the observation that the parallelogram in Fig. 1.1(b) is identical to the one in Fig. 1.11(a). Thus the difference may be formed by placing $\bar{A}$ and $\bar{B}$ tail to tail, so that $\bar{A} - \bar{B}$ extends from the tip of $\bar{B}$ to the tip of $\bar{A}$. The third construction forms $\bar{A} - \bar{B}$ by placing the tail of $-B$ at the head of $\bar{A}$, in which case $\bar{A} - \bar{B}$ extends from the tail of $\bar{A}$ to the tip of $-\bar{B}$. Regardless of how one goes about forming the difference, it is wise to verify that forming $\bar{B} + (\bar{A} - \bar{B})$ actually gives $\bar{A}$.

We will occasionally employ a diagrammatic approach to vector operations for derivations, but it is awkward and not easily implemented in mathematical software, especially for three-dimensional situations. Representation of vectorial quantities in component form addresses these issues. Let xyz denote a set of orthogonal Cartesian coordinates. Unit vectors $\bar{i}$, $\bar{j}$, and $\bar{k}$, whose magnitude is unity without dimensionality, are defined to be parallel to the x, y, and z axes, respectively. To represent its components, vector $\bar{A}$ in Fig. 1.2 has been situated with its tail at the origin of xyz.

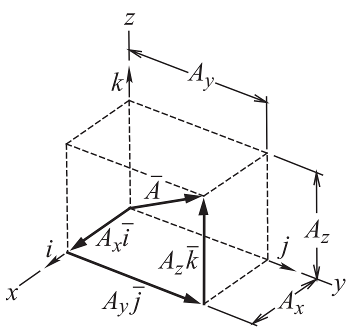


Figure 1.2. Unit vectors of a Cartesian coordinate system and the construction of vector components.

The edges of the box in the figure are constructed from the three lines that are perpendicular to a coordinate plane and intersect the tip of $\bar{A}$. The length of each line is the component of the vector, denoted with the subscript of the associated axis. (The length of a side would be the negative of the corresponding component's value if that side projected onto the negative coordinate axis.) Figure 1.2 shows that a vector along each edge of the box may be constructed by multiplying the component by the corresponding unit vector; see Eq. (1.1.3). The three such vectors depicted in the figure are situated head to tail, so their sum extends from the tail of the first, $A_x\bar{i}$, to the head of the third, $A_z\bar{k}$, but that is the original vector $\bar{A}$. Hence,

$$\boxed{\bar{A} = A_x\bar{i} + A_y\bar{j} + A_z\bar{k}.} \tag{1.1.5}$$

This is the *component representation of a vector*.

The utility of a component representation is that operations can be performed on the individual components without recourse to diagrams. By the Pythagorean theorem the magnitude of $\bar{A}$ is

$$|\bar{A}| = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}. \tag{1.1.6}$$

In many situations we need to construct a unit vector parallel to a vector. This is readily obtained from the preceding equation as

$$\boxed{\bar{A} = \left|\bar{A}\right| \bar{e}_A \iff \bar{e}_A = \frac{\bar{A}}{\left|\bar{A}\right|}.} \tag{1.1.7}$$

The operations of adding or subtracting vectors are performed by operating on the individual components in accord with the properties in Eqs. (1.1.2) and (1.1.4):

$$\begin{aligned}\bar{A} \pm \bar{B} &= \left(A_x\bar{i} + A_y\bar{j} + A_z\bar{k}\right) \pm \left(B_x\bar{i} + B_y\bar{j} + B_z\bar{k}\right) \\ &= \left(A_x\bar{i} \pm B_x\bar{i}\right) + \left(A_y\bar{j} \pm B_y\bar{j}\right) + \left(A_z\bar{k} \pm B_z\bar{k}\right),\end{aligned}$$

$$\boxed{\bar{A} \pm \bar{B} = (A_x \pm B_x)\,\bar{i} + (A_y \pm B_y)\,\bar{j} + (A_z \pm B_z)\,\bar{k}.} \tag{1.1.8}$$

There are two types of products of two vectors. The *dot product* is also known as the *scalar product* because it is a scalar result. It is defined in terms of the angle ϕ between the vectors when they are placed tail to tail, according to

$$\bar{A} \cdot \bar{B} \equiv \left|\bar{A}\right| \left|\bar{B}\right| \cos\phi. \tag{1.1.9}$$

To avoid ambiguity, we limit the angle to $0 \le \phi \le \pi$. It is clear from this definition that the order in which a product is taken does not affect the result, so a dot product is commutative:

$$\bar{A} \cdot \bar{B} = \bar{B} \cdot \bar{A}. \tag{1.1.10}$$

One of the reasons why a dot product is useful is described by Fig. 1.3, where $\left|\bar{B}\right| \cos\phi$ is shown to be the projection of $\bar{B}$ in the direction of $\bar{A}$, in other words, the component of $\bar{B}$ in the direction of $\bar{A}$. That figure also shows that $\left|\bar{A}\right| \cos\phi$ is the component of $\bar{A}$ in the direction of $\bar{B}$. Thus a dot product may be interpreted to be the magnitude of one vector multiplied by the parallel component of the other vector. In the event where they form an obtuse angle, $\pi/2 < \phi \le \pi$, the dot product will be negative, meaning that the component is in the opposite sense from the vector on which it is projected.

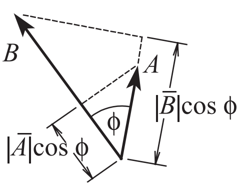


Figure 1.3. Dot product of two vectors, showing the component of each vector parallel to the other.

A dot product can be proven to be distributive, which may be stated as

$$\left(\alpha\bar{A} + \beta\bar{B}\right) \cdot \bar{C} = \alpha\bar{A} \cdot \bar{C} + \beta\bar{B} \cdot \bar{C}. \tag{1.1.11}$$

The significance of this property is that it enables us to evaluate a dot product directly in terms of the components of each vector. This comes about from the fact that $\bar{i}$, $\bar{j}$, and $\bar{k}$ are mutually orthogonal unit vectors, so that

$$\begin{gathered}\bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1, \\ \bar{i} \cdot \bar{j} = \bar{j} \cdot \bar{i} = \bar{j} \cdot \bar{k} = \bar{k} \cdot \bar{j} = \bar{k} \cdot \bar{i} = \bar{i} \cdot \bar{k} = 0.\end{gathered} \tag{1.1.12}$$

Combining these fundamental dot products with Eq. (1.1.11) leads to evaluation of a dot product according to

$$\begin{aligned}\bar{A}\cdot\bar{B} &= \left(A_x\bar{i}+A_y\bar{j}+A_z\bar{k}\right)\cdot\left(B_x\bar{i}+B_y\bar{j}+B_z\bar{k}\right)\\ &= \left(A_x\bar{i}\right)\cdot\left(B_x\bar{i}\right)+\left(A_x\bar{i}\right)\cdot\left(B_y\bar{j}\right)+\left(A_x\bar{i}\right)\cdot\left(B_z\bar{k}\right)\\ &\quad+\left(A_y\bar{j}\right)\cdot\left(B_x\bar{i}\right)+\left(A_y\bar{j}\right)\cdot\left(B_y\bar{j}\right)+\left(A_y\bar{j}\right)\cdot\left(B_z\bar{k}\right)\\ &\quad+\left(A_z\bar{k}\right)\cdot\left(B_x\bar{i}\right)+\left(A_z\bar{k}\right)\cdot\left(B_y\bar{j}\right)+\left(A_z\bar{k}\right)\cdot\left(B_z\bar{k}\right),\end{aligned}$$

$$\boxed{\bar{A}\cdot\bar{B} = A_xB_x + A_yB_y + A_zB_z.} \tag{1.1.13}$$

A useful corollary of the preceding is that the length of a vector may be evaluated from a dot product,

$$\left|\bar{A}\right| = \sqrt{\bar{A}\cdot\bar{A}}. \tag{1.1.14}$$

The *cross product* of two vectors is also known as the *vector product*, because it is defined to be a vector in the direction perpendicular to the plane formed when the vectors are brought tail to tail. The magnitude of a cross product is defined as

$$\left|\bar{A}\times\bar{B}\right| = \left|\bar{A}\right|\left|\bar{B}\right|\sin\phi, \tag{1.1.15}$$

where ϕ is the angle between the vectors, as it is for the dot product. As shown in Fig. 1.4, $\left|\bar{B}\right|\sin\phi$ is the magnitude of the component of $\bar{B}$ perpendicular to $\bar{A}$, and $\left|\bar{A}\right|\sin\phi$ is the component of $\bar{A}$ perpendicular to $\bar{B}$. Thus the magnitude of a cross product may be interpreted as the magnitude of one vector multiplied by the perpendicular component of the other vector. Figure 1.4 also shows that the sense of the cross-product direction is determined by the right-hand rule, in which the vectors are brought tail to tail, and the fingers of the right-hand curl from the first vector to the second, as indicated by the curling arrow. The extended thumb then gives the orientation of the cross product, which would be out of the plane depicted by Fig. 1.4.

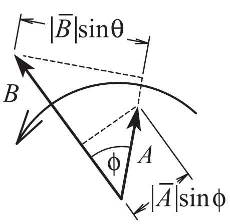


Figure 1.4. Construction of the cross product of two vectors showing the component of one vector perpendicular to the other. The curling arrow indicates the sense in which the fingers of the right hand should curl to form $\bar{A}\times\bar{B}$.

A cross product is not commutative because switching the sequence in which the product is formed reverses the sense of the curling arrow in Fig. 1.4. Thus,

$$\bar{B}\times\bar{A} = -\bar{A}\times\bar{B}. \tag{1.1.16}$$

The cross product does have the associative and distributive properties:

$$\begin{aligned}\left(\bar{A}\times\bar{B}\right)\times\bar{C} &= \bar{A}\times\left(\bar{B}\times\bar{C}\right),\\ \left(\alpha\bar{A}+\beta\bar{B}\right)\times\bar{C} &= \alpha\bar{A}\times\bar{C}+\beta\bar{B}\times\bar{C}.\end{aligned} \tag{1.1.17}$$

These properties lead to a rule for evaluating cross products in terms of vector components. We require that xyz be a right-handed coordinate system, so the fact that the unit vectors of the coordinate are mutually orthogonal gives

$$\begin{gathered}\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = 0, \\ \bar{i} \times \bar{j} = \bar{k}, \quad \bar{j} \times \bar{k} = \bar{i}, \quad \bar{k} \times \bar{i} = \bar{j}, \\ \bar{j} \times \bar{i} = -\bar{k}, \quad \bar{k} \times \bar{j} = -\bar{i}, \quad \bar{i} \times \bar{k} = -\bar{j}.\end{gathered} \tag{1.1.18}$$

A mnemonic device for remembering these products is to consider positive alphabetical order to proceed as $\bar{i}$ to $\bar{j}$ to $\bar{k}$, then back to $\bar{i}$. Applying these cross products in conjunction with the distributive law in Eqs. (1.1.17) leads to

$$\begin{aligned}\bar{A} \times \bar{B} &= \left(A_x\bar{i} + A_y\bar{j} + A_z\bar{k}\right) \times \left(B_x\bar{i} + B_y\bar{j} + B_z\bar{k}\right) \\ &= \left(A_x\bar{i}\right) \times \left(B_y\bar{j}\right) + \left(A_x\bar{i}\right) \times \left(B_z\bar{k}\right) + \left(A_y\bar{j}\right) \times \left(B_x\bar{i}\right) + \left(A_y\bar{j}\right) \times \left(B_z\bar{k}\right) \\ &\quad + \left(A_z\bar{k}\right) \times \left(B_x\bar{i}\right) + \left(A_z\bar{k}\right) \times \left(B_y\bar{j}\right) \\ &= A_xB_y\bar{k} - A_xB_z\bar{j} - A_yB_x\bar{k} + A_yB_z\bar{i} + A_zB_x\bar{j} - A_zB_y\bar{i},\end{aligned}$$

$$\boxed{\bar{A} \times \bar{B} = (A_yB_z - A_zB_y)\,\bar{i} + (A_zB_x - A_xB_z)\,\bar{j} + (A_xB_y - A_yB_x)\,\bar{k}.} \tag{1.1.19}$$

Some individuals, rather than carrying out a cross product term by term, as in the preceding evaluation, use a mnemonic device based on a determinant, specifically,

$$\bar{A} \times \bar{B} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ \bar{i} & \bar{j} & \bar{k} \end{vmatrix}. \tag{1.1.20}$$

A common analytical approach we will encounter entails describing a vector in different ways and then equating the different descriptions. A component description of vectors enables us to convert the vector equality to a set of scalar equations, based on the fact that if two vectors are equal their like components must be equal. Thus,

$$\boxed{\bar{A} = \bar{B} \iff A_x = B_x, \quad A_y = B_y, \text{ and } A_z = B_z.} \tag{1.1.21}$$

Position vectors are the fundamental kinematical quantities. In Fig. 1.5 the position vector extending from origin O to point P is labeled $\bar{r}_{P/O}$, which should be read as *the position vector to P from O*, or equivalently, *the position of P with respect to O*. Similarly, the position of point P with respect to point A is $\bar{r}_{P/A}$. The tail of $\bar{r}_{P/A}$ is situated at the head of $\bar{r}_{A/O}$, from which it follows that adding these vectors gives the position of point P with respect to point O:

$$\boxed{\bar{r}_{P/O} = \bar{r}_{A/O} + \bar{r}_{P/A}.} \tag{1.1.22}$$

This construction is fundamental to many operations in dynamics.

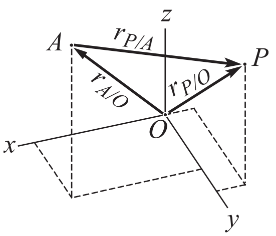


Figure 1.5. Observation of a moving point P by observers at points O and A.

The issue of how one carries out algebraic operations with vectors requires consideration of mathematical software. Three-dimensional vectors may be represented as matrices, which is the preferred data format for such popular programs as Matlab and Mathcad. Both programs allow one to carry out all vector operations using matrix notation. In Mathcad one proceeds by writing all vectors in matrix form and then carrying out the operations as indicated. For example, if $\bar{A} = 1\bar{i} + 2\bar{j} + 3\bar{k}$ and $\bar{B} = 3\bar{i} - \bar{j} - 5\bar{k}$, then the operation of constructing a unit vector parallel to $\bar{A} \times \bar{B}$, then verifying that this product is indeed perpendicular to $\bar{A}$ and $\bar{B}$, could proceed as

$$A := \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}, \quad B := \begin{Bmatrix} 3 \\ -1 \\ -5 \end{Bmatrix}, \quad C := A \times B, \quad e := \frac{C}{|C|}, \quad A_1 = A * e, \quad B_1 = B * e, \tag{1.1.23}$$

where := denotes Mathcad's equality operator, which is obtained by pressing the colon key, and the cross-product operator is obtained from the Ctrl-8 key combination. The dot product in matrix notation is obtained from the product of a three-element row matrix and a three-element column matrix, so one could evaluate the dot product in Mathcad by writing $A^{\mathrm{T}} * B$. An alternative is to simply multiply vectors to form a dot product, as was just done, which returns a scalar value.

Matlab proceeds similarly. The cross product is implemented with the "cross" function; a dot product can be obtained from the "dot" function, or more simply as a standard row–column product. Thus, the preceding example could be carried out in Matlab as

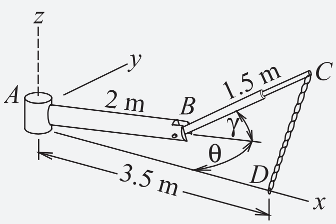
```
A=[1  2  3]; B=[3  -1  -5]; C=cross(A,B);

    e=C/norm(C), A_1=A *e'; B_1=B*e';
```

Note that the "norm" without other arguments is Matlab's function for evaluating the (Euclidean) length of a vector. If one wishes, the preceding operations could be carried out with A and B defined to be three-element columns, for example, $A = [1;\ 2;\ 3]$. Other mathematical software programs have similar capabilities. Also, it is possible to implement these operations symbolically in some programs by use of matrix notation.

Ultimately, how one carries out computations is a matter of personal choice. The procedure used in this text generally will implement the operations term by term using the associative and distributive properties. One reason for this choice is that the notation is somewhat more compact. The second has to do with a common situation that will frequently arise, in which it will be necessary to combine vectors that are defined in terms of components relative to different coordinate systems. It is awkward to indicate which coordinate system a matrix is associated with, whereas the symbols used for unit vectors display that information unambiguously.

EXAMPLE 1.1 Robotic arm ABC induces a tensile force of 5000 N in cable CD. The orientation angles are $\theta = 25^\circ$ for link AB, which lies in the horizontal plane, and $\gamma = 40^\circ$ for rotation of arm BC. Pin B for this rotation is horizontal and perpendicular to arm AB, so AB and BC lie in a common vertical plane. Let $\bar{F}$ denote the force the cable exerts on the robotic arm. Determine (a) the component of $\bar{F}$ parallel to link BC, (b) the moment of $\bar{F}$ about end A, (c) the moment of $\bar{F}$ about the vertical z axis, and (d) the moment of $\bar{F}$ about arm AB.



Example 1.1

SOLUTION This example reviews some basic evaluations of force properties, which call for most of the standard vector operations. The cable is in tension, so it pulls the robotic arm from point C to point D. We express this as $\bar{F} = 5000\bar{e}_{D/C}$ N, where $\bar{e}_{D/C}$ is the notation we use for the *unit vector to D from C*. The first task is to determine the coordinates of point C, which we can find by constructing position vectors along arms AB and BC. We project point B onto the x and y axes to evaluate $\bar{r}_{B/A}$. Similarly, we project point C onto the xy plane, and then project that point onto the x and y axes. This gives

$$\bar{r}_{B/A} = 2\left(\cos\theta\bar{i} + \sin\theta\bar{j}\right) = 1.8126\bar{i} + 0.8452\bar{j} \text{ m},$$

$$\bar{r}_{C/B} = 1.5\cos\gamma\left(\cos\theta\bar{i} + \sin\theta\bar{j}\right) + 1.5\sin\gamma\bar{k} = 1.0414\bar{i} + 0.4856\bar{j} + 0.9642\bar{k}.$$

The desired position vector is the sum of these vectors:

$$\begin{aligned}\bar{r}_{C/A} &= \bar{r}_{B/A} + \bar{r}_{C/B} = (1.8126 + 1.0414)\,\bar{i} + (0.8452 + 0.4856)\,\bar{j} + (0 + 0.9642)\,\bar{k} \\ &= 2.8540\bar{i} + 1.3309\bar{j} + 0.9642\bar{k} \text{ m}.\end{aligned}$$

Because $\bar{r}_{C/A}$ and $\bar{r}_{D/A} = 3.5\bar{i}$ m are tail to tail, it follows that $\bar{r}_{D/C} = \bar{r}_{D/A} - \bar{r}_{C/A}$, which leads to $\bar{e}_{D/C}$ according to

$$\begin{aligned}\bar{e}_{D/C} &= \frac{\bar{r}_{D/A} - \bar{r}_{C/A}}{\left|\bar{r}_{D/A} - \bar{r}_{C/A}\right|} = \frac{0.6460\bar{i} - 1.3309\bar{j} - 0.9642\bar{k}}{(0.6460^2 + 1.3309^2 + 0.9642^2)^{1/2}} \\ &= 0.3658\bar{i} - 0.7537\bar{j} - 0.5460\bar{k}.\end{aligned}$$

Thus the force applied to the arm is

$$\bar{F} = 5000\bar{e}_{D/C} = 1829\bar{i} - 3768\bar{j} - 2730\bar{k} \text{ N}.$$

The component of $\bar{F}$ parallel to arm BC may be obtained from a dot product with the unit vector $\bar{e}_{C/B}$, which is readily constructed from $\bar{r}_{C/B}$, whose value has already been determined. Thus,

$$\begin{aligned} F_{BC} &= \bar{F} \cdot \bar{e}_{C/B} = \bar{F} \cdot \frac{\bar{r}_{C/B}}{\left|\bar{r}_{C/B}\right|} = \bar{F} \cdot \left(0.6943\bar{i} + 0.3237\bar{j} + 0.6428\bar{k}\right) \\ &= (1829)(0.6943) + (-3768)(0.3237) + (-2730)(0.6428) = -1705 \text{ N}. \end{aligned} \quad \triangleleft$$

Negative F_{BC} indicates that the projection of $\bar{F}$ onto line BC is opposite the sense of $\bar{e}_{C/B}$.

The moment of a force may be evaluated from a cross product with a position vector from the reference point for the moment to the point where the force is applied. Hence,

$$\begin{aligned} \bar{M}_A &= \bar{r}_{C/A} \times \bar{F} = \left(2.8540\bar{i} + 1.3309\bar{j} + 0.9642\bar{k}\right) \times \left(1829\bar{i} - 3768\bar{j} - 2730\bar{k}\right) \\ &= (2.8540)(-3768)\,\bar{k} + (2.8540)(-2730)\left(-\bar{j}\right) + (1.3309)(1829)\left(-\bar{k}\right) \\ &\quad + (1.3309)(-2730)\,\bar{i} + (0.9642)(1829)\,\bar{j} + (0.9642)(-3768)\left(-\bar{i}\right) \\ &= 9555\bar{j} - 13189\bar{k} \text{ N-m}. \end{aligned} \quad \triangleleft$$

The moment of a force about an axis may be determined by forming the moment about any point on that axis, and then evaluating the component of that moment in the direction of the axis. Thus the moment of $\bar{F}$ about the z axis is merely the $\bar{k}$ component of $\bar{M}_A$,

$$M_{Az} = \bar{M}_A \cdot \bar{k} = -13189 \text{ N-m}. \quad \triangleleft$$

A negative value indicates that the sense of this moment is determined by aligning the extended thumb of the right hand in the $-\bar{k}$ direction. The same reasoning shows that the moment of $\bar{F}$ about arm AB is obtained with a dot product involving $\bar{e}_{B/A}$,

$$M_{AB} = \bar{M}_A \cdot \bar{e}_{B/A} = \bar{M}_A \cdot \frac{\bar{r}_{B/A}}{\left|\bar{r}_{B/A}\right|} = 4038 \text{ N-m}. \quad \triangleleft$$

1.1.2 Vector Calculus—Velocity and Acceleration

The primary kinematical variables for our initial studies are position, velocity, and acceleration. Velocity is defined to be the time derivative of position, and acceleration is the time derivative of velocity, so we need to establish how to handle derivatives of vectors. Because time derivatives are performed frequently, it is standard notation to use an overdot to denote each such operation. Overbars are used here to indicate that a quantity is a vector; the reader is encouraged to use the same notation.

Most of the laws for calculus operations are the same as those for scalar variables. Their adaptation requires that vector quantities be indicated unambiguously. In the following, $\bar{A}$ and $\bar{B}$ are time-dependent vector functions, and α and β are scalar functions of time.

Definition of a derivative:

$$\frac{d\bar{A}}{dt} \equiv \dot{\bar{A}} \equiv \lim_{\Delta t \to 0} \frac{\bar{A}(t+\Delta t) - \bar{A}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\bar{A}(t) - \bar{A}(t-\Delta t)}{\Delta t}. \tag{1.1.24}$$

Definite integration:

$$\text{If } \dot{\bar{A}} = \bar{B}, \text{ then } \bar{B}(t) = B(t_0) + \int_{t_0}^{t} A(\tau)\, d\tau. \tag{1.1.25}$$

Derivative of a sum:

$$\frac{d}{dt}\left(\bar{A} + \bar{B}\right) = \dot{\bar{A}} + \dot{\bar{B}}. \tag{1.1.26}$$

Derivative of products:

$$\begin{gathered} \frac{d}{dt}\left(\alpha \bar{A}\right) = \dot{\alpha}\bar{A} + \alpha\dot{\bar{A}}, \\ \frac{d}{dt}\left(\bar{A}\cdot\bar{B}\right) = \dot{\bar{A}}\cdot\bar{B} + \bar{A}\cdot\dot{\bar{B}}, \\ \frac{d}{dt}\left(\bar{A}\times\bar{B}\right) = \dot{\bar{A}}\times\bar{B} + \bar{A}\times\dot{\bar{B}}. \end{gathered} \tag{1.1.27}$$

As an immediate consequence of these properties, all calculus operations may be performed in terms of vector components. We consider here only situations in which xyz is a fixed coordinate system, so that $\bar{i}$, $\bar{j}$, and $\bar{k}$ are constant vectors, which means that $d\bar{i}/dt = d\bar{j}/dt = d\bar{k}/dt = 0$. We then find that

$$\frac{d\bar{A}}{dt} = \frac{d}{dt}\left(A_x\bar{i} + A_y\bar{j} + A_z\bar{k}\right) = \dot{A}_x\bar{i} + \dot{A}_y\bar{j} + \dot{A}_z\bar{k}. \tag{1.1.28}$$

A common situation that arises in many phases of our study of kinematics involves a vector that depends on some parameter α, which in turn varies with time. Differentiation of the vector with respect to time in this circumstance can be performed with the chain rule:

$$\frac{d\bar{A}}{dt} = \frac{d\bar{A}}{d\alpha}\frac{d\alpha}{dt} \equiv \dot{\alpha}\frac{d\bar{A}}{d\alpha}. \tag{1.1.29}$$

The chain rule may be extended by partial differentiation to situations in which the vector depends on two or more time-dependent parameters, according to

$$\frac{d\bar{A}}{dt} = \dot{\alpha}\frac{\partial\bar{A}}{\partial\alpha} + \dot{\beta}\frac{\partial\bar{A}}{\partial\beta} + \cdots. \tag{1.1.30}$$

In the present notation, where $\bar{r}_{P/O}$ denotes the position of point P with respect to the origin O of a fixed coordinate system, then the velocity and acceleration of that point are

$$\boxed{\bar{v} \equiv \dot{\bar{r}}_{P/O}, \quad \bar{a} \equiv \dot{\bar{v}} = \ddot{\bar{r}}_{P/O}.} \tag{1.1.31}$$

It should be noted that no subscripts have been used to denote the velocity and acceleration vectors. If there is any ambiguity as to the point whose velocity is under consideration, the notation will be $\bar{v}_P$ and $\bar{a}_P$, but even then there is no need to indicate in the velocity and acceleration what the origin is. This is so because different fixed observers all see the same motion. This may be proved from Fig. 1.5, where $\bar{r}_{P/O}$ is the position seen by an observer at the origin, and $\bar{r}_{P/A}$ is the position of point P as seen by an observer at point A. Equation (1.1.22) describes $\bar{r}_{P/O}$ as the sum of the other two vectors. If point A is stationary, then $\bar{r}_{A/O}$ is constant and $d\bar{r}_{A/O}/dt = \bar{0}$. It follows that

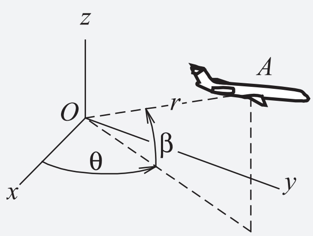
$$\bar{v}_P = \frac{d}{dt}\bar{r}_{P/O} = \frac{d}{dt}\left(\bar{r}_{A/O} + \bar{r}_{P/A}\right) = \frac{d}{dt}\bar{r}_{P/A}, \tag{1.1.32}$$

which shows that the velocity of a point is the derivative of the position vector to that point from any fixed point. The same must be true for acceleration because it is the derivative of the velocity.

In Chapter 3 we will treat situations in which the reference frame moves, in which case we will be interested in the motion relative to that reference frame. Equation (1.1.32) defines the *absolute velocity*, whereas the velocity seen by a moving observer is a *relative velocity*. The same terminology applies to the description of acceleration. If it is not specified otherwise, the words velocity and acceleration should be understood to mean absolute quantities.

Our initial studies are limited to situations in which the moving body may be considered to be a particle. By definition, a particle occupies only a single point in space. As time evolves, the particle will occupy a succession of positions. The locus of all positions occupied by the point is its *path.* One can obtain a visual representation of a path by illuminating the point and then taking a long-exposure photograph. Position, movement along a path, and velocity are inherently important because we can readily sense them. Acceleration is difficult for most individuals to observe without instrumentation. On the other hand, if we are subjected to an acceleration, our sensory system gives us an indication of its magnitude and direction primarily based on the internal forces that are generated. The time derivative of $\bar{a}$, which is called the *jerk*, primarily occurs in considerations of ride comfort for vehicles.

EXAMPLE 1.2 A radar station on the ground at origin O tracks airplane A by measuring the distance r, the angle θ in the horizontal xy plane, and the angle of elevation β. It is observed that these variables closely fit $r(t) = 2000 + 100t$ m, $\theta(t) = \pi/2\,[1 - \exp(-0.15t)]$ rad, and $\beta = \pi/3 - 0.1\sqrt{t}$ rad, where t is measured in seconds. (a) Determine the velocity of the airplane at $t = 2$ s according to a finite central difference approximation based on the change of the position vector $\bar{r}$ during an interval of 1 ms. (b) Determine the velocity of the airplane at $t = 2$ s by differentiating the xyz components of $\bar{r}$.



Example 1.2

SOLUTION The analysis brings to the fore many of the basic vector operations, but a more efficient evaluation of the velocity would use the spherical coordinate formulation we will develop in the next chapter. Both specified solution procedures require a description of the position vector. The projection of line OA onto the z axis is the the length of the vertical dashed line. We obtain the x and y components by projecting line OA onto the xy plane and then projecting that line onto the x and y axes. The result is that

$$\bar{r}\,(t) = (r\cos\beta\cos\theta)\,\bar{i} + (r\cos\beta\sin\theta)\,\bar{j} + (r\sin\beta)\,\bar{k}. \tag{1}$$

A central difference at $t = 2$ s covering a 1-ms interval is formed from $t = 2 \pm 0.0005$ s, so the velocity in this approximation is

$$\begin{aligned}\bar{v}\,(t) &\approx \frac{\bar{r}\,(2+0.0005) - \bar{r}\,(2-0.0005)}{0.001} \\ &= 1000[(1246.61294\bar{i} + 537.68542\bar{j} + 1731.20494\bar{k}) \\ &\quad - (1246.59391\bar{i} + 537.41916\bar{j} + 1731.17425\bar{k})],\end{aligned}$$

$$\bar{v}\,(t) = 19.0\bar{i} + 266.3\bar{j} + 30.7\bar{k}\ \text{m/s}. \tag{2} \triangleleft$$

To differentiate the position analytically, we recognize that the representation of $\bar{r}$ in Eq. (1) gives it as a function of $r(t)$, $\theta\,(t)$, and $\phi\,(t)$. Thus we employ the chain rule, which gives

$$\begin{aligned}\bar{v}\,(t) &= \dot{r}\frac{\partial\bar{r}}{\partial r} + \dot{\theta}\frac{\partial\bar{r}}{\partial\theta} + \dot{\beta}\frac{\partial\bar{r}}{\partial\beta} \\ &= \dot{r}\left[(\cos\beta\cos\theta)\,\bar{i} + (\cos\beta\sin\theta)\,\bar{j} + (\sin\beta)\,\bar{k}\right] \\ &\quad + \dot{\theta}\left[-(r\cos\beta\sin\theta)\,\bar{i} + (r\cos\beta\cos\theta)\,\bar{j}\right] \\ &\quad + \dot{\beta}\left[-(r\sin\beta\cos\theta)\,\bar{i} - (r\sin\beta\sin\theta)\,\bar{j} + (r\cos\beta)\,\bar{k}\right].\end{aligned} \tag{3}$$

At $t = 2$ s we have

$$r = 2200\ \text{m},\ \ \dot{r} = 100\ \text{m/s},$$

$$\theta = 0.40712\ \text{rad},\ \ \dot{\theta} = (\pi/2)\,(0.15)\exp(-0.15t) = 0.17455\ \text{rad/s},$$

$$\beta = 0.90578\ \text{rad},\ \ \dot{\beta} = -0.05/\sqrt{t} = -0.03536\ \text{rad/s}.$$

Substitution of these values into Eq. (3) gives the same result as that of Eqs. (2). The results of both analyses actually agree to within $4\left(10^{-7}\right)$ % if intermediate results are not truncated as the computations proceeded.

In other cases, the degree to which the two analyses would agree would depend on the duration of the interval for the finite difference, as well as the nature of the functions describing r, θ, and β. Interestingly, modern data acquisition systems are digital, so the data taken by a radar station are sampled discrete values. Any functions representing $r(t)$, $\theta(t)$, and $\beta(t)$ are likely in that situation to be approximations based on curve-fitting procedures. In such situations, it would be incorrect to decide that the finite difference approximation gives a less precise result than one obtained by analytical differentiation.

1.2 NEWTONIAN MECHANICS

A fundamental aspect of any kinetics laws is the reference frame from which the motion is observed. A reference frame will be depicted as a set of coordinate axes, such as xyz, with an additional specification of the body to which the axes are attached. However, it is important to realize that coordinate axes are also often used to represent the directions for the component description of vectorial quantities. The two uses for a coordinate system are not necessarily related. Indeed, we frequently describe a kinematical quantity relative to a specified frame of reference in terms of its components along the coordinate axes associated with a different frame of reference.

1.2.1 Newton's Laws

The kinetics laws associated with Sir Isaac Newton are founded on the concept of an absolute reference frame, which implies that somewhere in the universe there is an object that is stationary. This concept is abandoned in relativity theory, but the notion of a fixed reference frame introduces negligible errors for objects that move slowly in comparison with the speed of light. The corollary of this concept is the dilemma of what object should be considered to be fixed. Once again, considering the Sun to be fixed usually is sufficiently accurate. However, in most engineering situations it is preferable to use the Earth as our reference frame. The primary effect of the Earth's motion in most cases is to modify the (in vacuo) free-fall acceleration g resulting from the gravitational attraction between an object and the Earth. Other than that effect, it is usually permissible to consider the Earth to be an absolute reference frame. (A more careful treatment of the effects of the Earth's motion will be part of our study in Chapter 3 of motion relative to a moving reference frame.)

A remarkable feature of Newton's Laws is that they address only objects that can be modeled as a particle. Kinetics laws governing a body having finite dimensions were derived from Newton's Laws by considering a body to be a collection of particles. Thus Newton's Laws are fundamental to all aspects of our work. At the same time, we should recognize that these are axiomatic to our studies, as they are based on experimental observation without analytical proof. Indeed, relativity theory can be shown to reduce to Newton's Laws for bodies that move very slowly.

Newton's Laws have been translated in a variety of ways from their original statement in the *Principia* (1687), which was in Latin. We use the following version:

> FIRST LAW: The velocity of a particle can only be changed by the application of a force.
>
> SECOND LAW: The resultant force (that is, the sum of all forces) acting on a particle is proportional to the acceleration of the particle. The factor of proportionality is the mass.
>
> $$\boxed{\Sigma \bar{F} = m\bar{a}.} \tag{1.2.1}$$
>
> THIRD LAW: All forces acting on a body result from an interaction with another body, such that there is another force, called a *reaction*, applied to the other body. The action–reaction pair consists of forces having the same magnitude, and acting along the same line of action, but having opposite direction.

We realize that the First Law is included in the Second, but we retain it primarily because it treats systems in static equilibrium without the need to discuss acceleration. A number of individuals recognized this law prior to Newton. The Second Law is quite familiar, but it must be emphasized that it is a vector relation. Hence it can be decomposed into as many as three scalar laws, one for each component. The Third Law is very important to the modeling of systems. The models that are created in a kinetics study are free-body diagrams, in which the system is isolated from its surroundings. Careful application of the Third Law will assist identification of the forces exerted on the body.

The conceptualization of the First and Second Laws can be traced back to Galileo. Newton's revolutionary idea was the recognition of the Third Law and its implications for the first two. An interesting aspect of the Third Law is that it excludes the concept of an inertial force, $-m\bar{a}$, which is usually associated with d'Alembert, because there is no corresponding reactive body. We will address the inertial force concept in Chapter 7.

It is also worth noting that the class of forces described by the Third Law is limited – any force obeying this law is said to be a *central force*. An example of a noncentral force arises from the interaction between moving electric charges. Such forces have their origin in relativistic effects. Strictly speaking, the study of classical mechanics is concerned only with systems that fully satisfy all of Newton's laws. However, many of the principles and techniques are applicable either directly, or with comparatively minor modifications, to relativistic systems.

The acceleration employed in Newton's Second Law was stated to be observed from the hypothetical fixed reference frame. However, the same acceleration can also be observed from special moving reference frames. One may recognize this by returning to Fig. 1.5, in which the observer at point A is allowed to move. Differentiating twice the vector sum described by that figure gives

$$\bar{a}_P \equiv \frac{d^2}{dt^2}\bar{r}_{P/O} = \frac{d^2}{dt^2}\bar{r}_{A/O} + \frac{d^2}{dt^2}\bar{r}_{P/A}. \tag{1.2.2}$$

If the observer at point A is to see $\bar{a}_p$, it must be that $d^2\bar{r}_{A/O}/dt^2 = \bar{0}$, which means that $d\bar{r}_{A/O}/dt$ is constant. However, $d\bar{r}_{A/O}/dt$ is defined to be $\bar{v}_A$. Thus Newton's Second Law can be formulated in terms of the acceleration seen by an observer moving at a constant velocity. However, there is a further restriction to this statement. If the coordinate axes of the reference frame do not point in fixed directions, changes in velocity will be associated with the variability of these directions, as well as changes in the velocity components. An *inertial reference frame* is one that translates at a constant velocity. The translation condition, by definition, means that the coordinate axes point in fixed directions, so that we may interpret velocity and acceleration in the same way as we do for a fixed reference frame. The constant-velocity condition requires that both the magnitude and direction of the moving observer's velocity be constant, which we will soon see can be true only if the observer follows a straight path. The fact that Newton's Laws are valid in any inertial reference frame is the principle of *Galilean invariance*, or the principle of *Newtonian relativity*.

1.2.2 Systems of Units

Newton's Second Law brings up the question of the units to be used for describing the force and motion variables. Related to that consideration is dimensionality, which refers to the basic measures that are used to form the quantity. In dynamics, the basic measures are time T, length L, mass M, and force F. The law of dimensional homogeneity requires that these four quantities be consistent with the Second Law. Thus

$$\mathrm{F} = \mathrm{ML/T}, \tag{1.2.3}$$

which means that only three of the four basic measures are independent. Measures for time and length are easily defined, so this leaves the question of whether mass or force is the third independent quantity. Whichever unit is not taken as the basic measure is obtained from Eq. (1.2.3). An *absolute set of units* is defined such that the unit of mass is fundamental, whereas a *gravitational set of units* defines force to be the fundamental unit. This latter set of units is said to be "gravitational" because of the relation among the weight w, the mass m, and the free-fall acceleration g.

The only system of units to be employed in this text are SI (Standard International), which is a metric MKS system with standardized prefixes for powers of 10 and standard names for derived units. Newton's law of gravitation states that the magnitude of the attractive force exerted between the Earth and a body of mass* m is

$$F = \frac{GM_e m}{r^2}, \tag{1.2.4}$$

* It is implicit to this development that the inertial mass in Newton's Second Law is the same as the gravitational mass appearing in the law of gravitation. This fundamental assumption, which is known as the *principle of equivalence*, actually is owed to Galileo, who tested the hypothesis with his experiments on various pendulums. More refined experiments performed subsequently have continued to verify the principle.

where r is the distance between the centers of mass, G is the *universal gravitational constant*, and M_e is the mass of the Earth:

$$G = 6.6732(10^{-11})\ \text{m}^3/\text{kg-s}^2, \quad M_e = 5.976(10^{24})\ \text{kg}. \tag{1.2.5}$$

The weight w of a body usually refers to gravity's pull when a body is near the Earth's surface. If a body near the Earth's surface falls freely in a vacuum, its acceleration is g, which according to the Second Law is the ratio of w and m. In view of Eq. (1.2.4), it must be that

$$g = \frac{GM_e}{r_e^2}, \tag{1.2.6}$$

where $r_e = 6371$ km is the radius of the Earth.

The relationship between g and the gravitational pull of the Earth is actually far more complicated than Eq. (1.2.6). In fact, g depends on the location along the Earth's surface. One reason for such variation is the fact that the Earth is not perfectly homogeneous and spherical. In addition to these deviations of the gravitational force, the value of g is influenced by the motion of the Earth, because g is an acceleration measured relative to a noninertial reference frame. (This issue is explored in Section 3.6.) Consequently it is not exactly correct to employ Eq. (1.2.6).

The mass of a particle is constant (assuming no relativistic effects), so an absolute system of units is the same regardless of where they are measured. Prior to adoption of SI as a standard set of absolute units, many individuals used a gravitational metric system, in which grams or kilograms were used to specify the weight of a body. In SI units, where mass is basic, any body should be described in terms of its mass in kilograms. Its weight in newtons (1 N $\equiv$ 1 kg $\times$ 1 m/s = 1 kg-m/s) is mg. If an accurate measurement of g at the specific location on the Earth's surface is not available, one may use an average value

$$g = 9.807\ \text{m/s}. \tag{1.2.7}$$

The system now known as U.S. Customary is another gravitational system. Its basic unit is force, measured in pounds (lb). The body whose weight is defined as a pound must be at a specified location. If that body were to be moved to a different place, the gravitational force acting on it, and hence the unit of force, might be changed. The ambiguity as to a body's weight is one source of confusion in U.S. Customary units. Another stems from early usage of the pound as a mass unit. If one also employs a pound force unit, such that 1 lbf is the weight of a 1-lbm body at the surface of the Earth, then application of the law of dimensional homogeneity to $\bar{F} = m\bar{a}$ requires that acceleration be measured in multiples of g. This is an unnecessary complication that has been abandoned in most scientific work.

Even when one recognizes that mass is a derived unit in the U.S. Customary units, the mass unit is complicated by the fact that two length units, feet and inches, are in common use. Practitioners working in U.S. Customary units use the standard values

$$g = 32.17\ \text{ft/s} \quad \text{or} \quad g = 386.0\ \text{in./s}. \tag{1.2.8}$$

Hence, computing the mass as $m = w/g$ will give a value for m that depends on the length unit in use. The *slug* is a standard name for the U.S. Customary mass unit, with

$$1 \text{ slug} = 1 \text{ lb}/(1\text{ft/s}^2) = 1 \text{ lb-s}^2/\text{ft}. \tag{1.2.9}$$

This mass unit is not applicable when inches is the length unit. To emphasize this matter, it is preferable for anyone using U.S. Customary units to make it a standard practice to give mass in terms of the basic units. For example, a mass might be listed as 5.2 lb-s^2/ft, or a moment of inertia might be 125 lb-s^2-in. The SI system avoids all of these ambiguities.

1.2.3 Energy and Momentum

A basic application of the calculus of vectors in dynamics is the derivation of energy and momentum principles, which are integrals of Newton's Second Law. These integrals represent standard relations between velocity parameters and the properties of the force system. We derive these laws for particle motion here; the corresponding derivations for a rigid body appear in Chapter 6.

Energy principles are useful when we know how the resultant force varies as a function of the particle's position, in other words, when $\Sigma \bar{F}(\bar{r})$ is known. The displacement of a point is intimately associated with energy principles. The definition of a *displacement* is that it is the change in the position occupied by a point at two instants,

$$\Delta \bar{r} = \bar{r}(t + \Delta t) - \bar{r}(t). \tag{1.2.10}$$

To obtain a differential displacement $d\bar{r}$, we let Δt become an infinitesimal interval dt. A dot product of Newton's Second Law with a differential displacement of a particle yields

$$\Sigma \bar{F}(\bar{r}) \cdot d\bar{r} = m\bar{a} \cdot d\bar{r}. \tag{1.2.11}$$

The definition of velocity indicates that $d\bar{r} = \bar{v}dt$. Substitution of this and the definition of $\bar{a}$ into the preceding leads to

$$\Sigma \bar{F}(\bar{r}) \cdot d\bar{r} = m\frac{d\bar{v}}{dt} \cdot \bar{v}dt. \tag{1.2.12}$$

A dot product is commutative, so $(d\bar{v}/dt) \cdot \bar{v} \equiv \bar{v} \cdot (d\bar{v}/dt)$, from which it follows that

$$\Sigma \bar{F}(\bar{r}) \cdot d\bar{r} = m\frac{1}{2}\left[\frac{d}{dt}(\bar{v} \cdot \bar{v})\right] dt \equiv d\left[\frac{1}{2}m(\bar{v} \cdot \bar{v})\right]. \tag{1.2.13}$$

The right side is a perfect differential, whereas the left side is a function of only the position because of the assumed dependence of the force resultant. Hence we may integrate the differential relation between the two positions. The evaluation of the integral of the left side must account for the variation of the resultant force as the position changes when the particle moves along its path; this is a called a *path integral*. We therefore find that

$$\oint_1^2 \Sigma \bar{F}(\bar{r}) \cdot d\bar{r} = \frac{1}{2}m(\bar{v}_2 \cdot \bar{v}_2) - \frac{1}{2}m(\bar{v}_1 \cdot \bar{v}_1), \tag{1.2.14}$$

where "1" and "2" respectively denote the initial position and final position. The *kinetic energy* of a particle is

$$\boxed{T \equiv \frac{1}{2} m \left(\bar{v} \cdot \bar{v} \right) = \frac{1}{2} m \left| \bar{v} \right|^2,} \tag{1.2.15}$$

and the path integral is the *work* done by the resultant force in moving the particle from its initial to final position,

$$\boxed{W_{1\to 2} \equiv \oint_1^2 \Sigma \bar{F}(\bar{r}) \cdot d\bar{r}.} \tag{1.2.16}$$

The subscript notation for W indicates that the work is done in going from the starting position 1 to the final position 2 along the particles's path. Correspondingly, Eq. (1.2.14) may be written as

$$\boxed{T_2 = T_1 + W_{1\to 2},} \tag{1.2.17}$$

which is the *work–energy principle*. It states that the increase in kinetic energy between two positions equals the work that is done.

The operation of evaluating the work is depicted in Fig. 1.6. The angle between the resultant force $\Sigma \bar{F}$ and the infinitesimal displacement $d\bar{r}$ is θ. It follows from the definition of a dot product that the infinitesimal work done by $\Sigma \bar{F}$ in this displacement is $dW = \Sigma \bar{F} \cdot d\bar{r} = \left| \Sigma \bar{F} \right| \left| d\bar{r} \right| \cos\theta$. The figure indicates that the infinitesimal work is the product of the differential distance the particle moves, $\left| d\bar{r} \right|$, and the component of the resultant force in the direction of movement, $\left| \Sigma \bar{F} \right| \cos\theta$, or equivalently, the product of the magnitude of the resultant force, $\left| \Sigma \bar{F} \right|$, and the projection of the displacement in the direction of the force, $\left| d\bar{r} \right| \cos\theta$. Only in the simple case in which the force has a constant component in the direction of the displacement does the work reduce to the simple expression "force multiplied by distance displaced." Otherwise the work must be evaluated as a path integral, meaning that the value of $\Sigma \bar{F} \cdot d\bar{r}$ must be described as a function of the position along the path. The evaluation of the work is a major part of a formulation of the work–energy principle. We will find in Chapter 6 that this task is alleviated by introducing the concept of potential energy.

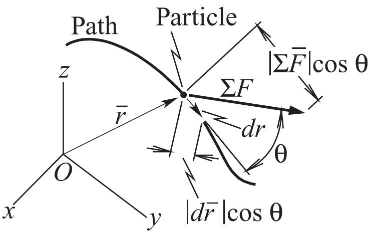


Figure 1.6. Work done by a resultant force $\Sigma \bar{F}$ in displacement $d\bar{r}$ of a particle.

In contrast to the situation covered by the work–energy principle, which is based on knowing how the resultant force depends on the particle's position, momentum principles are intended to address situations in which the resultant force is known as a function of time. Two such principles may be derived from Newton's Second Law. The *linear impulse–momentum principle* is a direct integration of $\Sigma \bar{F} = m\bar{a}$ with respect to time. Because $\bar{a} = d\bar{v}/dt$, multiplying the Second Law by dt and integrating over an interval $t_1 \leq t \leq t_2$ leads to

$$\int_{t_1}^{t_2} \Sigma \bar{F}\, dt = m\left(\bar{v}_2 - \bar{v}_1\right). \tag{1.2.18}$$

The quantity $m\bar{v}$ is the *momentum* of the particle, which we denote by the symbol $\bar{P}$. Thus we have

$$\boxed{\bar{P} = m\bar{v}, \quad \bar{P}_2 = \bar{P}_1 + \int_{t_1}^{t_2} \Sigma \bar{F}\, dt.} \tag{1.2.19}$$

The time integral of the resultant force is the *impulse*. Thus we have derived the *linear impulse–momentum principle,* where the word linear conveys the fact that the principle pertains to movement along a (possibly curved) line. Correspondingly, more precise names for the terms appearing in Eqs. (1.2.19) are the *linear impulse* and *linear momentum.*

This is a vector relation, so taking components in each of the coordinate directions will lead to three scalar equations, although some might be trivial, as in planar motion. There are few situations in which all forces acting on a body are known as functions solely of time, which is required if the impulse is to be evaluated. However, it might happen that the forces acting in a certain direction are known functions of time, in which case the linear impulse–momentum law may be invoked solely for that component.

A primary utility of the linear impulse–momentum principle is to treat systems excited by *impulsive forces*, that is, forces that impart a very large acceleration to a body over a very short time interval. We split the resultant force acting on a particle into two parts: $\bar{F}_{\text{imp}}$ is the resultant of the impulsive forces, and $\bar{F}_{\text{ord}}$ represents ordinary forces whose magnitude is not much larger than mg. The peak magnitude in the impulsive force, $F_{\max} = \max\left(\bar{F}_{\text{imp}}\right)$, must be much greater than the peak magnitude of the regular forces in order for it to qualify as an impulsive force. Because one of the regular forces is gravity, it follows that $F_{\max}/m$ must be much larger than g if a force is to be considered to be impulsive. The velocity at time t_2, when the impulsive force ceases, is related by the linear impulse–momentum principle to the velocity at t_1, when the force first began to act. Specifically,

$$m\bar{v}_2 = m\bar{v}_1 + \int_{t_1}^{t_2} \bar{F}_{\text{imp}} dt + \int_{t_1}^{t_2} \bar{F}_{\text{ord}} dt. \tag{1.2.20}$$

Because the impulsive forces are much bigger than the ordinary forces, we may ignore the impulse of the latter. Furthermore, in many situations, such as predicting the trajectory of an object after a collision, we are not very interested in specific manner in which the velocity changes between t_1 and t_2. This is especially true because of the brevity of this time interval. Both observations lead to the idealized model of an impulsive force, in which it is considered to act over a zero time interval, $t_2 = t_1$, but to still have the same impulse $\bar{G}$ as the actual force. The implication of a finite impulse being obtained over a zero time interval is that the force is infinite. We represent such a force as

$$\bar{F}_{\text{imp}} = \bar{G}\delta\,(t - t_1)\,, \qquad (1.2.21)$$

where $\delta\,(t - t_1)$ is a *Dirac delta function,* which is defined by two basic properties:

$$\delta\,(t - t_1) = 0 \text{ if } t \neq t_1, \quad \int_{t_0}^{t_2} \delta\,(t - t_1)\,dt = 1 \text{ if } t_0 < t_1 < t_2. \qquad (1.2.22)$$

The consequences of using this model to analyze the motion of a particle is that (a) the acceleration is infinite for one instant, (b) the velocity changes instantaneously, and (c) the position changes continuously. (The latter feature follows from the facts that the velocity is integrated to find the position and the velocity is finite.) As was mentioned, this representation of an impulsive force is satisfactory for predicting the motion of the particle at any instant outside the brief interval when the impulsive force acts. At the same time, the fact that the model considers an impulsive force to have an infinite magnitude makes it inappropriate for any stress analysis task, such as designing a golf club.

In statics, we know that the resultant moment is as important as the resultant force. Thus, let us investigate how the moment $\Sigma\bar{M}_O$ of the resultant force about origin O of a fixed reference frame is related to the acceleration. Application of the Second Law to the resultant moment leads to

$$\Sigma\bar{M}_O = \bar{r}_{P/O} \times \Sigma\bar{F} = \bar{r}_{P/O} \times m\bar{a} = \bar{r}_{P/O} \times m\frac{d\bar{v}}{dt}. \qquad (1.2.23)$$

We now bring the time derivative outside the cross product by compensating the equation with an appropriate term to maintain the identity, specifically,

$$\Sigma\bar{M}_O = \frac{d}{dt}\left(\bar{r}_{P/O} \times m\bar{v}\right) - \frac{d\bar{r}_{P/O}}{dt} \times m\bar{v}. \qquad (1.2.24)$$

However, the last term vanishes because $d\bar{r}_{P/O}/dt \equiv \bar{v}$ and the velocity is parallel to the momentum $m\bar{v}$. The remaining term on the right side of the equation is the time derivative of the moment about origin O of the linear momentum of the particle. We refer to $\bar{r}_{P/O} \times m\bar{v}$ as the *moment of momentum.* The more common name for this quantity is the *angular momentum*, which refers to the fact that a moment is associated with a rotational tendency. We use the symbol $\bar{H}_O$ to denote it. Thus,

$$\boxed{\bar{H}_O = \bar{r}_{P/O} \times m\bar{v}, \quad \Sigma\bar{M}_O = \dot{\bar{H}}_O,} \qquad (1.2.25)$$

which is the derivative form of the *angular impulse–momentum principle*. The integral form is obtained by integration over an arbitrary interval $t_1 \leq t \leq t_2$, which leads to

$$\boxed{\left(\bar{H}_O\right)_2 = \left(\bar{H}_O\right)_1 + \int_{t_1}^{t_2} \Sigma\bar{M}_O dt.} \qquad (1.2.26)$$

The time integral of the moment is called the *angular impulse* of the resultant force.

Situations in which the angular impulse–momentum principle is needed to study the motion of a particle are few. As is the case for its linear analog, the angular momentum principle for a particle might be useful to treat an impulsive force. Also, when the moment of the resultant force about an axis is zero, the principle yields a conservation principle. Specifically, if the unit vector parallel to this axis is $\bar{e}$, then

$$\Sigma \bar{M}_O \cdot \bar{e} = \bar{0} \iff \bar{H}_O \cdot \bar{e} \text{ is constant.} \tag{1.2.27}$$

This conservation principle was recognized by Kepler for planetary motion, and it is a key part of the analysis of orbits. For us, the primary utility of the angular momentum principle lies in the application of the derivative form, Eq. (1.2.25), to a rigid body. We will find in Chapter 5 that the angular momentum of a body is directly related to its rotation.

1.3 BIOGRAPHICAL PERSPECTIVE

As we proceed through the various topics, the names of some early scientists and mathematicians will be encountered in a variety of contexts. The magnitude of the contribution of these pioneers cannot be overstated. Indeed, it is a testimonial to their ingenuity that we continue to use so much of their work. A view of the historical relationship among these researchers can greatly enhance our insight. The following discussion is an informal chronological survey of deceased individuals whose names are associated with concepts we will discuss. The goal here is to provide a brief overview of their life and their technical contributions. As in all scientific endeavors, many others made important contributions leading to those concepts. One objective of this survey is to introduce the notion that the laws of dynamics are a natural philosophy, as well as an engineering discipline. Another perspective to be gained from this survey is that some of these pioneers were active in a broad range of subjects, whereas others were specialists, but all were important to the advancement of dynamics. The reader is encouraged to examine the references for this chapter to fully appreciate how the subject evolved.

Galileo Galilei

Born 15 February 1564 in Pisa, Italy; died 8 January 1642. Galileo's family moved to Florence when he was 10 years old. His father forced him to enroll in the University of Pisa for a medical degree in 1581, but Galileo focused on mathematics and natural philosophy. He left the university without receiving a degree and began teaching mathematics in Siena in 1585. He worked there on the concept of center of gravity and unsuccessfully sought an appointment at the University of Bologna on the basis of that work. Galileo was named the Chair of Mathematics at the University of Pisa in 1589. He became a professor of mathematics at the University of Padua in 1590, where he was elevated to the post of Chief Mathematician in 1610.

Galileo is popularly known for experiments on gravity at the leaning tower of Pisa, but there is no conclusive evidence that those experiments actually occurred. From his measurements of the motion of pendulums, which led him to propose the use of a pendulum to provide the time base for a clock, he deduced that gravitational and inertial masses are identical. He refuted Aristotle's ancient statements by observing that the state of motion can be altered only by the presence of other bodies and that there is no unique inertial reference frame. In astronomy, he developed the astronomical telescope, and used it for many pioneering observations. His last eight years were spent under house arrest for advocating the Copernican view of the solar system, which held that the Sun, rather than the Earth, is the center of the solar system.

Sir Isaac Newton

Born 4 January 1643 in Lincolnshire, England; died 31 March 1727. Newton's father was wealthy, but illiterate, and Newton was raised by his grandparents. He entered Trinity College of Cambridge University in 1661 with the intent of earning a law degree, but became interested in mathematics and natural philosophy. He earned a bachelor's degree from Cambridge University in 1665, but his academic record there was not particularly distinguished. He returned to Lincolnshire shortly after graduation because of an outbreak of plague. His brilliance emerged there when he developed the fundamentals of calculus. Newton returned to Cambridge University in 1667, where he was named the Lucasian Chair in 1669. He was elected to Parliament in 1689 and retired from research in 1693. He became Warden of the Mint in 1896 and was knighted in 1705.

In addition to his contributions in developing calculus, Newton made important contributions to the refraction and diffraction of light. For us, his most important work is the monumental *Philosophiae naturalis principia mathematica,* which is usually referred to as the *Principia.* In it, he brought together the basic laws of motion, the universal law of gravitation, the study of projectile motion, and of celestial orbits. Equally important, it introduced the world to the scientific method by tying together mathematical hypothesis and experimental observation. The revolutionary nature of Newton's contributions causes many to regard him as one of the two most important figures in science, rivaled only by Albert Einstein.

Leonhard Euler

Born 15 April 1707 in Basel, Switzerland; died 18 September 1783. Euler studied philosophy at the University of Basel, from which he earned a masters's degree in 1723. While there he became interested in mathematics, but much of his expertise in this subject was the result of self-instruction. He received an appointment at the University of St. Petersburg, Russia, at the age of 19, and served as a medical lieutenant in the Russian Navy from 1727 to 1730. In that year he was named a professor at the University of St. Petersburg, which enabled him to leave the navy. He was named the senior chair in 1733, but left to go to the University of Berlin in 1743 because of negative sentiment for foreigners in Russia. He became the Director of Mathematics when the Berlin Academy was created in 1744. Euler returned to St. Petersburg in 1763 because of disagreements with Frederich the Great. A failed operation led to his total blindness in 1771, but much of his technical contributions come from the subsequent period.

Euler was quite prolific, with more than 350 publications in such diverse areas as the calculus of variations, functional analysis, number theory, ordinary differential equations, differential geometry, cartography, orbital motion and trajectories, fluid mechanics, and solid mechanics. For the subject of dynamics, his primary contribution is the derivation of principles governing the kinematics and kinetics of rigid bodies. Euler was the most prolific mathematician of his century. By developing new mathematical principles in order to solve physically meaningful problems he fathered the discipline of mechanics. Unlike Newton, who created new concepts, much of Euler's work can be recognized as evolving from the efforts of his contemporaries.

Jean Le Rond d'Alembert

Born 17 November 1717 in Paris, France; died 29 October 1783. D'Alembert was abandoned by his unmarried mother on the steps of the Church of St. Jean Le Rond, which is the origin of his given name. He studied theology and mathematics at the Jansenist College of Quatre Nations, from which he graduated as a lawyer in 1735. He continued to study mathematics, with the result that he joined the Paris Académie des Sciences in 1741, where he remained. He wrote most of the mathematical sections of the *Encyclopédie*, published in 1754, which is the year he was appointed to the French Académie. D'Alembert's interests eventually turned to literature, philosophy, and music theory.

D'Alembert made significant contributions to the study of partial differential equations and their application to solid and fluid mechanics, as well as to functional analysis. Euler and d'Alembert initially held each other in high esteem, but the situation deteriorated in 1753. Nevertheless, many of Euler's works descended from d'Alembert's concepts. A notable aspect is that d'Alembert was uncomfortable with Newton's approach merging experiment and theory, with the result that the assumptions he made to initiate an analysis were often erroneous. D'Alembert is associated with the notion of an inertial force, $-m\bar{a}$, which is a key concept for the development of analytical dynamics principles. However, this attribution, and hence d'Alembert's presence in this survey, is questionable, which is an issue we will address in Chapter 7.

Joseph-Louis Lagrange

Born 25 January 1736 in Turin, Italy; died 10 April 1813. Lagrange initially studied classical Latin at the College of Turin, but turned to mathematics and physics in order to pursue a financially sound career. Much of his knowledge in these subjects was the result of self-study, and his correspondences with Euler were very important for this effort. One consequence is that the works of Euler and Lagrange are intimately related. Lagrange became a professor of mathematics at the Royal Artillery School in Turin in 1755. He was elected to the Berlin Academy in 1755 on the recommendation of Euler and succeeded Euler as Director of Mathematics of that institution in 1766. Lagrange joined the Paris Académie des Sciences in 1787. The following year saw the publication of his treatise *Mécanique analytique,* in which he used differential equations as the framework for all of the primary developments in mechanics since Newton. Lagrange became the first professor of analysis at the École Polytechnique in Paris in 1794. He was named a Count of the Empire and awarded the Legion of Honor by Napoleon in 1808. Although Lagrange's heritage is Italian, the French consider him to be one of their own.

Lagrange's major contributions were to the calculus of variations and its applications to mechanics, the theory of equations, probability theory, number theory, ordinary differential equations, wave propagation, and celestial mechanics. His dynamical equations of motion will be an essential element of our study here. Overall, Lagrange can be credited as being the individual who initiated the usage of applied mathematical tools to study dynamical systems.

Gaspard Gustave de Coriolis

Born 21 May 1792 in Paris, France; died 19 September 1843**.** Coriolis' family fled to Nancy, France, shortly after his birth to escape punishment during the French Revolution. He attended the École Polytechnique, then the École des Ponts et Chausées in Paris. His first position was an appointment in 1816 as a tutor in mathematical analysis at the École Polytechnique. In 1829 he became a professor of mechanics at the École Centrale des Artes et Manufactures, then returned to the École Polytechnique in 1832, where he became director of studies in 1838.

Coriolis' contributions to dynamics are less significant than those of the other researchers discussed here. His primary recognition stems from an explanation of the acceleration observed from the perspective of a rotating reference frame. In addition, he introduced the terms "work" and "energy" in their modern sense. He also contributed to the study of collisions of bodies, hydraulics, and machine design.

Sir William Rowan Hamilton

Born 4 August 1805 in Dublin, Ireland; died 2 September 1865. Hamilton knew Latin, Greek, and Hebrew by the age of five. He entered Trinity College in Dublin in 1823, where he earned the ranking of "optime" in both science and classics, which was unprecedented. At age 21, while still an undergraduate, he was named a professor of astronomy, which was accompanied by the honorary title of Royal Astronomer of Ireland. He wrote poetry, which he exchanged with his friend Wordsworth, who advised Hamilton to remain a scientist. Hamilton remained at Trinity College for his whole career. He was knighted in 1835 and, shortly before his death, became the first foreign member of the U.S. National Academy of Sciences.

One of Hamilton's specialties was the ray theory of optics, especially the phenomenon of caustics. A primary contribution to dynamics was an alternative formulation of equations of motion, which apparently grew out of an effort to apply ray theory to dynamical systems. We also will encounter "Hamilton's Principle," which draws on concepts from the calculus of variations to capture the Newtonian and Lagrangian forms of the equations of motion. It has been invaluable for the development of finite element analysis, yet can be extended to relativistic systems. Hamilton also developed the algebra of quarternions, which has been employed in some areas of kinematics.

Edward John Routh

Born 20 January 1831 in Quebec, Canada; died 7 June 1907. Routh's birth in Canada was a consequence of his father being posted there by the English army. Routh went to England in 1842 to enroll in University College, London, from which he earned a B.A. in 1849 and an M.A. in 1853. The latter was accompanied by gold medals in mathematics and natural philosophy. He received a B.A. from Cambridge University in 1854, at which time he was the Senior Wrangler in the Mathematics Tripos Exams. (The great physicist Maxwell came in second in that competition.) Routh was appointed to a lectureship at Cambridge in 1855, which was the position he held for his whole career. He became famous as a coach for the Tripos exams. For 22 years, starting in 1862, the Senior Wrangler was coached by him. Over a 30-year period, Routh coached 700 students, of whom 480 were named Wranglers out of a cohort of 900. Routh became a Fellow of the Cambridge Philosophical Society in 1854, of the Royal Astronomical Society in 1866, and of the Royal Society in 1872.

As a researcher, Routh contributed to geometry, astronomy, and wave propagation. His primary contribution for us is analysis of the dynamical behavior of rigid bodies. As part of that effort, he formulated general tools for analyzing dynamic stability, which proved also to be very useful for fluid mechanics.

Josiah Williard Gibbs

Born 11 February 1839 in New Haven, Connecticut; died 28 April 1903. Gibbs initially focused on mathematics and Latin when he entered Yale University, but switched to engineering. He received Yale's first doctorate in engineering in 1863. From 1866 to 1869 he studied in Europe, then became a professor of mathematical physics at Yale in 1871, where he remained.

In today's vernacular, he was a "late bloomer," in that his first published work appeared in 1873 when he was 34. His major contributions were in thermodynamics and statistical mechanics. He also worked on the electromagnetic theory of light. An alternative formulation of equations of motion, which is the aspect of his efforts that causes him to be included here, was part of his efforts to apply vector analysis to physical systems.

Paul Emile Appell

Born 27 September 1855 in Strasbourg, France; died 24 October 1930. Appell's family moved to Nancy, France, when he was 16 as a consequence of the German annexation of Alsace. There he became a lifelong friend of Poincaré. He entered in 1873 the École Normal Supérieure in Paris, and received a doctorate in mathematics in 1876. Appell became the Chair of Mathematics at the University of Sorbonne in Paris in 1885. He was ardently and actively patriotic for his whole life, as was manifested by his spying activities for France during his frequent visits to his family in Alsace. He was elected to the Académie des Sciences in 1892. He was dean of the Faculty of Science of the University of Paris from 1903 to 1920, then served as its rector from 1920 to 1923.

Appell's prolific research output emphasized the application of mathematical analysis to geometry and mechanics. His work tended to be refinements of existing concepts, rather than new ones. The contribution that brings him to our attention is the development of an alternative set of equations of motion, for which he shares attribution with Gibbs.

Walther Ritz

Born 22 February 1878 in Sion, Switzerland; died 7 July 1909. Ritz entered the Federal Polytechnic School in Zurich in 1897. His original intent was to study engineering, but he soon transferred to mathematics, where one of his classmates was Einstein. He moved to Göttingen, Germany, where he did his thesis. Between 1901 and 1904 Ritz held positions in Leiden, Bonn, and Paris. Despite tuberculosis, which caused him to return to Switzerland in 1907, he took on positions in Tübingen, Germany, in 1907, and Göttingen, in 1908. Ritz's most productive period of research came in his last two years. His life span of 31 years is the shortest of the great physicists and mathematicians through the early 20th century. The only remorse he expressed as he was dying was that he would no longer be able to advance science.

Ritz's contributions to spectroscopy were vital to the development of quantum theory, and his work in the general theory of electrodynamics was important to the development of relativity theory. To explain the vibrations of elastic plates, Ritz applied an analytical technique based on Hamilton's Principle that he developed in his thesis on atomic spectra. This technique proved to be quite general, and now is widely used. It provides the foundation for finite element analysis. Interestingly, the only reason Ritz was interested in plate vibrations was the possibility of winning a mathematical competition. The rest of his research was in atomic physics.

Perhaps the most remarkable aspect of the foregoing survey is the time span over which these pioneers lived. The basic principles were essentially finalized more than a century ago. However, the subject of mechanics is mature from only a philosophical view. Developments since then have transferred the works of our predecessors from the realm of physics and mathematics to engineering, thereby converting their contributions from fundamentally straightforward principles and concepts to sophisticated analytical tools capable of describing complex systems. One objective of this text is to show how far we have come, but expanding the versatility of the analytical tools and their level of sophistication continues to be a research focus.

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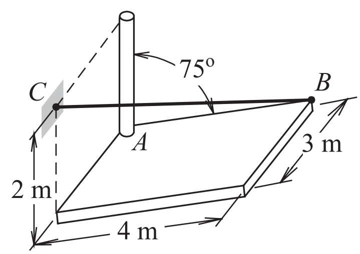
HOMEWORK PROBLEMS

EXERCISE 1.1 The rectangular plate is welded at corner A to the vertical shaft and braced by cable BC. The tension in the cable is 5 kN. (a) Determine the components parallel to the edges of the plate of the force the cable applies to the plate. (b) Determine the moment about corner A of the force applied by the cable. (c) Determine the moment about the axis of the shaft of the force applied by the cable.



Exercise 1.1

EXERCISE 1.2 The study of the kinematics of rigid bodies in Chapter 3 will show that the accelerations of two points in a rigid body are related by $\bar{a}_B = \bar{a}_A + \bar{\alpha} \times \bar{r}_{B/A} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{B/A})$. At a certain instant $\bar{r}_{A/O} = \bar{i} + \bar{j} + \bar{k}$ m, $\bar{r}_{B/O} = 4\bar{i} + 2\bar{j} - 3\bar{k}$ m, $\bar{a}_A = 4\bar{i} - 5\bar{j} + \bar{k}$ m/s^2, $\bar{\omega} = 5\bar{i} - 3\bar{j} + 2\bar{k}$ rad/s, $\bar{\alpha} = -20\bar{i} + 10\bar{j} - 40\bar{k}$ rad/s^2. Evaluate $\bar{a}_B$ by carrying out the calculation manually, then verify the result using mathematical software.

EXERCISE 1.3 The Gibbs–Appell function for a rigid body is related to the instantaneous angular velocity $\bar{\omega}$ and angular acceleration $\bar{\alpha}$ by

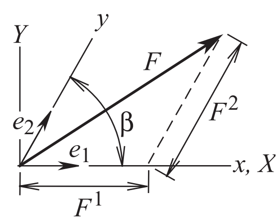
$$
\begin{aligned}
S &= \frac{1}{2}\bar{\alpha} \cdot \frac{\partial \bar{H}_A}{\partial t} + \bar{\alpha} \cdot \left(\bar{\omega} \times \bar{H}_A\right), \\
\bar{H}_A &= (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\,\bar{i} + (I_{yy}\omega_y - I_{xy}\omega_x - I_{yz}\omega_z)\,\bar{j} \\
&\quad + (I_{zz}\omega_z - I_{xz}\omega_x - I_{yz}\omega_y)\,\bar{k}, \\
\frac{\partial \bar{H}_A}{\partial t} &= (I_{xx}\alpha_x - I_{xy}\alpha_y - I_{xz}\alpha_z)\,\bar{i} + (I_{yy}\alpha_y - I_{xy}\alpha_x - I_{yz}\alpha_z)\,\bar{j} \\
&\quad + (I_{zz}\alpha_z - I_{xz}\alpha_x - I_{yz}\alpha_y)\,\bar{k},
\end{aligned}
$$

where I_{xx}, I_{yy}, and I_{zz} are the moments of inertia and I_{xy}, I_{yz}, and I_{xz} are products of inertia. Consider a situation in which $\bar{\omega} = -50\bar{i} - 20\bar{k}$ rad/s, $\bar{\alpha} = 1500\bar{i} - 500\bar{j} + 1000\bar{k}$ rad/s^2. The nonzero inertia values are $I_{xx} = 500$, $I_{yy} = 800$, $I_{zz} = 300$, and $I_{xz} = -200$ kg-m^2. Evaluate S by carrying out the calculation manually, then verify the result using mathematical software.

EXERCISE 1.4 Angular velocity $\bar{\omega}$ is a fundamental property of the motion of rigid bodies. In the Eulerian angle description, see Chapter 3, $\bar{\omega}$ is the sum of three contributions in directions defined by unit vectors $\bar{e}_1$, $\bar{e}_2$, and $\bar{e}_3 = \bar{k}$, such that $\bar{\omega} = c_1\bar{e}_1 + c_2\bar{e}_2 + c_3\bar{k}$, with $\bar{e}_2$ defined to be perpendicular to the plane containing $\bar{e}_1$ and $\bar{k}$. Consider the situation in which the angular velocity is $\bar{\omega} = 70\bar{i} + 110\bar{j} + 500\bar{k}$ rad/s and $\bar{e}_1 = -0.4913\bar{i} - 0.7651\bar{j} - 0.4161\bar{k}$. Determine the corresponding values of c_1, c_2, and c_3.

EXERCISE 1.5 The intersecting edges of a nonorthogonal parallelepiped are defined by the position vectors $\bar{r}_{B/A} = -20\bar{i} + 30\bar{j} + 5\bar{k}$, $\bar{r}_{C/A} = 8\bar{i} + 25\bar{j} + 10\bar{k}$, $\bar{r}_{D/A} = 4\bar{i} - 2\bar{j} - 15\bar{k}$ mm. Determine the volume of this object.

EXERCISE 1.6 An affine coordinate system has unit vectors that are not mutually orthogonal. The x and y axes in the sketch constitute such a system for a planar situation. A vector may be represented in terms of its *contravariant components* relative to xy by constructing lines parallel to the respective coordinate axes. Such a set of components are F^1 and F^2 for the force vector in the sketch. The unit vectors parallel to the x and y axes are $\bar{e}_1$ and $\bar{e}_2$. This force can also be represented in term of its F_X and F_Y components relative to the Cartesian XY coordinate system whose X axis is coincident with x. Given that $F_X = 500$ N, $F_Y = 350$ N, and $\beta = 65°$, use vector algebra to determine F^1 and F^2.

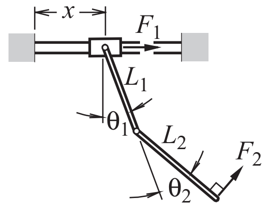


Exercise 1.6

EXERCISE 1.7 The mass flow rate per unit surface area is the product of the density and velocity at a point in space, and the mass flux per unit area is the component of the mass flow rate parallel to the unit vector $\bar{e}$ that is normal to a surface. Consider a square surface whose sides are 200 mm and whose normal is $\bar{e} = 0.6\bar{j} + 0.8\bar{k}$. The velocity everywhere on this surface is $\bar{v} = 80\cos(5\pi t)\,\bar{i} - 20\cos(10\pi t)\,\bar{j} + 40\sin(10\pi t)\,\bar{k}$ m/s, where t has units of seconds, and the density is 950 kg/m^3. Determine how much mass flows across the square in the interval $50 < t < 100$ ms.

EXERCISE 1.8 Polar coordinates in the xy plane are defined by the radial distance R and the polar angle θ measured from the x axis. The corresponding representation of position is $\bar{r}_{P/O} = R\cos\theta\bar{i} + R\sin\theta\,\bar{j}$. Suppose that $R = \rho + \varepsilon\sin(\alpha t)$ and $\theta = \alpha t^2/2$. Determine the velocity of point P, then evaluate the components of this velocity parallel and perpendicular to $\bar{r}_{P/O}$.

EXERCISE 1.9 Polar coordinates in the xy plane are defined by the radial distance R and the polar angle θ measured from the x axis. The corresponding representation of position is $\bar{r}_{P/O} = R\cos\theta\bar{i} + R\sin\theta\,\bar{j}$. Suppose that R and θ are arbitrary functions of time. Derive formulas for the velocity of point P in terms of xyz components, then evaluate the components of this velocity parallel and perpendicular to $\bar{r}_{P/O}$.



Exercise 1.10

EXERCISE 1.10 The collar slides to the left as the interconnected bars swing in the vertical plane. The position of the collar is $x = 20\sin(50t)$ mm, and the angles are $\theta_1 = 0.2\pi\cos(50t)$, $\theta_2 = 0.2\pi\sin(50t - \pi/3)$ rad. Determine the velocity of the lower end by differentiating its position in terms of horizontal and vertical components.

CHAPTER 2

Particle Kinematics

This chapter develops some basic techniques for describing the motion of a point and therefore of a particle. The procedures we follow are driven not merely by how the point's motion is described, but also by the information we seek. Each formulation is based on describing vector quantities with respect to a different set of unit vectors. Which description is best suited to a particular situation depends on a variety of factors, but a primary consideration is whether the associated quantities, such as the coordinates, naturally fit the known aspects of the motion. Ultimately we will find that it might be beneficial to combine a variety of descriptions.

The various kinematical description that we use fall into two general classes. The one that might seem to be the most natural is *extrinsic coordinates*, which means that the description is extrinsic to knowledge of the path followed by the point. A simple case is rectangular Cartesian coordinates, for which the position is know in terms of distances measured along three mutually orthogonal straight lines representing reference directions. A variety of other extrinsic coordinate systems are in use. However, we begin by studying *intrinsic coordinates*, in which knowledge of the path is fundamental to the description of the motion. For example, the unit vectors for intrinsic coordinates are defined in terms of the properties of the path. For this reason, intrinsic coordinates are more commonly referred to as *path variables.*

2.1 PATH VARIABLES

The idea that the motion of a point should be described in terms of the properties of its path may seem to be counterintuitive, in that the nature of the motion defines the path. However, this is precisely the way in which one thinks when using a road map in conjunction with the speedometer and odometer of an automobile. Another name for this type of description is *tangent and normal components,* because those are the primary component directions, as we shall see.

We assume that the path is known. The fundamental variable for a specified path is the arc length s along this curve, measured from some starting point to the point of interest. As shown in Fig. 2.1, measurement of s requires definition of positive sense along the path. Negative s means that the point has receded, rather than advanced, along its path. It is quite obvious from Fig. 2.1 that the position $\bar{r}_{P/O}$ is unambiguously defined by the value of s. Furthermore, because s changes with time, the position is an implicit function of time, $\bar{r}_{P/O} = \bar{r}\,(s)$ and $s = s\,(t)$. (Mathematically, the preceding

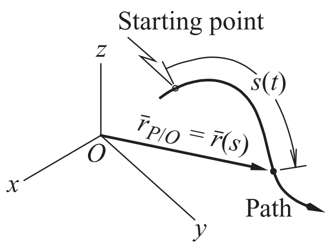


Figure 2.1. Position as a function of arc length.

should be understood to indicate that the position is a vector function of the arc length, which in turn is a scalar function of time.) It follows that the derivation of formulas for velocity and acceleration will involve the chain rule for differentiation.

By definition, velocity is the rate of change of the position vector. Differentiating the implicit description of position leads to

$$\bar{v} = \frac{d\bar{r}}{dt} = \frac{d\bar{r}}{ds}\frac{ds}{dt} = \dot{s}\frac{d\bar{r}}{ds}. \tag{2.1.1}$$

(In the present context only one point is under consideration, so subscripts can be omitted from the notation.) The quantity $d\bar{r}/ds$ occurring in Eq. (2.1.1) is determined solely by the nature of the dependence of the position vector on the path. Hence it is another of the path variables.

By definition, acceleration is the rate of change of the velocity, so

$$\bar{a} = \frac{d\bar{v}}{dt} = \ddot{s}\frac{d\bar{r}}{ds} + \dot{s}\frac{d}{dt}\left(\frac{d\bar{r}}{ds}\right). \tag{2.1.2}$$

To differentiate the last term we recall that $d\bar{r}/ds$ depends solely on s, so we invoke the chain rule to find

$$\bar{a} = \frac{d\bar{v}}{dt} = \ddot{s}\frac{d\bar{r}}{ds} + \dot{s}^2\left(\frac{d^2\bar{r}}{ds^2}\right). \tag{2.1.3}$$

The second derivative, $d^2\bar{r}/ds^2$, is another property of the path.

In the simple case of a straight path we can let the origin O be the starting point of the path without loss of generality. Let $\bar{e}$ denote the constant orientation of the straight path, so that $\bar{r} = s\bar{e}$, and $d\bar{r}/ds = \bar{e}$. Because $\bar{e}$ is invariant in this case, $d^2\bar{r}/ds^2 = \bar{0}$, from which it follows that $\bar{v} = \dot{s}\bar{e}$ and $\bar{a} = \ddot{s}\bar{e}$. Thus the velocity and acceleration are oriented parallel to the straight path. The key point here is that acceleration will not be parallel to the velocity for a smooth curvilinear path, unless $\dot{s} = 0$. Failure to recognize this elementary fact is probably the most common mistake in the application of the path variable approach to kinematics.

2.1.1 Tangent and Normal Components

To understand $d\bar{r}/ds$ for a curved path we consider Fig. 2.2, which shows the position vector at two locations that are separated by a small arc length Δs. The displacement $\Delta\bar{r}$

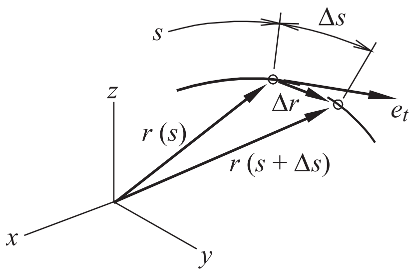


Figure 2.2. Tangent vector.

is the change of the point's position as it moves from position s to $s + \Delta s$,

$$\Delta \bar{r} = \bar{r}(s + \Delta s) - \bar{r}(s). \tag{2.1.4}$$

In the limit as $\Delta s \to ds$, the magnitude of $\Delta \bar{r}$ equals ds because a chord progressively approaches the curve. For the same reason, the direction of $\Delta \bar{r}$ approaches tangency to the curve, in the sense of increasing s. This *tangent direction* is described by the unit tangent vector $\bar{e}_t$. A unit vector has the dimensionless value 1 for magnitude, so

$$\frac{d\bar{r}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta \bar{r}}{\Delta s} = \bar{e}_t. \tag{2.1.5}$$

The tangent vector is one of three unit vectors used to describe vectorial quantities in terms of path variables. We encountered an aspect of the second unit vector in Eq. (2.1.3), which featured $d^2\bar{r}/ds^2 \equiv d\bar{e}_t/ds$. One basic property that is readily apparent comes from the fact that $\bar{e}_t$ is a unit vector, so $\bar{e}_t \cdot \bar{e}_t = 1$. Differentiation of this relation leads to

$$\bar{e}_t \cdot \frac{d\bar{e}_t}{ds} = 0. \tag{2.1.6}$$

In other words, $d\bar{e}_t/ds$ is always perpendicular to $\bar{e}_t$. (Perpendicularity of a unit vector and its derivative is a general property that will arise frequently.) We define the *normal direction*, whose unit vector is denoted as $\bar{e}_n$, to be parallel to to this derivative. Because parallelism of two vectors corresponds to their proportionality, this definition may be written as

$$\boxed{\bar{e}_n = \rho \frac{d\bar{e}_t}{ds}.} \tag{2.1.7}$$

Because $\bar{e}_n$ is a dimensionless unit vector, the factor of proportionality, ρ, may be found from

$$\rho = \frac{1}{\left| \dfrac{d\bar{e}_t}{ds} \right|}. \tag{2.1.8}$$

Dimensional consistency of Eq. (2.1.7) requires that ρ be a length parameter. It is the *radius of curvature*.

There is a simple construction that explains Eq. (2.1.7) for a circular path having radius R. In Fig. 2.3(a) tangent and normal vectors are placed at adjacent positions on the circle. To construct the increment $\Delta \bar{e}_t = \bar{e}_t\,(s + \Delta s) - \bar{e}_t\,(s)$, the tails of the two

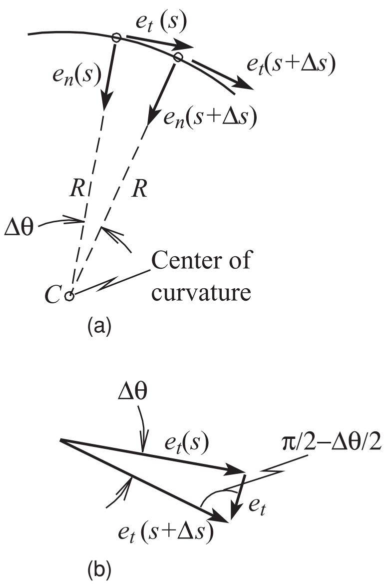


Figure 2.3. Relation between tangent and normal directions for a circular path.

tangent vectors are brought into coincidence in Fig. 2.3(b), so that $\Delta\bar{e}_t$ extends from the head of $\bar{e}_t\,(s)$ to $\bar{e}_t\,(s+\Delta s)$. The angle between $\Delta\bar{e}_t$ and either tangent vector is $\pi/2-\Delta\theta/2$. Because the triangle formed by the two unit vectors is isosceles with unit length, and $\Delta\theta$ is small, the length of $\Delta\bar{e}_t$ is essentially $\Delta\theta$. For a circular arc, the subtended angle is $\Delta\theta=\Delta s/R$, so $|\Delta\bar{e}_t|\approx\Delta s/R$. In the limit as $\Delta s\to ds$, the angle between $\Delta\bar{e}_t$ and either $\bar{e}_t$ approaches $\pi/2$. Multiplying the magnitude of $d\bar{e}_t$ by the unit vector for its direction gives $d\bar{e}_t=(ds/R)\,\bar{e}_n$. Dividing this relation by ds leads to an equation that exemplifies Eq. (2.1.7). In general, a quick way of visualizing the direction in which a unit vector changes is to follow the tip of the unit vector as it moves when the parameter θ changes.

In the case of an arbitrary path, $\bar{e}_t$ and $\bar{e}_n$ at any point along the curve form the *osculating plane.* The generalization of the center of a circle is the *center of curvature,* which is situated in the osculating plane at a distance ρ in the normal direction from the corresponding point on the curve. In some situations, such as one in which it is necessary to design a curve to meet a certain specification, the foregoing allows us to determine this location according to

$$\bar{r}_{\text{center}}=\bar{r}_{P/O}\,(s)+\rho\bar{e}_n. \tag{2.1.9}$$

When we substitute Eqs. (2.1.5) and (2.1.7) into the basic formulas for velocity and acceleration, we find that

$$\boxed{\begin{aligned}\bar{v}&=v\bar{e}_t,\qquad v=\dot{s},\\ \bar{a}&=\dot{v}\bar{e}_t+\frac{v^2}{\rho}\bar{e}_n.\end{aligned}} \tag{2.1.10}$$

The scalar quantity v is the *speed* of the particle. These relations, which apply to all paths, contain much information. The first of Eqs. (2.1.10) indicates that the speed is the magnitude of the velocity; the second equation states that it is the rate of increase of the distance traveled along the path. In the example of the automobile, the speedometer tells us the instantaneous speed of the vehicle. As noted, v is a scalar, and it is usually considered to be a positive number in common parlance. However, when we use v analytically, it can be either positive or negative. A negative value of v when s is positive indicates that the point is returning to the starting point after traveling a certain positive distance along the path. In contrast, a negative value of v when s is negative indicates that the point is receding farther from the starting point. The rate of change of the speed is readily obtained if v is known as a function of time, but in some situations, the speed might be known as a function of position, $v(s)$, in which case the chain rule for differentiation leads to

$$\dot{v} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}. \tag{2.1.11}$$

The velocity and acceleration always lie in the osculating plane formed by $\bar{e}_t$ and $\bar{e}_n$, but that plane is constant only if the path is planar. Otherwise, the osculating plane twists around as the point moves along its path. There is only a tangential velocity component, whereas both acceleration components exist unless v is constant, or the path is straight, in which case ρ is infinite. Recall that a vector is constant only if its magnitude *and* direction are constant. The normal acceleration component is the consequence of the changing direction of the velocity vector. This component is always oriented toward the center of curvature because that is the direction in which the tip of the velocity vector moves. The tangential acceleration results from changing the speed. Increasing v when v is positive, or decreasing the magnitude of negative v, produces a positive tangential acceleration.

The development thus far might seem to be paradoxical in light of Newton's Second Law, for we know that forces can act on a particle in three directions, but there are only two acceleration components. However, buried in the path variable derivation is the fact that there is another direction, perpendicular to the osculating plane. This is the *binormal direction,* which is denoted as $\bar{e}_b$. We can determine this direction in any situation by a cross product of the tangent and normal unit vectors,

$$\boxed{\bar{e}_b = \bar{e}_t \times \bar{e}_n.} \tag{2.1.12}$$

One of the uses of the binormal direction arises when we apply Newton's Second Law to a particle. Because two vectors can be equal only when their respective components match, we find that

$$\boxed{\begin{aligned} \Sigma F_t &\equiv \Sigma \bar{F} \cdot \bar{e}_t = m\dot{v}, \\ \Sigma F_n &\equiv \Sigma \bar{F} \cdot \bar{e}_n = m\frac{v^2}{\rho}, \\ \Sigma F_b &\equiv \Sigma \bar{F} \cdot \bar{e}_b = 0. \end{aligned}} \tag{2.1.13}$$

The last of Eqs. (2.1.13) shows that there must be a force balance perpendicular to the osculating plane. Also note that, according to the second equation, there must be a net force pushing inward toward the center of curvature, because the changing direction of acceleration is always directed toward that point.

Equations (2.1.13) form a set of *equations of motion.* Each type of kinematical description leads to a different form of these equations. In general, there are three classes of problems involving equations of motion. In the simplest, all aspects of the motion are specified, so that the forces required for the motion may be found algebraically after the acceleration has been determined by use of the appropriate kinematical formulas. The second class of problems occurs when all forces acting on a system are known. The equations of motion then represent differential equations, which may be solved by analytical or numerical methods. The third class of problems is a mixture of the first two, in that some forces are known and some aspects of the motion are specified.

In most situations in which path variables are useful there will be forces acting in the normal and binormal directions whose role is to prevent the particle from moving off the designated path. Both forces have unknown magnitudes that adjust to provide the centripetal acceleration in the presence of the other forces. Such forces are sometimes referred to as *reactions*, but we usually will use the more descriptive term *constraint forces.* If v is specified, the tangential equation of motion may be solved for a force causing the particle to move along the path. The more interesting condition is that in which the tangential forces are known, in which case that equation of motion is an ordinary differential equation. The tangential resultant force might depend on t, which could characterize a force that we control. This resultant force might also depend on s, which would be the case for a spring force, or it might depend on v, as in the case of any frictional resistance. Thus $\Sigma F_t = f(t, s, v)$ in the most general situation. The tangential equation of motion then is a second-order differential equation, $m\ddot{s} = f(t, s, \dot{s})$. This equation is readily solved if f is linear in s and $\dot{s}$ and the dependence on t is not too complicated. Other situations might require numerical methods, such as Runge–Kutta integration.

If f depends on only one variable, then we may obtain v and s by separating variables. Specifically, if the resultant force depends solely on t, we replace $\ddot{s}$ with dv/dt, so that separating variables leads to

$$m\,dv = f(t)\,dt. \tag{2.1.14}$$

A definite integration whose lower limits are the initial conditions yields the speed as a function of time, $v = g(t)$. Because $v = ds/dt$, separating variables here gives $ds = g(t)\,dt$, whose integration leads to s as a function of time. When the resultant force depends solely on s, we use Eq. (2.1.11) to replace $\ddot{s}$. The separated form of the tangential equation then is

$$mv dv = f(s)\,ds. \tag{2.1.15}$$

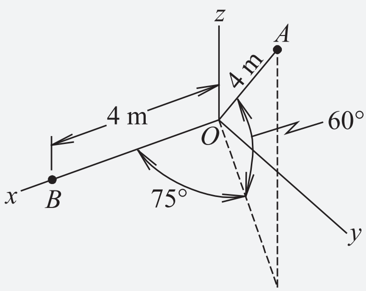
The definite integral of this differential form can be solved for v as a function of s, $v = g(s)$. Separation of variables by use of $v = ds/dt = g(s)$ leads to $ds/g(s) = dt$, which may be integrated to find the value of t for a specified s. The third case, in which

$\Sigma F_t = f(v)$, may also be addressed by use of Eq. (2.1.11). The separated form of the equation of motion then is

$$m\frac{v dv}{f(v)} = ds. \tag{2.1.16}$$

Definite integration of this expression gives s as a function of v, $s = h(v)$. Inversion gives $v = h^{-1}(s)$, which is the same type of dependency as the one obtained when $\Sigma F_t = f(s)$, so determination of s as a function of t can proceed as discussed for that case.

EXAMPLE 2.1 At the instant when the 5-kg particle is at position A, it has a velocity of 500 m/s directed from point A to point B and an acceleration of $10g$ directed from point A to point O. Determine the corresponding rate of change of the speed, the radius of curvature of the path, and the location of the center of curvature of the path. Also determine the tangent, normal, and binormal components of the resultant force acting on the particle.



Example 2.1

SOLUTION This example demonstrates how one can extract path variable information when the velocity and acceleration are known. The idea is to make the velocity and acceleration fit the fundamental formulas for path variables. The given vectors are

$$\bar{v} = 500\bar{e}_{B/A} \text{ m/s}, \quad \bar{a} = 10(9.807)\bar{e}_{O/A} \text{ m/s}^2.$$

The unit vectors are defined by the positions of the end points, according to

$$\begin{aligned}\bar{r}_{A/O} &= 4\cos 60^\circ(\cos 75^\circ\bar{i} + \sin 75^\circ\bar{j}) + 4\sin 60^\circ\bar{k}\\ &= 0.5176\bar{i} + 1.9319\bar{j} + 3.464\bar{k} \text{ m},\end{aligned}$$

$$\bar{r}_{B/O} = 4\bar{i},$$

$$\bar{e}_{B/A} = \frac{\bar{r}_{B/A}}{|\bar{r}_{B/A}|} = \frac{\bar{r}_{B/O} - \bar{r}_{A/O}}{|\bar{r}_{B/O} - \bar{r}_{A/O}|} = 0.6598\bar{i} - 0.3660\bar{j} - 0.6563\bar{k},$$

$$\bar{e}_{O/A} = -\frac{\bar{r}_{A/O}}{|\bar{r}_{A/O}|} = -0.1294\bar{i} - 0.4830\bar{j} - 0.8660\bar{k}.$$

In general, $\bar{v} = v\bar{e}_t$, from which it follows that

$$\bar{e}_t = \bar{e}_{B/A} = 0.6598\bar{i} - 0.3660\bar{j} - 0.6563\bar{k}.$$

Then, because $\dot{v}$ is the tangential component of acceleration, we find

$$\dot{v} = \bar{a} \cdot \bar{e}_t = 98.07\bar{e}_{O/A} \cdot \bar{e}_{B/A} = 64.70 \text{ m/s}^2. \qquad \triangleleft$$

We may evaluate the normal acceleration by forming the difference between $\bar{a}$ and $\dot{v}\bar{e}_t$, specifically,

$$\begin{aligned} \frac{v^2}{\rho}\bar{e}_n &= \bar{a} - \dot{v}\bar{e}_t = 98.07\bar{e}_{O/A} - 64.70\bar{e}_{B/A} \\ &= -55.38\bar{i} - 23.68\bar{j} - 42.47\bar{k} \text{ m/s}^2. \end{aligned}$$

The values of ρ and $\bar{e}_n$ come from the magnitude and direction of the normal acceleration component,

$$\frac{v^2}{\rho} = (55.38^2 + 23.69^2 + 42.47^2)^{1/2} = 73.70 \text{ m/s}^2,$$

$$\rho = \frac{v^2}{73.70} = 3392 \text{ m}, \qquad \triangleleft$$

$$\bar{e}_n = \frac{-55.38\bar{i} - 23.69\bar{j} - 42.47\bar{k}}{73.70} = -0.7515\bar{i} - 0.3213\bar{j} - 0.5762\bar{k}.$$

To locate the center of curvature C we recall Eq. (2.1.9):

$$\bar{r}_{C/O} = \bar{r}_{A/O} + \rho\bar{e}_n = -2549\bar{i} - 1088\bar{j} - 1951\bar{k} \text{ m}. \qquad \triangleleft$$

In general, if a problem involves forces we should draw a free-body diagram. However, the forces in the present case are straightforward. The gravity force is mg in the negative $\bar{k}$ direction. In addition, there is a force tangent to the path, which changes the speed, and there may be normal and binormal components of a contact force that prevent the particle from moving perpendicularly to the path. Thus Newton's Second Law is

$$\Sigma\bar{F} = -5\,(9.807)\,\bar{k} + F_t\bar{e}_t + F_n\bar{e}_n + F_b\bar{e}_b = 5\left(\dot{v}\bar{e}_t + \frac{v^2}{\rho}\bar{e}_n\right).$$

Expressions for $\bar{e}_t$ and $\bar{e}_n$ have been found. We evaluate $\bar{e}_b$ from the cross product:

$$\bar{e}_b = \bar{e}_t \times \bar{e}_n = 0.8734\bar{j} - 0.4871\bar{k}.$$

To find the force components we use a dot product of the resultant force and each of the unit vectors,

$$\begin{aligned} \Sigma\bar{F} \cdot \bar{e}_t &= -49.035\,(-0.6563) + F_t = 5\dot{v}, \\ \Sigma\bar{F} \cdot \bar{e}_n &= -49.035\,(-0.5763) + F_n = 5\frac{v^2}{\rho}, \\ \Sigma\bar{F} \cdot \bar{e}_b &= -49.035\,(-0.4871) + F_b = 0, \end{aligned}$$

from which we obtain

$$F_t = 291.3, \quad F_n = 340.2, \quad F_b = -23.9 \text{ N}. \qquad \triangleleft$$

2.1.2 Parametric Description of Curves

If we encountered only circular paths, the development thus far would suffice. However, many interesting motions occur along noncircular paths; indeed, the path might not even lie in a plane. A particularly useful description specifies a path in *parametric form*. This means that some algebraic variable, which we denote as α, is considered to cover a range of values, and that the rectangular Cartesian coordinates, x, y, z, of a point on the path are stated as functions of α. It is important to realize that α does not necessarily have any physical significance. (A special case is that in which α is the time t. In that case the formulation reduces to a straightforward Cartesian coordinate description, which is treated in the next section.) The position vector may be written in component form,

$$\boxed{\bar{r} = x(\alpha)\bar{i} + y(\alpha)\bar{j} + z(\alpha)\bar{k},} \tag{2.1.17}$$

where $x(\alpha)$, $y(\alpha)$, and $z(\alpha)$ are the stated parametric functions.

Because the position is indicated to be a function of α, evaluating $\bar{e}_t$ according to Eq. (2.1.5) requires the chain rule,

$$\boxed{\bar{e}_t = \frac{d\bar{r}}{ds} = \frac{d\alpha}{ds}\frac{d\bar{r}}{d\alpha} \equiv \frac{\bar{r}\,'}{s'},} \tag{2.1.18}$$

where it is convenient to use a prime to denote differentiation with respect to the parameter α, so that

$$\bar{r}\,' \equiv \frac{d\bar{r}}{d\alpha} = x'\bar{i} + y'\bar{j} + z'\bar{k}, \quad s' \equiv \frac{ds}{d\alpha}. \tag{2.1.19}$$

The quantity s' seldom has physical significance. Its value must be such that $\bar{e}_t$ is a unit vector, $|\bar{e}_t| = 1$, which yields

$$\bar{e}_t \cdot \bar{e}_t = \frac{\bar{r}\,' \cdot \bar{r}\,'}{(s')^2} = 1. \tag{2.1.20}$$

Consequently, we find that

$$\boxed{s' = (\bar{r}\,' \cdot \bar{r}\,')^{1/2} = \left[(x')^2 + (y')^2 + (z')^2\right]^{1/2}.} \tag{2.1.21}$$

In addition to enabling us to evaluate $\bar{e}_t$ according to Eq. (2.1.18), integrating this expression enables us to determine the arc length to any position on the path in terms of the parameter, according to

$$s(\alpha_P) = \int_{\alpha_0}^{\alpha_P} \left[(x')^2 + (y')^2 + (z')^2\right]^{1/2} d\alpha, \tag{2.1.22}$$

where α_0 is the value at the starting position at which $s = 0$, and α_P is the value at the position of interest.

The next step is to find $\bar{e}_n$ and ρ. Equation (2.1.18) gives $\bar{e}_t$ as a function of α, so forming Eq. (2.1.7) requires differentiation by use of the chain rule, such that

$$\bar{e}_n = \rho \frac{d\bar{e}_t}{ds} = \rho \frac{d\alpha}{ds} \frac{d\bar{e}_t}{d\alpha} = \frac{\rho}{s'} \frac{d\bar{e}_t}{d\alpha}. \tag{2.1.23}$$

We find $d\bar{e}_t / d\alpha$ by differentiating Eq. (2.1.18), with the result that

$$\bar{e}_n = \frac{\rho}{(s')^3} \left(\bar{r}\,'' s' - \bar{r}\,' s'' \right). \tag{2.1.24}$$

This may be simplified by using Eq. (2.1.21) to write

$$s'' = \left[\left(\bar{r}\,' \cdot \bar{r}\,' \right)^{1/2} \right]' = \frac{\bar{r}\,' \cdot \bar{r}\,''}{\left(\bar{r}\,' \cdot \bar{r}\,' \right)^{1/2}} = \frac{\bar{r}\,' \cdot \bar{r}\,''}{s'} \tag{2.1.25}$$

so that

$$\bar{e}_n = \frac{\rho}{(s')^4} \left[\bar{r}\,'' \left(s' \right)^2 - \bar{r}\,' \left(\bar{r}\,' \cdot \bar{r}\,'' \right) \right]. \tag{2.1.26}$$

This expression is close to what we seek, except that the radius of curvature remains to be determined. For this, we impose the requirement that $\bar{e}_n$ be a unit vector. Using a dot product to form the magnitude of this expression leads to

$$1 = \frac{\rho^2}{(s')^8} \left[\left(\bar{r}'' \cdot \bar{r}\,'' \right) (s')^4 - 2 \left(\bar{r}\,' \cdot \bar{r}\,'' \right)^2 (s')^2 + \left(\bar{r}\,' \cdot \bar{r}' \right) \left(\bar{r}\,' \cdot \bar{r}\,'' \right)^2 \right], \tag{2.1.27}$$

which simplifies to

$$\boxed{\rho = \frac{(s')^3}{\left[\left(\bar{r}\,'' \cdot \bar{r}\,'' \right) (s')^2 - \left(\bar{r}\,' \cdot \bar{r}\,'' \right)^2 \right]^{1/2}}.} \tag{2.1.28}$$

In turn, substitution of this expression into the preceding equation for $\bar{e}_n$ leads to

$$\boxed{\bar{e}_n = \frac{\bar{r}\,'' (s')^2 - \bar{r}\,' \left(\bar{r}\,' \cdot \bar{r}\,'' \right)}{(s') \left[\left(\bar{r}\,'' \cdot \bar{r}\,'' \right) (s')^2 - \left(\bar{r}\,' \cdot \bar{r}\,'' \right)^2 \right]^{1/2}}.} \tag{2.1.29}$$

Equations (2.1.18), (2.1.21), (2.1.28), and (2.1.29) are readily evaluated when the position function is specified in the form of Eq. (2.1.17). However, in some situations it might be somewhat easier to forego applying these formulas, and instead use the basic path variable formulas, Eqs. (2.1.5) and (2.1.7) in conjunction with the requirement that $\bar{e}_t$ and $\bar{e}_n$ be unit vectors.

In the parametric formulation the path variable unit vectors are described in terms of rectangular Cartesian coordinates. However, in some situations it might be sufficient to describe these directions merely by sketching them relative to the path. This would be especially true in cases in which the path is straight or circular. Overall, a key aspect is that knowledge of the path followed by a point does not necessarily mean that path variables are the appropriate formulation. Selection of an appropriate kinematical description requires recognition of how the point proceeds along the path. Path variables are likely to be useful if the path is specified *and* movement along the path is described in terms of the arc length or speed. This is a central theme of our further studies.

EXAMPLE 2.2 A standard description of a planar curve defines y as a function of x, that is, $y = y(x)$, and $z = 0$. Derive expressions for the tangent and normal unit vectors and the radius of curvature of the path.

SOLUTION This example is used to demonstrate the application of the general path variable expressions, rather than merely substituting into the parametric equations. The formulas that result can be quite useful. To match the given functional form to the standard parametric representation, we consider $\alpha = x$, so that $x' = 1$, $y' = dy/dx$, and $z' = 0$. It follows that $\bar{r}' = \bar{i} + (dy/dx)\,\bar{j}$, which combined with the requirement that $|\bar{e}_t| = 1$ leads to

$$\bar{e}_t = \frac{d\bar{r}}{ds} = \frac{\left(\dfrac{d\bar{r}}{dx}\right)}{\left(\dfrac{ds}{dx}\right)} = \frac{\bar{i} + \dfrac{dy}{dx}\bar{j}}{s'},$$
$$s' \equiv \frac{ds}{dx} = \left|\bar{i} + \frac{dy}{dx}\bar{j}\right| = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}. \tag{2.1.30}$$

Differentiation of $\bar{e}_t$ with respect to s by use of the chain rule gives

$$\bar{e}_n = \rho\frac{d\bar{e}_t}{ds} = \rho\frac{\left(\dfrac{d\bar{e}_t}{dx}\right)}{s'}$$
$$= \frac{\rho}{s'}\left[\frac{\dfrac{d^2y}{dx^2}\bar{j}}{s'} + \left(\bar{i} + \frac{dy}{dx}\bar{j}\right)\frac{(-s'')}{(s')^2}\right].$$

From the second of Eqs. (2.1.30) we find that

$$s'' = \frac{\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{1/2}},$$

which reduces $\bar{e}_n$ to

$$\bar{e}_n = \frac{\rho}{(s')^4}\frac{d^2y}{dx^2}\left(-\frac{dy}{dx}\bar{i} + \bar{j}\right).$$

Because $\bar{e}_n$ is a unit vector, it must be that

$$\bar{e}_n \cdot \bar{e}_n = \frac{\rho^2}{(s')^8}\left(\frac{d^2y}{dx^2}\right)^2\left[1 + \left(\frac{dy}{dx}\right)^2\right] = 1.$$

In view of the fact that $s' = \left[1 + (dy/dx)^2\right]^{1/2}$, solving the preceding expression for ρ gives

$$\rho = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\left|\dfrac{d^2y}{dx^2}\right|}, \tag{2.1.31}$$

which is the same as the formula derived in a course on calculus. Back substitution of ρ into the preceding expression for $\bar{e}_n$ leads to

$$\bar{e}_n = \frac{\dfrac{d^2y}{dx^2}}{\left|\dfrac{d^2y}{dx^2}\right|}\frac{1}{s'}\left(-\frac{dy}{dx}\bar{i} + \bar{j}\right) = \operatorname{sgn}\left(\frac{d^2y}{dx^2}\right)\frac{-\dfrac{dy}{dx}\bar{i} + \bar{j}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{1/2}}, \tag{2.1.32}$$

where sgn( ) denotes the signum function, that is, the sign of the argument. It tells us the sense in which the normal vector is perpendicular to the tangent vector. If d^2y/dx^2 is positive, which means that the slope increases with increasing x, then $\bar{e}_n$ will have a positive $\bar{e}_y$ component.

Two simple checks are available. The first verifies that both $\bar{e}_t$ and $\bar{e}_n$ have unit magnitudes, which may be done by comparing $\bar{e}_t \cdot \bar{e}_t$ and $\bar{e}_n \cdot \bar{e}_n$ to unity. The second check verifies that the unit vectors are orthogonal, which can be verified by ascertaining that $\bar{e}_t \cdot \bar{e}_n = 0$.

EXAMPLE 2.3 A particle follows the path defined by $x = 0.2\beta\cos(\beta)$, $y = 0.2\beta\sin(\beta)$, and $z = 0.1\beta^2$, where each coordinate is measured in meters. The speed depends on time t (in seconds) according to $v = 20t$ m/s, and the particle was at the origin when $t = 0$. Determine the velocity and acceleration of the particle when $t = 0.5$ s. Also describe the path.

SOLUTION This problem is quite intricate. It will serve to illustrate the application of most of the relevant relations and also to highlight the thought process needed to implement these relations. We begin by observing that the path is specified in parametric form, with β as the parameter. Furthermore, the value of β for the position of interest is not specified. To determine this value we need to follow an inverse process, in which we use the given function $v(t)$ to determine s at $t = 0.5$ s, then solve the relation for s in terms of β to determine the corresponding value of the parameter. After that, the remainder of the work reduces to straightforward calculations.

The speed increases at a constant rate and $s = 0$ at $t = 0$, so we have

$$v = 20t \implies s = 10t^2, \quad \dot{v} = 20\,.$$

Setting $t = 0.5$ s shows that the values at the position of interest are

$$s = 2.5 \text{ m}, \quad v = 10 \text{ m/s}, \quad \dot{v} = 20 \text{ m/s}^2.$$

The derivatives of the coordinate functions are

$$x' = 0.2\left[\cos(\beta) - \beta\sin(\beta)\right], \quad x'' = 0.2\left[-2\sin(\beta) - \beta\cos(\beta)\right],$$
$$y' = 0.2\left[\sin(\beta) + \beta\cos(\beta)\right], \quad y'' = 0.2\left[2\cos(\beta) - \beta\sin(\beta)\right],$$
$$z' = 0.2\beta, \quad z'' = 0.2.$$

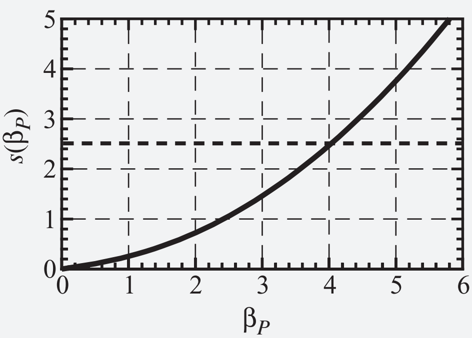
The first derivatives are the components of $\bar{r}\,'$, so

$$\begin{aligned} s' &= \left[(x')^2 + (y')^2 + (z')^2\right]^{1/2} \\ &= 0.2\left(2\beta^2 + 1\right)^{1/2}. \end{aligned}$$

We find s as a function of β by integrating the preceding. Setting $x = y = z = 0$ for the starting position reveals that $\beta = 0$ there, so that is the lower limit for a definite integral. Thus,

$$s(\beta_P) = \int_0^{\beta_P} 0.2\left(2\beta^2 + 1\right)^{1/2} d\beta = 0.1\beta\left(2\beta_P^2 + 1\right)^{1/2} + 0.05\sqrt{2}\sinh^{-1}\left(\sqrt{2}\beta_P\right).$$

The root-finding function of our mathematical software can be used to find the value of β_P for which $s = 2.5$ m. To do so we need a starting value of β_P, which we can obtain by graphing s for a range of β_P. From the graph, we see that a good starting value is $\beta_P = 4$. The root finder then yields $\beta_P = 4.026$. (It should be mentioned that if one encounters a situation for which it is not possible to integrate $s'd\beta$ analytically, numerical integration can be used to determine s at a set of β values.)



Graph of arc length as a function of the parameter β.

The values of $\bar{r}\,'$ and $\bar{r}\,''$ corresponding to this value of β_P are

$$\bar{r}\,' = x'\bar{i} + y'\bar{j} + z'\bar{k} = 0.4962\bar{i} - 0.6650\bar{j} + 0.8052\bar{k},$$
$$\bar{r}\,'' = x''\bar{i} + y''\bar{j} + z''\bar{k} = 0.8197\bar{i} + 0.3995\bar{j} + 0.20\bar{k},$$

and $s'(\beta_P) = 1.1489$. Substitution of these values into Eqs. (2.1.18), (2.1.29), and (2.1.28) yields

$$\bar{e}_t = 0.4292\bar{i} - 0.5751\bar{j} + 0.6965\bar{k},$$

$$\bar{e}_n = 0.7975\bar{i} + 0.6033\bar{j} + 0.0068\bar{k},$$

$$\rho = 1.5226 \text{ m}.$$

The basic kinematical formulas, Eqs. (2.1.10), correspondingly give

$$\bar{v} = 4.292\bar{i} - 5.751\bar{j} + 6.965\bar{k} \text{ m/s},$$

$$\bar{a} = 60.96\bar{i} + 28.123\bar{j} + 14.38\bar{k} \text{ m/s}^2. \quad \triangleleft$$

The last task is to describe the path. We observe that x and y are respectively proportional to $\cos(\beta)$ and $\sin(\beta)$, and recall that $(x^2 + y^2)^{1/2}$ is the distance to the z axis. This suggests that we consider polar coordinates, with the polar angle defined by $\tan\theta = y/x = \tan\beta$, so we take $\theta = \beta$. Thus the radial distance $R = (x^2 + y^2)^{1/2} = 0.2\theta$, which means that the projection of the path onto the xy plane is a linearly increasing spiral. The axial distance becomes $z = 0.1\theta^2 = 2.5R^2$, which tells us that the distance along the z axis increases as the square of the perpendicular distance from the z axis. In other words, the path seems to be a rapidly expanding helical-type path that wraps around the z axis.

2.1.3 Binormal Direction and Torsion of a Curve

The development thus far is adequate to determine the velocity and acceleration. However, additional study of the unit vectors will enhance our understanding of the properties of curves. Because $\bar{e}_t$ and $\bar{e}_n$ are situated in the osculating plane, the binormal direction $\bar{e}_b$ was introduced to resolve an arbitrary vector, such as the resultant force acting on a particle. This direction was defined by use of a cross product, $\bar{e}_b = \bar{e}_t \times \bar{e}_n$. However, it also is possible to express $\bar{e}_b$ in terms of derivatives of the other path variable unit vectors, which leads to an expression that parallels the definitions of $\bar{e}_t$ and $\bar{e}_n$

The component of an arbitrary vector in a specific direction may be obtained from a dot product with a unit vector in that direction. Multiplying each component by the corresponding unit vector and adding the individual contributions then reproduces the original vector. Applying this notion to the description of the derivative of $\bar{e}_n$ gives

$$\frac{d\bar{e}_n}{ds} = \left(\frac{d\bar{e}_n}{ds} \cdot \bar{e}_t\right)\bar{e}_t + \left(\frac{d\bar{e}_n}{ds} \cdot \bar{e}_n\right)\bar{e}_n + \left(\frac{d\bar{e}_n}{ds} \cdot \bar{e}_b\right)\bar{e}_b. \tag{2.1.33}$$

We obtain the tangential component in Eq. (2.1.33) from the orthogonality of the unit vectors, which requires that $\bar{e}_n \cdot \bar{e}_t = 0$, so that

$$\frac{d\bar{e}_n}{ds} \cdot \bar{e}_t = -\bar{e}_n \cdot \frac{d\bar{e}_t}{ds} = -\bar{e}_n \cdot \left(\frac{1}{\rho}\bar{e}_n\right) = -\frac{1}{\rho}. \tag{2.1.34}$$

Similarly, because $\bar{e}_n \cdot \bar{e}_n = 1$, we find that

$$\frac{d\bar{e}_n}{ds} \cdot \bar{e}_n = 0. \tag{2.1.35}$$

The binormal component in Eq. (2.1.33) is generally nonzero; its value is defined to be the reciprocal of the path's *torsion* τ:

$$\frac{d\bar{e}_n}{ds} \cdot \bar{e}_b \equiv \frac{1}{\tau}. \tag{2.1.36}$$

The reciprocal is used here for consistency with Eq. (2.1.7), such that τ has the dimension of length. Substitution of Eqs. (2.1.34)–(2.1.36) into Eq. (2.1.33) results in

$$\boxed{\frac{d\bar{e}_n}{ds} = -\frac{1}{\rho}\bar{e}_t + \frac{1}{\tau}\bar{e}_b.} \tag{2.1.37}$$

The derivative of $\bar{e}_b$ may be obtained by a similar approach. The decomposition of $d\bar{e}_b/ds$ is

$$\frac{d\bar{e}_b}{ds} = \left(\frac{d\bar{e}_b}{ds} \cdot \bar{e}_t\right)\bar{e}_t + \left(\frac{d\bar{e}_b}{ds} \cdot \bar{e}_n\right)\bar{e}_n + \left(\frac{d\bar{e}_b}{ds} \cdot \bar{e}_b\right)\bar{e}_b. \tag{2.1.38}$$

The fact that $\bar{e}_t$, $\bar{e}_n$, and $\bar{e}_b$ are mutually orthogonal, in combination with Eqs. (2.1.7) and (2.1.37), yields

$$\begin{gathered}
\bar{e}_b \cdot \bar{e}_t = 0 \implies \frac{d\bar{e}_b}{ds} \cdot \bar{e}_t = -\,\bar{e}_b \cdot \frac{d\bar{e}_t}{ds} = -\,\bar{e}_b \cdot \frac{1}{\rho}\bar{e}_n = 0, \\
\bar{e}_b \cdot \bar{e}_n = 0 \implies \frac{d\bar{e}_b}{ds} \cdot \bar{e}_n = -\,\bar{e}_b \cdot \frac{d\bar{e}_n}{ds} = -\bar{e}_b \cdot \left(-\frac{1}{\rho}\bar{e}_t + \frac{1}{\tau}\bar{e}_b\right) = -\frac{1}{\tau}, \\
\bar{e}_b \cdot \bar{e}_b = 0 \implies \frac{d\bar{e}_b}{ds} \cdot \bar{e}_b = 0.
\end{gathered} \tag{2.1.39}$$

It follows that

$$\boxed{\frac{d\bar{e}_b}{ds} = -\frac{1}{\tau}e_n.} \tag{2.1.40}$$

Because $\bar{e}_n$ is a unit vector, this relation provides an alternative to Eq. (2.1.36) for the torsion:

$$\frac{1}{\tau} = \left|\frac{d\bar{e}_b}{ds}\right|. \tag{2.1.41}$$

Equations (2.1.7), (2.1.37), and (2.1.40) are Frenet's formulas for a spatial curve. The first one shows that a small advancement along the path primarily changes the tangent vector in the normal direction. The osculating plane is formed from $\bar{e}_t$ and $\bar{e}_n$. We therefore may consider this plane to be the tangent plane that most closely fits the curve at the position of interest. Equation (2.1.40) shows that small increments in s primarily change the binormal vector in the direction of $\bar{e}_n$. This is equivalent to a rotation of the osculating plane about the tangent direction, which is the source of the terminology "torsion." The osculating plane is constant for a planar curve, which corresponds to an infinite value of τ. The greater the degree to which a curve is twisted in space, the smaller will be the value of τ. In a similar vein, ρ measures the amount by which the curve bends in the osculating plane. A small value of ρ corresponds to a highly bent curve, and ρ is infinite for a straight line.

2.2 RECTANGULAR CARTESIAN COORDINATES

We now turn our attention to extrinsic coordinate systems, in which all properties are defined independently of knowledge of the path. The simplest set of extrinsic coordinates is rectangular Cartesian coordinates. These are associated with orthogonal xyz axes that are right-handed by convention. Situations in which such coordinates might be suitable are recognizable by the fact that vectors (position, velocity, force, etc.) are described in terms of components with respect to fixed directions, such as left–right and up–down. Figure 2.4 shows that the components of the position vector are merely the (x, y, z) coordinates. This may be established by either of two alternatives. The position of the point may be projected onto each of the three coordinate planes, after which each projection point is itself projected onto the coordinate axes of that plane. The projection lines form a box, for which the position vector $\bar{r}$ is a main diagonal. The second construction drops a perpendicular from the point onto each of the coordinate axes. The perpendiculars form the diagonals of each face of the box formed in the first construction.

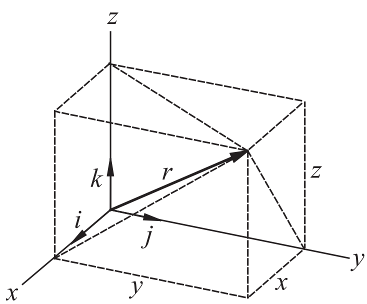


Figure 2.4. Rectangular Cartesian coordinates.

Each coordinate may be a function of time, so the position is given by

$$\boxed{\bar{r}_{P/O} = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}.} \tag{2.2.1}$$

Differentiating this expression is a simple matter because the unit vectors are constant. Thus, the velocity is given by

$$\boxed{\begin{gathered}\bar{v} = v_x\bar{i} + v_y\bar{j} + v_z\bar{k},\\ v_x = \dot{x}, \quad v_y = \dot{y}, \quad v_z = \dot{z},\end{gathered}} \tag{2.2.2}$$

from which it follows that the acceleration is

$$\boxed{\begin{gathered}\bar{a} = a_x\bar{i} + a_y\bar{j} + a_z\bar{k},\\ a_x = \dot{v}_x = \ddot{x}, \quad a_y = \dot{v}_y = \ddot{y}, \quad a_z = \dot{v}_z = \ddot{z}.\end{gathered}} \tag{2.2.3}$$

A notable feature of these relations is that the motions in the x, y, and z are *uncoupled inertially*. Specifically, the acceleration in one direction does not contain the other coordinates. One way of regarding this result conceptually is to think of it as a superposition of rectilinear (i.e., straight line) motions in each of the coordinate directions. However, one should not infer from this observation that the motions in the three directions are independent. This becomes evident when we formulate Newton's Second Law in terms of components relative to the xyz axes:

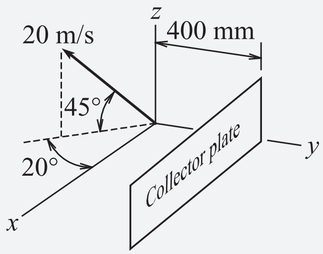
$$\boxed{\begin{aligned} \Sigma F_x &\equiv \Sigma \bar{F} \cdot \bar{i} = m\ddot{x}, \\ \Sigma F_y &\equiv \Sigma \bar{F} \cdot \bar{j} = m\ddot{y}, \\ \Sigma F_z &\equiv \Sigma \bar{F} \cdot \bar{k} = m\ddot{z}. \end{aligned}} \tag{2.2.4}$$

The force components might be known as functions of time, but they also can depend on the x, y, and/or z coordinates of the particle's location, as well as the $\dot{x}$, $\dot{y}$, and/or $\dot{z}$ velocity components. Thus, if these dependencies are known, Eqs. (2.2.4) can represent as many as three coupled ordinary differential equations whose solution is the position coordinates of the particle as a function of time.

In fully uncoupled situations the resultant force components depend solely on the corresponding position or velocity coordinate. For example, suppose $\ddot{x} = f(x, v_x, t)$. This constitutes a differential equation for x because $v_x = \dot{x}$. The techniques for separating variables that were described in the context of path variables, specifically Eqs. (2.1.14)–(2.1.16), are directly applicable here. If the force components couple motion in different directions, an analytical solution might be obtainable if all terms depend linearly on the position coordinates and their time derivatives. Of course, if analytical techniques do not seem promising, one can always solve the differential equations of motion numerically. Numerical solution techniques are discussed in Sections 7.6 and 8.2.

Not surprisingly, the simplicity of Cartesian coordinate formulation limits its usefulness. Practical situations in which the motion is given in terms of fixed directions are not abundant. The most common involves projectile motion near the Earth's surface. In that case the force of gravity is considered to act in the downward vertical direction, which means that the acceleration is always downward. Even this case breaks down when one wishes to treat the motion more accurately. For example, if it is desired to account for air resistance, the resistance force is always opposite the velocity. Such a force is readily described in path variables as $-f\bar{e}_t$. The description of projectile motion in terms of Cartesian coordinates also encounters difficulty when the motion covers a long range, as is the case for ballistic missiles. Then the gravitational force is always directed toward a fixed point, rather than having a fixed direction. A kinematical description using polar coordinates, see the next section, is more suitable to this type of consideration. The corresponding analysis is *orbital motion.*

EXAMPLE 2.4 A 10-mg dust particle is injected into an electrostatic precipitator with an initial velocity of 20 m/s, as shown. The z axis is vertical and the attractive force on the particle is $1.6 - 4y$ mN acting in the positive y direction, where y is measured in meters. Determine the location and velocity at which the dust particle will strike a collector plate that is situated in the vertical plane defined by $y = 400$ mm.



Example 2.4

SOLUTION In addition to showing how a decomposition of forces into a set of components can lead to differential equations governing the movement of a particle, a primary intent of this example is to emphasize that constant acceleration rates are the exceptional case. The forces acting on the particle are its weight and the electrostatic force, both of which act in fixed directions. It is for this reason that we employ Cartesian coordinates. Forming $\Sigma \bar{F} = m\bar{a}$ in units of newtons gives

$$(1.6 - 4y)\left(10^{-3}\right)\bar{j} - 10\left(10^{-6}\right)(9.807)\bar{k} = 10\left(10^{-6}\right)\bar{a}. \tag{1}$$

We proceed to take components of this equation in the three coordinate directions. We have an alternative here, depending on whether we consider $\bar{a}$ to be the first derivative of velocity or the second derivative of position. In the former viewpoint we substitute $\bar{a} = \dot{v}_x\bar{i} + \dot{v}_y\bar{j} + \dot{v}_z\bar{k}$, which yields

$$\dot{v}_x = 0, \quad \dot{v}_y = 160 - 400y, \ \dot{v}_z = -9.807. \tag{2}$$

These are first-order differential equations for the velocity components. The accelerations are constant in the first and third equations, so they may be integrated directly. In the second equation the acceleration rate depends on the corresponding distance, so we may solve this equation by changing s to y in Eq. (2.1.15). Note that integrating the equations in this manner yields solutions for the rate variables. Replacing each rate variable with its definition, that is, $v_x = \dot{x}$, $v_y = \dot{y}$, and $v_z = \dot{z}$, leads to another set of differential equations for the position coordinates.

The second of the aforementioned approaches entails writing $\bar{a} = \ddot{x}\bar{i} + \ddot{y}\bar{j} + \ddot{z}\bar{k}$, which leads to

$$\ddot{x} = 0, \quad \ddot{y} = 160 - 400y, \quad \ddot{z} = -9.807. \tag{3}$$

We now have three second-order differential equations of motion. Either approach is suitable in this problem because analytical solutions are readily obtained. We shall follow the latter one here because it directly yields the position coordinates as functions of time. In other situations, the first-order approach might be easier to implement.

Equations (3) are linear and second order, with constant coefficients. We obtain the initial conditions by setting $\bar{r}_0 = \bar{0}$ for launch at the origin, so the initial coordinates are $x_0 = y_0 = z_0 = 0$. Resolving the given initial velocity into components along the coordinate axes gives

$$\begin{aligned}\bar{v}_0 &= 20\cos 45^\circ(\cos 20^\circ i - \sin 20^\circ j) + 20\sin 45^\circ k\\ &= 13.289\bar{i} - 4.837\bar{j} + 14.142\bar{k}.\end{aligned} \tag{4}$$

The components of this velocity are the scalar initial values: $\dot{x}_0 = 13.289$ m/s, $\dot{y}_0 = -4.837$ ms/s, $\dot{z}_0 = 14.142$ m/s at $t = 0$.

Integrating the first of Eqs. (3) twice gives

$$x = c_1 t + c_2.$$

This expression must match the initial values x_0 and $\dot{x}_0$, which requires that $c_1 = 13.289$, $c_2 = 0$. Integrating the third of Eqs. (3) twice gives

$$z = -\frac{1}{2}(9.807)\,t^2 + c_3 t + c_4.$$

We find the integration constants by matching this expression to the initial values z_0 and $\dot{z}_0$, which gives $c_3 = 14.142$, $c_4 = 0$. The second of Eqs. (3) is a standard differential equation, $\ddot{y} + 400y = 160$, whose general solution is

$$y = c_5\cos(20t) + c_6\sin(20t) + 0.4.$$

Equating y and $\dot{y}$ from this relation to the initial conditions at $t = 0$ yields $c_5 = -0.4$, $c_6 = -0.2418$. Thus the Cartesian coordinates of the particle as functions of time are

$$\begin{gathered}x = 13.289t, \quad y = -0.4\cos(20t) - 0.2418\sin(20t) + 0.4,\\ z = -4.9035t^2 + 14.142t \text{ m}.\end{gathered} \tag{5}$$

Now that we have determined the response as a function of time, we may study its properties. We find the instant t_f when the particle hits the collector plate by setting $y = 0.4$ for $t > 0$, which occurs when

$$-0.4\cos(20t_f) - 0.2418\sin(20t_f) = 0 \implies \tan(20t_f) = -1.6543.$$

The smallest positive root corresponds to

$$\begin{gathered}20t_f = \tan^{-1}(-1.6543) = -1.0270 + \pi,\\ t_f = 0.10573 \text{ s}.\end{gathered}$$

Evaluating the position components for that instant yields

$$x = 1.4050, \quad y = 0.4, \quad z = 1.4404 \text{ m}. \qquad \triangleleft$$

We obtain the final velocity components by differentiating Eqs. (5) and then evaluating the results at t_f. This gives

$$\dot{x} = 13.289, \quad \dot{y} = 9.348, \quad \dot{z} = 13.105 \text{ m/s}. \qquad \triangleleft$$

EXAMPLE 2.5 A right circular cone is defined by $x^2 + y^2 = 9z^2$, (x, y, and z have units of millimeters). The vertical position of a block sliding along the interior of such a cone is observed to be $z = 480 - 80t^2$, and $x = y^2/200$. Also, $y > 0$ throughout the motion. Determine the velocity and acceleration of the block when $t = 2$ s.

SOLUTION In addition to demonstrating the application of the basic Cartesian coordinate formulas, this example shows how one can beneficially use implicit differentiation. Because the intersection of two functions relating x, y, and z is a curve, the functions for the conical surface and for x in terms of y specify the path of the particle. In the present situation $x = y^2/200$, which is a parabola whose axis of symmetry is x, describes the projection of the path onto the xy plane. The equation for the cone then gives the z value corresponding to a specified x, y pair. We elect to use Cartesian coordinates, rather than path variables, because the second of the given relations prescribes the movement in terms of the distance along the z axis. We could simplify the functional relationships by using the first and third equations to relate y solely to z, but this is not done in order to demonstrate how one could handle truly complicated functional relations.

The given position equations with x, y, and z in meters are

$$z = 0.480 - 0.080t^2, \quad x = 5y^2, \quad x^2 + y^2 = 9z^2 \text{ m.} \tag{1}$$

Differentiation of these expressions yields relations governing $\dot{x}$, $\dot{y}$, and $\dot{z}$:

$$\dot{z} = -0.16t, \quad \dot{x} = 10y\dot{y}, \quad x\dot{x} + y\dot{y} = 9z\dot{z}. \tag{2}$$

A second differentiation leads to

$$\begin{gathered} \ddot{z} = -0.16, \quad \ddot{x} = 10(y\ddot{y} + \dot{y}^2), \\ x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2 = 9(z\ddot{z} + \dot{z}^2). \end{gathered} \tag{3}$$

We may solve Eqs. (1) for x, y, and z at a specified t. Then Eqs. (2) become a set of linear equations for the corresponding first derivatives, which may be written in matrix form as

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -10y & 0 \\ x & y & -9z \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{y}\ \dot{z} \end{Bmatrix} = \begin{Bmatrix} -0.16t \\ 0 \\ 0 \end{Bmatrix}. \tag{4}$$

After these equations are satisfied, Eqs. (3) become simultaneous linear equations for the second derivatives, specifically,

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & -10y & 0 \\ x & y & -9z \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{Bmatrix} = \begin{Bmatrix} -0.16 \\ 10\dot{y}^2 \\ 9\dot{z}^2 - \dot{x}^2 - \dot{y}^2 \end{Bmatrix}. \tag{5}$$

We begin by substituting the first and second of Eqs. (1) into the third equation, which yields $25y^4 + y^2 = 9\left(0.08t^2\right)^2$. We retain the real root of this equation at $t = 2$ satisfying $y > 0$, and then evaluate the corresponding x,

$$x = 0.3903, \quad y = 0.2794, \quad z = 0.160 \text{ m}. \tag{6}$$

These values are substituted into Eqs. (4), whose solution at $t = 2$ is

$$\dot{x} = -0.9398, \quad \dot{y} = -0.3364, \quad \dot{z} = -0.32 \text{ m/s}. \tag{7}$$

Finally, we substitute Eqs. (6) and (7) into Eqs. (5), which yield

$$\ddot{x} = -0.3917, \quad \ddot{y} = -0.5452, \quad \ddot{z} = -0.160 \text{ m/s}^2.$$

The derivatives are the respective components of the velocity and acceleration, so

$$\begin{aligned}\bar{v} &= -0.9398\bar{i} - 0.3364\bar{j} - 0.320\bar{k} \text{ m/s},\\ \bar{a} &= -0.3917\bar{i} - 0.5452\bar{j} - 0.160\bar{k} \text{ m/s}^2.\end{aligned} \qquad \triangleleft$$

2.3 CURVILINEAR COORDINATES

Cartesian coordinates specify the location of a point by giving three numbers that are distances along the coordinate axes. Curvilinear coordinates also use a triad of parameters to locate a point, but they generalize the description by allowing the unit vectors associated with these parameters to be variable. Let α, β, and γ be the three parameters, such that there is a unique transformation between the (α, β, γ) values and the (x, y, z) rectangular Cartesian coordinates. The general form of this transformation is

$$x = x(\alpha, \beta, \gamma), \quad y = y(\alpha, \beta, \gamma), \quad z = z(\alpha, \beta, \gamma). \tag{2.3.1}$$

Occasionally, we need the inverse transformation, whose form would be

$$\alpha = \alpha(x, y, z), \quad \beta = \beta(x, y, z), \quad \gamma = \gamma(x, y, z). \tag{2.3.2}$$

When two of the parameters (α, β, γ) are held constant, and the third is given a range of values, Eqs. (2.3.1) specify a curve in space in parametric form. When the constant parameter pair is given an assortment of values, the result is a family of curves. Repeating this procedure with each of the other pairs of parameters held constant produces two more families of curves. These families of curves form a spatial mesh. The intersection of curves belonging to different families are orthogonal in the cases that we shall treat, in which case (α, β, γ) are said to be *orthogonal curvilinear coordinates*. The name for each set of coordinates usually corresponds to one of the types of surfaces on which one of the curvilinear coordinates is constant. We begin by studying cylindrical and spherical coordinates as special cases prior to tackling an arbitrary set of coordinates.

2.3.1 Cylindrical and Polar Coordinates

The vast majority of situations we encounter are well described in terms of cylindrical and spherical coordinates, both of which have meshes that consist of circles and straight lines. In cylindrical coordinates, we select one of the Cartesian coordinate axes as a reference. We then locate a point by perpendicularly projecting its position onto the coordinate plane formed by the other two axes and onto the reference axis. Without loss of generality, let z be the reference axis. The corresponding construction is illustrated in Fig. 2.5. The distance from the point to the z axis is the *transverse distance* R, and the distance from the point to the xy plane is the *axial distance* z. It still remains to locate the plane formed by the axial and transverse lines, for which we use the angle θ representing the rotation of the plane about the reference axis. Prior to the advent of inertial navigation, global positioning employed a sextant, which used a telescope on a swivel platform to locate the North Star. The angle the telescope was rotated about its platform was called the *azimuthal angle*. That is the name we shall use for θ, but some individuals prefer to call it the *circumferential angle*. To avoid ambiguity when it is necessary to select a value of θ corresponding to a specified position we shall limit the azimuthal angle range to $-\pi < \theta \leq \pi$. In Fig. 2.5, $\theta = 0$ places the shaded plane at the xz plane, and θ is measured counterclockwise looking down the z axis, but neither convention is mandatory.

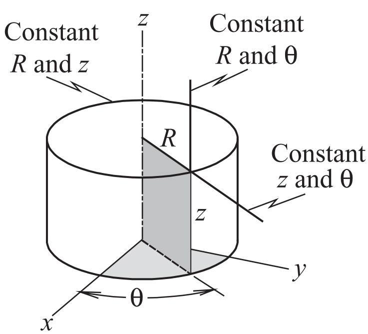


Figure 2.5. Definition of the cylindrical coordinates.

The values of (R, θ, z) are the *cylindrical coordinates*. By themselves, R and θ are *polar coordinates*. They locate the point in the xy plane, whereas the value of z tells us how far the point is from that plane. Geometrical constructions show the transformation from (R, θ, z) to (x, y, z) to be

$$\boxed{x = R\cos\theta, \quad y = R\sin\theta, \quad z = z.} \tag{2.3.3}$$

To construct the coordinate mesh, we observe that allowing R to change with θ and z held constant moves the point along a radial line, perpendicular to the z axis, whereas allowing z to change with R and θ held constant moves the point along a axial line, parallel to the z axis. The third coordinate curve is a circle parallel to the xy plane that is produced when θ changes with R and z is held constant. The curvilinear coordinate mesh corresponds to various transverse lines (different fixed θ and z values), axial lines (different fixed R and θ values), and azimuthal circles (different fixed R and z values).

To define unit vectors for vector components, we observe that Cartesian coordinate directions $\bar{i}$, $\bar{j}$, and $\bar{k}$ are the directions in which a point moves if two of the x, y, and z values are held constant, and the other coordinate value is increased. We define the curvilinear coordinate unit vectors in an analogous manner. We use $\bar{e}$ with the appropriate subscript to denote the unit vectors. Thus $\bar{e}_R$ is the direction in which a point moves if R is increased with θ and z constant; this is the *transverse direction*, which is outward along the transverse line from the z axis to the point. Increasing θ with R and z constant moves the point tangentially along the azimuthal circle; this is the *azimuthal direction* $\bar{e}_\theta$. Finally, increasing z with R and θ constant moves the point upward parallel to the z axis; this is the *axial direction* $\bar{e}_z$. The set of cylindrical coordinate unit vectors is depicted in Fig. 2.6. Avoidance of sign errors requires that we remember that the sense of the unit vectors is *always* defined according to the sense of increasing coordinate values. The unit vectors we have defined form a right-handed set according to

$$\bar{e}_R \times \bar{e}_\theta = \bar{e}_z. \tag{2.3.4}$$

However, if one needs to compute cross products, the foregoing should not be assumed because it is sometimes more convenient to define θ to be in a clockwise sense relative to the axial direction, which would reverse the sense of $\bar{e}_\theta$. Also, although we selected z to be the axial direction, it may be convenient in some situation to select a different axis. This is the primary reason for using $\bar{e}_z$ rather than $\bar{k}$. The unit vectors may be described in terms of $\bar{i}$, $\bar{j}$, $\bar{k}$ components by projecting them onto the xy plane. Doing so yields

$$\bar{e}_R = (\cos\theta)\,\bar{i} + (\sin\theta)\,\bar{j}, \quad \bar{e}_\theta = -(\sin\theta)\,\bar{i} + (\cos\theta)\,\bar{j}, \quad \bar{e}_z = \bar{k}. \tag{2.3.5}$$

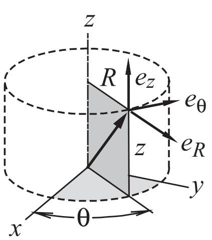


Figure 2.6. Definition of the unit vectors for cylindrical coordinates.

Figure 2.6 shows the instantaneous position $\bar{r}$ of a point. Decomposing this vector into its components relative to the cylindrical coordinate unit vectors reveals that

$$\boxed{\bar{r} = R\bar{e}_R + z\bar{e}_z.} \tag{2.3.6}$$

At first glance, this expression seems to be inconsistent with the fact that $\bar{r}$ depends on θ. However, the value of θ must be known in order to define the instantaneous orientation of $\bar{e}_R$. Differentiation of this expression to find the velocity leads to

$$\bar{v} = \dot{R}\bar{e}_R + R\frac{d\bar{e}_R}{dt} + \dot{z}\bar{e}_z. \tag{2.3.7}$$

Note that the radial unit vector is recognized as being a variable, because $\bar{e}_R$ depends on θ, and θ may depend on time. Hence, evaluating the derivative of this unit vector requires the chain rule,

$$\frac{d\bar{e}_R}{dt} = \dot{\theta}\frac{d\bar{e}_R}{d\theta} \tag{2.3.8}$$

We evaluate the preceding derivative by differentiating the first of Eqs. (2.3.5), which gives

$$\frac{d\bar{e}_R}{d\theta} = -(\sin\theta)\,\bar{i} + (\cos\theta)\,\bar{j} = \bar{e}_\theta. \tag{2.3.9}$$

We also will require the derivative of $\bar{e}_\theta$, which we obtain from the second of Eqs. (2.3.5):

$$\frac{d\bar{e}_\theta}{d\theta} = -(\cos\theta)\,\bar{i} - (\sin\theta)\,\bar{j} = -\bar{e}_R \tag{2.3.10}$$

Both derivatives could have been obtained pictorially. Recall that the derivative of a unit vector must be perpendicular to the vector. Figure 2.7(a) depicts the unit vectors at two close positions, and Fig. 2.7(b) places the vectors tail to tail. The angle between $\bar{e}_R(\theta)$ and $\bar{e}_R(\theta+\Delta\theta)$, and between $\bar{e}_\theta(\theta)$ and $\bar{e}_\theta(\theta+\Delta\theta)$, is $\Delta\theta$. The length of each unit vector is unity, so when $\Delta\theta$ is very small, the length of $\Delta\bar{e}_R$ is approximately $\Delta\theta$. The same is true for $\Delta\bar{e}_\theta$. In the limit as $\Delta\theta$ approaches $d\theta$, the approximation becomes exact. Furthermore, in this limit $d\bar{e}_R$ is parallel to $\bar{e}_\theta(\theta)$, so that $d\bar{e}_R = d\theta\,\bar{e}_\theta$. In the same limit, $d\bar{e}_\theta$ is parallel to $-\bar{e}_R(\theta)$, which leads to $d\bar{e}_\theta = -d\theta\,\bar{e}_R$. Division of these descriptions of $d\bar{e}_R$ and $d\bar{e}_\theta$ by $d\theta$ leads to Eqs. (2.3.9) and (2.3.10).

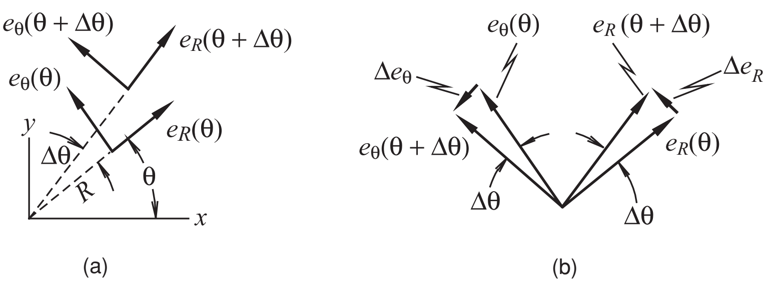


Figure 2.7. Differentiation of the radial and transverse unit vectors.

Substitution of Eqs. (2.3.9) and (2.3.8) into Eq. (2.3.7) gives the required expression for velocity:

$$\boxed{\bar{v} = \dot{R}\bar{e}_R + R\dot{\theta}\bar{e}_\theta + \dot{z}\bar{e}_z.} \tag{2.3.11}$$

The radial and axial velocity components have obvious meanings as rates of change of the corresponding distances. The second term tells us that the azimuthal velocity results from changing the azimuthal angle, with the effect growing in proportion to the radial distance. The latter matches our everyday experience, in that we rotate our head slowly when we track a faraway object moving at a high speed.

An expression for acceleration results from differentiating the velocity description. Let us start with the component representation of velocity $\bar{v} = v_R\bar{e}_r + v_\theta\bar{e}_\theta + v_z\bar{e}_z$.

We have already obtained $d\bar{e}_R/dt$, and the chain rule in conjunction with Eq. (2.3.10) gives $d\bar{e}_\theta/dt = -\dot{\theta}\, e_R$, from which we find that

$$\bar{a} = \left(\dot{v}_R - v_\theta\dot{\theta}\right)\bar{e}_r + \left(\dot{v}_\theta + v_R\dot{\theta}\right)\bar{e}_\theta + \dot{v}_z\bar{e}_z. \tag{2.3.12}$$

This description explicitly describes the fact that acceleration results from variability of the directions in which the velocity components are measured, as well as if the components are not constant. This form is useful when the velocity components are known as functions of time, but the more usual circumstance is that in which the cylindrical coordinates are known. In that case we differentiate Eq. (2.3.11) directly, to obtain

$$\begin{aligned}\bar{a} &= \frac{d}{dt}\left[\dot{R}\bar{e}_R\right] + \frac{d}{dt}\left[R\dot{\theta}\bar{e}_\theta\right] + \frac{d}{dt}\left[\dot{z}\bar{e}_z\right] \\ &= \left[\ddot{R}\bar{e}_R + \dot{R}\dot{\theta}\bar{e}_\theta\right] + \left[\dot{R}\dot{\theta}\bar{e}_\theta + R\ddot{\theta}\bar{e}_\theta - R\dot{\theta}^2\bar{e}_R\right] + \ddot{z}\bar{e}_z.\end{aligned} \tag{2.3.13}$$

The brackets in the preceding equation enable us to track which terms originate from the same velocity component. Both $\ddot{R}$ and $\ddot{z}$ are recognizable as acceleration rates in the respective directions, whereas $R\ddot{\theta}$ is an azimuthal acceleration that arises because an unsteady value of $\dot{\theta}$ will increase the transverse velocity component $R\dot{\theta}$. The fifth term, $-R\dot{\theta}^2\bar{e}_R$, results from the changing direction of $\bar{e}_\theta$; it occurs even if the azimuthal velocity $R\dot{\theta}$ is constant. Although the second and third terms are both $\dot{R}\dot{\theta}\bar{e}_\theta$, the brackets indicate that they originate from different velocity components, and therefore represent different effects. The term in the first bracket stems from the fact that, even if the radial speed $\dot{R}$ were constant, the radial direction is not constant. In contrast, the term in the second bracket is the rate at which the azimuthal speed, $R\dot{\theta}$, will change if R is not constant. Together, the two identical terms constitute the *Coriolis acceleration*, after G. Coriolis (1792–1843) who successfully explained the phenomenon. Grouping like terms leads to the desired acceleration expression.

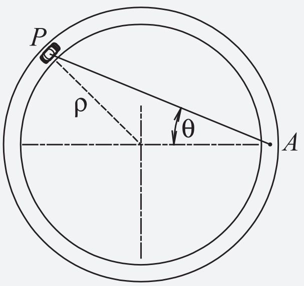
$$\boxed{\bar{a} = \left(\ddot{R} - R\dot{\theta}^2\right)\bar{e}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\bar{e}_\theta + \ddot{z}\bar{e}_z.} \tag{2.3.14}$$

The scalar form of Newton's Second Law that we obtain by decomposing it into the cylindrical coordinate directions is

$$\boxed{\begin{gathered}\Sigma F_R \equiv \Sigma\bar{F}\cdot\bar{e}_R = m\left(\ddot{R} - R\dot{\theta}^2\right), \\ \Sigma F_\theta \equiv \Sigma\bar{F}\cdot\bar{e}_\theta = m\left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right), \\ \Sigma F_z \equiv \Sigma\bar{F}\cdot\bar{e}_z = m\ddot{z}.\end{gathered}} \tag{2.3.15}$$

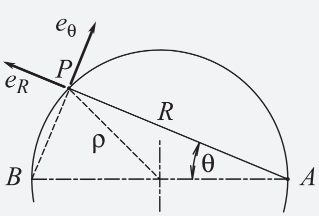
In general, one can expect cylindrical coordinates to be useful if some aspect of the force system or motion is best described in terms of a direction that perpendicularly intersects a fixed line, or a fixed point in the case of planar motion. It is clear that in situations in which it is desired to determine R and θ as functions of time resulting from application of a known force resultant, the preceding equation constitute nonlinear differential equations in which R and θ are strongly coupled.

EXAMPLE 2.6 An observer at point A watches automobile P follow the circular track. The angle between the diametral line to the observer and the radial line to the automobile is a measured function of time, $\theta(t)$. Derive expressions for the velocity and acceleration in terms of the radius ρ of the track and θ.



Example 2.6

SOLUTION One of the primary purposes of this example, in addition to illustrating the basic use of the formulas, is to emphasize that there is a variety of descriptions that might be useful for motion along a circular path. We use polar coordinates here because the motion is defined by an angle in a plane measured relative to a fixed line. The polar coordinate variables and the associated unit vectors are defined in a sketch, with the origin placed at point A, where the observer is.



Polar coordinates and unit vectors corresponding to origin A.

It is known from geometry that triangle ABP is a right triangle, with side AB as the hypotenuse. Thus the transverse distance is given by

$$R = 2\rho \cos\theta.$$

This expression may be differentiated with respect to t, but in doing so we must recognize that $\theta(t)$ is unspecified, so we cannot assume that $\ddot{\theta}$ is zero. Thus,

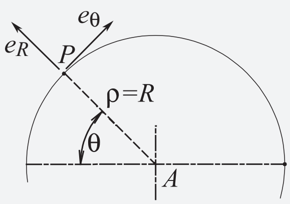
$$\dot{R} = (-2\rho \sin\theta)\,\dot{\theta}, \qquad \ddot{R} = (-2\rho\cos\theta)\,\dot{\theta}^2 + (-2\rho\sin\theta)\,\ddot{\theta}.$$

We substitute these expressions into Eqs. (2.3.11) and (2.3.14) to find

$$\bar{v} = -2\rho\dot{\theta}\sin\theta\,\bar{e}_R + 2\rho\dot{\theta}\cos\theta\,\bar{e}_\theta,$$
$$\bar{a} = -2\rho\left(2\dot{\theta}^2\cos\theta + \ddot{\theta}\sin\theta\right)\bar{e}_R + 2\rho\left(\ddot{\theta}\cos\theta - 2\dot{\theta}^2\sin\theta\right)\bar{e}_\theta. \quad \triangleleft$$

There is no need to convert these results to $\bar{i}$ and $\bar{j}$ components because the sketch of the polar coordinates fully describes the directions of these components.

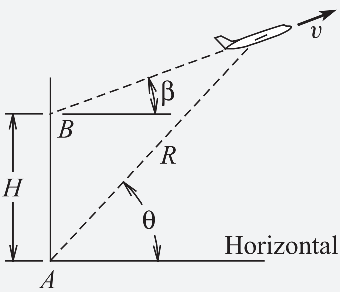
In closing, it is useful to observe that, if the observer following the automobile were situated at the center of the track, then the polar coordinate unit vectors would be as shown in the second sketch, with $\bar{e}_\theta$ tangent to the circle in the sense of increasing θ and $\bar{e}_R$ radially outward.



Polar coordinate unit vectors when the center *A* is the origin.

In comparison, the tangent direction could be either parallel to, or opposite, $\bar{e}_\theta$, depending on how the arc length s is measured, and $\bar{e}_n$ would be opposite $\bar{e}_R$, toward the center of curvature. Even though these alternative sets of directions are readily related here, the question of whether path variables or polar coordinates is more suitable depends on how movement along the path is defined. We use the former if we have information regarding s or v, whereas the latter should be selected if we have knowledge of R or θ or their derivatives as functions of time.

EXAMPLE 2.7 An airplane climbs at a constant speed v and constant climb angle β. The airplane is being tracked by a radar station at point A on the ground. Determine the radial velocity $\dot{R}$ and the angular velocity $\dot{\theta}$ as functions of the tracking angle θ.



Example 2.7

SOLUTION One objective of this example is to emphasize once again that the type of path is not the issue when one decides which kinematical description should be employed. The situation here is very much like the one in the previous example, in that movement along a given path is prescribed in terms of the rotation of a line. We implement a trigonometric approach here, in which the rate variables are obtained from differentiation of geometrical relations. (A simpler solution to this problem may be found in Example 2.11, where the Cartesian and polar coordinate descriptions of velocity and acceleration are compared.) First, we construct the distance vt the airplane has traveled after passing point B above the radar station. This forms one side of a triangle whose other sides are R and H. Then the law of sines yields

$$\frac{R}{\sin(\pi/2+\beta)} = \frac{vt}{\sin(\pi/2-\theta)} = \frac{H}{\sin(\theta-\beta)}.$$

For the purpose of differentiation it is preferable to write these two relations as

$$\begin{aligned} R\sin(\theta-\beta) &= H\sin(\pi/2+\beta) \equiv H\cos(\beta)\,, \\ vt\sin(\theta-\beta) &= H\sin(\pi/2-\theta) \equiv H\cos(\theta)\,. \end{aligned} \tag{1}$$

Differentiating each expression leads to

$$\dot{R}\sin(\theta-\beta) + R\dot{\theta}\cos(\theta-\beta) = 0,$$

$$v\sin(\theta-\beta) + vt\dot{\theta}\cos(\theta-\beta) = -H\dot{\theta}\sin(\theta)$$

These are simultaneous equations for $\dot{\theta}$ and $\dot{R}$, whose solutions are

$$\begin{aligned} \dot{\theta} &= -\frac{v\sin(\theta-\beta)}{vt\cos(\theta-\beta)+H\sin(\theta)}, \\ \dot{R} &= \frac{Rv\cos(\theta-\beta)}{vt\cos(\theta-\beta)+H\sin(\theta)}. \end{aligned} \tag{2}$$

The problem statement requested expressions for $\dot{R}$ and $\dot{\theta}$ as functions of θ, but the preceding equations also contain the variables R and t. We therefore solve Eqs. (1) for R and vt,

$$R = \frac{H\cos(\beta)}{\sin(\theta-\beta)},$$

$$vt = \frac{H\cos(\theta)}{\sin(\theta-\beta)},$$

and then substitute those results into Eqs. (2), which leads to

$$\begin{aligned} \dot{\theta} &= -\frac{v}{H}\frac{\sin(\theta-\beta)}{\cos(\theta)\cot(\theta-\beta)+\sin(\theta)}, \\ \dot{R} &= v\frac{\cos(\beta)\cot(\theta-\beta)}{\cos(\theta)\cot(\theta-\beta)+\sin(\theta)}. \end{aligned} \tag{3}$$

We may simplify these expressions by multiplying the numerator and denominator of each by $\sin(\theta-\beta)$, and then using the trigonometric identity that $\cos(\theta)\cos(\theta-\beta)+\sin(\theta)\sin(\theta-\beta) \equiv \cos[\theta-(\theta-\beta)] \equiv \cos(\beta)$, so that

$$\begin{aligned} \dot{\theta} &= -\frac{v}{H}\frac{\sin(\theta-\beta)^2}{\cos(\beta)}, \\ \dot{R} &= v\cos(\theta-\beta)\,. \end{aligned}$$

◁

Although not requested, it is useful to contemplate following the present approach to determine $\ddot{R}$ and $\ddot{\theta}$. Clearly, differentiating the results for $\dot{R}$ and $\dot{\theta}$ would be tedious. The remarkable aspect of the approach described in Section 2.4 is that it will not require explicit differentiation of any term.

2.3.2 Spherical Coordinates

Spherical coordinates locate a point in terms of one length parameter and two angles. The *radial distance* r, which is the length of the position vector from a fixed point to the point of interest, is the hallmark of the formulation. The fixed point is taken to be the origin of an xyz coordinate system, as depicted in Fig. 2.8. In an application the z axis would be selected to coincide with a relevant fixed direction. The z axis and the moving point form the *meridional plane*. The instantaneous orientation of that plane is measured by the azimuthal angle θ, just as it was in cylindrical coordinates. The second angle locating the point's position is the polar angle ϕ. The triad (r, ϕ, θ) constitutes *spherical coordinates*. To avoid ambiguity, we limit the azimuthal angle to $-\pi < \theta \leq \pi$ and the polar angle to $0 \leq \phi \leq \pi$. Note that motion in the xy plane corresponds to $\phi = \pi/2$, in which case (r, θ) constitute polar coordinates. Polar coordinates are also formed by r and ϕ for motion in any meridional plane defined by constant θ.

A modified version of spherical coordinates occurs in geography. Within the spherical Earth approximation, the radial distance from the center of the Earth to a point on the surface is the radius $R_e \approx 6370$ m. The fixed reference direction is the north–south polar axis. Position on the surface is specified by giving the longitude angle, which is the rotation angle of the radial line about the polar axis measured from the prime meridian at the Royal Observatory in Greenwich, England. The one difference is that, rather than measuring the polar angle from the North Pole, we use the latitude, which is measured from the equatorial plane. The latitude is the complement of the polar angle.

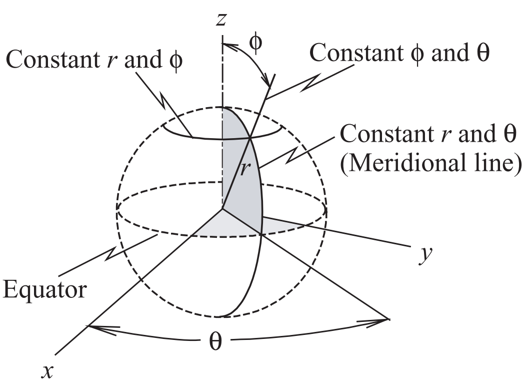


Figure 2.8. Definition of the spherical coordinates.

The transformation from (r, ϕ, θ) to (x, y, z) is found by dropping perpendiculars from the point onto the xy plane and onto the z axis. The distances from the origin to the projection points are $r \sin\phi$ and $r \cos\phi$, respectively, from which it follows that

$$\boxed{x = r \sin\phi \cos\theta, \quad y = r \sin\phi \sin\theta, \quad z = r \cos\phi.} \tag{2.3.16}$$

The spherical coordinate mesh consists of radial lines, formed by allowing r to change with ϕ and θ constant, circles of radius r lying in a meridional, and circles of radius $r \sin\phi$ parallel to the xy plane formed when θ is varied with r and ϕ constant. Different value combinations of the constant coordinate pairs associated with each type of curve produces the families of curves. Curves belonging to different families intersect perpendicularly.

The spherical coordinate unit vectors are formed by holding two of the three coordinates constant while the value of the third is increased. The unit vectors are tangent to the respective coordinate curves, as shown in Fig. 2.9. They are mutually orthogonal, with their sense being such that

$$\bar{e}_r \times \bar{e}_\phi = \bar{e}_\theta. \tag{2.3.17}$$

Note that the azimuthal angle is defined here to be positive according to the right-hand rule relative to the axial direction, that is, counterclockwise looking down the z axis. However, in some circumstances it might be convenient to define θ in the opposite sense. In that case the sign of the preceding cross product would be reversed.

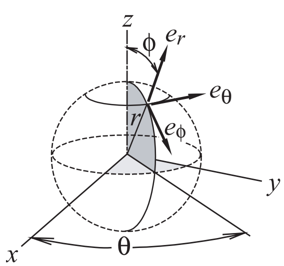


Figure 2.9. Unit vectors for spherical coordinates.

Projecting their unit length onto the respective coordinate axes shows the Cartesian components of the unit vectors to be

$$\begin{aligned} \bar{e}_r &= (\sin\phi\cos\theta)\,\bar{i} + (\sin\phi\sin\theta)\,\bar{j} + (\cos\phi)\,\bar{k}, \\ \bar{e}_\phi &= (\cos\phi\cos\theta)\,\bar{i} + (\cos\phi\sin\theta)\,\bar{j} - (\sin\phi)\,\bar{k}, \\ \bar{e}_\theta &= -(\sin\theta)\,\bar{i} + (\cos\theta)\,\bar{j}. \end{aligned} \tag{2.3.18}$$

Verification of this representation may be found in the fact that it satisfies Eq. (2.3.17). We see that the spherical coordinate unit vectors depend on both ϕ and θ. Derivation of formulas for velocity and acceleration from the position will require chain rule differentiation with respect to time, which in turn will require knowledge of the derivatives of the unit vectors with respect to both spherical angles. Differentiating Eqs. (2.3.18) leads to

$$\begin{aligned}
\frac{\partial \bar{e}_r}{\partial \phi} &= (\cos\phi\cos\theta)\,\bar{i} + (\cos\phi\sin\theta)\,\bar{j} - (\sin\phi)\,\bar{k}, \\
\frac{\partial \bar{e}_\phi}{\partial \phi} &= -(\sin\phi\cos\theta)\,\bar{i} - (\sin\phi\sin\theta)\,\bar{j} - (\cos\phi)\,\bar{k}, \\
\frac{\partial \bar{e}_\theta}{\partial \phi} &= 0, \\
\frac{\partial \bar{e}_r}{\partial \theta} &= -(\sin\phi\sin\theta)\,\bar{i} + (\sin\phi\cos\theta)\,\bar{j}, \\
\frac{\partial \bar{e}_\phi}{\partial \theta} &= -(\cos\phi\sin\theta)\,\bar{i} + (\cos\phi\cos\theta)\,\bar{j}, \\
\frac{\partial \bar{e}_\theta}{\partial \theta} &= -(\cos\theta)\,\bar{i} - (\sin\theta)\,\bar{j}.
\end{aligned} \tag{2.3.19}$$

We wish to use these derivatives to obtain formulas for velocity and acceleration that depend on only the spherical coordinate variables. A comparison of the derivatives just listed with Eqs. (2.3.18) shows that

$$\begin{gathered}
\frac{\partial \bar{e}_r}{\partial \phi} = \bar{e}_\phi, \quad \frac{\partial \bar{e}_\phi}{\partial \phi} = -\bar{e}_r, \quad \frac{\partial \bar{e}_\theta}{\partial \phi} = 0, \\
\frac{\partial \bar{e}_r}{\partial \theta} = (\sin\phi)\,\bar{e}_\theta, \quad \frac{\partial \bar{e}_\phi}{\partial \theta} = (\cos\phi)\,\bar{e}_\theta, \quad \frac{\partial \bar{e}_\theta}{\partial \theta} = -(\sin\phi)\,\bar{e}_r - (\cos\phi)\,\bar{e}_\phi.
\end{gathered} \tag{2.3.20}$$

One could alternatively derive these derivatives by constructing diagrams resembling Fig. 2.7, but doing so would be a more complicated process requiring several views.

The position vector in spherical coordinates is aligned with the radial direction, so that

$$\boxed{\bar{r} = r\bar{e}_r.} \tag{2.3.21}$$

The dependence of $\bar{r}$ on ϕ and θ is implicit, because the values of these coordinates must be known in order to locate $\bar{e}_r$. The chain rule for differentiation gives

$$\begin{aligned}
\bar{v} = \frac{d\bar{r}}{dt} &= \dot{r}\bar{e}_r + r\frac{d\bar{e}_r}{dt} \\
&= \dot{r}\bar{e}_r + r\left(\dot{\phi}\frac{\partial \bar{e}_r}{\partial \phi} + \dot{\theta}\frac{\partial \bar{e}_r}{\partial \theta}\right).
\end{aligned} \tag{2.3.22}$$

In view of Eqs. (2.3.20) this reduces to

$$\boxed{\bar{v} = \dot{r}\bar{e}_r + r\dot{\phi}\bar{e}_\phi + r\dot{\theta}\sin(\phi)\,\bar{e}_\theta.} \tag{2.3.23}$$

Each of these terms is readily explained as a superposition. If ϕ and θ were constant, there would be only a radial velocity component as a result of the changing radial distance. If r and θ were constant, a point would follow a meridional circle of radius r. The rotation rate of the radial line is $\dot{\phi}$, giving a speed of $r\dot{\phi}$, and the unit vector tangent to this circle is $\bar{e}_\phi$. The third component can be recognized by holding r and ϕ constant, in which case the point follows an azimuthal circle of radius $r\sin(\phi)$. The transverse line rotates about the z axis at $\dot{\theta}$, giving a speed $r\dot{\theta}\sin(\phi)$, and $\bar{e}_\theta$ is tangent to this circle.

The derivation of the formula for acceleration follows the same approach. For this, the time derivatives need to recognize that the unit vectors depend on ϕ and θ. As was done for cylindrical coordinates, brackets are used to track which acceleration terms originate from the same velocity component. Thus,

$$\begin{aligned}
\bar{a} &= \frac{d}{dt}\left[\dot{r}\bar{e}_r\right] + \frac{d}{dt}\left[r\dot{\phi}\bar{e}_\phi\right] + \frac{d}{dt}\left[r\dot{\theta}\sin(\phi)\,\bar{e}_\theta\right] \\
&= \left[\ddot{r}\bar{e}_r + \dot{r}\frac{d\bar{e}_r}{dt}\right] + \left[\dot{r}\dot{\phi}\bar{e}_\phi + r\ddot{\phi}\bar{e}_\phi + r\dot{\phi}\frac{d\bar{e}_\phi}{dt}\right] \\
&\quad + \left[\dot{r}\dot{\theta}\sin(\phi)\,\bar{e}_\theta + r\ddot{\theta}\sin(\phi)\,\bar{e}_\theta + r\dot{\theta}\dot{\phi}\cos(\phi)\,\bar{e}_\theta + \dot{r}\dot{\theta}\sin(\phi)\frac{d\bar{e}_\theta}{dt}\right] \\
&= \left[\ddot{r}\bar{e}_r + \dot{r}\left(\dot{\phi}\frac{\partial\bar{e}_r}{\partial\phi} + \dot{\theta}\frac{\partial\bar{e}_r}{\partial\theta}\right)\right] + \left[\dot{r}\dot{\phi}\bar{e}_\phi + r\ddot{\phi}\bar{e}_\phi + r\dot{\phi}\bar{e}_r\left(\dot{\phi}\frac{\partial\bar{e}_\phi}{\partial\phi} + \dot{\theta}\frac{\partial\bar{e}_\phi}{\partial\theta}\right)\right] \\
&\quad + \left[\dot{r}\dot{\theta}\,(\sin\phi)\,\bar{e}_\theta + r\ddot{\theta}\sin(\phi)\,\bar{e}_\theta + r\dot{\theta}\dot{\phi}\cos(\phi)\,\bar{e}_\theta + r\dot{\theta}\sin(\phi)\left(\dot{\phi}\frac{\partial\bar{e}_\theta}{\partial\phi} + \dot{\theta}\frac{\partial\bar{e}_\theta}{\partial\theta}\right)\right].
\end{aligned} \tag{2.3.24}$$

Substitution of Eqs. (2.3.20) leads to a profusion of terms:

$$\begin{aligned}
\bar{a} &= \left[\ddot{r}\bar{e}_r + \dot{r}\dot{\phi}\bar{e}_\phi + \dot{r}\dot{\theta}\sin(\phi)\,\bar{e}_\theta\right] + \left[\dot{r}\dot{\phi}\bar{e}_\phi + r\ddot{\phi}\bar{e}_\phi - r\dot{\phi}^2\bar{e}_r + r\dot{\phi}\dot{\theta}\cos(\phi)\,\bar{e}_\theta\right] \\
&\quad + [\dot{r}\dot{\theta}\sin(\phi)\,\bar{e}_\theta + r\ddot{\theta}\sin(\phi)\,\bar{e}_\theta + r\dot{\phi}\dot{\theta}\cos(\phi)\,\bar{e}_\theta \\
&\quad - r\dot{\theta}^2\sin(\phi)^2\,\bar{e}_r - r\dot{\theta}^2\sin(\phi)\cos(\phi)\,\bar{e}_\phi].
\end{aligned} \tag{2.3.25}$$

Although some terms are repeated, they originate from different effects. In general, one of the repeated terms results from the increase in a velocity component when one of the polar coordinates is increased, whereas the other results because acceleration occurs when the direction of a velocity component changes, even if the value of the component is constant. For example, the polar velocity is $r\dot{\phi}\bar{e}_\phi$. Changing the radial distance increases this velocity component, leading to an acceleration term $\dot{r}\dot{\phi}\bar{e}_\phi$. At the same time, the radial velocity is $\dot{r}\bar{e}_r$. Changing ϕ moves the tip of $\bar{e}_r$ in the direction of $\bar{e}_\phi$, so changing the direction of this velocity component also produces an acceleration term $\dot{r}\dot{\phi}\bar{e}_\phi$. Like cylindrical coordinates, the name given to this pair of equal acceleration terms, which lead to a factor of two in the formulas, is the Coriolis acceleration.

Other acceleration terms are either of two types. Some contain second derivatives of a spherical coordinate, such as $r\ddot{\phi}$. These arise when a rate variable increases. The others contain a product of r and the square of an angular rate. These correspond to centripetal acceleration effects. For example, moving along the polar circle of radius r at polar rate $\dot{\phi}$ produces a centripetal acceleration $r\dot{\phi}^2$. Such an acceleration is directed inward toward the center of the circle, which is the negative radial direction. The latter two effects can be recognized individually and then superposed.

Collecting like components yields the formula we shall employ:

$$\boxed{\begin{aligned}\bar{a} = {} & \left[\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2 \sin(\phi)^2\right]\bar{e}_r + \left[r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2 \sin(\phi)\cos(\phi)\right]\bar{e}_\phi \\ & + \left[r\ddot{\theta}\sin(\phi) + 2\dot{r}\dot{\theta}\sin(\phi) + 2r\dot{\phi}\dot{\theta}\cos(\phi)\right]\bar{e}_\theta.\end{aligned}} \tag{2.3.26}$$

We obtain an equivalent form, which is useful when the velocity components are known, by differentiating $\bar{v} = v_r\bar{e}_r + v_\phi\bar{e}_\phi + v_\theta\bar{e}_\theta$, with Eqs. (2.3.20) used to describe the derivatives of the unit vectors. The result is

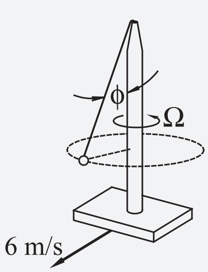
$$\begin{aligned}\bar{a} &= \dot{v}_r\bar{e}_r + v_r\left(\dot{\phi}\frac{\partial\bar{e}_r}{\partial\phi} + \dot{\theta}\frac{\partial\bar{e}_r}{\partial\theta}\right) + \dot{v}_\phi\bar{e}_\phi + v_\phi\left(\dot{\phi}\frac{\partial\bar{e}_\phi}{\partial\phi} + \dot{\theta}\frac{\partial\bar{e}_\phi}{\partial\theta}\right) \\ &\quad + \dot{v}_\theta\bar{e}_\theta + v_\theta\left(\dot{\phi}\frac{\partial\bar{e}_\theta}{\partial\phi} + \dot{\theta}\frac{\partial\bar{e}_\theta}{\partial\theta}\right) \\ &= \left[\dot{v}_r - v_\phi\dot{\phi} - v_\theta\dot{\theta}\sin(\phi)\right]\bar{e}_r + \left[\dot{v}_\phi + v_r\dot{\phi} - v_\theta\dot{\theta}\cos(\phi)\right]\bar{e}_\phi \\ &\quad + \left[\dot{v}_\theta + v_r\dot{\theta}\sin(\phi) + v_\phi\dot{\theta}\cos(\phi)\right]\bar{e}_\theta.\end{aligned} \tag{2.3.27}$$

One criterion for deciding to use spherical coordinates is that some aspect of the motion or force system is best described in terms of a line from the moving point to a designated fixed point. Scalar equations of motion featuring the spherical coordinates r, ϕ, and θ are obtained from Newton's Second Law when the force resultant is represented in terms of the spherical coordinate directions, specifically

$$\boxed{\begin{aligned}\Sigma F_r &\equiv \Sigma\bar{F}\cdot\bar{e}_r = m\left[\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\sin(\phi)^2\right], \\ \Sigma F_\phi &\equiv \Sigma\bar{F}\cdot\bar{e}_\phi = m\left[r\ddot{\phi} + 2\dot{r}\dot{\phi} - r\dot{\theta}^2\sin(\phi)\cos(\phi)\right], \\ \Sigma F_\theta &\equiv \Sigma\bar{F}\cdot\bar{e}_\theta = m\left[r\ddot{\theta}\sin(\phi) + 2\dot{r}\dot{\theta}\sin(\phi) + 2r\dot{\phi}\dot{\theta}\cos(\phi)\right].\end{aligned}} \tag{2.3.28}$$

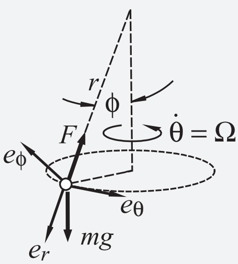
When the force resultant is known, these are nonlinear differential equations for the spherical coordinates in which all variables are strongly coupled.

EXAMPLE 2.8 The cable suspending a 400-g sphere is pulled in at a constant rate of 6 m/s. At the instant when the cable is 3 m long the angle of inclination is $\phi = 30^\circ$, $\Omega = 4$ rad/s, and $\dot{\phi} = 5$ rad/s. Determine the values of $\ddot{\phi}$ and $\dot{\Omega}$, and the tensile force in the cable at this instant.



Example 2.8

SOLUTION This example illustrates the usage of spherical coordinates to formulate Newton's Second Law. It also will make it apparent that drawing a good free-body diagram is a vital step when forces are to be analyzed. In this situation the cable length is measured from a fixed point to the sphere, and the other position parameters are the angle from the fixed vertical axis and the rotation rate about that axis. The former is the polar angle ϕ, and the latter corresponds to the azimuthal rotation rate $\Omega = \dot{\theta}$. These are spherical coordinates centered at the top of the post, with all motion parameters other than $\ddot{\phi}$ and $\ddot{\theta}$ given. Because it is required to evaluate a force acting on the particle, we draw a free-body diagram that also depicts the unit vectors of the spherical coordinate system we shall use.



Free-body diagram and spherical coordinates for the suspended sphere.

Each unit vector is defined to be positive in the sense of increasing values of the corresponding coordinate. Note that, because $\dot{\theta}$ has been defined to be Ω, we have $\bar{e}_r \times \bar{e}_\phi = -\bar{e}_\theta$, which is opposite the sign in Eq. (2.3.17). Also, the actual value of θ is irrelevant because there is no special position in regard to rotation about the vertical axis.

Because the cable is being pulled in at a constant rate, the instantaneous values are $r = 3$ m, $\dot{r} = -6$ m/s, and $\ddot{r} = 0$. Substitution of these values and $\phi = 30^\circ$, $\dot{\phi} = 5$ rad/s., and $\dot{\theta} = 4$ rad/s into Eq. (2.3.26) gives

$$\bar{a} = -87\bar{e}_r + \left(3\ddot{\phi} - 80.78\right)\bar{e}_\phi + \left(1.5\ddot{\theta} + 79.92\right)\bar{e}_\theta \text{ m/s}^2.$$

Newton's Second Law relates forces and acceleration variables, so we expect that it will yield the parameters we seek. We refer to the free-body diagram in order to describe the force components. The cable tension F pulls the sphere in the negative radial direction, and the weight, $0.4g$ N, acts parallel to the polar axis, so the spherical coordinate equations of motion, Eqs. (2.3.28), require that

$$\Sigma\bar{F}\cdot\bar{e}_r = -F + 0.4\,(9.807)\cos(\phi) = ma_r = 0.4\,(-87)\,,$$

$$\Sigma\bar{F}\cdot\bar{e}_\phi = -0.4\,(9.807)\sin(\phi) = ma_\phi = 0.4\left(3\ddot{\phi} - 80.78\right),$$

$$\Sigma\bar{F}\cdot\bar{e}_\theta = 0 = ma_\theta = 0.4\left(1.5\ddot{\theta} + 79.92\right).$$

The solution of these equations is

$$F = 38.72 \text{ N},$$

$$\ddot{\phi} = 25.29,\ \ \ddot{\theta} = -53.28 \text{ rad/s}. \qquad \triangleleft$$

2.3.3 Arbitrary Curvilinear Coordinates

Cylindrical and spherical coordinates are adequate for the majority of situations one might encounter. However, generalizing the formulation to handle any set of orthogonal curvilinear coordinates will vastly expand our capability. Doing so is also quite useful for topics in many related areas, including stress analysis, wave propagation, fluid mechanics, and acoustics. In the following subsection we consider a triad of curvilinear coordinates (α, β, γ) that are related to the Cartesian coordinates (x, y, z) by an arbitrary transformation in the form of Eqs. (2.3.1).

Coordinates and Unit Vectors

It is difficult to depict a three-dimensional situation, so Fig. 2.10 shows a two-dimensional grid associated with various values of the curvilinear coordinates α and β, with the third coordinate γ held constant. Each curve corresponds to constant values of two of the coordinates. We use the coordinate that varies to name the curve. For example, points on a specific α curve correspond to a range of α values with β and γ fixed. Neighboring curves for each family in the figure are separated by values of α or β that differ by an infinitesimal value. We shall consider only coordinate systems for which the curves of different families intersect orthogonally. The distance between intersection points on the grid is not the same as the value of the increment in that coordinate. The ratio of the differential arc length along a coordinate curve between intersections, and the increment in the coordinate corresponding to the intersections is the *stretch ratio* for that coordinate, denoted h_λ, with $\lambda = \alpha,\ \beta$, or γ. The arc length along a λ coordinate curve is denoted as s_λ, so

$$ds_\lambda = h_\lambda d\lambda, \quad \lambda = \alpha, \beta, \text{or } \gamma. \tag{2.3.29}$$

The relationship between the curvilinear coordinate transformation, Eqs. (2.3.1), and the stretch ratios will be established shortly.

Moving along any of the coordinate curves is very much like the situation in path variables. At any point, there are three unit vectors $\bar{e}_\lambda$ tangent to the α, β, and γ curves passing through that point. Recall that the tangent direction is defined by $\bar{e}_t = d\bar{r}/ds$. We use Eq. (2.3.29) to describe the differential arc length. Incrementing one coordinate

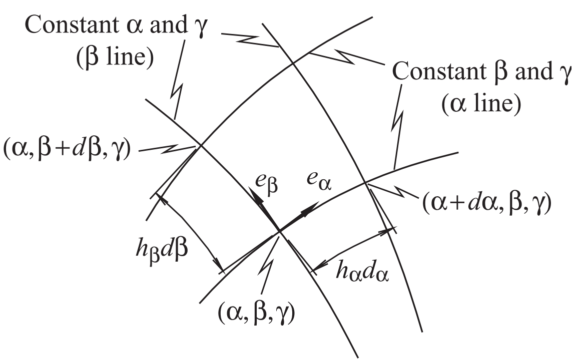


Figure 2.10. Two-dimensional curvilinear coordinate mesh.

with the other two fixed is a process of partial differentiation, so the unit vectors may be obtained from

$$\boxed{\bar{e}_\lambda = \frac{d\bar{r}}{ds_\lambda} = \frac{1}{h_\lambda}\frac{\partial \bar{r}}{\partial \lambda}, \qquad \lambda = \alpha, \beta, \text{ or } \gamma.} \tag{2.3.30}$$

It is conventional to employ a right-handed coordinate system in order to avoid sign errors in the evaluation of cross products. Consistency with this convention is obtained by ordering (α, β, γ) such that

$$\bar{e}_\alpha \times \bar{e}_\beta = \bar{e}_\gamma. \tag{2.3.31}$$

Equations (2.3.1) define the components of $\bar{r}$ with respect to xyz in terms of the curvilinear coordinates,

$$\bar{r} = x(\alpha, \beta, \gamma)\,\bar{i} + y(\alpha, \beta, \gamma)\,\bar{j} + z(\alpha, \beta, \gamma)\,\bar{k}. \tag{2.3.32}$$

Differentiation of this expression with respect to a specific curvilinear coordinate λ is straightforward because $\bar{i}$, $\bar{j}$, and $\bar{k}$ are constant, so that

$$\boxed{\frac{\partial \bar{r}}{\partial \lambda} = \left(\frac{\partial}{\partial \lambda} x(\alpha, \beta, \gamma)\right)\bar{i} + \left(\frac{\partial}{\partial \lambda} x(\alpha, \beta, \gamma)\right)\bar{j} + \left(\frac{\partial}{\partial \lambda} x(\alpha, \beta, \gamma)\right)\bar{k}.} \tag{2.3.33}$$

Enforcement of the requirement that the unit vector defined by Eq. (2.3.30) actually have unit magnitude then yields an expression for the stretch ratio h_λ:

$$\boxed{h_\lambda = \left|\frac{\partial \bar{r}}{\partial \lambda}\right| = \left[\left(\frac{\partial}{\partial \lambda} x(\alpha, \beta, \gamma)\right)^2 + \left(\frac{\partial}{\partial \lambda} y(\alpha, \beta, \gamma)\right)^2 + \left(\frac{\partial}{\partial \lambda} z(\alpha, \beta, \gamma)\right)^2\right]^{1/2}.} \tag{2.3.34}$$

Substitution of this expression and Eq. (2.3.33) into Eq. (2.3.30) gives the xyz components of each of the curvilinear unit vectors in terms of the curvilinear coordinate values. In other words, the result has the general form

$$\bar{e}_\lambda = e_{\lambda\alpha}(\alpha, \beta, \gamma)\,\bar{i} + e_{\lambda\beta}(\alpha, \beta, \gamma)\,\bar{j} + e_{\lambda\gamma}(\alpha, \beta, \gamma)\,\bar{k}. \tag{2.3.35}$$

The derivation of the acceleration equation will require differentiation of the unit vectors. There are two ways in which we may proceed. The first is an extension of the manner in which we treated spherical coordinates. It is a direct differentiation procedure. Let μ denote any of the curvilinear coordinates. Differentiation of Eq. (2.3.35) with respect to μ gives

$$\begin{aligned}\frac{\partial \bar{e}_\lambda}{\partial \mu} = &\left[\frac{\partial}{\partial \mu} e_{\lambda\alpha}(\alpha, \beta, \gamma)\right]\bar{i} + \left[\frac{\partial}{\partial \mu} e_{\lambda\beta}(\alpha, \beta, \gamma)\right]\bar{j} \\ &+ \left[\frac{\partial}{\partial \mu} e_{\lambda\gamma}(\alpha, \beta, \gamma)\right]\bar{k}, \quad \lambda, \mu = \alpha, \beta, \gamma.\end{aligned} \tag{2.3.36}$$

This expression describes the derivative in terms of $\bar{i}$, $\bar{j}$, and $\bar{k}$ components, but we seek kinematical formulas that describe velocity and acceleration relative to the curvilinear coordinate directions. In other words, we wish to represent the preceding derivative in the form of

$$\boxed{\frac{\partial \bar{e}_\lambda}{\partial \mu} = \Gamma_{\lambda\mu\alpha}\bar{e}_\alpha + \Gamma_{\lambda\mu\beta}\bar{e}_\beta + \Gamma_{\lambda\mu\gamma}\bar{e}_\gamma.} \tag{2.3.37}$$

The components in this expression may be found from a dot product with a specific curvilinear unit vector, from which we find that

$$\boxed{\Gamma_{\lambda\mu\nu} = \frac{\partial \bar{e}_\lambda}{\partial \mu}\cdot \bar{e}_\nu, \quad \lambda, \mu, \nu = \alpha, \beta, \text{or } \gamma.} \tag{2.3.38}$$

Note that the result of implementing the preceding equation by use of Eqs. (2.3.35) and (2.3.36) will be expressions for the $\Gamma_{\lambda\mu\nu}$ coefficients in terms of the curvilinear coordinate values. These coefficients are called *Christoffel symbols*. They play a prominent role in tensor analysis, which is a key mathematical tool for fields as diverse as solid mechanics and general relativity.

The shortcoming of Eq. (2.3.38) is that it does not shed any light on the fundamental nature of curvilinear coordinates in terms of recognizable parameters. Such information is obtained from the second derivation, which yields expressions in terms of stretch ratios. Although the derivation that follows is more circuitous, the result usually will be more efficient to evaluate, as we will see in an example.

We begin by observing that differentiation of a specified unit vector $\bar{e}_\lambda$ leads to different cases depending on whether μ and/or ν in Eq. (2.3.38) are the same as λ. All cases in which the unit vectors $\bar{e}_\lambda$ and $\bar{e}_\nu$ are the same are covered by

$$\bar{e}_\lambda \cdot \bar{e}_\lambda = 1 \implies \frac{\partial \bar{e}_\lambda}{\partial \mu}\cdot \bar{e}_\lambda = \Gamma_{\lambda\mu\lambda} = 0. \tag{2.3.39}$$

Cases not covered by Eq. (2.3.39) correspond to $\nu \neq \lambda$, that is, the component of the derivative of a unit vector in the direction of a different unit vector. We may evaluate these with the aid of a sequence of identities. It follows from the orthogonality of the unit vectors that

$$\bar{e}_\lambda \cdot \bar{e}_\nu = 0 \implies \frac{\partial \bar{e}_\lambda}{\partial \mu}\cdot \bar{e}_\nu = \Gamma_{\lambda\mu\nu} = -\frac{\partial \bar{e}_\nu}{\partial \mu}\cdot \bar{e}_\lambda = -\Gamma_{\nu\mu\lambda}, \quad \nu \neq \lambda. \tag{2.3.40}$$

The following relation originates from Eq. (2.3.30):

$$\frac{\partial}{\partial \mu}(h_\lambda \bar{e}_\lambda) = \frac{\partial}{\partial \lambda}(h_\mu \bar{e}_\mu). \tag{2.3.41}$$

Carrying out the derivatives leads to

$$\frac{\partial h_\lambda}{\partial \mu}\bar{e}_\lambda + h_\lambda \frac{\partial \bar{e}_\lambda}{\partial \mu} = \frac{\partial h_\mu}{\partial \lambda}\bar{e}_\mu + h_\mu \frac{\partial \bar{e}_\mu}{\partial \lambda}. \tag{2.3.42}$$

We may now consider the various $\Gamma_{\lambda\mu\nu}$ when $\nu \neq \lambda$. Because each of the symbols represents one of three possible coordinates, the only combinations fitting the restriction that $\nu \neq \lambda$ are $\mu = \nu \neq \lambda : \Gamma_{\lambda\mu\mu}$, $\mu = \lambda \neq \nu : \Gamma_{\lambda\lambda\nu}$, and $\mu \neq \nu \neq \lambda$. We begin by considering the first case. Such terms are obtained from the dot product of Eq. (2.3.42) with $\bar{e}_\mu$. Because $\bar{e}_\mu$ and $\bar{e}_\lambda$ are different, it follows that

$$h_\lambda \frac{\partial \bar{e}_\lambda}{\partial \mu} \cdot \bar{e}_\mu = \frac{\partial h_\mu}{\partial \lambda} + h_\mu \frac{\partial \bar{e}_\mu}{\partial \lambda} \cdot \bar{e}_\mu. \tag{2.3.43}$$

We use Eq. (2.3.39) to simplify the preceding equation, with the result that

$$\frac{\partial \bar{e}_\lambda}{\partial \mu} \cdot \bar{e}_\mu = \Gamma_{\lambda\mu\mu} = \frac{1}{h_\lambda} \frac{\partial h_\mu}{\partial \lambda}, \quad \mu \neq \lambda. \tag{2.3.44}$$

We obtain an expression for $\Gamma_{\lambda\lambda\nu} = (\partial \bar{e}_\lambda / \partial \lambda) \cdot \bar{e}_\nu$ when $\nu \neq \lambda$ by applying Eq. (2.3.40), and then using Eq. (2.3.44) with λ changed to ν and μ changed to λ. This gives

$$\Gamma_{\lambda\lambda\nu} = -\frac{1}{h_\nu} \frac{\partial h_\lambda}{\partial \nu} = -\Gamma_{\nu\lambda\lambda}, \quad \lambda \neq \nu. \tag{2.3.45}$$

The only remaining case is that for which λ, μ, and ν differ from each other. The dot product of Eq. (2.3.42) with $\bar{e}_\nu$ in this case yields

$$\frac{1}{h_\mu} \frac{\partial \bar{e}_\lambda}{\partial \mu} \cdot \bar{e}_\nu = \frac{1}{h_\lambda} \frac{\partial \bar{e}_\mu}{\partial \lambda} \cdot \bar{e}_\nu, \quad \lambda, \mu, \nu \text{ distinct.} \tag{2.3.46}$$

The next steps involve an alternative application of permutations of the properties in Eqs. (2.3.40) and (2.3.46), with the labels interchanged appropriately, to the *right-hand side* of Eq. (2.3.46). This gives

$$\begin{aligned} \frac{1}{h_\mu} \frac{\partial \bar{e}_\lambda}{\partial \mu} \cdot \bar{e}_\nu &= \frac{1}{h_\lambda} \left(-\frac{\partial \bar{e}_\nu}{\partial \lambda} \cdot \bar{e}_\mu \right) = -\frac{1}{h_\nu} \frac{\partial \bar{e}_\lambda}{\partial \nu} \cdot \bar{e}_\mu \\ &= \frac{1}{h_\nu} \left(\frac{\partial \bar{e}_\mu}{\partial \nu} \cdot \bar{e}_\lambda \right) = \frac{1}{h_\mu} \frac{\partial \bar{e}_\nu}{\partial \mu} \cdot \bar{e}_\lambda \\ &= -\frac{1}{h_\mu} \frac{\partial \bar{e}_\lambda}{\partial \mu} \cdot \bar{e}_\nu. \end{aligned} \tag{2.3.47}$$

The foregoing is a contradiction unless both terms vanish, so that

$$\frac{\partial \bar{e}_\lambda}{\partial \mu} \cdot \bar{e}_\nu = \Gamma_{\lambda\mu\nu} = 0, \quad \lambda, \mu, \nu \text{ distinct.} \tag{2.3.48}$$

Equations (2.3.39), (2.3.44), and (2.3.48) are the properties we seek. They describe the Christoffel symbols in terms of the stretch ratios and their derivatives. There are nine combinations of λ and μ values, whose individual components are evaluated by selecting the appropriate case from the identities. Only the results for $\lambda = \alpha$ are listed. The others follow by permutation of the symbols.

$$\boxed{\begin{aligned} &\frac{\partial \bar{e}_\alpha}{\partial \alpha} = -\frac{1}{h_\beta} \frac{\partial h_\alpha}{\partial \beta} \bar{e}_\beta - \frac{1}{h_\gamma} \frac{\partial h_\alpha}{\partial \gamma} \bar{e}_\gamma, \\ &\frac{\partial \bar{e}_\alpha}{\partial \beta} = \frac{1}{h_\alpha} \frac{\partial h_\beta}{\partial \alpha} \bar{e}_\beta, \quad \frac{\partial \bar{e}_\alpha}{\partial \gamma} = \frac{1}{h_\alpha} \frac{\partial h_\gamma}{\partial \alpha} \bar{e}_\gamma. \end{aligned}} \tag{2.3.49}$$

EXAMPLE 2.9 The two-dimensional hyperbolic–elliptic coordinate system is defined by

$$x = a\cosh(\alpha)\sin(\beta), \quad y = a\sinh(\alpha)\cos(\beta),$$

where a is a constant. Evaluate the unit vectors of this system in terms of components relative to the x and y axes, then describe the derivatives of the unit vectors.

SOLUTION This example illustrates the application of the basic curvilinear formulas to a coordinate system that is useful for some problems in acoustics, fluid mechanics, and electrodynamics. The name of this set of coordinates stems from the fact that lines of constant α are ellipses,

$$\frac{x^2}{a^2\cosh(\alpha)^2} + \frac{y^2}{a^2\sinh(\alpha)^2} = 1,$$

where $2a\cosh(\alpha)$ and $2a\sinh(\alpha)$ are the lengths of the major and minor diameters. Also, lines of constant β are hyperbolas,

$$\frac{x^2}{a^2\sin(\beta)^2} - \frac{y^2}{a^2\cos(\beta)^2} = 1,$$

where $x = \pm y\tan(\beta)$ are the asymptotes and $x = \pm a\sin(\beta)$ are the intercepts on the x axis.

To evaluate the stretch ratios and unit vectors we need the partial derivatives of the position vectors:

$$\begin{aligned}\frac{\partial\bar{r}}{\partial\alpha} &= \frac{\partial x}{\partial\alpha}\bar{i} + \frac{\partial y}{\partial\alpha}\bar{j} = a\sinh(\alpha)\sin(\beta)\,\bar{i} + a\cosh(\alpha)\cos(\beta)\,\bar{j},\\ \frac{\partial\bar{r}}{\partial\beta} &= \frac{\partial x}{\partial\beta}\bar{i} + \frac{\partial y}{\partial\beta}\bar{j} = a\cosh(\alpha)\cos(\beta)\,\bar{i} - a\sinh(\alpha)\sin(\beta)\,\bar{j}.\end{aligned} \tag{1}$$

According to Eq. (2.3.34), the stretch ratios are the magnitudes of the preceding, so

$$\begin{aligned}h_\alpha &= \left|\frac{\partial\bar{r}}{\partial\alpha}\right| = a\left[\sinh(\alpha)^2\sin(\beta)^2 + \cosh(\alpha)^2\cos(\beta)^2\right]^{1/2}\\ &= a\left[\left(\cosh(\alpha)^2 - 1\right)\sin(\beta)^2 + \cosh(\alpha)^2\cos(\beta)^2\right]^{1/2} = ah,\\ h_\beta &= \left|\frac{\partial\bar{r}}{\partial\beta}\right| = ah,\end{aligned} \tag{2}$$

where

$$h = \left[\cosh(\alpha)^2 - \sin(\beta)^2\right]^{1/2}. \tag{3}$$

We find the corresponding unit vectors by substituting Eqs. (1) and (2) into Eqs. (2.3.30):

$$\begin{aligned}\bar{e}_\alpha &= \frac{1}{h}\left[\sinh(\alpha)\sin(\beta)\,\bar{i} + \cosh(\alpha)\cos(\beta)\,\bar{j}\right],\\ \bar{e}_\beta &= \frac{1}{h}\left[\cosh(\alpha)\cos(\beta)\,\bar{i} - \sinh(\alpha)\sin(\beta)\,\bar{j}\right].\end{aligned} \tag{4} \triangleleft$$

The orthogonality of the mesh is confirmed by these unit vectors, because they show that $\bar{e}_\alpha \cdot \bar{e}_\beta = 0$.

The derivatives of the unit vectors involve partial derivatives of the stretch ratios, which we obtain by differentiating Eq. (3) to find

$$\frac{\partial h}{\partial \alpha} = \frac{\cosh(\alpha)\sinh(\alpha)}{\left(\cosh(\alpha)^2 - \sin(\beta)^2\right)^{1/2}} = \frac{\cosh(\alpha)\sinh(\alpha)}{h},$$
$$\frac{\partial h}{\partial \beta} = \frac{-\sin(\beta)\cos(\beta)}{\left(\cosh(\alpha)^2 - \sin(\beta)^2\right)^{1/2}} = \frac{-\sin(\beta)\cos(\beta)}{h}.$$

The corresponding expressions resulting from Eqs. (2.3.49), with $\partial h/\partial \gamma = 0$ for a two-dimensional situation, are

$$\frac{\partial \bar{e}_\alpha}{\partial \alpha} = -\frac{1}{h}\frac{\partial h}{\partial \beta}\bar{e}_\beta = \frac{\sin(\beta)\cos(\beta)}{h^2}\bar{e}_\beta,$$
$$\frac{\partial \bar{e}_\alpha}{\partial \beta} = \frac{1}{h}\frac{\partial h}{\partial \alpha}\bar{e}_\beta = \frac{\cosh(\alpha)\sinh(\alpha)}{h^2}\bar{e}_\beta,$$
$$\frac{\partial \bar{e}_\beta}{\partial \alpha} = \frac{1}{h}\frac{\partial h}{\partial \beta}\bar{e}_\alpha = -\frac{\sin(\beta)\cos(\beta)}{h^2}\bar{e}_\alpha,$$
$$\frac{\partial \bar{e}_\beta}{\partial \beta} = -\frac{1}{h}\frac{\partial h}{\partial \alpha}\bar{e}_\alpha = -\frac{\cosh(\alpha)\sinh(\alpha)}{h^2}\bar{e}_\alpha.$$

◁

It is useful to compare this derivation to the steps that are entailed in direct evaluation of the Christoffel symbols according to Eq. (2.3.38). The first step would entail differentiation of each unit vector in Eqs. (4) with respect to α and β. This would lead to four derivatives, each of whose form is somewhat lengthy because of the presence of h, which is a function of α and β, in the denominator. Taking a dot product of each derivative with $\bar{e}_\alpha$ and $\bar{e}_\beta$ in accord with Eq. (2.3.38) would yield eight Christoffel symbols. It should be apparent that such a derivation is more difficult to implement than the approach we pursued.

Kinematical Formulas

Our task in this subsection is to express the velocity and acceleration in terms of the parameters of a curvilinear coordinate system. For this development we consider the motion to be specified through the dependence of the curvilinear coordinates on time. The velocity is the time derivative of the position vector, which, in turn, is a function of the curvilinear coordinates (α, β, γ). Hence we employ the chain rule, in conjunction with the definition of the unit vectors in Eqs. (2.3.30), to differentiate $\bar{r}$:

$$\bar{v} = \dot{\alpha}\frac{\partial \bar{r}}{\partial \alpha} + \dot{\beta}\frac{\partial \bar{r}}{\partial \beta} + \dot{\gamma}\frac{\partial \bar{r}}{\partial \gamma} = h_\alpha \dot{\alpha}\bar{e} + h_\beta \dot{\beta}\bar{e}_\beta + h_\gamma \dot{\gamma}\bar{e}_\gamma. \tag{2.3.50}$$

This expression may be written in summation form as

$$\boxed{\bar{v} = \sum_{\lambda=\alpha,\beta,\gamma} h_\lambda \dot{\lambda} \bar{e}_\lambda.} \tag{2.3.51}$$

We derive the acceleration by differentiating Eq. (2.3.51) with respect to time. Each of the factors inside the preceding summation may vary with time, so

$$\bar{a} = \sum_{\lambda=\alpha,\beta,\gamma} h_\lambda \ddot{\lambda} \bar{e}_\lambda + \sum_{\lambda=\alpha,\beta,\gamma} \dot{\lambda} \left(\frac{dh_\lambda}{dt} \bar{e}_\lambda + h_\lambda \frac{d\bar{e}_\lambda}{dt} \right). \tag{2.3.52}$$

We consider only the rates of change of the curvilinear coordinates $\dot{\lambda}$ to depend explicitly on t, whereas the unit vectors $\bar{e}_\lambda$ and the stretch ratios h_λ depend on t implicitly through their dependence on the curvilinear coordinates. Application of the chain rule for differentiation to the latter two sets of parameters then yields

$$\boxed{\bar{a} = \sum_{\lambda=\alpha,\beta,\gamma} h_\lambda \ddot{\lambda} \bar{e}_\lambda + \sum_{\lambda=\alpha,\beta,\gamma} \sum_{u=\alpha,\beta,\gamma} \dot{\lambda} \dot{\mu} \left(\frac{\partial h_\lambda}{\partial \mu} \bar{e}_\lambda + h_\lambda \frac{\partial \bar{e}_\lambda}{\partial \mu} \right).} \tag{2.3.53}$$

Explicit expressions for a specific set of curvilinear coordinates may be obtained from the preceding equation by evaluating the stretch ratios and the derivatives of the unit vectors according to Eqs. (2.3.34) and (2.3.49), respectively.

It is apparent that each acceleration component might consist of several terms in the most general case. The situation for many common sets of curvilinear coordinates is simplified by the fact that the stretch ratios and unit vectors usually do not depend on all of the curvilinear coordinates. For example, in the case of spherical coordinates, all of the unit vectors are independent of the radial distance r and the radial stretch ratio $h_r = 1$. In the most general case Eq. (2.3.53) gives 21 different terms: 3 from the single summation and 9 from each term in the double summation corresponding to different pairs of λ and μ. Because there are only three curvilinear coordinate directions, it is clear that a variety of effects contributes to any acceleration component. Let us examine each type of effect.

The terms in the single summation of Eq. (2.3.53), that is, $h_\lambda \ddot{\lambda} \bar{e}_\lambda$, are intuitively obvious. They express an acceleration tangent to each λ coordinate curve that arises when $\dot{\lambda}$ is not constant. To understand the terms in the double sum we categorize them as to whether the indices for each sum are associated with the same curvilinear coordinate. If $\mu = \lambda$, three terms correspond to $\dot{\lambda}^2 (\partial h_\lambda / \partial \lambda) \bar{e}_\lambda$. This is another acceleration effect tangent to a λ coordinate curve; it arises because constant $\dot{\lambda}$ will not lead to a constant rate of movement along that curve if the stretch ratio h_λ changes along the curve. The other term corresponding to $\mu = \lambda$ is $\dot{\lambda}^2 h_\lambda (\partial \bar{e}_\lambda / \partial \lambda)$. Because the derivative of a unit vector is always perpendicular to the unit vector, this is an acceleration component perpendicular to the λ coordinate curve that arises from the changing direction of the unit vector $\bar{e}_\lambda$ as λ changes. Such a change is illustrated in Fig. 2.11 for the case in which $\lambda = \alpha$. Following the α curve from the point associated with specific (α, β, γ) values to an adjacent point at which the coordinates are $(\alpha + d\alpha, \beta, \gamma)$ causes the tip of $\bar{e}_\alpha$ to move perpendicularly to the original e_a. The fact that $\dot{\lambda} h_\lambda$ is the corresponding velocity component

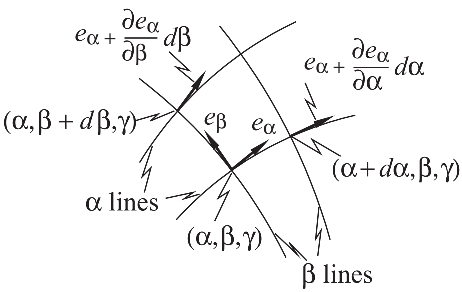


Figure 2.11. Change of unit vector $\bar{e}_\alpha$ along the α and β coordinate curves.

suggests that this acceleration component is analogous to the centripetal acceleration in path variables.

Let us now turn our attention to those terms in the double sum of Eq. (2.3.53) that correspond to $\mu \neq \gamma$. There are three combinations fitting this description, corresponding to $(\lambda, \mu) = (\alpha, \beta)$, (β, γ), or (α, γ) in either the listed or reversed order. Let us consider the combination $(\lambda, \mu) = (\alpha, \beta)$ or (β, α). The first term in the double sum leads to two terms: $[(\partial h_\alpha / \partial \beta)\bar{e}_\alpha + (\partial h_\beta / \partial \alpha)\bar{e}_\beta]\dot{\alpha}\dot{\beta}$. The first of the preceding terms exists if β changes and the stretch ratio h_α for the α curve depends on β. Similarly, the second term exists if α changes and the stretch ratio h_β depends on α. Both are representative of acceleration components tangent to a generic coordinate curve γ that results because the rate of movement along the curve, $\dot{\gamma} h_\gamma$, changes as a consequence of the nonconstancy of the stretch ratio. The second term in the double sum also leads to two terms for $(\lambda, \mu) = (\alpha, \beta)$ or (β, α) : $[h_\alpha \partial \bar{e}_\alpha / \partial \beta + h_\beta \partial \bar{e}_\beta / \partial \alpha]\,\dot{\alpha}\dot{\beta}$. These are acceleration terms perpendicular to a λ coordinate curve ($\lambda = \alpha$ for the first term and $\lambda = \beta$ for the second) that occur when $\bar{e}_\lambda$ depends on another curvilinear coordinate that is not constant in the motion. This is depicted in Fig. 2.11, where $\bar{e}_\alpha$ changes as a point moves by an infinitesimal amount along the β coordinate line. Correspondingly, the mesh point coordinates change from (α, β, γ) to $(\alpha, \beta + d\beta, \gamma)$. The tip of $\bar{e}_\alpha$ moves perpendicularly to the original $\bar{e}_\alpha$ direction, so the resulting acceleration effect is also perpendicular to $\bar{e}_\alpha$.

Let us focus on two terms that were just listed: $(\partial h_\beta / \partial \alpha)\bar{e}_\beta \dot{\alpha}\dot{\beta}$, which arose from the first term of the double sum, and $h_\alpha\,(\partial \bar{e}_\alpha / \partial \beta)\,\dot{\alpha}\dot{\beta}$, which arose from the second term of the double sum. According to Eqs. (2.3.49), the latter is the same as the former, which would lead to a factor of two in the ultimate acceleration formula. This is the general version of the Coriolis accelerations we encountered in the specific cases of cylindrical and spherical coordinates. Hence we have proven that, in general, Coriolis acceleration arises from two distinctly different effects associated with an interaction of motion along more than one coordinate curve.

To close this section it is emphasized that only in the special case of Cartesian coordinates does changing a single coordinate lead to velocity and acceleration solely in the direction associated with that coordinate. In the most general case of curvilinear coordinates, changing one coordinate can lead to acceleration tangentially to each coordinate curve. Furthermore, in Cartesian coordinates, acceleration results solely from nonconstancy of $\dot{x}$, $\dot{y}$, or $\dot{z}$. In a curvilinear coordinate, there can be acceleration, even if $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\gamma}$ are constant.

EXAMPLE 2.10 Derive Eqs. (2.3.11) and (2.3.14) for velocity and acceleration in terms of cylindrical coordinates by using stretch ratios.

SOLUTION The intent of applying the basic kinematical formulas for curvilinear coordinates to a coordinate system that has been analyzed by a different technique is to enhance one's recognition of the significance of the basic parameters and operations. The first step is to evaluate the unit vectors and stretch ratios. For the coordinate transformation in Eq. (2.3.3), we have

$$\bar{r} = R\cos(\theta)\,\bar{i} + R\sin(\theta)\,\bar{j} + z\bar{k}.$$

Then

$$h_R\bar{e}_R = \frac{\partial\bar{r}}{\partial R} = \cos(\theta)\,\bar{i} + \sin(\theta)\,\bar{j},$$
$$h_\theta\bar{e}_\theta = \frac{\partial\bar{r}}{\partial\theta} = -R\sin(\theta)\,\bar{i} + R\cos(\theta)\,\bar{j},$$
$$h_z\bar{e}_z = \frac{\partial\bar{r}}{\partial z} = \bar{k}.$$

Setting the magnitude of each unit vector to unity yields the stretch ratios,

$$h_R = \left|\frac{\partial\bar{r}}{\partial R}\right| = 1,$$
$$h_\theta = \left|\frac{\partial\bar{r}}{\partial\theta}\right| = R,$$
$$h_z = \left|\frac{\partial\bar{r}}{\partial z}\right| = 1,$$

which correspond to

$$\bar{e}_R = \cos(\theta)\,\bar{i} + \sin(\theta)\,\bar{j},$$
$$\bar{e}_\theta = -R\sin(\theta)\,\bar{i} + R\cos(\theta)\,\bar{j},$$
$$\bar{e}_z = \bar{k}.$$

We refer to Eqs. (2.3.49) and its permutations to obtain the derivatives of the unit vectors:

$$\frac{\partial\bar{e}_R}{\partial R} = -\frac{1}{h_\theta}\frac{\partial h_R}{\partial\theta}\bar{e}_\theta - \frac{1}{h_z}\frac{\partial h_R}{\partial_z}\bar{e}_z = 0,$$
$$\frac{\partial\bar{e}_R}{\partial\theta} = \frac{1}{h_R}\frac{\partial h_\theta}{\partial R}\bar{e}_\theta = \bar{e}_\theta, \qquad \frac{\partial\bar{e}_R}{\partial z} = \frac{1}{h_R}\frac{\partial h_z}{\partial R}\bar{e}_z = 0,$$
$$\frac{\partial\bar{e}_\theta}{\partial\theta} = -\frac{1}{h_R}\frac{\partial h_\theta}{\partial R}\bar{e}_R - \frac{1}{h_z}\frac{\partial h_\theta}{\partial z}\bar{e}_z = -\bar{e}_R,$$
$$\frac{\partial\bar{e}_\theta}{\partial R} = \frac{1}{h_\theta}\frac{\partial h_R}{\partial\theta}\bar{e}_R = 0, \qquad \frac{\partial\bar{e}_\theta}{\partial z} = \frac{1}{h_\theta}\frac{\partial h_z}{\partial\theta}\bar{e}_z = 0,$$

$$\frac{\partial \bar{e}_z}{\partial z} = -\frac{1}{h_R}\frac{\partial h_z}{\partial R}\bar{e}_R - \frac{1}{h_\theta}\frac{\partial h_z}{\partial \theta}\bar{e}_\theta = 0,$$
$$\frac{\partial \bar{e}_z}{\partial R} = \frac{1}{h_z}\frac{\partial h_R}{\partial z}\bar{e}_R = 0, \qquad \frac{\partial \bar{e}_z}{\partial \theta} = \frac{1}{h_z}\frac{\partial h_\theta}{\partial z}\bar{e}_\theta.$$

We obtain an expression for the velocity by expanding Eq. (2.3.51) and substituting the various terms. Thus,

$$\begin{aligned}\bar{v} &= h_R\dot{R}\bar{e}_R + h_\theta\dot{\theta}\bar{e}_\theta + h_z\dot{z}\bar{e}_z \\ &= \dot{R}\bar{e}_R + R\dot{\theta}\bar{e}_\theta + \dot{z}\bar{e}_z.\end{aligned} \quad \triangleleft$$

This is the same as Eq. (2.3.11). For acceleration, we expand Eq. (2.3.53) and omit terms that contain derivatives of stretch ratios or unit vectors that we have found to be zero. The remaining terms are

$$\begin{aligned}\bar{a} &= h_R\ddot{R}\bar{e}_R + h_\theta\ddot{\theta}\bar{e}_\theta + h_z\ddot{z}\bar{e}_z \\ &\quad + \dot{R}\dot{\theta}h_R\frac{\partial \bar{e}_R}{\partial \theta} + \dot{\theta}\dot{R}\frac{\partial h_\theta}{\partial R}e_\theta + \dot{\theta}^2 h_\theta\frac{\partial \bar{e}_\theta}{\partial \theta} \\ &= \ddot{R}\bar{e}_R + R\ddot{\theta}\bar{e}_\theta + \ddot{z}\bar{e}_z + \dot{R}\dot{\theta}\bar{e}_\theta + \dot{\theta}\dot{R}\bar{e}_\theta + \dot{\theta}^2 R(-\bar{e}_R).\end{aligned} \quad \triangleleft$$

Collecting like components shows that this expression is the same as Eq. (2.3.14).

2.4 MIXED KINEMATICAL DESCRIPTIONS

Thus far there was little ambiguity as to which kinematical description, path variables, Cartesian, or one of the curvilinear coordinate systems, we should use. Here we consider situations in which no single description leads to an optimal solution. The key factor to be considered in this regard is which description fits the aspects of the motion that are known and which fits the parameters we seek. For example, suppose that the path of a particle is known to be as shown in Fig. 2.12. If the rate of movement along that path is specified in terms of the speed v, we would certainly want to employ a path variable description. Now further suppose that, with v known, it is desired to determine the rotation

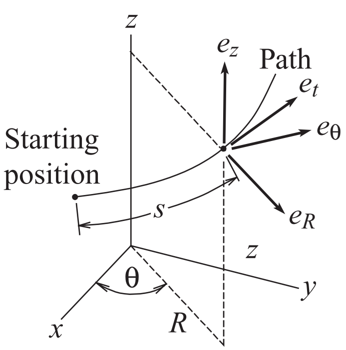


Figure 2.12. Mixed usage of path variables and cylindrical coordinates.

rate $\dot{\theta}$ of the azimuthal plane defined in the figure. Because θ is one of the cylindrical coordinates, it would seem wise to use that description. We could consider the kinematical description that best matches the parameters of the actual system to be the "natural" one, but sometimes no single formulation is entirely natural, although more than one have elements that are suitable. It is almost axiomatic that if one of the kinematical descriptions has some aspect that suits a problem, it should be employed. Thus, the task that confronts us here is to simultaneously implement two different descriptions. A precursor to this task arose in Example 2.1, where the velocity and acceleration were given in Cartesian coordinates and we needed to evaluate path variable parameters.

The general concept is to match the velocity and acceleration vectors obtained from the relevant kinematical descriptions. This matching is implemented by resolving unit vectors for one formulation into components relative to the unit vectors of the other formulation. For this purpose it is assumed that the position is known, so that all geometric quantities, including the angles between unit vectors, are known. For simplicity, let us consider planar motion. Let $\bar{e}_\alpha$ and $\bar{e}_\beta$ be the orthogonal planar unit vectors for one kinematical description (for example, $\bar{e}_\alpha$ and $\bar{e}_\beta$ are the tangent and normal directions), and let $\bar{e}_\lambda$ and $\bar{e}_\mu$ be the orthogonal planar unit vectors for the other description. These unit vectors have been depicted with their tails coinciding in Fig. 2.13 in order to expedite resolving a unit vector into components.

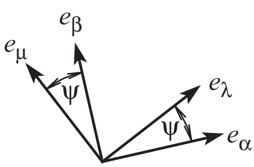


Figure 2.13. Relation between two sets of orthogonal unit vectors in a plane.

As shown in the figure, the orientation of one set of unit vectors relative to the other is defined by the angle ψ. (The definition of this angle as that between $\bar{e}_\alpha$ and $\bar{e}_\lambda$ is arbitrary.) The components of $\bar{e}_\lambda$ and $\bar{e}_\mu$ relative to $\bar{e}_\alpha$ and $\bar{e}_\beta$ are found from this figure to be

$$\bar{e}_\lambda = \cos(\psi)\,\bar{e}_\alpha + \sin(\psi)\,\bar{e}_\beta, \quad \bar{e}_\mu = -\sin(\psi)\,\bar{e}_\alpha + \cos(\psi)\,\bar{e}_\beta. \tag{2.4.1}$$

The velocity may be expressed in terms of components relative to either set of unit vectors. Thus,

$$\bar{v} = v_\alpha \bar{e}_\alpha + v_\beta \bar{e}_\beta = v_\lambda \bar{e}_\lambda + v_\mu \bar{e}_\mu. \tag{2.4.2}$$

It is implicit to the preceding equation that each velocity component would be expressed in terms of the parameters associated with its unit vectors. For example, if $\bar{e}_\alpha$ and $\bar{e}_\beta$ represented the tangent and normal unit vectors, respectively, then we would set $v_\alpha = v$ and $v_\beta = 0$.

Two vectors are equal if, and only if, their like components are equal. Thus Eqs. (2.4.2) constitute two scalar equations, which can be obtained by matching their components in two different coordinate directions. One approach for performing this operation is to substitute the unit vectors in Eqs. (2.4.1) into Eqs. (2.4.2), which yields

$$\begin{aligned}\bar{v} = v_\alpha \bar{e}_\alpha + v_\beta \bar{e}_\beta &= v_\lambda \bar{e}_\lambda + v_\mu \bar{e}_\mu \\ &= v_\lambda \left[\cos(\psi)\,\bar{e}_\alpha + \sin(\psi)\,\bar{e}_\beta\right] + v_\beta \left[-\sin(\psi)\,\bar{e}_\alpha + \cos(\psi)\,\bar{e}_\beta\right] \\ &= \left[v_\lambda \cos(\psi) - v_\beta \sin(\psi)\right]\bar{e}_\alpha + \left[v_\lambda \sin(\psi) + v_\beta \cos(\psi)\right]\bar{e}_\beta .\end{aligned} \tag{2.4.3}$$

Like components were grouped as the last operation in the preceding equation. Doing so assists equating like components on either side of the equality, with the result that

$$\begin{aligned} v_\alpha &= v_\lambda \cos(\psi) - v_\beta \sin(\psi), \\ v_\beta &= v_\lambda \sin(\psi) + v_\beta \cos(\psi). \end{aligned} \tag{2.4.4}$$

Each of the velocity components appearing in Eqs. (2.4.4) presumably has been represented in terms of the formula associated with its description. Furthermore, the position parameters are assumed to be known. Thus we have derived two scalar equations relating rate variables in either kinematical description.

An alternative way to obtain a component in a certain direction is to use a dot product. The dot product of the velocity in Eqs. (2.4.2) in the direction of two unit vectors gives two scalar equations representing the equality of the alternative component descriptions. The interesting aspect of this approach is that the selected unit vectors can be any pair. If we evaluate $\bar{v} \cdot \bar{e}_\alpha$ and $\bar{v} \cdot \bar{e}_\beta$, we obtain Eqs. (2.4.4). However, the unit vectors can also belong to different descriptions. For example, we could evaluate $\bar{v} \cdot \bar{e}_\beta$ and $\bar{v} \cdot \bar{e}_\lambda$, provided that $\bar{e}_\beta$ and $\bar{e}_\lambda$ are not parallel in the position of interest. This would give

$$\begin{aligned} \bar{v} \cdot \bar{e}_\beta &= v_\beta = v_\lambda \bar{e}_\lambda \cdot \bar{e}_\beta + v_\mu \bar{e}_\mu \cdot \bar{e}_\beta, \\ \bar{v} \cdot \bar{e}_\lambda &= v_\lambda = v_\alpha \bar{e}_\alpha \cdot \bar{e}_\lambda + v_\beta \bar{e}_\beta \cdot \bar{e}_\lambda. \end{aligned} \tag{2.4.5}$$

The dot products are readily described by referring to Fig. 2.13, which leads to

$$\begin{aligned} v_\beta &= v_\lambda \sin(\psi) + v_\mu \cos(\psi), \\ v_\lambda &= v_\alpha \cos(\psi) + v_\beta \sin(\psi). \end{aligned} \tag{2.4.6}$$

In practice, the procedure leading to Eqs. (2.4.6) is slightly more versatile, whereas following Eqs. (2.4.4) is somewhat less prone to computational errors. Either set represents two equations that may be used to solve for two unknown rate parameters. As an illustration of this procedure, suppose that (α, β) represents path variables and (λ, μ) represents polar coordinates. Substitution of the respective velocity components into Eqs. (2.4.4) then yields

$$\begin{aligned} v &= \dot{R} \cos(\psi) - R\dot{\theta} \sin(\psi), \\ 0 &= \dot{R} \sin(\psi) + R\dot{\theta} \cos(\psi). \end{aligned} \tag{2.4.7}$$

The values of the radial distance R and the angle of orientation ψ are known if the position is specified. Thus Eqs. (2.4.7) represent two relations among the three rate variables, v, $\dot{R}$, and $\dot{\theta}$.

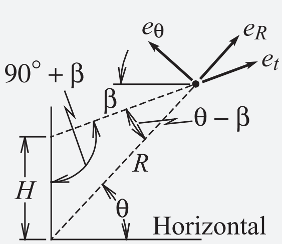
Equations derived by matching alternative velocity descriptions may be used in either of two general situations. It might be that the velocity is already known in terms of either the (α, β) or the (λ, μ) components. In that case, the equations provide the conversion to the parameters associated with the other set of components. The more interesting situation is that of a mixed description, that is, one in which the velocity is only partially known in terms of either of the two descriptions. In that case, the matching procedure provides the means to ascertain the velocity.

The same approach may be applied to treat acceleration. Specifically, the individual formulas for acceleration may be matched by employing either the unit vector transformation in Eqs. (2.4.1) or by taking dot products in the two different directions. The result will be two scalar equations for acceleration rate variables, such as $\dot{v}$ or $\ddot{R}$. Solving those equations requires that all velocity rate variables, such as v or $\dot{R}$, be evaluated first because they occur in the acceleration components. In other words, the velocity relations must be solved before accelerations can be addressed, which is not surprising because acceleration is the derivative of velocity. The remarkable aspect of the approach is there is no need to differentiate any quantity because the basic velocity and acceleration formulas represent standard derivatives.

The discussion treated the case of planar motion, but the same procedure also applies to three-dimensional motion. The kinematical formulas in that case have three components, so matching corresponding components will lead to three simultaneous equations. The primary difficulty that arises in this extension is the evaluation of the transformation of the unit vectors. The component representation in Eqs. (2.4.1) was derived by visual projections of one set of unit vector onto the other directions. The same procedure may be performed in a three-dimensional case if the geometry is not too complicated. An alternative approach for determining the unit vector components uses rotation transformation properties established in the next chapter.

EXAMPLE 2.11 Use the concept of a Mixed kinematical description to determine $\dot{R}$ and $\dot{\theta}$ for the airplane in Example 2.7.

SOLUTION This example demonstrates that viewing a system from multiple kinematical perspectives often can greatly simplify the analysis. The path and speed of the airplane are given, both of which are path variable parameters. We must determine the rates of change of polar coordinates. Thus we draw a sketch that depicts the unit vectors for both formulations at an arbitrary θ.



Example 2.11

The velocity in terms of each set of unit vectors is

$$\bar{v} = v\bar{e}_t = \dot{R}\bar{e}_R + R\dot{\theta}\bar{e}_\theta.$$

The speed v is the given parameter, so we may obtain an equation for $\dot{R}$ by forming

$$\bar{v} \cdot \bar{e}_R = v\bar{e}_t \cdot \bar{e}_R = \dot{R}.$$

From the sketch we see that the angle between $\bar{e}_t$ and $\bar{e}_R$ is $\theta - \beta$, so we find that

$$\dot{R} = v\cos(\theta - \beta)\,. \qquad \triangleleft$$

To find an expression for $\dot{\theta}$, we have

$$\bar{v} \cdot \bar{e}_\theta = v\bar{e}_t \cdot \bar{e}_\theta = R\dot{\theta}.$$

We evaluate the dot product by observing that the angle between $\bar{e}_t$ and $\bar{e}_\theta$ is $\theta - \beta + \pi/2$. Also, it was requested in the problem statement that the results be expressed in terms of the elevation angle θ, on which R depends. From the law of sines, we have

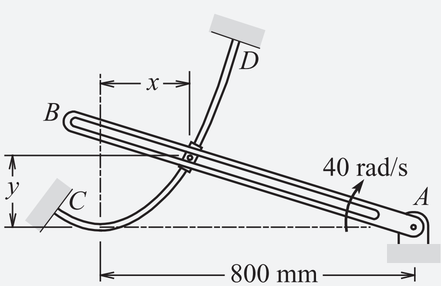
$$\frac{R}{\sin(\beta + \pi/2)} = \frac{H}{\sin(\theta - \beta)} \implies R = H\frac{\cos(\beta)}{\sin(\theta - \beta)}.$$

Consequently, the equation for $\bar{v} \cdot \bar{e}_\theta$ yields

$$\dot{\theta} = \frac{1}{R}v\cos(\theta - \beta + \pi/2) = -\frac{v}{H}\frac{\sin(\theta - \beta)^2}{\cos(\beta)}. \qquad \triangleleft$$

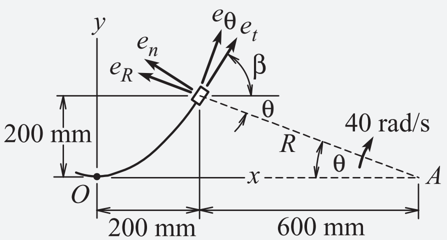
There is no doubt that this solution is easier than the one in Example 2.7.

EXAMPLE 2.12 Arm AB rotates clockwise at the constant rate of 40 rad/s as it pushes the slider along guide CD, which is described by $y = x^2/200$ (x and y are in millimeters). Determine the velocity and acceleration of the slider when it is at the position $x = 200$ mm.



Example 2.12

SOLUTION This example is a further demonstration that the concept of joint kinematical descriptions can make challenging problems quite tractable. The planar motion is specified by a rotation rate, but the path is not described in terms of polar coordinates. Hence we follow an approach that employs path variables and polar coordinates. We begin by evaluating the position parameters for both kinematical descriptions. A sketch shows both sets of unit vectors at $x = 200$ mm, which corresponds to $y = x^2/200 = 200$ mm. In this sketch the sense of $\bar{e}_t$ has been selected based on recognition that clockwise motion of bar AB will move the collar up and to the right. In more complicated situations, in which we are uncertain of the appropriate direction, we may guess the sense of $\bar{e}_t$. A wrong guess would lead to a negative value for v. Also, we have depicted $\bar{e}_n$ as the perpendicular to $\bar{e}_t$ that points toward the center of curvature.



Definiton of spherical coordinates and tangent and normal directions for describing the motion of the guided pin.

The polar coordinates are found from a right triangle to be

$$R = \left(600^2 + 200^2\right)^{1/2} = 632.5 \text{ mm} = 0.6325 \text{ m},$$

$$\theta = \tan^{-1}\left(\frac{200}{600}\right) = 18.435^\circ.$$

The slope of the guide bar at this location yields the angle of orientation of the tangent vector:

$$\beta = \tan^{-1}\left(\frac{dy}{dx}\right) = \tan^{-1}\left(\frac{y}{100}\right) = 63.435^\circ.$$

Matching like velocity components in each formulation is the next step. The relevant velocity formulas are

$$\bar{v} = \dot{R}\bar{e}_R + R\dot{\theta}\bar{e}_\theta = v\bar{e}_t.$$

For the sake of variety, we use the approach in which one set of unit vectors is expressed in terms of components relative to the other set. Hence we write

$$\bar{e}_t = -\cos(\theta + \beta)\,\bar{e}_R + \sin(\theta + \beta)\,\bar{e}_\theta,$$

$$\bar{e}_n = \sin(\theta + \beta)\,\bar{e}_R + \cos(\theta + \beta)\,\bar{e}_\theta.$$

Note that, although $\bar{e}_n$ is not needed to analyze the velocity, it has been described in anticipation of using it for acceleration. We substitute the expression for $\bar{e}_t$ into the velocity equation, and match like components:

$$\bar{v} = \dot{R}\bar{e}_R + R\dot{\theta}\bar{e}_\theta = v\left[-\cos(\theta + \beta)\,\bar{e}_R + \sin(\theta + \beta)\,\bar{e}_\theta\right],$$

$$\bar{v} \cdot \bar{e}_R = \dot{R} = -v\cos(\theta + \beta), \quad \bar{v} \cdot \bar{e}_\theta = R\dot{\theta} = v\sin(\theta + \beta).$$

The value of $\dot{\theta}$ is given to be 40 rad/s, and R, θ, and β have been evaluated, so the preceding equations represent two equations for $\dot{R}$ and v. Their solutions are

$$v = 25.56 \text{ m/s}, \quad \dot{R} = -3.614 \text{ m/s}.$$

Because we have evaluated all velocity parameters, we may now implement a similar procedure for acceleration. The relevant formulas are

$$\bar{a} = (\ddot{R} - R\dot{\theta}^2)\bar{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\bar{e}_\theta = \dot{v}\bar{e}_t + \frac{v^2}{\rho}\bar{e}_n.$$

Substitution of the earlier representations of $\bar{e}_t$ and $\bar{e}_n$ in terms of polar coordinates converts these equations to

$$\begin{aligned}\bar{a} &= (\ddot{R} - R\dot{\theta}^2)\bar{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\bar{e}_\theta \\ &= \dot{v}\,[-\cos(\theta+\beta)\,\bar{e}_R + \sin(\theta+\beta)\,\bar{e}_\theta] \\ &\quad + \frac{v^2}{\rho}\,[\sin(\theta+\beta)\,\bar{e}_R + \cos(\theta+\beta)\,\bar{e}_\theta]\,.\end{aligned}$$

The result of matching like acceleration components is

$$\bar{a}\cdot\bar{e}_R = \ddot{R} - R\dot{\theta}^2 = -\dot{v}\cos(\theta+\beta) + \frac{v^2}{\rho}\sin(\theta+\beta)\,,$$

$$\bar{a}\cdot\bar{e}_\theta = R\ddot{\theta} + 2\dot{R}\dot{\theta} = \dot{v}\sin(\theta+\beta) + \frac{v^2}{\rho}\cos(\theta+\beta)\,.$$

We know that $\dot{\theta}$ is constant at 40 rad/s, so $\ddot{\theta} = 0$. We evaluated R, θ, and β from the given position information. The radius of curvature, being a property of the path, also is a position parameter. We compute it from Eq. (2.1.31), which gives, for $x = 200$ mm,

$$\rho = \frac{\left[1 + \left(\dfrac{x}{100}\right)^2\right]^{3/2}}{\left|\dfrac{1}{100}\right|} = 1118.0 \text{ mm} = 1.1180 \text{ m}.$$

Substitution of all known quantities into the acceleration component equations gives

$$\bar{a}\cdot\bar{e}_R = \ddot{R} - 1011.9 = -\dot{v}\,(0.14142) + \frac{25.56^2}{1.1180}\,(0.9899)\,,$$

$$\bar{a}\cdot\bar{e}_\theta = 2\,(-3.614)\,(40) = \dot{v}\,(0.9899) + \frac{25.56^2}{1.1180}\,(0.14142)\,.$$

We solve the second equation for $\dot{v}$, then use that solution to calculate $\ddot{R}$, which yields

$$\dot{v} = -375.5 \text{ m/s}^2, \quad \ddot{R} = 1643.3 \text{ m/s}^2.$$

The problem statement requested the velocity and acceleration, which may be expressed in terms of either path variable or polar coordinate components:

$$\bar{v} = 25.56\,\bar{e}_t \text{ m/s} \quad \text{or} \quad \bar{v} = -3.614\bar{e}_R + 25.30\bar{e}_\theta \text{ m/s},$$
$$\bar{a} = -375.5\bar{e}_t + 584.1\bar{e}_n \text{ m/s}^2, \quad \text{or} \quad \bar{a} = 631.4\bar{e}_R - 289.1\bar{e}_\theta \text{ m/s}^2. \qquad \triangleleft$$

There usually is more than one way in which one may attack a problem, and sometimes the chosen path is not the most direct. This is the case here. The need to use polar coordinates was clearly indicated, but that description was matched to path variables primarily to give a full picture of the operations. The given information described the path in Cartesian coordinates, and there was no mention of speed. Thus it would have been more logical to match the polar and Cartesian coordinate descriptions of the slider's motion. Let us see how such a solution would evolve.

In units of meters the path is described by $y = 5x^2$. We do not know *a priori* how x or y depend on time, so the Cartesian coordinate descriptions of the slider's motion are

$$\bar{v} = \dot{x}\bar{i} + \dot{y}\bar{j} = \dot{x}\bar{i} + 10x\dot{x}\,\bar{j},$$
$$\bar{a} = \frac{d\bar{v}}{dt} = \ddot{x}\bar{i} + 10\left(x\ddot{x} + \dot{x}^2\right)\bar{j}.$$

At the position of interest $x = 0.2$ m, and we earlier determined the corresponding polar coordinates to be $R = 0.6325$ m, $\theta = 18.435°$. From the earlier sketch of the unit vectors we know that

$$\bar{i} = -\cos\theta\bar{e}_R + \sin\theta\bar{e}_\theta = -0.9487\bar{e}_R + 0.3162\bar{e}_\theta,$$
$$\bar{j} = \sin\theta\bar{e}_R + \cos\theta\bar{e}_\theta = 0.3162\bar{e}_R + 0.9487\bar{e}_\theta.$$

Equating the velocity descriptions leads to

$$\bar{v} = \dot{x}\bar{i} + 10\,(0.2)\,\dot{x}\,\bar{j} = \dot{x}\,(-0.9487\bar{e}_R + 0.3162\bar{e}_\theta) + 2\dot{x}\,(0.3162\bar{e}_R + 0.9487\bar{e}_\theta)$$
$$= \dot{R}\bar{e}_R + R\dot{\theta}\bar{e}_\theta = \dot{R}\bar{e}_R + (0.6325)\,(40)\,\bar{e}_\theta.$$

The algebraic equations obtained by matching like components are

$$\bar{v}\cdot\bar{e}_R = [-0.9487 + 2\,(0.3162)]\,\dot{x} = \dot{R},$$
$$\bar{v}\cdot\bar{e}_\theta = [0.3162 + 2\,(0.9487)]\,\dot{x} = 25.30,$$

from which we find that

$$\dot{x} = 11.429 \text{ m/s}, \quad \dot{R} = -3.6145 \text{ m/s}.$$

These values are required to evaluate the acceleration. Equating the Cartesian description of acceleration written earlier to the polar coordinate expression gives

$$\bar{a} = \ddot{x}\bar{i} + 10\left(x\ddot{x} + \dot{x}^2\right)\bar{j}$$
$$= \ddot{x}\,(-0.9487\bar{e}_R + 0.3162\bar{e}_\theta) + 10\left[(0.2)\,\ddot{x} + (11.429)^2\right](0.3162\bar{e}_R + 0.9487\bar{e}_\theta)$$
$$= (\ddot{R} - R\dot{\theta}^2)\bar{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\bar{e}_\theta = \left[\ddot{R} - (0.6325)\,(40)^2\right]\bar{e}_R + 2\,(-3.6145)\,(40)\,\bar{e}_\theta,$$

from which we have

$$\bar{a} \cdot \bar{e}_R = [-0.9487 + 2\,(0.3162)]\,\ddot{x} + 10\,(11.429)^2\,(0.3162) = \ddot{R} - (0.6325)\,(40)^2\,,$$

$$\bar{a} \cdot \bar{e}_\theta = [0.3162 + 2\,(0.9487)]\,\ddot{x} + 10\,(11.429)^2\,(0.9487) = 2\,(-3.6145)\,(40)\,.$$

These equations give

$$\ddot{x} = -690.4 \text{ m/s}^2, \quad \ddot{R} = 1643.3 \text{ m/s}^2.$$

The values of $\dot{R}$ and $\ddot{R}$ determined here are the same as those found previously, so the expression for $\bar{v}$ and $\bar{a}$ in terms of polar coordinate components would be unchanged. Using $\dot{x}$ and $\ddot{x}$ to form the Cartesian coordinate representation gives

$$\bar{v} = 11.429\bar{i} + 22.86\bar{j} \text{ m/s}, \quad \bar{a} = -690.4\bar{i} - 74.71\bar{j} \text{ m/s}^2 \qquad \lhd$$

This solution is more direct than the preceding one, but which approach will be more direct might not be apparent *a priori* in other situations.

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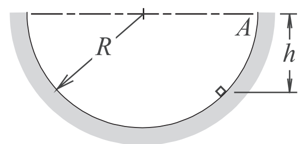
HOMEWORK PROBLEMS

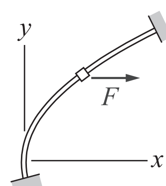


Exercise 2.1

EXERCISE 2.1 A small block slides in the interior of a smooth semicircular cylinder after being released from rest at the upper corner A. Because friction is negligible, the speed of the block is given by $v^2 = 2gh$, where h is the vertical distance the block has fallen. Determine the velocity and acceleration of the block as a function of h. Then sketch the acceleration when $h = R/2$.

EXERCISE 2.2 An automobile follows a circular road whose radius is 50 m. Let x and y respectively denote the eastern and northern directions, with origin at the center of the circle. Suppose the vehicle starts from rest at $x = 50$ m heading north, and its speed depends on the distance s it travels according to $v = 0.5s - 0.0025s^2$, where s is measured in meters and v is in meters per second. It is known that the tires will begin to skid when the total acceleration of the vehicle is $0.6g$. Where will the automobile be and how fast will it be going when it begins to skid? Describe the position in terms of the angle of the radial line relative to the x axis.

EXERCISE 2.3 A locomotive follows a circular track of radius R such that its speed depends on the distance it travels according to $v = v_0 \sin(\pi s/s_0)$, where s_0 is the maximum distance. (a) What value of s corresponds to the maximum tangential acceleration? (b) What value of s corresponds to the maximum normal acceleration? (c) What value of the radius of the track will lead to the maximum normal and tangential accelerations being equal? (d) If the radius is the value in Part (c), at what value of s is the magnitude of the acceleration a maximum?



Exercise 2.4

EXERCISE 2.4 The collar slides over the stationary guide defined by $x = ky^2$ in the vertical plane. The speed of the collar is the constant value v. This motion is implemented by application of a force $\bar{F}$ of variable magnitude parallel to the x axis. Derive expressions for the magnitude of $\bar{F}$ and of the reaction exerted by the guide as functions of the y coordinate of the collar.

EXERCISE 2.5 An old 5000-kg truck is traveling down a 5° incline at 20 km/h when its brakes and engine simultaneously fail. It is known that the air resistance is proportional to the square of the truck's speed and that the maximum speed the truck would obtain if the incline were sufficiently long and the truck did not become unstable is 160 km/h. Rolling resistance is negligible. Determine the truck's speed after it has traveled 1 km from the point of failure. How long does it take for the truck to arrive at this location?

EXERCISE 2.6 A particle follows a planar path defined by $x = k\xi,\ y = 2k[1 - \exp(\xi)]$, such that its speed is $v = \beta\xi$, where k and β are constants. Determine the velocity and acceleration at $\xi = 0.5$.

EXERCISE 2.7 An ellipse is defined by $(x/\alpha)^2 + (y/\beta)^2 = 1$, $z = 0$. Derive expressions for the tangent and normal unit vectors and the radius of curvature as functions of x.

EXERCISE 2.8 A helix is defined by $x = c\psi$, $y = L\sin(k\psi)$, $z = -L\cos(k\psi)$, where c, L, and k are constants. Determine the path variable unit vectors, the radius of curvature, and the torsion of this curve as functions of ψ.

EXERCISE 2.9 A slider moves over a curved guide whose shape in the vertical plane is given by $x = \beta\eta$, $y = \beta\cosh\eta$. Starting from $x = 0$, the speed is observed to vary as $v = v_0(1 - ks)$, where s is the distance traveled and k is a constant. Derive expressions for the velocity and acceleration of the slider as functions of x.

EXERCISE 2.10 A particle moves along the paraboloid of revolution $y = (x^2 + z^2)/L$, such that $x = L\sinh(k\xi)$, $z = -L\cosh(k\xi)$, where ξ is a parameter and L and k are constants. At the position where $\xi = 1/k$, its speed is $5Lk$ and its speed is decreasing at the rate $2Lk^2$. Determine the velocity and acceleration at this position.

EXERCISE 2.11 A particle slides along the hyperbolic paraboloidal surface $z = xy/2$ such that $x = 6\cos(ku)$, $y = -6\sin(ku)$, where x, y, and in z are in meters and u is a parameter. Determine the path variable unit vectors, the radius of curvature, and the torsion of the path at the position where $ku = 2\pi/3$.

EXERCISE 2.12 A roller coaster track is laid out by giving the Cartesian coordinates of its centerline, with x, y, and z respectively measured eastward, northward, and vertically relative to a reference point on the ground. For the track of interest $y = x^2/100$ and $z = 20[\cos(\pi x/50) + 1]$, where x, y, and z are in units of meters and $-50 < x < 50$. Determine and plot as functions of x the xyz components of the tangent, normal, and binormal unit vectors, as well as the dependence on x of the radius of curvature and torsion of the track.

EXERCISE 2.13 In Exercise 2.12 the speed of a car as it travels along the track is known to be $v = [2g(60 - z)]^{1/2}$ m/s. Determine and graph as a function of x the corresponding tangential and normal accelerations of a car.

EXERCISE 2.14 Derive expressions for the binormal unit vector and torsion when a curve is described in parametric form by $\bar{r}(\alpha)$.

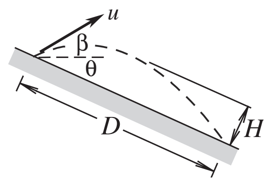
EXERCISE 2.15 The specification for laying out the transition from a straight to curved segment of a train track stipulates that the radius of curvature must change gradually from an infinite value according to $\rho = \rho_0 s_0/s$, where $0 < s < s_0$ is arc length from the beginning of the curve, after which the radius of curvature should be the constant value ρ_0. Consider a high-speed train that moves at a constant speed of 240 km/h. At the end of the curve, where $s = s_0 = 1$ km, the normal acceleration should be $0.5g$. Let x measure distance in the direction of the tangent to the track at $x = 0$, and let y be the offset distance. Use computer software to determine how y must depend on x in order to meet this specification. Also plot the acceleration of a car as a function of x. *Hint*: Use a parametric description of the path with s as the parameter, so that $s' = 1$ and

$y' = \left[1 - (x')^2\right]^{1/2}$. Equating the expression for ρ in this representation to the specified dependence of ρ will lead to a differential equation for $x(s)$ that can be solved numerically.

EXERCISE 2.16 A particle moves along the surface $z = (x^2 - y^2)/L$ such that $x = L\cos(\beta\xi)$, $x = L\sin(\beta\xi)$, where β and L are constants and ξ is a parameter. Consider the case in which $\xi = t$. Derive expressions for the velocity and acceleration.

EXERCISE 2.17 A particle slides along the hyperbolic paraboloidal surface $z = xy/2b$ such that $x = -\left(b^2 - y^2\right)^{1/2}$, $y = b\sin\left(kt^2\right)$, where h, b, and k are constants. Derive expressions for the velocity and acceleration as functions of elapsed time t.

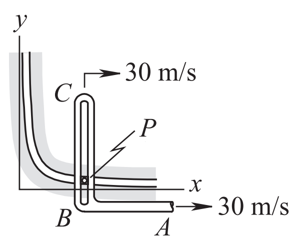
EXERCISE 2.18 A ball is thrown down an incline whose angle of elevation is θ. The initial velocity is u at an angle of elevation β. Derive an expression for the distance D measured along the incline at which the ball will return to the incline. Also determine the maximum height H, measured perpendicularly to the incline, of the trajectory, and the corresponding velocity of the ball at that position.



Exercise 2.18

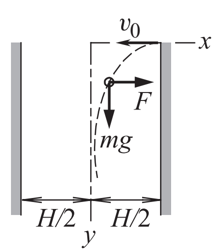
EXERCISE 2.19 A 200-g ball is thrown from the ground with the initial velocity $v_0 =$ 20 m/s at an angle of elevation β. In addition to its weight, there is a headwind that generates a horizontal resistance of 0.5 N. (a) For the case in which $\beta = 30^\circ$ find the horizontal distance at which the ball returns to the elevation from which it was thrown. Also find the velocity of the ball at that location. (b) Find the value of β that maximizes the range for a specified value of v_0.

EXERCISE 2.20 Pin P, whose mass is 10 g, moves in the horizontal plane within a groove defined by $xy = 2$, where x and y are in meters. The motion is actuated by arm ABC, which translates to the right at the constant speed of 30 m/s. (a) Determine the velocity and acceleration of the collar when $x = 2$ m. (b) Determine the forces exerted on the pin by the groove and arm ABC when $x = 2$ m.



Exercise 2.20

EXERCISE 2.21 Gravity causes a steel ball of mass m that is situated between positive and negative magnetic plates to fall as it is attracted toward one of the plates. The magnetic force acting on the ball is horizontal with a magnitude that increases as the square of the distance from the midplane between the plates. In the sketch the xz plane coincides with this midplane, so this force is $F = \alpha x^2$ sgn(x), where sgn(x) is the *signum* function. Suppose the ball is injected at the right plate, $x = H/2$, $y = 0$, with an initial horizontal velocity v_0 to the left. Derive an expression for the minimum v for



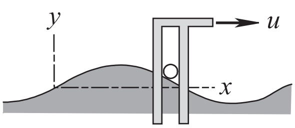
Exercise 2.21

which the ball will hit the left plate. Where will the ball hit the right plate if v is half this minimum value?

EXERCISE 2.22 For laminar flow at low Reynolds number, the air resistance on an object is $-c\bar{v}$, where c is a constant and $\bar{v}$ is the velocity of the object. A sphere of mass m is thrown from the ground with an initial speed v_0 at an angle of elevation β in the (vertical) xy plane. Derive algebraic expressions for the position and velocity of the sphere as a function of time. Then use mathematical software to evaluate the distance the sphere travels, i.e., the value of $x > 0$ at which $y = 0$, for the following sets of initial conditions: (a) initial velocity is 60 m/s at 45° angle of elevation, (b) initial velocity is 60 m/s at 30° angle of elevation. Compare the result in each case with what would be obtained if air resistance were neglected. The mass of the sphere is $m = 4.6$ g, and the value of the viscosity coefficient c is such that the maximum speed of the sphere in a vertical free fall is 60 m/s. (This condition is called the terminal velocity.)

EXERCISE 2.23 Solve Exercise 2.22 in the situation in which there is a steady headwind $v_h = 10$ m/s blowing horizontally. The air resistance in that case is proportional to the velocity of the sphere relative to the air, so that $\bar{f} = -c\left(\bar{v} + v_h\bar{i}\right)$.

EXERCISE 2.24 The diagram shows a small ball that is pushed in the vertical plane along a hill whose elevation is $y = H\sin(\pi x/L)$. The motion is actuated by an angle arm that translates horizontally at constant speed u. It may be assumed that the ball remains in contact with the hill. (a) Derive expressions for the velocity and acceleration of the ball as functions of its horizontal distance x from the origin. (b) Determine the maximum speed v of the ball and the value(s) of x at which it occurs. (c) Determine the maximum acceleration magnitude of the ball and the value(s) of x at which it occurs. (d) What is the largest value of u for which the ball will remain in contact with the hill when $x = L/2$? Friction is negligible, but gravity is not.



Exercise 2.24

EXERCISE 2.25 For the system in Exercise 2.24 determine as a function of x the forces exerted on the sphere by the arm and the hill.

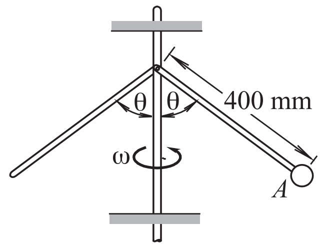
EXERCISE 2.26 The current flowing through the coiled wire sets up a magnetic field $\bar{B}$ that is essentially constant in magnitude and parallel to the axis of the coil, so $\bar{B} = B\bar{k}$. The force acting on a charged particle moving through this field at velocity $\bar{v}$ is given by $F = \beta\bar{v} \times \bar{B}$, where β is a constant. Suppose such a particle is injected into this field at the origin, with an arbitrary initial velocity. Derive an expression for the position of this particle as a function of time, and identify the corresponding path. Gravity is negligible.

EXERCISE 2.27 Use the formula for velocity in cylindrical coordinates to solve Exercise 1.8.

EXERCISE 2.28 A ball rolls on the interior of a paraboloid of revolution given by $x^2 + y^2 = cz$. The angle of rotation about the z axis is $\theta = \gamma\sin(\lambda t)$, and the elevation of the ball is $z = \beta\lambda^2 t^2$, where β, γ, and λ are constants. Determine the velocity and acceleration when $t = 4\pi/3\lambda$.

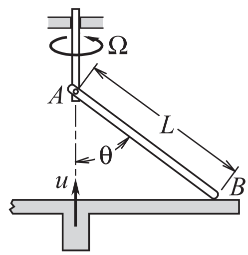
EXERCISE 2.29 In an Eulerian description of fluid flow, particle velocity components are described as functions of the current position of a particle. The polar velocity components of fluid particles in a certain flow are known to be $v_R = (A/R)\cos\theta$, $v_\theta = (A/R)\sin\theta$, where R, θ are the polar coordinates of the particle. Determine the corresponding expressions for the acceleration.

EXERCISE 2.30 The device in the sketch rotates about the vertical axis at $\omega = 1800$ rev/min, and the angle locating the arms relative to the vertical is known to vary as $\theta = (\pi/3)\sin(120t)$ rad, where t is in units of seconds. Determine the velocity and the acceleration of sphere A as a function of time. Then evaluate these expressions for the instants when the elevation of the sphere is a maximum and a minimum.



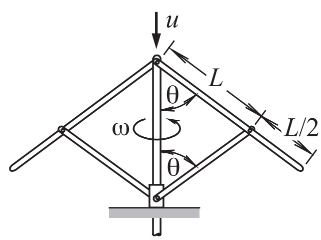
Exercise 2.30

EXERCISE 2.31 The vertical shaft rotates at the constant rate Ω, and the elevation of pin A is constant. End B of the bar slides over the base table, which translates upward at the constant speed u. Describe the velocity and acceleration of end B of the bar in terms of u, Ω, L, and θ.



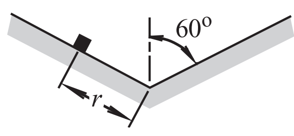
Exercise 2.31

EXERCISE 2.32 The device in the sketch is a flyball governor, which has been used to control the rotation rate of an engine. The concept is that increasing the rotation rate causes the spheres to move outward, thereby causing the vertical rod to move downward, which may be sensed magnetically. Consider a situation in which the angular speed ω is unsteady and the vertical velocity of the rod, u, is constant. Describe the acceleration of a ball in terms of ω, $\dot{\omega}$, θ, and u.



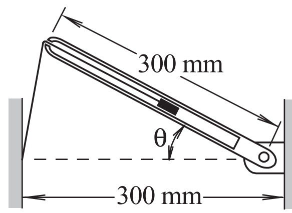
Exercise 2.32

EXERCISE 2.33 A small block whose mass is 400 g slides inside a right circular cone whose axis is vertical. At a certain instant the block is at $r = 200$ m, with $\dot{r} = -10$ m/s and $\ddot{r} = 0$. The block's rotation rate about the cone's axis is observed to be constant at 40 rad/s. Determine the components of the force tangent and normal to the surface required to obtain this motion.



Exercise 2.33

EXERCISE 2.34 The cable, whose length is 300 mm, is fastened to the 500-g block. Clockwise rotation of the arm at a constant angular speed of 5 rad/s causes the block to slide outward. The motion occurs in the vertical plane, and the coefficient of sliding friction is 0.4. Determine the tensile force in the cable and the force exerted by the block on the walls of the groove when $\theta = 53.1301^{\circ}$.

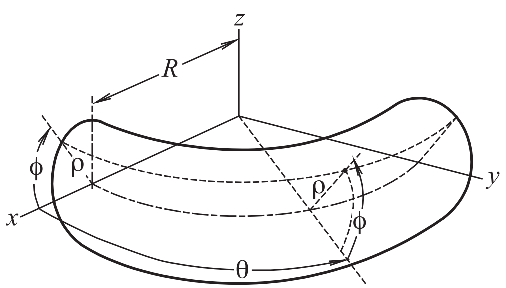


Exercise 2.34

EXERCISE 2.35 Derive the formulas for velocity and acceleration in spherical coordinates by following the formulation using stretch ratios.

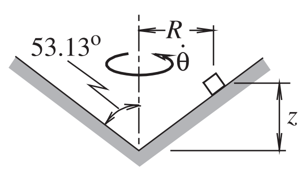
EXERCISE 2.36 Example 2.9 derived the unit vectors for hyperbolic–elliptic coordinates. Derive the corresponding formulas for velocity and acceleration.

EXERCISE 2.37 Toroidal coordinates $(\rho,\ \theta,\ \psi)$ are useful in situations in which it is desired to describe movement relative to a reference circle, which is the case for magnetohydrodynamic studies in the fusion reactor known as a tokamak. Let R be the radius of this reference circle. Then the transformation to Cartesian coordinates is $x = (R + \rho\cos\psi)\cos\theta$, $y = (R + \rho\cos\psi)\sin\theta$, $z = \rho\sin\psi$. Derive expressions for the unit vectors for this coordinate system and for the derivatives of the unit vectors with respect to each toroidal coordinate. Then obtain the toroidal coordinate expressions for velocity and acceleration.



Exercise 2.37

EXERCISE 2.38 A small block slides inside a cone whose apex angle is $\beta = 53.13^{\circ}$. Because angular momentum about the vertical axis of the cone is conserved, the block spins about the z axis such that $R^2\dot{\theta} = 5\ \text{m}^2/\text{s}$, and the vertical motion is observed to be a constant acceleration given by $z = 0.99 - 3t^2$ m. For the instant when $z = 0.24$ m, determine the velocity and acceleration of the block.



Exercise 2.38

EXERCISE 2.39 The instantaneous velocity of a point is $\bar{v} = 10\bar{i} - 4\bar{j} + 6\bar{k}$ m/s, and the acceleration is $\bar{a} = -30\bar{i} - 25\bar{j} + 15\bar{k}$ m/s^2. Determine the corresponding speed, rate of change of the speed, and the radius of curvature of the path.

EXERCISE 2.40 The elevation of the center of mass of an automobile following an extremely bumpy road is observed to be $y = 0.1\sin(\pi x/3)$, where x and y are the horizontal and vertical coordinates in meters. At $x = 1$ m the vehicle's speed is 20 m/s, and the speed at that position is decreasing at 5 m/s^2. Determine the horizontal and vertical components of the acceleration at that instant.

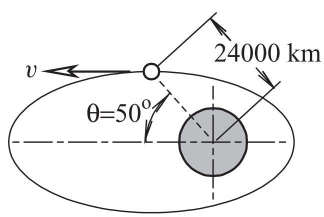
EXERCISE 2.41 Polar coordinates in the XY plane are defined by the radial distance R and the polar angle θ measured from the x axis. Derive expressions for the velocity and acceleration in terms of components relative to xyz by considering $\bar{r}_{P/O} = R\cos\theta\,\bar{i} + R\sin\theta\,\bar{j}$ to be a parametric description of the path based on knowing $R(\theta)$ and $\theta\,(t)$. Then obtain the polar coordinate representations of velocity and acceleration by transforming the xyz representation.

EXERCISE 2.42 At a certain instant, the position, velocity, and acceleration of a point are observed to be

$$\bar{r} = 2000\bar{i} - 1000\bar{j} + 2000\bar{k} \text{ m}, \bar{v} = 100\bar{i} + 150\bar{j} + 200\bar{k} \text{ m/s}, \quad \bar{a} = 30\bar{j} - 50\bar{k} \text{ m/s}^2.$$

Cylindrical coordinates for the system are (R, θ, z) with θ defined as the azimuthal angle in the xy plane, measured relative to the x axis. (a) For this instant determine the speed, the rate at which the speed is changing, and the direction of the normal vector pointing toward the path's center of curvature. (b) Determine the values of θ, $\dot{\theta}$, and $\ddot{\theta}$ at this instant.

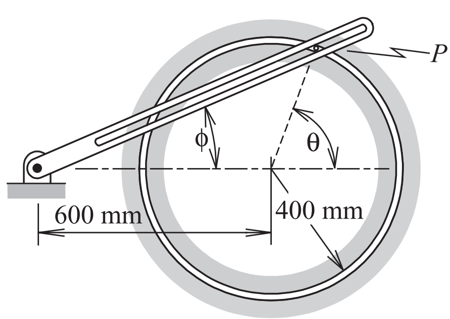
EXERCISE 2.43 A satellite is in an orbit about the Earth. The magnitude of the acceleration of this body is $g(R_e/R)^2$, where R is the distance from the body to the center of the Earth, $R_e = 6370$ km is the radius of the Earth, and $g = 9.807$ m/s^2. At the position shown, the speed of the body is $v = 27\,000$ km/h. (a) Determine the rate of change of the speed and the radius of curvature of the orbit at this position. (b) Determine $\dot{R}$, $\ddot{R}$, $\dot{\theta}$, and $\ddot{\theta}$ at this position.



Exercise 2.43

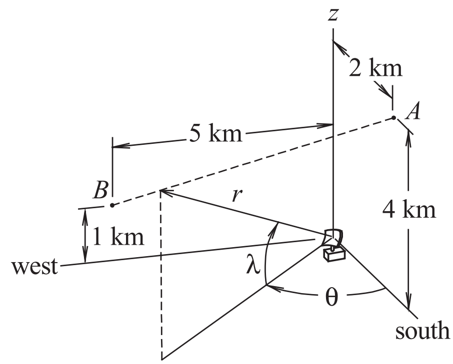
EXERCISE 2.44 Observation of a small mass attached to the end of the flexible bar reveals that the path of the particle is essentially an ellipse in the horizontal plane. The Cartesian coordinates for this motion are measured as $x = A\sin(\theta)$, $y = 2A\cos(\theta)$, $\theta = \omega t$, where A and ω are constants. Determine the speed, the rate of change of the speed, and the normal acceleration at the instants when $\omega t = 0$, $\pi/3$, and $\pi/2$.

EXERCISE 2.45 Pin P slides inside the 400-mm-radius groove at a constant rate of 8 m/s. This motion is actuated by arm AB. Determine the rotation rate of this arm and the rate of change of that rate when (a) $\theta = 90°$, (b) $\theta = 135°$.



Exercise 2.45

EXERCISE 2.46 A radar station at the origin measures the azimuth angle θ, the elevation angle λ, and the radial distance r to a target. At the instant when a high-performance aircraft is at point B it has a velocity of 500 m/s directed from point B to point A and an acceleration of $8g$ directed upward. Determine the values of $\dot{r}$, $\ddot{r}$, $\dot{\lambda}$, $\ddot{\lambda}$, $\dot{\theta}$, and $\ddot{\theta}$ that are observed at this location.



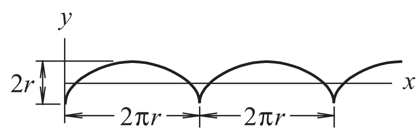
Exercises 2.46 and 2.47

EXERCISE 2.47 A radar station at the origin measures the azimuth angle θ, the elevation angle λ, and the radial distance r to a target, as shown in the sketch. At the instant when an aircraft is at location B, it is observed that $\dot{r} = -400$ m/s, $\ddot{r} = 20$ m/s^2, $\dot{\lambda} = 0.2$ rad/s, $\ddot{\lambda} = 0$, $\theta' = -0.1$ rad/s, and $\ddot{\theta} = 0$. Determine the corresponding speed, rate of change of the speed, and normal acceleration at this instant.

EXERCISE 2.48 An airplane heading eastward is observed to be in a 20° climb at a speed of 2400 km/h. At this instant its acceleration components are $2g$ eastward, $5g$ northward, and $1.5g$ downward. Determine the rate of change of the speed, as well as the radius of curvature and the location relative to the airplane of the center of curvature of the path.

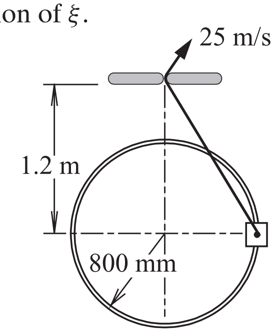
EXERCISE 2.49 A particle follows a planar path such that the azimuthal dependence of the radial distance from a fixed reference point is a known function $R(\theta)$. It is observed that $\dot{\theta}$ is constant. Derive expressions for the velocity and acceleration of the particle. Then use those results to derive an expression for the radius of curvature of a path in polar coordinates.

EXERCISE 2.50 A wheel, whose radius is r, rolls without slipping. A point on the perimeter of the wheel follows a cycloidal path, described in parametric form by $x = r(\xi - \sin \xi)$, $y = -r \cos \xi$. The parameter ξ is observed to depend on time according to $\xi = ct$. Derive expressions for the speed and rate of change of the speed of this point as functions of ξ. Also determine the radius of curvature of the cycloid as a function of ξ.



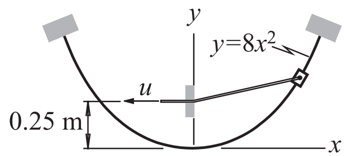
Exercise 2.50

EXERCISE 2.51 A cable that passes through a hole at point A is pulled inward at the constant rate of 25 m/s, thereby causing the 0.2-kg collar to move along the circular guide bar. The system is situated in the vertical plane. (a) Determine the speed and the rate of change of the speed of the slider at the instant shown in the sketch. (b) If the coefficient of sliding friction is $\mu = 0.4$, evaluate the corresponding tension in the cable.



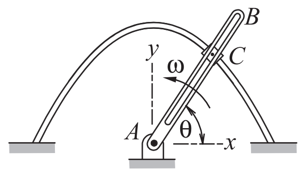
Exercise 2.51

EXERCISE 2.52 A collar slides along a guide bar that is bent to the shape of a parabola, $y = 8x^2$, where x and y are measured in meters. The system lies in the horizontal plane. The motion is actuated by pulling on a cable attached to the collar and passing though a slot. The rate at which the cable is pulled inward is a constant speed u. (a) For the position $x = 1/4$ m determine the velocity and acceleration of the collar in terms of u. (b) Determine the tensile force in the cable at $x = 1/4$ m.



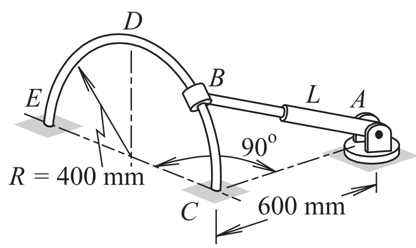
Exercise 2.52

EXERCISE 2.53 Collar C is pushed along the guide bar defined by $y = 2(1 - 0.25x^2)$, where x and y have units of meters. The angular speed of arm AB that actuates the motion is the constant value $\omega = 20$ rad/s, so $\theta = \omega t$. Determine the forces exerted on the collar by arm AB and the guide bar at $x = 1$ m. The mass of the collar is 2 kg, and gravitational effects are ignorable.



Exercise 2.53

EXERCISE 2.54 A hydraulic piston in arm AB controls the arm's length, thereby moving the collar. Guide bar CDE is circular, and it lies in the vertical plane. At the highest position D it is known that $\dot{L} = 10$ m/s and $\ddot{L} = 0$. Determine the speed of the collar and the rate of change of the speed at that location.



Exercise 2.54

CHAPTER 3

Relative Motion

When we ride in an automobile or airplane, the reference frame for our observations is moving. If we wish to use such observations to formulate Newton's Laws, we need to convert them to an inertial reference frame. More fundamentally, the basic fact that the points in a moving rigid body are stationary as viewed from that body is a vital aspect. In this chapter we develop the ability to correlate observations of position, velocity, and acceleration from fixed and moving reference frames.

3.1 COORDINATE TRANSFORMATIONS

It is standard terminology to refer to any quantity that is measured relative to a fixed reference frame as *absolute*, whereas quantities measured with respect to any moving reference frame are *relative*. Figure 3.1 depicts a general situation in which point P is being observed from a moving reference frame xyz whose motion we presumably know, whereas XYZ is a fixed reference frame. It is apparent from Fig. 3.1 that one can arrive at the absolute position $\bar{r}_{P/O}$ by proceeding first to the xyz origin along $\bar{r}_{O'/O}$, then following the relative position $\bar{r}_{P/O'}$, so that

$$\boxed{\bar{r}_{P/O} = \bar{r}_{O'/O} + \bar{r}_{P/O'}.} \tag{3.1.1}$$

Despite the simple appearance of this relation, it embodies many of the issues that we generally encounter. Both $\bar{r}_{P/O}$ and $\bar{r}_{P/O'}$ describe position as seen from a specific reference frame. Each vector could be represented in terms of components relative to the coordinate axes of its associated reference frame. However, if we are to evaluate the sum by adding like components, rather than by a graphically based procedure, then the components of each vector must be described with respect to a common set of unit vectors. In other words, although a vector might describe the perspective of an observer on a specific reference frame, that vector may be described in terms of components relative to the axes of any reference frame.

As an aid to representing each of the vectors in Eq. (3.1.1), we introduce the $x'y'z'$ reference frame in Fig. 3.1, whose origin always coincides with point O', but whose axes always remain parallel to the respective fixed axes of XYZ. Such a reference frame executes a *translational motion*. The $x'y'z'$ coordinates of point P are (x'_P, y'_P, z'_P), and the

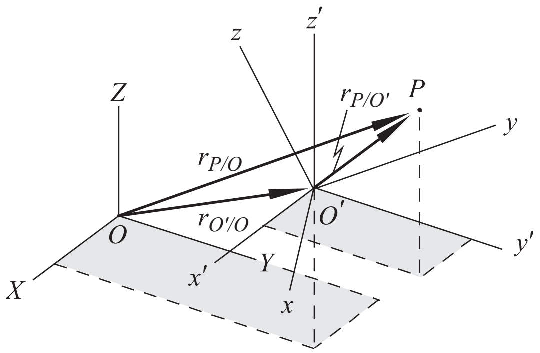


Figure 3.1. Fixed and moving reference frames for observing the position of a point.

XYZ coordinates of point O' are $(X_{O'}, Y_{O'}, Z_{O'})$. The corresponding position vectors therefore are

$$\bar{r}_{O'/O} = X_{O'}\bar{I} + Y_{O'}\bar{J} + Z_{O'}\bar{K}, \quad \bar{r}_{P/O'} = x'_P\bar{i}' + y'_P\bar{j}' + z'_P\bar{k}'. \tag{3.1.2}$$

Because of the parallelism of the $\bar{I}\bar{J}\bar{K}$ and $\bar{i}'\bar{j}'\bar{k}'$ directions, a component description of Eq. (3.1.1) leads to

$$\boxed{X_P = X_{O'} + x'_P, \quad Y_P = Y_{O'} + y'_P, \quad Z = Z_{O'} + z'_P.} \tag{3.1.3}$$

This conversion between coordinates is referred to as a *translation transformation*.

This transformation is useful if we know the coordinates of point P relative to the translating $x'y'z'$ reference frame. However, the more likely circumstance is that we would know the position coordinates (x_P, y_P, z_P) relative to the rotating axes of the xyz reference frame. Such would be the case when we describe position in terms of forward/back, left/right, and up/down relative to the cabin of an airplane, or when we locate a point in a piece of machinery by referring to the engineering drawings. This complicates the task of adding $\bar{r}_{O'/O}$ and $\bar{r}_{P/O'}$, because the directions used to represent the vectors are not parallel. Determining the $x'y'z'$ coordinates of point P corresponding to known xyz coordinates requires a rotation transformation.

3.1.1 Rotation Transformations

We consider a general situation in which a vector is described in terms of components relative to the axes of two coordinate systems, xyz and $x'y'z'$, that have a common origin. Figure 3.2 depicts the *direction angles* α, β, γ between the x' axis and each of the xyz axes. An examination of Fig. 3.2 shows that the values of the direction angles should be limited to the range $0 \le \alpha, \beta, \gamma \le \pi$ to avoid ambiguity. The components of $\bar{i}'$ are its projections onto $\bar{i}$, $\bar{j}$, and $\bar{k}$, which are determined from the direction angles according to

$$\begin{aligned} \bar{i}' &= \left(\bar{i}' \cdot \bar{i}\,\right)\bar{i} + \left(\bar{i}' \cdot \bar{j}\,\right)\bar{j} + \left(\bar{i}' \cdot \bar{k}\right)\bar{k} \\ &= (\cos\alpha)\,\bar{i} + (\cos\beta)\,\bar{j} + (\cos\gamma)\,\bar{k}. \end{aligned} \tag{3.1.4}$$

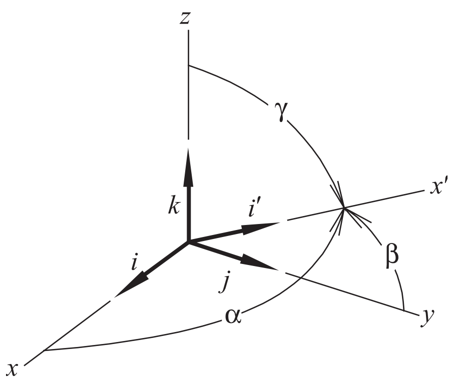


Figure 3.2. Direction angles α, β, and γ for a line.

This expression indicates that the cosines of the direction angles are more significant to our analysis: They are the *direction cosines*. We obviously are equally interested in all unit vectors, so we

Define $\ell_{p'q}$ *to be the cosine of the angle between axis* p' *and axis* q*, with* p *and* q *representing* x, y, *or* z.

Extending Eq. (3.1.4) to the other unit vectors then yields

$$\begin{aligned}\bar{i}' &= \ell_{x'x}\bar{i} + \ell_{x'y}\bar{j} + \ell_{x'z}\bar{k},\\ \bar{j}' &= \ell_{y'x}\bar{i} + \ell_{y'y}\bar{j} + \ell_{y'z}\bar{k},\\ \bar{k}' &= \ell_{z'x}\bar{i} + \ell_{z'y}\bar{j} + \ell_{z'z}\bar{k}.\end{aligned} \tag{3.1.5}$$

It is convenient to rewrite these equations in matrix form as

$$\boxed{\begin{bmatrix}\bar{i}' & \bar{j}' & \bar{k}'\end{bmatrix}^{\mathrm{T}} = [R]\begin{bmatrix}\bar{i} & \bar{j} & \bar{k}\end{bmatrix}^{\mathrm{T}},} \tag{3.1.6}$$

where

$$\boxed{[R] = \begin{bmatrix}\ell_{x'x} & \ell_{x'y} & \ell_{x'z}\\ \ell_{y'x} & \ell_{y'y} & \ell_{y'z}\\ \ell_{z'x} & \ell_{z'y} & \ell_{z'z}\end{bmatrix}.} \tag{3.1.7}$$

The matrix $[R]$ is the *rotation transformation*. It is a generalization of the conversion between coplanar pairs of unit vectors that we employed to discuss mixed kinematical descriptions in Section 2.4.

Several important properties of $[R]$ follow from the fact that $\bar{i}$, $\bar{j}$, and $\bar{k}$ are an orthogonal set of unit vectors, as are $\bar{i}'$, $\bar{j}'$, and $\bar{k}'$. Suppose that we were to follow parallel steps to the preceding in order to establish the transformation $[R']$ describing the unit vectors $\bar{i}$, $\bar{j}$, $\bar{k}$ in terms of their components with respect to $\bar{i}'$, $\bar{j}'$, $\bar{k}'$. By direct analogy with Eqs. (3.1.6) and (3.1.7) we find that the inverse transformation $[R']$ is described by

$$\begin{bmatrix}\bar{i} & \bar{j} & \bar{k}\end{bmatrix}^{\mathrm{T}} = [R']\begin{bmatrix}\bar{i}' & \bar{j}' & \bar{k}'\end{bmatrix}^{\mathrm{T}}, \tag{3.1.8}$$

where the elements of $[R']$ are the corresponding direction cosines between $\bar{i}$, $\bar{j}$, or $\bar{k}$ and $\bar{i}'$, $\bar{j}'$, or $\bar{k}'$. For example, $R'_{1,2} = \ell_{xy'}$. Because $\ell_{xy'} \equiv \ell_{y'z}$, it follows that $R'_{1,2} = R_{2,1}$. More generally, the definition of the direction cosines leads to $R'_{m,n} = R_{n,m}$, so that $[R'] = [R]^{\mathrm{T}}$.

A different description of $[R']$ results from solving Eq. (3.1.6) for $\bar{i}$, $\bar{j}$, $\bar{k}$, which gives

$$\begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \end{bmatrix}^{\mathrm{T}} = [R]^{-1} \begin{bmatrix} \bar{i}' & \bar{j}' & \bar{k}' \end{bmatrix}^{\mathrm{T}}. \tag{3.1.9}$$

A comparison of Eqs. (3.1.8) and (3.1.9) shows that $[R'] = [R]^{-1}$. Thus we find that

The transformation matrix $[R']$ converting $\bar{i}'$, $\bar{j}'$, and $\bar{k}'$ to $\bar{i}$, $\bar{j}$, and $\bar{k}$ is the inverse of the matrix $[R]$ converting $\bar{i}$, $\bar{j}$, and $\bar{k}$ to $\bar{i}'$, $\bar{j}'$, and $\bar{k}'$. This inverse transformation may be evaluated by taking the transpose of the original transformation,

$$\boxed{[R'] = [R]^{-1} = [R]^{\mathrm{T}}.} \tag{3.1.10}$$

Equation (3.1.7) defines the rows of $[R]$ to be the direction cosines of one of the primed unit vectors relative to the unprimed set. Similarly, the columns of $[R]$ consist of the direction cosines of an unprimed unit vector with respect to each of the primed set. Let $\{i\}$, $\{j\}$, etc., denote columns holding the direction cosines of the associated unit vector with respect to the other set of directions. Then we may write $[R]$ in partition form in either of two ways,

$$[R] = \begin{bmatrix} \{i'\}^{\mathrm{T}} \\ \{j'\}^{\mathrm{T}} \\ \{k'\}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \{i\} & \{j\} & \{k\} \end{bmatrix}. \tag{3.1.11}$$

The rules for products of partitioned matrices indicate that the partitions behave like single elements if the partitions are conformable (that is, consistently dimensioned). According to the first of Eqs. (3.1.11), it must be that

$$\begin{aligned} [R][R]^{\mathrm{T}} &= \begin{bmatrix} \{i'\}^{\mathrm{T}} \\ \{j'\}^{\mathrm{T}} \\ \{k'\}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \{i'\} & \{j'\} & \{k'\} \end{bmatrix} \\ &= \begin{bmatrix} \{i'\}^{\mathrm{T}}\{i'\} & \{i'\}^{\mathrm{T}}\{j'\} & \{i'\}^{\mathrm{T}}\{k'\} \\ \{j'\}^{\mathrm{T}}\{i'\} & \{j'\}^{\mathrm{T}}\{j'\} & \{j'\}^{\mathrm{T}}\{k'\} \\ \{k'\}^{\mathrm{T}}\{i'\} & \{k'\}^{\mathrm{T}}\{j'\} & \{k'\}^{\mathrm{T}}\{k'\} \end{bmatrix} = [U], \end{aligned} \tag{3.1.12}$$

where $[U]$ is the identity (unit) matrix. The final result stems from the fact that each element of the product matrix is the matrix representation of a dot product, and the unprimed unit vectors are mutually orthogonal. A similar expression results if we use the second form in Eq. (3.1.11) to form $[R]^{\mathrm{T}}[R]$, specifically

$$[R]^{\mathrm{T}}[R] = \begin{bmatrix} \{i\}^{\mathrm{T}}\{i\} & \{i\}^{\mathrm{T}}\{j\} & \{i\}^{\mathrm{T}}\{k\} \\ \{j\}^{\mathrm{T}}\{i\} & \{j\}^{\mathrm{T}}\{j\} & \{j\}^{\mathrm{T}}\{k\} \\ \{k\}^{\mathrm{T}}\{i\} & \{k\}^{\mathrm{T}}\{j\} & \{k\}^{\mathrm{T}}\{k\} \end{bmatrix} = [U], \tag{3.1.13}$$

where the final form is a consequence of the mutual orthogonality of the unprimed unit vectors. Thus a matrix having the property that $[R]^{-1} = [R]^{\mathrm{T}}$ is said to be *orthonormal.*

The fact that $[R][R]^{\mathrm{T}} = [U]$ gives rise to a useful property. Recall from matrix algebra that the determinant of a product of matrices is identical to the product of the individual determinants. Furthermore, the determinant of $[R]$ is identical to the determinant of $[R]^{\mathrm{T}}$. Simultaneous satisfaction of both properties, in combination with the requirement that both xyz and $x'y'z'$ are right-handed coordinate systems,* leads to the conclusion that

$$|[R]| = 1. \tag{3.1.14}$$

One use of this property is to check computations.

Because a dot product is independent of the order in which the product is formed, Eq. (3.1.12) consists of six independent elements, whose specific form is

$$\ell_{p'x}\ell_{q'x} + \ell_{p'y}\ell_{q'y} + \ell_{p'z}\ell_{q'z} = \delta_{pq}, \quad p, q = x,\ y,\ \text{or } z, \tag{3.1.15}$$

where δ_{pq} denotes the Kronecker delta; $\delta_{pq} = 1$ if $p = q$ and $\delta_{pq} = 0$ otherwise. Because there are six independent combinations of p and q in the preceding, it follows that there are six equations relating the nine direction cosines. Consequently, there are only three independent direction angles. However, the selection of which angles are independent is not entirely arbitrary. For example, the values of α, β, and γ in Fig. 3.2 are not independent because $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. This restriction arises because these three angles locate only one axis.

In Eqs. (3.1.6) the unit vectors of one coordinate system are described in terms of the unit vectors of another. However, $[R]$ also transforms the components of vectors. To see this we consider an arbitrary vector $\bar{A}$, which may be described in terms of its components with respect to either set of unit vectors,

$$\bar{A} = A_{x'}\bar{i}' + A_{y'}\bar{j}' + A_{z'}\bar{k}' = A_x\bar{i} + A_y\bar{j} + A_z\bar{k}. \tag{3.1.16}$$

A matrix representation of this expression is

$$\bar{A} = \begin{bmatrix} \bar{i}' & \bar{j}' & \bar{k}' \end{bmatrix} \begin{Bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{Bmatrix} = \begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \end{bmatrix} \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}. \tag{3.1.17}$$

* Equation (3.1.14) is also correct if both coordinate systems are left-handed, whereas $|[R]| = -1$ if xyz and $x'y'z'$ have different parities.

To eliminate the primed unit vectors, we substitute Eq. (3.1.9) into the right side of Eq. (3.1.17). To do so we must use the property that the transpose of a matrix product is the reversed product of the transpose of each matrix. According to the orthonormal property, $\left[[R]^{-1}\right]^{\mathrm{T}} = [R]$, so we have

$$\begin{bmatrix} \bar{i}' & \bar{j}' & \bar{k}' \end{bmatrix} \begin{Bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{Bmatrix} = \begin{bmatrix} \bar{i}' & \bar{j}' & \bar{k}' \end{bmatrix} [R] \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}. \tag{3.1.18}$$

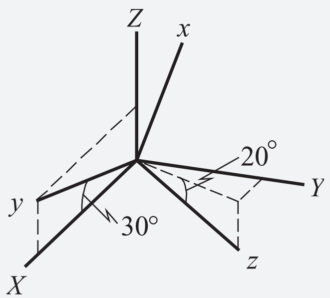
This must be true for an arbitrary $\bar{A}$, which permits us to cancel the row of unit vectors. Thus,

$$\boxed{\begin{Bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{Bmatrix} = [R] \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}.} \tag{3.1.19}$$

Kinematically, the position of a point with respect to the origin is of primary importance, in which case the vector components are the Cartesian coordinates of the point with respect to either the fixed XYZ or the moving xyz. In some situations we wish to follow a certain point P on the body, as we would when we monitor the motion of a point in a piece of machinery. In that case the $(x_P,\ y_P,\ z_P)$ values with respect to the body-fixed axes remain constant, and the $(X_P,\ Y_P,\ Z_P)$ coordinates change. In other situations it is necessary to determine how a point P that does not move is seen from the perspective of the moving body. Then it is $(X_P,\ Y_P,\ Z_P)$ that remain constant, and $(x_P,\ y_P,\ z_P)$ change. Either situation can be addressed once we have determined the transformation matrix $[R]$ that converts $\bar{I}\bar{J}\bar{K}$ components to $\bar{i}\,\bar{j}\bar{k}$ components.

We have seen that knowledge of $[R]$ enables us to fully characterize a vector in terms of alternative sets of components. The following example determines a transformation directly from the orthonormal properties. Sometimes there are simpler procedures for such a determination, which we will take up in the following sections.

EXAMPLE 3.1 The positions of two points are known to be $\bar{r}_A = -250\bar{i} + 400\bar{j} - 500\bar{k}$ mm relative to the xyz coordinate system described in the sketch and $\bar{r}_B = 400\bar{I} - 600\bar{J} + 200\bar{K}$ mm relative to the XYZ coordinate system. Determine the position coordinates of each point relative to the coordinate system not given, and also determine the distance between the points.



Example 3.1

SOLUTION This example illustrates most of the basic operations associated with rotation transformations. According to the sketch the y axis lies in the XZ plane such that

$$\bar{j} = \cos 30^\circ \bar{I} + \sin 30^\circ \bar{K}.$$

The only other piece of information conveyed by the sketch is that the z axis is depressed 20° below the XY plane. Thus the Z component of $\bar{k}$ is $-\sin 20^\circ$. Let θ denote the angle between the Y axis and the projection of the z axis onto the XY plane. Because $\bar{k}$ has unit magnitude, the length of its projection onto this plane is $\cos 20^\circ$, from which it follows that

$$\bar{k} = \cos 20^\circ \left(\sin\theta \bar{I} + \cos\theta \bar{J}\right) - \sin 20^\circ \bar{K}.$$

The angle θ is set by the condition that $\bar{j}$ and $\bar{k}$ are orthogonal, so that their dot product is zero,

$$\bar{j}\cdot\bar{k} = \cos 30^\circ \cos 20^\circ \sin\theta - \sin 30^\circ \sin 20^\circ = 0,$$

from which we find

$$\theta = \sin^{-1}(\tan 30^\circ \tan 20^\circ) = 0.2117 \text{ rad} = 12.131^\circ.$$

The preceding description of $\bar{k}$ correspondingly gives

$$\bar{k} = 0.19747\bar{I} + 0.91871\bar{J} - 0.34202\bar{k}.$$

The fact that $\bar{i}$, $\bar{j}$, and $\bar{k}$ are a set of orthonormal unit vectors enables us to directly determine $\bar{i}$ from a cross product:

$$\bar{i} = \bar{j}\times\bar{k} = -0.45936\bar{I} + 0.39493\bar{J} + 0.79563\bar{K}.$$

The rows of the transformation from XYZ to xyz are the components of the unit vectors of xyz relative to XYZ, so we have found that

$$[R] = \begin{bmatrix} -0.45936 & 0.39493 & 0.79563 \\ 0.86603 & 0 & 0.500 \\ 0.19747 & 0.91871 & -0.34202 \end{bmatrix}.$$

We use $[R]^{\mathrm{T}}$ to transform the xyz coordinates of point A, and $[R]$ to transform the XYZ coordinates of point B:

$$\begin{Bmatrix} X_A \\ Y_A \\ Z_A \end{Bmatrix} = [R]^{\mathrm{T}} \begin{Bmatrix} -250 \\ 400 \\ -500 \end{Bmatrix} = \begin{Bmatrix} 362.52 \\ -558.09 \\ 172.10 \end{Bmatrix} \text{ mm,}$$

$$\begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix} = [R] \begin{Bmatrix} 400 \\ -600 \\ 200 \end{Bmatrix} = \begin{Bmatrix} -261.58 \\ 446.41 \\ -540.64 \end{Bmatrix} \text{ mm.} \quad \lhd$$

The distance between the points is the magnitude of the position vector between them, which may be constructed from $\bar{r}_{B/A} = \bar{r}_{B/O} - \bar{r}_{A/O}$. We calculate this difference by taking the difference of components with respect to the XYZ coordinate system:

$$\{r_{B/A}\} = \begin{Bmatrix} X_B - X_A \\ Y_B - Y_A \\ Z_B - Z_A \end{Bmatrix} = \begin{Bmatrix} 37.48 \\ -41.91 \\ 27.90 \end{Bmatrix}.$$

The distance $|\bar{r}_{B/A}|$ may be determined from the matrix implementation of a dot product, which gives

$$|\bar{r}_{B/A}|^2 = [37.48 \quad -41.91 \quad 27.90] \begin{Bmatrix} 37.48 \\ -41.91 \\ 27.90, \end{Bmatrix},$$

$$|\bar{r}_{B/A}| = 62.77 \text{ mm.} \quad \lhd$$

This result should be the same as what would be obtained if we described $\bar{r}_{B/A}$ in terms of xyz components. Doing so would have given

$$\{r'_{B/A}\} = \begin{Bmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{Bmatrix} = \begin{Bmatrix} -11.58 \\ 46.41 \\ -40.64 \end{Bmatrix},$$

$$|\bar{r}'_{B/A}|^2 = [-11.58 \quad 46.41 \quad -40.64] \begin{Bmatrix} -11.58 \\ 46.41 \\ -40.64 \end{Bmatrix},$$

$$|\bar{r}'_{B/A}| = 62.77 \text{ mm.} \quad \lhd$$

3.1.2 Rotation Sequences

One use of a moving reference frame is to represent a rigid body in motion. A set of axes xyz that are attached to the body is said to be *body fixed*. To follow the rotation of the axes we designate XYZ as the orientation of the body-fixed axes prior to the initiation of motion. Because a translation transformation accounts for the motion of the origin of xyz, we temporarily consider the origins of XYZ and xyz to coincide. Our objective here is to characterize the transformation from XYZ to the current orientation of xyz in terms of rotations that the body undergoes. We do this by following successively more complicated types of rotations.

Simple Rotations

In a *simple rotation* one of the body-fixed coordinate axes remains stationary. We may picture such a rotation by looking down the stationary axis, because all points move in the plane perpendicular to that axis. To avoid ambiguity, we use the right-hand rule to define the positive sense of rotation. Specifically, one curls the fingers of the right hand in the sense of the rotation. If the extended thumb of that hand points in the positive sense of the rotation axis, then the rotation angle is positive. The three possibilities, involving positive rotation about the x, y, or z axis, are depicted in Fig. 3.3. We use a subscript to denote the axis for the simple rotation. Inspection of Fig. 3.3 gives the direction angles of xyz relative to XYZ, which leads to

$$\boxed{\begin{gathered}[R_x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix}, \quad [R_y] = \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix}, \\ [R_z] = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}.\end{gathered}} \tag{3.1.20}$$

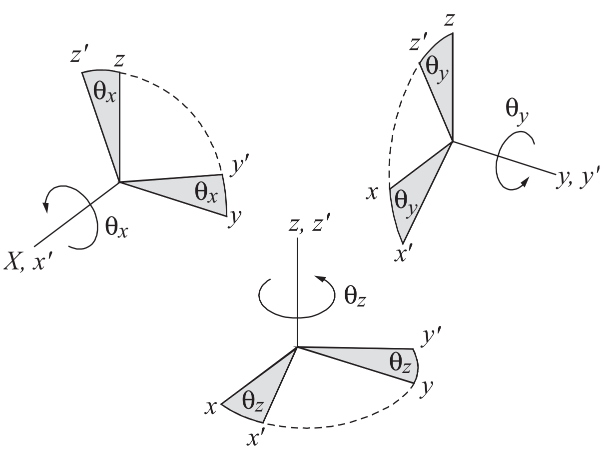


Figure 3.3. Simple rotations about each of the coordinate axes.

Note that a rotation that is opposite the sense of the right-hand rule corresponds to a negative value of the respective angle in the preceding equations. This leads to an interesting observation. Each transformation for the negative angle is the transpose of the transformation of the positive angle, that is

$$[R_x(-\theta_x)] = [R_x(+\theta_x)]^{\mathrm{T}}, \quad [R_y(-\theta_y)] = [R_y(+\theta_y)]^{\mathrm{T}}, \quad [R_z(-\theta_z)] = [R_z(+\theta_z)]^{\mathrm{T}}. \tag{3.1.21}$$

This demonstrates that the inverse transformation for a simple rotation merely corresponds to the opposite rotation, which is a fact that is apparent from Fig. 3.3.

Body-Fixed Rotations

A *spatial rotation* is one in which a new orientation does not result from rotation about a single coordinate axis. The first situation we treat is that in which the overall rotation can be pictured as a sequence of simple rotations. The ultimate orientation of a coordinate system that undergoes such a rotation clearly will depend on the orientation of each of the simple rotation axes, and the amount of rotation about each axis. It is less apparent that the final alignment of the coordinate axes also is dependent on the sequence in which the individual rotations occur. Two situations commonly arise. The simpler case to describe in words is a *space-fixed* rotation sequence, in which the simple rotation axes have fixed orientations in space. The contrasting situation is a *body-fixed* rotation sequence, in which each simple rotation is about one of the body-fixed axes at the preceding step in the sequence. For example, a body-fixed sequence θ_y, θ_z, θ_x occurs first about the initial position of the y axis, then about the z axis in its new orientation, then about the x axis in its orientation after the second rotation. Although a body-fixed rotation is more difficult than a space-fixed rotation to describe in words, the transformation for body-fixed rotations is easier to derive.

We begin by following a specific sequence of body-fixed rotations, after which we generalize the result. The first rotation θ_x occurs about the original orientation of the x axis, and the second rotation θ_z occurs about the position of the z axis after the first rotation. In Fig. 3.4(a), the stationary XYZ system marks the initial orientation of xyz.

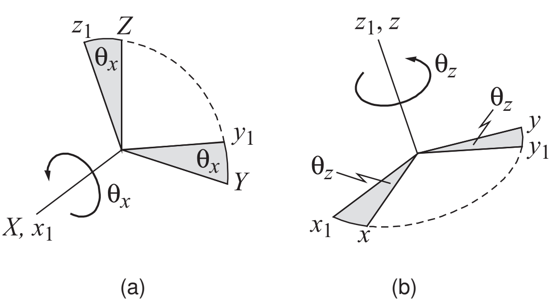


Figure 3.4. Body-fixed rotations: (a) rotation by θ_x about the original x axis, followed by (b) rotation by θ_z about the z axis resulting from the first rotation.

We mark the orientation of xyz after the θ_x rotation as $x_1y_1z_1$. The transformation describing the first rotation is given by

$$\begin{Bmatrix} A_{x_1} \\ A_{y_1} \\ A_{z_1} \end{Bmatrix} = [R_x] \begin{Bmatrix} A_X \\ A_Y \\ A_Z \end{Bmatrix}, \tag{3.1.22}$$

where we may readily construct $[R_x]$ because it describes a simple rotation.

The result of the second rotation is depicted in Fig. 3.4(b). The θ_z rotation moves xyz from its intermediate orientation coincident with $x_1y_1z_1$ to its final orientation. Because this corresponds to a single axis rotation about the z_1 axis, we have

$$\begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} = [R_z] \begin{Bmatrix} A_{x_1} \\ A_{y_1} \\ A_{z_1} \end{Bmatrix}. \tag{3.1.23}$$

Substitution of Eq. (3.1.22) into Eq. (3.1.23) leads to the overall transformation matrix $[R]$,

$$\begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} = [R] \begin{Bmatrix} A_X \\ A_Y \\ A_Z \end{Bmatrix}, \quad [R] = [R_z][R_x]. \tag{3.1.24}$$

Additional rotations about any of the xyz axes will merely extend the preceding by inserting additional premultiplication factors. Furthermore, we observe that the details of the individual transformations were not used, so the result is valid for any sequence of rotations, rather than being limited to simple rotations. We let $[R_i]$ denote the transformation describing the ith rotation. This enables us to conclude that

If xyz is a coordinate system that undergoes a sequence of rotations about body-fixed axes, and XYZ marks the initial orientation of xyz, then the transformation from $\bar{I}\bar{J}\bar{K}$ components to the final $\bar{i}\bar{j}\bar{k}$ components is obtained by premultiplying (from right to left) the sequence of transformation matrices for the individual rotations. For n rotations:

$$\boxed{[R] = [R_n] \cdots [R_2][R_1].} \tag{3.1.25}$$

It should be noted that although the preceding is valid for any type of body-fixed rotation, in practice, we usually apply it to simple rotations, that is, rotations about the axes of xyz. Doing so simplifies the description of the individual transformations.

EXAMPLE 3.2 Consider the coordinate systems in Example 3.1. The final orientation of xyz may be obtained from a set of body-fixed rotations, starting from an initial orientation in which xyz coincides with XYZ. Describe the axis and angle for each rotation.

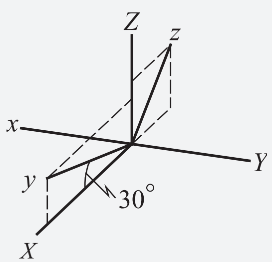
SOLUTION In addition to illustrating some basic operations, this example demonstrates the versatility provided by decomposing a transformation into a set of simple rotations. Initially xyz coincides with XYZ. Because we fully know the final orientation of the y axis, we begin by imparting a set of rotations that move this axis to where it should be. One way of doing this is first to rotate xyz about the negative Z axis by 90° in order to make the y axis align with the X axis. This leaves the z and Z axes coincident and the x axis aligned oppositely from the Y axis. Thus the first transformation is

$$[R_1] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Next we rotate by 30° about the current x axis, which places the y axis where it should be. This is a simple rotation transformation given by

$$[R_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30^\circ & \sin 30^\circ \\ 0 & -\sin 30^\circ & \cos 30^\circ \end{bmatrix}.$$

It is helpful to sketch the orientation of xyz after the second rotation.



x, y, z after notation of −90° about the original z axis, followed by rotation of 30° about the new x axis.

Any further rotation should keep the y axis in place, so we rotate xyz about its current y axis. Because the amount of this rotation cannot be ascertained by inspection, we let ϕ denote the angle. The associated simple rotation transformation is

$$[R_3] = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}.$$

The transformation corresponding to these three body-fixed rotations is

$$[R] = [R_3][R_2][R_1] = \begin{bmatrix} 0.5\sin\phi & -\cos\phi & -0.8660\sin\phi \\ 0.8660 & 0 & 0.5 \\ -0.5\cos\phi & -\sin\phi & 0.8660\cos\phi \end{bmatrix}.$$

The final orientation of the z axis is such that the angle between it and the Z axis is 110°, so that $\ell_{zZ} = \cos 110°$. Equating this to the (3, 3) element of $[R]$ gives

$$\cos\phi = \frac{\cos 110°}{0.8660} = -0.3949.$$

There are two roots of the preceding in the range $-180° < \phi \leq 180°$. To select the appropriate one we observe that the sketch accompanying Example 3.1 indicates that the direction angles from the X and Y axes to the z axis are both acute, which requires that both ℓ_{zX} and ℓ_{zY} be positive. Because these direction cosines are the (3,1) and (3,2) elements of $[R]$, the desired root must be such that $\cos\phi$ and $\sin\phi$ are both negative, so $\phi = -113.26°$ is the appropriate root. Evaluation of $[R]$ corresponding to this value of ϕ shows the result to be the same as $[R]$ derived in Example 3.1. Thus $[R]$ may be obtained by a sequence of three body-fixed rotations: $-90°$ about the z axis, followed by 30° about the x axis, concluding with $-113.26°$ about the y axis. ◁

Space-Fixed Rotations

The derivation of the transformation matrix for a sequence of space-fixed rotations follows a course that parallels the development in the previous section, in that we begin by considering a specific pair of rotations and then generalize the result. The rotations now occur about the axes of the stationary XYZ coordinate system. In the first rotation, θ_X, the x axis remains coincident with the X axis. In Fig. 3.5(a) $x_1y_1z_1$ marks the position of xyz after the first rotation. Thus the transformation from $\bar{I}\bar{J}\bar{K}$ components to $\bar{i}_1\bar{j}_1\bar{k}_1$ components is described by the simple rotation transformation $[R_X]$ corresponding to angle θ_X about the X axis, so that

$$\begin{Bmatrix} A_{x_1} \\ A_{y_1} \\ A_{z_1} \end{Bmatrix} = [R_x] \begin{Bmatrix} A_X \\ A_Y \\ A_Z \end{Bmatrix}. \tag{3.1.26}$$

The second rotation, which consists of θ_Z about the (fixed) Z axis, rotates $x_1y_1z_1$ to xyz. This movement is difficult to visualize because it does not occur about an axis of either $x_1y_1z_1$ or xyz. As an aid to following the rotation, Fig. 3.5(a) shows construction lines Ay_1 and Bz_1, which are formed by dropping perpendiculars from the tips of the respective axes to the Z axis. Coordinate system $x_2y_2z_2$ in Fig. 3.5(b) is defined to coincide with XYZ prior to the second rotation. The Z and z_2 axes remain coincident,

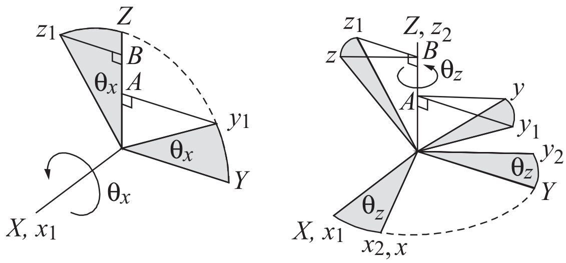


Figure 3.5. Space-fixed rotations: (a) θ_x about the fixed X axis, followed by (b) θ_z about the fixed Z axis.

so $\bar{i}_2\,\bar{j}_2\bar{k}_2$ components are related to $\bar{I}\bar{J}\bar{k}$ components by the simple rotation transformation $[R_Z]$:

$$\begin{Bmatrix} A_{x_2} \\ A_{y_2} \\ A_{z_2} \end{Bmatrix} = [R_z] \begin{Bmatrix} A_{x_1} \\ A_{y_1} \\ A_{z_1} \end{Bmatrix}. \tag{3.1.27}$$

Now comes the crucial observation: Because $x_1y_1z_1$ and $x_2y_2z_2$ undergo the second rotation in unison, their relative orientation is invariant. Before this rotation xyz coincided with $x_1y_1z_1$ and $x_2y_2z_2$ coincided with XYZ. Equation (3.1.26) gives the transformation from XYZ to $x_1y_1z_1$, so it also describes the transformation from $x_2y_2z_2$ to xyz:

$$\begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} = [R_x] \begin{Bmatrix} A_{x_2} \\ A_{y_2} \\ A_{z_2} \end{Bmatrix}. \tag{3.1.28}$$

To eliminate the components relative to $x_2y_2z_2$, we substitute Eq. (3.1.27), with the result that

$$\begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} = [R_x][R_z] \begin{Bmatrix} A_X \\ A_Y \\ A_Z \end{Bmatrix} = [R] \begin{Bmatrix} A_X \\ A_Y \\ A_Z \end{Bmatrix}. \tag{3.1.29}$$

We now observe that we never specifically treated $[R_X]$ and $[R_Z]$ as simple rotations. Furthermore, the pattern would repeat if we were to consider another space-fixed rotation. Consequently, the preceding is generally valid, provided that we use $[R_j]$ to denote the jth rotation about a space-fixed axis. We therefore conclude that

If xyz is a reference frame that undergoes a sequence of rotations about a set of axes that are fixed in space, and xyz is initially coincident with XYZ, then the transformation from $\bar{I}\bar{J}\bar{K}$ components to $\bar{i}\,\bar{j}\bar{k}$ components is obtained by postmultiplying *(from left to right) the sequence of transformation matrices for the individual rotations. For n rotations,*

$$\boxed{[R] = [R_1][R_2]\cdots[R_n].} \tag{3.1.30}$$

The similarity of Eqs. (3.1.25) and (3.1.30) is significant. We see that the result of a sequence of body-fixed rotations leads to the same orientation as the reverse sequence of space-fixed rotations, and vice versa. For example, the transformation in Eq. (3.1.29), which was obtained by consideration of a pair of space-fixed rotations θ_X followed by θ_Z, can be obtained alternatively by a pair of body-fixed rotations. The first such rotation would be θ_Z about the Z axis, and the second rotation would be θ_X about the x axis in its orientation following the first rotation. Both situations are depicted in Fig. 3.6. In this sketch $x_1y_1z_1$ marks the position of xyz after the first rotation in each case. It can be seen that, in both cases, the x axis is situated in the XY plane at an angle θ_Z from the X axis, the direction angle from the Z axis to the y axis is $\pi/2 - \theta_X$, and the direction angle from the Z axis to the z axis is θ_X. These properties lead to the conclusion that the orientation of xyz relative to XYZ is the same for each rotation case.

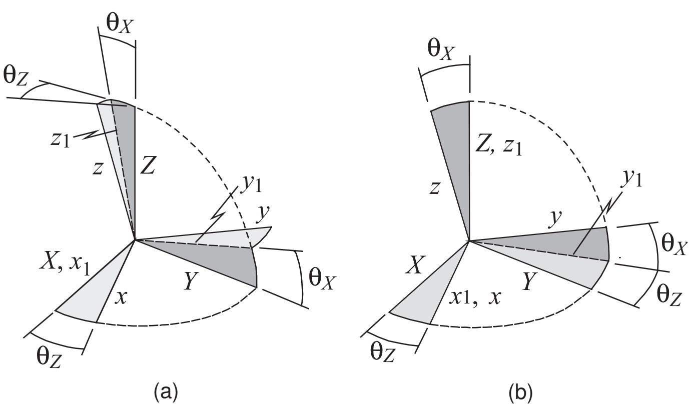


Figure 3.6. A pair of rotations about (a) space-fixed rotations θ_X followed by θ_Z, compared to (b) the body-fixed rotations θ_Z followed by θ_X.

For a given sequence of simple rotations, the order of multiplication in which $[R]$ is formed from the individual matrices must be consistent with the type of rotation: premultiplication for body-fixed rotations and postmultiplication for space-fixed rotations. In a situation in which the overall rotation involves both types of rotations, we may follow Eqs. (3.1.25) and (3.1.30) by premultiplying for the body-fixed rotations and postmultiplying for the space-fixed rotations. For example, a sequence described by $[R_1]$ and $[R_2]$ about body-fixed axes, followed by $[R_3]$ about a space-fixed axis, then $[R_4]$ about a body-fixed axis would lead to $[R] = [R_4][R_2][R_1][R_3]$. A fundamental property of vectors is that their sum is independent of the order of addition. Because the final rotation transformation depends on the sequence in which the rotations occur, *spatial rotations cannot be represented as vectors*.

EXAMPLE 3.3 An xyz coordinate system, which initially coincided with a stationary XYZ coordinate system, first undergoes a rotation $\theta_1 = 65^\circ$ about the Y axis, followed by $\theta_2 = -145^\circ$ about the Z axis. Determine (a) the coordinates relative to xyz in its final orientation of a stationary point at $X = 2,\ Y = -4,\ Z = 3$ m; (b) the coordinates relative to XYZ of the point that remains at $x = 2,\ y = -4,\ z = 3$ m throughout the motion.

SOLUTION In addition to illustrating the basic operations associated with space-fixed rotations, this example serves to emphasize the difference between points that are fixed in space and those that are stationary with respect to a moving reference frame. The transformation matrix for this pair of space-fixed rotations is $[R] = [R_1][R_2]$, where $[R_1]$ describes a simple rotation about the Y axis and $[R_2]$ is a simple rotation about the Z axis. Thus

$$[R] = \begin{bmatrix} \cos 65^\circ & 0 & -\sin 65^\circ \\ 0 & 1 & 0 \\ \sin 65^\circ & 0 & \cos 65^\circ \end{bmatrix} \begin{bmatrix} \cos(-145^\circ) & \sin(-145^\circ) & 0 \\ -\sin(-145^\circ) & \cos(-145^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.3462 & -0.2424 & -0.9063 \\ 0.5736 & -0.8192 & 0 \\ -0.7424 & -0.5198 & 0.4226 \end{bmatrix}.$$

For the first question, we know the position coordinates with respect to XYZ, and $[R]$ transforms from $\bar{I}\bar{J}\bar{K}$ components to $\bar{i}\bar{j}\bar{k}$ components, so we apply the direct transformation:

$$\begin{Bmatrix} x_a \\ y_a \\ z_a \end{Bmatrix} = [R] \begin{Bmatrix} 2 \\ -3 \\ 4 \end{Bmatrix} = \begin{Bmatrix} -2.442 \\ 4.424 \\ 1.862 \end{Bmatrix} \text{ m.} \qquad \triangleleft$$

In the second situation the coordinates of the point with respect to the xyz coordinate system are invariant, and we need to determine the XYZ coordinates. This is the inverse of the transformation described by $[R]$. In accord with the orthonormal property, we use $[R]^{\mathrm{T}}$ to find

$$\begin{Bmatrix} X_b \\ Y_b \\ Z_b \end{Bmatrix} = [R]^{\mathrm{T}} \begin{Bmatrix} 2 \\ -3 \\ 4 \end{Bmatrix} = \begin{Bmatrix} -5.214 \\ 1.232 \\ -0.545 \end{Bmatrix} \text{ m.} \qquad \triangleleft$$

Rotation About an Arbitrary Axis

We have seen that a sequence of simple rotations about various coordinate axes leads to a general rotation transformation. The question now is this: What should one do if

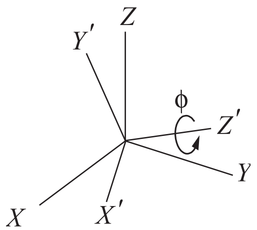


Figure 3.7. Rotation by angle ϕ about line Z' that does not coincide with one of the original coordinate axes.

there is a rotation about an axis that is not one of the coordinate axes? To address this question we follow the approach we employed to analyze space-fixed rotations. Such a situation appears in Fig. 3.7, where we have defined two fixed coordinate systems: XYZ, which is the one of interest, and $X'Y'Z'$, which is defined to have its Z' axis align with the rotation axis, but otherwise is arbitrary. The direction cosines $\ell_{Z'X}$, $\ell_{Z'Y}$, and $\ell_{Z'Z}$ define the orientation of the Z' axis.

The angle of rotation is ϕ. We denote as $[R']$ the transformation from XYZ components to $X'Y'Z'$ components. We seek to determine the transformation $[R]$ from $\bar{I}\bar{J}\bar{K}$ components to $\bar{i}\bar{j}\bar{k}$ components, where coordinate system xyz coincided with XYZ prior to the rotation. To assist in that determination we define another coordinate system $x'y'z'$ that also undergoes the rotation, with the property that its axes coincided with $X'Y'Z'$ prior to the rotation. Because xyz and $x'y'z'$ experience the same rotation, and therefore maintain their relative orientation, $[R']$ also describes the relation between these coordinate systems. Thus we have

$$\begin{Bmatrix} A_{X'} \\ A_{Y'} \\ A_{Z'} \end{Bmatrix} = [R'] \begin{Bmatrix} A_X \\ A_Y \\ A_Z \end{Bmatrix}, \quad \begin{Bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{Bmatrix} = [R'] \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix}. \tag{3.1.31}$$

Let ϕ denote the angle of rotation about the Z' axis. This is a simple rotation from the perspective of $X'Y'Z'$ and $x'y'z'$, so

$$\begin{Bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{Bmatrix} = [R_\phi] \begin{Bmatrix} A_{X'} \\ A_{Y'} \\ A_{Z'} \end{Bmatrix}, \quad [R_\phi] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.1.32}$$

Using Eqs. (3.1.31) to eliminate the primed components in the preceding equation leads to

$$[R'] \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} = [R_\phi][R'] \begin{Bmatrix} A_X \\ A_Y \\ A_Z \end{Bmatrix}. \tag{3.1.33}$$

The transformation from XYZ to xyz is readily obtained from the orthonormal property, which leads to an interesting conceptual picture:

The transformation matrix corresponding to rotation of a coordinate system about an arbitrary axis is equivalent to a sequence of body-fixed rotations. The first rotation, *corresponding to* $[R']$, *brings one of the coordinate system's axes into coincidence with the rotation axis. This is followed by a simple rotation* $[R_\phi]$ *about the designated rotation axis, after which the inverse of the first rotation is executed, so that*

$$\boxed{\begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} = [R] \begin{Bmatrix} A_X \\ A_Y \\ A_Z \end{Bmatrix}, \quad [R] = [R']^{\mathrm{T}} [R_\phi] [R'].} \tag{3.1.34}$$

The converse of the preceding is *Euler's theorem*, which states that any rotation is equivalent to a single rotation about an axis. Let us consider the task of finding the angle and axis of such a rotation given $[R]$. In the analysis leading to Eq. (3.1.34) we saw that the Z' axis has a constant orientation relative to xyz. Hence, the direction cosines of Z' with respect to xyz after the rotation are the same as its direction cosines with respect to XYZ. However, the direction cosines of any vector with respect to xyz and XYZ are related by $[R]$. Consequently, we have

$$\begin{Bmatrix} \ell_{Z'X} \\ \ell_{Z'Y} \\ \ell_{Z'Z} \end{Bmatrix} = \begin{Bmatrix} \ell_{Z'x} \\ \ell_{Z'y} \\ \ell_{Z'z} \end{Bmatrix} = [R] \begin{Bmatrix} \ell_{Z'X} \\ \ell_{Z'Y} \\ \ell_{Z'Z} \end{Bmatrix}. \tag{3.1.35}$$

Thus we obtain a set of simultaneous equations for the direction cosines of $\bar{K}'$:

$$[[R] - [U]] \{K'\} = \{0\}, \quad \{K'\} = \begin{Bmatrix} \ell_{Z'X} \\ \ell_{Z'Y} \\ \ell_{Z'Z} \end{Bmatrix}. \tag{3.1.36}$$

There must be a nontrivial solution for the direction cosines, which will be true only if $\left|[R] - [U]\right| = 0$. In other words, $[R] - [U]$ is rank deficient, which means that one or more of the elements of $\{K'\}$ is arbitrary. We can solve Eqs. (3.1.36) for two direction cosines in terms of the arbitrary one. Then all three values may be determined from the fact that $\bar{K}'$ is a unit vector, so that $\ell_{Z'X}^2 + \ell_{Z'Y}^2 + \ell_{Z'Z}^2 = 1$.

An alternative procedure for determining the direction cosines comes from solving the matrix eigenvalue problem described by

$$[[R] - \lambda [U]] \{\xi\} = \{0\}, \tag{3.1.37}$$

which gives nontrivial solutions if $\left|[R] - \lambda [U]\right| = 0$. Because $\left|[R] - [U]\right| = 0$, it follows that one of the eigenvalues must be unity. The eigenvector corresponding to $\lambda = 1$ and having a unit Euclidean norm will be $\{K'\}$. In other words

$$[[R] - \lambda_j [U]] \{\xi_j\} = \{0\} \text{ gives } \{\xi_j\} = \{K'\} \text{ if } \lambda_j = 1 \text{ and } \{\xi_j\}^{\mathrm{T}} \{\xi_j\} = 1. \tag{3.1.38}$$

Because $[R]$ is a 3×3 matrix, there are three eigenvalues. However, only one axis remains stationary in the rotation. This is manifested by the other two eigenvalues being complex, and therefore irrelevant, although their magnitude is unity. The primary virtue of Eqs. (3.1.38) as compared with Eq. (3.1.36) is that the former are readily implemented with mathematical software.

It still remains to determine the angle of rotation ϕ corresponding to $[R]$. A theorem of matrix algebra leads to an equation that ϕ must satisfy. The relation between $[R]$ and $[R_\phi]$ given by Eq. (3.1.34) is an orthogonal similarity transformation. (We will encounter this matter in greater detail in Chapter 5 when we discuss the inertia properties of rigid bodies.) An important property of such a transformation is constancy of the trace of the matrix, which is the sum of the diagonal terms. Thus $\mathrm{tr}[R] = \mathrm{tr}[R_\phi]$. In view of the definition of $[R_\phi]$, it follows that the angle of rotation must satisfy

$$1 + 2\cos\phi = \mathrm{tr}[R], \tag{3.1.39}$$

which is a relation that was derived by Euler.

The difficulty with Eq. (3.1.39) is that it does not uniquely determine ϕ because $\cos\phi = \cos(-\phi)$. This ambiguity in the sign of ϕ is associated with the fact that Eqs. (3.1.38) are satisfied by $-\{K'\}$, as well as by $+\{K'\}$. Physically, the same transformation will result from ϕ in the sense of $\bar{K}'$ and $-\phi$ in the sense of $-\bar{K}'$.

One way to determine ϕ involves first determining the transformation $[R']$ in Eq. (3.1.34). The direction cosines $\{K'\}$ are taken to be known from Eqs. (3.1.36) or (3.1.38). The orientation of the X' and Y' axes was unspecified in the development leading to Eq. (3.1.34). Let us impose the condition that the Y' axis should lie in the XY plane. This is the situation in Fig. 3.8, where ψ is the angle between the Y and Y' axes and θ is the angle between the Z and Z' axes. It is evident from the figure that all possible orientations of the Y' axis in the XY plane are covered by $-\pi < \psi \leq \pi$, whereas any Z' axis is described by $0 \leq \theta < \pi$.

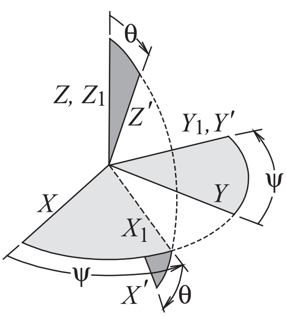


Figure 3.8. Sequence of rotations leading to a coordinate system whose Z' axis coincides with an arbitrary rotation axis.

It is possible to construct $[R']$ in terms of ψ and θ by use of Fig. 3.8 to project $\bar{I}'$, $\bar{J}$, and $\bar{K}'$ onto the XYZ axes. Alternatively, one can consider $[R']$ to be the result of a pair of body-fixed rotations: ψ about the Z' axis, followed by θ about the Y' axis. (This viewpoint will be used in Chapter 4, where ψ and θ will be identified as two of the Eulerian angles used to standardize the description of motion. The third Eulerian angle will be ϕ.) Either construction leads to

$$[R'] = \begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \cos\psi\sin\theta & \sin\psi\sin\theta & \cos\theta \end{bmatrix}. \tag{3.1.40}$$

The last row consists of the direction cosines of the Z' axis. Equating these to the corresponding elements of $\{K'\}$ leads to

$$\begin{aligned}\ell_{Z'Z} &= \cos\theta,\\ \ell_{Z'Y} &= \sin\psi\sin\theta,\\ \ell_{Z'X} &= \cos\psi\sin\theta.\end{aligned} \tag{3.1.41}$$

These relations may be solved for the values of θ and ψ. Only values $0 \le \theta \le \pi$ are meaningful, and the quadrant of ψ must be consistent with the last two relations.

Knowledge of ψ and θ allows us to fill in the missing elements of $[R']$ in Eq. (3.1.40). The corresponding $[R_\phi]$ obtained by solving Eq. (3.1.34) with the aid of the orthonormal property is

$$[R_\phi] = [R'][R][R']^{\mathrm{T}}. \tag{3.1.42}$$

Matching the result of this calculation to Eqs. (3.1.32) leads to values of $\cos\phi$ and $\sin\phi$, which together enable us to place ϕ in the proper quadrant.

EXAMPLE 3.4 Construct $[R]$ corresponding to angles $\psi = -30°$ and $\theta = 125°$ in Fig. 3.8, with $\phi = -143.13°$ being the angle of rotation about the Z' axis. Then test the procedure for ascertaining the angle and axis of rotation by extracting the values of ϕ, θ, and ψ from the resulting $[R]$.

SOLUTION This example entails application of all the concepts associated with rotation about an arbitrary axis. We begin by evaluating Eqs. (3.1.40) and (3.1.32) for the given angles,

$$[R'] = \begin{bmatrix} -0.4967 & 0.2868 & -0.8192 \\ 0.5 & 0.866 & 0 \\ 0.7094 & -0.4096 & -0.5736 \end{bmatrix},$$

$$[R_\phi] = \begin{bmatrix} -0.8 & -0.6 & 0 \\ 0.6 & -0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The overall transformation is found from Eq. (3.1.34) to be

$$[R] = \begin{bmatrix} 0.1059 & -0.1789 & -0.9782 \\ -0.8671 & -0.4980 & -0.0028 \\ -0.4867 & 0.8485 & -0.2078 \end{bmatrix}. \qquad \triangleleft$$

We now wish to work backwards to determine the angles associated with this transformation. We could determine $\{K'\}$ by solving Eqs. (3.1.38) with the aid of Matlab's *eigen* or Mathcad's *eigenvec* function. Instead we shall follow Eqs. (3.1.36).

We subtract the identity matrix from $[R]$ and recognize that only two of the rows are linearly independent because $\left|[R]-[U]\right|=0$. Using the first two rows leads to

$$\begin{bmatrix} 0.1059 & -0.1789 & -0.9782 \\ -0.8671 & -0.4980 & -0.0028 \end{bmatrix} \begin{Bmatrix} \ell_{Z'X} \\ \ell_{Z'Y} \\ \ell_{Z'Z} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

We have two equations for three unknowns, so we take $\ell_{Z'Z}$ to be arbitrary, and solve for the other direction cosines:

$$\begin{bmatrix} 0.8941 & -0.1789 \\ -0.8671 & -1.4980 \end{bmatrix} \begin{Bmatrix} \ell_{Z'X} \\ \ell_{Z'Y} \end{Bmatrix} = \begin{Bmatrix} 0.9782\ell_{Z'Z} \\ 0.0028\ell_{Z'Z} \end{Bmatrix},$$

$$\ell_{Z'X} = -1.2368\ell_{Z'Z}, \quad \ell_{Z'Y} = 0.7141\ell_{Z'Z}.$$

We find $\ell_{Z'Z}$ from the fact that $\bar{K}'$ is a unit vector, so that

$$\ell^2_{Z'X} + \ell^2_{Z'Y} + \ell^2_{Z'Z} = \left(1.2368^2 + 0.7141^2 + 1\right)\ell^2_{Z'Z} = 1.$$

The sign of $\ell_{Z'Z}$ is not defined by this relation. Because we require that $0 \le \theta < \pi$, setting $\ell_{Z'Z} = \cos\theta$ would lead to two alternative values for θ. Let us use the positive root, so that θ will be acute, specifically,

$$\ell_{Z'Z} = \frac{1}{(1.2368^2 + 0.7141^2 + 1)^{1/2}} = 0.5736 = \cos\theta,$$

$$\theta = 55^\circ. \qquad \triangleleft$$

The second and third of Eqs. (3.1.41) then lead to

$$\ell_{Z'Y} = \sin\psi\sin\theta = 0.4096, \quad \ell_{Z'X} = \cos\psi\sin\theta = -0.7094.$$

These relations place ψ in the second quadrant:

$$\psi = 150^\circ. \qquad \triangleleft$$

Equation (3.1.40) indicates that the transformation $[R']$ corresponding to the calculated ψ and θ is

$$[R'] = \begin{bmatrix} -0.4967 & 0.2868 & -0.8192 \\ -0.5 & -0.866 & 0 \\ -0.7094 & 0.4096 & 0.5736 \end{bmatrix}.$$

We use Eq. (3.1.42) to find the value of $[R_\phi]$ corresponding to the preceding, and match the result to the standard expression in Eqs. (3.1.32), which leads to

$$[R_\phi] = [R'][R][R']^{\mathrm{T}} = \begin{bmatrix} -0.8 & 0.6 & 0 \\ -0.6 & -0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$= \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0, \\ 0 & 0 & 1 \end{bmatrix},$$

$$\phi = 143.13^\circ. \qquad \triangleleft$$

The values of ψ and θ are 180° greater than the given values, which means that the identified Z' axis is oriented oppositely from the rotation axis that was specified in the problem statement. Correspondingly, the value of ϕ that was found is the negative of the specified value. In other words, we have identified the opposite rotation about the opposing axis. This situation is a consequence of using a positive sign for the square root leading to $\ell_{Z'Z}$.

3.2 DISPLACEMENT

The rotation transformation relates the components of a vector with respect to two coordinate systems having different orientation. For many vectors, such as force or velocity, only their direction angles relative to the reference directions are significant. This observation enabled us to develop the rotation transformation by depicting the coordinate systems as having a common origin, with the tail of the vector situated at that origin. The situation is different for position vectors. Figure 3.1 described the position of an arbitrary point P as it is observed from two different reference frames. The position with respect to the fixed reference frame is $\bar{r}_{P/O}$, whose components with respect to the stationary XYZ coordinate system can be designated as X_P, Y_P, and Z_P. The position with respect to a moving reference frame is $\bar{r}_{P/O'}$. Because the moving xyz coordinate system provides the viewpoint for this vector, we use the associated coordinates x_P, y_P, and z_P to describe $\bar{r}_{P/O'}$.

To implement Eq. (3.1.1) by adding like components, we need to describe all vectors in terms of components relative to the same coordinate directions. We may use a rotation transformation for this task by observing in Fig. 3.1 that $x'y'z'$ is parallel to XYZ. If the former coordinate system is to be useful, it must be that we know where its origin is, which means that we know $X_{O'}$, $Y_{O'}$, and $Z_{O'}$. To represent the components of $\bar{r}_{P/O'}$ with respect to XYZ, we use the rotation transformation $[R]$ between $x'y'z'$ and xyz.

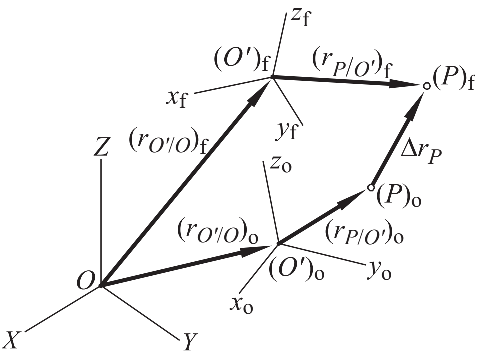


Figure 3.9. Displacement of a point as viewed from a moving reference frame. Points $(P)_o$ and $(P)_f$ are the original and final positions, respectively.

This lead to the *general positional transformation*, which is described by

$$\boxed{[X_P \;\; Y_P \;\; Z_P]^T = [X_{O'} \;\; Y_{O'} \;\; Z_{O'}]^T + [R]^T [x_P \;\; y_P \;\; z_P]^T .} \tag{3.2.1}$$

Proper application of this relation requires that we be cognizant of the fact that $[R]$ is defined to give components relative to xyz, given $x'y'z'$ components, and that $x'y'z'$ is parallel to XYZ. Obviously, when xyz coincides with $x'y'z'$, so that $[R]$ is the identity matrix, the preceding reduces to the translational transformation, Eq. (3.1.3).

A key aspect of the description of motion is evaluation of the *displacement* of a point, which is the position shift from an initial reference location to the current location,

$$\boxed{\Delta\bar{r}_P = (\bar{r}_{P/O})_f - (\bar{r}_{P/O})_o ,} \tag{3.2.2}$$

where the subscripts "o" and "f" are shorthand for original and final. Note that the origin O does not need to be indicated in the notation for $\Delta\bar{r}_P$ because the displacement vector merely extends from the initial to the final position, for which the location of the origin O is irrelevant.

The simplest evaluation of displacement is encountered when both positions are known in terms of their coordinates relative to a stationary XYZ reference frame. In that case, the displacement components are merely the difference of like coordinates. In matrix form, we have

$$\begin{Bmatrix} \Delta r_{PX} \\ \Delta r_{PY} \\ \Delta r_{PZ} \end{Bmatrix} \equiv \begin{Bmatrix} \Delta X_P \\ \Delta Y_P \\ \Delta Z_P \end{Bmatrix} = \begin{Bmatrix} (X_P)_f \\ (Y_P)_f \\ (Z_P)_f \end{Bmatrix} - \begin{Bmatrix} (X_P)_o \\ (Y_P)_o \\ (Z_P)_o \end{Bmatrix} . \tag{3.2.3}$$

The more usual case involves observation of some aspects of motion from a moving reference frame. This would be the case if we were in an airplane observing the displacement of another airplane. Consider the situation in Fig. 3.9, in which the position of point P is described by a position vector from the origin of a coordinate system xyz that executes a known motion. In this viewpoint, the original and final positions are each described by Eq. (3.2.1), with subscripts "o" and "f" used to designate the original and final states. Taking the difference of these expressions yields the displacement:

$$\begin{Bmatrix} \Delta r_{PX} \\ \Delta r_{PY} \\ \Delta r_{PZ} \end{Bmatrix} = \begin{Bmatrix} \Delta r_{O'X} \\ \Delta r_{O'Y} \\ \Delta r_{O'Z} \end{Bmatrix} + [R]_{\mathrm{f}}^{\mathrm{T}} \begin{Bmatrix} (x_P)_{\mathrm{f}} \\ (y_P)_{\mathrm{f}} \\ (z_P)_{\mathrm{f}} \end{Bmatrix} - [R]_{\mathrm{o}}^{\mathrm{T}} \begin{Bmatrix} (x_P)_{\mathrm{o}} \\ (y_P)_{\mathrm{o}} \\ (z_P)_{\mathrm{o}} \end{Bmatrix}. \tag{3.2.4}$$

The *relative displacement* $(\Delta\bar{r}_P)_{xyz}$ is the displacement that an observer moving in unison with xyz would see. Such an observer would consider the orientation of xyz to be invariant, so only the change in the values of (x_P, y_P, z_P) would be seen. Thus,

$$\boxed{(\Delta\bar{r}_P)_{xyz} = [(x_P)_{\mathrm{f}} - (x_P)_0]\,\bar{i} + [(y_P)_{\mathrm{f}} - (y_P)_0]\,\bar{j} + [(z_P)_{\mathrm{f}} - (z_P)_0]\,\bar{k}.} \tag{3.2.5}$$

To emphasize the difference between $(\Delta\bar{r}_P)_{xyz}$ and $\Delta\bar{r}_P$, the latter is sometimes called an *absolute displacement*. [Consistency of the notation suggests that the absolute displacement should be denoted as $(\Delta\bar{r}_P)_{XYZ}$, but we shall not do so for brevity. Instead, the absence of a subscript denoting a reference frame should be understood to mean that a displacement, and later, a velocity or acceleration, is relative to the stationary reference frame.] The matrix representation of Eq. (3.2.5) is

$$\begin{Bmatrix} (x_P)_{\mathrm{f}} \\ (y_P)_{\mathrm{f}} \\ (z_P)_{\mathrm{f}} \end{Bmatrix} = \begin{Bmatrix} (x_P)_{\mathrm{o}} \\ (y_P)_{\mathrm{o}} \\ (z_P)_{\mathrm{o}} \end{Bmatrix} + \begin{Bmatrix} (\Delta\bar{r}_P)_{xyz}\cdot\bar{i} \\ (\Delta\bar{r}_P)_{xyz}\cdot\bar{j} \\ (\Delta\bar{r}_p)_{xyz}\cdot\bar{k} \end{Bmatrix}. \tag{3.2.6}$$

Note that a dot product is used to denote which set of displacement components are under consideration in order to avoid later confusion. Substitution of the preceding equation into Eq. (3.2.4) leads to a useful expression for the $\bar{I}\bar{J}\bar{K}$ displacement components:

$$\boxed{\begin{aligned} \begin{Bmatrix} \Delta\bar{r}_P\cdot\bar{I} \\ \Delta\bar{r}_P\cdot\bar{J} \\ \Delta\bar{r}_P\cdot\bar{K} \end{Bmatrix} &= \begin{Bmatrix} \Delta\bar{r}_{O'}\cdot\bar{I} \\ \Delta\bar{r}_{O'}\cdot\bar{J} \\ \Delta\bar{r}_{O'}\cdot\bar{K} \end{Bmatrix} + [R]_{\mathrm{f}}^{\mathrm{T}} \begin{Bmatrix} (\Delta\bar{r}_P)_{xyz}\cdot\bar{i} \\ (\Delta\bar{r}_P)_{xyz}\cdot\bar{j} \\ (\Delta\bar{r}_p)_{xyz}\cdot\bar{k} \end{Bmatrix} \\ &\quad + \left[[R]_{\mathrm{f}}^{\mathrm{T}} - [R]_{\mathrm{o}}^{\mathrm{T}}\right] \begin{Bmatrix} (x_P)_{\mathrm{o}} \\ (y_P)_{\mathrm{o}} \\ (z_P)_{\mathrm{o}} \end{Bmatrix}. \end{aligned}} \tag{3.2.7}$$

Equation (3.2.7) highlights that when motion is described in terms of a moving reference frame, three effects combine to give the displacement relative to a stationary reference frame. If xyz were to translate, so that $[R]_\mathrm{f} \equiv [R]_\mathrm{o}$, and if the coordinates of point P relative to xyz were constant, so that $(\Delta\bar{r}_P)_{xyz} \equiv \bar{0}$, then only the displacement of the origin of xyz would matter. Thus $(\Delta\bar{r}_{O'})_{XYZ}$ is called the *translational displacement.* If xyz translates, but the position of point P relative to xyz is not constant, then the relative displacement is superposed onto the translational displacement. Finally, if xyz rotates, so that $[R]$ is no longer constant, then the last term, representing the *rotational displacement* expressed in terms of $\bar{I}\bar{J}\bar{K}$ components, superposes onto the other contributions. These three influences also will be encountered when we use a moving reference frame to describe velocity.

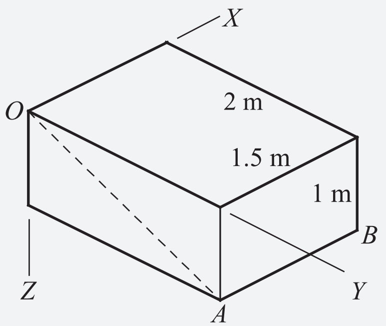
There are many situations in which it is more desirable to use the moving reference frame to describe vector components. For example, such a description corresponds to the position of another airplane from a pilot's perspective. Another general situation is one in which displacement is used to define strain in order to perform a stress analysis. Such a study is most relevant if viewed relative to favored directions defined with respect to the body. Equation (3.2.7) describes displacement in terms of $\bar{I}\bar{J}\bar{K}$ components. Multiplying that expression by $[R]_\mathrm{f}$ converts it to components relative to the final xyz orientation. We then use Eq. (3.2.6) to eliminate the original coordinates, with the result that

$$\boxed{\begin{aligned}\begin{Bmatrix}\Delta\bar{r}_P\cdot\bar{i}\\ \Delta\bar{r}_P\cdot\bar{j}\\ \Delta\bar{r}_P\cdot\bar{k}\end{Bmatrix} &= \begin{Bmatrix}\Delta\bar{r}_{O'}\cdot\bar{i}\\ \Delta\bar{r}_{O'}\cdot\bar{j}\\ \Delta\bar{r}_{O'}\cdot\bar{k}\end{Bmatrix} + [R]_\mathrm{f}\,[R]_\mathrm{o}^\mathrm{T}\begin{Bmatrix}(\Delta\bar{r}_P)_{xyz}\cdot\bar{i}\\ (\Delta\bar{r}_P)_{xyz}\cdot\bar{j}\\ (\Delta\bar{r}_p)_{xyz}\cdot\bar{k}\end{Bmatrix}\\ &\quad+\Big[[U]-[R]_\mathrm{f}\,[R]_\mathrm{o}^\mathrm{T}\Big]\begin{Bmatrix}(x_P)_\mathrm{f}\\ (y_P)_\mathrm{f}\\ (z_P)_\mathrm{f}\end{Bmatrix}.\end{aligned}} \tag{3.2.8}$$

A common source of error in the application of Eqs. (3.2.7) and (3.2.8) is confusion regarding the definition of $[R]$, which is that it converts $\bar{I}\bar{J}\bar{K}$ components to $\bar{i}\,\bar{j}\bar{k}$ components.

It is imperative to realize that this expression and Eq. (3.2.7) describe the same (absolute) displacement. Equation (3.2.7) gives displacement components relative to the axes of the fixed XYZ coordinate system in terms of the original coordinates of the point. This description, which is referred to as a *Lagrangian* description, represents the perspective of a fixed observer. In contrast, Eq. (3.2.8) is an *Eulerian* description giving displacement components relative to the axes of the moving xyz coordinate system in terms of the final coordinates. This is the perspective of an observer who moves in unison with xyz. Which is most suitable depends on the situation to be analyzed. The study of solid mechanics usually begins with an Lagrangian description, whereas fluid mechanics analyses are usually most easily carried out with an Eulerian description.

EXAMPLE 3.5 The sketch shows the initial position of a box, in which its edges coincided with fixed reference frame XYZ. The box is rotated 30° about axis OA, counterclockwise when viewed from O to A. Determine the displacement of point B in terms of components relative to xyz, and relative to XYZ.



Example 3.5

SOLUTION This example consolidates many of the developments for rotation transformations. To follow corner B as it moves, we attach the xyz coordinate system to the box, such that the axes coincide with XYZ prior to rotation. Thus the initial transformation is $[R]_o = [U]$, and the initial coordinates of point B with respect to xyz are

$$\begin{Bmatrix} (x_B)_o \\ (y_B)_o \\ (z_B)_o \end{Bmatrix} = \begin{Bmatrix} 1.5 \\ 2 \\ 1 \end{Bmatrix}.$$

We need to establish $[R]_f$, which transforms XYZ to xyz components in the final position. The rotation takes place about an axis that does not coincide with either set of axes, so we define an $X'Y'Z'$ coordinate system whose Z' axis coincides with the diagonal OA. This system may be obtained by a simple rotation about the negative X axis by $\theta_X = \tan^{-1}(2/1)$. The corresponding transformation from $\bar{I}\bar{J}\bar{K}$ to $\bar{I}'\bar{J}'\bar{K}'$ components is

$$[R'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_X & -\sin\theta_X \\ 0 & \sin\theta_X & \cos\theta_X \end{bmatrix}.$$

The specified rotation about the Z' axis is $\phi = -30°$, so

$$[R_\phi] = \begin{bmatrix} \cos 30° & -\sin 30° & 0 \\ \sin 30° & \cos 30° & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From Eq. (3.1.34) we find that

$$[R]_{\mathrm{f}} = [R']^{\mathrm{T}}[R_\phi][R'] = \begin{bmatrix} 0.8660 & -0.2236 & 0.4472 \\ 0.2236 & 0.9732 & 0.0536 \\ -0.4472 & 0.0536 & 0.8928 \end{bmatrix}.$$

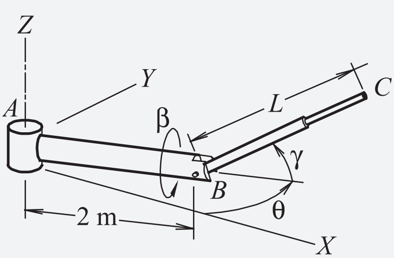
The origin O' was placed at a fixed point, so there is no translational displacement, $\Delta\bar{r}_{O'} = \bar{0}$. Point B is a part of the box, so it does not move relative to xyz. Thus its coordinates relative to xyz are constant and its relative displacement is $(\Delta\bar{r}_B)_{xyz} = \bar{0}$. We may directly employ either Eq. (3.2.7) or Eq. (3.2.8). Setting $[R]_{\mathrm{o}} = [U]$ in the latter shows that the displacement of point B in terms of xyz components is

$$\begin{Bmatrix} \Delta\bar{r}_B \cdot \bar{i} \\ \Delta\bar{r}_B \cdot \bar{j} \\ \Delta\bar{r}_B \cdot \bar{k} \end{Bmatrix} = [[U] - [R]_{\mathrm{f}}] \begin{Bmatrix} (x_B)_{\mathrm{f}} \\ (y_B)_{\mathrm{f}} \\ (z_B)_{\mathrm{f}} \end{Bmatrix} = \begin{Bmatrix} 0.2010 \\ -0.3354 \\ 0.6708 \end{Bmatrix} \text{ m.} \quad \triangleleft$$

We could use Eq. (3.2.7) to find the XYZ displacement components, but here it is simpler to use the rotation transformation $[R]_{\mathrm{f}}^{\mathrm{T}}$ to transform to XYZ components:

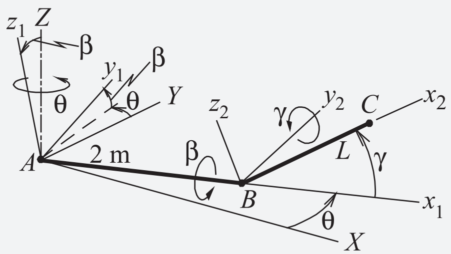
$$\begin{Bmatrix} \Delta\bar{r}_B \cdot \bar{I} \\ \Delta\bar{r}_B \cdot \bar{J} \\ \Delta\bar{r}_B \cdot \bar{K} \end{Bmatrix} = [R]_{\mathrm{f}}^{\mathrm{T}} \begin{Bmatrix} 0.2010 \\ -0.3354 \\ 0.6708 \end{Bmatrix} = \begin{Bmatrix} 0.5490 \\ -0.2455 \\ 0.4911 \end{Bmatrix} \text{ m.} \quad \triangleleft$$

EXAMPLE 3.6 The robotic linkage is reconfigured by rotating arm AB by angle θ about the vertical Z axis, and by angle β about its longitudinal axis. Link BC rotates about the pin at junction B through angle γ. The pin is horizontal when $\beta = 0$. The system is given a set of rotations consisting of $\theta = 50°$, $\beta = 30°$, $\gamma = 60°$. The length L of link BC changes from 0.5 m to 1.5 m in the course of this motion. Determine the coordinates of end C with respect to the fixed XYZ coordinate after these rotations.



Example 3.6

SOLUTION This example serves to illustrate the versatility of the general procedure for evaluating displacements, and the ease with which it may be extended. It should be noted that the sequence in which the rotations are applied is not specified. It will become apparent in the course of the solution that such information is irrelevant for this system. We define reference frames $x_1 y_1 z_1$ fixed to arm AB, and $x_2 y_2 z_2$ fixed to arm BC, with the x axis for each aligned with the corresponding arm. The z_1 axis is vertical when $\beta = 0$ and the y_2 axis is aligned with the pin at connection B, so that all rotations occur about coordinate axes.



Body-fixed coordinate systems for the robotic linkage.

The general approach is to use Eq. (3.2.7) to express the displacement of point B in terms of the rotation of $x_1 y_1 z_1$ and then to describe the displacement of point C in terms of the displacement of point B and the rotation of $x_2 y_2 z_2$. We use $\bar{I}\bar{J}\bar{K}$ components to facilitate relating vector quantities for each body. Thus,

$$\begin{Bmatrix} \Delta\bar{r}_B \cdot \bar{I} \\ \Delta\bar{r}_B \cdot \bar{J} \\ \Delta\bar{r}_B \cdot \bar{K} \end{Bmatrix} = \begin{Bmatrix} \Delta\bar{r}_A \cdot \bar{I} \\ \Delta\bar{r}_A \cdot \bar{J} \\ \Delta\bar{r}_A \cdot \bar{K} \end{Bmatrix} + [R_1]_\text{f}^\text{T} \begin{Bmatrix} (\Delta\bar{r}_B)_{x_1 y_1 z_1} \cdot \bar{i} \\ (\Delta\bar{r}_B)_{x_1 y_1 z_1} \cdot \bar{j} \\ (\Delta\bar{r}_B)_{x_1 y_1 z_1} \cdot \bar{k} \end{Bmatrix} + \left[[R_1]_\text{f}^\text{T} - [R_1]_\text{o}^\text{T} \right] \begin{Bmatrix} (x_{1B})_\text{o} \\ (y_{1B})_\text{o} \\ (z_{1B})_\text{o} \end{Bmatrix},$$

$$\begin{Bmatrix} \Delta\bar{r}_C \cdot \bar{I} \\ \Delta\bar{r}_C \cdot \bar{J} \\ \Delta\bar{r}_C \cdot \bar{K} \end{Bmatrix} = \begin{Bmatrix} \Delta\bar{r}_B \cdot \bar{I} \\ \Delta\bar{r}_B \cdot \bar{J} \\ \Delta\bar{r}_B \cdot \bar{K} \end{Bmatrix} + [R_2]_\text{f}^\text{T} \begin{Bmatrix} (\Delta\bar{r}_C)_{x_2 y_2 z_2} \cdot \bar{i} \\ (\Delta\bar{r}_C)_{x_2 y_2 z_2} \cdot \bar{j} \\ (\Delta\bar{r}_C)_{x_2 y_2 z_2} \cdot \bar{k} \end{Bmatrix} + \left[[R_2]_\text{f}^\text{T} - [R_2]_\text{o}^\text{T} \right] \begin{Bmatrix} (x_{2C})_\text{o} \\ (y_{2C})_\text{o} \\ (z_{2C})_\text{o} \end{Bmatrix}.$$

We derive the required rotation transformations as a sequence of simple rotations. In the initial state, like axes were parallel, so

$$[R_1]_\text{o} = [R_2]_\text{o} = [U].$$

For the transformation from XYZ to the final orientation of $x_1y_1z_1$ we may consider arm AB to rotate first about the Z axis by θ, followed by the β rotation about the x_1 axis, which corresponds to a sequence of body-fixed rotations. Alternatively, we may consider the β rotation to occur about the X axis, followed by the θ rotation about the Z axis, which is a space-fixed sequence. Either way, we find that

$$[R_1]_\mathrm{f} = [R_\beta][R_\theta] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

When $\gamma = 0$, like axes of $x_1y_1z_1$ and $x_2y_2z_2$ are parallel. Thus the final orientation of $x_2y_2z_2$ may be considered to be attained by rotating it about the negative y_2 axis by γ relative to $x_1y_1z_1$. This is a body-fixed rotation, so that

$$[R_2]_\mathrm{f} = [R_\gamma][R_1]_\mathrm{f} = \begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} [R_1]_\mathrm{f}.$$

Next we describe the relative positions and displacements. Because point A is stationary, $\Delta\bar{r}_A = \bar{0}$. Also, because point B remains at 2 m along the x_1 axis throughout the motion, $(\bar{r}_{B/A})_\mathrm{o} = 2\bar{i}_1$ and $(\Delta\bar{r}_B)_{x_1y_1z_1} = \bar{0}$. In contrast, point C remains on the x_2 axis, but its distance to the origin of $x_2y_2z_2$ increases from 0.5 m to 1.5 m, so it displaces 1 m in the x_2 direction relative to $x_2y_2z_2$. Thus $(\bar{r}_{C/B})_\mathrm{o} = 0.5\bar{i}_2$ and $(\Delta\bar{r}_C)_{x_2y_2z_2} = 1\bar{i}_2$. Substituting the given values of θ, β, and γ into the earlier expressions for $\Delta\bar{r}_B$ and $\Delta\bar{r}_C$ leads to

$$\begin{Bmatrix} \Delta\bar{r}_B\cdot\bar{I} \\ \Delta\bar{r}_B\cdot\bar{J} \\ \Delta\bar{r}_B\cdot\bar{K} \end{Bmatrix} = \begin{Bmatrix} -0.7144 \\ 1.5321 \\ 0 \end{Bmatrix} \text{ m},$$

$$\begin{Bmatrix} \Delta\bar{r}_C\cdot\bar{I} \\ \Delta\bar{r}_C\cdot\bar{J} \\ \Delta\bar{r}_C\cdot\bar{K} \end{Bmatrix} = \begin{Bmatrix} -0.2348 \\ 1.6891 \\ 1.1250 \end{Bmatrix} \text{ m}. \qquad \triangleleft$$

3.3 TIME DERIVATIVES

When we observe a movement over a reasonably long time interval, the change in position coordinates is measurable and the rotations about various axes are finite. Consequently, there is substantial change in a system's geometrical configuration. In contrast, the definition of velocity is that it is the ratio of the infinitesimal displacement to the infinitesimal time over which the displacement occurs. The change from finite to infinitesimal quantities actually simplifies many aspects of a kinematical analysis, essentially because the changing geometry will not be an issue.

We consider initial and final positions that are infinitesimally different and use Eq. (3.2.8) to describe the associated displacement in terms of components relative to the final orientation of xyz. The absolute displacements of the origin O' and of the observed point P are differential quantities, $d\bar{r}_{O'}$ and $d\bar{r}_P$, and the relative displacement is $(d\bar{r}_P)_{xyz}$. The orientation of xyz relative to XYZ in the initial position is described by $[R]_\mathrm{o}$, which is considered to be known.

To obtain $[R]_\mathrm{f}$ we recall that any transformation matrix has only three independent direction cosines and corresponding angles. This allows us to consider xyz to move from its initial to final orientation by executing infinitesimal rotations about each of its axes, $d\theta_x$, $d\theta_y$, $d\theta_z$, in the listed sequence. Because this is a sequence of body-fixed rotations, we have

$$[R]_\mathrm{f} = [R_z]\,[R_y]\,[R_x]\,[R]_\mathrm{o}\,. \tag{3.3.1}$$

For a differential angle $d\theta$, we have $\cos d\theta = 1$ and $\sin d\theta = d\theta$, so the simple rotation transformations are

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & d\theta_x \\ 0 & -d\theta_x & 1 \end{bmatrix},\quad [R_y] = \begin{bmatrix} 1 & 0 & -d\theta_y \\ 0 & 1 & 0 \\ d\theta_y & 0 & 1 \end{bmatrix},\quad [R_x] = \begin{bmatrix} 1 & d\theta_z & 0 \\ -d\theta_z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.3.2}$$

Evaluating the products in Eq. (3.3.1) yields

$$[R]_\mathrm{f} = \begin{bmatrix} 1 & d\theta_z + d\theta_y d\theta_x & -d\theta_y + d\theta_z d\theta_x \\ -d\theta_z & 1 - d\theta_z d\theta_y d\theta_x & d\theta_x + d\theta_z d\theta_y \\ d\theta_y & -d\theta_x & 1 \end{bmatrix} [R]_\mathrm{o}\,. \tag{3.3.3}$$

The second-order differential quantities in the preceding equation are unimportant to the evaluation of a derivative. Thus substituting Eq. (3.3.3) into Eq. (3.2.8) for the case of differential displacements gives

$$\begin{aligned} \begin{Bmatrix} d\bar{r}_P \cdot \bar{i} \\ d\bar{r}_P \cdot \bar{j} \\ d\bar{r}_P \cdot \bar{k} \end{Bmatrix} &= \begin{Bmatrix} d\bar{r}_{O'} \cdot \bar{i} \\ d\bar{r}_{O'} \cdot \bar{j} \\ d\bar{r}_{O'} \cdot \bar{k} \end{Bmatrix} + \begin{bmatrix} 1 & d\theta_z & -d\theta_y \\ -d\theta_z & 1 & d\theta_x \\ d\theta_y & -d\theta_x & 1 \end{bmatrix} \begin{Bmatrix} (d\bar{r}_P)_{xyz} \cdot \bar{i} \\ (d\bar{r}_P)_{xyz} \cdot \bar{j} \\ (d\bar{r}_p)_{xyz} \cdot \bar{k} \end{Bmatrix} \\ &\quad + \begin{bmatrix} 0 & -d\theta_z & d\theta_y \\ d\theta_z & 0 & -d\theta_x \\ -d\theta_y & d\theta_x & 0 \end{bmatrix} \begin{Bmatrix} (x_P)_\mathrm{f} \\ (y_P)_\mathrm{f} \\ (z_P)_\mathrm{f} \end{Bmatrix}. \end{aligned} \tag{3.3.4}$$

The second term on the right side also contains second-order differentials, which are unimportant. Another simplification is that because all displacements and rotations are infinitesimal, there is no significant difference between using the initial or final position coordinates in the last term. This allows us to denote the position of point P as $\bar{r}_{P/O'} = x_P\bar{i} + y_P\bar{j} + z_P\bar{k}$. Thus

$$\begin{Bmatrix} d\bar{r}_P \cdot \bar{i} \\ d\bar{r}_P \cdot \bar{j} \\ d\bar{r}_P \cdot \bar{k} \end{Bmatrix} = \begin{Bmatrix} d\bar{r}_{O'} \cdot \bar{i} \\ d\bar{r}_{O'} \cdot \bar{j} \\ d\bar{r}_{O'} \cdot \bar{k} \end{Bmatrix} + \begin{Bmatrix} (d\bar{r}_P)_{xyz} \cdot \bar{i} \\ (d\bar{r}_P)_{xyz} \cdot \bar{j} \\ (d\bar{r}_p)_{xyz} \cdot \bar{k} \end{Bmatrix} + \begin{bmatrix} 0 & -d\theta_z & d\theta_y \\ d\theta_z & 0 & -d\theta_x \\ -d\theta_y & d\theta_x & 0 \end{bmatrix} \begin{Bmatrix} x_P \\ y_P \\ z_P \end{Bmatrix}. \tag{3.3.5}$$

The important features of the preceding equation emerge when we recognize that it is the matrix representation of a vector equation, specifically,

$$d\bar{r}_P = d\bar{r}_{O'} + (d\bar{r}_P)_{xyz} + \overline{d\theta} \times \bar{r}_{P/O'}, \tag{3.3.6}$$

where

$$\overline{d\theta} = d\theta_x\bar{i} + d\theta_y\bar{j} + d\theta_z\bar{k} \tag{3.3.7}$$

is the *infinitesimal rotation* vector. It is evident from Eq. (3.3.6) that the only significant feature of an infinitesimal rotation is its vector sum, defined by Eq. (3.3.8). This would seem to conflict with the earlier observation that rotations cannot be represented as vectors, because the rotation transformation depends on the sequence in which rotations occur. To understand the difference between finite and infinitesimal rotations, consider altering the sequence in which the infinitesimal rotations were applied in Eq. (3.3.1), for example by performing $d\theta_z$, followed by $d\theta y$, then $d\theta_x$. Regardless of the sequence, only the second-order differentials in Eq. (3.3.3) would be different, and such terms are dropped. Furthermore, the transformations for space-fixed and body-fixed axes differ only by the sequence in which the individual rotation transformations are multiplied. Consequently, the same transformation is obtained if a set of infinitesimal rotations are imparted about body fixed or space fixed axes. Geometrically, the fact that an infinitesimal displacement depends on neither the type of rotation, body fixed or space fixed, nor the sequence of the rotations has a simple explanation: When rotations are infinitesimal, the difference between the initial and final orientation of any rotation axis is negligible.

A corollary of the fact that infinitesimal rotation is a vector is that the additive property applies, even if the individual rotations are not about coordinate axes. Specifically, when there are several rotations $d\theta_n$ about directions $\bar{e}_n$ that are parallel to the respective rotation axes in the sense of the right-hand rule (curl the fingers of the right hand about the axis in the sense of the rotation; the extended thumb gives the sense of the vector), then

$$\overline{d\theta} = \sum_n d\theta_n \bar{e}_n. \tag{3.3.8}$$

It should be noted that the overbar is placed above the entire symbol $\overline{d\theta}$ denoting the infinitesimal rotation in order to emphasize that there is no finite rotation vector from which the differential is formed. A principal advantage of Eq. (3.3.6) over Eq. (3.3.5) is that the vector form is independent of how the vectors are described—one can use whichever set of components is most suitable to the task of describing each quantity, then transform each set of components as necessary to sum the terms.

The underlying reason for studying differential displacement is that velocity is $\bar{v} = d\bar{r}/dt$. Thus dividing Eq. (3.3.6) by the time interval dt over which the displacement occurs gives us an expression for velocity of point P:

$$\boxed{\bar{v}_P = \bar{v}_{O'} + (\bar{v}_P)_{xyz} + \bar{\omega} \times \bar{r}_{P/O'},} \tag{3.3.9}$$

where $\bar{\omega}$ is the *angular velocity* of xyz, which is defined as

$$\bar{\omega} \equiv \frac{\overline{d\theta}}{dt}. \tag{3.3.10}$$

To explain $(\bar{v}_P)_{xyz}$ we recall the definition of relative displacement, Eq. (3.2.5), according to which the components of $(d\bar{r}_P)_{xyz}$ are the infinitesimal increments of the xyz coordinates of point P. Thus the components of $(\bar{v}_P)_{xyz}$ are the rates of change of those coordinates, that is,

$$\boxed{(\bar{v}_P)_{xyz} = \dot{x}_P \bar{i} + \dot{y}_P \bar{j} + \dot{z}_P \bar{k}.} \tag{3.3.11}$$

Because an observer on xyz would not see the coordinate axes' orientation change, we say that $(\bar{v}_P)_{xyz}$ is the *velocity relative to xyz*, or more simply, the *relative velocity*.

We will develop a consistent methodology for analyzing velocity and acceleration with the aid of moving reference frames, but one basic relation still remains to be derived. An alternative description of $\bar{v}_P$ in terms of the velocity of the origin comes about when we differentiate the description of position given by Eq. (3.1.1), which gives

$$\bar{v}_P = \bar{v}_{O'} + \frac{d}{dt}\left(\bar{r}_{P/O'}\right). \tag{3.3.12}$$

A comparison of this and Eq. (3.3.9) shows that

$$\frac{d}{dt}\left(\bar{r}_{P/O'}\right) = (\bar{v}_P)_{xyz} + \bar{\omega} \times \bar{r}_{P/O'}. \tag{3.3.13}$$

We arrive at an important generalization when we recall that Eq. (3.3.11) defines the $\bar{i}\,\bar{j}\bar{k}$ components of $(\bar{v}_P)_{xyz}$ to be the rates of change of the xyz coordinates of point P, which are, in turn, the $\bar{i}\,\bar{j}\bar{k}$ components of $\bar{r}_{P/O'}$. Thus $(\bar{v}_P)_{xyz}$ is like a partial time derivative of $\bar{r}_{P/O'}$ in which account is taken of the variability of the components of $\bar{r}_{P/O'}$, but the orientation of the $\bar{i}$, $\bar{j}$, $\bar{k}$ unit vectors is held constant. We shall generally use a partial derivative to denote such an operation. Specifically, if $\bar{A}$ is any vector, then

$$\boxed{\bar{A} = A_x\bar{i} + A_y\bar{j} + A_z\bar{k} \implies \frac{\partial \bar{A}}{\partial t} \equiv \dot{A}_x\bar{i} + \dot{A}_y\bar{j} + \dot{A}_z\bar{k},} \tag{3.3.14}$$

in which viewpoint, $(\bar{v}_P)_{xyz} = \partial\left(\bar{r}_{P/O'}\right)/\partial t$. We refer to the result of differentiating the components of a vector, without regard to the changing orientation of these components, as a *relative derivative*. (Many other texts, including earlier versions of this one, denoted this operation as $\delta/\delta t$. The δ symbol is reserved here for operations that will be encountered in Chapter 7 on analytical mechanics.) The analogy between $\bar{r}_{P/O'}$ and any other vector allows us to conclude from Eq. (3.3.13) that

The time rate of change of any vector $\bar{A}$ described in terms of components relative to reference frame xyz having angular velocity $\bar{\omega}$ is

$$\boxed{\dot{\bar{A}} = \frac{\partial \bar{A}}{\partial t} + \bar{\omega} \times \bar{A}.} \tag{3.3.15}$$

A less general case is that in which $\bar{A}$ is one of the unit vectors of xyz, which we denote by the generic symbol $\bar{e}$. Clearly, $\partial\bar{e}/\partial t = \bar{0}$, so we find that

$$\boxed{\bar{e} = \bar{i},\ \bar{j}, \text{ or } \bar{k} \implies \dot{\bar{e}} = \bar{\omega} \times \bar{e}.} \tag{3.3.16}$$

We will invoke this relation frequently as part of the methodology for analyzing angular acceleration. Figure 3.10(a) shows a typical unit vector $\bar{e}$ before and after an infinitesimal rotation. The axis of rotation is parallel to $\bar{\omega}$. The amount by which $\bar{e}$ changes is the difference $d\bar{e}$ between the final vector $(\bar{e})_{\mathrm{f}}$ and original vector $(\bar{e})_{\mathrm{o}}$. This difference is depicted in Fig. 3.10(b), where the tails of $(\bar{e})_{\mathrm{f}}$ and $(\bar{e})_{\mathrm{o}}$ have been brought to the axis represented by $\overline{d\theta}$. The sketch shows that only the portion of $\bar{e}$ that is perpendicular to the rotation axis changes; call this component $\bar{e}_\perp$. The line in Fig. 3.10(b) representing $\bar{e}_\perp$ rotates through the angle $\left|\overline{d\theta}\right|$. Hence the arc that represents $d\bar{e}$ has a length $|\bar{e}_\perp|\, d\theta$, and the direction of $d\bar{e}$ is perpendicular to both $\bar{e}$ and $\overline{d\theta}$. The magnitude of a cross product is defined to be the product of the magnitude of one vector and the perpendicular component of the other vector, and the direction of the product is perpendicular to the individual vectors in the sense of the right-hand rule. It follows that the pictorial analysis fully agrees with Eq. (3.3.16). The fact that the change of $\bar{e}$ is perpendicular to $\bar{e}$ is another manifestation of the general property derived in the study of tangent and normal components, as well as curvilinear coordinates.

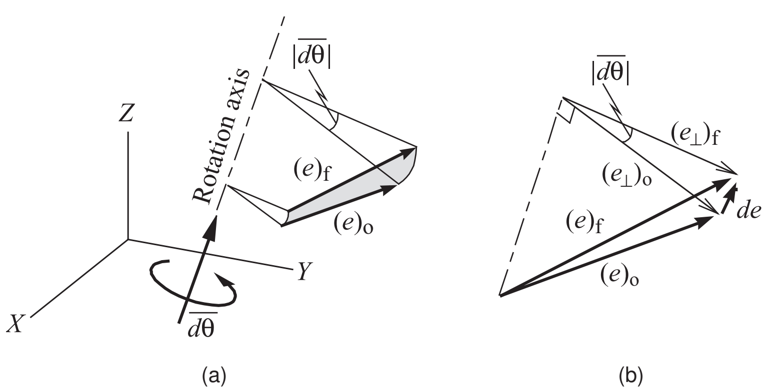


Figure 3.10. The change of a unit vector resulting from an infinitesimal rotation: (a) movement of the unit vector, (b) construction of the difference between the new and original unit vectors.

EXAMPLE 3.7 Consider the linkage in Example 3.6 when the final position specified there is the starting configuration, so that $L_o = 1.5$ m, $\theta_o = 50°$, $\beta_o = 30°$, $\gamma_o = 60°$. From that position the following increments are applied: $\Delta L = 12$ mm, $\Delta\theta = 0.5°$, $\Delta\beta = -0.5°$, $\Delta\gamma = -1°$. Determine the associated displacement using the exact matrix transformation technique, and compare it with the approximate value obtained by considering the rotation to be infinitesimal.

SOLUTION An objective of this example is to demonstrate the simplifications encountered in the transition from finite displacement to velocity. To apply the developments in the previous example here, we recognize that the final configuration obtained there is now the initial, so we have

$$[R_1]_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta_o & \sin\beta_o \\ 0 & -\sin\beta_o & \cos\beta_o \end{bmatrix} \begin{bmatrix} \cos\theta_o & \sin\theta_o & 0 \\ -\sin\theta_o & \cos\theta_o & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$[R_2]_o = \begin{bmatrix} \cos\gamma_o & 0 & \sin\gamma_o \\ 0 & 1 & 0 \\ -\sin\gamma_o & 0 & \cos\gamma_o \end{bmatrix} [R_1]_o.$$

The transformations $[R_1]_f$ and $[R_2]_f$ for the final location are obtained by replacing the angles in the preceding by $\theta_o + \Delta\theta$, $\beta_o + \Delta\beta$, and $\gamma_o + \Delta\gamma$. Relative to $x_2y_2z_2$, which is attached to arm BC, the position and displacement are $(\bar{r}_{C/B})_o = 1.5\bar{i}_2$ and $(\Delta\bar{r}_C)_{x_2y_2z_2} = (\Delta L)\bar{i}_2$. It still is true that $\bar{r}_{B/A} = 2\bar{i}$ and $\Delta\bar{r}_A = (\Delta\bar{r}_B)_{x_1y_1z_1} = \bar{0}$. Substitution of these terms into Eq. (3.2.7) to evaluate $\Delta\bar{r}_B$, and then $\Delta\bar{r}_C$, leads to

$$\begin{Bmatrix} \Delta\bar{r}_B \cdot \bar{I} \\ \Delta\bar{r}_B \cdot \bar{J} \\ \Delta\bar{r}_B \cdot \bar{K} \end{Bmatrix} = \begin{Bmatrix} 13.42 \\ 11.16 \\ 0 \end{Bmatrix} \text{ mm},$$

$$\begin{Bmatrix} \Delta\bar{r}_C \cdot \bar{I} \\ \Delta\bar{r}_C \cdot \bar{J} \\ \Delta\bar{r}_C \cdot \bar{K} \end{Bmatrix} = \begin{Bmatrix} -5.28 \\ 49.08 \\ 3.01 \end{Bmatrix} \text{ mm}. \quad \triangleleft$$

To perform the analysis based on the infinitesimal approximation we use Eq. (3.3.6) to relate the displacements of points A, B, and C. The rotation angles from the initial to final positions are small, but finite, so these relations are approximate. We let $\overline{\Delta\theta}_1$ and $\overline{\Delta\theta}_2$ denote small rotation vectors for the respective reference frames. Because $\Delta\bar{r}_A = (\Delta\bar{r}_B)_{x_1y_1z_1} = \bar{0}$, we have

$$\Delta\bar{r}_B \approx \overline{\Delta\theta}_1 \times \bar{r}_{B/A}, \quad \Delta\bar{r}_C \approx \Delta\bar{r}_B + (\Delta\bar{r}_C)_{x_2y_2z_2} + \overline{\Delta\theta}_2 \times \bar{r}_{C/B}.$$

Each small rotation vector is the vector sum of the scalar angles in the directions of the respective axes. The rotation of $x_1y_1z_1$ consists of $\Delta\theta$ about the Z axis, followed by $\Delta\beta$ about the x_1 axis. (Because β is not zero, the z_1 axis is not vertical, so we cannot say that $\Delta\theta$ is about the z_1 axis.) The vector sum of these rotations is

$$\overline{\Delta\theta}_1 = \Delta\theta\bar{K} + \Delta\beta\bar{i}_1.$$

Because $x_2y_2z_2$ rotates by β about the negative y_2 axis relative to $x_1y_1z_1$, its rotation vector is the sum of the rotation of $x_1y_1z_1$ and $\Delta\gamma$ in the direction of $-\bar{j}_2$:

$$\overline{\Delta\theta}_2 = \Delta\theta\bar{K} + \Delta\beta\bar{i}_1 - \Delta\gamma\,\bar{j}_2.$$

Note that the rotation angles must be expressed in units of radians.

To form the sums we need to express all quantities in terms of components relative to the same set of coordinate axes. We could use $\bar{I}\bar{J}\bar{K}$ components for this with the aid of rotation transformations. However, if we use $\bar{i}_1\bar{j}_1\bar{k}_1$ components we can find the required vectors by inspection. We refer back to the sketch of the coordinate axes in Example 3.6. Because $\bar{k}_1$ remains coincident with $\bar{K}$ in the θ rotation, $\bar{K}$ is situated in the y_1z_1 plane, such that $\bar{K} = \sin\beta_o\bar{j}_1 + \cos\beta_o\bar{k}_1$. (Note that, because β changes little, we may use the initial geometrical arrangement to describe components.) The y_1 and y_2 axes are parallel, so $\bar{j}_2 = \bar{j}_1$. Thus we find that

$$\overline{\Delta\theta}_1 = \Delta\theta\left(\sin\beta_o\bar{j}_1 + \cos\beta_o\bar{k}_1\right) + \Delta\beta\bar{i}_1,$$

$$\overline{\Delta\theta}_2 = \Delta\theta\left(\sin\beta_o\bar{j}_1 + \cos\beta_o\bar{k}_1\right) + \Delta\beta\bar{i}_1 - \Delta\gamma\,\bar{j}_1.$$

The positions and relative displacements were described earlier as $\bar{r}_{B/A} = 2\bar{i}_1$, $\bar{r}_{C/B} = 1.5\bar{i}_2$, $(\Delta\bar{r}_C)_{x_2y_2z_2} = (\Delta L)\,\bar{i}_2$. We convert the last two to $\bar{i}_1\bar{j}_1\bar{k}_1$ components by observing that the γ rotation leaves $\bar{i}_2$ in the x_1z_1 plane, so that $\bar{i}_2 = \cos\gamma_o\bar{i}_1 + \sin\gamma_o\bar{k}_1$. The result of substitution of the various terms corresponding to the given values of the angles and their increments into the earlier vector expressions for $\bar{u}_B$ and $\bar{u}_C$ is

$$\Delta\bar{r}_B = 0.01511\,\bar{j}_1 - 0.00873\bar{k}_1\ \text{m},$$

$$\Delta\bar{r}_C = 0.03434\bar{i}_1 + 0.03212\,\bar{j}_1 - 0.01470\bar{k}_1\ \text{m}.$$

The first evaluation of $\Delta\bar{r}_C$ gave the result in terms of components relative to XYZ. There is little difference between the initial and the final orientation of xyz in the present situation, so we may use either the initial or final $[R_1]$ to convert the preceding approximate displacement to those components. Using the initial transformation gives

$$\begin{Bmatrix} \Delta\bar{r}_C\cdot\bar{I} \\ \Delta\bar{r}_C\cdot\bar{J} \\ \Delta\bar{r}_C\cdot\bar{K} \end{Bmatrix} = [R_1]_o^{\mathrm{T}} \begin{Bmatrix} 0.03434 \\ 0.03212 \\ -0.01470 \end{Bmatrix} = \begin{Bmatrix} -4.86 \\ 48.91 \\ 3.33 \end{Bmatrix}\ \text{mm.} \qquad \triangleleft$$

These values are quite close to the previous; decreasing the values of $\Delta\theta$, $\Delta\beta$, $\Delta\gamma$, and ΔL would improve the agreement.

An interesting aspect of the analysis based on the infinitesimal displacement approximation is that the only place where a rotation transformation matrix was required was to convert the displacement to the components found from the exact transformation procedure. The steps we followed to carry out the approximate analysis are like those by which we will analyze velocity and acceleration.

EXAMPLE 3.8 The Frenet formulas give the derivatives of the path variable unit vectors with respect to the arc length s along an arbitrary curve. Because $\dot{s} = v$, these derivatives may be converted to time rates of change of the unit vectors. Furthermore, the orthonormal directions represented by these unit vectors form a moving reference frame. Determine the angular velocity of the $\bar{e}_t\bar{e}_n\bar{e}_b$ reference frame in terms of the path variable parameters.

SOLUTION It is useful to begin by recalling the Frenet formulas, which are

$$\frac{d\bar{e}_t}{ds} = \frac{1}{\rho}\bar{e}_n, \qquad \frac{d\bar{e}_n}{ds} = -\frac{1}{\rho}\bar{e}_t + \frac{1}{\tau}\bar{e}_b, \qquad \frac{d\bar{e}_b}{ds} = -\frac{1}{\tau}\bar{e}_n. \tag{1}$$

To convert these to time derivatives, we observe that, if $\bar{e}$ is a unit vector that depends on the arc length s locating a point, and $s = s(t)$, then the chain rule gives

$$\frac{d\bar{e}}{dt} = \frac{ds}{dt}\frac{d\bar{e}}{ds} = v\frac{d\bar{e}}{ds}.$$

Hence, multiplying each of Eqs. (1) by v gives

$$\frac{d\bar{e}_t}{dt} = \frac{v}{\rho}\bar{e}_n, \qquad \frac{d\bar{e}_n}{dt} = -\frac{v}{\rho}\bar{e}_t + \frac{v}{\tau}\bar{e}_b, \qquad \frac{d\bar{e}_b}{dt} = -\frac{v}{\tau}e_n. \tag{2}$$

Now let $\bar{\omega} = \omega_t\bar{e}_t + \omega_n\bar{e}_n + \omega_b\bar{e}_b$ be the angular velocity of $\bar{e}_t\bar{e}_n\bar{e}_b$. Each unit vector has constant components relative to the reference frame, so Eq. (3.3.16) applies. Thus,

$$\begin{aligned} \frac{d\bar{e}_t}{dt} &= \bar{\omega}\times\bar{e}_t = \omega_b\bar{e}_n - \omega_n\bar{e}_b, \\ \frac{d\bar{e}_n}{dt} &= \bar{\omega}\times\bar{e}_n = -\omega_b\bar{e}_t + \omega_t\bar{e}_b, \\ \frac{d\bar{e}_b}{dt} &= \bar{\omega}\times\bar{e}_b = \omega_b\bar{e}_t - \omega_t\bar{e}_n. \end{aligned} \tag{3}$$

Matching like components in Eqs. (2) and (3) leads to

$$\omega_b = \frac{v}{\rho}, \qquad \omega_n = 0, \qquad \omega_t = \frac{v}{\tau} \quad\Longrightarrow\quad \bar{\omega} = \frac{v}{\tau}\bar{e}_t + \frac{v}{\rho}\bar{e}_b. \qquad \triangleleft$$

We see from this result that a sharp bend in the curve (small ρ) causes a rapid rotation about the binormal direction, which is perpendicular to the osculating plane. Similarly, a sharp twist (small τ) causes a rapid rotation about the tangent direction. There is no rotation about the normal direction because the curve locally lies in the osculating plane.

3.4 ANGULAR VELOCITY AND ACCELERATION

We have seen that analysis of velocity by using a moving reference frame requires the current value of $\bar{\omega}$. A comparable analysis of acceleration requires knowledge of the *angular acceleration,* which is defined to be the rate of change of the angular velocity. The need to differentiate $\bar{\omega}$ requires that we describe this quantity in general terms, rather than merely ascertaining its instantaneous value.

3.4.1 Analytical Description

We always designate the angular velocity of the xyz reference frame as $\bar{\omega}$. The way in which we form this crucial quantity is a direct result of substituting Eq. (3.3.8), which adds infinitesimal rotations about various axes, into the definition of $\bar{\omega}$, Eq. (3.3.10). This leads to the recognition that

An angular velocity $\bar{\omega}$ is the sum of simple rotations described by angular velocities $\omega_n \bar{e}_n$, where $\bar{e}_n$ is a unit vector parallel to the respective rotation axis, in accord with the right-hand rule,

$$\boxed{\bar{\omega} = \sum_n \omega_n \bar{e}_n.} \tag{3.4.1}$$

Note that there is no rule as to how the rotations are numbered, because sequence is irrelevant to angular velocity.

The angular acceleration $\bar{\alpha}$ of xyz is the rate of change of the angular velocity:

$$\boxed{\bar{\alpha} \equiv \frac{d\bar{\omega}}{dt}.} \tag{3.4.2}$$

Because Eq. (3.4.1) is a general description of $\bar{\omega}$, it may be differentiated to obtain $\bar{\alpha}$. This operation requires that we evaluate the rate of change of the unit vectors $\bar{e}_n$. We use Eq. (3.3.16) for this purpose, but doing so requires a definition of the reference frame associated with each $\bar{e}_n$. Specifically,

For each simple rotation, define an auxiliary moving reference frame $x_n y_n z_n$, such that one of the axes of $x_n y_n z_n$ always coincides with that rotation axis. Hence, $\bar{e}_n$ is either $\bar{i}_n$, $\bar{j}_n$, or $\bar{k}_n$. Let $\bar{\Omega}_n$ denote the angular velocity of $x_n y_n z_n$.

Note that these auxiliary reference frames are quite unrestricted, other than the requirement that one of their coordinate axes should always align with a simple rotation axis. Any of them may actually be the xyz coordinate system, in which case the corresponding $\bar{\Omega}_n \equiv \bar{\omega}$. In general, a single auxiliary reference frame may be associated with more than one simple rotation. Also, if an axis of rotation is stationary, we may let the corresponding reference frame be XYZ, corresponding to $\bar{\Omega}_n = \bar{0}$.

We apply the standard rules for derivatives to Eq. (3.4.1) According to Eq. (3.3.16), the time derivative of the axis directions is $d\bar{e}_n/dt = \bar{\Omega}_n \times \bar{e}_n$, so we find that

$$\boxed{\bar{\alpha} = \sum_n \left(\dot{\omega}_n \bar{e}_n + \bar{\Omega}_n \times \omega_n \bar{e}_n \right).} \tag{3.4.3}$$

An interesting aspect of this description of $\bar{\alpha}$ is that the only derivatives we need to determine explicitly are the $\dot{\omega}_n$ values.

It is imperative to understand the meaning of the terms in Eq. (3.4.3). Two types of contributions are associated with each simple rotation. An unsteady rotation rate, $\dot{\omega}_n \neq 0$, gives rise to an angular acceleration $\dot{\omega}_n \bar{e}_n$ that is parallel to the rotation axis. In addition, even if all rotation rates are constant, there will be an angular acceleration term if the orientation of any rotation axis is not constant, corresponding to $\bar{\Omega}_n \times \omega_n \bar{e}_n \neq \bar{0}$. This type of angular acceleration is perpendicular to the rotation axis, as well as the angular velocity of that axis. Planar motion consists of a single simple rotation about an axis perpendicular to the plane. Because the orientation of this axis is constant, the angular acceleration in planar motion does not feature any effect associated with nonstationary rotation axes. This is one of the primary reasons why intuitive judgements based on experiences with planar motion are often incorrect.

After the individual terms in Eqs. (3.4.1) and (3.4.3) have been characterized, all vector quantities should be expressed in terms of a common set of components. This is necessary so that we may sum the terms by adding like components. The directions used for this purpose constitute the *global coordinate system.* One could use the stationary XYZ axes as the global system, but components relative to the xyz axes, whose angular motion is described by $\bar{\omega}$ and $\bar{\alpha}$, often are more meaningful. Another possibility is to use one of the auxiliary reference frames $x_n y_n z_n$ to describe components. A primary reason for the last choice is that expressing components relative to either XYZ or xyz might require evaluation of a rotation transformation matrix, whereas it might be possible to use visual inspection to construct the components relative to one of the auxiliary coordinate systems. This is especially so because, after we align one axis of $x_n y_n z_n$ with its associated rotation axis, we are free to align its other axes in a manner that expedites the description of all vector components.

3.4.2 Procedure

Correct evaluation of $\bar{\omega}$ and $\bar{\alpha}$ for a moving xyz reference frame is of primary importance in several contexts. For that reason, it is appropriate to formalize the concepts and developments thus far into a sequence of steps that will address most situations.

1. Examine the overall rotation of the body of interest, which is defined as reference frame xyz. Conceptually decompose it into a sequence of simple rotations ω_n,

$n = 1, 2, \ldots$, where the units of ω_n are radians per second. (These simple rotations are typically the rotation of one part of the system with respect to another part.)

2. For each rotation ω_n, define a reference frame $x_n y_n z_n$ such that one of its unit vectors $\bar{i}_n$, $\bar{j}_n$, or $\bar{k}_n$ *always* coincides with the corresponding rotation direction $\bar{e}_n$. Note that these reference frames may be fixed, or else execute some or all of the rotations associated with xyz, but they will never execute more rotations than xyz. Denote as $\bar{e}_n$ the unit vector that is aligned with the axis ω_n rotation axis.
3. Select a global coordinate system to be used for evaluating all vector components. The orientation of this coordinate system should facilitate describing all rotation directions $\bar{e}_n$ based on the manner in which linear and angular dimensions of the system are described. It usually is convenient to use as the global system one of the set of reference frame axes already defined.
4. Construct the angular velocity vector $\bar{\omega}$ of xyz by vectorially adding the simple rotation rates according to

$$\bar{\omega} = \omega_1 \bar{e}_1 + \omega_2 \bar{e}_2 + \cdots . \tag{3.4.4}$$

5. Vectorially add the simple rotation rates of each $x_n y_n z_n$ to construct the angular velocity vector $\bar{\Omega}_n$. The form of the superposition sum will be similar to Eq. (3.4.4).
6. Form the angular acceleration $\bar{\alpha}$ of xyz by differentiating $\bar{\omega}$ in Eq. (3.4.4). For this differentiation, use the fact that $\bar{e}_n$ is one of the unit vectors of $x_n y_n z_n$, so that $d\bar{e}_n/dt = \bar{\Omega}_n \times \bar{e}_n$. Thus this step gives

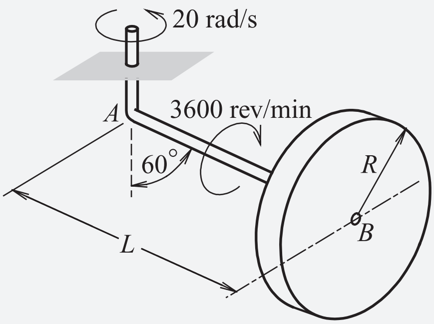
$$\bar{\alpha} = \dot{\omega}_1 \bar{e}_1 + \omega_1 \left(\bar{\Omega}_1 \times \bar{e}_1 \right) + \dot{\omega}_2 \bar{e}_2 + \omega_2 \left(\bar{\Omega}_2 \times \bar{e}_2 \right) + \cdots . \tag{3.4.5}$$

7. Express each unit vector $\bar{e}_n$ in terms of its components relative to the global coordinate system, in the form

$$\bar{e}_n = \ell_{nx} \hat{i} + \ell_{ny} \hat{j} + \ell_{nz} \hat{k} \tag{3.4.6}$$

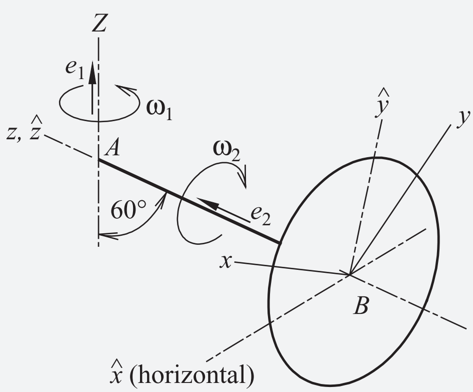
   where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are the unit vectors of whichever coordinate system was selected as the global one.
8. Substitute the component descriptions of the unit vectors given by Eq. (3.4.6) into the angular velocity of xyz described by Eq. (3.4.4), and collect like components. This gives $\bar{\omega}$ in terms of components with respect to the global coordinate system.
9. Substitute the component descriptions of the unit vectors given by Eq. (3.4.6) into each angular velocity $\bar{\Omega}_n$ formed in Step 5, and collect like components. This gives each $\bar{\Omega}_n$ in terms of components with respect to the global coordinate system.
10. Substitute the component descriptions of the unit vectors given by Eq. (3.4.6), and of each $\bar{\Omega}_n$ obtained in Step 9, into the angular acceleration given by Eq. (3.4.5). Collect like components. This gives $\bar{\alpha}$ in terms of components with respect to the global coordinate system.

EXAMPLE 3.9 The disk rotates about shaft AB at 3600 rev/min as the system rotates about the vertical axis at 20 rad/s. Determine the angular velocity and angular acceleration of the disk.



Example 3.9

SOLUTION This straightforward example illustrates the procedural steps leading to $\bar{\omega}$ and $\bar{\alpha}$. We attach xyz to the disk, so that $\bar{\omega}$ describes the angular velocity of that body. There are two rotations: $\omega_1 = 20$ rad/s, with $\bar{e}_1$ vertically upward by the right-hand rule, and $\omega_2 = 3600\,(2\pi/60)$ rad/s with $\bar{e}_2$ directed from center B to junction A. To describe the first rotation we select the fixed XYZ as the first auxiliary reference frame, with Z defined to be vertical, so that $\bar{e}_1 = \bar{K}$ and $\bar{\Omega}_1 = \bar{0}$. Because there are only two rotations, we may use xyz as $x_2y_2z_2$. Although xyz must be attached to the disk, we are free to orient it in a manner that expedites description of the second rotation axis. With this in mind, we observe that shaft AB has a constant orientation relative to the disk, which enables us to align the z axis with this shaft. Then $\bar{e}_2 = \bar{k}$ and $\bar{\Omega}_2 = \bar{\omega}$. We show the rotation direction vectors in a simple sketch as an aid to the task of evaluating components.



Coordinate systems for describing the angular motion of the disk.

Several features of this sketch should be noted. First, the location of the origin of each coordinate system is irrelevant to the task of evaluating the angular motion. Second, the orientations of the fixed X and Y axes are unimportant from the viewpoint of the disk, so these axes have been omitted from the sketch. Third, the only aspect of the orientation of xyz requiring specification is that the z axis coincide with shaft AB. The sketch depicts an arbitrary instant, so the z axis is aligned with shaft AB, but the x and y axes lie in neither the horizontal nor the vertical plane. The arbitrariness of these axes will be useful for later developments.

For a global coordinate system we define $\hat{x}\hat{y}\hat{z}$ to be attached to shaft AB such that the $\hat{z}$ axis is always aligned with the z axis, and the $\hat{x}$ axis is the horizontal diameter of the disk. This choice for the global system simultaneously facilitates describing $\bar{e}_1$ and $\bar{e}_2$. The sketch also depicts the $\hat{x}\hat{y}\hat{z}$ axes.

The general description of the angular velocity of xyz is

$$\bar{\omega} = \omega_1 \bar{e}_1 + \omega_2 \bar{e}_2 = \omega_1 \bar{K} + \omega_2 \bar{k}. \tag{1}$$

Because the first auxiliary reference frame is stationary, and the second one is xyz, we have

$$\bar{\Omega}_1 = \bar{0}, \quad \bar{\Omega}_2 = \bar{\omega}.$$

The rotation rates are constant, so the corresponding description of the angular acceleration of xyz is

$$\bar{\alpha} = \omega_1 \bar{\Omega}_1 \times \bar{K} + \omega_2 \bar{\Omega}_2 \times \bar{k} = \omega_2 \bar{\omega} \times \bar{k}. \tag{2}$$

We find the global components of the unit vectors by inspection of the sketch, which leads to

$$\bar{K} = \sin 60^\circ \hat{j} + \cos 60^\circ \hat{k}, \quad \bar{k} = \hat{k}. \tag{3}$$

Substitution of these unit vectors into Eqs. (1) and (2) yields

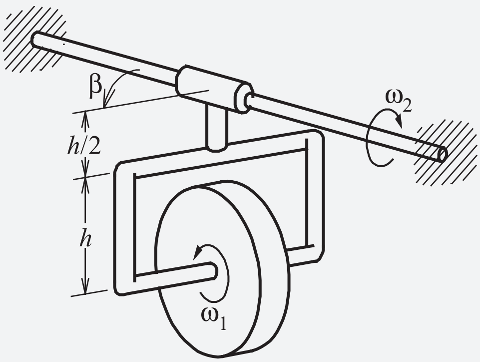
$$\bar{\omega} = 0.866\omega_1 \hat{j} + (0.5\omega_1 + \omega_2)\,\hat{k},$$
$$\bar{\alpha} = \omega_2 \left[0.866\omega_1 \hat{j} + (0.5\omega_1 + \omega_2)\,\hat{k}\right] \times \hat{k} = 0.866\omega_1\omega_2 \hat{i}.$$

Evaluation of these expressions at the specified rotation rates gives

$$\bar{\omega} = 17.32\hat{j} + 387.00\hat{k} \text{ rad/s},$$
$$\bar{\alpha} = 6530\hat{i} \text{ rad/s}^2. \qquad \triangleleft$$

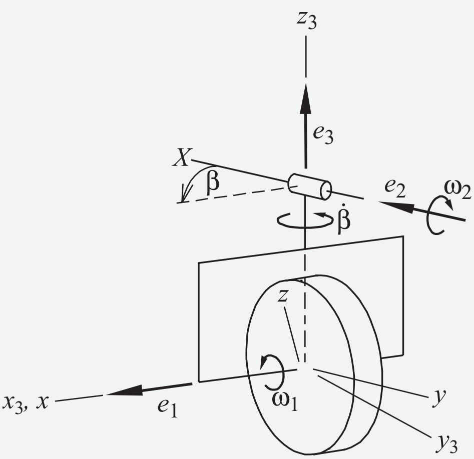
There is a ready explanation for the sense of the angular acceleration. The general expression for $\bar{\omega}$ shows that it is the sum of two terms: $\omega_1 \bar{K}$, which is constant, and $\omega_2 \bar{k}$, which rotates about the vertical axis. The rate of change of a vector that is due to its rotation is in the sense of the movement of the vector's tip when its tail is considered to be stationary. The tip of the z axis moves horizontally in the rotation about the Z axis, so the angular acceleration is horizontal in the sense of positive $\hat{i}$. In general, considering whether the result for $\bar{\alpha}$ makes sense is an excellent way to check one's work.

EXAMPLE 3.10 The gyroscopic turn indicator consists of a flywheel that spins about its axis of symmetry at the constant rate ω_1 relative to the gimbal, as the assembly rotates about the fixed horizontal shaft at the variable rate ω_2. The angle β locating the plane of the gimbal relative to the horizontal shaft is an arbitrary function of time. Determine the angular acceleration of the flywheel at an arbitrary instant.



Example 3.10

SOLUTION The angular motion is more complicated here than it was in the previous example, so this analysis will provide a more complete picture of the procedure. Let xyz be attached to the flywheel, so that its angular motion is identical to that of the flywheel. We begin by identifying the constituent simple rotations and draw a sketch depicting the rotation unit vectors and associated coordinate axes.



Coordinate systems for the turn indicator.

The ω_2 rotation is about the fixed horizontal shaft. Correspondingly, we let $x_2y_2z_2$ be the fixed reference frame XYZ defined such that $\bar{e}_2 = \bar{I}$. The ω_1 rotation is about the axis of the flywheel. This direction is fixed relative to the flywheel. We therefore select $x_1y_1z_1$ as xyz, with the specification that the x axis coincides with the flywheel's axis of symmetry, so that $\bar{e}_1 = \bar{i}$. The fact that β is variable means that there is a third rotation, $\dot{\beta}$, about an axis that is always perpendicular to the horizontal shaft. To describe it, we attach reference frame $x_3y_3z_3$ to the gimbal supporting the flywheel, with the z_3 axis defined such that $\bar{e}_3 = \bar{k}_3$. In the sketch the x_3 axis has been aligned with the gimbal's shaft, because doing so makes $x_3y_3z_3$ a convenient global coordinate system. (In fact, because $\bar{e}_1$ is parallel to the x_3 axis, this $x_3y_3z_3$ reference frame could also have been used to describe the ω_1 rotation.) To develop procedures suitable to later developments we observe that because xyz rotates at ω_1 relative to the gimbal, at the arbitrary instant depicted in the sketch the y and z axes have rotated away from y_3 and z_3, respectively.

The angular velocity of the flywheel is the sum of the simple rotations, so

$$\bar{\omega} = \omega_1\bar{e}_1 + \omega_2\bar{e}_2 + \dot{\beta}\bar{e}_3 = \omega_1\bar{i} + \omega_2\bar{I} + \dot{\beta}\bar{k}_3.$$

The auxiliary reference frame for $\bar{e}_2$ is XYZ, the frame for $\bar{e}_1$ is xyz, and the frame for $\bar{e}_3$ is $x_3y_3z_3$, which executes the ω_2 and $\dot{\beta}$ rotations. Thus we have

$$\bar{\Omega}_1 = \bar{\omega}, \quad \bar{\Omega}_2 = \bar{0}, \quad \bar{\Omega}_3 = \omega_2\bar{I} + \dot{\beta}\bar{k}_3.$$

The only rotation rate that is specified to be constant is ω_1, so the general description of $\bar{\alpha}$ corresponding to the preceding expression for $\bar{\omega}$ is

$$\begin{aligned}\bar{\alpha} &= \omega_1\left(\bar{\Omega}_1 \times \bar{i}\right) + \dot{\omega}_2\bar{I} + \omega_2\left(\bar{\Omega}_2 \times \bar{I}\right) + \ddot{\beta}\bar{k}_3 + \dot{\beta}\left(\bar{\Omega}_3 \times \bar{k}_3\right)\\ &= \omega_1\bar{\omega} \times \bar{i} + \dot{\omega}_2\bar{I} + \ddot{\beta}\bar{k}_3 + \dot{\beta}\left(\omega_2\bar{I} + \dot{\beta}\bar{k}_3\right) \times \bar{k}_3.\end{aligned}$$

We use geometrical projections to describe the global $\bar{i}_3\bar{j}_3\bar{k}_3$ components of the unit vectors. This gives

$$\bar{i} = \bar{i}_3, \quad \bar{I} = \cos\beta\,\bar{i}_3 - \sin\beta\,\bar{j}_3.$$

Substitution of these representations into the expressions for $\bar{\omega}$ and $\bar{\alpha}$ gives

$$\bar{\omega} = (\omega_1 + \omega_2\cos\beta)\,\bar{i}_3 - \omega_2\sin\beta\,\bar{j}_3 + \dot{\beta}\bar{k}_3,$$

$$\begin{aligned}\bar{\alpha} = &\left(\dot{\omega}_2\cos\beta - \dot{\beta}\omega_2\sin\beta\right)\bar{i}_3 + \left(-\dot{\omega}_2\sin\beta + \omega_1\dot{\beta} - \dot{\beta}\omega_2\cos\beta\right)\bar{j}_3\\ &+ \left(\ddot{\beta} + \omega_1\omega_2\sin\beta\right)\bar{k}_3.\end{aligned} \quad \triangleleft$$

Each term in $\bar{\alpha}$ can be explained physically. Unsteady values of ω_2 and $\dot{\beta}$ give rise to angular accelerations that are parallel to the respective rotation axes. Terms that are the products of two rotation rates, and therefore exist even if the rates are constant, represent angular acceleration effects that are perpendicular to the simple rotation axes. For example, the terms containing $\dot{\beta}\omega_2$ are a consequence of $\bar{e}_3$ rotating at $\omega_2\bar{e}_2$, which makes the tip of $\bar{e}_3$ moves perpendicularly to the X axis in the $\hat{x}_3\hat{y}_3$ plane. In contrast, $\bar{e}_1$ rotates at $\dot{\beta}\bar{e}_3$ and $\omega_2\bar{e}_2$. The $\dot{\beta}\bar{e}_3$ rotation makes the tip of $\bar{e}_1$ move in the $\hat{j}_3$ direction, which is the direction of the $\omega_1\dot{\beta}$ term in $\bar{\alpha}$. The term $\omega_1\omega_2\sin\beta$ in the $\hat{k}_3$ component of $\bar{\alpha}$ arises because the $\omega_2\bar{e}_2$ rotation makes the tip of $\bar{e}_1$ move in the $\hat{k}_3$ direction. The $\sin\beta$ coefficient in this term arises because only the component of $\bar{e}_1$ perpendicular to the $\bar{I}$ axis changes in this rotation.

3.5 VELOCITY AND ACCELERATION ANALYSIS USING A MOVING REFERENCE FRAME

This chapter began with the introduction of a moving reference frame as an aid to describing displacement. Specializing the relations to the case of infinitesimal movements led us to Eq. (3.3.9) for velocity. Application of that formula required evaluation of angular velocity, and we anticipated later developments by also considering angular acceleration. We now return to the study of point motion by deriving a formula for the acceleration of a point whose movement is observed from a moving reference frame.

Equation (3.3.9) provides a general description of velocity, so it may be differentiated. The time derivative of the origin's velocity $\bar{v}_{O'}$ is its acceleration $\bar{a}_{O'}$. The relative velocity $(\bar{v}_P)_{xyz}$ is defined by Eq. (3.3.11) in terms of components relative to the moving reference frame. Equation (3.3.15) gives the derivative of a vector described in such a manner, where the $\partial/\partial t$ operator defined in Eq. (3.3.14) denotes a time derivative that ignores the fact that the direction of the unit vectors is not constant. Because the relative velocity components are the derivatives of the position coordinates, the relative acceleration is given by

$$(\bar{a}_P)_{xyz} \equiv \frac{\partial}{\partial t}(\bar{v}_P)_{xyz} = \ddot{x}_P\bar{i} + \ddot{y}_P\bar{j} + \ddot{z}_P\bar{k}. \tag{3.5.1}$$

Correspondingly, we find from Eq. (3.3.15) that

$$\frac{d}{dt}(\bar{v}_P)_{xyz} = (\bar{a}_P)_{xyz} + \bar{\omega} \times (\bar{v}_P)_{xyz}. \tag{3.5.2}$$

Now consider the last term in Eq. (3.3.9). The time derivative of $\bar{\omega}$ is $\bar{\alpha}$, and Eq. (3.3.13) gives the time derivative of $\bar{r}_{P/O'}$. We therefore have

$$\frac{d}{dt}(\bar{\omega} \times \bar{r}_{P/O'}) = \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times \left[(\bar{v}_P)_{xyz} + \bar{\omega} \times \bar{r}_{P/O'}\right]. \tag{3.5.3}$$

We obtain the acceleration formula by adding Eqs. (3.5.2) and (3.5.3) to $\bar{a}_{O'}$. For later use the result is accompanied by the previously derived expression for velocity:

$$\boxed{\begin{aligned} \bar{v}_P &= \bar{v}_{O'} + (\bar{v}_P)_{xyz} + \bar{\omega} \times \bar{r}_{P/O'}, \\ \bar{a}_P &= \bar{a}_{O'} + (\bar{a}_P)_{xyz} + \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'}) + 2\bar{\omega} \times (\bar{v}_P)_{xyz}. \end{aligned}} \tag{3.5.4}$$

With one exception, the terms in these expressions could have been anticipated as a simple superposition of motion of the origin and motion relative to xyz. The velocity term $\bar{\omega} \times \bar{r}_{P/O'}$ and the acceleration term $\bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'})$ are described in Fig. 3.11. By definition, $\bar{\omega} \times \bar{r}_{P/O'}$ is perpendicular to the plane formed by $\bar{\omega}$ and $\bar{r}_{P/O'}$, and its magnitude is $|\bar{\omega}|\, r_\perp$, where $r_\perp$ is the component of $\bar{r}_{P/O'}$ perpendicular to $\bar{\omega}$. Thus this term is like the azimuthal velocity $R\dot{\theta}$ that occurred in cylindrical coordinates. Furthermore, it follows that the magnitude of $\bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'})$ is $|\bar{\omega}|^2\, r_\perp$ and that it is directed along the perpendicular line from point P toward the rotation axis. Thus this term is a centripetal acceleration, like the term $R\dot{\theta}^2$ in cylindrical coordinates. Another acceleration term associated with the rotational motion is $\bar{\alpha} \times \bar{r}_{P/O'}$. Although it appears to be analogous to the velocity term $\bar{\omega} \times \bar{r}_{P/O'}$, in spatial motion the angular acceleration is nonzero and

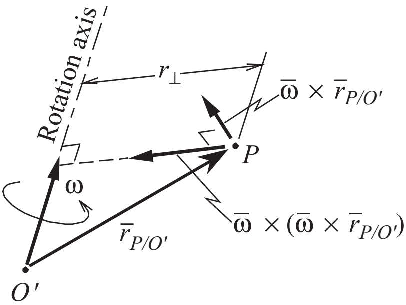


Figure 3.11. Construction of the centripetal acceleration of a point.

not parallel to the angular velocity, even if all rotation rates are constant. As a result, the direction of the corresponding acceleration term will generally not be perpendicular to the plane formed by the rotation axis and $\bar{r}_{P/O'}$.

The acceleration term $2\bar{\omega} \times (\bar{v}_P)_{xyz}$ is not intuitively obvious, but it is readily explained. As might be guessed from the two factor, it is the Coriolis acceleration that was encountered in the study of curvilinear coordinates, for example, $2\dot{R}\dot{\theta}$ in cylindrical coordinates. In fact, as we saw in the study of curvilinear coordinates, Coriolis acceleration actually arises from two distinct effects that have equal importance. Equation (3.5.2) indicates that half the Coriolis acceleration stems from the fact that the components of relative velocity have a variable orientation that is due to rotation of the xyz axes. The other half appears in Eq. (3.5.3), where it is associated with the fact that the transverse velocity $\bar{\omega} \times \bar{r}_{P/O'}$ is not constant if the coordinates of point P with respect to xyz are not constant. Thus it is to some extent a misnomer to use a single name to refer to $2\bar{\omega} \times (\bar{v}_P)_{xyz}$.

One aspect of the relative velocity $(\bar{v}_P)_{xyz}$ and relative acceleration $(\bar{a}_P)_{xyz}$ greatly facilitates their evaluation. These terms may be visualized as the effects that would remain if the reference frame were held stationary. They were described in Eqs. (3.3.11) and (3.5.1), respectively, in terms of a Cartesian coordinate description. However, other kinematical descriptions, such as path variables or curvilinear coordinates, might be more appropriate in some situations. If such an approach is employed, it is necessary to convert those components to the global set of components used to represent all vectors.

It is instructive to close this discussion by considering two special cases. The situation in which the xyz frame translates corresponds to $\bar{\omega}$ being identically zero. Hence, $\bar{\alpha}$ also is zero. The relative motion equations then reduce to

$$\begin{aligned} \bar{v}_P &= \bar{v}_{O'} + (\bar{v}_P)_{xyz}\,, \\ \bar{a}_P &= \bar{a}_{O'} + (\bar{a}_P)_{xyz}\,. \end{aligned} \tag{3.5.5}$$

The motion of the origin and of the point relative to the translating reference frame are additive—there are no corrections for direction changes that are due to rotation. If xyz, as well as XYZ, is fixed, so that $\bar{v}_{O'} = \bar{0}$ and $\bar{a}_{O'} = \bar{0}$, then the preceding relations show that the velocity and acceleration are the same, regardless of which fixed reference frame is selected. A more important observation arises when xyz is translating at a constant velocity, so that $\bar{a}_{O'} = \bar{0}$. (Note that this condition requires that the origin O' follow a straight path.) The second of Eqs. (3.5.5) shows that $\bar{a}_P = (\bar{a}_P)_{xyz}$ in this case. A reference frame translating at constant velocity is said to be an *inertial* or *Galilean reference frame*. The terminology arises from the fact that the absolute acceleration is observable from the reference frame, so the frame may be employed to formulate Newton's Laws.

The second special case arises when point P is fixed with respect to the moving reference frame. Because the position coordinates then are constant, $(\bar{v}_P)_{xyz}$ and $(\bar{a}_P)_{xyz}$ are both identically zero. This simplifies the velocity and acceleration relations to

$$\boxed{\begin{aligned} \bar{v}_P &= \bar{v}_{O'} + \bar{\omega} \times \bar{r}_{P/O'}, \\ \bar{a}_P &= \bar{a}_{O'} + \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'}) \,. \end{aligned}} \tag{3.5.6}$$

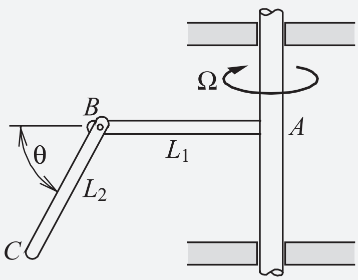
A primary reason for highlighting this situation is that it is descriptive of the motion of a rigid body. If xyz is attached to the body, then the position vectors between points in the body have constant components relative to the moving reference frame. Also, the angular motion of the body and of xyz are synonymous in this case. The motion of rigid bodies is the focus of the next chapter.

General equations (3.5.4) exemplify the notion that using a moving reference frame allows us to decompose a complicated motion into a set of simpler kinematical analyses associated with the individual terms in those equations. Application of these formulas requires definition of xyz. If it is required that $\bar{\omega}$ and $\bar{\alpha}$ of a body be determined, then xyz should be attached to that body. Otherwise, xyz seldom is specified *a priori*. Its selection affects the individual terms, although the ultimate results for velocity and acceleration will be unaffected. Depending on how xyz is defined, some terms will be easier to determine, whereas others will be more difficult. Some general criteria can be identified. The choice of the body to which xyz is attached dictates which simple rotations combine to form $\bar{\omega}$. Selecting xyz to execute many rotations complicates the analysis of $\bar{\omega}$ and $\bar{\alpha}$. However, the methodology laid out in the previous section is reasonably robust, so no selection is likely to be completely overwhelming. Furthermore, letting xyz execute many of the rotations is likely to simplify analysis of the relative motion terms $(\bar{v}_P)_{xyz}$ and $(\bar{a}_P)_{xyz}$. Thus, a guideline for selecting the attachment of xyz is that it should execute as many rotations as possible, provided that one can evaluate the corresponding $\bar{\omega}$ and $\bar{\alpha}$. (Once again, the exception is that if $\bar{\omega}$ and $\bar{\alpha}$ of a certain body must be determined, then xyz must be attached to that body.)

The selection of the origin O' affects the terms $\bar{v}_{O'}$ and $\bar{a}_{O'}$, as well as $\bar{r}_{P/O'}$ in Eqs. (3.2.4) and (3.2.7) for displacement. This selection is restricted by the requirement that point O' be one of the points of the body to which xyz is attached. Thus an optimal approach is to select the body to which xyz is attached by considering the difficulty entailed in describing $\bar{\omega}$ and $\bar{\alpha}$, simultaneously with considering whether some point in the body follows a relatively simple path so that the description of $\bar{v}_{O'}$ and $\bar{a}_{O'}$ will be manageable. (In complicated systems, evaluation of $\bar{v}_{O'}$ and $\bar{a}_{O'}$ might require a separate analysis with a different moving reference frame.)

As we did for the analysis of $\bar{\omega}$ and $\bar{\alpha}$, we use a global coordinate system to represent the components of all vectors, so that the various terms may be combined. The previous section stated a criterion that the orientation of these axes should be selected to facilitate representation of the unit vectors $\bar{e}_n$ for the rotations. Because other terms, notably $(\bar{v}_P)_{xyz}$ and $(\bar{a}_P)_{xyz}$, must now be represented in component form, it is best that the global system be selected to facilitate describing all terms. This requires striking a balance. For example, a certain selection might be ideal for describing the relative motion terms while simultaneously complicating the component description of the $\bar{e}_j$ rotation directions. If necessary, one can always employ rotation transformation matrices to describe the components of any troublesome vectors.

EXAMPLE 3.11 Bar BC is pinned to the T-bar, which is rotating about the vertical axis at constant rate Ω. Angle θ is an arbitrary function of time. Determine the velocity and acceleration of point C using the following alternative approaches: (a) attach the xyz reference frame to the T-bar; (b) attach the xyz reference frame to bar BC.

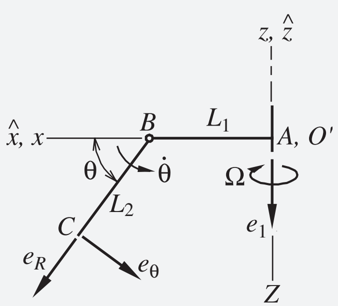


Example 3.11

SOLUTION By employing two different approaches, this example provides insight into the decisions one must make. In the first approach, attaching xyz to the T-bar means that the sole rotation of xyz is Ω about the fixed vertical axis. Thus we define the Z axis such that $\bar{e}_1 = \bar{K}$. Because Ω is specified to be constant, the general descriptions of angular motion are

$$\bar{\omega} = \Omega\bar{e}_1 = \Omega\bar{K}, \quad \bar{\Omega}_1 = \bar{0}, \quad \bar{\alpha} = \bar{\Omega}_1 \times \bar{K} = \bar{0}.$$

Any point on the vertical axis of rotation is stationary. Selecting the origin O' to be point A simplifies the description of $\bar{r}_{P/O'}$ and gives $\bar{v}_{O'} = \bar{a}_{O'} = \bar{0}$. We orient xyz consistently with the way in which the configuration of the system is specified and show our choice in a line sketch.



Kinematical analysis with *xyz* as a reference frame that is attached to the T-bar.

To analyze the relative motion we visualize bringing xyz to rest by setting $\Omega = 0$. Bar BC would still move in the vertical plane as it pivots about pin B. Thus end C moves in a circular path at angular speed $\dot{\theta}$ relative to xyz. Polar coordinates having origin at point B are suitable for describing this relative motion, with the radial distance $R = L_2$ and θ being the polar angle. The associated unit vectors are shown in the sketch. The corresponding descriptions of the relative velocity and acceleration are

$$(\bar{v}_C)_{xyz} = L_2\dot{\theta}\bar{e}_\theta, \quad (\bar{a}_C)_{xyz} = -L_2\dot{\theta}^2\bar{e}_R + L_2\ddot{\theta}\bar{e}_\theta.$$

The axes of xyz are convenient directions for representing vectors, so we use them as the global directions. (An alternative definition would align the $\hat{x}$ axis with bar BC, and place the $\hat{z}$ axis in the vertical plane. This would simplify the description of $\bar{e}_R$ and $\bar{e}_\theta$, but slightly complicate $\bar{K}$.) The global descriptions of the unit vectors are

$$\bar{K} = -\bar{k}, \quad \bar{e}_R = \cos\theta\bar{i} - \sin\theta\bar{k}, \quad \bar{e}_\theta = -\sin\theta\bar{i} - \cos\theta\bar{k}.$$

The corresponding relative position is

$$\bar{r}_{C/O'} = (L_1 + L_2\cos\theta)\,\bar{i} - L_2\sin\theta\bar{k}.$$

We may now assemble the individual terms. For velocity we have

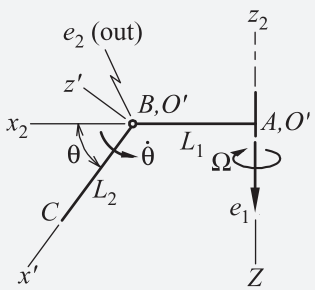
$$\begin{aligned}\bar{v}_C &= (\bar{v}_C)_{xyz} + \bar{\omega}\times\bar{r}_{C/O'}\\ &= L_2\dot{\theta}\left(-\sin\theta\bar{i} - \cos\theta\bar{k}\right) + \left(-\Omega\bar{k}\right)\times\left[(L_1 + L_2\cos\theta)\,\bar{i} - L_2\sin\theta\bar{k}\right],\end{aligned}$$

$$\bar{v}_C = -L_2\dot{\theta}\sin\theta\bar{i} - (L_1 + L_2\cos\theta)\,\Omega\,\bar{j} - L_2\dot{\theta}\cos\theta\bar{k}. \qquad \triangleleft$$

A shortcut for the evaluation of $\bar{\omega}\times(\bar{\omega}\times\bar{r}_{C/O'})$ is to retain $\bar{\omega}\times\bar{r}_{C/O'}$ from the velocity analysis. Thus we have

$$\begin{aligned}\bar{a}_C &= (\bar{a}_C)_{xyz} + \bar{\omega}\times(\bar{\omega}\times\bar{r}_{C/O'}) + 2\bar{\omega}\times(\bar{v}_C)_{xyz}\\ &= -L_2\dot{\theta}^2\left(\cos\theta\bar{i} - \sin\theta\bar{k}\right) + L\ddot{\theta}\left(-\sin\theta\bar{i} - \cos\theta\bar{k}\right) + \left(-\Omega\bar{k}\right)\\ &\quad \times\left[-(L_1 + L_2\cos\theta)\,\Omega\,\bar{j}\right] + 2\left(-\Omega\bar{k}\right)\times\left[L_2\dot{\theta}\left(-\sin\theta\bar{i} - \cos\theta\bar{k}\right)\right] \qquad \triangleleft\\ &= \left[-L_2\dot{\theta}^2\cos\theta - L\ddot{\theta}\sin\theta - (L_1 + L_2\cos\theta)\,\Omega^2\right]\bar{i} + 2L\Omega\dot{\theta}\sin\theta\,\bar{j}\\ &\quad + \left(L_2\dot{\theta}^2\sin\theta - L\ddot{\theta}\cos\theta\right)\bar{k}.\end{aligned}$$

Obviously, changing the selection of the moving reference frame should not alter the result. To avoid confusion with the preceding analysis, let $x'y'z'$ denote the moving reference frame that is attached to bar BC. The angular velocity of this bar is the sum of Ω about the vertical axis and $\dot{\theta}$ about an axis that is perpendicular to the plane of the T-bar. Both simple rotations are shown in a new sketch.



Kinematical analysis using $x'y'z'$ as a reference frame that is attached to the swinging bar BC.

As was done in the previous analysis, fixed XYZ can be used to describe $\bar{e}_1$. To describe the $\dot{\theta}$ rotation we attach $x_2y_2z_2$ to the T-bar, with the y_2 axis outward from the plane of the sketch, so that $\bar{e}_2 = \bar{j}_2$. The angular velocity $\bar{\Omega}_2$ then consists solely of the rotation about the vertical axis. Thus

$$\bar{\omega} = \Omega\bar{e}_1 + \dot{\theta}\bar{e}_2 = \Omega\bar{K} + \dot{\theta}\,\bar{j}_2, \quad \bar{\Omega}_1 = \bar{0}, \quad \bar{\Omega}_2 = \Omega\bar{K}.$$

The angular acceleration corresponding to constant Ω and variable $\dot{\theta}$ is

$$\bar{\alpha} = \ddot{\theta}\,\bar{j}_2 + \dot{\theta}\left(\bar{\Omega}_2 \times \bar{j}_2\right) = \ddot{\theta}\,\bar{j}_2 + \dot{\theta}\Omega\left(\bar{K} \times \bar{j}_2\right).$$

The guideline for the selection of the origin O' requires that it be a point in bar BC, which is the body to which $x'y'z'$ is attached. The only such point executing a simple motion is end B, which follows a circular path in the horizontal plane at constant angular speed Ω. We may describe this motion in cylindrical coordinates whose axis is z_2, with the x_2 axis parallel to arm AB, which is the $\bar{e}_R$ direction. Then the y_2 axis is opposite to $\bar{e}_\theta$. Correspondingly, the motion of the origin is given by

$$\bar{v}_{O'} = -L_1\Omega\,\bar{j}_2, \quad \bar{a}_{O'} = -L_1\Omega^2\bar{i}_2.$$

Because xyz has been defined to be attached to bar BC, point C remains fixed from the viewpoint of this reference frame. Thus,

$$(\bar{v}_C)_{xyz} = (\bar{a}_C)_{xyz} = \bar{0}.$$

Orienting $x'y'z'$ as depicted in the sketch facilitates the description of relative position. We correspondingly find that

$$\bar{K} = \sin\theta\bar{i}' - \cos\theta\bar{k}', \quad \bar{j}_2 = \bar{j}', \quad \bar{i}_2 = \cos\theta\bar{i}' + \sin\theta\bar{k}', \quad \bar{r}_{C/O'} = L_2\bar{i}'.$$

The global descriptions of the angular motion variables are

$$\bar{\omega} = \Omega\left(\sin\theta\bar{i}' - \cos\theta\bar{k}'\right) + \dot{\theta}\,\bar{j}',$$

$$\bar{\alpha} = \ddot{\theta}\,\bar{j}' + \dot{\theta}\Omega\left[\left(\sin\theta\bar{i}' - \cos\theta\bar{k}'\right) \times \bar{j}'\right] = \ddot{\theta}\,\bar{j}' + \dot{\theta}\Omega\left(\cos\theta\bar{i} + \sin\theta\bar{k}'\right).$$

The nonzero terms in the general velocity equation are

$$\begin{aligned} \bar{v}_C &= \bar{v}_{O'} + \bar{\omega} \times \bar{r}_{C/O'} \\ &= -L_1\Omega\,\bar{j}' + \left[\Omega\left(\sin\theta\bar{i}' - \cos\theta\bar{k}'\right) + \dot{\theta}\,\bar{j}'\right] \times L_2\bar{i}' \\ &= -\left(L_1 + L_2\cos\theta\right)\Omega\bar{j}' - L_2\dot{\theta}\bar{k}'. \end{aligned}$$

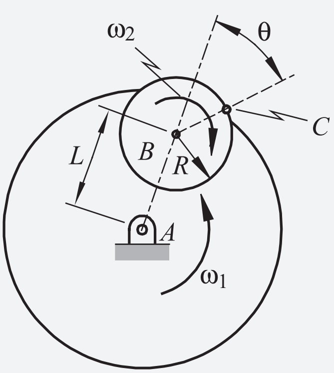
◁

The nonzero terms in the general acceleration equation are

$$\begin{aligned}\bar{a}_C &= \bar{a}_{O'} + \bar{\alpha} \times \bar{r}_{C/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{C/O'}) \\ &= -L_1\Omega^2 \left(\cos\theta \bar{i}' + \sin\theta \bar{k}'\right) + \left[\ddot{\theta}\, \bar{j}' + \dot{\theta}\Omega \left(\cos\theta \bar{i} + \sin\theta \bar{k}'\right)\right] \times L_2 \bar{i}' \\ &\quad + \left[\Omega \left(\sin\theta \bar{i}' - \cos\theta \bar{k}'\right) + \dot{\theta}\, \bar{j}'\right] \times \left(-L_2\Omega \cos\theta\, \bar{j}' - L_2\dot{\theta}\bar{k}'\right) \\ &= \left[-\left(L_1 + L_2\cos\theta\right)\Omega^2 \cos\theta - L_2\dot{\theta}^2\right]\bar{i}' + 2L_2\Omega\dot{\theta}\sin\theta\, \bar{j}' \\ &\quad + \left[-\left(L_1 + L_2\cos\theta\right)\Omega^2 \sin\theta - L_2\ddot{\theta}\right]\bar{k}'. \qquad \triangleleft\end{aligned}$$

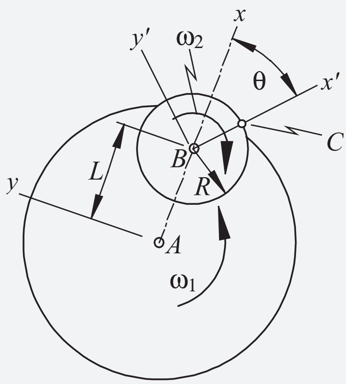
Different global coordinate systems were used to represent each set of results, so the components of $\bar{v}_C$ and $\bar{a}_C$ are not identical. One check that the vectors are consistent lies in the fact that the y' axis for the second analysis coincides with the $\hat{y}$ axis for the first. Correspondingly, we see that those velocity and acceleration components match. Another way to verify that both sets of results represent the same vectors is to apply the rotation transformation between $x'y'z'$ and $\hat{x}\hat{y}\hat{z}$ to transform one set of components to the other. A third approach is to sketch the vector resultant of similar terms. For example, the $(L_1 + L_2\cos\theta)\,\Omega^2$ terms obtained in either analysis represent a centripetal acceleration that is perpendicular to the vertical axis directed from point B to point A.

EXAMPLE 3.12 The turntable rotates at angular speed ω_1, and the disk rotates at angular speed ω_2 relative to the turntable. Both rates are constant. Determine the velocity and acceleration of point C on the perimeter of the disk using (a) a moving coordinate system that is attached to the turntable; (b) a moving coordinate system that is attached to the disk.



Example 3.12

SOLUTION As in the previous example, the intent here is to illustrate the decisions and trade-offs involved in using various moving reference frames. In the first analysis, we place the origin of xyz at the center of rotation of the turntable, point A. The reference frame for the second analysis is $x'y'z'$. We place its origin at the pivot point B, because this is the only point on the disk that follows a simple path. The system is in planar motion, so we orient the z and z' axes normal to the plane. We use xyz as the global directions for both analyses to track the differences between the two solutions. The alignments depicted in the sketch are consistent with the manner in which the system is described.



Analysis of the motion of point C using coordinate systems xyz attached to the turntable and $x'y'z'$ attached to the disk.

When xyz is attached to the turntable, it rotates at angular speed ω_1 about the z axis. This direction is constant, as is the rotation rate, so

$$\bar{\omega} = \omega_1 \bar{k}, \quad \bar{\alpha} = \bar{0}.$$

The origin is stationary, so $\bar{v}_A = \bar{a}_A = \bar{0}$. We could use polar coordinates to formulate the relative velocity and acceleration, but we employ the relative motion equations for this task as a way of emphasizing their utility. We visualize the motion that would remain if the turntable were stationary. The disk would then solely rotate at angular speed ω_2 about point B, which would be stationary. Points B and C have fixed positions when viewed from the disk, so we may employ Eqs. (3.5.6) with the angular motion being that of the disk in the relative motion, $\bar{\omega}_{\text{rel}} = \omega_2 \left(-\bar{k}\right)$, $\bar{\alpha}_{\text{rel}} = \dot{\omega}_2 \left(-\bar{k}\right) = \bar{0}$. Correspondingly, we have

$$(\bar{v}_C)_{xyz} = \bar{\omega}_{\text{rel}} \times \bar{r}_{C/B}, \quad (\bar{a}_C)_{xyz} = \bar{\omega}_{\text{rel}} \times (\bar{\omega}_{\text{rel}} \times \bar{r}_{C/B}).$$

The nonzero terms in Eqs. (3.5.4) for the motion of point C are

$$\bar{v}_C = (\bar{v}_C)_{xyz} + \bar{\omega} \times \bar{r}_{C/A},$$

$$\bar{a}_C = (\bar{a}_C)_{xyz} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{C/A}) + 2\bar{\omega} \times (\bar{v}_C)_{xyz}.$$

The xyz coordinate axes serve as a convenient global system. Evaluation of the velocity proceeds as follows:

$$(\bar{v}_C)_{xyz} = \left(-\omega_2 \bar{k}\right) \times \left(R\cos\theta\,\bar{i} - R\sin\theta\,\bar{j}\right) = \omega_2 R\left(-\sin\theta\,\bar{i} - \cos\theta\,\bar{j}\right),$$

$$\bar{v}_C = \omega_2 R\left(-\sin\theta\,\bar{i} - \cos\theta\,\bar{j}\right) + \left(\omega_1 \bar{k}\right) \times \left[(L + R\cos\theta)\,\bar{i} - R\sin\theta\,\bar{j}\right]$$

$$= (\omega_1 - \omega_2)\, R\sin\theta\,\bar{i} + [(\omega_1 - \omega_2)\, R\cos\theta + \omega_1 L]\,\bar{j}. \quad \triangleleft$$

The corresponding evaluation of acceleration gives

$$
\begin{aligned}
(\bar{a}_C)_{xyz} &= \left(-\omega_2\bar{k}\right) \times \left(-\omega_2 R\sin\theta\bar{i} - \omega_2 R\cos\theta\,\bar{j}\right) = \omega_2^2 R\left(-\cos\theta\bar{i} + \sin\theta\,\bar{j}\right), \\
\bar{a}_c &= \omega_2^2 R\left(-\cos\theta\bar{i} + \sin\theta\,\bar{j}\right) + \omega_1\bar{k} \times \left[\omega_1 R\sin\theta\bar{i} + (\omega_1 R\cos\theta + \omega_1 L)\,\bar{j}\right] \\
&\quad + 2\omega_1\bar{k} \times \omega_2 R\left(-\sin\theta\bar{i} - \cos\theta\,\bar{j}\right) \\
&= \left[-\left(\omega_1^2 + \omega_2^2\right) R\cos\theta - \omega_1^2 L + 2\omega_1\omega_2 R\cos\theta\right]\bar{i} \\
&\quad + \left[\left(\omega_1^2 + \omega_2^2\right) R\sin\theta - 2\omega_1\omega_2 R\sin\theta\right]\bar{j}. \qquad \triangleleft
\end{aligned}
$$

The second analysis uses $x'y'z'$ attached to the disk as the moving reference frame. A key aspect of this selection is recognizing that ω_2 is measured relative to the turntable, so the angular velocity of the disk is the vector sum of the two rotations, $\bar{\omega} = (\omega_1 - \omega_2)\,\bar{k}$. The rotation rates are constant, as is $\bar{k}$, so $\bar{\alpha} = \bar{0}$. Fixing reference frame $x'y'z'$ to the disk eliminates the relative velocity and relative acceleration, $(\bar{v}_C)_{x'y'z'} = (\bar{a}_C)_{x'y'z'} = \bar{0}$. This is balanced by the need to evaluate the velocity and acceleration of the origin B. We find these quantities by recognizing that points A and B are two points in the turntable, so me may employ Eqs. (3.5.6), with $\omega_1\bar{k}$ as the rotational velocity of the turntable, and $\bar{v}_A = \bar{a}_A = \bar{0}$. Thus,

$$
\bar{v}_B = \omega_1\bar{k} \times \bar{r}_{B/A}, \quad \bar{a}_B = \omega_1\bar{k} \times \left(\omega_1\bar{k} \times \bar{r}_{B/A}\right).
$$

The nonzero terms in the relative motion equations (3.5.4) for point C are

$$
\bar{v}_C = \bar{v}_B + \bar{\omega} \times \bar{r}_{C/B}, \quad \bar{a}_C = \bar{a}_B + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{C/B}).
$$

As mentioned earlier, we use xyz as the global coordinate system. Evaluation of the relations for velocity yields

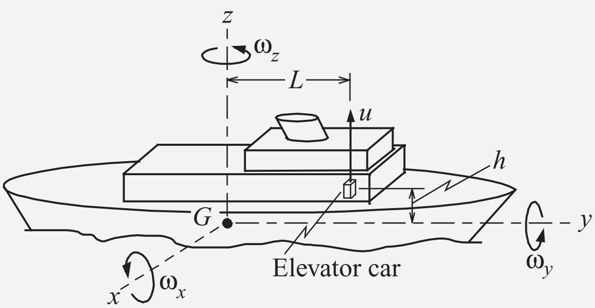
$$
\begin{aligned}
\bar{v}_B &= \omega_1\bar{k} \times L\bar{i} = \omega_1 L\bar{j}, \\
\bar{v}_C &= \omega_1 L\bar{j} + \omega_1\bar{k} \times \left(R\cos\theta\bar{i} - R\sin\theta\,\bar{j}\right) \\
&= (\omega_1 - \omega_2)\, R\sin\theta\bar{i} + \left[\omega_1 L + (\omega_1 - \omega_2)\, R\cos\theta\right]\bar{j}. \qquad \triangleleft
\end{aligned}
$$

Evaluation of the acceleration terms leads to

$$
\begin{aligned}
\bar{a}_B &= \omega_1\bar{k} \times \left(\omega_1\bar{k} \times L\bar{i}\right) = -\omega_1^2 L\bar{i}, \\
\bar{a}_C &= -\omega_1^2 L\bar{i} + (\omega_1 - \omega_2)\,\bar{k} \times \left[(\omega_1 - \omega_2)\, R\sin\theta\bar{i} + (\omega_1 - \omega_2)\, R\cos\theta\,\bar{j}\right] \\
&= \left[-\omega_1^2 L - (\omega_1 - \omega_2)^2\, R\cos\theta\right]\bar{i} + (\omega_1 - \omega_2)^2\, R\sin\theta\,\bar{j}. \qquad \triangleleft
\end{aligned}
$$

Because the global system is the same for both analyses, the resulting component representations should be the same, as they are. Overall, the second analysis is somewhat easier, but it requires recognizing that there are two contributions to the angular velocity of the disk.

EXAMPLE 3.13 Let ω_x, ω_y, and ω_z denote the pitch, roll, and yaw rates, respectively, of a ship about xyz axes that are attached to the ship with the orientations shown. All of these rotation rates are variable quantities. The origin of xyz coincides with the center of mass G of the ship. Consider an elevator car whose path perpendicularly intersects the centerline at a distance L forward from the center of mass. Let $h(t)$ denote the height of the car above the centerline. The velocity and acceleration of the center of mass at this instant are $\bar{v}_G$ and $\bar{a}_G$. Determine the corresponding velocity and acceleration of the car.



Example 3.13

SOLUTION This example brings to the fore an important feature of a body-fixed reference frame whose rotation about its own axes is known. We use xyz as the moving reference frame, as well as the global coordinate system. The given rotations are about body-fixed axes, so we have $\bar{e}_1 = \bar{i}, \bar{e}_2 = \bar{j}, \bar{e}_2 = \bar{k}$ corresponding to the rates ω_x, ω_y, and ω_z, respectively. Thus, the angular velocity of the ship is

$$\bar{\omega} = \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k}.$$

Because the rotation directions are the unit vectors of xyz, their angular velocity is $\bar{\omega}$, that is, $\bar{\Omega}_1 = \bar{\Omega}_2 = \bar{\Omega}_3 = \bar{\omega}$. The general description of the angular acceleration corresponding to variable rotation rates therefore is

$$\begin{aligned}
\bar{\alpha} &= \dot{\omega}_x \bar{i} + \omega_x \left(\bar{\omega} \times \bar{i}\right) + \dot{\omega}_y \bar{j} + \omega_y \left(\bar{\omega} \times \bar{j}\right) + \dot{\omega}_z \bar{k} + \omega_z \left(\bar{\omega} \times \bar{k}\right) \\
&= \dot{\omega}_x \bar{i} + \dot{\omega}_y \bar{j} + \dot{\omega}_z \bar{k} + \bar{\omega} \times \left(\omega_x \bar{i}\right) + \bar{\omega} \times \left(\omega_y \bar{j}\right) + \bar{\omega} \times \left(\omega_z \bar{k}\right) \\
&= \dot{\omega}_x \bar{i} + \dot{\omega}_y \bar{j} + \dot{\omega}_z \bar{k} + \bar{\omega} \times \bar{\omega} = \dot{\omega}_x \bar{i} + \dot{\omega}_y \bar{j} + \dot{\omega}_z \bar{k}.
\end{aligned}$$

This result for $\bar{\alpha}$ indicates that the angular acceleration components are always the time derivatives of the angular velocity components, provided that those components are relative to body-fixed axes. This observation is a key aspect to the development of kinetics principles in Chapter 5.

The elevator follows a straight path relative to the ship, so the relative motion is

$$\bar{r}_{P/G} = L\bar{i} + h\bar{k}, \quad (\bar{v}_P)_{xyz} = \dot{h}\bar{k} = u\bar{k}, \quad (\bar{a}_P)_{xyz} = \dot{u}\bar{k}.$$

We have described all the terms in Eqs. (3.5.4), so we find the velocity to be

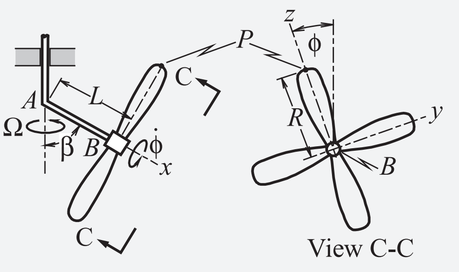
$$\begin{aligned}\bar{v}_P &= \bar{v}_G + (\bar{v}_P)_{xyz} + \bar{\omega} \times \bar{r}_{P/G} \\ &= \bar{v}_G + (\omega_y h - \omega_z L)\,\bar{i} - \omega_x h\,\bar{j} + (u + \omega_x L)\,\bar{k}, \qquad \triangleleft\end{aligned}$$

and the acceleration is

$$\begin{aligned}\bar{a}_P &= \bar{a}_G + (\bar{a}_P)_{xyz} + \bar{\alpha} \times \bar{r}_{P/G} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/G}) + 2\bar{\omega} \times (\bar{v}_P)_{xyz} \\ &= \bar{a}_G + \dot{u}\bar{k} + \left(\dot{\omega}_x\bar{i} + \dot{\omega}_y\bar{j} + \dot{\omega}_z\bar{k}\right) \times \left(L\bar{j} + h\bar{k}\right) + \left(\omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k}\right) \\ &\quad \times \left[(\omega_y h - \omega_z L)\,\bar{i} - \omega_x h\,\bar{j} + \omega_x L\bar{k}\right] + 2\left(\omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k}\right) \times u\bar{k} \\ &= \bar{a}_G + [\dot{\omega}_y h - \dot{\omega}_z L + \omega_x\omega_y L + \omega_x\omega_z h + 2\omega_y u]\,\bar{i} + [-\dot{\omega}_x h + \omega_y\omega_z h \\ &\quad - \left(\omega_x^2 + \omega_z^2\right) L - 2\omega_x u]\,\bar{j} + \left[\dot{u} + \dot{\omega}_x L - \left(\omega_x^2 + \omega_y^2\right) h + \omega_y\omega_z L\right]\bar{k}. \qquad \triangleleft\end{aligned}$$

Some of the terms in the acceleration were foreseeable. The acceleration of the elevator relative to the ship is represented by the $\dot{u}$ term, and the angular acceleration effects are contained in the $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z$ terms. In the same vein, the ω_x^2, ω_y^2, and ω_z^2 terms represent centripetal accelerations about the respective axes associated with each rotation being the only one present. The terms that are not intuitive are those containing products of rotation rates about different axes, as well as the Coriolis acceleration terms.

EXAMPLE 3.14 The cooling fan consists of a shaft that rotates about the vertical axis at angular speed Ω as the blades rotate around the shaft at angular rate $\dot{\phi}$, where ϕ is the angle of rotation of one of the blades from the top-center position. Both rotation rates are constant. Derive expressions for the velocity and acceleration of the blade tip P in terms of components relative to the body-fixed xyz reference system.



Example 3.14

SOLUTION This example synthesizes many of the concepts developed in this chapter, including rotation transformations. The problem statement requires that xyz be used as the global coordinate system. We also use it as the reference frame for the relative motion, as a precursor for the kinetics principles in Chapter 5, which require the usage of body-fixed axes. The rotation of xyz is the sum of the rotation Ω

about the vertical axis and the rotation $\dot{\phi}$ about shaft AB. To define the first rotation, we attach reference frame $x_1y_1z_1$ to the vertical shaft, with the z_1 axis vertically upward. The only meaningful horizontal direction is the one that lies in the vertical plane containing shaft AB, so we define the x_1 axis to lie in that plane, to the right for the given diagram. Then y_1 is inward relative to the plane of the diagram. The x axis always coincides with shaft AB, so we set $\bar{e}_2 = \bar{i}$. Thus general descriptions of the angular velocities of the reference frames are

$$\bar{\omega} = \Omega\bar{k}_1 + \dot{\phi}\bar{i}, \quad \bar{\Omega}_1 = \Omega\bar{k}_1, \quad \bar{\Omega}_2 = \bar{\omega}. \tag{1}$$

The rotation rates are constant, so the angular acceleration of xyz is

$$\bar{\alpha} = \Omega\left(\bar{\Omega}_1 \times \bar{k}_1\right) + \dot{\phi}\left(\bar{\Omega}_2 \times \bar{i}\right) = \dot{\phi}\left(\bar{\omega} \times \bar{i}\right). \tag{2}$$

Several approaches for expressing $\bar{k}_1$ in terms of global xyz components are available; we shall evaluate the rotation transformation from $x_1y_1z_1$ to xyz. The transformation may be visualized as the result of a pair of body-fixed rotations. The first is a rotation of $\pi/2 - \beta$ about the y' axis. This transforms $x_1y_1z_1$ to $x'y'z'$, where the x' axis is aligned with the x axis and the z' axis is the upward reference line in view C-C. A rotation by angle ϕ about the x' axis moves z' into alignment with the z axis without disturbing the x' axis. Thus the transformation is

$$\begin{Bmatrix}\bar{i} & \bar{j} & \bar{k}\end{Bmatrix}^{\mathrm{T}} = [R]\begin{Bmatrix}\bar{i}_1 & \bar{j}_1 & \bar{k}_1\end{Bmatrix}^{\mathrm{T}},$$

$$[R] = [R_x][R_y] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{2} - \beta\right) & 0 & -\sin\left(\frac{\pi}{2} - \beta\right) \\ 0 & 1 & 0 \\ \sin\left(\frac{\pi}{2} - \beta\right) & 0 & \cos\left(\frac{\pi}{2} - \beta\right) \end{bmatrix}$$

$$= \begin{bmatrix} \sin\beta & 0 & -\cos\beta \\ \sin\phi\cos\beta & \cos\phi & \sin\phi\sin\beta \\ \cos\phi\cos\beta & -\sin\phi & \cos\phi\sin\beta \end{bmatrix}.$$

The inverse transformation gives the unit vectors of $x_1y_1z_1$ in terms of the xyz unit vectors, so the last row of $[R]^{\mathrm{T}}$, which is the last column of $[R]$, gives the global components of the rotation direction $\bar{e}_1 = \bar{k}_1$,

$$\bar{k}_1 = -\cos\beta\bar{i} + \sin\phi\sin\beta\bar{j} + \cos\phi\sin\beta\bar{k}. \tag{3}$$

This enables us to evaluate the global components of $\bar{\omega}$ and $\bar{\alpha}$:

$$\bar{\omega} = \left(\dot{\phi} - \Omega\cos\beta\right)\bar{i} + \Omega\sin\phi\sin\beta\bar{j} + \Omega\cos\phi\sin\beta\bar{k}, \tag{4a}$$

$$\begin{aligned}\bar{\alpha} &= \left[\left(\dot{\phi} - \Omega\cos\beta\right)\bar{i} + \Omega\sin\phi\sin\beta\bar{j} + \Omega\cos\phi\sin\beta\bar{k}\right] \times \bar{i} \\ &= \Omega\dot{\phi}\left(\cos\phi\sin\beta\bar{j} - \sin\phi\sin\beta\bar{k}\right).\end{aligned} \tag{4b}$$

We may construct the velocity and acceleration of origin B by recognizing that this point follows a circular path in the horizontal plane. However, using an alternative approach based on its being a point on shaft AB is more suitable to expressing the results in terms of components relative to xyz. The angular velocity of shaft AB is $\bar{\Omega}_1$, which is constant, and point A is stationary, so we have

$$\bar{v}_B = \Omega\bar{k}_1 \times \bar{r}_{B/A}, \quad \bar{a}_B = \Omega\bar{k}_1 \times \left(\Omega\bar{k}_1 \times \bar{r}_{B/A}\right).$$

The global components of $\bar{k}_1$ are given by Eq. (3), whose substitution into the preceding yields

$$\begin{aligned} \bar{v}_B &= \Omega\left(-\cos\beta\bar{i} + \sin\phi\sin\beta\bar{j} + \cos\phi\sin\beta\bar{k}\right) \times L\bar{i} \\ &= \Omega L\left(\cos\phi\sin\beta\bar{j} - \sin\phi\sin\beta\bar{k}\right), \end{aligned} \tag{5a}$$

$$\begin{aligned} \bar{a}_B &= \Omega\left(-\cos\beta\bar{i} + \sin\phi\sin\beta\bar{j} + \cos\phi\sin\beta\bar{k}\right) \times \Omega L\left(\cos\phi\sin\beta\bar{j} - \sin\phi\sin\beta\bar{k}\right) \\ &= \Omega^2 L\left[-(\sin\beta)^2\bar{i} - \sin\phi\sin\beta\cos\beta\bar{j} - \cos\phi\sin\beta\cos\beta\bar{k}\right] \end{aligned} \tag{5b}$$

Because xyz is attached to the propeller, there is no relative velocity or acceleration. Hence $\bar{v}_P$ and $\bar{a}_P$ also are described by Eqs. (3.5.6), which indicate that

$$\bar{v}_P = \bar{v}_B + \bar{\omega} \times \bar{r}_{P/B}, \quad \bar{a}_P = \bar{a}_B + \bar{\alpha} \times \bar{r}_{P/B} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/B}).$$

Substitution of the global descriptions in Eqs. (4) and (5) yields

$$\begin{aligned} \bar{v}_P &= \Omega L\left(\cos\phi\sin\beta\bar{j} - \sin\phi\sin\beta\bar{k}\right) + [(\dot{\phi} - \Omega\cos\beta)\bar{i} + \Omega\sin\phi\sin\beta\bar{j} \\ &\quad + \Omega\cos\phi\sin\beta\bar{k}] \times R\bar{k} \qquad \triangleleft \\ &= \Omega\left[R\sin\phi\sin\beta\bar{i} + (L\cos\phi\sin\beta + R\cos\beta)\bar{j} - L\sin\phi\sin\beta\bar{k}\right]. \end{aligned}$$

The corresponding evaluation of acceleration gives

$$\begin{aligned} \bar{a}_P &= \Omega^2 L\left[-(\sin\beta)^2\bar{i} - \sin\phi\sin\beta\cos\beta\bar{j} - \cos\phi\sin\beta\cos\beta\bar{k}\right] \\ &\quad + \Omega\dot{\phi}\left(\cos\phi\sin\beta\bar{j} - \sin\phi\sin\beta\bar{k}\right) \times R\bar{k} \\ &\quad + [(\dot{\phi} - \Omega\cos\beta)\bar{i} + \Omega\sin\phi\sin\beta\bar{j} + \Omega\cos\phi\sin\beta\bar{k}] \\ &\quad \times \Omega R\left(\sin\phi\sin\beta\bar{i} + \cos\beta\bar{j}\right) \\ &= \Omega^2\left\{-\sin\beta\,(L\sin\beta + R\cos\phi\cos\beta)\bar{i} + \sin\phi\sin\beta\,(-L\cos\beta + R\cos\phi\sin\beta)\bar{j}\right. \\ &\quad \left. + \left[-L\cos\phi\sin\beta\cos\beta - R(\cos\beta)^2 - R(\sin\phi)^2(\sin\beta)^2\right]\bar{k}\right\} - \dot{\phi}^2 R\bar{k} \\ &\quad + 2\Omega\dot{\phi}R\left(\cos\phi\sin\beta\bar{i} + \cos\beta\bar{k}\right). \qquad \triangleleft \end{aligned}$$

These results are displayed in groups of like coefficients of the rate variables to highlight that the underlying physical phenomena are a superposition of effects. The terms in $\bar{v}_P$ that contain Ω and those in $\bar{a}_P$ that contain Ω^2 represent the motion that would be present if ϕ were constant, giving $\dot{\phi} \equiv 0$. In that case, point P would follow a circular path in the horizontal plane. Similarly, the terms in $\bar{v}_P$ that contain $\dot{\phi}$ and those in $\bar{a}_P$ that contain $\dot{\phi}^2$ correspond to a pure rotation about the x axis, in which case point P would follow a circular path of radius R. The effect that has no obvious description as a superposition is represented by the acceleration term that has the factor $2\Omega\dot{\phi}$. The two factor identifies it as a Coriolis acceleration that exists only if both rotation rates are nonzero. This physical Coriolis acceleration arises, even though the Coriolis acceleration term $2\bar{\omega} \times (\bar{v}_P)_{xyz}$ in relative acceleration equations (3.5.4) was identically zero in our analysis.

3.6 OBSERVATIONS FROM A MOVING REFERENCE FRAME

Thus far, our concern has been with situations in which the motion of some point could be more readily described in terms of a moving reference frame. However, sometimes the absolute motion is known, and the relative motion must be evaluated. For example, it might be necessary to ensure that one part of a machine merge with another part in a smooth manner, as in the case of gears. The influence of the Earth's motion on the dynamic behavior of a system is an important situation in which aspects of the absolute motion are known.

One approach is to interchange the absolute and relative reference frames, based on the fact that the kinematical relationships do not actually require that one of the reference frames be stationary. Thus in this viewpoint, if the angular velocity of xyz relative to XYZ is $\bar{\omega}$, then the angular velocity of XYZ as viewed from xyz is $-\bar{\omega}$. The difficulty with this approach is that it is prone to errors, particularly in signs, because of the need to change the observer's viewpoint for the formulation. The more reliable approach, which does not require redefinition of the basic quantities, manipulates the earlier relations.

The concept is quite straightforward. When the absolute velocity $\bar{v}_P$ and the absolute acceleration $\bar{a}_P$ are known, Eqs. (3.5.4) may be solved for the relative motion parameters. Specifically,

$$\boxed{\begin{aligned}(\bar{v}_P)_{xyz} &= \bar{v}_P - \bar{v}_{O'} - \bar{\omega} \times \bar{r}_{P/O'},\\ (\bar{a}_P)_{xyz} &= \bar{a}_P - \bar{a}_{O'} - \bar{\alpha} \times \bar{r}_{P/O'} - \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'}) - 2\bar{\omega} \times (\bar{v}_P)_{xyz}.\end{aligned}} \tag{3.6.1}$$

If it is appropriate, the relative velocity may be removed from the acceleration relation by substitution of the velocity relation. The result is

$$\boxed{(\bar{a}_P)_{xyz} = \bar{a}_P - \bar{a}_{O'} - \bar{\alpha} \times \bar{r}_{P/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/O'}) - 2\bar{\omega} \times (\bar{v}_P - \bar{v}_{O'}).}$$

The steps required for applying these relations are like those already established, because $\bar{\omega}$ and $\bar{\alpha}$ still describe the rotation of xyz relative to a fixed reference frame.

These relations are particularly useful when the rotation of the Earth must be considered. Newton's Second Law relates the forces acting on a particle to the acceleration relative to some hypothetical inertial reference frame. However, we commonly observe the motion with Earth-based instruments. We use the relative motion relations to reconcile Newton's Law with the motion we observe. Consider an observer at point O' on the Earth's surface. A natural definition for the reference frame employed by this observer is east–west and north–south for position along the surface and vertical for measurements off the surface. Such a reference frame is depicted in Fig. 3.12, where the x axis is northward and the y axis is westward. The observation point O' in the figure is located by the latitude angle λ measured from the equator and the longitude angle ϕ measured from some reference location, such as the prime meridian (the longitude of the Royal Observatory at Greenwich, England).

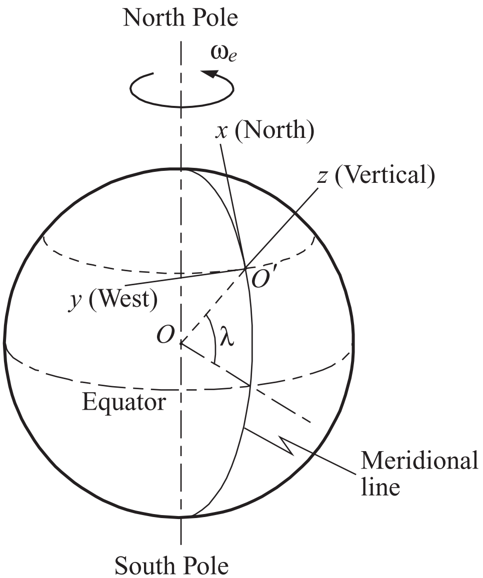


Figure 3.12. The Earth as a reference frame.

For the present purposes, it is adequate to employ an approximate model of the Earth. The Earth spins at one revolution about its polar axis in 23 h, 56 min, 4.06 s, which converts to $\omega_e = 2\pi$ rad/23.934 h $= 7.292(10^{-5})$ rad/s. For comparison, the orbital rate of rotation of the Earth about the Sun, ω_O, is smaller by an approximate factor of 365, because one such revolution requires a full year. To assess the relative importance of the two, let us consider the associated centripetal acceleration. The maximum that is due to the Earth's rotation occurs at the equator, where the distance from the polar axis is the Earth's radius, $R_e = 6370$ km. The centripetal acceleration at this location associated with the Earth's rotation is $\omega_e^2 R_e \approx 0.034$ m/s^2. The mean radius of the Earth's orbit is $R_O = 149.6(10^6)$ km, so the associated centripetal acceleration is $\omega_O^2 R_O \approx 0.0059$ m/s^2, which is 17.5% of the acceleration that is due to the Earth's spin, which is itself quite feeble in comparison with the free-fall acceleration g. Furthermore, the centripetal acceleration associated with the Earth's orbital motion is essentially balanced by the effect of the Sun's gravitational attraction because that balance produces the orbit. For these reasons, it is reasonable to consider the center of the Earth to be stationary and to ignore the Sun's gravitational attraction. If we also ignore the relatively minor wobble of the polar axis, our model of the Earth reduces to a sphere that rotates about the (fixed) polar axis at the constant rate ω_e.

Newton's Second Law gives the acceleration of a particle relative to a fixed reference frame corresponding to the resultant force. We decompose this force into two parts: $\bar{F}_g$ represents the gravitational attraction of the Earth, and $\bar{F}_a$ consists of all other forces, including applied loads and reactions. The angular acceleration of the Earth in our simple model is zero. Thus application of Eqs. (3.6.1) leads to a relation for the acceleration relative to the Earth-based xyz reference frame corresponding to a specified set of forces acting on a particle:

$$(\bar{a}_P)_{xyz} = \frac{\bar{F}_a + \bar{F}_g}{m} - \bar{a}_{O'} - \bar{\omega}_e \times (\bar{\omega}_e \times \bar{r}_{P/O'}) - 2\bar{\omega}_e \times (\bar{v}_P)_{xyz}. \tag{3.6.2}$$

Now consider a particle in free fall near the Earth's surface. Let point O' be close to the particle, so that $\bar{r}_{P/O'} \approx \bar{0}$. If air resistance is negligible, there are no applied forces, $\bar{F}_a \approx \bar{0}$. The definition of g is that it is the magnitude of the free-fall acceleration observed from the Earth. Furthermore, we interpret the direction of the free-fall acceleration as being vertically downward. Recall that the definition of the z axis in Fig. 3.12 was that it is the upward vertical, which now means that the observed free-fall acceleration is $(\bar{a}_P)_{xyz} = -g\bar{k}$. The magnitude of the actual gravitational force is given by the inverse square law, and this force is directed oppositely to the radial line from the center of the Earth to the particle. We use the position vector $\bar{r}_{O'/O}$ to construct the radial unit vector, so the gravitational force is described by

$$\bar{F}_g = \frac{GM_e m}{r_e^2}\left(-\frac{\bar{r}_{O'/O}}{|\bar{r}_{O'/O}|}\right) = -\frac{GM_e m}{r_e^3}\bar{r}_{O'/O}. \tag{3.6.3}$$

The origin O' follows a circular path parallel to the equatorial plane. The radius of this path is $R_e \cos\lambda$, and the rotation rate of a radial line is ω_e. The corresponding centripetal acceleration is $\bar{a}_{O'} = r_e\omega_e^2 (\cos\lambda)(-\bar{e}_\perp)$, where $\bar{e}_\perp$ is the radial unit vector perpendicular to the polar axis intersecting point O'. Unless the free fall occurs over a long time, the velocity relative to the Earth is small, which makes it permissible to neglect Coriolis acceleration effects. In this case Eq. (3.6.2) reduces to

$$\boxed{g\bar{k} = \frac{GM_e}{r_e^3}\bar{r}_{O'/O} - r_e\omega_e^2 (\cos\lambda)\,\bar{e}_\perp.} \tag{3.6.4}$$

There are two primary aspects of interest in this relation. The centripetal acceleration term is parallel to $\bar{r}_{O'/O}$ only at the equator, $\lambda = 0$, and at the poles, $\lambda = 0,\ \pi$, where it is zero. Hence, the vertical direction defined by $\bar{k}$ does not generally intersect the center of the Earth. However, it does lie in the meridional plane because that plane contains both $\bar{r}_{O'/O}$ and $\bar{e}_\perp$. Equally significant is the effect of the centripetal acceleration on the value of g that is obtained from measurements. This effect is largest at the equator, where $\bar{e}_\perp$ is parallel to $\bar{r}_{O'/O}$ and $\cos\lambda = 1$.

We may quantify both effects by resolving $\bar{r}_{O'/O}$ and $\bar{e}_\perp$ into components parallel and perpendicular to $\bar{k}$. Let the angle from $\bar{k}$ to $\bar{r}_{O'/O}$ be β, such that the angle from $\bar{k}$ to the equatorial plane is $\lambda + \beta$. Because all vectors are situated in the same meridional plane, this resolution is

$$\bar{r}_{O'/O} = r_e\left(\cos\beta\bar{k} - \sin\beta\bar{i}\right), \quad \bar{e}_\perp = \cos(\lambda+\beta)\,\bar{k} - \sin(\lambda+\beta)\,\bar{i}. \tag{3.6.5}$$

Substitution of these expressions into Eq. (3.6.4) followed by equating like components leads to two equations for the values of g and λ:

$$\begin{aligned} g &= \frac{GM_e}{r_e^2}\cos\beta - r_e\omega_e^2\,(\cos\lambda)\cos(\lambda+\beta)\,, \\ 0 &= -\frac{GM_e}{r_e^2}\sin\beta + r_e\omega_e^2\,(\cos\lambda)\sin(\lambda+\beta)\,. \end{aligned} \tag{3.6.6}$$

It is apparent that β is very small, so we may obtain the value of g by setting $\beta = 0$ in the first of Eqs. (3.6.6):

$$g = \frac{GM_e}{r_e^2} - r_e\omega_e^2\,(\cos\lambda)^2\,. \tag{3.6.7}$$

Thus we see that g decreases from the value associated with gravity at the poles to a minimum that is reduced by $r_e\omega_e^2 = 0.034\ \text{m/s}^2$ at the equator. The value $g = 9.807\ \text{m/s}^2$ represents a reasonable average value when the latitude is not specified. Smallness of β allows us to approximate $\cos(\lambda+\beta) \approx \cos\lambda$ in the second of Eqs. (3.6.6). We also approximate GM_e/r_e^2 as g, which leads to

$$\beta = \sin^{-1}\left(\frac{r_e\omega_e^2\sin(2\lambda)}{2g}\right). \tag{3.6.8}$$

The maximum deviation angle β occurs at a latitude of 45°, where $\beta = 0.099°$.

The definition of $g\bar{k}$, Eq. (3.6.4), allows us to simplify Eq. (3.6.2) slightly. The centripetal acceleration term $\bar{\omega}_e \times (\bar{\omega}_e \times \bar{r}_{P/O'})$ may be neglected, unless the magnitude of $\bar{r}_{P/O'}$ is a large fraction of the Earth's radius. [In fact, if this acceleration term is significant, we should not use Eq. (3.6.4).] Thus we have

$$(\bar{a}_P)_{xyz} = \frac{\bar{F}_a}{m} - g\bar{k} - 2\bar{\omega}_e \times (\bar{v}_P)_{xyz}\,, \tag{3.6.9}$$

which shows that the primary difference between the acceleration we observe from the Earth and the absolute acceleration associated with Newton's Second Law is the Coriolis term. We use the Earth-based xyz coordinate system to describe the preceding. The components of relative velocity and acceleration are respectively the first and second time derivatives of the Cartesian coordinates (x, y, z). Because the angular velocity of the Earth is parallel to the polar axis, and the deviation of the z axis from the line to the center of the Earth is small, the angular velocity is essentially

$$\bar{\omega}_e = \omega_e\left(\cos\lambda\bar{i} + \sin\lambda\bar{k}\right). \tag{3.6.10}$$

Correspondingly, Eq. (3.6.9) becomes

$$\begin{aligned} \ddot{x} - (2\omega_e\sin\lambda)\,\dot{y} &= \frac{\bar{F}_a\cdot\bar{i}}{m}, \\ \ddot{y} + (2\omega_e\sin\lambda)\,\dot{x} - (2\omega_e\cos\lambda)\,\dot{z} &= \frac{\bar{F}_a\cdot\bar{j}}{m}, \\ \ddot{z} + (2\omega_e\cos\lambda)\,\dot{y} &= \frac{\bar{F}_a\cdot\bar{k}}{m} - g. \end{aligned} \tag{3.6.11}$$

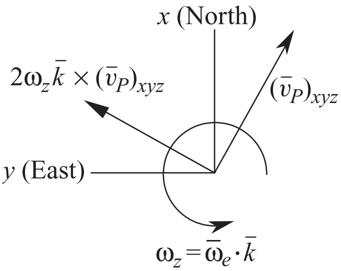


Figure 3.13. Coriolis acceleration associated with motion in the horizontal plane relative to the Earth.

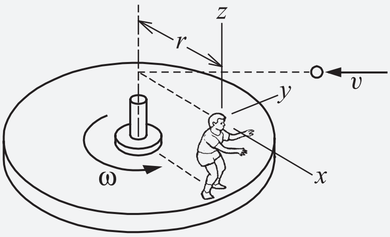
These equations may be solved for the forces required to have a specified motion relative to the Earth. Alternatively, they may be regarded as a set of coupled differential equations for the relative position in situations in which the forces are specified.

The fact that the Coriolis term is perpendicular to the velocity as seen by an observer on the Earth leads to some interesting anomalies. In the Northern Hemisphere, the component of $\bar{\omega}_e$ perpendicular to the Earth's surface is outward. If a particle is constrained to follow a horizontal path relative to the Earth in the Northern Hemisphere, the Coriolis term $2\bar{\omega}_e \times (\bar{v}_P)_{xyz}$ is as shown in Fig. 3.13. It follows that a horizontal force pushing to the left of the direction of motion is required if that direction is to be maintained.

A story that has been passed down from professor to student over the years, without substantiation, states that a railroad line had two sets of north–south tracks along which trains ran in only one direction. For the track along which trains ran northward, the inner surface of the east rail was supposedly more shiny, because of the westward Coriolis force it had to exert on the flange of the wheels. Correspondingly, the track for trains running south was more shiny on the inner surface of the west rail. The veracity of this statement is questionable, owing to the smallness of the force in comparison with other effects, such as the wind and elevation changes.

If a transverse force is not present to maintain a particle in a straight path relative to the Earth, as required by Eqs. (3.65), then the particle will deviate to the right. This observation leads to a qualitative explanation of the swirling of a liquid that is drained through the centered hole of a perfectly symmetrical cylindrical tank. As the fluid rushes to the hole, the tendency in the Northern Hemisphere to deviate to the right along any inward radial line induces a counterclockwise spiraling flow when viewed from above. (The flow will be clockwise in the Southern Hemisphere.) The same phenomenon acts on a much larger scale to set up the flow patterns in hurricanes and typhoons. Goldstein (1980) offers an excellent discussion of these effects. Meteorological models used to predict general weather patterns must account for the Coriolis acceleration effect.

EXAMPLE 3.15 Leah is standing stationary on a turntable rotating about the vertical axis at the constant rate ω. A ball traveling in the radial direction has constant speed v horizontally. It is timed to arrive at Leah's position so that she may catch it. Let xyz be Leah's reference frame with the x axis radially outward, as shown. Determine the horizontal components of position, velocity, and acceleration of the ball as seen by Leah as a function of time. Perform the analysis by (a) constructing the relative position vector geometrically, (b) using the relative motion formulas.



Example 3.15

SOLUTION This problem illustrates the application of the basic relations, as well as providing a further demonstration that there often is more than one way to perform a kinematical analysis. For both analyses we let $t = 0$ be the instant when Leah catches the ball, so $t < 0$ characterizes an arbitrary instant before the ball is caught. Correspondingly, the angle locating her relative to the path of the ball is $\theta = \omega(-t)$ and the radial distance R to the ball from the center of the turntable is $R = r + v(-t)$.

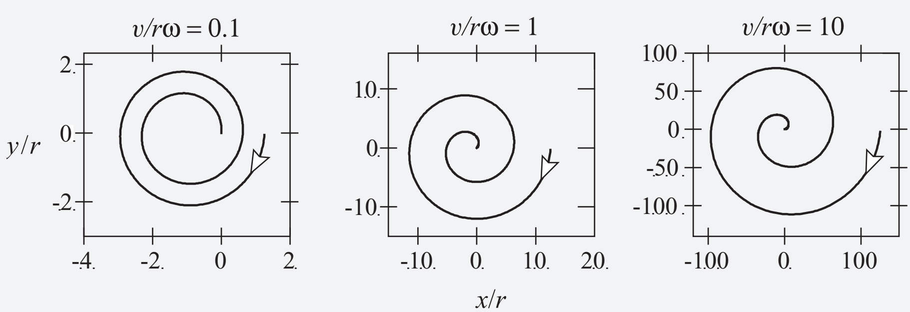
The first analysis involves geometrically constructing the xyz components of the ball's position relative to the child, $\bar{r}_{B/A}$. Projecting the radial line from the center of the turntable to the ball onto the x and y axes gives

$$\begin{aligned}\bar{r}_{B/A} &= x\bar{i} + y\bar{j} = (R\cos\theta - r)\bar{i} + R\sin\theta\,\bar{j} \\ &= [(r - vt)\cos(\omega t) - r]\,\bar{i} - (r - vt)\sin(\omega t)\,\bar{j}.\end{aligned} \quad \triangleleft$$

We may find the path of the ball in the xy plane as seen by Leah by plotting the x and y components at various t. Let $\tau = \omega t$ be a nondimensional time parameter. Then a nondimensional description of the relative path coordinates is

$$\frac{x}{r} = \left(1 - \frac{v}{r\omega}\tau\right)\cos\tau - 1, \quad \frac{y}{r} = -\left(1 - \frac{v}{r\omega}\tau\right)\sin\tau.$$

These expressions show that the relative path, scaled by r, depends solely on the ratio $v/r\omega$. Plots for several values of this ratio for the interval $-4\pi \le \tau \le 0$, corresponding to two revolutions of the turntable, show that the path is an inward spiral.



Path of the ball in the horizontal plane as seen by the child on the turntable

By definition, we may obtain the relative velocity by differentiating the components of the relative position:

$$(\bar{v}_B)_{xyz} \equiv \frac{\partial}{\partial t}\bar{r}_{B/A} = -\left[v\cos(\omega t) - (r - vt)\omega\sin(\omega t)\right]\bar{i} + \left[v\sin(\omega t) - (r - vt)\omega\cos(\omega t)\right]\bar{j}. \quad \triangleleft$$

Similarly, we may obtain the relative acceleration by differentiating the relative velocity components:

$$(\bar{a}_B)_{xyz} \equiv \frac{\partial}{\partial t}(\bar{v}_B)_{xyz} = \left[2v\omega\sin(\omega t) - (r - vt)\omega^2\cos(\omega t)\right]\bar{i} + \left[2v\omega\cos(\omega t) + (r - vt)\omega^2\sin(\omega t)\right]\bar{j}. \quad \triangleleft$$

The solution obtained with the moving reference formulation barely resembles the operations in the previous solution. The position as seen by Leah is

$$\bar{r}_{B/A} = \bar{r}_{B/O} - \bar{r}_{A/O}.$$

We know that the ball is at distance $R = r - vt$ in the fixed radial direction, which we define to be the direction of the stationary X axis. Thus $\bar{r}_{B/O} = (r - vt)\,\bar{I}$. The child is at distance r in the radial direction along which the x axis is aligned, so $\bar{r}_{A/O} = r\bar{i}$. We use these descriptions to form $\bar{r}_{B/A}$ in terms of components relative to the xyz axes, which is Leah's viewpoint. This requires that we express $\bar{r}_{B/O}$ in terms of $\bar{i}\,\bar{j}\bar{k}$ components. A visual inspection shows that $\bar{I} = \cos\theta\bar{i} - \sin\theta\,\bar{j}$, which leads to

$$\bar{r}_{B/A} = (r - vt)\left(\cos\theta\bar{i} + \sin\theta\,\bar{j}\right) - r\bar{i}. \quad \triangleleft$$

Because $\theta = \omega(-t)$, this is the same description as that derived in the first analysis.

To analyze velocity and acceleration, we begin with the angular motion of the xyz reference frame. This is a simple rotation about the vertical axis at constant rate, so $\bar{\omega} = \omega\bar{k}$, $\bar{\alpha} = \bar{0}$. The origin of xyz is point A, which follows a circular path relative to the fixed reference frame. Thus $\bar{v}_A = r\omega\bar{j}$, $\bar{a}_A = -r\omega^2\bar{i}$. We also know that the ball's motion in the horizontal plane is a straight path at constant speed v, so $\bar{v}_B = -v\bar{I}$, $\bar{a}_B = \bar{0}$, where $\bar{I}$ was previously described in terms of $\bar{i}\,\bar{j}\bar{k}$ components. Substitution of these expressions into the first of Eqs. (3.6.1) yields the relative velocity:

$$\begin{aligned}(\bar{v}_B)_{xyz} &= \bar{v}_B - \bar{v}_A - \bar{\omega}\times\bar{r}_{B/A}\\ &= -v\left(\cos\theta\bar{i} + \sin\theta\,\bar{j}\right) - r\omega\bar{j} - \omega\bar{k}\times\left[(r - vt)\left(\cos\theta\bar{i} + \sin\theta\,\bar{j}\right) - r\bar{i}\right],\end{aligned}$$

$$(\bar{v}_B)_{xyz} = \left[-v\cos\theta + \omega(r - vt)\sin\theta\right]\bar{i} + \left[-v\sin\theta - \omega(r - vt)\cos\theta\right]\bar{j}. \quad \triangleleft$$

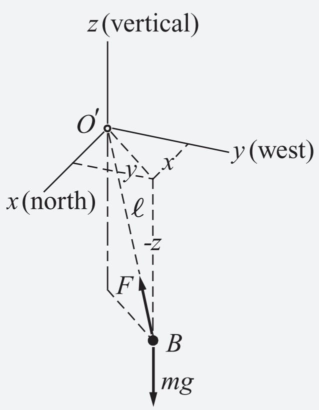
Using this last expression to form the second of Eqs. (3.6.1) gives the relative acceleration:

$$(\bar{a}_B)_{xyz} = \bar{a}_B - \bar{a}_A - \bar{\alpha}\times\bar{r}_{B/A} - \bar{\omega}\times(\bar{\omega}\times\bar{r}_{B/A}) - 2\bar{\omega}\times(\bar{v}_B)_{xyz}$$
$$= -\left(-r\omega^2\bar{i}\right) - \omega\bar{k}\times\left\{\omega\bar{k}\times\left[(r-vt)\left(\cos\theta\,\bar{i}+\sin\theta\,\bar{j}\right)-r\bar{i}\right]\right\}$$
$$-\,2\omega\bar{k}\times\left\{[-v\cos\theta+\omega(r-vt)\sin\theta]\,\bar{i}+[-v\sin\theta-\omega(r-vt)\cos\theta]\,\bar{j}\right\},$$
$$(\bar{a}_B)_{xyz} = \left[-\omega^2(r-vt)\cos\theta - 2\omega v\sin\theta\right]\bar{i} + \left[-\omega^2(r-vt)\sin\theta + 2\omega v\cos\theta\right]\bar{j}. \quad \triangleleft$$

Substitution of $\theta = -\omega t$ into the second set of results would show that they are identical to the results of the first analysis. We could employ either approach with equal ease in this problem because the motion is planar. The relative motion formulation becomes increasingly advantageous as the rotation of the reference frame becomes more complicated.

EXAMPLE 3.16 When a small ball is suspended by a stiff cable from an ideal swivel joint that permits three-dimensional motion, the system is called a spherical pendulum. Suppose such a pendulum, whose cable length is ℓ, is released from rest relative to the Earth with the ball at a distance $b \ll \ell$ north of the point below the pivot. Analyze the effect of the Earth's rotation on the motion. It may be assumed that the angle between the suspending cable and the vertical is always very small.

SOLUTION In addition to demonstrating the solution of the coupled equations of motion, Eqs. (3.6.11), this example explains an interesting phenomenon regarding motion relative to the Earth. A free-body diagram of the ball shows the weight mg and the tensile force $\bar{F}$ exerted by the cable.



Example 3.16

In terms of the xyz coordinates sketch defined in the diagram, we have

$$\bar{F} = F\frac{\bar{r}_{O'/B}}{\left|\bar{r}_{O'/B}\right|} = F\left(-\frac{x}{\ell}\bar{i} - \frac{y}{\ell}\bar{j} + \frac{z}{\ell}\bar{k}\right). \tag{1}$$

When the values of x and y are known, the value of z is given by

$$z = -\left(\ell^2 - x^2 - y^2\right)^{1/2}. \tag{2}$$

Because ℓ is constant, differentiating the preceding gives

$$\dot{z} = \frac{x\dot{x} + y\dot{y}}{\left(\ell^2 - x^2 - y^2\right)^{1/2}}.$$

The specification that the angle with the vertical is small means that $|x| \ll \ell$ and $|y| \ll \ell$. Introducing this approximation above yields

$$\dot{z} \approx \frac{x\dot{x} + y\dot{y}}{\ell}, \quad \ddot{z} \approx \frac{x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2}{\ell} \approx \frac{\dot{x}^2 + \dot{y}^2}{\ell}. \tag{3}$$

The preceding indicates that $\dot{z}$ is much less than $\dot{x}$ and $\dot{y}$. Furthermore, smallness of $\dot{z}$ means that the speed of the ball is essentially $\dot{x}^2 + \dot{y}^2$. This leads to the interpretation of $\ddot{z}$ as the centripetal acceleration associated with motion in a circle of radius ℓ. The result of substituting the preceding descriptions of $\bar{F}$ and $\ddot{z}$ into the general equations of motion, Eqs. (3.6.11), is

$$\ddot{x} - (2\omega_e \sin\lambda)\,\dot{y} = -\frac{F}{m\ell}x, \tag{4}$$

$$\ddot{y} + (2\omega_e \sin\lambda)\,\dot{x} = -\frac{F}{m\ell}y, \tag{5}$$

$$\frac{\dot{x}^2 + \dot{y}^2}{\ell} + (2\omega_e \cos\lambda)\,\dot{y} = \frac{F}{m\ell}\left(\ell^2 - x^2 - y^2\right)^{1/2} - g, \tag{6}$$

where the Coriolis acceleration term containing $\dot{z}$ in Eq. (5) has been dropped because $\dot{z}$ is small compared with $\dot{x}$ and $\dot{y}$.

There are three variables in Eqs. (4)–(6). Elimination of F from these equations leads to a pair of ordinary differential equations governing x and y. We use Eq. (6) for that purpose. To estimate the order of magnitude of the first term in Eq. (6) we recall that $\dot{x}^2 + \dot{y}^2 \approx v^2$. Because the Coriolis effect is quite small, the motion should be essentially like the result for a simple pendulum whose pivot is stationary, so we may employ the principle of conservation of energy to estimate v^2. Let the elevation of the pivot be the datum, so the gravitational potential energy is mgz. At the lowest position, $z = -\ell$ and $v = v_{\max}$, whereas at the highest position, $z = z_{\max}$, and $v = 0$. It follows that

$$\frac{1}{2}mv_{\max}^2 - mg\ell \approx mgz_{\max}. \tag{7}$$

The value of $z_{\max}$ is related to the other coordinates by Eq. (2), to which we apply a binomial series expansion to find

$$z_{\max} = -\left(\ell^2 - x_{\max}^2 - y_{\max}^2\right)^{1/2} \approx -\ell + \frac{x_{\max}^2 + y_{\max}^2}{2\ell}.$$

Substitution of this expression into Eq. (7) gives

$$\frac{v_{\max}^2}{\ell} \approx g \frac{x_{\max}^2 + y_{\max}^2}{\ell^2}.$$

Because x and y are stated to remain small compared with ℓ throughout the motion, it follows that v^2/ℓ is always small compared with g. Accordingly, we may neglect the first term in Eq. (6). Furthermore, the Coriolis acceleration is a small effect. Thus Eq. (6) indicates that the cable tension is well approximated as the ball's weight, $F/m = g$. This approximation reduces Eqs. (4) and (5) to

$$\begin{aligned} \ddot{x} - 2p\dot{y} + \Omega^2 x &= 0, \\ \ddot{y} + 2p\dot{x} + \Omega^2 y &= 0, \end{aligned} \tag{8}$$

where

$$\Omega = \left(\frac{g}{\ell}\right)^{1/2}, \quad p = \omega_e \sin\lambda.$$

Evaluating the motion requires that we solve this pair of linear, coupled, ordinary differential equations. We could use the method of characteristic exponents, but an examination of the equations leads to a much briefer solution. We observe that if the Coriolis effect were not present, $p = 0$, then the equations for x and y would be uncoupled, and the fundamental solutions for both variables would be combinations of $\sin(\Omega t)$ and $\cos(\Omega t)$. Furthermore, in either equation with $p \neq 0$, the order of the derivatives of y is one different from the order of the derivatives of x. The combination of these two features suggests that both x and y vary sinusoidally at some frequency μ, with a 90° phase difference between them. We therefore consider a trial solution whose form is

$$x = A\cos(\mu t + \phi), \quad y = B\sin(\mu t + \phi). \tag{9}$$

We obtain relations for the amplitudes A and B, the frequency μ, and the phase angle ϕ by substituting the trial forms into Eqs. (8), which leads to

$$\begin{aligned} \left(\Omega^2 - \mu^2\right) A - 2p\mu B &= 0, \\ -2p\mu A + \left(\Omega^2 - \mu^2\right) B &= 0. \end{aligned} \tag{10}$$

For A and B to be nonzero, the determinant of this pair of homogeneous equations must vanish. This leads to a characteristic equation:

$$\left(\Omega^2 - \mu^2\right)^2 - (2p\mu)^2 = 0 \implies \Omega^2 - \mu^2 = \pm 2p\mu.$$

This pair of quadratic equations for μ has a total of four roots, but we need only the positive values, which are

$$\mu_1 = \left(\Omega^2 + p^2\right)^{1/2} - p, \quad \mu_2 = \left(\Omega^2 + p^2\right)^{1/2} + p. \tag{11a}$$

In view of the smallness of ω_e, any conceivable value of ℓ will lead to $\Omega \gg p$, so the characteristic roots are well approximated as

$$\mu_1 \approx \Omega - p, \quad \mu_2 \approx \Omega + p. \tag{11b}$$

There is a solution of Eqs. (10) for each characteristic root. Because the determinant vanishes when $\mu = \mu_1$ or $\mu = \mu_2$, only one of those equations is independent. This means that both equations will be satisfied if A is arbitrarily set to some value A_j, provided that we then obtain B_j by solving either of Eqs. (10). The second of these equations gives

$$B_j = \frac{2p\mu_j}{\left(\Omega^2 - \mu_j^2\right)} A_j.$$

Substitution of the respective values of μ_j from Eqs. (11a) leads to $B_1 = A_1$ and $B_2 = -A_2$. The corresponding general solution of the equations of motion therefore is

$$\begin{aligned} x &= A_1 \cos(\mu_1 t + \phi_1) + A_2 \cos(\mu_2 t + \phi_2), \\ y &= A_1 \sin(\mu_1 t + \phi_1) - A_2 \sin(\mu_2 t + \phi_1). \end{aligned} \tag{12}$$

The coefficients A_1, A_2, ϕ_1, and ϕ_2 must satisfy initial conditions. It was given in the problem statement that the ball was released from rest relative to xyz at a distance b to the north of the pivot. Thus the initial conditions are $x = b$, $\dot{x} = y = \dot{y} = 0$ at $t = 0$. We match these values to the result of evaluating the general solution, from which we find that

$$A_1 = b\frac{\mu_2}{\mu_1 + \mu_2} \approx \frac{b}{2}, \quad A_2 = b\frac{\mu_1}{\mu_1 + \mu_2} \approx \frac{b}{2}.$$

The general solution corresponding to the preceding when approximations (11b) are used for the characteristic exponents is

$$\begin{aligned} x &= \frac{b}{2}\left\{\cos[(\Omega - p)t] + \cos[(\Omega + p)t]\right\}, \\ y &= \frac{b}{2}\left\{\sin[(\Omega - p)t] - \sin[(\Omega + p)t]\right\}. \end{aligned} \tag{13}$$

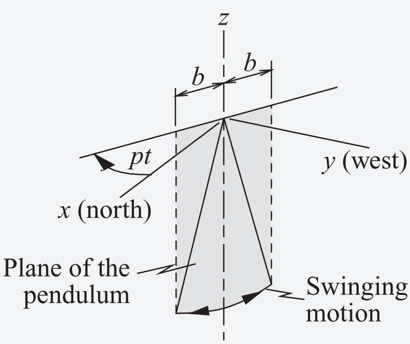
Trigonometric identities simplify this to

$$x = b\cos(pt)\cos(\Omega t), \quad y = -b\sin(pt)\cos(\Omega t). \qquad \triangleleft$$

The nature of the path becomes obvious when we observe that $\sin(pt)$ and $\cos(pt)$ vary much more slowly than $\cos(\Omega t)$ because $p \ll \Omega$. Hence we may consider x and y to oscillate at frequency Ω with an amplitude that slowly oscillates at frequency p. The preceding solution satisfies

$$y = -x\tan(pt),$$

which is the equation of a straight line whose slope is $-\tan(pt)$, if we neglect the variation in the value of pt. As shown in the diagram, the path seems to lie in a vertical plane that is situated at angle pt relative to the xz plane, measured clockwise when viewed downward.



The shaded plane shows the short-term swinging motion of the Foucault pendulum.

The vertical plane in which the cable lies therefore seems to an observer on the Earth to rotate about the upward vertical axis at angular speed $-p = -\omega_e \sin\lambda$. This is exactly opposite the vertical component of the angular velocity of the Earth.

It is interesting to consider the movement of the local plane of the Foucault pendulum's path from the perspective of an observer in outer space who is not experiencing the Earth's rotation. The angular velocity of this plane is the sum of the Earth's rotation $\bar{\omega}_e$ and the rotation $-\omega_e \sin\lambda$ about the $-z$ axis seen by an Earth-based observer. Because the latter cancels the vertical component of $\bar{\omega}_e$, the angular velocity that is seen from outer space at any instant is $\omega_e \cos\lambda$ about the x axis, which points northward along the Earth's surface.

The movement of the plane of a spherical pendulum relative to the Earth was used in 1851 by the French physicist Jean Louis Foucault (1819–1869) to demonstrate the Earth's rotation. The most famous Foucault pendulum may be found in the General Assembly building at United Nations headquarters in New York City.

In closing, we should note that the spherical pendulum for arbitrary, small initial values of x, $\dot{x}$, $\dot{y}$, and $\dot{y}$ would seem to follow an elliptical path in the xy plane. The major and minor axes of the ellipse would rotate about the z axis relative to the Earth at angular speed $-\omega_e \sin\lambda$.

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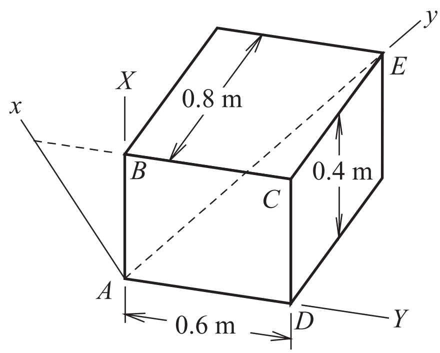
HOMEWORK PROBLEMS

EXERCISE 3.1 The XYZ coordinate system coincides with the edges of the box, the y axis coincides with the main diagonal for the box, and the x axis coincides with face $ABCD$. Use the orthonormal properties to determine the transformation that converts vector components relative to XYZ to components relative to xyz. Then use this transformation to determine the xyz coordinates of corner C.

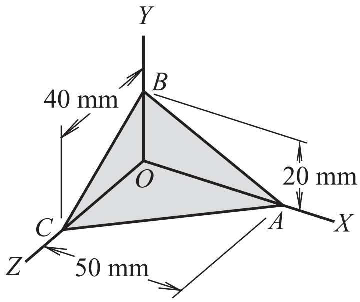


Exercise 3.1

EXERCISE 3.2 Solve Exercise 3.1 by considering the transformation from XYZ to xyz to be the result of a sequence of body-fixed rotations.

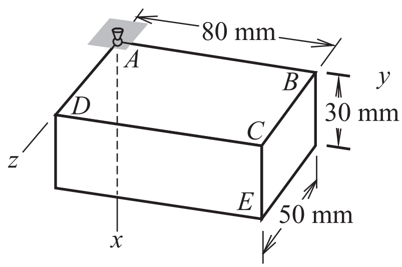
EXERCISE 3.3 At a certain instant gyrosensors on an airplane report that it is heading 40° west of north, climbing at 20° and that its wings are banked at an angle of 10° clockwise as viewed looking forward. At this instant, the aircraft's accelerometers indicate that the center of mass has acceleration components relative to the aircraft of $2g$ downward and $0.5g$ forward. What are the acceleration components in terms of north–south, east–west, and vertical?

EXERCISE 3.4 The corners of triangular plate ABC are situated along the axes of coordinate system XYZ as shown. Another coordinate system, whose origin is at corner A, is defined such that $\bar{i} = \bar{e}_{B/A}$ and $\bar{k}$ is perpendicular to plane ABC with a positive component in the direction of $\bar{J}$. Determine the rotation transformation from XYZ to xyz. Then determine the coordinates of the origin O with respect to xyz. *Hint:* Define a coordinate system parallel to XYZ with its origin at point A.



Exercise 3.4

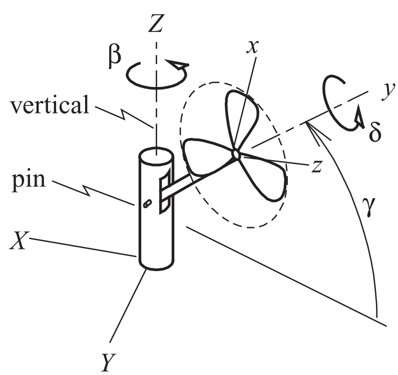
EXERCISE 3.5 The xyz coordinate system is attached to the box, and XYZ is a parallel stationary coordinate system. The box undergoes a pair of rotations: First, $\theta_1 = 65^\circ$ about the y axis, followed by $\theta_2 = -145^\circ$ about the z axis. For this rotation determine (a) the coordinates relative to xyz in its final orientation of the stationary point that was at the location of point E prior to the rotations, (b) the coordinates relative to XYZ of corner E of the box after both rotations.



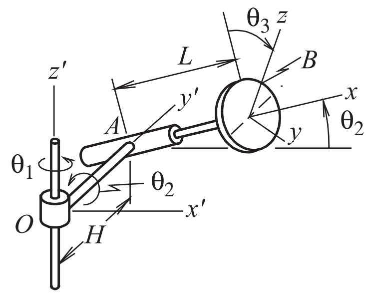
Exercise 3.5

EXERCISE 3.6 Solve Exercise 3.5 if the rotations are $\theta_1 = 65^\circ$ about the Y axis, followed by $\theta_2 = -145^\circ$ about the Z axis.

EXERCISE 3.7 The sketch shows an electric fan that may rotate about three axes. In this sketch XYZ constitute a set of fixed axes, and xyz are attached to the fan blades. The rotations are defined as follows. When the rotation angle β about the fixed vertical Z axis is zero, the pin's axis aligns with the fixed horizontal Y axis. When the rotation angle γ about the pin is zero, the shaft about which the fan blades spin is horizontal. When the spin angle δ is zero, the x axis is aligned with the axis of the pin. Consider a sequence of rotations in which γ occurs first, followed by β, then δ. Describe the transformation matrix $[R]$ for which $\begin{bmatrix}\bar{i} & \bar{j} & \bar{k}\end{bmatrix}^{\mathrm{T}} = [R]\begin{bmatrix}\bar{I} & \bar{J} & \bar{K}\end{bmatrix}^{\mathrm{T}}$ as a set of simple rotation transformations about specific axes. Then identify an alternative sequence in which the rotations β, γ, δ may be executed and still arrive at the same final orientation of xyz relative to XYZ.

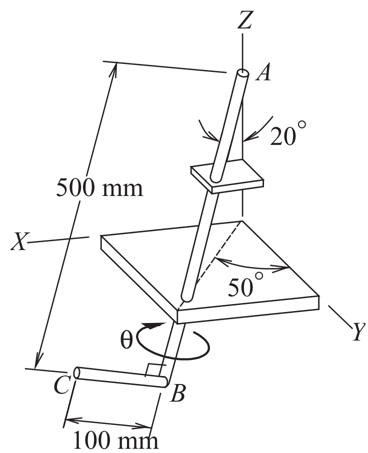


Exercise 3.7



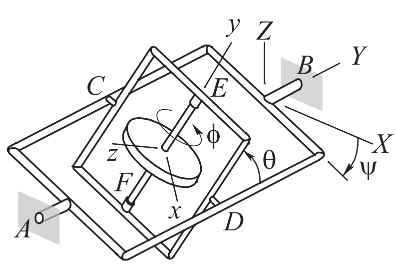
Exercise 3.8

EXERCISE 3.8 Motor A in the sketch can be rotated through angle θ_1 about the vertical shaft, as well as through angle θ_2 about shaft OA. The rotation of the flywheel relative to the motor housing is θ_3. In this diagram xyz is attached to the flywheel, and $x'y'z'$ is attached to shaft OA, such that the y' axis always coincides OA and the z' axis is always vertical. (a) What is the transformation by which vector components defined with respect to $x'y'z'$ may be expressed in terms of xyz components? (b) One may obtain the rotated position of xyz with respect to $x'y'z'$ by rotating first by θ_2, and then by θ_3, or else by applying θ_3, followed by θ_2. Prove that the transformation in Part (a) is independent of the sequence in which these rotations are performed. (c) Consider point B, which is located on the z axis at the perimeter of the flywheel. The flywheel's radius is ρ. Use the transformation in Part (a) to describe the displacement of this point relative to $x'y'z'$. (You may express this result in terms of any convenient set of coordinate directions, but state what your choice is.)



Exercise 3.9

EXERCISE 3.9 Rod ABC is such that segment BC is perpendicular to segment AB. The brackets align segment AB at 20° from the vertical Z axis, as shown. The rod rotates by angle θ about axis AB, which is defined such that segment BC is situated in the vertical plane formed by segment AB and the Z axis when when $\theta = 0$. Determine and graph the angle between the centerline of segment BC and the X axis as a function of θ.

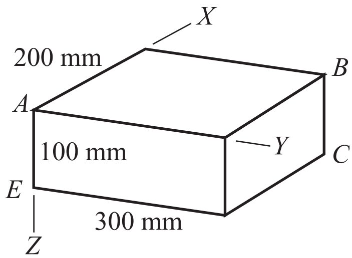


Exercise 3.10

EXERCISE 3.10 The three-axis gyroscope consists of an outer gimbal that may rotate by angle ψ about the AB axis relative to a fixed reference frame XYZ and an inner gimbal that may rotate by angle θ about the CD axis relative to the outer gimbal. The spin of the flywheel relative to the inner gimbal is the angle ϕ. (These are, respectively, the Eulerian angles of precession, nutation, and spin, which will be discussed in Chapter 4.) When these angles are zero, the body-fixed xyz system coincides with the respective axes of XYZ. There are six possible sequences in which the rotations may take place. Prove that the transformation from XYZ to xyz components depends on only the values of these angles, but not the sequence in which the rotations occur.

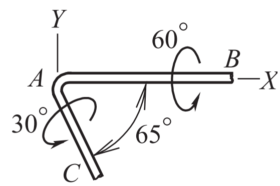
EXERCISE 3.11 Starting from the position shown, the box is rotated by 40° about face diagonal AB, clockwise as viewed from corner B toward corner A. Determine the coordinates of corner C relative to the fixed reference frame after this rotation.

EXERCISE 3.12 Starting from the position shown, the box is rotated by angle θ about main diagonal AC, counterclockwise as viewed from corner C toward corner A. The angle between the fixed Y axis and the unit vector $\bar{e}_{E/A}$ after the rotation is 110°. Determine θ.



Exercises 3.11 and 3.12

EXERCISE 3.13 The bent rod is given a pair of rotations, first by 60° about line AB, and then 30° about line AC, with the sense of each rotation as shown in the sketch. Let xyz be a coordinate system fixed to the rod that initially aligned with the fixed XYZ system shown. Determine the transformation by which vector components with respect to XYZ may be converted to components with respect to xyz.

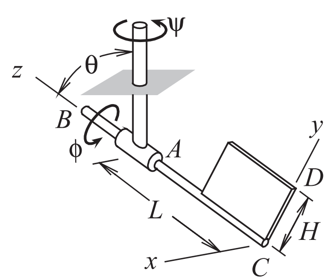


Exercise 3.13

EXERCISE 3.14 Consider the rotation of the bent rod in Exercise 3.13. Determine the orientation of the axis and the angle of rotation of the single rotation that would be equivalent to the pair of rotations specified there.

EXERCISE 3.15 It is desired to impart to the box in Exercise 3.5 a rotation about a single axis that is equivalent to the rotations specified there. Determine the orientation of that axis and the angle of rotation about that axis.

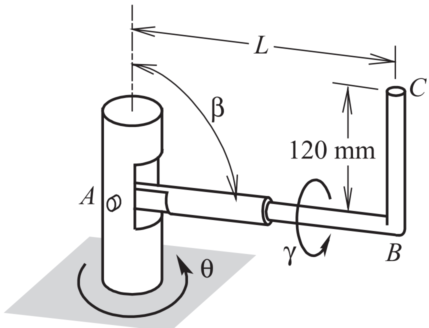
EXERCISE 3.16 Collar A is welded to the vertical shaft, so θ is constant. The rectangular plate is welded to this shaft, and xyz is a coordinate system that is attached to the plate. Splines prevent shaft BC from rotating relative to the collar, so $\phi = 0$, in which case the yz plane is always situated in the vertical plane. The system is rotated by $\psi = 75°$ about the vertical axis; the value of θ is 30°. Determine the rotation transformation from relating components relative to the initial and final xyz coordinate system. From this result determine the angle between the initial and final orientations of the y axis.



Exercises 3.16 and 3.17

EXERCISE 3.17 Collar A is welded to the vertical shaft, so θ is constant. The rectangular plate is welded to this shaft, and xyz is a coordinate system that is attached to the plate. Shaft BC may rotate relative to the collar by angle ϕ, with $\phi = 0$ when the yz plane coincides with the vertical plane, as depicted in the sketch. Starting from the illustrated position the system is rotated by $\psi = 75^{\circ}$ about the vertical axis, and $\phi = 115^{\circ}$, with $\theta = 30^{\circ}$. Determine the rotation transformation from relating components relative to the initial and final xyz coordinate system. From this result determine the angle between the initial and final orientations of the y axis.

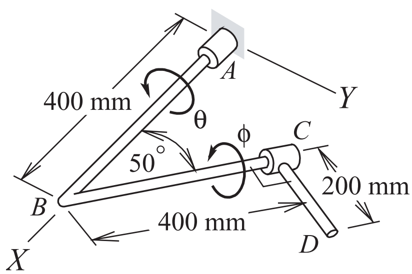
EXERCISE 3.18 A hydraulic cylinder allows the length of arm AB to vary, and servomotors control the rotation angles θ about the vertical, β about pin A, and γ about axis AB, with $\gamma = 0$ corresponding to bar BC being situated in the vertical plane as shown. In the initial position $L = 250$ mm, $\theta = 0$, $\beta = 90^{\circ}$, and $\gamma = 0$. In the final position, $\theta = \beta = 120^{\circ}$, $\gamma = -90^{\circ}$, and $L = 500$ mm. Determine the corresponding displacement of end C.



Exercise 3.18

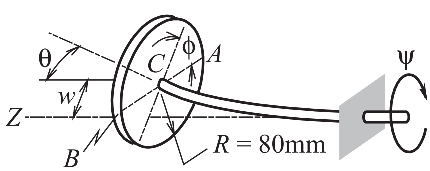
EXERCISE 3.19 Solve Exercise 3.18 for the case the initial state is such that $L = 250$ mm, $\theta = 0$, $\beta = 30^{\circ}$, and $\gamma = 50^{\circ}$. The final position is as specified there.

EXERCISE 3.20 Bar ABC rotates through angle θ about the fixed X axis, and collar C enables bar CD to rotate by angle ϕ about segment BC of bar ABC. When $\theta = \phi = 0$, both bars are situated in the XY plane. Determine the displacement of end D from this reference position to one where $\phi = -70^{\circ}$ and $\theta = 120^{\circ}$.



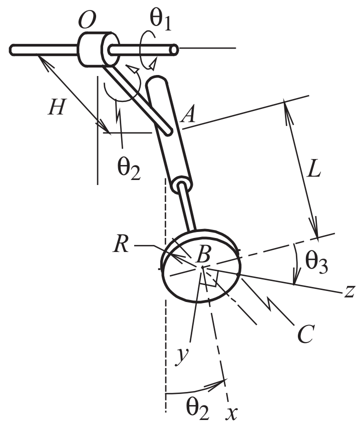
Exercise 3.20

EXERCISE 3.21 Flexure of a cantilevered shaft supporting a rotating flywheel causes the center C of the flywheel to undergo transverse displacement w, and the centerline of the flywheel to rotate relative to the bearing axis, which is marked as Z in the diagram. The rotation of the plane formed by the bearing axis and center C is ψ, and the flywheel's centerline lies in this rotated plane. The disk is welded to the shaft, so the angle θ between this centerline and the bearing axis is the flexural angle by which the tangent to the shaft rotates because of deformation. Torsional deformation of the shaft is described by the rotation ϕ of diametral line ACB; when $\phi = 0$, this line lies in the plane containing the flywheel's centerline and the bearing axis. In the initial position $w = \psi = \theta = \phi = 0$, whereas $w = 50$ mm, $\psi = 460^{\circ}$, $\theta = 10^{\circ}$, and $\phi = 8^{\circ}$ in the final position. Determine the corresponding displacement of points A and B on the perimeter of the flywheel.



Exercise 3.21

EXERCISE 3.22 The disk centered at point B is an optical mirror that is positioned by servo-controlled arms. Angle θ_1 is the rotation about the stationary horizontal shaft, θ_2 is the rotation of the motor housing about shaft OA, and θ_3 is the rotation of the flywheel relative to the motor housing. When $\theta_1 = \theta_2 = \theta_3 = 0$ shaft OA is horizontal, shaft AB is vertical, and the line from center B to point C on the perimeter of the flywheel is parallel to the shaft for the θ_1 rotation. In the position of interest the angles are $\theta_1 = 75^\circ$, $\theta_2 = -120^\circ$, and $\theta_3 = 210^\circ$. Determine the displacement of point C relative to its location when $\theta_1 = \theta_2 = \theta_3 = 0$. The length dimensions are $H = 300$ mm, $L = 500$ mm, $R = 100$ mm.



Exercise 3.22

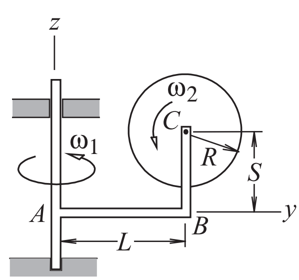
EXERCISE 3.23 In Exercise 3.16 collar A, which is welded to the vertical shaft, allows shaft BC to slide rotate relative to it, so the distance L and rotation angle ϕ are not constant. The rectangular plate is welded to this shaft, and it is situated in the vertical plane when $\phi = 0$. In the initial state $L = 3H$ and $\psi = \phi = 0$. In the final position $\psi = 60^\circ$, $\phi = -75^\circ$, and $L = 5H$. Determine the displacement of corner D in terms of components relative to the final orientation of the body-fixed xyz coordinate system.

EXERCISE 3.24 In Exercise 3.18 the rates are $\dot{L} = 5$ m/s, $\dot{\theta} = 40$ rad/s, $\dot{\beta} = 0$ rad/s, and $\dot{\gamma} = 10$ rad/s. At $t = 0$ it is known that $L = 200$ mm, $\theta = 0$, $\beta = 75^\circ$, and $\gamma = -30^\circ$. Determine the displacement of end C in the interval from $t = 0$ to $t = 10$ ms. Compare that result with the approximate value obtained by considering the angular velocity to be constant over the interval.

EXERCISE 3.25 Consider the linkage in Exercise 3.20 when the initial position corresponds to $\theta = 30^\circ$ and $\phi = 50^\circ$, and $\theta = 28^\circ$ and $\phi = 47^\circ$ in the final position. Compare the displacement of end D obtained by considering these rotations to be infinitesimal with the result obtained from an analysis based on rotation transformations.

EXERCISE 3.26 The bladed disk of a gas turbine in an aircraft is spinning at the constant rate of 30 000 rev/min while the aircraft travels at 1200 km/h in a 2-km radius turn to the left. Determine the angular velocity and angular acceleration of the disk.

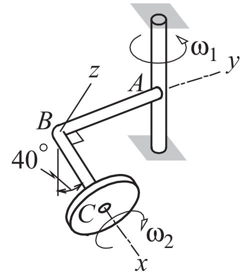
EXERCISE 3.27 The rotation rates ω_1 of the bracket supporting the spinning disk are constant. The spin rate of the disk is increasing linearly with elapsed time, so that $\omega_2 = \xi t$. (a) Describe the angular velocity of the disk in terms of a superposition of simple rotations. (b) Solely from an examination of the description in Part (a), predict the direction(s) in which the angular acceleration of the disk will be situated relative to the xyz axes defined in the sketch, which rotates in unison with the support



Exercise 3.27

bracket. Briefly explain your answer. (c) Describe the angular velocity and angular acceleration of the disk in terms of components relative to xyz coordinate system.

EXERCISE 3.28 The entire system rotates about the vertical axis at constant angular speed ω_1, and the rotation rate ω_2 of the rotor relative to bar BC also is constant. (a) Describe the angular velocity of the rotor in terms of a superposition of simple rotations. (b) Solely from an examination of the description in Part (a), predict the direction(s) in which the angular acceleration of the rotor will be situated relative to the xyz axes defined in the sketch. Briefly explain your answer. (c) Describe the angular velocity and angular acceleration of the rotor in terms of components relative to xyz.



Exercise 3.28

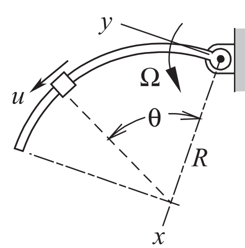
EXERCISE 3.29 The flywheel of the gyroscope in Exercise 3.10 rotates about its own axis at $\dot{\phi} = 50\,000$ rev/min, and the outer gimbal is rotating about the horizontal axis at $\dot{\psi} = 20$ rad/s, with $\ddot{\psi} = -100$ rad/s^2. Determine the angular velocity and angular acceleration of the flywheel if the orientation of the inner gimbal relative to the outer one is constant at (a) $\theta = 90^\circ$, (b) $\theta = 60^\circ$.

EXERCISE 3.30 The flywheel of the gyroscope in Exercise 3.10 rotates about its own axis at $\dot{\phi} = 50\,000$ rev/min, and the outer gimbal is rotating about the horizontal axis at $\dot{\psi} = 20$ rad/s, with $\ddot{\psi} = -100$ rad/s^2. At the instant when the angle θ locating the inner gimbal relative to the outer one is $\theta = 75^\circ$, it is changing at $\dot{\theta} = -2$ rad/s, $\ddot{\theta} = 50$ rad/s^2. Determine the angular velocity and angular acceleration of the flywheel at this instant.

EXERCISE 3.31 The orientation angles θ, β, and γ in Exercise 3.18 each change at a constant rate. At the instant of interest, $\beta = 90^\circ$ and $\gamma = 0$. Determine the angular velocity and angular acceleration of arm BC at this instant in terms of components relative to a coordinate system that is attached to the vertical shaft.

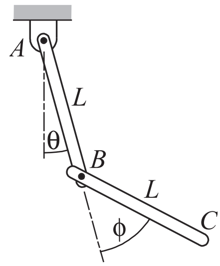
EXERCISE 3.32 The orientation angles θ, β, and γ in Exercise 3.18 each change at a constant rate. At the instant of interest, $\beta = 60^\circ$ and $\gamma = 30^\circ$. Determine the angular velocity and angular acceleration of arm BC at this instant in terms of components relative to a coordinate system that is attached to this arm with $\bar{k} = \bar{e}_{C/B}$.

EXERCISE 3.33 The collar moves at the constant speed u relative to the guide bar, which rotates in the horizontal plane at the constant rate Ω. Derive expressions for the velocity and acceleration of the collar as functions of the angle θ, locating where the collar is situated along its guide. Describe the results in terms of components relative to the xyz coordinate system appearing in the sketch.



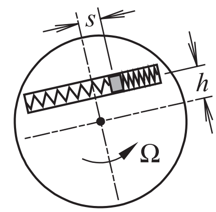
Exercise 3.33

EXERCISE 3.34 A servomotor maintains the angle of bar BC relative to bar AB at $\phi = 2\theta$, where θ is the angle of inclination of bar AB. Determine the acceleration of end C corresponding to arbitrary values of θ, $\dot{\theta}$, and $\ddot{\theta}$.



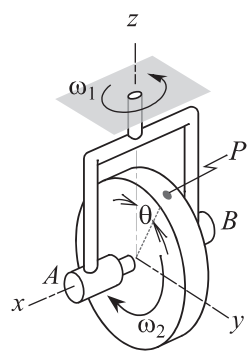
Exercise 3.34

EXERCISE 3.35 A speed governor consists of a block of mass m that slides within a smooth groove in a turntable that rotates about its center point O at angular speed Ω. The identical opposing springs, whose stiffness is k, are precompressed. Consequently the springs maintain their contact with the block regardless of the displacement s. The system lies in the horizontal plane. Derive an expression for the normal force exerted by the groove wall on the block and the differential equation governing s as a function of time in the case where Ω is an arbitrary function of time. Then determine the natural frequency of vibratory motion of the block within the groove for the case in which Ω is constant, and explain how that result can be used to monitor when Ω exceeds a critical value.



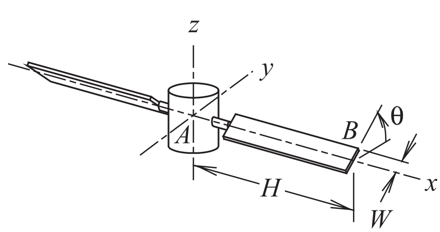
Exercise 3.35

EXERCISE 3.36 The disk rotates at ω_2 about its axis, and the rotation rate of the forked shaft is ω_1. Both rates are constant. It is desired to determine the velocity and acceleration of point P in the perimeter of the disk, which is oriented on the radial line that has rotated by θ relative to the upward vertical. Perform this analysis using a reference frame attached to the forked shaft; then compare that analysis with one that uses a reference frame attached to the disk. Describe the results in terms of components relative to the xyz axes in the sketch.



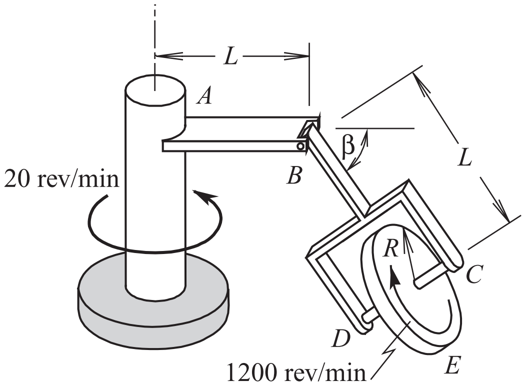
Exercise 3.36

EXERCISE 3.37 The angle θ describing the rotation of a reconnaissance satellite's solar panels about the body-fixed x axis is an arbitrary function of time. The satellite spins about the z axis at the constant rate Ω. Derive expressions for the absolute velocity and acceleration of point B relative to the origin of xyz.



Exercise 3.37

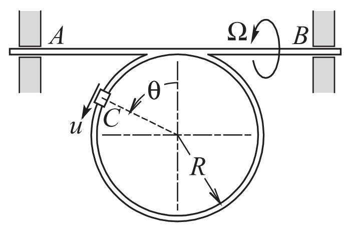
EXERCISE 3.38 The disk spins about its axis CD at 1200 rev/min as the system rotates about the vertical axis at 20 rev/min. Both rates are constant. It is desired to determine the velocity and acceleration of point E, which is the lowest point on the perimeter of the disk, in the situation in which β is constant at 60°. (a) Carry out the analysis by using a moving reference frame attached to gimbal BCD. (b) Carry out the analysis by using a moving reference frame attached to the disk.



Exercises 3.38 and 3.39

EXERCISE 3.39 The disk spins about its axis CD at 1200 rev/min as the system rotates about the vertical axis at 20 rev/min. Both rates are constant. The angle of elevation of the arm supporting the disk is such that $\dot{\beta} = 10$ rad/s and $\ddot{\beta} = -500$ rad/s^2 when $\beta = 36.87°$. Determine the velocity and acceleration of point E, which is the lowest point on the perimeter of the disk.

EXERCISE 3.40 Collar C slides relative to the curved rod at a constant speed u. The rotation rate about bearing axis AB is constant at Ω. Determine the acceleration of the collar in terms of the angle θ locating the collar. Also derive expressions for the dynamic forces exerted on the collar by the rod and the tangential force required to hold u constant. Gravity may be assumed to be unimportant.



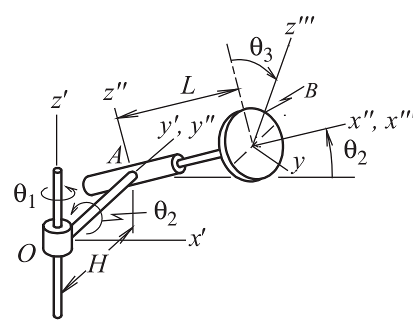
Exercise 3.40

EXERCISE 3.41 The following questions pertain to the application of the relative motion formulas:

$$\bar{v}_B = \bar{v}_{O'} + (\bar{v}_B)_{xyz} + \bar{\omega} \times \bar{r}_{B/O'},$$

$$\bar{a}_B = \bar{a}_{O'} + (\bar{a}_B)_{xyz} + \bar{\alpha} \times \bar{r}_{B/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{B/O'}) + 2\bar{\omega} \times (\bar{v}_B)_{xyz}.$$

In each question, xyz in these formulas is equated to one of the coordinate systems in the diagram, and you are to describe one of the corresponding relative motion terms. Each answer should be expressed in component form relative to any of the three coordinate systems, but a different coordinate system may be used to describe each answer. (a) If $xyz = x'''y'''z'''$, what is $\bar{\omega}$? (b) If $xyz = x''y''z''$, what is $\bar{\alpha}$? (c) If $xyz = x'''y'''z'''$, what is $\bar{\alpha}$? (d) If $xyz = x''y''z''$, what is $(\bar{v}_B)_{xyz}$? (e) If $xyz = x''y''z''$, what is $(\bar{a}_B)_{xyz}$? (f) If $xyz = x'''y'''z'''$, what is $\bar{v}_{O'}$? (g) If $xyz = x'y'z'$, what is $\bar{r}_{B/O'}$?

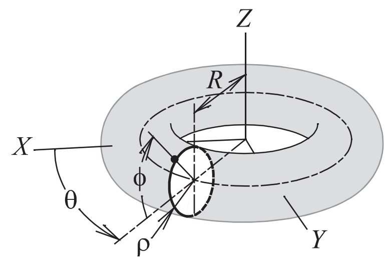


Exercise 3.41

EXERCISE 3.42 The rotation rates of the bars in Exercise 3.20 are constant at $\dot{\theta} = 20$ rad/s, $\dot{\phi} = 30$ rad/s. For the instant when $\theta = 120^\circ$ and $\phi = 50^\circ$, determine the velocity and acceleration of end D of the bar.

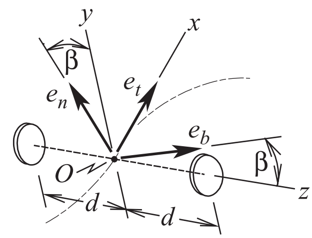
EXERCISE 3.43 Use the concepts of relative motion to derive the formulas for velocity and acceleration of a point in terms of a set of spherical coordinates.

EXERCISE 3.44 The sketch defines an orthogonal curvilinear coordinate system ρ, θ, ϕ known as toroidal coordinates. The radius R is a constant parameter on which these coordinates are based. Use the concepts of relative motion to derive the corresponding formulas for velocity and acceleration of point P in terms of the unit vectors of this coordinate system.



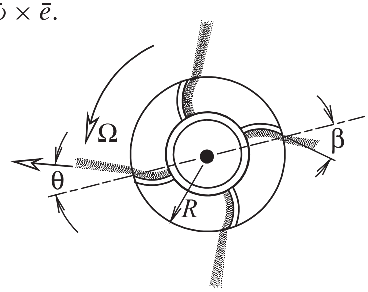
Exercise 3.44

EXERCISE 3.45 The track of a roller coaster is described in terms of a reference centerline that is defined in parametric form by a function $\bar{r}(s)$ locating points along it, with s being the arc length from the start. In turn this defines a set of path variable unit vectors $\bar{e}_t,\ \bar{e}_n,\ \bar{e}_b$ at each point. These unit vectors depend on the location, so they define a moving reference frame $x'y'z'$. As shown in the diagram, the track is laid out by defining a line that is rotated by angle β, which can depend on s, about $\bar{e}_t$. This line is used to situate the tracks at a fixed distance d on either side of the reference centerline. Thus coordinate system xyz is a moving reference frame whose x axis is $\bar{e}_t$ and whose y axis indicates the orientation of an axle of a car as it moves along the track, with $\bar{\omega}_{xyz} = \bar{\omega}_{x'y'z'} + \dot{\beta}\bar{e}$. (a) Suppose a car moves along the track at a speed v that depends on s. Based on the approximation that the longitudinal axis of the car is parallel to $\bar{e}_t$, derive an expression for the angular velocity and angular acceleration of the car in terms of the radius of curvature $\rho(s)$ and torsion $\tau(s)$ of the path. (b) Consider a point at distance h above the axle, $\bar{r}_{P/O} = h\bar{j}$. Determine the velocity and acceleration of such a point corresponding to the expressions derived in Part (a). *Hint:* Use the Frenet formulas and the fact that $ds/dt \equiv v$ to describe the rate of change of the path variable unit vectors, then equate these derivatives to the general property that $d\bar{e}/dt = \bar{\omega} \times \bar{e}$.



Exercise 3.45

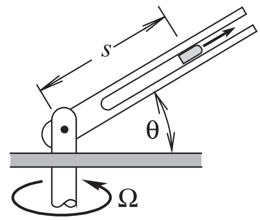
EXERCISE 3.46 The blades of a centrifugal flow pump are attached to the central hub such that their tangent at their outer radius is at angle β relative to the radial line at that location. The radius of curvature of the blades at the tip is ρ. Suppose water flows outward along a blade at a constant relative speed u and the rotation rate Ω is constant. Determine the velocity and acceleration of a water particle immediately



Exercise 3.46

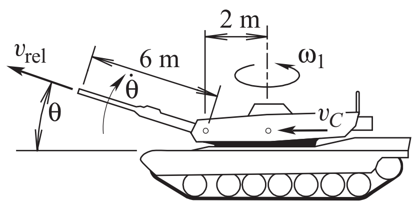
before it flows off a blade. Also determine the exit angle θ at which the water is directed as it leaves the blade.

EXERCISE 3.47 A pellet of mass m moves through the smooth barrel. At the instant before the pellet emerges, its speed relative to the barrel is u. At that instant, the magnitude of the propulsive force F, which acts parallel to the barrel, is a factor of 50 times greater than the weight of the pellet. The barrel rotates about the vertical axis at angular speed Ω as the angle of elevation of the barrel is increased at the rate $\dot{\theta}$. Both rates are constant. Derive expressions for the acceleration term $\dot{u}$ and the force the pellet exerts on the walls of the barrel at this instant.



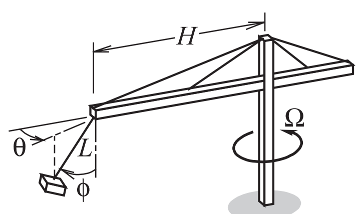
Exercise 3.47

EXERCISE 3.48 While the tank is moving forward at a constant speed $v_c = 30$ km/h, the turret is rotating at the constant rate $\omega_1 = 0.3$ rad/s and the barrel is being raised at the constant rate $\dot{\theta} = 0.5$ rad/s. At a certain instant the barrel is facing forward and $\theta = 15°$. At this instant a shell whose mass is 80 kg is about to emerge from the barrel with a muzzle velocity $v_{\text{rel}} = 5500$ km/h that has reached a maximum because the internal propulsive pressure within the barrel has been dissipated. Determine the force exerted by the shell on the wall of barrel at this instant.



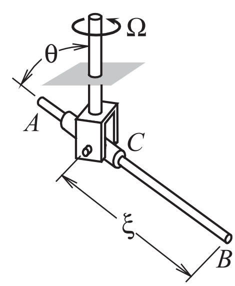
Exercise 3.48

EXERCISE 3.49 A shipping container is suspended from a crane by an inextensible cable. The crane rotates in the vertical plane at the constant angular speed Ω. It may be assumed that the cable remains taut, so its orientation is describable in terms of the angle θ locating the vertical plane in which it is situated relative to the plane of the crane and the angle of elevation ϕ from a vertical line. Based on a model of the container as a small particle, derive differential equations of motion in which the only unknowns are θ and ϕ.



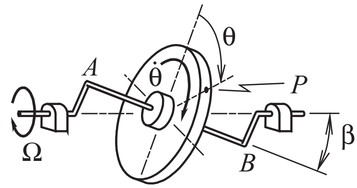
Exercise 3.49

EXERCISE 3.50 The vertical shaft rotates at the constant rate Ω. Collar C is attached to this shaft by a fork-and-clevis, so the angle of inclination θ of bar AB is an unknown time function. The bar is free to slide through the collar, so the distance ξ from the pivot point to end B is an arbitrary function of t. Derive expressions for the velocity and acceleration of point B in terms of ξ, θ, and their derivatives, as well as Ω.



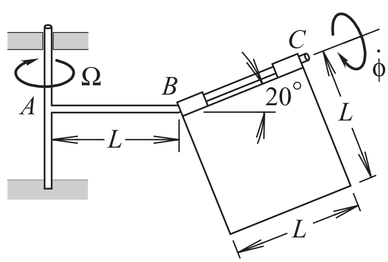
Exercise 3.50

EXERCISE 3.51 The disk, whose radius is R, spins about axis AB relative to the bent shaft. The angle of rotation of a radial line to point P on the perimeter of the disk about this axis is θ, so $\dot{\theta}$ is the spin rate. This angle is defined such that line OP is situated in the plane containing the bearing axis and axis AB when $\theta = 0$. The whole assembly precesses at angular speed Ω about the bearing axis, with the disk's center situated on the bearing axis. Derive expressions for the velocity and acceleration of point P on the perimeter of the disk. Describe the results in terms of components relative to an xyz system that is attached to the disk, with $\bar{i}$ aligned from the disk's center to point P.



Exercise 3.51

EXERCISE 3.52 Bent shaft ABC is welded to the vertical shaft, which rotates at constant speed Ω. The angle of rotation of the square plate about axis BC is ϕ, which is an arbitrary function of time. The configuration depicted in the sketch, in which the plate is situated in the vertical plane, corresponds to $\phi = 0$. The xyz coordinate system is attached to the plate. (a) Derive expressions for the angular velocity and angular acceleration of the plate, valid for arbitrary ϕ, in terms of components with respect to xyz. (b) Derive expressions for the velocity and acceleration of the center of the plate in terms of xyz components.

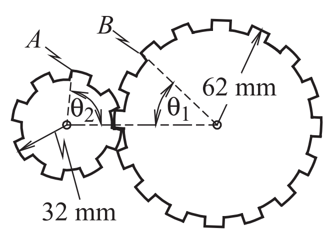


Exercise 3.52

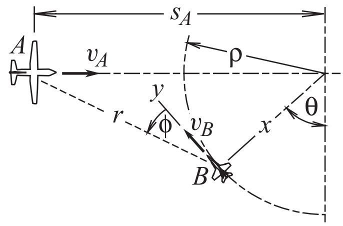
EXERCISE 3.53 Instantaneous parameters for the robotic linkage in Example 3.6 are $\theta = 40^\circ$, $\dot{\theta} = 5$ rad/s, $\ddot{\theta} = -200$ rad/s^2, $\beta = 0$, $\dot{\beta} = -3$ rad/s, $\ddot{\beta} = 0$, $\gamma = 20^\circ$, $\dot{\gamma} = 10$ rad/s, $\ddot{\gamma} = 0$, and L is constant at 0.8 m. Determine the velocity and acceleration of end C at this instant.

EXERCISE 3.54 Instantaneous parameters for the robotic linkage in Example 3.6 are $\theta = 70^\circ$, $\dot{\theta} = 5$ rad/s, $\ddot{\theta} = -200$ rad/s^2, $\beta = -60^\circ$, $\dot{\beta} = 4$ rad/s, $\ddot{\beta} = 40$ rad/s^2, $\gamma = 20^\circ$, $\dot{\gamma} = 10$ rad/s, $\ddot{\gamma} = 0$, $L = 0.8$ m, $\dot{L} = 20$ m/s, $\ddot{L} = 100$ m/s^2. Determine the velocity and acceleration of end C at this instant.

EXERCISE 3.55 The larger gear rotates at half the angular speed of the smaller gear, which is rotating clockwise at a constant rate of 4800 rev/min. Points A and B are corners of gear teeth that will mesh. At a certain instant $\theta_1 = 85^\circ$ and $\theta_2 = 42^\circ$. For this instant determine the velocity and acceleration of corner B with respect to corner A as seen from a reference frame that is attached to the smaller gear.



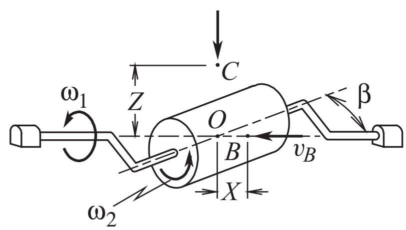
Exercise 3.55



Exercise 3.56

EXERCISE 3.56 Airplane A travels eastward at constant speed $v_B = 560$ km/h, while airplane A executes a constant-radius turn, $\rho = 3.2$ km, in the horizontal plane at constant speed $v_A = 1440$ km/h. At $t = 0$ the angle θ locating airplane A was zero, and s_A at that instant was 5.2 km. Radar equipment on aircraft A can measure the separation distance r and the angle ϕ relative to the aircraft's longitudinal axis, as well as the rates of change of these parameters. Derive expressions for r, $\dot{r}$, ϕ, and $\dot{\phi}$. Graph these parameters as functions of time.

EXERCISE 3.57 At a certain instant the absolute velocity and acceleration of an aircraft's center of mass in terms of a body-fixed coordinate system are $\bar{v} = 900\bar{i}$ km/h, $\bar{a} = 5\bar{j}$ m/s^2, where $\bar{i}$ is the longitudinal forward direction and $\bar{k}$ is the direction perceived to be upward. The longitudinal axis is pitched upward at 10° from horizontal. The airplane has rolled to 25° in order to execute a left turn and the roll rate at this instant is 2 rad/s about the positive x axis. The corresponding yaw rate is 0.5 rad/s about the positive z axis, and both the roll and yaw rates are constant. At this instant an attendant drops a beverage container from a height of 800 mm above a passenger. Determine the acceleration of the container as seen by the passenger at the instant of release.



Exercises 3.58 and 3.59

EXERCISE 3.58 A test chamber for astronauts rotates about its centerline at constant angular speed ω_1 as the entire assembly rotates about the bearing axis at angular speed ω_2, which also is constant. An astronaut is seated securely in the chamber at center point O, which is collinear with both axes of rotation. At a certain instant a ball that was thrown toward the astronaut is at position B, which is at distance X along the fixed rotation axis. Its speed at this instant is v_B, and $\dot{v}_B = 0$. Derive expressions for the velocity and acceleration of the ball as seen by the astronaut at this instant.

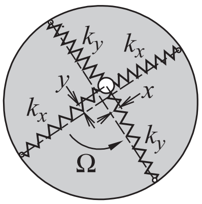
EXERCISE 3.59 A test chamber for astronauts rotates about its centerline at constant angular speed ω_1 as the entire assembly rotates about the bearing axis at angular speed ω_2, which also is constant. An astronaut is seated securely in the chamber at center point O, which is collinear with both axes of rotation. Ball C falls freely after being released at $t = 0$ from height H, so the vertical distance from the observer to the ball is $Z = H - gt^2/2$. At $t = 0$ the plane containing the two rotation axes is horizontal. Determine as a function of elapsed time t the velocity and acceleration of the ball as seen by the astronaut.

EXERCISE 3.60 A small disk slides with negligible friction on a horizontal sheet of ice. The initial velocity of the disk was u in the southerly direction. Determine the distance and sense of the east–west shift s in the position after the disk has traveled distance d

southward. How would this result have changed if the initial velocity was northward or eastward?

EXERCISE 3.61 A ball is thrown vertically from the ground at speed v. Assuming that air resistance is negligible and that the distance the ball travels is sufficiently small to consider the gravitational force to be mg downward, derive an expression for the shift that is due to the Coriolis effect in the position where it returns to the ground. Evaluate the result for $v = 4000$ m/s at a latitude of 45°.

EXERCISE 3.62 An object falls in a vacuum after being released at a distance H above the surface of the Earth. The line extending from the center of the Earth to this object is at latitude λ, and point O' on the Earth's surface is concurrent with this line. Determine the location (east–west and north–south relative to point O') at which the object strikes the ground in each of the following cases: (a) The object is initially at rest relative to the Earth, (b) the object was initially at rest relative to a reference frame that translates with the center of the Earth but does not execute the Earth's spinning rotation. For the sake of simplicity, the gravitational attraction may be considered to be constant at mg. Explain the difference between the results in cases (a) and (b).



Exercise 3.63

EXERCISE 3.63 A small disk of mass m is attached to a horizontal turntable by two pairs of opposing springs having stiffnesses k_x and k_y that are unstretched length when the block coincides with the axis of the turntable. The (x, y) coordinates of the block relative to the turntable are much less than the radius of the turntable, which means that the force exerted by each spring on the disk may be taken to be $k_x x$ or $k_y y$ opposite the direction of the respective displacement. Derive differential equations for x and y for the case in which the turntable's rotation rate Ω is an arbitrary function of time. Then solve those equations for the case in which Ω is constant. The initial state for this solution is one in which the block is released from rest relative to the turntable at $x = b$, $y = 0$. Discuss how this system is analogous to the Foucault pendulum.

CHAPTER 4

Kinematics of Constrained Rigid Bodies

The concept of a rigid body is an artificial one, in that all materials deform when forces are applied to them. Nevertheless, this artifice is very useful when we are concerned with an object whose shape changes little in the course of its motion. In addition, it often is convenient to decompose the motion of a flexible body into rigid-body and deformational contributions.

Most engineering systems feature bodies that are interconnected. Each body must move consistently with the restrictions imposed on it by the other bodies. We refer to these restrictions as *constraints*. Constraint conditions are the kinematical manifestations of the reaction forces. Indeed, a synonym for reactions is constraint forces. A keystone of analytical dynamics, whose treatment begins in Chapter 7, is the duality of constraint forces and constraint conditions, which enable us to describe one if we know the other. However, in a kinematics analysis one is not concerned with the forces required to attain a specified state of motion.

4.1 GENERAL EQUATIONS

When an object is modeled as a *rigid body,* the distance separating any pair of points in that object is considered to be invariant. This approximation is quite useful because it leads to greatly simplified kinematical and kinetic analyses. Because the distance between points cannot change, any set of coordinate axes xyz that is scribed in the body will maintain its orientation relative to the body. Such a coordinate system forms a body-fixed reference frame. The orientation of xyz relative to the body and the location of its origin are arbitrary. A typical situation is depicted in Fig. 4.1, where points O', A, and B are arbitrarily selected points in the body.

A corollary of the rigidity of the body is that no point in the body can displace relative to xyz, so (x_A, y_A, z_A) are both the initial and final xyz coordinates of point A and there is no displacement relative to xyz, $(\Delta\bar{r}_A)_{xyz} = \bar{0}$. Correspondingly, Eq. (3.2.7) shows that the $\bar{I}\bar{J}\bar{K}$ displacement components of points A are related to those of point O' by

$$\begin{Bmatrix} \Delta\bar{r}_A \cdot \bar{I} \\ \Delta\bar{r}_A \cdot \bar{J} \\ \Delta\bar{r}_A \cdot \bar{K} \end{Bmatrix} = \begin{Bmatrix} \Delta\bar{r}_{O'} \cdot \bar{I} \\ \Delta\bar{r}_{O'} \cdot \bar{J} \\ \Delta\bar{r}_{O'} \cdot \bar{K} \end{Bmatrix} + \left[[R]_\mathrm{f}^\mathrm{T} - [R]_\mathrm{o}^\mathrm{T}\right] \begin{Bmatrix} x_A \\ y_A \\ z_A \end{Bmatrix}, \tag{4.1.1}$$

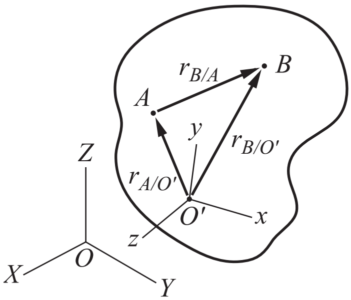


Figure 4.1. Relating the motion of three points in a rigid body.

where $[R]_o$ and $[R]_f$ are the rotation transformations of xyz with respect to XYZ in the original and final locations, respectively. The fact that points in the body cannot move relative to the body-fixed reference frame also simplifies the velocity and acceleration relations. Because $(\bar{v}_A)_{xyz} = \bar{0}$ and $(\bar{a}_A)_{xyz} = \bar{0}$, we have

$$\begin{aligned} \bar{v}_A &= \bar{v}_{O'} + \bar{\omega} \times \bar{r}_{A/O'}, \\ \bar{a}_A &= \bar{a}_{O'} + \bar{\alpha} \times \bar{r}_{A/O'} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{A/O'}) \,. \end{aligned} \tag{4.1.2}$$

Although we placed the origin of xyz at point O', there is nothing special about the choice of points. Thus, relations like Eqs. (4.1.1) and (4.1.2) exist, relating the motion of any pair of points. However, all such relations are not independent. In the case of displacement we observe by analogy to Eq. (4.1.1) that the displacement of point B is related to that of point O' by

$$\begin{Bmatrix} \Delta\bar{r}_B \cdot \bar{I} \\ \Delta\bar{r}_B \cdot \bar{J} \\ \Delta\bar{r}_B \cdot \bar{K} \end{Bmatrix} = \begin{Bmatrix} \Delta\bar{r}_{O'} \cdot \bar{I} \\ \Delta\bar{r}_{O'} \cdot \bar{J} \\ \Delta\bar{r}_{O'} \cdot \bar{K} \end{Bmatrix} + \left[[R]_f^T - [R]_o^T \right] \begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix}. \tag{4.1.3}$$

On the other hand, we could use an $x'y'z'$ coordinate system that is parallel to xyz with origin at point A. Because $\bar{r}_{B/A} = \bar{r}_{B/O} - \bar{r}_{A/O}$, the coordinates of the points are related by a translation transformation, such that

$$x'_B = x_B - x_A, \quad y'_B = y_B - y_A, \quad z'_B = z_B - z_A. \tag{4.1.4}$$

Because $x'y'z'$ is specified to always be parallel to xyz, $[R]_f$ and $[R]_o$ also are the transformations from $\bar{I}\bar{J}\bar{K}$ to $\bar{i}'\bar{j}'\bar{k}'$ components. Thus the analog to Eq. (4.1.1) relating the displacements of points A and B is

$$\begin{Bmatrix} \Delta\bar{r}_B \cdot \bar{I} \\ \Delta\bar{r}_B \cdot \bar{J} \\ \Delta\bar{r}_B \cdot \bar{K} \end{Bmatrix} = \begin{Bmatrix} \Delta\bar{r}_A \cdot \bar{I} \\ \Delta\bar{r}_A \cdot \bar{J} \\ \Delta\bar{r}_A \cdot \bar{K} \end{Bmatrix} + \left[[R]_f^T - [R]_o^T \right] \begin{Bmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{Bmatrix}. \tag{4.1.5}$$

The significant aspect of this relation is that it is the same as the result of subtracting Eq. (4.1.1) from Eq. (4.1.3). Thus we deduce that the displacements of three points in a rigid body are related by two independent equations. We would arrive at a similar conclusion if we were to follow similar steps to relate the velocity or acceleration of points O', A, and B by using Eqs. (4.1.2). For example, for velocity we would have

$$\begin{aligned}\bar{v}_A &= \bar{v}_{O'} + \bar{\omega} \times \bar{r}_{A/O'},\\ \bar{v}_B &= \bar{v}_{O'} + \bar{\omega} \times \bar{r}_{B/O'},\\ \bar{v}_B &= \bar{v}_A + \bar{\omega} \times \bar{r}_{B/A}.\end{aligned} \tag{4.1.6}$$

Because $\bar{r}_{B/A} = \bar{r}_{B/O'} - \bar{r}_{A/O'}$, the third of the preceding equations is the same as the difference between the second and first equations. If we consider relating additional points, we come to the realization that only one additional equation for displacement, velocity, or acceleration exists for each point added. This leads us to a general conclusion:

> Given a set of n points in a rigid body, there are $n-1$ independent equations relating their displacement, velocity, or acceleration. We may obtain these equations by selecting one point as the reference for the description of the other $n-1$ points. Let A designate the reference point and P denote any of the other points. Then the displacement components relative to the fixed reference frame are related by
>
> $$\boxed{\begin{Bmatrix} \Delta\bar{r}_P \cdot \bar{I} \\ \Delta\bar{r}_P \cdot \bar{J} \\ \Delta\bar{r}_P \cdot \bar{K} \end{Bmatrix} = \begin{Bmatrix} \Delta\bar{r}_A \cdot \bar{I} \\ \Delta\bar{r}_A \cdot \bar{J} \\ \Delta\bar{r}_A \cdot \bar{K} \end{Bmatrix} + \left[[R]_{\mathrm{f}}^{\mathrm{T}} - [R]_{\mathrm{o}}^{\mathrm{T}}\right] \begin{Bmatrix} x_P - x_A \\ y_P - y_A \\ z_P - z_A \end{Bmatrix}.} \tag{4.1.7}$$
>
> The velocities of these points are related by
>
> $$\boxed{\bar{v}_P = \bar{v}_A + \bar{\omega} \times \bar{r}_{P/A},} \tag{4.1.8}$$
>
> and the accelerations are related by
>
> $$\boxed{\bar{a}_P = \bar{a}_A + \bar{\alpha} \times \bar{r}_{P/A} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/A}).} \tag{4.1.9}$$

It is important to realize that the preceding expressions for displacement, velocity, and acceleration, being vectorial in nature, each represent three scalar equations associated with the respective components. It follows that if one relates the motion of n points in a body, $3(n-1)$ scalar equations will result. Such equations will be solved in the following sections to perform kinematical analyses of the motion of systems of rigid bodies.

Equations (4.1.7)–(4.1.9) show that the velocity and the acceleration of any point in a rigid body are the superposition of the movement of an arbitrarily selected reference point A and a rotational motion about point A. These observations are manifestations of Chasle's theorem:

> The *general motion* of a rigid body is a superposition of a translation and a pure rotation. In the *translation*, all points follow the movement of an arbitrary point A in the body, and the orientation of any line scribed in the body remains constant. In the rotation the selected point A is at rest.

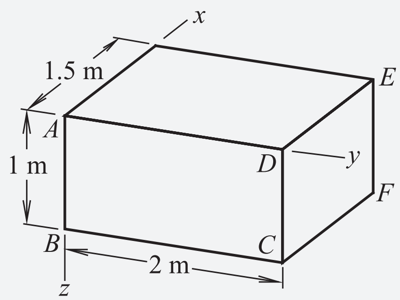
As a consequence of the arbitrariness of the point selected for the translation, changing the reference point changes the translational part of the motion, except for the case of pure translational motion. This means that the only global property of a rigid body's motion is its rotation, as described by the current orientation of a set of xyz axes scribed in it, its angular velocity, and its angular acceleration. Various methods for locating a point by means of intrinsic and extrinsic coordinates were discussed in Chapter 2. The next section presents a standardized way for describing orientation.

A basic tool in the analysis of velocity for a body in planar motion is the instant center method. In essence, this technique is based on considering a body in general motion (translation plus rotation) to be rotating about a rest point, which is called the instantaneous center of zero velocity, or more briefly, the *instant center*. If there is a point C in a body for which $\bar{v}_C = \bar{0}$ at some instant, then Eq. (4.1.8) indicates that $\bar{v}_P = \bar{\omega} \times \bar{r}_{P/C}$. This means that point C must be along a line perpendicular to $\bar{w}$ and intersecting point P. If the direction of the velocity of two points in a body is known at some instant, the instant center, if it exists, will be at the intersection of the two perpendiculars that are constructed according to this specification. (In the case of a translating body, the perpendiculars to the velocity of all points will be parallel, which leads to $\bar{r}_{P/C}$ being infinite. This is consistent with ω being zero.)

The instant center for planar motion leads to a simplified visualization of the velocity of points. Each point's velocity is like what would be obtained if the point were following a circular path whose radius is the perpendicular distance to the axis of rotation intersecting the instant center. However, the usefulness of the instant center concept is quite narrow. The condition that $\bar{v}_P = \bar{\omega} \times \bar{r}_{P/C}$ for all points in a body requires that the velocity of all points be perpendicular to $\bar{\omega}$, which is not true for general spatial motion. In addition, the instant center concept is not valid for acceleration analysis because the instant center will have an acceleration unless the body is in pure rotation. Also, $\bar{\alpha}$ is not parallel to $\bar{\omega}$ in a spatial motion. Thus, for the purpose of analyzing acceleration, we cannot visualize points as following a circular path about a stationary center. For these reasons we shall not invoke instant center concept as an analytical tool, but it might be useful for explaining some feature.

Chasle's theorem could be used to represent the velocity as the superposition of a translation that follows a special point C and a pure rotation about point C at angular speed $\bar{\omega}$. Point C has the property that its velocity is parallel to $\bar{\omega}$. This is a *screw motion*, whose terminology stems from an analogy with the movement of a screw with a right-handed thread, which is to advance in the direction of the outstretched thumb of the right hand when the fingers of that hand are curled in the sense that the screw turns. We shall not pursue such a representation because it does little to improve our ability to perform a kinematical analysis. However, some people do find it to be a useful way to visualize spatial motion.

EXAMPLE 4.1 Observation of the motion of the block reveals that at a certain instant the velocity of corner B is parallel to the diagonal BE. At this instant components relative to the body-fixed xyz coordinate system of the velocities of the other corners are believed to be $(v_A)_y = 10$, $(v_C)_y = 20$, $(v_D)_z = 10$, $(v_E)_x = 5$ m/s. Determine whether these values are possible, and if so, evaluate the velocity of corner F.



Example 4.1

SOLUTION In addition to illustrating application of the basic equation relating the velocity of points in a rigid body, the objective here is to emphasize that Chasle's theorem is embedded in any motion. It is given that $\bar{v}_B = v_B \bar{e}_{E/B}$, $\bar{v}_A \cdot \bar{j} = 10$, $\bar{v}_C \cdot \bar{j} = 20$, $\bar{v}_D \cdot \bar{k} = 10$, and $\bar{v}_E \cdot \bar{i} = 5$ m/s. There are four independent equations in the form of Eq. (4.1.8) relating the five corner points. Decomposing each into components would yield 12 scalar equations. The associated unknown scalar quantities are the speed of point B, the two unspecified components of the velocities of points A, C, D, and E, and the three components of $\bar{\omega}$, which are a set of 12 values. This reasoning suggests that the number of equations and unknowns will match in the contemplated analysis, so we proceed.

We select point B as the reference point for the translational motion, because the only unknown aspect of its velocity is the speed, that is,

$$\bar{v}_B = v_B \bar{e}_{E/B} = v_B \frac{1.5\bar{i} + 2\bar{j} - 1\bar{k}}{(1.5^2 + 2^2 + 1^2)^{1/2}} = v_B \left(0.5571\bar{i} + 0.7428\bar{j} - 0.3714\bar{k}\right).$$

The angular velocity is unknown, so we let $\bar{\omega} = \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k}$. The velocity equations relating point B to the other points are

$$\bar{v}_A = \bar{v}_B + \bar{\omega} \times \bar{r}_{A/B}, \quad \bar{r}_{A/B} = -1\bar{k},$$

$$\bar{v}_C = \bar{v}_B + \bar{\omega} \times \bar{r}_{C/B}, \quad \bar{r}_{C/B} = 2\bar{j},$$

$$\bar{v}_D = \bar{v}_B + \bar{\omega} \times \bar{r}_{D/B}, \quad \bar{r}_{D/B} = 2\bar{j} - 1\bar{k},$$

$$\bar{v}_E = \bar{v}_B + \bar{\omega} \times \bar{r}_{E/B}, \quad \bar{r}_{E/B} = 1.5\bar{i} + 2\bar{j} - 1\bar{k}.$$

Rather than forming the full set of scalar equations resulting from matching like components in each of the preceding vector equations, focusing on those velocity components that are specified lessens the number of equations to be solved. Thus we have

$$\bar{v}_A \cdot \bar{j} = 10 = 0.7428 v_B + \omega_x (1),$$

$$\bar{v}_C \cdot \bar{j} = 20 = 0.7428 v_B,$$

$$\bar{v}_D \cdot \bar{k} = 10 = -0.3714 v_B + \omega_x (2),$$

$$\bar{v}_E \cdot \bar{i} = 5 = 0.5571 v_B - \omega_y (1). - \omega_z (2).$$

Although these are four equations for the four unknown parameters, the equations are not solvable. The first three equations contain only two unknowns: ω_x and v_B. If we solve the second equation for v_B, the value of ω_x obtained from the first equation differs from the value obtained from the third equation. This means that the motion is *overconstrained*. In the exceptional situation in which the velocity components $\bar{v}_A \cdot \bar{j}$, $\bar{v}_C \cdot \bar{j}$, and $\bar{v}_D \cdot \bar{k}$ are selected such that there is a consistent solution for ω_x and v_B, we would still be unable to solve the problem because the fourth equation would be the sole relation for the two remaining unknowns: ω_y and ω_z. ◁

4.2 EULERIAN ANGLES

Three independent direction angles define the orientation of a set of xyz axes. Because there are a total of nine direction angles locating xyz with respect to an absolute reference frame XYZ, an independent set of angles may be selected in a variety of ways. Eulerian angles treat this matter as a specific sequence of rotations.

Let us follow the intermediate orientations of a moving reference frame as it is rotated away from its initial alignment with XYZ. The first rotation, called the *precession*, is about the fixed Z axis. The angle of rotation in the precession is denoted ψ, as depicted in Fig. 4.2. The orientation of the moving reference frame after it has undergone only the precession is denoted as $x'y'z'$. The transformation from $\bar{I}\bar{J}\bar{K}$ components to $\bar{i}'\bar{j}'\bar{k}'$ components is that of a single axis rotation about the Z (or z') axis, specifically

$$\begin{Bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{Bmatrix} = [R_\psi] \begin{Bmatrix} A_X \\ A_Y \\ A_Z \end{Bmatrix}, \tag{4.2.1}$$

where

$$[R_\psi] = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.2.2}$$

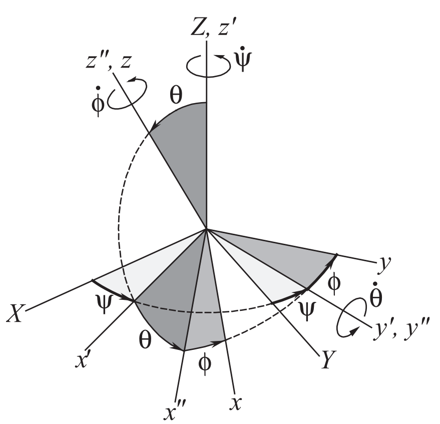


Figure 4.2. Definition of the Eulerian angles.

The second rotation is the *nutation,* and the nutation angle is θ. It is defined to be about an axis that is perpendicular to the precession axis. We label the nutation axis as y'. The orientation of the moving reference frame after the nutation is denoted $x''y''z''$ in Fig. 4.2. The second transformation, from $\bar{i}'\bar{j}'\bar{k}'$ components to $\bar{i}''\bar{j}''\bar{k}''$ components, is that of a simple rotation about the y' (or y'') axis, so

$$\begin{Bmatrix} A_{x''} \\ A_{y''} \\ A_{z''} \end{Bmatrix} = [R_\theta] \begin{Bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{Bmatrix}, \tag{4.2.3}$$

where

$$[R_\theta] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}. \tag{4.2.4}$$

The last rotation is the *spin* ϕ, which moves the reference frame from $x''y''z''$ to its final orientation xyz. It is executed about the z'' (or z) axis, so this is a simple rotation transformation from $\bar{i}''\bar{j}''\bar{k}''$ components to $\bar{i}\,\bar{j}\bar{k}$ components given by

$$\begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} = [R_\phi] \begin{Bmatrix} A_{x''} \\ A_{y''} \\ A_{z''} \end{Bmatrix}, \tag{4.2.5}$$

where

$$[R_\phi] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.2.6}$$

The sequence of rotations, precession, nutation, then spin, constitute a set of body-fixed rotations. Consequently, we can use the transformation properties of such a sequence to relate any two sets of unit vectors. For example, the overall transformation from $\bar{I}\bar{J}\bar{K}$ components to $\bar{i}\,\bar{j}\bar{k}$ components is obtained by postmultiplication, according to

$$\begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} = [R] \begin{Bmatrix} A_X \\ A_Y \\ A_Z \end{Bmatrix}, \quad [R] = [R_\phi][R_\theta][R_\psi]. \tag{4.2.7}$$

The angular velocity and angular acceleration are readily expressed by adding the precession, nutation, and spin rates about the respective axes. The precession is about the Z axis, so the first rotation rate is $\dot{\psi}$, the direction is $\bar{e}_1 = \bar{K}$, and the angular velocity of XYZ is $\bar{\Omega}_1 = \bar{0}$. The nutation occurs about the y' axis. (The term *line of nodes* is sometimes used to refer to the y' axis because points on this axis do not move in the nutation.) The nutation rate is $\dot{\theta}$, the direction is $\bar{e}_2 = \bar{j}' \equiv \bar{j}''$, and the angular velocity of $x'y'z'$ is $\bar{\Omega}_2 = \dot{\psi}\bar{K}$. Finally, the spin rate is $\dot{\phi}$ about the z'' axis, so the third rotation direction is $\bar{e}_3 = \bar{k}''$, and the angular velocity of $x''y''z''$ is $\bar{\Omega}_3 = \dot{\psi}\bar{K} + \dot{\theta}\bar{j}''$. The angular velocity of xyz is the vector sum of the individual rotation rates, so

$$\boxed{\bar{\omega} = \dot{\psi}\bar{K} + \dot{\theta}\bar{j}'' + \dot{\phi}\bar{k}''.} \tag{4.2.8}$$

The general expression for angular acceleration obtained by using each angular velocity $\bar{\Omega}_j$ to differentiate the corresponding rotation direction $\bar{e}_j$ is

$$\boxed{\begin{aligned}\bar{\alpha} &= \ddot{\psi}\bar{K} + \ddot{\theta}\bar{j}'' + \dot{\theta}\left(\bar{\Omega}_2 \times \bar{j}''\right) + \ddot{\phi}\bar{k}'' + \dot{\phi}\left(\bar{\Omega}_3 \times \bar{k}''\right) \\ &= \ddot{\psi}\bar{K} + \ddot{\theta}\bar{j}'' + \dot{\psi}\dot{\theta}\left(\bar{K} \times \bar{j}''\right) + \ddot{\phi}\bar{k}'' + \dot{\psi}\dot{\phi}\left(\bar{K} \times \bar{k}''\right) + \dot{\theta}\dot{\phi}\left(\bar{j}'' \times \bar{k}''\right).\end{aligned}} \tag{4.2.9}$$

To use these expressions in computations, they must be transformed to a common set of components. Many situations involve bodies that have an axisymmetric shape, with the z axis defined to be the axis of symmetry. In such cases there is no special orientation of the x and y axes relative to the body, so using $x''y''z''$ as the global coordinate system will yield a description of vector quantities from the viewpoint of the body that is generally descriptive. Inspection of Fig. 4.2 shows that $\bar{K}$ lies in the $x''z''$ plane, such that

$$\bar{K} = -\sin\theta\bar{i}'' + \cos\theta k''. \tag{4.2.10}$$

We thereby find that

When the angular velocity $\bar{\omega}$ and angular acceleration $\bar{\alpha}$ of reference frame xyz are described in terms of the Eulerian angles of precession ψ, nutation θ, and spin ϕ, then

$$\boxed{\begin{aligned}\bar{\omega} &= -\dot{\psi}\sin\theta\bar{i}'' + \dot{\theta}\bar{j}'' + \left(\dot{\psi}\cos\theta + \dot{\phi}\right)\bar{k}'', \\ \bar{\alpha} &= \left(-\ddot{\psi}\sin\theta - \dot{\psi}\dot{\theta}\cos\theta + \dot{\theta}\dot{\phi}\right)\bar{i}'' + \left(\ddot{\theta} + \dot{\psi}\dot{\phi}\sin\theta\right)\bar{j}'' \\ &\quad + \left(\ddot{\psi}\cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta}\sin\theta\right)\bar{k}'',\end{aligned}} \tag{4.2.11}$$

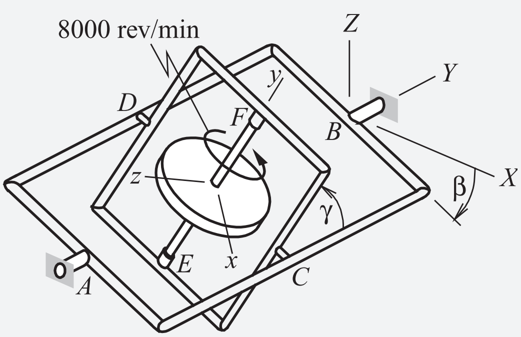
where $x''y''z''$ is a reference frame that executes only the precession and nutation. If appropriate, the preceding expressions may be transformed to $\bar{i}\,\bar{j}\bar{k}$ components by applying Eq. (4.2.5). This operation yields

$$\boxed{\begin{aligned}\bar{\omega} &= \left(-\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi\right)\bar{i} + \left(\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi\right)\bar{j} + \left(\dot{\psi}\cos\theta + \dot{\phi}\right)\bar{k}, \\ \bar{\alpha} &= \left(-\ddot{\psi}\sin\theta\cos\phi + \ddot{\theta}\sin\phi - \dot{\psi}\dot{\theta}\cos\theta\cos\phi + \dot{\psi}\dot{\phi}\sin\theta\sin\phi + \dot{\theta}\dot{\phi}\cos\phi\right)\bar{i} \\ &\quad + \left(\ddot{\psi}\sin\theta\sin\phi + \ddot{\theta}\cos\phi + \dot{\psi}\dot{\theta}\cos\theta\sin\phi + \dot{\psi}\dot{\phi}\sin\theta\cos\phi - \dot{\theta}\dot{\phi}\sin\phi\right)\bar{j} \\ &\quad + \left(\ddot{\psi}\cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta}\sin\theta\right)\bar{k}.\end{aligned}} \tag{4.2.12}$$

Utilization of Eulerian angles requires recognition of the appropriate axes of rotation. This involves identifying a fixed axis of rotation as the precession axis. Then the nutation axis precesses orthogonally to the precession axis. Finally, the spin axis precesses and nutates while it remains perpendicular to the nutation axis. In many cases, the nutation or spin rates may be zero, in which case either of the respective angles is constant. This results in a degree of arbitrariness in the selection of the axes. Indeed, a simple rotation can be considered to be solely precession, nutation, or spin, as one wishes.

From one perspective it is sufficient to rely on Eqs. (4.2.12) because they reduce to Eqs. (4.2.11) when $\phi = 0$, which corresponds to xyz being coincident with $x''y''z''$. On the other hand, the raw expressions in Eqs. (4.2.8) and (4.2.9) are readily adapted to a variety of situations. This is especially true if the axes are labeled differently from the convention used here. For example, some texts define the x' axis to be the line of nodes for nutation. Also, a different definition of the Eulerian angles often is used by aerodynamicists. (This is the topic of Exercise 4.6.) Another consideration is that the Eulerian angle formulas may not be directly applicable. For example, the representation addresses a motion featuring no more than three rotations, so one set of Eulerian angles cannot describe a situation that features more than three rotations. Such situations could be treated by defining multiple sets of transformations. Another case in which the Eulerian angle formulation is inadequate by itself arises when a motion consists of three rotations in which no two rotation axes are orthogonal. No line of nodes is evident in that case. This is another situation that can be treated with more than one set of Eulerian angles.

EXAMPLE 4.2 A free gyroscope consists of a flywheel that rotates relative to the inner gimbal at the constant angular speed of 8000 rev/min, and the rotation of the inner gimbal relative to the outer gimbal is $\gamma = 0.2\sin(100\pi t)$ rad. The rotation of the outer gimbal is $\beta = 0.5\sin(50\pi t)$ rad. Use the Eulerian angle formulas to determine the angular velocity and angular acceleration of the flywheel at $t = 4$ ms. Express the results in terms of components relative to the body-fixed xyz and space-fixed XYZ reference frames, where the x axis was directed from bearing D to bearing C at $t = 0$.



Example 4.2

SOLUTION This example explains how one can apply the Eulerian angle formulas when none of the rotations are explicitly stated to be those parameters. The primary task in applying the formulas is identification of the precession, nutation, and spin. The angle β is the rotation about a fixed axis, so it is the precession. Thus we set $\psi = \beta = 0.5\sin(50\pi t)$ in the formulas. By definition, the nutation axis is perpendicular to the precession axis. This matches the characteristic of line CD for the γ rotation, so we identify $\bar{e}_{C/D}$ as the line of nodes, and set $\theta = \gamma = 0.2\sin(100\pi t)$ in the formulas. The 8000 rev/min rotation occurs in the direction of $\bar{e}_{F/E}$, which is perpendicular to the line of nodes, so it fits the spinning rotation. Furthermore, it is specified that the x axis was aligned with $\bar{e}_{c/D}$ at $t = 0$, so we set $\phi = 8000\,(2\pi/60)\,t$ rad. The Eulerian angles and their derivatives at $t = 0.004$ are

$$\begin{gathered}\psi = 0.2939 \text{ rad},\ \dot{\psi} = 63.54 \text{ rad/s},\ \ddot{\psi} = -7\,252 \text{ rad/s}^2,\\ \theta = 0.19021 \text{ rad},\ \dot{\theta} = 19.416 \text{ rad/s},\ \ddot{\theta} = -18\,773 \text{ rad/s}^2,\\ \phi = 3.351 \text{ rad},\ \dot{\phi} = 837.8 \text{ rad/s},\ \ddot{\phi} = 0.\end{gathered} \tag{1}$$

To avoid confusing the xyz coordinate system in the definition of Eulerian angles with the coordinate system defined here, let us use a subscript f to denote the unit vectors in the formulas, Eqs. (4.2.12). Then the result of substituting the values in Eqs. (1) into those formulas is

$$\begin{gathered}\bar{\omega} = 7.71\bar{i}_f - 21.49\bar{j}_f + 900.15\bar{k}_f \text{ rad/s},\\ \bar{\alpha} = -14256\bar{i}_f + 11934\bar{j}_f - 7354\bar{k}_f \text{ rad/s}^2.\end{gathered} \tag{2}$$

The rotation transformation $[R]$ in Eq. (4.2.7) relates vector components in the body-fixed system to the standard fixed reference frame $X_f Y_f Z_f$. For the angles in Eqs. (1), this transformation is

$$\begin{gathered}\begin{bmatrix}\bar{i}_f & \bar{j}_f & \bar{k}_f\end{bmatrix}^{\mathrm{T}} = [R]\begin{bmatrix}\bar{I}_f & \bar{J}_f & \bar{K}_f\end{bmatrix}^{\mathrm{T}},\\ [R] = \begin{bmatrix}-0.85909 & -0.47723 & 0.18494\\ 0.47876 & -0.87707 & -0.03931\\ 0.18096 & 0.05477 & 0.98196\end{bmatrix}.\end{gathered}$$

We use the inverse transformation to obtain the fixed coordinate system components of $\bar{\omega}$ and $\bar{\alpha}$, as follows:

$$\begin{gathered}\{\omega\}_f = [R]^{\mathrm{T}}\begin{Bmatrix}7.71\\ -21.49\\ 900.15\end{Bmatrix} = \begin{Bmatrix}145.98\\ 64.47\\ 886.19\end{Bmatrix},\\ \{\alpha\}_f = [R]^{\mathrm{T}}\begin{Bmatrix}-14256\\ 11934\\ -7354\end{Bmatrix} = \begin{Bmatrix}16\,630\\ -4\,066\\ -10\,327\end{Bmatrix}.\end{gathered} \tag{3}$$

It still remains to identify the components corresponding to the coordinate systems that were defined in the problem statement. Toward that end we observe that the spin occurs about the y axis, so it must be that $\bar{k}_f = \bar{j}$. We also observe that the x axis aligned with $\bar{e}_{C/D}$ at $t = 0$, which is the line of nodes. In the Eulerian definition the y_f axis lines up with the line of nodes when $\phi = 0$, which matches the alignment of the x axis here. Thus we set $\bar{j}_f = \bar{i}$. Then, because $\bar{j}_f \times \bar{k}_f \equiv \bar{i} \times \bar{j}$, it must be that $\bar{i}_f = \bar{k}$. Substitution of these equivalencies into Eqs. (2) yields

$$\begin{aligned}\bar{\omega} &= -21.49\bar{i} + 900.2\bar{j} + 7.71\bar{k} \text{ rad/s},\\ \bar{\alpha} &= 11934\bar{i} - 7354\bar{j} - 14256\bar{k} \text{ rad/s}^2.\end{aligned} \qquad (4) \triangleleft$$

In regard to the fixed coordinate components, we observe in the present situation that $\bar{j} = \bar{J}$ and $\bar{i} = \bar{I}$ when all of the Eulerian angles are zero. Hence it must be that the fixed unit vectors are permuted in the same way as the moving unit vectors. In other words, $\bar{K}_f = \bar{J}$, $\bar{J}_f = \bar{I}$, and $\bar{I}_f = \bar{K}$. Selecting the elements in Eqs. (3) accordingly then leads to

$$\begin{aligned}\bar{\omega} &= 64.47\bar{I} + 886.19\bar{J} + 145.98K \text{ rad/s},\\ \bar{\alpha} &= -4\,066\bar{I} - 10\,327\bar{J} + 16\,630\bar{K} \text{ rad/s}^2.\end{aligned} \qquad \triangleleft$$

As an aside, it should be mentioned that if one is confronted in practice with the situation posed here, an approach based on returning to the basic formulation might be easier, and less prone to error, than one that relies on formulaic subsitutions.

4.3 INTERCONNECTIONS AND LINKAGES

According to Eq. (4.1.7), the displacement of any point in a rigid body from a known initial state can be evaluated if the displacement of a reference point and the rotation transformation in the final state are known. Point displacement is a three-component vector, and a rotation transformation is defined completely by three angles, such as the Eulerian angles. Thus a rigid body in free motion has six degrees of freedom.

Most bodies are restricted in their movement because they are connected to adjoining bodies. These connections give rise to *kinematical constraint equations*, which are mathematical statements of conditions that are imposed on the motion of a point or on the angular motion of a body. Such equations are relations between kinematical variables that must be satisfied under any circumstance, regardless of the nature of the forces that actuate the motion. The kinematical constraints are imposed by *constraint forces* (and couples), which are more commonly known as *reactions*. The role of constraint forces and their relation to the kinematical conditions they impose will be treated in the chapters on kinetics.

A simple, though common, constraint condition arises when a body is permitted to execute only a planar motion. By definition, planar motion means that all points in the body move in parallel planes, which can only happen if the angular velocity is always perpendicular to these planes. Let the XY plane of the fixed reference frame and the

xy plane of the body fixed reference frame be coincident planes of motion. Points that differ only in their z coordinate execute the same motion in this case, so they may be considered to be situated in the xy plane. Hence the kinematical equations for planar motion are

$$\begin{gathered}\bar{\omega} = \omega\bar{k} = \omega\bar{K}, \quad \bar{\alpha} = \dot{\omega}\bar{k} = \dot{\omega}\bar{K}, \\ r_{P/A} = x\bar{i} + y\bar{j} = X\bar{I} + Y\bar{J}, \\ \bar{v}_P = \bar{v}_A + \bar{\omega} \times \bar{r}_{P/A}, \\ \bar{a}_P = \bar{a}_A + \bar{\alpha} \times \bar{r}_{P/A} - \omega^2\bar{r}_{P/A},\end{gathered} \tag{4.3.1}$$

where point P is an arbitrary point in the body and point A is any convenient reference point in the same body. Note that the preceding centripetal acceleration term is simplified from $\bar{\omega} \times (\bar{\omega} \times \bar{r}_{P/A})$ to $-\omega^2\bar{r}_{P/A}$ by an identity that is valid only when $\bar{r}_{P/A}$ is perpendicular to $\bar{\omega}$. These relations are depicted in Fig. 4.3.

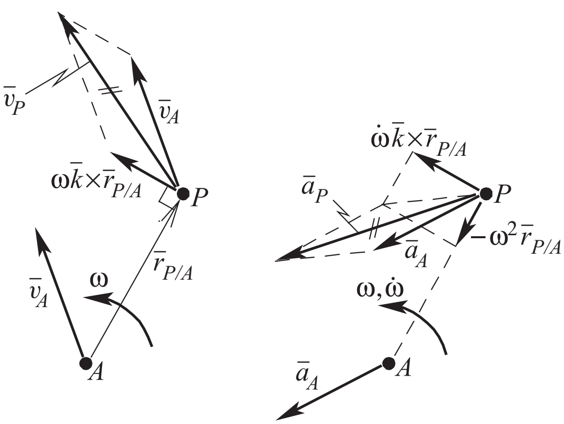


Figure 4.3. Relation of the velocity and acceleration of two points in a rigid body undergoing planar motion.

The restriction to planar motion contains an implicit assumption that the connections to other bodies permit rotation only about an axis perpendicular to the plane. An analysis of spatial motion requires explicit consideration of the constraint conditions arising from connections. These may be identified by characterizing the nature of the connection using the fundamental concepts developed thus far. For example, the *ball-and-socket joint* connecting bodies 1 and 2 in Fig. 4.4 allows each body to rotate freely about its center point B, so it does not impose a constraint on the orientation of either

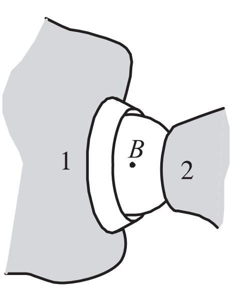


Figure 4.4. Ball-and-socket joint.

body. However, the center B of the ball is common to both bodies, so the bodies must move in unison at this junction. This means that the displacement, velocity, and acceleration of point B on each body must match, so that

$$(\Delta\bar{r}_B)_1 = (\Delta\bar{r}_B)_2\,,\quad (\bar{v}_B)_1 = (\bar{v}_B)_2\,,\quad (\bar{a}_B)_1 = (\bar{a}_B)_2 \tag{4.3.2}$$

The case of a *pin connection* between bodies, depicted in Fig. 4.5, has some elements in common with a ball-and-socket joint. In the figure, $x_1y_1z_1$ and $x_2y_2z_2$ represent reference frames attached to each of the joined bodies, with their shared z axis aligned along the axis of the pin. We define each x axis to align with a convenient reference line in the respective body, such as the centerline for a bar. The direction of the pin's axis may be evaluated from a cross product:

$$\bar{e}_{\text{pin}} = \bar{k}_1 = \bar{k}_2 = \frac{\bar{i}_1 \times \bar{i}_2}{\left|\bar{i}_1 \times \bar{i}_2\right|} \tag{4.3.3}$$

As is true of the ball-and-socket joint, both bodies have the same motion at their point of commonality. Consequently, Eqs. (4.3.2) must be satisfied. However, the pin also introduces a constraint on rotation. The only rotation of body 2 relative to body 1 permitted by the pin connection is a spin about the z_1 or z_2 axis. We denote the spin angle as ϕ, with $\phi = 0$ defined as the configuration in which $x_1y_1z_1$ and $x_2y_2z_2$ coincide. A simple rotation transformation about the z axis gives $\bar{i}_2\bar{j}_2\bar{k}_2$ components in terms of $\bar{i}_1\bar{j}_1\bar{k}_1$ components:

$$\begin{Bmatrix} A_{x_2} \\ A_{y_2} \\ A_{z_2} \end{Bmatrix} = [R_\phi] \begin{Bmatrix} A_{x_1} \\ A_{y_1} \\ A_{z_1} \end{Bmatrix}, \quad [R_\phi] = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1. \end{bmatrix} \tag{4.3.4}$$

The constraints introduced by a pin connection on the angular velocity and angular acceleration may be expressed in vector form. Because $x_2y_2z_2$ spins at $\dot{\phi}$ about the $\bar{k}_1 = \bar{k}_2$ axis, the respective angular velocities are related by

$$\bar{\omega}_2 = \bar{\omega}_1 + \dot{\phi}\bar{k}_1. \tag{4.3.5}$$

The angular velocity of $x_1y_1z_1$ is $\bar{\omega}_1$, so the time derivative of the preceding is

$$\bar{\alpha}_2 = \bar{\alpha}_1 + \ddot{\phi}\bar{k}_1 + \dot{\phi}\left(\bar{\omega}_1 \times \bar{k}_1\right). \tag{4.3.6}$$

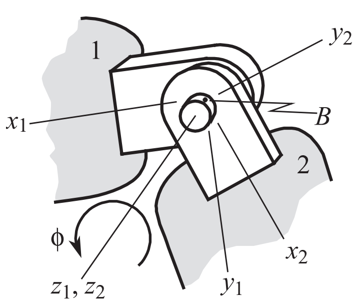


Figure 4.5. Pin connection.

Equations (4.3.4)–(4.3.6) are constraint equations on the angular motion that must be satisfied in addition to Eqs. (4.3.2) for the connecting points. Each of these relations may be transformed to any convenient global set of components. It is interesting to observe that, if a system is arranged such that all pin connections are parallel, then the rotational constraint equations lead to the conclusion that the z axis of each pin is invariant. In other words, using pins aligned perpendicular to a fixed plane will lead to planar motion of a system in that plane.

Another common method for connecting bodies consists of a *collar* that slides over a bar, as depicted in Fig. 4.6. (This connection is also known as a *slider*.) Similar to the treatment of a ball-and-socket joint, point C in the figure denotes the center of the ball. For most purposes, the distance to the adjacent point B on the centerline of bar 1 is small relative to the overall dimensions of a system. We take this distance to be zero for the sake of simplicity. (The modifications required to account for the finiteness of this distance are the topic of Exercise 4.7.) The collar is free to slide over bar 1. We characterize the constraint condition in this case by attaching reference frame $x_1y_1z_1$ to bar 1, with the x_1 axis aligned with the centerline of that bar. With respect to this reference frame, the collar can move along only the x_1 axis. We let u denote the amount of this relative displacement, so $(\Delta\bar{r}_C)_{x_1y_1z_1} = u\bar{i}_1$, and approximate $\bar{r}_{C/B}$ to be zero. Taking point O' in Eq. (3.2.8) to be point B in the present situation leads to the conclusion that the collar's displacement in terms of components relative to the axes of $x_1y_1z_1$ is given by

$$\begin{Bmatrix} \Delta\bar{r}_C\cdot\bar{i} \\ \Delta\bar{r}_C\cdot\bar{j} \\ \Delta\bar{r}_C\cdot\bar{k} \end{Bmatrix} = \begin{Bmatrix} \Delta\bar{r}_B\cdot\bar{i} \\ \Delta\bar{r}_B\cdot\bar{j} \\ \Delta\bar{r}_B\cdot\bar{k} \end{Bmatrix} + \begin{Bmatrix} u \\ 0 \\ 0 \end{Bmatrix}.$$

Alternatively, the general description in Eq. (3.2.7) may be used to describe the absolute displacement, which leads to

$$\begin{Bmatrix} \Delta\bar{r}_C\cdot\bar{I} \\ \Delta\bar{r}_C\cdot\bar{J} \\ \Delta\bar{r}_C\cdot\bar{K} \end{Bmatrix} = \begin{Bmatrix} \Delta\bar{r}_B\cdot\bar{I} \\ \Delta\bar{r}_B\cdot\bar{J} \\ \Delta\bar{r}_B\cdot\bar{K} \end{Bmatrix} + [R]_{\mathrm{f}}^{\mathrm{T}} \begin{Bmatrix} u \\ 0 \\ 0 \end{Bmatrix},$$

where $[R]$ describes the orientation of $x_1y_1z_1$. The velocity of point C relative to $x_1y_1z_1$ is $\dot{u}\bar{k}_1$, so we have

$$\bar{v}_C = \bar{v}_B + \dot{u}\bar{i}_1, \quad \bar{a}_C = \bar{a}_B + \ddot{u}\bar{i}_1 + 2\bar{\omega}_1 \times \dot{u}\bar{i}_1, \tag{4.3.7}$$

where the sign of u and its derivatives gives the sense of the sliding motion.

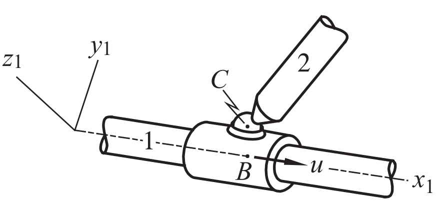


Figure 4.6. Collar with a ball-and-socket joint.

It should be noted that the analyses of the collar's displacement, velocity, and acceleration assumed that bar 1 is straight. If this bar were curved, it would be necessary to augment the relative acceleration with the centripetal acceleration term $\left(\dot{u}^2/\rho\right)\bar{e}_n$, where ρ is the radius of curvature of bar 1 at point B and $\bar{e}_n$ is the normal direction (pointing toward the center of curvature) for the segment at point B.

The collar in Fig. 4.6 introduces no angular motion restrictions because bar 2 is connected to the collar by a ball-and-socket joint. However, a common connection method is a pin, Fig. 4.7(a), or a *fork-and-clevis* joint, Fig. 4.7(b). If the cross section of bar 1 is not circular, interference prevents the collar from spinning about that bar. In that case the constraints on angular motion are the same as Eqs. (4.3.4)–(4.3.6) for a pin connection. However, if the cross section of bar 1 is circular, then bar 1 acts as a pin that permits rotation about its axis. We treat the angular motion constraints for this connection by attaching $x_2y_2z_2$ to bar 2; the z_2 axis is chosen to align with the axis of the pin connecting bar 2 and the collar, and the x_2 axis coincides with a convenient reference line of bar 2.

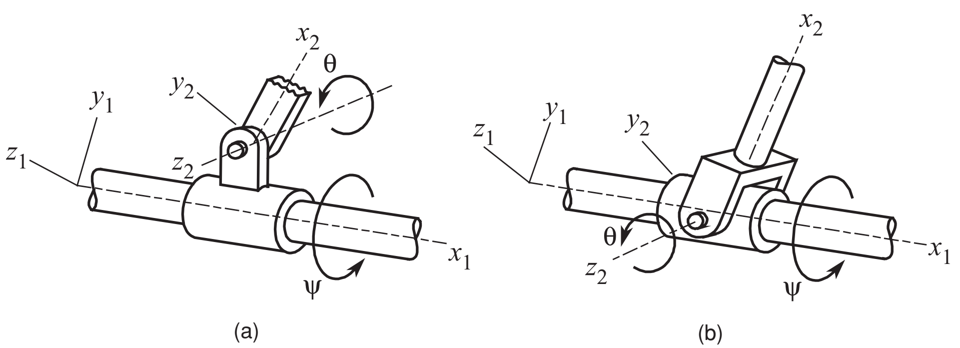


Figure 4.7. Collar connections that impose rotational constraints: (a) collar with a pin connection, (b) collar with a fork-and-clevis joint.

Let ψ denote the angle by which the collar rotates relative to bar 1 about the x_1 axis, and let θ be the rotation of bar 2 about the pin relative to the collar. (As implied by their labels, ψ represents precession of bar 2 relative to bar 1, and θ represents a relative spin.) Define these angles such that $\psi = \theta = 0$ corresponds to $x_1y_1z_1$ and $x_2y_2z_2$ being parallel. These are a pair of body-fixed rotations, so the rotation transformation from $\bar{i}_1\bar{j}_1\bar{k}_1$ to $\bar{i}_2\bar{j}_2\bar{k}_2$ is given by

$$\begin{Bmatrix} A_{x_2} \\ A_{y_2"} \\ A_{z_2} \end{Bmatrix} = [R_\theta][R_\psi] \begin{Bmatrix} A_{x_1} \\ A_{y_1} \\ A_{z_1} \end{Bmatrix}, \qquad (4.3.8)$$

where $[R_\psi]$ and $[R_\theta]$ are simple rotation transformations for ψ about the x_1 axis and by θ about the z_2 axis, respectively. The rotation rates of bar 2 relative to bar 1 are $\dot{\psi}$ about the x_1 axis, and by $\dot{\theta}$ about the z_2 axis, so

$$\bar{\omega}_2 = \bar{\omega}_1 + \dot{\psi}\bar{i}_1 + \dot{\theta}\bar{k}_2. \qquad (4.3.9)$$

Using the angular velocities of each reference frame to describe the time derivative of the respective unit vectors leads to

$$\bar{\alpha}_2 = \bar{\alpha}_1 + \ddot{\psi}\bar{i}_1 + \dot{\psi}\left(\bar{\omega}_1 \times \bar{i}_1\right) + \ddot{\theta}\bar{k}_2 + \dot{\theta}\left(\bar{\omega}_2 \times \bar{k}_2\right). \tag{4.3.10}$$

Equations (4.3.7)–(4.3.10) constitute the constraint equations for a collar sliding over bar 1 whose cross section is circular. As always, these conditions may be represented in terms of any convenient global coordinate system.

It should be obvious from the discussion thus far that the connections need to be examined in detail to identify all constraints on the motion. If all of the permutations and novel features of various types of connections were to be tabulated, it would not aid our understanding. It is preferable to consider each connection on a case-by-case basis, and then to employ the type of reasoning developed thus far to identify the constraint equations.

As noted at the outset, it follows from Chasle's theorem that a rigid body in free motion has six degrees of freedom. If a system is composed of N bodies, its movement could be described by as many as $6N$ kinematical variables. However, the constraint equations for the appropriate types of connection and the fact that some connections are part of the same body reduce the number of free variables. The overall approach is to use the rotational constraints, if any exist, to characterize the angular motion of the body, simultaneously with employing kinematical equations (4.1.7)–(4.1.9) to relate the motion of constrained points in a body. When such relations are broken down into components, one obtains simultaneous equations for the kinematical variables describing the motion of each body.

If the motion of the system is *fully constrained*, then this system of equations will be solvable such that, for each body, the linear motion of a point and the angular motion may be evaluated. The system is only *partially constrained* if the number of kinematical variables exceeds the number of kinematical equations. The simultaneous equations may then be solved for a set of excess variables in terms of the other. The excess variables in this case depend on the nature of the force system, so their evaluation requires a kinetics study. The number of excess kinematical variables is the system's *number of degrees of freedom*. Another possibility is that the kinematical equations are not solvable. In that case, there are too many constraints on the motion of the system. This means that no motion is possible – such a system is rigid.

In principle, it is possible to analyze the constrained motion of a system in terms of its displacements and angles of rotation. However, the occurrence of these angles as sines and cosines in component descriptions and in rotation transformations, as well as the complicated nature of the spatial geometry in many situations, combine to make it quite difficult to formulate and solve the associated kinematical equations. Furthermore, we will see in our later studies, beginning in Chapter 7, that some motion constraints restrict velocity, without any associated positional restrictions. For both reasons, our efforts here focus on the kinematical analysis of velocity and acceleration when the system is at a given position. In Chapter 8 we will see that such an analysis can be used to obtain differential equations whose solution will be the position variables as a function of time.

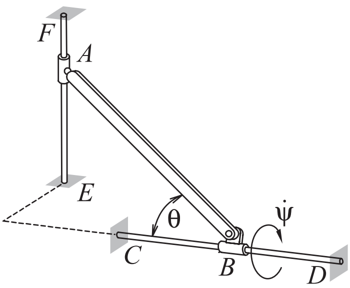


Figure 4.8. Spatial motion of a bar subject to constraints.

To demonstrate these matters, consider bar AB in Fig. 4.8, which is constrained by collars that follow noncoplanar guide bars CD and EF. The connection at collar A is a ball-and-socket joint, whereas the one at collar B is a pin. Because points A and B are part of the same rigid body, their velocities are related by

$$\bar{v}_A = \bar{v}_B + \bar{\omega} \times \bar{r}_{A/B}. \tag{4.3.11}$$

We next address the constraint conditions associated with the connections. Because the guide bars are fixed, velocity constraint equations (4.3.7) state that

$$\bar{v}_A = v_A \bar{e}_{E/F}, \quad \bar{v}_B = v_B \bar{e}_{D/C}, \tag{4.3.12}$$

where positive values of v_A and v_B indicate that the respective velocities are taken to be downward and to the right. (If the actual sense of each motion is contrary to the assumed one, then the associated rate will be negative.) Let us for the moment ignore the constraint on the rotation of bar AB that is introduced by the pin on collar B. Then the angular velocity of the bar is an unknown $\bar{\omega}_{AB}$ having three components. Equation (4.3.11) reduces to three scalar equations, one for each component, and there are five unknowns: v_A, v_B, and the three components of $\bar{\omega}_{AB}$. (It is assumed that we have worked out whatever geometrical relations are required to describe distances and angles, so there are no unknown positional parameters.)

To further characterize the system, we need to account for the rotational constraint on $\bar{\omega}_{AB}$ imposed by having a pin at collar B, which is described by Eq. (4.3.9). (This assumes that guide bar CD has a round cross section.) The guide bar is fixed, so it serves as the precession axis for rotation ψ, which equivalently is the angle by which plane ABC has rotated away from vertical. Thus the precession direction is $\bar{k}_1 \equiv \bar{e}_{C/D}$, based on the sense with which ψ is depicted in Fig. 4.8. In addition, the angle θ between bar AB and guide bar CD is equivalent to the angle of rotation about the axis of the pin, which is assumed to be perpendicular to the plane containing AB and CD. We represent this direction with a cross product, with the sequence of terms based on the right-hand rule for when θ increases:

$$\bar{j}_2 = \frac{\bar{e}_{C/D} \times \bar{r}_{A/B}}{\left|\bar{e}_{C/D} \times \bar{r}_{A/B}\right|}. \tag{4.3.13}$$

When we use Eq. (4.3.9) to describe the angular velocity of bar AB, we find that

$$\bar{\omega}_{AB} = \dot{\psi} \bar{e}_{C/D} + \dot{\theta} \frac{\bar{e}_{C/D} \times \bar{r}_{A/B}}{\left|\bar{e}_{C/D} \times \bar{r}_{A/B}\right|}. \tag{4.3.14}$$

This reduces the number of unknown scalars in $\bar{\omega}_{AB}$ from three to two, thereby reducing the number of unknowns in the system to four: v_A, v_B, $\dot{\psi}$, and $\dot{\theta}$. Two possibilities arise now. In a fully constrained situation, the overall motion will be defined through some kinematical input, such as a specification of v_B, which removes the corresponding velocity parameter from the list of unknowns. In contrast, if the motion is induced by a given set of forces, so that none of the four kinematical parameters are specified, then the system is partially constrained. In that case, the three scalar equations obtained from Eq. (4.3.11) can be used to describe three of the unknowns in terms of the fourth. The system then has one degree of freedom, and kinetics principles would relate the remaining unknown to the force system.

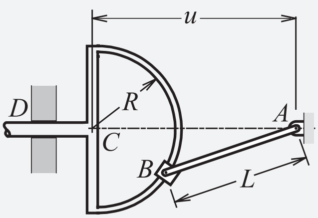
A different condition of partial constraint is obtained when bar AB is connected to both collars by ball-and-socket joints, because Eq. (4.3.14) then does not apply. The lack of constraint in such a case is associated with the ability of bar AB to spin about its own axis. Such a rotation does not affect the motion of either collar. The kinematical equations therefore can be solved for a relation between v_A and v_B, although there will be no unique solution for $\bar{\omega}_{AB}$. If it is desired that the number of equations and unknowns match, one could consider the spin of bar AB about its own axis to be zero, in which case one may invoke Eq. (4.3.14).

Other conditions are possible. Suppose that bar AB was connected to both collars by pins. That would introduce another constraint on $\bar{\omega}_{AB}$ having similar form to that of Eq. (4.3.14). It would not be possible to satisfy simultaneously both angular motion constraints, which would mean that the system is rigid. (The exception to this condition occurs if both guide bars are coplanar and the pins are perpendicular to that plane, in which case motion in that plane is possible.)

Thus far the discussion has only addressed the analysis of velocities. The treatment of acceleration follows a parallel development with the same logic as that by which the velocity was analyzed. It is essential to recognize that the velocities must be analyzed before accelerations can be addressed. One reason for this is that the angular velocities occur in the acceleration relation between two points in a body, Eq. (4.1.9). Also, angular accelerations in spatial motion contain terms that are the products of rotation rates. A third place where velocity parameters arise in an acceleration analysis is the characterization of the acceleration of a collar sliding along a curved bar, as was noted following Eqs. (4.3.7).

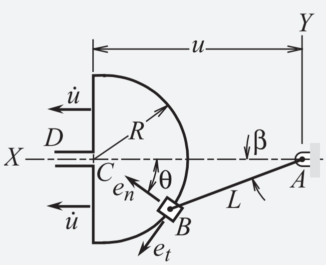
An area of special interest in kinematics is concerned with *linkages*, in which bars are interconnected sequentially in order to convert an input motion to a different output motion. From the standpoint of our general approach to rigid-body motion, the treatment of linkages presents no special problems. The constrained points in the system are the ends of the linkage, the connection points, and any point whose motion is specified. The velocity analysis is performed by using Eq. (4.1.8) to relate different connection points on each link. The linear velocity constraint equation appropriate to each connection is introduced into these relations between the velocity of points. Simultaneously, the angular velocity constraint equation associated with each connection is used to characterize the angular velocity of each link. In the special case of a robot, it is likely that the rotations about some or all of pins are controlled by servomotors, which remove these rotations from the list of unknowns. After the velocity analysis has been completed, one may carry out an acceleration analysis by following the same logic using the analogous acceleration equations. In comparison with the system in Fig. 4.8, the primary complication encountered in analyzing linkages is that the description of the position vectors and vector components is likely to be more difficult.

EXAMPLE 4.3 Collar B is pinned to arm AB as it slides over a circular guide bar. The guide bar translates to the left such that the distance from pivot A to the center C is an arbitrary function of time $u(t)$. Derive expressions for the angular velocity and angular acceleration of arm AB in terms of u.



Example 4.3

SOLUTION Although we have focused on three-dimensional motion, this example demonstrates that the procedures work equally well for planar problems. We begin with a sketch depicting the global XYZ coordinate system and the basic features of the geometrical configuration.



Coordinate systems and unit vectors for the planar linkage

The law of cosines gives expressions for the angles locating collar B:

$$\theta = \cos^{-1}\left(\frac{u^2 + R^2 - L^2}{2uR}\right), \quad \beta = \cos^{-1}\left(\frac{u^2 + L^2 - R^2}{2uL}\right) \tag{1}$$

These relations are valid at any instant so we could differentiate them to obtain relations for $\dot{\theta}$ and $\dot{\beta}$, and then $\ddot{\theta}$ and $\ddot{\beta}$. Such a procedure exemplifies a procedure for analyzing linkages based on describing the position of constrained points at an arbitrary instant and then differentiating such expressions. The geometrical complexity of many linkages often makes such an approach too difficult to implement.

Our general approach does not require differentiation of position vectors, because it recognizes that the kinematical relations for velocity and acceleration represent standard derivatives of position. Guide bar CD forms a translating reference frame for the collar's motion whose translational velocity is $\dot{u}$ to the left. The vectors $\bar{e}_t$ and $\bar{e}_n$ in the sketch are the path variable unit vectors for the motion of the collar relative to this reference frame, and we let v_{rel} denote the speed of this relative motion. We also know that the collar is pinned to bar AB. These two views of the velocity of the collar must match, so that

$$\bar{v}_B = \bar{v}_{CD} + v_{\text{rel}}\bar{e}_t = \bar{\omega}_{AB} \times \bar{r}_{B/A}.$$

Resolving the vectors into global components relative to XYZ leads to

$$\bar{v}_B = \dot{u}\bar{I} + v_{\text{rel}}\left(\sin\theta\bar{I} - \cos\theta\bar{J}\right) = \left(-\dot{\beta}\bar{K}\right) \times L\left(\cos\beta\bar{I} - \sin\beta\bar{J}\right). \tag{2}$$

We match like components:

$$\begin{aligned} \bar{v}_B \cdot \bar{I} &= \dot{u} + v_{\text{rel}}\sin\theta = -\dot{\beta}L\sin\beta, \\ \bar{v}_B \cdot \bar{J} &= -\,v_{\text{rel}}\cos\theta = -\dot{\beta}L\cos\beta. \end{aligned}$$

The solution of these component equations is

$$\begin{aligned} \dot{\beta} &= -\frac{\dot{u}\cos\theta}{L(\sin\beta\cos\theta + \cos\beta\sin\theta)} \equiv -\frac{\dot{u}}{L}\frac{\cos\theta}{\sin(\beta+\theta)}, \\ v_{\text{rel}} &= -\dot{u}\frac{\cos\beta}{\sin(\beta+\theta)}. \end{aligned} \tag{3}$$

Solution of Eqs. (1) gives the values of β and θ as functions of u, so the first of Eqs. (3) gives the angular velocity of bar AB as an implicit function of u and $\dot{u}$.

The analysis of acceleration follows the same logical outline. We match the acceleration of the collar from the viewpoint of the translating bar CD to the acceleration based on the collar being attached to rotating bar AB. The translational acceleration is $\ddot{u}\bar{I}$, and the collar follows a circular path of radius R relative to bar CD, so we have

$$\bar{a}_B = \ddot{u}\bar{I} + \dot{v}_{\text{rel}}\bar{e}_t + \frac{v_{\text{rel}}^2}{R}\bar{e}_n = \bar{\alpha}_{AB} \times \bar{r}_{B/A} - \omega_{AB}^2\bar{r}_{B/A}.$$

We resolve all vectors into $\bar{I}$ and $\bar{J}$ components:

$$\begin{aligned} \bar{a}_B &= \ddot{u}\bar{I} + \dot{v}_{\text{rel}}\left(\sin\theta\bar{I} - \cos\theta\bar{J}\right) + \frac{v_{\text{rel}}^2}{R}\left(\cos\theta\bar{I} + \sin\theta\bar{J}\right) \\ &= \left(-\ddot{\beta}\bar{K}\right) \times L\left(\cos\beta\bar{I} - \sin\beta\bar{J}\right) - \dot{\beta}^2 L\left(\cos\beta\bar{I} - \sin\beta\bar{J}\right). \end{aligned} \tag{4} \triangleleft$$

The component form of this relation is

$$\begin{aligned} \bar{a}_B \cdot \bar{I} &= \ddot{u} + \dot{v}_{\text{rel}}\sin\theta + \frac{v_{\text{rel}}^2}{R}\cos\theta = -\ddot{\beta}L\sin\beta - \dot{\beta}^2 L\cos\beta, \\ \bar{a}_B \cdot \bar{J} &= -\,\dot{v}_{\text{rel}}\cos\theta + \frac{v_{\text{rel}}^2}{R}\sin\theta = -\ddot{\beta}L\cos\beta + \dot{\beta}^2 L\sin\beta. \end{aligned} \tag{5}$$

We eliminate $\dot{v}_{\text{rel}}$ from these equations, which gives

$$\ddot{u}\cos\theta + \frac{v_{\text{rel}}^2}{R} = -\ddot{\beta}L(\sin\beta\cos\theta + \cos\beta\sin\theta) - \dot{\beta}^2 L(\cos\beta\cos\theta - \sin\beta\sin\theta)\,.$$

The identities for the sine and cosine of the sum of two angles simplify this to

$$\ddot{\beta} = -\frac{\ddot{u}}{L}\frac{\cos\theta}{\sin(\beta+\theta)} - \dot{\beta}^2\cot(\beta+\theta) - \frac{v_{\text{rel}}^2}{RL\sin(\beta+\theta)}.$$

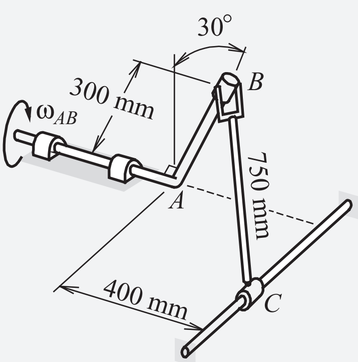
Equations (3) give the values of v_{rel} and $\dot{\beta}$, so the preceding becomes

$$\ddot{\beta} = -\frac{\ddot{u}}{L}\frac{\cos\theta}{\sin(\beta+\theta)} - \frac{\dot{u}^2}{L^2}\frac{(\cos\theta)^2\cos(\beta+\theta)}{[\sin(\beta+\theta)]^3} - \frac{\dot{u}^2}{RL}\frac{(\cos\beta)^2}{[\sin(\beta+\theta)]^3}. \qquad \triangleleft$$

Once again, because Eqs. (1) give β and θ as functions of u, this result describes the angular acceleration of bar AB as an implicit function of u, $\dot{u}$, and $\ddot{u}$.

Before we leave this example, it is useful to observe that the primary complication for the preceding analysis was the need to carry out algebraic and trigonometric manipulations. The analysis would have been much simpler if a specific value of u had been specified, because numerical values would have occurred instead.

EXAMPLE 4.4 Arm AB is turned by a motor at a constant rate of 1800 rev/min. Cap B connects this bar to bar BC with a fork-and-clevis joint, and the cap is free to rotate about the AB axis. The connection between bar BC and collar C is a ball-and-socket joint. Determine the velocity and acceleration of collar C and the angular velocity and angular acceleration of bar BC when the system is in the position shown.



Example 4.4

SOLUTION This example applies the general procedures to a system that is sometimes referred to as a four-bar linkage: rotating arm AB, connecting rod BC, slider C, and the fixed guide bar for the slider. The process is readily extended to multiple links. The global coordinate system we use has a vertical Z axis, and Y is aligned with the rotation axis of arm AB, with $\bar{J}$ pointing rightward. We begin by expressing the constraints on the motion of the connection points. Point A is stationary, collar C follows a straight path parallel to the X axis at an unknown speed v_C, and cap B is attached to both arm AB and connecting rod CD. Thus we have

$$\begin{aligned} \bar{v}_B &= \bar{\omega}_{AB} \times \bar{r}_{B/A} = v_C \bar{I} + \bar{\omega}_{BC} \times \bar{r}_{B/C}, \\ \bar{a}_B &= \bar{\alpha}_{AB} \times \bar{r}_{B/A} + \bar{\omega}_{AB} \times (\bar{\omega}_{AB} \times \bar{r}_{B/A}) \\ &= \dot{v}_C \bar{I} + \bar{\alpha}_{BC} \times \bar{r}_{B/C} + \bar{\omega}_{BC} \times (\bar{\omega}_{BC} \times \bar{r}_{B/C}). \end{aligned} \tag{1}$$

The position vectors at the instant depicted in the diagram are

$$\begin{aligned} \bar{r}_{B/A} &= 0.3\left(-\sin 30^\circ \bar{I} + \cos 30^\circ \bar{K}\right) = -0.15\bar{I} + 0.2598\bar{K} \text{ m}, \\ \bar{r}_{B/C} &= -\left[0.75^2 - 0.4^2 - (0.3\cos 30^\circ)^2\right]^{1/2} \bar{I} - 0.4\bar{J} + 0.3\cos 30^\circ \bar{K} \\ &= -0.5788\bar{I} - 0.4\bar{J} + 0.2598\bar{K} \text{ m}. \end{aligned} \tag{2}$$

Next, we describe the constraints on angular motion. It is given that

$$\bar{\omega}_{AB} = 1800\left(\frac{2\pi}{60}\right)\left(-\bar{J}\right) \text{ rad/s}, \quad \bar{\alpha}_{AB} = \bar{0}. \tag{3}$$

Cap C rotates at an unknown rate $\dot{\psi}$ about axis AB relative to arm AB. Relative to the cap, the fork-and-clevis joint allows bar BC only to rotate at an unknown rate $\dot{\theta}$ about the axis of that joint's pin. Thus we have

$$\bar{\omega}_{\text{cap}} = \bar{\omega}_{AB} + \dot{\psi}\bar{e}_{B/A}, \quad \bar{\omega}_{BC} = \bar{\omega}_{\text{cap}} + \dot{\theta}\bar{e}_{\text{pin}} = \bar{\omega}_{AB} + \dot{\psi}\bar{e}_{B/A} + \dot{\theta}\bar{e}_{\text{pin}}. \tag{4}$$

To describe the angular acceleration we observe that $\bar{e}_{B/A}$ rotates at $\bar{\omega}_{AB}$ while $\bar{e}_{\text{pin}}$ rotates at $\bar{\omega}_{\text{cap}}$. Because $\bar{\alpha}_{AB} = \bar{0}$, we have

$$\bar{\alpha}_{BC} = \ddot{\psi}\bar{e}_{B/A} + \dot{\psi}\left(\bar{\omega}_{AB} \times \bar{e}_{B/A}\right) + \ddot{\theta}\bar{e}_{\text{pin}} + \dot{\theta}\left(\bar{\omega}_{\text{cap}} \times \bar{e}_{\text{pin}}\right). \tag{5}$$

We obtain $\bar{e}_{B/A}$ from $\bar{r}_{B/A}$ and evaluate $\bar{e}_{\text{pin}}$ as the normal to the plane formed by $\bar{r}_{B/A}$ and $\bar{r}_{B/C}$, where both position vectors are given by Eqs. (2). Thus,

$$\begin{aligned} \bar{e}_{B/A} &= \frac{\bar{r}_{B/A}}{\left|\bar{r}_{B/A}\right|} = -0.5\bar{I} + 0.866\bar{K}, \\ \bar{e}_{\text{pin}} &= \frac{\bar{r}_{B/A} \times \bar{r}_{B/C}}{\left|\bar{r}_{B/A} \times \bar{r}_{B/C}\right|} = 0.6347\bar{I} - 0.6804\bar{J} + 0.3664\bar{K}. \end{aligned} \tag{6}$$

Now that we have characterized the constraint conditions, we proceed to analyze the velocities. This will yield the unknown rotation rates, which we will need to know in order to evaluate the angular acceleration. The first of Eqs. (1) gives

$$\begin{aligned} \bar{v}_B &= \left(-60\pi\bar{J}\right) \times \left(-0.15\bar{I} + 0.2598\bar{K}\right) \\ &= v_C\bar{I} + [-60\pi\bar{J} + \dot{\psi}\left(-0.5\bar{I} + 0.866\bar{K}\right) + \dot{\theta}(0.6347\bar{I} - 0.6804\bar{J} \\ &\quad + 0.3664\bar{K})] \times \left(-0.5788\bar{I} - 0.4\bar{J} + 0.2598\bar{K}\right). \end{aligned}$$

The component equations obtained from this vector equation are

$$\bar{v}_B \cdot \bar{I} = -48.97 = v_C + 0.3464\dot{\psi} - 0.0302\dot{\theta} - 48.97,$$
$$\bar{v}_B \cdot \bar{J} = 0 = -0.3713\dot{\psi} - 0.3770\dot{\theta},$$
$$\bar{v}_B \cdot \bar{K} = -28.27 = 0.200\dot{\psi} - 0.6477\dot{\theta} - 109.10.$$

The solution of this set of linear equations is

$$v_C = -36.28 \text{ m/s}, \;\; \dot{\psi} = 96.46 \text{ rad/s}, \;\; \dot{\theta} = -95.01 \text{ rad/s}. \qquad (7) \lhd$$

The corresponding angular velocity obtained from the second of Eqs. (4) is

$$\bar{\omega}_{BC} = -\,108.53\bar{I} - 123.86\bar{J} + 48.72\bar{K} \text{ rad/s}. \qquad (8)$$

We may now proceed to analyze the acceleration. We begin by using the parameters obtained thus far to form $\bar{\alpha}_{BC}$. From the value of $\dot{\psi}$ we know that

$$\bar{\omega}_{\text{cap}} = \bar{\omega}_{AB} + 96.45\bar{e}_{B/A} = -48.23\bar{I} - 188.50\bar{J} + 83.53\bar{K}.$$

We substitute into Eq. (5) this expression, the unit vectors in Eqs. (6), and the results for $\dot{\psi}$ and $\dot{\theta}$, which gives

$$\begin{aligned}\bar{\alpha}_{BC} &= \ddot{\psi}\bar{e}_{B/A} + 96.45\left(\bar{\omega}_{AB} \times \bar{e}_{B/A}\right) + \ddot{\theta}\bar{e}_{\text{pin}} - 95.01\left(\bar{\omega}_{\text{cap}} \times \bar{e}_{\text{pin}}\right) \\ &= \left(-0.50\ddot{\psi} + 0.6347\ddot{\theta} - 14582\right)\bar{I} + (-0.6804\ddot{\theta} - 6716)\bar{J} \\ &\quad + \left(0.8660\ddot{\psi} + 0.3664\ddot{\theta} - 23575\right)\bar{K} \text{ rad/s}^2.\end{aligned} \qquad (9)$$

Substitution of Eqs. (8) and (9), as well as the position vectors in Eqs. (2), into the second of Eqs. (1) governing the acceleration of cap B gives

$$\begin{aligned}\bar{a}_B &= \left(-60\pi\bar{J}\right) \times \left[\left(-60\pi\bar{J}\right) \times \left(-0.15\bar{I} + 0.2598\bar{K}\right)\right] \\ &= \dot{v}_C\bar{I} + \left[\left(-0.50\ddot{\psi} + 0.6347\ddot{\theta} - 14582\right)\bar{I} + (-.6804\ddot{\theta} - 6716)\bar{J}\right. \\ &\quad \left. + \left(0.8660\ddot{\psi} + 0.3664\ddot{\theta} - 23575\right)\bar{K}\right] \times \left(-0.5788\bar{I} - 0.4\bar{J} + 0.2598\bar{K}\right) \\ &\quad + \left(-\,108.53\bar{I} - 123.86\bar{J} + 48.72\bar{K}\right) \times \left[\left(-\,108.53\bar{I} - 123.86\bar{J}\right.\right. \\ &\quad \left.\left. + 48.72\bar{K}\right) \times \left(-0.5788\bar{I} - 0.4\bar{J} + 0.2598\bar{K}\right)\right].\end{aligned}$$

The corresponding component equations are

$$\begin{aligned}\bar{a}_B \cdot \bar{I} &= 5330 = \dot{v}_C + 0.3464\ddot{\psi} - 0.0302\ddot{\theta} - 7673, \\ \bar{a}_B \cdot \bar{J} &= 0 = -0.3713\ddot{\psi} - 0.3770\ddot{\theta} + 13747, \\ \bar{a}_B \cdot \bar{K} &= -9231 = 0.200\ddot{\psi} - 0.6477\ddot{\theta} + 373.6,\end{aligned} \qquad (10)$$

whose solution is

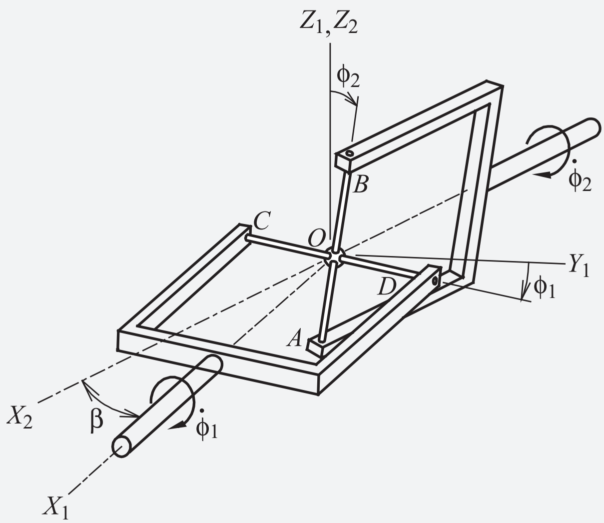
$$\dot{v}_C = 7813 \text{ m/s}, \;\; \ddot{\psi} = 16722 \text{ rad/s}^2, \;\; \ddot{\theta} = 19994 \text{ rad/s}^2. \qquad \lhd$$

The angular acceleration corresponding to these acceleration rates is

$$\bar{\alpha}_{BC} = \left(-10.253\bar{I} + 20.319\bar{J} - 1.77\bar{K}\right)\left(10^3\right) \text{ rad/s}^2. \quad \triangleleft$$

In closing, it should be noted that the operations we performed are readily implemented in mathematical software.

EXAMPLE 4.5 The *cardan joint* depicted in the sketch is used to connect shafts that intersect, but are not collinear. The angle between the shafts is β, and the respective rotation rates are $\dot{\phi}_1$ and $\dot{\phi}_2$. The cross-link $OABCD$ connecting the shafts is able to rotate about axis CD relative to shaft 1, and about axis AB relative to shaft 2. In effect, each shaft is terminated by a fork-and-clevis joint, with the cross-link being the clevis for both forks. Derive an expression for the rotation rate $\dot{\phi}_2$ in terms of the instantaneous values of ϕ_1 and $\dot{\phi}_1$. Also determine the corresponding angular velocity of the cross-link.



Example 4.5

SOLUTION The analysis of this seemingly simple device actually will bring to the fore many of the concepts in both this chapter and the previous one. Thus this example will give a broader perspective of the procedures for characterizing the kinematics of connections. There are a variety of approaches for establishing the relation between ϕ_2 and ϕ_1. The first is quite direct. It is founded on the fact that arm AB is perpendicular to arm CD, so that $\bar{e}_{B/A} \cdot \bar{e}_{D/C} = 0$.

The X_j axes depicted in the sketch are aligned with the respective shafts, and both Z_j axes are perpendicular to the plane containing the shafts. At an arbitrary instant $\bar{e}_{D/C}$ lies in the $Y_1 Z_1$ plane at angle ϕ_1 below the Y_1 axis, and $\bar{e}_{B/A}$ is in the

$Y_2 Z_2$ plane at angle ϕ_2 from the Z_2 axis. Thus,

$$\bar{e}_{D/C} = \cos\phi_1 \bar{J}_1 - \sin\phi_1 \bar{K}_1, \quad \bar{e}_{B/A} = \sin\phi_2 \bar{J}_2 + \cos\phi_2 \bar{K}_2. \tag{1}$$

Because both coordinate systems share the same Z axis it must be that Y_2 lies in the $X_1 Y_1$ plane so that

$$\bar{J}_2 = \sin\beta \bar{I} + \cos\beta \bar{J}.$$

It follows that

$$\bar{e}_{B/A} = \sin\phi_2 \left(\sin\beta \bar{I}_1 + \cos\beta \bar{J}_1\right) + \cos\phi_2 \bar{K}_1. \tag{2}$$

Setting the dot product of the unit vectors to zero yields

$$\bar{e}_{B/A} \cdot \bar{e}_{D/C} = \sin\phi_2 \cos\beta \cos\phi_1 - \cos\phi_2 \sin\phi_1 = 0,$$

which may be rewritten as

$$\tan\phi_2 = \frac{\tan\phi_1}{\cos\beta}. \tag{3}$$

Because this is a general relation between the angles, it may be differentiated with respect to time to obtain a relation between the rotation rates. Thus

$$\dot{\phi}_2 \frac{1}{(\cos\phi_2)^2} = \dot{\phi}_1 \frac{1}{\cos\beta\,(\cos\phi_1)^2}.$$

To remove the dependence on ϕ_2, we employ Eq. (3) in conjunction with the identity that $(\cos\phi_2)^2 = 1/\left[1 + (\tan\phi_2)^2\right]$. The result is

$$\dot{\phi}_2 = \dot{\phi}_1 \frac{\cos\beta}{(\sin\phi_1)^2 + (\cos\beta\cos\phi_1)^2}. \tag{4} \triangleleft$$

This derivation does not examine the nature of the constraints imposed on the rotation of the cross-link, and therefore does not address the angular velocity of the cross-link. To determine that quantity we pursue an alternative analysis, which also will lead to Eqs. (3) and (4). If we consider the cross-link to be connected to the first shaft, we would say that the cross-link rotates through angle ϕ_1 together with that shaft about the negative X_1 axis, and it also rotates relative to that shaft about $\bar{e}_{D/C}$ by an unknown angle θ_1. The ϕ_1 rotation is a precession (rotation about a fixed axis), whereas θ_2 is a nutation, with $\bar{e}_{D/C}$ being the line of nodes. Similarly, from the perspective of shaft 2, ϕ_2 is the precession about the negative X_2 axis, and unknown angle θ_2 is the nutation about the line of nodes defined by $\bar{e}_{B/A}$.

Because the legs of the cross-link are orthogonal, we may consider them to represent a rotating coordinate system. Let $\bar{e}_{D/C} = \bar{j}$ and $\bar{e}_{B/A} = \bar{k}$. The alternative viewpoints enables us to derive two rotation transformations from components relative to the fixed directions to $\bar{i}\,\bar{j}\bar{k}$ components. For shaft 1 we have a pair of body-fixed rotations $-\phi_1$ about the X_1 axis, followed by θ_1 about the y axis, so that

$$\begin{Bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{Bmatrix} = \begin{bmatrix} \cos\theta_1 & 0 & -\sin\theta_1 \\ 0 & 1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_1 & -\sin\phi_1 \\ 0 & \sin\phi_1 & \cos\phi_1 \end{bmatrix} \begin{Bmatrix} \bar{I}_1 \\ \bar{J}_1 \\ \bar{K}_1 \end{Bmatrix}$$
$$= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1\sin\phi_1 & -\sin\theta_1\cos\phi_1 \\ 0 & \cos\phi_1 & -\sin\phi_1 \\ \sin\theta_1 & \cos\theta_1\sin\phi_1 & \cos\theta_1\cos\phi_1 \end{bmatrix} \begin{Bmatrix} \bar{I}_1 \\ \bar{J}_1 \\ \bar{K}_1 \end{Bmatrix}. \tag{5}$$

From the perspective of shaft 2, the body-fixed rotations are $-\phi_2$ about the X_2 axis, followed by θ_2 about the z axis, so

$$\begin{Bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{Bmatrix} = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_2 & -\sin\phi_2 \\ 0 & \sin\phi_2 & \cos\phi_2 \end{bmatrix} \begin{Bmatrix} \bar{I}_2 \\ \bar{J}_2 \\ \bar{K}_2 \end{Bmatrix}$$
$$= \begin{bmatrix} \cos\theta_2 & \sin\theta_2\cos\phi_2 & -\sin\theta_2\sin\phi_2 \\ -\sin\theta_2 & \cos\theta_2\cos\phi_2 & -\cos\theta_2\sin\phi_2 \\ 0 & \sin\phi_2 & \cos\phi_2 \end{bmatrix} \begin{Bmatrix} \bar{I}_2 \\ \bar{J}_2 \\ \bar{K}_2 \end{Bmatrix}. \tag{6}$$

Because the Z_1 and Z_2 axes coincide, the rotation transformation between the fixed coordinate systems is

$$\begin{Bmatrix} \bar{I}_2 \\ \bar{J}_2 \\ \bar{K}_2 \end{Bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{I}_1 \\ \bar{J}_1 \\ \bar{K}_1 \end{Bmatrix}. \tag{7}$$

Substituting this transformation into Eq. (6) leads to a transformation from $\bar{I}_1\bar{J}_1\bar{K}_1$ components to $\bar{i}\,\bar{j}\bar{k}$ components. Matching it to Eq. (5) requires that

$$\begin{bmatrix} \cos\theta_2 & \sin\theta_2\cos\phi_2 & -\sin\theta_2\sin\phi_2 \\ -\sin\theta_2 & \cos\theta_2\cos\phi_2 & -\cos\theta_2\sin\phi_2 \\ 0 & \sin\phi_2 & \cos\phi_2 \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1\sin\phi_1 & -\sin\theta_1\cos\phi_1 \\ 0 & \cos\phi_1 & -\sin\phi_1 \\ \sin\theta_1 & \cos\theta_1\sin\phi_1 & \cos\theta_1\cos\phi_1 \end{bmatrix}.$$

Although there are nine elements to match in the preceding equation, the orthonormal property leads to six identities, so only three elements are independent. The simplest equations come from the last row, specifically

$$\begin{aligned} (3,1):&\quad \sin\phi_2 \sin\beta = \sin\theta_1, \\ (3,2):&\quad \sin\phi_2 \cos\beta = \cos\theta_1 \sin\phi_1, \\ (3,3):&\quad \cos\phi_2 = \cos\theta_1 \cos\phi_1. \end{aligned} \tag{8}$$

These are three equations for θ_1, θ_2, and ϕ_2 in terms of ϕ_1. The ratio of the second to the third gives

$$\tan\phi_2 \cos\beta = \tan\phi_1, \tag{3$'$}$$

which is the same as the relation obtained in the first derivation.

The analysis leading to this relation was less direct than the procedure by which it was derived in the first analysis. However, pursuing it enables us to recognize how the cross-link rotates. In fact, once we have evaluated ϕ_2 corresponding to a specified value of ϕ_1, the first two of Eqs. (8) can be solved for θ_1 and θ_2.

Just as there are two viewpoints for the orientation of the cross-link, the angular velocity can be formulated from alternative perspectives. For shaft 1 the precession rate is $-\dot{\phi}_1$ about the X_1 axis and the nutation rate is $\dot{\theta}_1$ in the direction of $\bar{e}_{D/C}$, so the angular velocity of the cross-link is

$$\bar{\omega} = -\dot{\phi}_1 \bar{I}_1 + \dot{\theta}_1 \bar{e}_{D/C}. \tag{9}$$

For shaft 2 the precession rate is $-\dot{\phi}_2$ about the X_2 axis and the nutation rate is θ_2 in the direction $\bar{e}_{B/A}$. This leads to a description of the angular velocity as

$$\bar{\omega} = -\dot{\phi}_2 \bar{I}_2 + \dot{\theta}_2 \bar{e}_{B/A}. \tag{10}$$

Matching Eqs. (9) and (10) leads to three scalar component equations for the values of $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\phi}_2$ in terms of $\dot{\phi}_1$. The most direct set of equations for further manipulation is obtained by taking components in the mutually orthogonal directions $\bar{e}_{D/C}$, $\bar{e}_{B/A}$, and $\bar{e}_{D/C} \times \bar{e}_{B/A}$. Because $\bar{e}_{D/C}$ is perpendicular to the X_1 axis and $\bar{e}_{B/A}$ is perpendicular to the X_2 axis, we have

$$\begin{aligned} \bar{\omega} \cdot \bar{e}_{D/C} = \dot{\theta}_1 &= -\dot{\phi}_2 \bar{I}_2 \cdot \bar{e}_{D/C}, \\ \bar{\omega} \cdot \bar{e}_{B/A} = -\dot{\phi}_1 \bar{I}_1 \cdot \bar{e}_{B/A} &= \dot{\theta}_2, \\ \bar{\omega} \cdot (\bar{e}_{D/C} \times \bar{e}_{B/A}) = -\dot{\phi}_1 \bar{I}_1 \cdot (\bar{e}_{D/C} \times \bar{e}_{B/A}) &= -\dot{\phi}_2 \bar{I}_2 \cdot (\bar{e}_{D/C} \times \bar{e}_{B/A}). \end{aligned} \tag{11}$$

To evaluate these terms we express the unit vectors in $\bar{I}_1 \bar{J}_1 \bar{K}_1$ components. Equations (1) and (2) describe $\bar{e}_{D/C}$ and $\bar{e}_{B/A}$, and the first row of the rotation transformation in Eq. (7) shows that $\bar{I}_2$ is

$$\bar{I}_2 = \cos\beta \, \bar{I}_1 - \sin\beta \, \bar{J}_1.$$

The result of substitution of these representations of the unit vectors into Eqs. (11) is

$$\begin{aligned}\dot{\theta}_1 &= -\,\dot{\phi}_2 \cos\beta \cos\phi_1, \\ & -\dot{\phi}_1 \sin\phi_2 \sin\beta = \dot{\theta}_2, \\ & -\,\dot{\phi}_1 \left(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2 \cos\beta\right) \\ &= -\,\dot{\phi}_2 \left[\cos\beta \left(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2 \cos\beta\right)\right. \\ & \left. -\ \sin\beta \left(-\sin\phi_1 \sin\phi_2 \sin\beta\right)\right] \\ &\equiv -\dot{\phi}_2 \left(\cos\beta \cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2\right).\end{aligned} \tag{12}$$

Dividing the last of these equations by $\cos\phi_1 \cos\phi_2$ gives

$$\dot{\phi}_2 \left(\cos\beta + \tan\phi_1 \tan\phi_2\right) = \dot{\phi}_1 \left(1 + \tan\phi_1 \tan\phi_2 \cos\beta\right).$$

We can eliminate the dependence on ϕ_2 by using Eq. (3′). Further manipulation by use of $\tan\phi_1 \equiv \sin\phi_1 / \cos\phi_1$ shows that this relation for $\dot{\phi}_2$ is identical to Eq. (4). Once the value of $\dot{\phi}_2$ has been determined, the first of Eqs. (12) gives the corresponding value of $\dot{\theta}_1$. Substitution of that result and Eqs. (1) into Eq. (9) gives

$$\bar{\omega} = -\dot{\phi}_1 \left[\bar{I}_1 + \frac{(\cos\beta)^2 \cos\phi_1}{(\sin\phi_1)^2 + (\cos\beta \cos\phi_1)^2} \left(\cos\phi_1 \bar{J}_1 - \sin\phi_1 \bar{K}_1\right) \right]. \qquad \triangleleft$$

It is interesting to observe that the maximum value of $\dot{\phi}_2$ is $\dot{\phi}_1 / \cos\beta$ at $\phi_1 = 0$ and π, whereas the minimum value of $\dot{\phi}_2$ is $\dot{\phi}_1 \cos\beta$ at $\phi_2 = 0$ and $3\pi/2$. This oscillation relative to the input speed $\dot{\phi}_1$ makes the cardan joint by itself unsuitable as a constant-velocity joint. In conventional front-engine, rear-wheel-drive automobiles, two cardan joints are employed in the drive train in opposition. The reciprocal arrangement produces a final speed that matches the input.

4.4 ROLLING

When two bodies contact each other, one kinematical condition is that they not penetrate each other. In rolling motion, the contacting surfaces have no corners, so the surfaces share an identifiable tangential contact plane. Figure 4.9 shows two surfaces in contact, as viewed edgewise along their plane of contact. The z axis in the figure is

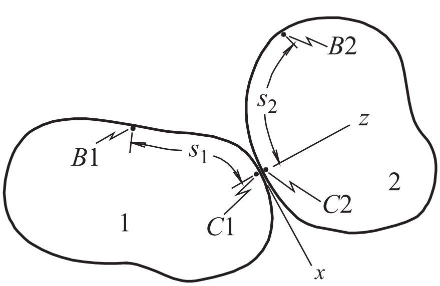


Figure 4.9. Equal-arc-length rule for rolling without slipping.

defined to be normal to the plane of contact. Because the surface of each body is impenetrable, the velocity components normal to the contact plane must match. Let $C1$ and $C2$ be contacting points on each body. Then

$$\bar{v}_{C1} \cdot \bar{k} = \bar{v}_{C2} \cdot \bar{k}. \tag{4.4.1}$$

The special case of *rolling without slipping* imposes an additional constraint associated with the condition that the contacting surfaces have no relative motion parallel to the contact plane. One way of characterizing this condition is to consider arc lengths on the perimeter of each body. In Fig. 4.9 points $B1$ and $B2$ were the points of contact at an earlier instant. The absence of slipping means that the arc length s_1 along the perimeter of body 1 between points $B1$ and $C1$ is the same as the arc length s_2 along body 2 between points $B2$ and $C2$.

We restrict our attention to situations in which the rolling bodies are circular, because the round shape makes it substantially simpler to perform a kinematical analysis. For example, arc lengths are measured along circles or flat surfaces. Describing the arc-length constraint imposed by the absence of slippage leads to a description of the position of points in the rolling bodies. The most common application of this approach is a wheel rolling along the ground. The path followed by a point on the circumference of the wheel is a *cycloid*. The geometrical parameters needed to characterize this path are depicted in Fig. 4.10.

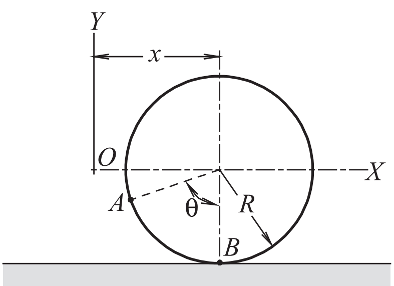


Figure 4.10. Rolling of a wheel on a flat surface.

Point A on the cylinder contacted the ground initially, at which position the center of the wheel was at the origin O of the fixed XYZ reference frame. Point B is the current contact point, so the horizontal distance x from point O to the center of the wheel equals the arc length along the ground between the initial and the current contact points. When there is no slippage, the arc length between points A and B on the wheel is the same as x, which gives a relation for the angle by which the wheel rotates:

$$x = R\theta. \tag{4.4.2}$$

Thus the position of point A is described in parametric form as a function of θ according to

$$\bar{r}_{A/O} = X\bar{I} + Y\bar{J}, \quad X = R(\theta - \sin\theta), \quad Y = -R\cos\theta, \tag{4.4.3}$$

which is the parametric description of a cycloid depicted in Fig. 4.11.

In addition to enabling us to describe the path followed by a point on the perimeter of the wheel, the circular shape makes it easy to describe the motion of the wheel's

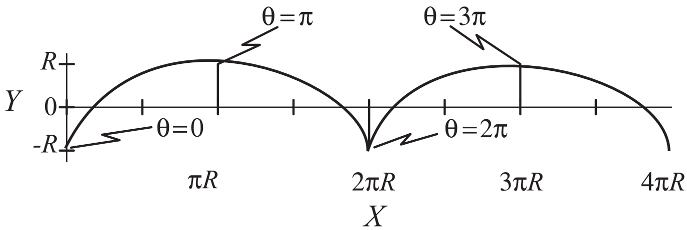


Figure 4.11. Cycloidal path followed by a point on the perimeter of a wheel rolling without slipping over a flat surface.

center. Such knowledge may be used to obtain expressions for the velocity and acceleration of a point on the wheel's perimeter. The center point in Fig. 4.10 follows a straight path, and x is the distance traveled. Thus the speed of the center is $v = \dot{x} = R\dot{\theta}$. We use this expression to eliminate the angular velocity when we take the time derivative of Eqs. (4.4.3), which leads to expressions for the velocity and acceleration of an arbitrary point on the perimeter of a wheel:

$$\begin{aligned}\bar{v}_A &= v(1-\cos\theta)\bar{i} + v\sin\theta\,\bar{j},\\ \bar{a}_A &= \left[\dot{v}(1-\cos\theta) + \frac{v^2}{R}\sin\theta\right]\bar{i} + \left(\dot{v}\sin\theta + \frac{v^2}{R}\cos\theta\right)\bar{j}.\end{aligned} \tag{4.4.4}$$

An aspect of the velocity and acceleration of particular relevance to further developments arises at $\theta = 0$, at which position $\bar{v}_A = \bar{0}$ and $\bar{a}_A = (v^2/R)\bar{j}$. In other words, a point on the perimeter of a wheel comes to rest when it comes in contact with the ground, and its acceleration is upward at that instant. This corresponds to the cusp in the cycloidal path.

One difficulty with a formulation in terms of arc lengths is that it becomes increasingly difficult to use as the complexity of the motion increases. This is particularly true for spatial motion. We therefore develop an alternative method in which constraint conditions on velocity and acceleration are formulated. Consider the limiting situation in which the points of contact Bi and Ci in Fig. 4.9 correspond to instants that are very close. The points of contact on each body then seem to have the same displacement along the contact plane. Dividing this displacement by the small time interval shows that the tangential velocity components must be the same. Because the contact condition between the bodies requires that the normal velocity components are equal, it must be that

> The velocities of contacting points of bodies that roll over each other without slipping must match in all directions. The constraint condition is
>
> $$\bar{v}_{C1} = \bar{v}_{C2} \text{ for no slipping.} \tag{4.4.5}$$

Acceleration is more complicated because the contacting points on each body come together and then separate. This means that they have different accelerations in the normal direction. A common misconception arises from the case of the rolling wheel in Fig. 4.10, as well as other planar situations. As was noted after Eqs. (4.4.4), the acceleration of a point on the perimeter of a wheel in planar motion is upward when the

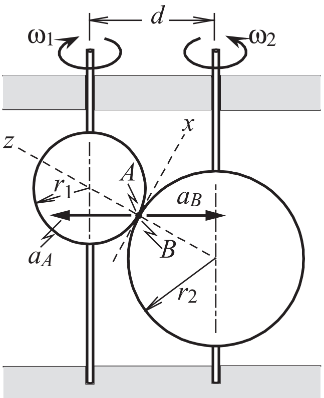


Figure 4.12. Acceleration of contact points on two spheres that have no relative slip as they rotate about fixed axes.

point comes in contact with the ground. This is often incorrectly interpreted to mean that, in all situations of rolling without slipping, the contact points may accelerate relative to each other only perpendicularly to the contact plane. This cannot be assumed to be true in spatial motion, as may be recognized by considering a simple case. In Fig. 4.12, two spheres rotate at constant rates ω_1 and ω_2 about fixed parallel axes, such that there is no slipping between the contacting points A and B. The plane of contact is xy. Points on each sphere follow circular paths in the horizontal plane whose radii are $\rho_1 = r_1 d/(r_1 + r_2)$ and $\rho_2 = r_2 d/(r_1 + r_2)$. The tangent direction of the paths followed by points A and B at the contact location are $(\bar{e}_t)_A = (\bar{e}_t)_B = -\bar{j}$, so the no-slip velocity condition that $\bar{v}_A = \bar{v}_B$ is satisfied if $\omega_2/\omega_1 = r_1/r_2$. For constant rotation rate, each point's acceleration is solely centripetal, being $a_A = \rho_1 \omega_1^2$ and $a_B = \rho_2 \omega_2^2$ in the horizontal directions displayed in the figure. It is apparent that $\bar{a}_A \cdot \bar{i} \neq \bar{a}_B \cdot \bar{i}$, so the acceleration components parallel to the xy plane do not match. This demonstrates that contact points on rolling bodies can have different accelerations parallel to the contact plane, as well as perpendicularly to it.

The lack of a direct constraint condition for the acceleration presents a dilemma. We remedy it by recalling that we have limited our attention to rolling bodies that are round in some sense. For such bodies the distance from the center to the point of contact on each rolling body is constant. This constancy leads to a velocity constraint condition that may be differentiated, thereby yielding to the additional kinematical conditions required for analyzing acceleration. In effect, in addition to considering kinematical conditions at the points of contact, we *consider the center point to be subject to a kinematical constraint* that must be identified.

To explore this idea, consider the planar situation of a planetary gear rolling over a sun gear. In Fig. 4.13, ω_2 is the angular speed of the sun gear, v_A is the speed of the center of the planetary gear, and ω_1 is the angular speed of that gear. Because the distance from the center A of the planetary gear to the point of contact C is constant, point A follows a circular path of radius $r_1 + r_2$. Thus, for the xyz coordinate system depicted in the figure, the velocity and acceleration of point A are described according to path variables as

$$\bar{v}_A = v_A \bar{i}, \quad \bar{a}_A = \dot{v}_A \bar{i} - \frac{v_A^2}{r_1 + r_2} \bar{j} \tag{4.4.6}$$

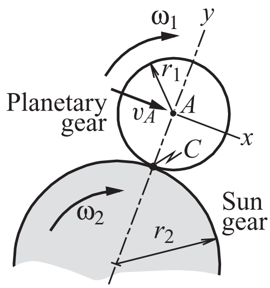


Figure 4.13. Rolling without slipping of a planetary gear over a fixed inner gear.

In addition, for planar motion we know that $\bar{\omega}_1 = -\omega_1 \bar{k}$ and $\bar{\omega}_2 = -\omega_2 \bar{k}$. Because there is no slipping at point C, the velocity constraint requires that $\bar{v}_C = \omega_2 r_2 \bar{i}$ be the same for both gears. Using this to construct the velocity of center A gives

$$\bar{v}_A = \bar{v}_C + \bar{\omega}_1 \times \bar{r}_{A/C} = (\omega_2 r_2 + \omega_1 r_1)\,\bar{i}. \tag{4.4.7}$$

The descriptions of $\bar{v}_A$ in Eqs. (4.4.6) and (4.4.7) agree in direction. Matching the speeds in these two equations leads to

$$v_A = \omega_2 r_2 + \omega_1 r_1. \tag{4.4.8}$$

To obtain a kinematical constraint on acceleration, we recognize that the round shape of the gears and the fact that the path followed by point A is circular make the preceding relation valid at any instant. Consequently, we may differentiate it with respect to time. Doing so yields

$$\dot{v}_A = \dot{\omega}_2 r_2 + \dot{\omega}_1 r_1. \tag{4.4.9}$$

Equations (4.4.8) and (4.4.9) are constraint equations relating the three rate variables. This enables us to describe the velocity and acceleration of the center point A, as well as the angular velocity and angular acceleration of the planetary gear, in terms of the two rate variables we choose to retain. From such knowledge we can determine the velocity and acceleration of any other point on the planetary gear in terms of those variables.

The same approach may be extended directly to cases of spatial motion. A relative simple situation is that of steady precession of a disk that rolls along the ground, as depicted by the side view in Fig. 4.14. The inclination angle β is constant, as is the

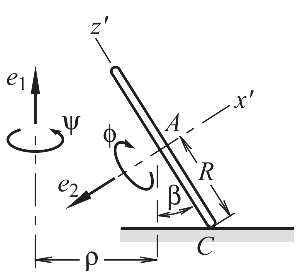


Figure 4.14. Steady precession of a rolling disk.

precession rate $\dot{\psi}$. In this case the center point follows a circular path of radius ρ, so the velocity and acceleration of the center of the disk are

$$\bar{v}_A = \rho\dot{\psi}\,\bar{j}', \quad \bar{a}_A = \rho\dot{\psi}^2\left(-\cos\beta\bar{i}' + \sin\beta\bar{k}'\right), \tag{4.4.10}$$

where $x'y'z'$ is a coordinate system that precesses with its origin at the center of the disk. The angular velocity and acceleration of the disk are given by

$$\begin{gathered}\bar{\omega} = \dot{\psi}\bar{e}_1 + \dot{\phi}\bar{e}_2 = \left(\dot{\psi}\sin\beta - \dot{\phi}\right)\bar{i}' + \dot{\psi}\cos\beta\bar{k}', \\ \bar{\alpha} = \dot{\phi}\left(\bar{\omega}\times\bar{e}_2\right) = -\dot{\psi}\dot{\phi}\cos\beta\,\bar{j}'.\end{gathered} \tag{4.4.11}$$

The preceding assumes that $\dot{\phi}$ is constant, which is something we will verify. This rate is not yet known, but we have not imposed the condition that there is no slippage at the contact point C. Because $\bar{v}_C = \bar{0}$, the velocity of the center of the disk is $\bar{\omega}\times\bar{r}_{A/C}$. Equating this to the expression for $\bar{v}_A$ corresponding to circular motion leads to

$$\bar{v}_A = \rho\dot{\psi}\,\bar{j}' = \bar{\omega}\times R\bar{k}' = \left(\dot{\psi}\sin\beta - \dot{\phi}\right)R\bar{j}'. \tag{4.4.12}$$

Matching the $\bar{j}'$ components yields

$$\dot{\phi} = \left(\sin\beta - \frac{\rho}{R}\right)\dot{\psi}. \tag{4.4.13}$$

Because β and ρ are constant, this expression shows that $\dot{\phi}$ is proportional to $\dot{\psi}$, which confirms our assumption that $\ddot{\phi}$ is zero. Substitution of $\dot{\phi}$ into Eqs. (4.4.11) gives

$$\bar{\omega} = \dot{\psi}\left(\frac{\rho}{R}\bar{i}' + \cos\beta\bar{k}'\right), \quad \bar{\alpha} = -\dot{\psi}^2\left(\sin\beta - \frac{\rho}{R}\right)\cos\beta\,\bar{j}'. \tag{4.4.14}$$

Equations (4.4.10) and (4.4.14) show that knowledge of the Eulerian angles is sufficient to characterize the motion of the center, as well as the angular motion, of a rolling disk in the steady precession case. The same is true for unsteady motion, in which the spin rate is not proportional to the precession rate. The analysis is more difficult, but it still uses the basic concept that the center is a constrained point whose velocity may be described in a general manner by relating it to the contact point on the disk. Figure 4.15 depicts a disk that is rolling without slipping over a flat surface in a wobbly manner. We use Eulerian angles to represent the orientation of the disk, with precession angle ψ measured about the upward vertical $\bar{e}_1$ and nutation θ measured from the vertical to the center line of the disk. The line of nodes is the y' axis, which is defined to be the

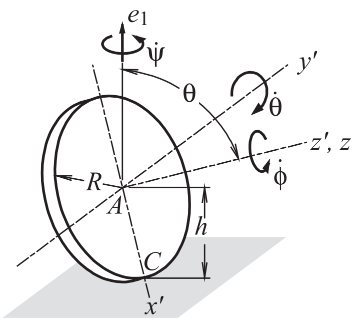


Figure 4.15. Unsteady rolling without slipping of a disk on a flat surface.

horizontal diameter of the disk. Body-fixed xyz axes execute a spin ϕ about the z' axis relative to $x'y'z'$.

The importance of the round shape in this case is that, if all other quantities are held fixed, the motion of the system will not be altered by changing ϕ. As a consequence of this invariance, a description of the velocity of the center of disk in terms of $\bar{i}'\bar{j}'\bar{k}'$ components, with the Eulerian angles represented algebraically, will be valid at any instant. Such a representation may be differentiated to analyze acceleration. In terms of the Eulerian angles the angular velocities $\bar{\Omega}'$ of $x'y'z'$ and $\bar{\omega}$ of xyz are

$$\begin{aligned}\bar{\Omega}' &= \dot{\psi}\bar{e}_1 + \dot{\theta}\bar{j}' = -\dot{\psi}\sin\theta\bar{i}' + \dot{\theta}\bar{j}' + \dot{\psi}\cos\theta\bar{k}', \\ \bar{\omega} &= \bar{\Omega}' + \dot{\phi}\bar{k}' = -\dot{\psi}\sin\theta\bar{i}' + \dot{\theta}\bar{j}' + \left(\dot{\phi} + \dot{\psi}\cos\theta\right)\bar{k}'.\end{aligned} \tag{4.4.15}$$

At point C, where the disk comes in contact with the ground, the no-slip condition requires that $\bar{v}_C = \bar{0}$. Thus the velocity of the center A must be

$$\bar{v}_A = \bar{\omega}\times\bar{r}_{A/C} = -R\left(\dot{\phi} + \dot{\psi}\cos\theta\right)\bar{j}' + R\dot{\theta}\bar{k}'. \tag{4.4.16}$$

This is a general relation for $\bar{v}_A$ in terms of of the Eulerian angles. It therefore may be differentiated to determine $\bar{a}_A$. The relative derivative concept described by Eq. (3.3.15) is appropriate to this task. Equation (4.4.16) gives the $\bar{i}'\bar{j}'\bar{k}'$ components of $\bar{v}_A$, and the first of Eqs. (4.4.15) is the angular velocity of $x'y'z'$. We thereby find that

$$\begin{aligned}\bar{a}_A &= \frac{\partial}{\partial t}\left[-R\left(\dot{\phi} + \dot{\psi}\cos\theta\right)\bar{j}' + R\dot{\theta}\bar{k}'\right] + \bar{\Omega}'\times\left[-R\left(\dot{\phi} + \dot{\psi}\cos\theta\right)\bar{j}' + R\dot{\theta}\bar{k}'\right] \\ &= R\left[\dot{\theta}^2 + \left(\dot{\phi} + \dot{\psi}\cos\theta\right)\dot{\psi}\cos\theta\right]\bar{i}' + R\left[-\ddot{\phi} - \ddot{\psi}\cos\theta + 2\dot{\psi}\dot{\theta}\sin\theta\right]\bar{j}' \\ &\quad + R\left[\ddot{\theta} + \left(\dot{\phi} + \dot{\psi}\cos\theta\right)\dot{\psi}\sin\theta\right]\bar{k}'.\end{aligned} \tag{4.4.17}$$

The angular acceleration is found by differentiating the second of Eqs. (4.2.11):

$$\begin{aligned}\bar{\alpha} &= \left(-\ddot{\psi}\sin\theta - \dot{\psi}\dot{\theta}\cos\theta + \dot{\theta}\dot{\phi}\right)\bar{i}' + \left(\ddot{\theta} + \dot{\psi}\dot{\phi}\sin\theta\right)\bar{j}' \\ &\quad + \left(\ddot{\psi}\cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta}\sin\theta\right)\bar{k}'.\end{aligned} \tag{4.4.18}$$

Equations (4.4.15)–(4.4.18) describe the motion of the center point and the angular motion of the rolling disk in terms of the Eulerian angles. The velocity and acceleration of any other point in the disk may be determined through the kinematical formulas relating points in a rigid body.

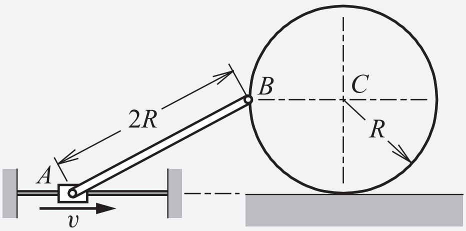
Although Eqs. (4.4.16) and (4.4.17) contain a variety of effects, one is readily identifiable. The vertical unit vector in Fig. 4.15 is $\bar{e}_1 = -\sin\theta\bar{i} + \cos\theta\bar{k}'$. Correspondingly, we find that the components of $\bar{v}_A$ and $\bar{a}_A$ perpendicular to the contact plane are

$$\bar{v}_A\cdot\bar{e}_1 = R\dot{\theta}\cos\theta, \quad \bar{a}_A\cdot\bar{e}_1 = R\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right). \tag{4.4.19}$$

We now observe from Fig. 4.15 that the elevation of point A above the ground is $h = R\sin\theta$. Successive differentiation of h shows that $\dot{h} \equiv \bar{v}_A\cdot\bar{e}_1$ and $\ddot{h} \equiv \bar{a}_A\cdot\bar{e}_1$. In other words, the procedure we followed implicitly recognized a fundamental geometric property stemming from the fact that the distance from the contact point to the center is always R.

Although the roundness of the disk played a less obvious role in this motion, it was crucial. If the disk were elliptical, it would have been necessary to describe the velocity of the center point as a function of the spin angle and the properties of the ellipse. Differentiating such a representation would have been substantially more difficult than the corresponding tasks in the case of a circular disk.

EXAMPLE 4.6 Piston A has constant velocity v to the right. The wheel, which is connected to the piston by connecting rod AB, rolls without slipping over the ground. Determine the velocity and acceleration of the center of the wheel for the position shown in the sketch.



Example 4.6

SOLUTION This example is intended to reinforce our ability to analyze a planar linkage while simultaneously accounting for the special features of a rolling body. It is imperative to realize that, even though the input speed v is constant, none of the other acceleration parameters can be assumed to be zero because the geometrical configuration changes when connecting pin B moves. Also, we could analyze this system when pin B is at an arbitrary position, but doing so would merely complicate the geometrical description of position variables without modifying the basic operations.

For a global coordinate system, let the x axis be horizontal to the right and the y axis be vertically upward. The angular velocity of each body is perpendicular to the plane, with unknown rates, so we have

$$\bar{\omega}_{AB} = \omega_{AB}\bar{k}, \quad \bar{\omega}_w = \omega_w\bar{k}.$$

We next account for the fact that both points A and C follow straight paths, so that

$$\bar{v}_A = v\bar{i}, \quad \bar{a}_A = \bar{0}, \quad \bar{v}_C = v_C\bar{i}, \quad \bar{a}_C = \dot{v}_C\bar{i}. \tag{1}$$

Because there is no slippage between the wheel and the stationary ground, relating the velocity of center C to that of the contact point yields

$$\bar{v}_C = v_C\bar{i} = \omega_w\bar{k} \times R\bar{j} \implies v_C = -R\omega_w. \tag{2}$$

This relation is true at any instant, so differentiating it with respect to t gives

$$\dot{v}_C = -R\dot{\omega}_w. \tag{3}$$

From this juncture, the analysis proceeds like that for any other planar linkage. We describe the velocity of the connecting pin B in terms of the parameters for bar AB and for the wheel. Thus,

$$\bar{v}_B = \bar{v}_A + \omega_{AB}\bar{k} \times \bar{r}_{B/A} = \bar{v}_C + \omega_w\bar{k} \times \bar{r}_{B/C}. \tag{4}$$

The instantaneous positions are

$$\bar{r}_{B/A} = \sqrt{3}R\bar{i} + R\bar{j}, \quad \bar{r}_{B/C} = -R\bar{i}. \tag{5}$$

Substitution of Eqs. (1), (2), and (5) into Eqs. (4) results in

$$\bar{v}_B = v\bar{i} + \omega_{AB}\bar{k} \times \left(\sqrt{3}R\bar{i} + R\bar{j}\right) = -R\omega_w\bar{i} + \omega_w\bar{k} \times \left(-R\bar{i}\right). \tag{6}$$

Matching like components yields two simultaneous equations for ω_{AB} and ω_w in terms of v,

$$\bar{v}_B \cdot \bar{i} = v - R\omega_{AB} = -R\omega_w,$$
$$\bar{v}_B \cdot \bar{j} = \sqrt{3}R\omega_{AB} = -R\omega_w,$$

from which we find that the instantaneous rotation rates are

$$\omega_{AB} = 0.3660\frac{v}{R}, \quad \omega_w = -0.6340\frac{v}{R}. \tag{7}$$

The corresponding velocity of center C is

$$\bar{v}_C = -R\omega_w\bar{i} = 0.6340v\bar{i}. \qquad \triangleleft$$

Now that the velocity has been fully analyzed, we may proceed to acceleration. We begin by writing the acceleration analogs to Eqs. (4):

$$\begin{aligned}\bar{a}_B &= \bar{a}_A + \dot{\omega}_{AB}\bar{k} \times \bar{r}_{B/A} - \omega_{AB}^2\bar{r}_{B/A} \\ &= \bar{a}_C + \dot{\omega}_w\bar{k} \times \bar{r}_{B/C} - \omega_w^2\bar{r}_{B/C}.\end{aligned} \tag{8}$$

From Eqs. (1) and (3), we know that $\bar{a}_A = \bar{0}$ and $\bar{a}_C = -R\dot{\omega}_w\bar{i}$. The current rotation rates are given by Eqs. (7), and Eqs. (5) give the instantaneous position vectors. Substitution of these values into the preceding results in

$$\begin{aligned}\bar{a}_B &= \dot{\omega}_{AB}\bar{k} \times \left(\sqrt{3}R\bar{i} + R\bar{j}\right) - \left(0.3660\frac{v}{R}\right)^2\left(\sqrt{3}R\bar{i} + R\bar{j}\right) \\ &= -R\dot{\omega}_w\bar{i} + \dot{\omega}_w\bar{k} \times \left(-R\bar{i}\right) - \left(-0.6340\frac{v}{R}\right)^2\left(-R\bar{i}\right).\end{aligned}$$

The simultaneous equations obtained from matching like components of $\bar{a}_B$ are

$$\bar{a}_B \cdot \bar{i} = -R\dot{\omega}_{AB} - 0.2321\frac{v^2}{R} = -R\dot{\omega}_w + 0.4019\frac{v^2}{R},$$
$$\bar{a}_B \cdot \bar{j} = \sqrt{3}R\dot{\omega}_{AB} - 0.1340\frac{v^2}{R} = -R\dot{\omega}_w.$$

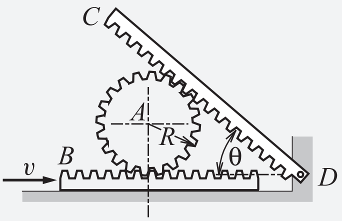
The angular acceleration rates obtained from these equations are

$$\dot{\omega}_{AB} = -0.1830\frac{v^2}{R^2}, \quad \dot{\omega}_w = 0.4510\frac{v^2}{R^2}.$$

The corresponding acceleration of center C is

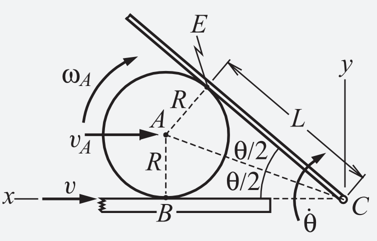
$$\bar{a}_C = -R\dot{\omega}_w\bar{i} = -0.4510\frac{v^2}{R}\bar{i}. \qquad \triangleleft$$

EXAMPLE 4.7 Rack CD pivots about pin D such that the angle of elevation θ is a known function of time. Rack B translates to the right in order to keep gear A in contact with rack CD. Determine the required velocity v, the corresponding angular velocity of gear A, and the velocity of the gear's center. Express all results in terms of the instantaneous values of θ and $\dot{\theta}$.



Example 4.7

SOLUTION This example concerns a situation in which contacting surfaces involved in rolling have a normal velocity component, so its analysis will generalize our understanding of rolling in planar motion. Our approach treats the system as a linkage, in which some of the constrained points are subject to the no-slip condition. The sketch shows the global xyz coordinate system we shall use, as well as the geometrical parameters we will need to describe position vectors.



Global coordinate system and geometrical parameters for analyzing the rolling motion of the gear.

The constrained points are the center A, which follows a straight horizontal path, point B, at which there can be no slipping between gear A and the horizontal rack, point E, at which there can be no slipping between gear A and rack CD, and point C, which is stationary. We set $\bar{v}_A = -v_A\bar{i}$, $\bar{v}_C = \bar{0}$, and equate the velocities of point E obtained by considering it to be a point on rack CD, or on gear A. Thus,

$$\bar{v}_E = \dot{\theta}\bar{k} \times \bar{r}_{E/C} = -v_A\bar{i} + \omega_A\bar{k} \times \bar{r}_{E/A}. \tag{1}$$

Matching the velocity descriptions of contact point B based on considering it to be a point in the rack or in the gear gives

$$\bar{v}_B = -v\bar{i} = -v_A\bar{i} + \omega_A\bar{k} \times \bar{r}_{B/A}. \tag{2}$$

The length L from the pivot is related to the angle θ by

$$L = R\cot\left(\frac{\theta}{2}\right), \tag{3}$$

so that

$$\bar{r}_{E/C} = R\cot\left(\frac{\theta}{2}\right)\left(\cos\theta\bar{i} + \sin\theta\,\bar{j}\right), \quad \bar{r}_{E/A} = R\left(-\sin\theta\bar{i} + \cos\theta\,\bar{j}\right), \quad \bar{r}_{B/A} = -R\bar{j}.$$

Substitution of these position vectors into Eqs. (1) and (2) gives

$$\begin{gathered}\bar{v}_E = R\dot{\theta}\cot\left(\frac{\theta}{2}\right)\left(-\sin\theta\bar{i} + \cos\theta\,\bar{j}\right) = -v_A\bar{i} + \omega_A R\left(-\cos\theta\bar{i} - \sin\theta\,\bar{j}\right),\\ \bar{v}_B = -v\bar{i} = \dot{v}_A\bar{i} + \omega_A R\bar{i}.\end{gathered} \tag{4}$$

Matching like components in each of these equations leads to

$$\begin{gathered}\bar{v}_B\cdot\bar{i} = -v = -v_A + \omega_A R,\\ \bar{v}_E\cdot\bar{i} = -R\dot{\theta}\cot\left(\frac{\theta}{2}\right)\sin\theta = -v_A - \omega_A R\cos\theta,\\ \bar{v}_E\cdot\bar{j} = R\dot{\theta}\cot\left(\frac{\theta}{2}\right)\cos\theta = -\omega_A R\sin\theta.\end{gathered}$$

The solution of these equations is

$$\begin{gathered}\omega_A = -\dot{\theta}\cot\left(\frac{\theta}{2}\right)\cot\theta \equiv -\dot{\theta}\frac{\cos\theta}{(1-\cos\theta)},\\ v_A = -\frac{\omega_A R}{\cos\theta} = R\dot{\theta}\frac{1}{(1-\cos\theta)},\\ v = -\omega_A R\left(\frac{1}{\cos\theta} + 1\right) = R\dot{\theta}\frac{(1+\cos\theta)}{(1-\cos\theta)}.\end{gathered} \tag{5 $\triangleleft$}$$

An interesting aspect of the result for v_A is that it could have been obtained much more simply by a different approach. A geometrical analysis shows that L also is the distance from pivot C to the point of contact B. Because this point is always directly below the center A, we have $\bar{r}_{A/C} = L\bar{i} + R\bar{j}$. Differentiating this vector, with L given by Eq. (3), leads to

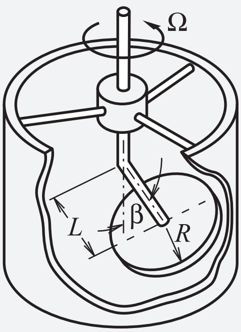
$$\bar{v}_A = \dot{L}\bar{i} = \frac{dL}{d\theta}\dot{\theta}\bar{i} = -R\dot{\theta}\frac{1}{2\left(\sin(\theta/2)\right)^2}. \tag{6}$$

Because $2\left(\sin(\theta/2)\right)^2 \equiv 1-\cos\theta$, this result is the same as the second of Eqs. (5). Clearly the second approach represents a more direct route for analyzing $\bar{v}_A$, but it gives neither the angular velocity of gear A nor the velocity of rack B.

Suppose it had been requested to determine the accelerations of rack B and center A and the angular acceleration of the gear. As a consequence of the circular geometry, the derived expressions for ω_A, v_A, and v are valid at any θ. Thus expressions for $\dot{\omega}_A$, $\dot{v}_A$, and $\dot{v}$ can be obtained by direct differentiation. For example,

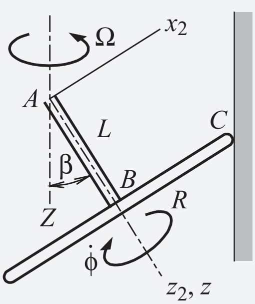
$$\begin{aligned}\dot{\omega}_A &= -\ddot{\theta}\frac{\cos\theta}{(1-\cos\theta)} - \dot{\theta}^2\frac{d}{\partial\theta}\left[\frac{\cos\theta}{(1-\cos\theta)}\right]\\ &= -\ddot{\theta}\frac{\cos\theta}{(1-\cos\theta)} + \dot{\theta}^2\frac{\sin\theta}{(1-\cos\theta)^2}.\end{aligned}$$

EXAMPLE 4.8 The disk rotates freely about its shaft, as the shaft rotates about the vertical axis at the constant rate Ω. There is no slippage at the point where the disk contacts the inner wall of the stationary cylindrical tank. Determine the angular velocity and angular acceleration of the disk, and the acceleration of the point on the disk that is in contact with the cylinder.



Example 4.8

SOLUTION A key objective of this example of rolling in spatial motion is to clarify some misconceptions that individuals have as a result of using intuition, rather than the formal analytical concepts. We begin by treating this system like any other in which it is necessary to evaluate the angular velocity and angular acceleration. Thus we draw a sketch of the system as a side view that shows both rotation axes in true view. Reference frame xyz is fixed to the disk, so only its z axis lies in the plane of the sketch at all instants, whereas the $x_2y_2z_2$ is attached to the shaft so both the x_2 and z_2 axes are always situated in the vertical plane. We define the Z axis of the fixed reference frame to coincide with the axis for the Ω rotation. The unknown spin rate is denoted as $\dot{\phi}$, consistent with the notation for Eulerian angles.



Coordinate systems for analyzing the spatial rolling motion of the disk

A general expression for the angular velocity of the disk is

$$\bar{\omega} = \Omega\bar{e}_1 + \dot{\phi}\bar{e}_2, \quad \bar{e}_1 = -\bar{K}, \quad \bar{e}_2 = \bar{k}_2.$$

Because XYZ is fixed and $x_2y_2z_2$ rotates about the vertical axis, we have

$$\bar{\Omega}_1 = \bar{0}, \quad \bar{\Omega}_2 = \Omega\bar{e}_1.$$

It will turn out that there is a constant ratio between Ω and $\dot{\phi}$, which means that $\ddot{\phi}$ will be zero because Ω is constant. However, that might not be true in other situations, so we carry out the analysis of angular acceleration allowing for the possibility that $\ddot{\phi} \neq 0$. Thus differentiating $\bar{\omega}$ leads to

$$\bar{\alpha} = \ddot{\phi}\bar{e}_2 + \dot{\phi}\left(\bar{\Omega}_2 \times \bar{e}_2\right).$$

The axes of $x_2y_2z_2$ form a convenient global coordinate system, so that $\bar{e}_1 = \sin\beta\bar{i}_2 - \cos\beta\bar{k}_2$. Correspondingly, we find that

$$\bar{\omega} = \Omega\sin\beta\bar{i}_2 + \left(\dot{\phi} - \Omega\cos\beta\right)\bar{k}_2,$$
$$\bar{\Omega}_2 = \Omega\sin\beta\bar{i}_2 - \Omega\cos\beta\bar{k}_2,$$
$$\bar{\alpha} = -\Omega\dot{\phi}\sin\beta\,\bar{j}_2 + \ddot{\phi}\bar{k}_2.$$

At this juncture, we have not addressed the constraint imposed by the no-slip condition at the wall, which requires that the instantaneous velocity at the contact point C be zero. The center point B follows a horizontal circular path of radius $L\sin\beta$, and it is moving inward relative to the plane of the sketch, so $\bar{v}_B = -L\Omega\sin\beta\,\bar{j}_2$. The kinematical relation between the velocities of these points based on both belonging to the disk then requires that

$$\bar{v}_B = -L\Omega\sin\beta\,\bar{j}_2 = \bar{\omega} \times \bar{r}_{B/C/} = \left[\Omega\sin\beta\bar{i}_2 + \left(\dot{\phi} - \Omega\cos\beta\right)\bar{k}_2\right] \times \left(-R\bar{i}_2\right).$$

The only nonzero component of the preceding is

$$\bar{v}_B \cdot \bar{j}_2 = -L\Omega\sin\beta = -R\left(\dot{\phi} - \Omega\cos\beta\right),$$

from which we find that

$$\dot{\phi} = \left(\frac{L}{R}\sin\beta + \cos\beta\right)\Omega.$$

Because the analysis has been carried out for an arbitrary instant, this relation is generally valid. It shows that $\dot{\phi}$ is proportional to Ω, so we have $\ddot{\phi} = 0$, as was anticipated earlier. Substitution of the spin rate into the general expressions for $\bar{\omega}$ and $\bar{\alpha}$ yields

$$\bar{\omega} = \Omega\sin\beta\left(\bar{i}_2 + \frac{L}{R}\bar{k}_2\right), \quad \bar{\alpha} = -\Omega^2\left[\frac{L}{R}(\sin\beta)^2 + \sin\beta\cos\beta\right]\bar{j}_2. \qquad \triangleleft$$

A key aspect of the rotation is the fact that both $\dot{\phi}$ and Ω are responsible for the rotation about the z axis. Failure to recognize the contribution of Ω leads to a common error for novices, who state that as a result of the no-slip condition the speed of point B is $\dot{\phi}R$. As we have seen, the angular velocity component about the z axis actually is $\dot{\phi} - \Omega\cos\beta$, which led to $v_B = \left(\dot{\phi} - \Omega\cos\beta\right) R$.

Now that we have described the angular motion we can proceed to evaluate the acceleration of the contact point. We do this by relating $\bar{a}_C$ to the acceleration of a reference point in the disk, for which the center B is convenient. Because this point follows a horizontal circular path, we have

$$\bar{a}_B = (L\sin\beta)\,\Omega^2\bar{e}_n = (L\sin\beta)\,\Omega^2\left(-\cos\beta\bar{i}_2 - \sin\beta\bar{k}_2\right).$$

[An alternative to using path variables to describe $\bar{a}_B$ is to recognize that point B is also a point on the shaft, whose angular velocity is the constant $\bar{\Omega}_1$ and whose point A is stationary, so that $\bar{a}_B = \bar{\Omega}_1 \times \left(\bar{\Omega}_1 \times \bar{r}_{B/A}\right)$.] With $\bar{a}_B$, $\bar{\omega}$, and $\bar{\alpha}$ established, we evaluate

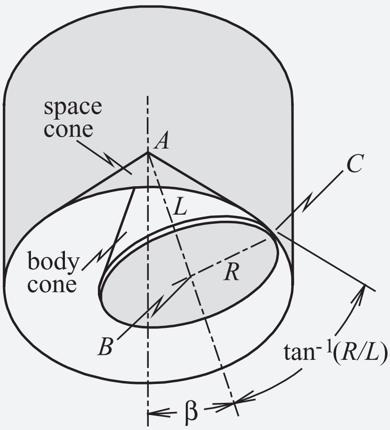
$$\begin{aligned}\bar{a}_C &= \bar{a}_B + \bar{\alpha}\times\bar{r}_{C/B} + \bar{\omega}\times\left(\bar{\omega}\times\bar{r}_{C/B}\right)\\ &= \Omega^2\left[\frac{L}{R}(\sin\beta)^2 + \sin\beta\cos\beta\right]\left(-L\bar{i}_2 + R\bar{k}_2\right).\end{aligned} \quad \triangleleft$$

An interesting interpretation of these results follows from the observation that $\bar{r}_{C/A} = R\bar{i}_2 + L\bar{k}_2$. The expressions we have derived indicate that $\bar{\omega}$ is parallel to $\bar{r}_{C/A}$ and $\bar{a}_C$ is perpendicular to $\bar{r}_{C/A}$. Both features would have been apparent if we had slightly altered our analysis based on the observation that point A is always at distance L along the disk's centerline. Consequently, we can consider point A to belong to the disk, which allows us to use point A as the reference point for describing the motion of the contact point. Because point A is stationary, we could have written

$$\bar{v}_C = \bar{0} = \bar{\omega}\times\bar{r}_{C/A}, \quad \bar{a}_C = \bar{\alpha}\times\bar{r}_{C/A} + \bar{\omega}\times\left(\bar{\omega}\times\bar{r}_{C/A}\right).$$

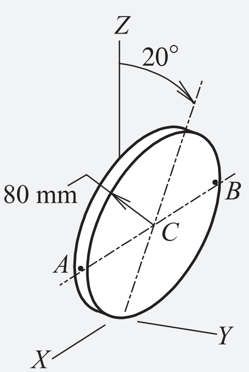
The first equation establishes the parallelism of $\bar{\omega}$ and $\bar{r}_{C/A}$, and it simplifies the second equation to $\bar{a}_C = \bar{\alpha}\times\bar{r}_{C/A}$, from which it follows that $\bar{a}_C\cdot\bar{r}_{C/A} = 0$.

This leads to a visual model for the motion of the disk. Regardless of the angles of rotation about the Z and z axes, the line connecting points A and C is always situated at a constant angle from the vertical. The sketch shows that this angle is $\beta + \tan^{-1}(R/L)$. Thus the locus of this line is a cone. Because it represents the view of line AC from the perspective of a fixed observer, it is called the *space cone.* On the other hand, from the perspective of an observer on the disk, who considers xyz to be stationary, the angle from the x axis to line AC is always $\tan^{-1}(L/R)$. However, line AC may be situated arbitrarily relative to the xz plane. Thus the locus of line AC relative to an observer on xyz is the *body cone.* The last sketch depicts both cones. The motion of the disk is equivalent to the body cone rolling without slipping over the space cone. The instantaneous axis of rotation is the line of contact between the cones. The acceleration of any point on the body cone that is on this line of contact is normal to the rotation axis. The concept of space and body cones is particularly useful for the treatment of the rotation of bodies in free flight, which is a topic in Chapter 10.



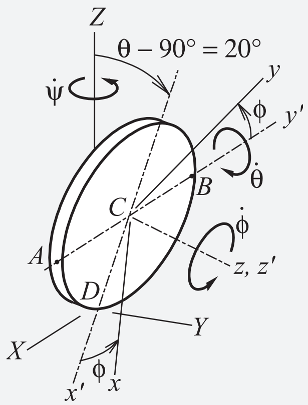
Body and space cones for the disk that rolls without slipping over the interior of the cylinder.

EXAMPLE 4.9 A disk rolls without slipping on the XY plane. At the instant shown, the horizontal diameter ACB is parallel to the X axis. Also, at this instant, the horizontal components of the velocity of the center C are known to be 5 m/s in the X direction and 3 m/s in the Y direction, and the Y component of the velocity of point B is 6 m/s. Determine the precession, nutation, and spin rates for the Eulerian angles in Fig. 4.15.



Example 4.9

SOLUTION We begin with a sketch that is like Fig. 4.15, except that the nutation angle exceeds $\pi/2$ because the disk is tilted to the right. The y' axis coincides with the horizontal diameter AB, which is the line of nodes. Because an increase of θ is defined to correspond to rotation about the positive y' axis, we have $\bar{j}' = -\bar{I}$ at the given instant.



Eulerian angles and coordinate systems for analyzing arbitrary rolling motion of a disk

The axes and angle defined here match those in Fig. 4.15, so we may directly employ the description of $\bar{\omega}$ in Eqs. (4.2.12). We set $\theta = 110^\circ$ for the instant of interest, which yields

$$\bar{\omega} = -0.9397\dot{\psi}\bar{i}' + \dot{\theta}\,\bar{j}' + (\dot{\phi} - 0.3420\dot{\psi})\bar{k}'.$$

There is no slipping at the contact point D, so $\bar{v}_D = \bar{0}$. When we refer the velocities of points B and C to this point, we find that

$$\begin{aligned}\bar{v}_C &= \bar{\omega} \times \left(-0.08\bar{i}'\right) = -0.08(\dot{\phi} - 0.3420\dot{\psi})\,\bar{j}' + 0.08\dot{\theta}\bar{k}',\\ \bar{v}_B &= \bar{\omega} \times \left(-0.08\bar{i}' + 0.08\,\bar{j}'\right)\\ &= -0.08(\dot{\phi} - 0.3420\dot{\psi})\left(\bar{i}' + \bar{j}'\right) + 0.08\left(\dot{\theta} - 0.9397\dot{\psi}\right)\bar{k}'.\end{aligned}$$

These velocities must match the given components. The fact that $\bar{j}' = -\bar{I}$ at this instant substantially expedites the evaluation of dot products, which we find to be

$$\begin{aligned}\bar{v}_C \cdot \bar{I} = 5 &= -0.08(\dot{\phi} - 0.3420\dot{\psi})\left(\bar{j}' \cdot \bar{I}\right) + 0.08\dot{\theta}\left(\bar{k}' \cdot \bar{I}\right) = 0.08(\dot{\phi} - 0.3420\dot{\psi}),\\ \bar{v}_C \cdot \bar{J} = 3 &= -0.08(\dot{\phi} - 0.3420\dot{\psi})\left(\bar{j}' \cdot \bar{J}\right) + 0.08\dot{\theta}\left(\bar{k}' \cdot \bar{J}\right) = 0.08\dot{\theta}\cos 20^\circ,\\ \bar{v}_B \cdot \bar{J} = 6 &= -0.08(\dot{\phi} - 0.3420\dot{\psi})\left(\bar{i}' \cdot \bar{J} + \bar{j}' \cdot \bar{J}\right) + 0.08\left(\dot{\theta} - 0.9397\dot{\psi}\right)\bar{k}' \cdot \bar{J}\\ &= -0.08(\dot{\phi} - 0.3420\dot{\psi})(-\sin 20^\circ) + 0.08\left(\dot{\theta} - 0.9397\dot{\psi}\right)\cos 20^\circ.\end{aligned}$$

The solution of these simultaneous equations is

$$\dot{\psi} = -18.260, \quad \dot{\theta} = 39.907 \quad \dot{\phi} = 56.255 \text{ rad/s.} \qquad \triangleleft$$

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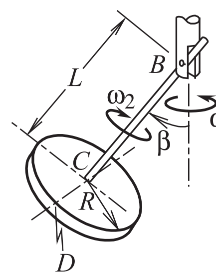
HOMEWORK PROBLEMS

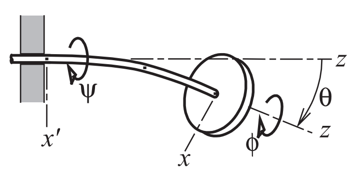
EXERCISE 4.1 A gyropendulum consists of a flywheel that rotates at constant angular speed ω_1 relative to shaft BC. This shaft is pinned to the vertical shaft, which rotates at constant angular speed ω_2. The angle β measuring the inclination of shaft BC is an arbitrary function of time. Use the Eulerian angle formulas for angular velocity and angular acceleration to derive expressions for the velocity and acceleration of point D, which coincides with the horizontal diameter at the instant of interest.



Exercise 4.1

EXERCISE 4.2 Consider a body whose orientation is described by Eulerian angles. Derive the transformation from space-fixed to body-fixed axes for a sequence beginning with precession $\psi = 20^\circ$, followed by nutation $\theta = -60^\circ$, then spin $\phi = 140^\circ$. Is it possible to obtain the same transformation with a different sequence beginning with nutation θ', followed by spin ϕ', then precession ψ'? If so, determine the values of θ', ϕ', and ψ'.

EXERCISE 4.3 A rigid disk is welded to the end of a flexible shaft that rotates about bearing A. The bending deformation of the shaft is such that its centerline forms a curve in a plane that always contains the bearing's axis. The rotation of this plane about the bearing's axis is the precession ψ. The tangent to this curve at end B is the axis of symmetry of the disk, and



Exercise 4.3

the angle between the bearing's axis and the disk's axis is the nutation angle θ. Torsional deformation of the shaft produces a spin ϕ about the disk's axis. Let xyz be a set of axes attached to the disk, and let $x'y'z'$ be a set of axes that undergo only the precessional motion. The z' axis is coincident with the bearing axis, and the curved centerline of the shaft is always situated in the $x'z'$ plane. It is observed that at some instant $\theta = 10^\circ$, $\phi = -5^\circ$, and the angular velocity of the disk is $\omega = 17\bar{i}' - 20\bar{j}' + 48\bar{k}'$ rad/s. Determine the corresponding precession, nutation, and spin rates. Then express the angular velocity in terms of components relative to xyz.

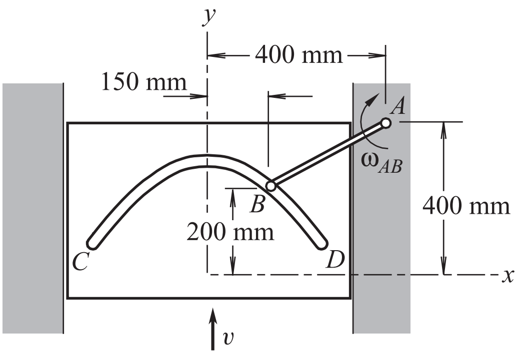
EXERCISE 4.4 Measurements of the rotational motion of an orbiting satellite indicate that, at a certain instant, the angular velocity with respect to body-fixed coordinate system xyz is $\bar{\omega} = 19.24\bar{i} + 6.68\bar{j} + 303.21\bar{k}$ rad/s. The laws for free rotation of a body, developed in Chapter 10, indicate that $\bar{k}$ is the spin axis and the precession axis is $\bar{K} = -0.3830\bar{i} - 0.6634\bar{j} + 0.6428\bar{k}$. Determine $\dot{\psi}$, θ, $\dot{\theta}$, ϕ, and $\dot{\phi}$ at this instant.

EXERCISE 4.5 Measurements of the rotational motion of an orbiting satellite indicate that, at a certain instant, the angular velocity with respect to body-fixed coordinate system xyz is $\bar{\omega} = -34.64\bar{i} + 10\bar{j} - 820\bar{k}$ rad/s and the angular acceleration is $\bar{\alpha} = -7800\bar{i} - 28713\bar{j} - 346\bar{k}$ rad/s^2. The laws for free rotation of a body, developed in Chapter 10, indicate that $\bar{k}$ is the spin axis and the precession axis is $\bar{K} = -0.8660\bar{i} + 0.50\bar{k}$. Determine $\dot{\psi}$, $\ddot{\psi}$, θ, $\dot{\theta}$, $\ddot{\theta}$, ϕ, $\dot{\phi}$, and $\ddot{\phi}$ at this instant.

EXERCISE 4.6 An alternative set of Eulerian angles is often employed to describe the rotation of aircraft and spacecraft. Let xyz be a reference frame that is attached to the vehicle, with the x axis aligned with the longitudinal axis of the vehicle and z aligned in a meaningful orthogonal direction, such as the direction of the aerodynamic lift for an aircraft. The fixed XYZ reference frame is defined such that its axes coincide with the orientation of the respective axes of xyz when the vehicle is in its nominal operational condition. The *yaw* angle ψ takes place about the Z axis, followed by the *pitch* angle θ about the new y axis, followed by the *roll* angle ϕ about the final x axis. (a) Derive the rotation transformation that converts $\bar{I}\bar{J}\bar{K}$ components to $\bar{i}\bar{j}\bar{k}$ components. (b) Describe the angular velocity and angular acceleration of the airplane in terms of body-fixed $\bar{i}\bar{j}\bar{k}$ components.

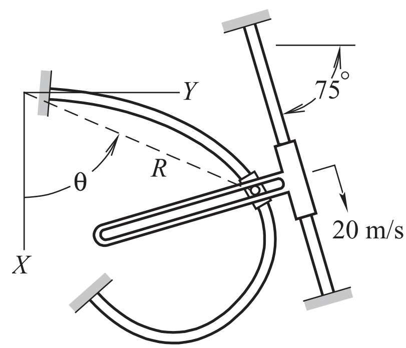
EXERCISE 4.7 Let the distance between points B and C in Fig. 4.6 be h, and let the y_1 axis coincide with the line connecting these points. Let ψ denote the angle that the collar rotates relative to bar 1. Describe the constraint equations relating the displacements, velocities, and accelerations of points B and C.

EXERCISE 4.8 Pin B slides through groove CD in a plate that translates upward at speed v. The groove forms the parabolic curve $y = 300 - x^2/400$, where x and y have units of millimeters. In the position shown, bar AB is rotating clockwise at 40 rad/s, and that rate is decreasing at 160 rad/s^2. Determine the corresponding values of v and $\dot{v}$.



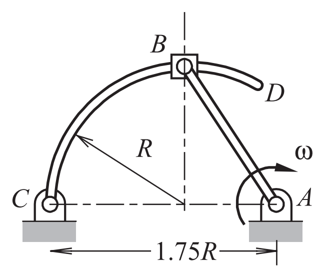
Exercise 4.8

EXERCISE 4.9 A collar slides in the horizontal plane over a curved rod defined in polar coordinates by $R = 0.1 \sin(2\theta)$ m. The motion is actuated by the translating arm, which contains a groove that pushes a pin in the collar. The speed of the arm is constant at 20 m/s. Determine the velocity and acceleration of the collar in the position where $\theta = 1$ rad.



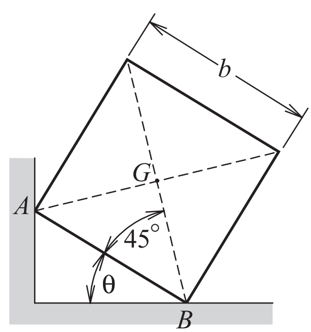
Exercise 4.9

EXERCISE 4.10 Bar AB rotates at the constant rate ω, which causes collar B to slide along curved bar CD. For the instant depicted in the diagram, determine the angular velocity and angular acceleration of bar CD and velocity and acceleration of the collar.



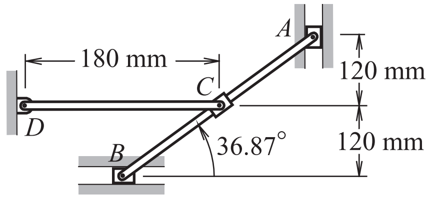
Exercise 4.10

EXERCISE 4.11 The cubic box slides along the wall and floor. The motion of the box is fully specified if the angle θ is determined as a function of time. Derive expressions for the velocity and acceleration of corners A and B and of the central point G in terms of θ, $\dot{\theta}$, and $\ddot{\theta}$.



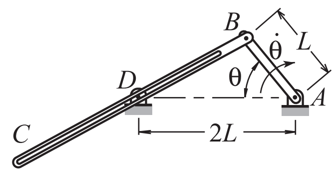
Exercise 4.11

EXERCISE 4.12 Collar C slides over bar AB. When the system is in the position shown, slider A is moving downward at 600 mm/s and its speed is decreasing at 15 m/s^2. Determine the corresponding angular velocity and angular acceleration of each bar.



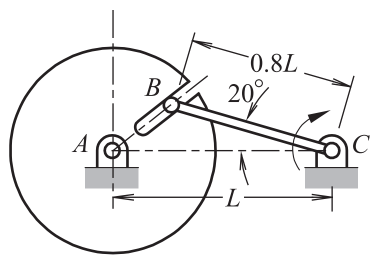
Exercise 4.12

EXERCISE 4.13 The rotation rate $\dot{\theta}$ of crank AB is constant. Determine the angular velocity and angular acceleration of bar BC and the velocity and acceleration of end C when $\theta = 60°$ and when $\theta = 120$.



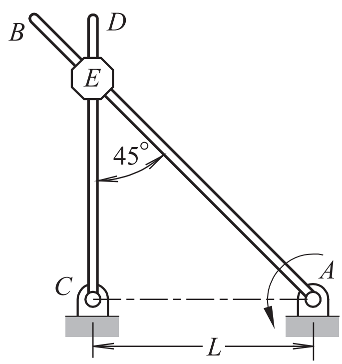
Exercise 4.13

EXERCISE 4.14 The slotted disk rotates at a constant angular speed ω_A. Determine the angular velocity and angular acceleration of connecting rod BC in the illustrated position.



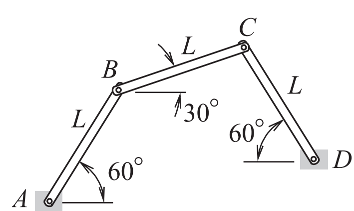
Exercise 4.14

EXERCISE 4.15 Holes bored through block E maintain the angle between bars AB and CD at 45°. At the instant when bar CD is in the upright position shown, bar AB is rotating counterclockwise at 10 rad/s and that rate is decreasing at 50 rad/s^2. For this instant determine the velocity and acceleration of the point on block E at which the centerlines of the bars intersect.



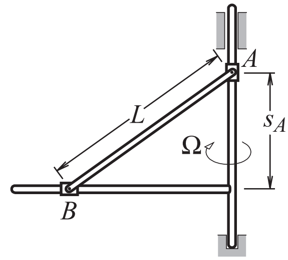
Exercise 4.15

EXERCISE 4.16 Bar AB rotates clockwise at the constant angular speed ω_1. Determine the angular velocity and angular acceleration of the other bars when the linkage is at the position shown in the sketch.



Exercise 4.16

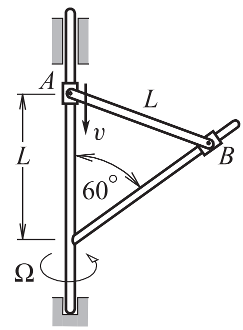
EXERCISE 4.17 Starting from $s_A = 0$ at $t = 0$, collar A is given a constant upward acceleration $\dot{v}_A$. The rotation rate Ω about the vertical axis is constant. Derive expressions for the angular velocity and acceleration of connecting rod AB as functions of the elapsed time t. Also derive expressions for the time dependence of the velocity and acceleration of collar B. Describe all results in terms of components relative to a coordinate system that rotates in unison with the T-bar.



Exercise 4.17

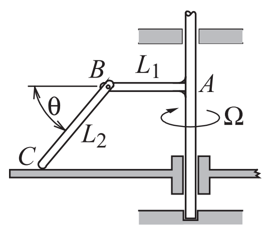
EXERCISE 4.18 Collar A is pushed downward at speed v, while the entire system precesses about the vertical axis at constant angular speed Ω. Determine the velocity of collar B and of the midpoint of bar AB at the instant depicted in the sketch.

EXERCISE 4.19 Collar A is pushed downward at speed v, while the entire system precesses about the vertical axis at angular speed Ω. Determine the velocity and acceleration of collar B and of the midpoint of bar AB at the instant depicted in the sketch.



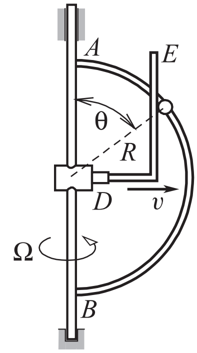
Exercises 4.18 and 4.19

EXERCISE 4.20 Bar BC is pinned to the T-bar, which rotates at constant rate Ω about the vertical axis. The bottom of this bar contacts the platform, which translates upward at constant speed v. Determine the angular velocity and angular acceleration of bar BC as functions of the angle of elevation θ and the value of Ω and ξ.



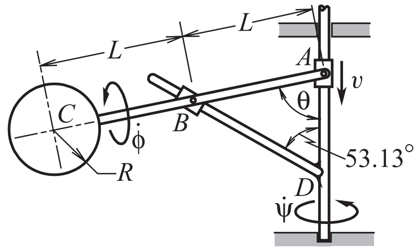
Exercise 4.20

EXERCISE 4.21 Bead C slides relative to the curved guide bar AB, which rotates about the vertical axis at the constant rate Ω. The movement of the slider is actuated by arm DE, which pushes the collar outward from the vertical axis at a constant rate v. Determine the velocity and acceleration of the slider as a function of θ.



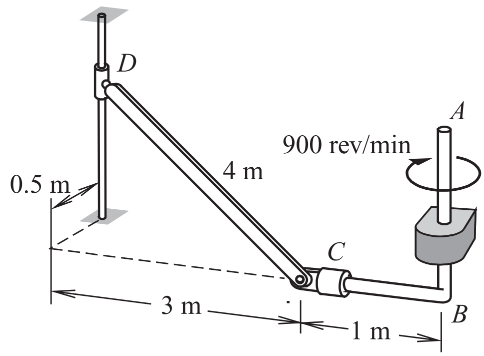
Exercise 4.21

EXERCISE 4.22 The spin rate $\dot{\psi}$ of sphere C is constant, as is the precession rate $\dot{\phi}$ and the speed v of collar A. Derive expressions for the angular velocity and angular acceleration of the sphere as functions of θ, v, $\dot{\psi}$, and $\dot{\phi}$.



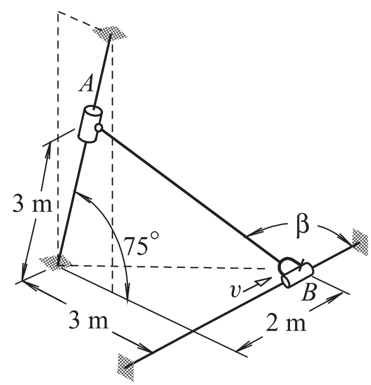
Exercise 4.22

EXERCISE 4.23 Crankshaft ABC rotates at the constant rate of 900 rev/min. The connecting rod CD is pinned to cap C, which is free to rotate about axis BC. The connection at collar D is a ball-and-socket joint. For the instant depicted in the sketch, determine the velocity and acceleration of collar D, and the corresponding angular velocity and angular acceleration of the connecting rod.



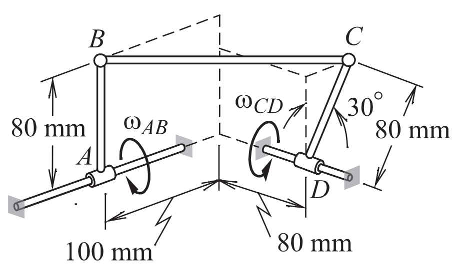
Exercise 4.23

EXERCISE 4.24 Collar A is connected to bar AB by a ball-and-socket joint, whereas the connection between collar B and bar AB is a fork-and-clevis. The speed of collar B is $v = 30$ m/s and $\dot{v} = -500$ m/s^2 at the position shown. (a) Determine the velocity of slider A and the value of $\dot{\beta}$ at this position, where β is the angle between bar AB and the horizontal guide. (b) Determine the acceleration of slider A and the value of $\ddot{\beta}$ at this position.



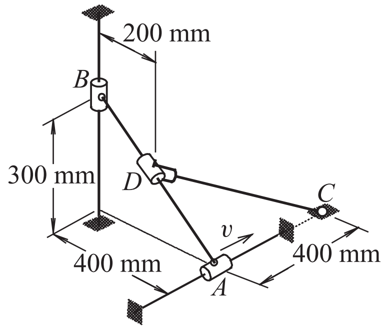
Exercise 4.24

EXERCISE 4.25 The axes of bearings A and D lie in the same horizontal plane, and intersect orthogonally. Connections B and C are ball-and-socket joints, and bars AB and CD are welded to collars A and D, respectively. Bar AB rotates at the constant rate of 200 rev/min. Determine the velocity and acceleration of joint C at the instant shown.



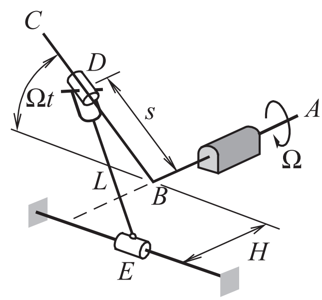
Exercise 4.25

EXERCISE 4.26 Bar CD is connected to collar D by a clevis joint, and all other connections are ball-and-socket joints. Collar A moves toward point C at constant speed v. Determine the angular velocity of both bars at the instant depicted in the sketch in the case in which bar AB has a circular cross section. Then repeat the analysis for the case in which the cross section of bar AB is square and there is a close sliding fit between collar D and this bar.



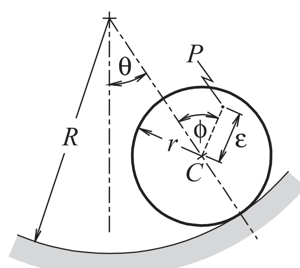
Exercise 4.26

EXERCISE 4.27 Bar DE is attached to collar E by a ball-and-socket joint, and its connection to collar D is a fork-and-clevis joint. The cross section of crankshaft ABC is circular, so collar D may rotate about axis BC relative to the crankshaft. The crankshaft rotates at constant angular speed Ω, so specification of the position of all parts of the linkage at a specified instant requires knowledge of the distance s locating collar D. Derive expressions for the angular velocity of bar DE and the speed of collar E in terms of s and $\dot{s}$ for the instant at which $\Omega t = \pi/3$. The horizontal distance $H = 0.3L$.



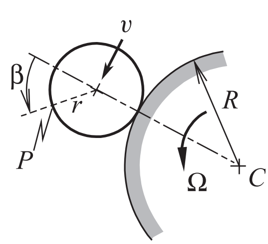
Exercise 4.27

EXERCISE 4.28 A cylinder of radius r rolls without slipping inside a semicylindrical cavity. Point P is collinear with the vertical centerline when the vertical angle θ locating the cylinder's center C is zero. Derive expressions for the velocity and acceleration of point P in terms of θ, $\dot{\theta}$, and $\ddot{\theta}$.



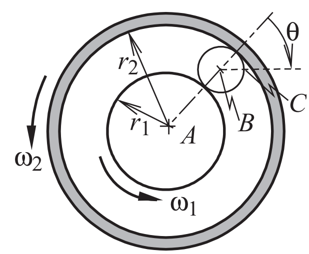
Exercise 4.28

EXERCISE 4.29 A disk rolls without slipping over the exterior of a large drum. The rotation rate Ω of the drum is constant. In the position shown the center of the disk has a speed v, which is increasing at the rate $\dot{v}$. Derive expressions for the velocity and acceleration of point P, which is situated at an arbitrary angle β relative to the line of centers.



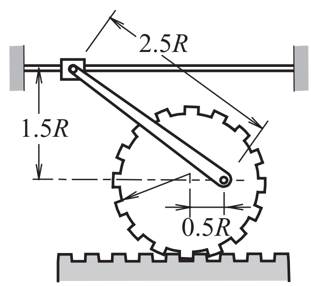
Exercise 4.29

EXERCISE 4.30 The angular velocities of the inner and outer gears are counterclockwise at the constant values ω_1 and ω_2, respectively. Determine the velocity and acceleration of point C on the perimeter of the planetary gear as a function of the angle θ locating the instantaneous position of point C relative to the radial line.



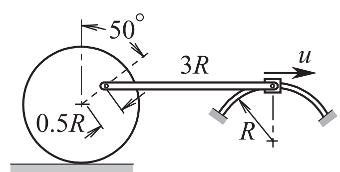
Exercise 4.30

EXERCISE 4.31 The collar has a constant speed v to the right, and the rack is stationary. Determine the angular velocity and angular acceleration of the gear at the instant depicted in the sketch.



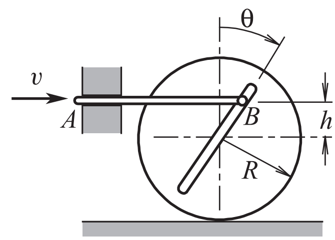
Exercise 4.31

EXERCISE 4.32 The wheel rolls without slipping over the ground as the collar slides at constant speed u over the curved guide bar. Determine the velocity and acceleration of the center of the wheel in terms of u when the linkage is in the position shown.



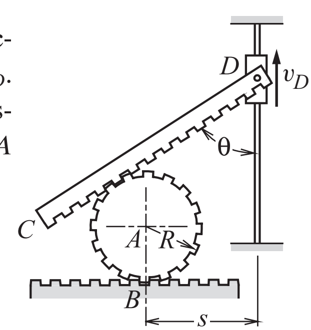
Exercise 4.32

EXERCISE 4.33 Movement of the actuating rod AB at constant speed v pushes the connecting pin through the groove in the wheel, thereby causing the wheel to roll over the ground. Determine the angular velocity and angular acceleration of the gear as a function of θ if there is no slippage in the rolling motion.



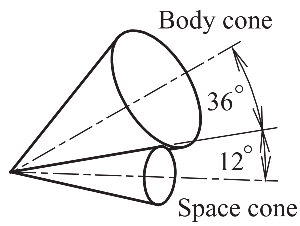
Exercise 4.33

EXERCISE 4.34 Rack CD, which meshes with gear A, is actuated by moving collar D upward at the constant speed v_D. Rack B, over which gear A rolls, is stationary. Derive expressions for the velocity and acceleration of the center of gear A as functions of the current values of s and θ.



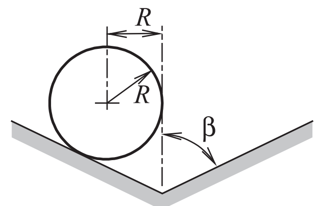
Exercise 4.34

EXERCISE 4.35 The body cone rolls without slipping over the stationary space cone. It is observed that the axis of the body cone requires 200 ms to complete one revolution. Determine the angular velocity and angular acceleration of the body cone.



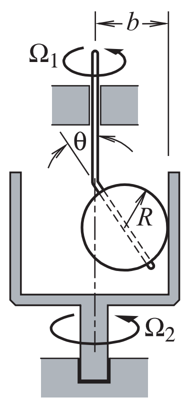
Exercise 4.35

EXERCISE 4.36 A sphere of radius R rolls without slipping in the interior of a cone such that R also is the distance from the axis of the cone to the center of the sphere. The speed of the center of the sphere is the constant value v. The rotation of the sphere is observed to consist of a precession about a vertical axis and a spin about an axis parallel to the cone generator. Derive expressions for the angular velocity and angular acceleration of the sphere in terms of v, R, and the apex angle θ.



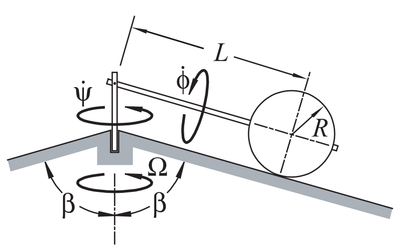
Exercise 4.36

EXERCISE 4.37 The sphere rolls without slipping over the interior wall of a hollow cylinder that rotates about its axis at Ω_2. The angular speed of the vertical shaft driving the sphere is Ω_1. Both rotation rates are constant. Determine the angular velocity and angular acceleration of the sphere.



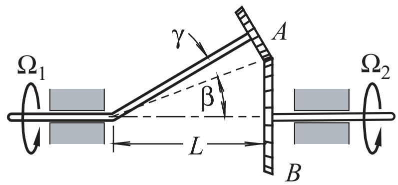
Exercise 4.37

EXERCISE 4.38 The sketch shows the cross section of a cone whose axis is vertical and whose vertex angle is 2β. This cone rotates about its axis at the constant angular speed Ω. The sphere spins freely at angular speed $\dot{\phi}$ about the shaft that intersects its center. This shaft precesses about the vertical axis at the variable angular speed $\dot{\psi}$, so $\ddot{\psi} \neq 0$. (a) Determine the ratio $\dot{\phi}/\Omega$ for which there is no slippage between the sphere and the cone. (b) Determine the angular velocity and angular acceleration of the sphere when there is no slippage.



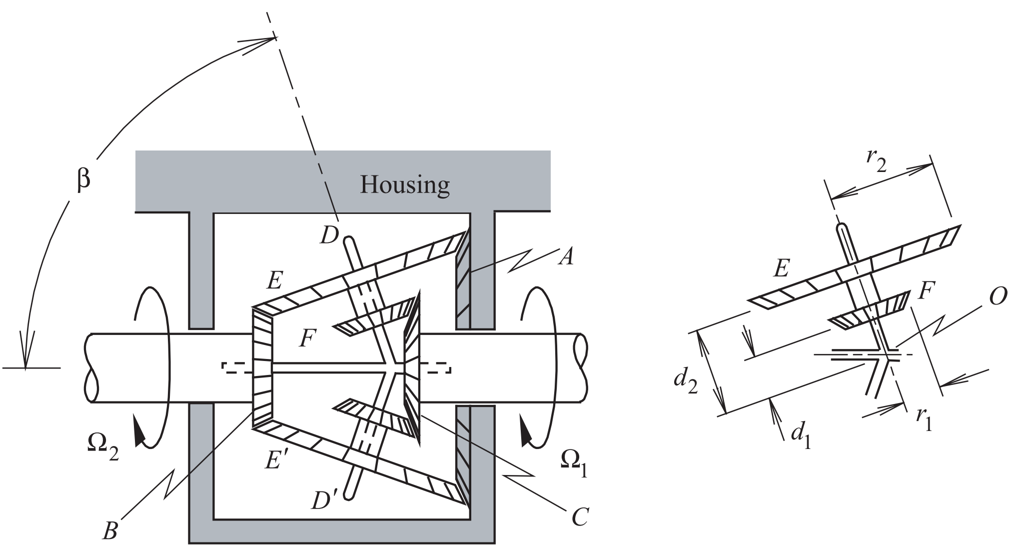
Exercise 4.38

EXERCISE 4.39 Gear A spins relative to its shaft, which rotates at variable rate Ω_1 about the horizontal axis. Gear B rotates at the variable rate Ω_2. Determine the angular velocity and angular acceleration of gear A.



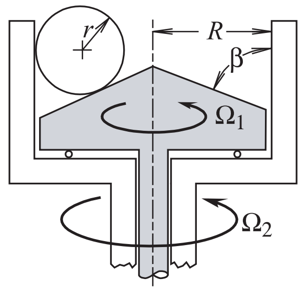
Exercise 4.39

EXERCISE 4.40 The sketch shows a reduction gear assembly that converts an input angular speed Ω_1 to an output speed Ω_2. Gear A is welded to the stationary housing, gears B and C are welded to their respective shafts, and spider arm D rotates about the horizontal axis at a different angular speed. Gears E and F are a single planetary body that spins about its axis of symmetry relative to the angled arm of the spider. (a) Derive an expression for the gear ratio Ω_2/Ω_1 in terms of the length dimensions and the angle β. (b) Derive expressions for the angular velocity and angular acceleration of planetary gear EF. *Hint*: The intersection O of the axis of the planetary gear and of the drive shaft is stationary.



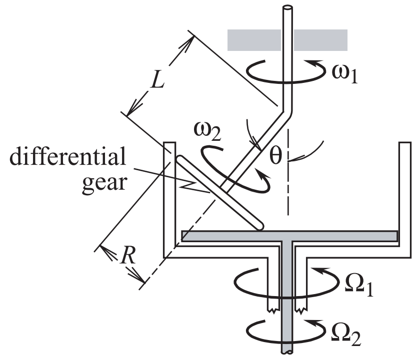
Exercise 4.40

EXERCISE 4.41 The cylindrical drum rotates about the vertical axis at the constant rate Ω_2, and the conical floor of the tank rotates at the constant rate Ω_1. In the situation of interest the radial line to the center of the sphere also rotates at Ω_1. There is no slippage between either the interior wall of the tank or the spinning conical floor in the rolling motion. Determine the angular velocity and acceleration of the sphere.



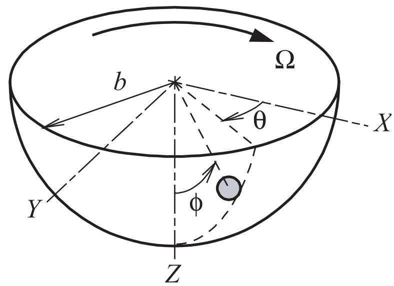
Exercise 4.41

EXERCISE 4.42 The diagram shows the cross section of a differential gear of radius R that spins relative to the bent shaft at angular speed ω_2. The rotation rate of this shaft about the vertical axis is ω_1. The cylindrical drum rotates at constant angular speed Ω_1 about the vertical axis, and the base gear rates at constant angular speed Ω_2. The differential gear rotates without slipping relative to both the base and the cylinder. (a) Derive expressions for ω_1 and ω_2 corresponding to specified values of Ω_1 and Ω_2. (b) Determine the angular velocity and angular acceleration of the differential gear.



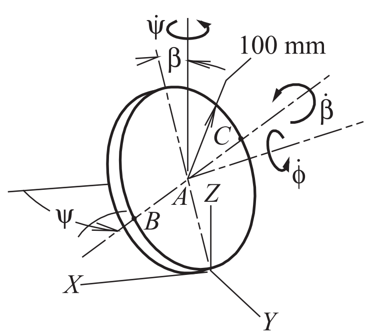
Exercise 4.42

EXERCISE 4.43 The system in the sketch is analogous to a roulette wheel, in that a sphere of radius R rolls without slipping over the interior of a hemispherical shell of radius b that rotates about the vertical axis at constant rate Ω. The polar and azimuth angles locating the center of the sphere are ϕ and θ, defined with respect to the fixed XYZ coordinate system. Both angles are arbitrary functions of time. Derive expressions for the angular velocity and angular acceleration of the sphere in terms of ϕ, θ, and their derivatives.



Exercise 4.43

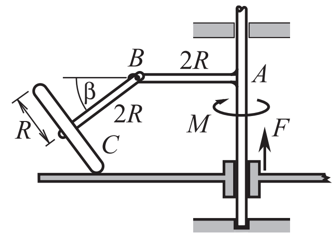
EXERCISE 4.44 The disk rolls without slipping over the horizontal XY plane. At the instant when $\beta = 36.87^\circ$, the X and Y components of the velocity of point B on the horizontal diameter of the disk are 8 m/s and −4 m/s, respectively, and the corresponding velocity components of center A at this instant are 4 m/s and 2 m/s. Determine the precession angle ψ between the horizontal diameter BAC and the X axis, and also evaluate the precession, nutation, and spin rates.



Exercises 4.44 and 4.45

EXERCISE 4.45 The disk is rolling without slipping over the horizontal XY plane. At the instant when the angle of inclination is $\beta = 30^\circ$, the disk is observed to be spinning at $\dot{\phi} = 5$ rad/s. At this instant, the speed of points B and C on its horizontal diameter are 1 and 2 m/s, respectively. Determine the corresponding precession and nutation rates.

EXERCISE 4.46 Shaft BC is pinned to the T-bar, which rotates at the constant angular speed ω_1. Wheel C rotates freely relative to shaft BC. The platform over which wheel C rolls is raised at the constant speed u, causing angle β to decrease. The wheel does not slip relative to the platform in the direction transverse to the diagram, but slipping in the radial direction is observed to occur. Derive expressions for the angular velocity and the angular acceleration of the wheel.



Exercise 4.46

CHAPTER 5

Inertial Effects for a Rigid Body

Chasle's theorem states that the general motion of a rigid body can be represented as a superposition of a translation following any point in a body and a pure rotation about that point. The kinematics tools we have developed provide the capability to describe these motions in terms of a few parameters. In this chapter we begin to characterize the relationship between forces acting on a rigid body and kinematical parameters for that body. The resultant of a set of forces may be regarded intuitively as the net tendency of the force system to push a body, so one should expect it to be related to the translational effect. Similarly, it is reasonable to expect that the resultant moment of a set of forces represents the rotational influence. We shall confirm and quantify these expectations.

From a philosophical perspective, the shift from statics, in which one equilibrates forces, to kinetics, in which the forces must match an inertial effect, is rather drastic. For a particle, Newton's Second and Third Laws are readily understood in this regard. However, the corresponding shift for the rotational effect will lead to effects associated with the angular momentum of a rigid body that sometimes are counterintuitive. This is especially true for those who try to examine spatial motion from a planar motion viewpoint. This chapter focuses on the determination and evaluation of angular momentum. In the course of the development we derive basic laws governing the relationship between a body's motion and the forces that act on the body. The development will be the extension to rigid bodies of Newton's Laws for particles, following concepts associated with Euler. Hence we refer to the resulting kinetics principles as the Newton–Euler equations of motion. The application of those principles will be addressed in Chapter 6.

5.1 LINEAR AND ANGULAR MOMENTUM

We begin by considering an arbitrary system of particles, which can represent anything from a region within a fluid to a deformable solid body. After we derive some general principles, we will specialize them to the particular case of a rigid body. Because we are cognizant of the type of information required by Chasle's theorem, our focus in all cases will be on using some point A in the system as a reference for the motion. Identification of criteria for selecting this point is one of the issues to be addressed. The fundamental axioms for the development are Newton's Second Law, which governs the motion of each particle in the system, and the Third Law, which tells us how each particle interacts with its surroundings.

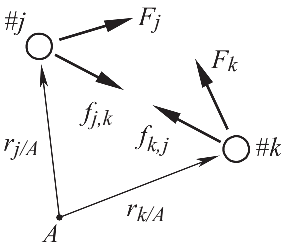


Figure 5.1. External and internal forces acting on a two-particle system.

5.1.1 System of Particles

Figure 5.1 shows a system of two particles, numbered j and k. Two types of forces are described there. Both $\bar{F}_j$ and $\bar{F}_k$ are *external forces,* that is, the forces exerted on the respective particles by anything other than those particles that constitute the system of interest. Both $\bar{f}_{j,k}$ and $\bar{f}_{k,j}$ are the *interaction forces*, where the first subscript denotes the particle to which the force is applied and the second subscript indicates which particle exerted the force. According to Newton's Third Law, a pair of interaction forces such as these are equal in magnitude and oppositely directed:

$$\bar{f}_{k,j} = -\bar{f}_{j,k}. \tag{5.1.1}$$

Furthermore, the Third Law states that the interaction forces are collinear, meaning that they have the same line of action. The significance of this feature becomes apparent when we consider the moment exerted by each interaction force about origin A, that is, $\bar{r}_{j/A} \times \bar{f}_{j,k}$ and $\bar{r}_{k/A} \times \bar{f}_{k,j}$. Because the forces act along a common line, the perpendicular distance from point A to their line of action is identical. In combination with Eq. (5.1.1), we conclude that the moment these forces exert about the origin are equal in magnitude, but directed oppositely:

$$\bar{r}_{k/A} \times \bar{f}_{k,j} = -\bar{r}_{j/A} \times \bar{f}_{j,k}. \tag{5.1.2}$$

Now let us consider Newton's Second Law for each particle. The resultant force acting on each is the sum of the external and internal contributions, so

$$\bar{F}_j + \bar{f}_{j,k} = m_j \bar{a}_j, \quad \bar{F}_k + \bar{f}_{k,j} = m_k \bar{a}_k. \tag{5.1.3}$$

Our interest here is the role of the force system on the ensemble of particles. To find the resultant of all forces, which we denote as $\Sigma \bar{F}$, we add the preceding equations. In view of Eq. (5.1.1), the internal forces cancel in this sum, so we have

$$\Sigma \bar{F} = \bar{F}_j + \bar{F}_k = m_j \bar{a}_j + m_k \bar{a}_k. \tag{5.1.4}$$

A similar result arises when we consider the total moment $\Sigma \bar{M}_A$ exerted by all forces about origin A. We use the position vector from the origin to each particle to evaluate this moment, and use Eqs. (5.1.3) to characterize the forces. This leads to

$$\Sigma \bar{M}_A = \bar{r}_{j/A} \times \left(\bar{F}_j + \bar{f}_{j,k}\right) + \bar{r}_{k/A} \times \left(\bar{F}_k + \bar{f}_{k,j}\right) = \bar{r}_{j/A} \times m_j \bar{a}_j + \bar{r}_{k/A} \times m_k \bar{a}_k. \tag{5.1.5}$$

Equation (5.1.2) simplifies this to

$$\Sigma \bar{M}_A = \bar{r}_{j/A} \times \bar{F}_j + \bar{r}_{k/A} \times \bar{F}_k = \bar{r}_{j/A} \times m_j \bar{a}_j + \bar{r}_{k/A} \times m_k \bar{a}_k. \tag{5.1.6}$$

This shows that only the external forces contribute to the resultant force and to the resultant moment about point A.

If we were to add a third particle to the system addressed thus far, there would be additional internal forces exerted between this particle and each of particles in the original system. Each additional pair of interaction forces would give no net contribution to the resultant force and the moment about point A. Thus the extension of the system from two particles to an arbitrary collection of particles does not alter the fact that *only the external forces contribute to the resultant force and to the resultant moment about point* A. To quantify this fact we let N denote the number of particles contained in the system and number the particles consecutively from $j = 1$ to $j = N$. The forms analogous to Eqs. (5.1.4) and (5.1.6) are

$$\begin{aligned} \Sigma \bar{F} &= \sum_{j=1}^{N} \bar{F}_j = \sum_{j=1}^{N} m_j \bar{a}_j, \\ \Sigma \bar{M}_A &= \sum_{j=1}^{N} \bar{r}_{j/A} \times \bar{F}_j = \sum_{j=1}^{N} \left(\bar{r}_{j/A} \times m_j \bar{a}_j \right). \end{aligned} \tag{5.1.7}$$

We now turn our attention to the inertial effects described by the right-hand side of the preceding equations. For the resultant force we replace the acceleration with the second derivative of position. Because the mass of each particle is constant, we may form the sum before differentiating, specifically,

$$\Sigma \bar{F} = \sum_{j=1}^{N} \bar{F}_j = \sum_{j=1}^{N} m_j \frac{d^2}{dt^2} \bar{r}_{i/O} = \frac{d^2}{dt^2} \left(\sum_{j=1}^{N} m_j \bar{r}_{j/O} \right). \tag{5.1.8}$$

The term in parentheses is the *first moment of mass*. To understand it consider a set of particles near the surface of the Earth, with gravity acting in the negative Z direction, as depicted in Fig. 5.2. The gravitational attraction force on all particles may be replaced with a single resultant $\bar{F}$ acting parallel to the individual forces. The magnitude of this result is the sum of the individual forces,

$$F = m_{\text{system}} g, \quad m_{\text{system}} = \sum_{j=1}^{N} m_j. \tag{5.1.9}$$

The point through which the resultant force acts is the *center of mass,* which is denoted as point G. To locate the X and Y coordinates of this point we equate the total moment of the gravitational forces acting on all particles to the moment of the resultant. After cancellation of a common g factor, the moments about the Y and X axes reduce to

$$m_{\text{system}} X_G = \sum_{j=1}^{N} m_j X_j, \quad m_{\text{system}} Y_G = \sum_{j=1}^{N} m_j Y_j. \tag{5.1.10}$$

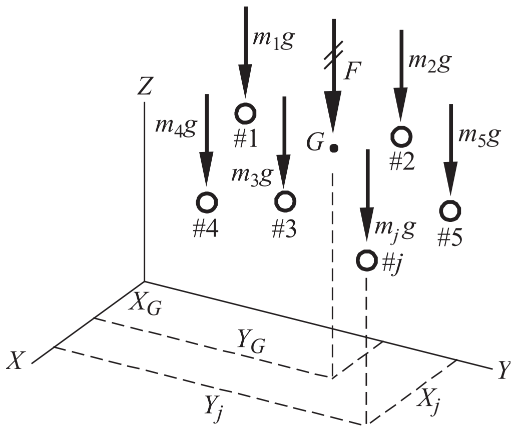


Figure 5.2. Resultant of a the gravity forces acting on a set of particles close to the Earth.

These are first moments of mass with respect to the X and Y coordinates. A similar form featuring Z coordinates would result if gravity were to act parallel to the X or Y axes. In view of the fact that $\bar{r}_{G/O} = X_G\bar{I} + Y_G\bar{J} + Z_G\bar{K}$, the vector form of the first moment of mass is

$$\boxed{m_{\text{system}}\bar{r}_{G/O} = \sum_{j=1}^{N} m_j\left(X_j\bar{I} + Y_j\bar{J} + Z_j\bar{K}\right) = \sum_{j=1}^{N} m_j\bar{r}_{j/O}.} \tag{5.1.11}$$

Substitution of this relation into Eq. (5.1.8) leads to

$$\Sigma\bar{F} = \frac{d^2}{dt^2}\left(m_{\text{system}}\bar{r}_{G/O}\right). \tag{5.1.12}$$

The system's mass is constant, so the preceding reduces to

$$\boxed{\Sigma\bar{F} = m_{\text{system}}\bar{a}_G.} \tag{5.1.13}$$

From this expression, we recognize that, although he posed the Second Law for a particle, Newton actually captured the behavior of the center of mass of any system of particles. If the particles move independently, it is one of many equations of motion for the various particles. The primary value of this relation lies in its application to the collection of particles forming a rigid body, in which case it addresses the portion of Chasle's theorem that requires description of the motion of one point in a body.

There are several formulations of the kinetic moment equation; they differ by the reference point that is selected. We now specify that the reference point for the kinematical description of velocity and acceleration should be the same as the point A about which we evaluate the moment sum. At this juncture, the collection of particles can move without kinematical constraints, so there is no overall rotational motion to consider. Thus we define the reference point A to be the origin of a translating reference frame xyz. For shorthand we designate the relative velocity and acceleration of point P with respect to the moving origin A as $\bar{v}_{P/A}$ and $\bar{a}_{P/A}$, respectively, where

$$\begin{aligned} \bar{v}_{P/A} &= (\bar{v}_{P/A})_{xyz} = \bar{v}_P - \bar{v}_A, \\ \bar{a}_{P/A} &= (\bar{a}_{P/A})_{xyz} = \bar{a}_P - \bar{a}_A. \end{aligned} \tag{5.1.14}$$

The second of Eqs. (5.1.7) describes the moment exerted by all forces acting on the system. In that relation we use the preceding to relate the acceleration of each particle to the acceleration of point A, which leads to

$$\Sigma \bar{M}_A = \sum_{j=1}^{N} \bar{r}_{j/A} \times m_j \left(\bar{a}_A + \bar{a}_{j/A}\right). \tag{5.1.15}$$

The sum may be broken into two parts. The first contains $\bar{a}_A$ as a common factor. When it is taken out of the summation, what remains is the first moment of mass relative to origin A. In view of Eq. (5.1.11), the expression becomes

$$\Sigma \bar{M}_A = m_{\text{system}} \bar{r}_{G/A} \times \bar{a}_A + \sum_{j=1}^{N} \left[\bar{r}_{j/A} \times m_j \bar{a}_{j/A}\right]. \tag{5.1.16}$$

To convert the remaining sum to a more useful form, we make the time derivative entailed in acceleration explicit, and then invoke the rule for differentiating a cross product:

$$\begin{aligned} \Sigma \bar{M}_A &= m_{\text{system}} \bar{r}_{G/A} \times \bar{a}_A + \sum_{j=1}^{N} \bar{r}_{j/A} \times m_j \frac{d}{dt}\left(\bar{v}_{j/A}\right) \\ &\equiv m_{\text{system}} \bar{r}_{G/A} \times \bar{a}_A + \sum_{j=1}^{N} \left[\frac{d}{dt}\left(\bar{r}_{j/A} \times m_j \bar{v}_{j/A}\right) - \frac{d}{dt}\left(\bar{r}_{j/A}\right) \times m_j \bar{v}_{j/A}\right]. \end{aligned} \tag{5.1.17}$$

By definition, $d\left(\bar{r}_{j/A}\right)/dt = \bar{v}_{j/A}$, so the last term vanishes. Taking the remaining time derivative outside the sum then gives

$$\Sigma \bar{M}_A = m_{\text{system}} \bar{r}_{G/A} \times \bar{a}_A + \frac{d}{dt} \bar{H}_A, \tag{5.1.18}$$

where $\bar{H}_A$ is the *angular momentum* of the system about point A:

$$\boxed{\bar{H}_A = \sum_{j=1}^{N} \left(\bar{r}_{j/A} \times m_j \bar{v}_{j/A}\right).} \tag{5.1.19}$$

Note that $m_j \bar{v}_{j/A}$ is the (linear) momentum of particle i relative to the translating xyz reference frame whose origin is point A. Hence a more descriptive name for $\bar{H}_A$ is *moment of momentum* relative to point A. In the special case where point A is stationary, these equations reduce to the sum of the angular impulse–momentum equations for each particle relative to an inertial reference point, as described by Eqs. (1.2.25).

Equations (5.1.13) and (5.1.18) can be considered to govern respectively the average translation and rotation of any system of particles. The details of the interaction forces affect how each particle in the system moves relative to these averages. For example, if our concern were with the effects of deformation of an elastic body, the internal forces would be stress resultants, and we would need to characterize how those quantities are related to the positions of the various particles. In the case of a rigid body, the kinematical condition of rigidity, which led to Chasle's theorem, is such that knowledge of the "average" motions is all that is required.

5.1.2 Rigid Body—Basic Equations

In the absence of deformation, the center of mass has a stationary position relative to a body. Thus, application of the extended version of Newton's Second Law, Eq. (5.1.13), governs the acceleration of a point in that body. Integrating this equation would enable us to determine the velocity and position of a point in the body, which is the first part of the information required according to Chasle's theorem.

Up to now, point A could be any that we wish. Now *we require that point A have a fixed position relative to the body*. The difference between the velocities of any two points in a rigid body is solely due to the body's angular velocity $\bar{\omega}$, so this restriction enables us to assert that

$$\bar{v}_{j/A} = \bar{\omega} \times \bar{r}_{j/A}. \tag{5.1.20}$$

Correspondingly, the angular momentum becomes

$$\boxed{\bar{H}_A = \sum_{j=1}^{N} m_j \left[\bar{r}_{j/A} \times (\bar{\omega} \times \bar{r}_{j/A})\right].} \tag{5.1.21}$$

It is important to observe at this juncture that the mass and position of each particle relative to a body-fixed reference frame are constant for a specified rigid body. It follows that, after the preceding terms are summed over all particles, the resulting $\bar{H}_A$ will depend solely on the angular velocity $\bar{\omega}$ of the body. Because Eq. (5.1.18) features the rate of change of $\bar{H}_A$, this equation will lead to an equation governing the rotation of the body. From this perspective, the fact that Eq. (5.1.18) also contains $\bar{a}_A$ is a complication, because the analysis of rotational motion requires simultaneous consideration of point motion. We avoid this complication in our initial studies by foregoing the freedom we have in a statics study to select a point for summing moments in favor of a restricted choice having the virtue of simplifying the kinetics terms. We refer to such points as "allowable" for forming the moment equation, which indicates that the simplified version of the moment equation is applicable. At the closure of the next chapter, we will reassert our freedom to sum moments about any point.

There are three possibilities for selecting point A such that the term $\bar{r}_{G/A} \times \bar{a}_A$ coupling the translational and rotational effects in Eq. (5.1.18) will vanish:

1. Select point A to be the center of mass G, so that $\bar{r}_{G/A} \equiv \bar{0}$. *The center of mass is always an allowable point*. This is the point we shall use whenever the body of interest executes a general motion or a translation.

2. Select point A such that $\bar{a}_A = \bar{0}$. By definition, a body in pure rotation has some point that is stationary. Thus *the pivot point for a pure rotation is always an allowable point.*
3. Select point A to be the point that is accelerating directly toward or away from the center of mass, in which case $\bar{a}_A$ is parallel to $\bar{r}_{G/A}$, so that $\bar{r}_{G/A} \times \bar{a}_A = \bar{0}$. This is a highly specialized case, so we will not employ such a point to formulate the equations governing rotation.

Formulation of the force equation of motion requires that we identify the center of mass. Thus, selecting the center of mass as the focal point for the moment equation would lead to a general procedure. However, there is an important reason why the second selection is preferable for pure rotation. Preventing the pivot point from moving requires that reaction forces be exerted at that point. For example, a ball-and-socket joint exerts an arbitrary reaction force having three components. By definition, reactions are not known in advance, so summing moments about the pivot avoids the occurrence of these unknowns in the rotational equations of motion.

In contrast to the first two possibilities, there seldom is any point in a body that is accelerating directly toward or away from the center of mass. Even if there were, locating it would be difficult. One exception occurs in planar motion, when a disk rolls without slipping over a stationary surface. The contact point's acceleration in that case is normal to the contact plane, and therefore directed toward the center of the disk. Thus the contact point would fit the allowability specification, but only if the wheel were balanced, so that the center of mass and geometric centroid coincide. Furthermore, if we wish to study the effect of slippage, the contact point would no longer be acceptable.

Translational motion is interesting because it is sometimes mistaken for a static situation. Because $\bar{\omega} \equiv \bar{0}$ for pure translation, Eq. (5.1.21) indicates that $\bar{H}_A \equiv \bar{0}$. Thus the moment equation for a translating body reduces to $\Sigma \bar{M}_A = m_{\text{system}} \bar{r}_{G/A} \times \bar{a}_A$. If point A is not selected to be the center of mass, the resultant moment vanishes only if $\bar{a}_A = \bar{0}$. However, all points in a translating body experience the same acceleration. Hence the condition $\bar{a}_A = \bar{0}$ here corresponds to rectilinear translation at a constant speed. A body executing such a motion represents an inertial reference frame, so the laws of statics apply. *If points in a translating body do not move along a straight path, or their speed is not constant, then the center of mass is the only allowable point for formulating the simplified moment equation of motion.*

When point A is allowable, the moment equation reduces to

$$\boxed{\Sigma \bar{M}_A = \frac{d}{dt} \bar{H}_A.} \tag{5.1.22}$$

This form is analogous to the equation for the motion of the center of mass. To recognize this we differentiate the first moment of mass, Eq. (5.1.11), which shows that the total linear momentum of any system is

$$\boxed{\bar{P} = \sum_{j=1}^{N} m_j \bar{v}_j = \sum_{j=1}^{N} \frac{d}{dt} \left(m_j \bar{r}_{j/O} \right) = \frac{d}{dt} \left(m_{\text{system}} \bar{r}_{G/O} \right) = m_{\text{system}} \bar{v}_G.} \tag{5.1.23}$$

Thus Eq. (5.1.13) is equivalent to

$$\boxed{\Sigma \bar{F} = \frac{d}{dt} \bar{P}.} \tag{5.1.24}$$

In other words, the linear or angular effect of the external force system equals the rate of change of the corresponding type of momentum for the body. Thus, the equations of motion described by this chapter are said to be momentum based. An alternative energy-based procedure for formulating equations of motion will be addressed in Chapter 7.

5.1.3 Kinetic Energy

In addition to being the fundamental kinetic quantity for our studies in the later chapters, kinetic energy appears in work–energy principles that are sometimes a useful adjunct to the momentum-based equations of motion. Also, kinetic energy will play a prominent role for one aspect of the evaluation of inertia properties of a rigid body. We begin by describing the kinetic energy of a system of independently moving particles.

Because kinetic energy is a scalar, we obtain the total energy T of the system by adding the values for each particle:

$$T = \sum_{j=1}^{N} \frac{1}{2} m_j \bar{v}_j \cdot \bar{v}_j. \tag{5.1.25}$$

Let B denote an arbitrary reference point to which the velocity of all particles is referred, so that $\bar{v}_j = \bar{v}_B + \bar{v}_{j/B}$. The corresponding form for the system's kinetic energy is

$$\begin{aligned} T &= \sum_{j=1}^{N} \frac{1}{2} m_j \left(\bar{v}_B + \bar{v}_{j/B} \right) \cdot \left(\bar{v}_B + \bar{v}_{j/B} \right) \\ &= \sum_{j=1}^{N} \frac{1}{2} m_j \left(\bar{v}_B \cdot \bar{v}_B + 2\bar{v}_B \cdot \bar{v}_{j/B} + \bar{v}_{j/B} \cdot \bar{v}_{j/B} \right). \end{aligned} \tag{5.1.26}$$

We factor out of each sum terms that are independent of the particle number, which yields

$$T = \frac{1}{2} m_{\text{system}} \bar{v}_B \cdot \bar{v}_B + \bar{v}_B \cdot \sum_{j=1}^{N} \frac{d}{dt} \left(m_j \bar{r}_{j/B} \right) + \frac{1}{2} \sum_{j=1}^{N} m_j \bar{v}_{j/B} \cdot \bar{v}_{j/B}. \tag{5.1.27}$$

We recognize the first sum as the time derivative of the first moment of mass relative to point B. In view of Eq. (5.1.11), we find that

$$T = \frac{1}{2} m_{\text{system}} \bar{v}_B \cdot \bar{v}_B + m_{\text{system}} \bar{v}_B \cdot \bar{v}_{G/B} + \frac{1}{2} \sum_{j=1}^{N} m_j \bar{v}_{j/B} \cdot \bar{v}_{j/B}. \tag{5.1.28}$$

One viewpoint of this expression is that the kinetic energy of any system of particles is associated with three effects: translation of all particles following the reference point (the first term above), motion of the particles relative to the reference point (the third term), and a coupling of the motions of the reference point and of the center of mass relative to the reference point.

Our interest is the kinetic energy of a rigid body. We restrict point B to be stationary relative to the body, so that the relative velocity of any material point is $\bar{v}_{j/B} = \bar{\omega} \times \bar{r}_{j/B}$. The kinetic energy correspondingly becomes

$$T = \frac{1}{2} m\bar{v}_B \cdot \bar{v}_B + m\bar{v}_B \cdot (\bar{\omega} \times \bar{r}_{G/B}) + \frac{1}{2} \sum_{j=1}^{N} m_j (\bar{\omega} \times \bar{r}_{j/B}) \cdot (\bar{\omega} \times \bar{r}_{j/B}). \tag{5.1.29}$$

The sum may be written in a more recognizable form by use of an identity for the scalar triple product:

$$\left(\bar{a} \times \bar{b}\right) \cdot \bar{c} \equiv \bar{a} \cdot \left(\bar{b} \times \bar{c}\right). \tag{5.1.30}$$

We employ this identity with $\bar{a} = \bar{\omega}$, $\bar{b} = \bar{r}_{j/B}$, and $\bar{c} = \bar{\omega} \times \bar{r}_{j/B}$, which yields

$$T = \frac{1}{2} m\bar{v}_B \cdot \bar{v}_B + m\bar{v}_B \cdot (\bar{\omega} \times \bar{r}_{G/B}) + \frac{1}{2} \sum_{j=1}^{N} m_j \bar{\omega} \cdot [\bar{r}_{j/B} \times (\bar{\omega} \times \bar{r}_{j/B})]. \tag{5.1.31}$$

Because $\bar{\omega}$ is an overall property of the motion, it may be factored out of the sum. The terms that remain are recognizable from Eq. (5.1.21) as the angular momentum relative to point B, so that

$$T = \frac{1}{2} m\bar{v}_B \cdot \bar{v}_B + m\bar{v}_B \cdot (\bar{\omega} \times \bar{r}_{G/B}) + \frac{1}{2} \bar{\omega} \cdot \bar{H}_B. \tag{5.1.32}$$

To simplify this expression we restrict point B to fit either of the first two criteria for an allowable point, that is, the center of mass or the pivot point for a body in pure rotation. Either choice cancels the second term. (Other choices for point B would achieve the same simplification, such as selecting it to be the instant center for a general planar motion, but there is little need to consider them.) Thus the alternatives we employ to evaluate the kinetic energy of a rigid body are

$$\boxed{\begin{aligned} &T = \frac{1}{2} m\bar{v}_G \cdot \bar{v}_G + \frac{1}{2} \bar{\omega} \cdot \bar{H}_G\text{: any type of motion,} \\ &T = \frac{1}{2} \bar{\omega} \cdot \bar{H}_O\text{: pure rotation about point } O. \end{aligned}} \tag{5.1.33}$$

The fact that there are two ways to describe the kinetic energy of a body in pure rotation leads to a relation whereby the angular momentum may be transferred between points. Suppose we were to hold some point P in a body stationary, in which case the velocity of the center of mass would be $\bar{\omega} \times \bar{r}_{G/P}$. The alternative descriptions of the kinetic energy in that case would require that

$$T = \frac{1}{2} \bar{\omega} \cdot \bar{H}_P = \frac{1}{2} m (\bar{\omega} \times \bar{r}_{G/P}) \cdot (\bar{\omega} \times \bar{r}_{G/P}) + \frac{1}{2} \bar{\omega} \cdot \bar{H}_G. \tag{5.1.34}$$

A rearrangement of terms based on Eq. (5.1.30) leads to

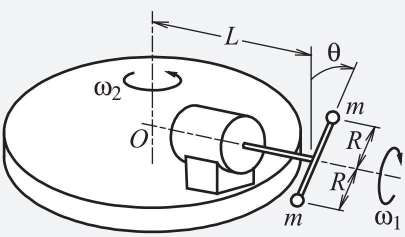
$$T = \frac{1}{2}\bar{\omega}\cdot\bar{H}_P = \frac{1}{2}m\bar{\omega}\cdot[\bar{r}_{G/P}\times(\bar{\omega}\times\bar{r}_{G/P})] + \frac{1}{2}\bar{\omega}\cdot\bar{H}_G. \tag{5.1.35}$$

This relation must be true for any angular velocity, so the factor of $\bar{\omega}$ in the dot product on either side of the equality sign must match. It is useful for later work to use $\bar{r}_{P/G} = -\bar{r}_{G/P}$, from which it follows that

$$\bar{H}_P = \bar{H}_G + m\bar{r}_{P/G}\times(\bar{\omega}\times\bar{r}_{P/G}). \tag{5.1.36}$$

There seldom is need to transfer angular momentum between points. However, Eq. (5.1.36) will soon prove to be useful for our exploration of inertial properties.

EXAMPLE 5.1 Identical small spheres having mass m are welded to the ends of a rigid bar that spins about the axis of the motor at angular speed ω_1. The motor is mounted on the horizontal turntable, which rotates at angular speed ω_2. Determine the angular momentum of this pair of particles about point C where the connecting bar is welded to the motor's shaft. Then use the angular momentum to characterize the force system exerted on the connecting bar at point C, as well as the kinetic energy of these spheres. Express the result in terms of the angle θ from the bar's centerline to vertical.



Example 5.1

SOLUTION The intent of this problem is to provide visualizations that will lessen the abstract nature of the development thus far, and also bring to the fore aspects that distinguish the kinetics of spatial motion from the simpler case of planar motion. Both spheres, being small, are treated as particles. They are fastened to the massless rigid body formed by the connecting bar and the motor's shaft. Point O is also a point in this body, and $\bar{v}_O \equiv \bar{0}$. Thus point O is an allowable point for the moment equation. The center of mass G of the pair of particles is halfway between them, which is the attachment point of the bar to the motor's shaft. This is always an allowable point.

The angular velocity of the bar/shaft system is

$$\bar{\omega} = \omega_1\bar{e}_1 + \omega_2\bar{e}_2.$$

To describe the unit vector for each rotation, we let $x'y'z'$ be a coordinate system attached to the turntable with its origin O at the center of the turntable. The z' axis is vertical and the x' axis is aligned with the shaft, such that $\bar{e}_1 = -\bar{i}'$ and $\bar{e}_2 = \bar{k}'$. The angular velocity of the connecting bar is then given by

$$\bar{\omega} = -\omega_1 \bar{i}' + \omega_2 \bar{k}'. \tag{1}$$

To evaluate the angular momentum with respect to point C, we form the position vector to each sphere. In terms of components relative to $x'y'z'$, these are

$$\bar{r}_{1/C} = R\sin\theta\,\bar{j}' + R\cos\theta\bar{k}' = -\bar{r}_{2/C}. \tag{2}$$

Equation (5.1.21) for this two-particle system is

$$\bar{H}_C = 2m_1\left[\bar{r}_{1/C} \times (\bar{\omega} \times \bar{r}_{1/C})\right].$$

The result of substituting Eqs. (1) and (2) into this expression is

$$\bar{H}_C = -2mR^2\omega_1\bar{i}' - 2mR^2\omega_2(\sin\theta)(\cos\theta)\,\bar{j}' + 2mR^2\omega_2(\sin\theta)^2\,\bar{k}'. \qquad (3) \triangleleft$$

The first feature to note is that some components of $\bar{H}_C$ are not constant, even though $\bar{\omega}$ has constant components relative to $x'y'z'$. Furthermore, the orientations of both $\bar{i}'$ and $\bar{j}'$ are variable. This feature typifies the fundamental fact that, even though the rotation rates in spatial motion might be constant, the angular momentum will not be a constant vector. Moments are required to change the angular momentum, regardless of whether the change of the angular momentum is a result of components having a variable direction or magnitude. It follows that the shaft must exert a moment on the connecting bar in order to produce the specified motion.

To evaluate the force system that the motor's shaft must apply, we consider a rigid body consisting of the two spheres and the connecting bar. If the rotation rates are sufficiently high, the gravitational forces will be negligible, in which case the only significant forces acting on this body are exerted by the motor's shaft. Point C is the body's center of mass, and this point follows a circular path of radius L, so the resultant force exerted by the shaft is found from Eq. (5.1.13) to be

$$\bar{F}_{\text{shaft}} = m_{\text{body}}\bar{a}_C = 2m\left(-L\omega_2^2\bar{i}'\right). \qquad (4) \triangleleft$$

Similar reasoning indicates that the only significant moment acting on the two-sphere rigid body is a couple exerted by the shaft. According to Eq. (5.1.22), this couple must equal the rate at which $\bar{H}_C$ changes. To differentiate the components of the angular momentum in Eq. (3) we observe that $\dot{\theta} = \omega_1$, whereas the derivatives of the unit vectors follow from the fact that the angular velocity of $x_1y_1z_1$ is $\omega_2\bar{k}'$, so that

$$\begin{aligned}\bar{M}_{\text{shaft}} = \frac{d\bar{H}_C}{dt} &= \dot{\omega}_1\frac{\partial}{\partial\theta}\left[-2mR^2\omega_2(\sin\theta)(\cos\theta)\,\bar{j}' + 2mR^2\omega_2(\sin\theta)^2\,\bar{k}'\right] \\ &\quad - 2mR^2\omega_1\left(\omega_2\bar{k}' \times \bar{i}'\right) - 2mR^2\omega_2(\sin\theta)(\cos\theta)\left(\omega_2\bar{k}' \times \bar{j}'\right) \\ &\quad + 2mR^2\omega_2(\sin\theta)^2\left(\omega_2\bar{k}' \times \bar{k}'\right).\end{aligned}$$

Carrying out these operations yields

$$\begin{aligned}\bar{M}_{\text{shaft}} &= 2mR^2\omega_2^2 \sin\theta \cos\theta \bar{i} - 4mR^2\omega_1\omega_2 (\cos\theta)^2 \bar{j}' \\ &\quad + 4mR^2\omega_1\omega_2 (\sin\theta)(\cos\theta) \bar{k}'.\end{aligned} \tag{5} \triangleleft$$

The kinetic energy of the two-particle rigid body may be evaluated from Eq. (5.1.33). Point C is the center of mass, and we already have $\bar{\omega}$ and $\bar{H}_C$. The speed of the center of mass is $L\omega_2$, so we find that

$$\begin{aligned}T &= \frac{1}{2}(2m)(L\omega_2)^2 + \frac{1}{2}\left(-\omega_1\bar{i}' + \omega_2\bar{k}'\right)\cdot \bar{H}_C \\ &= mR^2\omega_1^2 + m\omega_2^2\left[L^2 + R^2(\sin\theta)^2\right].\end{aligned} \tag{6} \triangleleft$$

It is not necessary to introduce angular momentum to derive these results. Instead, we can consider each sphere individually. We use point C as the reference for the kinematical analysis. Because $\bar{r}_{1/C} = -\bar{r}_{1/C}$, the velocities are

$$\begin{aligned}\bar{v}_1 &= \bar{v}_C + \bar{\omega} \times \bar{r}_{1/C} \\ &= -R\omega_2 \sin\theta \bar{i}' + (\omega_1 R\cos\theta + L\omega_2)\,\bar{j}' - \omega_1 R \sin\theta \bar{k}', \\ \bar{v}_2 &= \bar{v}_C + \bar{\omega} \times \bar{r}_{2/C} \\ &= R\omega_2 \sin\theta \bar{i}' + (-\omega_1 R\cos\theta + L\omega_2)\,\bar{j}' + \omega_1 R \sin\theta \bar{k}'.\end{aligned}$$

The result of using these expressions to evaluate the kinetic energy according to $\frac{1}{2}m\bar{v}_1 \cdot \bar{v}_1 + \frac{1}{2}m\bar{v}_2 \cdot \bar{v}_2$ would be identical to Eq. (6).

To examine the force system exerted by the shaft we describe each sphere's acceleration relative to point C, according to

$$\begin{aligned}\bar{a}_1 = \bar{a}_C + \bar{a}_{1/C}, \quad \bar{a}_2 = \bar{a}_C + \bar{a}_{2/C}, \\ \bar{a}_{1/C} = -\bar{a}_{2/C} = \bar{\alpha} \times \bar{r}_{1/C} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{1/C}).\end{aligned} \tag{7}$$

The angular velocity is given in Eq. (1), and the angular acceleration is

$$\bar{\alpha} = -\omega_1\left(\omega_2\bar{k}' \times \bar{i}'\right) = -\omega_1\omega_2\bar{j}',$$

which leads to

$$\bar{a}_{1/C} = -2R\omega_1\omega_2 \cos\theta \bar{i}' - R\left(\omega_1^2 + \omega_2^2\right)\sin\theta\,\bar{j}' - \omega_1^2 R\cos\theta \bar{k}'. \tag{8}$$

We use these expressions to form the force and moment sums for the system consisting of the two spheres and the connecting bar. Specifically, we apply $\Sigma \bar{F}_n = m\bar{a}_n$ to each particle and use Eqs. (7) to describe each acceleration. Because the force exerted by the connecting bar on each sphere is internal to this system, the result is

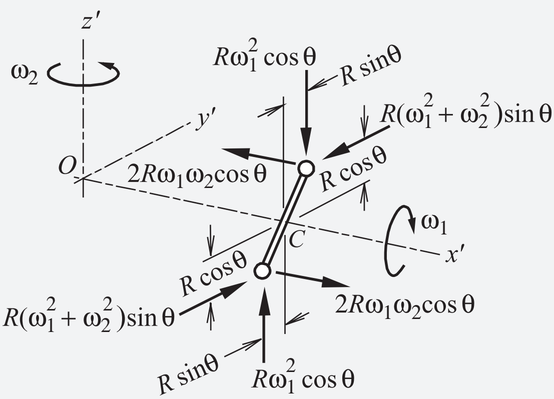
$$\Sigma \bar{F} = \bar{F}_{\text{shaft}} = m_1\bar{a}_1 + m_2\bar{a}_2 = 2m\bar{a}_C.$$

This is the same as the force described by Eq. (4).

The forces exerted by the connecting bar also disappear from a moment sum for the system. Application of Eqs. (7) in conjunction with $\bar{r}_{2/C} = -\bar{r}_{1/C}$ leads to

$$\Sigma \bar{M}_C = \bar{M}_{\text{shaft}} = \bar{r}_{1/C} \times m_1 \bar{a}_1 + \bar{r}_{2/C} \times m_2 \bar{a}_2 = m\bar{r}_{1/C} \times \bar{a}_{1/C} + m\bar{r}_{2/C} \times \bar{a}_{2/C}.$$

The sketch shows the acceleration of each sphere relative to point C, which when multiplied by m become the forces that contribute to the moment sum. Inspection of this diagram shows that for each particle the sum of the z' component and the portion of the y' component that is due to ω_1 is a vector that acts from each sphere toward point C. It follows that these forces do not contribute to the moment sum. Consequently, the moment exerted by the shaft is formed from two couples. The x' components form a couple that is $(2mR\omega_1\omega_2 \cos\theta)(2R)$ acting perpendicular to the connecting bar. The unit vector for this couple is $-\cos\theta\,\bar{j}' + \sin\theta\,\bar{k}'$. The other couple, which is formed from the remaining portion of the y' components of relative acceleration, is $\left(mR\omega_2^2 \sin\theta\right)(2R\cos\theta)$. This couple acts about the x' axis. The sum of these couples is the same as Eq. (5).



Position and acceleration components of each sphere relative to point C at which the connecting rod is joined to the motor's shaft

This discussion explains in fundamental terms why moments are required to sustain a spatial motion in which all rotation rates are constant. Because particles have acceleration components that do not lie in a common plane, the forces required to accelerate these particles exert moments about several axes. For the system of two interconnected spheres the analysis using angular momentum was merely an alternative approach, but it will be the only viable one for bodies whose size is not negligible.

5.2 INERTIA PROPERTIES

It is inconceivable to evaluate the angular momentum of a rigid body by adding the contribution of each of its atomic particles. Our approach is to model the rigid body as a continuous distribution of mass, which will permit us to apply the principles of calculus. The result will be an expression for the angular momentum that features a set of numbers characterizing the manner in which mass is distributed in the body.

5.2.1 Moments and Products of Inertia

In a continuum model of a rigid body the particles are differential elements of mass dm having infinitesimal dimensions. These elements fill the region occupied by the body. In this viewpoint, any summation over the particles forming the body becomes an integral over the body's domain. In Fig. 5.3, xyz is a global coordinate system whose origin A is an allowable point for the moment equation of motion.

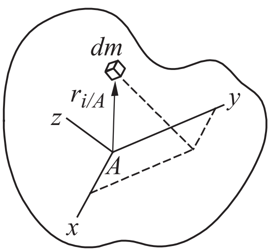


Figure 5.3. Differential element of mass dm relative to a body-fixed xyz reference frame.

In terms of components relative to xyz the position vector $\bar{r}_{i/A}$ and angular velocity $\bar{\omega}$ are

$$\bar{r}_{i/A} = x\bar{i} + y\bar{j} + z\bar{k}, \quad \bar{\omega} = \omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k}. \tag{5.2.1}$$

We substitute these expressions into Eq. (5.1.21) for the angular momentum and convert the summation to an integral. This transforms the general relation to

$$\bar{H}_A = \iiint \left(x\bar{i} + y\bar{j} + z\bar{k}\right) \times \left[\left(\omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k}\right) \times \left(x\bar{i} + y\bar{j} + z\bar{k}\right)\right] dm. \tag{5.2.2}$$

The result of evaluating the cross products is an integrand that consists of $\bar{i}$, $\bar{j}$, and $\bar{k}$ components. Each component may be integrated individually. Furthermore, the rotation rates are overall properties of the motion, rather than functions of the position within the body. Consequently, ω_x, ω_y, and ω_z may be factored out of each integral. The result is

$$\boxed{\begin{aligned}\bar{H}_A = {} & (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\,\bar{i} + (I_{yy}\omega_y - I_{yx}\omega_x - I_{yz}\omega_z)\,\bar{j} \\ & + (I_{zz}\omega_z - I_{zx}\omega_x - I_{zy}\omega_y)\,\bar{k},\end{aligned}} \tag{5.2.3}$$

where

$$\boxed{\begin{aligned} & I_{xx} = \iiint \left(y^2 + z^2\right) dm, \quad I_{yy} = \iiint \left(x^2 + z^2\right) dm, \quad I_{zz} = \iiint \left(x^2 + y^2\right) dm, \\ & I_{xy} = I_{yx} = \iiint xy\,dm, \quad I_{xz} = I_{zx} = \iiint xz\,dm, \quad I_{yz} = I_{zy} = \iiint yz\,dm.\end{aligned}} \tag{5.2.4}$$

The terms I_{pp} (repeated subscripts) are *moments of inertia* about the three coordinate axes, and the terms I_{pq} (nonrepeated subscripts) are *products of inertia*. The vector description of $\bar{H}_A$ may be written alternatively in matrix form as

$$\boxed{\{H_A\} = [I]\{\omega\},} \tag{5.2.5}$$

where $\{H_A\}$ and $\{\omega\}$ are formed from the components of $\bar{H}_A$ and $\bar{\omega}$, respectively, and $[I]$ is the *inertia matrix*:

$$\boxed{[I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}.} \tag{5.2.6}$$

The matrix representation will be seen to be quite useful in conjunction with mathematical software. The inertia matrix, combined with the mass and the location of the center of mass, fully characterizes the inertia properties of a rigid body.*

The moments of inertia are properties encountered in planar motion. The similarity of any I_{pp} to the parameter for planar motion may be realized by looking down the p axis. Such a view for I_{zz} is shown in Fig. 5.4. The distance $R = (x^2 + y^2)^{1/2}$ is the perpendicular distance from the z axis to the mass element dm. Thus I_{zz} is the sum for all mass elements of the R^2 values weighted by dm.

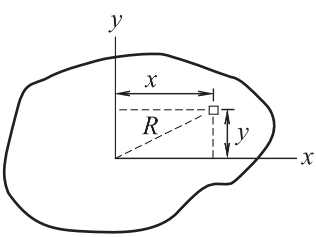


Figure 5.4. Contribution of an element of mass to the moment of inertia about the z axis.

A common way to prescribe a moment of inertia is to give its *radius of gyration,*

$$\kappa_p = \left(\frac{I_{pp}}{m}\right)^{1/2}. \tag{5.2.7}$$

Consider a thin ring whose mass is situated on a circle of radius κ_p, with p being the axis perpendicular to the plane of the ring and intersecting the center. The distance from this axis to any mass element is κ_p, so the integral of $R^2 dm$ reduces to κ_p^2 multiplied by the integral of the mass, in other words, $I_{pp} = m\kappa_p^2$. Thus a radius of gyration describes a circular ring whose mass is the same as the body of interest and whose moment of inertia about its axis of symmetry p is identical to I_{pp} for the body. Because I_{pp} is the sum of the $R^2 dm$ values, the radius of gyration cannot exceed the largest distance from axis p to

* The definitions of products of inertia in Eqs. (5.2.4) are opposite in sign to those used by some individuals. The present definitions are based on indicating quadrants in which mass is dominant. The alternative definition gives an inertia matrix whose off-diagonal elements equal the products of inertia.

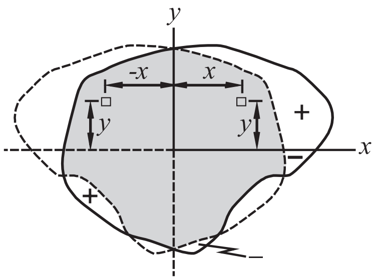


Figure 5.5. The significance of a product of inertia as a measure of deviation from symmetry.

a mass point in the body. This knowledge can be quite useful when one wishes to make an order-of-magnitude estimate of the moments of inertia of a given body.

In contrast to the moments of inertia, which depend on how far mass is situated from each coordinate axis, the products of inertia describe the degree to which mass is distributed symmetrically relative to the three coordinate planes. Figure 5.5 shows the cross section of a body at an arbitrary value of z. The dotted region is the mirror image of the cross section, which one obtains by flipping it about the y axis. The shaded area is the union of the cross section and its mirror image. Two mass elements on either side of the yz plane are depicted in Fig. 5.5. The x coordinate for the left element is the negative of the x coordinate for the right element, whereas their y and z coordinates are the same. If the density is the same for both elements, then they have the same mass dm. In that case the values of $xy\,dm$ and $xz\,dm$ for the left element are the negative of the corresponding values for the right element, so their combined contributions to I_{xy} and I_{xz} are zero. It follows that the total contribution to I_{xy} and I_{xz} of all mass elements situated in the shaded region of Fig. 5.5 is zero. The unshaded portions of the cross section have no mirror image on the other side of the yz plane. The sign marking each such region indicates whether the product xy is positive or negative within that region. The situation in the figure is one in which the unbalanced regions correspond mostly to positive xy. Thus, if the density is constant over the cross section, the net contribution of this cross section to I_{xy} would be positive. Similar reasoning applies for I_{xz}. One should note, however, that the actual values of I_{xy} and I_{xz} for this body cannot be judged solely from the figure, because they depend on the combined contribution of all cross sections.

From the preceding considerations, we conclude that a positive value of I_{pq} indicates that the mass of the body is predominantly situated in either or both of the quadrants where the p and q coordinates have the same sign, whereas negative I_{pq} means that mass is predominant in the quadrants where p and q have opposite sign. We obtain an important corollary of the discussion by considering a situation in which the shaded region in Fig. 5.5 is the actual cross section and the density at the mirror image points is the same. In that case the contributions to I_{xy} and I_{xz} of this cross section will vanish. If the same is true for all cross sections, then the body is symmetric with respect to the yz plane, and $I_{xy} = 0$ and $I_{xz} = 0$. Similar conclusions would result if we were to consider symmetry with respect to the xz or xy plane. The fact that the z axis is normal to the plane of symmetry for the situation in the figure leads to this generalization:

> If two coordinate axes form a plane of symmetry for a body, then all products of inertia involving the coordinate normal to that plane are zero.

A further corollary is

> If at least two of the three coordinate planes are planes of symmetry for a body, then all products of inertia are zero.

The condition of perpendicular planes of symmetry is attained for any body of revolution if the axis of symmetry coincides with a coordinate axis. Whenever the coordinate axes correspond to vanishing values of all products of inertia, they are said to be *principal axes*. We will soon see that it is possible to identify principal axes for any body, regardless of it symmetry properties.

EXAMPLE 5.2 Evaluate the inertia properties of the pair of particles in Example 5.1 relative to the $x'y'z'$ coordinate system defined there. Then use those properties to determine $\bar{H}_A$.

SOLUTION This example uses a simple system to illustrate the significance of the inertia properties. It also provides some insight that will be useful to later developments concerning the rate of change of $\bar{H}_A$. The two particles and their massless connecting rod form the rigid body of interest here. Each particle may be considered to be an element of mass. Correspondingly, the integral reduces to a single term for each particle. The z' axis in Example 5.1 was defined to be vertical, and the x' axis is aligned with the motor's rotation axis. The position of the particles in terms of components relative to $x'y'z'$ was found previously to be

$$\bar{r}_{1/O} = L\bar{i}' + R\sin\theta\,\bar{j}' + R\cos\theta\,\bar{k}',\ \ \bar{r}_{2/O} = L\bar{i}' - R\sin\theta\,\bar{j}' - R\cos\theta\,\bar{k}'.$$

Because the components are the respective values of x', y', and z' for each particle, Eqs. (5.2.4) for the present situation reduce to

$$\begin{aligned}
I_{x'x'} &= m\left(y_1^2 + z_1^2\right) + m\left(y_2^2 + z_2^2\right) = 2mR^2, \\
I_{y'y'} &= m\left(x_1^2 + z_1^2\right) + m\left(x_2^2 + z_2^2\right) = 2mL^2 + 2mR^2\left(\cos\theta\right)^2, \\
I_{z'z'} &= m\left(x_1^2 + y_1^2\right) + m\left(x_2^2 + y_2^2\right) = 2mL^2 + 2mR^2\left(\sin\theta\right)^2, \\
I_{x'y'} &= I_{y'x'} = mx_1y_1 + mx_2y_2 = 0, \\
I_{x'z'} &= I_{z'x'} = mx_1z_1 + mx_2z_2 = 0, \\
I_{y'z'} &= I_{z'y'} = my_1z_1 + my_2z_2 = 2mR^2\sin\theta\cos\theta = mR^2\sin\left(2\theta\right).
\end{aligned} \quad \triangleleft$$

The products of inertia indicate that there is a balanced mass distribution relative to the $x'y'$ and $x'z'$ quadrants, and the mass is situated in the two quadrants where $y'z'$ are positive whenever $\sin(2\theta)$ is positive, that is, $0 < \theta < \pi/2$ and $\pi < \theta < 3\pi$.

To evaluate $\bar{H}_A$ we recall from the previous example that the angular velocity of the assembly of the two particles and the connecting rod is the sum of the rotation rates ω_2 about the vertical and ω_1 of the motor. We previously found that

$$\bar{\omega} = -\omega_1\bar{i}' + \omega_2\bar{k}'.$$

The components are $\omega_{x'} = -\omega_1$, $\omega_{y'} = 0$, $\omega_{z'} = \omega_2$. Correspondingly, Eq. (5.2.3) gives

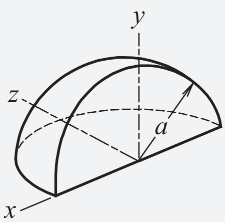
$$\begin{aligned}\bar{H}_A &= \left[2mR^2(-\omega_1) - 0(0) - 0(\omega_2)\right]\bar{i}' \\ &\quad + \left\{\left[2mL^2 + 2mR^2(\cos\theta)^2\right](0) - 0(-\omega_1) - \left(2mR^2\sin\theta\cos\theta\right)(\omega_2)\right\}\bar{j}' \\ &\quad + \left\{\left[2mL^2 + 2mR^2(\sin\theta)^2\right](\omega_2) - 0 - \left(2mR^2\sin\theta\cos\theta\right)(0)\right\}\bar{k}'.\end{aligned}$$

The preceding exemplifies a general observation that many terms in the standard formula for angular momentum usually vanish because either a product of inertia or a component of $\bar{\omega}$ is zero. The present result reduces to

$$\bar{H}_A = -2mR^2\omega_1\bar{i}' - 2mR^2\omega_2\sin\theta\cos\theta\,\bar{j}' + 2m\left[L^2 + R^2(\sin\theta)^2\right]\omega_2\bar{k}'. \qquad \triangleleft$$

This is the same as the expression we derived in Example 5.1 by actually evaluating moments of the relative momenta.

EXAMPLE 5.3 Derive the inertia matrix of the quarter-sphere about the xyz axes; then use that result to obtain the inertia matrix for a quarter-spherical shell whose skin thickness is $d \ll a$. Express each result in terms of the mass m of that body.



Example 5.3

SOLUTION Although we usually use other techniques to evaluate the inertia matrix, integration is a basic tool. This example illustrates some procedures that were used to derive the properties tabulated in the Appendix. Spherical coordinates with an origin at the center are ideal here because the quarter-sphere's surfaces correspond to constant values of one of these coordinates. Any coordinate axis may be employed as the reference for the polar angle ϕ; we select the y axis, so the azimuthal angle θ is measured in the xz plane relative to the x axis. The coordinate transformation is

$$y = r\cos\phi, \quad x = r\sin\phi\cos\theta, \quad z = r\sin\phi\sin\theta.$$

The body occupies the domain $0 \le r \le a$, $0 \le \phi \le \pi/2$, $0 \le \theta \le \pi$, and a differential element of mass is $dm = \rho(dr)(r\,d\phi)(r\,d\theta\sin\phi)$. The body is symmetric with respect to the yz plane, so the two products of inertia containing x as the subscript are zero:

$$I_{xy} = I_{xz} = 0. \qquad \triangleleft$$

Also, the symmetry of the quarter-sphere is such that mass is situated in the same manner relative to the y and z axes, so $I_{zz} = I_{yy}$. Thus it is necessary to compute only I_{xx}, I_{yy}, and I_{yz}.

The integral definitions, Eqs. (5.2.4), give

$$\begin{aligned} I_{xx} &= \int_0^a \int_0^{\pi/2} \int_0^{\pi} \left(y^2 + z^2\right) \rho r^2 \sin\phi \, d\theta \, d\phi \, dr \\ &= \rho \int_0^a \int_0^{\pi/2} \int_0^{\pi} \left[(r\cos\phi)^2 + (r\sin\phi\sin\theta)^2\right] r^2 \sin\phi \, d\theta \, d\phi \, dr, \end{aligned}$$

$$I_{yy} = \int_0^a \int_0^{\pi/2} \left[(r\sin\phi\cos\theta)^2 + (r\sin\phi\sin\theta)^2\right] \rho r^2 \sin\phi \, d\theta \, d\phi \, dr,$$

$$I_{zz} = \int_0^a \int_0^{\pi/2} \int_0^{\pi} \left[(r\sin\phi\cos\theta)^2 + (r\cos\phi)^2\right] \rho r^2 \sin\phi \, d\theta \, d\phi \, dr,$$

$$\begin{aligned} I_{yz} &= \int_0^a \int_0^{\pi/2} \int_0^{\pi} yz\rho r^2 \sin\phi \, d\theta \, d\phi \, dr \\ &= \rho \int_0^a \int_0^{\pi/2} \int_0^{\pi} (\cos\phi)(\sin\phi\sin\theta) \, r^4 \sin\phi \, d\theta \, d\phi \, dr. \end{aligned}$$

The results are

$$I_{xx} = I_{yy} = I_{zz} = \frac{2\pi}{15}\rho a^5, \quad I_{yz} = \frac{2}{15}\rho a^5.$$

To express the inertia properties in terms of the mass m, the density is expressed as the ratio of the mass to the volume of a quarter-sphere:

$$\rho = \frac{m}{V} = \frac{m}{\frac{1}{4}\left(\frac{4\pi a^3}{3}\right)} = \frac{3m}{\pi a^3}.$$

We substitute for ρ in each of the inertia properties and recall from Eq. (5.2.6) that the off-diagonal terms of the inertia matrix are the negative of the products of inertia. The result is

$$[I] = \frac{2}{5} m a^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/\pi \\ 0 & -1/\pi & 1 \end{bmatrix}. \qquad \triangleleft$$

A shell is a body whose mass is concentrated at its surface. There are two procedures for obtaining the inertia properties of the quarter-spherical shell. The general one specializes the differential element of mass. Let σ be the mass per unit surface area. A differential element of surface area in spherical coordinates is $dS = (a\,d\phi)\,(a\,\sin\phi\,d\theta)$. The surface is defined by $0 \le \phi \le \pi/2$, $0 \le \theta \le \pi$. We modify the integrals for the quarter-sphere, with $r = a$ for points on the surface and $dm = \sigma\,dS$, which leads to

$$I_{xx} = \int_0^{\pi/2}\int_0^{\pi}\left[(a\cos\phi)^2 + (a\sin\phi\sin\theta)^2\right]\sigma a^2\sin\phi\,d\theta\,d\phi = \frac{2}{3}\pi\sigma a^4,$$
$$I_{yy} = \int_0^{\pi/2}\int_0^{\pi}\left[(a\sin\phi\cos\theta)^2 + (a\sin\phi\sin\theta)^2\right]\sigma a^2\sin\phi\,d\theta\,d\phi = \frac{2}{3}\pi\sigma a^4,$$
$$I_{zz} = \int_0^{\pi/2}\int_0^{\pi}\left[(a\sin\phi\cos\theta)^2 + (a\cos\phi)^2\right]\sigma a^2\sin\phi\,d\theta\,d\phi = \frac{2}{3}\pi\sigma a^4,$$
$$I_{yz} = \int_0^{\pi/2}\int_0^{\pi}(a\cos\phi)\,(a\sin\phi\sin\theta)\,\sigma a^3\sin\phi\,d\theta\,d\phi = \frac{2}{3}\sigma a^4.$$

The surface area of a sphere is $4\pi a^2$, so the mass per unit surface area is related to the mass of the quarter-spherical shell by

$$\sigma = \frac{m_{\text{shell}}}{\frac{1}{4}(4\pi a^2)}.$$

Substitution of this expression into the inertia values leads to

$$[I] = \frac{2}{3}m_{\text{shell}}a^2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -1/\pi\\ 0 & -1/\pi & 1\end{bmatrix}. \qquad \triangleleft$$

An alternative is to derive the properties of the shell from those of the full body. We may obtain the shell by removing from the quarter-sphere a concentric quarter-sphere whose radius is $a - h$, where h is the thickness of the shell. Because the origins of xyz for both the original and removed bodies coincide, the inertia properties of the shell are the difference of the values for the two full bodies. We form the differences by using the original forms, which featured the density, so that

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{15}\pi\rho a^5 - \frac{2}{15}\pi\rho\,(a-h)^5 = \frac{2}{15}\pi\rho\left(5a^4h + \cdots\right),$$
$$I_{yz} = \frac{2}{15}\rho a^5 - \frac{2}{15}\rho\,(a-h)^5 = \frac{2}{15}\rho\left(5a^4h + \cdots\right).$$

Higher powers of h are omitted because the definition of a shell is that its thickness is very small compared with the overall dimensions. The density is the ratio of the shell's mass to its volume:

$$\rho = \frac{m_{\text{shell}}}{V} = \frac{m_{\text{shell}}}{\frac{1}{4}\left[\frac{4\pi a^3}{3} - \frac{4\pi\,(a-h)^3}{3}\right]} = \frac{m_{\text{shell}}}{\pi\,(a^2h + \cdots)}.$$

Substitution of this description of ρ into the last set of expressions for the inertia properties would yield the same results as those obtained by integration.

EXAMPLE 5.4 A first-order correction for the mass distribution of the Earth is to take it to be an oblate spheroid with the polar axis as the axis of symmetry. A consequence of this deviation from sphericity, combined with the tilt of the polar axis relative to the Earth's orbital plane about the Sun, and the Moon's orbital plane about the Earth, is that the forces exerted by those bodies do not exactly act at the Earth's center. The task here is to characterize the gravitational attraction of the Sun and the Moon as force–couple systems acting at the Earth's center.

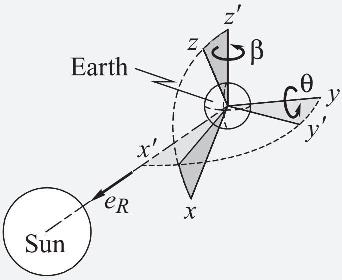
SOLUTION This example will lead to a different perspective to the meaning of "center of mass," as well as recognition that the inertia properties can occur in contexts other than evaluation of angular momentum. The usage of the first moment of mass to locate the point at which the resultant gravitational force acts originates from considering the force of gravity to be constant. The analysis we carry out here, in which the inverse square law is used to describe the force exerted by the Sun on each of particle of the Earth's mass, is required to explain some features of the Earth's rotation. In particular, the result will be a crucial piece of our analysis in Chapter 10 of *precession of the equinoxes*, which is manifested by variability in the dates when the seasons change.

We begin by recalling the inverse square law for gravitational force exerted by a large spherical body on a particle:

$$\bar{F} = \frac{Gm_s m}{r^2}\bar{e}_r,$$

where m and m_s are the masses of the particle and the attracting body, r is the distance from the particle to the center of the other body, $\bar{e}_r$ is the unit vector from the particle to the center of the attracting body, and G is the universal gravitational constant. The Earth's diameter is much smaller than the distance to either the Sun or the Moon, so the Earth appears to be a dot when viewed from either body, especially the Sun. If we were to set r equal to the distance R between the respective centers and take $\bar{e}_r$ to be constant at $\bar{e}_R$, which is the unit vector from the center of the Earth to the center of the other body, we would obtain the conventional representation of the effect of gravity, in which the Earth's center of gravity, at which the gravitational attraction acts, coincides with its center of mass, which is essentially the center of the Earth.

We perform the analysis for the case of the interaction between the Earth and the Sun, then consider what modifications are required to describe the role of the Moon. Let xyz be a coordinate system whose origin is at the center of the Earth such that the z axis is the polar axis; this axis is inclined by angle θ relative to the normal to the Earth's orbital plane. The y axis is defined to coincide with the diametral line at the Earth's equator that lies in the orbital plane. To describe how the Earth is oriented relative to the Sun define $x'y'z'$ whose origin also is at the Earth's center. The z' axis is defined to be normal to the Earth's orbital plane, and x' is aligned with the unit vector from the center of the Earth to the center of the Sun, so that $\bar{e}_R = \bar{i}'$.



Coordinate system for describing the graviational force exerted by the Sun on the Earth

This arrangement is depicted in the sketch. The transformation between these coordinates systems is

$$[x \;\; y \;\; z]^{\mathrm{T}} = [R]\,[x' \;\; y' \;\; z']^{\mathrm{T}},$$

$$[R] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

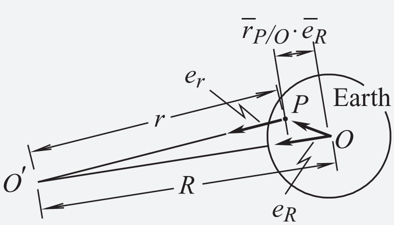
where β is the angle between the y and y' axes. The first row of $[R]^{\mathrm{T}}$ consists of the direction cosines of the x' axis with respect to xyz, which enables us to describe the direction of the Sun's center in terms of xyz components,

$$\bar{e}_R = \ell_{x'x}\bar{i} + \ell_{x'y}\bar{j} + \ell_{x'z}\bar{k},$$

where the direction cosines are

$$\ell_{x'x} = \cos\theta\cos\beta, \quad \ell_{x'y} = -\sin\beta, \quad \ell_{x'z} = \sin\theta\cos\beta.$$

The near parallelism of the lines from all particles to the Sun's center assists us in describing the distance and direction of the line from a particle of the Earth to the Sun. In the second sketch, which shows the plane containing a mass particle P, the center O of the Earth, and the center O' of the Sun, r is the distance from the center of the Sun to the particle and $\bar{e}_r$ is the direction to point O'.



Gravitational force exerted by the Sun on a particle of the Earth

It is evident that

$$r\bar{e}_r = R\bar{e}_R - \bar{r}_{P/O}, \tag{1}$$

so the distance is

$$r \equiv (r\bar{e}_r \cdot r\bar{e}_r)^{1/2} = \left(R^2 + \bar{r}_{P/O} \cdot \bar{r}_{P/O} - 2R\bar{r}_{P/O} \cdot \bar{e}_R\right)^{1/2}.$$

Because $\left|\bar{r}_{P/O'}\right|$ is much smaller than R, a binomial series expansion of the preceding gives

$$r \approx R - \bar{r}_{P/O} \cdot \bar{e}_R. \tag{2}$$

In turn, substituting this expression into Eq. (1) yields

$$\begin{aligned} \bar{e}_r &= \frac{R\bar{e}_R - \bar{r}_{P/O}}{r} \approx \frac{R}{R - \bar{r}_{P/O} \cdot \bar{e}_R}\bar{e}_R - \frac{\bar{r}_{P/O}}{R} \\ &\approx \left(1 + \frac{\bar{r}_{P/O} \cdot \bar{e}_R}{R}\right)\bar{e}_R - \frac{\bar{r}_{P/O}}{R}. \end{aligned} \tag{3}$$

Equation (2) is used to express the distance in the inverse square law for the gravitational force $d\bar{F}$ acting on a differential element of mass situated at point P, and Eq. (3) gives the direction of that force. Application of a binomial series expansion then leads to

$$\begin{aligned} d\bar{F} &= \frac{Gm_s dm}{(R - \bar{r}_{P/O} \cdot \bar{e}_R)^2}\bar{e}_r \\ &\approx \frac{Gm_s dm}{R^2}\left(1 + 2\frac{\bar{r}_{P/O} \cdot \bar{e}_R}{R}\right)\left[\left(1 + \frac{\bar{r}_{P/O} \cdot \bar{e}_R}{R}\right)\bar{e}_R - \frac{\bar{r}_{P/O}}{R}\right] \\ &\approx Gm_s dm\left(\frac{1}{R^2}\bar{e}_R + 3\frac{\bar{r}_{P/O} \cdot \bar{e}_R}{R^3}\bar{e}_R - \frac{\bar{r}_{P/O}}{R^3}\right). \end{aligned} \tag{4}$$

We obtain the resultant force by integrating the contribution associated with each mass element. For this integration, we observe that the only quantity in Eq. (3) that depends on the position of the mass element is $\bar{r}_{P/O}$. The integral of $\bar{r}_{P/O}dm$ is identically zero because it is the first moment of mass and point O is the center of mass. Thus the resultant force is

$$\bar{F} = \iiint d\bar{F} = \iiint \frac{Gm_s dm}{R^2}\bar{e}_R = \frac{Gm_s m_e}{R^2}\bar{e}_R, \tag{5}$$

where m_e is the Earth's total mass. The preceding expression is the same as the force derived from an approximation that considers the gravitational field to be constant across the Earth.

The equivalent couple must have the same moment as the resultant moment of the gravitational forces about point O, so we set

$$\bar{M} = \iiint \bar{r}_{P/O} \times d\bar{F}.$$

Using the approximation of $d\bar{F}$ in Eq. (4) leads to

$$\begin{aligned}\bar{M} &= Gm_s \iiint \bar{r}_{P/O} \times \left(\frac{1}{R^2}\bar{e}_R + 3\frac{\bar{r}_{P/O} \cdot \bar{e}_R}{R^3}\bar{e}_R - \frac{\bar{r}_{P/O}}{R^3} \right) dm \\ &\approx \frac{Gm_s}{R^2} \iiint \bar{r}_{P/O} \times \bar{e}_R dm + 3\frac{Gm_s}{R^3} \iiint (\bar{r}_{P/O} \times \bar{e}_R)(\bar{r}_{P/O} \cdot \bar{e}_R)\, dm.\end{aligned}$$

The first integral vanishes because $\bar{e}_R$ is independent of the position of the mass element, so the integral is proportional to the first moment of mass relative to the center of mass. Therefore the gravitational couple is described by

$$\bar{M} = 3\frac{Gm_s}{R^3} \iiint (\bar{e}_R \cdot \bar{r}_{P/O})(\bar{r}_{P/O} \times \bar{e}_R)\, dm. \tag{6}$$

To understand this integral we represent the position vector in terms of the position coordinates of the mass element, $\bar{r}_{P/O} = x\bar{i} + y\bar{j} + z\bar{k}$. The corresponding representation of the integrand is

$$(\bar{e}_R \cdot \bar{r}_{P/O})(\bar{r}_{P/O} \times \bar{e}_R) = (\ell_{x'x}x + \ell_{x'y}y + \ell_{x'z}z)\left[(\ell_{x'z}y - \ell_{x'y}z)\,\bar{i} + \cdots\right],$$

where only the x component is listed because the others can be obtained by permuting the symbols. Carrying out the product shows that the integrand contains various quadratic products of the point coordinates, so the integral may be expressed in terms of the moments and products of inertia. The result is simplified if xyz are principal axes because the terms in the integrand that contain mixed products, xy, xz, or yz, vanish in the integration. The nonzero terms are

$$\begin{aligned}(\bar{e}_R \cdot \bar{r}_{P/O})(\bar{r}_{P/O} \times \bar{e}_R) &= \ell_{x'y}\ell_{x'z}\left(y^2 - z^2\right)\bar{i} + \ell_{x'x}\ell_{x'z}\left(z^2 - x^2\right)\bar{j} \\ &\quad + \ell_{x'x}\ell_{x'y}\left(x^2 - y^2\right)\bar{k} + \cdots.\end{aligned}$$

Substitution of this representation of the integrand into Eq. (6) leads to

$$\begin{aligned}M = 3\frac{Gm_s}{R^3}\Bigg[&\ell_{x'y}\ell_{x'z} \iiint \left(x^2 + y^2 - x^2 - z^2\right) dm\,\bar{i} \\ &+ \ell_{x'x}\ell_{x'z} \iiint \left(y^2 + z^2 - x^2 - y^2\right) dm\,\bar{j} \\ &+ \ell_{x'x}\ell_{x'y} \iiint \left(x^2 + z^2 - y^2 - z^2\right) dm\,\bar{k}\Bigg] \\ = 3\frac{Gm_s}{R^3}&\left[\ell_{x'y}\ell_{x'z}(I_{zz} - I_{yy})\,\bar{i} + \ell_{x'x}\ell_{x'z}(I_{xx} - I_{zz})\,\bar{j} + \ell_{x'x}\ell_{x'y}(I_{yy} - I_{xx})\,\bar{k}\right].\end{aligned} \tag{7}$$

Equations (5) and (7) are valid for the gravitational force–couple system exerted on any large body when xyz are principal axes. Axisymmetry of the present approximation of the Earth's shape further simplifies the couple expression because $I_{xx} = I_{yy}$. When we use Eq. (4) for $\bar{F}$ to describe the coefficient of the bracketed term and recognize that $\ell_{x'z}$ is the direction cosine between the radial line to the

Sun and the polar axis, we find that

$$\begin{gathered}\bar{F} = F\bar{e}_R, \quad \bar{M} = 3\frac{F\left(I_{zz} - I_{xx}\right)}{m_e R}\left(\bar{e}_R \cdot \bar{k}\right)\left(\bar{e}_R \times \bar{k}\right), \\ F = \frac{Gm_s m_e}{R^2}, \quad \bar{e}_R = \ell_{x'x}\bar{i} + \ell_{x'y}\bar{j} + \ell_{x'z}\bar{k}.\end{gathered} \tag{7} \triangleleft$$

These expressions indicate that the moment acts about an axis that is perpendicular to the plane formed by the radial line to the Sun and the polar axis. To get a sense of the scale of $\bar{M}$, consider a pair of forces $\bar{F}$ and $-\bar{F}$ whose couple moment Fh equals $\left|\bar{M}\right|$. The corresponding separation distance is

$$h = \frac{3\left(I_{zz} - I_{xx}\right)}{m_e R}\left(\bar{e}_R \cdot \bar{k}\right)\left|\bar{e}_R \times \bar{k}\right|.$$

The ratio of a moment of inertia to the mass is the square of the corresponding radius of gyration, κ_z and κ_x. Because both $\bar{e}_R$ and $\bar{k}$ are unit vectors, it must be that

$$h < \frac{3\left|\kappa_z^2 - \kappa_x^2\right|}{R}. \tag{8}$$

The Earth is nearly spherical, so $\kappa_x \approx \kappa_z$. Furthermore, both radii of gyration are smaller than the radius R_e of the Earth, so $\left|\kappa_z^2 - \kappa_x^2\right| \ll R_e^2$, which leads to the conclusion that $h/R_e \ll R_e/R$.

A significant aspect of the derivation is that it is readily adapted to treat the role of the Moon. Obviously, R must be interpreted as the distance between the centers of the Earth and the Moon, and m_s must be changed to the mass of the Moon. We also must define $x'y'z'$ to lie in the Moon's orbital plane relative to the Earth. An interesting aspect of the fact that the Moon is much smaller but much closer than the Sun is that their relative significance for the gravitational moment is opposite their significance for the attractive force. Their respective masses are $m_s = 1.98892\left(10^{30}\right)$ kg and $m_{\text{moon}} = 7.348\left(10^{22}\right)$ kg, whereas the average orbital distances are $R_s = 1.4960\left(10^{11}\right)$ m and $R_{\text{moon}} = 3.844\left(10^{8}\right)$ m. Thus, the relative magnitudes are

$$\begin{gathered}\frac{F_{\text{moon}}}{F_s} = \frac{m_{\text{moon}}}{m_s}\left(\frac{R_s}{R_{\text{moon}}}\right)^2 = 0.56\%, \\ \frac{\left|M_{\text{moon}}\right|}{\left|M_s\right|} = \frac{m_{\text{moon}}}{m_s}\left(\frac{R_s}{R_{\text{moon}}}\right)^3 = 218\%.\end{gathered}$$

In other words, a first-order analysis of the movement of the Earth's center of mass is well justified in neglecting the role of the Moon, but understanding the rotational motion of the Earth requires that the effect of both bodies be considered. Although both moments are relatively small, they are ever present. This leads to precession of the tilted Earth's polar axis that has a period of many millennia, as well as a wobble of the polar axis, as we will see in Chapter 10. [The variability of the gravitational attraction of the moon also is a large part of the cause of oceanic tides. The text by Sverdrup *et al.* (2005) is a good entry to this topic.]

5.2.2 Transformations

Several resources, including the Appendix, tabulate the inertia properties of homogeneous bodies having common shapes. The formulas appearing in the Appendix were obtained by carrying out the integrals in Eqs. (5.2.4). The inertia properties of shapes could be evaluated from the integral definitions, but it often is easier to consider a body to be a composite of tabulated shapes. Integrals over different domains are additive. This makes it possible to decompose the moments and products of inertia of a composite shape into contributions of the constituents, that is,

$$\boxed{I_{\xi\eta} = (I_{\xi\eta})_1 + (I_{\xi\eta})_2 + \cdots, \quad \xi, \eta = x,\ y, \text{or } z.} \tag{5.2.8}$$

The same decomposition applies to the mass and first moment of mass, from which we can locate the center of mass G of the composite shape:

$$\boxed{\begin{aligned} m &= m_1 + m_2 + \cdots, \\ m\xi_G &= m_1 (\xi_G)_1 + m_2 (\xi_G)_2 + \cdots, \quad \xi = x,\ y,\ \text{or } z. \end{aligned}} \tag{5.2.9}$$

Although these relations for the properties of the composite shape appear to be straightforward, one aspect substantially complicates the task. In particular, applying Eq. (5.2.8) requires that the inertia properties of the constituent bodies all be relative to the desired xyz coordinate system. All inertia properties in the Appendix are for centroidal axes, whereas the constituent parts of most composite shapes seldom have coincident centroids. It also is possible that the orientation of coordinate systems for the basic shapes will be different. A rotation transformation of the inertial properties will allow us to bring the coordinate axes for each shape into parallel alignment with the coordinate system of interest. Then the parallel axis transformation of inertia properties will allow us to bring the origin of the coordinate system for each shape into coincidence with the desired origin. Only after such transformations are evaluated may the individual inertia properties be combined according to Eq. (5.2.8).

Composite shapes are not the only reason for studying transformations of the inertia properties. If a body is in pure rotation about a noncentroidal point, it is desirable to sum moments about the pivot point. This would require knowing the inertia properties for a set of axes whose origin is at the pivot, rather than the center of mass. A situation requiring a rotation transformation might arise if the coordinate system we use is selected to facilitate description of the components of $\bar{\omega}$, without considering which coordinate system is used to describe the tabulated inertia properties.

Parallel Axis Transformation

There are several ways in which we may transfer the inertia properties between points. The approach we will use is based on Eq. (5.1.36). In Fig. 5.6 xyz are a set of centroidal coordinate axes for which the inertia properties are known, and $x'y'z'$ is a parallel coordinate system having origin B for which we wish to determine the properties. The distances x_B, y_B, and z_B are the coordinates of origin B with respect to the centroidal

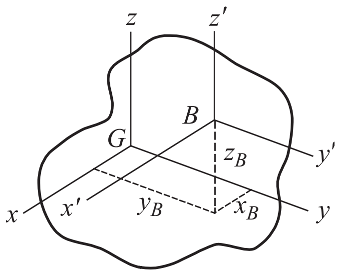


Figure 5.6. Parallel coordinate axes for transforming inertia properties.

coordinate system. Writing the position of origin B with respect to xyz is one way to ensure that the signs of these coordinates are correctly identified:

$$\boxed{\bar{r}_{B/G} = x_B \bar{i} + y_B \bar{j} + z_B \bar{k}.} \tag{5.2.10}$$

We use this expression together with a component representation of $\bar{\omega}$ to evaluate $\bar{r}_{B/G} \times (\bar{\omega} \times \bar{r}_{B/G})$, which changes Eq. (5.1.36) to

$$\begin{aligned}\bar{H}_B = \bar{H}_G &+ m\left[\left(y_B^2 + z_B^2\right)\omega_x - x_B y_B \omega_y - x_B z_B \omega_z\right]\bar{i}\\ &+ m\left[\left(x_B^2 + z_B^2\right)\omega_y - x_B y_B \omega_x - y_B z_B \omega_z\right]\bar{j}\\ &+ m\left[\left(x_B^2 + y_B^2\right)\omega_z - x_B z_B \omega_x - y_B z_B \omega_y\right]\bar{k}.\end{aligned}$$

Let the inertia properties with respect to the parallel coordinate systems xyz and $x'y'z$ be $[I_G]$ and $[I_B]$, respectively. Then the preceding may be written as

$$[I_B]\{\omega\} = [I_G]\{\omega\} + m\begin{bmatrix}\left(y_B^2 + z_B^2\right) & -x_B y_B & -x_B z_B\\ -x_B y_B & \left(x_B^2 + z_B^2\right) & -y_B z_B\\ -x_B z_B & -y_B z_B & \left(x_B^2 + y_B^2\right)\end{bmatrix}\{\omega\}. \tag{5.2.11}$$

This relation must apply for any $\{\omega\}$, so the factor of $\{\omega\}$ on each side of the equality must match. The result is the *parallel axis transformation of inertia properties*:

$$[I_B] = [I_G] + m\begin{bmatrix}\left(y_B^2 + z_B^2\right) & -x_B y_B & -x_B z_B\\ -x_B y_B & \left(x_B^2 + z_B^2\right) & -y_B z_B\\ -x_B z_B & -y_B z_B & \left(x_B^2 + y_B^2\right)\end{bmatrix}. \tag{5.2.12}$$

Matching like elements in this relation leads to the scalar form of this parallel axis transformation. The diagonal terms transform moments of inertia,

$$\boxed{\begin{aligned}I_{x'x'} &= I_{xx} + m\left(y_B^2 + z_B^2\right),\\ I_{y'y'} &= I_{yy} + m\left(x_B^2 + z_B^2\right),\\ I_{z'z'} &= I_{zz} + m\left(x_B^2 + y_B^2\right),\end{aligned}} \tag{5.2.13}$$

whereas products of inertia transform according to

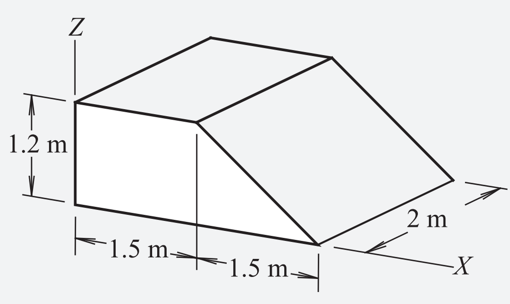
$$\boxed{\begin{aligned} I_{x'y'} &= I_{yx} = I_{xy} + m x_B y_B, \\ I_{x'z'} &= I_{zx} = I_{xz} + m x_B z_B, \\ I_{y'z'} &= I_{zy} = I_{yz} + m y_B z_B. \end{aligned}} \tag{5.2.14}$$

Equations (5.2.13) add a positive quantity to the centroidal values, which means that the moments of inertia for centroidal axes are smaller than those about any parallel noncentroidal axes. Also, it is not necessary to actually implement the transformation to obtain any product of inertia that can be recognized as being zero because of symmetry.

It is imperative to remember that the preceding transformations apply *only if xyz are centroidal axes*, which is the condition under which Eq. (5.1.36) is valid. Otherwise, additional terms featuring first moments of mass would arise. Another common error is failure to remember that x_B, y_B, and z_B are the coordinates of the origin B of the noncentroidal coordinate system. One can avoid the latter error by locating this point vectorially, as in Eq. (5.2.10). Also, if it is desired to transform between two parallel noncentroidal coordinate systems, one can convert from the set of known inertia properties to centroidal values by solving the aforementioned equations, and then transfer from the latter values to the coordinate system for which the properties are desired.

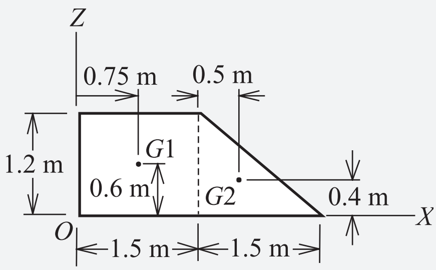
An interesting interpretation of the parallel axis transformation results from considering the case of a single particle. The particle model considers a body's dimensions to be negligible, so that the centroidal inertia properties vanish, $[I] = [0]$. The moments of inertia for noncentroidal axes are then simply the particle's mass multiplied by the square of its distance from the respective coordinate axes, and the products of inertia are the mass multiplied by the products of the respective coordinates of the noncentroidal coordinate system. Thus the general parallel axis theorems indicate that transferring from centroidal to noncentroidal axes increments the inertia properties as though all of the body's mass were situated at its center of mass.

EXAMPLE 5.5 Locate the center of mass of the trapezoidal parallelepiped. Then determine the moments and products of inertia corresponding to centroidal coordinate system xyz whose axes are parallel to XYZ defined in the sketch.



Example 5.5

SOLUTION This is an illustration of the procedures for evaluating the inertia properties of a body that is a composite of basic shapes. The decomposition we use is a rectangular parallelepiped and a rectangular prism, both of whose properties are described in the Appendix. We begin with a sketch that shows the location of the center of mass for each constituent shape.



Parallel centroidal coordinate systems for the individual shapes

The first moment of mass is used to locate the center of mass. We may perform this calculation in one step by writing the positions of the respective centers of mass vectorially. The masses are

$$m_1 = \rho\,(1.5)\,(2)\,(1.2)\,, \quad m_2 = \rho\frac{1}{2}\,(1.5)\,(2)\,(1.2)\,, \quad m = m_1 + m_2 = 5.4\rho.$$

Relative to XYZ the centers of mass are located at

$$\bar{r}_{G1/O} = 0.75\bar{I} + 1\bar{J} + 0.6\bar{k}, \quad \bar{r}_{G2/O} = 2\bar{I} + 1\bar{J} + 0.4\bar{k} \text{ m.}$$

The corresponding moment of mass is

$$m\bar{r}_{G/O} = m_1\bar{r}_{G1/O} + m_2\bar{r}_{G2/O},$$
$$5.4\rho\bar{r}_{G/O} = 3.6\rho\left(0.75\bar{I} + 1\bar{J} + 0.6\bar{k}\right) + 1.8\rho\left(2\bar{I} + 1\bar{J} + 0.4\bar{K}\right),$$

from which we find that

$$\bar{r}_{G/O} = 1.1667\bar{I} + 1\bar{J} + 0.5333\bar{K} \text{ m.} \qquad \triangleleft$$

Obviously we could have identified the Y coordinate of the center of mass by symmetry.

Let xyz be the coordinate system that is parallel to XYZ with origin at point G. The xz plane cuts the body in half, so it is a plane of symmetry. Because y is the axis perpendicular to this plane, we have

$$I_{xy} = I_{yz} = 0. \qquad \triangleleft$$

Thus we need to evaluate each moment of inertia, as well as I_{xz}.

For each basic shape let $x_j y_j z_j$ denote the centroidal axis that is aligned parallel to XYZ. For body 1 we find from the Appendix that

$$(I_{x_1x_1})_1 = \frac{1}{12} m_1 \left(2^2 + 1.2^2\right) = 1.6320\rho,$$

$$(I_{y_1y_1})_1 = \frac{1}{12} m_1 \left(1.5^2 + 1.2^2\right) = 1.1070\rho,$$

$$(I_{z_1z_1})_1 = \frac{1}{12} m_1 \left(1.5^2 + 2^2\right) = 1.8750\rho,$$

$$(I_{x_1z_1})_1 = 0.$$

The coordinates of point G with respect to $x_1 y_1 z_1$ are the components of the position of the relative position vector,

$$\bar{r}_{G/G1} = \bar{r}_{G/O} - \bar{r}_{G1/O} = 0.4167\bar{I} - 0.0667\bar{K}.$$

The parallel axis transformation for shape 1 therefore is

$$(I_{xx})_1 = (I_{x_1x_1})_1 + m_1 (-0.0667)^2 = 1.6480\rho,$$

$$(I_{yy})_1 = (I_{y_1y_1})_1 + m_1 \left[(0.4167)^2 + (-0.0667)^2\right] = 1.7481\rho,$$

$$(I_{zz})_1 = (I_{z_1z_1})_1 + m_1 (0.4167)^2 = 2.500\rho,$$

$$(I_{xz})_1 = (I_{x_1z_1})_1 + m_1 (0.4167)(-0.0667) = -0.100\rho.$$

For shape 2 care must be taken to properly permute the axis labels from those depicted in the Appendix. The centroidal inertia properties of shape 2 are thereby found to be

$$(I_{x_2x_2})_2 = \frac{1}{36} m_2 \left[3\left(2^2\right) + 2\left(1.2^2\right)\right] = 0.7440\rho,$$

$$(I_{y_2y_2})_2 = \frac{1}{18} m_2 \left(1.5^2 + 1.2^2\right) = 0.3690\rho,$$

$$(I_{z_2z2})_2 = \frac{1}{36} m_2 \left[3\left(2^2\right) + 2\left(1.5^2\right)\right] = 0.8250\rho,$$

$$(I_{x_2z_2})_1 = -\frac{1}{36} m_2 (1.5)(1.2) = -0.0900\rho.$$

The position vector for point G relative to the centroid of shape 2 is

$$\bar{r}_{G/G2} = \bar{r}_{G/O} - \bar{r}_{G2/O} = -0.8333\bar{I} + 0.1333\bar{K}.$$

The corresponding parallel axis transformation is

$$(I_{xx})_2 = (I_{x_2x_2})_2 + m_2 (0.1333)^2 = 0.7760\rho,$$

$$(I_{yy})_2 = (I_{y_2y_2})_2 + m_2 \left[(-0.8333)^2 + (0.1333)^2\right] = 1.6510\rho,$$

$$(I_{zz})_2 = (I_{z_1z_1})_2 + m_2 (-0.8333)^2 = 2.0750\rho,$$

$$(I_{xz})_2 = (I_{x_2z_2})_2 + m_2 (-0.8333)(0.1333) = -0.2900\rho.$$

Now that the inertia properties of each shape are known with respect to xyz their contributions may be added. Thus,

$$\begin{aligned} I_{xx} &= (I_{xx})_1 + (I_{xx})_2 = 2.4240\rho, \\ I_{yy} &= (I_{yy})_1 + (I_{yy})_2 = 3.3990\rho, \\ I_{zz} &= (I_{zz})_1 + (I_{zz})_2 = 4.5750\rho, \\ I_{xz} &= (I_{xz})_1 + (I_{xz})_2 = -0.3900\rho. \end{aligned} \quad \triangleleft$$

Rotational Transformation

Our concern here is with a situation in which we know $[I]$ relative to xyz, and we wish to determine $[I']$ corresponding to coordinate system $x'y'z'$ whose origin coincides with the origin of xyz. From the rotation transformation we have

$$\{\omega'\} = [R]\{\omega\}, \tag{5.2.15}$$

where $\{\omega\}$ and $\{\omega'\}$ consist of the components of the respective vectors.

The derivation of the parallel axis transformations was based on a property of the kinetic energy. Similarly, the rotational transformation will employ the rotational contribution to the kinetic energy of a body. Energy is a scalar quantity, so formulating it in terms of angular velocity components with respect to either coordinate system must yield the same result.

Either of Eqs. (5.1.33) indicates that the kinetic energy associated with rotation about point A is

$$T_{\text{rot}} = \frac{1}{2}\bar{\omega} \cdot \bar{H}_A. \tag{5.2.16}$$

The components of the angular momentum $\bar{H}_A$ of a rigid body relative to the designated point are described in matrix form in Eq. (5.2.5), which leads to

$$T_{\text{rot}} = \frac{1}{2}\{\omega\}^{\text{T}}[I]\{\omega\}. \tag{5.2.17}$$

The same value of T_{rot} should result if the angular velocity and the inertia properties are referred to the $x'y'z'$ axes, so

$$T_{\text{rot}} = \frac{1}{2}\{\omega'\}^{\text{T}}[I']\{\omega'\}. \tag{5.2.18}$$

Equation (5.2.15) allows us to replace the $x'y'z'$ components of $\bar{\omega}$ in the preceding with those relative to xyz, with the result that

$$T_{\text{rot}} = \frac{1}{2}\{\omega\}^{\text{T}}[R]^{\text{T}}[I'][R]\{\omega\}. \tag{5.2.19}$$

The value of T_{rot} obtained from this equation must match the result of Eq. (5.2.17), regardless of what the angular velocity actually is. The only way this equivalence can be attained is if the inner product matches $[I]$, that is,

$$[I] = [R]^{\mathrm{T}} [I'] [R]. \tag{5.2.20}$$

In the scenario of interest we presumably know $[I]$. To solve for $[I']$ we recall the orthonormal property, $[R][R]^{\mathrm{T}} = [R]^{\mathrm{T}}[R] = [U]$. Thus we find that

$$\boxed{[I'] = [R][I][R]^{\mathrm{T}}.} \tag{5.2.21}$$

Any quantity transforming in the manner described by this relation is said to be a tensor of the second rank. In this viewpoint, vectors, whose components transform according to Eq. (5.2.15), are tensors of the first rank. Note that transforming a vector involves premultiplication by the matrix $[R]$ that transforms from the known to unknown components. Transformation of a second-rank tensor involves premultiplication by $[R]$ and postmultiplication by $[R]^{\mathrm{T}}$. In the case of the symmetric second-rank tensors like the inertia properties, such a transformation preserves the symmetry of the tensor.

The transformation in Eq. (5.2.21) may be decomposed into individual inertia values. Toward that end we recall Eq. (3.1.11), which expresses $[R]$ as a sequence of row partitions consisting of the direction cosines of an axis of $x'y'z'$ relative to xyz, according to

$$[R] = \begin{bmatrix} \{e_{x'}\}^{\mathrm{T}} \\ \{e_{y'}\}^{\mathrm{T}} \\ \{e_{z'}\}^{\mathrm{T}} \end{bmatrix}. \tag{5.2.22}$$

The rows of $[R]$ are denoted as $\{e_{\xi'}\}^{\mathrm{T}}$, $\xi' = x',\ y'$, or z', because each column vector consists of the components of a unit vector $\bar{e}_{\xi'}$ relative to xyz, that is,

$$\{e_{\xi'}\}^{\mathrm{T}} = \begin{bmatrix} \ell_{\xi' x} & \ell_{\xi' y} & \ell_{\xi' z} \end{bmatrix}. \tag{5.2.23}$$

The utility of the partitioned form of $[R]$ is that it enables us to condense the operations, because each partition may be treated as a single element. The columns of $[R]^{\mathrm{T}}$ are rows of $[R]$, so

$$[R]^{\mathrm{T}} = \begin{bmatrix} \{e_{x'}\} & \{e_{y'}\} & \{e'_z\} \end{bmatrix}. \tag{5.2.24}$$

Thus, substituting the partitioned form of $[R]$ into the rotation transformation, Eq. (5.2.21), leads to

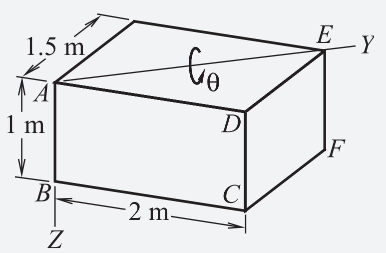
$$
\begin{aligned}
[I'] &= \begin{bmatrix} \{e_{x'}\}^{\mathrm{T}} \\ \{e_{y'}\}^{\mathrm{T}} \\ \{e_{z'}\}^{\mathrm{T}} \end{bmatrix} [I] \Big[\{e_{x'}\} \quad \{e_{y'}\} \quad \{e_z'\} \Big] \\
&= \begin{bmatrix} \{e_{x'}\}^{\mathrm{T}} \\ \{e_{y'}\}^{\mathrm{T}} \\ \{e_{z'}\}^{\mathrm{T}} \end{bmatrix} \Big[[I]\{e_{x'}\} \quad [I]\{e_{y'}\} \quad [I]\{e_z'\} \Big] \\
&= \begin{bmatrix} \{e_{x'}\}^{\mathrm{T}}[I]\{e_{x'}\} & \{e_{x'}\}^{\mathrm{T}}[I]\{e_{y'}\} & \{e_{x'}\}^{\mathrm{T}}[I]\{e_z'\} \\ \{e_{y'}\}^{\mathrm{T}}[I]\{e_{x'}\} & \{e_{y'}\}^{\mathrm{T}}[I]\{e_y'\} & \{e_{y'}\}^{\mathrm{T}}[I]\{e_z'\} \\ \{e_{z'}\}^{\mathrm{T}}[I]\{e_x'\} & \{e_{z'}\}^{\mathrm{T}}[I]\{e_{y'}\} & \{e_{z'}\}^{\mathrm{T}}[I]\{e_z'\} \end{bmatrix}.
\end{aligned}
\tag{5.2.25}
$$

The diagonal elements of the product are the moments of inertia, whereas the off-diagonal terms are the negative of the products of inertia. Thus we find that

$$
\left.\begin{aligned} I_{\xi'\xi'} &= \{e_{\xi'}\}^{\mathrm{T}}[I]\{e_{\xi'}\} \\ I_{\xi'\eta'} &= -\{e_{\xi'}\}^{\mathrm{T}}[I]\{e_{\eta'}\} \end{aligned}\right\}, \quad \xi',\ \eta' = x,\ y,\ \text{or } z,\quad \xi \neq \eta. \tag{5.2.26}
$$

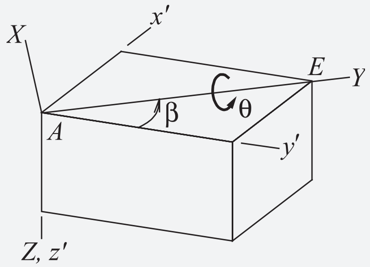
The purpose of deriving these relations is to understand how a specific inertia property is altered by a rotation. In most situations we would need all of the transformed inertia properties. In that case it is much simpler to evaluate Eq. (5.2.21) directly.

EXAMPLE 5.6 The 5-kg homogeneous box rotates through angle θ about the Y axis of the stationary XYZ coordinate system. Edge AB coincides with the Z axis when $\theta = 0$. Determine the moments and products of inertia with respect to XYZ as functions of θ.



Example 5.6

SOLUTION This problem has the obvious purpose of illustrating the transformation of inertia properties, but the results also help us understand some general aspects. In the second drawing $x'y'z'$ is a body-fixed coordinate system having origin A whose axes are aligned with the edges of the box. It is parallel to the centroidal system of a rectangular parallelepiped described in the Appendix.



Coordinate systems used to derive the rotation transformation

The coordinates of corner A relative to the parallel centroidal coordinate system are the components of the relative position vector,

$$\bar{r}_{A/G} = -0.75\bar{i}' - 1\bar{j}' - 0.5\bar{k}' \text{ m.}$$

Applying the parallel axis transformation to the tabulated properties gives

$$I_{x'x'} = \frac{1}{12}(5)\left(2^2 + 1^2\right) + 5\left[(-1)^2 + (-0.5)^2\right] = 8.3333,$$

$$I_{y'y'} = \frac{1}{12}(5)\left(1.5^2 + 1^2\right) + 5\left[(-0.75)^2 + (-0.5)^2\right] = 5.4167,$$

$$I_{z'z'} = \frac{1}{12}(5)\left(1.5^2 + 2^2\right) + 5\left[(-0.75)^2 + (-1)^2\right] = 10.4167,$$

$$I_{x'y'} = I_{y'x'} = 0 + 5(-0.75)(-1) = 3.7500,$$

$$I_{x'z'} = I_{z'x'} = 0 + 5(-0.75)(-0.5) = 1.8750,$$

$$I_{y'z'} = I_{z'y'} = 0 + 5(-1)(-0.5) = 2.5 \text{ kg-m}^2.$$

The rotation transformation from $x'y'z'$ to XYZ is a simple rotation by $\beta = \tan^{-1}(1.5/2) = 36.87^\circ$ about the negative z' axis:

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = [R_1]\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}, \quad [R_1] = \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us define the body-fixed coordinate system xyz such that it coincides with XYZ when $\theta = 0$. Thus, transforming the inertia matrix from $x'y'z'$ to XYZ gives the constant properties relative to xyz. We designate this as $[I_{\theta=0}]$. Using the preceding rotation matrix gives

$$[I_{\theta=0}] = [R_1]\begin{bmatrix} 8.3333 & -3.7500 & -1.8750 \\ -3.7500 & 5.4167 & -2.5 \\ -1.8750 & -2.5 & 10.4167 \end{bmatrix}[R_1]^{\mathrm{T}}$$

$$= \begin{bmatrix} 10.8833 & 0.3500 & 0 \\ 0.3500 & 2.8667 & -3.1250 \\ 0 & -3.1250 & 10.4167 \end{bmatrix}.$$

Another rotation transformation is needed to find the properties relative to space-fixed XYZ axes when θ is not zero. Because xyz rotates about the Y axis relative to XYZ, the transformation from XYZ to xyz is

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = [R_2]\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}, \quad [R_2] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

The transformation from xyz to XYZ is described by $[R_2]^{\mathrm{T}}$, so the inertia matrix for the XYZ axes is given by

$$[I] = [R_2]^{\mathrm{T}}[I_{\theta=0}][R_2].$$

The diagonal elements are the moments of inertia relative to XYZ, and the off-diagonal elements are the negative of the products of inertia:

$$I_{XX} = 0.4667(\cos\theta)^2 + 10.4167, \quad I_{ZZ} = 0.4667(\sin\theta)^2 + 10.4167,$$
$$I_{YY} = 2.8667,$$
$$I_{XY} = I_{YX} = -0.3500\cos\theta + 3.1250\sin\theta, \qquad \triangleleft$$
$$I_{XZ} = I_{ZX} = 0.4667(\sin\theta)(\cos\theta),$$
$$I_{YZ} = I_{ZY} = 0.3500\sin\theta + 3.1250\cos\theta.$$

The moment of inertia about the Y axis does not depend on θ because the distance from this axis to each mass point is unaffected by the rotation. Both I_{XX} and I_{ZZ} are periodic in $\Delta\theta = \pi$, but they vary little as θ changes, essentially because the dimensions of the box in both directions transverse to the Y axis are comparable. The products of inertia I_{XY} and I_{YZ} are periodic in $\Delta\theta = 2\pi$, whereas I_{XZ} is periodic in $\Delta\theta = \pi$. The periodic nature of the inertia properties stems from the fact that a rotation by $\Delta\theta = \pi$ changes the sign of the X and Z coordinates of each mass element, whereas the Y coordinate remains constant. An important observation is that there are values of θ for which each product of inertia vanishes, but the angle is different for each term.

5.2.3 Inertia Ellipsoid

Equation (5.2.17) for the rotational kinetic energy and Eq. (5.2.26) for the moment of inertia about an arbitrary axis have similar forms. We exploit that similarity here to develop a pictorial representation of the dependence of moments of inertia on the orientation of the associated axis. In the present context the construction is primarily of qualitative interest, but it will be quite useful in Chapter 10 for a detailed study of the rotation of a body in free motion.

Suppose a body is made to rotate at angular speed ω about a specified axis $p' = x'$, y', or z' whose direction is $\bar{e}_{p'}$.The matrix representation of the angular velocity is

$$\{\omega\} = \omega \left\{e_{p'}\right\}, \tag{5.2.27}$$

where $\left\{e_{p'}\right\}$ contains the components of one of the unit vectors of $x'y'z'$ relative to xyz. The construction we seek considers a situation in which the angular speed is adjusted such that, regardless of how the rotation axis is oriented relative to body-fixed coordinates xyz, the rotational kinetic energy is always $T_{\text{rot}} = 1/2$. Let $\bar{\rho}$ denote a vector extending from the origin of xyz to a point whose coordinates are the components of this special $\bar{\omega}$. In matrix notation we have

$$\{\rho\} = [x \;\; y \;\; z]^{\text{T}} = \rho \left\{e_{p'}\right\}, \quad \rho = \left(x^2 + y^2 + z^2\right)^{1/2}. \tag{5.2.28}$$

Let $[I]$ denote the inertia matrix relative to xyz. Setting $T_{\text{rot}} = 1/2$ in Eq. (5.2.17) shows that these coordinates satisfy

$$[x \;\; y \;\; z]\,[I]\,[x \;\; y \;\; z]^{\text{T}} = 1. \tag{5.2.29}$$

Expansion of this product yields

$$I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 - 2I_{xy}xy - 2I_{xz}xz - 2I_{yz}yz = 1. \tag{5.2.30}$$

This is the equation for an ellipsoidal surface whose centroid coincides with the origin of the xyz set of axes. This surface is called the *ellipsoid of inertia*.

One interpretation of the ellipsoid of inertia is associated with its construction, that is, it is the locus of points for which an angular velocity equal to the vector from the origin to any point gives $T_{\text{rot}} = 1/2$. A more useful interpretation is obtained by using Eq. (5.2.28) to eliminate the position coordinates in Eq. (5.2.29). In view of the first of Eqs. (5.2.26), this operation yields

$$\rho^2 \left\{e_{p'}\right\}^{\text{T}} [I] \left\{e_{p'}\right\} = \rho^2 I_{p'p'} = 1. \tag{5.2.31}$$

Thus the distance ρ from the origin to a point on the ellipsoid of inertia is the reciprocal of the square root of the moment of inertia about the axis intersecting the origin and that point. In other words, the distance is inversely proportional to the radius of gyration about that axis.

If we know $[I]$, we can construct the inertia ellipsoid according to Eq. (5.2.30), as depicted in Fig. 5.7. The major, minor, and intermediate axes of this ellipsoid of inertia, along which the distance from the origin is an extreme value, are mutually orthogonal.

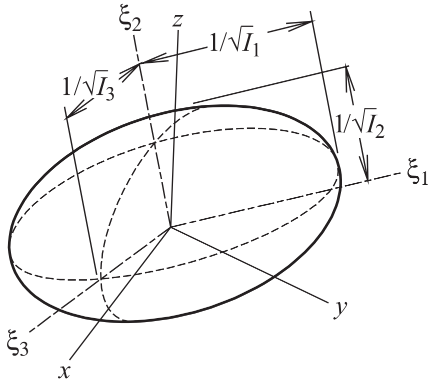


Figure 5.7. A typical ellipsoid of inertia, whose major, intermediate, and minor axes align with ξ_1, ξ_2, and ξ_3, respectively.

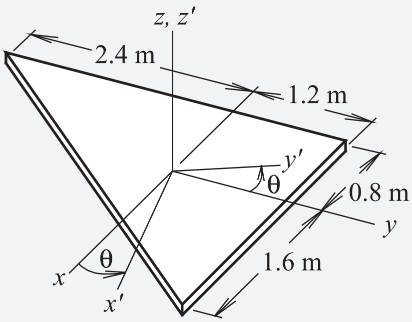
However, they do not necessarily coincide with the xyz axes. In the figure $\xi_1\xi_2\xi_3$ is a coordinate system whose axes coincide respectively with the major, intermediate, and minor axes of the ellipsoid of inertia. The canonical equation for an ellipsoidal surface relative to such a coordinate system is

$$I_1\xi_1^2 + I_2\xi_2^2 + I_3\xi_3^2 = 1, \tag{5.2.32}$$

where $I_1 \leq I_2 \leq I_3$ when ξ_1 aligns with the major axis and ξ_3 align with the minor axis. The semidiameters of the ellipsoid are $1/\sqrt{I_1}$, $1/\sqrt{I_2}$,and $1/\sqrt{I_3}$, respectively, as shown in the figure. Equation (5.2.32) is like Eq. (5.2.30), except that there are no terms associated with products of inertia. It follows that I_1, I_2, and I_3 are the principal moments of inertia and the $\xi_1\xi_2\xi_3$ coordinate system is a set of principal axes.

This discussion proves that, for a specified origin, there always is a set of principal axes. We could contemplate locating these axes by evaluating $[I]$ for nonprincipal axes and then constructing the inertia ellipsoid. The principal axes would then be located by graphically identifying the major, intermediate, and minor axes. However, such a procedure would be imprecise, as well as challenging to implement. The following example shows how mathematical analysis of the inertia ellipsoid can locate the principal axes in a two-dimensional situation.

EXAMPLE 5.7 In the sketch xyz and $x'y'z'$ are centroidal coordinate systems for the right triangular plate, with the former aligned parallel to the parallel edges and the latter rotated about the z axis through angle θ. The mass of the plate is 400 kg. Because the plate is bisected by the $x'y'$ plane, $I_{x'z'} = I_{y'z'} = 0$. There are values of θ for which $I_{x'y'} = 0$, thereby making $x'y'z'$ principal axes for the plate. Determine θ and the corresponding principal moments of inertia by using a rotation transformation. Then perform the same evaluation by using the properties of the inertia ellipsoid.



Example 5.7

SOLUTION The primary intent here is to lessen the abstract nature of the ellipsoid of inertia by seeing its features for an actual body. We may find the inertia properties relative to xyz by letting the length a of the right triangular prism in the Appendix approach zero. The z and x axes are swapped with the corresponding axes in the tabulation, but the y axis here is reversed from the sense described there. This means that we should interchange the subscripts of the inertia properties to match the present definitions and also set I_{xy} and I_{yz} equal to the negative of the tabulated expressions. Thus we have

$$I_{xx} = \frac{1}{36}(400)(2)\left(3.6^2\right) = 288, \quad I_{yy} = \frac{1}{36}(400)(2)\left(2.4^2\right) = 128,$$

$$I_{zz} = I_{xx} + I_{yy} = 416 \text{ kg-m}^2,$$

$$I_{xy} = +\frac{1}{36}(400)(3.6)(2.4) = 96, \quad I_{yz} = I_{xz} = 0.$$

It is worth noting I_{zz} was evaluated as the sum of the other moments of inertia. This is identically true for any thin flat slab when the origin of xyz lies in the plane of the slab and the z axis is normal to that plane.

The rotation transformation from xyz to $x'y'z'$ is

$$[R] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We wish to determine all inertia properties with respect to $x'y'z'$, so we implement the full rotation transformation:

$$[I'] = [R]\begin{bmatrix} 288 & -96 & 0 \\ -96 & 128 & 0 \\ 0 & 0 & 416 \end{bmatrix}[R]^{\mathrm{T}}$$

$$= \begin{bmatrix} 288\,(c\theta)^2 + 128\,(s\theta)^2 - 192 s\theta c\theta & 96 - 192\,(c\theta)^2 - 160 s\theta c\theta & 0 \\ 96 - 192\,(c\theta)^2 - 160 s\theta c\theta & 288\,(s\theta)^2 + 128\,(c\theta)^2 + 192 s\theta c\theta & 0 \\ 0 & 0 & 416 \end{bmatrix},$$

where sθ and cθ are abbreviations for $\sin\theta$ and $\cos\theta$. We find the value of θ that makes $I_{x'y'}$ vanish by applying the identities for the sine and cosine of 2θ to the (1,2) element of $[I']$, which leads to

$$-96\cos(2\theta) - 80\sin(2\theta) = 0 \Longrightarrow \theta = 64.903^\circ,\ -25.097^\circ. \qquad \triangleleft$$

These two possible angles differ by 90°, so they merely correspond to alternate labeling of the x' and y' axes. The principal moments of inertia for $\theta = 64.903^\circ$ are

$$I_{x'x'} = I_1 = 83.04, \quad I_{y'y'} = I_2 = 332.96, \quad I_{z'z'} = I_3 = 416\ \text{kg-m}^2. \qquad \triangleleft$$

Solution of this problem by use of the ellipsoid of inertia barely resembles the preceding operations. Because we know that z is a principal axis, we focus on the xy plane by writing the defining equation, Eq. (5.2.30), for the case in which $z = 0$, which gives

$$288x^2 + 128y^2 - 192xy = 1.$$

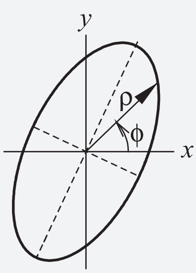
This is an ellipse. The multivalued nature of the value of y as a function of x makes it convenient to use polar coordinates. By definition, the distance from the origin to a point on the ellipsoid of inertia is ρ, so we define

$$x = \rho\cos\phi, \quad y = \rho\sin\phi.$$

The equation for the xy ellipse then becomes

$$\rho^2\left[288(\cos\phi)^2 + 128(\sin\phi)^2 - 192(\sin\phi)(\cos\phi)\right] = 1.$$

We obtain the graph of this ellipse by solving the preceding equation for the value of ρ corresponding to numerous values of ϕ covering a 2π range.



Intersection of the triangular plates's inertia ellipsoid with the *xy* plane

The major and minor diameters depicted in this graph could be located by several methods, each of which is based on the same property. In general, ρ has an extreme value at the principal axes. Hence the major and minor diameters correspond to points at which ρ has a maximum or minimum value. We could locate both lines by visually searching for the points on the ellipse that are farthest and closest to the origin. Another procedure is to search through a table of values of ϕ and ρ for the points where ρ is largest and smallest. Both procedures have limited precision. A mathematical approach is to use calculus to locate values of ϕ for which

$d\rho/d\phi = 0$. To expedite such an evaluation we introduce the identities for $\cos(2\phi)$ and $\sin(2\phi)$ into the equation for the xy ellipse, which gives

$$\rho^2\,(208 + 80\cos 2\phi - 96\sin 2\phi) = 1.$$

Implicit differentiation of this equation with respect to ϕ yields

$$2\rho\frac{d\rho}{d\phi}(208 + 80\cos 2\phi - 96\sin 2\phi) + \rho^2\,(-160\sin 2\phi - 192\cos 2\phi) = 0.$$

Setting $d\rho/d\phi = 0$ shows that the principal axes correspond to $160\sin 2\phi + 192\cos 2\phi = 0$, which is the same as the condition identified from the rotation transformation. Thus the values of ϕ, and the corresponding ρ, are

$$\begin{aligned} \phi &= 64.903^\circ \implies \rho_1 = 0.109740, \\ \phi &= -25.097^\circ \implies \rho_2 = 0.054803. \end{aligned} \qquad \triangleleft$$

The principal moments of inertia are the reciprocals of the squares of the extreme values of ρ, so

$$I_1 = \frac{1}{\rho_1^2} = 83.04, \quad I_2 = \frac{1}{\rho_2^2} = 332.96\ \text{kg-m}^2. \qquad \triangleleft$$

These are the same principal values as those found earlier.

The key aspect to bear in mind is that both procedures, explicitly setting the off-diagonal elements of $[I']$ to zero and geometrically analyzing the inertia ellipsoid, were expedited by the fact that z was known to be a principal axis. Consequently, only a single angle needed to be evaluated. Either approach would be substantially more difficult to implement if multiple direction angles needed to be found. Also worth noting is the fact that the orientation of the principal axes has no simple geometrical explanation, unlike the case for symmetrical bodies.

5.2.4 Principal Axes

Identification of principal axes from the properties of the inertia ellipsoid is unwieldy in a general situation where none of the axes of the original coordinate system are principal. Here we establish a mathematical procedure that achieves the same goal in a straightforward manner. We seek to determine a rotation transformation $[R]$ that converts the inertia matrix $[I]$ associated with nonprincipal axes xyz to a diagonal matrix $[I']$. Let $I_1 \le I_2 \le I_3$ denote the principal moments of inertia corresponding to axes whose unit vectors are respectively $\bar{e}_1$, $\bar{e}_2$, and $\bar{e}_3$. Equation (5.2.22) describes a rotation transformation as a column of partitions that consist of the components of unit vectors. In terms of the principal directions, that description is

$$[R] = \begin{bmatrix} \{e_1\}^{\mathrm{T}} \\ \{e_2\}^{\mathrm{T}} \\ \{e_3\}^{\mathrm{T}} \end{bmatrix}. \tag{5.2.33}$$

When we replace $[I']$ and the arbitrary unit vectors in the first line of Eq. (5.2.25) with the quantities for principal axes, we find that

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = \begin{bmatrix} \{e_1\}^{\mathrm{T}} \\ \{e_2\}^{\mathrm{T}} \\ \{e3\}^{\mathrm{T}} \end{bmatrix} [I] \begin{bmatrix} \{e_1\} & \{e_2\} & \{e_3\} \end{bmatrix}. \tag{5.2.34}$$

Our task now is to determine the unit vector components and principal moments of inertia that satisfy the preceding relation when $[I]$ is given. The first step is to premultiply by $[R]^{\mathrm{T}}$ and apply the orthonormal property. This gives

$$\begin{bmatrix} \{e_1\} & \{e_2\} & \{e_3\} \end{bmatrix} \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = [I] \begin{bmatrix} \{e_1\} & \{e_2\} & \{e_3\} \end{bmatrix}. \tag{5.2.35}$$

The multiplication property for partitioned matrices simplifies this to

$$\begin{bmatrix} I_1 \{e_1\} & I_2 \{e_2\} & I_3 \{e_3\} \end{bmatrix} = \begin{bmatrix} [I] \{e_1\} & [I] \{e_2\} & [I] \{e_3\} \end{bmatrix}. \tag{5.2.36}$$

Like columns on either side of this equation must match, from which it follows that each principal direction and its associated moment of inertia are solutions of

$$I_j \left\{e_j\right\} = [I] \left\{e_j\right\}, \quad j = 1, 2, 3. \tag{5.2.37}$$

In other words, I_j are the eigenvalues λ and $\left\{e_j\right\}$ are the eigenvectors $\{e\}$ satisfying

$$[[I] - \lambda [U]] \{e\} = \{0\}, \tag{5.2.38}$$

where $[U]$ is the identity matrix.

This is called a standard eigenvalue problem. Routines for solving such problems are contained in most mathematical software. Only the concepts are highlighted here. Equation (5.2.38) represents three simultaneous equations for the components of $\{e\}$, which are the direction cosines between a principal axis and the axes associated with inertia matrix $[I]$. These equations are homogeneous. Consequently, the only solution is the trivial one, $\{e\} = \{0\}$, unless the coefficient matrix $[I] - \lambda [U]$ cannot be inverted. Hence nontrivial solutions for $\{e\}$ require that λ satisfy the characteristic equation corresponding to vanishing of the determinant of the coefficients:

$$\left|[I] - \lambda [U]\right| = 0. \tag{5.2.39}$$

Evaluation of this determinant with λ as an algebraic parameter leads to a cubic equation. The eigenvalues, which are the three roots of the characteristic equation, are the principal moments of inertia, $\lambda = I_1, I_2, I_3$.

Once the eigenvalues are known, we may proceed to determine the eigenvectors, which will be the principal directions. The fact that the determinant of $[I] - I_j [U]$ vanishes means that its rank has been reduced, so that the elements of an eigenvector $\left\{e_j\right\}$ cannot be determined uniquely from Eq. (5.2.38). In the standard situation the principal moments of inertia are distinct values. In that case the rank of $[I] - I_j [U]$ is two, so that one of the three simultaneous equations represented by Eq. (5.2.38) is a linear combination of the others. Because of this loss of an independent equation, any nonzero component of $\{e_j\}$ may be chosen arbitrarily. The other components may then be found in terms of the arbitrary one by solving the independent equations.

The eigenvectors satisfying Eq. (5.2.38) have an arbitrary element, but that is so because consideration has not been given to the definition of $\{e_j\}$ as the components of a unit vector oriented parallel to the jth principal axis. The condition that such a vector has a unit magnitude is written in matrix form as

$$\{e_j\}^{\mathrm{T}}\{e_j\} = 1. \tag{5.2.40}$$

Enforcement of this condition yields the additional equation required to evaluate uniquely $\{e_j\}$. Note that the solution of the preceding leads to an ambiguity in sign. Even though any eigenvector multiplied by -1 is still an eigenvector, assembling the three eigenvectors to form $[R]$ might lead to a left-handed system, identifiable by the fact that $|[R]| = -1$. Such a condition is readily corrected by multiplying one of the eigenvectors by -1.

The eigenvectors $\{e_j\}$ form an orthogonal set. To prove this property, consider Eq. (5.2.37) for two different principal values, $j = m$ and $j = n$. Premultiplying each equation by the transpose of the other eigenvector leads to

$$\begin{aligned} \{e_n\}^{\mathrm{T}}[I]\{e_m\} &= I_m\{e_n\}^{\mathrm{T}}\{e_m\}, \\ \{e_m\}^{\mathrm{T}}[I]\{e_n\} &= I_n\{e_m\}^{\mathrm{T}}\{e_n\}. \end{aligned} \tag{5.2.41}$$

The quantities that are equated are scalars, so we may transpose the products without altering the result. Because $[I]$ is symmetric, performing this operation on the second of the preceding equations gives

$$\{e_n\}^{\mathrm{T}}[I]\{e_m\} = I_n\{e_n\}^{\mathrm{T}}\{e_m\}. \tag{5.2.42}$$

If the moments of inertia are distinct values, $I_n \neq I_m$, subtracting this equation from the first of Eqs. (5.2.41) leads to the conclusion that

$$\{e_n\}^{\mathrm{T}}\{e_m\} = 0 \quad \text{if} \quad I_m \neq I_n. \tag{5.2.43}$$

This is the matrix form of the dot product $\bar{e}_n \cdot \bar{e}_n = 0$, which proves the orthogonality of the unit vectors. It then follows from Eq. (5.2.42) that

$$\{e_n\}^{\mathrm{T}}[I]\{e_m\} = 0 \quad \text{if} \quad I_m \neq I_n. \tag{5.2.44}$$

According to the second of Eqs. (5.2.26), the left-hand side of this relation is the negative of the product of inertia associated with axes $\bar{e}_m$ and $\bar{e}_n$. Hence the preceding merely proves that the directions derived by solving the eigenvalue problem will be principal axes.

Both Eqs. (5.2.43) and (5.2.44) exclude the case in which the moments of inertia about axes $\bar{e}_m$ and $\bar{e}_n$ are equal. In general, the rank of $[I] - I_m[U]$ is reduced by the number of times the eigenvalue $\lambda = I_m$ occurs as a root of the characteristic equation. Thus, if $I_m = I_n$ and the third eigenvalue is different from I_m, then $[I] - I_m[U]$ has a rank of one. This means that $[[I] - I_m[U]]\{e\} = \{0\}$ consists of one independent equation for $\{e\}$, so that $\{e\}$ has two arbitrary elements. It follows that, in addition to requiring that $\{e\}$ represent a unit vector, Eq. (5.2.40), another condition must be identified.

The method in which to proceed when this ambiguity arises becomes apparent when we consider a body of revolution. Any coordinate system having an axis that coincides with the axis of symmetry will be principal, and the moments of inertia about the axes perpendicular to the axis of symmetry will be the same. This suggests that, when two principal moments of inertia are equal, the ratio of two arbitrary elements in the first eigenvector can be set to any convenient value. (For example, we might wish that the first eigenvector have no component parallel to the z axis, in which case we would equate the third element of $\{e_m\}$ to zero.) The principal direction described by $\{e_n\}$ should be orthogonal to the direction described by $\{e_m\}$, so we require that

$$\{e_n\}^{\mathrm{T}}\{e_m\} = 0 \quad \text{if} \quad m \neq n. \tag{5.2.45}$$

This, in combination with Eq. (5.2.40), gives two conditions required to determine uniquely the second eigenvector.

The greater degree of arbitrariness associated with identical principal moments of inertia arises because the corresponding principal directions are not unique. The case in which all three principal values are identical merely means that any set of axes are principal. There is then no need to solve an eigenvalue problem. This feature is exemplified by a homogeneous sphere or cube when the origin is placed at the centroid.

The ellipsoid of inertia provides an interesting geometrical interpretation of the eigenvalue problem, which was derived from the mathematical properties of the rotation transformation of inertia properties. We saw in the previous section that principal axes correspond to locations on the inertia ellipsoid at which the distance to the center is a local extremum. At such locations the normal to the ellipsoid's surface will be parallel to the line to the origin. The vector from the origin to a point on the surface is $\bar{\rho} = x\bar{i} + y\bar{j} + z\bar{k}$. If a surface is described in functional form as $f(x, y, z) = C$, a constant, then the gradient of f is normal to the surface.

Thus the condition that $\bar{\rho}$ is parallel to the surface normal when it is aligned with a principal axis is described by

$$\bar{\rho} = x\bar{i} + y\bar{j} + z\bar{k} = \sigma \nabla f, \tag{5.2.46}$$

where σ is a factor of proportionality. Equation (5.2.30) is the function for the ellipsoid. The gradient of this function is

$$\begin{aligned} \nabla f = {} & 2(I_{xx}x - I_{xy}y - I_{xz}z)\,\bar{i} + 2(I_{yy}y - I_{xy}x - I_{yz}z)\,\bar{j} \\ & + 2(I_{zz}z - I_{xz}x - I_{yz}y)\,\bar{k}. \end{aligned} \tag{5.2.47}$$

We substitute this expression into Eq. (5.2.46), and match like components, which leads to

$$\begin{aligned} x &= 2\sigma(I_{xx}x - I_{xy}y - I_{xz}z), \\ y &= 2\sigma(I_{yy}y - I_{xy}x - I_{yz}z), \\ z &= 2\sigma(I_{zz}z - I_{xz}x - I_{yz}y). \end{aligned} \tag{5.2.48}$$

The matrix form of this set of simultaneous equations is

$$[I]\,[x \;\; y \;\; z]^{\mathrm{T}} = \frac{1}{2\sigma}\,[x \;\; y \;\; z]^{\mathrm{T}}. \tag{5.2.49}$$

This is equivalent to the eigenvalue problem described by Eq. (5.2.37) when $\sigma = 1/2\lambda_j$.

EXAMPLE 5.8 Determine the principal moments of inertia and associated rotation transformation for the triangular plate in Example 5.7 by solving an eigenvalue problem.

SOLUTION This example demonstrates the formal mathematical eigensolution. In Example 5.7 the inertia matrix of the triangular plate relative to centroidal axes matching those in the Appendix was found to be

$$[I] = \begin{bmatrix} 288 & -96 & 0 \\ -96 & 128 & 0 \\ 0 & 0 & 416 \end{bmatrix} \text{kg-m}^2.$$

The corresponding eigenvalue problem for principal axes is

$$\begin{bmatrix} (288-\lambda) & -96 & 0 \\ -96 & (128-\lambda) & 0 \\ 0 & 0 & 416-\lambda \end{bmatrix} \{e\} = \{0\}.$$

The characteristic equation is

$$\begin{aligned} \left|[I] - \lambda\,[U]\right| &= (416-\lambda)\left[(288-\lambda)(128-\lambda) - 96^2\right] \\ &= (416-\lambda)\left(\lambda^2 - 416\lambda + 27648\right) = 0. \end{aligned}$$

One eigenvalue makes the first factor vanish, whereas the other two roots are solutions of the quadratic equation we obtain by setting the second factor to zero. These values are

$$\lambda_1 = I_1 = 83.036, \quad \lambda_2 = I_2 = 332.964, \quad \lambda_3 = I_3 = 416\ \text{kg-m}^2. \qquad \triangleleft$$

Note that the eigenvalues have been sequenced from smallest to largest.

We denote the elements of eigenvector $\{e\}$ corresponding to I_j as e_{nj}. Thus the first eigenvector must satisfy

$$[[I] - \lambda_1\,[U]] \begin{Bmatrix} e_{11} \\ e_{21} \\ e_{31} \end{Bmatrix} = \begin{bmatrix} 204.964 & -96 & 0 \\ -96 & 44.964 & 0 \\ 0 & 0 & 332.964 \end{bmatrix} \begin{Bmatrix} e_{11} \\ e_{21} \\ e_{31} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

This eigenvalue is a single root of the characteristic equation, so one row of the coefficient matrix is a linear combination of the others. The third row clearly is different from the other two, so we discard the second row. Thus we need to find the solution of

$$204.964e_{11} - 96e_{21} = 0,$$
$$332.964e_{31} = 0.$$

The last of the preceding equations requires that $e_{31} = 0$, which means that it cannot be considered arbitrary. The first equation defines the ratio of e_{21} to e_{11}, so we have

$$e_{21} = 2.1350e_{11}, \quad e_{31} = 0.$$

The condition that $\{e\}$ represent a unit vector requires that

$$e_{11}^2 + e_{21}^2 = \left(1 + 2.1350^2\right) e_{11}^2 = 1.$$

We select the positive root, so that the first principal axis will have a component in the positive x direction. This leads to

$$e_{11} = 0.4242, \quad e_{21} = 0.9056, \quad e_{31} = 0.$$

The evaluation of the second principal direction follows similar steps. We have

$$[[I] - \lambda_2 [U]] \begin{Bmatrix} e_{12} \\ e_{22} \\ e_{32} \end{Bmatrix} = \begin{bmatrix} -44.964 & -96 & 0 \\ -96 & -204.964 & 0 \\ 0 & 0 & 83.036 \end{bmatrix} \begin{Bmatrix} e_{12} \\ e_{22} \\ e_{32} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

Once again, we discard the second equation as the one that is not independent, which leads to

$$\begin{matrix} -44.964e_{12} - 96e_{22} = 0 \\ 83.036e_{32} = 0 \end{matrix} \implies e_{22} = -0.4684e_{12}, \quad e_{32} = 0.$$

Making this eigenvector represent a unit vector gives

$$e_{12} = 0.9056, \quad e_{22} = -0.4242, \quad e_{32} = 0.$$

The situation for the third eigenvector is slightly different. In this case, we have

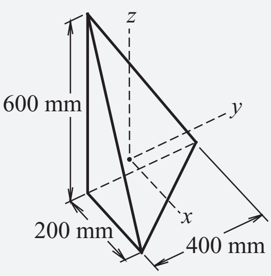
$$[[I] - \lambda_3 [U]] \begin{Bmatrix} e_{13} \\ e_{23} \\ e_{33} \end{Bmatrix} = \begin{bmatrix} -128 & -96 & 0 \\ -96 & -128 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} e_{13} \\ e_{23} \\ e_{33} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

The first two rows of the coefficient matrix are independent of each other, and the third coefficient in each is zero, so the associated equations require that $e_{13} = e_{23} = 0$, but place no conditions on e_{33}. The last row of the coefficient matrix represents a trivial equation, so any value of e_{33} will constitute an eigenvector. For it to represent a unit vector it must be that $|e_{33}| = 1$. If $e_{33} = +1$ and the first two rows of $[R]$ are the values we have determined, then $\left|[R]\right| = -1$. Multiplying an eigenvector by -1, which corresponds to reversing the associated unit vector, does not alter the fact that it is an eigensolution. Thus we set $e_{33} = -1$. According to Eq. (5.2.33), the eigenvectors are the rows of the transformation matrix, so we have found that

$$[R] = \begin{bmatrix} 0.4242 & 0.9056 & 0 \\ 0.9056 & -0.4242 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \qquad \lhd$$

A comparison of this result with the one found in Example 5.7 shows that the second and third principle directions are reversed.

EXAMPLE 5.9 In the sketch, xyz are centroidal axes of the 60-kg homogeneous orthogonal tetrahedron. Determine the principal moments of inertia and the rotation transformation for the principal axes whose origin also is the centroid.



Example 5.9

SOLUTION This example addresses how mathematical software can be used to evaluate the principal inertia properties.† For the dimensions of the given body with $m = 60$ kg, we find from the Appendix that

$$[I] = \begin{bmatrix} 1.170 & 0.060 & 0.090 \\ 0.060 & 0.900 & 0.180 \\ 0.090 & 0.180 & 0.450 \end{bmatrix} \text{ kg-m}^2.$$

Most mathematical software packages have functions or subroutines that can solve the standard eigenvalue problem $[[I] - \lambda\,[U]]\,\{e\} = \{0\}$. In Matlab, one obtains the desired result by writing `[e_vecs, I_princ] = eigs(I);` where `I` is the preceding 3×3 array, `I_princ` is a 3×3 diagonal array whose elements are the principal

† One should check the behavior of their software version. In particular, the manner in which eigenvectors are normalized is somewhat arbitrary. The results presented here were obtained from Matlab 7.1 and Mathcad 13.1.

moments of inertia, and `e_vecs` is a 3×3 array whose jth column is the eigenvector corresponding to `I_princ(j,j)`. The Matlab result is

$$I\_princ(1,1) = 1.2059 \quad I\_princ(2,2) = 0.9327 \quad I\_princ(3,3) = 0.3814,$$

$$e\_vecs = \begin{bmatrix} -0.9395 & -0.3322 & -0.0836 \\ -0.2908 & 0.9023 & -0.3181 \\ -0.1811 & 0.2746 & 0.9444 \end{bmatrix}.$$

The algorithm employed by Matlab returns `e_vecs` as an orthonormal transformation, that is, $[e\_vecs][e\_vecs]^{\mathrm{T}} = [U]$, but the principal values are not sequenced from smallest to largest, as we prefer. We therefore swap the first and third principal moments of inertia, and correspondingly swap the first and third columns of $[e\_vecs]$. However, swapping a pair of columns of a 3×3 matrix reverses the sign of its determinant, which brings to the fore the question of whether the transformation initially computed by Matlab was right-handed. The raw output previously listed has a determinant of -1, so the result of swapping columns one and three yields the correct result. After these steps, the transformation matrix $[R]$ is readily obtained by recalling that the individual eigenvectors constituting the columns of $[e\_vecs]$ are the rows of $[R]$, so that $[R] = [e\_vecs]^{\mathrm{T}}$. After these adjustments, the results obtained from Matlab are

$$I_1 = 0.3814,\ I_2 = 0.9327,\ I_3 = 1.2059 \text{ kg-m}^2, \qquad \triangleleft$$

$$[R] = \begin{bmatrix} -0.0836 & -0.3181 & 0.9444 \\ -0.3322 & 0.9023 & 0.2746 \\ -0.9395 & -0.2908 & -0.1811 \end{bmatrix}. \qquad \triangleleft$$

An excellent check for any solution, whether it is obtained by solving the characteristic equation or from computer software, is to verify that $[R][I][R]^{\mathrm{T}}$ yields a diagonal array whose nonzero elements are the principal moments of inertia in the correct sequence.

Mathcad uses two functions to solve the eigenvalue problem corresponding to a specified 3×3 array I. The principal moments of inertia are computed by `I`$_{\mathtt{princ}}$`:= eigenvals(I);` and the eigenvectors are obtained from `e`$_{\mathtt{princ}}$`:= eigenvecs(I);` where `I`$_{\mathtt{princ}}$ is a three-element column vector of principal moments of inertia, and `e`$_{\mathtt{princ}}$ is a 3×3 array whose columns are the eigenvectors sequenced to match `I`$_{\mathtt{princ}}$. The result is

$$\{I_{\text{princ}}\} = \begin{Bmatrix} 1.2059 \\ 0.9327 \\ 0.3814 \end{Bmatrix}, \quad [e_{\text{princ}}] = \begin{bmatrix} 0.9395 & 0.3322 & -0.0836 \\ 0.2908 & -0.9023 & -0.3181 \\ 0.1811 & -0.2746 & 0.9444 \end{bmatrix}.$$

The eigenvectors returned by Mathcad's algorithm, like Matlab's, have the orthonormal property. Mathcad, like Matlab, does not address the handedness of the transformation, and it does not necessarily sequence the principal moments of inertia from smallest to largest. The preceding values are like the raw output from Matlab, except that the first two columns of $[e_{\text{princ}}]$ are multiplied by -1 relative to their Matlab correspondences, so $|[e_{\text{princ}}]| = -1$. Because the principal moments of inertia are reversed from the desired order, as they were in Matlab, we swap the first and third values of $\{I_{\text{princ}}\}$ and swap the first and third columns of $[e_{\text{princ}}]$. This leads to $|[e_{\text{princ}}]| = +1$, as required. The result is that the principal moments of inertia are the same as those obtained from Matlab, whereas the last two rows of $[R]$ obtained from Mathcad are the negative of the Matlab values.

5.3 RATE OF CHANGE OF ANGULAR MOMENTUM

The angular momentum of a rigid body is used in several contexts. Our focus here is its usage to form Eq. (5.1.22), which describes the effect of the resultant moment. The angular momentum was shown in Eq. (5.2.3) to be describable in terms of the body's angular velocity and inertia properties. Evaluation of the inertia matrix was discussed in detail in the preceding section, and the techniques for evaluating angular velocity should be familiar by now. We now turn to the task of taking the time derivative of the angular momentum.

A careful reading of the development thus far will reveal that we have not fully defined how the xyz reference frame rotates. To use the simplified moment equation, we stipulated that point A for the moment sum shall be an allowable point. We then placed the origin of the xyz coordinate system, which was used to describe the position of each mass element, at point A. These conditions do not address how xyz rotates. We remove this ambiguity by imposing a restriction that expedites differentiating $\bar{H}_A$.

In the context of kinematics, xyz has played the role of a global coordinate system. We use it to describe the body's distribution of mass, as well as its angular velocity components. If xyz has an arbitrary angular velocity $\bar{\Omega}$, then the derivatives of its unit vectors are $d\bar{i}/dt = \bar{\Omega} \times \bar{i}$, *etc.* Further, because the body's angular velocity differs from $\bar{\Omega}$, the orientation of the body relative to xyz is not constant, so that the inertia properties are not constant. Consequently, every term in Eq. (5.2.3) might change with time. The time derivative of that equation therefore is

$$\boxed{\begin{aligned}\frac{d\bar{H}_A}{dt} &= \left[\frac{d}{dt}\left(I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z\right)\right]\bar{i} + \left[\frac{d}{dt}\left(I_{yy}\omega_y - I_{yx}\omega_x - I_{yz}\omega_z\right)\right]\bar{j} \\ &\quad + \left[\frac{d}{dt}\left(I_{zz}\omega_z - I_{zx}\omega_x - I_{zy}\omega_y\right)\right]\bar{k} + \bar{\Omega} \times \bar{H}_A.\end{aligned}} \tag{5.3.1}$$

Clearly, it would be preferable if the inertia properties were constant. This condition is achieved if the body's orientation relative to xyz is invariant. For this reason we introduce the requirement that

$$\boxed{xyx \text{ should be a body-fixed coordinate system.}}$$

In addition to simplifying evaluation of the derivatives in Eq. (5.3.1), such a requirement sets $\bar{\Omega} = \bar{\omega}$, so the only angular velocity we need to evaluate is that of the body. A third simplification that results is more subtle. Differentiating the components of $\bar{H}_A$ in accord with Eq. (5.3.1) results in the occurrence of terms containing $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z$, which suggests that we need to describe the components of $\bar{\omega}$ as functions of time. In contrast, the procedures we established for evaluating angular acceleration did not require such a description. When $\bar{\omega}$ is described in terms of components relative to coordinate system xyz that rotates at $\bar{\Omega}$, its time derivative is given by

$$\bar{\alpha} = \frac{d\bar{\omega}}{dt} = \frac{\partial \bar{\omega}}{\partial t} + \bar{\Omega} \times \bar{\omega} \equiv \left(\dot{\omega}_x \bar{i} + \dot{\omega}_y \bar{j} + \dot{\omega}_z \bar{k}\right) + \bar{\Omega} \times \bar{\omega}. \tag{5.3.2}$$

By deciding to attach xyz to the body, we always have $\bar{\Omega} = \bar{\omega}$, in which case the partial and total derivatives of $\bar{\omega}$ are identical, so that

$$\bar{\alpha} = \frac{\partial \bar{\omega}}{\partial t} \implies \alpha_p = \dot{\omega}_p, \quad p = x, y, z \text{ if } xyz \text{ is body fixed.} \tag{5.3.3}$$

In other words, the angular acceleration components are the rates of change of the angular velocity components when those components are relative to body-fixed axes.

The overall consequence of requiring that xyz be body fixed is that differentiation of $\bar{H}_A$ becomes reasonably straightforward. We apply the partial derivative technique in Eq. (3.3.15) to the angular momentum. In that equation $\bar{\omega}$ was the angular velocity of the reference frame. It now also is the angular velocity of the body, so we have

$$\boxed{\dot{\bar{H}}_A = \frac{\partial \bar{H}_A}{\partial t} + \bar{\omega} \times \bar{H}_A.} \tag{5.3.4}$$

The components of $\partial \bar{H}_A / \partial t$ are the derivatives of the respective components of $\bar{H}_A$, which are indicated by Eq. (5.2.3) to consist of products of inertia coefficients and $\bar{\omega}$ components. The former are constant, and derivatives of the latter are described by Eq. (5.3.3). Thus,

$$\boxed{\begin{aligned} \frac{\partial \bar{H}_A}{\partial t} &= \left(I_{xx}\alpha_x - I_{xy}\alpha_y - I_{xz}\alpha_z\right)\bar{i} + \left(I_{yy}\alpha_y - I_{yx}\alpha_x - I_{yz}\alpha_z\right)\bar{j} \\ &\quad + \left(I_{zz}\alpha_z - I_{zx}\alpha_x - I_{zy}\alpha_y\right)\bar{k}. \end{aligned}} \tag{5.3.5}$$

It is evident that evaluation of $d\bar{H}_A/dt$ requires prior determination of $[I]$ for the body and analysis of $\bar{\omega}$ and $\bar{\alpha}$ for the body's motion. These quantities are substituted into Eq. (5.2.3) for $\bar{H}_A$ and the preceding equation for $\partial \bar{H}_A / \partial t$, which are then assembled to form Eq. (5.3.4). The matrix representation of $\bar{H}_A$ in Eq. (5.2.5) leads to a compact representation of the full expression:

$$\boxed{\left\{\dot{H}_A\right\} = [I]\{\alpha\} + \{\omega\} \otimes \left([I]\{\omega\}\right),} \tag{5.3.6}$$

where $\otimes$ denotes the matrix implementation of a cross product.

If xyz are principal axes, there are fewer terms to compute in $\bar{H}_A$ and $\partial \bar{H}_A/\partial t$. This simplifies the vector form of the various quantities, with the result that

$$\begin{aligned}\dot{\bar{H}}_A &= [I_{xx}\alpha_x - (I_{yy} - I_{zz})\,\omega_y\omega_z]\,\bar{i} + [I_{yy}\alpha_y - (I_{zz} - I_{xx})\,\omega_x\omega_z]\,\bar{j} \\ &\quad + [I_{zz}\alpha_z - (I_{xx} - I_{yy})\,\omega_x\omega_y]\,\bar{k}.\end{aligned} \tag{5.3.7}$$

This standard form is known as *Euler's equations.* Their simpler appearance makes it quite attractive to select xyz as principal axes. However, it is important to realize that the orientation of these axes affects the ease with which $\bar{\omega}$ and $\bar{\alpha}$ are determined. Also, as we have seen, principal axes have no special orientation relative to a nonsymmetrical body, in which case their identification requires additional mathematical operations. This leads to the recognition that

$$\boxed{\begin{array}{c}\text{It is best to select the } xyz \text{ reference frame based} \\ \text{on its suitability for describing } [I]\,,\ \bar{\omega}, \text{ and } \bar{\alpha}.\end{array}}$$

If our choice for xyz happens to correspond to principal axes, then we may use Eq. (5.3.7).

Many mechanical systems feature axisymmetric bodies that spin about their axis of symmetry. Any coordinate system having an axis that coincides with the axis of symmetry represents a principal set of axes. In Fig. 5.8 xyz is attached to axisymmetric body 1 such that its z axis coincides with the axis of symmetry. The spinning rotation $\dot{\phi}$ occurs about the z axis relative to body 2, whose angular velocity is an arbitrary quantity $\bar{\Omega}$. Thus the angular velocity of body 1, and of xyz, is

$$\bar{\omega} = \bar{\Omega} + \dot{\phi}\bar{k}. \tag{5.3.8}$$

In the figure $x'y'z'$ is introduced as a reference frame that is attached to body 2 such that its z' axis also coincides with the axis of symmetry of body 1. This leaves x' and y' axes unspecified, so we may orient those axes to facilitate description of $\bar{\Omega}$. For example, we might choose to let the y' axis coincide with the line of nodes for nutation in an Eulerian angle formulation.

There are two ways in which we may use $x'y'z'$ to formulate the problem. In the first we observe that, at any instant, there is a body-fixed coordinate system that is parallel to $x'y'z'$. This is evident from the fact that the reference line used to measure ϕ may be defined arbitrarily. Thus, in this special circumstance, using $x'y'z'$ as a global coordinate

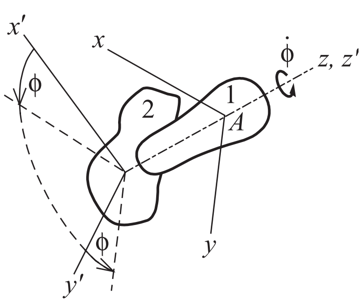


Figure 5.8. Reference frames for describing the rotation of axisymmetric body 1 relative to arbitrarily rotating body 2.

system leads to a description of vectors that is valid at any instant. In other words we may consider $x'y'z'$ to be the body-fixed xyz coordinate system that is situated at $\phi = 0$ at the instant of interest. This leads to a general procedure for axisymmetric bodies:

> To evaluate $d\bar{H}_A/dt$ for a body spinning about its axis of the symmetry, we may define the body-fixed xyz reference frame such that one axis coincides with the axis of symmetry. Then an expression for $d\bar{H}_A/dt$ that is valid at any instant will be derived, regardless of how the other axes of xyz are aligned. Thus the other axes should be aligned instantaneously to expedite the component description of $\bar{\omega}$ and $\bar{\alpha}$.

The invariance of the inertia properties with respect to angle ϕ has led some to employ an alternative formulation of $d\bar{H}_A/dt$ based on truly using $x'y'z'$ as a global coordinate system. Evaluating $d\bar{H}_A/dt$ in this approach requires that we return to Eq. (5.2.3). We form $\bar{\omega}$ according to Eq. (5.3.8). Because the z' axis always coincides with the axis of symmetry of body 2, it must be that $x'y'z'$ constitute principal axes regardless of ϕ. Thus the angular momentum of body 1 is

$$\bar{H}_A = I_{xx}\Omega_{x'}\bar{i}' + I_{yy}\Omega_{y'}\bar{j}' + I_{zz}\left(\Omega_{z'} + \dot{\phi}\right)\bar{k}'. \tag{5.3.9}$$

Because the components in this expression are relative to $x'y'z'$, the unit vector derivatives are described by $\bar{\Omega} \times \bar{e}'$. Thus the total and partial derivatives are related by

$$\boxed{\dot{\bar{H}}_A = \frac{\partial \bar{H}_A}{\partial t} + \bar{\Omega} \times \bar{H}_A.} \tag{5.3.10}$$

Differentiation of the component representation of $\bar{H}_A$ given by Eq. (5.3.9) leads to

$$\boxed{\frac{\partial \bar{H}_A}{\partial t} = I_{xx}\dot{\Omega}_{x'}\bar{i}' + I_{yy}\dot{\Omega}_{y'}\bar{j}' + I_{zz}\left(\dot{\Omega}_{z'} + \ddot{\phi}\right)\bar{k}'.} \tag{5.3.11}$$

As we did previously, we observe that, because $\Omega_{x'}$, $\Omega_{y'}$, and $\Omega_{z'}$ are the angular velocities of a reference frame relative to its own axes, their derivatives are the components of the angular acceleration of $x'y'z'$, that is,

$$\bar{\alpha}' = \dot{\Omega}_{x'}\bar{i}' + \dot{\Omega}_{y'}\bar{j}' + \dot{\Omega}_{z'}\bar{k}'.$$

The result of assembling these expressions is

$$\begin{aligned}\dot{\bar{H}}_A = {} & \left[I_{xx}\alpha'_x - \left(I_{yy} - I_{zz}\right)\Omega_{y'}\left(\Omega_{z'} + \dot{\phi}\right)\right]\bar{i} + \left[I_{yy}\alpha'_{y'} - \left(I_{zz} - I_{xx}\right)\Omega_{x'}\left(\Omega_{z'} + \dot{\phi}\right)\right]\bar{j} \\ & + \left[I_{zz}\left(\alpha'_{z'} + \ddot{\phi}\right) - \left(I_{xx} - I_{yy}\right)\omega_{x'}\omega_{y'}\right]\bar{k}.\end{aligned} \tag{5.3.12}$$

These are sometimes referred to as the *modified Euler equations.* The primary advantage of this approach lies in the fact that it makes the role of the spinning motion apparent. Countering it is the fact that the procedure is not generally applicable and therefore prone to misapplication. In contrast, the fundamental methodology is valid regardless of the inertia properties.

In relatively simple circumstances featuring axisymmetric bodies, yet another method may be available to evaluate $d\bar{H}_A/dt$. It is mostly employed to explain phenomena qualitatively, but sometimes it is suitable for quantitative analysis. Suppose we evaluate the angular momentum according to Eq. (5.2.3) by using a body-fixed xyz reference frame. That description should be representative of an arbitrary position of the body. Depending on the motion it might be that an alternative global coordinate system $\hat{x}\hat{y}\hat{z}$ executing fewer rotations than xyz is equally suitable for describing the components of $\bar{H}_A$. Then, either by graphical projections or through a rotation transformation, we can write

$$\bar{H}_A = (H_A)_{\hat{x}}\,\bar{e}_{\hat{x}} + (H_A)_{\hat{y}}\,\bar{e}_{\hat{y}} + (H_A)_{\hat{z}}\,\bar{e}_{\hat{z}}. \tag{5.3.13}$$

We consider each component individually. If we are following a pictorial approach, we evaluate the derivative of each component as the sum of a term parallel to the unit vector that is due to the change of its magnitude plus the change perpendicular to the unit vector resulting from its rotation at the angular velocity $\hat{\omega}$ of $\hat{x}\hat{y}\hat{z}$. Mathematically, this is equivalent to writing

$$\dot{\bar{H}}_A = \sum_{\hat{p}=\hat{x},\hat{y},\hat{z}} \left[\left(\dot{H}_A\right)_p \bar{e}_{\hat{p}} + (H_A)_{\hat{p}}\,\hat{\omega}\times\bar{e}_{\hat{p}}\right]. \tag{5.3.14}$$

This representation might not seem to be easier than any of the other approaches developed thus far, but that depends on the characteristics of the system and the choice for a global coordinate system. To explore this, consider the situation in Fig. 5.9, in which a disk sander spinning at ω_1 must be rotated at angular speed ω_2 about the handle, which is transverse to the disk's centerline.

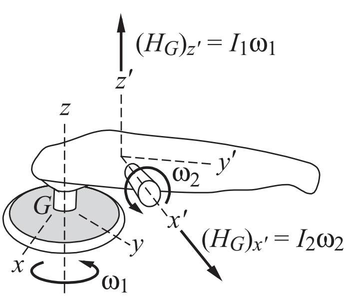


Figure 5.9. Disk sander illustrating the evaluation of the rate of change of angular momentum of a spinning axismmetric body.

Our focus is on the angular momentum of the rotating disk about its center of mass G. Let z denote the body-fixed axis for ω_1 and z' be a parallel axis for a reference frame attached to the motor housing. The x' axis is aligned with the handle, so it is the axis about which ω_2 occurs. We make use of the simplification afforded by axisymmetry by defining the body-fixed centroidal x axis at the instant of interest to be parallel to x'. Because xyz are principal axes, the angular momentum at any instant is correspondingly indicated by Eq. (5.2.3) to be

$$\bar{H}_G = I_2\omega_2\bar{i} + I_1\omega_1\bar{k}, \tag{5.3.15}$$

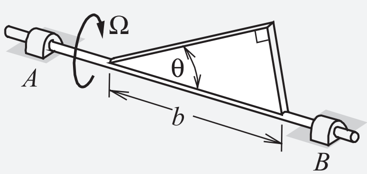
where I_1 and I_2 are respectively the centroidal moments of inertia of the disk about its centerline and any transverse axis. This expression is generally valid. It follows that we may replace the unit vectors in Eq. (5.3.15) with those of $x'y'z'$, so at any instant the two components of angular momentum are as depicted in Fig. 5.9

We know that $\bar{i}'$ is stationary, whereas $\bar{k}'$ rotates at $\omega_2\bar{i}'$. Furthermore, the values of the moments of inertia and rotation rates are constant. Thus, in the figure the $\bar{i}'$ component of $\bar{H}_G$ is constant, and the sole variability of the $\bar{k}'$ component of $\bar{H}_G$ is the changing direction of $\bar{k}'$. We can visualize that the tip of $I_1\omega_1\bar{k}'$ moves in the $-\bar{j}'$ direction, so we can anticipate that $d\bar{H}_G/dt$ will be oriented in that direction. This is confirmed by the fact that the only variable in Eq. (5.3.15) is $\bar{k}'$, whose derivative is $\omega_2\bar{i}' \times \bar{k}'$. Thus we have

$$\dot{\bar{H}}_A = -I_2\omega_1\omega_2\bar{j}'.$$

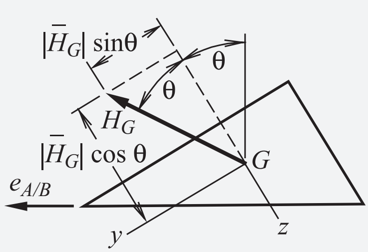
The moment equation of motion requires that a couple equal to $d\bar{H}_G/dt$ be applied to the rotor. This must ultimately be applied by the operator to the sander's handle. Thus a moment perpendicular to both rotation axes is required to sustain the motion, which is not what an inexperienced operator would anticipate.

EXAMPLE 5.10 The right triangular plate is welded along its hypotenuse to a shaft that rotates at the variable rate Ω. Determine $d\bar{H}_G/dt$ for the plate. For the special case in which Ω is constant, predict which way the dynamic reactions generated at the bearings will be oriented.



Example 5.10

SOLUTION One purpose of this example is to emphasize that bodies rotating about a fixed axis sometimes cannot be fully described by a planar formulation. Also, the simplicity of this system will enable us to assess fully the analytical results from a qualitative viewpoint. The axes of the body-fixed xyz coordinate system depicted in the sketch match those described in the Appendix for a right triangular prism whose width is zero.



Angular momentum components of the spinning plate relative to the body-fixed coordinate system defined in the Appendix.

The tabulation gives

$$I_{xx} = \frac{1}{18}mb^2, \quad I_{yy} = \frac{1}{18}mb^2 (\sin\theta)^2, \quad I_{zz} = \frac{1}{18}mb^2 (\cos\theta)^2,$$

$$I_{xy} = I_{yx} = I_{xz} = I_{zx} = 0, \quad I_{yz} = I_{zy} = -\frac{1}{36}mb^2 \sin\theta\cos\theta.$$

It is given that Ω is not constant, so the angular velocity and angular acceleration are parallel to the shaft. We resolve both quantities into components with respect to xyz, so

$$\bar{\omega} = \Omega\bar{e}_{A/B} = \Omega\left(\cos\theta\,\bar{j} - \sin\theta\bar{k}\right),$$

$$\bar{\alpha} = \dot{\Omega}\bar{e}_{A/B} = \dot{\Omega}\left(\cos\theta\,\bar{j} - \sin\theta\bar{k}\right).$$

Substitution of the inertia values and the components of $\bar{\omega}$ and $\bar{\alpha}$ into Eqs. (5.2.3) and (5.3.5) leads to

$$\bar{H}_G = \frac{1}{36}mb^2\Omega\sin\theta\cos\theta\left(\sin\theta\,\bar{j} - \cos\theta\bar{k}\right),$$

$$\frac{\partial\bar{H}_G}{\partial t} = \frac{1}{36}mb^2\dot{\Omega}\sin\theta\cos\theta\left(\sin\theta\,\bar{j} - \cos\theta\bar{k}\right).$$

Correspondingly, Eq. (5.3.4) yields

$$\dot{\bar{H}}_G = \frac{1}{36}mb^2\Omega^2\sin\theta\cos\theta\left[-(\cos\theta)^2 + (\sin\theta)^2\right]\bar{i}$$
$$+\frac{1}{36}mb^2\dot{\Omega}\sin\theta\cos\theta\left(\sin\theta\,\bar{j} - \cos\theta\bar{k}\right),$$

which is identically

$$\dot{\bar{H}}_G = \frac{1}{72}mb^2\sin 2\theta\left[-\Omega^2\cos 2\theta\bar{i} + \dot{\Omega}\left(\sin\theta\,\bar{j} - \cos\theta\bar{k}\right)\right]. \qquad \triangleleft$$

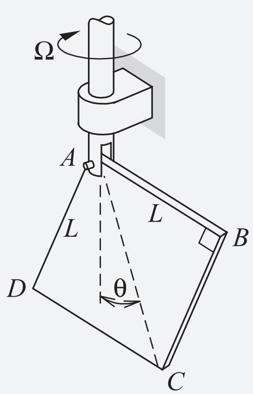
It is evident from this result that the forces acting on the plate must exert a moment about point G to sustain the rotation, even if Ω is constant. The only exception is $\theta = \pi/4$, in which case $d\bar{H}_G/dt = \bar{0}$ if $\dot{\Omega} = 0$.

To understand this result, the $\bar{H}_G$ vector appears in the sketch defining xyz. The preceding expression for $\bar{H}_G$ indicates that its direction cosines are $\sin\theta$ and $-\cos\theta$ with respect to the y and z axes, respectively. This leads to the observation that at any instant the angle between $\bar{H}_G$ and the shaft is $\pi/2 - 2\theta$, as shown in the sketch. Rotation of the system changes the direction of $\bar{H}_G$, but the angle between $\bar{H}_G$ and the shaft remains constant. Thus the rotation of $\bar{H}_G$ changes only the component of $\bar{H}_G$ perpendicular to the shaft, $(H_G)_\perp = \left|\bar{H}_G\right|\sin(\pi/2 - 2\theta) = (1/36)\,mb^2\Omega\sin\theta\cos\theta\cos(2\theta)$. The rotation causes the tip of this vector component to move into the plane of the sketch, which is the $-\bar{i}$ direction, and the magnitude of the rate of change is $\Omega\,(H_G)_\perp$, which matches the $\bar{i}$ component in the derived expression. The $\bar{j}$ and $\bar{k}$ components in that expression are proportional to $\dot{\Omega}$. An increase in the value of Ω increases the length of $\bar{H}_G$ in the sketch, which is manifested as terms in $d\bar{H}_G/dt$ that are parallel to $\bar{H}_G$, which are the $\bar{j}$ and $\bar{k}$ components of the result.

The fact that the portion of $d\bar{H}_G/dt$ that is independent of $\dot{\Omega}$ vanishes when $\theta = \pi/4$ is readily explained. In that case a centroidal coordinate system having one axis, say y', parallel to the shaft symmetrically bisects the plate, so it represents principal axes. Using these principal axes to form the angular momentum shows that $\bar{H}_G$ reduces to $I_{y'y'}\Omega\bar{j}'$. Thus in this case, $\bar{H}_G$ is always oriented parallel to the axis of rotation, whose orientation is constant.

When Ω is constant our analysis indicates that $d\bar{H}_G/dt$ is in the negative $\bar{i}$ direction, which is inward in the perspective of the sketch depicting $\bar{H}_G$. Thus the external forces acting on the system must exert a net moment that is clockwise relative to the sketch. Such a moment is provided by the bearings, which exert forces situated in the plane of the diagram. Their total moment about point G must equal $d\bar{H}_G/dt$. In addition, when Ω is constant, the center-of-mass acceleration is directed toward the shaft. The sum of the bearing forces must equal $m\bar{a}_G$, which also is situated in the plane of the sketch. From these observations we conclude that the bearing forces required to sustain a constant Ω act perpendicularly to the shaft in the plane of the plate. Both $\left|d\bar{H}_G/dt\right|$ and $\left|m\bar{a}_g\right|$ are proportional to Ω^2, which means that these bearing forces will increase as the square of the rotation rate. We say that these are the *dynamic reactions* because they are generated to support the rotation. In contrast the static reactions counter the gravity force. They act in the fixed vertical direction. The direction of the dynamic reactions rotates at Ω, so the components of the dynamic reactions in the fixed horizontal and vertical directions vary sinusoidally as $\cos(\Omega t)$ and $\sin(\Omega t)$. Thus we see that the system supporting the bearings will experience fluctuating forces having an amplitude that increases as Ω^2. Such forces could induce strong vibration in the supporting system.

EXAMPLE 5.11 The square plate is pinned at corner A to the vertical shaft, which rotates at the constant angular speed Ω. The angle θ is an arbitrary function of time. Determine $d\bar{H}_A/dt$ for the plate as a function of θ.



Example 5.11

SOLUTION The rotation in this example occurs about two axes, so this solution will give a more complete picture of the general procedures. Nevertheless, the motion is sufficiently simple that we will be able to understand the physical meaning of the result. To form $\bar{H}_A$, the origin of xyz must coincide with point A. We align the x axis with edge AD and the y axis with edge AB, which are parallel to the centroidal axes described in the Appendix. The inertia properties obtained from the parallel axis theorems are

$$I_{xx} = I_{yy} = \frac{1}{3}mL^2, \quad I_{zz} = \frac{2}{3}mL^2,$$
$$I_{xy} = \frac{1}{4}mL^2, \quad I_{xz} = I_{yz} = 0.$$

The Ω rotation occurs about the fixed vertical axis, which we designate as $\bar{K}$. The θ rotation occurs about the pin, whose axis is perpendicular to the plate; this is the body-fixed z axis. The general expressions for the angular velocity of xyz and of the reference frames for the rotation directions are

$$\bar{\omega} = \Omega\bar{e}_1 + \dot{\theta}\bar{e}_2, \quad \bar{e}_1 = \bar{K}, \quad \bar{\Omega}_1 = \bar{0}, \quad \bar{e}_2 = \bar{k}, \quad \bar{\Omega}_2 = \bar{\omega}.$$

The angular acceleration corresponding to constant Ω and variable $\dot{\theta}$ are

$$\bar{\alpha} = \ddot{\theta}\bar{e}_2 + \dot{\theta}\left(\bar{\omega} \times \bar{e}_2\right).$$

It is convenient to let $\gamma = \pi/4 + \theta$, so that

$$\bar{\omega} = \Omega\left(\sin\gamma\,\bar{i} + \cos\gamma\,\bar{j}\right) + \dot{\theta}\bar{k},$$
$$\bar{\alpha} = \Omega\dot{\theta}\left(\cos\gamma\,\bar{i} - \sin\gamma\,\bar{j}\right) + \ddot{\theta}\bar{k}.$$

Equation (5.2.3) for the present rotation gives

$$\begin{aligned}\bar{H}_A &= (I_{xx}\omega_x - I_{xy}\omega_y)\,\bar{i} + (I_{yy}\omega_y - I_{yx}\omega_x)\,\bar{j} + I_{zz}\omega_x\bar{k} \\ &= mL^2\left[\Omega\left(\frac{1}{3}\sin\gamma - \frac{1}{4}\cos\gamma\right)\bar{i} + \Omega\left(\frac{1}{3}\cos\gamma - \frac{1}{4}\sin\gamma\right)\bar{j} + \frac{2}{3}\dot{\theta}\bar{k}\right],\end{aligned}$$

and the corresponding expression for the body-fixed derivative of $\bar{H}_A$ is

$$\begin{aligned}\frac{\partial\bar{H}_A}{\partial t} &= (I_{xx}\alpha_x - I_{xy}\alpha_y)\,\bar{i} + (I_{yy}\alpha_y - I_{yx}\alpha_x)\,\bar{j} + I_{zz}\alpha_x\bar{k} \\ &= mL^2\left[\Omega\dot{\theta}\left(\frac{1}{3}\cos\gamma + \frac{1}{4}\sin\gamma\right)\bar{i} + \Omega\dot{\theta}\left(-\frac{1}{3}\sin\gamma - \frac{1}{4}\cos\gamma\right)\bar{j} + \frac{2}{3}\ddot{\theta}\bar{k}\right].\end{aligned}$$

The rate at which angular momentum changes is correspondingly found to be

$$\begin{aligned}\dot{\bar{H}}_A &= \frac{\partial\bar{H}_A}{\partial t} + \bar{\omega} \times \bar{H}_A \\ &= mL^2\Omega\dot{\theta}\left(\frac{2}{3}\cos\gamma + \frac{1}{2}\sin\gamma\right)\bar{i} + mL^2\Omega\dot{\theta}\left(-\frac{2}{3}\sin\gamma - \frac{1}{2}\cos\gamma\right)\bar{j} \\ &\quad + mL^2\left[\frac{2}{3}\ddot{\theta} + \frac{1}{4}\Omega^2\left((\cos\gamma)^2 - (\sin\gamma)^2\right)\right]\bar{k}.\end{aligned}$$

The result was requested as a function of θ, which we can recover by setting $\gamma = \pi/4 + \theta$ in the trigonometric terms. The identities for the sine and cosine of a sum lead to

$$\begin{aligned}\dot{\bar{H}}_A = {} & \frac{\sqrt{2}}{12} mL^2 \Omega \dot{\theta} \left(7\cos\theta - \sin\theta\right) \bar{i} + \frac{\sqrt{2}}{12} mL^2 \Omega \dot{\theta} \left(-7\cos\theta - \sin\theta\right) \bar{j} \\ & + mL^2 \left[\frac{2}{3}\ddot{\theta} - \frac{1}{4}\Omega^2 \sin 2\theta\right] \bar{k}.\end{aligned} \quad \triangleleft$$

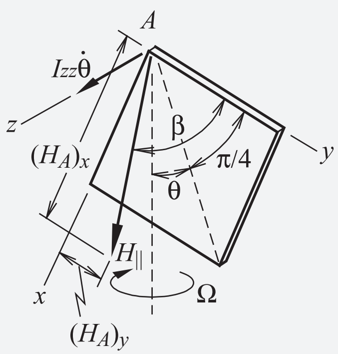
To understand this result it is useful to eliminate γ in the preceding expression for $\bar{H}_A$. Doing so yields

$$\bar{H}_A = \frac{\sqrt{2}}{24} mL^2 \Omega \left(\cos\theta + 7\sin\theta\right) \bar{i} + \frac{\sqrt{2}}{24} mL^2 \Omega \left(\cos\theta - 7\sin\theta\right) \bar{j} + \frac{2}{3} mL^2 \dot{\theta} \bar{k}.$$

We may separately examine the manner in which Ω and $\dot{\theta}$ cause $\bar{H}_A$ to change. Let us designate the portion of $\bar{H}_A$ that lies in the plane of the plate as $\bar{H}_{\|} = (H_A)_x \bar{i} + (H_A)_y \bar{j}$. When $\theta = 0$ the components of $\bar{H}_{\|}$ have equal magnitude. In that case $\bar{H}_{\|}$ is along the diagonal to corner C, which is aligned with vertical rotation axis. When $\theta > 0$, we see that $(H_A)_x > (H_A)_y$, which means that the angle from the y axis to $\bar{H}_{\|}$ exceeds $\pi/4$. In fact, this angle, which is given by

$$\beta = \tan^{-1}\left[\frac{(H_A)_x}{(H_A)_y}\right] = \tan^{-1}\left(\frac{\cos\theta + 7\sin\theta}{\cos\theta - 7\sin\theta}\right),$$

can be shown to be greater than $\gamma = \pi/4 + \theta$, so the angle from the y axis to $\bar{H}_{\|}$ exceeds the angle to the Ω rotation vector. Thus $\bar{H}_{\|}$ is situated as shown in the second sketch.



Angular momentum components parallel and perpendicular to the plane of the square plate.

Rotation about the vertical axis causes the tip of $\bar{H}_{\|}$ to move in the negative $\bar{k}$ direction. This portion combines with the term $I_{zz}\ddot{\theta}$ corresponding to the changing magnitude of $\bar{H}_A \cdot \bar{k}$ to produce the $\bar{k}$ component of $d\bar{H}_A/dt$.

This is not the sole effect of the Ω rotation on $\bar{H}_A$. The sketch shows $\bar{H}_A \cdot \bar{k} = I_{zz}\dot{\theta}$. Rotation about the vertical axis at Ω causes the tip to this vector to move in the horizontal direction, parallel to the plane of the plate. This is one of the effects producing the $\bar{i}$ and $\bar{j}$ components of $d\bar{H}_A/dt$. The other source of these components is the fact that changing θ changes $(H_A)_x$ and $(H_A)_y$.

The fact that two rotations are present in this system makes it more difficult than in the previous example to visualize the reasons why the angular momentum changes. As system complexity increases, we will come to rely on the analytical result, and only occasionally seek a physical explanation for some particularly interesting aspect.

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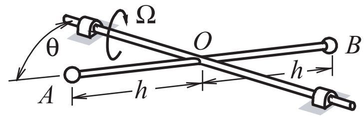
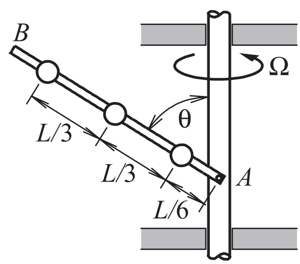
HOMEWORK PROBLEMS

EXERCISE 5.1 Spheres A and B are welded to a connecting bar of negligible mass that is mounted obliquely on a shaft that rotates at the constant rate Ω. The mass of the shaft also is negligible, so the center of mass O of this system is situated on the shaft. Thus the system is statically balanced. Evaluate the angular momentum of the two spheres about the center of mass when the connecting bar is horizontal and when the shaft has rotated by 1° from that position. The difference between the value of the angular momentum at these two positions is approximately $d\bar{H}_O$. Evaluate that difference, and from that result deduce the sense of the moment the bearing forces must exert to sustain a constant rotation rate.



Exercise 5.1

EXERCISE 5.2 The bar is pinned to the vertical shaft, whose rate of rotation is Ω. The mass distribution of the bar is represented by three small spheres having mass $m/3$ shown in the sketch. Consider the situation in which the bar is falling at $\dot{\theta} = 2\Omega$ at the instant when $\theta = 53.13^\circ$. Determine and sketch the angular velocity and angular momentum of the bar relative to pin A. Then use these quantities to evaluate the kinetic energy of the bar and verify that this result is the same as that obtained by adding the kinetic energy of each sphere.



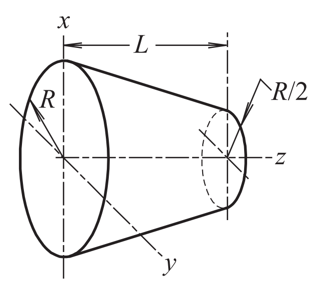
Exercise 5.2

EXERCISE 5.3 Use integration to determine the inertia properties of the truncated parallelepiped in Example 5.5 relative to the XYZ system defined there.

EXERCISE 5.4 Derive the centroidal location and centroidal inertia properties of a homogeneous prism, as tabulated in the Appendix.

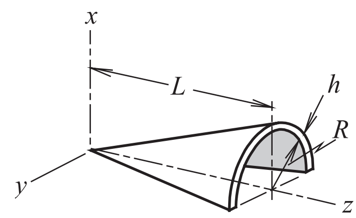
EXERCISE 5.5 Use integration to determine the inertia properties of the semicone appearing in the Appendix.

EXERCISE 5.6 The body in the sketch is a truncated cone. Use integration to determine its inertia properties relative to the xyz coordinate system in the sketch.



Exercise 5.6

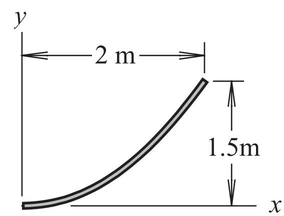
EXERCISE 5.7 A constant thickness shell is a body whose dimension measured perpendicular to its curved surface is an invariant value that is a small fraction of the other dimensions. Consider a semiconical shell whose thickness is h. The origin of xyz is placed at the apex and the z axis is aligned with the cone's axis. (a) Use cylindrical coordinates to evaluate the inertia properties. (b) Use spherical coordinates to evaluate the inertia properties.



Exercise 5.7

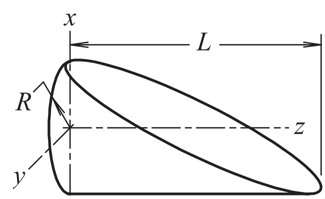
EXERCISE 5.8 Use integration to determine the inertia properties of a spheroid tabulated in the Appendix.

EXERCISE 5.9 The thin wire is bent to a parabolic shape such that its centerline is defined by the generic equation $y = kx^2$. The wire is steel, $\rho = 7800$ kg/m^3, and its circular cross section has a 20-mm diameter. Use numerical integration to determine its mass, the location of its center of mass, and its moments and products of inertia relative to the xyz coordinate system in the sketch.



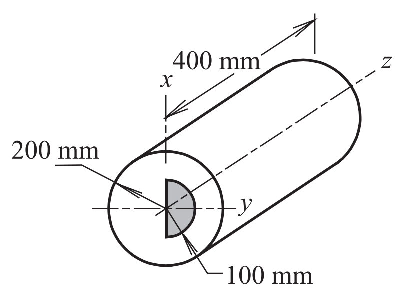
Exercise 5.9

EXERCISE 5.10 A cylinder is sliced in half along its diagonal. Determine the location of the center of mass and the inertia properties relative to a coordinate system whose z axis coincides with the axis of the cylinder and whose origin is situated at the circular end.



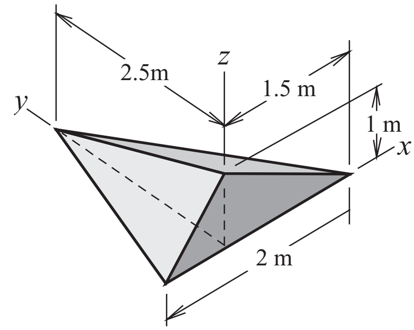
Exercise 5.10

EXERCISE 5.11 The semicircular cutout in the steel cylinder is filled with lead. Determine the centroidal location and the inertia properties of this body with respect to the centroidal coordinate system whose axes are parallel to the xyz system shown.



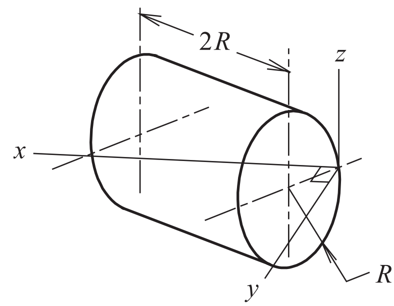
Exercise 5.11

EXERCISE 5.12 Use the inertia properties of an orthogonal tetrahedron given in the Appendix to determine the location of the center of mass and inertia properties of the tetrahedron relative to the xyz coordinate specified in the sketch.



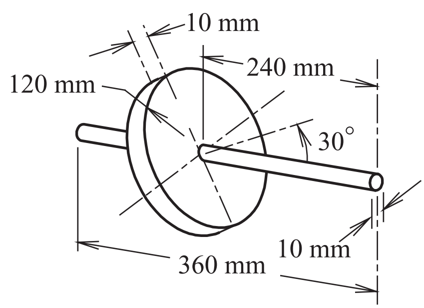
Exercise 5.12

EXERCISE 5.13 The x axis forms a diagonal intersecting the centroid of the homogeneous cylinder. Determine the inertia properties of the cylinder with respect to xyz.



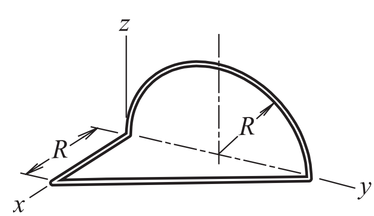
Exercise 5.13

EXERCISE 5.14 A rotor in the shape of a disk is welded obliquely to a shaft, such that its centerline does not coincide with the axis of the shaft. The mass of the shaft is 5 kg, and the rotor's mass is 20 kg. Determine the location of the center of mass of this assembly, then evaluate the centroidal moments and products of inertia for a coordinate system whose x axis coincides with the axis of the shaft.



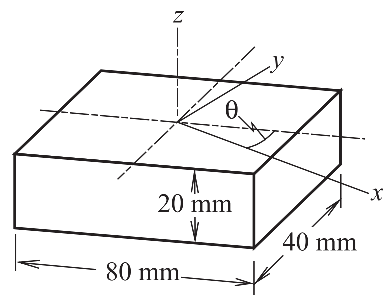
Exercise 5.14

EXERCISE 5.15 The mass per unit length of the wire frame is the constant value σ. Determine its inertia properties relative to the xyz coordinate system in the sketch.



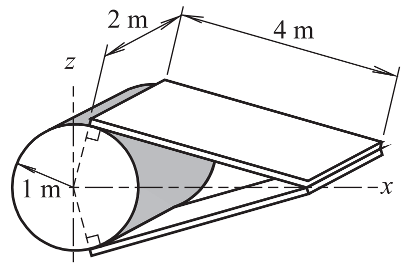
Exercise 5.15

EXERCISE 5.16 The origin of the xyz coordinate system coincides with the centroid of the upper face of the box. The mass of the box is 10 kg. Graph I_{xx}, I_{yy}, and I_{xy} as functions of θ. Also determine the values of θ for which $|I_{xy}|$ has its maximum and minimum values.



Exercise 5.16

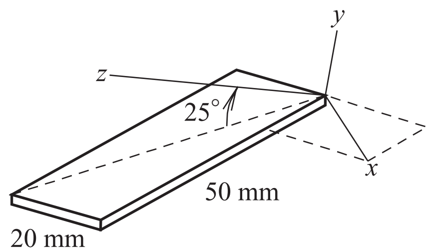
EXERCISE 5.17 Two metal plates are welded to the aluminum cylinder as shown in the sketch. Each plate has a mass of 20 kg, and the mass of the cylinder is 200 kg. Determine the location of the center of mass and the moments and products of inertia of this body relative to the xyz coordinate system shown in the sketch.



Exercise 5.17

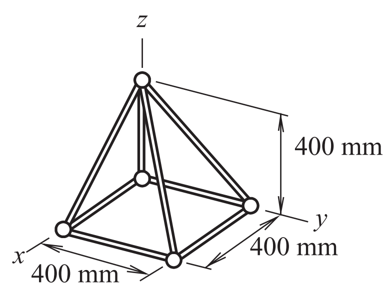
EXERCISE 5.18 The x axis lies in the plane of the 500-g plate, and the y axis is elevated at 25° above the diagonal. Determine the inertia matrix of the plate relative to the xyz coordinate system in the sketch.

EXERCISE 5.19 The mass of the plate is 500 g. Determine the principal moments of inertia relative to a coordinate system whose origin coincides with the origin of xyz depicted in the sketch.



Exercises 5.18 and 5.19

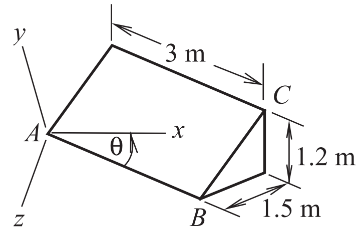
EXERCISE 5.20 A rigid body consists of five small spheres of mass m mounted at the corners of a lightweight wire frame in the shape of an orthogonal pyramid. Determine the principal moments of inertia and the direction angle between each principal axis and the xyz coordinate axes.



Exercise 5.20

EXERCISE 5.21 The y axis is normal to the inclined face of the 400-kg homogeneous prism. Consider the situation in which the x axis coincides with diagonal AC. Determine the inertia matrix of the prism relative to this coordinate system.

EXERCISE 5.22 The y axis is normal to the inclined face of the 400-kg homogeneous prism. Is there any value of the angle θ for which the product of inertia $I_{xz} = 0$? If so, what is the angle and what is the corresponding inertia matrix?



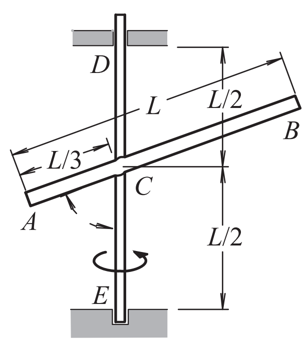
Exercises 5.21, 5.22, and 5.23

EXERCISE 5.23 The prism's mass is 400 kg. Determine the principal moments of inertia with respect to a coordinate system whose origin is coincident with the origin of xyz in the sketch. Also determine the rotation transformation for the principal axes relative to coordinate axes aligned with the orthogonal edges.

EXERCISE 5.24 Consider the tetrahedron in Exercise 5.12. Determine the principal moment of inertia for the origin specified there. Also determine the direction angle of the principal axes relative the given xyz coordinate system.

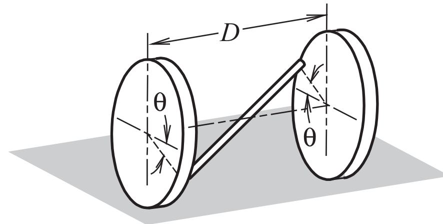
EXERCISE 5.25 In the case of a flat plate having an arbitrary shape, any coordinate system whose xy plane coincides with the midplane is a plane of symmetry, so that I_{xz} and I_{yz} are identically zero. In such situations the principal axes are obtained by a rotation transformation by angle θ about the z axis, which one can evaluate by considering only the inertia properties with respect to the xy plane. A Mohr's circle construction, typically encountered in stress analysis, may be used to perform such an evaluation. Derive expressions for $I_{x'x'}$, $I_{y'y'}$, and $I_{x'y'}$ corresponding to arbitrary values of I_{xx}, I_{yy}, and I_{xy}. Show that if the points $(I_{x'x'}, I_{x'y'})$ and $(I_{y'y'}, -I_{x'y'})$ are plotted relative to orthogonal axes, the distance between these points is constant, regardless of θ. From this, prove that the points lie on a circle. Then explain the significance of the angle from the abscissa to the line connecting the plotted points. Also explain how the principal moments of inertia may be evaluated from the properties of the circle.

EXERCISE 5.26 Thin bar ACB is welded to a shaft that rotates at the constant angular speed Ω, so the angle θ between the bar and the shaft is constant. (a) Derive expressions for the angular momentum $\bar{H}_C$ and the kinetic energy of the bar. Draw a sketch of $\bar{H}_C$. (b) Based on an analysis of the manner in which $\bar{H}_C$ in Part (a) rotates, derive an expression for $d\bar{H}_C/dt$. (c) Use Eq. (5.3.4) to evaluate $d\bar{H}_C/dt$, and compare it with the result of Part (b).



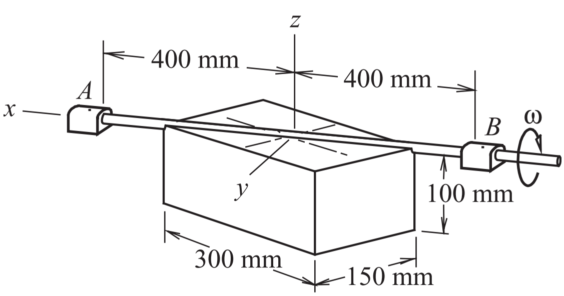
Exercise 5.26

EXERCISE 5.27 Two identical disks having radius R and mass m are welded together by a rigid bar. The ends of the connecting bar are welded at diametrically opposite points on the perimeter of each wheel. The mass of the rod also is m. The disks roll without slipping at constant speed v. Derive an expression for the angular momentum of this assembly about its center of mass as a function of the angle θ for the rolling. Depict this quantity in a sketch for $\theta = 0$ and $\theta = 90^{\circ}$, and describe the corresponding friction and normal forces that are required to maintain the steady rolling. *Hint*: Use a body-fixed coordinate system whose y axis aligns with the disks' center line, and whose z axis is perpendicular to the bar.



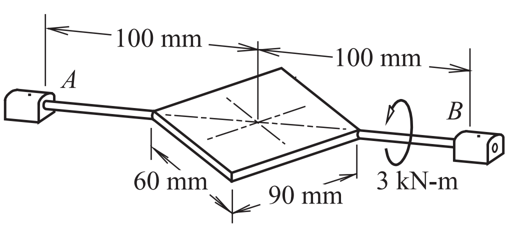
Exercise 5.27

EXERCISE 5.28 The 24-kg block is welded to a shaft that rotates about bearings A and B at a constant rate ω. The shaft is collinear with the diagonal to a face of the block. Determine the inertia properties of the block relative to the xyz coordinate system in the sketch, whose x axis coincides with the shaft and whose z axis is normal to the face of the block. Also determine the inertia properties for a set of principal axes sharing the same origin as xyz. Then evaluate the angular momentum about the origin of xyz and the kinetic energy of the block by using each set of inertia properties.



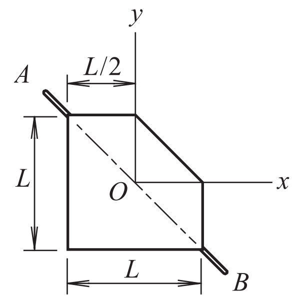
Exercise 5.28

EXERCISE 5.29 A 200-g rectangular plate is mounted diagonally on a shaft whose mass is negligible. The system has an unsteady rotation rate Ω. Determine the force–couple system acting at bearing B that is equivalent to the external forces that must be applied to attain such a motion. What portion of that couple represents the torque required to change Ω?



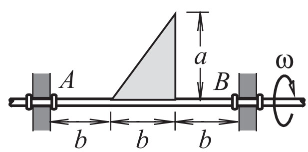
Exercise 5.29

EXERCISE 5.30 The sketch shows a steel plate whose thickness perpendicular to the plane of the diagram is very small. The mass per unit surface area of this plate is σ. The plate rotates about axis AB at constant angular speed ω. (a) Determine the mass and the location of the center of mass of this plate. (b) Determine I_{xx}, I_{yy}, and I_{xy} for the xyz coordinate system shown in the sketch. (c) Determine $\bar{H}_O$ and $d\bar{H}_O/dt$ for the plate about origin O on the axis of rotation. Draw a sketch $\bar{H}_O$ and explain the sense of $d\bar{H}_O/dt$ by considering how $\bar{H}_O$ rotates. (d) Predict the direction of the dynamic reactions exerted by bearings at ends A and B.



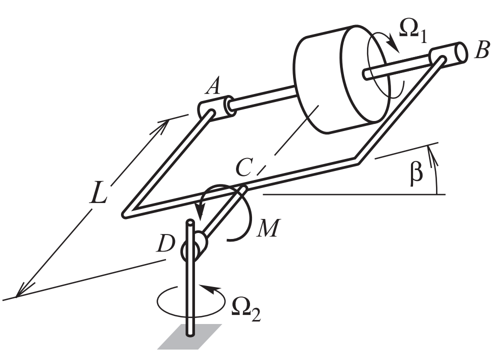
Exercise 5.30

EXERCISE 5.31 The right triangular plate is welded to the shaft, which rotates at constant speed ω. Determine the force–couple acting at bearing A that is equivalent to the force system the bearings must exert to sustain this motion.



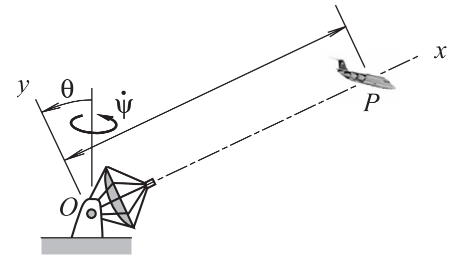
Exercise 5.31

EXERCISE 5.32 The gyroscopic turn indicator consists of a 1-kg flywheel whose principal radii of gyration are $\kappa_x = 50$ mm, $\kappa_y = \kappa_z = 35$ mm, where the x axis is the flywheel's axis of symmetry. The center of mass of the flywheel coincides with the intersection of axes AB and CD. The flywheel spins relative to the gimbal at the constant rate $\Omega_1 =$ 50 000 rev/min, while the whole system rotates about the vertical axis at $\Omega_2 = 1.2$ rad/s. The pivot angle β is constant. Determine the force–couple system acting on the flywheel at its center that is equivalent to the actual forces acting on it, assuming that only the mass of the flywheel is significant. What portion of this couple represents the torque M that must be applied to shaft CD?



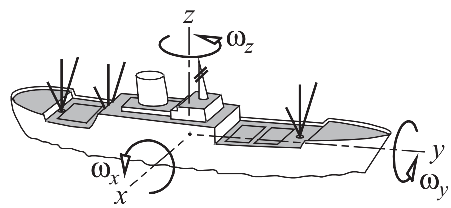
Exercise 5.32

EXERCISE 5.33 The radar antenna tracks airplane P by rotating about the vertical axis at $\dot{\psi}$ while the elevation angle θ is adjusted. Assuming that the body-fixed xyz axes are principal, what is the force–couple system acting at the stationary pivot O that must be applied to overcome the inertial resistance when ψ and θ are arbitrary time functions?



Exercise 5.33

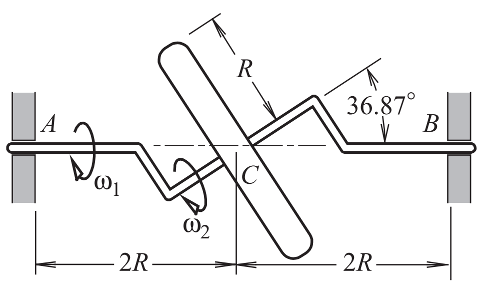
EXERCISE 5.34 The rotation rates of the ship with respect to the body-fixed centroidal xyx coordinate system in the sketch are the pitch ω_x, roll ω_y, and yaw ω_z. Consider the case in which these rates simultaneously attain their maximum magnitudes, with $\omega_x = 0.5$ rad/s, $\omega_y = -1.1$ rad/s, $\omega_z = 0.2$ rad/s. The accelerations of the center of mass at this instant are $a_x = 5$ m/s^2, $a_y = -12$ m/s^2, $a_z = 15$ m/s^2. The mass of the ship is $40\left(10^6\right)$ kg, and the radii of gyration are $\kappa_x = 90$ m, $\kappa_y = 10$ m, $\kappa_z = 15$ m; it may be assumed that xyz are principal axes. Determine the force–couple system acting at the origin of xyz that is equivalent to the forces exerted on the ship by the ocean.



Exercise 5.34

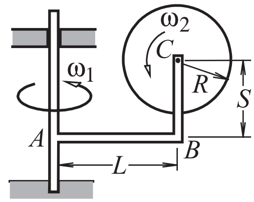
EXERCISE 5.35 A very thin circular disk rolls without slipping relative to the ground such that its plane is oriented vertically throughout the motion. Consider the situation in which the center C of the disk follows a circular path of radius ρ. Determine $\bar{H}_C$ and $d\bar{H}_C/dt$. From those results explain why the condition that the plane of the disk be vertical can be satisfied only if the center follows a straight path, unless forces other than gravity and the contact force are exerted on the disk.

EXERCISE 5.36 The device shown is a wobble plate, in which the precession rate ω_1 of shaft AB and the spin rate ω_2 of the disk relative to the shaft are constant. The mass of the shaft is negligible. Let λ denote the ratio of the angular speeds, such that $\omega_2 = \lambda\omega_1$. (a) In terms of ω_1 and λ, derive expressions for the angular velocity, angular momentum H_C relative to the center of the disk, and $d\bar{H}_C/dt$. (b) Evaluate the results in Part (a) for the case in which $\lambda = 3$, and draw a sketch depicting these quantities. Determine the magnitude of each and the angle between each quantity and the bearing axis AB. (c) Determine whether there is any value of λ for which no



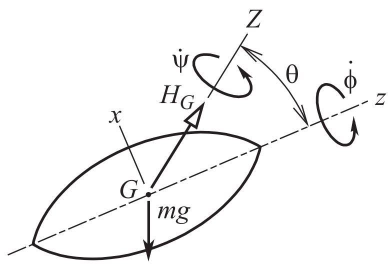
Exercise 5.36

dynamic reactions are generated at bearings A and B. Explain your answer in terms of the properties of H_C.



Exercise 5.37

EXERCISE 5.37 Arm ABC rotates about the vertical axis at constant rate ω_1, and the disk rotates relative to the arm at constant rate ω_2. The mass of the disk is m, and its centroidal moments of inertia are $I_1 = 0.5mR^2$ about its centerline and $I_2 = 0.25mR^2$ about a diameter. (a) Draw one or more sketches depicting the angular momentum $\bar{H}_C$ of the disk about its center. (b) Based on the sketch(es) in Part (a) and the manner in which the system rotates, evaluate $d\bar{H}_C/dt$. (c) Compare the result in Part (b) with what is obtained by evaluating $\partial\bar{H}_C/\partial t + \bar{\omega} \times \bar{H}_C$.



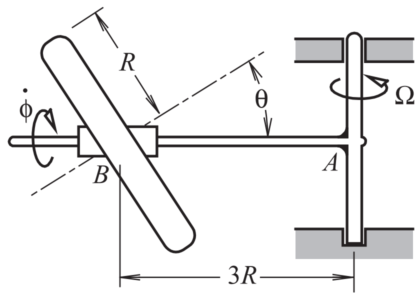
Exercise 5.38

EXERCISE 5.38 The topic of rotation of a body in free motion is treated extensively in Chapter 10. Some key aspects of that study are described in the sketch, which shows a body that is axisymmetric about the body-fixed z axis. The moment of inertia of the body about this axis is I_1, and the moment of inertia about any axis intersecting the center of mass and perpendicular to z is I_2. The body is in free flight and air resistance is negligible, so the only force acting on the body is its weight. Because this force acts at the center of mass G, $\Sigma \bar{M}_G = d\bar{H}_G/dt \equiv \bar{0}$, so the angular momentum $\bar{H}_G$ is constant. Let the fixed Z axis denote the constant direction of $\bar{H}_G$. Eulerian angles are used to describe the rotation of the body, with precession angle ψ being defined as the rotation about the fixed Z axis and the spin angle ϕ being the rotation about the z axis of symmetry. The nutation angle θ is the angle between these two axes, as shown in the sketch. (a) Describe the angular velocity of the body in terms of ψ, θ, and ϕ. Use this description of $\bar{\omega}$ to derive an expression for $\bar{H}_G$. (b) Derive a component description for $\bar{H}_G$ based on the fact that $\bar{H}_G$ is parallel to the Z axis. Match this description to the expression for $\bar{H}_G$ in Part (a). Show that the two descriptions are consistent only if the nutation angle is constant, that is, $\dot{\theta} = 0$. (c) From the fact that $\dot{\theta} = 0$, it follows that at any instant the angular velocity $\bar{\omega}$ must lie in the xz plane, so that $\bar{\omega} = \omega \sin\beta\,\bar{i} + \omega\cos\beta\,\bar{k}$, where β is the angle between $\bar{\omega}$ and the z axis, and the x axis lies in the plane formed by Z and z. Compare the descriptions of $\bar{\omega}$ and $\bar{H}_G$ in terms of ω and β with the corresponding expressions in terms of ψ, θ, and ϕ. From that comparison, derive the expression

$$\tan\beta = \frac{I_1}{I_2}\tan\theta.$$

(d) Derive an expression for $\dot{\psi}$ in terms of $\dot{\phi}$ and θ.

EXERCISE 5.39 The disk is mounted obliquely on its hub, which spins at angular speed Ω about the horizontal arm AB of the T-bar. Consequently, the disk's center line forms a constant angle θ relative to arm AB. This rotation rate is $\dot{\phi}$, with $\phi = 0$ corresponding to the instant depicted in the sketch, where the disk's axis is situated in the vertical plane. Both this rotation rate and the precession rate Ω are constant. The disk's mass is m, whereas the hub and both shafts have negligible mass. Derive an expression for the force–couple system exerted on the disk by hub B when $\phi = 0$.



Exercise 5.39

EXERCISE 5.40 Determine the force–couple systems in Exercise 5.39 as a function of the rotation angle ϕ.

CHAPTER 6

Newton–Euler Equations of Motion

The previous chapter focused on describing and understanding the variability of angular momentum. We now apply those concepts to relate the motion of a system to the forces driving that motion. The formulation is based on the linear and angular momentum principles of Newton and Euler. These principles govern the motion of a single rigid body, but practical applications feature many bodies. In such situations, individual equations of motion may be written for each body. If one pursues such an analysis, careful attention must be given to accounting for the forces exerted between bodies, so the construction of free-body diagrams will play a prominent role in this chapter's development. As a supplement to this approach, a following section develops a momentum-based concept for systems of rigid bodies that sometimes can lead to the desired solution without considering all of the interaction forces. Ultimately, the energy-based concepts associated with Lagrange, whose development is taken up in the next chapter, provide a more robust alternative approach. However, they are mathematical in nature and afford little physical insight. For this reason, particular attention is given here to providing physical explanations for the results derived from the Newton–Euler formulation of equations of motion.

6.1 FUNDAMENTAL EQUATIONS

6.1.1 Basic Considerations

The basic laws governing each rigid body in a system are Eq. (5.1.13) for the resultant force and Eq. (5.1.22) for the resultant moment, which are repeated here as a single reference:

$$\boxed{\begin{aligned} \Sigma \bar{F} &= m\bar{a}_G, \\ \Sigma \bar{M}_A &= \frac{\partial \bar{H}_A}{\partial t} + \bar{\omega} \times \bar{H}_A. \end{aligned}} \tag{6.1.1}$$

It cannot be emphasized too much that this moment equation should be applied only when point A is either

1. the center of mass of the body, or
2. a fixed point in a body that is in a state of pure rotation.

From a philosophical viewpoint, each expresses a dynamic equilibrium, in which either the forcing or the inertial effect can be considered to be the causative agent. For example, in some situations we might consider the resultant moment to change the angular momentum of the body, whereas in others it might be better to consider that changing the linear momentum requires application of the required moment. The same is true for angular momentum. The key to understanding is recognizing that one must accompany the other. A conceptual picture reinforcing this perspective is provided in Fig. 6.1, where the inertial effects are depicted as a force–couple system acting at the center of mass G. The equations of motion state that this combination is equivalent to the actual force system acting on the body. One side of the figure cannot exist without the other.

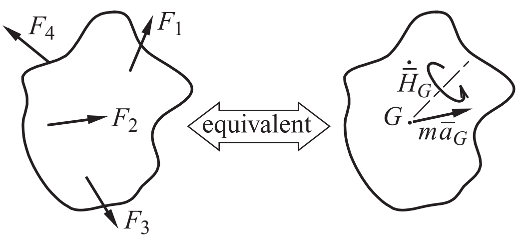


Figure 6.1. The force system acting on a body and its equivalent inertial effect.

Some individuals use a diagram such as Fig. 6.1 as the basis for a slightly modified set of equations of motion based on summing moments about an arbitrary point. Because the forces and inertial effects in Fig. 6.1 have equivalent effect, the moment of each about the selected point must be equal, which leads to

$$\Sigma \bar{M}_B = \frac{d\bar{H}_G}{dt} + \bar{r}_{G/B} \times m\bar{a}_G. \tag{6.1.2}$$

This equation of motion is reminiscent of the general principle in Eq. (5.1.18), except that the kinematical reference point is the center of mass, rather than the point for the moment sum. The formulation is like the treatment of a static system, for which one can select the point for a moment sum to avoid the occurrence of some unknown reactions in the moment equation. From a practical standpoint, the simplification of possibly eliminating an unknown variable is balanced by the need to include a linear acceleration in the moment equation. This increases the chance for error, especially in regard to signs. An analysis following this approach is seldom necessary for a single rigid body. However, we will see later in this chapter that it can be quite useful for some systems containing several interconnected bodies.

Both the force and moment equations of motion are vector relations, whose formulation requires definition of an xyz coordinate system for components. For the moment equation this coordinate system will always be attached to the body. The basic relations for $\bar{H}_A$ and its relative derivative were found to be

$$\boxed{\begin{aligned}\bar{H}_A &= (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\,\bar{i} + (I_{yy}\omega_y - I_{yx}\omega_x - I_{yz}\omega_z)\,\bar{j} \\ &\quad + (I_{zz}\omega_z - I_{zx}\omega_x - I_{zy}\omega_y)\,\bar{k}, \\ \frac{\partial\bar{H}_A}{\partial t} &= (I_{xx}\alpha_x - I_{xy}\alpha_y - I_{xz}\alpha_z)\,\bar{i} + (I_{yy}\alpha_y - I_{yx}\alpha_x - I_{yz}\alpha_z)\,\bar{j} \\ &\quad + (I_{zz}\alpha_z - I_{zx}\alpha_x - I_{zy}\alpha_y)\,\bar{k}.\end{aligned}} \tag{6.1.3}$$

In some situations it might be easier to analyze the acceleration of the center of mass or the resultant force components with respect to a global coordinate system $\hat{x}\hat{y}\hat{z}$ that is not attached to the body. Such a coordinate system can be used for formulate the force equations. To emphasize this aspect, the components of force and moment are indicated explicitly as dot products in the following equations:

$$\boxed{\begin{gathered}\Sigma\bar{F}\cdot\hat{i} = m\left(\bar{a}_G\cdot\hat{i}\right), \quad \Sigma\bar{F}\cdot\hat{j} = m\left(\bar{a}_G\cdot\hat{j}\right), \\ \Sigma\bar{F}\cdot\hat{k} = m\left(\bar{a}_G\cdot\hat{k}\right), \\ \Sigma\bar{M}_A\cdot\bar{i} = \left(\frac{\partial\bar{H}_A}{\partial t} + \bar{\omega}\times\bar{H}_A\right)\cdot\bar{i}, \quad \Sigma\bar{M}_A\cdot\bar{j} = \left(\frac{\partial\bar{H}_A}{\partial t} + \bar{\omega}\times\bar{H}_A\right)\cdot\bar{j}, \\ \Sigma\bar{M}_A\cdot\bar{k} = \left(\frac{\partial\bar{H}_A}{\partial t} + \bar{\omega}\times\bar{H}_A\right)\cdot\bar{k}.\end{gathered}} \tag{6.1.4}$$

Matrix notation offers a compact scheme for performing calculations, and several symbolic mathematics software packages are well attuned to such notation. The angular momentum was written in this form in Eq. (5.2.5). The corresponding forms of the equations of motion are

$$\begin{gathered}\begin{Bmatrix}\Sigma\bar{F}\cdot\hat{i} \\ \Sigma\bar{F}\cdot\hat{j} \\ \Sigma\bar{F}\cdot\hat{k}\end{Bmatrix} = m\begin{Bmatrix}\left(\bar{a}_G\cdot\hat{i}\right) \\ \left(\bar{a}_G\cdot\hat{j}\right) \\ \left(\bar{a}_G\cdot\hat{k}\right)\end{Bmatrix}, \\ \begin{Bmatrix}\Sigma\bar{M}_A\cdot\bar{i} \\ \Sigma\bar{M}_A\cdot\bar{j} \\ \Sigma\bar{M}_A\cdot\bar{k}\end{Bmatrix} = [I]\begin{Bmatrix}\alpha_z \\ \alpha_y \\ \alpha_z\end{Bmatrix} + \begin{Bmatrix}\omega_x \\ \omega_y \\ \omega_z\end{Bmatrix} \otimes \left\{[I]\begin{Bmatrix}\omega_x \\ \omega_y \\ \omega_z\end{Bmatrix}\right\},\end{gathered} \tag{6.1.5}$$

where $\otimes$ denotes the cross-product operator. The moment equations simplify considerably when xyz are principal axes. The result is *Euler's equations of motion,* which explicitly display the role of the angular velocity and angular acceleration components:

$$\boxed{\begin{aligned}\Sigma\bar{M}_A\cdot\bar{i} &= I_{xx}\alpha_x - (I_{yy} - I_{zz})\,\omega_y\omega_z, \\ \Sigma\bar{M}_A\cdot\bar{j} &= I_{yy}\alpha_y - (I_{zz} - I_{xx})\,\omega_x\omega_z, \\ \Sigma\bar{M}_A\cdot\bar{k} &= I_{zz}\alpha_z - (I_{xx} - I_{yy})\,\omega_x\omega y.\end{aligned}} \tag{6.1.6}$$

One can use the repetitive pattern of Euler's equations to remember the individual components by a mnemonic algorithm based on permutations of the alphabetical order. Euler's equations are particularly useful when it is only necessary to address the moment exerted about one axis.

A basic aspect of the force and moment components is their dual interpretation. One way of evaluating them is to form the resultant as the vector sum of the contribution of each force acting on the body. Alternatively, we may sum the contribution of each force to a specific force or moment component. The latter is very useful for the moment components, which are the moments each force exerts about each of the xyz coordinate axes.

The previous section made it evident that a spatial rotation will require that $\bar{H}_A$ change, even if the all rates of rotation are constant, as a consequence of changing the orientation of $\bar{H}_A$. The moment equation merely requires that the force system apply a moment that balances the rate at which the angular momentum changes. The moment required to balance the portion of $d\bar{H}_A/dt$ that features products of rotation rates, and therefore is present even if the rotation rates are constant, is often referred to as the *gyroscopic moment.*

Various questions may be investigated with the equations of motion. In the simplest case, the motion of a rigid body is fully specified. This permits complete evaluation of the right side of the translational and rotational equations. The forcing effects, which appear on the left side of the equations, originate from known excitations, as well as reactions. The latter are particularly important to characterize. A free-body diagram, in which the body is isolated from its surroundings, is essential to the correct description of the reactions. As an aid in drawing a free-body diagram, recall that reactions are the physical manifestations of kinematical constraints. Thus, if a support prevents a point in the body from moving in a certain direction, then at that point there must be a reaction force exerted on the body in that direction. Similarly, a kinematical constraint on rotation about an axis is imposed by a reaction couple exerted about that axis. The reactions are not known in advance—they are unknown values that will appear in some or all of the equations of motion.

In real applications the system usually contains multiple interconnected bodies. Because the translational and rotational equations of motion describe a single isolated body, it is necessary in such cases to isolate each body whose mass is significant. Care must be taken in the respective free-body diagrams to account for Newton's Third Law in the depiction of the connective forces exerted between bodies. There are only six scalar equations of motion for each body contained in a system (three force and three moment components). The unknowns appearing in these equations might be kinematical parameters or parameters describing the forces. Some equations might be trivial as a consequence of the basic nature of the system. For example, planar motion, which is treated in the next section, reduces to two force equations and one moment equation per body. In any event, a system composed of N bodies can have no more than $6N$ component equations of motion. It is possible for the number of unknown reactions to exceed the number of available equations, yet for the equations to have a consistent solution for the motion variables. Assuming that this condition does not result from erroneous omission of some characteristic of the supports, it represents a condition of *redundant constraint*. That is the dynamic analog of the condition of static indeterminacy, whose resolution requires consideration of deformation effects.

In some situations, analysis of a system's motion entails determination of a few unknown constant parameters, such as an angle of orientation or a rotation rate. This is the case, for example, if it is necessary to determine the conditions for a steady precession. The equations of motion in this case are algebraic. The more difficult situation arises when the nature of the motion is not fully known in advance. The orientation of each body may then be described in terms of Eulerian angles (precession, nutation, and spin), and the position of the center of mass of each body may be described by any of the formulations used to describe point motion. In addition to these kinematical variables, parameters describing any reactions will appear in the equations of motion. The reaction parameters enter into the equations of motion algebraically through the force and moment sums. Hence their elimination involves a process of simultaneous solution of algebraic equations. (This, of course assumes that a condition of redundant constraint does not exist.) Algebraic elimination of the force parameters will lead to a set of differential equations of motion for the kinematical variables. We saw that $d\bar{H}_A/dt$ generally features products of rotation rates, and the description of the motion of a point in terms of curvilinear coordinates or moving reference frames also contains products of rate variables. As a result, the equations of rotational motion will be coupled second-order differential equations. Analytical solutions of such equations are available in limited situations, such as when the equations are linear in the dependent variables. Numerical techniques are always available as an alternative. Procedures for numerically solving differential equations of motion are discussed in Chapters 7 and 8.

6.1.2 Procedural Steps

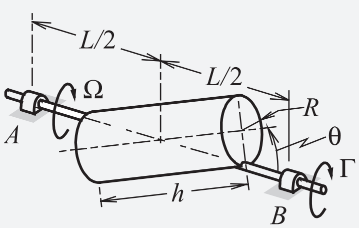
Although a specific system may have distinctive features, many aspects of the formulation of its equations of motion are generic. The following list assembles the operations into a standard procedure that we shall follow in the examples that follow.

1. Draw a free-body diagram describing each body whose mass is significant. To be sure that all constraint forces are properly described, remember that, if the motion of a point is restricted in a certain direction, then an unknown force acts in that direction to impose the condition. Similarly, if the rotation of a body about a certain axis is restricted, then there must be an unknown reaction couple about that axis.
2. Choose a point, designated here as A, for summing moments for the isolated body. This point should be the center of mass when the body is translating or in general motion. For the case of pure rotation, use the fixed point in order that the reaction forces holding that point stationary not occur in the moment equation of motion.
3. Attach xyz to the body with origin at point A. The first consideration in selecting the orientation of xyz is minimizing computations of inertia properties. Thus, if possible, orient xyz parallel to the axes used in the Appendix to describe the inertia properties. If this does not lead to a full specification of the coordinate axes, as would be the case for an axisymmetric body, then finalize the definition by considering the

axes of rotation and the manner in which the geometry of the system is described. Show the coordinate axes of xyz in the free-body diagram.

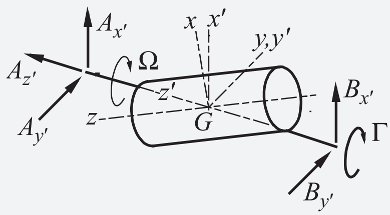
4. Write down, in equation form, all given information that does not appear in the free-body diagram. List all quantities to be determined in the solution.
5. Follow the standard kinematical procedure to describe $\bar{\omega}$ and $\bar{\alpha}$ of the body in terms of components relative to xyz. Be sure to satisfy any auxiliary kinematical conditions that are known, such as that certain points have known motions, or that there is no slipping at some location.
6. Evaluate the inertia properties of the body with respect to xyz.
7. Compute the moment about point A exerted by all forces appearing in the free-body diagram. Equate this moment to $d\bar{H}_A/dt$. This may be done as a vector equation with Eqs. (6.1.1) and (6.1.3), but the matrix form in Eqs. (6.1.5) might be preferred for numerical problems. Also, if it can be recognized that only the moment about a specific axis x, y, or z is needed, and xyz are principal axes, then the corresponding component of Euler's equations, Eqs. (6.1.6), offers a shortcut.
8. Examine whether the moment equation(s) obtained from the preceding step are sufficient to solve the problem. If additional equations are required, form the force equation of motion. In that event describe the center-of-mass acceleration $\bar{a}_G$ in terms of the basic kinematical parameters. The components of $\bar{a}_G$ may be described in terms of components with respect to any convenient coordinate system $x'y'z'$, although xyz often will be sufficient for this purpose. Equate like components of $\Sigma\bar{F}$ and $m\bar{a}_G$.
9. Count the number of scalar equations J and number of unknowns N. If $J > N$, something such as a kinematical relation has been assumed incorrectly, or else a reaction might have been omitted from the free-body diagram. If $J < N$, some information that was given might not have been used, or else some kinematical condition imposed on the body, such as rolling without slipping, might not have been satisfied.

EXAMPLE 6.1 The cylinder, whose mass is m, is welded to the shaft such that its center is situated on the axis of rotation. Application of a constant torque Γ at $t = 0$ causes the rotation rate Ω to increase from zero. Derive expressions for Ω and the reactions at bearings A and B as functions of the elapsed time.



Example 6.1

SOLUTION This example reinforces our recognition that a planar analysis might not suffice, even though the motion is a simple rotation. Examination of the solution will lead us to a general discussion of the topic of dynamic imbalance. The forces of interest are the reactions the bearings apply to the shaft. These are represented in the free-body diagram as components transverse to the shaft at both bearings and a thrust component parallel to the shaft at bearing A. We may consider the shaft and the cylinder to constitute a single body because they are welded together. Doing so makes the forces exerted between them internal to the system. We ignore the weight because it is a static force. It is balanced by a static force acting in the (stationary) vertical direction at each bearing that superposes on the dynamic reactions we shall evaluate.



Free-body diagram of the rigid body consisting of the cylinder and the shaft.

The cylinder executes a simple rotation, so any point along shaft AB would be an allowable point for summing moments. We use the center of mass G. We align the xyz axes with the centerline of the cylinder in accord with the Appendix, and exploit the axisymmetry to define the y axis to be perpendicular to the plane formed by the shaft and the centerline. We also define an $x'y'z'$ coordinate system aligned with the shaft and with its y' axis coincident with the y axis to facilitate describing the bearing forces and the rotation.

The angular velocity is parallel to the shaft, and the direction of this rotation is constant, so

$$\bar{\omega} = \Omega \bar{k}', \quad \bar{\alpha} = \dot{\Omega} \bar{k}', \tag{6.1.7}$$

which when resolved into xyz components become

$$\bar{\omega} = \Omega \left(\sin\theta \bar{i} + \cos\theta \bar{k} \right), \quad \bar{\alpha} = \dot{\Omega} \left(\sin\theta \bar{i} + \cos\theta \bar{k} \right). \tag{6.1.8}$$

We know from symmetry that xyz are principal axes, and the tabulated inertia properties are

$$I_{xx} = I_{yy} = m \left(\frac{1}{4} R^2 + \frac{1}{12} h^2 \right), \quad I_{zz} = \frac{1}{2} m R^2. \tag{6.1.9}$$

We may employ Euler's equations because xyz are principal axes. To form the moment resultants the applied couple Γ must be resolved into components. The moment of the bearing forces may be computed vectorially. For example, for bearing A the moment is $\bar{r}_{A/G} \times \left(A_{x'} \bar{i}' + A_{z'} \bar{k}' \right)$. Alternatively the moment sums can be computed as moments about the coordinate axes. Thus the bearing forces $A_{x'}$ and $B_{x'}$ exert moments about only the y axis, for which the lever arms are $L/2$, whereas $A_{y'}$ and $B_{y'}$ have lever arms of $(L/2)\sin\theta$ about the z axis and $(L/2)\cos\theta$ about the

x axis. Applying the right-hand rule to ascertain the sign of each moment yields

$$\begin{aligned}
\Sigma \bar{M}_A \cdot \bar{i} &= \Gamma \sin\theta + (-A_{y'} + B_{y'})\frac{L}{2}\cos\theta = I_{xx}\alpha_x \\
&= m\left(\frac{1}{4}R^2 + \frac{1}{12}h^2\right)\dot{\Omega}\sin\theta, \\
\Sigma \bar{M}_A \cdot \bar{j} &= (A_{x'} - B_{x'})\frac{L}{2} = -(I_{zz} - I_{xx})\,\omega_x\omega_z \\
&= m\left(-\frac{1}{4}R^2 + \frac{1}{12}h^2\right)\Omega^2\sin\theta\cos\theta, \\
\Sigma \bar{M}_A \cdot \bar{k} &= \Gamma\cos\theta + (A_{y'} - B_{y'})\frac{L}{2}\sin\theta = I_{zz}\alpha_x \\
&= m\left(\frac{1}{2}R^2\right)\dot{\Omega}\cos\theta.
\end{aligned}$$

The unknowns in these three equations are the four transverse bearing forces and Ω. The force equation of motion provides the additional equations required for determining these parameters and the thrust bearing force. The center of mass is on the axis of rotation, so $\bar{a}_G = \bar{0}$. In terms of components relative to $x'y'z'$, the force resultants are

$$\begin{aligned}
\Sigma \bar{F} \cdot \bar{i}' &= A_{x'} + B_{x'} = 0, \\
\Sigma \bar{F} \cdot \bar{j}' &= A_{y'} + B_{y'} = 0, \\
\Sigma \bar{F} \cdot \bar{k}' &= A_{z'} = 0.
\end{aligned}$$

The solution of these equations is

$$\dot{\Omega} = \frac{12\Gamma}{m\left[6R^2 + (h^2 - 3R^2)(\sin\theta)^2\right]},$$

$$A_{x'} = -B_{x'} = \frac{m\Omega^2}{24L}\left(h^2 - 3R^2\right)\sin 2\theta, \qquad \triangleleft$$

$$A_{y'} = -B_{y'} = -\frac{m\dot{\Omega}}{24L}\left(h^2 - 3R^2\right)\sin 2\theta,$$

$$A_{z'} = 0.$$

The torque is constant, so it follows that $\dot{\Omega}$ is constant. Because $\Omega = 0$ initially, integrating $\dot{\Omega}$ leads to a rotation rate that increases proportionally to elapsed time:

$$\Omega = \frac{12\Gamma t}{m\left[6R^2 + (h^2 - 3R^2)(\sin\theta)^2\right]}. \qquad \triangleleft$$

Substitution of these results into the expressions for the bearing forces shows that $A_{x'}$ and $B_{x'}$ increase as t^2, whereas $A_{y'}$ and $B_{y'}$ are independent of the elapsed time.

Each of these results may be readily explained by following the manner in which the angular momentum $\bar{H}_G$ changes. Such an examination would show that the rotation moves the head of $\bar{H}_G$ in the $-y'$ direction, which requires a couple about the negative y' axis. Such a moment is generated by the bearing forces $A_{x'}$ and $B_{x'}$, which depend solely on the current rotation rate. The bearing forces rotate at Ω, which means that the forces the shaft exerts in the bearing are experienced by the foundation as rotating forces. Such forces cause vibration in the foundation because their components relative to a fixed reference frame oscillate in magnitude. This is a condition of *dynamic imbalance*. The other transverse bearing forces, $A_{y'}$ and $B_{y'}$, are proportional to $\dot{\Omega}$. They are associated with the changing magnitude of $\bar{H}_G$ and remain constant as long as $\dot{\Omega}$ does not change. They too rotate at Ω relative to a fixed reference frame, and therefore also produce oscillations in the foundation, although their value usually is small compared with that of the dynamic imbalance forces.

The nature of the forces would be more apparent if we had formulated the equations of motion by using $x'y'z'$ as the body-fixed reference frame. This would have required the rotation transformation to evaluate $[I']$. We may generalize such an analysis. Consider an arbitrary body that executes a pure rotation at angular speed Ω about a shaft as the result of a torque Γ. The center of mass is situated at an arbitrary distance from this rotation axis. Let $x'y'z'$ be a body-fixed coordinate system whose z' axis coincides with the shaft, and let $[I']$ be the corresponding inertia matrix. In this case the angular motion is $\bar{\omega} = \Omega\bar{k}'$, $\bar{\alpha} = \dot{\Omega}\bar{k}'$. Because $x'y'z'$ is not necessarily principal, we use Eqs. (6.1.3) to form $\bar{H}_G$ and its relative derivative, which gives

$$\bar{H}_G = -I_{x'z'}\Omega\bar{i}' - I_{y'z'}\Omega\bar{j}' + I_{z'z'}\Omega\bar{k}',$$
$$\partial\bar{H}_G/\partial t = -I_{x'z'}\dot{\Omega}\bar{i}' - I_{y'z'}\dot{\Omega}\bar{j}' + I_{z'z'}\dot{\Omega}\bar{k}'.$$

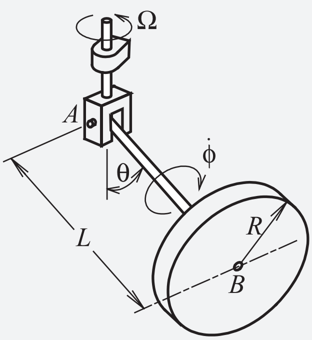
We refer to the free-body diagram to sum forces and moments relative to $x'y'z'$. The equations of motion, Eqs. (6.1.1), then require that

$$(A_{x'} + B_{x'})\bar{i}' + (A_{y'} + B_{y'})\bar{j}' + A_{z'}\bar{k}' = m\bar{a}_G,$$
$$(-A_{y'} + B_{y'})\frac{L}{2}\bar{i} + (A_{x'} - B_{x'})\frac{L}{2}\bar{j}' + \Gamma\bar{k}'$$
$$= \left(-I_{x'z'}\dot{\Omega} + I_{y'z'}\Omega^2\right)\bar{i}' + \left(-I_{y'z'}\dot{\Omega} - I_{x'z'}\Omega^2\right)\bar{j}' + I_{z'z'}\dot{\Omega}\bar{k}'.$$

A dynamically balanced system is one in which any rotation can occur without generating dynamic reactions at either bearing, $A_{x'} = A_{y'} = A_{z'} = B_{x'} = B_{y'} = 0$. The preceding equations of motion indicate that such a condition requires that $\bar{a}_G = 0$ and $I_{x'z'} = I_{y'z'} = 0$. In other words, dynamic balancing requires that the center of mass be situated on the rotation axis and that the rotation axis be a principal axis of inertia. In contrast, the condition of *static balance* merely requires the former.

This general observation is manifested by the specific results for the skewly mounted cylinder of the present system. When $\theta = 0$ the z axis coincides with the rotation axis, whereas $\theta = 90^\circ$ corresponds to alignment of x and the rotation axis. Both x and z are principal axes, and the results confirm that the bearing forces are zero in both cases. We also see that the forces are zero for any θ if $h = \sqrt{3}R$. In that case $I_{xx} = I_{yy} = I_{zz}$, so that the cylinder is inertially equivalent to a sphere, for which all centroidal axes are principal. In any other case, bearing forces are required to sustain the motion, even though the center of mass coincides with the shaft.

EXAMPLE 6.2 A servomotor maintains a constant value of the spin rate $\dot{\phi}$ at which the disk rotates relative to the pivoted shaft AB. The precession rate Ω is held constant by a torque $M(t)$ applied to the vertical shaft. Derive the differential equation governing the unsteady nutation angle θ, and also derive an expression for M as a function of θ. Then determine all possible states of steady precession, in which θ is constant, and evaluate the stability of each state.

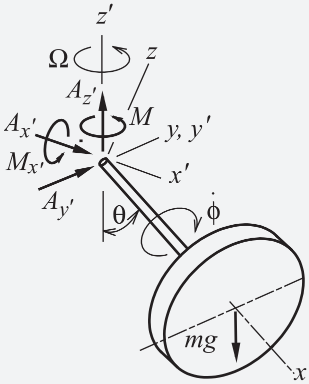


Example 6.2

SOLUTION This system is truly in spatial motion, so the analysis will bring to the fore some of the issues that one might typically encounter. In addition, it will address the characterization of connections in terms of the kinematical restrictions they impose and the constraint forces they exert. The only body having significant inertia is the disk, but the fork-and-clevis joint at point A must be considered because we know how it constrains motion. We assume that the shaft's mass is negligible, which enables us to consider it and the disk to be a single rigid body. (Without this assumption we would need to consider each body individually, and account for the forces exerted between them at bearing B.)

Point A is at a constant distance from the disk along the disk's centerline, which means that the disk is in pure rotation about this point. Consequently we choose point A as the origin of the body-fixed xyz coordinate system. We align the x axis with the centerline to employ directly the tabulated inertia properties of a disk. The axisymmetry of the disk gives us some freedom to define the body-fixed xyz system in a manner that suits the kinematical features. Thus we align the y axis at the instant of interest horizontally to facilitate describing the θ rotation. To describe

$\bar{\omega}$ and the reactions at point A we also define $x'y'z'$ to precess at Ω with z' vertical and x' situated in the vertical plane formed by the x and z' axes.



Free-body diagram of the spinning disk and its shaft.

Joint A prevents movement of the shaft, so we show the corresponding force reactions as components parallel to $x'y'z'$. Also, this connection allows shaft AB to rotate only about the y' axis relative to the vertical shaft. Thus the free-body diagram depicts couple reactions about the x' and z' axes, but not about the y' axis. The vertical component is the torque M applied to the vertical shaft to sustain Ω. The free-body diagram also includes the weight of the disk, whose role is not static because its moment is not constant in magnitude and direction. The angular velocity and the angular acceleration of the disk are described by

$$\bar{\omega} = -\dot{\phi}\bar{i} + \Omega\bar{k}' - \dot{\theta}\,\bar{j}', \quad \bar{\omega}' = \Omega\bar{k}',$$
$$\bar{\alpha} = -\dot{\phi}\left(\bar{\omega} \times \bar{i}\right) - \ddot{\theta}\,\bar{j}' - \dot{\theta}\left(\bar{\omega}' \times \bar{j}'\right).$$

At the instant in the free-body diagram $\bar{j}' = \bar{j}$ and $\bar{k}' = -\cos\theta\bar{i} + \sin\theta\bar{k}$, so the angular motion is

$$\bar{\omega} = \left(-\dot{\phi} - \Omega\cos\theta\right)\bar{i} - \dot{\theta}\,\bar{j} + \Omega\sin\theta\bar{k}, \quad \bar{\omega}' = -\Omega\cos\theta\bar{i} + \Omega\sin\theta\bar{k},$$
$$\bar{\alpha} = \dot{\theta}\Omega\sin\theta\bar{i} + \left(-\ddot{\theta} - \dot{\phi}\Omega\sin\theta\right)\bar{j} + \left(-\dot{\phi}\dot{\theta} + \dot{\theta}\Omega\cos\theta\right)\bar{k}. \tag{1}$$

We use the Appendix and the parallel axis theorems to find the inertia properties of the disk relative to xyz, which are principal:

$$I_{xx} = \frac{1}{2}mR^2, \quad I_{yy} = I_{zz} = m\left(\frac{1}{4}R^2 + L^2\right).$$

Inspection of the free-body diagram reveals that the force equations of motion will contain the reaction forces at the pin, which are of no interest here. Thus we focus on the moment equation. Euler's equations are applicable because xyz are principal axes. We refer to the free-body diagram to determine the moment sums, which leads to

$$\begin{aligned}
\Sigma \bar{M}_A \cdot \bar{i} &= -M\cos\theta + M_{x'}\sin\theta = I_{xx}\alpha_x - (I_{yy} - I_{zz})\,\omega_y\omega_z, \\
&= \left(\frac{1}{2}mR^2\right)\dot{\theta}\Omega\sin\theta, \\
\Sigma \bar{M}_A \cdot \bar{j} &= mg\,(L\sin\theta) = I_{yy}\alpha_y - (I_{zz} - I_{xx})\,\omega_x\omega_z \\
&= \left(\frac{1}{4}R^2 + L^2\right)\left(-\ddot{\theta} - \dot{\phi}\Omega\sin\theta\right), \\
&\quad -m\left(\frac{1}{4}R^2 - L^2\right)\left(-\dot{\phi} - \Omega\cos\theta\right)(\Omega\sin\theta), \\
\Sigma \bar{M}_A \cdot \bar{k} &= M\sin\theta + M_{x'}\cos\theta = I_{zz}\alpha_z - (I_{xx} - I_{yy})\,\omega_x\omega y \\
&= m\left(\frac{1}{4}R^2 + L^2\right)\left(-\dot{\phi}\dot{\theta} + \dot{\theta}\Omega\cos\theta\right) \\
&\quad -\left(\frac{1}{4}R^2 - L^2\right)\left(-\dot{\phi} - \Omega\cos\theta\right)\left(-\dot{\theta}\right).
\end{aligned}$$

The sole unknown in the second equation is θ, so it yields the desired differential equation, and the first and third equations combine to give an expression for M:

$$\begin{aligned}
\left(\frac{1}{4}R^2 + L^2\right)\ddot{\theta} - \left(L^2 - \frac{1}{4}R^2\right)\Omega^2\sin\theta\cos\theta + \left(\frac{1}{2}R^2\dot{\phi}\Omega + gL\right)\sin\theta = 0, \\
M = -\frac{1}{2}mR^2\dot{\phi}\dot{\theta}\sin\theta + 2\left(L^2 - \frac{1}{4}R^2\right)\dot{\theta}\Omega\sin\theta\cos\theta.
\end{aligned} \qquad (2)\triangleleft$$

It is interesting to note that couples acting about both the x' and z' axes are required for sustaining the precession and spin rates.

To find the possible steady precession states we take θ to be constant in the differential equation, which leads to two possibilities:

$$\sin\theta = 0 \quad \text{or} \quad \left(L^2 - \frac{1}{4}R^2\right)\Omega^2\cos\theta = \frac{1}{2}R^2\dot{\phi}\Omega + gL. \qquad (3)$$

The diagram describing the system indicates that the construction of the fork-and-clevis joint makes $\theta = \pi$ impossible, but we allow for it in order to get a full picture of the behavior. Thus there are three possible constant nutation angles:

$$\theta_1 = 0, \quad \theta_2 = \pi, \quad \theta_3 = \cos^{-1}\left[\frac{2R^2\dot{\phi}\Omega + 4gL}{\Omega^2\,(4L^2 - R^2)}\right]. \qquad (4)\triangleleft$$

The nutated state represented by θ_3 arises for only a range of parameters. The analysis is most easily carried out if Ω is considered to be fixed, whereas $\dot{\phi}$ is adjustable.

Let us consider $L > R/2$. Then the condition that $\left|\cos(\theta_3)\right| \le 1$ is satisfied when

$$\dot{\phi}_{\min} \le \dot{\phi} \le \dot{\phi}_{\max},$$
$$\dot{\phi}_{\min} = -2\frac{gL}{R^2\Omega} - \Omega\frac{4L^2 - R^2}{2R^2}, \quad \dot{\phi}_{\max} = -2\frac{gL}{R^2\Omega} + \Omega\frac{4L^2 - R^2}{2R^2}. \tag{5}$$

Note that $L > R/2$ leads to $\dot{\phi}_{\min}$ being negative, whereas $\dot{\phi}_{\max}$ can be either positive or negative.

Nutation angles θ_1 and θ_2 correspond to the precession and spin rates both being about vertical axes, so the angular momentum $\bar{H}_A$ also is vertical, and therefore constant. The gravity force acting on the disk intersects joint A in this case, so there is no moment about the y axis to be balanced by $d\bar{H}_A/dt$. In the third state, $\bar{H}_A$ lies in the xz plane, which means that it precesses at Ω about the vertical axis. The moment of gravity about the pin of joint A balances the rate at which $\bar{H}_A$ is changed by this rotation. The value of θ_3 ranges from zero at $\dot{\phi}_{\max}$ to π at $\dot{\phi}_{\min}$.

Although the three roots for constant θ are mathematically possible, determining whether they will actually occur requires consideration of the stability of the steady precession. This involves investigating whether the steady motion will change drastically if it is disturbed. For our purpose we limit consideration to a small disturbance that changes one of the constant θ_n solutions by a small amount Δ. The smallness restriction enables us to simplify $\sin\theta$ and $\cos\theta$ with a Taylor series:

$$\theta = \theta_n + \Delta \implies \sin\theta \approx \sin\theta_n + \Delta\cos\theta_n, \quad \cos\theta \approx \cos\theta_n - \Delta\sin\theta_n.$$

We set $\ddot{\theta} = \ddot{\Delta}$ in the differential equation (2) and substitute the preceding expressions, which leads to

$$\left(\frac{1}{4}R^2 + L^2\right)\ddot{\Delta} - \left(L^2 - \frac{1}{4}R^2\right)\Omega^2(\sin\theta_n + \Delta\cos\theta_n)(\cos\theta_n - \Delta\sin\theta_n)$$
$$+ \left(\frac{1}{2}R^2\dot{\phi}\Omega + gL\right)(\sin\theta_n + \Delta\cos\theta_n) = 0.$$

This equation is further simplified by the fact that Δ is very small, so quadratic terms in Δ may be dropped. Furthermore, the terms that do not contain Δ correspond to the steady precession solution, so they cancel. Thus we are led to

$$\left(\frac{1}{4}R^2 + L^2\right)\ddot{\Delta} + \left[-\left(L^2 - \frac{1}{4}R^2\right)\Omega^2\cos(2\theta_n) + \left(\frac{1}{2}R^2\dot{\phi}\Omega + gL\right)\cos\theta_n\right]\Delta = 0. \tag{6}$$

The preceding is the equation for a one-degree-of-freedom undamped linear oscillator, so the stability condition is governed by the sign of the coefficient of Δ, which is

$$K\left(\theta_n, \Omega, \dot{\phi}\right) = -\left(L^2 - \frac{1}{4}R^2\right)\Omega^2\cos(2\theta_n) + \left(\frac{1}{2}R^2\dot{\phi}\Omega + gL\right)\cos\theta_n. \tag{7}$$

Positive K corresponds to an oscillatory Δ, which means that θ always remains close to the steady value, whereas negative K corresponds to an exponentially increasing Δ, which indicates that the precession will diverge from the steady value. For $\theta = \theta_1 = 0$, we have

$$K\left(0, \Omega, \dot{\phi}\right) = -\left(L^2 - \frac{1}{4}R^2\right)\Omega^2 + \frac{1}{2}R^2\dot{\phi}\Omega + gL.$$

When we consider $K\left(0, \Omega, \dot{\phi}\right)$ to be a function of $\dot{\phi}$ for fixed Ω, we see that the condition $K\left(0, \Omega, \dot{\phi}\right) > 0$ is satisfied when $\dot{\phi} > \dot{\phi}_{\text{max}}$. The steady-state precession at $\theta = \theta_2 = \pi$ leads to

$$K\left(\pi, \Omega, \dot{\phi}\right) = -\left(L^2 - \frac{1}{4}R^2\right)\Omega^2 - \frac{1}{2}R^2\dot{\phi}\Omega - gL.$$

Stability of this steady nutation, corresponding to $K\left(\pi, \Omega, \dot{\phi}\right) > 0$, occurs when $\dot{\phi} < \dot{\phi}_{\text{min}}$. Analysis of the case in which $\theta = \theta_3$ is slightly more complicated. The identity $\cos(2\theta_3) \equiv 2(\cos\theta_3)^2 - 1$ gives

$$K\left(\theta_3, \Omega, \dot{\phi}\right) = \left(L^2 - \frac{1}{4}R^2\right)\Omega^2\left[1 - 2(\cos\theta_3)^2\right] + \left(\frac{1}{2}R^2\dot{\phi}\Omega + gL\right)\cos\theta_3.$$

Rather than using the last of Eqs. (3) to replace θ_3 in this expression, let us use it to eliminate $\dot{\phi}$. The result is

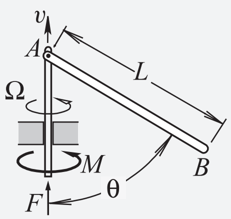
$$K\left(\theta_3, \Omega, \dot{\phi}\right) = \left(L^2 - \frac{1}{4}R^2\right)\Omega^2(\sin\theta_3)^2.$$

The value of $K\left(\theta_3, \Omega, \dot{\phi}\right)$ is positive whenever θ_3 is real, which means that this state is stable if it exists.

An overview of the results gives a clearer picture. We find that, if $\dot{\phi} > \dot{\phi}_{\text{max}}$, then the vertically suspended state, $\theta = 0$, is the only stable steady precession. If the spin rate falls below $\dot{\phi}_{\text{max}}$, but exceeds $\dot{\phi}_{\text{min}}$, then the only stable steady precession is one in which the shaft is tilted at θ_3. Further decrease of $\dot{\phi}$ below $\dot{\phi}_{\text{min}}$ makes the inverted position $\theta_2 = \pi$ the only stable steady precession. Proper interpretation of this result requires that one recall that $\dot{\phi}_{\text{min}}$ is negative, so the condition that $\dot{\phi} < \dot{\phi}_{\text{min}}$ corresponds to a large spin rate in the sense of the positive x axis. ◁

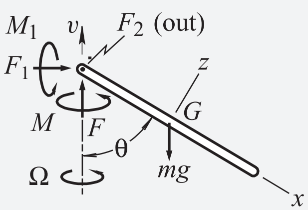
The system in this example is referred to as a *spinning top*, because it effectively is identical to the toy that spins with its apex in contact with the ground. This system has been widely studied as a way of explaining many of the physical principles of spatial kinetics. Further exploration of the behavior of a spinning top may be found in Chapter 10.

EXAMPLE 6.3 The vertical force F causes the vertical bar to translate upward at a speed v that is a specified function of time, and the whole system precesses about the vertical axis at the constant speed Ω as a result of the action of the torque M. Derive the equation of motion for the nutation angle θ, as well as an expression for the value of F required to attain this motion. It may be assumed that the mass of the vertical shaft is negligible.



Example 6.3

SOLUTION This exercise, which requires the simultaneous application of the linear and angular equations of motion, highlights the need to pick the moment reference point appropriately. We begin with a free-body diagram. Because the vertical shaft is massless, it effectively is in static equilibrium. The bearing can resist neither the vertical force F nor the torque M, so these are transmitted to the bar, as shown in the diagram. The couple M_1 acts horizontally perpendicular to the pin, and F_1 and F_2 are horizontal forces that the pin can exert on the bar.



Free-body diagram of the swinging bar.

It is tempting to the novice to place the origin of the body-fixed xyz coordinate system at pin A, because doing so eliminates the pin forces from the moment equation. However, this point is not allowable for using the moment equation of motion in Eqs. (6.1.1) because it is accelerating upward at $\dot{v}$. We therefore must place the origin at the center of mass G. The orientation of xyz shown in the free-body diagram allows us to find the inertia properties of the bar directly from the Appendix, with $I_{xx} = 0$, $I_{yy} = I_{zz} = (1/12)\, mL^2$ based on considering the bar to be slender.

The angular velocity is the sum of the precession and the rotation θ, which actually is a nutation. The precession direction $\bar{e}_1$ is constant, whereas the nutation direction $\bar{e}_2$ precesses at Ω, so that

$$\bar{\omega} = \Omega \bar{e}_1 + \dot{\theta} \bar{e}_2, \quad \bar{\alpha} = \ddot{\theta} \bar{e}_2 + \dot{\theta} \left(\Omega \bar{e}_1 \times \bar{e}_2 \right).$$

The rotation directions are

$$\bar{e}_1 = -\cos\theta \bar{i} + \sin\theta \bar{k}, \quad \bar{e}_2 = -\bar{j},$$

which leads to

$$\bar{\omega} = -\Omega \cos\theta \bar{i} - \dot{\theta} \bar{j} + \Omega \sin\theta \bar{k}, \;\; \bar{\alpha} = \Omega\dot{\theta} \sin\theta \bar{i} - \ddot{\theta} \bar{j} + \Omega\dot{\theta} \cos\theta \bar{k}.$$

Because we cannot eliminate the pin forces from the moment equation, we need the additional relations provided by the force equation of motion, which requires that we describe the acceleration of the center of mass. Pin A is in rectilinear motion, so we have

$$\begin{aligned}\bar{a}_G &= \dot{v}\bar{e}_1 + \bar{\alpha} \times \bar{r}_{G/A} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{G/A}) \\ &= -\left[\dot{v}\cos\theta + \frac{L}{2}\Omega^2(\sin\theta)^2 + \frac{L}{2}\dot{\theta}^2\right]\bar{i} + L\Omega\dot{\theta}\cos\theta\,\bar{j} \\ &\quad + \left[\dot{v}\sin\theta + \frac{L}{2}\ddot{\theta} - \Omega^2\frac{L}{2}\sin\theta\cos\theta\right]\bar{k}.\end{aligned}$$

The Euler equations of motion are applicable because xyz are principal axes. This is an aid because the y axis is the only axis about which neither M nor M_1 exerts a moments. Consequently, only the Euler equation for moment about the y axis is relevant. We add the moment of each pin force about the respective axes, and substitute that sum, along with the inertia properties and components of $\bar{\omega}$ and $\bar{\alpha}$, into Eqs. (6.1.6). Because we have approximated I_{xx} as being zero, we have

$$\begin{aligned}\Sigma\bar{M}\cdot\bar{j} &= F\frac{L}{2}\sin\theta + F_1\frac{L}{2}\cos\theta \\ &= I_{yy}\alpha_y - I_{zz}\omega_x\omega_z = \frac{1}{12}mL^2\left(-\ddot{\theta} + \Omega^2\sin\theta\cos\theta\right).\end{aligned}$$

We are not interested in the value of F_2, which means that we can omit the force equation of motion in the $\bar{j}$ direction. The useful force equations are

$$\begin{aligned}\Sigma\bar{F}\cdot\bar{i} &= -F\cos\theta + F_1\sin\theta + mg\cos\theta = -m\left[\dot{v}\cos\theta + \frac{L}{2}\Omega^2(\sin\theta)^2 + \frac{L}{2}\dot{\theta}^2\right], \\ \Sigma\bar{F}\cdot\bar{k} &= F\sin\theta + F_1\cos\theta - mg\sin\theta = m\left[\dot{v}\sin\theta + \frac{L}{2}\ddot{\theta} - \Omega^2\frac{L}{2}\sin\theta\cos\theta\right].\end{aligned}$$

These equations give the force values corresponding to a specified motion:

$$F = m\left[\dot{v} + g + \frac{L}{2}\ddot{\theta}\sin\theta + \frac{L}{2}\dot{\theta}^2\cos\theta\right], \qquad \triangleleft$$

$$F_1 = m\left[\frac{L}{2}\ddot{\theta}\cos\theta - \frac{L}{2}\dot{\theta}^2\sin\theta - \Omega^2\frac{L}{2}\sin\theta\right].$$

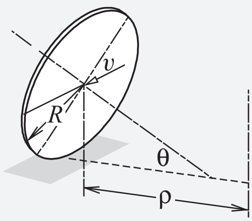
Substitution of these expressions into the moment equation gives a differential equation governing the rotation:

$$\frac{1}{3}\ddot{\theta} - \frac{2}{3}\Omega^2\sin\theta\cos\theta + \frac{\dot{v}+g}{L}\sin\theta = 0. \qquad \triangleleft$$

This equation resembles the one for a pendulum, with the effective gravitational acceleration being $g + \dot{v}$. The term containing Ω^2 represents the influence of the centripetal acceleration associated with precession. Similarly, the result for F shows that it must overcome the effective gravitational acceleration and the vertical components of the transverse and centripetal accelerations of the center of mass that are due to the unsteady value of θ.

It is evident that solving this problem would have been much easier if we were able to form the moment relative to the pin. In Section 6.3 we will see how to do so.

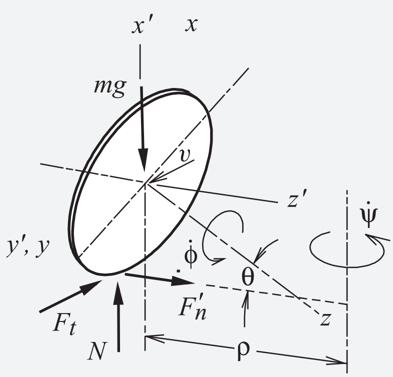
EXAMPLE 6.4 A thin homogeneous disk of mass m rolls without slipping on a horizontal plane such that its center has a constant speed v as it follows a circular path of radius ρ. The angle of elevation of the disk's axis is a constant value θ. Derive an expression relating v to the value of θ.



Example 6.4

SOLUTION The problem of a rolling disk has classical interest. Here we explore the special case of steady precession as an illustration of the full set of Newton–Euler equations of motion. The kinematics of this type of motion was discussed in Section 4.4. That development is an important aspect of the present analysis.

In addition to the gravitational force, there are reactions at the contact between the disk and the ground. They are depicted in the free-body diagram as the normal force N and two frictional force components lying in the horizontal plane: F_n toward the vertical axis about which the disk rotates and F_t opposite the velocity of point G. The former represents the force required to make the center of mass follow a circular path, whereas the latter force anticipates that friction opposes the sliding tendency of the contact point. The normal force N prevents the contact point from penetrating the ground.



Free-body diagram of the rolling disk.

We place the origin of the body-fixed coordinate system xyz at the center G, which is valid regardless of how the disk moves. (The intersection of the centerline of the disk with the vertical axis for the precession also is an allowable point for the moment equation, because it is a stationary point whose position relative to the disk is fixed. However, there is no advantage to using this point.) We exploit the axisymmetry of the disk by defining the body-fixed x and y axes to lie in the plane of the disk, with the y axis at the instant of interest aligned horizontal, in the sense of the velocity of point G. The precession $\dot{\psi}$ is the only rotation of the $x'y'z'$ coordinate system depicted in the free-body diagram. This coordinate system expedites description of $\bar{\omega}$, as well as the acceleration of the center of mass.

The rotation of the disk consists of a precession $\dot{\psi}$ about the vertical axis and a spin $\dot{\phi}$ about the z axis. The angular velocity of the disk is a superposition of the rotations about the two axes,

$$\bar{\omega} = \dot{\psi}\bar{i}' + \dot{\phi}\bar{k}, \quad \bar{\omega}' = \dot{\psi}\bar{i}'.$$

All rotation rates are constant in steady precession, so the corresponding angular acceleration is

$$\bar{\alpha} = \dot{\psi}\left(\bar{\omega}' \times \bar{i}'\right) + \dot{\phi}\left(\bar{\omega} \times \bar{k}\right) = \dot{\psi}\dot{\phi}\left(\bar{i}' \times \bar{k}\right).$$

At the instant depicted in the free-body diagram,

$$\bar{i}' = \cos\theta\bar{i} - \sin\theta\bar{k},$$

which leads to instantaneous expressions for the angular motion of the disk,

$$\bar{\omega} = \dot{\psi}\cos\theta\bar{i} + \left(\dot{\phi} - \dot{\psi}\sin\theta\right)\bar{k}, \quad \bar{\alpha} = -\dot{\psi}\dot{\phi}\cos\theta\,\bar{j}.$$

The velocity of the center is $v\bar{j}$, and the rotation rates are related to the speed v by the no-slip condition at the contact point C, which requires that $\bar{v}_C = 0$. Points C and G belong to the disk, so we have

$$\bar{v}_G = v\bar{j} = \rho\dot{\psi}\bar{j} = \bar{\omega} \times \bar{r}_{G/C} = R\left(\dot{\phi} - \dot{\psi}\sin\theta\right)\bar{j},$$

from which we find that

$$\dot{\psi} = \frac{v}{\rho}, \quad \dot{\phi} = \frac{v}{R} + \frac{v}{\rho}\sin\theta.$$

Correspondingly, the angular motion expressions become

$$\bar{\omega} = \frac{v}{\rho}\cos\theta\bar{i} + \frac{v}{R}\bar{k}, \quad \bar{\alpha} = -\frac{v^2}{\rho}\left(\frac{1}{R} + \frac{1}{\rho}\sin\theta\right)\cos\theta\,\bar{j}.$$

The xyz axes are principal, with

$$I_{xx} = I_{yy} = \frac{1}{4}mR^2, \quad I_{zz} = \frac{1}{2}mR^2.$$

For the sake of variety, we shall formulate $d\bar{H}_A/dt$ from the basic relations, rather than invoke Euler's equations. Combining the inertia properties with the angular rotation variables leads to

$$\bar{H}_G = mR^2\left(\frac{1}{4}\frac{v}{\rho}\cos\theta\,\bar{i} + \frac{1}{2}\frac{v}{R}\bar{k}\right),$$

$$\frac{\partial \bar{H}_G}{\partial t} = -\frac{1}{4}mR^2\frac{v^2}{\rho}\left(\frac{1}{R} + \frac{1}{\rho}\sin\theta\right)\cos\theta\,\bar{j}.$$

The reaction forces applied by the ground are unknown, so there the moment equations of motion will not be sufficient to determine all the unknown quantities. We also need the force equation of motion. We know that point G follows a circular path at constant speed v, and the z' axis points toward the center of curvature. Thus the acceleration of the center of mass is

$$\bar{a}_G = \frac{v^2}{\rho}\bar{k}'.$$

The corresponding Eqs. (6.1.1) are

$$\Sigma\bar{F} = m\frac{v^2}{\rho}\bar{k}',$$

$$\Sigma\bar{M}_G = -\frac{1}{4}mv^2\frac{R}{\rho}\left(2 + \frac{R}{\rho}\sin\theta\right)\cos\theta\,\bar{j}.$$

We form the components of the force sums relative to the $x'y'z'$ axes, and the moments of the forces at the ground about the axes of xyz. Equating each to the corresponding inertial term yields

$$\Sigma\bar{F}\cdot\bar{i}' = N - mg = 0,\quad \Sigma\bar{F}\cdot\bar{j}' = F_t = 0,\quad \Sigma\bar{F}\cdot\bar{k}' = F_n = m\frac{v^2}{\rho},$$

$$\Sigma\bar{M}_G\cdot\bar{i} \equiv 0,$$

$$\Sigma\bar{M}_G\cdot\bar{j} = F_n\,(R\cos\theta) - N\,(R\sin\theta) = -\frac{1}{4}mv^2\frac{R}{\rho}\left(2 + \frac{R}{\rho}\sin\theta\right)\cos\theta,$$

$$\Sigma\bar{M}_G\cdot\bar{k} = F_t R = 0.$$

We solve the force equations for N and F_n, and substitute those expressions into $\Sigma\bar{M}_G\cdot\bar{j}$. The result is

$$m\frac{v^2}{\rho}\,(R\cos\theta) - mg\,(R\sin\theta) = -\frac{1}{4}mv^2\frac{R}{\rho}\left(2 + \frac{R}{\rho}\sin\theta\right)\cos\theta,$$

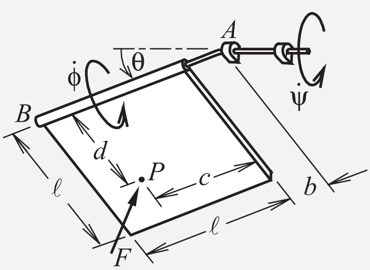
which reduces to

$$v^2 = \frac{4g\rho^2\tan\theta}{6\rho + R\sin\theta}. \qquad \triangleleft$$

There is a simple explanation for this steady motion. The gravitational force and normal reaction form a couple about the horizontal diameter of the disk because the disk is tilted. The frictional reaction required to impart the centripetal acceleration to the center of mass also exerts a moment about this line. The net moment must be matched by a change in the angular momentum. This effect is achieved by the precession, which alters the true direction of the angular momentum, even though its components relative to the xyz axes are constant.

It will be noted that f_t was found to be zero. This is comparable to planar rolling of a disk on level ground. The moment equation of motion in that case indicates that the friction force is zero unless another force exerts a moment about the center of mass, which leads to the anomaly that a disk rolling freely should never slow. This is the fault of the rigid-body model for contacting surfaces. The rolling friction model [see, for example, Ginsberg and Genin (1984)] addresses this anomaly. It would lead to the correct conclusion that a steady precessional motion of the rolling disk is not possible.

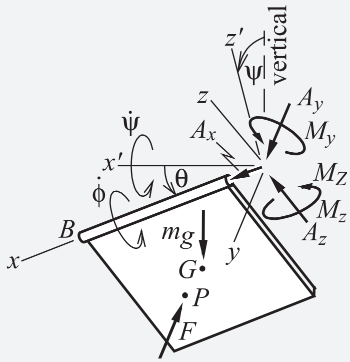
EXAMPLE 6.5 An experiment in aerodynamics features a square plate that is free to spin at rate $\dot{\phi}$ about axis AB of the bent shaft, while the precession rate $\dot{\psi}$ about the horizontal shaft is held constant by application of a torque Γ about the shaft. The angles are defined such that $\psi = \phi = 0$ when the plate coincides with the vertical plane. This system is situated in a wind tunnel whose flow is horizontal. The resultant of the aerodynamic pressure is a known force $F(t)$ acting at the center of pressure P and always normal to the plane of the plate. Derive the differential equation of motion for ϕ. Include the effect of gravity in the derivation.



Example 6.5

SOLUTION This example addresses situations in which a nonaxisymmetric body executes a spatial rotation. Aligning the body-fixed coordinate system to match the tabulated inertia properties fully specifies each axis, so we are not free to align the axes to simplify the kinematical analysis. Rather, we need to use rotation transformations.

To construct the free-body diagram we recognize that the bent shaft applies a distributed force to the plate along edge AB. The resultant of this is arbitrary, except that it cannot exert a torque about that edge because the plate rotates freely. Thus we represent the reaction exerted by the shaft on the plate as a force–couple system at corner A, in which the couple has no component parallel to edge AB. In addition to the aerodynamic force, the free-body diagram shows the gravity force acting vertically downward.



Free-body diagram of the plate showing the supporting forces as a force–couple system at point *A*.

In the free-body diagram the origin of body-fixed xyz coordinate system is placed at bearing A. This point has a fixed position relative to the plate, and it is stationary, which means that the plate executes a pure rotation about point A. The axes of xyz are aligned with the edges of the plate in order to use the tabulated inertia properties. The $x'y'z'$ coordinate systems depicted in the free-body diagram is introduced to aid in the description of the gravity force and the angular motion. The sole rotation it undergoes is the precession about the horizontal x' axis. We define $\psi = 0$ to correspond to the z' axis being vertical.

The transformation from $x'y'z'$ to xyz may be pictured as a pair of body-fixed rotations: θ about the y' axis, followed by $-\phi$ about the x axis. The associated simple rotation transformations are

$$[R_\theta] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$[R_\phi] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix},$$

so the transformation of the unit vectors is

$$\begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \end{bmatrix}^{\mathrm{T}} = [R] \begin{bmatrix} \bar{i}' & \bar{j}' & \bar{k}' \end{bmatrix}^{\mathrm{T}},$$

$$[R] = [R_\phi][R_\theta] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ -\sin\phi\sin\theta & \cos\phi & -\sin\phi\cos\theta \\ \cos\phi\sin\theta & \sin\phi & \cos\phi\cos\theta \end{bmatrix}.$$

We may now form the angular velocity and angular acceleration. Adding the precessional motion about the horizontal x' axis and the spin about the x axis leads to

$$\bar{\omega} = -\dot{\psi}\bar{i}' - \dot{\phi}\bar{i}.$$

The precession rate is specified to be constant and it is about a fixed axis, so the angular acceleration of the plate is

$$\bar{\alpha} = -\ddot{\phi}\bar{i} - \dot{\phi}\left(\bar{\omega} \times \bar{i}\right).$$

We need the components of $\bar{\omega}$ and $\bar{\alpha}$ relative to xyz, and we need to express $\bar{i}$ in terms of its xyz components. The transformation from $x'y'z'$ to xyz is described by $[R]$, so we extract the components of $\bar{i}'$ from the first row of $[R]^{\mathrm{T}}$, with the result that

$$\bar{i}' = \cos\theta\bar{i} - \sin\phi\sin\theta\bar{j} + \cos\phi\sin\theta\bar{k},$$

which leads to

$$\begin{aligned}
\bar{\omega} &= \left(-\dot{\psi}\cos\theta - \dot{\phi}\right)\bar{i} + \dot{\psi}\sin\phi\sin\theta\bar{j} - \dot{\psi}\cos\phi\sin\theta\bar{k},\\
\bar{\omega}' &= -\dot{\psi}\cos\theta\bar{i} + \dot{\psi}\sin\phi\sin\theta\bar{j} - \dot{\psi}\cos\phi\sin\theta\bar{k},\\
\bar{\alpha} &= -\ddot{\phi}\bar{i} + \dot{\phi}\dot{\psi}\cos\phi\sin\theta\bar{j} + \dot{\phi}\dot{\psi}\sin\phi\sin\theta\bar{k}.
\end{aligned}$$

We find the inertia properties for the square plate by setting to zero the dimension along the y edge of the rectangular parallelepiped in the Appendix, and then invoking the parallel axis transformation. The result is

$$I_{xx} = \frac{1}{3}m\ell^2, \quad I_{yy} = m\left(\frac{2}{3}\ell^2 + \ell b + b^2\right), \quad I_{zz} = m\left(\frac{1}{3}\ell^2 + \ell b + b^2\right),$$
$$I_{xy} = I_{yz} = 0, \quad I_{xz} = m\left(\frac{1}{4}\ell^2 + \frac{1}{2}\ell b\right).$$

We combine these properties and the components of $\bar{\omega}$ and $\bar{\alpha}$ to form $d\bar{H}_A/dt$. For this we must use the the full equations (6.1.3) because some products of inertia are nonzero. Thus the angular momentum is

$$\begin{aligned}
\bar{H}_A &= (I_{xx}\omega_x - I_{xz}\omega_z)\bar{i} + I_{yy}\omega_y\bar{j} + (I_{zz}\omega_z - I_{zx}\omega_x)\bar{k}\\
&= \left[I_{xx}\left(-\dot{\psi}\cos\theta - \dot{\phi}\right) - I_{xz}\left(-\dot{\psi}\cos\phi\sin\theta\right)\right]\bar{i} + I_{yy}\dot{\psi}\sin\phi\sin\theta\bar{j}\\
&\quad + \left[I_{zz}\left(-\dot{\psi}\cos\phi\sin\theta\right) - I_{xz}\left(-\dot{\psi}\cos\theta - \dot{\phi}\right)\right]\bar{k},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\bar{H}_A}{\partial t} &= (I_{xx}\alpha_x - I_{xz}\alpha_z)\bar{i} + I_{yy}\alpha_y\bar{j} + (I_{zz}\alpha_z - I_{zx}\alpha_x)\bar{k}\\
&= \left[I_{xx}\left(-\ddot{\phi}\right) - I_{xz}\left(\dot{\phi}\dot{\psi}\sin\phi\sin\theta\right)\right]\bar{i} + I_{yy}\dot{\phi}\dot{\psi}\cos\phi\sin\theta\bar{j}\\
&\quad + \left[I_{zz}\left(\dot{\phi}\dot{\psi}\sin\phi\sin\theta\right) - I_{xz}\left(-\ddot{\phi}\right)\right]\bar{k}.
\end{aligned}$$

The algebraic operations required to form $d\bar{H}_A/dt$ according to the second of Eqs. (6.1.1) are somewhat tedious, but symbolic mathematical software lessens the difficulty. The result is

$$\begin{aligned}\dot{\bar{H}}_A = &\left\{-I_{xx}\ddot{\phi} + \left[(I_{yy} - I_{zz}) \sin\phi \cos\phi\, (\sin\phi)^2 + I_{xz} \sin\phi \sin\theta \cos\theta\right] \dot{\psi}^2\right\} \bar{i} \\ &+ \left\{\left[(I_{xx} - I_{yy}) \cos\phi \sin\theta \cos\theta + I_{xz} \left((\cos\theta)^2 - (\cos\phi)^2 (\sin\theta)^2\right)\right] \psi^2\right. \\ &+ \left.[(I_{xx} + I_{yy} - I_{zz}) \cos\phi \sin\theta + 2I_{xz} \cos\theta]\, \psi\dot{\phi} - I_{xz}\dot{\phi}^2\right\} \bar{j} \\ &+ \left\{I_{xz}\ddot{\phi} + \left[(I_{xx} - I_{yy}) \sin\phi \sin\theta \cos\theta - I_{xz} \sin\phi \cos\phi\, (\sin\theta)^2\right] \psi^2\right. \\ &+ \left.[(I_{xx} - I_{yy} + I_{zz}) \sin\phi \sin\theta]\, \psi\dot{\phi}\right\} \bar{k}.\end{aligned}$$

We now are ready to form the equations of motion. We seek an equation of motion for ϕ that does not contain unknown reactions. The force equations of motion will contain A_x, A_y, and A_z, so we will gain nothing by actually forming those equations. We also can see that the unknown couple reactions M_y and M_z would appear in the equations governing the moment sums about the y and z axes. In contrast, none of the reactions exert a moment about the x axis, so we focus solely on that term. Two forces exert a moment about that axis: the aerodynamic force F, whose lever arm about the x axis is d, and gravity, which acts in the (fixed) vertical direction. Rather than trying to visualize the moment of gravity about the x axis, we proceed formally by using a cross product. The angle between the vertical and the z' axis is ψ, so

$$m\bar{g} = mg\left(\sin\psi\, \bar{j}' - \cos\psi\, \bar{k}'\right).$$

The position of the center of mass is readily described in terms of components relative to xyz:

$$\bar{r}_{G/A} = \left(\frac{\ell}{2} + b\right)\bar{i} - \frac{\ell}{2}\bar{k}.$$

To evaluate the cross product of these vectors, both need to be described in terms of components relative to the same coordinate system. We select xyz for this purpose, because those are the axes for the moment sums. The transformation from $x'y'z'$ to xyz is described by $[R]$, so we have

$$\{mg\} = mg\,[R]\begin{Bmatrix} 0 \\ -\sin\psi \\ \cos\psi \end{Bmatrix} = mg\begin{Bmatrix} \sin\theta\cos\psi \\ \cos\phi\sin\psi + \sin\phi\cos\theta\cos\psi \\ \sin\phi\sin\psi - \cos\phi\cos\theta\cos\psi \end{Bmatrix}.$$

The gravitational moment is $\bar{r}_{G/A} \times m\bar{g}$. We require only the x component of this moment. We add that result to the moment of $\bar{F}$ about the x axis and equate that result to the x component of $d\bar{H}_A/dt$ obtained previously. This gives

$$\Sigma \bar{M}_A \cdot \bar{i} = -Fd + mg\frac{\ell}{2}(\cos\phi \sin\psi + \sin\phi \cos\theta \cos\psi) = \dot{\bar{H}}_A \cdot \bar{i}$$
$$= -I_{xx}\ddot{\phi} + \left[(I_{yy} - I_{zz}) \sin\phi \cos\phi (\sin\theta)^2 + I_{xz} \sin\phi \sin\theta \cos\theta\right] \dot{\psi}^2.$$

Substituting the values of the inertia properties yields

$$\frac{1}{3}\ddot{\phi} - \left[\frac{1}{3}\cos\phi \sin\theta + \left(\frac{1}{4} + \frac{1}{2}\frac{b}{\ell}\right)\cos\theta\right](\sin\phi \sin\theta)\,\dot{\psi}^2$$
$$+ \frac{g}{2\ell}(\cos\phi \sin\psi + \sin\phi \cos\theta \cos\psi) = \frac{Fd}{m\ell}. \quad \triangleleft$$

It was stated that $\dot{\psi}$ is constant, so we may set $\psi = \dot{\psi}t$. The value of θ presumably is known, so the preceding is the differential equation of motion governing ϕ. It is nonlinear, and its coefficients depend on time as a consequence of the known variation of ψ. The solution of this differential equation satisfying specified initial conditions could be obtained numerically if all parameter values were provided. The case in which the shaft is straight, $\theta = 0$, provides a check for this expression. The equation of motion then reduces to $\ddot{\phi}/3 + (g/2\ell)\sin\phi = Fd/m\ell$. This is the equation of a pendulum subjected to an external moment, which is what the system reduces to when $\theta = 0$, because precession of the shaft becomes irrelevant.

6.2 PLANAR MOTION

In planar motion there is a single axis of rotation, whose direction is the normal to the plane. To the extent that it is a special case of arbitrary spatial motion, there is no reason to consider planar motion separately. However, many systems are limited to planar motion, and the simple nature of this type of motion affords a good opportunity to delve into some interesting effects, such as friction.

To derive the equations of planar motion from the general set, we define xyz to be a body-fixed coordinate system whose origin is an allowable point for summing moments, with the further specification that the z axis be perpendicular to the plane of motion. To describe the motion of the center of mass we define a convenient coordinate system $x'y'z'$ whose z' axis also is perpendicular to the plane of motion. The angular motion and the acceleration of the center of mass correspondingly are

$$\bar{\omega} = \omega\bar{k}, \quad \bar{\alpha} = \dot{\omega}\bar{k}, \quad \bar{a}_G = a_{Gx'}\bar{i}' + a'_{Gy}\bar{j}'. \tag{6.2.1}$$

For a body having arbitrary inertia properties the angular momentum is

$$\bar{H}_G = -I_{xz}\omega\bar{i} - I_{yz}\omega\bar{j} + I_{zz}\omega\bar{k}. \tag{6.2.2}$$

The corresponding equations of motion are

$$\Sigma \bar{F} \cdot \bar{i}' = ma_{Gx'}, \quad \Sigma \bar{F} \cdot \bar{j}' = ma_{Gy'}, \quad \Sigma \bar{F} \cdot \bar{k}' = 0,$$
$$\Sigma \bar{M}_A \cdot \bar{i} = -I_{xz}\dot{\omega} + I_{yz}\omega^2, \quad \Sigma \bar{M}_A \cdot \bar{j} = -I_{yz}\dot{\omega} + I_{xz}\omega^2, \tag{6.2.3}$$
$$\Sigma \bar{M}_A \cdot \bar{k} = I_{zz}\dot{\omega}.$$

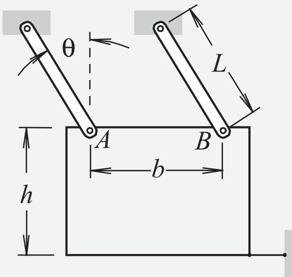
The three force equations and the equation governing the moment about the z axis are the same as those developed in elementary dynamics courses. If the z axis is not principal, that is, if $I_{xz} \neq 0$ or $I_{yz} \neq 0$, then the remaining moment equations describe the resultant moments about axes lying in the plane required to constrain the body from rotating about those axes. We could have anticipated the need for these moment resultants by considering Eq. (6.2.2), which shows that the angular momentum is not parallel to $\bar{\omega}$, and therefore not constant, if the z axis is not principal. The portion of the moments about the x and y axes that are proportional to ω^2 are gyroscopic moments that are the consequence of unsymmetrical distributions of mass relative to the xy plane.

For situations concerning dynamic imbalance of rotating machinery, such as the system treated in Example 6.1, consideration of these restraining moments is vital to the analysis. However, the force equations and the moment equation about the z axis are independent of the the values of I_{xz} and I_{yz}. Thus the same motion in the xy plane will occur regardless of how the body's mass is distributed in the z direction. For this reason most planar motion analyses implicitly assume that the distribution is such that the body is symmetric relative to the xy plane. Then the orientations of the x and y axes are irrelevant to the moment equation. This leaves us free to let these axes be parallel to the x' and y' axes, whose orientations are selected to expedite the description of $\bar{a}_G$ and the force components. Thus the equations of planar motion reduce to

$$\boxed{\begin{gathered} \Sigma \bar{F} \cdot \bar{i}' = ma_{Gx}, \quad \Sigma \bar{F} \cdot \bar{j}' = ma_{Gy}, \\ \Sigma \bar{M}_A \cdot \bar{k} = I_{zz}\dot{\omega}. \end{gathered}} \tag{6.2.4}$$

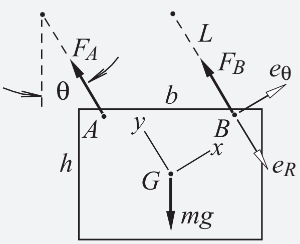
It follows that a system in planar motion is governed by three scalar equations of motion for each body in the system.

EXAMPLE 6.6 The rectangular plate, whose mass is m, serves as a fire door. In case of an emergency, the cable holding the plate is severed and the door swings down under the restraint of the rigid links that suspend the plate from the ceiling. Derive a differential equation of motion governing the angle of inclination θ of the links. Also derive expressions for the forces exerted by the links on the plate. The mass of each link is negligible.



Example 6.6

SOLUTION This plate undergoes a translational motion because the links remain parallel for any θ. Thus it might seem that the system is unremarkable, but the solution will serve to emphasize the fundamental changes relative to statics concepts that are required to analyze any system in motion. In the free-body diagram of the plate, the links are considered to exert tensile forces. This is a consequence of neglecting the mass of the links, which makes them two-force members. Note that the forces on the left and right are taken to be different because the arrangement is not symmetrical when θ is nonzero.



Free-body diagram of the translating fire door.

In general, all points in a translating rigid body have the same acceleration. Hence the acceleration of the center of mass matches that of either point where a link is attached. The latter follow circular paths centered at the respective upper pivot, so we have

$$\bar{a}_G = \bar{a}_A = \bar{a}_B = -L\dot{\theta}^2\bar{e}_R + L\ddot{\theta}\bar{e}_\theta.$$

Because the box translates, we must sum moments about the center of mass. As shown in the free-body diagram, we align the x and y axes with the polar directions for $\bar{a}_G$, so that

$$\bar{a}_G = \bar{a}_A = \bar{a}_B = L\dot{\theta}^2\bar{j} + L\ddot{\theta}\bar{i}.$$

The angular momentum in translation is identically zero, so moments of inertia need not be computed. The corresponding equations of motion are

$$\Sigma\bar{F}\cdot\bar{i} = -mg\sin\theta = mL\ddot{\theta},$$

$$\Sigma\bar{F}\cdot\bar{j} = F_A + F_B - mg\cos\theta = mL\dot{\theta}^2,$$

$$\Sigma\bar{M}_G\cdot\bar{k} = F_B\cos\theta\left(\frac{b}{2}\right) + F_B\sin\theta\left(\frac{h}{2}\right) - F_A\cos\theta\left(\frac{b}{2}\right) + F_A\sin\theta\left(\frac{h}{2}\right) = 0.$$

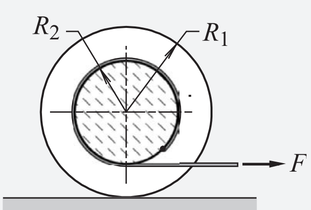
The solution of these equations is

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0,$$
$$F_A = \frac{1}{2}\left(1 + \frac{h}{b}\tan\theta\right)\left(mg\cos\theta + mL\dot{\theta}^2\right),$$
$$F_B = \frac{1}{2}\left(1 - \frac{h}{b}\tan\theta\right)\left(mg\cos\theta + mL\dot{\theta}^2\right).$$

◁

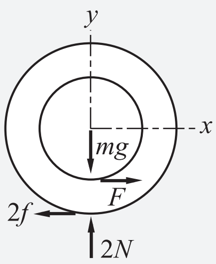
Note that the differential equation for θ is identical to that for a simple pendulum formed by attaching a particle to the end of a cable of length L.

EXAMPLE 6.7 A cable drum, which is shown in cross section, consists of a cylinder having radius R_2 that is capped at both ends by circular plates whose radius is R_1. A cable is wrapped around the cylinder and then pulled out horizontally by a force $\bar{F}$, as shown. The system was at rest when the force was applied. The mass of the wrapped wire and the drum is m, the mass of the unwrapped segment of cable is negligible, and the centroidal radius of gyration is κ. The static and kinetic coefficients of friction between the end plates and the ground are μ_s and μ_k, respectively. (a) Determine the acceleration of the center of the drum assuming that the end plates roll without slipping. (b) Determine the maximum magnitude of $\bar{F}$ for which slipping will not occur. (c) Determine the angular acceleration and the acceleration of the center of the drum when the magnitude of $\bar{F}$ exceeds the value obtained in Part (b).



Example 6.7

SOLUTION Along the way to the solution of this problem, which highlights the various aspects of the Coulomb friction model, we arrive at a result that demonstrates once again that intuition based on experience with static systems cannot be trusted. A free-body diagram shows $\bar{F}$ and the weight, as well as normal forces $2\bar{N}$ and friction forces $2\bar{f}$ exerted between the two end plates and the ground. The relationship between $\bar{N}$ and $\bar{f}$ is governed by Coulomb's laws, which we will address separately. Here the friction force $\bar{f}$ is depicted as acting to the left, based on the assumption that it opposes the action of $\bar{F}$, but we will be able to verify that assumption. We select the center of mass as the reference point for summing moments because the drum is in general motion. The axes of the body-fixed xyz coordinate system are defined to be horizontal and vertical, which matches the direction of the motion.



Free-body diagram of the cylinder with rigid end caps.

We know that the drum is in planar motion and that the center follows a horizontal path. Thus, regardless of the slipping condition, we know that

$$\bar{\alpha} = \dot{\omega}\bar{k}, \quad \bar{a}_G = \dot{v}\bar{i}. \tag{1}$$

Note that we have selected the sense of positive v and ω to match the direction of the respective axis, but any other selection is acceptable if it is implemented consistently. In the special case where the drum rolls without slipping, the rotation will occur about the negative z axis, with

$$v = -\omega R_1 \text{ for no slippage.} \tag{2}$$

The centroidal moment of inertia corresponding to the given radius of gyration is $I_{zz} = m\kappa^2$. The equations of motion are

$$\begin{aligned} \Sigma \bar{F} \cdot \bar{i} &= F - 2f = m\dot{v}, \\ \Sigma \bar{F} \cdot \bar{j} &= 2N - mg = 0, \\ \Sigma \bar{M}_A \cdot \bar{k} &= FR_2 - 2fR_1 = m\kappa^2\dot{\omega}. \end{aligned} \tag{3}$$

There are four unknowns, f, N, $\dot{v}$, and $\dot{\omega}$, in these three equations. In general, the only available relations between force and kinematical variables in a kinetics problem are the equations of motion, so the additional equation required for solving Eqs. (3) must relate f and N or $\dot{v}$ and $\dot{\omega}$. This is where the question of slippage enters. It is given that the drum starts from rest, so initially the drum rolls without slipping. This means that Eq. (2) applies. The contacting surfaces then are described by Coulomb's law for static friction, because the surfaces do not move relative to each other. Thus we know *a priori* only that the magnitude of $\bar{f}$ is less than $\mu_s N$ times. The actual magnitude and sense of $\bar{f}$ are dictated by the laws of motion. In effect, the friction force is a constraint force—it prevents the relative movement of the contacting surfaces.

When Eq. (2) is substituted into Eqs. (3), the result is

$$\dot{v} = \frac{FR_1(R_1 - R_2)}{m\left(R_1^2 + \kappa^2\right)}, \quad f = \frac{1}{2}\left(\frac{R_1 R_2 + \kappa^2}{R_1^2 + \kappa^2}\right)F. \tag{4} \triangleleft$$

The signs of $\dot{v}$ and f are both positive, so the sense initially assumed for both quantities is correct. This means that the drum rolls to the right and rotates clockwise. Kinematically, this implies that the cable gets wound around the drum, even though $\bar{F}$ seems to act to pull it off the drum. The explanation of this behavior lies in the result for $\dot{v}$. We know that $F(R_1 - R_2)$ is the moment of $\bar{F}$ about the contact point, and $m\left(R_1^2 + \kappa^2\right)$ is the moment of inertia about an axis parallel to z intersecting the contact point. Replacing $\dot{v}$ with $-\dot{\omega}R_1$ shows that the motion is a consequence of angular acceleration caused by $\bar{F}$ exerting a moment about the contact point. (Recall that the contact point is an allowable point when there is no slipping, provided that the wheel is balanced.) The maximum magnitude of $\bar{F}$ that can be sustained corresponds to impending slippage, at which $\left|\bar{f}\right| = \mu_s N$. From the force equation of motion in the y direction we have $N = \frac{1}{2}mg$, so the second of Eqs. (4) correspondingly gives

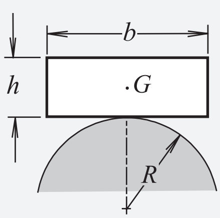
$$F_{\max} = \frac{R_1^2 + \kappa^2}{R_1 R_2 + \kappa^2}\mu_s mg. \qquad \triangleleft$$

When slipping occurs because $\left|\bar{F}\right| > F_{\max}$, the friction force is governed by the kinetic portion of Coulomb's friction law, which states that $\left|\bar{f}\right| = \mu_k N$ in opposition to the relative sliding motion. We know this sense from the solution of the no-slip case, which indicated that $\bar{f}$ acts to the left. Thus we substitute $f = \mu_k N = \mu_k mg$ into Eqs. (3). Note that, when slippage occurs, the friction force no longer acts to constrain the sliding, so there is no kinematical relation between ω and v. The solution of Eqs. (3) now is

$$\dot{v} = \frac{F}{m} - \mu_k g, \quad \dot{\omega} = \frac{FR_2}{m\kappa^2} - \frac{\mu_k g R_1}{\kappa^2}. \qquad \triangleleft$$

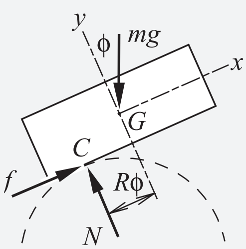
The positive sign for $\dot{\omega}$ indicates that the rotation is counterclockwise. Thus the drum will accelerate to the right and the cable will unwrap when F is sufficiently large to cause slippage.

EXAMPLE 6.8 The homogeneous box, whose mass is m, is placed horizontally on the semicylindrical surface, such that contact is below the centroid G. Assuming that the box does not slip, derive the differential equation of motion governing the angle ϕ by which the box rotates away from horizontal.



Example 6.8

SOLUTION The center of the wheel in the previous problem followed a straight path, whereas the motion of the center of the box had no obvious feature. Thus this example will give a more complete description of the analysis of bodies that roll. The free-body diagram of the bar must show the bar at an arbitrary angle of elevation ϕ. The friction and normal forces act at the contact point C. In the absence of slippage, the distance along the bottom from point C to the centerline of the box must equal the arc length $R\phi$ along the circle from the contact point to the top from the center. Note that neither the magnitude nor sense of the friction force is known, because it acts as a constraint force that prevents relative motion of the contacting surfaces. The only allowable point for summing moments is the center of mass, and it makes sense from a kinematical viewpoint to orient the x and y axes consistently with the local plane of contact with the cylindrical surface.



Free-body diagram of the box showing the contact forces that prevent slippage.

The position of the box clearly is specified by the value of ϕ. (In the terminology of the analytical dynamics concepts of the following chapters, ϕ is a generalized coordinate.) We must express the acceleration of point G in terms of ϕ consistent with the no-slip condition. The rate of rotation is $\dot{\phi}$, and we know that point C has zero velocity, so the velocity of point G is given by

$$\bar{v}_G = \bar{\omega} \times \bar{r}_{G/C}, \quad \bar{\omega} = \dot{\phi}\bar{k}.$$

For the arbitrary position described by the free-body diagram, we have

$$\bar{r}_{G/C} = R\phi\bar{i} + \frac{1}{2}h\bar{j},$$

which gives

$$\bar{v}_G = -\frac{1}{2}h\dot{\phi}\bar{i} + R\phi\dot{\phi}\bar{j}. \tag{1}$$

Because this expression describes an arbitrary position, it may differentiated. The unit vectors are not constant, so we use the partial differential technique for relative motion. This gives

$$\begin{aligned}\bar{a}_G &= \frac{\partial \bar{v}_G}{\partial t} + \bar{\omega} \times \bar{v}_G \\ &= -\frac{1}{2}h\ddot{\phi}\bar{i} + R\left(\dot{\phi}^2 + \phi\ddot{\phi}\right)\bar{j} + \dot{\phi}\bar{k} \times \left(-\frac{1}{2}h\dot{\phi}\bar{i} + R\phi\dot{\phi}\bar{j}\right) \qquad (2) \\ &= -\left(\frac{1}{2}h\ddot{\phi} + R\phi\dot{\phi}^2\right)\bar{i} + \left(R\dot{\phi}^2 + R\phi\ddot{\phi} - \frac{1}{2}h\dot{\phi}^2\right)\bar{j}.\end{aligned}$$

The Appendix gives the centroidal moment of inertia. The corresponding equations of motion are

$$\Sigma \bar{F} \cdot \bar{i} = f - mg \sin\phi = ma_{Gx} = -m\left(\frac{1}{2}h\ddot{\phi} + R\phi\dot{\phi}^2\right),$$
$$\Sigma \bar{F} \cdot \bar{j} = N - mg\cos\phi = ma_{Gy} = m\left(R\dot{\phi}^2 + R\phi\ddot{\phi} - \frac{1}{2}h\dot{\phi}^2\right), \tag{3}$$
$$\Sigma \bar{M}_G \cdot \bar{k} = f\left(\frac{h}{2}\right) - N(R\phi) = I_{zz}\dot{\omega} = \frac{1}{12}m\left(b^2 + h^2\right)\ddot{\phi}.$$

There are three unknowns, ϕ, N, and f, in these three equations, so we may proceed to their solution. To obtain the desired differential equation we eliminate the reactions. The force equations give

$$f = m\left(g\sin\phi - \frac{1}{2}h\ddot{\phi} + R\phi\dot{\phi}^2\right), \quad N = m\left(g\cos\phi + R\dot{\phi}^2 + R\phi\ddot{\phi} - \frac{1}{2}h\dot{\phi}^2\right), \tag{4}$$

which, when substituted into the moment equations, yield

$$\left[\frac{h^2}{3} + \frac{b^2}{12} + R^2\phi^2\right]\ddot{\phi} + R^2\phi\dot{\phi}^2 = \frac{gh}{2}\sin\phi - gR\phi\cos\phi. \tag{5} \triangleleft$$

If the initial values of ϕ and $\dot{\phi}$ are specified, this differential equation could be solved for ϕ as a function of t. We would use numerical techniques to obtain the solution, because an analytical solution would be difficult. When one performs a numerical analysis it is useful to have test solutions to verify the analysis. We may derive one such solution by considering ϕ to be small, which enables us to linearize Eq. (5). We introduce the approximations $\cos\phi \approx 1$, $\sin\phi \approx \phi$, and drop any terms that have quadratic or higher powers of ϕ. The resulting equation for small rotations is

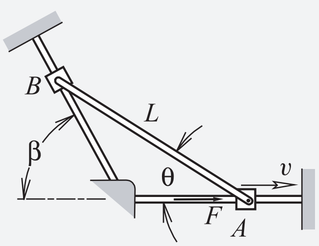
$$\left(\frac{h^2}{3} + \frac{b^2}{12}\right)\ddot{\phi} + g\left(R - \frac{h}{2}\right)\phi = 0. \tag{6}$$

When $h < 2R$, the response obtained from this equation is sinusoidal, corresponding to oscillations about a stable static equilibrium position. The results obtained from solving Eq. (6) should then be a good approximation of those for Eq. (5). In contrast, when $h > 2R$, the solution of the linearized equation of motion is exponential, corresponding to continuous movement away from an unstable static equilibrium position. The solutions of Eqs. (5) and (6) then will be consistent only in the early phase of the response, when ϕ is small.

The transition from stability to instability has a simple explanation. In the case where the bar is slender, $h < 2R$, the center of mass rises as ϕ increases. Thus $\phi = 0$ is a position of minimum potential energy. In contrast, when $h > 2R$, the center of mass descends with movement away from the equilibrium position, which means that $\phi = 0$ corresponds to maximum potential energy. Note that the stability transition is independent of the value of the length b, whose only effect is its influence on the effective moment of inertia of the bar, which is the coefficient of the angular acceleration term in the equation of motion. Thus, when the equilibrium position is stable, the value of b solely affects the frequency of the stable oscillation.

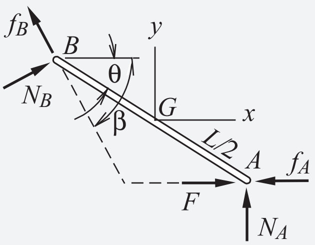
The derivation of Eq. (5) assumed that the box does not slip over the cylinder, but that might not be true. At every instant at which the differential equation has been solved, the no-slip condition should be verified. This requires that, in addition to knowing the instantaneous value of ϕ, we would need to determine $\dot{\phi}$, either analytically or by finite differences. The corresponding $\ddot{\phi}$ may be found from Eq. (5), which then allows us to evaluate f and N according to Eqs. (4). Coulomb's friction laws state that the maximum friction force that can be developed between surfaces that rub against each other is $\mu_s N$. It follows that, if the computed value of $|f|$ is less than $\mu_s N$, then the solution is acceptable. Conversely, if $\left|\bar{f}\right|$ exceeds $\mu_s N$, slippage will occur. In addition, one should monitor the value of $\left|\bar{N}\right|$. Negative N indicates that the surface must pull on the box, which is not possible. When the criteria for no-slip or positive contact fail to be met, the problem must be reformulated. In the case of slippage, kinematical equations (1) and (2) are no longer valid, but the friction force is known. Occurrence of a negative N would indicate that the box has left the surface, in which case it would be in free motion.

EXAMPLE 6.9 The slender bar moves in the horizontal plane under the constraint of collars A and B. The horizontal force F acting on collar A is such that the velocity of this collar is observed to be a constant value v to the right. The mass of bar AB is m, and the collars have negligible mass. The coefficient of sliding friction between each collar and its guide bar is μ. Derive an expression for the value of F as a function of the angle of elevation θ and the other parameters of the system.



Example 6.9

SOLUTION This example highlights some of the issues entailed in applying the Newton–Euler equations of motion to analyze linkages, without introducing the complications associated with multibody systems. Properly accounting for the friction forces is another aspect to be covered. The condition that the collars follow the guide bars is enforced by the normal constraint forces $\bar{N}_A$ and $\bar{N}_B$. The friction forces $\bar{f}_A$ and $\bar{f}_B$ oppose the movement of the collars relative to the respective guide bars, as shown in the free-body diagram. The sense in which the free-body diagram depicts the friction forces is set by the condition that these forces act in opposition to the movement of each collar. However, either normal force might act in the opposite sense from the one that appears in the diagram, depending on which direction either end would move if there were no normal force. We will deal with this complication when we solve the equations of motion. The applied force $\bar{F}$ also acts as a constraint force because it imposes the kinematical condition that end A has a constant velocity. Gravity has been omitted from the free-body diagram, which implies that the system lays in the horizontal plane or that the applied force $\bar{F}$ is large compared to mg.



Free-body diagram of the sliding bar.

Bar AB is in general motion, so we place the origin of xyz at the center of mass. It will be necessary to relate the motions of ends A and B. Doing so is assisted by aligning the axes such that at the instant described by the equations of motion the axes are horizontal and vertical.

It is apparent that knowledge of the value of the elevation angle θ fully specifies the position of the bar. Furthermore, $\dot{\theta}$ is the rotation rate of the bar, which means that we need to establish the kinematical relationship between $\ddot{\theta}$ and the given speed v. As we saw in Chapter 4, we could obtain this relationship by differentiating algebraic expressions derived from trigonometry. Instead, to convey a picture of how multibody linkages could be treated, we pursue a kinematical analysis relating the velocity and acceleration of constrained points. The velocity of end A is stated to be constant v to the right, so end B must move downward parallel to its guide bar. Because $\dot{\theta}$ is the rotation rate about the z axis, we have

$$\bar{v}_A = v\bar{i}, \quad \bar{v}_B = v_B\left(\cos\beta\bar{i} - \sin\beta\bar{j}\right), \quad \bar{\omega} = -\dot{\theta}\bar{k},$$
$$\bar{a}_A = \bar{0}, \quad \bar{a}_B = \dot{v}_B\left(\cos\beta\bar{i} - \sin\beta\bar{j}\right), \quad \bar{\alpha} = -\ddot{\theta}\bar{k}.$$

We relate the velocities of both ends, using θ to describe the relative position. This gives

$$\bar{v}_B = \bar{v}_A + \bar{\omega}\times\bar{r}_{B/A},$$
$$v_B\left(\cos\beta\bar{i} - \sin\beta\bar{j}\right) = v\bar{i} + \left(-\dot{\theta}\bar{k}\right)\times\left(-L\cos\theta\bar{i} + L\sin\theta\bar{j}\right).$$

The scalar equations we obtain by matching like components are

$$\bar{v}_B\cdot\bar{i} = v_B\cos\beta = v + \dot{\theta}L\sin\theta, \quad \bar{v}_B\cdot\bar{j} = -v_B\sin\beta = \dot{\theta}L\cos\theta.$$

Elimination of v_B from these equations and use of a trigonometric identity lead to

$$\dot{\theta} = -\frac{v\sin\beta}{L\cos(\beta-\theta)}. \tag{1}$$

We now apply the analogous analysis to acceleration,

$$\bar{a}_B = \bar{a}_A + \bar{\alpha} \times \bar{r}_{B/A} - \omega^2 \bar{r}_{B/A},$$

$$\dot{v}_B \left(\cos\beta \bar{i} - \sin\beta \bar{j}\right) = \left(-\ddot{\theta}\bar{k}\right) \times \left(-L\cos\theta \bar{i} + L\sin\theta \bar{j}\right) - \dot{\theta}^2 \left(-L\cos\theta \bar{i} + L\sin\theta \bar{j}\right).$$

Matching like components gives

$$\bar{a}_B \cdot \bar{i} = \dot{v}_B \cos\beta = \ddot{\theta} L\sin\theta + \dot{\theta}^2 L\cos\theta,$$

$$\bar{a}_B \cdot \bar{j} = -\acute{v}_B \sin\beta = \ddot{\theta} L\cos\theta - \dot{\theta}^2 L\sin\theta.$$

Elimination of $\dot{v}_B$ leads to

$$\ddot{\theta} = -\dot{\theta}^2 \frac{\sin(\beta-\theta)}{\cos(\beta-\theta)} = -\left(\frac{v}{L}\right)^2 \frac{(\sin\beta)^2 \sin(\beta-\theta)}{\cos(\beta-\theta)^3}. \tag{2}$$

Note that we could have obtained the same expression directly by differentiating Eq. (1) with respect to time.

Equations (1) and (2) allow us to describe the acceleration of the center of mass in terms of v and θ. Relating this point to the constrained point A gives

$$\begin{aligned}\bar{a}_G &= \bar{a}_A + \bar{\alpha} \times \bar{r}_{G/A} - \bar{\omega}^2 \times \bar{r}_{G/A} \\ &= \frac{L}{2}\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\bar{i} + \frac{L}{2}\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\bar{j}.\end{aligned}$$

Substitution of Eqs. (1) and (2) and a trigonometric identity eventually lead to

$$\bar{a}_G = \frac{v^2}{2L}\frac{(\sin\beta)^2}{\cos(\beta-\theta)^3}\left(\cos\beta \bar{i} - \sin\beta \bar{j}\right). \tag{3}$$

[The preceding expression indicates that $\bar{a}_G$ is parallel to the left guide bar. This is a consequence of a general property of rigid-body motion. If G is the midpoint between points A and B, describing $\bar{a}_G$ in terms of $\bar{a}_A$ and $\bar{a}_B$ leads to $\bar{a}_G = \frac{1}{2}(\bar{a}_A + \bar{a}_B)$.]

We now are ready to form the equations of motion. The centroidal moment of inertia of the bar is $(1/12)\,mL^2$, so we have

$$\Sigma \bar{F} \cdot \bar{i} = F - f_A + N_B \sin\beta - f_B \cos\beta = m\left(\bar{a}_G \cdot \bar{i}\right) = \frac{mv^2}{2L}\frac{(\sin\beta)^2\cos\beta}{\cos(\beta-\theta)^3},$$

$$\Sigma \bar{F} \cdot \bar{j} = N_A + N_B \cos\beta + f_B \sin\beta = m\left(\bar{a}_G \cdot \bar{j}\right) = -\frac{mv^2}{2L}\frac{(\sin\beta)^2\sin\beta}{\cos(\beta-\theta)^3},$$

$$\begin{aligned}\Sigma \bar{M}_A \cdot \bar{k} &= N_A\left(\frac{L}{2}\cos\theta\right) - f_A\left(\frac{L}{2}\sin\theta\right) - \left[N_B\cos(\beta-\theta)\right]\frac{L}{2} \\ &\quad - \left[f_B\sin(\beta-\theta)\right]\frac{L}{2} = I_{zz}\left(-\ddot{\theta}\right) = -\frac{1}{12}mL^2\left(\frac{v}{L}\right)^2\frac{(\sin\beta)^2\sin(\beta-\theta)}{\cos(\beta-\theta)^3}.\end{aligned} \tag{4}$$

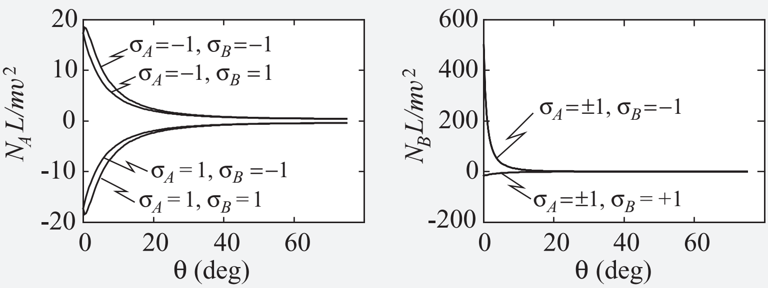
These three equations of motion contain five unknown variables: F, N_A, N_B, f_A, and f_B. The required additional equations relate the sliding friction and normal forces at each collar. Coulomb's law asserts that, when two surfaces slide over each other, then $\left|\bar{f}\right| = \mu_k \left|\bar{N}\right|$. The free-body diagram properly showed the sense of each friction force, but it was noted that either normal force might actually be in the opposite sense from the one that was used to construct the diagram. Equations (4) may be employed if either normal force is reversed, provided that N_A or N_B is replaced with its negative value. In other words, we may obtain a set of equations that are valid in all cases by defining N_A and N_B to be positive values, and replacing N_A with $\sigma_A N_A$ and N_B with $\sigma_B N_B$ in Eqs. (4), with $\sigma_A = \pm 1$ and $\sigma_B = \pm 1$. Correspondingly, Eqs. (4) may be written as

$$[H(\sigma_A, \sigma_B)] \begin{Bmatrix} F \\ N_A \\ N_B \end{Bmatrix} = \frac{mv^2}{L}\{K\},$$

$$[H(\sigma_A, \sigma_B)] = \begin{bmatrix} 1 & -\mu & (\sigma_B \sin\beta - \mu\cos\beta) \\ 0 & \sigma_A & (\sigma_B\cos\beta + \mu\sin\beta) \\ \sin(\theta) & \sigma_A\cos(\theta) - \mu\sin(\theta) & -\sigma_B\cos(\beta-\theta) - \mu\sin(\beta-\theta) \end{bmatrix},$$

$$\{K\} = \frac{(\sin\beta)^2}{\cos(\beta-\theta)^3}\left[0.5\ \cos\beta \quad -0.5\sin\beta \quad -\sin(\beta-\theta)/6\right]^{\mathrm{T}}. \tag{5}$$

A corrollary of this modification is that solutions of Eqs. (5) are meaningful only if N_A and N_B are both positive. Thus we select an angle of inclination in the range $0 < \theta < \beta$ and solve Eqs. (5) for each permutation of $\sigma_A = 1$ or -1 and $\sigma_B = 1$ or -1. We discard the solution for a set of signs if either N_A or N_B is found to be negative. This leads to the possiblity of multiple solutions, which is a common occurrence in nonlinear systems. However, the normal forces found for $\beta = 75^\circ$ and $\mu = 0.25$, which are depicted nondimensionally in the first set of graphs, indicate that $\sigma_A = \sigma_B = -1$ is the only valid case over the entire range of θ.

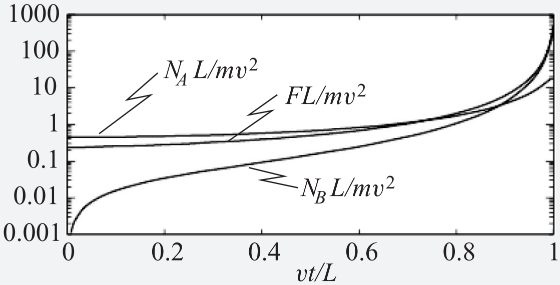


Values of the normal forces N_A and N_B as functions of θ when $\beta = 75^\circ$, $\mu_k = 0.25$, for each permutation of their sign factors

The concluding graph shows F and the reaction forces as functions of the nondimensional time vt/L. This parameter is related to θ by the law of sines. If we take $t = 0$ to correspond to $\theta = \beta$, then we have

$$\frac{vt}{\sin(\beta - \theta)} = \frac{L}{\sin\beta}.$$

One reason for the rising value of F as t increases (decreasing θ) is the presence of $\cos(\beta - \theta)^3$ in the denominators of both $\bar{a}_G$ and $\ddot{\theta}$.



Forces as functions of nondimensional elapsed time corresponding to $\sigma_A = \sigma_B = -1$; $\beta = 75°$, $\mu_k = 0.25$

6.3 NEWTON–EULER EQUATIONS FOR A SYSTEM

The reduction from an arbitrary collection of particles to a rigid body had important benefits, in that doing so reduced enormously the number of kinematical variables, simultaneously with enabling us to ignore the interaction forces exerted between the particles. Both gains result from the recognition that the particles forming a rigid body are mutually constrained. In the same manner, it sometimes is useful to consider interacting rigid bodies as a unified system. The concepts we develop here consider several bodies to act in unison. Doing so lessens the need to consider the forces associated with the interaction of these bodies. However, in doing so, fewer equations of motion will be available. Thus the concepts that follow should be considered to supplement, rather than replace, the basic Newton–Euler equations for each body in a system.

To assemble a system from its constituent rigid bodies, consider the pair of bodies in Fig. 6.2, which are loaded by a set of external forces that are not labeled. In addition, Body 2 exerts force $\bar{f}_{1,2}$ and couple $\bar{M}_{1,2}$ on Body 1 as a result of their interaction, which can be the result of their being connected or as a consequence of field effects such as gravity. The influence of Body 1 on Body 2 consists of force $\bar{f}_{2,1}$ and couple $\bar{M}_{2,1}$. These

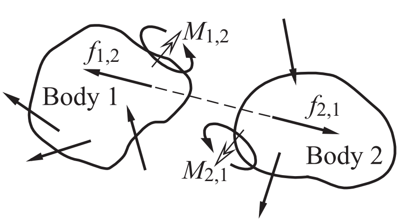


Figure 6.2. Forces acting on a pair of interconnected rigid bodies.

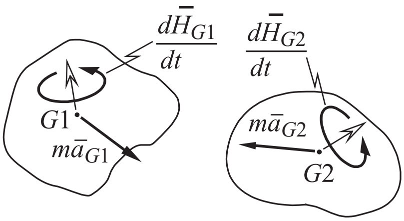


Figure 6.3. Inertial force–couple systems equivalent to the forces acting on a pair of interacting rigid bodies.

interaction effects are governed by Newton's Third Law, which means that $\bar{f}_{2,1} = -\bar{f}_{1,2}$, $\bar{M}_{2,1} = -\bar{M}_{1,2}$, and $\bar{f}_{2,1}$ and $\bar{f}_{1,2}$ share the same line of action.

The method by which we combine these two bodies into a system appeared in Fig. 6.1, which depicted the force system acting on a rigid body as being equivalent in its effect to a dynamic $m\bar{a}_G$ applied at the center of mass and a dynamic couple $d\bar{H}_G/dt$. Figure 6.3 applies the same representation to each body considered in the previous figure. The forces and couples acting on each body in Figs. 6.2 and 6.3 are equivalent, so their combined effect must also be equivalent. It follows that their sum, which represents the resultant force acting on the system, is the same, and their moment about any point B also is the same. The same conclusion would be reached if the system were composed of N bodies, so the *Newton–Euler equations for the system* are

$$\boxed{\begin{aligned} \Sigma\bar{F} &= \sum_{j=1}^{N} m_j \bar{a}_{Gj}, \\ \Sigma\bar{M}_B &= \sum_{j=1}^{N} \frac{d\bar{H}_{Gj}}{dt} + \sum_{j=1}^{N} \bar{r}_{Gj/B} \times m_j \bar{a}_{Gj}. \end{aligned}} \tag{6.3.1}$$

These equations of motion share with Eq. (6.1.2) the cumbersome feature of having the center-of-mass accelerations appear in the moment equations. For a single body, which was the scope of Eq. (6.1.2), there was no advantage to such an approach. The same is not true here because of an aspect of the resultant force and moment of the actual force system. As was noted, the forces and couples exerted between any pair of rigid bodies satisfies Newton's Third Law. The contributions of each pair to the force and moment sums cancel, so we may ignore any forces or couples that are internal to the system when we formulate the resultants in Eqs. (6.3.1). Another useful aspect of these equations lies in the arbitrariness of point B. The main consideration in selecting the point for a moment sum in the static case is the ability to avoid the appearance of unknown forces (usually reactions) in the moment equilibrium equations. We have developed here a comparable ability for dynamic systems.

To see the possible advantage in the system viewpoint consider two interacting rigid bodies in spatial motion. Describing the motion of each rigid body in isolation from the other yields a total of 12 scalar Newton–Euler equations: three force component equations and three moment component equations for each body. These equations contain the forces and/or couples exerted between each body as unknown reactions, so the solution of the equations must determine, or at least eliminate, those reactions. Considering the two bodies as a single system, so that the interaction forces exerted between them become internal to the system, leads to three scalar force equations and three scalar moment equations that do not contain these forces.

The systems viewpoint does not necessarily lead to a solvable set of equations, because that question depends on the number of external reaction forces and kinematical variables that arise. In the event that Eqs. (6.3.1) are not sufficient to find the desired variables, the equations may be supplemented by Newton–Euler equations of motion for any of the rigid bodies. However, it is important to recognize that Eqs. (6.3.1) are essentially the same as the result of summing the equations of motion for each isolated body, so they do not contain any new information. If there are N bodies in a system, then there are no more than $6N$ independent scalar Newton–Euler equations of motion. Up to six of these may be obtained from Eqs. (6.3.1).

The ability to consider a set of moving parts as a system enables us to qualitatively explain how a bicyclist can maintain balance and maneuver without falling. To avoid going into details that would obscure the discussion, we employ a simplified model of the steering configuration. Our model, which is shown in top view in Fig. 6.4, considers the axis of the steering fork to be perpendicular to the longitudinal axis and to intersect vertically the axis of the front wheel. Under perfect conditions, to follow a straight path the bicycle would be oriented vertically, with the rider's center of mass situated directly over the line connecting the centers of the wheels. The angular momenta of the forward and rear wheels, $\bar{H}_f$ and $\bar{H}_r$, are horizontal in this case, as shown. If it were not for rolling resistance, this motion could be sustained without any effort on the rider's part.

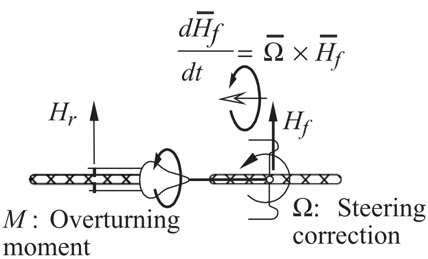


Figure 6.4. Balancing of a bicycle as seen in a top view.

Such ideal conditions cannot be maintained. For example, the rider might lean over or a gust of wind might arise. Such a disturbance, in combination with the reaction of the wheels, creates an overturning moment $\bar{M}$ that acts about the longitudinal axis. (The situation in the figure corresponds to the rider leaning to the left.) This moment must be matched by a corresponding change in the angular momentum. If the rider makes no adjustments and remains stationary relative to the bike, the result will be an angular acceleration in the direction of $\bar{M}$. In other words, the bike would fall over. Instead, the rider turns the handlebars, which causes the steering fork to rotate at some angular speed Ω. This rotation causes the tip of $\bar{H}_f$ to move in the sense of $\bar{\Omega} \times \bar{H}_f$. The consequence of $d\bar{H}_f/dt$ equaling $\bar{M}$ is that the sense of $\bar{\Omega}$ induces a turn to the side in which the bike is tending to lean (left in the case of the figure).

If the rider wishes to return to the direction initially set, then this correctional maneuver must be reversed. The rider turns the steering wheel in the opposite sense, thereby reversing the sense of $d\bar{H}_f/dt$. Thus, in the scenario of Fig. 6.4, after compensating for $\bar{M}$, the rider would turn the handlebars clockwise as viewed from above, thereby generating a $d\bar{H}_f/dt$ effect that is oriented forward. To generate a force system whose resultant moment about the longitudinal axis matches this rate of change of the angular momentum, the rider simultaneously leans to the right, thereby shifting the center mass to the other side. Thus riding in a straight line is actually a sequence of corrective steering maneuvers and shifts of the center of mass. In essence, the bicyclist is both the actuator and controller of a feedback control system. This feature is most evident in children who have just begun to ride a bicycle. For a very experienced bicyclist, the corrective maneuvers are barely perceptible. Also, if the rider's hands are not placed on the handlebars, then the steering wheel turns of its own accord in the manner required to change the angular momentum at a rate that matches the unbalanced moment. In any event the ability to steer is essential to maintaining one's balance.

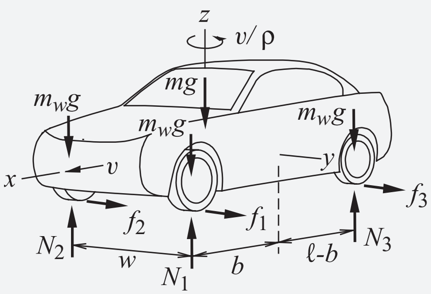
When a rider wishes to execute a steady turn the handlebars are not returned after their initial rotation, so the bicycle is leaning to the side in which the turn is being executed. This situation is essentially as described by Example 6.4, except that it is now convenient to use a reference frame whose origin is the center of mass of the whole bicycle, with the y axis intersecting the center of both wheels. There is a normal force and a friction force acting on each wheel, the resultant of the latter being what is required to make the center of mass of the bicycle follow a circular path at speed v. The angular momenta of both wheels no longer are oriented in fixed directions, because the bicycle is rotating about the vertical axis at v/ρ, where ρ is the radius of the circular path. For a left turn in Fig. 6.4, both $d\bar{H}_f/dt$ and $d\bar{H}_r/dt$ will be rearward. Regardless of the direction of the turn, we know that $d\bar{H}_f/dt + d\bar{H}_r/dt$ must equal the overturning moment of the friction, gravity, and normal forces about the longitudinal axis.

A performer riding a unicycle exploits these same phenomena to maintain left–right balance. Thus falling to the left is controlled by twisting the wheel left, and vice versa. Forward–rear balance requires a different control strategy. This relies on the fact that a falling stick can be kept at a constant angle of tilt if it is given the correct translational acceleration in the horizontal direction. Thus the unicyclist compensates for a tendency to fall forward or back by accelerating in the direction of that tendency. Obviously riding a unicyle is substantially more difficult than riding a bicycle.

This discussion of a bicycle simplifies much, and it also ignores some important effects. For example, if the bicycle begins to tip over, the angular momentum of each wheel will change in the vertical direction. The matching moment is generated at the front and rear wheels in the form of transverse friction forces that form a vertical couple. These forces influence the handling and stability of the bicycle. Nevertheless, the discussion does give a reasonable picture.

EXAMPLE 6.10 An automobile moving at a constant speed v follows a circular path of radius ρ. The track (distance between a pair of wheels) is w and the wheelbase (distance from front to rear axle) is ℓ. The mass of the automobile is m, and the center of mass is on the midplane, at distance b behind the front axle and height h above the ground. Each wheel has radius R, mass m_w, moment of inertia J about its axle, and coefficient of static friction μ. Determine the maximum speed v that may be sustained by the vehicle without skidding or tipping over.

SOLUTION This example explores an interesting effect of the inertia of a wheel as a way of illustrating the analysis of an assembly of bodies. The system consists of the vehicle chassis and the four wheels. The xyz coordinate system is defined in the free-body diagram to have its origin at the center of mass of the chassis, with the x axis forward. Acting at each wheel is the normal force $\bar{N}_j$, which is assumed to be collinear with the weight of the wheel acting at its center of mass of the wheel. Each wheel also has a friction force $\bar{f}_j$, which is depicted in the free-body diagram as acting to the left relative to the forward direction, as they would for a left turn.



Free-body diagram of an automobile executing a turn to the left.

It is reasonable to neglect rolling resistance, so each friction force should be shown as acting perpendicular to the plane of the wheel. However, this introduces a complication regarding the front wheels, because they must be rotated to execute the turn, so the friction forces at the front and rear wheels are not parallel. One consequence is that there is a net component of the friction force in the longitudinal x direction, which means that a friction force representing traction must be applied to maintain a constant speed. To simplify the analysis all friction forces are shown in the diagram as though the front wheels were not steering, so that they act in the transverse y direction.

Another simplification we employ is to consider all centers of mass to be following circles whose normal direction are $\bar{i}$. For the wheels on the left side, which are closer to the center of curvature, the radius of the circle is $\rho - w/2$, whereas the radius for wheels on the right side is $\rho + w/2$. Thus the center-of-mass accelerations are approximated as

$$\bar{a}_{G0} = \rho\Omega^2 \bar{j}, \quad \bar{a}_{G1} = \bar{a}_{G3} = \left(\rho - \frac{w}{2}\right)\Omega^2 \bar{j}, \quad \bar{a}_{G2} = \bar{a}_{G4} = \left(\rho + \frac{w}{2}\right)\Omega^2 \bar{j}, \tag{1}$$

where $\Omega = v/\rho$ is the rotation rate about the vertical axis. The angular momentum of the chassis is due to Ω, and it is constant. The rotation rate of a wheel about its axle is v/R, so the angular momentum of a wheel has two components: $J\,(v/R)$ about its axle, which is parallel to the y axis, and a constant component in the vertical direction that is due to Ω. The horizontal component rotates at $\bar{\Omega}$, so the rates of change of the angular momenta are

$$\dot{\bar{H}}_{G0} = \bar{0}, \quad \dot{\bar{H}}_{Gj} = \bar{\Omega} \times \left(J\frac{v}{R}\bar{j}\right) = -J\Omega^2 \frac{\rho}{R}\bar{i}. \tag{2}$$

We sum moments about the chassis' center of mass. In view of Eqs. (1) and (2) and the fact that each $m\bar{a}$ vector is in the y direction, Eqs. (6.3.1) become

$$\Sigma\bar{F} = \sum_{j=1}^{N} m_j \bar{a}_{Gj} = (m + 4m_w)\,\rho\Omega^2\,\bar{j},$$

$$\begin{aligned}\Sigma\bar{M}_B &= \sum_{j=0}^{4}\frac{d\bar{H}_{Gj}}{dt} + \sum_{j=0}^{4}\bar{r}_{Gj/B}\times m_j\bar{a}_{Gj} = -4J\Omega^2\frac{\rho}{R}\bar{i} + \left[b\bar{i} - (h-R)\,\bar{k}\right] \\ &\quad\times m_w\,(\bar{a}_{G1} + \bar{a}_{G2}) + \left[-(\ell - b)\,\bar{i} - (h-R)\,\bar{k}\right]\times m_w\,(\bar{a}_{G1} + \bar{a}_{G2}) \\ &= \left[-4J\Omega^2\frac{\rho}{R} + 4\,(h-R)\,\rho m_w\Omega^2\right]\bar{i} + (2b - \ell)\,\rho m_w\Omega^2\bar{k}.\end{aligned} \tag{3}$$

The simplified construction of the friction force enables us to evaluate the moments by inspection, so matching like components in Eqs. (1) and (2) leads to

$$\Sigma\bar{F}\cdot\bar{i} \equiv \bar{0},$$

$$\Sigma\bar{F}\cdot\bar{j} = (f_1 + f_2 + f_3 + f_4) = (m + 4m_w)\,\rho\Omega^2, \tag{4}$$

$$\Sigma\bar{F}\cdot\bar{k} = N_1 + N_2 + N_3 + N_4 - mg - 4m_w g = 0, \tag{5}$$

$$\begin{aligned}\Sigma\bar{M}_G\cdot\bar{i} &= (N_1 - N_2 + N_3 - N_4)\,\frac{w}{2} + (f_1 + f_2 + f_3 + f_4)\,h \\ &= \Omega^2\left[-J\frac{\rho}{R} + 4\,(h-R)\,\rho m_w\right],\end{aligned} \tag{6}$$

$$\Sigma\bar{M}_G\cdot\bar{j} = (-N_1 - N_2 + 2m_w g)\,b + (N_3 + N_4 - 2m_w g)\,(\ell - b) = 0, \tag{7}$$

$$\Sigma\bar{M}_G\cdot\bar{k} = (f_1 + f_2)\,b - (f_3 + f_4)\,(\ell - b) = (2b - \ell)\,\rho m_w\Omega^2\bar{k}. \tag{8}$$

There are more unknown forces than the number of available equations of motion. It is possible to solve Eqs. (4) and (8) for the combinations $f_1 + f_2$ and $f_3 + f_4$, and simultaneous solution of Eqs. (5) and (7) leads to a solution for $N_1 + N_2$ and $N_3 + N_4$. Doing so shows that

$$\begin{aligned}N_1 + N_2 &= \left(\frac{\ell - b}{\ell}m + 2m_w\right)g, \\ N_3 + N_4 &= \left(\frac{b}{\ell}m + 2m_w\right)g, \\ f_1 + f_2 &= \left[\frac{\ell - b}{\ell}(m + 4m_w) + \frac{2\,(2b - \ell)}{\ell}m_w\right]\frac{v^2}{\rho}, \\ f_3 + f_4 &= \left[\frac{b}{\ell}(m + 4m_w) - \frac{2\,(2b - \ell)}{\ell}m_w\right]\frac{v^2}{\rho}, \\ (N_1 - N_2 + N_3 - N_4)\,\frac{w}{2} &= \left[-\frac{J}{R} - hm - 4Rm_w\right]\frac{v^2}{\rho}.\end{aligned} \tag{9}$$

We recover the static normal forces from these equations by setting $v = 0$. It is reasonable to assume that the dynamic effect is such that N_1 and N_3 on the left side of the automobile decrease by the same amount Δ relative to their values, whereas N_2 and N_4 increase by that same amount. Hence we set

$$\begin{aligned} N_1 &= \left(\frac{\ell - b}{2\ell} m + m_w\right) g - \Delta \quad N_2 = \left(\frac{\ell - b}{2\ell} m + m_w\right) g + \Delta, \\ N_3 &= \left(\frac{b}{2\ell} m + m_w\right) g - \Delta, \quad N_4 = \left(\frac{b}{2\ell} m + m_w\right) g + \Delta. \end{aligned} \tag{10}$$

The last of Eqs. (9) then yields

$$\Delta = \frac{J + hRm + 4R^2 m_w}{2wR\rho} v^2. \tag{11}$$

Even though we cannot determine the individual friction forces, we have sufficient information to identify the critical conditions. In the skidding limit, each wheel has attained its maximum possible friction force, so we analyze it by setting $f_j = \mu N_j$. From the first and third of Eqs. (9) we find that

$$v_{\text{skid}} = \left[\frac{(\ell - b)\, m + 2m_w \ell}{(\ell - b)\,(m + 4m_w) + 2\,(2b - \ell)\, m_w}\right]^{1/2} (\mu\rho g)^{1/2}. \qquad \triangleleft$$

The same calculation made with the second and fourth of Eqs. (9) yields

$$v_{\text{skid}} = \left[\frac{bm + 2m_w \ell}{b\,(m + 4m_w) - 2\,(2b - \ell)\, m_w}\right]^{1/2} (\mu\rho g)^{1/2}. \qquad \triangleleft$$

Note that $f_1 + f_2$ and $f_3 + f_4$ both increase monotonically with increasing v, whereas the corresponding normal force sums are independent of speed. Thus the first value of v_{skid} is the maximum speed for which the front wheels will not skid, and the second value is the maximum speed for which the rear wheels do not skid. Safe operation requires that we remain below the lower speed. In the special case in which the center of mass is midway, $b = \ell/2$, both speeds reduce to $(\mu\rho g)^{1/2}$. The same result is obtained in the limiting case in which each wheels's mass vanishes. Thus we deduce that the primary factors influencing whether the wheels will skid are the coefficient of static friction and the radius of curvature of the turn.

The condition in which the vehicle is about to tip over occurs if the normal force at both inner wheels is zero, $N_1 = N_3 = 0$, but the vehicle is still horizontal. In a strict sense, Eqs. (9) do not apply in this case because Eqs. (10) indicate that if $b > \ell/2$ then the value of Δ for which $N_3 = 0$ gives $N_1 < 0$, which is not possible. Similarly, if $b < \ell/2$, zero N_1 corresponds to negative N_3. Rather than reanalyze the basic equations, we argue that any condition where a wheel loses contact with the ground is dangerous. In that case we take Δ to be the smaller of the values for which N_1 or N_3 vanishes, then substitute that value into Eq. (11) to determine the corresponding value of the speed. The result is

$$v_{\text{tip}} = \left[\frac{\min(b, \ell - b)\, m + 2\ell m_w}{J + hRm + 4R^2 m_w}\right]^{1/2} \left(\frac{wR\rho g}{\ell}\right)^{1/2}. \qquad \triangleleft$$

In the limiting case in which the mass of the wheels is negligible, setting $m_w = J = 0$ in the preceding equation gives

$$v_{\text{tip}} = \left[\frac{\min(b, \ell - b)}{\ell}\right]^{1/2} \left(\frac{w\rho g}{h}\right)^{1/2}.$$

Our analysis indicated that the speed at which the automobile will begin to skid is not strongly dependent on the vehicle's inertial properties, but that is not the case for tipping over. One is more likely to recover control of a vehicle that has begun to skid than one that is about to tip over, so we wish that $v_{\text{tip}} > v_{\text{skid}}$. Because any value of J decreases v_{tip}, it is desirable to minimize the mass of the wheels. In the limit as m and J vanish, we find that this design criterion on the speed is satisfied if

$$\frac{w}{h} > \frac{\ell}{\min(b, \ell - b)}\mu.$$

This is consistent with our intuition that an automibile will be less likely to tip over than a truck or SUV having the same track w.

6.4 MOMENTUM AND ENERGY PRINCIPLES

The force and moment equations discussed thus far govern the linear and angular acceleration of a body. Momentum and energy principles, which represent standard integrals of these equations, may be used to relate the linear and angular velocity of the body at successive instants or locations. Because these principles are derived directly from the Newton–Euler equations of motion, these integral relations should be considered to supplement, rather than replace, the basic acceleration equations. Furthermore, evaluation of the associated impulse and work quantities often requires knowledge of some aspects of a body's motion, in which case these principles cannot be used to predict the motion. One place where momentum principles are particularly useful is understanding the manner in which any projectile rotates, which is the focus of Section 10.1.

6.4.1 Impulse–Momentum Principles

Equations (5.1.22) and (5.1.24) are the time derivative forms of impulse–momentum relations. Definite integration of each between any two instants t_1 and t_2 leads to

$$\boxed{\begin{aligned} \bar{P}_2 &= \bar{P}_1 + \textstyle\int_{t_1}^{t_2} \Sigma \bar{F}\, dt, \\ (\bar{H}_A)_2 &= (\bar{H}_A)_1 + \textstyle\int_{t_1}^{t_2} \Sigma \bar{M}_A\, dt. \end{aligned}} \qquad (6.4.1)$$

These relations state that the final value of a body's linear or angular momentum exceeds the initial value by the corresponding type of impulse, which is defined to be the time integral of the resultant force or moment. The linear momentum of a rigid body is $m\bar{v}_G$ and the angular momentum is described by Eqs. (6.1.3).

Both momentum principles are vector equations, so they each yield three scalar equations obtained from equating like components. If the impulses can be evaluated, then the scalar equations fully define the corresponding change in the linear or angular velocities. The difficulty lies in that evaluation. The resultant force and moment acting on a body are seldom known in advance because the reactions are unknown. Furthermore, it is not sufficient to know the force or moment in terms of components relative to the body because the corresponding unit vectors are not constant. Evaluating the impulse integrals in that case would require knowledge of the orientation of the unit vectors, and of the components, as functions of time. Such information usually is not available, because it depends on the bodily motion being studied.

Momentum–impulse relations are particularly useful when *impulsive forces* act on a body. Examples of such forces are the those generated by an impact, such as between a golf club and a ball, and explosions. Impulsive forces are defined to impart very large accelerations to a body over a very short time interval. The notion of an impulsive force is that it is sufficiently large that the influence of nonimpulsive forces, such as those associated with gravity and springs, may be ignored during the brief interval of the impulse. (Note in this regard that reactions can act impulsively, because they must be as large as necessary to impose the associated motion constraint.) A corollary of the brevity of the time interval is that the system's position cannot change much because the velocity is finite. These observations lead to a simplified model of the action of impulsive forces, which is quite useful if we are interested in the macroscopic aspects of a system's motion. We represent the linear and angular impulses by the average values of the resultant force and moment, and take the velocity to change instantly from time t_0^- to time t_0^+, while the position is unaltered. Let $t_i < t < t_i + \Delta t$ denote the interval in which the impulsive forces act, and let a subscript "imp" denote average values of the resultant force and moment of the impulsive forces. The simplified model of impulsive action then states that

$$\boxed{\begin{gathered}\bar{P}\left(t = t_i^+\right) = \bar{P}\left(t = t_i^-\right) + \left(\Sigma \bar{F}\right)_{\text{imp}} \Delta t,\\ \bar{H}_A\left(t = t_i^+\right) = \bar{H}_A\left(t = t_i^-\right) + \left(\Sigma \bar{M}_A\right)_{\text{imp}} \Delta t,\\ \bar{r}_{P/O}\left(t = t_i^+\right) = \bar{r}_{P/O}\left(t = t_i^-\right),\end{gathered}} \tag{6.4.2}$$

where point P referred to in the last relation is any point in the body. It is obvious that these relations should be used judiciously. For example, the corollary of taking the velocity to change instantaneously is that the corresponding acceleration is infinite. Thus the simplified model is of no use if one wishes to study the details of the impulsive process, such as what stresses are generated by the collision of bodies. The role of impulse forces in accounting for collisions between bodies is treated in Subsection 6.4.3.

The extension of impulse–momentum principles to systems of bodies is readily obtained from Eqs. (6.3.1). We seek time integrals of those relations in which the final result requires knowledge of motion parameters at only the initial and final instants. Toward that end we operate on the last term in the angular momentum equation by referring the acceleration of each body's center of mass to the motion of point B. Because the summation of $m_j \bar{r}_{Gj/B}$ for each body is the first moment of mass, which defines the system's center of mass G, this operation leads to

$$\begin{aligned}\sum_{j=1}^{N} \bar{r}_{Gj/B} \times m_j \bar{a}_{Gj} &= \sum_{j=1}^{N} m_j \bar{r}_{Gj/B} \times \bar{a}_B + \sum_{j=1}^{N} \bar{r}_{Gj/B} \times m_j \bar{a}_{Gj/B} \\ &= m_{\text{system}} \bar{r}_{G/B} \times \bar{a}_B + \sum_{j=1}^{N} \bar{r}_{Gj/B} \times m_j \bar{a}_{Gj/B}.\end{aligned} \tag{6.4.3}$$

The next step mirrors the derivation of the angular momentum principle for a particle, in which the acceleration term in the summation is converted to a velocity-dependent term by taking the time derivative of the cross product. Because $d/dt\,(\bar{r}_{Gj/B}) \equiv \bar{v}_{Gj/B}$, the result is

$$\sum_{j=1}^{N} \bar{r}_{Gj/B} \times m_j \bar{a}_{Gj} = m_{\text{system}} \bar{r}_{G/B} \times \bar{a}_B + \sum_{j=1}^{N} \frac{d}{dt} \left(\bar{r}_{Gj/B} \times m_j \bar{v}_{Gj/B} \right). \tag{6.4.4}$$

Thus an alternative form of the second equation of Eqs. (6.3.1) governing the moment acting on a system is

$$\Sigma \bar{M}_B = \sum_{j=1}^{N} \frac{d \bar{H}_{Gj}}{dt} + m_{\text{system}} \bar{r}_{G/B} \times \bar{a}_B + \sum_{j=1}^{N} \frac{d}{dt} \left(\bar{r}_{Gj/B} \times m_j \bar{v}_{Gj/B} \right). \tag{6.4.5}$$

We seek a form of this relation in which the momentum terms are time derivatives. We therefore require that *the point B about which we sum moments be an inertial point,* for which $\bar{a}_B = \bar{0}$, *or else the system's center of mass*, in which case $\bar{r}_{G/B} \equiv \bar{0}$. In either case the right side is an exact derivative, whose integration yields the principle of angular impulse and momentum for a system. Integration of the first of Eqs. (6.3.1) yields the corresponding linear impulse–momentum principle, so we have found that

$$\boxed{\begin{gathered}\sum_{j=1}^{N} m \left(\bar{v}_{Gj} \right)_2 = \sum_{j=1}^{N} m \left(\bar{v}_{Gj} \right)_1 + \int_{t_1}^{t_2} \Sigma \bar{F} dt, \\ \sum_{j=1}^{N} \left(\bar{H}_{Gj} \right)_2 + \sum_{j=1}^{N} \left(\bar{r}_{Gj/B} \times m_j \bar{v}_{Gj/B} \right)_2 \\ = \sum_{j=1}^{N} \left(\bar{H}_{Gj} \right)_1 + \sum_{j=1}^{N} \left(\bar{r}_{Gj/B} \times m_j \bar{v}_{Gj/B} \right)_1 + \int_{t_1}^{t_2} \Sigma \bar{M}_B dt.\end{gathered}} \tag{6.4.6}$$

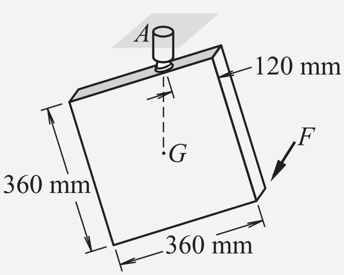
Proper application of this principle requires that one be cognizant that the moment equation is valid only if the reference point B is either stationary, in which case the term $\bar{v}_{Gj/B}$ is the absolute velocity, that is, $\bar{v}_{Gj/B} = \bar{v}_{Gj}$, or else it must be the center of mass for the entire system, which corresponds to $\bar{v}_{Gj/B} = \bar{v}_{Gj} - \bar{v}_B$. One simplification arises if it should happen that the center of mass is stationary relative to a body. In that case $\bar{v}_{Gj/B} = \bar{\omega}_j \times \bar{r}_{Gj/B}$, but $\bar{r}_{Gj/B} \equiv -\bar{r}_{B/Gj}$, so that

$$\begin{aligned} \bar{H}_{Gj} + \bar{r}_{Gj/B} \times m_j \bar{v}_{Gj/B} &= \bar{H}_{Gj} + m_j \bar{r}_{B/Gj} \times (\bar{\omega}_j \times \bar{r}_{B/Gj}) \\ &= \bar{H}_{Bj} \text{ if point } B \text{ is stationary relative to body } j, \end{aligned} \tag{6.4.7}$$

where the final form follows from Eq. (5.1.36).

A primary aspect of the system momentum principles is associated with their vectorial nature. Although we might not have a complete idea of the time dependence of all forces acting on a body, it might be that the component of the resultant force in a specific fixed direction $\bar{e}_F$ is known as a function of time. Similarly, we might know the resultant moment about a fixed axis in direction $\bar{e}_M$ intersecting an allowable point for the moment equation. The component of the corresponding type of impulse may be evaluated and equated to the momentum change in each direction. The most common situation fitting this specification is that in which an impulse component vanishes. In that case, the associated component of Eqs. (6.4.6) becomes a conservation principle, stating that a system's linear momentum in a certain direction, or angular momentum about a certain axis, is constant.

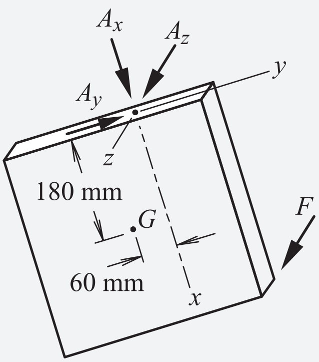
EXAMPLE 6.11 A 10-kg square plate suspended by ball-and-socket joint A is at rest when it is struck by a hammer. The impulsive force $\bar{F}$ generated by the hammer is normal to the surface of the plate, and its average value during the 4-ms interval that it acts is 5000 N. Determine the angular velocity of the plate at the instant following the impact and the average reaction at the support.



Example 6.11

SOLUTION This straightforward example once again highlights the fact that the behavior of bodies in spatial motion is often counterintuitive. We ignore the weight of the plate, which is much smaller than the average applied force. In contrast, the reaction exerted by the ball-and-socket joint, which consists of forces acting in each of the coordinate directions, is impulsive, because it must be as large as necessary to prevent movement of point A. The plate pivots about the stationary point A

where the ball-and-socket joint is located, so we place the origin of the body-fixed xyz coordinate system there and align the axes with the edges of the plate.



Free-body diagram showing only the impulsive forces that act on the square plate.

We know that $\bar{\omega}_1 = \bar{0}$, but have no idea what angular velocity results from the action of $\bar{F}$. If we knew $\bar{\omega}_2$ we could find $\bar{v}_G$ kinematically according to

$$\bar{\omega}_2 = \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k}, \quad (\bar{v}_G)_2 = \bar{\omega}_2 \times \bar{r}_{G/A}.$$

We set $\bar{r}_{G/A} = 0.18\bar{i} - 0.06\bar{j}$ m, which leads to

$$(\bar{v}_G)_2 = 0.06\omega_z \bar{i} + 0.18\omega_z \bar{j} + (-0.06\omega_x - 0.18\omega_y)\,\bar{k}.$$

The inertia properties relative to xyz are obtained from the Appendix and the parallel axis theorems. The coordinates of point A relative to centroidal axes parallel to xyz are $(-0.18, 0.06, 0)$ m, which leads to

$$I_{xx} = 0.144 \text{ kg-m}^2, \quad I_{yy} = 0.432 \text{ kg-m}^2, \quad I_{zz} = 0.576 \text{ kg-m}^2,$$

$$I_{xy} = -0.108 \text{ kg-m}^2, \quad I_{xz} = I_{yz} = 0.$$

The final angular momentum about pivot A corresponding to these properties is

$$\left(\bar{H}_A\right)_2 = (0.144\omega_x + 0.108\omega_y)\,\bar{i} + (0.432\omega_y + 0.108\omega_x)\,\bar{j} + 0.576\omega_z \bar{k}.$$

The initial angular momentum is zero, so the final angular momentum must equal the moment impulse. The sole impulsive force exerting a moment about point A is $\bar{F}$. The moment impulse is the average moment of this force multiplied by the 4-ms interval or, equivalently, the moment of the linear impulse of $\bar{F}$ during this interval. Thus,

$$\left(\bar{H}_A\right)_2 = \bar{r}_{F/A} \times \bar{F}_{\text{av}}\,(\Delta t) = \left(0.36\bar{i} + 0.12\bar{j}\right) \times \left(5000\bar{k}\right)(0.004)\,.$$

Matching like components in the two descriptions of $\left(\bar{H}_A\right)_2$ gives

$$\left(\bar{H}_A\right)_2 \cdot \bar{i} = (0.144\omega_x + 0.108\omega_y) = 2.4,$$

$$\left(\bar{H}_A\right)_2 \cdot \bar{j} = (0.432\omega_y + 0.108\omega_y) = -7.2,$$

$$\left(\bar{H}_A\right)_2 \cdot \bar{k} = 0.576\omega_z = 0.$$

We solve these equations for the rotation rates, from which we obtain

$$\bar{\omega}_2 = 35.90\bar{i} - 25.64\bar{j} \text{ rad/s.} \qquad \triangleleft$$

The next step is to form the linear impulse–momentum principle in order to determine the reaction. The velocity of the center of mass corresponding to $\bar{\omega}_2$ is

$$(\bar{v}_G)_2 = \bar{\omega}_2 \times \bar{r}_{G/A} = 2.462\bar{k} \text{ m/s.}$$

The initial linear momentum was zero, so the final momentum must equal the impulse of all forces:

$$m(\bar{v}_G)_2 = \left[A_x\bar{i} + A_y\bar{j} + (A_z + F)\bar{k}\right]\Delta t.$$

The solution of these equations is

$$A_x = A_y = 0, \quad A_z = 1153 \text{ N.} \qquad \triangleleft$$

These are average values over the 4-ms interval. The maximum values exceed these.

It might surprise you that the reaction is in the same sense as the impulsive force $\bar{F}$. This result indicates that, if the ball-and-socket joint were not present, the plate would rotate about its mass center because of the moment of the force, such that point A moves in the negative z direction. It is possible to locate a curve on the plate representing the locus of points at which the force can be applied without generating a dynamic reaction at the joint. Any such point is sometimes referred to as a *center of percussion*.

6.4.2 Work–Energy Principles

We begin the derivation of work–energy principles for rigid bodies by considering an isolated rigid body as a collection of particles. Equation (1.2.14) states the principle for a single particle. We use it to describe particle number j in a system of N particles. As we did previously, we denote each force acting on this particle as $\bar{f}_{j,k}$ if it is exerted by any other particle k within the body, while $\bar{F}_j$ represents the resultant of all forces exerted on particle j by bodies not included in the system. The resultant force acting on the particle is the sum of $\bar{F}_j$ and all of the $\bar{f}_{j,k}$. The work done by this resultant is the integral of the resultant's component in the direction of the displacement, multiplied by the differential displacement. The work–energy principle for this particle states that this work increases the kinetic energy:

$$\frac{1}{2}m_j\left(v_j^2\right)_2 = \frac{1}{2}m_j\left(v_j^2\right)_1 + \oint_1^2 \left(\sum_{\substack{k=1\\k\neq j}}^{N} \bar{f}_{j,k} + \bar{F}_j\right)\cdot d\bar{r}_j. \tag{6.4.8}$$

This relation describes an arbitrary particle within a system, but we are interested in the specific case in which the particles constitute a rigid body. We recall Chasle's theorem, and select any convenient point B as the reference point for the motion. The differential displacement $d\bar{r}_j$ is the result of the displacement of point B and the infinitesimal rotation $\overline{d\theta}$ about point B, see Eq. (3.3.6), so that

$$d\bar{r}_j = d\bar{r}_B + \overline{d\theta} \times \bar{r}_{j/B}. \tag{6.4.9}$$

We know from the Newton–Euler equations of motion that the forces internal to a rigid body do not directly affect its motion, so the same must be true of any energy principle. Thus let us focus on the term in Eq. (6.4.8) that contains these force. The identity for the scalar triple product, $\bar{a} \cdot (\bar{b} \times \bar{c}) \equiv (\bar{c} \times \bar{a}) \cdot \bar{b}$, in conjunction with the preceding representation of $d\bar{r}_j$, leads to

$$\sum_{\substack{k=1\\k\neq j}}^{N} \bar{f}_{jk} \cdot d\bar{r}_j = \sum_{\substack{k=1\\k\neq j}}^{N} \bar{f}_{j,k} \cdot d\bar{r}_B + \sum_{\substack{k=1\\k\neq j}}^{N} \bar{r}_{j/B} \times \bar{f}_{j,k} \cdot \overline{d\theta}. \tag{6.4.10}$$

Note that $\bar{r}_{j/B} \times \bar{f}_{j,k}$ is the moment of $\bar{f}_{j,k}$ about point B. Thus the right side of this expression replaces the work done by internal forces acting on particle j with the work done by an equivalent force–couple system acting at point B.

Now consider the combination of all particles obtained by adding Eq. (6.4.10) for $j = 1, 2, \ldots, N$. The first term on the right side will add all of the internal forces at point B, and the second term will add all moments. Specifically,

$$\sum_{j=1}^{N} \sum_{\substack{k=1\\k\neq j}}^{N} \bar{f}_{jk} \cdot d\bar{r}_j = \left(\sum_{j=1}^{N} \sum_{\substack{k=1\\k\neq j}}^{N} \bar{f}_{j,k} \right) \cdot d\bar{r}_B + \left(\sum_{j=1}^{N} \sum_{\substack{k=1\\k\neq j}}^{N} \bar{r}_{j/B} \times \bar{f}_{j,k} \right) \cdot \overline{d\theta}. \tag{6.4.11}$$

For every occurrence of a specific j in the first sum and a specific k in the second, there is a matching occurrence in the opposite sequence. According to Newton's Third Law, $\bar{f}_{j,k}$ and $\bar{f}_{k,j}$ are equal in magnitude, opposite in sense, and collinear. The consequence of these properties was seen in Eqs. (5.1.7) to be that the internal forces have a zero resultant, and they exert no net moment about any point. Thus the preceding sum reduces to zero.

It follows that adding work–energy equation (6.4.8) for each particle of the rigid body will result in cancellation of the contribution of all internal forces. Because $\frac{1}{2} m_j v_j^2$ is the kinetic energy of one particle of the rigid body, and kinetic energy is a scalar, adding the work–energy equation for each particle in a rigid body leads to the *work–energy equation* for a rigid body,

$$\boxed{T_2 = T_1 + W_{1\to 2},} \tag{6.4.12}$$

where T_1 and T_2 are the kinetic energy of the rigid body at any two instants and $W_{1\to 2}$ is the work done by all forces acting on the body as it moves from its position at t_1 to its position at t_2:

$$W_{1\to 2} = \sum_{j=1}^{N} \oint_1^2 \bar{F}_j \cdot d\bar{r}_j. \tag{6.4.13}$$

Recall that the kinetic energy of a rigid body corresponding to specified $\bar{v}_G$ and $\bar{\omega}$ is

$$\boxed{T = \begin{cases} \dfrac{1}{2}mv_G^2 + \dfrac{1}{2}\bar{\omega}\cdot\bar{H}_G : \text{ any motion} \\ \dfrac{1}{2}\bar{\omega}\cdot\bar{H}_O : \text{ pure rotation about point } O \end{cases}.} \tag{6.4.14}$$

Evaluation of the work according to Eq. (6.4.13) requires that we describe functionally how each external force depends on the position of the point at which it is applied. This can be quite cumbersome for two reasons: There might be many points to consider, and some or all of the points might follow intricate paths. An alternative is to shift the forces from the point at which they are applied to any convenient point B in the body. When Eq. (6.4.9) is used to relate the displacement of a force's point of application to the displacement of the reference point, the work done by force $\bar{F}_j$ in a differential displacement $d\bar{r}_j$ becomes

$$\bar{F}_j\cdot d\bar{r}_j = \bar{F}_j\cdot d\bar{r}_B + \bar{F}_j\cdot\left(\overline{d\theta}\times\bar{r}_{j/B}\right) \equiv \bar{F}_j\cdot d\bar{r}_B + \left(\bar{r}_{j/B}\times\bar{F}_j\right)\cdot\overline{d\theta}, \tag{6.4.15}$$

where the final form stems from the identity for the scalar triple product. Because $\bar{r}_{j/B}\times\bar{F}_j$ is the moment of $\bar{F}_j$ about point B and both $d\bar{r}_B$ and $\overline{d\theta}$ are independent of which force is under consideration, adding the work done by each force, as required by Eq. (6.4.13), will lead to two terms containing the resultant force and resultant moment about point B, specifically,

$$\boxed{W_{1\to 2} = \oint_1^2 \Sigma\bar{F}\cdot d\bar{r}_B + \oint_1^2 \Sigma\bar{M}_B\cdot\overline{d\theta}.} \tag{6.4.16}$$

This expression could have been anticipated from Chasle's theorem. It shows that the total work is the sum of the work done by the resultant of the external forces in moving an arbitrary point B and the work done by the moment of the external forces in the rotation about that point.

Another alternative to direct evaluation of the work done by a force arises when a force is *conservative*. The term "conservative" is a corollary of the property that such a force does no net work when the point at which it is applied follows an arbitrary closed path. Consequently, whatever work it does in going from position 1 to position 2 on a closed path is the negative of the work it will do to return to position 1, so that work is not lost. This property is expressed by

$$W_{1\to 2} = -W_{1\to 2}. \tag{6.4.17}$$

This must be true for any closed path containing the specified positions $\bar{r}_1$ and $\bar{r}_2$ of the point of application. Furthermore, it must be true for any pair of points $\bar{r}_1$ and $\bar{r}_2$ on a specific path. These conditions can only be satisfied if the work is determined by the change in the value of a function of position. This function, termed the *potential energy*, is defined such that

$$W_{1\to 2} = V(\bar{r}_1) - V(\bar{r}_2) \equiv V_1 - V_2. \tag{6.4.18}$$

Strictly speaking, all requirements would be met if the opposite sign was used to define $V(\bar{r})$. The definition we use enables us to interpret the work done by a conservative force as the amount by which its potential energy is depleted or, equivalently, as the ability of a force to do work.

One method by which we may determine whether a specific force is conservative, and if so, determine the corresponding form of the potential energy, is to evaluate the work the force does when the point at which it is applied follows an arbitrary path. If it is found that the work depends solely on the initial and final position coordinates of its point of application, then $V(\bar{r}_1)$ and $V(\bar{r}_2)$ are merely $V(\bar{r})$ at either location. Clearly, forces imparted with a specified time history are not conservative because the work they do will depend explicitly on the time interval. In the same vein, velocity-dependent forces such as friction are not conservative.

This approach is readily implemented in the case of gravity close to the Earth's surface. Because the force acts in a constant direction, we use Cartesian coordinates to represent the force and the position, which leads to

$$\bar{F} = -mg\bar{K}, \quad d\bar{r} = dX\bar{I} + dY\bar{J} + dZ\bar{K}. \tag{6.4.19}$$

The work done by this force is

$$W_{1\to 2} = \oint_1^2 \left(-mg\bar{K}\right)\cdot\left(dX\bar{I} + dY\bar{J} + dZ\bar{K}\right) = -\int_{Z_1}^{Z_2} mg\,dZ = mgZ_1 - mgZ_2. \tag{6.4.20}$$

Matching this to Eq. (6.4.18) leads to

$$\boxed{V_{\text{grav}} = mgZ,} \tag{6.4.21}$$

which usually is applied by taking Z to be the height of the center of mass above some arbitrarily selected reference elevation known as the *datum*.

When we are concerned with a gravity force exerted by a body other than the Earth, or the motion is known to result in large changes in the distance to the center of the Earth, we need to apply the universal law of gravitation, which states that

$$\bar{F} = -\frac{GMm}{r^2}\bar{e}_r, \tag{6.4.22}$$

where r is the distance to the center of mass of the interacting body and $\bar{e}_r$ is the unit vector oriented from the attracting body to the body to which $\bar{F}$ is applied. For the Earth, one can use $GM = 5.990(10^4)$ m^3/s^2. For the determination of $V(\bar{r})$ we may consider the attracting body to be stationary, so that $\bar{e}_r$ is the radial unit vector for spherical coordinates. Correspondingly, the differential displacement is $d\bar{r} = \bar{v}dt = dr\bar{e}_r + rd\phi\bar{e}_\phi + (r\sin\phi)\,d\theta\bar{e}_\theta$, so the work is given by

$$\begin{aligned} W_{1\to 2} &= \oint_1^2 \left(-\frac{GmM}{r^2}\bar{e}_r\right)\cdot(dr\bar{e}_r + rd\phi\bar{e}_\phi + rd\theta\sin\phi\bar{e}_\theta) \\ &= -\int_{r_1}^{r_2}\frac{GmM}{r^2}dr = -\frac{GmM}{r_1} + \frac{GmM}{r_2}. \end{aligned} \tag{6.4.23}$$

The terms on the right side are the same function of r evaluated at each position, so the potential energy is given by

$$\boxed{V_{\text{grav}} = -\frac{GmM}{r}.} \tag{6.4.24}$$

Springs are important as actual devices, as well as models for elastic bodies. If we take one end of the spring to be stationary, the force $\bar{F}$ applied to the moving point is oriented along the radial line toward the fixed end. In the model of a spring that behaves linearly, the magnitude of the force is $k\Delta$, where k is the spring stiffness and Δ is the elongation. The latter is defined as the difference between the current length ℓ of the spring and the undeformed length ℓ_0:

$$\Delta = \ell - \ell_0 \tag{6.4.25}$$

Spherical coordinates centered at the fixed end are suitable for describing the orientation of $\bar{F}$. Then $r = \ell$ and $dr = d\ell = d\Delta$, so that

$$\begin{aligned} &\bar{F} = -k\Delta\bar{e}_r, \quad d\bar{r} = d\Delta\bar{e}_r + \ell d\phi\bar{e}_\phi + (\ell\sin\phi)\,d\theta\bar{e}_\theta, \\ &W_{1\to 2} = \oint_1^2 (-k\Delta\bar{e}_r)\cdot(d\Delta\bar{e}_r + \ell d\phi\bar{e}_\phi + \ell d\theta\sin\phi\bar{e}_\theta) \\ &\quad = -\int_{\Delta_1}^{\Delta_2} k\Delta d\Delta = \frac{1}{2}k\Delta_1^2 - \frac{1}{2}k\Delta_2^2. \end{aligned} \tag{6.4.26}$$

Comparison of the latter expression with Eq. (6.4.18) shows that the potential energy of a spring is

$$\boxed{V_{\text{spr}} = \frac{1}{2}k\Delta^2.} \tag{6.4.27}$$

At any position, Δ is the elongation of the spring referenced to the unstretched length of the spring, as described, by Eq. (6.4.25). Failure to properly describe Δ is a common error in the evaluation of a spring's potential energy.

An alternative approach for examining the conservative nature of a force employs vector calculus. Consider two positions that differ by an infinitesimal amount, so that $\bar{r}_2 = \bar{r}_1 + d\bar{r}$. The infinitesimal work dW done by a conservative force in such a displacement is given by

$$dW = V(\bar{r}_1) - V(\bar{r}_1 + d\bar{r}) = -dV. \tag{6.4.28}$$

Let us use Cartesian coordinates to define the position, so that V is a considered to be a known function of the XYZ coordinates of the point where the force is applied. The chain rule for differentiation yields

$$dW = -\frac{\partial V}{\partial X}dX - \frac{\partial V}{\partial Y}dY - \frac{\partial V}{\partial Z}dZ. \tag{6.4.29}$$

In Cartesian coordinates an infinitesimal displacement is $d\bar{r} = dX\bar{I} + dY\bar{J} + dZ\bar{K}$, so the preceding equation is the scalar form of $dW = -\nabla V \cdot d\bar{r}$. However, the work done by any force in an infinitesimal displacement is $dW = \bar{F} \cdot d\bar{r}$. Either form is equally correct if $\bar{F}$ is conservative, so it must be that a conservative force is the negative of the gradient of its potential-energy function:

$$\boxed{\bar{F}_{\text{cons}} = -\nabla V.} \tag{6.4.30}$$

This leads to a simple way of checking that a force is conservative. The curl of the gradient is identically zero, so it must be that

$$\text{if } \nabla \times \bar{F} \equiv \bar{0}, \text{ then } \bar{F} \text{ is conservative.} \tag{6.4.31}$$

If some forces are not conservative, we may use Eq. (6.4.18) to account for the conservative effects, whereas the work done by the nonconservative forces, which we denote as $W_{1\to 2}^{\text{nc}}$, is found according to either Eq. (6.4.13) or Eq. (6.4.16). Thus,

$$W_{1\to 2} = V_1 - V_2 + W_{1\to 2}^{\text{nc}}. \tag{6.4.32}$$

Substitution of this expression into the basic work–energy equation yields

$$\boxed{T_2 + V_2 = T_1 + V_1 + W_{1\to 2}^{\text{nc}}.} \tag{6.4.33}$$

The quantity $T + V$ is called the *mechanical energy.* In any motion in which the nonconservative forces do no work, there is *conservation of energy,* meaning that $T + V$ is constant throughout the motion.

The development thus far is sufficient for the purpose of analyzing specific systems. However, we may garner a different perspective for the role of energy by deriving another energy principle. The motion of the center of mass is governed by Newton's Second Law. The derivation in Chapter 1 of the work–energy principle for a particle is equally valid when applied to the center of mass, so it must be that

$$\frac{1}{2}m\left(v_G^2\right)_2 = \frac{1}{2}m\left(v_G^2\right)_1 + \oint_1^2 \Sigma\bar{F} \cdot d\bar{r}_G. \tag{6.4.34}$$

This equation has a simple explanation when we recall Chasle's theorem to consider the body's motion to be a superposition of a translation following the center of mass and a rotation about the center of mass. As indicated by Eq. (6.4.14), $\frac{1}{2}mv_G^2$ is the translational kinetic energy. Thus Eq. (6.4.34) states that the work done by the resultant force to move the center of mass of a rigid body increases the translational kinetic energy.

An interesting aspect of Eq. (6.4.34) is that the work term on the right side also appears in the alternative description of work given by Eq. (6.4.16). In view of Eqs. (6.4.14) and (6.4.16), taking the difference between the work–energy principle of Eq. (6.4.12) and Eq. (6.4.34) leaves

$$\frac{1}{2}\bar{\omega}_2 \cdot \left(\bar{H}_G\right)_2 = \frac{1}{2}\bar{\omega}_1 \cdot \left(\bar{H}_G\right)_1 + \oint_1^2 \Sigma \bar{M}_G \cdot \overline{d\theta}. \tag{6.4.35}$$

This relation shows that the work done by the resultant moment about the center of mass has the effect of increasing the rotational kinetic energy. Thus replacing the actual forces by an equivalent force–couple system acting at the center of mass leads to an uncoupling of effects, with the resultant force increasing the translational kinetic energy and the resultant moment increasing the rotational kinetic energy.

It might seem that Eqs. (6.4.34) and (6.4.35) provide an alternative to the work–energy principle stated by Eq. (6.4.33). However, this seldom is true, because of a subtle aspect of constraint forces. We will see in the next chapter that such forces often do no work. However, if we transfer a constraint force from its actual point of application to the center of mass, this force will contribute to the work terms in both Eqs. (6.4.34) and (6.4.35). Constraint forces are not known in advance, so neither the translational nor rotational work–energy equation would be useful by themselves for evaluating the motion.

Constraint forces of particular concern are those associated with connections between moving bodies. It sometimes is possible to avoid the occurrence of such forces by considering the assembly of bodies to form a system. The mechanical energies T and V, and the work done by forces, are scalars, so the addition of Eq. (6.4.33) for each body in a system yields the same form for the assembly:

$$\boxed{(T_2)_{\text{total}} + (V_2)_{\text{total}} = (T_1)_{\text{total}} + (V_1)_{\text{total}} + \left(W^{\text{nc}}_{1\to 2}\right)_{\text{total}}.} \tag{6.4.36}$$

The value of this principle is that some forces exerted between the bodies will do no work when bodies are considered as a system, even though they do work when each body is considered individually. To see how this might be, consider the planar situation in Fig. 6.5, where two bodies are connected by pins to massless link AB. Because the link has no mass, it effectively is in static equilibrium, which means that it can sustain only axial force F. The forces exerted to each body are equal and opposite. The differential amount of work done by these forces when both bodies move is

$$dW = (F\bar{e}_{B/A}) \cdot d\bar{r}_B + (-F\bar{e}_{B/A}) \cdot d\bar{r}_A. \tag{6.4.37}$$

If the connecting link is considered to be a rigid bar, then the length L is taken to be constant. In that case the displacements are related by $d\bar{r}_B = d\bar{r}_A + \overline{d\theta}_{AB} \times \bar{r}_{B/A}$. The work is zero because the rotational part of the displacement is perpendicular to $\bar{e}_{B/A}$.

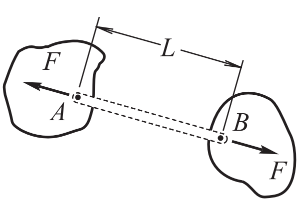


Figure 6.5. Forces exerted by a massless rigid connecting link.

If the link is regarded as an elastic spring, the length L is treated as a variable. The displacements in this case are related by $d\bar{r}_B = d\bar{r}_A + dL\bar{e}_{B/A} + \overline{d\theta}_{AB} \times \bar{r}_{B/A}$, and the work is $dW = F dL$. The axial force F in a spring is known to be $k\Delta$, where Δ is the change in the value of L. Thus we may compute the work done by an elastic connecting force. (It would be simpler to merely account for the work of the spring in terms of its potential energy.) The problematic case occurs when the length L is a variable that is controlled, as in the case of a servo-actuated hydraulic cylinder. The force F in this case is not known. Rather, it is a constraint force whose role is to change L in the prescribed manner. Considering both bodies to form a system will not enable us to avoid considering the unknown connective force in this situation.

The work–energy principle relates velocity parameters at two different positions. If we differentiate this expression with respect to time we obtain the *power balance law*,

$$\boxed{\dot{T} + \dot{V} = \mathcal{P}^{\text{nc}},} \tag{6.4.38}$$

where $\mathcal{P}^{\text{nc}}$ represents the power input to the body by the nonconservative forces. This is equivalent to the first law of thermodynamics for a rigid body, for it states that the rate of increase of the body's mechanical energy equals the rate at which power is provided to the body. By definition, power is the rate at which work is done, so the work done by a force in an infinitesimal displacement $d\bar{r}$ during an interval dt can be computed as either $\bar{F} \cdot d\bar{r}$ or $\mathcal{P}dt$. It follows from Eq. (6.4.16) that

$$\boxed{\mathcal{P}^{\text{nc}} = \sum_{j=1}^{N} \bar{F}_j^{\text{nc}} \cdot \bar{v}_j = \Sigma \bar{F}^{\text{nc}} \cdot \bar{v}_B + \oint_1^2 \Sigma \bar{M}_B^{\text{nc}} \cdot \bar{\omega}.} \tag{6.4.39}$$

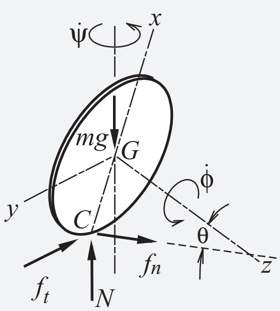
Some individuals use the power balance law to obtain a differential equation of motion corresponding to known energy expressions for a system, but we will see in the next chapter that there is a better way of achieving that end.

As is true for momentum principles, the work–energy principles have inherent limitations. Most profound of these is the necessity to evaluate the work done by nonconservative forces. Equation (6.4.16), which replaces any external force by an equivalent force–couple system acting at an arbitrary point, is an aid. Obviously, the motion of such a point must be known in order to evaluate the path integrals. It is equally important to know how the resultant force varies as the position of the selected point changes and how the moment depends on the angle of orientation. Also, as noted previously, the work–energy principle is not likely to be useful if any force acting on the body is a specified function of time or is velocity dependent. Even if it might seem that the work could be evaluated, doing so might be quite complicated.

Another reason why we cannot rely on momentum and energy principles comes from the fact that there usually are many position variables to evaluate. For example, we can locate the position of a rigid body with three position coordinates for its center of mass and three Eulerian angles. In general, the number of momentum and energy equations not containing unknown reactions will be fewer than the number of unknown position and rate variables. The important thing to remember is that momentum and energy principles are derived as standard integrals of the Newton–Euler equations of motion. Thus any information we obtain from these first integrals could be obtained by solving the differential equations of motion. Although it might be more difficult to solve such equations, formulation and solution of the equations of motion is often the only useful approach.

EXAMPLE 6.12 A disk is rolling without slipping in an unsteady manner, such that the angle θ at which the plane of the coin is inclined is not constant. Prove that the work done by the friction and normal forces is zero.

SOLUTION The main objective here is to demonstrate how one can evaluate work when a force acts at a point whose motion is uncertain. Also, the result that the friction force does no work if the body rolls without slipping has important implications for our later studies. We draw a free-body diagram of the disk in which xyz is a centroidal body-fixed reference frame whose y axis at the instant in the diagram is the horizontal diameter of the disk. The contact force exerted by the ground is decomposed into three components: $\bar{N}$ is the normal force, $\bar{f}_t$ is the tangential friction force, which is parallel to the y axis, and $\bar{f}_n$ is the friction force transverse to the y axis.



Free-body diagram of the disk rolling in an unsteady precession.

At first glance, the fact that the velocity of the contact point C is zero might seem to make it obvious that the contact forces do no work. Such thinking is based on the belief that, because $\bar{v}_C = \bar{0}$, the contact force acts at a stationary point, so it cannot do work. The difficulty with such reasoning is that the contact point is at rest for only an instant. After even an infinitesimal rotation, a different point is in contact with the ground, so one cannot assert with certainty that the forces act at a single point that does not move. We therefore transfer the forces at the contact point to the center G, whose motion is easier to understand. The normal and friction forces at point C are equivalent to a force $\bar{F}$ and couple $\bar{M}_G$ at point G, where $\bar{F}$ is the resultant force and $\bar{M}_G$ is the resultant moment about point G:

$$\bar{F} = \bar{N} + \bar{f}_t + \bar{f}_n, \quad \bar{M}_G = \bar{r}_{C/G} \times \left(\bar{N} + \bar{f}_t + \bar{f}_n\right) = \bar{r}_{C/G} \times \bar{F}.$$

In an infinitesimal time interval dt, point G displaces by $\bar{v}_G\,dt$ and the disk rotates by $\bar{\omega}\,dt$. The work done by the equivalent force–couple system in this infinitesimal movement is

$$dW = \bar{F}\cdot(\bar{v}_G\,dt) + \bar{M}_G\cdot(\bar{\omega}\,dt) = \left[\bar{F}\cdot\bar{v}_G + \left(\bar{r}_{C/G}\times\bar{F}\right)\cdot\bar{\omega}\right]dt.$$

If there is slippage at the contact point, then $\bar{v}_G$ and $\bar{\omega}$ are kinematically unrelated, which leads to a nonzero value for dW. The absence of slippage requires that $\bar{v}_G = \bar{\omega}\times\bar{r}_{G/C}$. Correspondingly, we have

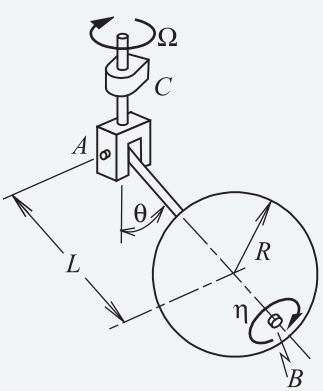
$$dW = \left[\bar{F}\cdot(\bar{\omega}\times\bar{r}_{G/C}) + \left(\bar{r}_{C/G}\times\bar{F}\right)\cdot\bar{\omega}\right]dt.$$

We apply the identity $\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)\equiv(\bar{c}\times\bar{a})\cdot\bar{b}$ to the first term inside the brackets, which gives

$$dW = \left[\left(\bar{r}_{G/C}\times\bar{F}\right)\cdot\bar{\omega} + \left(\bar{r}_{C/G}\times\bar{F}\right)\cdot\bar{\omega}\right]dt \equiv 0,$$

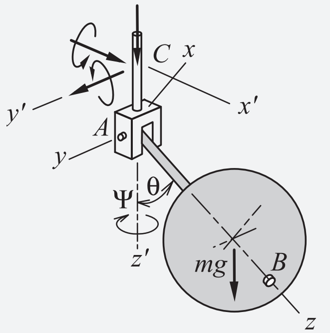
where the result is a consequence of the fact that $\bar{r}_{G/C}\equiv-\bar{r}_{C/G}$. The present analysis is completely general, other than considering the ground to be stationary. Intuitively, it makes sense that no work is done when there is no relative slippage between contacting surfaces, because we associate rubbing with heat generation, which depletes the mechanical energy.

EXAMPLE 6.13 The sphere, whose mass is m, spins relative to shaft AB at the constant rate $\eta = 10$ rad/s. This shaft is attached to the vertical shaft AC by fork-and-clevis joint A. The vertical shaft rotates freely, with no torque applied to its rotation axis. The mass of bar AB is $m/2$, and the other parameters are $R = 150$ mm and $L = 400$ mm. Initially, θ is held constant at 90° by a cable, and the precession rate is $\Omega = 5$ rad/s. Determine the maximum and minimum values of θ in the motion following breakage of the cable and the corresponding precession rates. Then determine the values of $\dot{\theta}$ and Ω at the instant when θ is the average of these extreme angles.



Example 6.13

SOLUTION We see here that momentum and energy principles can be used simultaneously, thereby enabling us to analyze an intricate motion. We consider the sphere and shaft AB to form a system, because the forces exerted between them do no net work. We include the massless vertical shaft in this system to utilize our knowledge of how bearing C acts.



Free-body diagram of the sphere and its supporting shafts.

The free-body diagram of this system shows that bearing C exerts an arbitrary force and couple, except that the couple has no component about the vertical axis. This force–couple system does no work, and gravity is conservative. Therefore mechanical energy is conserved throughout the motion. We also observe that none of the forces or couples depicted in the free-body diagram exert no moment about the z' axis, which means that the system's angular momentum about the z' axis also is conserved.

Both the sphere and shaft AB are in pure rotation about stationary point A, so the xyz coordinate system shown in the the free-body diagram, whose z axis coincides with AB, will be convenient for describing the angular momentum of both bodies. We also define another reference frame $x'y'z'$ that solely precesses about its z' axis.

Point A is on the z' axis, and it is the pivot point for the pure rotation of both the shaft and the sphere. We develop an expression for $\bar{H}_A$ of each body under arbitrary conditions in order to address all aspects of the problem. The inertia properties in the Appendix and the parallel axis theorems indicate that

$$(I_{zz})_s = \frac{2}{5}mR^2 = 0.009m \equiv I_1, \quad (I_{xx})_s = (I_{yy})_s = \frac{2}{5}mR^2 + mL^2 = 0.169m \equiv I_2,$$

$$(I_{xx})_{AB} = (I_{yy})_{AB} = \frac{1}{3}\left(\frac{m}{2}\right)L^2 = 0.02667m \equiv I_3. \tag{1}$$

The angular velocity of the sphere is these bodies is sum of the precession rate Ω and the nutation rate $\dot{\theta}$:

$$\bar{\omega}_s = \Omega\bar{k}' + \dot{\theta}\,\bar{j}' - \eta\bar{k} = -\Omega\sin\theta\bar{i} + \dot{\theta}\,\bar{j} + (\Omega\cos\theta - \eta)\,\bar{k},$$

$$\bar{\omega}_{AB} = \Omega\bar{k}' + \dot{\theta}\,\bar{j}' = -\Omega\sin\theta\bar{i} + \dot{\theta}\,\bar{j} + \Omega\cos\theta\bar{k}.$$

The corresponding angular momenta and kinetic energies corresponding to pure rotation are

$$\begin{gathered}
\left(\bar{H}_A\right)_s = -I_2\Omega\sin\theta\bar{i} + I_2\dot{\theta}\bar{j} + I_1\left(\Omega\cos\theta - \eta\right)\bar{k}, \\
\left(\bar{H}_A\right)_{AB} = -I_3\Omega\sin\theta\bar{i} + I_3\dot{\theta}\bar{j}, \\
(T)_s = \frac{1}{2}\bar{\omega}_s\cdot\left(\bar{H}_A\right)_s = \frac{1}{2}\left[I_2\Omega^2\left(\sin\theta\right)^2 + I_2\dot{\theta}^2 + I_1\left(\Omega\cos\theta - \eta\right)^2\right], \\
(T)_{AB} = \frac{1}{2}\bar{\omega}_{AB}\cdot\left(\bar{H}_A\right)_{AB} = \frac{1}{2}\left[I_3\Omega^2\left(\sin\theta\right)^2 + I_3\dot{\theta}^2\right].
\end{gathered} \tag{2}$$

Because no resultant moment about the z' axis acts on the system, the angular momentum about this axis is conserved. Point A lies on this axis, so we assert that

$$\left[\left(\bar{H}_A\right)_s + \left(\bar{H}_A\right)_{AB}\right]_2\cdot\bar{k}' = \left[\left(\bar{H}_A\right)_s + \left(\bar{H}_A\right)_{AB}\right]_1\cdot\bar{k}'.$$

In terms of xyz components we have $\bar{k}' = -\sin\theta\bar{i} + \cos\theta\bar{k}$, which in combination with Eqs. (2) gives

$$\left[\left(\bar{H}_A\right)_s + \left(\bar{H}_A\right)_{AB}\right]\cdot\bar{k}' = \left(I_2 + I_3\right)\Omega\left(\sin\theta\right)^2 + I_1\left(\Omega\cos\theta - \eta\right)\cos\theta.$$

We take instant t_1 to be the initial condition, at which it is given that $\Omega_1 = 5$ rad/s, $\theta_1 = 90^\circ$, and $\dot{\theta}_1 = 0$. The value of η is specified and the inertia properties are listed in Eqs. (1). Thus, conservation of angular momentum about the vertical shaft requires that

$$\left(I_2 + I_3\right)\Omega_2\left(\sin\theta_2\right)^2 + I_1\left(\Omega_2\cos\theta_2 - \eta\right)\cos\theta_2 = 0.9783m. \tag{3}$$

This is one relation between Ω_2 and θ_2 that must be satisfied. Another is obtained from conservation of energy. The only force acting on the system of rigid bodies that does work is gravity. We take the elevation of point A to be the datum, so that

$$V = mg\left(-L\cos\theta\right) + \frac{mg}{2}\left(-\frac{L}{2}\cos\theta\right).$$

Correspondingly, conservation of energy requires that

$$\begin{gathered}
T_2 + V_2 = T_1 + V_1, \\
\frac{1}{2}\left[\left(I_2 + I_3\right)\Omega_2^2\left(\sin\theta_2\right)^2 + \left(I_2 + I_3\right)\dot{\theta}_2^2 + I_1\left(\Omega_2\cos\theta_2 - \eta\right)^2\right] \\
-1.25mgL\cos\theta_2 = 2.896m.
\end{gathered} \tag{4}$$

Thus we have two conservation principles relating Ω and θ at any instant t_2.

The value of θ_2 is a maximum or minimum when $\dot{\theta}_2 = 0$. To determine the corresponding values of θ_2 and Ω_2, we observe that Eq. (3) may be solved for Ω_2 as a function of θ_2:

$$\Omega_2\left(\theta\right) = \frac{0.9783m + I_1\eta\cos\theta_2}{\left(I_2 + I_3\right)\left(\sin\theta_2\right)^2 + I_1\cos\left(\theta_2\right)^2}. \tag{5}$$

When this expression is substituted into Eq. (4), the result is an equation that contains only θ_2. To determine the roots of this equation, let $E\left(\theta_2, \dot{\theta}_2, \Omega_2\right)$ denote the left side of Eq. (4). Thus we seek the roots of a function $F(\theta_2)$ defined symbolically by

$$F(\theta_2) \equiv \frac{1}{m} E(\theta_2, 0, \Omega_2(\theta_2)) - 2.896.$$

A plot of this function would show that it is positive at $\theta_2 = 0$ and $\theta_2 = \pi$ and is negative slightly below $\theta_2 = \pi/2$. Thus there are two roots. Numerical software readily yields these values if we use $\theta_2 = \pi/4$ and $\theta_2 = \pi/2$ as the initial guesses. The results, and the corresponding values of Ω at each position, are

$$(\theta_2)_{\min} = 36.288^\circ, \quad \Omega_2((\theta_2)_{\min}) = 14.128 \text{ rad/s},$$
$$(\theta_2)_{\max} = 90^\circ, \quad \Omega_2((\theta_2)_{\max}) = 5 \text{ rad/s}.$$

The fact that the system does not rise higher than the angle at which it was released is not surprising. When the cable is severed, the system falls. As it does so, the value of Ω must increase to conserve angular momentum, because the mass is situated closer to the vertical axis. The inertia of this descent carries the bar past the value of θ at which a steady precession is possible. At the minimum θ_2 the gyroscopic rotational effects cause the system to swing upward, slowing Ω. When the system reaches its initial elevation, it has the value of Ω with which it was released. Because the mechanical energy is conserved, and both θ and Ω match the conditions when the system was released, it must be that $\dot{\theta}$ also matches the initial value.

The second part seeks the values of Ω_2 and $\dot{\theta}_2$ corresponding when $\theta_2 = 0.5[(\theta_2)_{\min} + (\theta_2)_{\max} = 63.14^\circ$. To find Ω_2 we substitute this value of θ_2 into Eq. (3) to find

$$\Omega_2 = 6.467 \text{ rad/s} \quad \text{at} \quad \theta_2 = 63.14^\circ. \qquad \triangleleft$$

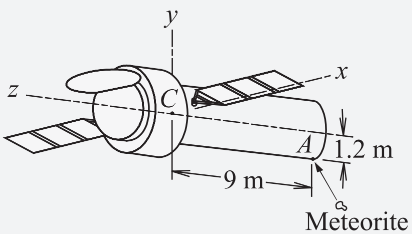
Substitution of this value into Eq. (4) converts it to a single equation for $\dot{\theta}_2$, whose solution is

$$\dot{\theta}_2 = \pm 4.0807 \text{ rad/s at } \theta_2 = 63.14^\circ. \qquad \triangleleft$$

The alternative sign, which results from the fact that $\dot{\theta}_2$ is obtained by taking a square root, indicates that the magnitude of the nutation rate does not depend on whether the sphere is rising or falling. This is another consequence of the conservative nature of this system.

EXAMPLE 6.14 The orbiting satellite is spinning about its z axis at 3 rad/s, and its velocity is 8 km/s parallel to that axis. The mass of the satellite is 5000 kg, and its inertia properties relative to its center of mass C are $I_{xx} = 32\,000$, $I_{yy} = 40\,000$, $I_{zz} = 3\,600$, $I_{xy} = I_{xz} = I_{yz} = 0$ kg-m^2. A 2-kg meteorite, whose velocity is

$-9\bar{i} + 12\bar{j}$ km/s, impacts the satellite at point A in the yz plane and then is embedded in the satellite's wall. The embedding process is completed in an interval of 400 ms. Determine the velocity of the satellite's center of mass and its angular velocity immediately after the collision. Also determine the average impact force exerted by the meteorite and how much energy is dissipated in the collision. (The deposition of energy is one of the damage mechanisms.)



Example 6.14

SOLUTION This example explores how to treat impulsive forces by use of systems concepts. The only force that is significant during the impact is the interaction force between the meteorite and the satellite, which is very large because the embedding process is very short and there is a large velocity difference at the point of impact. Both bodies move freely, so the interaction force $\bar{F}$ is the only force to consider. The body-fixed reference frame is already defined, so there is no need to draw a free-body diagram.

The external force and moment resultants vanish for this system. Consequently both linear and angular momentum are conserved. There is no fixed point in this situation, so we formulate the angular momentum with respect to the center of mass of both bodies, which we designate as point G. The meteorite's small size makes it permissible to consider it to be a particle, so its angular momentum relative to its own center of mass may be ignored. The linear momentum part of Eqs. (6.4.6) therefore requires that

$$m_s\,(\bar{v}_C)_2 + m_m\,(\bar{v}_m)_2 = m_s\,(\bar{v}_C)_1 + m_m\,(\bar{v}_m)_1\,, \tag{1}$$

whereas the angular momentum equation gives

$$\begin{aligned}&\left(\bar{H}_C\right)_2 + (\bar{r}_{C/G})_2 \times m_s\,(\bar{v}_{C/G})_2 + (\bar{r}_{m/G})_2 \times m_m\,(\bar{v}_{m/G})_2\\ &\quad= \left(\bar{H}_C\right)_1 + (\bar{r}_{C/G})_1 \times m_m\,(\bar{v}_{C/G})_1 + (\bar{r}_{m/G})_1 \times m_m\,(\bar{v}_{m/G})_1\,.\end{aligned} \tag{2}$$

The positions and velocities relative to the center of mass are governed by the first moment of mass. Using the satellite's center and the meteorite's location as alternative reference points for this evaluation leads to

$$(m_s + m_m)\,\bar{r}_{G/C} = m_m\bar{r}_{m/C} \implies \bar{r}_{C/G} = -\bar{r}_{G/C} = -\frac{m_m}{(m_s+m_m)}\bar{r}_{m/C},$$

$$(m_s + m_m)\,\bar{r}_{G/m} = m_s\bar{r}_{C/m} \implies \bar{r}_{m/G} = -\bar{r}_{G/m} = -\frac{m_s}{(m_s+m_m)}\left(-\bar{r}_{m/C}\right).$$

These relations are general, so they may be differentiated to describe the relative velocity terms in Eq. (2). The result is

$$\bar{v}_{C/G} = -\frac{m_m}{(m_s + m_m)}\bar{v}_{m/C}, \quad \bar{v}_{m/G} = \frac{m_s}{(m_s + m_m)}\bar{v}_{m/C}.$$

Substitution of these relations into Eq. (2) simplifies that equation to

$$\left(\bar{H}_C\right)_2 + \frac{m_s m_m}{m_s + m_m}\left(\bar{r}_{m/C}\right)_2 \times \left(\bar{v}_{m/C}\right)_2 = \left(\bar{H}_C\right)_1 + \frac{m_s m_m}{m_s + m_m}\left(\bar{r}_{m/C}\right)_1 \times \left(\bar{v}_{m/C}\right)_1. \quad (3)$$

By definition, $\bar{v}_{m/C} = \bar{v}_m - \bar{v}_C$. The initial values of both velocities are given. To determine the final relative velocity we observe that embedding of the meteorite means that its velocity will match that of point A on the satellite. Therefore its final velocity is related to the velocity of point C by the fact that points A and C belong to the same rigid body. Also, as a consequence of taking the embedding process to be impulsive, positions are the same at the beginning and end of the impact, so we take $(\bar{r}_{m/C})_2 = (\bar{r}_{m/C})_1 = \bar{r}_{A/C}$. Another aspect of ignoring position changes is that the given body-fixed xyz coordinate system may be used to describe vectors throughout the duration of the impact. Thus we have

$$\begin{gathered}(\bar{v}_m)_1 = -9000\bar{i} + 12000\bar{j}, \quad (\bar{v}_C)_1 = 8000\bar{k} \text{ m/s}, \quad \bar{\omega}_1 = 3\bar{k} \quad \text{rad/s}, \\ (\bar{v}_{m/C})_1 = (\bar{v}_m)_1 - (\bar{v}_C)_1 = -9000\bar{i} + 12000\bar{j} - 8000\bar{k} \text{ m/s}, \\ (\bar{v}_m)_2 = (\bar{v}_C)_2 + \bar{\omega}_2 \times \bar{r}_{A/C}, \quad (\bar{v}_{m/C})_2 = \bar{\omega}_2 \times \bar{r}_{A/C}.\end{gathered} \quad (4)$$

The initial values enables us to compute the right side of Eqs. (1) and (3), which represent the constant values of the respective momenta. For this evaluation we know $\bar{H}_C$ from the standard relation for angular momentum in conjunction with the given centroidal inertia properties. The result is

$$\begin{aligned}(m_s + m_m)(\bar{v}_C)_2 + m_m\bar{\omega}_2 \times \bar{r}_{A/C} &= 5000(\bar{v}_C)_1 + 2(\bar{v}_m)_1 \\ &= \left(-0.018\bar{i} + 0.024\bar{j} + 40.000\bar{k}\right)\left(10^6\right) \text{ kg-m/s},\end{aligned} \quad (5)$$

$$\begin{aligned}\left(\bar{H}_C\right)_2 + \frac{m_s m_m}{m_s + m_m}\bar{r}_{A/C} \times (\bar{\omega}_2 \times \bar{r}_{A/C}) &= \left(\bar{H}_C\right)_1 + \frac{10\,000}{5002}\bar{r}_{A/C} \times (\bar{v}_{m/C})_1 \\ &= 235106\bar{i} + 161935\bar{j} - 10791\bar{k} \text{ kg-m}^2\text{/s}.\end{aligned} \quad (6)$$

In view of the vectorial nature of Eqs. (5) and (6), their components give six scalar equations. The unknowns are the components of $(\bar{v}_C)_2$ and ω_2, but Eq. (6) does not depend on $(\bar{v}_C)_2$. Thus we first solve Eq. (6) for $\bar{\omega}_2$. To assist that solution, we evaluate the cross product term by representing $\bar{\omega}_2$ in terms of its components, ω_x, ω_y, and ω_z:

$$\begin{aligned}\frac{m_s m_m}{m_s + m_m}\bar{r}_{A/C} \times (\bar{\omega}_2 \times \bar{r}_{A/C}) &= 164.8140\omega_x\bar{i} + (161.9352\omega_y - 21.5914\omega_z)\bar{j} \\ &\quad + (-21.5914\omega_y + 2.8788\omega_z)\bar{k}\end{aligned} \quad (7)$$

Solving Eq. (6) for the components of $\bar{\omega}_2$ is expedited by writing the equation in matrix form. In matrix notation Eq. (7) may be written as

$$\frac{m_s m_m}{m_s + m_m} \left\{r_{A/G}\right\} \otimes \left\{\{\omega\}_2 \otimes \left\{r_{A/G}\right\}\right\} = [J]\{\omega\}_2 ,$$

$$[J] = \begin{bmatrix} 164.8140 & 0 & 0 \\ 0 & 161.9352 & -21.5914 \\ 0 & -21.5914 & 2.8788 \end{bmatrix}.$$

The corresponding form of Eq. (6) is

$$[[I_C] + [J]]\{\omega\}_2 = [235\,106 \;\; 161\,935 \;\; -10\,791]^{\mathrm{T}} ,$$

which is readily solved for $\{\omega\}_2$. We then can find $(\bar{v}_C)_2$ by substituting the result into Eq. (5), such that

$$(m_s + m_m)\{v_C\}_2 = m_s\{v_C\}_1 + m_m(\bar{v}_m)_1 - m_m\{\omega\}_2 \otimes \left\{r_{A/C}\right\}$$
$$= [-17.920321 \;\; 23.868431 \;\; 40\,000]^{\mathrm{T}} \text{ rad/s.}$$

In vector notation the solutions are

$$\bar{\omega}_2 = 7.30941\bar{i} + 4.03046\bar{j} - 2.97105\bar{j} \text{ rad/s,}$$
$$(\bar{v}_C)_2 = -3.58263\bar{i} + 4.77178\bar{j} + 7996.805\bar{k} \text{ m/s.} \qquad \triangleleft$$

We readily find the impact force $\bar{F}$ by applying the linear impulse–momentum principle to the meteorite isolated from its surroundings. Toward that end we use Eq. (4) to evaluate the velocity of the meteorite after it is embedded, which yields

$$(\bar{v}_m)_2 = -43.422034\bar{i} + 70.556508\bar{j} + 7988.033489\bar{k}.$$

The only force whose impulse is important is that of $\bar{F}$, which we represent by its average value over the 400-ms duration, from which we find that

$$\left(\bar{F}\right)_{\mathrm{av}} = \frac{1}{\Delta t} m_m [(\bar{v}_m)_2 - (\bar{v}_m)_1] = 44.78\bar{i} - 59.65\bar{j} + 39.94\bar{k} \text{ kN.} \qquad \triangleleft$$

This is the average force acting on the meteorite; the maximum is likely to be substantially greater. The force acting on the satellite is opposite.

Conservative forces, being position dependent, cannot become large during the interval of the impact, so the change in potential energy must be negligible during an impact. Thus the work–energy principle for the system states that

$$W_{1\to 2} = T_2 - T_1.$$

We compute the kinetic energy of the satellite by adding the translational part that is due to the motion of its center of mass to rotational energy about the center of mass. We then find the kinetic energy of the system by adding the contribution of the meteorite:

$$T = \frac{1}{2} m_s \{v_C\}^{\mathrm{T}} \{v_C\} + \frac{1}{2} \{\omega\}^{\mathrm{T}} [I_C] \{\omega\} + \frac{1}{2} m_m \{v_m\}^{\mathrm{T}} \{v_m\} .$$

Substitution of the velocity parameters at the initial and final instants yields

$$T_1 = 1.602250\left(10^{11}\right), \quad T_2 = 1.599373\left(10^{11}\right) \text{ J},$$
$$W_{1\to 2} = -2.877\left(10^{8}\right) \text{ J}. \qquad \triangleleft$$

It will be noted that T_1 and T_2 are quite close, so an accurate determination of the energy change necessitated extra precision in the calculations. The fact that $W_{1\to 2}$ is negative means that mechanical energy is lost. It is converted to other types of energy, such as heat and plastic material deformation. Although the value of $W_{1\to 2}$ is small relative to the total kinetic energy, it is an enormous amount of energy transfer. These observations, in combination with the large value of $\bar{F}_{\text{av}}$, make it evident that this impact would be devastating for the satellite.

The momentum conservation equations, Eqs. (5) and (6), have an interesting interpretation. At the conclusion of the impulse process, the meteorite is embedded in the satellite, so they are essentially one body. The left side of the linear momentum equation, Eq. (5), is actually the product of the total mass and the velocity of the center of mass of this body. Similarly, the left side of the angular momentum equation, Eq. (6), is the product of the inertia matrix of this new body and its angular velocity. In both cases, the motion variables change because the inertial properties of the combined body are different from those of the satellite before the impact.

In closure, it is useful to observe that formulating the angular momentum with respect to the system's center of mass considerably complicated the analysis. The smallness of the meteorite's mass causes that center of mass to be nearly coincident with that of the satellite. If we had ignored the difference between these two points, it would not have been necessary to consider first moments of mass, and the general angular impulse–momentum equation would have reduced to

$$\left(\bar{H}_C\right)_2 + \bar{r}_{A/C} \times m_m\left(\bar{\omega}_2 \times \bar{r}_{A/C}\right) \approx \left(\bar{H}_C\right)_1 + \bar{r}_{A/C} \times m_m\left[(\bar{v}_m)_1 - (\bar{v}_C)_1\right],$$

which yields

$$\bar{\omega}_2 \approx 7.31232\bar{i} + 4.03206\bar{j} - 2.97343\bar{k} \text{ m/s}.$$

This value is extremely close to the value obtained from the full analysis. The average force and mechanical-energy loss would be essentially unchanged. However, the distinction between the center of mass of the system and of the satellite would have been important if m_s and m_m were of similar orders of magnitude.

6.4.3 Collisions of Rigid Bodies

To the naked eye a collision seems to cause the velocity of a body to change instantaneously, which is a characteristic of an impulsive force. This is a fundamental feature of the model for the action of impulsive forces. In many situations, such as the study of crash dynamics for vehicles, deformation effects are of primary importance, which obviates analysis by use of rigid-body concepts. In other situations, such as the collision of billiard balls, the presence of deformation is not apparent. Although deformation is always important, we consider here a concept based on empirical observations that makes it possible to adapt the concepts of rigid-body dynamics. The scope of the treatment is limited to collisions that leave the bodies essentially unchanged, which is possible if the relative speed of the colliding bodies is sufficiently low.

We restrict our attention to cases in which there is an identifiable plane of contact for each body that is tangent to each body at a single point. (For planar motion, this would correspond to contact along a line perpendicular to the plane.) In the strictest sense a corner is marked by a slope discontinuity, so no unique tangent point exists at such a location. However, this ambiguity is removed if we consider the corner to be slightly rounded. We define a coordinate system xyz whose origin is the point at which the bodies are in contact and whose yz plane coincides with the contact point. The configuration is depicted in Fig. 6.6, where points A and B are the centers of mass of bodies 1 and 2, and points $C1$ and $C2$ mark the contact point on the respective body.

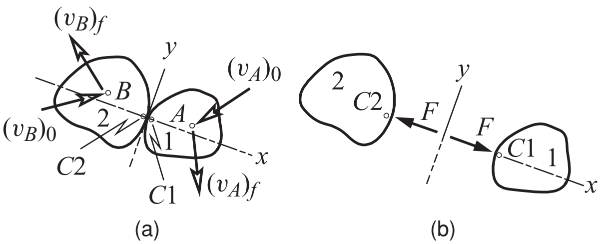


Figure 6.6. Collision of rigid bodies. Points A and B are the centers of mass, and points $C1$ and $C2$ are the respective contact points.

The bodies in Fig. 6.6(a) are free of external restrictions on their movement, so the only force to consider during the impact is the one exerted between the bodies at the contact point, as shown in the free-body diagrams of Fig. 6.6(b). We begin with the idealization that the contacting surfaces are very smooth, so the impact force $\bar{F}$ on body A acts in the positive $\bar{i}$ direction, whereas it acts on body B in the negative $\bar{i}$ direction. In addition to requiring determination of the final velocities, we need to account for the angular motion of each body. Thus the initial condition consists of the initial velocities $(\bar{v}_A)_0$ and $(\bar{v}_B)_0$ of the centers of mass, as well as the angular velocities $(\bar{\omega}_1)_0$ and $(\bar{\omega}_2)_0$. The quantities to be determined are these variables at the termination of the impact, whose values are denoted with the subscript f, and the impulse of the contact force.

The impulsive force model is appropriate, so we represent the unknown impulse as the product $F\Delta t$ of the average force magnitude and the duration of the impact. An important aspect of the impulsive force model is that the brevity of the contact interval makes it permissible to consider all points to have a constant position during the impact. Hence we can consider xyz to maintain its orientation, and the contact points on each body are indistinguishable with regard to their position. Thus we may write the position vector from each center of mass to the respective contact point simply as $\bar{r}_{C/A}$ and $\bar{r}_{C/B}$.

Equations that are available to determine the unknown linear and angular velocity parameters come from both the linear and angular impulse–momentum principles. We formulate these relative to each body's center of mass because doing so decouples the linear and angular velocities in the momentum principles. Thus we have four vector equations representing the kinetics of the system:

$$\boxed{\begin{gathered} m_1 (\bar{v}_A)_f = m_1 (\bar{v}_A)_0 + F\Delta t\bar{i}, \\ m_2 (\bar{v}_B)_f = m_2 (\bar{v}_B)_0 - F\Delta t\bar{i}, \\ \left(\bar{H}_A\right)_f = \left(\bar{H}_A\right)_0 + \bar{r}_{C/A} \times \left(F\Delta t\bar{i}\right), \\ \left(\bar{H}_B\right)_f = \left(\bar{H}_B\right)_0 + \bar{r}_{C/B} \times \left(-F\Delta t\bar{i}\right). \end{gathered}} \tag{6.4.40}$$

The final angular momenta contain the components of the respective angular velocities. Thus, if the value of $F\Delta t$ were known, it would be a simple matter to determine the unknown velocity parameters. It follows that we need one more scalar equation.

A proper analysis would examine the manner in which each body deforms, but doing so is quite challenging. Instead, we introduce a concept that allows for deformation in the vicinity of the point of contact while retaining the elements of rigid-body dynamics. The deformation of the surfaces that is produced by the large collision force may be considered to cause each body's surface to move relative to its nominal position. Thus, in the time interval following the initial contact t_0 until the instant t_f immediately before the bodies separate, the velocity of the contact points $C1$ and $C2$ are generally not describable by rigid-body kinematics. However, there is one exceptional instant when this is not true. At an instant that we denote as $t_{\max}$ the local deformation has reached a maximum. This means that the inward displacement of each contact point relative to the undeformed state has reached its largest value. It follows that the velocity of the contact points at this instant relative to the nominal rigid-body configuration is zero. Hence, at this instant, each contact point's velocity is describable by rigid-body kinematics in terms of the center-of-mass velocities $(\bar{v}_A)_{\max}$ and $(\bar{v}_B)_{\max}$ and angular velocities $(\bar{\omega}_A)_{\max}$ and $(\omega_B)_{\max}$. It follows that, at $t = t_0,\ t_{\max}$, or t_f, the contact point velocities are given by

$$\bar{v}_{C1} = \bar{v}_A + \bar{\omega}_1 \times \bar{r}_{C/A}, \quad \bar{v}_{C2} = \bar{v}_B + \bar{\omega}_2 \times \bar{r}_{C/B}. \tag{6.4.41}$$

The values of $(\bar{v}_{C1})_0$ and $(\bar{v}_{C2})_0$ are set by the initial conditions, whereas $(\bar{v}_{C1})_f$ and $(\bar{v}_{C2})_f$ are a consequence of the system's kinetics. However, at $t_{\max}$ the contact point velocities are also related by

$$(\bar{v}_{C1})_{\max} \cdot \bar{i} = (\bar{v}_{C2})_{\max} \cdot \bar{i}. \tag{6.4.42}$$

The importance of the instant of maximum deformation is that it is featured in the definition of the *coefficient of restitution*, ε. This empirical parameter may be considered to be the result of observation of many collisions in which the bodies do not display significant permanent deformation. It is the ratio of the impulse of the contact force during the restitution interval $t_{\max} < t < t_f$ to the impulse in the deformation interval $t_0 < t < t_{\max}$. In other words,

$$\boxed{\varepsilon = \frac{\int_{t_{\max}}^{t_f} F dt}{\int_{t_0}^{t_{\max}} F dt}.} \tag{6.4.43}$$

If the collision is not energetic, then the value of ε is a number between zero and unity that depends on the materials, shape, and size of the body. (An energetic collision releases an internal energy source, such as a spring with a trigger release.)

Our task now is to convert this definition of ε to one that features the same kinematical variables as those appearing in Eqs. (6.4.40). Such a derivation for the case of an arbitrary spatial motion is complicated by the fact that each angular momentum component contains all angular velocity components if each body has an arbitrary shape. We therefore consider the case of bodies in planar motion. The result will be a form that is applicable for spatial motion.

For planar motion, we set $\bar{H}_A = I_A\omega_1\bar{k}$ and $\bar{H}_B = I_B\omega_2\bar{k}$ in Eqs. (6.4.40). Also, because the centers of mass and the contact point lie in the xy plane for planar motion, the $\bar{k}$ components of the cross products in the moments of $\bar{F}$ about points A and B can be written as

$$\bar{k} \cdot \left(\bar{r} \times F\bar{i}\right) \equiv F\bar{r} \cdot \left(\bar{i} \times \bar{k}\right) = -F\bar{r} \cdot \bar{j} \quad \text{if} \quad \bar{r} \cdot \bar{k} = 0. \tag{6.4.44}$$

We use this relation to break each of Eqs. (6.4.40) into portions covering the deformation interval $t_0 < t < t_{\max}$ and the restitution period $t_{\max} < t < t_f$. Doing so for body A yields

$$\begin{aligned}
m_1\left(\bar{v}_A\right)_{\max} &= m_1\left(\bar{v}_A\right)_0 + \textstyle\int_{t_0}^{t_{\max}}\left(F\bar{i}\right)dt,\\
m_1\left(\bar{v}_A\right)_f &= m_1\left(\bar{v}_A\right)_{\max} + \textstyle\int_{t_{\max}}^{t_f}\left(F\bar{i}\right)dt,\\
I_A\left(\omega_1\right)_{\max}\bar{k} &= I_A\left(\omega_1\right)_0\bar{k} + \bar{r}_{C/A} \times \textstyle\int_{t_0}^{t_{\max}}\left(F\bar{i}\right)dt,\\
I_A\left(\omega_1\right)_f\bar{k} &= I_A\left(\omega_1\right)_{\max}\bar{k} + \bar{r}_{C/A} \times \textstyle\int_{t_{\max}}^{t_f}\left(F\bar{i}\right)dt.
\end{aligned} \tag{6.4.45}$$

Notice that the position vector was brought outside the integral in both angular impulses because positions are taken to not change during the short interval of the impulsive action. We take the $\bar{i}$ component of the linear momentum equations and the $\bar{k}$ component of the angular momentum equations and apply Eq. (6.4.44) to the latter, with the result that

$$\begin{aligned}
\textstyle\int_{t_0}^{t_{\max}} F dt &= m_1\left[\left(\bar{v}_A\right)_{\max}\cdot\bar{i} - \left(\bar{v}_A\right)_0\cdot\bar{i}\right],\\
\textstyle\int_{t_{\max}}^{t_f} F dt &= m_1\left[\left(\bar{v}_A\right)_f\cdot\bar{i} - \left(\bar{v}_A\right)_{\max}\cdot\bar{i}\right],\\
\left(\bar{r}_{C/A}\cdot\bar{j}\right)\textstyle\int_{t_0}^{t_{\max}} F dt &= -I_A\left[\left(\omega_1\right)_{\max} - \left(\omega_1\right)_0\right],\\
\left(\bar{r}_{C/A}\cdot\bar{j}\right)\textstyle\int_{t_0}^{t_{\max}}\left(F\bar{i}\right)dt &= -I_A\left[\left(\omega_1\right)_f - \left(\omega_1\right)_{\max}\right].
\end{aligned} \tag{6.4.46}$$

We ratio the second equation to the first and the fourth to the third. In view of Eq. (6.4.43), each ratio is ε. The result of clearing denominators is

$$\begin{aligned}
\left[\left(\bar{v}_A\right)_f\cdot\bar{i} - \left(\bar{v}_A\right)_{\max}\cdot\bar{i}\right] &= \varepsilon\left[\left(\bar{v}_A\right)_{\max}\cdot\bar{i} - \left(\bar{v}_A\right)_0\cdot\bar{i}\right],\\
\left[\left(\omega_1\right)_f - \left(\omega_1\right)_{\max}\right] &= \varepsilon\left[\left(\omega_1\right)_{\max} - \left(\omega_1\right)_0\right].
\end{aligned} \tag{6.4.47}$$

Comparable equations result from following the same procedures for the momenta of body B:

$$\begin{gathered}\left[(\bar{v}_B)_f\cdot\bar{i}-(\bar{v}_B)_{\max}\cdot\bar{i}\right]=\varepsilon\left[(\bar{v}_B)_{\max}\cdot\bar{i}-(\bar{v}_B)_0\cdot\bar{i}\right],\\ \left[(\omega_2)_f-(\omega_2)_{\max}\right]=\varepsilon\left[(\omega_2)_{\max}-(\omega_2)_0\right].\end{gathered}\tag{6.4.48}$$

Neither set of equations is quite in the form we would like because they contain the linear and angular velocities at the instant $t_{\max}$, which are not parameters we need to know. We remedy this by recalling that the normal velocity components match at $t_{\max}$, as described in Eq. (6.4.42). We find an expression for these velocity components by taking the dot product of the velocities in Eqs. (6.4.41), which gives

$$\begin{aligned}\bar{v}_{C1}\cdot\bar{i}&=\bar{v}_A\cdot\bar{i}+\left(\omega_1\bar{k}\times\bar{r}_{C/A}\right)\cdot\bar{i}\equiv\bar{v}_A\cdot\bar{i}-\omega_1\left(\bar{r}_{C/A}\cdot\bar{j}\right),\\ \bar{v}_{C2}\cdot\bar{i}&=\bar{v}_B\cdot\bar{i}+\left(\omega_2\bar{k}\times\bar{r}_{C/B}\right)\cdot\bar{i}\equiv\bar{v}_B\cdot\bar{i}-\omega_2\left(\bar{r}_{C/B}\cdot\bar{j}\right).\end{aligned}\tag{6.4.49}$$

In light of the form of this relation, we multiply the second of Eqs. (6.4.47) by $\bar{r}_{C/A}\cdot\bar{j}$ and subtract it from the first of those equations. We also apply similar operations to Eqs. (6.4.48) for body B. The result is

$$\begin{aligned}\left[(\bar{v}_{C1})_f\cdot\bar{i}-(\bar{v}_{C1})_{\max}\cdot\bar{i}\right]&=\varepsilon\left[(\bar{v}_{C1})_{\max}\cdot\bar{i}-(\bar{v}_{C1})_0\cdot\bar{i}\right],\\ \left[(\bar{v}_{C2})_f\cdot\bar{i}-(\bar{v}_{C2})_{\max}\cdot\bar{i}\right]&=\varepsilon\left[(\bar{v}_{C2})_{\max}\cdot\bar{i}-(\bar{v}_{C2})_0\cdot\bar{i}\right].\end{aligned}\tag{6.4.50}$$

In view of Eqs. (6.4.48), the velocity components at $t_{\max}$ are identical, so subtraction of the first of the preceding equations from the second leads to

$$\boxed{\left[(\bar{v}_{C2})_f\cdot\bar{i}-(\bar{v}_{C1})_f\cdot\bar{i}\right]=\varepsilon\left[(\bar{v}_{C1})_0\cdot\bar{i}-(\bar{v}_{C2})_0\cdot\bar{i}\right].}\tag{6.4.51}$$

This is the additional equation to be used in conjunction with Eqs. (6.4.40) to evaluate the velocities with which the bodies rebound. Although it was derived by considering planar motion, it is equally applicable to spatial motion. Equation (6.4.51) is useful for an experiment in which the velocities are measured and the value of ε is computed. When we need to determine how bodies move after they collide, it will be necessary to eliminate the contact velocities in favor of the center-of-mass velocities and the angular velocities. Equations (6.4.41) are available for that purpose. Evaluation of the $\bar{i}$ component of the cross-product term is somewhat simplified by the identity that $(\bar{\omega}\times\bar{r})\cdot\bar{i}=\bar{\omega}\cdot\left(\bar{r}\times\bar{i}\right)$. This leads to

$$\begin{aligned}&\left[(\bar{v}_B)_f\cdot\bar{i}-(\bar{v}_A)_f\cdot\bar{i}\right]+(\bar{\omega}_2)_f\cdot\left(\bar{r}_{C/B}\times\bar{i}\right)-(\bar{\omega}_1)_f\cdot\left(\bar{r}_{C/A}\times\bar{i}\right)\\ &\quad=\varepsilon\left\{\left[(\bar{v}_A)_0\cdot\bar{i}-(\bar{v}_B)_0\cdot\bar{i}\right]+(\bar{\omega}_1)_f\cdot\left(\bar{r}_{C/A}\times\bar{i}\right)-(\bar{\omega}_2)_f\cdot\left(\bar{r}_{C/B}\times\bar{i}\right)\right\}.\end{aligned}\tag{6.4.52}$$

Note that this form is convenient for spatial motion, but planar motion problems are readily formulated by use of Eq. (6.4.51).

This is a scalar equation. For an arbitrary spatial motion, decomposing Eqs. (6.4.40) into their components leads to 12 additional scalar equations. The 13 unknowns are the three components of each center-of-mass velocity, the three components of each angular velocity, and the impulse $F\Delta t$. The number of equations and variables reduces to seven for planar motion.

As was mentioned at the outset, the coefficient of restitution should be considered to be an empirical quantity obtained from experimental measurement. This would entail measuring the initial and final velocities of each contact point, either directly or by calculation from measurements relative to any convenient point in each body. Substitution into Eq. (6.4.51) would then provide the value of ε. Special names are used to describe the limiting values of ε. A collision is said to be *perfectly plastic* if $\varepsilon = 0$. In that case the contact points have identical velocities in the normal direction after the impact. A common misconception is to say that the bodies stick together, which is not true because such a result would require constraint forces acting tangentially to the contact plane. The case in which $\varepsilon = 1$ is said to be *perfectly elastic*. This term originates from the fact that it is the only case in which energy is conserved. It obviously is an idealized case that is never obtained in a real system.

The development has treated an *eccentric impact,* which is the term used to describe a collision in which the centers of mass are not collinear with the line through the contact point and normal to the plane of contact. Elementary texts typically introduce the topic of collisions by considering spheres. In that case the centers of mass and point of contact are collinear with the normal to the tangent plane. That is the characteristic of *central impact,* which is always the case for spheres. When the bodies are spheres, $\bar{r}_{C/A} = -R_A\bar{i}$ and $\bar{r}_{C/B} = R_B\bar{i}$, so the moment impulses in Eqs. (6.4.40) are zero and the angular velocity does not appear in Eq. (6.4.52). This means that the angular velocities of the spheres are unchanged and therefore irrelevant to the collision process.

In some cases, one of the bodies involved in the collision is massive, like the Earth. The preceding formulation is equally valid in such cases, but it will merely tell us that the very large body's motion is unaltered. It is reasonable in such cases to merely say that a large body's motion is unaffected by the collision, and correspondingly ignore the momentum equations for large body.

Often, when bodies collide, one or both are restricted kinematically in the movement they can undergo. Any motion restriction, such as the requirement that a block slide along the ground, is imposed by a reaction force that must be as large as necessary to enforce that restriction. In the case of collisions, the impact forces are impulsive, so the reactions must have that same characteristic. This means that reaction forces cannot be ignored when one constructs free-body diagrams and that the impulses of these forces are additional unknowns to be evaluated as part of the analysis of the collision. Balancing this is the fact that the motion restriction reduces the number of unknown velocity parameters that will need to be determined.

Figure 6.7 depicts a simple system that makes this aspect apparent. It shows a ball that strikes the inclined face of a wedge that can slide over a smooth horizontal surface. The ground prevents the wedge from moving vertically, which means that the horizontal velocity component is the sole unknown velocity variable for this body. The impact force $\bar{F}$ acts normally to the face of the wedge. The normal force $\bar{N}$ prevents the wedge from penetrating the horizontal surface, so it too must act impulsively. Thus the fact that the horizontal surface prevents movement of the block in the vertical direction reduces by one the number of velocity components to determine, while also increasing by one

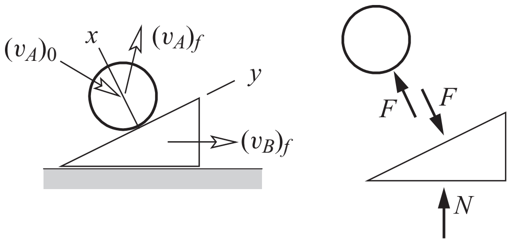


Figure 6.7. An example of a collision in which a body is constrained. The constraint force $\bar{N}$ and the collision force $\bar{F}$ are impulsive.

the number of force impulses that are unknown. Suppose $(\bar{v}_A)_0$ were given. Then the unknowns to be determined in this planar situation are the two components of $(\bar{v}_A)_f$, the value of $(v_B)_f$, F, and N. The angular velocity of the sphere is irrelevant because this is a central impact for that body. The five available equations are the linear impulse–momentum equations for the sphere and the wedge, and the relation between velocity components in the x direction associated with the coefficient of restitution. (A minor shortcut is to formulate only the horizontal component of the impulse–momentum equation for the wedge, which would avoid the occurrence of N in the equations to solve.)

A bar that executes a pure rotation is a common configuration featuring constrained motion. The reactions at the pivot must act impulsively to prevent that point from moving. These unknown reactions can be avoided in the formulation if the angular impulse–momentum equation is formulated about the fixed pivot. If the impulse associated with each component of the pivot reaction is required, such information can be extracted from the linear impulse–momentum after the rebound velocity of the center of mass has been determined.

Examination of the collision of unconstrained bodies reveals a general weakness of this model of collisions. The contact force is taken to act solely in the normal direction. Consequently the linear impulse–momentum equations indicate that the velocity of each body's center of mass tangent to the contact plane is unaltered by the impact. This does not fit our observation of certain situations in which the contact surfaces are rough, for example, when a tennis ball touches the ground. This shortcoming can be remedied by use of a standard friction model. At the initial contact the contact points have different tangential velocities, which means that a sliding friction model should be appropriate. The magnitude of the friction force in that model is proportional to the normal force magnitude F, and the latter acts impulsively.

Consequently, we can insert a tangential force $f = \mu_k F$ in each free-body diagram. The direction of this force would be opposite the initial tangential velocity of one surface relative to the other at the initial instant t_0. Thus we can say that the force on body A would be

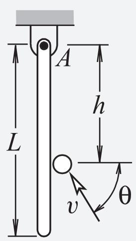
$$\bar{f} = -\mu_k F \frac{(\bar{v}_{CA})_0 \cdot \bar{i} - (\bar{v}_{CB})_0 \cdot \bar{i}}{\left|(\bar{v}_{CA})_0 \cdot \bar{i} - (\bar{v}_{CB})_0 \cdot \bar{i}\right|}. \tag{6.4.53}$$

The friction force applied to body B would be oppositely directed. The validity of the coefficient of restitution is unaltered by the friction force, so Eq. (6.4.52) still applies.

The impulse–momentum equations would be modified by replacing the contact force $F\bar{i}$ acting on body A with $F\bar{i} + \bar{f}$, whereas the force acting on body B would be changed from $-F\bar{i}$ to $-F\bar{i} - \bar{f}$. No new variables are introduced by these modifications, so the solvability of the equations is not altered.

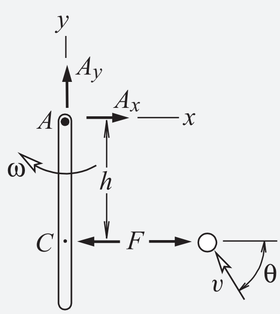
This preceding model of friction effects has limitations. For example, it is hypothetically possible that the analysis will lead to a rebound in which the tangential relative velocity at the contact points is reversed from its initial sense, which would indicate that the sense of $\bar{f}$ reverses at some intermediate instant between t_0 and t_f. As explained in Example 6.17, using this simple friction model would require determination of the instant at which slipping ceases, which is not part of the analysis.

EXAMPLE 6.15 The bar, whose mass is m_1 and whose moment of inertia about its pivot is I_A, is at rest in the vertical position when it is obliquely struck by the ball, whose mass is m_2. The coefficient of restitution is an unspecified value. Determine the angular velocity of the bar and the velocity of the ball immediately after the collision. Also determine the impulse of the pivot force during the collision. Is there a distance h for the impact that minimizes this impulse?



Example 6.15

SOLUTION This example addresses the role of constraint forces during a collision. In addition to the impact force, which is horizontal, the free-body diagram shows the forces exerted by the pin to prevent movement of point A. The x axis is defined to be normal to the contact plane, and the origin has been placed at the pivot point A, which is an allowable point for summing moments, because doing so will eliminate the reaction impulses from the angular momentum equation.



Individual free-body diagrams of the bar and the ball.

We carry out the analysis by considering the ball to be a particle, which means that we need not consider its rotation. Because the bar executes a pure rotation about point A, the rebound velocities of the center of mass G and of the impact point C are kinematically related to the angular velocity of the bar at the conclusion of the impact. The duration of the impact is brief, so we may use the initial position of points to characterize this relationship. It is convenient to consider a clockwise rotation in rebound to be positive, so we have

$$\begin{aligned}(\bar{v}_G)_f &= \left(-\omega_f\bar{k}\right)\times\bar{r}_{G/A} = -\omega_f\frac{L}{2}\bar{i},\\ (\bar{v}_C)_f &= \left(-\omega_f\bar{k}\right)\times\bar{r}_{C/A} = -\omega_f h\bar{i}.\end{aligned} \tag{1}$$

We now turn to the momentum equations. It is evident that the linear momentum equations for the bar will feature A_x and A_y. As noted at the outset, these forces will not appear in the equation describing the angular momentum of the bar about the pivot point. We therefore delay consideration of the bar's linear momentum. As we did for the kinematical equations, we consider the bar's position to not change during the impact, so equating the bar's angular momentum change about point A to the corresponding angular impulse gives

$$I_A(-\omega_f) = -(F\Delta t)\,h. \tag{2}$$

The linear impulse–momentum equations for the ball relate the ball's rebound velocity to the impact force. It is helpful to write these equations in component form:

$$m_2(\bar{v}_2)_f\cdot\bar{i} = m_2(-v\cos\theta)+(F\Delta t), \tag{3a}$$

$$m_2(\bar{v}_2)_f\cdot\bar{j} = m_2 v\cos\theta. \tag{3b}$$

The second of the preceding equations shows that the velocity of the ball parallel to the plane of contact is unaltered by the collision. Three unknowns, ω_f, $(\bar{v}_2)_f\cdot\bar{i}$, and $F\Delta t$, appear in Eqs. (2) and (3a). The third equation is obtained from the coefficient of restitution. Application of Eq. (6.4.51) in the present case gives

$$(\bar{v}_C)_f\cdot\bar{i}-(\bar{v}_2)_f\cdot\bar{i} = \varepsilon\left[(-v\cos\theta)-0\right].$$

Subsitution of the second of Eqs. (1) into this relation converts it to

$$-\omega_f h-(\bar{v}_2)_f\cdot\bar{i} = \varepsilon\left[(-v\cos\theta)-0\right]. \tag{4}$$

The simultaneous solution of Eqs. (2), (3a), and (4) yields

$$\begin{gathered}\omega_f = (\varepsilon+1)\frac{m_2h^2}{I_A+m_2h^2}\left(\frac{v}{h}\cos\theta\right),\quad F\Delta t = \frac{I_A}{h}\omega_f,\\ (\bar{v}_2)_f\cdot\bar{i} = \frac{\varepsilon I_A-m_2h^2}{I_A+m_2h^2}(v\cos\theta).\end{gathered} \quad \triangleleft$$

Now that the rebound velocity of the bar is known, we may evaluate the impulse of the reaction forces. We use Eqs. (1) to describe the velocity of the center of mass when we form the linear impulse–momentum equation for the bar, whose component form is

$$m_1 (\bar{v}_G)_f \cdot \bar{i} = m_1 \left(-\frac{L}{2}\omega_f\right) = A_x \Delta t - F\Delta t,$$

$$m_1 (\bar{v}_G)_f \cdot \bar{j} = 0 = 0 + A_y \Delta t.$$

The result is

$$A_x \Delta t = \left(\frac{I_A}{h} - \frac{m_1 L}{2}\right)\omega_f, \quad A_y \Delta t = 0. \quad \triangleleft$$

Several aspects of these results are interesting. The vertical reaction impulse is negligible because the impact force acts horizontally. The horizontal reaction impulse also is negligible if $h = 2I_A/m_1 L = 2\kappa^2/L$, where κ is the radius of gyration. For the case of a bar whose cross section is uniform, $h = 2L/3$. This distance is called the *center of percussion,* which previously was mentioned in Example 6.11. When a bar is struck at this distance from an end, it will execute a pure rotation about that end, even if it is unsupported. $\triangleleft$

Another interesting aspect of the results is the change in mechanical energy. The initial kinetic energy is solely stored in the ball, $T_0 = m_2 v^2/2$. The kinetic energy after the collision is found to be

$$T_f = \frac{1}{2} m_2 (\bar{v}_2)_f \cdot (\bar{v}_2)_f + \frac{1}{2} I_A \omega_f^2 = \frac{1}{2} m_2 v^2 \left[\frac{I_A \varepsilon^2 + m_2 h^2}{I_A + m_2 h^2} (\cos\theta)^2 + (\sin\theta)^2\right].$$

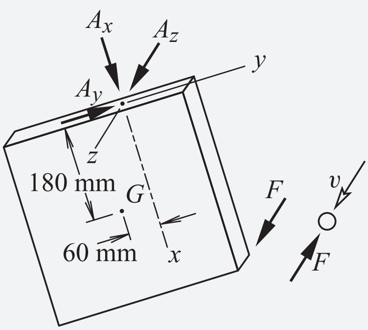
Thus the energy lost is

$$\Delta T = T_0 - T_2 = T_0 \left(1 - \varepsilon^2\right) (\cos\theta)^2 \left(\frac{I_A}{I_A + m_2 h^2}\right).$$

Therefore, except for this idealized condition of a a perfectly elastic impact, $\varepsilon = 1$, mechanical energy is dissipated as a result of the collision. This behavior is generally true. Because the amount of energy that is dissipated can be determined only after the collision dynamics has been evaluated, one should not use the work–energy principle to analyze collisions. Another interesting feature of the expression for ΔT is the $(\cos\theta)^2$ factor, which shows that the energy loss decreases significantly if the ball arrives obliquely. We refer to such an impact as a "grazing blow."

EXAMPLE 6.16 Suppose the square plate in Example 6.11 is struck at corner A by a 500-g ball, rather than a hammer. The impact velocity of the ball is v normal to the plate's surface, and the plate is initially at rest. The coefficient of restitution is $\varepsilon = 0.6$. In Example 6.11 the impulsive force had an average value of 5000 N over a 4-ms duration. Determine the initial speed v of the ball bearing that will generate an impact equal to the hammer blow. For this initial speed, determine the angular velocity of the plate and the velocity of the ball at the instant following the impact, and the average reaction at the support.

SOLUTION This example introduces techniques for solving collision problems featuring spatial motion. It also is intended to shed some light on the quantitative magnitude of collision effects. The free-body diagram of the plate is unchanged from Example 6.11, but we also need to consider the ball. Contact occurs along the xy plane, so the impact force $\bar{F}$ acts parallel to the z axis.



Free-body diagram showing the impulsive forces that act on the square plate.

Much of the solution resembles the one followed previously, except that the linear impulse–momentum equation for the ball and the equation for the coefficient of restitution need to be considered.

Let point C designate the corner where the ball strikes the plate. We designate the plate as body 1 and the ball as body 2. The initial state is characterized by $(\bar{\omega}_1)_0 = \bar{0}$, $(\bar{v}_B)_0 = v\bar{k}$, whereas the rebound is described by arbitrary an arbitrary $\bar{\omega}_2$, so that

$$\begin{aligned}(\bar{\omega}_1)_f &= \omega_x\bar{i} + \omega_y\bar{j} + \omega_z\bar{k},\\ (\bar{v}_G)_f &= (\bar{\omega}_1)_f \times \bar{r}_{G/A} = 0.06\omega_z\bar{i} + 0.18\omega_z\bar{j} + (-0.06\omega_x - 0.18\omega_y)\,\bar{k}.\end{aligned} \tag{1}$$

Because the plate executes a pure rotation about the pivot, whereas Eq. (6.4.52) refers to point G, we use Eq. (6.4.51) to describe the coefficient of restitution's role. Toward that end we form the velocity of the contact point on the plate:

$$(\bar{v}_C)_f = (\bar{\omega}_1)_f \times \bar{r}_{C/A} = -0.12\omega_z\bar{i} + 0.36\omega_z\bar{j} + (0.12\omega_x - 0.36\omega_y)\,\bar{k}. \tag{2}$$

The impulse–momentum equations for the plate are like those previously written in Example 6.11:

$$\begin{aligned}\left(\bar{H}_A\right)_2 \cdot \bar{i} &= (0.144\omega_x + 0.108\omega_y) = 2.4,\\ \left(\bar{H}_A\right)_2 \cdot \bar{j} &= (0.432\omega_y + 0.108\omega_x) = -7.2,\\ \left(\bar{H}_A\right)_2 \cdot \bar{k} &= 0.576\omega_z = 0.\end{aligned} \tag{3}$$

In view of $(\bar{v}_G)_f$ in Eqs. (1), the linear impulse–momentum principle for the plate gives

$$\begin{aligned}&10\left[0.06\omega_z\bar{i} + 0.18\omega_z\bar{j} + (-0.06\omega_x - 0.18\omega_y)\,\bar{k}\right]\\&= \left[A_x\bar{i} + A_y\bar{j} + (A_z + 5000)\,\bar{k}\right](0.004)\,.\end{aligned} \tag{4}$$

The linear impulse–momentum equation for the ball is

$$0.5\,(\bar{v}_2)_f = 0.5\left(v\bar{k}\right) - \left(5000\bar{k}\right)0.004. \tag{5}$$

The last of the basic equations is the one featuring the coefficient of restitution. Here $\bar{k}$ is the normal to the contact plane, so we have

$$(\bar{v}_2)_f \cdot \bar{k} - (\bar{v}_C)_f \cdot \bar{k} = \varepsilon\,[0 - v]\,.$$

Substitution of Eq. (2) gives

$$(\bar{v}_2)_f \cdot \bar{k} - (0.12\omega_x - 0.36\omega_y) = -0.6v. \tag{6}$$

We find the final angular velocity components and normal velocity of the ball by simultaneously solving Eqs. (3), which yield

$$\omega_x = 35.90, \quad \omega_y = -25.64 \text{ rad/s}.$$

These values are substituted into Eq. (6), which is solved simultaneously with the $\bar{k}$ component of Eq. (5) to find

$$(\bar{v}_2)_f \cdot \bar{k} = -16.54, \quad v = 23.46 \text{ m/s}. \qquad \triangleleft$$

According to Eq. (5), there is no change in the velocity components parallel to the plate because the contact force does not act in those directions, so the rebound velocity of the ball is

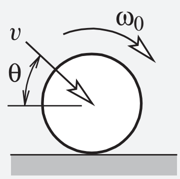
$$(\bar{v}_2)_f = -16.54\bar{k} \text{ m/s}. \qquad \triangleleft$$

Because F here is stated to match the impulsive force in Example 6.11, the average reaction forces, which are obtained from Eq. (4), are the same as those of the previous results:

$$A_x = A_y = 0, \quad A_z = 1153 \text{ N}. \qquad \triangleleft$$

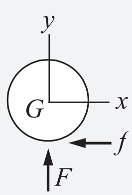
When one considers that the impact was generated by a 0.5-kg ball moving at an initial speed of almost 24 m/s, these average reaction forces are not exceptionally large. However, these are average values that vary inversely to the time duration, corresponding to a fixed impulse.

EXAMPLE 6.17 A ball obliquely hits the ground with speed v at angle of elevation θ. The coefficient of restitution is an unspecified value ε. The rotation rate at the instant of impact is ω_0 about the horizontal diameter that is perpendiular to the initial velocity. The ground is rough, with the effect of friction described by an impulsive friction force corresponding to coefficient of sliding friction μ. Derive an expression for the velocity with which the ball bounces off the ground in terms of v, ω, ε, μ, θ, and the ball's radius R. Compare the result with the rebound velocity that would result if the surface were smooth, so that $\mu = 0$.



Example 6.17

SOLUTION This example explores how one can account for friction along the surface where bodies collide. We consider the ground to be stationary, so only a free-body diagram of the ball is required. In that diagram the friction force is merely labeled as f, in the sense it would act if the contact point on the ball were moving to the left. The only impulsive forces are generated at the contact plane because the ball's motion is unconstrained.



Free-body diagram of the spinning ball showing the impulsive contact forces.

The impulse–momentum equations are

$$m\left(\bar{v}_G\right)_f \cdot \bar{i} = mv\cos\theta - f\Delta t,$$
$$m\left(\bar{v}_G\right)_f \cdot \bar{j} = m\left(-v\sin\theta\right) + F\Delta t, \tag{1}$$
$$I_G\left(-\omega_f\right) = I_G\left(-\omega_0\right) - fR\Delta t.$$

The velocity of the contact point is

$$\bar{v}_C = \bar{v}_G + \omega\left(-\bar{k}\right) \times \left(-R\bar{j}\right) = \bar{v}_G - \omega R\bar{i}, \tag{2}$$

so the normal velocity at the point of contact is $\bar{v}_G \cdot \bar{j}$. Thus the proportionality of normal velocity components imposed by the coefficient of restitution gives

$$\left(\bar{v}_G\right)_f \cdot \bar{j} = \varepsilon\left[0 - \left(-v\sin\theta\right)\right]. \tag{3}$$

The unknown variables in Eqs. (1) and (2) are the components of $\left(\bar{v}_G\right)_f$, the normal impulse $F\Delta t$, the friction impulse $f\Delta t$, and the final rotation rate ω_f.

To use the sliding friction model we need to know the direction in which the sliding motion occurs. For this, we turn to Eq. (2), which shows that the tangential velocity is $\bar{v}_C \cdot \bar{i} = \bar{v}_G \cdot \bar{i} - \omega R$. Thus, if $v \cos\theta > \omega_0 R$, the contact point is initially moving to the right, so the sliding force generated when the ball contacts the ground is $f = \mu F$ acting to the left, which is the sense shown in the free-body diagram. Equation (3) gives the vertical velocity component in any case. When that result and $f = \mu F$ are substituted into the Eqs. (1), we find that

$$\text{if } v\cos\theta > \omega_0 R: \begin{cases} (\bar{v}_G)_f \cdot \bar{i} = v\cos\theta - \mu(1+\varepsilon)v\sin\theta \\ (\bar{v}_G)_f \cdot \bar{j} = \varepsilon v \sin\theta \\ \omega_f = \omega_0 + \mu(1+\varepsilon)\dfrac{mR}{I_G} v\sin\theta \\ F\Delta t = (1+\varepsilon) m v \sin\theta \end{cases}. \tag{4}$$

The case in which $v \cos\theta < \omega_0 R$ is analyzed similarly, except that the friction force is reversed because the contact point is initially moving to the left. Thus we set $f = -\mu F$, and proceed as previously, which yields

$$\text{if } v\cos\theta < \omega_0 R: \begin{cases} (\bar{v}_G)_f \cdot \bar{i} = v\cos\theta + \mu(1+\varepsilon)v\sin\theta \\ (\bar{v}_G)_f \cdot \bar{j} = \varepsilon v \sin\theta \\ \omega_f = \omega_0 - \mu(1+\varepsilon)\dfrac{mR}{I_G} v\sin\theta \\ F\Delta t = (1+\varepsilon) m v \sin\theta \end{cases}. \tag{5}$$

Inspection of these results, which are valid if $\mu = 0$, shows that the presence of friction affects neither the normal contact force nor the vertical component of the rebound velocity. If the initial spin rate ω_0 exceeds $(v/R)\cos\theta$, friction increases the horizontal component of the rebound velocity. The angle of elevation for the rebound velocity is $\theta_f = \tan^{-1}\left[(\bar{v}_G)_f \cdot \bar{j} / (\bar{v}_G)_f \cdot \bar{i}\right]$, so this angle is decreased relative to what it would be if there were no friction. Both trends are reversed in the slow-rotation case, which is characterized by $\omega_0 < (v/R)\cos\theta$. In either case the rebound angle is smaller than it would be for a perfectly elastic collision, with the exception that there is a value of μ in the slow-rotation case beyond which $\left[(\bar{v}_G)_f \cdot \bar{j}\right] > (\tan\theta)(\bar{v}_G)_f \cdot \bar{i}$, which leads to $\theta_f > \theta$.

The velocity of the contact point may be found from Eq. (2). In the slow-rotation case, the friction force decreases $\bar{v}_G \cdot \bar{i}$ and and it increases ω, so the contact point is moving more slowly to the right at the conclusion of the impact. If μ is sufficiently large, the implication is that the contact point at the conclusion of the impact is moving to the left. This would contradict the assumed direction of the friction force. In effect, there would be an instant between t_0 and t_f when the contact point comes to rest. At that instant, static friction would take over, and there is nothing that would cause the condition to change during the remainder of the impact. Therefore the velocity at rebound would be such that $(\bar{v}_G)_f \cdot \bar{i} = \omega_f R$, corresponding to the contact point having no tangential velocity. The friction force can be set to zero for $t_{\max} < t < t_f$. The difficulty is that ω_f is uncertain in this case because the moment impulse in the second of Eqs. (1) becomes $\mu F(t_{\max} - t_0)$ and we do not know $t_{\max}$.

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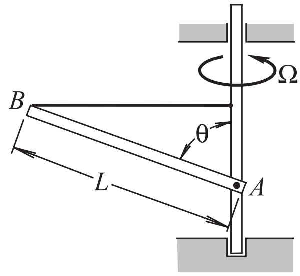
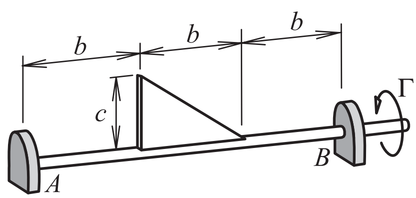
HOMEWORK PROBLEMS

EXERCISE 6.1 The cable holds the angle θ for the bar constant as the vertical shaft rotates at the constant speed Ω. Determine the value of Ω for which the tension in the cable is twice as large as it would be if Ω were zero.



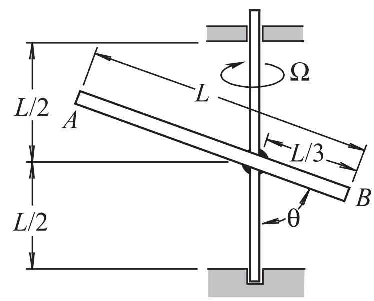
Exercise 6.1

EXERCISE 6.2 Torque Γ causes the horizontal shaft to rotate. The triangular plate's thickness is negligible, as is the mass of the shaft. A torque Γ is applied to the shaft, thereby inducing a time-dependent rotation rate Ω. The system was at rest at $t = 0$. Derive expressions for Ω and the bearing reactions as functions of time. The influence of gravity may be ignored.



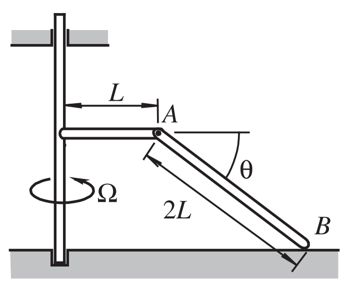
Exercise 6.2

EXERCISE 6.3 The vertical shaft rotates at the constant speed Ω. Determine the forces, including those required to balance the effect of gravity, exerted by the bearings on this shaft. The shaft and the bar have equal mass m.



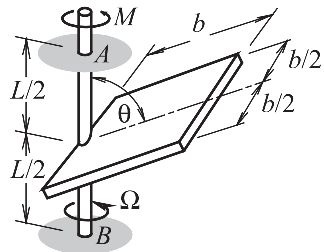
Exercise 6.3

EXERCISE 6.4 Bar AB is pinned to the T-bar and rubs along the ground. The coefficient of friction is μ. A torque acting about the vertical axis imposes a constant rotation rate Ω. Determine the contact forces exerted on bar AB by the ground and the force–couple system exerted on the bar at pin A. (It may be assumed that the friction force acts perpendicular to the vertical plane depicted in the sketch.)



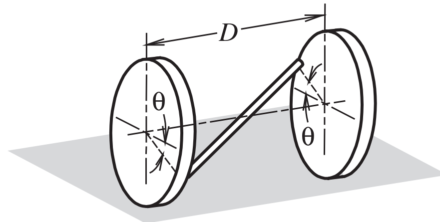
Exercise 6.4

EXERCISE 6.5 The rectangular plate is welded to the vertical shaft, whose mass is negligible. An unsteady precession rate Ω is generated by the application of torque M to the shaft. Derive expressions for M and the total reactions exerted by bearings A and B, including the influence of gravity, as functions of the instantaneous values of Ω and $\dot{\Omega}$.



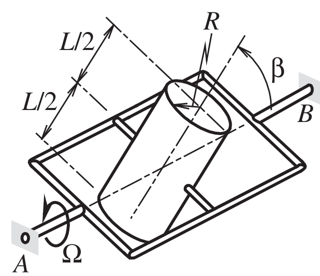
Exercise 6.5

EXERCISE 6.6 Two identical disks having radius R and mass m are joined by a rigid bar whose ends are welded at diametrically opposite points on the perimeter of each wheel. The mass of the rod also is m. The disks roll without slipping at constant speed v. Prove that such a motion is consistent with the equations of motion, and also determine the corresponding friction and normal forces as a function of the angle θ through which the assembly has rotated. *Hint*: Define a body-fixed coordinate system such that the y axis aligns with the disks' center line and the z axis is perpendicular to the bar.



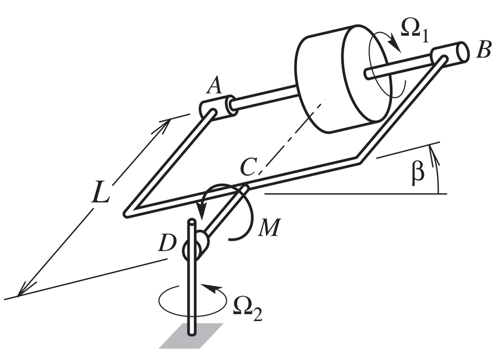
Exercise 6.6

EXERCISE 6.7 The cylinder is mounted on a gimbal that rotates about the horizontal axis at constant rate Ω induced by a torque Γ that acts about shaft AB. Derive the differential equation governing the angle β between the bar and the horizontal axis, and also derive an expression for Γ.



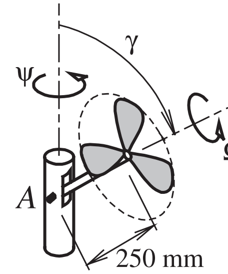
Exercise 6.7

EXERCISE 6.8 The torque M acting on the gimbal of the gyroscopic turn indicator is exerted by a torsional spring, so $M = -k\beta$. The precession rate Ω_2 is a specified function of time, and the spin rate Ω_1 is held constant by a servomotor. Let I_1 denote the moment of inertia of the flywheel about axis AB, and let I_2 be the centroidal moment of inertia perpendicular to axis AB. Derive the differential equation of motion for β.



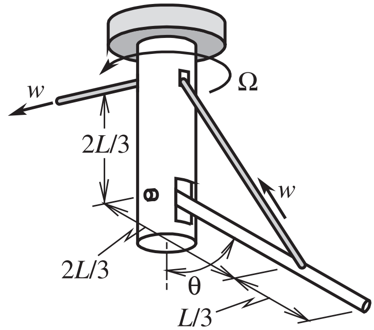
Exercise 6.8

EXERCISE 6.9 Procession angle ψ of the electric fan is unsteady, and the spin rate Ω is constant at 1800 rev/min. The fan blade assembly has a mass of 10 kg, and its radii of gyration are 200 mm about the spin axis and 150 mm about a centroidal axis perpendicular to the spin axis. This assembly may be considered to be axisymmetric. Determine the dynamic force–couple system that must be exerted on the motor housing at pin A to sustain this motion.



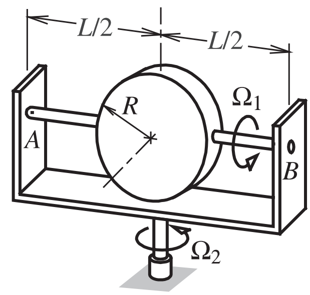
Exercise 6.9

EXERCISE 6.10 Pin A supporting the 50-kg thin bar has ideal properties. The angle θ is adjusted by pulling the cable inward at the constant rate $w = 20$ m/s. The precession rate is constant at $\Omega = 5$ rad/s. Determine the tensile force in the cable when $\theta = 35°$. The bar's length is $L = 600$ mm.



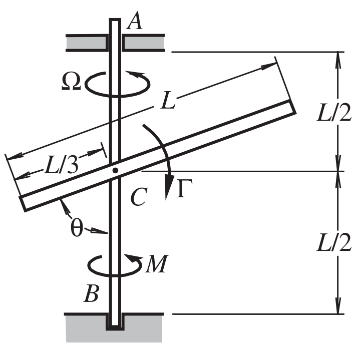
Exercise 6.10

EXERCISE 6.11 The thin disk is mounted on shaft AB, which coincides with a diametral line. Servomotors sustain the rotation rates Ω_1 about this axis and Ω_2 about the vertical are maintained at constant values by servomotors that exert torques about the respective shafts. Derive expressions for the dynamic reactions at bearings A and B and the required moment about shaft AB as functions of θ. It may be assumed that the bearings exert forces, but not couples, and that only the mass of the disk is significant.



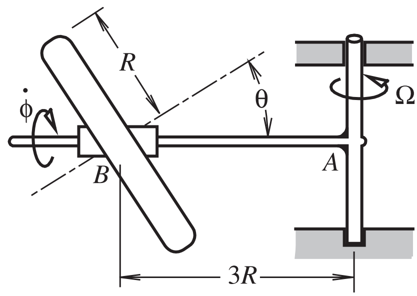
Exercise 6.11

EXERCISE 6.12 The uniform bar is pinned to the vertical shaft AB. A servomotor applies torque Γ about the axis of pin C, with the result that the angle θ oscillates periodically, $\theta = \pi/2 \sin(\mu t)$ rad. Torque M applied to the vertical shaft maintains the rotation rate about the vertical shaft at the constant value Ω. The mass of the bar is m and the mass of the shaft is negligible. For the instants when $\theta = \pi/2$ and $\theta = \pi/3$, derive expressions for M and Γ.



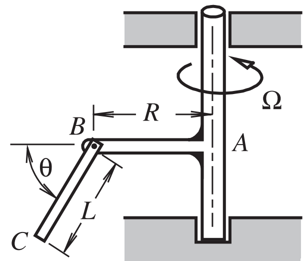
Exercise 6.12

EXERCISE 6.13 The disk of mass m is obliquely mounted on shaft AB, such that the angle between the center line of the disk and the shaft always is θ. The disk spins freely relative to this shaft at the unknown angular rate $\dot{\phi}$. The illustrated position, in which the diametral line perpendicular to the shaft is horizontal, corresponds to $\phi = 0$. A torque Γ acting about the vertical shaft maintains the precession rate at the constant value Ω. Derive the differential equation of motion governing ϕ and an expression for Γ in terms of ϕ. Only the mass of the disk is significant.



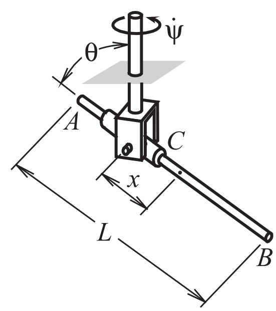
Exercise 6.13

EXERCISE 6.14 Bar BC is pivoted from the end of the T-bar. A torque Γ applied to the vertical shaft is such that the system rotates about the vertical axis at the constant speed Ω. Derive the differential equation of motion for the angle of elevation θ.



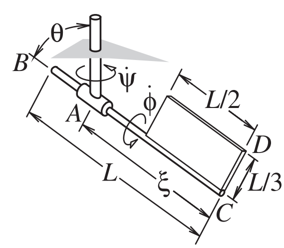
Exercise 6.14

EXERCISE 6.15 Collar C is attached to the vertical shaft by a fork-and-clevis, so the angle of inclination θ of bar AB is arbitrary. Because this bar slides through the collar, the distance x from the pivot point to the center of mass is variable, but it may be assumed that the bar does not spin about its own axis. The vertical shaft rotates at the constant rate $\dot{\psi}$. Determine the differential equations of motion for x and θ, as well as expressions for the force–couple system exerted by the collar on the bar in terms of these responses.



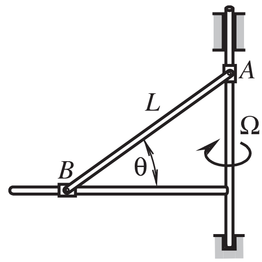
Exercise 6.15

EXERCISE 6.16 Collar A is welded to the vertical shaft, so the angle of inclination θ is constant. The rectangular plate, whose mass is m, is welded to bar BC. This bar may slide through the collar, as well as spin about its own axis at angular speed $\dot{\phi}$, with $\phi = 0$ defined to be the position where the plate is upright in the vertical plane. The mass of the plate is m and the mass of the bar may be neglected. Consider the situation in which the system precesses at the constant rate $\dot{\psi}$. Derive the corresponding differential equations of motion for ϕ and the distance ξ to the end C of the bar.



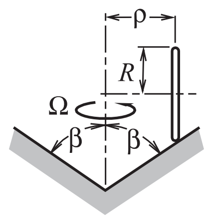
Exercise 6.16

EXERCISE 6.17 The vertical shaft rotates at constant speed Ω. Bar AB has mass m. Both collars have negligible mass, and friction between the collars and their respective guides is negligible. Derive the differential equation of motion governing θ.



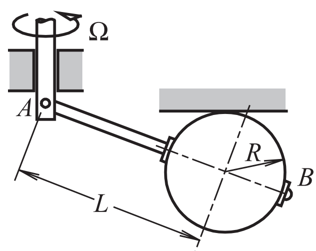
Exercise 6.17

EXERCISE 6.18 The cross-sectional view shows a disk of radius R that rolls without slipping over the interior surface of a cone whose apex angle is 2β. The axis of the cone is vertical. The motion depicted in the sketch is such that the disk remains upright as it precesses at constant speed Ω, with its center following a circular path of radius ρ. Prove that there is no combination of Ω and β for which this motion is possible.



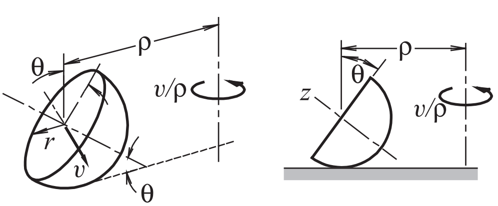
Exercise 6.18

EXERCISE 6.19 The sphere, whose mass is m, spins freely relative to shaft AB, whose mass is negligible. The system precesses about the vertical axis at constant rate Ω, and joint A is an ideal pin. Consider the possibility that the sphere rolls over the ceiling without slipping. Determine whether there is a range of values of Ω for which such a motion can occur. It is permissible to assume that the radial component of the friction force at the ceiling is zero.



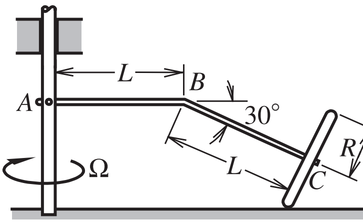
Exercise 6.19

EXERCISE 6.20 A hemisphere is observed to roll without slipping on a horizontal surface. The motion is such that the center of the hemisphere follows a horizontal circle of radius ρ at constant speed v, and the angle of inclination θ of the centerline of the hemisphere is constant. Derive an algebraic equation for the speed v corresponding to specified values of θ and ρ. Also determine friction force f.



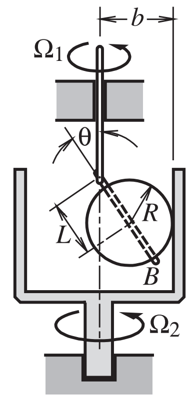
Exercise 6.20

EXERCISE 6.21 A thin disk of mass m rolls over the ground without slipping as it rotates freely relative to bent shaft ABC. The connection at end A is an ideal pin. The precession rate of the bent shaft about the vertical axis is the constant value Ω. Determine the magnitude of the normal force N exerted between the disk and the ground.



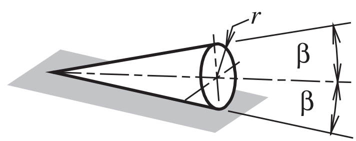
Exercise 6.21

EXERCISE 6.22 The sphere spins freely about its shaft, which is connected by pin A to the vertical shaft whose rotation rate is Ω_1. The cylindrical housing, which is shown in cross section, rotates at Ω_2. Derive an expression for the normal force exerted on the sphere by the cylinder wall based on the assumption that the sphere rolls over the wall without slippage.



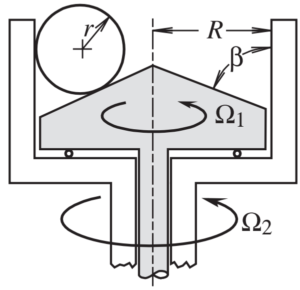
Exercise 6.22

EXERCISE 6.23 The cone, whose mass is m and apex angle is 2β, rolls without slipping over the horizontal surface. The rolling motion of the cone is such that it is observed to rotate about a fixed vertical axis intersecting its apex at constant angular rate ω_1. Determine the maximum value of ω_1 for which the cone will not tip over its rim in this motion. Also, determine the minimum coefficient of static friction corresponding to that value of ω_1. *Hint:* The line force along the edge where the cone contacts the ground may be replaced with a normal force and radial friction force. The magnitude of these forces and the location along the edge where the normal force acts may be determined from the equations of motion.



Exercise 6.23

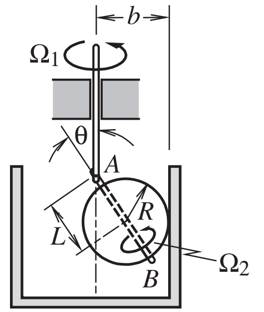
EXERCISE 6.24 The sphere, whose radius is r, rolls without slipping relative to the wall of the cylindrical tank and the conical floor. The tank and the floor rotate about the vertical axis at the constant rates of Ω_1 and Ω_2, respectively. It is observed that the radial line to the center of the sphere also rotates at Ω_2. This behavior arises when Ω_2 is very large, which also has the consequence that the friction force exerted by the floor is negligible. (a) Determine the forces exerted on the sphere by each surface it contacts in the special



Exercise 6.24

case in which $\beta = 90°$. (b) Determine the forces exerted on the sphere by each surface it contacts if β is an arbitrary acute angle.

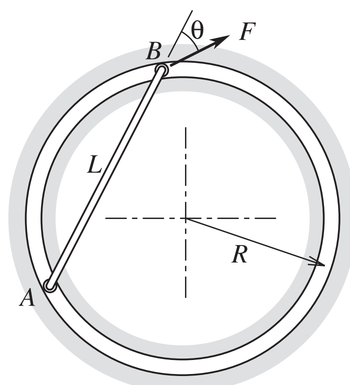
EXERCISE 6.25 The sketch shows the cross section of a stationary cylindrical tank of radius b. A torque, which is not depicted, acting on the vertical shaft maintains the precession rate at a constant angular speed Ω_1 that is very large. The connection between this shaft and shaft AB is an ideal pin, so the angle θ can have any value in the range $0 \le \theta \le \theta_{\max}$, where $L \sin\theta_{\max} + R = b$. The sphere spins freely at angular speed Ω_2 relative to shaft AB. For $t < 0$, a torque applied to shaft AB keeps the sphere in close proximity to the tank's wall, that is, θ is slightly less than $\theta_{\max}$. The spin rate in this condition is $\Omega_2 = 0$. At $t = 0$, the shaft is released, causing the sphere to immediately contact the cylinder. The coefficient of sliding friction between the sphere and the cylinder wall is μ, and all other frictional effects are negligible. Derive the differential equation governing Ω_2 for $t > 0$. The mass of the sphere is m, which is much larger than the mass of shaft AB. *Hint*: The moment exerted on the sphere by shaft AB has a zero component in the direction of the shaft.



Exercise 6.25

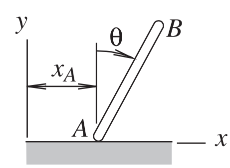
EXERCISE 6.26 Consider an automobile that is following a straight path at speed v. Its wheelbase is L and its center of mass is located at distance b behind the front wheels and distance h above the ground. The coefficient of friction between the tires and the ground is μ. Determine the maximum possible acceleration rate $\dot{v}$ for cases of front-wheel, rear-wheel, and all-wheel drive.

EXERCISE 6.27 Bar AB is attached at both ends to rollers that follow a horizontal circular groove of radius R. Frictional resistance is negligible. At what angle θ relative to the bar should force $\bar{F}$ be applied to maximize the angular acceleration of the bar? What is the corresponding angular acceleration?



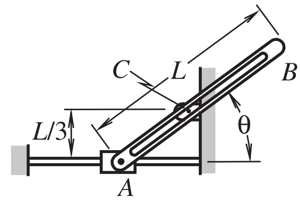
Exercise 6.27

EXERCISE 6.28 The bar of mass m is falling toward the horizontal surface as it slides over the ground. The coefficient of sliding friction is μ. Derive differential equations of motion for the position coordinate x_A of the lower end and the angle of inclination θ.



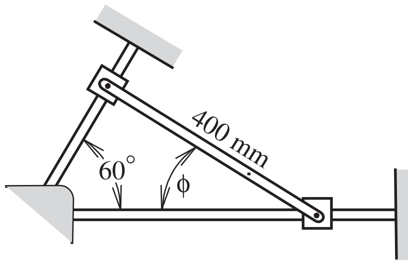
Exercise 6.28

EXERCISE 6.29 Bar AB is pinned at its lower end to collar A, and its groove slides over the stationary pin C. Friction effects are negligible, as is the mass of the collar. The bar's center of mass is situated at its midpoint, and its centroidal radius of gyration is $0.4L$. At $\theta = 30^\circ$ it is observed that $\dot{\theta} = -2(g/L)^{1/2}$. What is $\ddot{\theta}$ this position?



Exercise 6.29

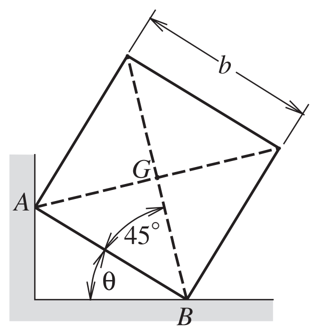
EXERCISE 6.30 The system lies in the vertical plane. The bar is released from rest at $\phi = 70^\circ$. Determine the angular acceleration of the bar at the instant of release. Friction between the collars and their guides is negligible, as is the mass of the collars.



Exercises 6.30 and 6.31

EXERCISE 6.31 When the bar is at $\phi = 20^\circ$, it is falling with an instantaneous angular velocity of $\dot{\phi} = -8$ rad/s. The coefficients of sliding friction at the ground and the incline are both $\mu = 0.10$, and the collars have negligible mass. Determine the angular acceleration $\ddot{\phi}$ of the bar in this position.

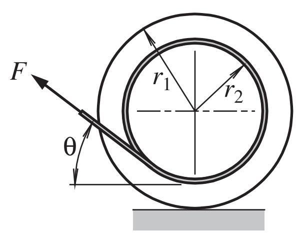
EXERCISE 6.32 Corner A of the homogeneous square box remains in contact with the wall as the box falls. Frictional resistance at both the wall and the floor is negligible. Derive the differential equation of motion for the angle θ at which the box is tilted. Also derive algebraic equations for the contact forces exerted by the wall in terms of instantaneous values of θ and its derivatives.



Exercise 6.32

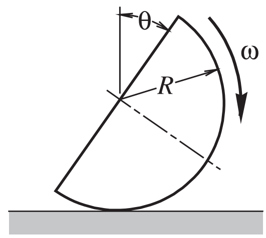
EXERCISE 6.33 At $t = 0$ a bowling ball is thrown onto the ground with a velocity v that is essentially horizontal. It is not spinning at the instant it first comes into contact with the ground. Observation of the ball's motion reveals that it begins to roll without slipping at a certain time $t = \tau$. Derive an expression for the coefficient of friction in terms of τ and the system's parameters.

EXERCISE 6.34 A cable is wrapped around the drum of the stepped cylinder and held at angle of elevation θ by the tensile force $\bar{F}$. The system was at rest before the application of $\bar{F}$. The mass of the cylinder is m, its centroidal radius of gyration is κ, and the coefficients of static and kinetic friction with the ground are both μ. The elevation angle can be any value in the range $0 \le \theta \le 90°$. (a) Derive an expression for the maximum value of F for which the cylinder will roll without slipping at a specified θ. Also determine the corresponding acceleration of the center of the cylinder. (b) Graph the results in Part (a) for the case in which $m = 500$ kg, $r_1 = 400$ mm, $r_2 = \kappa = 300$ mm, and $\mu = 0.2$. (c) Derive expressions for the acceleration of the center of cylinder and the angular acceleration when F at a specified θ is 10% greater than the value found in Part (a).



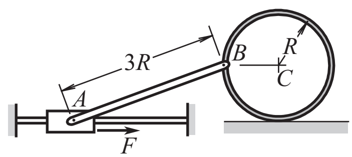
Exercise 6.34

EXERCISE 6.35 The homogeneous semicylinder has an angular speed $\omega = 5$ rad/s when $\theta = 30°$. The cylinder's mass is 20 kg, and $R = 150$ mm. Determine the minimum coefficient of static friction between the ground and the semicylinder for which slipping between the semicylinder and the ground will not occur in this position. What is the corresponding angular acceleration $\dot{\omega}$ of the semicylinder?



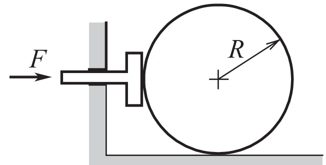
Exercise 6.35

EXERCISE 6.36 Bar AB, which is pinned to collar A and to the perimeter of the cylindrical pipe, is pushed to the right by application of force F to the collar. The mass of the cylinder is m; the influence of the mass of the bar and of the collar may be neglected. Consider the situation in which $F = 0.5mg$. From the assumption that the cylinder rolls without slipping, determine the acceleration of the center of the cylinder when the system is at the instantaneous position depicted in the sketch. Also determine the minimum value of the coefficient of static friction required to prevent slippage at this position.



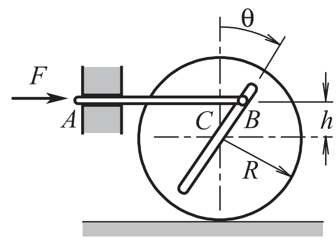
Exercise 6.36

EXERCISE 6.37 Horizontal force F is applied to the piston, whose mass is small compared with that of the circular cylinder of mass m. The coefficients of friction are μ and η for static and kinetic friction, respectively. (a) For the case in which there is no slipping relative to the ground, determine the acceleration of the cylinder's center. (b) Determine the largest value of F for which the motion in Part (a) is possible. (c) Determine the translational and angular acceleration of the disk when the magnitude of F exceeds the value in Part (b).



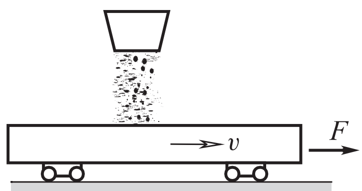
Exercise 6.37

EXERCISE 6.38 Horizontal force F causes the actuating rod to move to the left at the constant speed v. This rod is connected to the wheel by pin B, which may slide through the groove. The mass of the wheel is m, the radius of gyration is κ, and μ_s and μ_k are the coefficients of static and kinetic friction, respectively, between the wheel and the ground. Friction between the pin and the groove is negligible, as is the mass of the rod. Consider the instant when $\theta = 30°$. Derive expressions for the acceleration of center C, the angular acceleration of the gear, and the force F under the assumption that (a) the wheel rolls without slipping, (b) there is slippage between the wheel and the ground.



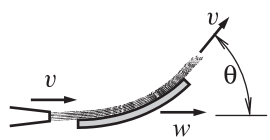
Exercise 6.38

EXERCISE 6.39 The mass of the ore truck when empty is m_0. The traction force F propels it at constant speed v while ore is dropped into it at constant mass flow rate σ. It may be assumed that the ore falls vertically. Derive an expression for F. *Hint:* Follow the ore particles that fall into the carrier in an interval dt.



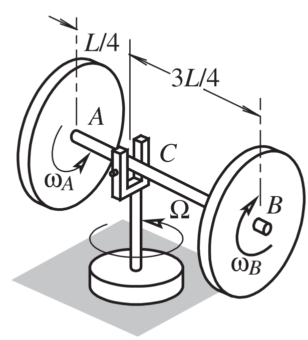
Exercise 6.39

EXERCISE 6.40 Water that flows out of a hose whose nozzle area is A is moving at velocity v horizontally when it is incident upon a curved vane. This deflects the flow such that the stream is elevated at angle θ. The vane translates horizontally at constant speed w. What is the force the water exerts on the vane?



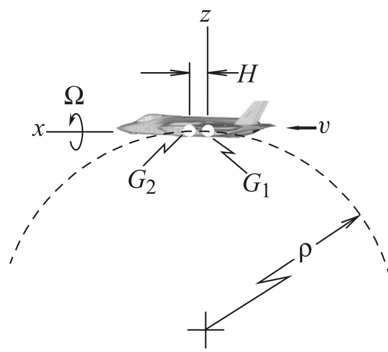
Exercise 6.40

EXERCISE 6.41 Identical disks A and B spin at the constant rates ω_A and ω_B, respectively, about shaft AB, which is horizontal. The entire system precesses about the vertical axis at the constant rate Ω. Determine the relationship between the spin rates ω_A and ω_B for which this motion can occur without application of a torque acting about the axis of pin C.



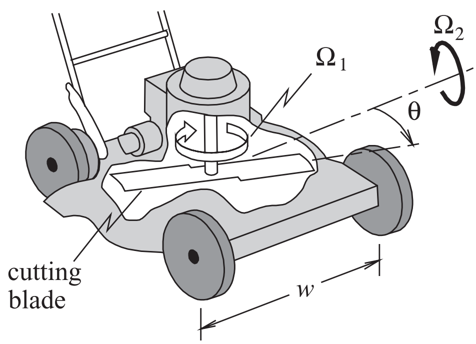
Exercise 6.41

EXERCISE 6.42 A single-engine turbojet airplane has its minimum speed v at the top of a vertical circle of radius ρ. At that instant, the airplane is executing a roll, clockwise as viewed by the pilot, at angular speed ω_1. The engine turns at angular speed Ω about the longitudinal axis, which is labeled x in the diagram. This rotation is clockwise from the pilot's viewpoint. The rotating parts of the engine have mass m_2, and centroidal moments of inertia J about the rotation axis and J' transverse to the rotation axis. The mass of the airplane, excluding the rotating parts of the engine, is m_1, and the corresponding moments of inertia about centroidal xyz axes are I_x, I_y, and I_z. The spin axis of the engine is collinear with the longitudinal x axis of the airplane, and the centers of mass G_1 and G_2, associated respectively with m_1 and m_2, both lie on this axis. Derive expressions for the aerodynamic force and moment about the center of mass of the airplane required to execute this maneuver.



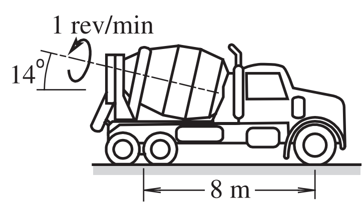
Exercise 6.42

EXERCISE 6.43 While Abby pushes a lawn mower at constant speed v along level ground, she pushes down on the handle, causing it to rotate about its rear axle at constant speed $\Omega_2 = \dot{\psi}$. The motor's rotation rate $\Omega_1 = \dot{\theta}$ is constant. Derive an expression for the dynamic force at each wheel associated with these rotations when the motor's axis is inclined at angle ψ from the vertical. Do these forces depend on the angular position of the blade? The mass of the cutting blade is m, the length of the blade is L, and the blade may be considered to be a slender bar.



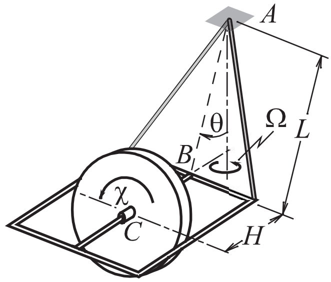
Exercise 6.43

EXERCISE 6.44 The barrel of a truck transporting wet-mix concrete rotates at 1 rev/min. The axis of rotation is elevated 14° from horizontal. The barrel and its contents have a mass of 30 000 kg, and the concrete may be considered to be a solid that fills the rotating barrel. The centroidal radii of gyration of the rotating parts are 1.5 m about the rotation axis and 2.5 m transversely to that axis. Determine the dynamic force at each wheel associated with rotation of the barrel when the truck executes a 40-m radius turn at 50 km/h. The track (distance between a pair of wheels on opposite sides) is 3 m.



Exercise 6.44

EXERCISE 6.45 The gimbal supporting the flywheel is suspended from pivot A by two cables of equal length. The flywheel, which may be modeled as a thin disk of mass m_1, spins at the constant rate χ. The mass of the gimbal is m_2. It is observed that under certain conditions the system will undergo a steady precession Ω about the vertical axis through pivot A with the angle of inclination θ constant and the spin axis BC of the flywheel horizontal. Derive expressions for χ and Ω under these circumstances as functions of the angle θ and the other system parameters.



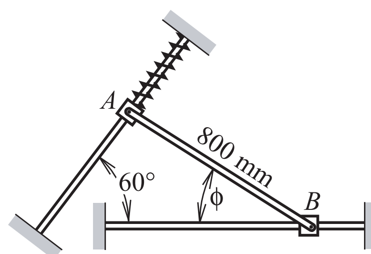
Exercise 6.45

EXERCISE 6.46 The force $\bar{F}$ in Exercise 6.38 is constant and frictional resistance between pin B and the groove is negligible. Initially, $\theta = -60^\circ$ and $\dot{\theta} = 0$. The distance $h = 0.6R$. Derive an expression for the angular speed of the wheel when $\theta = 30^\circ$, based on the assumption that there is no slippage when the wheel rolls.

EXERCISE 6.47 The bar in Exercise 6.29 is released from rest at $\theta = 60^\circ$. Determine its angular velocity when $\theta = 10^\circ$. The mass of the collar is half the mass of the bar.

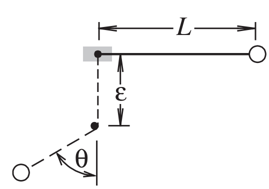
EXERCISE 6.48 The semicylinder in Exercise 6.35 is released with angular speed ω_0 at $\theta = 0$. Derive expressions for the normal and friction forces exerted by the ground as functions of ω_0 and θ if the semicylinder rolls without slipping.

EXERCISE 6.49 Bar AB has a mass of 40 kg, and the stiffness of the spring is 9 kN/m. The bar is released from rest at $\phi = 0$, at which position the spring is elongated by 300 mm. The linkage lies in the vertical plane. (a) Determine the largest value of ϕ attained in the subsequent motion. (b) Determine $\dot{\phi}$ when ϕ is 5° less than the value found in Part (a).



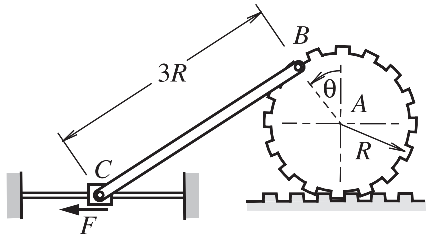
Exercise 6.49

EXERCISE 6.50 The small ball is released from rest with the cable horizontal. At its lowest point the cable encounters the fixed peg. Describe the speed of the sphere as a function of the angle θ.



Exercise 6.50

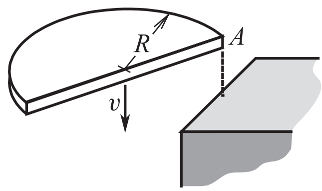
EXERCISE 6.51 Gear A has mass m and centroidal radius of gyration κ. It rolls over the horizontal rack because of the constant horizontal force F acting on collar C. The connecting bar and collar C have negligible mass. The system was at rest at $\theta = 0$. Derive an expression for the speed v of the center of gear A as a function of θ.



Exercise 6.51

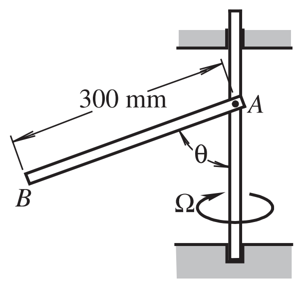
EXERCISE 6.52 Solve Exercise 6.51 for the case in which the mass of rod BC is $m/2$.

EXERCISE 6.53 The semicircular plate is falling at speed v with its plane oriented horizontally. It strikes the ledge at corner A, and the impact is perfectly elastic (that is, the recoil velocity of corner A is v upward). The interval of the collision is Δt. Derive expressions for the velocity of the center of mass and the angular velocity at the instant following the collision. Also, derive an expression for the collision force exerted between the plate and the ledge.



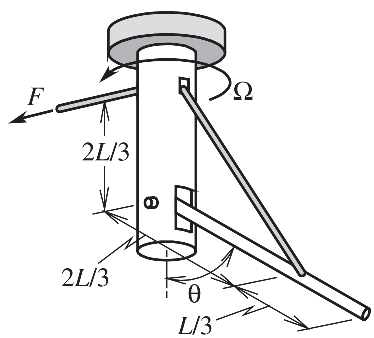
Exercise 6.53

EXERCISE 6.54 Bar AB is pinned to the vertical shaft, which rotates freely. When the bar is inclined at $\theta = 10^\circ$, the rotation rate about the vertical axis is $\Omega = 10$ rad/s, and $\dot{\theta} = 4$ rad/s at that instant. Determine the maximum value of θ in the subsequent motion. The mass of the vertical shaft may be neglected.



Exercise 6.54

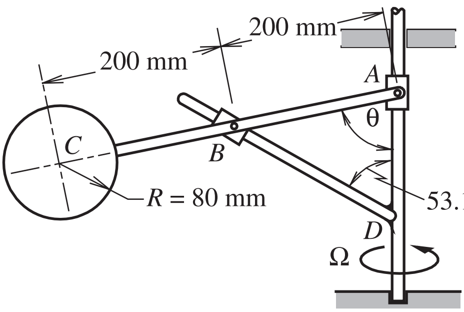
EXERCISE 6.55 The pin allows the angle of inclination θ of the 50-kg bar to change when a constant tensile force of $F = 2$ kN is applied. No torque is applied to the vertical shaft, so the precession rate Ω varies as θ is altered. At the initial position $\theta = 20^\circ$, $\dot{\theta} = 0$, and $\Omega = 5$ rad/s. Determine the values of $\dot{\theta}$ and Ω when $\theta = 90^\circ$.



Exercise 6.55

EXERCISE 6.56 The system in Exercise 6.14 spins freely about the vertical axis because the torque Γ is not present. Initially $\Omega = 4\,(g/L)^{1/2}$ with $\theta = 150^\circ$ and $\dot{\theta} = 0$. The moment of inertia of the T-bar about its rotation axis is $0.5mL^2$, where m is the masses of bar BC. (a) Determine the value of $\dot{\theta}$ when $\theta = 90^\circ$. (b) Determine the minimum value of θ in the ensuing motion.

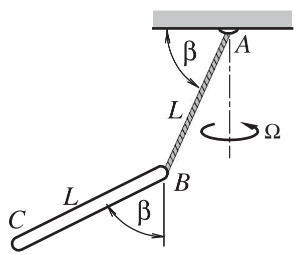
EXERCISE 6.57 The system rotates freely about the vertical axis. Collars A and B are pinned to the bar that is welded to sphere C. The mass of the sphere is 12 kg, and the masess of the collars and the bar are negligible. Initially, the system is precessing at $\Omega = 100$ rad/s with θ held constant at 20°. It then is released, causing θ to increase and Ω to change. Determine whether $\theta = 90°$ is attained in the subsequent motion. If so, what is the angular velocity of the sphere at that position?



Exercise 6.57

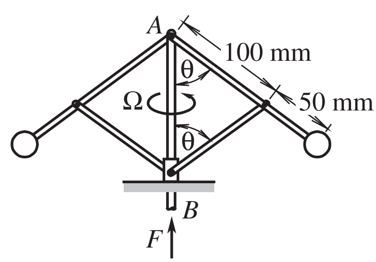
EXERCISE 6.58 The system in Exercise 6.57 was rotating at rate 50 rad/s with θ constant at $\theta = 30°$ when it was released. Determine whether in the ensuing motion there is another value of θ at which $\dot{\theta}$ is zero. If so, what is the value of Ω at that position?

EXERCISE 6.59 A slender bar of mass m, which is suspended by a cable from pivot A, executes a steady precession about the vertical axis at angular speed Ω as it maintains the orientation shown. (a) Determine Ω and the angle of inclination β. (b) An impulsive force $\bar{F}$ at end B, parallel to the initial velocity of that end, acts over a short time interval Δt. Determine the magnitude of $\bar{F}$ for which $\Omega = 0$ at the conclusion of the impulsive action. What are the corresponding velocity of the center of mass G and angular velocity of the bar? *Hint:* If $\Omega = 0$, the angular velocity of the bar must be horizontal.



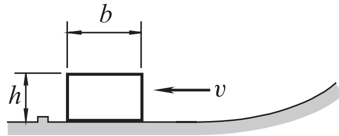
Exercise 6.59

EXERCISE 6.60 The flyball speed governor consists of two 500-g spheres connected to the vertical shaft by a parallelogram linkage. This shaft, which passes through the collar supporting the linkage, rotates freely. The mass of the links is negligible. The system is initially rotating steadily at 900 rev/min about the vertical axis, with $\theta = 75°$. A constant upward force $\bar{F}$ is applied to the vertical shaft at end B, causing point A to move upward and θ to decrease. Derive algebraic equations for Ω and $\dot{\theta}$ as functions of the magnitude of $\bar{F}$ and θ.



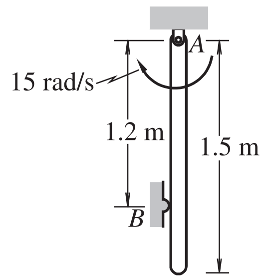
Exercise 6.60

EXERCISE 6.61 The package was translating to the left at speed v when it hit cleat A. The collision is perfectly plastic. Determine the angular velocity of the package immediately after the collision.



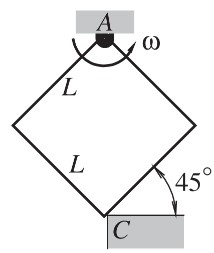
Exercise 6.61

EXERCISE 6.62 The 100-kg bar was released from rest in the horizontal position. The coefficient of restitution between the bar and bumper B is 0.4, and the duration of the impact is 5 ms. Determine the maximum angle of inclination of the bar subsequent to the impact and the average force exerted by the bumper during the impact.



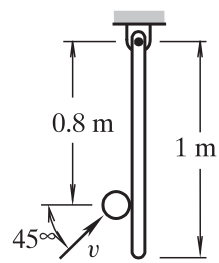
Exercise 6.62

EXERCISE 6.63 The square plate, whose mass is m, strikes corner C of ledge at angular speed ω. The impact point is slightly above the corner of the plate. The coefficient of restitution is ε and the impact's duration is 8 ms. Determine the angular velocity at which the plate rebounds and the average forces that are exerted on the plate by the ledge and pin A during the impact.



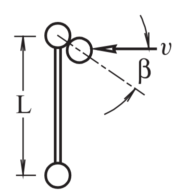
Exercise 6.63

EXERCISE 6.64 The 20-kg bar was at rest when it was hit by the 1-kg ball whose initial velocity was $v = 30$ m/s in the direction shown in the sketch. The coefficient of restitution is 0.75. Determine the angular velocity of the bar and the velocity of the ball immediately after the collision.

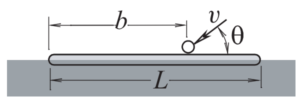


Exercise 6.64

EXERCISE 6.65 The sketch shows the velocity of a small disk at the instant before its collision with a rigid body consisting of the same disks connected by a rigid massless bar. Determine the rebound velocity of each disk in terms of the coefficient of restitution ε.



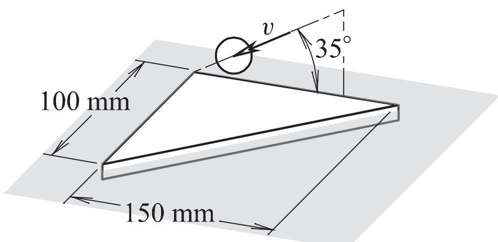
Exercise 6.65



Exercise 6.66

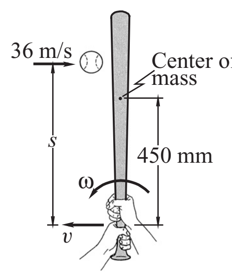
EXERCISE 6.66 The slender wooden board was at rest on the surface of a large body of water immediately before it was hit by a ball having the initial speed v appearing in the sketch. The masses are 2 kg for the board and 0.5 kg for the ball. The dimensions are $L = 2m$, $b = 1.4m$, and the coefficient of restitution is $\varepsilon = 0.3$. Determine the velocities of the rock and of the center of the board, and the angular velocity of the board, at the instant when the bodies separate.

EXERCISE 6.67 The sketch shows a right triangular plate floating on water at the instant before it is struck by a ball whose speed is $v = 25$ m/s in the direction shown. The point of contact is on the flat surface adjacent to the right-angle corner. The masses are 6 kg for the plate and 1 kg for the ball. The coefficient of restitution is $\varepsilon = 0.3$. For the instant when the bodies separate, determine the velocity of the ball, the velocity of the plate's center of mass, and the angular velocity of the plate.



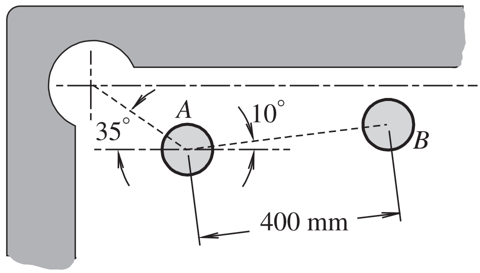
Exercise 6.67

EXERCISE 6.68 A baseball moving at a speed of 36 m/s is struck by a bat in the orientation depicted in the sketch. At this instant the batter's hands are moving forward at $v = 10$ m/s and the bat is rotating at $\omega = 20$ rad/s, which is equivalent to the bat being in pure rotation about some point O along its centerline. The distance from the player's hands to the point of impact is s. What value of s will minimize the impulsive force $\bar{F}$ the player must exert to maintain the bat in a pure rotation about point O throughout the impact? What are the corresponding velocity of the ball and the values of v and ω after the impact if s has that value? The bat's radius of gyration about its center of mass is 300 mm, and the coefficient of restitution is 0.6. Also, the player may be considered to be holding the bat at the end.



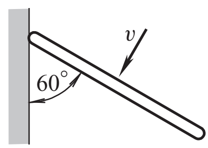
Exercise 6.68

EXERCISE 6.69 The sketch shows the position of two billiard balls. It is desired to give billiard ball B an initial velocity such that, after the impact, ball A falls into the pocket by following the path described by the 35° angle. The coefficient of restitution is $\varepsilon = 0.75$ and the effects of friction may be neglected. Determine the angle at which ball B should be aimed, and the direction of the velocity with which it rebounds from the collision. The diameter of a billiard ball is 57.1 mm.



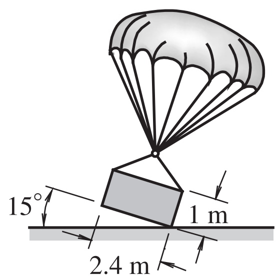
Exercise 6.69

EXERCISE 6.70 At the instant before the bar hit the wall with the orientation shown in the sketch, it was translating at velocity $\bar{v}$. Derive expressions for the velocity of the center of the stick and the angular velocity of the bar at the instant it rebounds from the wall. The coefficient of restitution is an arbitrary value ε.



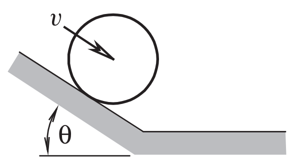
Exercise 6.70

EXERCISE 6.71 The crate, which was dropped from an aircraft, hits the ground in the position shown. At the instant of impact the crate was translating downward at 25 m/s. The coefficient of restitution for the impact is 0.10, the mass of the crate is 20 000 kg, and the crate may be approximated as a homogeneous body. Determine the velocity of the center of the crate and the angular velocity of the crate immediately after the collision.



Exercise 6.71

EXERCISE 6.72 The sphere rolls without slipping down the incline. At the instant it contacts the ground, its speed was v_0. The coefficient of restitution is ε. Derive expressions for the velocity of its center and its angular velocity at the instant it rebounds from the collision based on the assumption that friction between the ball and the ground is negligible.



Exercises 6.72 and 6.73

EXERCISE 6.73 Solve Exercise 6.72 corresponding to a nonzero coefficient of friction.

EXERCISE 6.74 A 100-mm-diameter spinning ball lands on the ground with an initial speed $\bar{v}_0 = 20\bar{i} - 12\bar{k}$ m/s, where the xy plane is horizontal. The angular velocity of the ball at this instant is $\bar{\omega}_0 = -100\bar{i} + 150\bar{j} + 50\bar{k}$ rad/s. The coefficient of restitution is $\varepsilon = 0.5$ and the coefficient of friction is $\mu = 0.2$. Determine the ball's angular velocity and the velocity of its center of mass at the instant the ball rebounds from the surface.

CHAPTER 7

Introduction to Analytical Mechanics

Any formulation of equations of motion requires characterization of the role of the physical restrictions that are imposed on a system's movement. These restrictions lead to kinematical relations between motion variables, and they also are manifested as reaction forces. When a system consists of interconnected bodies, the standard Newton–Euler formulation isolates individual bodies. The need to account for the kinematical constraints and corresponding reaction forces associated with each connection substantially enhances the level of effort entailed in deriving equations of motion.

The Lagrangian formulation implicitly recognizes the dual role of motion constraints. Indeed, it is recognition of this duality that has made it preferable to use the term *constraint force* rather than reaction. A primary benefit of the Lagrangian formulation is the ability to automatically account for constraint forces in the equations of motion. The formulation will allow us to treat connected bodies as a single system, rather than individual entities. The primary kinetic quantity for Lagrange's equations of motion is mechanical energy (kinetic and potential), whereas the Newtonian equations of motion are time derivatives of momentum principles.

The term *analytical mechanics*, which encompasses the developments of Lagrange, Hamilton, and many others who followed Euler, refers to the fact that the procedures that we shall develop are more mathematical than those of Newton and Euler. They also are more abstract. In fact, we often will find that features of the equations of motion, as well as of the physical responses predicted by those equations, are most readily explained in terms of Newton–Euler concepts.

7.1 BACKGROUND

As a preview to where we are headed, it is useful to consider a historical development that preceded Lagrange and provided an important impetus to his work. Some popular undergraduate textbooks, for example, Hibbeler (2003), state that d'Alembert reformulated Newton's Second Law as a static principle by defining an inertial force to be the negative of the product of mass and acceleration, that is,

$$\Sigma \bar{F} + \bar{F}_{\text{inertial}} = \bar{0} \quad \text{where} \quad \bar{F}_{\text{inertial}} = -m\bar{a}. \tag{7.1.1}$$

Such a viewpoint would be problematic, in that the so-called inertial force is not exerted by another body, as postulated by Newton's Third Law. A different perspective, which one may find in more advanced works such as Goldstein (1980) and Rosenberg (1977), asserts that d'Alembert introduced the inertial force concept as a vehicle for applying the principle of virtual work, which had often been used for static systems, to the analysis of dynamic systems. In fact, it is not clear how d'Alembert came to be associated with the inertial force concept. Dugas' (1988) extensive treatment of d'Alembert's work does not mention either an inertial force or virtual work. Truesdell (1960) attributes the inertial force concept to Euler and credits Lagrange with using the concept to apply the principle of virtual work. We avoid this question of attribution by referring to the concept as the *principle of dynamic virtual work*, without mention of d'Alembert. Development of this principle is how we will begin, but doing so requires that we first consider the principle of virtual work for static systems.

7.1.1 Principle of Virtual Work

Were it not for the requirement that one consider movement of a static system, the principle of virtual work would be quite straightforward. The statics laws state that the actual forces acting on a body are self-equilibrating. Because any set of forces may be replaced with an equivalent force–couple system acting at an arbitrary point P, static equilibrium requires that there be a force balance and a moment balance about point P:

$$\Sigma \bar{F} = \bar{0}, \quad \Sigma \bar{M}_P = \bar{0}. \tag{7.1.2}$$

Physically moving a body that is in static equilibrium requires that the forces acting on it be altered, such that their resultant does not vanish. Thus let us introduce an abstraction called a *virtual movement;* the word virtual is used here in the same sense as virtual reality, to denote something that does not really occur. When we introduce the virtual movement, we consider all forces, including those that constrain motion, to be unchanged, even though we ignore kinematical restrictions.

The virtual movement results in altering the position of the point P chosen for moment equilibrium, and it also leads to rotation of the body. The virtual movement we impart could have any magnitude we wish. However, if it leads to finite changes in the position of points and the orientation of the body, it will be necessary to evaluate the work as a path integral that accounts for the dependence of the force components as the position changes. For this reason, a virtual movement is limited to be such that the virtual displacements of points and the virtual rotations of bodies are infinitesimal. Correspondingly, we denote the virtual displacement of point P as $\delta\bar{r}_P$ and the virtual rotation of the body as $\overline{\delta\theta}$, where we use the symbol δ, rather than the differential symbol d, to emphasize that we are not dealing with an increment that actually occurs.

The work done by the resultant force in Eq. (7.1.2) is $\Sigma \bar{F} \cdot \delta\bar{r}_P$, and the work done by the resultant moment is $\Sigma \bar{M}_P \cdot \overline{\delta\theta}$. Because both the resultant force and resultant moment vanish, each work term is zero. Furthermore, work is additive as a scalar quantity, so the *virtual work* δW also is zero for a system of rigid bodies. A further consequence of work being a scalar is that we may form it from the individual contributions of each force $\bar{F}_j$ and couple $\bar{M}_n$ that acts on the system. Thus, we are led to the *principle of* *virtual work*, which states that the work done by the set of forces and couples acting on a static system vanishes in a virtual displacement,

$$\boxed{\delta W = \sum_j \bar{F}_j \cdot \delta\bar{r}_j + \sum_n \bar{M}_n \cdot \overline{\delta\theta}_n = 0,} \tag{7.1.3}$$

where $\delta\bar{r}_j$ is the virtual displacement of the point where force $\bar{F}_j$ is applied and $\overline{\delta\theta}_n$ is the virtual rotation of the body on which couple $\bar{M}_n$ acts.

At this juncture, it might seem strange that the principle of virtual work can be useful, because it is a single scalar equation, yet there are likely to be many unknown forces to determine. Such thoughts ignore a key feature, specifically that we may impose any virtual movement we wish. The virtual work is different for different movements, thereby leading to different equilibrium equations. Furthermore, we may select the virtual movement such that the only unknown forces doing work are the forces we wish to determine. This enables us to form one or more equilibrium equations that contain only the forces we wish to determine.

These concepts can be understood by considering the frame structure in Fig. 7.1, where it is desired to determine the horizontal constraint force C_x. To avoid the occurrence of the reactions A_x, A_y, and C_y in the virtual work, the virtual movement of the system should be such that end A remains stationary and end C moves horizontally. Such a movement will alter the angle θ. Let us denote the virtual increment of this angle as $\delta\theta$.

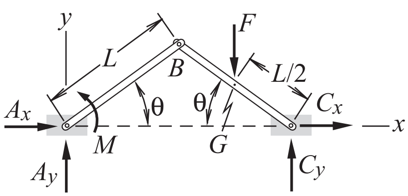


Figure 7.1. Free-body diagram of a frame structure in static equilibrium.

The work done by the couple M will be $M\delta\theta$, but we also must determine the work done by $\bar{F}$ and $\bar{C}_y$, which means that we need to determine the virtual displacement of their points of application. This actually is a familiar task, because ignoring the constraint against horizontal motion of point C effectively converts the system to a linkage. To determine infinitesimal displacements associated with $\delta\theta$, we first describe both positions in terms of θ:

$$\bar{r}_{C/A} = 2L\cos\theta\,\bar{i}, \quad \bar{r}_{G/A} = \frac{3}{2}L\cos\theta\,\bar{i} + \frac{1}{2}L\sin\theta\,\bar{j}. \tag{7.1.4}$$

The virtual displacements, being infinitesimal, may be obtained from the chain rule, so that

$$\begin{aligned}\delta\bar{r}_{C/A} &= \frac{\partial\bar{r}_{C/A}}{\partial\theta}\delta\theta = \left(-2L\sin\theta\,\bar{i}\right)\delta\theta,\\ \delta\bar{r}_{G/A} &= \frac{\partial\bar{r}_{G/A}}{\partial\theta}\delta\theta = \left(-\frac{3}{2}L\sin\theta\,\bar{i} + \frac{1}{2}L\cos\theta\,\bar{j}\right)\delta\theta.\end{aligned} \tag{7.1.5}$$

We add the virtual work done by both forces and $\bar{M}$, and equate it to zero. This gives

$$\begin{aligned}\delta W &= C_x \bar{i} \cdot \delta \bar{r}_{C/A} + \left(-F\bar{j}\right) \cdot \delta \bar{r}_{G/A} + M\delta\theta \\ &= C_x\left(-2L\sin\theta\right)\delta\theta + (-F)\left(\frac{1}{2}L\cos\theta\right)\delta\theta + M\delta\theta = 0.\end{aligned} \tag{7.1.6}$$

According to the principle of virtual work, δW must vanish, even though $\delta\theta$ is not zero, so we find that

$$C_x = \frac{M}{2L\sin\theta} - \frac{F\cot\theta}{4}. \tag{7.1.7}$$

The same result could be obtained by equilibrating bars AB and BC individually. However, doing so would lead to several simultaneous equations in which the internal forces exerted between the bars at pin B appear. The ability to focus on the variables of interest is what makes the principle of virtual work attractive for both static and dynamic systems.

7.1.2 Principle of Dynamic Virtual Work

The first step in extending the principle of virtual work to dynamic systems entails a slight alteration of the concept of dynamic equivalence, which we used in Section 6.3 to obtain equations of motion for a system of rigid bodies. This alteration involves drawing a single free-body diagram of the system. In addition to the actual forces, an inertial force–couple system, $-m\bar{a}_G$ and $-d\bar{H}_A/dt$, respectively, is shown for each body. (The former is the d'Alembert inertial force discussed at the beginning of this chapter, and the second term is the analogous moment effect.) The point A at which this force–couple system is applied is the same as the reference point for the Newton–Euler equations of motion. Thus we limit the specific point A to be each body's center of mass, unless a body is in pure rotation, in which case it may be the stationary pivot point. These inertial forces and couples must equilibrate the actual forces and couples that act on the bodies. Consequently, the total virtual work done by the inertial and actual forces and couples must be zero, as though the system were in static equilibrium. Also, as in the static case, the virtual movement may be defined in any manner. Selecting a virtual movement in which constraint forces do no virtual work will lead to an equation of motion whose only unknowns are position variables.

A slight modification of the system in Fig. 7.1 serves well to illustrate this concept. In Fig. 7.2 the pin connection at end C has been replaced with a collar riding in a horizontal guide, so the system is not rigid. Bar BC executes a general motion, so its center of mass G is the reference point for the Euler moment equation. Correspondingly, the inertial force $-m\bar{a}_G$ is applied at the center of mass, and the inertial couple applied to this bar is $I_G\ddot{\theta}\bar{k}$, where the sense is opposite the angular acceleration associated with positive $\ddot{\theta}$. In contrast, bar AB executes a pure rotation about pin A, so it is advantageous to use this point. Thus the inertial force $-m\bar{a}_D$ is applied at end A, and the inertial couple is $-I_G\ddot{\theta}\bar{k}$. These forces and couples are depicted in the free-body diagram.

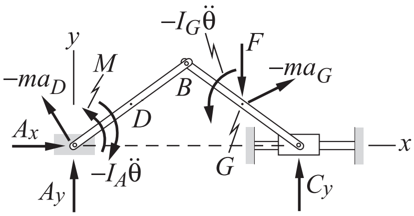


Figure 7.2. Free-body diagram showing the actual forces acting on a linkage, as well as d'Alembert's inertial forces.

We select the same virtual movement that we used to solve the static version of this system. Doing so avoids the occurrence of the constraint forces $\bar{A}_x,\ \bar{A}_y$, and $\bar{C}_y$ in the virtual work. We must add the virtual work of the inertial forces and couples to that of the actual force system. Because $-m\bar{a}_D$ acts at the stationary pin A, it does no work. We described the virtual displacement of point G in Eqs. (7.1.5). For the virtual rotations, we observe that increasing θ by $\delta\theta$ causes bar AB to rotate by $\delta\theta\bar{k}$ and bar BC to rotate by $-\delta\theta\bar{k}$. Thus the total virtual work done by the actual and inertial effects is

$$\delta W = \left(M\bar{k} - I_A\bar{\alpha}_{AB}\right)\cdot\delta\theta\bar{k} + \left(-m\bar{a}_G - F\bar{j}\right)\cdot\delta\bar{r}_{G/A} + \left(-I_G\bar{\alpha}_{BC}\right)\cdot\left(-\delta\theta\bar{k}\right) = 0. \quad (7.1.8)$$

Although we have equated the virtual work to zero, we cannot yet obtain an equation of motion because we must account for the dependence of $\bar{a}_G$ on θ. Differentiating $\bar{r}_{G/A}$ in Eqs. (7.1.4) gives

$$\bar{a}_G = \ddot{\bar{r}}_{G/A} = -\frac{3}{2}L\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\bar{i} + \frac{1}{2}L\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\bar{j}. \quad (7.1.9)$$

Substitution of this expression and the angular accelerations into the preceding equation for δW results in

$$\begin{aligned}\delta W = \left(M - I_A\ddot{\theta}\right)\delta\theta + \frac{3}{2}mL\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\left(-\frac{3L}{2}\delta\theta\sin\theta\right)\\ -\left[\frac{1}{2}mL^2\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) + F\right]\left(\frac{L}{2}\delta\theta\cos\theta\right) + I_G\ddot{\theta}\left(-\delta\theta\right) = 0.\end{aligned} \quad (7.1.10)$$

Because $\delta\theta$ is not zero, it may be factored out. What remains is the differential equation of motion:

$$\left[I_A + I_G + \frac{9}{4}mL^2\left(\sin\theta\right)^2 + \frac{1}{4}mL^2\left(\cos\theta\right)^2\right]\ddot{\theta} + 2mL^2\dot{\theta}^2\sin\theta\cos\theta = M - \frac{FL}{2}\cos\theta. \quad (7.1.11)$$

The analysis leading to this equation of motion exemplifies the direct application of the principle of dynamic virtual work. Each body is treated as though it were in static equilibrium under a force–couple system composed of the actual and inertial forces, where the latter are as given by the Newton–Euler equations of motion. Regardless of the type of motion, the center of mass is always an appropriate reference point for the moment equations, so we begin with

$$\Sigma\bar{F}_i + \left(-m_i\bar{a}_{Gi}\right) = \bar{0}, \quad \Sigma\bar{M}_{Gi} + \left(-\frac{d\bar{H}_{Gi}}{dt}\right) = \bar{0}, \quad i = 1, \ldots, K. \quad (7.1.12)$$

The virtual work of each force resultant is obtained from a dot product with the virtual displacement of the corresponding body's center of mass, and the the virtual work of each couple resultant is obtained from a dot product with the virtual rotation of the corresponding body. Summing the individual contributions gives

$$\sum_{i=1}^{K}\left[\sum \bar{F}_i + (-m_i\bar{a}_{Gi})\right]\cdot\delta\bar{r}_{Gi} + \left[\sum \bar{M}_{Gi} + \left(-\frac{d\bar{H}_{Gi}}{dt}\right)\right]\cdot\overline{\delta\theta}_i = 0. \tag{7.1.13}$$

(In the case in which a body is in pure rotation, the inertial force–couple system for that body may be taken to be $-m\bar{a}_G$ acting at the pivot point O and $-d\bar{H}_O/dt$, in which case the inertial force will not contribute to the virtual work if point O does not displace in the virtual movement.) An alternative to forming the virtual work done by the resultant force and moment is to consider each force individually, as described by Eq. (7.1.3). Thus the principle of dynamic virtual work may be written as

$$\boxed{\begin{aligned}&\sum_{i=1}^{K}\left[(-m_i\bar{a}_{Gi})\cdot\delta\bar{r}_{Gi} + \left(-\frac{d\bar{H}_{Gi}}{dt}\right)\cdot\overline{\delta\theta}_i\right]\\ &\quad+\sum_j \bar{F}_j\cdot\delta\bar{r}_j + \sum_n \bar{M}_n\cdot\overline{\delta\theta}_n = 0.\end{aligned}} \tag{7.1.14}$$

The process of extracting an equation of motion from the virtual work requires that one formulate a virtual movement of the system in which constraint forces do no work. If several position variables are required, a variety of virtual movements fitting this criterion can be used to find the appropriate number of equations of motion.

Our exploration here has served to introduce the important concepts of virtual movement and virtual work, which are keystones of the Lagrangian formulation. The main points are that virtual displacements are not the same as how a system actually moves, that virtual displacements are limited to being infinitesimal in order to simplify as much as possible the description of work, and that the concept of virtual work is motivated by the desire to avoid the occurrence of constraint forces in the derivation of differential equations of motion. Based on the example of a linkage, the principle of dynamic virtual work might seem to be an attractive tool, but there are several reasons why it has not been widely adopted. Selection of suitable virtual movements can be problematic, and the manner in which the inertia force–couple systems appear offers substantial opportunity for sign errors. More important, we will find that formulating Lagrange's equations of motion does not require that we describe accelerations, which substantially simplifies the effort in the kinematical portion of the analysis. To derive those equations we need to formalize the manner in which a system's position and velocity are described.

7.2 GENERALIZED COORDINATES AND KINEMATICAL CONSTRAINTS

One of the most important features of the preceding application of the principle of dynamic virtual work is the prominent role of the angle θ locating the bars. Both the actual accelerations and the virtual displacements were described in terms of this variable, yet θ is not the only variable that could have been selected for this purpose. For example, we could have used the horizontal distance ℓ from the fixed pin A to the collar C. In fact, because $\ell = 2L\cos\theta$, we could switch between the two variables. The variables that we select to locate the current position of a system are called *generalized coordinates.*

7.2.1 Selection of Generalized Coordinates

Suppose the reference location of a system is given. (Such a location might be the starting position or the static equilibrium position.) The generalized coordinate values must uniquely define any possible position of the system relative to the initial position. For example, it should be possible to draw a diagram of the system in its current position by knowing only values of the generalized coordinates and the fixed dimensions. The minimum number of generalized coordinates required to specify the position of the system is the *number of degrees of freedom* of that system. The linkage in Fig. 7.2 has one degree of freedom because we are able to locate the current configuration of the linkage solely from knowledge of θ. As mentioned earlier, the choice of generalized coordinates is not unique. Thus, different individuals analyzing the same system might select different generalized coordinates, and therefore derive different equations of motion.

When a body moves in space without any restrictions, its current location is defined by the position of any point, such as the center of mass, and a set of three independent direction angles locating lines in the body. Thus, in the absence of kinematical constraints, any rigid body moving in space has six degrees of freedom, and a possible set of generalized coordinates are three position coordinates of the center of mass relative to a convenient fixed reference frame and three Eulerian angles defined relative to that reference frame. Now suppose a body is constrained to execute a planar motion. This restriction reduces the number of degrees of freedom to three, because only two position coordinates are required to locate a point in the plane of motion, and the only angle of rotation is about the axis perpendicular to that plane.

Consider the rigid bar in Fig. 7.3(a), where position coordinates x_A and y_A locate one end of the bar, and angle θ measured from the horizontal locates the current orientation. Only geometrical parameters that change as the system moves are candidates for generalized coordinates, whereas fixed parameters, such as L, are considered to be known system properties. Thus the generalized coordinates defined in Fig. 7.3(a) are (x_A, y_A, θ).

Figure 7.3(b) defines an alternative set of three generalized coordinates. The bar's location is there described in terms of (x_B, y_B, ϕ), which define the position of end B and the rotation relative to the vertical direction. In general, it must be possible to express

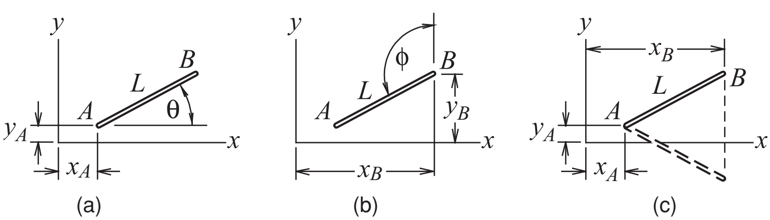


Figure 7.3. Alternative generalized coordinates for a bar moving in a plane.

one set of generalized coordinates in terms of another, for each must be capable of describing the position of all points in the system. The transformation from the set in Fig. 7.3(a) to Fig. 7.3(b) is

$$x_B = x_A + L\cos\theta, \quad y_B = y_A + L\sin\theta, \quad \phi = \theta + \pi/2. \tag{7.2.1}$$

Another choice, shown in Fig. 7.3(c), leads to a difficulty. The three generalized coordinates depicted there are (x_A, y_A, x_B). The difficulty is depicted in the figure, where it is evident that, for given values of x_B, the bar can have one of two orientations. Specifically, because the length L is a fixed parameter, the vertical position of end B is given by $y_A \pm \left[L^2 - (x_B - x_A)^2\right]^{1/2}$. Recall that the generalized coordinates must uniquely specify the location. This means that (x_A, y_A, x_B) can serve as generalized coordinates only in situations in which we know that end B will remain either above or below end A throughout the motion. In most cases in which the orientation of a body is significant, it is best to use angles as the generalized coordinates defining the orientation.

The situations in Fig. 7.3 correspond to cases in which the number of generalized coordinates equals the number of degrees of freedom. The generalized coordinates in such cases are *unconstrained*. This means that their values may be set independently, without violating any kinematical conditions. (Indeed, unconstrained generalized coordinates are sometimes alternatively called *independent coordinates*.) When the number of generalized coordinates exceeds the number of degrees of freedom, the generalized coordinates are *constrained*, because they must satisfy additional kinematical conditions, independently of any other considerations. The kinematical conditions that must be satisfied are called *constraint equations.*

A set of constrained generalized coordinates for the bar in Fig. 7.3 could be the position coordinates of each end and the angle of orientation $(x_A, y_A, x_B, y_B, \theta)$. Rather than merely offering a way to transform between alternative sets of generalized coordinates, Eqs. (7.2.1) then become two relationships between the generalized coordinates that must be satisfied, regardless of how the bar moves. Thus $(x_A, y_A, x_B, y_B, \theta)$ constitute a set of five generalized coordinates that are related by two constraint equations. This is consistent with the fact that the bar has three degrees of freedom, because the existence of two relations among five variables means that only three variables may be selected independently. The number of degrees of freedom equals the number of generalized coordinates minus the number of constraint equations.

Other than the restriction to planar motion, the bar in Fig. 7.3 is free to move. Any constraint imposed on the bar's motion by supporting it in some manner alters the number of degrees of freedom and therefore the selection of unconstrained generalized coordinates. If end A of this bar is pinned to the ground, then the number of degrees of freedom is reduced to one, because the position of the bar is now completely specified by the value of θ. A different way of locating the bar when end A is a pin is to use (x_A, y_A, θ) as a set of three generalized coordinates that are constrained to satisfy two constraint equations stating that x_A and y_A are constants. In this viewpoint, three constrained generalized coordinates are used to represent a one-degree-of-freedom system, so there are two constraint equations.

The linkage in Fig. 7.2 connects pinned bar AB to bar CD. That system also has one degree of freedom because the position of joint B is specified by the orientation of bar AB, and the position of collar C is then set by the requirement that it be situated on the horizontal guidebar at a distance L from joint B. Thus attaching bar BC to bar AB does not allow bar BC any freedom to move independently. This situation is altered drastically if the bars are joined by a spring rather than a pin, as depicted in Fig. 7.4. Any connection by means of a spring does not impose kinematical restrictions because the spring's length depends on the force it carries. Thus the movements of points B and D are kinematically unrelated. It is apparent from Fig. 7.4 that the location of both bars is fully specified by the values of the angles of elevation θ and ϕ and the horizontal distance x_C from fixed pin A to collar C. Furthermore, these three variables can have any value, depending on how forces move the system.* In other words the system has three degrees of freedom, and (θ, ϕ, x_C) are a suitable set of unconstrained generalized coordinates.

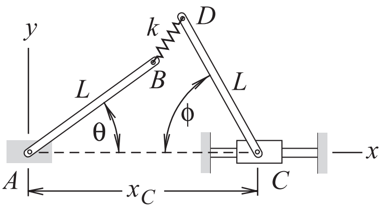


Figure 7.4. Generalized coordinates for a linkage whose members are connected by a spring.

An alternative would be to use (θ, x_D, y_D) as a set of unconstrained generalized coordinates, because, once we know the position of end D, collar C will be situated on the horizontal guide bar at the point that is distance L from end D. However, there is a potential difficulty in the selection of y_D as a generalized coordinate, because none of the generalized coordinates directly specifies the orientation of bar CD. Thus it is preferable to use (θ, ϕ, x_C) as the generalized coordinates. Of course, we also could use a constrained set of generalized coordinates, such as (θ, ϕ, x_D, y_D). The associated kinematical constraint equation relating the generalized coordinates is $y_D = L \sin\phi$, which confirms that there are only three independent variables in the set of generalized coordinates.

The definition of generalized coordinates as geometrical variables locating all parts of a system has a subtle aspect. In some cases motion is imposed on a point, or a body is made to rotate in a certain manner. It is implicit to such a specification that the movement is imposed by some force or couple that may be altered as necessary to obtain that movement. For example, suppose we wish to make horizontal distance x_C in Fig. 7.4 change in a specified manner, so that $x_C = f(t)$. We may conceive of doing so with a very large servomechanism that exerts on the collar an instantaneously adjustable horizontal force. The magnitude of this force will be whatever is required to make collar C move in the required manner, regardless of the influence of other forces. The condition $x_C = f(t)$ is a constraint equation, and the force exerted by the servomechanism is a

* Auxiliary conditions, such as the fact that the movement of collar C cannot be so large that it hits the stops on the guide bar, may be examined after the equations of motion are formulated.

constraint force. In this case the linkage is reduced to two degrees of freedom, and (θ, ϕ) serve as convenient unconstrained generalized coordinates. Alternatively, we could use (θ, ϕ, x_C) as a set of constrained generalized coordinates.

It might be troubling that generalized coordinates can be either constrained or unconstrained. We will see that in some situations it is not possible to identify an unconstrained set. Furthermore, our studies will show that, when the alternative is available, there are specific advantages to selecting either type. It also might be troubling that there is no unique set of generalized coordinates. However, this is the essence of the modeling process. When we select the variables with which to characterize the motion of a system, we are concurrently making decisions about the features of the system that will need to be considered. Different individuals can use different variables to represent these features and therefore obtain different equations of motion. However, if their models are equivalent, they will have the same number of degrees of freedom, and the physical motion predicted by solution of those equations will be identical.

7.2.2 Constraint Equations

To progress in our study of analytical mechanics we need to fit the notions of generalized coordinates and constraint equations into a mathematical framework. Suppose we select a set of N constrained generalized coordinates to represent a system. We reserve the symbol q_j to denote the jth generalized coordinate. There might be several constraint equations that these generalized coordinates must satisfy. If these equations are like those arising in the previous subsection, each may be written in the functional form

$$\boxed{f_i\,(q_n, t) = 0,} \tag{7.2.2}$$

where the subscript i denotes which of the several constraints is being considered and the appearance in the argument list of q_n with n unspecified serves to indicate dependence on all of the generalized coordinates. A relation such as Eq. (7.2.2) is referred to as a *configuration constraint* equation. This term stems from the fact that, by limiting the values the generalized coordinates may have, it restricts the overall arrangement of the system at any instant. Examples of configuration constraint equations are Eqs. (7.2.1), which constitute three equations that must be satisfied by the six generalized coordinates $(x_A, y_A, x_B, y_B, \theta, \phi)$. Given the value of three of these variables, the values of the other three may be determined by simultaneously solving these equations.

Constraint equations like Eqs. (7.2.1) represent relations that are time invariant. Solving such equations for a subset of generalized coordinates would yield values that depend on only the current values of the other coordinates. In contrast, the general configuration constraint described by Eq. (7.2.2) contains t in its argument list. This is done to accommodate situations in which physical features of a system change in a known manner. For example, instead of being fixed, suppose the bar in Fig. 7.3 is replaced with a hydraulic cylinder whose length L is controlled. In that case Eqs. (7.2.1) would still apply, except that L would be the length at a specified value of t.

Rather than directly constraining the position of a system, we can impose conditions on the velocity of its points or the angular velocity of its bodies. Such restrictions are described by *velocity constraint* equations. Any configuration constraint may be converted into an equivalent velocity constraint by differentiating it with respect to time. The chain rule for differentiation must be employed to differentiate Eq. (7.2.2) because the generalized coordinates are (unknown) functions of time. Thus the velocity constraint equation corresponding to Eq. (7.2.2) is

$$\dot{f}_i = \sum_{j=1}^{N} \left[\frac{\partial}{\partial q_j} f_i\left(q_n, t\right) \right] \dot{q}_j + \frac{\partial}{\partial t} f_i\left(q_n, t\right) = 0. \tag{7.2.3}$$

One way of viewing the preceding velocity constraint equation is that it is a differential equation that the generalized coordinates must satisfy. In view of its derivation, multiplying Eq. (7.2.3) by dt converts it to a perfect differential that may be integrated. The constant of integration that arises in this process may be evaluated by substitution of initial values of the generalized coordinates. If these initial values satisfy the configuration constraint equation, the restriction on velocity represented by Eq. (7.2.3) is fully equivalent in its effect to the configuration constraint.

A different perspective results if we ignore the fact that a *generalized velocity* $\dot{q}_j$ is known if the corresponding q_j is a known function of time. Instead, suppose we were to begin to observe a system at an arbitrary time t. We could deduce the corresponding values of the q_j variables at that instant from measurements of the physical position. In this view, Eq. (7.2.3) constrains the possible values of the generalized velocities when the system is at a specific position. Such a view is comparable with using $\bar{v}_B = \bar{v}_A + \bar{\omega} \times \bar{r}_{B/A}$ to relate the velocities of points in a rigid body, rather than differentiating a general expression for $\bar{r}_{B/O}$. Another example of this dual viewpoint is to consider using as constrained generalized coordinates the x and y coordinates of the centroid of a collar that slides along a circular of radius R. If we place the center of the guide bar at the origin, then the configuration constraint equation for the motion is $x^2 + y^2 - R^2 = 0$. Differentiation of this relation with respect to time yields the velocity constraint equation $2x\dot{x} + 2y\dot{y} = 0$. The position and velocity in Cartesian coordinates are $\bar{r} = x\bar{i} + y\bar{j}$ and $\bar{v} = \dot{x}\bar{i} + \dot{y}\bar{j}$. Thus the velocity constraint equation requires that $\bar{v} \cdot \bar{r} = 0$, that is, the velocity of the collar must always be perpendicular to the position vector from the origin to the collar. As shown in Fig. 7.5, we could arrive at the same velocity constraint directly through a kinematical analysis, in which we adjust the velocity components to make $\bar{v}$ tangent to the circle, and then observe that the angle θ locating $\bar{v}$ also defines the radial line. Note that the radius R does not appear in the velocity constraint equation. However, integrating the velocity constraint equation would require that the initial values of x and y correspond to a point on a circle of radius R.

Figure 7.5 exemplifies a primary reason to consider velocity constraints. In many cases, especially those involving complicated systems, it is easier to identify constraint conditions by analyzing the velocity of a system at a specific system. Furthermore, there are important situations in which the velocity of a system is constrained, but there is no corresponding configuration constraint. Such cases may be represented by modifying

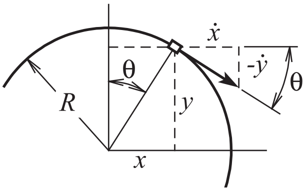


Figure 7.5. Kinematical relationship of generalized velocities for motion along a circular path.

Eq. (7.2.3) such that the coefficients of the generalized velocities are arbitrary functions, rather than being restricted to be partial derivatives of a configuration function f_n. Specifically, this more general form is

$$\boxed{\sum_{j=1}^{N} a_{ij}\left(q_n, t\right)\dot{q}_j + b_i\left(q_n, t\right) = 0.} \tag{7.2.4}$$

Constraint conditions that have this form are said to be *linear velocity constraints*, because they represent linear algebraic equations for the generalized velocities when the the values of the generalized coordinates and time are given.†

From a philosophical standpoint, considering a generalized coordinate and the corresponding generalized velocity to be independent variables is an important development. A configuration constraint represents an algebraic/transcendental equation that governs how a system arrived at its current state. In contrast, a velocity constraint represents a restriction on the manner in which the system can move given its current state. Movements that are consistent with the constraint conditions are said to be *kinematically admissible.*

A kinematical analysis of a system will often lead to one or more velocity constraint equation. It is useful in such cases to determine whether these equations are equivalent to configuration constraints. To carry out such an analysis, we convert the velocity form to a differential by multiplying it by dt, which leads to

$$\sum_{j=1}^{N} a_{ij}\left(q_n, t\right) dq_j + b_i\left(q_n, t\right) dt = 0. \tag{7.2.5}$$

This is the *Pfaffian form* of a constraint equation. Instead of restricting the generalized velocities, it represents a restriction on the amount by which the generalized coordinates may change in a time interval dt. The Pfaffian form of the configuration constraint may be obtained similarly by multiplying Eq. (7.2.3) by dt:

$$df_i = \sum_{j=1}^{N}\left[\frac{\partial}{\partial q_j} f_i\left(q_n, t\right)\right] dq_j + \frac{\partial}{\partial t} f_i\left(q_n, t\right) dt = 0. \tag{7.2.6}$$

† Equation (7.2.4) does not represent the most general type of kinematical constraint that can be imposed on the motion of a system. Some types of motion restrictions cannot be treated in any general manner. One such situation arises in treating inequality relationships, such as the limitation that the wheels of a car reaching the top of a hill remain in contact with the ground only if the normal contact force is positive. Also, it is conceivable that generalized velocities might occur nonlinearly in a constraint equation.

Equations (7.2.5) and (7.2.6) are equivalent if they differ by only a multiplicative factor. This *integrating factor* may be a function of the generalized coordinates and time, so we denote it as $g_i\left(q_1, q_2, \ldots, q_N, t\right)$. Thus we deduce that a given velocity constraint equation is derivable from a configuration constraint if there exists an integrating factor and constraint function such that

$$g_i\left(q_n, t\right) a_{ij}\left(q_n, t\right) = \frac{\partial}{\partial q_j} f_i\left(q_n, t\right), \quad j = 1, 2, \ldots, N, \tag{7.2.7a}$$

$$g_i\left(q_n, t\right) b_i\left(q_n, t\right) = \frac{\partial}{\partial t} f_i\left(q_n, t\right). \tag{7.2.7b}$$

If a velocity constraint equation can be shown to satisfy Eqs. (7.2.7) for every j, it is said to be *holonomic,* which is a Greek word for "integrable." Constraint equations that do not satisfy these conditions are *nonholonomic.* This terminology arises from the fact that differentiating a configuration constraint yields a velocity constraint, and integration is the inverse process. The terms $\partial f_i/\partial q_j$ constitute the Jacobian of a set of holonomic constraints. The coefficients a_{ij} are referred to as the *Jacobian constraint matrix*, even when the constraint conditions are not holonomic. We will encounter this matrix in several contexts.

Ascertaining whether a velocity constraint is holonomic requires that we determine whether, for a given a set coefficient functions a_{ij}, $j = 1, 2, \ldots, N$ and b_i, there is a single integrating factor g_i and single constraint function f_i that satisfy Eqs. (7.2.7). Such a determination might not be trivial. If we are lucky, we might be able to do it by inspection. For example, consider the velocity constraint equation $y\dot{x} - x\dot{y} = 0$, whose Pfaffian form is $ydx - xdy = 0$. The left side is recognizable as the numerator of the perfect differential $d\left(x/y\right)$. Thus we multiply the Pfaffian form by the integrating factor $g = 1/y^2$. The altered form may be integrated, which leads to $x/y = C$, where C is the integration constant. This is the equation of a straight line that intersects the origin, which was not obvious from the original velocity constraint equation.

If we do not succeed in identifying a suitable integrating factor, there is another process we can pursue. Let us multiply Eq. (7.2.7a) for each j by the corresponding dq_j, and multiply Eq. (7.2.7b) by dt, then take the indefinite integral of each. Note that all other variables other than the one for the integral are held constant in the integration. Also, the constant of integration is replaced with a function that can depend on all of the variables, other than the integration variable. Let us denote these functions as $h_{ij}\left(q_n, t\right)$, where the first subscript, i, indicates the constraint equation to which it pertains, and the second subscript, j, indicates the generalized coordinate on which it does *not depend,* with $h_{it}\left(q_n\right)$ being independent of t. Thus the result of integrating each of Eqs. (7.2.7) may be written as

$$\begin{aligned} f_i\left(q_n, t\right) &= \int g_i\left(q_n, t\right) a_{ij}\left(q_n, t\right) dq_j + h_{ij}\left(q_n, t\right), \quad j = 1, 2, \ldots, N, \\ f_i\left(q_n, t\right) &= \int g_i\left(q_n, t\right) b_i\left(q_n, t\right) dt + h_{it}\left(q_n\right). \end{aligned} \tag{7.2.8}$$

If a single configuration function f_i and integrating factor $g_i\left(q_n, t\right)$ consistently satisfying each of these $N+1$ constraint can be identified, then the velocity constraint is indeed derivable from a configuration constraint. Conversely, if attempting to satisfy conditions associated with each j leads to an inconsistency, it must be that the constraint is nonholonomic.

The occurrence of nonholonomic constraint conditions serves to limit our options when we select generalized coordinates. Suppose there are J_c configuration constraints and J_v nonholonomic velocity constraints to be satisfied by our choice of N generalized coordinates. Let D denote the number of degrees of freedom, so the number of generalized coordinates must be

$$N = D + J_c + J_v. \tag{7.2.9}$$

In principle, we can solve the configuration constraints for J_C generalized coordinates in terms of the other. Such a solution may be used to eliminate the solved generalized coordinates from the formulation. In contrast, nonholonomic velocity equations cannot be integrated, unless we know the time history of the generalized coordinates, which requires solution of the equations of motion. Thus the presence of nonholonomic constraints means that generalized coordinates must be a constrained set, with the minimum number of generalized coordinates being $D + J_v$. Because it might be difficult to determine whether some velocity constraints are holonomic, we treat any system represented by velocity constraint equations as though it were nonholonomic.

The action of the edge of an ice skate blade serves to illustrate these matters. In Fig. 7.6 point P is the idealized single point of contact between the ice and the curved blade. The blade's position is set by the coordinates x_p and y_P of this point and the angle θ from the x axis along which the blade is aligned. Thus we have $N = 3$. As shown in the figure, the velocity of the point of contact must be aligned with the blade. Thus it must be that $\bar{v}_P = \dot{x}_P \bar{i} + \dot{y}_P \bar{j} = v_P \left(\cos\theta \bar{i} + \sin\theta \bar{j}\right)$. Eliminating the speed v_P leads to $\dot{x}_P \sin\theta - \dot{y}_p \cos\theta = 0$, which matches the standard velocity constraint form, with $a_{11} = \cos\theta$, $a_{12} = -\sin\theta$, $a_{13} = 0$, and $b_1 = 0$. This constraint is nonholonomic. To prove this we observe that, if it were holonomic, then $a_{13} g_1 = 0 = \partial f_1/\theta$. This requires that the constraint be independent of θ, which contradicts the dependence of a_{11} and a_{12}.

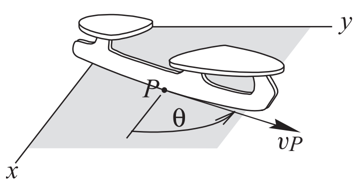
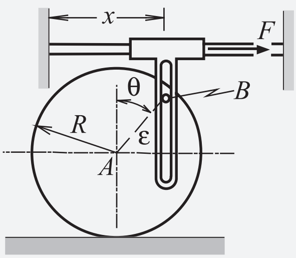


Figure 7.6. Velocity constraint for an ice skate blade.

The skate blade exemplifies a fundamental characteristic of systems with nonholonomic constraints. The blade may be set down in any location, which means that any set of values for x_P, y_P, and θ is possible. However, when the blade is brought into contact with the ice, the equivalent Pfaffian form of the constraint equations requires that $dy_P = dx_P \tan\theta$, which restricts what new position point P can attain.

Further consideration of this ice skate blade points out a problematic aspect of the term "degrees of freedom" when it is applied to a system having nonholonomic constraints. In Subsection 7.2.1 N was defined as the minimum number of generalized coordinates required to locate a system. Here, setting down the skate corresponds to defining the initial conditions of the three generalized coordinates, so it might seem that $N = 3$. However, on specification of the three generalized coordinates, their values at the next instant are restricted to be consistent with the Pfaffian form of the differential constraint equations. Because there is one such equation for the skate blade, only two of the three generalized coordinates are independent. Provided one recognizes this aspect, it is consistent to say that the number of degrees of freedom is $D = N - J_c - J_v$ in any situation.

EXAMPLE 7.1 The disk rolls without slipping as it is pulled to the right by the yoke. Generalized coordinates are the horizontal distance x to pin B and the angle θ by which the radial line to pin B rotates. Derive the velocity constraint equation relating these two variables and show that it is holonomic. What is the corresponding configuration constraint?



Example 7.1

SOLUTION This example demonstrates that the kinematical tools developed previously are readily adapted for analytical mechanics formulation. We sequence the generalized coordinates as $q_1 = x$, $q_2 = \theta$. There is no doubt that these variables fully locate all parts of the system, but picking both values arbitrarily will lead to a motion in which there is slippage at the point where the disk is in contact with the ground. Let the XY axes be horizontal and vertical, respectively. The angular velocity of the disk is $\bar{\omega} = -\dot{\theta}\bar{K}$, and the condition that there is no slip requires that $\bar{v}_A = R\dot{\theta}\bar{I}$. The velocity of pin B perpendicular to the wall of the groove must match the velocity of the yoke in that direction, so that $\bar{v}_B \cdot \bar{I} = \dot{x}$. Because points A and B are part of the disk, we therefore have

$$\dot{x} = \bar{v}_B \cdot \bar{I} = (\bar{v}_A + \bar{\omega} \times \bar{r}_{B/A}) \cdot \bar{I}.$$

Substitution of $\bar{r}_{B/A} = e \sin\theta\,\bar{I} + e\cos\theta\,\bar{J}$ into the preceding equation shows that the constraint equation relating $\dot{x}$ and $\dot{\theta}$ is

$$\dot{x} - (R + \varepsilon\cos\theta)\,\dot{\theta} = 0. \qquad \triangleleft$$

All terms have been brought to the left side in this equation in order to match it to the standard linear velocity constraint, Eq. (7.2.4). From this we identify the Jacobian constraint coefficients to be

$$a_{11} = 1, \quad a_{12} = -(R + \varepsilon\cos\theta), \quad b_1 = 0.$$

To determine whether the constraint equation is holonomic we multiply it by dt to convert it to Pfaffian form,

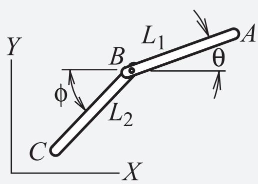
$$dx - (R + \varepsilon \cos\theta)\, d\theta = 0.$$

There is no need to introduce an integrating factor because $\cos\theta\, d\theta = d(\sin\theta)$. Thus the Pfaffian form is a perfect differential, whose integration yields

$$x - R\theta - \varepsilon \sin\theta = C. \qquad \triangleleft$$

The constant C may be determined by substituting the values of x and θ at some reference location. For example, if $x = 0$ corresponds to $\theta = 0$, then $C = 0$. We could have derived this configuration constraint directly by observing that, if there is no slip, then the horizontal distance displaced by the center is $R\theta$. Adding the horizontal distance $\varepsilon \sin\theta$ from the center to the yoke gives x.

EXAMPLE 7.2 Two bars, pinned at joint B, move in the horizontal plane subject only to the restriction that the velocity of end C must be directed toward end A. Determine the corresponding velocity constraint. Is this constraint holonomic?



Example 7.2

SOLUTION The objective here is to demonstrate the procedure for analyzing constraint conditions that are more complicated than the one in the previous example. Determining whether the velocity constraint is holonomic will require more ingenuity on our part. The position of each bar is uniquely specified by the coordinates of pin B and the angle of rotation of each bar. Hence we define $q_1 = x_B$, $q_2 = y_B$, $q_3 = \theta$, $q_4 = \phi$. The given condition on the velocity of point C may be written in vector form as

$$\bar{v}_C = v_C \bar{e}_{A/C}.$$

We must express this condition in terms of the generalized coordinates, which means that the speed v_c should be eliminated. Thus we restate the constraint condition as

$$\bar{v}_C \times \bar{r}_{A/C} = \bar{0}. \tag{1}$$

Points B and C are common to the same body, so their velocities are related by

$$\bar{v}_C = \bar{v}_B + \bar{\omega}_{BC} \times \bar{r}_{C/B}.$$

We express each of the terms on the right side in terms of the generalized coordinates to find

$$\begin{aligned}\bar{v}_C &= \dot{X}_B\bar{I} + \dot{Y}_B\bar{J} + \dot{\phi}\bar{K} \times \left(-L_2\cos\phi\,\bar{I} - L_2\sin\phi\,\bar{J}\right)\\ &= \left(\dot{X}_B + \dot{\phi}L_2\sin\phi\right)\bar{I} + \left(\dot{Y}_B - \dot{\phi}L_2\cos\phi\right)\bar{J}.\end{aligned} \tag{2}$$

Also, the position vector is

$$\bar{r}_{A/C} = \left(L_1\cos\theta + L_2\cos\phi\right)\bar{I} + \left(L_1\sin\theta + L_2\sin\phi\right)\bar{J}. \tag{3}$$

Substitution of Eqs. (2) and (3) into Eq. (1) leads to

$$\begin{aligned}\left(\bar{v}_C \times \bar{r}_{A/C}\right)\cdot\bar{K} = &\left(\dot{X}_B + \dot{\phi}L_2\sin\phi\right)\left(L_1\sin\theta + L_2\sin\phi\right)\\ &- \left(\dot{Y}_B - \dot{\phi}L_2\cos\phi\right)\left(L_1\cos\theta + L_2\cos\phi\right) = 0.\end{aligned}$$

Collecting the coefficients of each generalized velocity converts this equation to the standard form of a linear velocity constraint:

$$\begin{aligned}&a_{11}\dot{X}_B + a_{12}\dot{Y}_B + a_{13}\dot{\theta} + a_{14}\dot{\phi} = 0,\\ &a_{11} = L_1\sin\theta + L_2\sin\phi, \quad a_{12} = -\left(L_1\cos\theta + L_2\cos\phi\right), \quad a_{13} = 0,\\ &a_{14} = L_2\sin\phi\left(L_1\sin\theta + L_2\sin\phi\right) + L_2\cos\phi\left(L_1\cos\theta + L_2\cos\phi\right)\\ &\quad\equiv L_1L_2\cos\left(\theta - \phi\right) + L_2^2.\end{aligned} \qquad \lhd$$

To determine whether the constraint equation is integrable we convert it to Pfaffian form by multiplying by dt, which gives

$$\begin{aligned}&\left(L_1\sin\theta + L_2\sin\phi\right)dX_B - \left(L_1\cos\theta + L_2\cos\phi\right)dY_B\\ &\quad + \left[L_1L_2\cos\left(\theta - \phi\right) + L_2^2\right]d\phi = 0.\end{aligned}$$

In view of the appearance of θ and ϕ in the terms containing dX_B and dY_B and the absence of X_B and Y_B in the coefficient of $d\phi$, it certainly does not appear that this differential constraint equation can be integrated. This, however, is not a proof. We proceed to the alternative strategy described by Eqs. (7.2.8). Thus we assume that the velocity constraint equation is equivalent to $f_1\left(q_n, t\right) = 0$, and then seek a configuration function that is consistent with all of the conditions. Equations (7.2.7) in the present case require that

$$\begin{aligned}\frac{\partial f_1}{\partial X_B} &= g_1\left(X_B, Y_B, \theta, \phi\right)\left(L_1\sin\theta + L_2\sin\phi\right),\\ \frac{\partial f_1}{\partial Y_B} &= -g_1\left(X_B, Y_B, \theta, \phi\right)\left(L_1\cos\theta + L_2\cos\phi\right),\\ \frac{\partial f_1}{\partial\theta} &= 0,\\ \frac{\partial f_1}{\partial\phi} &= g_1\left(X_B, Y_B, \theta, \phi\right)\left[L_1L_2\cos\left(\theta - \phi\right) + L_2^2\right].\end{aligned}$$

Integrating the first two conditions gives

$$f_1 = (L_1 \sin\theta + L_2 \sin\phi) \int g_1 (X_B, Y_B, \theta, \phi)\, dX_B + \beta_1 (Y_B, \theta, \phi),$$

$$f_1 = -(L_1 \cos\theta + L_2 \cos\phi) \int g_1 (X_B, Y_B, \theta, \phi)\, dY_B + \beta_2 (X_B, \theta, \phi),$$

where β_n with associated arguments indicates the functions of integration. The integration factor g_1 cannot be zero, so both conditions state that f_1 must depend on θ. However, the third condition, $\partial f_1/\partial\theta = 0$, requires that it not depend on θ. This is a contradiction, so we conclude that the constraint equation is not holonomic. ◁

7.2.3 Configuration Space

Everything we need to know about the motion of a particle is conveyed by the time dependence of its Cartesian coordinates. The *configuration space,* which is defined to be an N-dimensional rectangular Cartesian space in which the distance along each axis is measured by a generalized coordinate, provides a unified perspective for any system. One of its primary uses is providing a pictorial representation of some abstract mathematical concepts. To distinguish vectors that are defined in the configuration space from those that are defined in the physical world, we use a caret to denote the former. Thus the unit vectors of the configuration space are $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_N$. The *generalized position* is defined in this space as

$$\hat{r} = q_1\hat{e}_1 + q_2\hat{e}_2 + \cdots + q_N\hat{e}_N. \tag{7.2.10}$$

This position represents the location of every part of a physical system at a given instant of time. The generalized coordinates are time dependent, so the motion of a system appears in the configuration space as a path that is the locus of the instantaneous positions. This is the *configuration path.*

The displacement of the entire system in an infinitesimal time interval dt is represented by the increment of the position in that interval. By virtue of defining the configuration space to be Cartesian, the unit vectors $\hat{e}_j$ are fixed directions. Thus,

$$d\hat{r} = \hat{r}(t + dt) - \hat{r}(t) = dq_1\hat{e}_1 + \cdots + dq_N\hat{e}_N. \tag{7.2.11}$$

Correspondingly, the *generalized velocity vector* has components that are the rates of change of the generalized coordinates:

$$\hat{v} = \frac{d\hat{r}}{dt} = \dot{q}_1\hat{e}_1 + \dot{q}_2\hat{e}_2 + \cdots + \dot{q}_N\hat{e}_N. \tag{7.2.12}$$

As is true for motion of a point in the physical world, the generalized velocity $\hat{v}$ and displacement $d\hat{r}$ are tangent to the configuration path at the position associated with the current position $\hat{r}(t)$.

The manner in which a system moves obviously depends on the forces that are applied to it. Thus the path of the system in the configuration space depends on the forces acting on the system – applying forces at different locations or altering the time dependence of the forces will change the configuration path. Let us consider two alternative

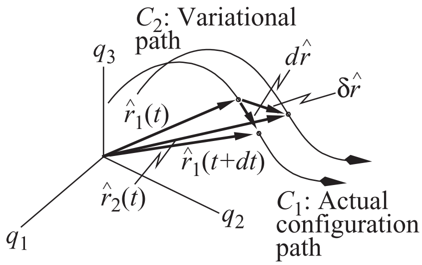


Figure 7.7. System paths in the configuration space.

configuration paths. Path C_1 corresponds to the actual set of forces for the case in which we are interested, whereas path C_2 represents the motion that is due to a slightly different set of forces. Let $\hat{r}_1(t)$ be a point on C_1 at a specific instant t, and let $\hat{r}_2(t)$ be the corresponding point on C_2 at that instant. Clearly, $\hat{r}_2(t) - \hat{r}_1(t)$ depends somehow on the difference between the actual and alternative forces. Let us denote this position difference as $\delta\hat{r}$. We restrict the difference between the actual and alternative force systems to be infinitesimal solely because doing so will enable us to employ the tools of differential calculus. Consequently C_2 is infinitesimally close to C_1, so that $\delta\hat{r}$ also is infinitesimal. It follows from the definition of $\hat{r}$ that

$$\delta\hat{r} = \delta q_1\hat{e}_1 + \cdots + \delta q_N\hat{e}_N. \tag{7.2.13}$$

An alternative configuration path that is infinitesimally close to the actual path is called the *variational path.* Figure. 7.7 depicts the situation for a three-degree-of-freedom system described by three unconstrained generalized coordinates. We see there that the vector $d\hat{r}$ represents the actual movement of the system over an elapsed time dt, whereas $\delta\hat{r}$ represents the shift in the position at instant t that would arise if the system followed the variational path rather than the actual path. Because this position shift does not represent an actual movement, we say that $\delta\hat{r}$ represents a *virtual displacement.* Despite the similar appearance of $d\hat{r}$ and $\delta\hat{r}$ in Eqs. (7.2.11) and (7.2.13), respectively, one should recall the very different nature of their definitions. In particular, $d\hat{r}$ is tangent to the actual configuration path, whereas $\delta\hat{r}$ is not. The symbol δ is used to convey the fact that the associated change in any quantity does not represent the actual manner in which the system evolves.

If the generalized coordinates are unconstrained, then any point in the configuration space is attainable, assuming one has the ability to generate the forces required to cause the configuration path to pass through that point. In contrast, a constraint equation represents a restriction on the values that the generalized coordinates may have. Thus, if a system is described by a set of constrained generalized coordinates, then there are restrictions on the possible paths the system may follow through the configuration space.

The restrictions imposed by a configuration constraint that is independent of time are the easiest to visualize. Such a constraint has the mathematical form $f_i(q_1, \ldots, q_N) = 0$. Imposing a single mathematical relation on the generalized coordinates restricts the position to lie on a surface in the N-dimensional configuration space. Because the constraint does not depend on time, this *constraint surface* has a shape that is time invariant.

For this reason the constraint equation, $f_i\,(q_1, \ldots, q_N) = 0$, is said to be *scleronomic,* which is a Greek word conveying the idea of a constraint surface frozen in time. Figure 7.8 depicts a scleronomic constraint that must be satisfied by three generalized coordinates.

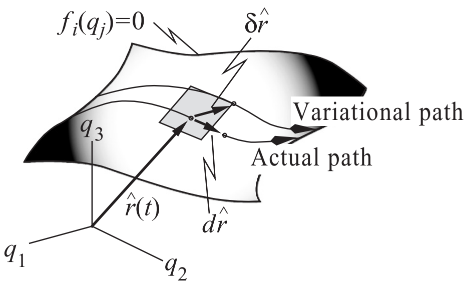


Figure 7.8. Virtual and real displacements in the configuration space satisfying a scleronomic constraint condition.

At a specified time instant the system's position is represented by the current value of $\hat{r}$ to a specific point on the constraint surface. Because the generalized coordinates must satisfy the scleronomic constraint equation, which represents an invariant relation, the configuration path must be situated on the constraint surface. The motion resulting from any other set of applied forces must be consistent with the constraints that are imposed on the system's motion. Thus it must be that any variational path consistent with a scleronomic constraint must also lie on the constraint surface. In contrast, if one were to ignore the constraint condition, any variational path would be acceptable.

A virtual displacement is said to be *kinematically admissible* if the point defining $\delta\hat{r}$ on the variational path is consistent with all constraint conditions. Both the actual displacement $d\hat{r}$ and the virtual displacement $\delta\hat{r}$ are infinitesimal in magnitude because the former occurs over an infinitesimal time interval and the variational path is defined to be infinitesimally close to the actual path. Thus the property that the actual and variational paths lie on a scleronomic constraint surface leads to the conclusion that the actual displacement $d\hat{r}$ and any kinematically admissible $\delta\hat{r}$ are situated in the plane that is tangent to that surface. The condition that a vector lie in a tangent plane is equivalent to saying that the vector is perpendicular to the normal to that plane at the current location. The gradient of a surface along which a function of several variables is constant is perpendicular to the surface. Thus the normal $\hat{a}_i$ to a scleronomic constraint surface defined by $f_i\,(q_n) = 0$ may be constructed by taking its gradient, according to

$$\hat{a}_i = \hat{\nabla} f_i = \frac{\partial}{\partial q_1} f_i\,(q_n)\,\hat{e}_1 + \cdots + \frac{\partial}{\partial q_N} f_i\,(q_n)\,\hat{e}_N. \tag{7.2.14}$$

The condition that both $d\hat{r}$ and $\delta\hat{r}$ are perpendicular to $\hat{a}_i$ requires that

$$d\hat{r} \cdot \hat{a}_i = dq_1 \frac{\partial}{\partial q_1} f_i\,(q_n) + \cdots + dq_N \frac{\partial}{\partial q_N} f_i\,(q_n) = 0, \tag{7.2.15a}$$

$$\delta\hat{r} \cdot \hat{a}_i = \delta q_1 \frac{\partial}{\partial q_1} f_i\,(q_n) + \cdots + \delta q_N \frac{\partial}{\partial q_N} f_i\,(q_n) = 0. \tag{7.2.15b}$$

The first relation is the Pfaffian form of a configuration constraint, Eq. (7.2.6), in the case of a scleronomic constraint for which $\partial f_i/\partial t$ is identically zero. It follows from the definition of the generalized velocity vector in Eq. (7.2.12) that a scleronomic velocity constraint requires that $\hat{v}$, like $d\hat{r}$ and $\delta\hat{r}$, always be perpendicular to the constraint surface's normal direction.

The situation changes if we consider a *rheonomic constraint,* which is the term used to describe a configuration constraint that is time dependent, $f_i\,(q_n, t) = 0$. Figure 7.9 depicts a rheonomic constraint imposed on three generalized coordinates. At a specific instant t, the rheonomic constraint requires that any point be situated on a plane in the configuration surface. Unlike a scleronomic constraint surface, a rheonomic constraint surface changes with time. That is, each instant t corresponds to a different surface. The prefix "rheo" is Greek for flowing, which is appropriate because the surface seems to flow through the configuration space in a pictorial representation like Fig. 7.9.

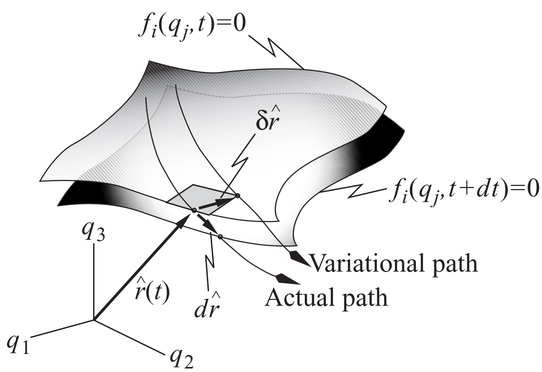


Figure 7.9. Virtual and real displacements in the configuration space satisfying a rheonomic constraint condition.

One consequence of the evolving nature of a rheonomic constraint surface is that the actual configuration path cannot lie in the surface corresponding to a specific instant. Rather, each instant $\hat{r}\,(t)$ must locate a point on the constraint surface associated with that instant t. Thus neither the actual displacement $d\hat{r}$ nor the generalized velocity $\hat{v}$ is tangent to the normal to the constraint surface at the instant associated with $\hat{r}\,(t)$.

The character of the virtual movement $\delta\hat{r}$ is quite different. By definition, if the alternative motion described by the variational path is consistent with the rheonomic constraint, then the point $\hat{r}\,(t)$ on a variational path will be situated somewhere on the configuration surface associated with that t, as does the vector $\hat{r}\,(t)$ for the actual path. By definition, $\delta\hat{r}$ extends from the point on the configuration path to the point on a variational path associated with the same instant. It follows that a kinematically admissible virtual displacement $\delta\hat{r}$ will be situated in the local tangent plane. The condition of tangency is described by stating that $\delta\hat{r}$ is perpendicular to the normal to the surface. Because t is constant for the constraint surface at a specific instant, the normal to the surface is still described by Eq. (7.2.14), except that t must be added to the argument list for the constraint function. Thus we have

$$\begin{aligned}\hat{a}_i &= \hat{\nabla} f_i = \frac{\partial}{\partial q_1} f_i\,(q_n, t)\,\hat{e}_1 + \cdots + \frac{\partial}{\partial q_N} f_i\,(q_n, t)\,\hat{e}_N,\\ \delta\hat{r}\cdot\hat{a}_i &= \delta q_1 \frac{\partial}{\partial q_1} f_i\,(q_n, t) + \cdots + \delta q_N \frac{\partial}{\partial q_N} f_i\,(q_n, t) = 0.\end{aligned} \tag{7.2.16}$$

The velocity and Pfaffian forms of a rheonomic constraint are given by Eqs. (7.2.3) and (7.2.6), respectively. These may be written in terms of configuration space vectors as

$$\hat{v}\cdot\hat{a}_i + b_i\,(q_n, t) = 0 \iff d\hat{r}\cdot\hat{a}_i + b_i\,(q_n, t)\,dt = 0. \tag{7.2.17}$$

By demonstrating that neither $\hat{v}\cdot\hat{a}_i$ nor $d\hat{r}\cdot\hat{a}_i$ is zero, the preceding relation emphasizes that neither $\hat{v}$ nor $d\hat{r}$ is tangent to the constraint surface at a specific t.

There is no constraint surface in the case of a nonholonomic constraint, which can only be stated as a velocity condition, Eq. (7.2.4), or in Pfaffian form, Eq. (7.2.5). Time is held constant in a virtual displacement. Hence the Pfaffian form of a constraint equation requires that virtual displacement satisfy

$$\sum_{j=1}^{N} a_{ij}\,(q_n, t)\,\delta q_j = 0. \tag{7.2.18}$$

Although the a_{ij} functions in this case are not derivable from a configuration constraint, we nevertheless can consider them to be the components in the configuration space of a vector $\hat{a}_i$:

$$\hat{a}_i = a_{i1}\,(q_n, t)\,\hat{e}_1 + \cdots + a_{iN}\,(q_n, t)\,\hat{e}_N. \tag{7.2.19}$$

It follows that a nonholonomic constraint requires that the virtual displacement satisfy

$$\boxed{\hat{a}_i\cdot\delta\hat{r} = 0.} \tag{7.2.20}$$

An overview of the discussion of scleronomic, rheonomic, and nonholonomic constraints shows that, in each case, a kinematically admissible virtual displacement must satisfy Eq. (7.2.20), which is a statement that at each location the virtual displacement vector must lie in a plane whose normal is $\hat{a}_i$. For holonomic constraints, this normal is perpendicular to the plane that is tangent to the associated constraint surface. However, the existence of a constraint surface, and whether the generalized position vector $\hat{r}$ and generalized velocity vector $\hat{v}$ lie in this plane, will not be crucial issues for the procedures that follow. For this reason, two terms used to describe nonholonomic constraints, *catastatic* when $b_i \equiv 0$ and *acatastatic* when $b_i \neq 0$, are introduced for reference purposes only.

EXAMPLE 7.3 An insect walks along the surface of an expanding spherical balloon whose radius is $r = ct$. The insect's position can be described in terms of Cartesian, cylindrical, or spherical coordinates, each of which constitute three generalized coordinates. Describe in terms of each set of variables the constraint imposed on the motion of the insect by the condition that it remain on the surface. Describe the constraint condition in each case as a velocity equation and as a configuration constraint. Identify the constraint surface in the configuration space, and the corresponding normal $\hat{a}_1$ to a point on that surface.

SOLUTION The intent of this example is to provide a visualization of the mathematical relations that we have developed. In practice, one seldom would actually examine a constraint surface. We begin with generalized coordinates that are Cartesian coordinates, $q_1 = x$, $q_2 = y$, $q_3 = z$. The origin is not specified, so we can select the center of the sphere. The distance from the origin to a point on the surface is the current value of r, so the configuration constraint corresponding to a set of Cartesian coordinates may be expressed as

$$f_1(x, y, z, t) = x^2 + y^2 + z^2 - c^2t^2 = 0. \qquad \triangleleft$$

We obtain the corresponding velocity constraint by differentiating this equation with respect to t, which gives

$$x\dot{x} + y\dot{y} + z\dot{z} - c^2 t = 0. \qquad \triangleleft$$

This is a linear velocity constraint whose Jacobian constraint coefficients are

$$a_{11} = x, \quad a_{12} = y, \quad a_{13} = z.$$

The generalized coordinates in this case are the physical coordinates, so the configuration space is the physical space. Thus the constraint surface is the expanding sphere. The Jacobian coefficients are the components of $\hat{a}_1$, so the normal to this constraint surface is

$$\hat{a}_1 = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3. \qquad \triangleleft$$

This is identical to the radial vector $\hat{r}$ from the origin to the point on the sphere represented by the current values of the generalized coordinates, which verifies that $\hat{a}_1$ is normal to the constraint surface.

Let z be the axial direction for cylindrical coordinates, so the second case uses $q_1 = R$, $q_2 = \theta$, $q_3 = z$. The transverse distance R is related to the Cartesian coordinates by $R = \left(x^2 + y^2\right)^{1/2}$, so the configuration constraint equation now is

$$f_1(R, \theta, z, t) = R^2 + z^2 - c^2t^2 = 0. \qquad \triangleleft$$

Differentiating this with respect to t yields the linear velocity constraint,

$$R\dot{R} + z\dot{z} - c^2 t = 0, \qquad \triangleleft$$

for which the Jacobian constraint coefficients are

$$a_{11} = R, \quad a_{12} = 0, \quad a_{13} = z.$$

The corresponding normal to the constraint surface at any instant is

$$\hat{a}_1 = R\hat{e}_1 + z\hat{e}_3. \qquad \triangleleft$$

To identify the nature of the configuration constraint surface we replace the physical variables with their equivalent generalized coordinate label, so we have $q_1^2 + q_3^2 = c^2t^2$. The axes of the configuration space are Cartesian, so the surface at any t now is a cylinder whose radius is ct and whose axial direction is $\hat{e}_2$. In this view, the normal direction is $\hat{a}_1 = q_1\hat{e}_1 + q_3\hat{e}_3$, which is the line from the origin to the projection of the current position onto the q_1q_3 plane. The fact that $\hat{a}_1 \cdot \hat{e}_2 = 0$ verifies that $\hat{a}_1$ actually is normal to the cylindrical constraint surface.

Let $x = y = z = 0$ be the origin of the spherical coordinates, so the third case is $q_1 = r$, $q_2 = \phi$, $q_3 = \theta$. The configuration constraint now is

$$f_1(r, \phi, \theta, t) = r^2 - c^2t^2 = 0, \qquad \triangleleft$$

which corresponds to the velocity constraint

$$r\dot{r} - c^2t = 0. \qquad \triangleleft$$

The Jacobian constraint coefficients in this case are

$$a_{11} = r, \quad a_{12} = a_{13} = 0,$$

so the normal to the constraint surface at any instant is

$$\hat{a}_1 = r\hat{e}_1.$$

In the $q_1q_2q_3$ Cartesian space of this set of variables, the configuration surface at any instant is the plane, $r = q_1 = ct$; the other two generalized coordinates are unconstrained. The normal to the constraint surface is $\hat{e}_1$, which is parallel to $\hat{a}_1$.

7.3 EVALUATION OF VIRTUAL DISPLACEMENTS

When we applied the principle of dynamic virtual work earlier in this chapter, the linkage we considered was a holonomic, one-degree-of-freedom system. The notion of a virtual movement was introduced there as the differential position shift that would occur if the single generalized coordinate was given an infinitesimal increment. The analysis of the corresponding point displacements was expedited by the geometrical simplicity of the system. The introduction of the configuration space significantly extends the variety of situations that can be considered. An important aspect of this generalization is the definition of a virtual displacement as the difference between a system's actual position at some instant and the position that would be obtained at that same instant if the forces acting on the system were altered. In the configuration space the virtual displacement $\delta\hat{r}$ appears as a vector from a point on the configuration path to a point on the variational path, with both points corresponding to the same t. The symbol δ serves to distinguish this displacement from the one that actually occurs. The vector $\delta\hat{r}$ is taken to be infinitesimal in order to use differential calculus to describe the difference of various quantities when they are evaluated on the actual and variational paths, so its components are differential increments in the generalized coordinate values; see Eq. (7.2.13). In words, this definition states that:

> *A virtual movement of a system is what would result if, with time held constant, the generalized coordinates were incremented by infinitesimal amounts. The increments δq_j are arbitrary values that may vary from instant to instant. Virtual displacements are the corresponding changes in the positions of points and angles of orientation.*

It will be necessary to evaluate the virtual work of the actual forces applied to the system. Such an evaluation is carried out in the physical world in which these forces are known. Our first task is to describe the amount by which points in the system move and bodies rotate when the generalized coordinates are incremented by virtual amounts. The similarity between $\delta\hat{r}$, whose components are δq_j, and the actual displacement $d\hat{r}$, whose components are dq_j corresponding to the actual position change during an infinitesimal interval, will be seen to be a significant aid to the evaluation of virtual displacements.

7.3.1 Analytical Method

If all generalized coordinates are measured from stationary reference locations, then knowledge of the q_n values will enable us to locate the position of any point in the system. However, in some cases the generalized coordinates might be defined as distances or angles measured relative to reference locations that undergo a known motion. In such situations we must know the specific value of t in order to place these references. Thus, in the most general case, the absolute position of any point depends on the generalized coordinates and time:

$$\bar{r}_{P/O} = \bar{r}_{P/O}(q_n, t)\,. \tag{7.3.1}$$

The analytical method for virtual displacements differentiates algebraic descriptions of position. Recall that time is held fixed at arbitrary t, whereas the generalized coordinate values are incremented by $\delta q_1, \ldots, \delta q_N$. Correspondingly, the chain rule for partial differentiation indicates that the virtual displacement of any point is given by

$$\delta\bar{r}_P = \sum_{j=1}^{N}\left[\frac{\partial}{\partial q_j}\bar{r}_{P/O}(q_n, t)\right]\delta q_j. \tag{7.3.2}$$

It is important to recognize that because t is arbitrary, the generalized coordinates have arbitrary values. Thus an evaluation of the virtual displacement of a specific point according to the analytical method begins by describing that point's position algebraically in terms of the generalized coordinates. Any geometrical variables that are not generalized coordinates must be eliminated from that description. Equation (7.3.2) may then be applied directly.

It will be necessary in many cases to express the amount by which angles that define the orientation of bodies, but are not generalized coordinates, change in a virtual movement. The analytical method for virtual displacement requires that we first establish the functional dependence of those angles on the generalized coordinates, for which trigonometric relations such as the laws of sines and cosines often are useful. The result is that, if β represents an angle of orientation, then we have determined the functional dependence $\beta = f(q_n, t)$. The virtual rotation is then found by the chain rule to be

$$\delta\beta = \sum_{j=1}^{N} \left[\frac{\partial}{\partial q_j} f(q_n, t) \right] \delta q_j. \tag{7.3.3}$$

Before we implement the analytical method it is instructive to develop Eq. (7.3.2) in a manner that relates to the earlier discussion of alternative paths in the configuration space. Consider a system at time $t - dt$. In the actual movement of the system from this time to time t the generalized coordinates are incremented by the infinitesimal amounts dq_j. Time is not constant in the true displacement, so the chain rule for the differential displacement gives

$$d\bar{r}_P = \sum_{j=1}^{N} \left[\frac{\partial}{\partial q_j} \bar{r}_{P/O}(q_n, t) \right] dq_j + \frac{\partial}{\partial t} \bar{r}_{P/O}(q_n, t)\, dt. \tag{7.3.4}$$

The values of dq_j appearing here are those for the true motion.

Suppose we were to alter at time $t - dt$ and onward the forces applied to the system. This would cause the system to execute a different motion subsequent to time $t - dt$. (This different motion corresponds to a variational path in the configuration space that branches off from the true path at time $t - dt$.) The set of values of dq_j in this alternative motion would be different from those of the true motion. Let superscript a denote a quantity associated with this alternative force system. Thus the displacement corresponding to altering the force system at $t - dt$ and onward is given by

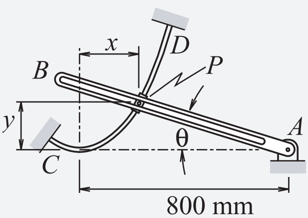
$$d\bar{r}_P^{(a)} = \sum_{j=1}^{N} \left[\frac{\partial}{\partial q_j} \bar{r}_{P/O}(q_n, t) \right] dq_j^{(a)} + \frac{\partial}{\partial t} \bar{r}_{P/O}(q_n, t)\, dt. \tag{7.3.5}$$

The true position of point P at time t is given by $\bar{r}_{P/O}(q_n, t - dt) + d\bar{r}_P$, whereas the position at the same instant of this point in the alternative motion would be $\bar{r}_{P/O}(q_n, t - dt) + d\bar{r}_P^{(a)}$. By definition, the virtual displacement of point P is the difference between its position in the actual motion and the position it would have at the same instant with a slightly different set of forces. Thus the virtual displacement of the point at time t is given by $\delta\bar{r}_P = d\bar{r}_P^{(a)} - d\bar{r}_P$. Using Eqs. (7.3.4) and (7.3.5) to describe the differential displacements leads to

$$\delta\bar{r}_P = \sum_{j=1}^{N} \left[\frac{\partial}{\partial q_j} \bar{r}_{P/O}(q_n, t) \right] \left(dq_j^{(a)} - dq_j \right). \tag{7.3.6}$$

By definition, the difference between a generalized coordinate value on the actual and variational paths is δq_j, so the preceding equation is identical to Eq. (7.3.2). This strengthens the earlier observation that the study of virtual movement and its associated effects highlights the differences between alternative responses that could possibly occur under the action of different sets of forces.

EXAMPLE 7.4 Collar P slides over guide bar CD that is bent in the shape of a parabola given by $y = x^2/300$, where x and y are measured in millimeters. This system has one degree of freedom. Determine the virtual displacement of the collar and the virtual rotation of bar AB corresponding to the following alternatives: (a) x is a single unconstrained generalized coordinate; (b) θ is a single unconstrained generalized coordinate; (c) x and θ are two constrained generalized coordinates.



Example 7.4

SOLUTION In addition to demonstrating the procedure for implementing the analytical method, this example illustrates how the choice of generalized coordinates can significantly alter the analysis. That this system has one degree of freedom is demonstrated by the fact that, if x is given, then y is known from the equation of bar CD. Also, the horizontal distance from pivot A to the collar is $0.8 - x$ m, which then sets the value of θ. Thus we have

$$y = \frac{x^2}{0.3}, \text{ where } x \text{ and } y \text{ are in meters,}$$

$$\tan\theta = \frac{y}{0.8 - x} = \frac{x^2}{0.24 - 0.3x}.$$

In the first approach, $q_1 = x$. Point O is the origin at which $x = y = 0$. We express the position of point P and angle θ as functions of x:

$$\bar{r}_{P/O} = x\bar{i} + \frac{x^2}{0.3}\bar{j}, \quad \theta = \tan^{-1}\left(\frac{x^2}{0.24 - 0.3x}\right).$$

Differentiating each with respect to x yields

$$\delta\bar{r}_P = \left(\bar{i} + \frac{x}{0.15}\bar{j}\right)\delta x,$$

$$\overline{\delta\theta}_{AB} = -\delta\theta\bar{k} = -\frac{d}{dx}\left[\tan^{-1}\left(\frac{x^2}{0.24 - 0.3x}\right)\right] \quad \triangleleft$$

$$\times\ \delta x\bar{k} = -\left[\frac{2x\,(0.24 - 0.3x) + 0.3x^2}{(0.24 - 0.3x)^2 + x^4}\right]\bar{k}\delta x.$$

The second formulation requires that we express $\bar{r}_{P/O}$ in terms of θ. The inverse of the preceding expression for θ is a quadratic equation for x at a given θ,

$$x^2 + (0.3\tan\theta)\,x - 0.24\tan\theta = 0,$$

from which we find

$$x = -0.15\tan\theta + \left[(0.15\tan\theta)^2 + 0.24\tan\theta\right]^{1/2},$$

where the positive sign is associated with the square root because $x > 0$. We differentiate this with respect to θ to find

$$\delta x = \left\{-0.15 + \frac{0.0225\tan\theta + 0.12}{\left[(0.15\tan\theta)^2 + 0.24\tan\theta\right]^{1/2}}\right\}\frac{1}{(\cos\theta)^2}\delta\theta.$$

The virtual displacement in the y direction is given by

$$\delta y = \left(\frac{dy}{dx}\right)\delta x = \frac{x}{0.15}\delta x.$$

We replace x and δx with their representations in terms of the generalized coordinate θ, which yields

$$\begin{aligned}\delta\bar{r}_P &= \delta x\bar{i} + \delta y\bar{j} = \left(\bar{i} + \frac{x}{0.15}\bar{j}\right)\delta x \\ &= \left(\bar{i} + \frac{\left\{-0.15\tan\theta + \left[(0.15\tan\theta)^2 + 0.24\tan\theta\right]^{1/2}\right\}}{0.15}\bar{j}\right) \\ &\quad\times\left\{-0.15 + \frac{0.045\tan\theta + 0.24}{\left[(0.15\tan\theta)^2 + 0.24\tan\theta\right]^{1/2}}\right\}\frac{1}{(\cos\theta)^2}\delta\theta.\end{aligned} \quad \triangleleft$$

The virtual rotation of bar AB is $\overline{\delta\theta}_{AB} = -\delta\theta\bar{k}$, so that evaluation is simpler than the first case. Nevertheless, there is no doubt that using θ as the generalized coordinate leads to much more complicated mathematical relations than those obtained when x is used.

Using $q_1 = x$, $q_2 = \theta$ as constrained generalized coordinates actually is the simplest approach. These variables are related by the configuration constraint equation obtained at the outset of this analysis. What makes the use of constrained generalized coordinates easier is that, after their constraint equation is identified, the coordinates are considered to be independent until all elements of the equations of motion are synthesized. Thus the position of the collar is described as it was in the first case, $\bar{r}_{P/O} = x\bar{i} + \left(x^2/0.3\right)\bar{j}$, and θ gives the orientation of bar AB. Then the virtual displacement and rotation are given by

$$\delta\bar{r}_P = \left(\bar{i} + \frac{x}{0.15}\bar{j}\right)\delta x,$$
$$\overline{\delta\theta}_{AB} = -\delta\theta\bar{k}. \quad \triangleleft$$

A conclusion to be carried away from this example is that some choices for generalized coordinates are better than others, but that any set can lead to a successful analysis if one perseveres. Also, in certain circumstances it might be advantageous to use constrained generalized coordinates.

7.3.2 Kinematical Method

The analytical method for evaluating virtual displacements requires that we explicitly differentiate the representation of position as a function of the generalized coordinates. Simple systems, such as those whose parts form isosceles triangles or right angles, are relatively easy to describe geometrically, and the resulting expressions are likely to be quite easy to differentiate. However, increasing the complexity of the geometry can lead to substantial complication. One of the primary reasons for developing kinematical formulas is to address such difficulties. Those formulas represent standard derivatives.

The essence of the kinematical approach for virtual displacement is to exploit similarities between the mathematical forms of velocity and virtual displacement. Equation (7.3.4) expresses the true differential displacement of a point in terms of the generalized coordinates and their increments dq_j over an infinitesimal time interval. Dividing that expression by dt gives an expression for the velocity of a point, which is accompanied in the following by Eq. (7.3.2) to highlight the similarity between velocity and virtual displacement:

$$\begin{aligned} \bar{v}_P &= \sum_{j=1}^{N}\left[\frac{\partial}{\partial q_j}\bar{r}_{P/O}(q_n, t)\right]\dot{q}_j + \frac{\partial}{\partial t}\bar{r}_{P/O}(q_n, t), \\ \delta\bar{r}_P &= \sum_{j=1}^{N}\left[\frac{\partial}{\partial q_j}\bar{r}_{P/O}(q_n, t)\right]\delta q_j. \end{aligned} \tag{7.3.7}$$

These equations indicate that $\bar{v}_P$ depends linearly on the generalized velocities $\dot{q}_j$, whereas the virtual displacement depends linearly on the virtual increments δq_j. The significant aspect of this similarity is that the coefficient associated with a specific $\dot{q}_j$ is the same as the coefficient of the corresponding δq_j and that this coefficient is a vector function of generalized coordinates and time. The fact that the coefficients are derivatives of the position is unimportant, so we denote them as $\bar{v}_{Pj}(q_n, t)$ and let $\bar{v}_{pt}(q_n, t)$

denote the partial time derivative term. Then the general forms for the velocity and virtual displacement of a point are

$$\begin{aligned}
\bar{v}_P &= \sum_{j=1}^{N} \bar{v}_{Pj}(q_n, t)\,\dot{q}_j + \bar{v}_{pt}(q_n, t), \\
\delta\bar{r}_P &= \sum_{j=1}^{N} \bar{v}_{Pj}(q_n, t)\,\delta q_j, \\
\bar{v}_{Pj} &= \frac{\partial \bar{r}_P}{\partial q_j}.
\end{aligned} \tag{7.3.8}$$

The methodology for implementing the kinematical method follows from the fact that both of the preceding relations are always valid, regardless of how one goes about evaluating the velocity. Thus,

Description of a virtual displacement of a point P in the kinematical method begins with a velocity analysis of the system, for which the system must be at an arbitrary position, corresponding to algebraic values of all generalized coordinates and time. The result of that analysis should be an expression for $\bar{v}_P$ that depends solely on the generalized coordinates, generalized velocities, and time, which requires elimination of any other variables. To obtain an expression for $\delta\bar{r}_P$ from the result for $\bar{v}_P$, one merely replaces each $\dot{q}_j$ factor with the corresponding δq_j, and drops any term that does not contain a generalized velocity. (The latter type of term represents the velocity of the point that would be obtained even if the generalized coordinates were constant. Such a term will not arise if the generalized coordinates are measured from a fixed reference location.)

As a simple illustration of this procedure, consider a single particle whose position is defined by spherical coordinates. If that particle moves freely in space then the spherical coordinates can be used as three unconstrained generalized coordinates. The velocity in spherical coordinates is $\bar{v}_P = \dot{r}\bar{e}_r + r\dot{\phi}\bar{e}_\phi + r\dot{\theta}\sin\phi\bar{e}_\theta$. There is no $\bar{u}_p$ term here, because each term contains a generalized velocity factor. To obtain an expression for $\delta\bar{r}_P$, we replace $\dot{r}$ with δr, $\dot{\phi}$ with $\delta\phi$, and $\dot{\theta}$ with $\delta\theta$. Thus the virtual displacement of this particle would be $\delta\bar{r}_P = (\bar{e}_r)\,\delta r + (r\bar{e}_\phi)\,\delta\phi + (r\sin\phi\bar{e}_\theta)\,\delta\theta$. Now suppose that, instead of a particle moving freely in space, point P is the insect in Example 7.3. The radial distance in that case is controlled to be $r = ct$, so only ϕ and θ are generalized coordinates. The velocity in this case is $\bar{v}_P = c\bar{e}_r + r\dot{\phi}\bar{e}_\phi + r\dot{\theta}\sin\phi\bar{e}_\theta$. The radial term does not contain a generalized velocity, so it is ignored. Thus, replacing generalized velocities with the corresponding virtual increments in this case leads to $\delta\bar{r}_P = (r\bar{e}_\phi)\,\delta\phi + (r\sin\phi\bar{e}_\theta)\,\delta\theta$.

The kinematical method may be applied to describe the three-dimensional virtual rotation of a body. Such an evaluation would begin by evaluating the angular velocity $\bar{\omega}_n$ of nth body in terms of a convenient set of components, for example, a body-fixed xyz coordinate system. In this description the only allowable variables are the generalized coordinates and velocities, and t. The rates of rotation occur linearly in the angular velocity. Consequently, $\bar{\omega}_n$ will depend linearly on the generalized velocities, so its general form will be

$$\bar{\omega}_n = \sum_{j=1}^{N} \bar{\omega}_{nj}\left(q_n, t\right) \dot{q}_j + \bar{\omega}_{nt}\left(q_n, t\right), \tag{7.3.9}$$

where $\bar{\omega}_{nt}\left(q_n, t\right)$ represents the angular velocity when all generalized velocities are zero. The virtual rotation is obtained by dropping $\bar{\omega}_{nt}$ and replacing each $\dot{q}_j$ with the corresponding δq_j, so that

$$\overline{\delta\theta}_n = \sum_{j=1}^{N} \bar{\omega}_{nj}\left(q_n, t\right) \delta q_j. \tag{7.3.10}$$

Virtual rotation arises in any situation in which it is convenient to use relative motion concepts to relate the velocity of two points. The relative velocity formula is

$$\bar{v}_P = \bar{v}_{O'} + \left(\bar{v}_P\right)_{xyz} + \bar{\omega} \times \bar{r}_{P/O'}, \tag{7.3.11}$$

where $\bar{\omega}$ is the angular velocity of xyz and $\left(\bar{v}_P\right)_{xyz} = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k}$. We then obtain the virtual displacement of point P by replacing all rate variables with virtual increments, which leads to

$$\delta\bar{r}_P = \delta\bar{r}_{O'} + \delta x\bar{i} + \delta y\bar{j} + \delta z\bar{k} + \overline{\delta\theta} \times \bar{r}_{B/A}. \tag{7.3.12}$$

This expression may be adapted to a variety of situations, but in any case it is necessary that the only variables that appear are the q_j, $\dot{q}_j$, and δq_j. In the special case in which points P and O' belong to a rigid body, the values of δx, δy, and δz are identically zero.

An example of this representation arises when the orientation of an axisymmetric rigid body is described by the three Eulerian angles. Let $x'y'z'$ be a reference frame that executes the precession ψ and nutation θ of the body, but not the spin, with the z' axis defined to coincide with the body's axis of symmetry and y' aligned with the line of nodes. The angular velocity of the body is described by the first of Eqs. (4.2.11). Correspondingly, virtual rotation would be

$$\overline{\delta\theta} = \left(-\sin\theta\bar{i}' + \cos\theta\bar{k}'\right)\delta\psi + \bar{j}'\delta\theta + \bar{k}'\delta\phi. \tag{7.3.13}$$

If the body is rotating freely, then each Eulerian angle may be used as a generalized coordinate. In that case, replacing the rate variables with virtual increments in the relation between the velocity of two points in a rigid body leads to a relation between the virtual displacements of the points:

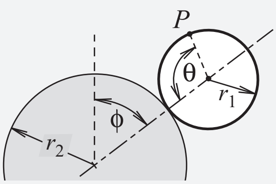
$$\delta\bar{r}_B = \delta\bar{r}_A + \left[\left(-\sin\theta\bar{i}' + \cos\theta\bar{k}'\right)\delta\psi + \left(\bar{j}'\right)\delta\theta + \left(\bar{k}'\right)\delta\phi\right] \times \bar{r}_{B/A}. \tag{7.3.14}$$

Now suppose that the precession rate is controlled by a servomotor, such that $\dot{\psi}$ is a specified function of t. This removes ψ from the list of generalized coordinates. To use the preceding expression in that case we merely set $\delta\psi = 0$, which reduces the preceding relation between virtual displacements to

$$\delta\bar{r}_B = \delta\bar{r}_A + \left[\left(\bar{j}'\right)\delta\theta + \left(\bar{k}'\right)\delta\phi\right] \times \bar{r}_{B/A}. \tag{7.3.15}$$

Equations (7.3.8) and (7.3.10) are also useful as visualization tools. They indicate that any virtual displacement or rotation is a superposition of individual contributions resulting from incrementing each generalized coordinate with the others held stationary. The virtual displacement in that case is like the velocity that would be obtained if only that generalized coordinate were variable. This knowledge enables us use our experience and understanding of kinematics to qualitatively assess the correctness of a virtual displacement we have evaluated.

EXAMPLE 7.5 The disk rolls without slipping over the stationary cylinder. The rotation ϕ of the line of centers from vertical is selected as the generalized coordinate for the disk. Point P contacted the cylinder when $\phi = 0$. Determine the virtual displacement of this point.



Example 7.5

SOLUTION The kinematical method is particularly useful for systems in which bodies roll without slipping, as seen here. We begin by deriving an expression for the velocity of point P as a function of the generalized coordinate ϕ and generalized velocity $\dot{\phi}$. This requires that we determine the angle θ, which locates point P relative to the radial line, as a function of ϕ. According to the no-slip condition, the arc length from the current point of contact to point P along the rolling disk must equal the arc length from the contact point to the top of the stationary cylinder, which gives

$$\theta = \frac{r_2}{r_1}\phi.$$

A convenient global coordinate system has its y axis radially outward through the center of the disk and its x axis down and to the right. The angular velocity of the disk is $\omega\bar{k}$ for plane motion. To express ω in terms of the generalized coordinate ϕ, we use the no-slip condition for velocity and the fact that the center C follows a circular path. The radius of this path is $r_1 + r_2$, and $\dot{\phi}$ is the angular speed of the radial line to that point. Thus we have

$$\bar{v}_C = \omega\bar{k} \times r_1\bar{j} = -r_1\omega\bar{i} = (r_1 + r_2)\,\dot{\phi}\bar{i},$$

from which we obtain

$$\omega = -\frac{(r_1 + r_2)}{r_1}\dot{\phi}.$$

The angular motion and the velocity of the center have been expressed in terms of the generalized coordinate, so we proceed to the unconstrained point P. The position of this point is given by

$$\bar{r}_{P/C} = r_1\left(-\sin\theta\,\bar{i} - \cos\theta\,\bar{j}\right) = r_1\left[-\sin\left(\frac{r_2}{r_1}\phi\right)\bar{i} - \cos\left(\frac{r_2}{r_1}\phi\right)\bar{j}\right],$$

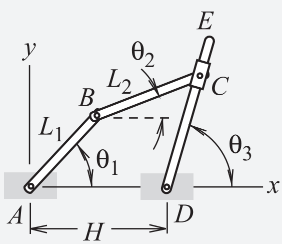
so that

$$\bar{v}_P = \bar{v}_C + \omega\bar{k}\times\bar{r}_{P/C}$$
$$= (r_1+r_2)\dot{\phi}\bar{i} + \left[-\frac{(r_1+r_2)}{r_1}\dot{\phi}\right]\left[r_1\cos\left(\frac{r_2}{r_1}\phi\right)\bar{i} - r_1\sin\left(\frac{r_2}{r_1}\phi\right)\bar{j}\right].$$

The kinematical method indicates that $\bar{v}_P$ should be replaced with $\delta\bar{r}_P$ and $\dot{\phi}$ with $\delta\phi$. Furthermore, we observe that $\bar{v}_P = 0$ if $\dot{\phi} = 0$, which means that there are no terms to drop. Thus we find that the virtual displacement of point P is

$$\delta\bar{r}_P = (r_1+r_2)\left\{\left[1-\cos\left(\frac{r_2}{r_1}\phi\right)\right]\bar{i} + \sin\left(\frac{r_2}{r_1}\phi\right)\bar{j}\right\}\delta\phi. \qquad \lhd$$

EXAMPLE 7.6 The angles θ_1, θ_2, and θ_3 locate the links of the mechanism, but there is a geometrical relation that these angles must satisfy. Determine the virtual displacement of collar C, which slides along bar DE under each of the following conditions: (a) θ_1 and θ_2 are unconstrained generalized coordinates; (b) θ_1 and θ_3 are unconstrained generalized coordinates; and (c) θ_1 is driven by a motor such that it is a known function of time, and θ_2 is an unconstrained generalized coordinate; and (d) θ_3 is driven by a motor such that it is a known function of time, and θ_1 is an unconstrained generalized coordinate.



Example 7.6

SOLUTION The purpose here is to illustrate the variety of conditions that can arise in a system and the significant effect that these conditions can have on the formulation of virtual displacement. In Part (a) we have $q_1 = \theta_1$ and $q_2 = \theta_2$. These variables uniquely locate the collar, so the fact that θ_3 depends on the generalized coordinates is irrelevant to the evaluation of the virtual displacement of the collar. We follow the kinematical method by describing the velocity of the collar in terms of the generalized coordinates:

$$\bar{v}_C = \bar{v}_B + \bar{\omega}_{BC}\times\bar{r}_{C/B} = \dot{\theta}_1\bar{k}\times\bar{r}_{B/A} + \dot{\theta}_2\bar{k}\times\bar{r}_{C/B}$$
$$= -\left(L_1\dot{\theta}_1\sin\theta_1 + L_2\dot{\theta}_2\sin\theta_2\right)\bar{i} + \left(L_1\dot{\theta}_1\cos\theta_1 + L_2\dot{\theta}_2\cos\theta_2\right)\bar{j}. \qquad (1)$$

We obtain the virtual displacement by replacing the rate variables with the corresponding virtual increments. It is useful to group the coefficients of $\delta\theta_1$ and $\delta\theta_2$, so we find

$$\begin{aligned}\delta\bar{r}_C = &\left(-L_1\dot{\theta}_1 \sin\theta_1 \bar{i} + L_1\dot{\theta}_1 \cos\theta_1 \bar{j}\right)\delta\theta_1 \\ &+ \left(-L_2\dot{\theta}_2 \sin\theta_2 \bar{i} + L_2\dot{\theta}_2 \cos\theta_2 \bar{j}\right)\delta\theta_2.\end{aligned} \qquad (2) \triangleleft$$

In Part (b) we have $q_1 = \theta_1$ and $q_2 = \theta_3$. We cannot describe the position of the collar C in terms of only the generalized coordinates without considering how θ_2 depends on θ_1 and θ_3. To that end we form the tangent of θ_3 in terms of horizontal and vertical distances constructed along bars AB and BC. This gives

$$\tan\theta_3 = \frac{L_1 \sin\theta_1 + L_2 \sin\theta_2}{L_1 \cos\theta_1 + L_2 \cos\theta_2 - H},$$

which may be written as

$$\sin\theta_2 = \tan\theta_3 \cos\theta_2 + \tan\theta_3 \left(\frac{L_1}{L_2}\cos\theta_1 - \frac{H}{L_2}\right) - \frac{L_1}{L_2}\sin\theta_1. \qquad (3)$$

We need to solve this relation for θ_2 in terms of θ_1 and θ_3, which we can do by using symbolic software or by using $\sin\theta_2 \equiv \left[1 - (\cos\theta_2)^2\right]^{1/2}$ to obtain a quadratic equation for $\cos\theta_2$. The specific solution is unimportant for the present purpose, so we merely denote it in functional form as

$$\theta_2 = f(\theta_1, \theta_3). \qquad (4)$$

We proceed to the analysis of $\bar{v}_C$, for which we recognize that we need to determine $\dot{\theta}_2$ in terms of the generalized coordinates and velocities. We could obtain this result by differentiating Eq. (4), but the function is quite complicated. Instead we carry out a velocity analysis of the collar, based on its being constrained to move radially outward at an unknown rate u relative to bar DE. This means that $\bar{v}_C = u\bar{e}_{E/D} + \bar{\omega}_{DE} \times \bar{r}_{C/D}$. We describe $\bar{r}_{C/D}$ in terms of θ_1 and θ_2 to avoid introducing the distance from pin D to the collar as another geometrical variable to eliminate. Comparing this viewpoint for $\bar{v}_C$ with Eq. (1) gives

$$\begin{aligned}\bar{v}_C = &\, u\left(\cos\theta_3 \bar{i} + \sin\theta_3 \bar{j}\right) + \dot{\theta}_3 \bar{k} \times \left[(L_1 \cos\theta_1 + L_2 \cos\theta_2)\,\bar{i}\right. \\ &\left. + (L_1 \sin\theta_1 + L_2 \sin\theta_2)\,\bar{j}\right] \\ = &-\left(L_1\dot{\theta}_1 \sin\theta_1 + L_2\dot{\theta}_2 \sin\theta_2\right)\bar{i} + \left(L_1\dot{\theta}_1 \cos\theta_1 + L_2\dot{\theta}_2 \cos\theta_2\right)\bar{j}.\end{aligned}$$

Matching like components in the preceding equation gives two simultaneous equations that may be solved for u and $\dot{\theta}_2$ in terms of θ_1, θ_2, θ_3, $\dot{\theta}_1$, and $\dot{\theta}_3$. Equation (4) describes θ_2. Thus the result of this kinematical analysis has the form

$$\dot{\theta}_2 = g_1(\theta_1, \theta_3)\,\dot{\theta}_1 + g_3(\theta_1, \theta_3)\,\dot{\theta}_3. \qquad (5)$$

The detailed expressions are intricate, so we merely monitor how the analysis builds on them. Substitution of Eq. (5) into Eq. (1) gives $\bar{v}_C$ in terms of the generalized velocities for this case:

$$\bar{v}_C = \{-[L_1 \sin\theta_1 + L_2 g_1(\theta_1,\theta_3)\sin(f(\theta_1,\theta_3))]\,\bar{i} + [L_1\cos\theta_1 + L_2 g_1(\theta_1,\theta_3)\cos(f(\theta_1,\theta_3))]\,\bar{j}\}\,\dot{\theta}_1 + \{-L_2 g_3(\theta_1,\theta_3)\sin(f(\theta_1,\theta_3))\,\bar{i} + L_2 g_3(\theta_1,\theta_3)\cos(f(\theta_1,\theta_3))\,\bar{j}\}\,\dot{\theta}_3.$$

To obtain the virtual displacement of the collar we replace the generalized velocities with the corresponding virtual increments, which gives

$$\delta\bar{r}_C = \{-[L_1 \sin\theta_1 + L_2 g_1(\theta_1,\theta_3)\sin(f(\theta_1,\theta_3))]\,\bar{i} + [L_1\cos\theta_1 + L_2 g_1(\theta_1,\theta_3)\cos(f(\theta_1,\theta_3))]\,\bar{j}\}\,\delta\theta_1 + \{-L_2 g_3(\theta_1,\theta_3)\sin(f(\theta_1,\theta_3))\,\bar{i} + L_2 g_3(\theta_1,\theta_3)\cos(f(\theta_1,\theta_3))\,\bar{j}\}\,\delta\theta_3. \tag{6}$$

It is obvious that analyzing $\delta\bar{r}_C$ with θ_1 and θ_3 as generalized coordinates is much more difficult than it is if θ_1 and θ_2 are the generalized coordinates. The same would be true for most other steps ultimately leading to the equations of motion. Thus it is wise to assess at the outset of a solution the difficulty of eliminating variables that are not generalized coordinates.

The situation in Part (c) is like that of Part (a), except that θ_1 is specified, rather than a generalized coordinate. Thus the terms in Eq. (1) that contain $\dot{\theta}_1$ correspond to the $\bar{v}_{P_t}$ term in Eq. (7.3.8). In accord with the kinematical method, in Eq. (1) we replace $\bar{v}_C$ with $\delta\bar{r}_C$ and $\dot{\theta}_2$ with $\delta\theta_2$, and ignore the $\dot{\theta}_1$ term. The resulting virtual displacement is

$$\delta\bar{r}_C = \left(-L_2\dot{\theta}_2 \sin\theta_2 \bar{i} + L_2\dot{\theta}_2\cos\theta_2\,\bar{j}\right)\delta\theta_2. \qquad \triangleleft$$

This is the same as Eq. (2) when $\delta\theta_1 = 0$, which is consistent with the definition of a virtual displacement as corresponding to fixed t, so that θ_1 does not change.

Similar to the preceding case, the analysis for Part (d) may be performed by referring back to Part (b). Because θ_3 is not a generalized coordinate in this case, we set $\delta\theta_3 = 0$ in Eq. (6), which reduces the virtual displacement to

$$\delta\bar{r}_C = \{-[L_1 \sin\theta_1 + L_2 g_1(\theta_1,\theta_3)\sin(f(\theta_1,\theta_3))]\,\bar{i} + [L_1\cos\theta_1 + L_2 g_1(\theta_1,\theta_3)\cos(f(\theta_1,\theta_3))]\,\bar{j}\}\,\delta\theta_1. \qquad \triangleleft$$

An overview of the cases considered here shows that eliminating constrained variables can lead to significant complications. An alternative not listed in the problem statement is to use all three angles as constrained generalized coordinates, rather than trying to eliminate one variable. In each case a constraint equation relating the angles, specifically Eq. (3) or its equivalent, would need to be enforced as a separate condition. Specification of the rotation of θ_1 or θ_3 as a function of time would lead to an additional constraint equation.

7.4 GENERALIZED FORCES

The selection of a set of generalized coordinates, and the evaluation of the virtual displacements in terms of those quantities, are primary aspects of the Lagrangian approach to the derivation of equations of motion for a system. It is necessary to recognize what parameters are appropriate to the kinematical description. By doing so, we create the model on which the rest of the analysis will be based. The kinematics phase of the formulation is essentially complete when the physical velocities and virtual displacements have been related to the generalized coordinates. The aspects of the formulation that are concerned with the role of forces and inertia are more straightforward. The first task is to represent the effect of the forces exerted on a system.

7.4.1 Definition of Generalized Forces

Virtual work is done by the actual forces when a system is given a virtual movement. Because time is constant in such a movement, forces and couples do not change in magnitude or direction. Let $\delta\bar{r}_n$ denote the virtual displacement of the point where force $\bar{F}_n$ acts. Adding the virtual work done by each load gives

$$\delta W = \sum_n \bar{F}_n \cdot \delta\bar{r}_n. \tag{7.4.1}$$

A virtual displacement is related to the increments of the generalized coordinates by Eqs. (7.3.8), whose substitution gives

$$\delta W = \sum_n \bar{F}_n \cdot \sum_{j=1}^{N} \bar{v}_{nj}\delta q_j \equiv \sum_n \bar{F}_n \cdot \sum_{j=1}^{N} \frac{\bar{r}_n}{\partial q_j}\delta q_j. \tag{7.4.2}$$

We move the sum over the generalized coordinates outside the sum over the forces, which allows us to collect coefficients of each δq_j. The result is

$$\boxed{\delta W = \sum_{j=1}^{N} Q_j \delta q_j,} \tag{7.4.3}$$

where

$$Q_j = \sum_n \bar{F}_n \cdot \bar{v}_{nj}\,(q_k, t) \equiv \sum_n \bar{F}_n \cdot \frac{\partial \bar{r}_n}{\partial q_j}. \tag{7.4.4}$$

The terminology is to say that Q_j are *generalized forces.*

To describe situations in which various couples $\bar{\Gamma}_k$ are applied to the system, we recall that the differential work done by a couple is obtained from a dot product with the differential rotation. Let $\overline{\delta\theta}_k$ be the virtual rotation of the body to which couple $\bar{\Gamma}_k$ is applied. The virtual work is the sum of the contributions of each force and each couple:

$$\boxed{\delta W = \sum_n \bar{F}_n \cdot \delta\bar{r}_n + \sum_k \bar{\Gamma}_k \cdot \overline{\delta\theta}_k.} \tag{7.4.5}$$

Substitution of the kinematical descriptions of virtual displacement and rotation, Eqs. (7.3.8) and (7.3.10), respectively, into the preceding equation for δW gives

$$\delta W = \sum_n \bar{F}_n \cdot \sum_{j=1}^{N} \bar{v}_{nj}(q_n, t)\,\delta q_j + \sum_k \bar{\Gamma}_k \cdot \sum_{j=1}^{N} \bar{\omega}_{kj}(q_n, t)\,\delta q_j. \tag{7.4.6}$$

Moving the sum over the generalized coordinates outside the sum over the forces or couples still allows us to collect coefficients of each δq_j, so the virtual work still must have the standard form in Eq. (7.4.3). We match that expression to Eq. (7.4.6) and recall that the δq_j values may be arbitrarily selected. Consequently, like coefficients of δq_j must match, which leads to the conclusion that

$$Q_j = \sum_n \bar{F}_n \cdot \bar{v}_{nj}(q_n, t) + \sum_k \bar{\Gamma}_k \cdot \bar{\omega}_{Pj}(q_n, t). \tag{7.4.7}$$

One reason for referring to Q_j as a generalized force may be recognized from the case of free motion of a particle. If the Cartesian coordinates X, Y, and Z measured relative to a fixed reference frame are selected as the set of q_j, then $\delta\bar{r}_P = \delta X\bar{I} + \delta Y\bar{J} + \delta Z\bar{K}$. Correspondingly the virtual work done by the resultant force $\Sigma\bar{F}$ acting on this particle is $\delta W = \Sigma F_X \delta X + \Sigma F_Y \delta Y + \Sigma F_Z \delta Z$. Comparing this with the standard form in Eq. (7.4.3) shows that $Q_1 = \Sigma F_X$, $Q_2 = \Sigma F_Y$, and $Q_3 = \Sigma F_Z$. In other words, each generalized force is the net force acting in the direction of the corresponding generalized coordinate. Now suppose we use spherical coordinates r, ϕ, and θ as the generalized coordinates for the particle. Converting the velocity in spherical coordinates to virtual displacement according to the kinematical method gives $\delta\bar{r}_P = \delta r\bar{e}_r + r\delta\phi\bar{e}_\phi + r(\sin\phi)\,\delta\theta\bar{e}_\theta$. Correspondingly, the virtual work done by $\Sigma\bar{F}$ is $\delta W = \Sigma F_r \delta r + \Sigma F_\phi r\delta\phi + \Sigma F_\theta r(\sin\phi)\,\delta\theta$. Matching this expression to Eq. (7.4.3) gives $Q_1 = \Sigma F_r$, $Q_2 = r\Sigma F_\phi$, and $Q_3 = r\sin\phi\,\Sigma F_\theta$. The first generalized coordinate is the net force in the radial direction, but the other two are moments that represent the net effect causing ϕ and θ to change positively. In any situation we will find that

> *Each generalized force is the net effect of the force system to cause the associated generalized coordinate to increase.*

Note that this observation is confirmed by the units of the generalized force. The units of work are force times length. Thus, if q_j has units of length, then Q_j has units of force, whereas when q_j is an angle, then Q_j has moment units, force times length.

Another reason to refer to the coefficient of δq_j in Eq. (7.4.3) as a generalized force may be seen by returning to the configuration space. A system's position appears there as a point following its configuration path. Let us define a force vector $\hat{Q}$ in the configuration space such that the virtual work it does equals δW for the actual forces acting on the system. Equation (7.2.13) describes the virtual displacement $\delta\hat{r}$ in the configuration space. The virtual work done by $\hat{Q}$ therefore is

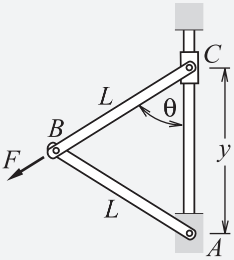
$$\delta W = \hat{Q} \cdot \delta\hat{r} = (Q_1\hat{e}_1 + \cdots + Q_N\hat{e}_N) \cdot (\delta q_1\hat{e}_1 + \cdots + \delta q_N\hat{e}_N). \tag{7.4.8}$$

Because the configuration space is Cartesian, the unit vectors are orthonormal. Hence evaluating the dot product yields the same expression for δW as Eq. (7.4.3). From this we conclude that in the configuration space a generalized force Q_j is the force component acting in the direction of the corresponding q_j.

We seldom use Eq. (7.4.7) to evaluate the generalized forces for a specific system. Instead, we follow the same procedure as the one that led to that expression. Specifically, we first evaluate the virtual displacement of each point where a force is applied and the virtual rotation of each body that carries a torque. We then form the virtual work of these forces and couples by following Eq. (7.4.5). Matching the coefficient of each δq_j in the actual expression for δW to corresponding coefficient in the standard form of δW, Eq. (7.4.3), leads to identification of each generalized force.

In principle, the evaluation of virtual work needs to include each force and couple that acts on a system, whether it is external or internal. However, we will see in the following sections that two special classes of forces, constraint forces and conservative forces, may be accounted for without explicitly evaluating their contribution to δW.

EXAMPLE 7.7 Force F acts parallel to link AB. Alternative choices for the generalized coordinate are the angle θ or the distance y. Determine the generalized force corresponding to each selection.



Example 7.7

SOLUTION The basic procedure for evaluating generalized forces is covered here, as well as the association between the nature of a generalized force and the definition of the generalized coordinate. The virtual work done by $\bar{F}$ is $\bar{F} \cdot \delta \bar{r}_B$, and the choice of generalized coordinate affects the description of both vectors. Let the x axis be horizontal to the left and the y axis be the upward vertical. When θ is the generalized coordinate, the analytical method for virtual displacement gives

$$\bar{r}_{B/A} = L \sin\theta \bar{i} + L \cos\theta \bar{j},$$

$$\delta \bar{r}_B = \frac{\partial \bar{r}_{B/A}}{\partial \theta} \delta\theta = \left(L \cos\theta \bar{i} - L \sin\theta \bar{j} \right) \delta\theta.$$

The description of $\bar{F}$ in terms of the generalized coordinate is $\bar{F} = F \sin\theta \bar{i} - F \cos\theta \bar{j}$, so the virtual work is

$$\delta W = \left(F \sin\theta \bar{i} - F \cos\theta \bar{j} \right) \cdot \left(L \cos\theta \bar{i} - L \sin\theta \bar{j} \right) \delta\theta = 2FL \sin\theta \cos\theta \, \delta\theta.$$

The standard form of the virtual work is $\delta W = Q_1 \delta\theta$, so we find that

$$q_1 = \theta \implies Q_1 = FL\sin 2\theta. \quad \triangleleft$$

Note that 2θ is the angle between $\bar{F}$ and the line from point B to point A, which leads to the observation that Q_1 actually is the moment of $\bar{F}$ about the fixed pin A.

When y is the generalized coordinate, constructing a right triangle shows that

$$\cos\theta = \frac{y}{2L}, \quad \sin\theta = \frac{\left(4L^2 - y^2\right)^{1/2}}{2L},$$

so that

$$\bar{r}_{B/A} = \left(L^2 - \frac{y^2}{4}\right)^{1/2} \bar{i} + \frac{y}{2}\bar{j}.$$

The analytical method for virtual displacement then gives

$$\delta\bar{r}_B = \frac{\partial \bar{r}_{B/A}}{\partial y}\delta y = \left[-\frac{y}{2\left(4L^2 - y^2\right)^{1/2}}\bar{i} + \frac{1}{2}\bar{j}\right]\delta y.$$

The force must also be described in terms of y, for which we use the expressions for $\cos\theta$ and $\sin\theta$:

$$\bar{F} = F\left[\frac{\left(4L^2 - y^2\right)^{1/2}}{2L}\bar{i} - \frac{y}{2L}\bar{j}\right].$$

The corresponding virtual work is

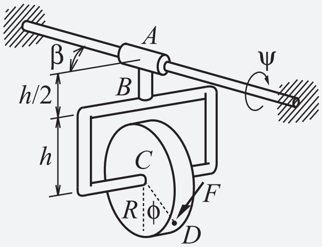
$$\delta W = F\left[\frac{\left(4L^2 - y^2\right)^{1/2}}{2L}\bar{i} - \frac{y}{2L}\bar{j}\right] \cdot \left[-\frac{y}{2\left(4L^2 - y^2\right)^{1/2}}\bar{i} + \frac{1}{2}\bar{j}\right]\delta y = -\frac{Fy}{2L}\delta y.$$

Matching this to the standard form $\delta W = Q_1 \delta y$ gives

$$Q_1 = -\frac{Fy}{2L}. \quad \triangleleft$$

To interpret this we recall that $y/2L = \cos\theta$, from which it follows that Q_1 is the component of $\bar{F}$ downward, which makes sense because y is a variable describing movement in the vertical direction.

EXAMPLE 7.8 Force $\bar{F}$ acts tangentially to the flywheel at point P. Generalized coordinates are selected to be the rotations ψ about the fixed horizontal shaft and β about shaft AB, which describe the orientation of the gimbal supporting the flywheel, and ϕ, which locates the flywheel's orientation relative to the gimbal. Determine the contribution of $\bar{F}$ to the corresponding generalized forces.



Example 7.8

SOLUTION This example of spatial motion uses the kinematical method for virtual displacements. There is no need to evaluate accleration, so we may use any convenient coordinate system. We attach xyz to the gimbal with its origin at center C, such that the y axis is aligned with shaft AB, upward, and the z axis is aligned with the axis of the disk, leftward in the sketch. The angular velocities of the gimbal and of the disk are

$$\bar{\omega}_{\text{gimbal}} = \dot{\psi}\bar{e}_{\psi} + \dot{\beta}\bar{e}_{A/B}, \quad \bar{\omega}_{\text{disk}} = \bar{\omega}_{\text{gimbal}} + \dot{\phi}\bar{k}.$$

We describe the unit vectors in terms of the generalized coordinates,

$$\bar{e}_{\psi} = -\sin\beta\bar{i} + \cos\beta\bar{k}, \quad \bar{e}_{A/B} = \bar{j},$$

from which we find that

$$\bar{\omega}_{\text{gimbal}} = -\dot{\psi}\sin\beta\bar{i} + \dot{\beta}\bar{j} + \dot{\psi}\cos\beta\bar{k}, \quad \bar{\omega}_{\text{disk}} = -\dot{\psi}\sin\beta\bar{i} + \dot{\beta}\bar{j} + \left(\dot{\psi}\cos\beta + \dot{\phi}\right)\bar{k}.$$

To describe the velocity of point D where $\bar{F}$ is applied, we proceed along the gimbal from collar A, whose center is stationary, to center C, and then onward to D along the disk, which leads to

$$\bar{v}_C = \bar{\omega}_{\text{gimbal}} \times \bar{r}_{C/A}, \ \bar{v}_D = \bar{v}_C + \bar{\omega}_{\text{disk}} \times \bar{r}_{D/C}.$$

The positions are $\bar{r}_{C/A} = -(3h/2)\,\bar{j}$, $\bar{r}_{D/C} = R\sin\phi\bar{i} - R\cos\phi\,\bar{j}$, so that

$$\bar{v}_D = \left[\dot{\psi}\cos\beta\left(\frac{3h}{2} + R\cos\phi\right) + \dot{\phi}R\cos\phi\right]\bar{i} + \left[\left(\dot{\psi}\cos\beta + \dot{\phi}\right)R\sin\phi\right]\bar{j}$$
$$+ \left[\dot{\psi}\left(\frac{3h}{2} + R\cos\phi\right)\sin\beta - \dot{\beta}R\sin\phi\right]\bar{k}.$$

We obtain the virtual displacement by replacing rate variables with virtual increments:

$$\delta\bar{r}_D = \left[\delta\psi\cos\beta\left(\frac{3h}{2} + R\cos\phi\right) + \delta\phi R\cos\phi\right]\bar{i} + \left[\left(\delta\psi\cos\beta + \delta\phi\right)R\sin\phi\right]\bar{j}$$
$$+ \left[\delta\psi\left(\frac{3h}{2} + R\cos\phi\right)\sin\beta - \delta\beta R\sin\phi\right]\bar{k}.$$

The force is perpendicular to $\bar{r}_{D/C}$ in the xy plane, so its component representation is

$$\bar{F} = -F\cos\phi\bar{i} - F\sin\phi\bar{j},$$

which leads to the virtual work being

$$\begin{aligned}\delta W = \bar{F}\cdot\delta\bar{r}_D &= (-F\cos\phi)\left[\delta\psi\cos\beta\left(\frac{3h}{2} + R\cos\phi\right) + \delta\phi R\cos\phi\right] \\ &\quad + (-F\sin\phi)\left(\delta\psi\cos\beta + \delta\phi\right)R\sin\phi \\ &= \left[-F\cos\beta\left(\frac{3h}{2}\cos\phi + R\right)\right]\delta\psi - FR\delta\phi.\end{aligned}$$

The standard form of virtual work in this case is $\delta W = Q_1\delta\psi + Q_2\delta\beta + Q_3\delta\phi$. Matching this to the actual δW yields

$$Q_1 = -F\cos\beta\left(\frac{3h}{2}\cos\phi + R\right), \quad Q_2 = 0, \quad Q_3 = -FR. \qquad \triangleleft$$

Observe that Q_3 is the moment of $\bar{F}$ about the axis of the disk, which is sensible because that moment is what causes ϕ to change. The value of Q_2 is zero because $\bar{F}$ intersects the line of shaft AB, so it does not have a tendency to change β. The first generalized force represents the tendency of $\bar{F}$ to change ψ, which we would expect to be the moment of $\bar{F}$ about the horizontal shaft. It is not difficult to verify that $Q_1 = \left(\bar{r}_{D/A}\times\bar{F}\right)\cdot\bar{e}_\psi$.

7.4.2 Relation Between Constraint Forces and Conditions

The terms "reactions" and "constraint forces" are synonyms describing the forces and couples that enforce constraint equations, either by preventing motion or by imposing a motion that is a specified function of time. Whereas the Newton–Euler formulation requires separate consideration of the forcing effect and the kinematical restriction of a constraint, we will find here that knowledge of the velocity constraint equation will suffice to describe the effect of the associated constraint force. We begin by considering the case in which a particle is constrained to move along a specified curve. A kinematical description using tangent-normal components indicates that the position of this particle is given by the arc length $s(t)$ measured along the path, which we select to be the generalized coordinate. The resultant force may be resolved into components ΣF_t, ΣF_n, and ΣF_b in the tangent, normal, and binormal directions, respectively. The latter two include constraint forces because they prevent the particle from moving perpendicularly to the path. In a virtual movement that increases s by δs, the particle moves in the tangential direction by that amount, so $\delta\bar{r} = \delta s\bar{e}_t$. The corresponding virtual work is $\delta W = \Sigma F_t\,\delta s$. In this simple situation a virtual displacement that is kinematically admissible, meaning that it obeys the motion constraints imposed on the system, leads to the constraint forces doing no work.

We encountered a similar result when we applied the principle of dynamic virtual work in Subsection 7.1.2. The free-body diagram in Fig. 7.2 showed constraint forces $\bar{A}_x$ and $\bar{A}_y$ that prevent movement of pin A, and vertical constraint force $\bar{C}_y$ exerted by the horizontal guidebar on collar C to prevent movement in the vertical direction. The virtual movement used there kept the pin stationary and moved the collar horizontally. Such a movement is kinematically admissible, and the virtual work done by the constraint forces again was found to be zero.

This observation that the virtual work done by a reaction is zero is not a chance occurrence. In fact, it provides a consistent definition of a constraint force. (Once again, constraint force is the preferred term, rather than reaction.) Specifically,

> *A force or couple is a constraint force associated with a kinematical constraint condition if, and only if, it does no work in a virtual movement that is consistent with the constraint.*

An important corollary applies when the generalized coordinates are unconstrained, which can be the case only for a holonomic system. In such a situation, any set of generalized coordinate values is kinematically possible, so any virtual displacement will be kinematically admissible. Consequently,

> *The virtual work done by constraint forces always is zero when unconstrained generalized coordinates are used to describe a holonomic system.*

Let us now consider what happens when we use constrained generalized coordinates. A virtual displacement entails incrementing each generalized coordinate by an arbitrary amount δq_j, whereas Eq. (7.2.18) indicates that the δq_j values cannot all be selected arbitrarily if the virtual displacement is to be consistent with the ith constraint equation. It follows that

> *When constrained generalized coordinates are used to describe the position of a system, constraint forces will do virtual work.*

It is necessary in that case to account for the contribution of constraint forces to each of the generalized forces. One way of doing so is to treat constraint forces as forces that are known when the virtual work is formed. However, a simpler alternative is available.

In the physical world each constraint equation has an associated constraint force or couple. Let us denote as $R_j^{(i)}$, $i = 1, 2 \ldots J$, the contribution of the ith constraint force to the jth generalized force. In the configuration space the generalized forces are the components of the vector $\hat{Q}$ representing the forcing effect on the system. Let $\hat{R}^{(i)}$ denote the portion that is attributable to the constraint, so that

$$\hat{R}^{(i)} = R_1^{(i)} \hat{e}_1 + \cdots + R_N^{(i)} \hat{e}_N. \tag{7.4.9}$$

As noted earlier, if the δq_j values are selected arbitrarily, the virtual displacement will not be kinematically admissible. In the configuration space, this would be represented by a displacement vector $\delta \hat{r}$ that may point in any direction. Let us consider the special situation in which the δq_j values are selected such that the virtual displacement is consistent with the ith constraint equation. Equation (7.2.20) indicates that $\delta \hat{r}$ then must be perpendicular to the normal direction $\hat{a}_i$ associated with the Jacobian constraint coefficients, that is, $\hat{a}_i \cdot \delta\hat{r} = 0$. Furthermore, if $\delta\hat{r}$ represents a movement that is consistent with the ith constraint equation, the definition of a constraint force requires that the virtual work done by the ith constraint force be zero. The virtual work of this force in the configuration space is given by $\hat{R}^{(i)} \cdot \delta\hat{r}$, so we find that

$$\text{if} \quad \hat{a}_i \cdot \delta\hat{r} = 0, \text{ then } \hat{R}^{(i)} \cdot \delta\hat{r} = 0. \tag{7.4.10}$$

An important aspect of this condition is that it must be true for any $\delta\hat{r}$ that is perpendicular to $\hat{a}_i$, but there is a limitless number of such vectors. Hence Eq. (7.4.10) can be satisfied only if $\hat{R}^{(i)}$ is parallel to $\hat{a}_i$. Parallelism of two vectors means that they are proportional, so we have

$$\hat{R}^{(i)} = \hat{a}_i \lambda_i. \tag{7.4.11}$$

The factor of proportionality λ_i is a *Lagrange multiplier*. When we substitute Eqs. (7.4.9) and (7.2.19) into this expression, and equate like components in each $\hat{e}_j$ direction, we find that

$$\boxed{R_j^{(i)} = a_{ij} \lambda_i.} \tag{7.4.12}$$

This equation enables us to describe the contributions of constraints to generalized forces solely by knowing the Jacobian constraint matrix, a_{ij}, which clearly is easier than actually evaluating the virtual work done by constraint forces.

In some situations it is desirable to see the role of the actual constraint force, rather than representing it with a Lagrange multiplier. We may do this by evaluating the virtual work of the force, rather than by using Eq. (7.4.12). Let $\bar{e}_i$ denote the known direction in which motion is restricted by a constraint condition. This also is the direction in which the constraint force acts, so the ith constraint force (or couple) is $C_i \bar{e}_i$, where the magnitude C_i is unknown. According to Eqs. (7.3.8) and (7.3.10), the virtual displacement is linear in the δq_j. It follows that the virtual work done by $C_i \bar{e}_i$ will have the general form

$$\delta W_i = C_i \sum_{j=1}^{N} c_{ij} \delta q_j, \tag{7.4.13}$$

where the c_{ij} coefficients may be functions of the generalized coordinates and time. Equating this expression to Eq. (7.4.3), which defines generalized forces, shows that

$$R_j^{(i)} = c_{ij} C_i. \tag{7.4.14}$$

A comparison of Eqs. (7.4.12) and (7.4.14) shows that a Lagrange multiplier is proportional to the amplitude of the associated constraint force according to

$$C_i = \frac{a_{ij}}{c_{ij}} \lambda_i. \tag{7.4.15}$$

Because this relation must hold for any j, it must be that

$$C_i = \sigma_i \lambda_i, \quad \sigma_i = \frac{a_{ij}}{c_{ij}}. \tag{7.4.16}$$

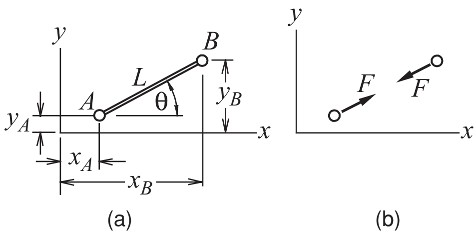


Figure 7.10. Generalized coordinates for a system consisting of two spheres connected by a rigid bar.

Establishing the proportionality factor σ_i in a specific case requires determination of the c_{ij} coefficients.

Let us examine a few situations that illustrate when constraint forces may be avoided in the formulation. First, consider Fig. 7.10(a), where two tiny spheres, modeled as particles, are constrained to move in the xy plane. The spheres are connected by a massless rigid bar. As a result, the distance L is constant, so (x_A, y_A, ϕ) constitute a set of unconstrained generalized coordinates.

Because the bar is massless, it can exert only an axial force on each particle, as shown in the free-body diagram for each particle, Fig. 7.10(b). To evaluate the virtual work done by this force, we employ the kinematical method to relate the virtual displacements of the spheres. The velocities are related by

$$\bar{v}_B = \bar{v}_A + \left(\dot{\phi}\bar{k}\right) \times \bar{r}_{B/A} = \bar{v}_A + L\dot{\phi}\bar{e}_\phi, \tag{7.4.17}$$

so the virtual displacements are related by

$$\delta\bar{r}_B = \delta\bar{r}_A + L\delta\phi\bar{e}_\phi. \tag{7.4.18}$$

The virtual work done by the axial force is therefore

$$\delta W = (-F\bar{e}_R) \cdot \delta\bar{r}_A + F\bar{e}_R \cdot \delta\bar{r}_B = F\bar{e}_R \cdot L\delta\phi\bar{e}_\phi = 0. \tag{7.4.19}$$

The axial force does no work in this situation because it is the constraint force required to keep L constant. If we employ a set of constrained generalized coordinates, this constraint will be violated. For example, suppose that (x_A, y_A, x_B, y_B) are used as generalized coordinates. The constraint condition that the bar distance between the spheres must be L gives a configuration constraint equation, $(x_B - x_A)^2 + (y_B - y_A)^2 = L^2$. The velocity form of this equation is

$$\begin{gathered} a_{11}\dot{x}_A + a_{12}\dot{y}_A + a_{13}\dot{x}_B + a_{14}\dot{y}_B = 0, \\ a_{11} = -a_{13} = x_A - x_B, \quad a_{12} = -a_{14} = y_A - y_B, \end{gathered} \tag{7.4.20}$$

where the leading subscript 1 denotes the first constraint equation. The virtual displacements of the spheres now are unrelated, being given by

$$\delta\bar{r}_A = \delta x_A\bar{i} + \delta y_A\bar{j}, \quad \delta\bar{r}_B = \delta x_B\bar{i} + \delta y_B\bar{j}. \tag{7.4.21}$$

The virtual work done by the axial force in this case is

$$\delta W = (-F\bar{e}_R) \cdot \delta\bar{r}_A + F\bar{e}_R \cdot \delta\bar{r}_B = F\cos\phi\,(\delta x_B - \delta x_A) + F\sin\phi\,(\delta y_B - \delta y_A). \tag{7.4.22}$$

We must eliminate ϕ from this expression because it is not a generalized coordinate. Forming a right triangle whose hypotenuse is L shows that $\cos\phi = (x_B - x_A)/L$ and $\sin\phi = (y_B - y_A)/L$. We substitute these expressions into δW and match the result to the standard form, $\delta W = R_1^{(1)}\delta x_A + R_2^{(1)}\delta y_A + \ldots$, from which we find that the contributions of the axial force to the generalized forces are

$$R_1^{(1)} = -R_3^{(1)} = -\frac{x_B - x_A}{L}F,$$
$$R_2^{(1)} = -R_4^{(1)} = -\frac{y_B - y_A}{L}F. \tag{7.4.23}$$

The virtual work is not zero, as expected. Furthermore, we observe that these expressions fit Eq. (7.4.14), with the constraint force $F = C_1$ and

$$c_{11} = -c_{13} = -\frac{x_B - x_A}{L}, \quad c_{12} = -c_{14} = -\frac{y_B - y_A}{L}. \tag{7.4.24}$$

This is consistent with Eqs. (7.4.16), for we see that

$$\frac{a_{j1}}{c_{j1}} = \sigma_1 = L, \quad j = 1, 4. \tag{7.4.25}$$

It follows from Eqs. (7.4.16) that $\lambda_1 = F/L$.

The case in which the spheres in Fig. 7.10 are connected by a spring offers an instructive contrast. Instead of being an unknown reaction that restricts motion, the axial force within a spring is known in terms of the deformation, $\Delta = L - L_0$, where L_0 is the undeformed length. There would be no constraints on the planar motion of the spheres in this case, so the system would have four degrees of freedom, and (x_A, y_A, x_B, y_B) would be unconstrained generalized coordinates. The contributions of the axial spring force to each generalized coordinate in this case are still given by Eqs. (7.4.23), except that $F = k\Delta$, which means that the force is now known in terms of the generalized coordinates. This exemplifies the general fact that

> *Forces and couples are either unknown constraint effects that impose a kinematical condition, or else they are known as function of time and/or the generalized coordinates, in which case they do not kinematically restrict the motion.*

The two-particle system provides an important analogy for the general task of modeling. Let the spheres represent two adjacent particles in a body. Correspondingly, the force $\bar{F}$ represents the stress resultant exerted between them. When a body is considered to be rigid, then the internal stress resultants are equivalent to reactions that maintain the particles at fixed relative distances. These forces do no virtual work. In contrast, when deformation of a body must be considered, the internal stress resultants are equivalent to spring forces. Thus derivation of the equations of motion for deformable bodies requires consideration of the effects of internal stresses.

Connections between bodies are an important element in most dynamic systems. They impose kinematical conditions on the relative movement of the parts they connect. According to the definition of a constraint force, the force or couple associated with a connection will do no virtual work if the virtual movement does not violate the condition imposed by that connection. An important example of this aspect is described

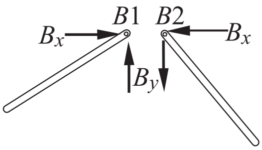


Figure 7.11. Constraint forces associated with a pin connection.

by Fig. 7.11, where two bars are connected by a pin. Let $B1$ and $B2$ denote the connection point on the respective bars. The pin exerts opposing horizontal and vertical forces on each bar, which enforce the condition that points $B1$ and $B2$ move in unison. The corresponding virtual work is

$$\delta W = \left(B_x\bar{i} + B_y\bar{j}\right)\cdot\delta\bar{r}_{B1} + \left(-\,B_x\bar{i} - B_y\bar{j}\right)\cdot\delta\bar{r}_{B2}. \tag{7.4.26}$$

If the constraint imposed by the pin is not violated in a virtual displacement, then the virtual displacements of the connection points will be identical, $\delta\bar{r}_{B2} = \delta\bar{r}_{B1}$, and $\delta W = 0$. Thus, if the virtual movement of a system maintains the integrity of a pin connection, the forces exerted by the pin will not contribute to the generalized forces.

Another connection that is commonly encountered is a sliding collar, such as the one in Fig. 7.12. The collar can move only inward or outward relative to bar CD, corresponding to changing distance R measured along bar CD. In a virtual movement that is consistent with this constraint, the virtual displacements of points $B1$ on bar CD and $B2$ on the collar are related by

$$\delta\bar{r}_{B2} = \delta\bar{r}_{B1} + \delta R\bar{e}_R. \tag{7.4.27}$$

If friction is negligible, $\bar{f} \equiv \bar{0}$, then the only force that is developed between the collar and bar CD is the normal reaction $\bar{N}$. The virtual work done by this force is

$$\delta W = (N\bar{e}_\phi)\cdot\delta\bar{r}_{B1} + (-N\bar{e}_\phi)\cdot\delta\bar{r}_{B2}. \tag{7.4.28}$$

If the virtual displacement is consistent with the constraint imposed by the collar, so that Eq. (7.4.27) holds, then we find that $\delta W = (-N\bar{e}_\phi)\cdot\delta R\bar{e}_R = 0$. Thus the force associated with a collar connection will not appear in the generalized forces if the virtual movement satisfies the kinematical restriction imposed by the collar.

Not all forces associated with a connection are associated with a constraint condition. For example, the friction force $\bar{f}$ in Fig. 7.12 acts in opposition to the direction in which each contacting surface moves relative to the other. (The situation appearing in

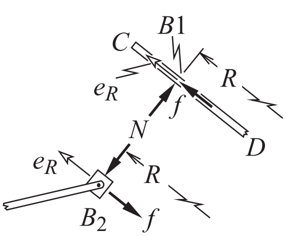


Figure 7.12. Forces associated with a collar connection.

this figure corresponds to the collar sliding outward, $\dot{R} > 0$.) The virtual work in this case would be

$$\delta W = (N\bar{e}_\phi + f\bar{e}_R) \cdot \delta\bar{r}_{B1} + (-N\bar{e}_\phi - f\bar{e}_R) \cdot \delta\bar{r}_{B2}. \tag{7.4.29}$$

Even if the the virtual displacement satisfies Eq. (7.4.27), so that the virtual displacement is consistent with the constraint imposed by the collar, the virtual work is not zero. Rather, it is

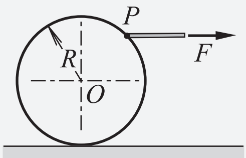
$$\delta W = (-N\bar{e}_\phi - f\bar{e}_R) \cdot \delta R\bar{e}_R = -f\,\delta R. \tag{7.4.30}$$

Note that kinetic friction is not a constraint force, because it does not prevent sliding motion. Rather it is known by Coulomb's kinetic friction model to be $\mu_k\left|\bar{N}\right|$. (A static friction force does prevent relative motion. Such a force in the present case would have the effect of making the collar act as a pin connection, because the collar would not move relative to the bar.)

Many other types of constraint forces could be considered at this juncture, for example, the constraint forces associated with rolling motion. The normal force prevents interpenetration of the contacting surfaces. In the case of no slipping, the tangential force, developed by friction or gear teeth, makes the points of contact on the two bodies move by the same amount. Hence it also is a constraint force. It follows that the forces exerted between two rolling bodies will do no virtual work if there is no slippage in the rolling motion, provided that the virtual movement does not lead to relative displacement of the contacting points. (This was proven for a true displacement in Example 6.12.) Conversely, the tangential force will do virtual work when there is slippage, because it then does not constrain motion.

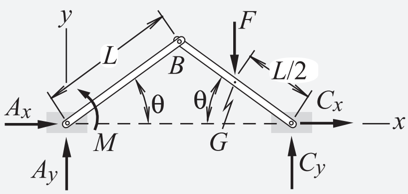
One should realize that, when a system is represented by a set of constrained generalized coordinates that must satisfy a certain number of constraint equations, it is only the forces associated with these conditions that contribute to the virtual work. There may be numerous other kinematical restrictions that are always satisfied, regardless of the values of the generalized forces. The forces associated with the latter conditions will not appear in the virtual work. For example, if one uses constrained generalized coordinates to model a linkage, but a pin connection remains intact when the δq_j values are arbitrary, then the virtual work of the forces associated with the pin will be zero.

EXAMPLE 7.9 Tensile force $\bar{F}$ is applied to the free end of the cable, such that the cable remains horizontal regardless of the position of the pin to which it is attached. First consider the case in which the disk slips as it rolls over the ground. The coefficient of kinetic friction is μ_k. Select generalized coordinates, evaluate the virtual work done by all forces acting on the disk, and identify the corresponding generalized forces. Then repeat the analysis for the case in which there is no slippage in the rolling motion.



Example 7.9

SOLUTION This example helps us understand the different roles of static and kinetic friction, and the way in which each is described. In the first case there is slippage between the disk and the ground, which means that the motion of the center of the disk is independent of its rotation. Let us use x_O to measure horizontal distance rightward from some fixed reference to the center O, and let the angle θ of the radial line from the center to the pin be the generalized coordinate for rotation of the disk. We show these variables in a free-body diagram of the disk.



Free-body diagram of the rolling disk showing the friction force $\bar{f}$.

The normal force $\bar{N}$ is a constraint force that prevents the contact point from moving vertically, and that constraint will not be violated by incrementing either generalized coordinate. We therefore could ignore this force in the evaluation of the virtual work. However, the friction force $\bar{f}$ does not constrain motion when there is slippage, so we must evaluate its contribution to δW. The sense of $\bar{f}$ depicted in the free-body diagram corresponds to the velocity of the contact point being to the right, so we have $\bar{f} = -f\,\bar{i}$, with $f = \mu_k N \operatorname{sgn}\left(\bar{v}_C \cdot \bar{i}\right)\bar{i}$. As we did previously for bodies that roll, we replace the actual forces with a resultant force $\bar{S}$ acting at the center of the disk, and a couple $\bar{M}$, where

$$\bar{S} = (F - f)\,\bar{i} + (N - mg)\,\bar{j}, \quad \bar{M} = -(FR\cos\theta + fR)\,\bar{k}. \tag{1}$$

In terms of the generalized coordinates the velocity of the center and the angular velocity are

$$\bar{v}_O = \dot{x}_O\bar{i}, \quad \bar{\omega} = -\dot{\theta}\bar{k}.$$

Converting velocity variables to virtual increments leads to

$$\delta\bar{r}_O = \delta x_O\bar{i}, \quad \overline{\delta\theta} = -\delta\theta\bar{k}.$$

The corresponding virtual work is

$$\delta W = \bar{S}\cdot\delta\bar{r}_O + \bar{M}\cdot\overline{\delta\theta} = (F - f)\,\delta x_0 + (FR\cos\theta + fR)\,\delta\theta. \tag{2}$$

The coefficient of δx_O in this expression is Q_1, and Q_2 is the coefficient of $\delta\theta$, so we have

$$Q_1 = F - f, \quad Q_2 = FR\cos\theta + fR. \qquad \triangleleft$$

When slipping occurs, the velocity of the contact point is $\bar{v}_C = \left(\dot{x}_O - R\dot{\theta}\right)\bar{i}$ and $N = mg$, so the friction force is $f = \mu_k mg\ \text{sgn}\left(\dot{x}_O - R\dot{\theta}\right)$.

We see that Q_1 is the resultant horizontal force, which is the force tending to increase x_O, whereas Q_2 is the clockwise resultant moment about the center of the disk, which is the forcing effect causing θ to increase. As expected, the normal force does not appear in any generalized force, and the absence of gravity is a consequence of the center moving horizontally.

When there is no slippage, the angle of rotation is $\theta = x_O/R$, which is a configuration constraint. We use it to eliminate θ, so x_O is now the single unconstrained generalized coordinate. Differentiating the configuration constraint gives $\delta\theta = \delta x_O/R$. Equation (2) for the virtual work is generally valid. When the rotation is consistent with the no-slip condition, this equation becomes

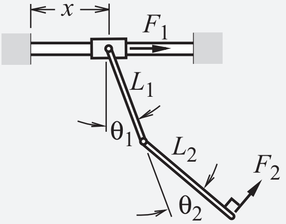
$$\delta W = F\delta x_O\left[1 + \cos\left(\frac{x_O}{R}\right)\right].$$

Matching this to $\delta W = Q_1\delta x_O$ shows that the generalized force is

$$Q_1 = F\left[1 + \cos\left(\frac{x_O}{R}\right)\right]. \qquad \triangleleft$$

This is the moment of $\bar{F}$ about the contact point divided by the radius R. The friction force does not appear in the generalized force when there is no slippage because it then is a constraint force that prevents relative movement of the contacting surfaces.

EXAMPLE 7.10 Force F_1 causes the collar to translate such that its horizontal position x is known as a function of t. Force F_2 is known as a function of t. Generalized coordinates are the absolute angle of rotation θ_1 for the upper bar and the relative angle θ_2 for the lower bar. Determine the corresponding generalized forces. The weight of each bar is negligible in comparison with the magnitude of $\bar{F}_2$.



Example 7.10

SOLUTION The intent here is to clarify the role of constraint forces in comparison with applied forces for a system with prescribed motion. The fact that x is a specified time function means that it is not a generalized coordinate. It also means that force $\bar{F}_1$ imposing the motion is a constraint force. The normal force acting on the collar also is a constraint force. We know that neither will contribute to the virtual work because θ_1 and θ_2 are unconstrained generalized coordinates. However, let us verify that property by explicitly evaluating the virtual work done by all forces.

We employ the analytical method to determine the virtual displacement. In terms of horizontal and vertical components, the positions of the points where forces act are

$$\bar{r}_1 = x\bar{i}, \quad \bar{r}_2 = [x + L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2)]\,\bar{i} - [L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2)]\,\bar{j}.$$

Because t is constant in a virtual displacement, x remains constant in the evaluation of $\delta\bar{r}_1$ and $\delta\bar{r}_2$, which are found to be

$$\begin{aligned}
\delta\bar{r}_1 &= \frac{\partial\bar{r}_1}{\partial\theta_1}\delta\theta_1 + \frac{\partial\bar{r}_1}{\partial\theta_2}\delta\theta_2 = \bar{0}, \\
\delta\bar{r}_2 &= \frac{\partial\bar{r}_2}{\partial\theta_1}\delta\theta_1 + \frac{\partial\bar{r}_2}{\partial\theta_2}\delta\theta_2 \\
&= \left\{[L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2)]\,\bar{i} + [L_1\cos\theta_1 + L_2\sin(\theta_1+\theta_2)]\,\bar{j}\right\}\delta\theta_1 \\
&\quad + \left\{[L_2\cos(\theta_1+\theta_2)]\,\bar{i} + [L_2\sin(\theta_1+\theta_2)]\,\bar{j}\right\}\delta\theta_2.
\end{aligned}$$

The virtual work is

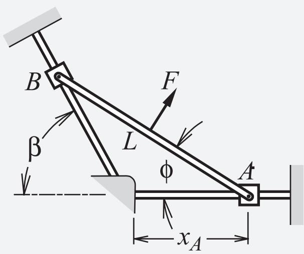
$$\begin{aligned}
\delta W &= \left(F_1\bar{i} + N\bar{j}\right)\cdot\delta\bar{r}_1 + \left[F_2\cos(\theta_1+\theta_2)\,\bar{i} + F_2\sin(\theta_1+\theta_2)\right]\cdot\delta\bar{r}_2 \\
&= F_2\left\{[L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2)]\cos(\theta_1+\theta_2)\right. \\
&\quad \left. + [L_1\cos\theta_1 + L_2\sin(\theta_1+\theta_2)]\sin(\theta_1+\theta_2)\right\}\delta\theta_1 \\
&\quad + F_2\left\{[L_2\cos(\theta_1+\theta_2)]\cos(\theta_1+\theta_2) + [L_2\sin(\theta_1+\theta_2)]\sin(\theta_1+\theta_2)\right\}\delta\theta_2 \\
&= F_2\,(L_1\cos\theta_2 + L_2)\,\delta\theta_1 + F_2 L_2\delta\theta_2.
\end{aligned}$$

As expected, neither F_1 nor N appears in δW. The standard form is $\delta W = Q_1\delta\theta_1 + Q_2\delta\theta_2$, which matches the actual δW with

$$Q_1 = F_2\,(L_1\cos\theta_2 + L_2)\,, \quad Q_2 = F_2 L. \qquad \triangleleft$$

It is apparent that the second generalized force is the moment of F_2 about the pin connecting the bars, and it is not difficult to show that Q_1 is the moment of F_2 about the pivot in the collar. Both forces exempify the notion that a generalized force captures the tendency of the force system to alter the associated generalized coordinate.

EXAMPLE 7.11 The bar is constrained by collars A and B that slide over smooth guide bars. Force $\bar{F}$ acts perpendicularly to the bar at its midpoint. (a) Determine the generalized force when the angle of elevation ϕ is selected as the single unconstrained generalized coordinate. (b) Determine the generalized forces when the angle of elevation ϕ and the horizontal distance x_A are selected as constrained generalized coordinates. Perform the analysis by evaluating the virtual work done by the constraint forces. (c) Use Lagrange multipliers to carry out the determination in Part (b). From this analysis, identify the meaning of the Lagrange multiplier. (d) Prove that the constraint forces do no virtual work if one selects $\delta\phi$ and δx_A in Part (b) to be kinematically admissible.



Example 7.11

SOLUTION This example addresses most of the situations we encounter in the evaluation of generalized forces. This system clearly has one degree of freedom, so using a single generalized coordinate ϕ in the first analysis automatically leads to a virtual movement of the system that is consistent with all constraints. Consequently, no reactions do work, so we need to evaluate δW only for the applied force $\bar{F}$. We must describe $\delta \bar{r}_D$ for the point where $\bar{F}$ is applied solely in terms of ϕ, without reference to x_A. To eliminate the latter variable we apply the law of sines to triangle OAB, where point O is the intersection of the guide bars. This gives

$$\frac{x_A}{\sin(\beta - \phi)} = \frac{L}{\sin\beta}.$$

We solve this for x_A, then differentiate the result to determine $\dot{x}_A$:

$$x_A = L\frac{\sin(\beta - \phi)}{\sin\beta}, \quad \dot{x}_A = -L\dot{\phi}\frac{\cos(\beta - \phi)}{\sin\beta}. \tag{1}$$

The velocity of collar A is now known in terms of ϕ and $\dot{\phi}$, and the angular velocity of bar AB is $-\dot{\phi}\bar{k}$, so we proceed to describe the velocity of the midpoint D according to

$$\bar{v}_D = \bar{v}_A + \left(-\dot{\phi}\bar{k}\right) \times \bar{r}_{D/A} = \dot{x}_A\bar{i} + \left(-\dot{\phi}\bar{k}\right) \times \frac{L}{2}\left(-\cos\phi\bar{i} + \sin\phi\bar{j}\right)$$
$$= L\dot{\phi}\left[\left(\frac{1}{2}\sin\phi - \frac{\cos(\beta - \phi)}{\sin\beta}\right)\bar{i} + \frac{1}{2}\cos\phi\bar{j}\right].$$

We obtain $\delta\bar{r}_D$ by replacing $\dot{\phi}$ with $\delta\phi$, so

$$\delta\bar{r}_D = L\delta\phi\left[\left(\frac{1}{2}\sin\phi - \frac{\cos(\beta-\phi)}{\sin\beta}\right)\bar{i} + \frac{1}{2}\cos\phi\,\bar{j}\right]. \tag{2}$$

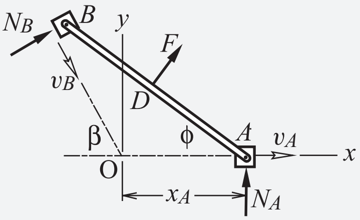
The applied force is $\bar{F} = F\sin\phi\bar{i} + F\cos\phi\,\bar{j}$, so the virtual work done by this force is

$$\delta W = \bar{F}\cdot\delta\bar{r} = FL\delta\phi\left[\left(\frac{1}{2}\sin\phi - \frac{\cos(\beta-\phi)}{\sin\beta}\right)\sin\phi + \frac{1}{2}(\cos\phi)^2\right].$$

The standard form is $\delta W = Q_1\delta\phi$, so the preceding equation leads to

$$Q_1 = FL\left(\frac{1}{2} - \frac{\cos(\beta-\phi)\sin\phi}{\sin\beta}\right). \quad \triangleleft$$

In the second analysis $q_1 = \phi$ and $q_2 = x_A$ are constrained generalized coordinates. The first of Eqs. (1) is the configuration constraint, and the second is the corresponding velocity constraint. Reaction forces contribute to the virtual work in such cases, and we are requested to evaluate these contributions explicitly. As an aid to this analysis, we draw a free-body diagram of the system.



Free-body diagram showing the bar in a position corresponding to arbitrary values of the two constrained generalized coordinates.

The important aspect of using constrained generalized coordinates is that, after the constraint equations have been characterized, the subsequent analytical steps consider these variables to be independent. Thus, in the free-body diagram, collar A is situated at an arbitrary distance x_A, whereas bar AB is attached to collar A and elevated to an arbitrary ϕ. This places end B at a location that is not situated on the inclined guidebar. We may directly describe in terms of ϕ and x_A the velocities of the points where forces are applied:

$$\bar{v}_A = \dot{x}_A\bar{i},\ \ \bar{v}_B = \dot{x}_A\bar{i} + \left(-\dot{\phi}\bar{k}\right)\times\bar{r}_{B/A} = \left(\dot{x}_A + L\dot{\phi}\sin\phi\right)\bar{i} + L\dot{\phi}\cos\phi\,\bar{j},$$

$$\bar{v}_D = \dot{x}_A\bar{i} + \left(-\dot{\phi}\bar{k}\right)\times\bar{r}_{D/A} = \left(\dot{x}_A + \frac{L}{2}\dot{\phi}\sin\phi\right)\bar{i} + \frac{L}{2}\dot{\phi}\cos\phi\,\bar{j}.$$

Changing the rate variables to virtual increments gives

$$\delta\bar{r}_A = \delta x_A\bar{i},\ \ \delta\bar{r}_B = (\delta x_A + L\delta\phi\sin\phi)\,\bar{i} + L\delta\phi\cos\phi\,\bar{j},$$

$$\delta\bar{r}_D = \left(\delta x_A + \frac{L}{2}\delta\phi\sin\phi\right)\bar{i} + \frac{L}{2}\delta\phi\cos\phi\,\bar{j}. \tag{3}$$

Correspondingly, the virtual work done by $\bar{F}$ and the normal force applied to each collar is

$$\begin{aligned}\delta W &= \left(F\sin\phi\bar{i} + F\cos\phi\bar{j}\right)\cdot\delta\bar{r}_D + N_A\bar{j}\cdot\delta\bar{r}_A + \left(N_B\sin\beta\bar{i} + N_B\cos\beta\bar{j}\right)\cdot\delta\bar{r}_B \\ &= (F\sin\phi)\,\delta x_A + \frac{FL}{2}\delta\phi + (N_B\sin\beta)\,\delta x_A + N_B L\delta\phi\,(\sin\beta\sin\phi + \cos\beta\cos\phi)\,.\end{aligned} \tag{4}$$

We match this expression to $\delta W = Q_1\delta\phi + Q_2\delta x_A$ to find that

$$\begin{aligned}Q_1 &= \frac{FL}{2} + N_B L\cos(\beta-\phi)\,, \\ Q_2 &= F\sin\phi + N_B\sin\beta.\end{aligned} \tag{5} \triangleleft$$

The third analysis also uses $q_1 = \phi$ and $q_2 = x_A$, but the contributions of N_A and N_B to the generalized forces will be described by Lagrange multipliers. The velocity constraint equation, the second of Eqs. (1), has the standard form of a linear velocity constraint, $a_{11}\dot{\phi} + a_{12}\dot{x}_A + b_1 = 0$, with the Jacobian constraint coefficients being

$$a_{11} = L\frac{\cos(\beta-\phi)}{\sin\beta}, \quad a_{12} = 1. \tag{6}$$

Equation (7.4.12) indicates that the contributions of the associated constraint force to the generalized forces are

$$R_1^{(1)} = \lambda_1 a_{11} = \lambda_1 L\frac{\cos(\beta-\phi)}{\sin\beta}, \quad R_2^{(1)} = \lambda_1 a_{12} = \lambda_1. \tag{7}$$

A comparison of these expressions with the terms in Eqs. (5) that contain the N_B shows that

$$\lambda_1 = N_B\sin\beta. \tag{8} \triangleleft$$

In other words, the Lagrange multiplier associated with the single constraint equation is the horizontal component of the associated constraint force. The same relation could have been derived by matching the N_B terms in Eq. (4) to Eq. (7.4.13), which would lead to the recognition that

$$c_{11} = L\cos(\beta-\phi)\,, \quad c_{12} = \sin\beta.$$

Substitution of the preceding equation and Eqs. (6) into Eqs. (7.4.16) reveals that

$$\sigma_1 = \frac{a_{11}}{c_{11}} = \frac{a_{12}}{c_{12}} = \frac{1}{\sin\beta}.$$

Thus Eq. (8) is consistent with $N_B = \sigma_1\lambda_1$, as indicated by Eqs. (7.4.16).

The contribution to δW of standard applied like $\bar{F}$ proceeds in the same way regardless of whether one employs unconstrained or constrained generalized coordinates. Thus the portion of the generalized forces associated with $\bar{F}$ is described in Eq. (4). Adding those terms to the respective $R_j^{(1)}$ in Eqs. (6) gives

$$Q_1 = \frac{FL}{2} + \lambda_1 L\frac{\cos(\beta-\phi)}{\sin\beta}, \quad Q_2 = F\sin\phi + \lambda_1. \qquad \triangleleft$$

The last task is to consider a situation in which $\delta\phi$ and δx_A are consistent with the constraint equation. This requires that $a_{11}\delta\phi + a_{12}\delta x_A = 0$. For the Jacobian coefficients in Eqs. (6) this leads to

$$\delta x_A = -\frac{a_{11}}{a_{12}}\delta\phi = -L\frac{\cos(\beta-\phi)}{\sin\beta}\delta\phi.$$

When δx_A is related to $\delta\phi$ in this way the virtual work in Eq. (4) becomes

$$\begin{aligned}\delta W &= Q_1\delta\phi + Q_2\delta x_A \\ &= \left[\frac{FL}{2} + N_B L\cos(\beta-\phi)\right]\delta\phi + (F\sin\phi + N_B\sin\beta)\left(-L\frac{\cos(\beta-\phi)}{\sin\beta}\delta\phi\right) \\ &= FL\delta\phi\left(\frac{1}{2} - \frac{\cos(\beta-\phi)\sin\phi}{\sin\beta}\right).\end{aligned}$$

◁

We see that neither reaction appears in δW for this special choice of virtual increments. We also see that the resulting δW is identical to the virtual work in Part (a).

7.4.3 Conservative Forces

The work done by a conservative force equals the amount by which the associated potential energy is depleted, that is, $W_{1\to2} = V_1 - V_2$. This property enables us to characterize a conservative force's contribution to the generalized forces solely in terms of the properties of the potential energy. The potential energy depends on only the position of the system, so it is an explicit function of the generalized coordinates. It is also possible that V will explicitly depend on time, which would be the case if generalized coordinates are measured from reference locations that move in a prescribed manner. Thus the potential energy is described functionally as $V(q_j, t)$.

We obtain a virtual movement by adding a virtual increment δq_j to each generalized coordinate and holding time constant. The corresponding virtual work is the amount by which V decreases in the shift from the original to the displaced position:

$$\delta W = V(q_j, t) - V(q_j + \delta q_j, t). \tag{7.4.31}$$

Because the δq_j values are infinitesimal, the chain rule for partial differentiation indicates that

$$\boxed{\delta W = \sum_{j=1}^{N}\left[-\frac{\partial}{\partial q_j}V(q_j, t)\right]\delta q_j.} \tag{7.4.32}$$

The definition of a generalized force is that it is the coefficient of the corresponding δq_j in an expression for δW. Hence the bracketed term in the preceding expression must be the contribution of conservative forces to the generalized forces:

$$(Q_j)_{\text{conservative}} = -\frac{\partial}{\partial q_j} V(q_j, t). \tag{7.4.33}$$

Of course, not all forces are conservative. The virtual work in a general situation may be apportioned between conservative and nonconservative effects, such that

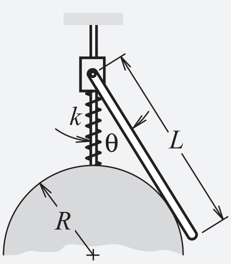
$$\delta W = (\delta W)_{\text{nonconservative}} - \delta V. \tag{7.4.34}$$

The corresponding expression for the generalized forces may be written as

$$\boxed{(Q_j)_{\text{total}} = Q_j - \frac{\partial}{\partial q_j} V(q_j, t).} \tag{7.4.35}$$

Here, and in all future developments, the symbol Q_j denotes generalized forces associated with forces and couples that are not described by the potential energy. This provides a degree of flexibility. It is not necessary to formulate the potential energy of a conservative force. If the nature of a force is uncertain, or if it is straightforward to evaluate the virtual work of a force that is known to be conservative, then that force may be considered to be nonconservative. Thus the generalized forces Q_j should be understood to describe all forces, conservative and nonconservative, whose effect is derived from an analysis of the virtual work. Obviously, it would not be correct to account for a conservative force by including it in the potential energy and also evaluating its virtual work.

EXAMPLE 7.12 The stiffness of the spring is k, and the unstretched length is R. Friction between the collar and the vertical guide bar, and between the bar and the cylinder is negligible. The mass of the bar is m and the mass of the collar is $m/2$. Determine the generalized force corresponding to θ's being the generalized coordinate.



Example 7.12

SOLUTION We see here that recognizing that a force is conservative simplifies the evaluation of generalized forces. The fact that the bar is tangent to the cylinder means that its position is uniquely specified by θ, so this system has one degree of freedom and there is no need to define other generalized coordinates. The contact point, the center of the cylinder, and the collar form a right triangle whose hypotenuse is $R+\ell$, where ℓ is the current length of the spring. We find from this triangle that

$$\ell = \frac{R}{\sin\theta} - R.$$

A free-body diagram of the bar and collar would show the spring and gravity forces, which are conservative. The diagram also would show a horizontal force exerted by the guide bar and a normal force exerted by the cylinder. The latter two are constraint forces because they prevent movement in the direction in which they act. The generalized coordinate for this system is unconstrained. These forces may be ignored in the evaluation of δW because the associated constraint conditions will not be violated in a virtual movement.

The potential energy of the spring is $\frac{1}{2}k\Delta^2$, where Δ is the elongation. We express this quantity in terms of θ by subtracting the unstretched length R from ℓ:

$$V_{\text{spring}} = \frac{1}{2}k\left(\ell - R\right)^2 = \frac{1}{2}k\left(\frac{R}{\sin\theta} - 2R\right)^2.$$

The potential energy of a gravity force near the Earth's surface is defined in terms of the elevation above a reference location, which should be a physical stationary point in the system in order to avoid confusion. A convenient location here is the center of the cylinder. The distance of the collar above this datum is $R+\ell$. The bar's center of mass is $(L/2)\cos\theta$ below the collar, so the potential energy of gravity is given by

$$V_{\text{gravity}} = \frac{mg}{2}\left(R+\ell\right) + mg\left(R+\ell - \frac{L}{2}\cos\theta\right).$$

We add the potential energy of each conservative force:

$$V = \frac{1}{2}k\left(\frac{R}{\sin\theta} - 2R\right)^2 + \frac{3}{2}mg\left(\frac{R}{\sin\theta}\right) - \frac{mgL}{2}\cos\theta.$$

These are the only forces that do virtual work, so the generalized force is found from Eq. (7.4.35) to be

$$Q_1 = -\frac{\partial V}{\partial\theta} = \left[kR^2\left(\frac{1}{\sin\theta} - 2\right) + \frac{3}{2}mgR\right]\frac{\cos\theta}{(\sin\theta)^2} - \frac{mgL}{2}\sin\theta. \qquad \triangleleft$$

This evaluation certainly is easier than one based on determining the virtual work done by the gravity and spring forces. Perhaps the only negative aspect of the procedure is that it sheds little light on the way the forces act.

7.5 LAGRANGE'S EQUATIONS

The generalized forces describe the actual force system's effects. We now turn our attention to characterizing the inertial features of a system. The outcome will be a standard set of equations of motion, which were first derived by Lagrange and therefore bear his name. We begin the derivation by considering a single particle, numbered n, in this system. Newton's Second Law describes the relationship between the resultant force $\Sigma \bar{F}_n$ acting on this particle and the particle's motion variable, acceleration $\bar{a}_n$. The principle of dynamic virtual work to this particle states that

$$\left(m_n \bar{a}_n - \Sigma \bar{F}_n\right) \cdot \delta \bar{r}_n = m_n \bar{a}_n \cdot \delta \bar{r}_n - \delta W_n = 0, \tag{7.5.1}$$

where δW_n denotes the contribution of the resultant force to the virtual work for the whole system.

Lagrange's primary contribution is to account for the dependence of kinematical variables on the generalized coordinates. Two key relations are Eqs. (7.3.7) for velocity and virtual displacement. Let $\bar{r}_n$ denote the position of the particle. The aforementioned equations then give

$$\bar{v}_n = \sum_{j=1}^{N} \left(\frac{\partial \bar{r}_n}{\partial q_j}\right) \dot{q}_j + \frac{\partial \bar{r}_n}{\partial t}, \tag{7.5.2}$$

$$\delta \bar{r}_n = \sum_{j=1}^{N} \left(\frac{\partial \bar{r}_n}{\partial q_j}\right) \delta q_j. \tag{7.5.3}$$

A crucial identity may be derived from Eq. (7.5.2). Because position depends solely on the values of the generalized coordinates and time, the partial derivatives $\partial \bar{r}_n / \partial \dot{q}_j$ depend on only those variables. Hence only the kth term the summation for $\bar{v}_n$ contains a specific $\dot{q}_k$. It follows that evaluating $\partial \bar{v}_n / \partial \dot{q}_k$ will filter out all terms other than the kth term, so it must be that

$$\frac{\partial \bar{v}_n}{\partial \dot{q}_k} \equiv \frac{\partial \bar{r}_n}{\partial q_k}, \quad k = 1, 2, \ldots, N. \tag{7.5.4}$$

Each of the preceding relations will be needed at an appropriate juncture as we manipulate the principle of dynamic virtual work. First we substitute Eq. (7.5.3) into Eq. (7.5.1), and bring the inertial term inside the sum, which gives

$$\sum_{j=1}^{N} m_n \bar{a}_n \cdot \left(\frac{\partial \bar{r}_n}{\partial q_j}\right) \delta q_j - \delta W_n = 0. \tag{7.5.5}$$

The next step is suggested by the fact that $\bar{a}_n \equiv d\bar{v}_n / dt$. Rather than differentiating only the velocity, let us introduce the rule for the time derivative of a product. Doing so results in

$$\begin{aligned} &\sum_{j=1}^{N} m_n \frac{d\bar{v}_n}{dt} \cdot \left(\frac{\partial \bar{r}_n}{\partial q_j}\right) \delta q_j - \delta W_n \\ &\equiv \sum_{j=1}^{N} \left\{ \frac{d}{dt} \left[m_n \bar{v}_n \cdot \left(\frac{\partial \bar{r}_n}{\partial q_j}\right) \right] - m_n \bar{v}_n \cdot \frac{d}{dt} \left(\frac{\partial \bar{r}_n}{\partial q_j}\right) \right\} \delta q_j - \delta W_n = 0. \end{aligned} \tag{7.5.6}$$

Consistent with the occurrence of $\bar{v}_n$, we use Eq. (7.5.4) to remove $\partial \bar{r}_n / \partial q_j$ from the first term in the summation. Furthermore, various derivatives of a quantity may be performed in any sequence, which means that

$$\frac{d}{dt}\left(\frac{\partial \bar{r}_n}{\partial q_j}\right) \equiv \frac{\partial}{\partial q_j}\left(\frac{d}{dt}\bar{r}_n\right) \equiv \frac{\partial \bar{v}_n}{\partial q_j}. \tag{7.5.7}$$

These operations convert the expression for virtual work to

$$\sum_{j=1}^{N}\left\{\frac{d}{dt}\left[m_n\bar{v}_n\cdot\left(\frac{\partial \bar{v}_n}{\partial \dot{q}_j}\right)\right] - m_n\bar{v}_n\cdot\frac{\partial \bar{v}_n}{\partial q_j}\right\}\delta q_j - \delta W_n = 0. \tag{7.5.8}$$

Each term in the summation may be represented as a derivative of $\bar{v}_n \cdot \bar{v}_n$ because a dot product is commutative. Thus we have

$$\sum_{j=1}^{N}\left\{\frac{d}{dt}\left[m_n\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\bar{v}_n\cdot\bar{v}_n\right)\right] - m_n\frac{\partial}{\partial q_j}\left(\frac{1}{2}\bar{v}_n\cdot\bar{v}_n\right)\right\}\delta q_j - \delta W_n = 0. \tag{7.5.9}$$

When we bring m_n inside each pair of parentheses, we recognize that the term being differentiated is the kinetic energy T_n of the particle, so that

$$\sum_{j=1}^{N}\left\{\frac{d}{dt}\left(\frac{\partial T_n}{\partial \dot{q}_j}\right) - \frac{\partial T_n}{\partial q_j}\right\}\delta q_j - \delta W_n = 0, \quad T_n = \frac{1}{2}m_n\bar{v}_n\cdot\bar{v}_n. \tag{7.5.10}$$

A relation such as this applies to each particle in the system. When all such equations are added, the virtual work terms describe the effect of all forces, which is described by Eq. (7.4.35). Also, the summation over all particles may be taken inside the sum over the generalized coordinate index, but the sum of T_n for all particles is merely the kinetic energy T of the system. Thus we find that

$$\sum_{j=1}^{N}\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} - Q_j\right\}\delta q_j = 0. \tag{7.5.11}$$

Although this is a single equation, it leads to multiple equations governing each of the N generalized coordinates. In a virtual displacement the generalized coordinates are given arbitrary increments. This means that the values of the δq_j are an arbitrary set of infinitesimal values. Suppose we select the first increment, δq_1, to be nonzero and all other δq_j values to be zero. Then the summation consists of only the $j = 1$ term. Because δq_1 is not zero, the coefficient contained within the brace for $j = 1$ must be zero. Now repeat the process, except make δq_2 the only nonzero virtual increment. Then the term inside the brace for $n = 2$ must be zero. Continuing this procedure to $j = N$ leads to the conclusion that the coefficient of each δq_j must vanish.

A different way to arrive at the same conclusion is to observe that δq_j are the components of the configuration space vector $\delta\hat{r}$. Correspondingly, Eq. (7.5.11) may be considered to be a dot product in the configuration space of $\delta\hat{r}$ and a vector whose components are contained inside the braces. Let $\hat{A}$ denote the latter vector. Equation (7.5.11) requires $\hat{A}\cdot\delta\hat{r} = 0$, which could simply mean that the vectors are orthogonal. However, the direction of $\delta\hat{r}$ is completely arbitrary. It is not possible for a vector to be orthogonal simultaneously to vectors in all directions, so it must be that $\hat{A}$ is a null vector. The result of either line of reasoning is that

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial}{\partial q_j} V(q_j, t) = Q_j, \quad j = 1, 2, \ldots, N,} \tag{7.5.12}$$

which are *Lagrange's equations*.

In this version of Lagrange's equations any contributions of constraint forces is contained in the generalized forces Q_j. The remainder of this chapter is limited to derivation of the equations of motion for holonomic systems. Constraint forces do not appear in the generalized forces in that case, so the q_j in the case of a holonomic system are the only dependent variables contained in these equations. Because Eq. (7.5.12) must apply for each index j, we will obtain N ordinary differential equations for the N generalized coordinates, in which the Q_j contain the nonconservative forces causing the system to move.

When the generalized coordinates are constrained, the generalized forces contain contributions from the constraint force associated with each kinematical constraint that must be enforced explicitly. Section 8.1 develops the techniques by which Lagrange's equations are employed in such situations. In any event, derivation of the equations of motion is only the first part of a dynamic analysis. Determining the system's response requires that the differential equations be solved. If these equations are not amenable to analytical solution, numerical techniques can be employed. This phase of an analysis is taken up in Sections 7.6 and 8.2.

An alternative form of Lagrange's equations features the *Lagrangian* function, which is defined to be

$$\boxed{\mathscr{L} = T - V.} \tag{7.5.13}$$

The potential energy cannot depend on the generalized velocities, so it it must be that

$$\frac{\partial \mathscr{L}}{\partial \dot{q}_j} \equiv \frac{\partial T}{\partial \dot{q}_j}, \tag{7.5.14}$$

which allows us to rewrite Lagrange's equations as

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathscr{L}}{\partial q_j} = Q_j, \quad j = 1, 2, \ldots, N.} \tag{7.5.15}$$

Although this form has one fewer term than Eq. (7.5.12), both equations require equivalent mathematical evaluations. The primary reason for introducing the Lagrangian function is its utility for the development of other kinetics principles, some of which are treated in Chapter 9.

The actual evaluation of Lagrange's equations for a specific system is straightforward, provided that one is cognizant of the difference between partial and total derivatives. For the partial derivatives the generalized coordinates q_j and generalized velocities $\dot{q}_j$ are treated as independent variables. In the total derivative with respect to time, both q_j and $\dot{q}_j$ are time-dependent quantities that must be differentiated.

A simple example that demonstrates the equivalence of Lagrange's equations and Newton's Second Law is a particle whose spatial motion is described in terms of cylindrical coordinates. We consider this motion to be the result of application of a known force $\bar{F}$, as well as gravity. Correspondingly, let the axial direction be vertical. The cylindrical coordinates for this particle are (R, θ, z), and the corresponding velocity is $\bar{v} = \dot{R}\bar{e}_R + R\dot{\theta}\bar{e}_\theta + \dot{z}\bar{e}_z$, so the kinetic energy is

$$T = \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2\right). \tag{7.5.16}$$

According to the kinematical method, the virtual displacement is

$$\delta\bar{r} = \delta R\bar{e}_R + R\delta\theta\bar{e}_\theta + \delta z\bar{e}_z, \tag{7.5.17}$$

so that the virtual work done by the nonconservative force $\bar{F}$ is

$$\delta W = F_R\delta R + F_\theta R\delta\theta + F_z\delta z. \tag{7.5.18}$$

Hence the generalized forces are

$$Q_1 = F_R, \quad Q_2 = F_\theta R, \quad Q_3 = F_z. \tag{7.5.19}$$

The gravity force was not included in the generalized forces, so we define the datum for gravity to be the xy plane, which gives $V = mgz$.

Now that all terms appearing in Lagrange's equations have been defined, we proceed to evaluate the derivatives. The generalized coordinates and velocities are independent variables for these operations, so we have

$$\begin{gathered}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{R}}\right) = \frac{d}{dt}\left(m\dot{R}\right) = m\ddot{R}, \quad \frac{\partial T}{\partial R} = mR\dot{\theta}^2, \quad \frac{\partial V}{\partial R} = 0, \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(mR^2\dot{\theta}\right) = 2mR\dot{R}\dot{\theta} + mR^2\ddot{\theta}, \quad \frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial V}{\partial \theta} = 0, \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) = \frac{d}{dt}\left(m\dot{z}\right) = m\ddot{z}, \quad \frac{\partial T}{\partial z} = 0, \quad \frac{\partial V}{\partial z} = mg.
\end{gathered} \tag{7.5.20}$$

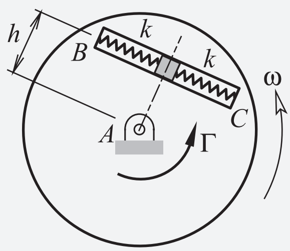
The three Lagrange equations of motion obtained from Eq. (7.5.12) are

$$\begin{aligned}
m\ddot{R} - mR\dot{\theta}^2 &= F_R, \\
2mR\dot{R}\dot{\theta} + mR^2\ddot{\theta} &= F_\theta R, \\
m\ddot{z} + mg &= F_Z.
\end{aligned} \tag{7.5.21}$$

It is evident that each of these is merely Newton's Second Law in terms of polar coordinates, with the exception that the second equation, which is associated with $q_j = \theta$, has an additional factor R. That form results from the fact that $Q_2 = RF_R$ represents the moment of the external force system about the z axis. Correspondingly, the left side of the second of Lagrange's equations is the derivative of the angular momentum, $mR^2\dot{\theta}$, about that axis.

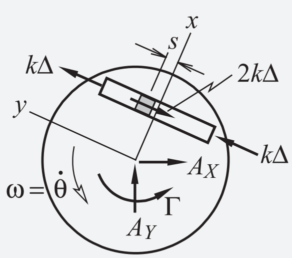
The steps we followed in this simple example parallel those for all systems for which unconstrained generalized coordinates have been selected. The bulk of the effort usually is devoted to the kinematical analysis of velocity and virtual displacement. After that, the process of forming the kinetic- and potential-energy functions, describing the generalized forces, and carrying out the derivatives required for Lagrange's equations is quite straightforward. In fact, these operations can be readily automated with the aid of symbolic mathematical software. At the same time, Lagrange's equations do not provide one with physical insight. This became apparent when we interpreted Eqs. (7.5.21) in terms of Newton's Second Law and angular momentum.

EXAMPLE 7.13 The table rotates in a horizontal plane about bearing A due to a torque Γ whose time dependence is known. The mass of the table is m_1, and its radius of gyration about its center is κ. The slider, whose mass is m_2, moves within groove BC under the restraint of two springs that are unstretched in the position shown. Derive the equations of motion for this system.



Example 7.13

SOLUTION The objective here is to gain experience in the basic operations associated with forming Lagrange's equations. The position of the table is defined by its rotation about its bearing, and the slider executes a rectilinear motion relative to the table, so suitable generalized coordinates are the angle of rotation, $q_1 = \theta$, and the displacement of the slider relative to the unstretched position of the springs, $q_2 = s$. We show these in a free-body diagram of the system.



Free-body diagram of the system formed by the slider and the turntable.

Note that, although the spring forces exerted between the slider and the table are internal to the system, they are not constraint forces, so they are depicted in this diagram.

The kinetic energy is the sum of the values for the table and for the slider:

$$T = \frac{1}{2}\left(m_1\kappa^2\right)\dot{\theta}^2 + \frac{1}{2}m_2 v_s^2.$$

The velocity of the slider may be related to the generalized coordinates by use of moving reference frame xyz attached to the table. The velocity of the slider relative to the table is $\dot{s}$ parallel to the groove, so

$$\bar{v}_s = \dot{s}\,\bar{j} + \dot{\theta}\bar{k} \times \left(h\bar{i} + s\,\bar{j}\right) = -s\dot{\theta}\bar{i} + \left(\dot{s} + h\dot{\theta}\right)\bar{j}.$$

This gives

$$\begin{aligned} T &= \frac{1}{2}\left(m_1\kappa^2\right)\dot{\theta}^2 + \frac{1}{2}m_2\left[s^2\dot{\theta}^2 + \left(\dot{s} + h\dot{\theta}\right)^2\right] \\ &= \frac{1}{2}\left(m_1\kappa^2 + m_2 s^2 + m_2 h^2\right)\dot{\theta}^2 + \frac{1}{2}m_2\dot{s}^2 + m_2 h\dot{\theta}\dot{s}. \end{aligned}$$

The deformation of each spring is $\Delta = \pm s$, with the sign indicating whether positive s corresponds to the spring being elongated or compressed. This sign does not matter because Δ is squared to obtain potential energy. Thus

$$V = 2\left(\frac{1}{2}ks^2\right).$$

The bearing forces constrain point A from moving, so they do not contribute to the generalized forces. The remaining forcing effect is the couple Γ. A virtual movement increases θ by $\delta\theta$, so the virtual force is

$$\delta W = \Gamma\delta\theta = Q_1\delta\theta + Q_2\delta s.$$

The corresponding generalized forces are

$$Q_1 = \Gamma, \quad Q_2 = 0.$$

All terms appearing in Lagrange's equations have been described, so we proceed to evaluate the various derivatives:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) &= \frac{d}{dt}\left[\left(m_1\kappa^2 + m_2 s^2 + m_2 h^2\right)\dot{\theta} + m_2 h\dot{s}\right] \\ &= \left(m_1\kappa^2 + m_2 s^2 + m_2 h^2\right)\ddot{\theta} + 2m_2 s\dot{s}\dot{\theta} + m_2 h\ddot{s}, \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{s}}\right) &= \frac{d}{dt}\left(m_2\dot{s} + m_2 h\dot{\theta}\right) = m_2\left(\ddot{s} + h\ddot{\theta}\right), \end{aligned}$$

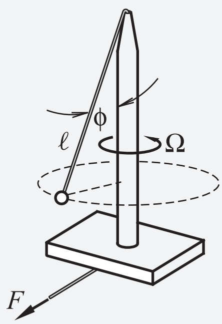
$$\frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial T}{\partial s} = m_2 s\dot{\theta}^2, \quad \frac{\partial V}{\partial \theta} = 0, \quad \frac{\partial V}{\partial s} = 2ks.$$

The corresponding Lagrange's equations are

$$\begin{aligned} \left(m_1\kappa^2 + m_2 s^2 + m_2 h^2\right)\ddot{\theta} + 2m_2 s\dot{s}\dot{\theta} + m_2 h\ddot{s} &= \Gamma, \\ m_2\left(\ddot{s} + h\ddot{\theta} - s\dot{\theta}^2\right) + 2ks &= 0. \end{aligned} \quad \triangleleft$$

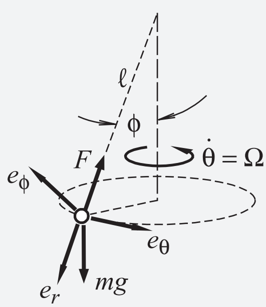
Some effects associated with these terms are readily identified. For example, $m_1\kappa^2 + m_2 s^2 + m_2 h^2$ represents the total moment of inertia of the system about bearing A. If we wished to explain other terms, we could consider the Newton–Euler formulation, in which we would isolate the slider and the turntable in individual free-body diagrams. The corresponding equations of motion would be $m_2\left(\bar{a}_s \cdot \bar{i}\right) = \Sigma F_x$ and $m_2\left(\bar{a}_s \cdot \bar{j}\right) = \Sigma F_y$ for the slider, and $m_1\kappa^2\ddot{\theta} = \Sigma M_z$ for the table. Such a derivation would bring into the analysis the normal force exerted between the slider and the table.

EXAMPLE 7.14 A small sphere of mass m is suspended from the top of a hollow pole through which the cable passes. The cable's free end is pulled inward by the tensile force $\bar{F}$, such that the length of the cable is a specified function $\ell(t)$. The sphere is given an initial velocity that causes it to rotate about the pole, as well as to swing outward from the pole, so that neither the precession rate Ω nor the polar angle ϕ are constant. Determine the equations of motion for the sphere.



Example 7.14

SOLUTION Although the requested equations of motion could be readily obtained by applying Newton's Second Law in conjunction with a description of the sphere's motion in terms of spherical coordinates, using Lagrange's equations to solve this example will highlight the differences in the way that known and unknown position parameters are treated. Spherical coordinates describing the position of the sphere are depicted in the free-body diagram.



Free-body diagram of the suspended mass.

The radial distance ℓ is a specified function of time, so we do not treat it as a generalized coordinate. We therefore have $q_1 = \phi$, $q_2 = \theta$, with $\dot{\theta} \equiv \Omega$.

The fact that ℓ is not a generalized coordinate is irrelevant to the description of the kinetic energy. The sphere's velocity in terms of spherical coordinates is

$$\bar{v} = \dot{\ell}\bar{e}_r + \ell\dot{\phi}\bar{e}_\phi + \ell\dot{\theta}\sin\phi\bar{e}_\theta.$$

The corresponding kinetic energy is

$$T = \frac{1}{2}m\left[\dot{\ell}^2 + \ell^2\dot{\phi}^2 + \ell^2\dot{\theta}^2(\sin\phi)^2\right].$$

The top of the post is a convenient datum for gravitational potential energy, so

$$V = mg(-\ell\cos\phi).$$

Other than gravity, the only force acting on the suspended sphere is the cable force. This force does not contribute to the generalized forces. One way of recognizing this is to evaluate the virtual work. To convert the preceding expression for $\bar{v}$ to the virtual displacement, we replace $\dot{\phi}$ with $\delta\phi$ and $\dot{\theta}$ with $\delta\theta$, and drop the $\dot{\ell}$ term because it is not a generalized velocity. Thus,

$$\delta\bar{r} = \ell\delta\phi\bar{e}_\phi + \ell\delta\theta\sin\phi\bar{e}_\theta.$$

The tensile force acts radially, so $\delta W = (-F\bar{e}_r)\cdot\delta\bar{r} = 0$, giving $Q_1 = Q_2 = 0$. Of course, rather than evaluating the virtual work, we could have observed that $\bar{F}$ is the force that makes ℓ change in the specified manner. This means that it is a constraint force, and constraint forces do not appear in the generalized forces when the generalized coordinates are unconstrained.

We now proceed to evaluate the derivatives for Lagrange's equations. Observe that, in these derivatives, ℓ is held constant in the partial differentiations, whereas the variability of ℓ must be recognized when total derivatives with respect to t are evaluated. Thus,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}}\right) = \frac{d}{dt}\left(m\ell^2\dot{\phi}\right) = m\left(\ell^2\ddot{\phi} + 2\ell\dot{\ell}\dot{\phi}\right),$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) = \frac{d}{dt}\left[m\ell^2\dot{\theta}(\sin\phi)^2\right] = m\left[\ell^2\ddot{\theta}(\sin\phi)^2 + 2\ell\dot{\ell}\dot{\theta}(\sin\phi)^2 + 2\ell^2\dot{\theta}\dot{\phi}\sin\phi\cos\phi\right],$$

$$\frac{\partial T}{\partial\phi} = m\ell^2\dot{\theta}^2\sin\phi\cos\phi, \qquad \frac{\partial T}{\partial\theta} = 0, \qquad \frac{\partial V}{\partial\phi} = mg\ell\sin\phi, \qquad \frac{\partial V}{\partial\theta} = 0.$$

The corresponding Lagrange's equations are

$$\begin{aligned}\ell^2\ddot{\phi} + 2\ell\dot{\ell}\dot{\phi} - \ell^2\dot{\theta}^2\sin\phi\cos\phi + g\ell\sin\phi &= 0,\\ \ell^2\ddot{\theta}(\sin\phi)^2 + 2\ell\dot{\ell}\dot{\theta}(\sin\phi)^2 + 2\ell^2\dot{\phi}\dot{\theta}\sin\phi\cos\phi &= 0.\end{aligned} \qquad \triangleleft$$

We may verify that these equations are correct by recalling the relations for acceleration in terms of spherical coordinates. The preceding equations are merely $\Sigma F_\phi = ma_\phi$, multiplied by ℓ/m, and $\Sigma F_\phi = ma_\phi$, multiplied by $(\ell/m)\sin\phi$.

It is possible to remove one equation of motion by a procedure that anticipates the treatment of ignorable generalized coordinates in Subsection 9.2.3. Because T and V do not explicitly depend on θ, and $Q_2 = 0$, the second Lagrange equation states that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = 0,$$

from which it follows that $\partial T/\partial \dot{\theta}$ is a constant value we denote as $m\beta$. For the T function of this system, we therefore have

$$\ell^2 \dot{\theta} (\sin\phi)^2 = \beta,$$

which is what we would find if we actually integrated the preceding second Lagrange equation. The value of β may be determined by substituting the values of the generalized coordinates and generalized velocities at the initial instant, which must be specified if the response is to be uniquely defined. Once that value is known, we may solve the preceding relation for $\dot{\theta}$, and then use that result to eliminate $\dot{\theta}$ from the Lagrange equation for ϕ. This yields

$$\dot{\theta} = \frac{\beta}{\ell^2 (\sin\phi)^2},$$

$$\ell\ddot{\phi} + 2\dot{\ell}\dot{\phi} - \frac{\beta^2 \cos\phi}{\ell^3 (\sin\phi)^3} + g\sin\phi = 0.$$

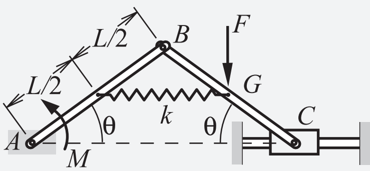
The fact that ℓ is not constant would necessitate solution by numerical methods. However, it is slightly less difficult to solve the second equation than the two Lagrange equations originally obtained. Also, having a single differential equation makes it somewhat easier to apply formal mathematical analysis tools to identify fundamental properties.

It is useful at this juncture to compare the situations in the preceding two examples. In the second, ℓ was a length that depended on time in a specified manner as a result of the unknown tensile force F. This force did not enter into the analysis because it is a constraint force whose kinematical condition is not violated when the generalized coordinates ϕ and θ are given virtual increments. In contrast, in the first example the torque causing the rotation was a specified function of time and the corresponding angle of rotation was unknown. That analysis led to an equation of motion for the rotation angle. Similarly, specifying F in the second example would lead to ℓ being unknown, and therefore a suitable generalized coordinate. The result would be three equations of motion.

These observations lead to an important alternative for systems in which a force (or couple) causes a length (or angle) parameter ξ to change in a specified manner. We can consider the force to be an unknown constraint force and treat ξ as we would any other parameter. In that approach the unknown force will not appear in any of the equations of motion. On the other hand, we can defer accounting for the fact that the time dependence of ξ is specified. In that case ξ becomes a generalized coordinate, and the force imposing the motion contributes to the generalized forces in the manner of a standard applied force. This leads to an additional equation of motion associated with ξ. On substitution of the specified dependence of ξ, the other equations of motion become the same as those derived in the first approach. After we solve those equations, we can substitute the resulting generalized coordinates into the ξ equation. Doing so will yield an algebraic equation for the force imposing the motion.

This alternative is understandable from a more general perspective. When we derive equations of motion ignoring the fact that position parameters are known, we are using constrained generalized coordinates. The constraint equations are carried through the analysis, and enforced explicitly as additions to the equations of motion. In such an analysis the associated constraint forces enter into the equations of motion. The situation in the preceding example, in which one position variable is specified, is a special case of constrained generalized coordinates. The first section of Chapter 8 addresses situations in which several generalized coordinates are related by multiple constraint equations.

EXAMPLE 7.15 The linkage in the sketch is like the one in Fig. 7.2, except for the addition of spring k. The spring may sustain compressive as well as tensile forces, and its unstretched length is $L/2$. The mass of each bar is m_1, and m_2 is the mass of collar C. Use Lagrange's equations to derive the equation of motion for θ.



Example 7.15

SOLUTION This analysis demonstrates the efficacy of the Lagrange equation approach for a simple linkage. Similar procedures could be applied to linkages whose geometry is more complicated, but one should also examine the use of constrained generalized coordinates for such situations, as will be discussed in Chapter 8. We can use Fig. 7.2 as the system's free-body diagram by ignoring the inertial effects depicted there and allowing for spring forces on each bar. We use θ as the generalized coordinate for this one-degree-of-freedom system.

To express T solely in terms of θ, we must describe the velocities of collar C and of the center of mass of bar BC in terms of this variable. The required expressions are readily found by differentiating the respective position vectors:

$$\bar{r}_{G/A} = \frac{3}{2}L\cos\theta\bar{i} + \frac{1}{2}L\sin\theta\bar{j}, \quad \bar{r}_{C/A} = 2L\cos\theta\bar{i},$$
$$\bar{v}_G = \frac{L}{2}\dot{\theta}\left(-3\sin\theta\bar{i} + \cos\theta\bar{j}\right), \quad \bar{v}_C = -2L\dot{\theta}\sin\theta\bar{i}.$$

Using these expressions to form the kinetic-energy function of the bars and collar C gives

$$T = \frac{1}{2}I_A\dot{\theta}^2 + \frac{1}{2}m_1 v_G^2 + \frac{1}{2}I_G\dot{\theta}^2 + \frac{1}{2}m_2 v_C^2$$

$$= \frac{1}{2}(I_A + I_G)\dot{\theta}^2 + \frac{1}{2}m_1\left(\frac{L}{2}\dot{\theta}\right)^2\left[(3\sin\theta)^2 + (\cos\theta)^2\right] + \frac{1}{2}m_2\left(2L\dot{\theta}\right)^2(\sin\theta)^2$$

$$= \frac{1}{2}\left[I_A + I_G + \frac{1}{4}m_1 L^2 + (2m_1 + 4m_2)L^2(\sin\theta)^2\right]\dot{\theta}^2.$$

Subtracting the unstretched length of the spring from its length at an arbitrary θ shows that the elongation is $L\cos\theta - L/2$. We place the datum for gravitational potential energy at pin A, so the potential-energy function is

$$V = \frac{1}{2}k(L\cos\theta - L/2)^2 + mg\left(\frac{L}{2}\sin\theta\right)(2).$$

The effects of the applied force $\bar{F}$ and torque Γ are contained in the virtual work. We find the virtual displacement of point G where $\bar{F}$ acts by applying the kinematical method to the expression for $\bar{v}_G$. Thus,

$$\delta\bar{r}_G = \frac{L}{2}\delta\theta\left(-3\sin\theta\,\bar{i} + \cos\theta\,\bar{j}\right).$$

The virtual work is

$$\delta W = \left(-F\bar{j}\right)\cdot\delta\bar{r}_G + M\delta\theta = Q_1\delta\theta,$$

so the generalized force is

$$Q_1 = -\frac{L}{2}F\cos\theta + M.$$

The derivatives in Lagrange's equations for this system are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) = \frac{d}{dt}\left\{\left[I_A + I_G + \frac{1}{4}m_1L^2 + (2m_1 + 4m_2)L^2(\sin\theta)^2\right]\dot{\theta}\right\}$$

$$= \left[I_A + I_G + \frac{1}{4}m_1L^2 + (2m_1 + 4m_2)L^2(\sin\theta)^2\right]\ddot{\theta}$$

$$+ (4m_1 + 8m_2)L^2(\sin\theta\cos\theta)\dot{\theta}^2,$$

$$\frac{\partial T}{\partial\theta} = (2m_1 + 4m_2)L^2(\sin\theta\cos\theta)\dot{\theta}^2,$$

$$\frac{\partial V}{\partial\theta} = k(L\cos\theta - L/2)(-L\sin\theta) + mgL\cos\theta.$$

The corresponding Lagrange equation is

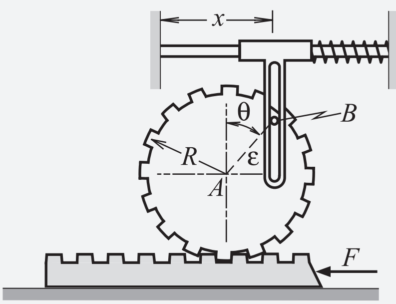
$$\left[I_A + I_G + \frac{1}{4}m_1L^2 + (2m_1 + 4m_2)L^2(\sin\theta)^2\right]\ddot{\theta}$$

$$+ (m_1 + 2m_2)L^2(\sin 2\theta)\dot{\theta}^2 + \frac{1}{2}kL^2[\sin(\theta) - \sin(2\theta)]$$

$$+ mgL\cos\theta = \frac{FL}{2}\cos\theta + M.$$

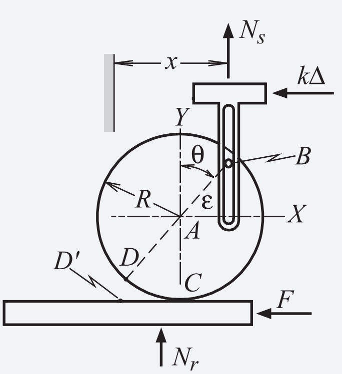
Setting $k = 0$ and $m_2 = 0$ in this equation of motion gives the same result as Eq. (7.1.11). A comparison of the procedures discloses that, although implementation of the principle of dynamic virtual work and Lagrange's equations both involve virtual displacement, the latter only requires such an evaluation to describe nonconservative applied forces. Furthermore, forming Lagrange's equations does not require a kinematical analysis of acceleration, and the analysis is performed in the context of a general methodology.

EXAMPLE 7.16 Motion of the system in the sketch is actuated by the horizontal force $\bar{F}(t)$ applied to the rack. The spring, whose stiffness is k, is unstretched when $x = 2R$, at which location $\theta = 0$. The masses are m_A for the gear, m_s for the slider, and m_r for the rack, and the radius of gyration of the gear about its center is κ. Friction between pin B and the rack is negligible. Derive the differential equations of motion.



Example 7.16

SOLUTION This example gives a comprehensive view regarding how one treats planar systems in which bodies roll without slipping. It is useful to begin with a free-body diagram of the entire system, in which the relevant kinematical variables also are defined. The gravity forces are omitted because they act vertically and the centers of mass move horizontally.



Free-body diagram of the system formed by the gear, the fork, and the rack.

This system has two degrees of freedom, which is indicated by the observation that x locates the slider and θ locates the gear relative to the slider. There is no slippage between the gear and the rack, so point D on the perimeter of the gear diametrically opposite pin B was the point of contact with the gear when $\theta = 0$. The arc length from point D to the current contact point C is $R\theta$, which must equal the distance along the rack from point C to point D' on the rack, which contacted point D. This enables us to locate any point on the rack. Another line of reasoning leading to the same conclusion evaluates velocities, which we will need to do to form the kinetic energy. Let x_r denote the distance the rack moves rightward from its starting position. We relate the velocities of points C, A, and B by using the facts that they are common to the gear, that point A moves horizontally, and that the horizontal movement of pin B must match the slider's motion. Thus,

$$\bar{v}_C = \dot{x}_r \bar{I}, \quad \bar{v}_A = v_A \bar{I} = \bar{v}_C + \left(-\dot{\theta}\bar{K}\right) \times R\bar{J},$$
$$\bar{v}_B = \bar{v}_A + \left(-\dot{\theta}\bar{K}\right) \times \left(\varepsilon \sin\theta \bar{I} + \varepsilon \cos\theta \bar{J}\right), \quad \bar{v}_s = \dot{x}\bar{I}, \quad \dot{x} = \bar{v}_B \cdot \bar{I}.$$

Matching like components in the preceding conditions leads to

$$v_A = \dot{x}_r + R\dot{\theta}, \quad \dot{x} = v_A + \varepsilon\dot{\theta}\cos\theta, \tag{1}$$

from which we find that

$$\dot{x}_r = \dot{x} - (R + \varepsilon\cos\theta)\,\dot{\theta}. \tag{2}$$

This is a linear velocity constraint relating x, θ, and x_r, which proves that the system has two degrees of freedom. Furthermore, it is holonomic. Multiplying it be dt and integrating leads to

$$x_r = x - R\theta - \varepsilon\sin\theta + S, \tag{3}$$

where S is a constant of integration that may be evaluated if the values of x_r, x, and θ at some reference location, such as the initial position, are given. Thus we may take $q_1 = x$ and $q_2 = \theta$, and use Eqs. (2) and (3) to eliminate any dependence on x_r.

We form the kinetic energy of the system by adding the energies of the three bodies, with Eqs. (1) and (2) used to describe v_A and $\dot{x}_r$. Thus,

$$\begin{aligned} T &= \frac{1}{2}m_s\dot{x}^2 + \frac{1}{2}m_A v_A^2 + \frac{1}{2}\left(m_A\kappa^2\right)\dot{\theta}^2 + \frac{1}{2}m_r\dot{x}_r^2 \\ &= \frac{1}{2}m_s\dot{x}^2 + \frac{1}{2}m_A\left(\dot{x} - \varepsilon\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}\left(m_A\kappa^2\right)\dot{\theta}^2 + \frac{1}{2}m_r\left[\dot{x} - (R + \varepsilon\cos\theta)\,\dot{\theta}\right]^2 \\ &= \frac{1}{2}\left(m_s + m_A + m_r\right)\dot{x}^2 + \frac{1}{2}\left[m_A\kappa^2 + m_A\varepsilon^2\left(\cos\theta\right)^2 + m_r\left(R + \varepsilon\cos\theta\right)^2\right]\dot{\theta}^2 \\ &\quad - \left[m_r R + (m_A + m_r)\,\varepsilon\cos\theta\right]\dot{x}\dot{\theta}. \end{aligned} \tag{4}$$

Note that the terms in the kinetic-energy function are grouped according to their dependence on the generalized velocities in order to expedite evaluation of the derivatives in Lagrange's equations.

The spring is the only conservative force requiring consideration. It is given that the spring is unstretched when $x = 2R$, so the spring is shortened at a general position by $x - 2R$, and the potential energy is

$$V = \frac{1}{2}k\,(x - 2R)^2\,.$$

To account for the nonconservative force $\bar{F}$, we evaluate its virtual work. We may describe $\delta\bar{r}_r$ by applying the kinematical method to Eq. (2), which gives

$$\delta\bar{r}_r = [\delta x - \,(R + \varepsilon\cos\theta)\,\delta\theta]\,\bar{I},$$
$$\delta W = \left(-F\bar{I}\right)\cdot\delta\bar{r}_r = -F\delta x + F\,(R + \varepsilon\cos\theta)\;\delta\theta.$$

Matching the corresponding virtual work to the standard form, $\delta W = Q_1\delta x + Q_2\delta\theta$, gives

$$Q_1 = -F,\quad Q_2 = F\,(R + \varepsilon\cos\theta)\,. \tag{5}$$

For the kinetic energy described by Eq. (4), the derivatives appearing in Lagrange's equations are

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{x}}\right) &= \frac{d}{dt}\left\{(m_s + m_A + m_r)\,\dot{x} - [m_r R + (m_A + m_r)\,\varepsilon\cos\theta]\,\dot{\theta}\right\}\\ &= (m_s + m_A + m_r)\,\ddot{x} - [m_r R + (m_A + m_r)\,\varepsilon\cos\theta]\,\ddot{\theta}\\ &\quad + (m_A + m_r)\,\varepsilon\,(\sin\theta)\,\dot{\theta}^2,\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) &= \frac{d}{dt}\left\{\left[m_A\kappa^2 + m_A\varepsilon^2\,(\cos\theta)^2 + m_r\,(R + \varepsilon\cos\theta)^2\right]\dot{\theta}\right.\\ &\quad \left. - [m_r R + (m_A + m_r)\,\varepsilon\cos\theta]\,\dot{x}\right\}\\ &= \left[m_A\kappa^2 + m_A\varepsilon^2\,(\cos\theta)^2 + m_r\,(R + \varepsilon\cos\theta)^2\right]\ddot{\theta}\\ &\quad - [m_r R + (m_A + m_r)\,\varepsilon\cos\theta]\,\ddot{x} - 2\left[(m_A + m_r)\,\varepsilon^2\cos\theta\sin\theta\right.\\ &\quad \left. + m_r R\varepsilon\sin\theta\right]\dot{\theta}^2 + (m_A + m_r)\,\varepsilon\,(\sin\theta)\,\dot{x}\dot{\theta},\end{aligned}$$

$$\frac{\partial T}{\partial x} = 0,$$

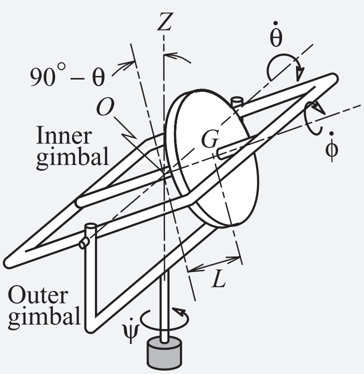
$$\frac{\partial T}{\partial\theta} = \left[-\,(m_A + m_r)\,\varepsilon^2\,(\cos\theta\sin\theta) + m_r R\varepsilon\sin\theta\right]\dot{\theta}^2 + (m_A + m_r)\,\varepsilon\,(\sin\theta)\,\dot{x}\dot{\theta},$$

$$\frac{\partial V}{\partial x} = k\,(x - 2R)\,,\quad \frac{\partial V}{\partial\theta} = 0.$$

The corresponding set of Lagrange's equations is

$$\begin{aligned}&(m_s + m_A + m_r)\,\ddot{x} - [m_r R + (m_A + m_r)\,\varepsilon\cos\theta]\,\ddot{\theta}\\&\quad + (m_A + m_r)\,\varepsilon\,(\sin\theta)\,\dot{\theta}^2 + k\,(x - 2R) = -F,\\&\left[m_A\kappa^2 + m_A\varepsilon^2\,(\cos\theta)^2 + m_r\,(R + \varepsilon\cos\theta)^2\right]\ddot{\theta} - [m_r R + (m_A + m_r)\,\varepsilon\cos\theta]\,\ddot{x}\\&\quad - \left[(m_A + m_r)\,\varepsilon^2\cos\theta\sin\theta + m_r R\varepsilon\sin\theta\right]\dot{\theta}^2 = F\,(R + \varepsilon\cos\theta).\end{aligned} \qquad (6) \triangleleft$$

EXAMPLE 7.17 The device depicted in the sketch is a simple model of a gyroscope. Servomotors make the flywheel spin at a constant rate $\dot{\phi}$ and maintain the precession rate $\dot{\psi}$ at a constant value. The Z axis is vertical, and the center of mass of the flywheel is situated on the spin axis at distance L from point O, which is the intersection of the precession axis and the line of nodes. The flywheel's centroidal moments of inertia are I_1 about the spin axis and I_2 transverse to that axis, and the inertia of the gimbals may be neglected. Derive the differential equations of motion for the system.



Example 7.17

SOLUTION We see here that Lagrange's equations are readily employed to analyze systems in spatial motion. This system is quite similar to the one in Example 6.2, so a comparison of the present and previous solutions affords one an opportunity to judge the merits of the Newton–Euler and Lagrange formulations. Because point O is stationary, the orientation of the disk, which is the only inertial body in the model, is completely specified by its three Eulerian angles. However, it is specified that only the nutation angle is uncontrolled, so this system has one degree of freedom, which is well represented by selecting $q_1 = \theta$. The system that is best considered includes the flywheel and gimbals, because doing so enables us to avoid consideration of all bearing effects other than the one at the base. There, we see that the base does not move and the torques that are exerted prevent rotation perpendicular to the precession axis. Thus the virtual work is zero, so $Q_1 = 0$. Note that the role of the servomotors is to maintain a constant rotation rate, so the associated couple moments do not contribute to the virtual work.

To describe the kinetic energy we need a general description of the angular velocity of the disk. The axisymmetry of the disk makes it suitable to use a coordinate system attached to the inner gimbal with origin O; the z axis is defined such that $\bar{k} = \bar{e}_{G/O}$ and the x axis is upward in the vertical plane. Then the angular velocity of the disk at any instant is given by

$$\begin{aligned}\bar{\omega} &= \dot{\psi}\bar{K} - \dot{\theta}\bar{j} - \dot{\phi}\bar{k} \\ &= \dot{\psi}\sin\theta\bar{i} - \dot{\theta}\bar{j} + \left(\dot{\psi}\cos\theta - \dot{\phi}\right)\bar{k}.\end{aligned}$$

The inertia properties with respect to xyz are

$$I_{xx} = I_{yy} = I_2 + mL^2, \quad I_{zz} = I_1.$$

Because $\bar{v}_O = \bar{0}$, the disk is in pure rotation about that point, so we have

$$\begin{aligned}T &= \frac{1}{2}\left[\left(I_2 + mL^2\right)\left(\left(\dot{\psi}\sin\theta\right)^2 + \dot{\theta}^2\right) + I_1\left(\dot{\psi}\cos\theta - \dot{\phi}\right)^2\right] \\ &= \frac{1}{2}\left[\left(I_2 + mL^2\right)(\sin\theta)^2 + I_1(\cos\theta)^2\right]\dot{\psi}^2 \\ &\quad + \frac{1}{2}\left(I_2 + mL^2\right)\dot{\theta}^2 - I_1\dot{\psi}\dot{\phi}\cos\theta + \frac{1}{2}I_1\dot{\phi}^2.\end{aligned}$$

The gravity force acts parallel to the Z axis through point G, so the potential energy is

$$V = mgL\sin\theta.$$

In the derivatives for Lagrange's equations the values of $\dot{\psi}$ and $\dot{\phi}$ are treated as constants, so that

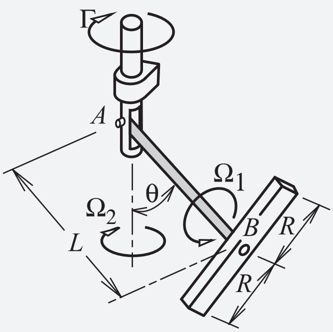
$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) &= \left(I_2 + mL^2\right)\ddot{\theta}, \\ \frac{\partial T}{\partial\theta} &= \left(I_2 + mL^2 - I_1\right)\sin\theta\cos\theta + I_1\dot{\psi}\dot{\phi}\sin\theta, \\ \frac{\partial V}{\partial\theta} &= mgL\sin\theta.\end{aligned}$$

The corresponding equation of motion is

$$\left(I_2 + mL^2\right)\ddot{\theta} - \left(I_2 + mL^2 - I_1\right)\sin\theta\cos\theta + \left(mgL - I_1\dot{\psi}\dot{\phi}\right)\sin\theta = 0.$$

An interesting aspect of the analysis is that it is readily generalized to handle the case in which the precession or spin rates are specified time functions. Doing so would merely require that the derivatives occurring in Lagrange's equations recognize the dependence of those rates.

EXAMPLE 7.18 A servomotor makes the bar spin about shaft AB at angular speed Ω_1 that is a specified function of time, whereas the known torque Γ induces the unknown precession rate Ω_2. Pin A allows the angle θ to change. The bar, which may be considered to be thin, has mass are m_1, whereas the mass of shaft AB is m_2, and the moment of inertia of the vertical shaft about its axis of rotation is I_3. Derive the equations of motion and an expression for the required torque.



Example 7.18

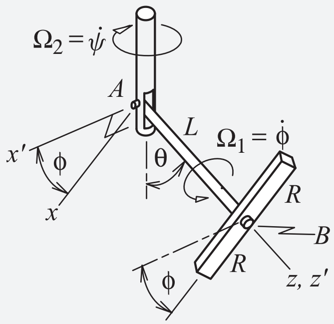
SOLUTION This example generalizes the techniques for treating spatial motion by introducing rotation transformations to describe the motion of a nonaxisymmetric body. In terms of Eulerian angles $\Omega_1 = \dot{\phi}$ is the spin rate, $\Omega_2 = \dot{\psi}$ is the precession rate, and θ is the nutation angle. Integrating the spin rate subject to the initial condition that $\phi = 0$ at $t = 0$ gives

$$\phi = \int_0^t \Omega_1(\tau)\, d\tau. \tag{1}$$

Hence, the position of all parts is specified by knowing t, which gives ϕ, and the corresponding values of the other Eulerian angles. Thus we select as the generalized coordinates, $q_1 = \psi$ and $q_2 = \theta$. To ensure that we recognize the functional dependence of all terms, we denote the respective rotation rates as $\dot{\psi}$ and $\dot{\phi}$, rather than Ω_1 and Ω_2.

The vertical shaft's bearing permits rotation only about its axis. Thus a free-body diagram of the system would show, at that bearing, a force–couple reaction that is arbitrary, except that the couple would have no vertical component. The free-body diagram also would show the conservative gravity forces acting on the flywheel and shaft AB, and the nonconservative torque Γ.

Unlike the previous example, the present system is not axisymmetric, so describing the kinetic energy will require somewhat more effort. The sketch defines two moving coordinate systems: xyz is attached to the bar, whereas $x'y'z'$ only precesses and nutates. Both the z' and z axes coincide with the spin axis, and x' is defined to align with the line of nodes, whereas the x axis is parallel to the bar's longitudinal axis. The angle between x' and x is ϕ.



Moving coordinate systems and generalized coordinates for the spinning bar and the shafts that support it.

The xyz coordinate system executes a simple rotation about the z' axis relative to $x'y'z'$, so the associated rotation transformation is

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix}. \tag{4}$$

The angular velocity of the bar is readily found in terms of $x'y'z'$ components to be $\bar{\omega}_{\text{bar}} = \dot{\theta}\bar{i}' + \dot{\psi}\left(\sin\theta\,\bar{j}' + \cos\theta\bar{k}'\right) + \dot{\phi}\bar{k}'$. The introduction of rotation transformations makes it convenient to switch to matrix notation. Then the xyz components are given by

$$\{\omega_{\text{bar}}\} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{\psi}\sin\theta \\ \dot{\psi}\cos\theta + \dot{\phi} \end{Bmatrix} = \begin{Bmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\ -\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi \\ \dot{\psi}\cos\theta + \dot{\phi} \end{Bmatrix}. \tag{5}$$

The inertia matrix for the bar relative to xyz is obtained from the tabulated properties and the parallel axis theorems, which give

$$[I_{\text{bar}}] = \begin{bmatrix} m_1 L^2 & 0 & 0 \\ 0 & \frac{1}{12}m_1(2R)^2 + m_1 L^2 & 0 \\ 0 & 0 & \frac{1}{12}m_1(2R)^2 \end{bmatrix}.$$

The kinetic energy of the bar is $T_{\text{bar}} = \frac{1}{2}\{\omega_{\text{bar}}\}^{\text{T}}[I_{\text{bar}}]\{\omega_{\text{bar}}\}$. The $x'y'z'$ axes are principal axes for shaft AB, with $I_{x'x'} = I_{y'y'} = (1/3)\,m_2L^2$, $I_{z'z'} = 0$, and the angular velocity of this shaft is $\bar{\omega}_{AB} = \dot{\theta}\bar{i}' + \dot{\psi}\left(\sin\theta\,\bar{j}' + \cos\theta\bar{k}'\right)$. The vertical shaft is in pure rotation at rate $\dot{\psi}$, and its moment of inertia about its rotation axis is I_3. The sum of the kinetic energy of each body is

$$T_{\text{bar}} = \frac{1}{2}\Big[\left(m_1 L^2\right)\left(\dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi\right)^2 + \left(\frac{1}{3}m_1 R^2 + m_1 L^2\right)\left(-\dot{\theta}\sin\phi + \dot{\psi}\sin\theta\cos\phi\right)^2 + \left(\frac{1}{3}m_1 R^2\right)\left(\dot{\psi}\cos\theta + \dot{\phi}\right)^2\Big] + \frac{1}{2}\left(\frac{1}{3}m_2 L^2\right)\Big[\left(\dot{\psi}\sin\theta\right)^2 + \dot{\theta}^2\Big] + \frac{1}{2}I_3\dot{\psi}^2. \tag{6}$$

Collecting like coefficients of the generalized velocities in T expedites evaluation of the derivatives in Lagrange's equations, so we write the expression as

$$T = \frac{1}{2}\Big\{\left(I_2' + I_3'\right)(\sin\theta)^2 + I_1'\Big[1 - (\sin\theta)^2(\sin\phi)^2\Big] + I_3\Big\}\dot{\psi}^2 + \frac{1}{2}\Big[I_2' + I_3' + I_2'(\sin\phi)^2\Big]\dot{\theta}^2 + \frac{1}{2}I_1'\dot{\phi}^2 - \frac{1}{2}I_1'\dot{\psi}\dot{\theta}(\sin\theta)(\sin 2\phi) + I_1'\dot{\psi}\dot{\phi}(\cos\theta), \tag{7}$$

where

$$I_1' = \frac{1}{3}m_1 R^2, \quad I_2' = m_1 L^2, \quad I_3' = \frac{1}{3}m_2 L^2.$$

The elevations of the centers of mass depend only on θ. The corresponding potential energy is

$$V = -\left(m_1 + \frac{1}{2}m_2\right) g L\cos\theta \tag{8}$$

We may now form Lagrange's equations for each generalized coordinate. The derivatives of T are more intricate than they were in the previous problem because the precession angle now is a generalized coordinate and the spin rate is variable. The derivatives are

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\psi}}\right) &= \frac{d}{dt}\Big\{\Big[\left(I_2' + I_3'\right)(\sin\theta)^2 + I_1'\Big[1 - (\sin\theta)^2(\sin\phi)^2\Big] + I_3\Big]\dot{\psi} \\
&\qquad - \frac{1}{2}I_1'\dot{\theta}(\sin\theta)(\sin 2\phi) + I_1'\dot{\phi}(\cos\theta)\Big\} \\
&= \Big\{\left(I_2' + I_3'\right)(\sin\theta)^2 + I_1'\Big[1 - (\sin\theta)^2(\sin\phi)^2\Big] + I_3\Big\}\ddot{\psi} \\
&\qquad - \Big[\frac{1}{2}I_1'(\sin\theta)(\sin 2\phi)\Big]\ddot{\theta} + I_1'\ddot{\phi}(\cos\theta) \\
&\qquad + \Big[\Big(I_2' + I_3' - I_1'(\sin\phi)^2\Big)(\sin 2\theta)\Big]\dot{\psi}\dot{\theta} - \Big[I_1'(\sin\theta)^2(\sin 2\phi)\Big]\dot{\psi}\dot{\phi} \\
&\qquad - \Big[\frac{1}{2}I_1'(\cos\theta)(\sin 2\phi)\Big]\dot{\theta}^2 - \Big[2I_1'(\sin\theta)(\cos\phi)^2\Big]\dot{\theta}\dot{\phi},
\end{aligned}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left\{\left[I_2' + I_3' + I_2'(\sin\phi)^2\right]\dot{\theta} - \frac{1}{2}I_1'\dot{\psi}(\sin\theta)(\sin 2\phi)\right\}$$
$$= \left[I_2' + I_3' + I_2'(\sin\phi)^2\right]\ddot{\theta} - \left[\frac{1}{2}I_1'(\sin\theta)(\sin 2\phi)\right]\ddot{\psi}$$
$$+\left[I_2'(\sin 2\phi)\right]\dot{\theta}\dot{\phi} - \frac{1}{2}\left[I_1'(\cos\theta)(\sin 2\phi)\right]\dot{\psi}\dot{\theta}$$
$$-\left[I_1'(\sin\theta)(\cos 2\phi)\right]\dot{\psi}\dot{\phi},$$

$$\frac{\partial T}{\partial \psi} = 0, \quad \frac{\partial T}{\partial \theta} = \frac{1}{2}\left[I_2' + I_3' - I_1'^2(\sin\phi)^2\right](\sin 2\theta)\dot{\psi}^2$$
$$-\frac{1}{2}I_1'\dot{\psi}\dot{\theta}(\cos\theta)(\sin 2\phi) - I_1'\dot{\psi}\dot{\phi}(\sin\theta),$$

We could proceed to substitute the preceding terms into Lagrange's equations, but there is little to be gained from doing so because we will not solve the resulting equations of motion. It is obvious that the loss of axisymmetry complicated the analysis. At the same time, it also is obvious that these complications entail mathematical operations, rather than the basic formulation. As was mentioned earlier, symbolic mathematical software can significantly alleviate these complications.

7.6 SOLUTION OF EQUATIONS FOR HOLONOMIC SYSTEMS

When the motion of a system is known, the equations of motion may be solved algebraically for the forces, such as applied loads, required for sustaining that motion. The more interesting situation arises when some aspect of the motion is unknown. In that case the generalized coordinates are governed by differential equations of motion. In some situations, notably those involving vibration relative to a static equilibrium position, the generalized coordinates remain sufficiently small to permit linearizing problematic terms in the equations of motion. For example, when a pendulum is released from a small initial angle, the rotation angle θ will be small. The gravity moment for a pendulum is $mg\ell\sin\theta$, which may be simplified under the small-angle restriction to to $mg\ell\theta$. Linearization of equations of motion and the solution of the resulting linearized equations of motion are discussed extensively by Ginsberg (2001).

Our concern here is situations for which it is inappropriate to simplify the equations of motion. When these equations are derived by direct application of Lagrange's equations, they have a standard form that follows as a consequence of the fact that the kinetic energy has a standard functional dependence on the generalized velocities. To demonstrate this, consider the general form of the kinetic energy of a particle. The position $\bar{r}_k$ of particle k in a system may be an explicit function of the generalized coordinates and time, that is, $\bar{r}_k = \bar{r}_k(q_i, t)$. The corresponding velocity expression is

$$\bar{v}_k \equiv \frac{d}{dt}\bar{r}_k(q_i,t) = \sum_j \left(\frac{\partial}{\partial q_j}\bar{r}_k(q_i,t)\right)\dot{q}_j + \frac{\partial}{\partial t}\bar{r}_k(q_i,t). \tag{7.6.1}$$

The kinetic energy of this particle therefore is

$$
\begin{aligned}
T_k &= \frac{1}{2} m_k \bar{v}_k \cdot \bar{v}_k \\
&= \frac{1}{2} \sum_j \sum_n m_k \left(\frac{\partial}{\partial q_j} \bar{r}_k (q_i, t) \right) \left(\frac{\partial}{\partial q_n} \bar{r}_k (q_i, t) \right) \dot{q}_j \dot{q}_n \\
&\quad + \sum_j m_k \left(\frac{\partial}{\partial q_j} \bar{r}_k (q_i, t) \right) \left(\frac{\partial}{\partial t} \bar{r}_k (q_i, t) \right) \dot{q}_j + \frac{1}{2} m_k \left(\frac{\partial}{\partial t} \bar{r}_k (q_i, t) \right)^2 .
\end{aligned}
\tag{7.6.2}
$$

The preceding expression indicates that the terms forming the kinetic energy of a particle fall into one of three categories: They either contain the generalized velocities as quadratic products or as linear terms, or they are independent of the generalized velocities; no term in the kinetic energy contains the $\dot{q}_i$ in any other manner. We also see that the terms multiplying each occurrence of a generalized velocity may depend on the generalized coordinates and time. This basic form is not altered if we add the kinetic energy of the various particles forming a system. Furthermore, the kinetic energy of a rigid body contains quadratic products of the velocity of the center of mass and the angular velocity, and both types of velocities depend on the generalized coordinates in a manner similar to that of Eq. (7.6.1). It follows that the kinetic energy of a system of particles and rigid bodies is a *quadratic sum* in the generalized velocities, which means that it consists of three groups of terms: T_2 is quadratic in $\dot{q}_i$, T_1 is linear in $\dot{q}_i$, and T_0 is independent of the $\dot{q}_i$. The general form is

$$
\boxed{T = T_2 + T_1 + T_0 (q_i, t) ,}
\tag{7.6.3}
$$

where

$$
T_2 = \frac{1}{2} \sum_j \sum_n M_{jn} (q_i, t) \dot{q}_j \dot{q}_n, \quad T_1 = \sum_j N_j (q_i, t) \dot{q}_j .
\tag{7.6.4}
$$

A useful property obeyed by the coefficients of the quadratic terms, M_{jn}, is symmetry:

$$
M_{nj} = M_{jn}, \quad j, n = 1, 2, \ldots, N,
\tag{7.6.5}
$$

which follows from the fact that the order in which the product $\dot{q}_j \dot{q}_n$ is formed is unimportant.

It will be crucial for some later developments to recognize situations in which all terms in the kinetic energy are quadratic in the generalized velocities, so that $T = T_2$. To identify when such a condition arises, we note that the terms contributing to T_1 and T_0 originated from $\partial \bar{r}_k / \partial t$ in Eq. (7.6.1). This term vanishes if all generalized coordinates are measured from a stationary position, in which case the system is at rest if all generalized coordinates are maintained at zero.

Let us consider the result of using Eqs. (7.6.4) to form the Lagrange's equations associated with a specific generalized coordinate q_s. We begin with $\partial T_2 / \partial \dot{q}_s$. We know that $\partial M_{jn} / \partial \dot{q}_s \equiv 0$. Partial differentiation of the product $\dot{q}_j \dot{q}_n$ with respect to $\dot{q}_s$ will give zero unless either or both of the indices j and n matches s. We use the Kronecker delta

δ_{js}, which is zero if $j \neq s$ and one if $j = s$, to describe this property. This symbol allows us to write

$$\begin{aligned}\frac{\partial T_2}{\partial \dot{q}_s} &= \frac{1}{2}\sum_j \sum_n M_{jn} \frac{\partial}{\partial \dot{q}_s}(\dot{q}_j \dot{q}_n) = \frac{1}{2}\sum_j \sum_n M_{jn}(\delta_{sj}\dot{q}_n + \dot{q}_j \delta_{sn}) \\ &= \frac{1}{2}\sum_n M_{sn}\dot{q}_n + \frac{1}{2}\sum_j M_{js}\dot{q}_j = \sum_n (M_{sn} + M_{ns})\dot{q}_n = \sum_n M_{sn}\dot{q}_n,\end{aligned} \tag{7.6.6}$$

where the last step is a consequence of the symmetry of the M_{jn} coefficients. Similar operations applied to T_1 and T_0 lead to

$$\frac{\partial T_1}{\partial \dot{q}_s} = N_s, \quad \frac{\partial T_0}{\partial \dot{q}_s} = 0, \tag{7.6.7}$$

from which it follows that

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) = \sum_n \left[M_{sn}\ddot{q}_n + \left(\frac{dM_{sn}}{dt}\right)\dot{q}_n\right] + \frac{dN_s}{dt}. \tag{7.6.8}$$

Proper evaluation of the total time derivative requires that we recognize that the coefficients may depend explicitly on the generalized coordinates, which are time dependent. Accounting for this dependence leads to

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) = \sum_n M_{sn}\ddot{q}_n + \sum_n \sum_j \frac{\partial M_{sn}}{\partial q_j}\dot{q}_j \dot{q}_n + \sum_n \dot{M}_{sn}\dot{q}_n \\ + \sum_j \frac{\partial N_s}{\partial q_j}\dot{q}_j + \dot{N}_{sn},\end{aligned} \tag{7.6.9}$$

where the overdot indicates partial differentiation with respect to time. We proceed similarly to evaluate $\partial T/\partial q_s$. Because the generalized coordinates and velocities constitute independent variables for the partial differentiation, we find that

$$\frac{\partial T}{\partial q_s} = \frac{1}{2}\sum_n \sum_j \frac{\partial M_{jn}}{\partial q_s}\dot{q}_j \dot{q}_n + \sum_j \frac{\partial N_j}{\partial q_s}\dot{q}_j + \frac{\partial T_0}{\partial q_s}. \tag{7.6.10}$$

Substitution of the preceding derivatives into Lagrange's equations leads to

$$\begin{aligned}\sum_n M_{sn}\ddot{q}_n + \sum_n \sum_j \left(\frac{\partial M_{sn}}{\partial q_j} - \frac{1}{2}\frac{\partial M_{jn}}{\partial q_s}\right)\dot{q}_j \dot{q}_n + \sum_n \left(\dot{M}_{sn} + \frac{\partial N_s}{\partial q_n} - \frac{\partial N_n}{\partial q_s}\right)\dot{q}_n \\ + \dot{N}_{sn} + \frac{\partial T_0}{\partial q_s} + \frac{\partial V}{\partial q_s} = Q_s, \quad s = 1, 2, \ldots, N.\end{aligned} \tag{7.6.11}$$

This expression is an analytical confirmation that a wide variety of terms might be generated when one forms the Lagrange equations for a system. For our present purpose the main feature is that the highest derivative of a generalized coordinate that can occur is second order and that these derivatives occur linearly. The other terms may be grouped on the right side of the equality sign as a single function $F_s(q_i, \dot{q}_i, t)$. The term containing the second derivative is the summation form of the product of a matrix $[M]$ and a vector of generalized coordinates $\{q\}$. Thus the standard form of Lagrange's equations for a holonomic system described by unconstrained generalized coordinates is

$$[M(q_i, t)]\{\ddot{q}\} = \{F(q_i, \dot{q}_i, t)\}. \tag{7.6.12}$$

This set of second-order differential equations is likely to be highly nonlinear and $[M]$ seldom is constant. An analytical solution of these equations is not likely to be found, so evaluation of the response will require a numerical procedure. Numerous numerical methods and associated standard computerized routines have been developed to assist us to solve sets of coupled differential equations, but most require that we express the differential equations of motion in first-order form. The standard form of such equations is

$$\frac{d}{dt}\{z\} = \{G(z_i, t)\}. \tag{7.6.13}$$

A discussion of numerical algorithms by which a set of equations in this form may be solved is beyond the scope of this book. A good starting point to learn about possible techniques is the text by Press *et al.* (1992). Most computational software packages, such as Matlab and Mathcad, as well as libraries of compiled languages such as C and FORTRAN, provide functions or subroutines that can solve these equations. The procedures step forward in time, that is, they obtain the value of $\{z\}$ at time $t + \Delta t$, in some cases proceeding in an automatic manner for as many time steps as required. Implementing this capability requires that one specify how to compute $\{G\}$ from the current values of the elements of $\{z\}$ and t. From this juncture onward, we assume that a reliable technique capable of solving a system of differential equations in the form of Eq. (7.6.13) is available. The task we address here is how we convert Eq. (7.6.12) to a form that is compatible with Eq. (7.6.13).

At any instant t, the state of a system consists of the values of its generalized coordinates $\{q\}$ and velocities $\{\dot{q}\}$. From such data the generalized accelerations may be computed from Eq. (7.6.12), and higher derivatives may be computed by differentiating that equation. For this reason the values of $\{q\}$ and $\{\dot{q}\}$ are said to constitute the *state space*. Conversion of the governing equations to state-space form will lead to the requisite form of the equations of motion.

The key step is to treat the generalized velocities as variables that are independent of the generalized coordinates by defining a set of $2N$ variables x_i. The first group of N variables comprises the generalized coordinates, whereas the second group comprises the generalized velocities:

$$x_i = q_i, \quad x_{i+N} = \dot{q}_i, \quad i = 1, \ldots, N. \tag{7.6.14}$$

This may be described in matrix form with $\{q\}$ and $\{\dot{q}\}$ used as upper and lower partitions of a vector, according to

$$\{x\} = \begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix}. \tag{7.6.15}$$

The derivative of $\{q\}$ obviously is $\{\dot{q}\}$. We may write this identity for the derivative in partitioned form by recognizing that $\{q\}$ is the upper partition of $\{x\}$ and $\{\dot{q}\}$ is the lower partition. Recall that, when matrices are partitioned conformably, then a product may be formed by treating the partitions as though they were individual elements. Thus we have

$$\begin{bmatrix}[U]_{N\times N} & [0]_{N\times N}\end{bmatrix}\{x\} = \{q\}\,,\quad \begin{bmatrix}[0]_{N\times N} & [U]_{N\times N}\end{bmatrix}\{x\} = \{\dot{q}\}\,, \tag{7.6.16}$$

where $[U]$ is an identity matrix. The following set of N differential equations results from using these relations to enforce the identity that the time derivative of $\{q\}$ is $\{\dot{q}\}$:

$$\begin{bmatrix}[U]_{N\times N} & [0]_{N\times N}\end{bmatrix}\{\dot{x}\} = \begin{bmatrix}[0]_{N\times N} & [U]_{N\times N}\end{bmatrix}\{x\}\,. \tag{7.6.17}$$

We obtain another set of N equations by solving the equations of motion, Eq. (7.6.12), for the generalized accelerations, which yields

$$\left\{\ddot{q}^*\right\} = [M(q_i,t)]^{-1}\left\{F(q_i,\dot{q}_i,t)\right\}. \tag{7.6.18}$$

The asterisk is used to indicate that the generalized accelerations are obtained by solving a set of equations that is evaluated at the current instant. Note that the inverse is used here solely to indicate solution of the matrix equation. Unless $[M]$ is constant, it is far more efficient in practice to merely solve Eq. (7.6.12) numerically by any convenient algorithm, such as LU decomposition or Gauss elimination. To fully transform to the state-space variables we use Eqs. (7.6.14) to replace any occurrence of a q_i or $\dot{q}_i$ variable appearing in the preceding equation with the corresponding variable x_i or x_{i+N}, respectively. The last step is to assemble the full set of equations by stacking the derivative identity above the equations of motion. The result is a form of the equations of motion for a holonomic system suitable for solution by most numerical algorithms, specifically

$$\frac{d}{dt}\{x\} = \left\{\begin{matrix}\begin{bmatrix}[0]_{N\times N} & [U]_{N\times N}\end{bmatrix}\{x\} \\ [M(x_i,t)]^{-1}\left\{F(x_i,x_{i+N},t)\right\}\end{matrix}\right\}. \tag{7.6.19}$$

The foregoing, whose form is the same as that of Eq. (7.6.13), represents a set of $2N$ first-order differential equations for the $2N$ elements of $\{x\}$. To solve them numerically one needs to identify to the software how to compute the vector on the right side. In addition, initial values are required to start the numerical solver. Thus we need to specify the initial values of the generalized coordinates and velocities at an initial time t_0, from which we may form the initial state-space vector according to

$$\{x_0\} = \left\{\begin{matrix}\{q(t=t_0)\} \\ \{\dot{q}(t=t_0)\}\end{matrix}\right\}. \tag{7.6.20}$$

The preceding equation describes the analytical steps required to solve numerically the equations of motion for a holonomic system. Software packages typically offer several functions/subroutines based on different algorithms. Each has its benefits for certain classes of problems. A good starting point for dynamical systems is the fourth-order-Runge–Kutta algorithm, but one should be prepared to switch if irregularities, such as numerical instability, severely decreased time step in an automated algorithm, or error build-up, are observed as the program marches forward. Methods for stiff systems, such as the Gear method, often are useful if such difficulties are encountered.

EXAMPLE 7.19 Parameters for the rack and gear in Example 7.16 are $m_r = 20$ kg, $m_A = 10$ kg, $m_s = 2$ kg, $R = 200$ mm, $\varepsilon = 0.175$ mm, $\kappa = 150$ mm, and $k = 40$ kN/m. The system starts from rest at $x = 2R$, $\theta = 0$, at which location x_r is defined to be zero. A constant actuating force $F = 200$ N is applied at $t = 0$. Determine and graph x, x_r, and θ as functions of time.

SOLUTION The emphasis for this example is the setup of the differential equations and the application of a software package. The discussion is framed in terms of Matlab, but it is readily modified to treat the reader's choice of software. The generalized coordinates are $q_1 = x$ and $q_2 = \theta$, so the state-space vector is[‡]

$$\{x\} = \begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^{\mathrm{T}}.$$

It is easier to program the solution if the equations of motion are written in terms of the elements of $\{x\}$. Thus the coefficient matrix multiplying generalized accelerations in Eqs. (6) of Example 7.16 is written as

$$[M] = \begin{bmatrix} (m_s + m_A + m_r) & -[m_r R + (m_A + m_r)\,\varepsilon\cos(x_2)] \\ -[m_r R + (m_A + m_r)\,\varepsilon\cos(x_2)] & \begin{bmatrix} m_A\kappa^2 + m_A\varepsilon^2(\cos(x_2))^2 \\ \qquad + m_r\,(R + \varepsilon\cos(x_2))^2 \end{bmatrix} \end{bmatrix}. \tag{1}$$

All terms not containing generalized accelerations are placed on the right side of the first-order equations, so that

$$\{F\} = \begin{Bmatrix} -(m_A + m_r)\,\varepsilon\,(\sin(x_2))\,(x_4)^2 - k\,(x_1 - 2R) - F \\ \left\{\left[(m_A + m_r)\,\varepsilon^2\cos(x_2)\sin(x_2) + m_r R\varepsilon\sin(x_2)\right](x_4)^2 \right. \\ \left. \qquad + F\,(R + \varepsilon\cos(x_2))\right\} \end{Bmatrix}. \tag{2}$$

[‡] The parameters in the problem statement are given as dimensional quantities, so the equations are programmed in that form. However, nondimensionalizing a system of equations can sometimes beneficially affect both the accuracy and stability of a solution. One can obtain such a form in the present problem by using R as a length scale, $m_s + m_A + m_r$ as a mass scale, and $[(m_s + m_A + m_r)/K]^{1/2}$ as a time scale.

The state-space equations to be solved are

$$\frac{d}{dt}\{x\} = \left\{ \begin{matrix} x_3 \\ x_4 \\ [M]^{-1}\{F\} \end{matrix} \right\}. \qquad (3)$$

It is specified that the system starts from rest with $x = 2R$, $\theta = 0$, so the initial state-space vector is $\{x(0)\} = [2R \;\; 0 \;\; 0 \;\; 0]^{\mathrm{T}}$.

The ODE suite of functions in Matlab (`ODE45`, `ODE23`, etc.) differ in their internal algorithmic steps, but they all require that one define in an M-file a function that evaluates the right side of Eq. (3). The first two input arguments for this function are required to be the current values of t and $\{x\}$,after which one can place other system parameters. The function that was used to obtain the following solution is

```
function dx_dt = dx_dt_rack(t, x, m_r, m_A, m_s, R, eps, ...
        kappa, K, Force)
    M(1,1) = m_r + m_A + m_s;
    M(1,2) = -(m_r * R + (m_A + m_r) * eps * cos(x(2)));
    M(2,1) = M(1,2);
    M(2,2) = m_A * (kappa^2 + (eps * cos(x(2)))^2) ...
            + m_r * (R + eps * cos(x(2)))^2;
    F(1,1) = - (m_A + m_r) * eps * sin(x(2)) * x(4)^2 ...
            - K * (x(1) - 2 * R) - Force ;
    F(2,1) = ((m_A + m_r) * eps^2 * sin(x(2)) * cos(x(2)) ...
            + m_r * R * eps * sin(x(2))) * x(4)^2 ...
            + Force * (R + eps * cos(x(2)));
    dx_dt = [x(3:4); M\F];
```

Note that the "\", or left divide, operation in the last line implements an algebraic solution, which is more efficient than actually evaluating $[M]^{-1}\{F\}$.

The ODE suite implement adaptive algorithms with built-in error controls, which means that they have the capability of automatically evaluating the output at a succession of instants in a single invocation of the ODE function. However, the author's experience is that using this automatic capability is unreliable, and requires tuning of the error tolerances. Such was found to be the case here for some combinations of parameters other than those specified in the problem statement. When the code appearing after this discussion was modified to call `ODE45` once for the same error parameters (stored in `options`)with an initial $t_0 = 0$ and a final $t_f = 1$, significant deviations from the results displayed here were found as t increased. In contrast, the displayed results were unaltered when the error parameters were reduced by a factor of 100 or alternative algorithms were used. In general, if one is analyzing a system with relatively few degrees of freedom and does not need to perform multiple computations with a variety of system parameters, it is preferable to limit the time steps to be sufficiently small that error build-up is not an issue. This is achieved here with a program loop in which the new start time is set to the final time for the previous step. Although 100 integration steps are used here, far fewer would suffice. Another item to note in the following program fragment is the manner in which the argument list of ODE45 specifies which function should evaluate $d\{x\}/dt$. An anonymous function, which is a feature of Matlab 7, provides the capability of passing system parameters other than t and $\{x\}$ to the function dx_dt_rack. Here dx_dt_anon is the anonymous function that is passed as a “function handle” to ODE45. The definition of this function, which must appear before it is used, tells the differential equation solver to look for the variables it needs in the function dx_dt_rack. Note that all parameters other than those contained in the argument list, which are t and x in the present case, have been defined. The main steps stored in a Matlab script file are

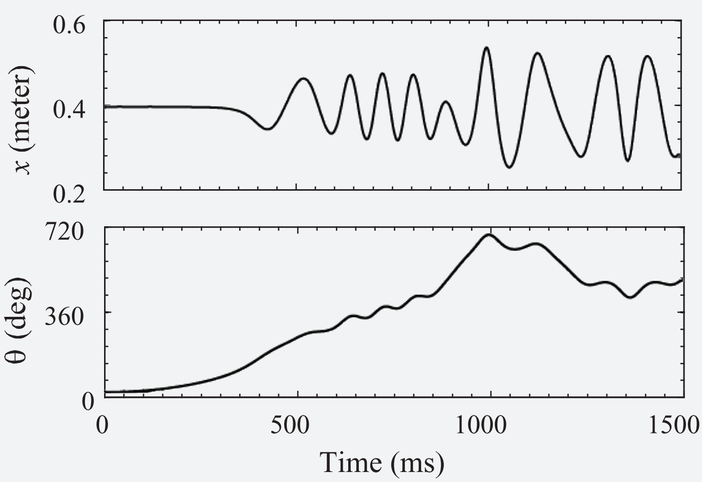
```
options = odeset('RelTol',1e-6,'AbsTol',[1e-6 1e-6 1e-6 1e-6]);
x_state(1, 1:4) = [2*R   0   0   0]'; t_state(1,1) = 0;
dx_dt_anon = @(t, x) dx_dt_rack(t, x, m_r, m_A, m_s, R, eps, ...
      kappa, K, Force);
t_max = 1; N_step = 100; t = [0:N_step]/N_step;
for j = 2:N_step + 1
      x_0 = x_state(j - 1,:)';
      [t_ode, x] = ode45(dx_dt_anon, [t(j - 1) t(j)], x_0,
options);
% Determine how many steps ODE45 took.
%Save the values at the last step.
% Row j of x_state is the state vector at t(j)
      n_steps = size(x, 1);
      x_state(j,:) = x(n_steps,1:4);
      t_state(j,1) = t(j);

   end
```

The state-space vectors provide all of the response information required to evaluate any feature of the response. The first two columns of x_state contain the values of x and θ at the instants in t_state, whereas the third and fourth columns are the corresponding generalized velocity values. The problem statement requested the response x_r, for which we recall Eq. (3) in Example 7.16. It is stated here that $x_r = 0$ when $x = 2R$ and $\theta = 0$. Substituting these values into Eq. (3) gives $S = -2R$. It also is useful to consider where the center of the gear is situated. This quantity is given by $x_A = x - \varepsilon \sin\theta$. We compute these displacements by writing `x_rack = x_state(:,1) - 2 * R - R * x_state(:,2) - eps * sin(x_state(:,2));` `x_A = x_state(:,1) - eps * sin(x_state(:,2));`. If it is desired to evaluate forces, such as the one exerted between pin B and the groove, one can employ the Newton–Euler equations of motion for isolated bodies. Such an evaluation would require the accelerations at each instant, which can be obtained from the equations of motion, which indicate that

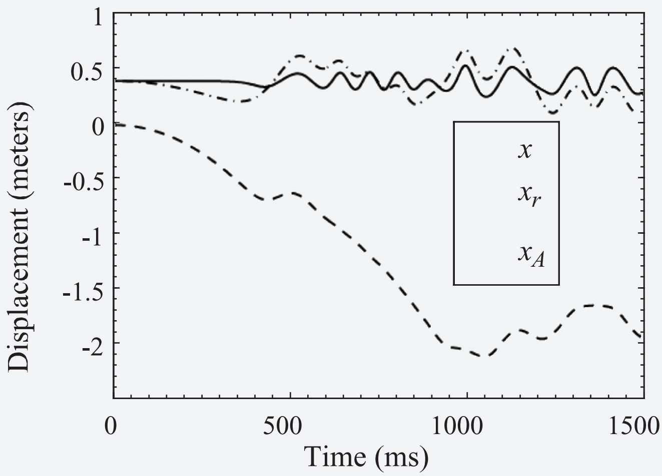
$$\left[\ddot{x}\ \ \ddot{\theta}\right]^{\mathrm{T}} = [M(\{x\})]^{-1}\left\{F(\{x\})\right\},$$

where the state vector $\{x\}$ would be the state vector at any instant. (An alternative method for finding internal forces by use of constrained generalized coordinates is discussed in the next chapter.)

The first graph shows that θ increases almost monotonically to a peak value and then falls off. The slider moves little in the early stages of the motion, after which it begins to oscillate, with an amplitude that increases after the gear has attained its maximum rotation. The mean value for x is approximately $2R$, which is its starting value. The second graph compares the positions of the rack, the slider, and the center of the gear. The rack moves to the left (negative x_r) in a nearly monotonic manner until the gear attains its maximum rotation, after which it begins to move rightward. In contrast, the slider and the gear execute similar oscillations. The physical interpretation of this behavior is that force $\bar{F}$ pushes the rack to the left, and the inertia of the gear resists this movement, thereby causing it to rotate clockwise (positive θ) about its center. The result is that pin B and the follower move to the right. When θ is small in the early stage, this movement is small, but x becomes eventually becomes large. This increases the compression of the spring, until a position is reached at which the follower's velocity reverses. Then the follower moves to the left, which eventually induces a tensile force in the spring that is sufficient to pull the follower back to the right, and so on.



Time dependence of the generalized coordinates.



Positions of the slider, rack, and gear center as functions of time.

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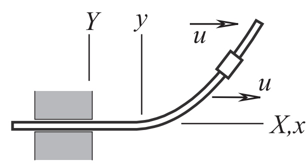
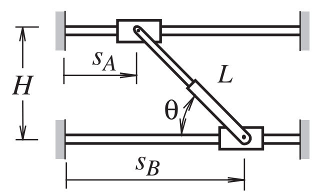
HOMEWORK PROBLEMS

EXERCISE 7.1 The slider descends along a curved guide as the guide translates to the right at the constant speed u. The shape of the guide bar in terms of a body-fixed set of coordinates is $y = \beta x^2$. Generalized coordinates selected for this system are the fixed X and Y coordinates of the collar. Independently derive the velocity and configuration constraint equations relating X and Y. Then show that integration of the velocity constraint yields the configuration constraint.



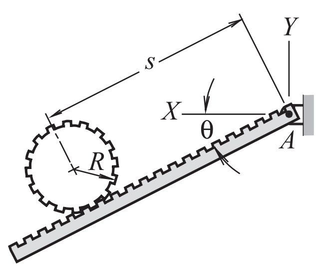
Exercise 7.1

EXERCISE 7.2 The length of rod AB connecting the collars is a controlled function $L(t)$. Generalized coordinates selected for this system are the distances s_A and s_B and the inclination angle θ. Derive the velocity constraint equations governing these variables, then integrate those constraint equations to obtain the corresponding configuration constraints. Show that the configuration constraint equations could have alternatively been obtained from a geometrical analysis.



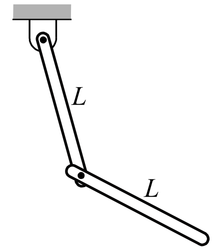
Exercise 7.2

EXERCISE 7.3 The gear rolls without slipping over the rack, which pivots about pin A. Generalized coordinates for this system are selected to be the angle of rotation θ of the rack, the distance s from the pivot to the center of the gear, and the (X, Y) coordinates of the center of the gear. Determine the velocity constraints relating these generalized coordinates. Are these constraints holonomic? How many degrees of freedom does this system have?



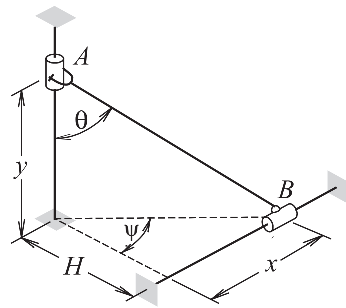
Exercise 7.3

EXERCISE 7.4 The generalized coordinates selected for the compound pendulum are the x, y coordinates of the lower end and the rotation angle of the upper bar. Perform a kinematical analysis to derive the velocity constraint equation relating these variables. Then perform a geometrical analysis of position to derive the configuration constraint. Prove that differentiating the configuration constraint leads to the velocity constraint.



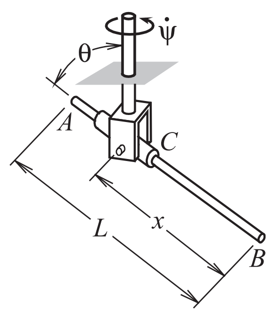
Exercise 7.4

EXERCISE 7.5 Collar A slides along the vertical guide bar, and collar B follows the horizontal guide. Bar AB is connected to collar A by a fork-and-clevis, and the connection to collar B is a ball-and-socket joint. It has been decided to use y, θ, and ψ as constrained generalized coordinates for this system. (a) Derive all applicable configuration constraints relating these variables based on a geometrical analysis. (b) Derive all applicable velocity constraints relating these variables based on a velocity analysis that uses these generalized coordinates to represent the position. (c) Show that differentiating the results of Part (a) yields velocity constraints that are equivalent to the results of Part (b).



Exercise 7.5

EXERCISE 7.6 The yoke permits collar C to precess through angle $\dot{\psi}$ and to nutate through angle θ. Bar AB slides freely relative to the collar, so the distance x to end B is an unknown function of time. What velocity constraint(s) must apply in order that the velocity of end B always be horizontal. Are they holonomic? If so, determine the corresponding configuration constraints.



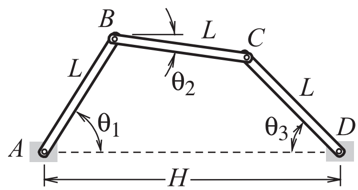
Exercise 7.6

EXERCISE 7.7 The Cartesian coordinates (x, y, z) of a particle relative to a fixed reference frame are related by

$$\left[ay^2\cos\left(\frac{ax}{y^2}\right) - czy^2\right]\dot{x} + \left[-2axy\cos\left(\frac{ax}{y^2}\right) + 2byz\sin\left(\frac{bz}{y^2}\right) + 2cxyz\right]\dot{y}$$
$$-\left[by^2\sin\left(\frac{bz}{y^2}\right) + cxy^2\right]\dot{z} = 0,$$

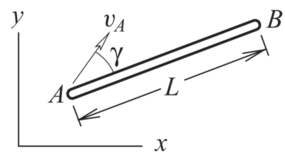
where a, b, and c are constants. Prove that this constraint is holonomic, and derive the corresponding configuration constraint.

EXERCISE 7.8 The angles θ_1, θ_2, and θ_3 have been selected as generalized coordinates for the linkage. Perform a velocity analysis of the linkage to derive the velocity constraint equations that must satisfied by these angles. Then prove that these equations are holonomic.



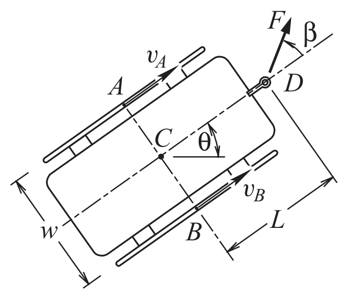
Exercise 7.8

EXERCISE 7.9 The bar is made to slide along the horizontal plane such that the velocity of end A is always directed at a constant angle γ relative to the bar. Generalized coordinates are the x and y coordinates of end B and the angle θ. Describe the corresponding velocity constraint. Then determine whether the constraint is holonomic.



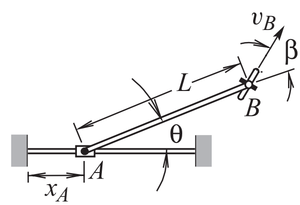
Exercise 7.9

EXERCISE 7.10 The sketch shows the top view of a sled that is towed over the ice by fastening a cable to loop D. The ice constrains the velocity of points A and B to be parallel to the rails, but the speeds of each point may be different. Generalized coordinates have been selected as the east and north coordinates of point C in the horizontal plane and the heading angle θ. What are the associated constraint equations?



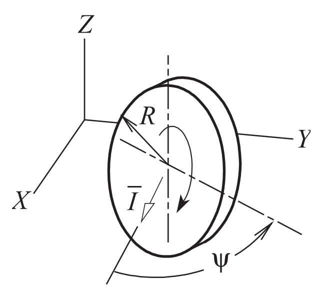
Exercise 7.10

EXERCISE 7.11 The illustrated linkage lies in the horizontal plane. End A of the bar follows the horizontal guide, and a wheel at end B is aimed at an angle β that is a specified function of time. The wheel rolls without slipping over the ground, so the velocity $\bar{v}_B$ must be in the direction indicated in the diagram. Generalized coordinates are selected to be the horizontal position x_A and the rotation angle θ. Determine the velocity constraint relating these variables. Prove that this constraint equation is nonholonomic unless β is constant.



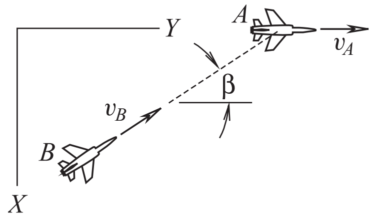
Exercise 7.11

EXERCISE 7.12 The figure shows a disk that is constrained to roll without slipping on a horizontal XY plane, such that its plane remains vertical. Let the position coordinates X and Y of the geometric center, the heading angle ψ, and the spin angle ϕ be generalized coordinates. Describe the velocity constraints between these generalized coordinates. From those results, determine the number of degrees of freedom, and whether the system is holonomic.



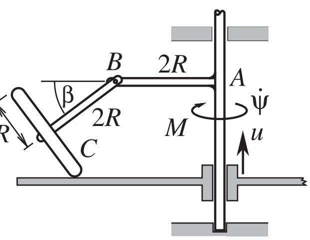
Exercise 7.12

EXERCISE 7.13 An interesting example of a nonholonomic constraint, which does not usually arise in a course in mechanics, is a pursuit problem. The sketch depicts airplane A that flies eastward at a constant velocity v_A. Airplane B has a laser that is mounted parallel to its axis, and therefore parallel to $\bar{v}_B$. It is necessary to maneuver this airplane such that the laser is always aimed at airplane A. Derive the constraint equations that must be satisfied by the coordinates X_B and Y_B and the heading angle β.



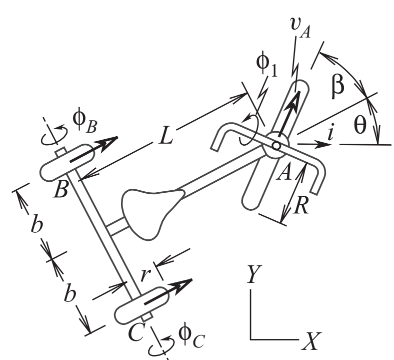
Exercise 7.13

EXERCISE 7.14 The horizontal platform translates upward at constant speed u as the T-bar assembly rotates about the vertical axis at angular speed $\dot{\psi}$. Disk C, which spins freely about its shaft, rolls over the platform without slipping in the direction perpendicular to the diagram, although slippage in the direction transverse to the vertical shaft does occur. Derive the velocity constraint equation(s) relating the precession angle ψ, the elevation angle β, and the angle ϕ by which the disk spins about shaft BC.



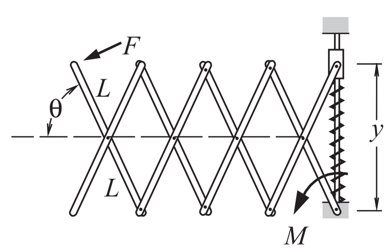
Exercise 7.14

EXERCISE 7.15 The figure shows a child's tricycle as viewed from above. When the wheels do not slip over the ground, the velocity of each wheel's center must be perpendicular to the wheel's shaft in the horizontal plane, as shown. Consider a set of generalized coordinates consisting of the position coordinates X_A and Y_A of the steering joint, the angle of orientation θ of the frame, the steering angle β, and the spin angles ϕ_A, ϕ_B, and ϕ_C of the wheels. Derive the velocity constraints among these seven generalized coordinates. From that result, determine the number of degrees of freedom.



Exercise 7.15

EXERCISE 7.16 The parallelogram linkage, which lies in the vertical plane, is loaded by force F that acts perpendicular to the link to which it is applied, and torque M. The mass per unit length of each bar is σ. The spring can support compressive or tensile axial force, and is unstretched when $y = 1.5L$. Determine the conservative and nonconservative portions of the generalized force associated with the selection of the vertical distance y as the generalized coordinate. Then repeat the analysis using the angle θ as the generalized coordinate.



Exercise 7.16

EXERCISE 7.17 The distances s_A and s_B have been selected as generalized coordinates for the system in Exercise 7.2. Determine the virtual work done by all forces acting on the collars in an arbitrary virtual displacement. Then show that the virtual work done by the forces exerted by the hydraulic cylinder vanishes if the virtual displacement is kinematically admissible.

EXERCISE 7.18 Suppose the angle θ and vertical distance y are selected as generalized coordinates for the system in Example 7.12. Determine the virtual work done by all forces acting on the collar and bar when δy and $\delta\theta$ are arbitrary values. Then show that the result is equivalent to the expression obtained in Example 7.12 when the virtual displacement is kinematically admissible.

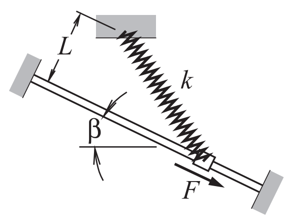
EXERCISE 7.19 Consider the force that constrains the velocity of the wheel in Exercise 7.11. Evaluate the virtual work done by this force when the generalized coordinates x_A and θ are given arbitrary virtual increments. Then show that this work vanishes if the increments δx_A and $\delta\theta$ are kinematically admissible.

EXERCISE 7.20 Generalized coordinates for the linkage in Exercise 7.5 are the vertical distance y and the angles θ and ψ. Determine the virtual work done by gravity and the constraint force exerted by the horizontal guide on collar B. Then show that the virtual work of the constraint force vanishes when δy, $\delta\theta$, and $\delta\psi$ are kinematically admissible.

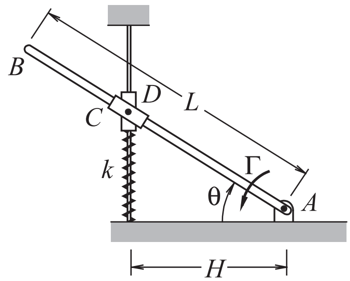
EXERCISE 7.21 A force is observed to depend on the x and y coordinates of the point at which it is applied according to $\bar{F} = \left(x\bar{i} - y\bar{j}\right) / \left(x^2 - y^2\right)^{1/2}$. Show that this force is conservative, and determine its potential-energy function. Is there a limitation on the range of values of x and y for which this function applies?

EXERCISE 7.22 The slider in Exercise 7.1 has mass m, and the curved guide bar is situated in the vertical plane. Determine the differential equations of motion. Friction has negligible effect.

EXERCISE 7.23 The collar of mass m slides over the smooth guide, which is elevated by β from the horizontal. The spring's stiffness is k, and its unstretched length $0.8L$. The force F acting tangentially to the guide varies in a known manner. Determine the equations of motion for the system.

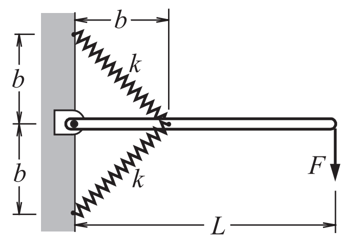


Exercise 7.23



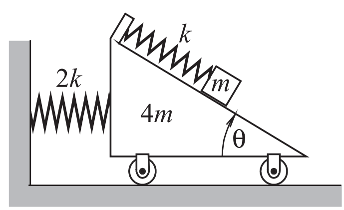
Exercise 7.24

EXERCISE 7.24 Rod AB slides through collar C, which is pinned to collar D that slides over the vertical guide. The bar's mass is m, and the mass of each collar is $m/2$. The motion is actuated by torque $\Gamma(t)$. The system lies in the vertical plane, and the spring is unstretched in the position where $\theta = 20°$. Determine the equations of motion for the system. Then determine θ for static equilibrium when $(g/H)^{1/2} = 4$ rad/s, $(k/m)^{1/2} = 5$ rad/s, $L = 2H$, and $\Gamma = 0$.



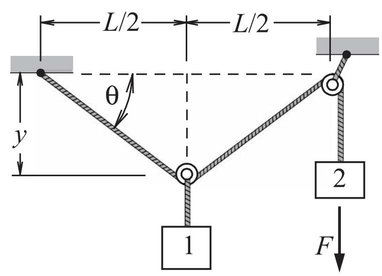
Exercise 7.25

EXERCISE 7.25 The bar is supported by two springs whose stiffness is k. The springs are unstretched when the bar is horizontal. Determine the equations of motion. The force F, which acts perpendicularly to the bar, is a known excitation.



Exercise 7.26

EXERCISE 7.26 The stiffness of the horizontal spring is $2k$, whereas the spring holding the small block has stiffness k. The masses are $4m$ and m for the cart and the block, respectively. Frictional resistance at all contacting surfaces is negligible. Determine the equations of motion for the system.

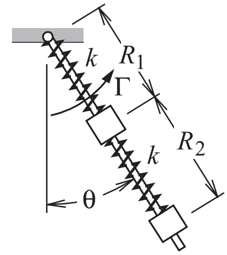


Exercise 7.27

EXERCISE 7.27 A downward force $F(t)$ is applied to block 2. The masses are $m_1 = 2m$, $m_2 = 3m$. (a) Derive the equation of motion corresponding to using y as the generalized coordinate for the system. (b) Derive the equation of motion corresponding to using θ as the generalized coordinate for the system.

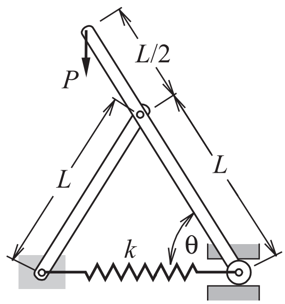
EXERCISE 7.28 A horizontal force $H(t)$ is applied at the low end of the compound pendulum in Exercise 7.4. Both bars have mass m. Determine the corresponding differential equations of motion.

EXERCISE 7.29 Identical small collars of mass m slide with negligible frictional resistance over the bar that is made to rotate in the vertical plane by torque Γ that acts about its pivot. The result is that the angle θ is made to vary as a specified function of time. The unstretched length of the identical springs is L. Derive differential equations of motion governing the radial distances R_1 and R_2.



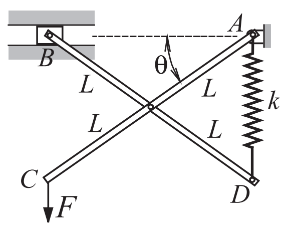
Exercise 7.29

EXERCISE 7.30 The linkage is braced by a spring of stiffness k in order to support the force P that acts perpendicularly to the long link. The system lies in the vertical plane, and σ is the mass per unit length of both bars. The spring is unstretched when $\theta = 45°$. Derive the equation of motion governing θ.



Exercise 7.30

EXERCISE 7.31 Determine the equations of motion for the linkage shown in the diagram when the horizontal distance from pin A to piston B is used as the generalized coordinate. The mass of each bar is m, the mass of collar B is negligible, and the unstretched length of the spring is $3L/4$.

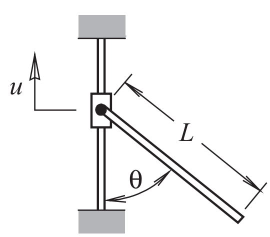


Exercise 7.31

EXERCISE 7.32 Derive the differential equation governing the motion of the parallelogram linkage in Exercise 7.16. The mass per unit length of each bar is σ, and the unstretched length of the spring is $2L$.

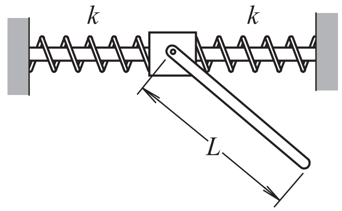
EXERCISE 7.33 Use Lagrange's equations to derive the differential equation of motion for the angle θ at which the box in Exercise 6.32 is tilted.

EXERCISE 7.34 The collar to which the bar is pinned is given a specified displacement $u(t)$. The collar and the bar have equal mass m. Use Lagrange's equations to derive the equations of motion for this system.



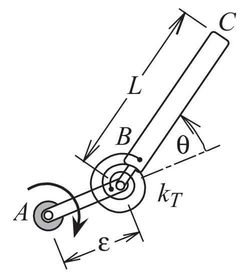
Exercise 7.34

EXERCISE 7.35 The collar, whose mass is m_1, supports a bar whose mass is m_2. The springs restraining the collar have identical properties. Determine the equations of motion for this system.



Exercise 7.35

EXERCISE 7.36 A simplified model of one blade of a helicopter is shown in the sketch. The short segment AB is driven at a constant rotational speed Ω. The blade BC is connected to AB by a pin, about which is wrapped a torsional spring having stiffness k_T. The spring is unstressed when the lag angle ϕ is zero. Derive the equation of motion governing ϕ. Blade BC may be considered to be a homogeneous bar whose cross section is uniform.

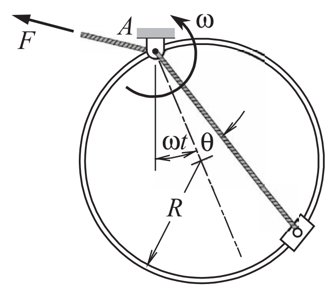


Exercise 7.36

EXERCISE 7.37 Consider the system treated in Example 7.12. Derive the differential equation of motion when θ is used as the unconstrained generalized coordinate.

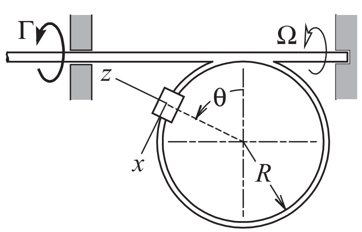
EXERCISE 7.38 Use Lagrange's equations to solve Exercise 6.29.

EXERCISE 7.39 The circular hoop rotates in the vertical plane about its pivot at the constant angular speed ω, so ωt is the angle from the diametral line to the vertical. The collar, whose mass is m, slides over the guide bar under the influence of the cable, which carries a known tensile force F at its free end. Frictional resistance between the collar and the guide is negligible. Derive the equation of motion for this system.



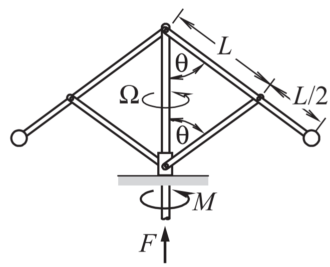
Exercise 7.39

EXERCISE 7.40 A known torque Γ acts about the horizontal shaft, resulting in rotation Ω. The collar slides without friction over the hoop. The collar is essentially a hollow circular cylinder whose axis of symmetry is the tangential x axis. The mass of the collar is m, and its centroidal moments of inertia are I_1 and I_2 about the x and z axes, respectively. The moment of inertia of the hoop–shaft assembly about its rotation axis is I_3. Derive the equations of motion based on the assumption that the collar does not spin about the x axis relative to the hoop.



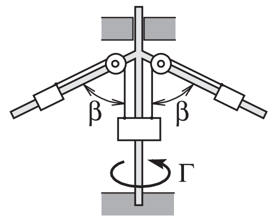
Exercise 7.40

EXERCISE 7.41 Application of the force F to the vertical control rod in the flyball speed governor causes the angle θ to change. Simultaneously, a servomotor applies torque M to maintain a constant precession rate Ω. Determine the differential equations of motion for this system. The mass of each sphere is m, and the links have negligible mass.



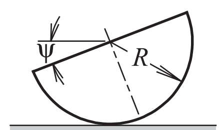
Exercise 7.41

EXERCISE 7.42 Each collar has mass m and is sufficiently small to consider it to be a particle. The bar assembly on which the collars ride rotates about the vertical axis due to a known torsional load $\Gamma(t)$. The moment of inertia of the bar assembly about its axis of rotation is I. Derive the differential equations of motion for the system.



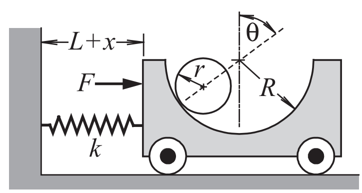
Exercise 7.42

EXERCISE 7.43 The homogeneous semicylinder rolls without slipping. Derive the equation of motion governing the angle of rotation ψ.



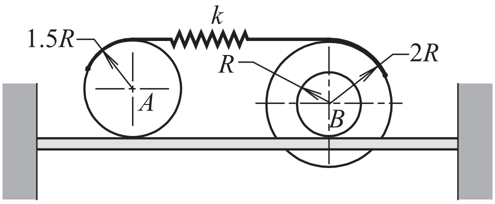
Exercise 7.43

EXERCISE 7.44 The known excitation force F pushes the cart to the right, which causes the homogeneous cylinder to roll without slipping relative to the cylindrical cavity in the cart. The mass of the cart is m_1 and the mass of the cylinder is m_2. The spring's unstretched length is L, so x is the displacement from the static equilibrium position in the absence of F. It may be assumed that the cart's wheels are massless and friction at the wheel bearings is ineligible. Derive differential equations of motion governing x and the angle θ locating the center of the cylinder.



Exercise 7.44

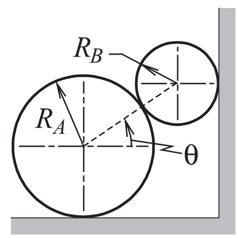
EXERCISE 7.45 The pulleys roll over the rack without slipping. The masses and centroidal radii of gyration are m_i and κ_i ($i = 1$ for the left pulley and $i = 2$ for the right). The spring, whose stiffness is k, is capable of sustaining both compressive and tensile forces. Determine the equations of motion.



Exercise 7.45

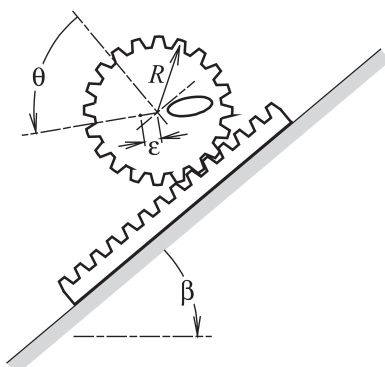
EXERCISE 7.46 Use Lagrange's equations to derive the differential equation of motion governing the elevation angle for the box in Example 6.8.

EXERCISE 7.47 The cylinders roll in the vertical plane such that there is no slipping between them, nor between cylinder A and the ground. The vertical surface is sufficiently smooth to assume that there is negligible frictional resistance there. The masses of the cylinders are m_A and m_B. Derive the equation of motion governing the unconstrained generalized coordinate θ.



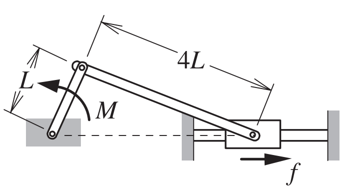
Exercise 7.47

EXERCISE 7.48 Because of the cutout section, the center of mass of the gear is situated at a distance ε from the geometrical center. The gear has mass m_1, its centroidal radius of gyration is κ, and the mass of the rack is m_2. Friction between the rack and the incline is negligible. Derive differential equations of motion governing the angle θ locating the center of mass and the distance s that the rack has translated.



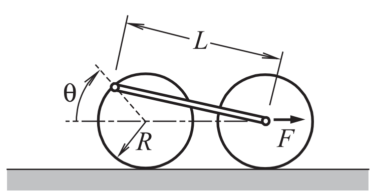
Exercise 7.48

EXERCISE 7.49 The couple M acting causing the crankshaft to rotate is resisted by the drag force f acting on the piston. The masses are m_1, m_2, and m_3 for the piston. Gravity is unimportant. Derive the differential equation governing the angle of rotation of the crankshaft.



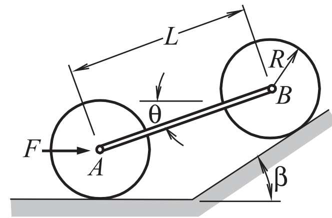
Exercise 7.49

EXERCISE 7.50 Two cylinders, each having mass m, are linked by a connecting rod whose mass is negligible. A known horizontal force $F(t)$ is applied to the right cylinder, and neither cylinder slips in its rolling motion. In the initial position, the angle θ locating the connecting pin is zero. Derive the equation of motion for this angle.



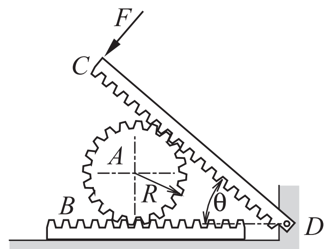
Exercise 7.50

EXERCISE 7.51 The horizontal force F pushes cylinder A to the right, thereby pushing cylinder B up the hill. Both cylinders have identical mass m and radius of gyration κ about their axis of symmetry. The mass of the connecting rod also is m. Determine the equation of motion corresponding to using the elevation angle θ of the connecting rod as the generalized coordinate. It may be assumed that there is no slippage when the cylinders roll.



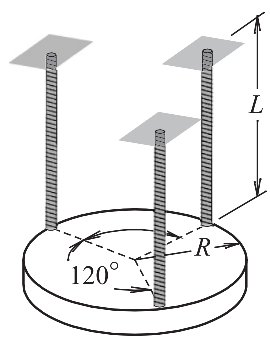
Exercise 7.51

EXERCISE 7.52 Force P acts normal to rack CD, which pivots about pin D. Because gear A must roll without slipping, the result is that rack B is forced to move to the left. The mass of the racks is m, and they may be approximated as homogeneous bars. The mass of the gear is $2m$ and its radius of gyration about its center is $0.8R$. Derive the equations of motion for the system assuming that frictional resistance is unimportant.



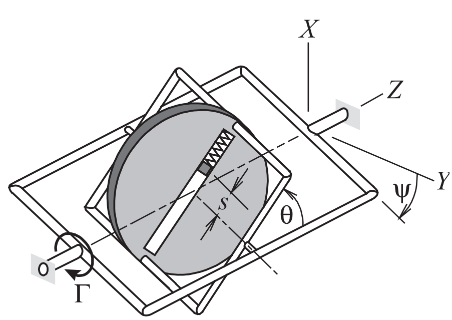
Exercise 7.52

EXERCISE 7.53 A circular disk of mass m is suspended in the horizontal plane by three cables of equal length L. The cables are vertical when the system is at its equilibrium position. Derive the equation of motion for the angle θ by which the disk rotates about its vertical axis of symmetry. Assume that all cables remain taut.



Exercise 7.53

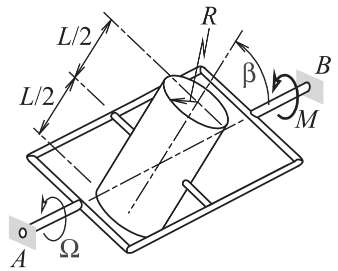
EXERCISE 7.54 The slider, whose mass is m_1, oscillates within the groove in the turntable, so the distance s from the center of the block to the axis of the shaft is an undefined function of time. In turn, the turntable rotates freely relative to the shaft, so the nutation angle θ also is variable, and the torque $\Gamma(t)$ induces the precessional rotation ψ. The shaft is horizontal, and $\psi = 0$ corresponds to the diagram depicting the vertical plane. The centroidal moments of inertia of the turntable are I_1 and I_2 about axes that are respectively perpendicular to, and parallel to, the shaft, and the inertia of the shaft may be ignored. The spring restraining the slider has stiffness k and it is unstretched when $s = 0$. Derive the Lagrange equations



Exercise 7.54

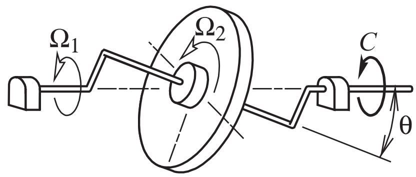
of motion governing the generalized coordinates s, θ, and ψ. Note that gravity is important because the rotation θ has no other force inducing it.

EXERCISE 7.55 The orientation of the homogeneous cylinder relative to the gimbal is described by the angle β. The torque M is such that the rotation rate Ω of the gimbal about the horizontal axis is constant. The gimbal's mass is negligible. Derive the equation of motion for β.



Exercise 7.55

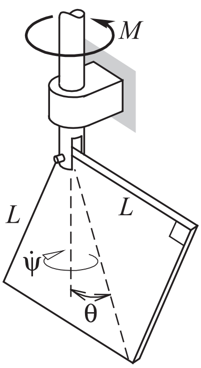
EXERCISE 7.56 The couple $C(t)$ induces rotation at rate Ω_1 of the system about the horizontal shaft. A servomotor causes the disk to spin at an unsteady rate Ω_2 whose time dependence is known. Derive the differential equation governing Ω.



Exercise 7.56

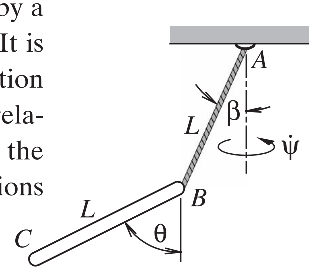
EXERCISE 7.57 Find the equations of motion for the rotordynamic system in Exercise 7.56 for the case in which the servomotor torque Γ driving is specified and the spin rate Ω_2 is unknown.

EXERCISE 7.58 The square plate is pinned to the vertical shaft, which is made to rotate by a known torque $M(t)$. Derive differential equations of motion for the precession angle ψ and nutation angle θ.



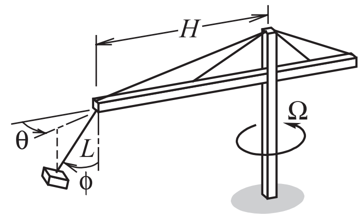
Exercise 7.58

EXERCISE 7.59 Slender bar BC is suspended from pivot A by a cable. A precessional motion about the vertical is induced. It is desired to investigate the ensuing motion under the assumption that the cable remains taut, in which case it forms angle β relative to the vertical. The bar lies in the same vertical plane as the cable, at angle θ from vertical. Derive the differential equations of motion governing β, θ, and the precession angle ψ.



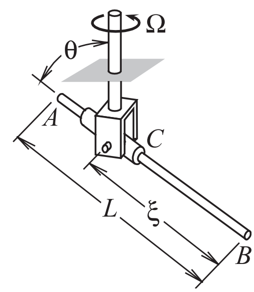
Exercise 7.59

EXERCISE 7.60 A shipping container is suspended from a crane by an inextensible cable. The crane rotates in the vertical plane at angular speed Ω whose time dependence is known. It may be assumed that the cable remains taut, so its orientation is describable in terms of the angle θ locating the vertical plane in which it is situated relative to the plane of the crane, and the angle of elevation ϕ from a vertical line. Based on a model of the container as a small particle, derive differential equations of motion in which the only unknowns are θ and ϕ.



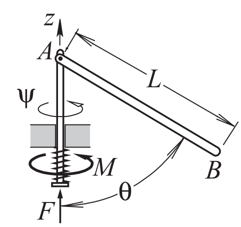
Exercise 7.60

EXERCISE 7.61 Collar C is attached to the vertical shaft by a fork-and-clevis, so the angle of inclination θ of bar AB is arbitrary. Because this bar slides through the collar, the distance ξ from the pivot point to the end of the bar is variable, but it may be assumed that the bar does not spin about its own axis. The vertical shaft rotates at the constant rate Ω. Derive the differential equations governing ξ and θ.



Exercise 7.61

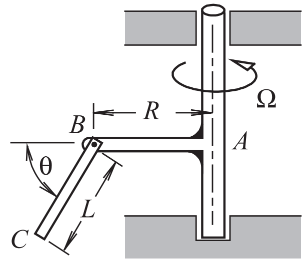
EXERCISE 7.62 The force–couple system $\bar{F}$ and $\bar{M}$ causes the vertical shaft to move upward and precess. The couple is such that the precession rate $\dot{\psi}$ is constant, but the vertical displacement z of the shaft is not known *a priori*. The spring's extensional stiffness (in opposition to displacement z) is k_E, and $z = 0$ corresponds to the spring being undeformed. The mass of the shaft is m_1, and the mass of bar AB is m_2, and both bodies may be approximated as being very thin. Pin A is ideal, so the angle of inclination θ is an unknown function of time. Derive the differential equations of motion for z and θ.



Exercises 7.62 and 7.63

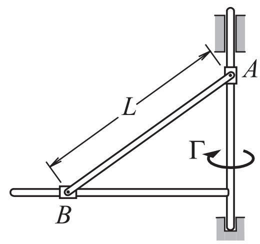
EXERCISE 7.63 A specified force–couple system $\bar{F}$ and $\bar{M}$ causes the vertical shaft to move upward and precess. The spring's extensional stiffness k_E acts in opposition to displacement z, and the torsional stiffness k_T opposes precession ψ. The spring is undeformed in the position where $z = \psi = 0$. The mass of the shaft is m_1, the mass of bar AB is m_2. and both bodies may be approximated as being very thin. Pin A is ideal. Derive the differential equations of motion for z, ψ and θ.

EXERCISE 7.64 A known torque Γ acting about the vertical shaft causes the T-bar to precess at angular speed $\dot{\psi} = \Omega$. Pin B is ideal, so the angle of elevation of bar BD also is unknown. The mass of bar BC is m, and the moment of inertia of the T-bar about its rotation axis is I_{T}. Derive the differential equations of motion governing θ and ψ.



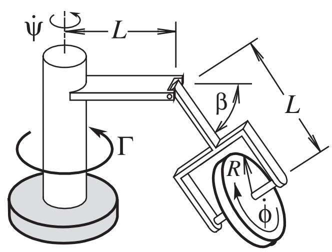
Exercise 7.64

EXERCISE 7.65 A known couple $\Gamma(t)$ induces rotation of the system about the vertical axis. Collars A and B, each of whose mass is m, are interconnected by a rigid bar whose mass is $4m$. The moment of inertia of the T-bar about the vertical axis is I_T. Derive the equations of motion for this system.



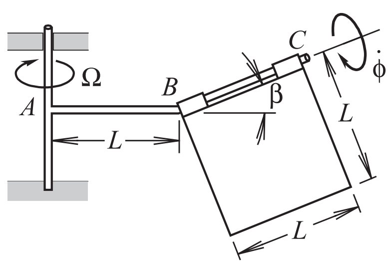
Exercise 7.65

EXERCISE 7.66 A known torque Γ is applied to the vertical post, which causes the system to rotate about the vertical axis at angular speed $\dot{\psi}$. Pin B allows the angle β to change freely, but the spin rate of the flywheel is the constant value $\dot{\phi}$. The moment of inertia of the vertical post about its axis of rotation is I_P, the mass of the forked arm may be neglected, and the flywheel may be considered to be a thin disk with mass m and moment of inertia I_ϕ about its spin axis. Derive the equations of motion for $\dot{\psi}$ and β. Under what conditions will β be constant?



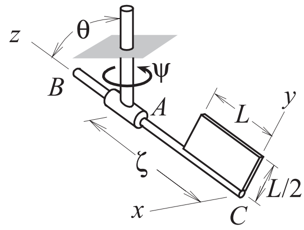
Exercise 7.66

EXERCISE 7.67 Bent arm ABC is welded to the vertical shaft, which rotates at the constant rate Ω. The square plate rotates freely about axis BC at angular speed $\dot{\phi}$, with $\phi = 0$ corresponding to the plate being situated in the vertical plane, as shown in the sketch. Determine the differential equation governing ϕ.



Exercise 7.67

EXERCISE 7.68 Collar A is welded to the vertical shaft, so the angle θ is constant. The rectangular plate is welded to bar BC, which may slide through the collar and rotate by angle ϕ about its own axis relative to the collar. When $\phi = 0$, the plate is in the upright position shown. The whole system freely precesses about the vertical axis, and frictional resistance is negligible. Only the mass of the plate is significant. Derive the differential equations of motion for this system.



Exercise 7.68

EXERCISE 7.69 The semicylinder in Exercise 7.43 is released from rest with its flat side in the vertical plane, $\psi = \pi/2$. Determine and graph the ensuing response $\psi(t)$. From that result deduce the period of oscillation. Compare that result with the period when the semicylinder is released from rest at $\psi = \pi/20$. *Hint:* Use the nondimensional time $\tau = (g/R)^{1/2}\,t$ to remove algebraic parameters from the differential equations.

EXERCISE 7.70 Consider the system in Example 7.15 when there is no applied couple, $M = 0$. The system is initially at rest at $\theta = 80^\circ$ when a constant force $F = 2mg$ is applied. The spring stiffness is $k = 0.5mg/L$. Determine whether the linkage attains the horizontal position, $\theta = 0$. If so, what is the elapsed time and angular speed $\dot{\theta}$ for that condition?

EXERCISE 7.71 The parameters of the system in Exercise 7.64 are $R = L$, $I_T = 0.5mL^2$, $M = 4mgL$, and $(L/g)^{1/2} = 0.1$ s. The system is released from rest at $t = 0$ with $\theta = 90^\circ$. Determine θ as function of time in the ensuing motion, as well as the torque Γ. What is the minimum value of θ during the course of the motion? Does θ approach a steady-state value?

EXERCISE 7.72 Prior to $t = 0$ the spherical pendulum in Example 7.14 was undergoing a steady precession at angle of inclination $\theta = 15^\circ$, with the cable length ℓ constant at a specified value L. The angular speed Ω in this condition was the value associated with steady precession at a constant cable length. At $t = 0$ the tensile force is suddenly changed from the value required for steady precession to $F = 4mg$. Determine and graph as functions of time the length ℓ, angle θ, and precession rate Ω. Compare the instants when each of these variables have their maximum and minimum values. *Hint:* Use the nondimensional time $\tau = (g/L)^{1/2}\,t$ to convert the equations of motion to a form suitable for numerical integration.

EXERCISE 7.73 The compound pendulum in Exercise 7.4 is released from rest with both bars inclined at 75° from the vertical position. The bars are identical with $(g/L)^{1/2} =$ 2 rad/s. Examine the equations of motion to determine whether it is possible for the system to oscillate with both bars remaining aligned. Then solve the equations of motion to determine each angle of inclination as a function of time. Is the response periodic?

EXERCISE 7.74 The system in Exercise 7.54 is initially at rest in the vertical plane, $\psi = 0$, with the groove tilted at $\theta = \pi/4$ and the block coincident with the axis of the shaft, $s = 0$. At that instant a constant torque of $M = 40$ N-m is applied. Determine and graph the displacement s and rotation θ as functions of time. Also graph the precession rate $\dot{\psi}$. Properties of the system are $m_1 = 500$ g, $k = 128$ N/m, and $I_1 = 2.5$ kg-m^2, $I_2 =$ 1.5 kg-m^2.

EXERCISE 7.75 The system of interest is the flyball governor in Exercise 7.41 in the case where the torque M acting about the vertical axis is zero. Suitable generalized coordinates are $q_1 = \psi$ and $q_2 = \theta$, where ψ is the precession angle. The initial conditions for the system are

$$\psi = 0, \quad \dot{\psi} = 1500 \text{ rev/min}, \quad \theta = 85^\text{o}, \quad \dot{\theta} = 0 \text{ @ } t = 0,$$

and the system parameters are $m = 250$ g, and $L = 400$ mm. (a) If properly derived, the Lagrange equation associated with ψ is a perfect differential. The result of integrating that equation is the constant angular momentum about the z axis, which relates the values of $\dot{\psi}$ and θ corresponding to the specified initial conditions. Solve this equation for $\dot{\psi}$ at $\theta = 30^\text{o}$. (b) Let F have the constant value obtained from the Lagrange equation when $\theta = 30^\text{o}$, $\dot{\theta} \equiv 0$, and $\dot{\psi}$ has the value found in Part (a). Use numerical methods to solve the equations of motion corresponding to this value of F and the specified initial conditions. Plot ψ and θ as functions of t for an interval sufficiently long to ascertain whether $\theta = 30^\text{o}$ is a dynamic equilibrium state. Check the correctness of the integration by monitoring whether the angular momentum about the vertical axis is constant. (c) An alternative value of F is found by simultaneously solving the angular momentum equation in Part (a) and the work–energy principle $\Delta T + \Delta V = F \Delta y_B$, where Δy_B is the upward displacement of the vertical bar to which F is applied. The initial conditions for this evaluation of F are those specified and $\theta = 30^\circ$ and $\dot{\theta} = 0$ at the second position. The value of F obtained in this manner does the minimum amount of work required to attain the second state, but it is possible that this value of F would not be adequate to move the system beyond some intermediate position. Solve the equations of motion with the initial conditions set at the specified values and F equated to this minimum-work value. Does the system actually arrive at $\theta = 30^\circ$ with $\dot{\theta} = 0$?

CHAPTER 8

Constrained Generalized Coordinates

The basic development of Lagrange's equations in the preceding chapter is suitable for many important engineering applications. However, up to now these equations have only been employed when the generalized coordinates constitute an unconstrained set. The first part of this chapter removes this limitation. For nonholonomic systems the use of constrained generalized coordinates is mandatory. However, it might be desirable to use constrained coordinates to analyze holonomic systems, as will be seen. This is the situation when the effect of sliding friction is an important feature, which will be treated in depth.

Regardless of whether one follows the Lagrangian or Newton–Euler approach, derivation of the differential equations of motion is only the first phase of a dynamic analysis. Solution of those equations to simulate a system's response is usually the ultimate objective. As several examples have already demonstrated, the equations of motion can be quite complicated, and therefore not amenable to analytical solution. The basic state-space approach to solving the differential equations of motion associated with holonomic systems was developed in Section 7.6. The occurrence of constraint equations and Lagrange multipliers requires modification of that formulation. The second part of this chapter develops and implements several numerical algorithms that may be used to solve the equations of motion governing constrained generalized coordinates.

8.1 LAGRANGE'S EQUATIONS—CONSTRAINED CASE

Lagrange's equations for unconstrained coordinates constitute a set of differential equations of motion whose number equals the number of generalized coordinates. Thus the only other information required to solve these equations is an appropriate set of initial conditions. When generalized coordinates are constrained, the equations of motion are supplemented by kinematical constraint equations. Another difference is that an arbitrary virtual movement will not be consistent with the stated constraint conditions, which has the consequence of causing constraint forces to be manifested in the generalized forces. Our first task is to see how to synthesize a solvable set of equations in this situation. We then consider some general situations in which constrained generalized coordinates are either necessary or useful, and specific procedures for analyzing each situation.

As we saw in the previous chapter, scleronomic (time-invariant) and rheonomic (time-dependent) configuration constraints can be written as linear velocity constraint equations, which is the only form of nonholonomic constraint we consider here. Writing all constraint equations in velocity form is more than a matter of convenience. The equations that result turn out to be much more amenable to standard numerical analysis than they would be if configuration constraints equations were used. Thus we consider a set of N generalized coordinates that are required to satisfy J velocity constraint equations described by

$$\boxed{\sum_{j=1}^{N} a_{ij}\dot{q}_j + b_i = 0, \quad i = 1, 2, \ldots, J.} \tag{8.1.1}$$

The Jacobian constraint coefficients a_{ij}, which multiply the generalized velocities, as well as the b_i coefficients, may depend on the values of the q_j variables and time t.

By definition, the symbol Q_j represents the role of all nonconservative forces, as well as any conservative forces that have not been represented by the potential energy. Let us apportion Q_j into two parts: $Q_j^{(a)}$ is associated with the applied loads and R_j is associated with the unknown reactions,

$$Q_j = Q_j^{(a)} + R_j. \tag{8.1.2}$$

A constraint force enforces each constraint equation that is described by Eq. (8.1.1). The influence of each such force will appear in the R_j terms. Virtual work is additive as a scalar, which means that R_j is a scalar sum of the contribution $R_j^{(i)}$ associated with each constraint:

$$R_j = \sum_{i=1}^{J} R_j^{(i)}, \quad j = 1, 2, \ldots, N. \tag{8.1.3}$$

We saw in Subsection 7.4.2 that knowledge of the Jacobian constraint coefficients is sufficient to characterize each $R_j^{(i)}$ term. According to Eq. (7.4.12), a Lagrange multiplier λ_i is proportional to $R_j^{(i)}$. When we use that equation to represent each $R_j^{(i)}$ term in Eq. (8.1.3), and combine that result with $Q_j^{(a)}$, the resulting Lagrange's equations are

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j^{(a)} + \sum_{i=1}^{J} a_{ij}\lambda_i, \quad j = 1, 2, \ldots, N.} \tag{8.1.4}$$

In some situations it is necessary to determine the actual constraint forces C_i, as well as the generalized coordinates. We may obtain these quantities from the associated Lagrange multipliers λ_i by invoking Eqs. (7.4.16), which stated that $C_i = \sigma_i\lambda_i$. We may identify the proportionality factors σ_i by explicitly evaluating the virtual work of the constraint forces. Alternatively, we can express the virtual work explicitly in terms of the constraint forces. This approach will be demonstrated in Example 8.7.

The requirement that one eliminate all variables other than the q_j and t from the potential and kinetic energies applies equally for constrained and unconstrained generalized coordinates. Such an analysis leads to the functions $T(q_n, \dot{q}_n, t)$ and $V(q_n, t)$. It follows that the terms on the left side of Lagrange's equations will contain generalized coordinates, velocities, and accelerations. Thus these equations constitute a set of N differential equations governing the N generalized coordinates. They are not a sufficient set of equations because they also contain J unknown λ_i parameters. The necessary J additional equations are provided by the kinematical constraint equations. Thus evaluation of system response when a system is described by constrained generalized coordinates entails simultaneously satisfying N differential equations of motion, given by Eqs. (8.1.4) and J differential equations of constraint, Eq. (8.1.1). The result of such a solution would be the value of each q_j and each λ_i as a function of t.

When a system has nonholonomic constraints, we have no choice but to use constrained coordinates. However, we might decide to use constrained generalized coordinates to describe holonomic systems. One reason for doing so is to simplify the synthesis of the equations of motion. This aspect may be explained by considering a one-degree-of-freedom system. Suppose the positions of various parts of such a system are conveniently described by two generalized coordinates that are related by a scleronomic constraint, $f(q_1, q_2) = 0$. If we wish to use q_1 as a single unconstrained generalized coordinate, we must algebraically solve this constraint equation for q_2 in terms of q_1. It might be that this solution, $q_2 = g(q_1)$, is extremely difficult to obtain, but if we succeed in doing so, we must differentiate it to describe the generalized velocity, $\dot{q}_2 = \dot{q}_1 (dg/dq_1)$. Substitution of this relationship and $q_2 = g(q_1)$ into the energy expressions yields T and V as functions of only q_1 and $\dot{q}_1$. These functions are likely to be quite complicated unless the g function has a simple form. Now consider the alternative of using both q_2 and q_1 to describe the system. The velocity constraint equation obtained by differentiating the configuration constraint will be enforced explicitly as part of the solution, so q_1 and q_2 are treated as completely independent variables in the formulation. Thus we leave the energy expressions in basic forms containing both q_1 and q_2. Because the expressions for T and V will be functionally simpler, the complexity of the operations required to form Lagrange's equations will be lessened. At the same time, there is a penalty, because there are two Lagrange equations and a velocity constraint in this case, as opposed to the single Lagrange equation when q_1 solely is used. In essence, the philosophy underlying the use of constrained generalized coordinates for complicated holonomic systems is to shift the burden from formulating the equations of motion to solving them. Computer methods are quite adept at the latter, as we will see.

Another reason to use constrained generalized coordinates to describe a holonomic system is to evaluate constraint forces. If we were to use unconstrained generalized coordinates, reaction forces could be determined by a multiphase process that begins with the solution of the equations of motion, and saving the values of q_j, $\dot{q}_j$, and $\ddot{q}_j$ at each instant. These results would be used to compute the linear and angular positions, velocities, and accelerations of all parts of the system. Substituting those values into the Newton–Euler equations of motion for individual bodies would yield the constraint force of interest. Such a procedure is suitable to situations in which most or all of the reaction forces need to be evaluated, but it is inefficient if only a few reactions are of interest. In contrast, it is possible to solve for the instantaneous reactions simultaneously with the generalized coordinates when we use constrained generalized coordinates.

We previously encountered in Example 7.14 a simple situation suggesting how constrained generalized coordinates can be used for this purpose. We found there that we could obtain an expression for the cable force in the spherical pendulum by considering the cable length to be a generalized coordinate subject to a rheonomic constraint, rather than a specified length variable. The general concept underlying the treatment of such situations uses a set of generalized coordinates that do not automatically satisfy the kinematical condition imposed by the desired constraint force. This force, in the form of the associated Lagrange multiplier, will appear in the equations of motion. Solution of the differential equations of motion formulated in this manner would yield the time history of the Lagrange multiplier. The actual constraint force can then be determined from $C_i = \sigma_i \lambda_i$, as described by Eqs. (7.4.16).

The need to evaluate a constraint force is often a discretionary matter. However, in situations involving Coulomb sliding friction, the evaluation of the normal force is an intrinsic part of the solution process, because the magnitude of the tangential friction force depends on the normal force. The sliding friction force does not prevent motion. Therefore it acts like an applied force that does virtual work. Hence, even if we use unconstrained generalized coordinates, the magnitude of the normal force will always occur in some of the generalized forces. The procedure we follow is to use constrained generalized coordinates that are selected such that the constraint against movement perpendicular to the contact surface must be enforced explicitly. The associated normal force will appear in the virtual work in addition to the friction force. The constraint equations and the additional Lagrange equations arising from using more generalized coordinates than necessary will provide the requisite number of equations.

A system illustrating this aspect is the linkage in Fig. 8.1, which was the subject of Example 7.15. In the previous analysis there was no friction between collar C and the guide bar, so the angle θ could be used as an unconstrained coordinate, with the angle ϕ for bar BC equated to θ. If there is sliding friction at the collar, that analysis will not suffice, because the friction force will do virtual work when θ is incremented. The friction force f is proportional to the magnitude of the normal force, so the single equation of motion would contain two unknowns, θ and N_C. A constrained formulation of this problem could use θ and ϕ as generalized coordinates that must satisfy the scleronomic constraint, $\theta - \phi = 0$. Because it is necessary to violate the kinematical constraint of the collar without having other constraint forces contribute to the virtual work, the

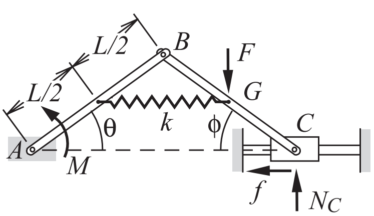


Figure 8.1. A linkage in which there is friction at collar C.

configuration of the system for arbitrary θ and ϕ is taken to be such that the connection at pin B remains intact. Thus the position of the collar is given by

$$\bar{r}_{C/A} = L(\cos\theta + \cos\phi)\,\bar{i} + L(\sin\theta - \sin\phi)\,\bar{j}. \tag{8.1.5}$$

The corresponding virtual displacement is

$$\begin{aligned}\delta\bar{r}_C &= \frac{\partial\bar{r}_{C/A}}{\partial\theta}\delta\theta + \frac{\partial\bar{r}_{C/A}}{\partial\phi}\delta\phi \\ &= L\big(-\sin\theta\bar{i} + \cos\theta\,\bar{j}\big)\,\delta\theta + L\big(-\sin\phi\bar{i} - \cos\phi\,\bar{j}\big)\,\delta\phi.\end{aligned} \tag{8.1.6}$$

Rather than using a Lagrange multiplier to describe $\bar{N}_C$, we treat this force as any other nonconservative force, because $|\bar{N}_C|$ will occur anyway when we describe the friction force. The latter force may be written as $\bar{f} = -\mu_k\,|N_C|\mathrm{sgn}\big(\bar{v}_C\cdot\bar{i}\big)\,\bar{i}$, which captures the fact that the friction force is proportional to the normal force's magnitude and its sense is opposite that of $\bar{v}_C$. The virtual work done by the forces at collar C in this formulation is

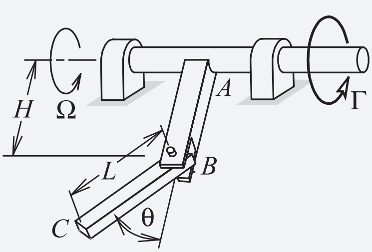
$$\delta W = \big[-\mu_k\,|N_C|\,\mathrm{sgn}\,(\bar{v}_C\cdot\bar{i})\,\bar{i} + N_C\,\bar{j}\big]\cdot\delta\bar{r}_C. \tag{8.1.7}$$

Substitution of the expression for $\delta\bar{r}_C$ leads to the recognition that the generalized forces are

$$\begin{aligned}Q_1 &= \mu_k\,|N_C|\,L\sin\theta\mathrm{sgn}\,(\bar{v}_C\cdot\bar{i}) + N_C L\cos\theta, \\ Q_2 &= \mu_k\,|N_C|\,L\sin\phi\mathrm{sgn}\,(\bar{v}_C\cdot\bar{i}) - N_C L\cos\phi.\end{aligned} \tag{8.1.8}$$

Thus, using θ and ϕ as constrained coordinates leads to two Lagrange equations, in which the normal force occurs in both generalized forces. In combination with the constraint equation, we have three equations governing θ, ϕ, and N_C. These considerations will be an important aspect of Example 8.7, which will derive and then solve the equations of motion when friction effects are important.

EXAMPLE 8.1 A servomotor applies torque Γ to the horizontal shaft of the T-bar such that the rotation rate Ω increases in proportion to the angle θ by which bar BC swings away from the radial position, that is, $\Omega = b\theta^2$. The masses are m_1 for bar AB and m_2 for bar BC, and the cylindrical shaft's moment of inertia about its axis of rotation is I_3. Determine the equations of motion for the system.

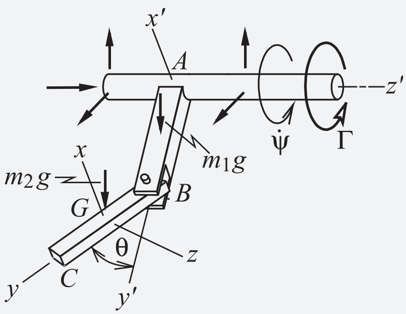


Example 8.1

SOLUTION This example is a straightforward demonstration of how constraint conditions are incorporated into an analysis of spatial motion. The location of bar BC is fully specified by the angle the T-bar rotates about its bearings, and the angle θ by which bar BC rotates about pin B relative to bar AB. Rotation about the bearings is a precession, which we denote to be angle ψ, with $\dot{\psi} = \Omega$ and $\psi = 0$ defined to correspond to bar AB being upright. The given constraint on the motion is $\dot{\psi} = c\theta$, which is nonholonomic. Hence we employ both angles as generalized coordinates, $q_1 = \theta$ and $q_2 = \psi$. Comparing the stated constraint equation with the standard form for a linear velocity constraint shows that

$$a_{11} = 0, \quad a_{12} = 1, \quad b_1 = -b\theta^2. \qquad \triangleleft$$

It is useful to draw a free-body diagram of the system in which we define coordinate systems for describing the motion. The $x'y'z'$ coordinate system is attached to the T-bar, whereas xyz is a centroidal coordinate system for bar AB with y aligned with the bar and x aligned with pin B.



Free-body diagram of the swinging bar and its supporting shaft

In addition to the transverse and axial forces exerted by the bearings, the applied loads are the conservative gravity force and the nonconservative torque Γ.

All motion variables required for constructing the kinetic energy must be described solely in terms of the generalized coordinates. The angular velocity of the T-bar is $\dot{\psi}\bar{k}'$. Adding the two simple rotations of bar BC gives

$$\bar{\omega}_{BC} = \dot{\psi}\bar{k}' + \dot{\theta}\left(-\bar{i}\right) = -\dot{\theta}\bar{i} - \dot{\psi}\sin\theta\,\bar{j} + \dot{\psi}\cos\theta\bar{k}.$$

We obtain an expression for $\bar{v}_G$ by recognizing that points G and B are on the swinging bar, whereas points B and A are on the T-bar, so that

$$\bar{v}_G = \bar{v}_B + \bar{\omega}_{BC} \times \bar{r}_{G/B} = \dot{\psi}\bar{k}' \times H\bar{j}' + \bar{\omega}_{BC} \times \frac{L}{2}\bar{j} = -\dot{\psi}\left(H + \frac{L}{2}\cos\theta\right)\bar{i} - \frac{L}{2}\dot{\theta}\bar{k}.$$

We decompose the kinetic energy of the T-bar into contributions of the shaft and bar AB, which is the manner in which the inertia properties are specified.

Considering the thickness of both bars to be negligible gives

$$T = \frac{1}{2} I_3 \dot{\psi}^2 + \frac{1}{2}\left(\frac{1}{3} m_1 H^2\right)\dot{\psi}^2 + \frac{1}{2} m_2 v_G^2 + \frac{1}{2}\bar{\omega}_{BC}\cdot \bar{H}_G$$
$$= \frac{1}{2}\left(I_3 + \frac{1}{3} m_1 H^2\right)\dot{\psi}^2 + \frac{1}{2} m_2 \left[\dot{\psi}^2 \left(H + \frac{L}{2}\cos\theta\right)^2 + \frac{L^2}{4}\dot{\theta}^2\right]$$
$$+ \frac{1}{2}\left(\frac{1}{12} m_2 L^2\right)\left[\dot{\theta}^2 + \dot{\psi}^2 (\cos\theta)^2\right]$$
$$= \frac{1}{2}\left(\frac{1}{3} m_2 L^2\right)\dot{\theta}^2 + \frac{1}{2}\left[I_3 + \left(\frac{1}{3} m_1 + m_2\right) H^2\right.$$
$$\left. + \left(\frac{1}{3} m_2 L^2\right)(\cos\theta)^2 + m_2 H L \cos\theta\right]\dot{\psi}^2.$$

We define the datum for gravitational potential energy to coincide with the shaft. We may find the elevation of point G by geometrically projecting this point's position onto the vertical direction, or else by coordinate transformations. Either way, the result is

$$V = m_1 g \frac{H}{2}\cos\psi + m_2 g\left(H + \frac{L}{2}\cos\theta\right)\cos\psi.$$

Note that the positive sign for the elevation applies because $\psi = 0$ was defined to correspond to bar AB being upright.

To evaluate the virtual work we observe that Γ imposes the constraint on $\dot{\psi}$. We can represent its effect with a Lagrange multiplier. (If we wished to derive an equation for Γ we would evaluate how it contributes to the virtual work, rather than using a Lagrange multiplier.) The bearing forces constrain the shaft from moving, and those constraints are not violated by a virtual movement, regardless of the values of $\delta\psi$ and $\delta\theta$. Thus we have $\delta W = 0$, so that $Q_1 = Q_2 = 0$. (Depending on how the bearings are designed, it is possible that they also exert couples orthogonally to the shaft. Those couples too would not appear in δW because they would impose rotational constraints that are not violated in the virtual movement.) We form the derivatives of T and V appearing in Lagrange's equations and substitute the result into the Lagrange multiplier form of the equations of motion, Eq. (8.1.4). This yields

$$\frac{1}{3} m_2 L^2 \ddot{\theta} + \left[\frac{1}{3} m_2 L^2 (\cos\theta) + \frac{1}{2} m_2 H L\right](\sin\theta)\,\dot{\psi}^2 - \frac{1}{2} m_2 g L \sin\theta\cos\psi = a_{11}\lambda_1 = 0$$

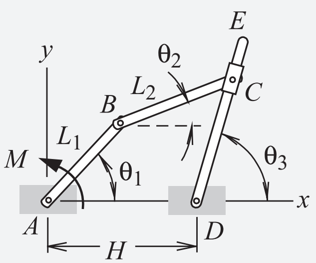
$$\left[I_3 + \left(\frac{1}{3} m_1 + m_2\right) H^2 + \left(\frac{1}{3} m_2 L^2\right)(\cos\theta)^2 + m_2 H L\cos\theta\right]\ddot{\psi}$$
$$- \left[\frac{2}{3} m_2 L^2 (\cos\theta) + m_2 H L\right](\sin\theta)\,\dot{\theta}\dot{\psi} - \left(\frac{1}{2} m_1 + m_2\right) g\sin\psi$$
$$- \frac{1}{2} m_2 g L\cos\theta\sin\psi = a_{12}\lambda_1 = \lambda_1.$$

◁

There are three unknowns in these two equations; the third equation is the constraint, $\dot{\psi} = b\theta^2$.

The simple nature of some constraint equations, such as $\dot{\psi} = c\theta$ in the present case, sometimes leads inexperienced individuals to a fundamental error. It might seem that we could simplify the energy expressions if we were to use the constraint equations to eliminate one or more generalized coordinates. For example, here we could completely eliminate θ by substituting $\theta = \left(\dot{\psi}/b\right)^{1/2}$. However, this is a forbidden operation. The reason for this prohibition goes back to the derivation of Lagrange's equations, which requires an arbitrary virtual displacement, whereas using a constraint equation in the aforementioned manner limits the adjacent variational paths.

EXAMPLE 8.2 A known torque $M(t)$ actuates the linkage. The masses are m_1, m_2, and m_3 for bars AB, BC, and DE, respectively, and collar C has mass m_4. The system lies in the vertical plane, and friction is negligible. Derive the equations of motion.



Example 8.2

SOLUTION The benefits of using constrained generalized coordinates to lessen the complications of geometrical complexity are demonstrated in this example. We encountered this system previously in Example 7.6. It has two degrees of freedom, because the position of all bars is specified by their respective angles, and one angle is algebraicially related to the other two. If we were to use θ_1 and θ_2 as unconstrained generalized coordinates, we would find that the expression for θ_3 is fairly complicated, so that the kinetic-energy expression is intricate. We therefore use all three angles as a set of constrained generalized coordinates.

The first task is to establish the constraint equation these variables must satisfy. We could obtain the velocity equation directly by following a linkage analysis, according to

$$\bar{v}_C = v_{\text{rel}}\bar{e}_{E/D} + \dot{\theta}_3\bar{k} \times \bar{r}_{C/D} = \dot{\theta}_1\bar{k} \times \bar{r}_{B/A} + \dot{\theta}_2\bar{k} \times \bar{r}_{C/B}.$$

However, Example 7.6 derived the configuration constraint based on the horizontal and vertical distances proceeding from pin A to pin B and then pin C. Doing so gave

$$\tan\theta_3 = \frac{L_1\sin\theta_1 + L_2\sin\theta_2}{L_1\cos\theta_1 + L_2\cos\theta_2 - H},$$

which may be rewritten as

$$\left(\cos\theta_1 + \frac{L_2}{L_1}\cos\theta_2 - \frac{H}{L_1}\right)\tan\theta_3 - \sin\theta_1 - \frac{L_2}{L_1}\sin\theta_2 = 0.$$

Differentiation of this expression yields the velocity constraint equations:

$$a_{11}\dot{\theta}_1 + a_{12}\dot{\theta}_2 + a_{13}\dot{\theta}_3 = 0,$$

$$a_{11} = -\sin\theta_1\tan\theta_3 - \cos\theta_1, \quad a_{12} = -\frac{L_2}{L_1}\left(\sin\theta_2\tan\theta_3 + \cos\theta_2\right), \qquad \triangleleft$$

$$a_{13} = \frac{L_1\cos\theta_1 + L_2\cos\theta_2 - H}{L_1\left(\cos\theta_3\right)^2}.$$

The kinetic energy of the system is the sum of contributions of the three bars and the sliding collar. Each term may be described in terms of whichever generalized coordinates are appropriate, without concern for the fact that they are constrained. Thus the angular velocities of the bars are $\dot{\theta}_1\bar{k}$, $\dot{\theta}_2\bar{k}$, and $\dot{\theta}_3\bar{k}$. Bars AB and DE are in pure rotation. We may obtain expressions for the velocity of the center of mass of bar BC and of the collar by differentiating the respective position vectors described in terms of θ_1 and θ_2, which are

$$\bar{r}_{G2/A} = \left(L_1\cos\theta_1 + \frac{L_2}{2}\cos\theta_2\right)\bar{i} + \left(L_1\sin\theta_1 + \frac{L_2}{2}\sin\theta_2\right)\bar{j},$$

$$\bar{r}_{C//A} = \left(L_1\cos\theta_1 + L_2\cos\theta_2\right)\bar{i} + \left(L_1\sin\theta_1 + L_2\sin\theta_2\right)\bar{j},$$

which leads to

$$\bar{v}_{G2} = \left(-L_1\dot{\theta}_1\sin\theta_1 - \frac{L_2}{2}\dot{\theta}_2\sin\theta_2\right)\bar{i} + \left(L_1\dot{\theta}_1\cos\theta_1 + \frac{L_2}{2}\dot{\theta}_2\cos\theta_2\right)\bar{j},$$

$$\bar{v}_C = \left(-L_1\dot{\theta}_1\sin\theta_1 - L_2\dot{\theta}_2\sin\theta_2\right)\bar{i} + \left(L_1\dot{\theta}_1\cos\theta_1 + L_2\dot{\theta}_2\cos\theta_2\right)\bar{j}.$$

Let L_3 be the length of bar DE. Then the kinetic energy derived from these expressions is

$$\begin{aligned} T = {} & \frac{1}{2}\left(\frac{1}{3}m_1L_1^2\right)\dot{\theta}_1^2 + \frac{1}{2}m_2\left[\left(-L_1\dot{\theta}_1\sin\theta_1 - \frac{L_2}{2}\dot{\theta}_2\sin\theta_2\right)^2\right. \\ & \left. + \left(L_1\dot{\theta}_1\cos\theta_1 + \frac{L_2}{2}\dot{\theta}_2\cos\theta_2\right)^2\right] + \frac{1}{2}\left(\frac{1}{12}m_2L_2^2\right)\dot{\theta}_2^2 \\ & + \frac{1}{2}\left(\frac{1}{3}m_3L_3^2\right)\dot{\theta}_3^2 + \frac{1}{2}m_4\left[\left(-L_1\dot{\theta}_1\sin\theta_1 - L_2\dot{\theta}_2\sin\theta_2\right)^2\right. \\ & \left. + \left(L_1\dot{\theta}_1\cos\theta_1 + L_2\dot{\theta}_2\cos\theta_2\right)^2\right]. \end{aligned}$$

Collecting like coefficients of the generalized velocities converts this to

$$\begin{aligned} T = {} & \frac{1}{2}\left(\frac{1}{3}m_1 + m_2 + m_4\right)L_1^2\dot{\theta}_1^2 + \frac{1}{2}\left(\frac{1}{3}m_2 + m_4\right)L_2^2\dot{\theta}_2^2 + \frac{1}{2}\left(\frac{1}{3}m_3L_3^2\right)\dot{\theta}_3^2 \\ & + \frac{1}{2}\left(m_2 + 2m_4\right)L_1L_2\cos\left(\theta_1 - \theta_2\right)\dot{\theta}_1\dot{\theta}_2. \end{aligned}$$

The system is stated to lie in the vertical plane, which implies that the analysis should account for gravity. The elevation of the center of mass of each body above the datum for gravity, which we take to coincide with pins A and D, may be represented in terms of whichever of the generalized coordinates is most suitable. Hence the potential energy of gravity is

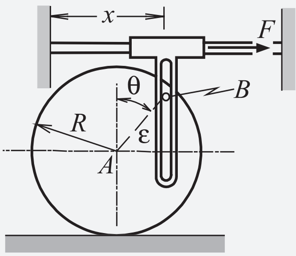
$$\begin{aligned} V &= m_1 g \frac{L_1}{2} \sin\theta_1 + m_2 g \left(L_1 \sin\theta_1 + \frac{L_2}{2} \sin\theta_2 \right) \\ &\quad + m_4 g \left(L_1 \sin\theta_1 + L_2 \sin\theta_2 \right) + m_3 g \frac{L_3}{2} \sin\theta_3 \\ &= \left(\frac{m_1}{2} + m_2 + m_4 \right) g L_1 \sin\theta_1 + \left(\frac{m_2}{2} + m_4 \right) g L_2 \sin\theta_2 + \frac{m_3}{2} g L_3 \sin\theta_3. \end{aligned}$$

The constraints imposed by pin A and D are not violated when the angle of each bar is incremented. Therefore the only nonconservative load contributing to the virtual work is the torque M, which gives $\delta W = M\delta\theta_1 \Longrightarrow Q_1 = M,\ Q_2 = Q_3 = 0$. The Jacobian coefficients were listed earlier. The three Lagrange equations that result are

$$\begin{aligned} &\left(\frac{1}{3} m_1 + m_2 + m_4 \right) L_1^2 \ddot{\theta}_1 + \frac{1}{2} (m_2 + 2m_4) L_1 L_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) \right] \\ &\quad + \left(\frac{m_1}{2} + m_2 + m_4 \right) g L_1 \cos\theta_1 = Q_1 + a_{11}\lambda_1 = M - \left(\sin\theta_1 \tan\theta_3 + \cos\theta_1 \right) \lambda_1, \\ &\left(\frac{1}{3} m_2 + m_4 \right) L_2^2 \ddot{\theta}_2 + \frac{1}{2} (m_2 + 2m_4) L_1 L_2 \left[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) \right] \\ &\quad + \left(\frac{m_2}{2} + m_4 \right) g L_2 \cos\theta_2 = Q_2 + a_{12}\lambda_1 = -\frac{L_2}{L_1} \left(\sin\theta_2 \tan\theta_3 + \cos\theta_2 \right) \lambda_1, \\ &\frac{1}{3} m_3 L_3^2 \ddot{\theta}_3 = Q_3 + a_{13}\lambda_1 = \frac{L_1 \cos\theta_1 + L_2 \cos\theta_2 - H}{L_1 (\cos\theta_3)^2} \lambda_1. \qquad \triangleleft \end{aligned}$$

Evaluation of the response would entail simultaneously solving these three equations of motion and the constraint equation for the three generalized coordinates and the Lagrange multiplier λ_1. We could eliminate the Lagrange multiplier by solving one equation of motion for λ_1, and substituting that expression into the other two equations of motion. This would reduce the problem to three simultaneous differential equations (two equations of motion and the velocity constraint), whose form would be significantly more complicated than the original set of four equations.

EXAMPLE 8.3 A known force $\bar{F}$ is applied to the yoke, causing the system to move rightward. The disk rolls without slipping, and the coefficient of kinetic friction between pin B and the yoke is μ. Masses are m_1 for the yoke and m_2 for the homogeneous disk. Derive the equations of motion.



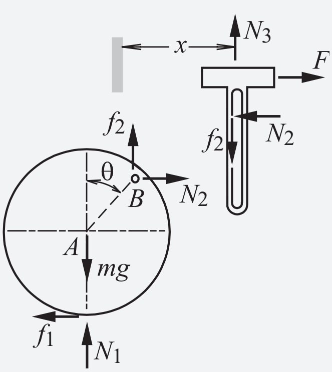
Example 8.3

SOLUTION We see here how constrained generalized coordinates are used to analyze systems with Coulomb friction. As was shown in Example 7.1, this system has a single degree of freedom. That example derived the velocity constraint and corresponding configuration constraint that must be satisfied by x and θ in order that the pin remain situated in the groove of the yoke. The result was

$$\dot{x} - \dot{\theta}\,(R + \varepsilon\cos\theta) = 0 \iff x - R\theta - \varepsilon\sin\theta = C, \tag{1}$$

where C depends on where the disk is situated when $x = \theta = 0$. We need to obtain equations of motion in which the normal force exerted between pin B and the yoke appears. That force imposes Eq. (1). We select $q_1 = \theta$ and $q_2 = x$ as generalized coordinates because arbitrary increments of each will result in the normal force appearing in δW. The disk is always situated on the ground, and the yoke is on the guide bar, so that the only constraint not automatically satisfied is the one associated with pin B.

To capture the effect of the forces exerted at pin B, we draw a free-body diagram of each body. The forces at the ground are constraint forces associated with the no-slip condition, and $\bar{N}_3$ prevents vertical movement of the yoke; these conditions are not violated when x and θ are incremented. Also, the center of mass of each body displaces horizontally in this virtual movement. Thus the only forces that do virtual work are the normal force $\bar{N}_2$, the friction force $\bar{f}_2$ associated with contact between pin B and the yoke, and the applied force $\bar{F}$.



Free-body diagrams of the rolling disk and the yoke showing the pin force exerted between them.

The collar translates horizontally, so the virtual displacement of all points on the yoke are $\delta\bar{r}_{\text{yoke}} = \delta x\bar{i}$. We obtain the virtual displacement of pin B on the disk by the kinematical method. For no slippage at the contact, we have $v_A = \dot{\theta}R$, so that

$$\bar{v}_B = \bar{v}_A + \left(-\dot{\theta}\bar{k}\right) \times \varepsilon\left(\sin\theta\,\bar{i} + \cos\theta\,\bar{j}\right) = (R + \varepsilon\cos\theta)\,\dot{\theta}\bar{i} - \varepsilon\dot{\theta}\sin\theta\,\bar{j}. \tag{2}$$

The corresponding virtual displacement is

$$\delta\bar{r}_B = (R + \varepsilon\cos\theta)\,\delta\theta\bar{i} - \varepsilon\delta\theta\sin\theta\,\bar{j}, \tag{3}$$

from which we find the virtual work to be

$$\begin{aligned}\delta W &= \left(N_2\bar{i} + f_2\bar{j}\right)\cdot\delta\bar{r}_B + \left(-N_2\bar{i} - f_2\bar{j}\right)\cdot\delta x\bar{i} + F\bar{i}\cdot\delta x\bar{i} \\ &= \left[N_2(R + \varepsilon\cos\theta) - f_2\varepsilon\sin\theta\right]\delta\theta + (F - N_2)\,\delta x.\end{aligned}$$

The generalized forces obtained by matching δW to the standard form are

$$Q_1 = N_2(R + \varepsilon\cos\theta) - f_2\varepsilon\sin\theta, \quad Q_2 = F - N_2. \tag{4}$$

As an aside, it should be noted that if the virtual movement were kinematically admissible, then the Pfaffian form of the velocity constraint would lead to $\delta x = (R + \varepsilon\cos\theta)\,\delta\theta$. In that case the constraint force N_2 would not appear in δW, which is what would happen if we were to use θ or x as the sole generalized coordinate.

The expression for kinetic energy is particularly easy to obtain because we may use θ and/or x to describe each term. Thus

$$T = \frac{1}{2}m_1 v_A^2 + \frac{1}{2}\left(\frac{1}{2}m_1 R^2\right)\dot{\theta}^2 + \frac{1}{2}m_2\dot{x}^2 = \frac{1}{2}\left(\frac{3}{2}m_1 R^2\right)\dot{\theta}^2 + \frac{1}{2}m_2\dot{x}^2.$$

The corresponding Lagrange's equations are

$$\begin{aligned}\frac{3}{2}m_1 R^2\ddot{\theta} &= N_2(R + \varepsilon\cos\theta) - f_2\varepsilon\sin\theta, \\ m_2\ddot{x} &= F - N_2.\end{aligned} \tag{5} \triangleleft$$

These equations are the same as what one would obtain by summing moments about the point where the disk contacts the ground, and summing forces horizontally for the yoke. They are not yet ready to solve because they contain f_2. The kinetic-friction law states that the magnitude of f_2 is $\mu|N_2|$ and that it opposes the sliding motion. In the free-body diagram of the yoke, f_2 is shown acting downward, which corresponds to pin B moving downward relative to the yoke. Thus we set $f_2 = \mu|N_2|\text{sgn}\left(-\bar{v}_B\cdot\bar{j}\right)$. Substitution of the expression for $\bar{v}_B$ in terms of the generalized coordinates converts this to $f_2 = \mu|N_2|\text{sgn}\left(\dot{\theta}\sin\theta\right)$. The full set of differential equations of motion to solve are the Lagrange equations, Eqs. (5), with this description of f_2, and the velocity constraint, Eq. (1). Specifically

$$\begin{aligned}\frac{3}{2}m_1 R^2\ddot{\theta} &= N_2(R + \varepsilon\cos\theta) - \mu|N_2|\,\varepsilon\text{sgn}\left(\dot{\theta}\sin\theta\right)\sin\theta, \\ m_2\ddot{x} &= F - N_2, \\ \dot{x} - (R + \varepsilon\cos\theta)\,\dot{\theta} &= 0.\end{aligned} \tag{6} \triangleleft$$

These constitute three differential equations for θ, x, and N_2. Their relatively simple form allows us to contemplate reducing the number of equations. Two different approaches are possible. We can eliminate N_2 by solving the second of Eqs. (6) to obtain

$$N_2 = F - m_2 \ddot{x}. \tag{7}$$

If we substitute Eq. (7) into the first of Eqs. (6), we can ignore the second of Eqs. (6) when we solve the equations of motion. Alternatively, we can eliminate x by differentiating the constraint equation with respect to t, which yields

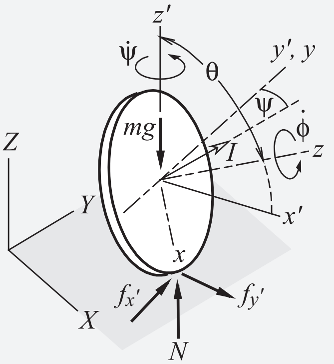
$$\ddot{x} = \ddot{\theta}\,(R + \varepsilon \cos\theta) - \varepsilon\dot{\theta}^2 \sin\theta. \tag{8}$$

Substitution of Eq. (8) into the second of Eqs. (6) allows us to ignore the third equation in the solution process. Furthermore, substitution of both Eqs. (7) and (8) into the first of Eqs. (6) reduces the problem to a single, rather complicated, differential equation for θ. In the ideal case, in which $\mu = 0$, that equation would be found to be identical to the one we would have obtained by using θ as a single unconstrained generalized coordinate.

EXAMPLE 8.4 A thin disk wobbles as it rolls without slipping along the ground. Consequently, the plane of the disk is inclined at an unsteady angle. Derive the equations of motion for the system. Then specialize the result to the steady precession case, in which the disk is tilted at a constant angle and the disk's center follows a circular path. The radius of gyration of the disk about its axis of symmetry is κ.

SOLUTION This problem is a comprehensive illustration of the use of Lagrange multipliers. We will solve the resulting equations of motion in the next section. The position of any rigid body may always be described in terms of three position coordinates for any point, such as the center of mass, and three Eulerian angles. The circular shape of the disk and the absence of slipping constrain some of these variables, so we recall from Section 4.4 the kinematical analysis of a disk that wobbles as it rolls without slipping.

The first step is to draw a free-body diagram of the disk, in which several reference frames are depicted. The axes of XYZ are fixed, and $x'y'z'$ is defined to have its origin at the center of the disk with the z' axis always vertical and the y' axis always coincident with the horizontal diametral line for the disk. The angle between y' and $\bar{J}$ for the fixed reference frame is defined to be the precession angle ψ, and y' is the line of nodes for nutation angle θ measured from vertical to the axis of symmetry. The xyz reference frame is defined to execute the nutation relative to $x'y'z'$, with the z and z' axes always coincident. The $x'y'$ plane is horizontal, so the friction force has been represented in the free-body diagram in terms of components parallel to the x' and y' axes. The only other forces are the normal reaction and gravity.



Free-body diagram and Eulerian angles for a disk that rolls unsteadily.

To ascertain the kinematical restrictions imposed on the Eulerian angles and position coordinates, we relate the velocities of the constrained points on the disk: the contact point and the center. Because of the rotational symmetry of the disk, a kinematical description using xyz components will yield a general representation. The angular velocity of the disk is the vector sum of precession $\dot\psi$, nutation $\dot\theta$, and spin $\dot\phi$:

$$\bar\omega = \dot\psi \bar k' + \dot\theta \bar j' + \dot\phi \bar k = -\dot\psi \sin\theta \bar i + \dot\theta \bar j + \left(\dot\psi \cos\theta + \dot\phi\right) \bar k. \tag{1}$$

It is stated that there is no slippage at the ground, so the velocity of the center of the disk is

$$\bar v_C = \bar\omega \times \left(-R\bar i\right) = -R\left(\dot\psi \cos\theta + \dot\phi\right) \bar j + R\dot\theta \bar k. \tag{2}$$

The velocity of the center may also be described in terms of the coordinates relative to the fixed reference frame:

$$\bar v_C = \dot X \bar I + \dot Y \bar J + \dot Z \bar k. \tag{3}$$

Matching Eqs. (2) and (3) will yield relations between the position coordinates and the Eulerian angles. This operation requires that both descriptions be converted to a common set of components, which we can do by using rotation transformations or by projecting the unit vectors as follows:

$$\bar i = \cos\theta \left(\cos\psi \bar I + \sin\psi \bar J\right) - \sin\theta \bar k,$$
$$\bar j = -\sin\psi \bar I + \cos\psi \bar J,$$
$$\bar k = \sin\theta \left(\cos\psi \bar I + \sin\psi\right) \bar J + \cos\theta \bar k.$$

We use these expressions to match the components of Eqs. (2) and (3), which yields

$$\begin{aligned} \dot X &= R\left(\dot\psi \cos\theta + \dot\phi\right) \sin\psi + R\dot\theta \cos\psi \sin\theta, \\ \dot Y &= -R\left(\dot\psi \cos\theta + \dot\phi\right) \cos\psi + R\dot\theta \sin\psi \sin\theta, \\ \dot Z &= R\dot\theta \cos\theta. \end{aligned} \tag{4}$$

These relations are three velocity constraints that the six position variables must satisfy, so the disk has only three degrees of freedom. The first two equations are nonholonomic. However, the one governing Z may be integrated. Multiplying each rate variable in the last equation by dt makes both sides perfect differentials. When $\theta = 0$, the disk lies in the XY plane, so integrating the third velocity equation and setting $Z = 0$ at $\theta = 0$ leads to

$$Z = R\sin\theta. \tag{5}$$

This configuration constraint enables us to eliminate Z from the formulation. Hence we employ five generalized coordinates in the sequence: X, Y, ψ, θ, ϕ. These are constrained generalized coordinates because they must satisfy the first two of Eqs. (4), which have the standard form of linear velocity constraints:

$$a_{i1}\dot{X} + a_{i2}\dot{Y} + a_{i3}\dot{\psi} + a_{i4}\dot{\theta} + a_{i5}\dot{\phi} = 0, \quad i = 1, 2, \tag{6} \triangleleft$$

where

$$\begin{gathered} a_{11} = 1, \quad a_{12} = 0, \quad a_{13} = -R\sin\psi\cos\theta, \quad a_{14} = -R\cos\psi\sin\theta, \quad a_{15} = -R\sin\psi, \\ a_{21} = 0, \quad a_{22} = 1, \quad a_{23} = R\cos\psi\cos\theta, \quad a_{24} = -R\sin\psi\sin\theta, \quad a_{25} = R\cos\psi. \end{gathered} \tag{7} \triangleleft$$

We now proceed to formulate the mechanical energies. We use the fact that the disk is thin to set $I_{xx} = I_{yy} = I_{zz}/2$, with $I_{zz} = m\kappa^2$. The velocity of the center in terms of the generalized coordinates is described by Eq. (3) with $\dot{Z} = R\dot{\theta}\cos\theta$, and Eq. (1) gives the corresponding angular velocity, so that

$$\begin{aligned} T &= \frac{1}{2}mv_C^2 + \frac{1}{2}\bar{\omega}\cdot\bar{H}_C \\ &= \frac{1}{2}m\left[\dot{X}^2 + \dot{Y}^2 + R^2\dot{\theta}^2(\cos\theta)^2\right] + \frac{1}{2}m\kappa^2\left[\frac{1}{2}\left(\dot{\psi}\sin\theta\right)^2 + \frac{1}{2}\dot{\theta}^2 + \left(\dot{\psi}\cos\theta + \dot{\phi}\right)^2\right]. \end{aligned} \tag{8}$$

The potential energy of gravity is

$$V = mgZ = mgR\sin\theta. \tag{9}$$

We are not specifically interested in the reactions at the ground, which enforce the constraint against slippage. Therefore we employ the Lagrange multiplier formulation. There are no other nonconservative forces to describe, so we set all $Q_j = 0$. The constrained Lagrange's equations, Eqs. (8.1.4), are formed from the kinetic energy in Eq. (8), the potential energy in Eq. (9), and the Jacobian constraint coefficients in Eqs. (7). There are two constraint equations, each of which has an associated Lagrange multiplier. The result is a set of five differential equations:

$$m\ddot{X} = \lambda_1, \quad m\ddot{Y} = \lambda_2, \tag{10, 11}$$

$$\begin{gathered} m\kappa^2 \left\{ \frac{1}{2}\ddot{\psi}\left[1 + (\cos\theta)^2\right] - \dot{\psi}\dot{\theta}\sin\theta\cos\theta + \ddot{\phi}\cos\theta - \dot{\theta}\dot{\phi}\sin\theta \right\} \\ = -\lambda_1 R\sin\psi\cos\theta + \lambda_2 R\cos\psi\cos\theta, \end{gathered} \tag{12 }$$

$$\begin{gathered} m\left\{\ddot{\theta}\left[\frac{1}{2}\kappa^2 + R^2(\cos\theta)^2\right] + \left(\frac{1}{2}\kappa^2\dot{\psi}^2 - R^2\dot{\theta}^2\right)\sin\theta\cos\theta \right. \\ \left. + \kappa^2\dot{\psi}\dot{\phi}\sin\theta + gR\cos\theta\right\} = -\lambda_1 R\cos\psi\sin\theta - \lambda_2 R\sin\psi\sin\theta, \end{gathered} \tag{13 }$$

$$m\kappa^2\left(\ddot{\psi}\cos\theta + \ddot{\phi} - \dot{\psi}\dot{\theta}\sin\theta\right) = -\lambda_1 R\sin\psi + \lambda_2 R\cos\psi. \tag{14} \triangleleft$$

There are seven unknowns: the five generalized coordinates and the two Lagrange multipliers. These variables must satisfy Eq. (6) for $i = 1$ and $i = 2$, and Eqs. (10)–(14).

When the disk precesses steadily, such that θ is constant and the center follows a circular path, the precession rate will be the rotation rate of the radial line from the center of the path to the center of the disk. Also, $\ddot{\psi} = 0$ for steady precession. Let ρ denote the radius of the path, so that $v_c = \rho\left|\dot{\psi}\right|$. Setting $\dot{\theta} = 0$ in Eq. (2) shows that $\bar{v}_c = \rho\dot{\psi}\,\bar{j} = -R\left(\dot{\psi}\cos\theta + \dot{\phi}\right)\bar{j}$, from which it follows that

$$\dot{\phi} = -\left(\frac{\rho}{R} + \cos\theta\right)\dot{\psi}. \tag{15}$$

Substitution of this expression into Eqs. (4) leads to

$$\dot{X} = -\rho\dot{\psi}\sin\psi, \quad \dot{Y} = \rho\dot{\psi}\cos\psi.$$

These relations may be integrated directly to find

$$X = \rho\cos\psi + X_0, \quad Y = \rho\sin\psi + Y_0, \tag{16}$$

where X_0 and Y_0 are constants of integration. The fact that these coordinates satisfy $(X - X_0)^2 + (y - Y_0)^2 = \rho^2$ verifies that the center follows a circular path, and leads to recognition of X_0 and Y_0 as the coordinates of the center of the path.

With the constraint equations now satisfied, we use the Lagrange equations to relate ρ and $\dot{\psi}$. Substitution of Eqs. (16) into Eqs. (10) and (11) gives expressions for the Lagrange multipliers:

$$\lambda_1 = m\left(-\rho\dot{\psi}^2\cos\psi\right), \quad \lambda_2 = m\left(-\rho\dot{\psi}^2\sin\psi\right). \tag{10′, 11′}$$

We substitute these expressions, together with $\ddot{\psi} = 0$, $\dot{\theta} = 0$, and Eq. (15) for $\dot{\phi}$, into Eqs. (12)–(14). The only one that is not identically satisfied is Eq. (13), which becomes

$$\left(\frac{1}{2}\kappa^2\dot{\psi}^2\right)\sin\theta\cos\theta - \kappa^2\dot{\psi}^2\left(\frac{\rho}{R} + \cos\theta\right)\sin\theta + gR\cos\theta = \rho R\dot{\psi}^2\sin\theta.$$

Solving this for $\dot{\psi}$ yields an expression for the steady precession rate corresponding to a constant nutation angle:

$$\dot{\psi} = \left[\frac{2gR^2 \cot\theta}{2\rho\,(R^2 + \kappa^2) + \kappa^2 R\cos\theta} \right]^{1/2}. \qquad \triangleleft$$

If the disk is homogeneous, then $\kappa = R/\sqrt{2}$, which leads to

$$\dot{\psi} = \left[\frac{4g\cot\theta}{6\rho + R\cos\theta} \right]^{1/2}.$$

This solution is in complete agreement with the result derived in Example 6.4 with the Newton–Euler formulation. The earlier methods provide greater physical insight, but it would have been more difficult to use them to derive the equations of motion for the unsteady case treated here.

8.2 COMPUTATIONAL METHODS

Our focus here is on techniques for finding solutions of the differential equations of motion, which are derived from kinetics principles, that also satisfy the constraint equations, which represent kinematical restrictions intrinsic to the system. Although the discussion is centered on the application of Lagrange's equations, the development is readily altered to treat situations in which the equations of motion are derived from a Newton–Euler formulation or from principles discussed in the next chapter. We have seen that numerical methods are often required to solve the equations of motion associated with unconstrained generalized coordinates. The need to satisfy constraint equations in addition to the equations of motion makes it even likely that we will need to invoke such methods when the generalized coordinates are kinematically constrained.

We found in Section 7.6 that it is convenient to write the equations of motion in matrix form. The presence of constraint equations does not alter the fact that the kinetic energy is a quadratic sum in the generalized velocities, so that the terms appearing in Lagrange's equations are as described by Eqs. (7.6.9) and (7.6.10). In fact, the only alteration relative to the standard form in Eq. (7.6.12) is the addition of terms to account for the contribution of constraint forces to the Q_j. Equations (8.1.4) describe these terms as a linear sum of Lagrange multipliers. The matrix representation of that sum is $[a]^{\mathrm{T}}\{\lambda\}$. Thus the general form of the equations of motion will be

$$\boxed{[M(q_j,t)]\{\ddot{q}\} = [a(q_j,t)]^{\mathrm{T}}\{\lambda\} + \{F(q_j,\dot{q}_i,t)\},} \tag{8.2.1}$$

where $\{F(q_i,\dot{q}_i,t)\}$ consists of the generalized forces $\{Q\}$ and any terms from the left side of Lagrange's equations that do not contain generalized accelerations. The corresponding matrix form of the velocity constraint equations, Eq. (8.1.1), is

$$\boxed{[a(q_i,t)]\{\dot{q}\} + \{b(q_i,t)\} = \{0\}.} \tag{8.2.2}$$

One should note that the preceding equations are the forms we shall use to discuss the implementation of numerical algorithms. However, when Coulomb friction forces are present, it is preferable to let the constraint forces appear explicitly, rather than using Lagrange multipliers to represent them. The modifications required for such systems are addressed in Subsection 8.2.4.

Usage of a computer routine to solve the differential equations of motion requires input of the current generalized acceleration values, denoted as $\{\ddot{q}^*\}$. When the generalized coordinates are unconstrained, these values may be obtained directly by solving Eq. (7.6.19). When the generalized coordinates form a constrained set, the occurrence of Lagrange multipliers and constraint equations means that the $\{\ddot{q}^*\}$ values cannot be found solely from the equations of motion. Unfortunately, the manner in which $\{\lambda\}$ occurs prevents a direct solution.

To recognize the difficulty, recall that standard numerical algorithms handle a set of first-order differential equations whose form is

$$\frac{d}{dt}\{x\} = \left\{G(x_i, t)\right\}. \tag{8.2.3}$$

The generalized velocities are treated as though they are independent coordinates, and the Lagrange multipliers also are unknown, so we form the state vector from these variables. We stack the derivative identity, the Lagrange equations, and the velocity constraint equations to form

$$\begin{bmatrix} [U]_{N\times N} & [0]_{N\times N} & [0]_{N\times J} \\ [0]_{N\times N} & [M]_{N\times N} & -\left([a]_{J\times N}\right)^{\mathrm{T}} \\ [a]_{J\times N} & [0]_{J\times N} & [0]_{J\times J} \end{bmatrix} \begin{Bmatrix} \{\dot{q}\} \\ \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{\dot{q}\} \\ \{F\} \\ -\{b\} \end{Bmatrix}. \tag{8.2.4}$$

We need to solve these equations for the generalized accelerations, but the first and third partition rows of the coefficient matrix merely differ by the factor $[a]$. Consequently, this matrix is not full rank, so the equations cannot be solved for the values of $\{\ddot{q}\}$.

Because Eq. (8.2.4) does not contain derivatives of the λ variables, it is said to constitute a set of *differential-algebraic equations* (DAEs). Several solution strategies have been developed to solve them. One approach is to use a DAE solver, which is implemented programmatically in the same manner as one uses a routine such as Runge–Kutta for solving unconstrained equations of motion. Brenan *et al.* (1989) provide a comprehensive discussion of this topic. These methods do not seem to have been widely implemented by dynamics researchers, because they are more prone to numerical difficulties and often are less computationally efficient than the procedures we develop here. However, a popular dynamic simulation software package known as ADAMS relies on a DAE solver. Rather than attacking the problem with a general solution process, the procedures we consider exploit the special nature of the equations for a dynamical system.

8.2.1 Algorithms

The fundamental difficulty is the fact that the derivative identity and the constraint equations govern the generalized velocities rather than the accelerations. To circumvent this aspect, the latter is differentiated with respect to time, thereby converting them to acceleration constraint equations. This leads to

$$[a]\{\ddot{q}\} + \left[\frac{da}{dt}\right]\{\dot{q}\} = -\left\{\frac{db}{dt}\right\}. \tag{8.2.5}$$

The use of the total derivative notation to denote the time derivative of $[a]$ serves as a reminder that the differentiation needs to recognize that the Jacobian constraint matrix may depend on the generalized coordinates. Consequently, the derivatives are evaluated according to

$$\frac{da_{in}}{dt} = \sum_{j=1}^{N} \frac{\partial a_{in}}{\partial q_j}\dot{q}_j + \frac{\partial a_{in}}{\partial t}, \quad \frac{db_i}{dt} = \sum_{i=1}^{N} \frac{\partial b_i}{\partial q_j}\dot{q}_j + \frac{\partial b_i}{\partial t}, \quad i = 1, .., J, \quad n = 1, .., N. \tag{8.2.6}$$

The most direct solution strategy for solving the differential equations is to define *integrated multipliers.* Let μ_i denote the definite integral of λ_i:

$$\mu_i(t) = \int_0^t \lambda_i(\tau)\, d\tau \iff \lambda_i = \dot{\mu}_i. \tag{8.2.7}$$

When the Lagrange multipliers are replaced with the respective $\dot{\mu}_i$ parameter, the assembly of the derivative identity, the equations of motion, and the acceleration constraint equations yields

$$\begin{bmatrix} [U] & [0] & [0] \\ [0] & [M] & -[a]^{\mathrm{T}} \\ [0] & -[a] & [0] \end{bmatrix} \frac{d}{dt} \begin{Bmatrix} \{q\} \\ \{\dot{q}\} \\ \{\mu\}. \end{Bmatrix} = \begin{Bmatrix} \{\dot{q}\} \\ \{F\} \\ \left[\frac{da}{dt}\right]\{\dot{q}\} + \left\{\frac{db}{dt}\right\} \end{Bmatrix}. \tag{8.2.8}$$

(The negative of the acceleration constraint was used to form the third partition row in order that the coefficient matrix be symmetric.) This set of equations has the form described by Eq. (8.2.3) with

$$\boxed{\begin{gathered} \{z\} = \left[\{q\}^{\mathrm{T}} \quad \{\dot{q}\}^{\mathrm{T}} \quad \{\dot{\mu}\}^{\mathrm{T}}\right]^{\mathrm{T}}, \\ \left\{G(z_i, t)\right\} = \begin{bmatrix} [U] & [0] & [0] \\ [0] & [M] & -[a]^{\mathrm{T}} \\ [0] & -[a] & [0] \end{bmatrix}^{-1} \begin{Bmatrix} \{\dot{q}\} \\ \{F\} \\ \left[\frac{da}{dt}\right]\{\dot{q}\} + \left\{\frac{db}{dt}\right\} \end{Bmatrix}. \end{gathered}} \tag{8.2.9}$$

Note that the matrix inverse appears in the preceding equation for notational purposes. The value of $\left\{G(z_i, t)\right\}$ at the desired instant may be obtained more efficiently by a standard technique for linear algebraic equations, such as Gauss elimination.

The alternative formulation known as the *augmented method* proceeds similarly to the integrated multiplier procedure, but it is more efficient from a computational standpoint. If we omit the derivative identity when we assemble the equations, we find that

$$\begin{bmatrix} [M] & -[a]^{\mathrm{T}} \\ -[a] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ \left[\dfrac{da}{dt}\right]\{\dot{q}\} + \left\{\dfrac{db}{dt}\right\} \end{Bmatrix}. \tag{8.2.10}$$

This constitutes a set of $N+J$ simultaneous algebraic equations for the values $\{\ddot{q}\}$ and $\{\lambda\}$ corresponding to any instant that the values of $\{q\}$ and $\{\dot{q}\}$ are known. Thus the state-space vector here is defined to contain only the generalized coordinates and velocities. In combination with the current value of $\{\dot{q}\}$, the values of $\{\ddot{q}\}$ obtained from the preceding equation provide the input for the differential equation solver. Thus the augmented algorithm entails implementing Eq. (8.2.3), with

$$\boxed{\begin{gathered} \{z\} = \begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix}, \quad \{G(z_i, t)\} = \begin{Bmatrix} \{\dot{q}\} \\ \{\ddot{q}\} \end{Bmatrix}, \\ \begin{Bmatrix} \{\ddot{q}\} \\ \{\lambda\} \end{Bmatrix} = \begin{bmatrix} [M] & -[a]^{\mathrm{T}} \\ -[a] & [0] \end{bmatrix}^{-1} \begin{Bmatrix} \{F\} \\ \left[\dfrac{da}{dt}\right]\{\dot{q}\} + \left\{\dfrac{db}{dt}\right\} \end{Bmatrix}. \end{gathered}} \tag{8.2.11}$$

Once again, the $\{\ddot{q}\}$ values should be obtained by a standard technique for linear algebraic equations, rather than with a matrix inverse.

Both the integrated multiplier and augmented methods evaluate the Lagrange multipliers at each instant. A different approach to solving the coupled set of dynamical equations of motion and constraint is sometimes implemented manually in an ad hoc manner. We refer to it as the *elimination method,* because it entails eliminating the multipliers by direct substitution. We may see the essence of the approach by considering Eqs. (10) and (11) in Example 8.4. Those equations may be solved for the two Lagrange multipliers in terms of the q_j, $\dot{q}_j$, and $\ddot{q}_j$. Substitution of those expressions into the remaining equations of motion, (12)–(14), would lead to three differential equations of motion. When these equations are combined with the acceleration form of the constraint equations, the result is a set of five differential equations governing the five generalized coordinates.

We may formalize this approach by decomposing the equations of motion into two partitions: an upper set of J equations and a lower set of $N-J$ equations,

$$\begin{bmatrix} [M_1]_{J\times N} \\ [M_2]_{(N-J)\times N} \end{bmatrix} \{\ddot{q}\} = \begin{Bmatrix} \{F_1\}_{J\times 1} \\ \{F_2\}_{(N-J)\times 1} \end{Bmatrix} + \begin{bmatrix} [a_1]^{\mathrm{T}} \\ [a_2]^{\mathrm{T}} \end{bmatrix} \{\lambda\}_{J\times 1}, \tag{8.2.12}$$

where the Jacobian constraint matrix is partitioned as

$$[a]_{J\times N} = \Big[[a_1]_{J\times J} \quad [a_2]_{J\times(N-J)} \Big]. \tag{8.2.13}$$

It is necessary that the equations used to form the first group be such that $[a_1]$ is full rank, that is, $\left|[a_1]\right| \neq 0$. Then the upper partition of Eq. (8.2.12) may be solved for $\{\lambda\}$,

$$\{\lambda\} = [a_1]^{-\mathrm{T}}[M_1]\{\ddot{q}\} - [a_1]^{-\mathrm{T}}\{F_1\}, \tag{8.2.14}$$

where $[a_1]^{-\mathrm{T}}$ is the transpose of the inverse of $[a_1]$. Substitution of this expression for $\{\lambda\}$ into the second partition row of Eq. (8.2.12) leads to

$$\Big[[M_2] - [a_2]^{\mathrm{T}}[a_1]^{-\mathrm{T}}[M_1]\Big]\{\ddot{q}\} = \{F_2\} - [a_2]^{\mathrm{T}}[a_1]^{-\mathrm{T}}\{F_1\}. \tag{8.2.15}$$

All quantities appearing in this expression other than $\{\ddot{q}\}$ are known at any instant in terms of the current values of $\{q\}$, $\{\dot{q}\}$, and t. We then obtain a solvable set of equations by stacking these equations above the acceleration constraint equations. Thus the elimination algorithm is summarized by

$$\boxed{\begin{gathered}\{z\} = \begin{Bmatrix}\{q\}\\ \{\dot{q}\}\end{Bmatrix}, \quad \{G(z_i,t)\} = \begin{Bmatrix}\{\dot{q}\}\\ \{\ddot{q}\}\end{Bmatrix},\\ \{\ddot{q}\} = \begin{bmatrix}[M_2] - [a_2]^{\mathrm{T}}[a_1]^{-\mathrm{T}}[M_1]\\ -[a]\end{bmatrix}^{-1}\begin{Bmatrix}\{F_2\} - [a_2]^{\mathrm{T}}[a_1]^{-\mathrm{T}}\{F_1\}\\ \left[\dfrac{da}{dt}\right]\{\dot{q}\} + \left\{\dfrac{db}{dt}\right\}\end{Bmatrix}.\end{gathered}} \tag{8.2.16}$$

The preceding expression for $\{\ddot{q}\}$ contains an important subtlety. The manner in which $[a_1]^{-\mathrm{T}}$ appears requires actual evaluation of an inverse. In contrast, it is solely for notational purposes that an inverse is used to indicate solution for $\{\ddot{q}\}$ values, whereas the actual implementation should use a more efficient solution technique such as Gauss elimination.

One can perform Eq. (8.2.14), as well as subsequent operations, algebraically if the number of equations is not too large. Alternatively, one can carry out the operations numerically at each time instant using the current values of $\{q\}$ and $\{\dot{q}\}$. In any event, care must be taken to avoid grouping the equations of motion in a manner that causes $[a_1]$ to be uninvertible or ill-conditioned. If such a situation occurs subsequently to initiation of the integration process, it is necessary to interrupt the solver in order to resequence the columns of $[a]$ and the elements of $\{q\}$ in a consistent manner, such that the submatrix $[a_1]$ of the rearranged $[a]$ is well behaved.

Such resequencing can be carried out manually, but an automated procedure can be quite useful when there are many equations. The process of shifting columns of $[a]$ may be represented by an $N \times N$ sorting matrix $[P]$ that satisfies

$$[\tilde{a}] = [a][P], \tag{8.2.17}$$

where a rearranged matrix is denoted with a tilde, $\sim$. The corresponding rearrangement of the elements of $\{q\}$ is described by

$$\{q\} = [P]\{\tilde{q}\}. \tag{8.2.18}$$

The elements of $[P]$ are a set of ones and zeros, and $[P]$ would obviously be the identity matrix in the event that resequencing was not necessary. As an example, suppose $N = 3$,

$J = 2$, and we wish for $[\tilde{a}_1]$, which is the left $J \times J$ partition of $[\tilde{a}]$, to consist of the first and third columns of $[a]$. This requires that

$$\begin{bmatrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}. \tag{8.2.19}$$

The nonzero elements of $[P]$ satisfying this condition are $P_{11} = P_{32} = P_{23} = 1$. In general, if we wish for column j of $[\tilde{a}]$ to match column k of $[a]$, then $P_{kj} = 1$, whereas the unspecified elements of $[P]$ will be zero.

Because $[P]$ is constant, we have

$$\left[\frac{d\tilde{a}}{dt}\right] = \left[\frac{da}{dt}\right][P], \tag{8.2.20}$$

so substituting Eq. (8.2.18) into the acceleration constraint equations leads to

$$[\tilde{a}]\left\{\ddot{\tilde{q}}\right\} = -\left[\frac{d\tilde{a}}{dt}\right]\left\{\dot{\tilde{q}}\right\} - \left\{\frac{db}{dt}\right\}, \tag{8.2.21}$$

which has the same form as that of the original constraint equations. Because the generalized coordinates are now represented by $\{\tilde{q}\}$, we also substitute Eq. (8.2.18) into the Lagrange equations. To maintain the symmetry of the coefficient matrix, we premultiply the result of that substitution by $[P]^{\mathrm{T}}$, which leads to

$$\left[\tilde{M}\right]\left\{\ddot{\tilde{q}}\right\} = [\tilde{a}]^{\mathrm{T}}\{\lambda\} + \left\{\tilde{F}\right\}, \tag{8.2.22}$$

where

$$\left[\tilde{M}\right] = [P]^{\mathrm{T}}[M][P], \quad \left\{\tilde{F}\right\} = [P]^{\mathrm{T}}\{F\}. \tag{8.2.23}$$

Equations (8.2.21) and (8.2.22) have the same form as the respective original equations, so the solution process can resume after resorting by use of the rearranged matrices in place of the respective original versions. The solution vector returned by the differential equation solver may be restored to the original $\{q\}$ by application of Eq. (8.2.18).

In the elimination method, one can evaluate the Lagrange multipliers at any instant by substituting the $\{\ddot{q}\}$ values obtained from Eqs. (8.2.16) into Eq. (8.2.14). In contrast, the *orthogonal complement method* completely removes the Lagrange multipliers from consideration. The orthogonal complement of the Jacobian constraint matrix is an $(N - J) \times N$ matrix $[C]$ that satisfies the condition that

$$[C][a]^{\mathrm{T}} = [0]. \tag{8.2.24}$$

Note that $[C]$ is not unique, as one can recognize by considering the foregoing to be a set of simultaneous equations obtained on an element-by-element basis. There are $(N - J)\,N$ elements of $[C]$, but the product has only $(N - J)\,J$ elements.

Let us for the moment assume that we can find a suitable $[C]$. When we multiply the Lagrange equations by $[C]$, we find that

$$[C][M]\{\ddot{q}\} = [C]\{F\} + [C][a]^{\mathrm{T}}\{\lambda\} = [C]\{F\}. \tag{8.2.25}$$

We see that the consequence of this operation is to remove the Lagrange multipliers from the equations to be integrated. Because $[C]$ has $N - J$ rows, we may compute values of the N generalized accelerations at any instant by combining Eq. (8.2.25) with the acceleration constraint equations. Thus the orthogonal complement algorithm is capsulized by

$$\boxed{\begin{aligned} &\{z\} = \begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix}, \quad \{G(z_i, t)\} = \begin{Bmatrix} \{\dot{q}\} \\ \{\ddot{q}\} \end{Bmatrix}, \\ &\{\ddot{q}\} = \begin{bmatrix} [C]\,[M] \\ -[a] \end{bmatrix}^{-1} \begin{Bmatrix} [C]\,\{F\} \\ \left[\dfrac{da}{dt}\right]\{\dot{q}\} + \left\{\dfrac{db}{dt}\right\} \end{Bmatrix}. \end{aligned}} \tag{8.2.26}$$

These values and the associated current values of the generalized velocities constitute the inputs required for implementing the differential equation solver associated with Eq. (8.2.3).

Implementation of the orthogonal complement method requires that the orthogonal complement to the current value of $[a]$ be determined prior to each call to the differential equation solver. Several techniques for determining the orthogonal complement are discussed in the text by Amirouche (1992) under the category of coordinate reductions. One method not discussed there is suggested by the recognition that the condition for the orthogonal complement can be rewritten as

$$[a][C]^{\mathrm{T}} = [0]. \tag{8.2.27}$$

Thus we may consider the columns of $[C]^{\mathrm{T}}$, that is, the rows of $[C]$, to be nonzero solutions of

$$[a]\,\{y\} = \{0\}\,. \tag{8.2.28}$$

A numerical procedure for determining all possible solutions for $\{y\}$ is the *singular value decomposition* (SVD), which is described by Press *et al.* (1992). The method is briefly summarized here.

If $[A]$ is a known $I \times J$ array, with $I \geq J$, then the SVD of $[A]$ is

$$[A]_{I\times J} = [L]_{I\times J}\,[w]_{J\times J}\,[R]^{\mathrm{T}}_{J\times J}\,, \tag{8.2.29}$$

where $[w]$ is a diagonal array that holds the singular values of $[A]$, and $[L]$ and $[R]$ consist of columns that are orthonormal:

$$[L]^{\mathrm{T}}[L] = [U]_{J\times J}, \quad [R]^{\mathrm{T}}[R] = [U]_{J\times J}. \tag{8.2.30}$$

It is important for our development that the singular values are real and nonnegative, $w_j \geq 0$. Press *et al.* give a reliable subroutine for carrying out the process, and comparable routines are contained in most mathematical software packages. Only the singular values $[w]$ are unique, so $[L]$ and $[R]$ obtained from different pieces of software might not be the same.

Now consider the situation in which $[A]$ is a square $m \times m$ matrix, and we seek to solve

$$[A]\{y\} = \{0\}. \tag{8.2.31}$$

Clearly, the only solution is $\{y\} = \{0\}$ unless $\left|[A]\right| = 0$. Any $\{y\} \neq \{0\}$ satisfying this equation is said to be in the null space of $[A]$. The number k of independent solutions lying in the null space of $[A]$ is the nullity of $[A]$, and the rank of $[A]$ is $m - k$. It can be shown that the nullity of a matrix is the number of zero singular values obtained in its SVD. (In general, the ratio of the largest to the smallest singular value is the condition number of a matrix.) For our present purpose, the significance of zero singular values is a theorem that states that, if $w_j = 0$, then column $\# j$ of $[R]$ is an independent vector in the null space of $[A]$. The singular values may be arranged in any sequence without violating Eq. (8.2.29), provided that the columns of $[L]$ and $[R]$ are rearranged in the same manner. It is convenient to use this property to arrange the singular values such that the w_j values occur in ascending order, which leads to $w_1 = \cdots = w_k = 0$. Let $\{R_j\}$ denote the jth column of $[R]$. It follows that

$$[A]\{R_j\} = \{0\}, \quad j = 1, 2, \ldots, k. \tag{8.2.32}$$

These may be assembled into a single matrix relation:

$$[A]\left[\{R_1\} \quad \{R_2\} \quad \ldots \quad \{R_k\}\right] = [0]. \tag{8.2.33}$$

To employ SVD to determine the orthogonal complement $[C]$ satisfying Eq. (8.2.27), we form the square array $[A]$ by stacking the $J \times N$ matrix $[a]$ above an array of $(N - J) \times N$ zeros:

$$[A]_{N\times N} = \begin{bmatrix} [a]_{J\times N} \\ [0]_{(N-J)\times N} \end{bmatrix}. \tag{8.2.34}$$

Aside from degenerate conditions that might occur at some instant, the J constraint conditions are independent, so the rank of $[A]$ is J. It follows that the nullity of $[A]$ formed as previously defined is $N - J$. Hence, when we perform a SVD of the previously defined $[A]$, and resequence the singular values in the prescribed manner, then the first $N - J$ columns of the resulting $[R]$ are in the null space of $[A]$. In view of the partitioned manner in which $[A]$ is defined, it follows that

$$[a]\left[\{R_1\} \quad \{R_2\} \quad \ldots \quad \{R_{(N-J)}\}\right] = [0]. \tag{8.2.35}$$

A comparison of this relation with Eq. (8.2.27) indicates that we should use these $\{R_j\}$ to form the rows of $[C]$:

$$[C] = \begin{bmatrix} \{R_1\}^{\mathrm{T}} \\ \{R_2\}^{\mathrm{T}} \\ \vdots \\ \{R_{(N-J)}\}^{\mathrm{T}} \end{bmatrix}. \tag{8.2.36}$$

Using SVD may not be the most efficient method for finding the orthogonal complement, but it is reliable. Example 8.5 illustrates the SVD method for a typical step in the solution of a set of constrained equations of motion by the orthogonal complement method.

Each of the algorithms discussed thus far require that the constraint equations be satisfied explicitly as a supplement to the Lagrange equations. A different approach that has received some advocacy uses the constraint equations to algebraically eliminate the dependent constrained generalized velocities and accelerations. Amirouche (1992) refers to this as the *embedding method*, because the constraint equations become intertwined with the Lagrange equations of motion. The notion is to select $N-J$ of the generalized coordinates to be unconstrained, whereas the other J generalized coordinates are required to satisfy the constraint equations. The method begins by assigning a set of $N-J$ generalized coordinates to $\{q_u\}$, which consists of the generalized coordinates that are considered to be unconstrained. The remaining J generalized coordinates form the constrained variables, which are placed in $\{q_c\}$. This partitioning can be implemented by use of a sorting matrix $[P]$ in the manner of Eq. (8.2.18), such that

$$\{q\} = [P]\begin{Bmatrix} \{q_u\}_{(N-J)\times 1} \\ \{q_c\}_{J\times 1} \end{Bmatrix}. \tag{8.2.37}$$

Of course, if $\{q_c\}$ is formed from the last J elements of $\{q\}$, then $[P]$ would be an $N\times N$ identity matrix. As indicated by Eq. (8.2.17), $[P]$ also leads to partitioning of $[a]$ according to

$$\begin{bmatrix} [a_u]_{J\times(N-J)} & [a_c]_{J\times J} \end{bmatrix} = [a]\,[P], \tag{8.2.38}$$

from which it follows that

$$[a]\,\{q\} \equiv [[a_u] \quad [a_c]]\begin{Bmatrix} \{q_u\} \\ \{q_c\} \end{Bmatrix}. \tag{8.2.39}$$

The task now is to express the constrained variables in terms of the unconstrained ones. Toward that end we observe that $[P]$ is constant, so differentiation of Eq. (8.2.37) gives

$$\{\dot{q}\} = [P]\begin{Bmatrix} \{\dot{q}_u\}_{(N-J)\times 1} \\ \{\dot{q}_c\}_{J\times 1} \end{Bmatrix}. \tag{8.2.40}$$

In view of Eq. (8.2.38), the result of substitution of this expression into the velocity constraint equations is

$$[[a_u] \quad [a_c]]\begin{Bmatrix} \{\dot{q}_u\} \\ \{\dot{q}_c\} \end{Bmatrix} \equiv [a_u]\,\{\dot{q}_u\} + [a_c]\,\{\dot{q}_c\} = -\,\{b\}\,. \tag{8.2.41}$$

The only condition imposed on the selection of which generalized coordinates form $\{q_c\}$ is that the resulting $[a_c]$ be full rank and well conditioned. Then we may solve the preceding equation for the constrained generalized velocities:

$$\{\dot{q}_c\} = -[a_c]^{-1}[a_u]\{\dot{q}_u\} - [a_c]^{-1}\{b\}. \tag{8.2.42}$$

This expression enables us to evaluate the full set of generalized velocities from the unconstrained values according to

$$\{\dot{q}\} = [P]\begin{Bmatrix}\{\dot{q}_u\}\\ \{\dot{q}_c\}\end{Bmatrix} = [D]\{\dot{q}_u\} + [B]\{b\}, \tag{8.2.43}$$

where

$$[D] = [P]\begin{bmatrix}[U]_{(N-J)\times(N-J)}\\ -[a_c]^{-1}[a_u]\end{bmatrix}, \quad [B] = [P]\begin{bmatrix}[0]_{(N-J)\times J}\\ -[a_c]^{-1}\end{bmatrix}. \tag{8.2.44}$$

A similar expression giving the full set of generalized accelerations is obtained from the acceleration constraint equations, which may be written as

$$[a_u]\{\ddot{q}_u\} + [a_c]\{\ddot{q}_c\} = -\left[\frac{da}{dt}\right]\{\dot{q}\} - \left\{\frac{db}{dt}\right\}, \tag{8.2.45}$$

so that

$$\{\ddot{q}_c\} = -[a_c]^{-1}[a_u]\{\ddot{q}_u\} - [a_c]^{-1}\left\{\left[\frac{da}{dt}\right]\{\dot{q}\} + \left\{\frac{db}{dt}\right\}\right\}. \tag{8.2.46}$$

When we use this expression to reconstruct the full set of generalized accelerations, we find that

$$\begin{aligned}\{\ddot{q}\} = [P]\begin{Bmatrix}\{\ddot{q}_u\}\\ \{\ddot{q}_c\}\end{Bmatrix} &= [P]\begin{bmatrix}[U]_{(N-J)\times(N-J)}\\ -[a_c]^{-1}[a_u]\end{bmatrix}\{\ddot{q}_u\}\\ &\quad + [P]\begin{bmatrix}[0]_{(N-J)\times J}\\ -[a_c]^{-1}\end{bmatrix}\left\{\left[\frac{da}{dt}\right]\{\dot{q}\} + \left\{\frac{db}{dt}\right\}\right\}.\end{aligned} \tag{8.2.47}$$

In view of Eq. (8.2.43), this reduces to

$$\{\ddot{q}\} = [D]\{\ddot{q}_u\} + [B]\left[\frac{da}{dt}\right][D]\{\dot{q}_u\} + \{E\}, \tag{8.2.48}$$

where

$$\{E\} = [B]\left[\frac{da}{dt}\right][B]\{b\} + [B]\left\{\frac{db}{dt}\right\}. \tag{8.2.49}$$

An important identity is that

$$\begin{aligned}[a][D] \equiv [a][P]\begin{bmatrix}[U]\\ -[a_c]^{-1}[a_u]\end{bmatrix} &= [[a_u] \quad [a_c]]\begin{bmatrix}[U]\\ -[a_c]^{-1}[a_u]\end{bmatrix}\\ &\equiv [a_u] - [a_c][a_c]^{-1}[a_u] \equiv [0].\end{aligned} \tag{8.2.50}$$

The significance of this property becomes apparent when we substitute Eq. (8.2.48) into the Lagrange equation, Eq. (8.2.1), which leads to

$$[M]\left\{[D]\{\ddot{q}_u\} + [B]\left[\frac{da}{dt}\right][D]\{\dot{q}_u\} + \{E\}\right\} = \{F\} + [a]^{\mathrm{T}}\{\lambda\}. \tag{8.2.51}$$

We premultiply this equation by $[D]^{\mathrm{T}}$. According to Eq. (8.2.50), we have*

$$[D]^{\mathrm{T}}[a]^{\mathrm{T}} \equiv [[a][D]]^{\mathrm{T}} = [0]. \tag{8.2.52}$$

Thus the equations of motion reduce to

$$[D]^{\mathrm{T}}[M][D]\{\ddot{q}_u\} = [D]^{\mathrm{T}}\left\{\{F\} - [M][B]\left[\frac{da}{dt}\right][D]\{\dot{q}_u\} - [M]\{E\}\right\}. \tag{8.2.53}$$

The constraint equations are embedded in these equations, rather than enforced explicitly. Thus, as indicated by Amirouche (1992), it appears that the procedure has reduced the equations of motion to a form comparable with those for a holonomic system having $N - J$ degrees of freedom. However, this view misses the fact that $[M]$, $[a]$, and the other coefficient matrices are likely to depend on all generalized coordinates, not just on the unconstrained variables. Furthermore, $\{F\}$, $[da/dt]$, and $\{db/dt\}$ are likely to depend on all of the generalized velocities. Thus, at every time step in the integration process, it will be necessary to also evaluate the current values of $\{q_c\}$ and $\{\dot{q}_c\}$. Toward that end, we can consider Eq. (8.2.42) to be an auxiliary set of differential equations that are to be integrated simultaneously with the basic equations of motion. Implementation of the embedding algorithm requires that the solver implementing Eq. (8.2.3) be formulated with

$$\boxed{\begin{gathered}
\{z\} = \begin{Bmatrix} \{q_u\} \\ \{\dot{q}_u\} \\ \{q_c\} \end{Bmatrix}, \quad \{G(z_i, t)\} = \begin{Bmatrix} \{\dot{q}_u\} \\ \{\ddot{q}_u\} \\ \{\dot{q}_c\} \end{Bmatrix}, \\
\{q\} = [P]\begin{Bmatrix} \{q_u\}_{(N-J)\times 1} \\ \{q_c\}_{J\times 1} \end{Bmatrix}, \quad \left[[a_u]_{J\times(N-J)} \quad [a_c]_{J\times J}\right] = [a][P], \\
\{\dot{q}_c\} = -[a_c]^{-1}[a_u]\{\dot{q}_u\} - [a_c]^{-1}\{b\}, \\
\left[[D]^{\mathrm{T}}[M][D]\right]\{\ddot{q}_u\} = [D]^{\mathrm{T}}\left\{\{F\} - [M][B]\left[\frac{da}{dt}\right][D]\{\dot{q}_u\} - [M]\{E\}\right\},
\end{gathered}} \tag{8.2.54}$$

where $[D]$ and $[B]$ are defined in Eqs. (8.2.44), and Eq. (8.2.49) gives $\{E\}$.

At any time step the value of $\{z\}$ will be known. To perform the next integration step one needs to extract $\{q_u\}$ and $\{q_c\}$ from the current $\{z\}$. Knowledge of the generalized coordinate values allows one to evaluate the current values of $[a]$, $[da/dt]$, and $\{b\}$, which then allows for evaluation of $[a_c]$ and $[a_u]$. The values of $\{\dot{q}_c\}$ and $\{\ddot{q}_u\}$ may then be

* According to Eq. (8.2.50), $[D]^{\mathrm{T}}$ is the orthogonal complement of $[a]$. The primary difficulty with using this, rather than SVD, to implement the orthogonal complement method lies in the need to be certain that $[a_c]$ is invertible.

found by direct substitution. These values in conjunction with the $\{\dot{q}_u\}$ values extracted from the current $\{z\}$ are used to evaluate the current value of $\{G\}$, which is the input to the differential equation solver. In this process it is important to monitor $[a_c]$ for invertibility. If this matrix were to become ill-conditioned at some integration step, or actually become rank deficient, it would be necessary to change the definition of $\{q_c\}$. This is readily achieved because it merely entails changing the definition of $[P]$.

Greenwood (2003) offers a different procedure that also embeds the constraint equations. It avoids the need to split the generalized coordinates into constrained and unconstrained sets, so that the question of invertiblity of $[a_c]$ does not arise. The modified procedure begins by solving the N Lagrange equations for the generalized accelerations, and then substituting that solution into the acceleration constraint equations, which leads to

$$\begin{aligned} \{\ddot{q}\} &= [M]^{-1}\left\{[a]^{\mathrm{T}}\{\lambda\} + \{F\}\right\}, \\ [a][M]^{-1}\left\{[a]^{\mathrm{T}}\{\lambda\} + \{F\}\right\} &+ \left[\frac{da}{dt}\right]\{\dot{q}\} + \left\{\frac{db}{dt}\right\} = \{0\}. \end{aligned} \tag{8.2.55}$$

The expression for $\{\lambda\}$ obtained by solving the second set of equations is used to eliminate the multipliers from the first set. This leads to an alternative embedding algorithm:

$$\boxed{\begin{gathered} \{z\} = \begin{Bmatrix} \{q\} \\ \{\dot{q}\} \end{Bmatrix}, \quad \{G(z_i, t)\} = \begin{Bmatrix} \{\dot{q}\} \\ \{\ddot{q}\} \end{Bmatrix}, \\ \{\ddot{q}\} = [M]^{-1}\{F\} - [M]^{-1}[a]^{\mathrm{T}}\left[[a][M]^{-1}[a]^{\mathrm{T}}\right]^{-1}\left\{[a][M]^{-1}\{F\} + [\dot{a}]\{\dot{q}\} + \{\dot{b}\}\right\}. \end{gathered}}$$

The associated Lagrange multipliers values are not needed to invoke the differential equation solver, but their value at any instant can be obtained from

$$\{\lambda\} = -\left[[a][M]^{-1}[a]^{\mathrm{T}}\right]^{-1}\left\{[a][M]^{-1}\{F\} + [\dot{a}]\{\dot{q}\} + \{\dot{b}\}\right\}. \tag{8.2.56}$$

An overview of the methods that have been presented shows that preparing the equations to be passed to the differential equation solver at each time instant requires several operations. Thus the issue of which is more efficient depends on more than the number of equations to be integrated in each method. Furthermore, in addition to the number of operations associated with each approach, it might happen that the equations for one of these methods are better behaved, in the sense that the method allows for a larger time step without loss of numerical stability. Which of these methods will be best in a specific situation cannot be stated *a priori*. One factor that can be used as a guideline is that some of the techniques are preferable if the values of the Lagrange multipliers/constraint forces are required. In the author's experience, the augmented method provides a reliable approach that is easy to implement, reasonably efficient, and provides all the information one might seek in a dynamics simulation.

8.2.2 Numerical Error Checking

Regardless of the algorithm one uses, a potential difficulty lies in the fact that the velocity constraint equations are not satisfied directly at every instant. Rather, they occur only in differentiated form as acceleration constraint equations. Numerical methods are not exact, so it is possible that a small error in satisfying the acceleration constraint equations at each time step will accumulate to a substantial error relative to the velocity constraint equations. A similar issue arises if it was decided to ignore the fact that a constraint condition is holonomic by expressing it in velocity form, rather than using unconstrained generalized coordinates. It is wise to implement an auxiliary step to monitor this error periodically as the integration process marches forward. The associated operations are simple. Let an asterisk denote a quantity that is a current value returned by the equation solver. When we substitute the instantaneous values $\{q^*\}$ and $\{\dot{q}^*\}$ into each velocity constraint equation, the resulting value of each equation leads to an error vector $\{\varepsilon\}$:

$$\{\varepsilon_v\} = [a(q_i^*, t)]\left\{\dot{q}^*\right\} + \left\{b(q_i^*, t)\right\}. \tag{8.2.57}$$

In the same manner, another error vector may be formed from the values of any configuration constraints that apply:

$$\left\{\varepsilon_f\right\} = \left\{f(q_i^*, t)\right\}. \tag{8.2.58}$$

Other error measures are available from any conservation theorems that pertain, such as mechanical energy, a linear or angular momentum component, or the Hamiltonian, which is discussed in Subsection 9.2.1. The value of a conserved quantity is known from the initial conditions, so the difference between the current and initial values of a conserved quantity is the error. The value of an error measure by itself is not significant. Rather, the magnitude of each scalar error needs to be compared with the largest term from which it is formed. If such a comparison suggests that the solution is drifting from its correct condition, one would be wise to halt the numerical solution, at least to adjust the parameters passed to the differential equation solver.

Baumgarte's (1972) *constraint stabilization method* is sometimes used to compensate for the tendency of error in the velocity constraints to accumulate. In it, the acceleration constraint is modified by the addition of the velocity constraint, multiplied by an appropriate constant. This modified constraint equation is

$$[a]\{\ddot{q}\} + [\dot{a}]\{\dot{q}\} + \left\{\dot{b}\right\} + 2\alpha\left\{[a]\{\dot{q}\} + \{b\}\right\} = \{0\}. \tag{8.2.59}$$

Clearly, the added term would have no effect if the velocity constraint equations actually have been satisfied. To understand the rationale behind this alteration, designate the constraint equation as $\{h\} = [a]\{\dot{q}\} + \{b\} = \{0\}$. The acceleration constraint equation requires that $d\{h\}/dt = 0$, but the numerical approximations and round-off errors in effect convert this to $d\{h\}/dt = \{\varepsilon_h\}$. Depending on the nature of $\{\varepsilon_h\}$, the integral of $\{\varepsilon_h\}\,dt$ might grow, or even be unstable. The stabilized constraint in the presence of an error term has the form $d\{h\}/dt + 2\alpha\{h\} = \{\varepsilon_h\}$. This is like the equation of motion for a damped oscillator, in which 2α is the damping rate. The effect of the damper is to remove energy and thereby inhibit growth of the solution. In fact, the $2\alpha\{h\}$ term acts like artificial viscosity, which is often introduced when partial differential equations are solved by finite differences. Nevertheless, the constraint stabilization method does have limitations, a discussion of which may be found in the text by Haug (1989).

The constraint stabilization method can be altered to account for any constraints that are holonomic. Also, other procedures have been developed to correct for errors in quantities that should be conserved. None of these are infallible. One reason for having several numerical algorithms for solving differential equations is the fact that, for a given set of equations with a given time increment, some might be more efficient, whereas others might be able to go farther in time with lower errors. Thus, if one encounters difficulty, it might be appropriate to change the method by which the equations are solved. This might entail restarting the simulation, or switching to a different algorithm at an intermediate instant in the integration process.

8.2.3 Initial Conditions

Determination of a unique solution of the differential equations of motion and constraint requires that the initial conditions be specified. As was mentioned, the left side of Lagrange's equations contains generalized accelerations. Thus we may consider these equations to give the values of the $\ddot{q}_j$ variables when all q_j and $\dot{q}_j$, as well as all forces, are specified at some time t. It follows that the required initial conditions are the values of each q_j and $\dot{q}_j$ at an initial time t_0. When the generalized coordinates are unconstrained, $J = 0$, the initial value of each q_j and $\dot{q}_j$ may be selected arbitrarily. In contrast, constrained generalized coordinates must satisfy the constraint equations at every instant, including the initial one. If one fails to do so, the solution of the equations of motion will be meaningless.

Satisfaction of the velocity constraint equations is necessary to ensure that the system initially is moving in a kinematically admissible manner. A simple example of this consideration is the ice skate blade described by Fig. 7.6. The constraint that must be enforced gives a velocity that is oriented in the direction in which the blade cuts a groove in the ice. Initial generalized velocities that are inconsistent with this constraint correspond to the blade skidding sideways over the ice, which is a condition that is not described by the corresponding equations of motion.

Similarly, it is necessary that the initial conditions satisfy all configuration constraints, because failure to do so would start the system from an unattainable position. Recall that our approach is to express all constraint equations in velocity form. It is irrelevant to the algorithms described in Subsection 8.2.1 whether some of those equations actually are holonomic. However, it is crucial for setting the initial conditions that all configuration constraints be satisfied. Thus, setting the initial conditions is the only point in the solution process at which it is essential to know that a velocity constraint equation actually is holonomic. Let $J_c \le J$ denote the number of holonomic constraint equations, so the N initial values $(q_j)_0$ must satisfy

$$f_k\left((q_j)_0, t = 0\right) = 0, \quad k = 1, 2 \ldots, J_c. \tag{8.2.60}$$

It follows that one may assign $N - J_c$ of the $(q_j)_0$ to match what is known about the initial position of the system. The remaining $(q_j)_0$ values must then be found as solutions of these (algebraic) configuration constraints.

After initial values of the generalized coordinates consistent with Eq. (8.2.60) have been determined, initial generalized velocities consistent with the velocity constraint equations may be evaluated. For this evaluation there is no need to distinguish between holonomic and nonholonomic equations Thus the initial generalized velocities must satisfy

$$\left[a\left((q_0)_j, t=0\right)\right]\{\dot{q}_0\} + \left\{b\left((q_0)_j, t=0\right)\right\} = \{0\}. \tag{8.2.61}$$

There are J velocity constraints. Hence only $N - J$ elements of $\{\dot{q}_0\}$ may be selected as appropriate to specific initial velocity of the system. The unassigned J elements of $\{\dot{q}_0\}$ velocities must satisfy Eq. (8.2.61). These equations are linear in the generalized velocities, and one may use a sorting matrix $[P]$ as described in the previous section to isolate the initial velocities that are computed.

EXAMPLE 8.5 At a particular instant, the generalized coordinates and generalized velocities for the rolling disk in Example 8.4 are

$$X = 2 \text{ m}, Y = -1 \text{ m}, \psi = -0.6109 \text{ rad}, \quad \theta = 0.4363 \text{ rad}, \quad \phi = 3.6652 \text{ rad},$$

$$\dot{\psi} = 0.4000 \text{ rad/s}, \quad \dot{\theta} = -0.2000 \text{ rad/s}, \quad \dot{\phi} = 80.0000 \text{ rad/s}.$$

The disk's radius is $R = 0.25$ m. First determine the values of $\dot{X}$ and $\dot{Y}$ at this instant. Then describe programming steps and carry out the evaluation of $\{z\}$ and $\{\dot{z}\}$ at this instant according to the (a) integrated multiplier method, (b) augmented method, (c) the elimination method, (d) the orthogonal complement method, and (e) the embedding method.

SOLUTION By demonstrating the actual operations required to implement a single step of each of the algorithms, this example makes us more aware of the relative merits of each technique. In addition, converting the given initial values to a state vector $\{z\}$ exemplifies the operations required for initializing each algorithm. We sequence the generalized coordinates like the scheme in Example 8.4:

$$\{q\} = [X \;\; Y \;\; \psi \;\; \theta \;\; \phi]^{\mathrm{T}}. \tag{1}$$

One should note that only three of the five initial generalized velocities are given in the problem statement. The values of the other two are obtained by satisfying the velocity constraint equations.

The first step for each algorithm is to express the equations of motion and constraint in matrix form, with elements described in terms of q_j symbols. Equations

(10)–(14) in the previous example can be placed into the standard matrix form, Eq. (8.2.1). Thus the inertia matrix is

$$[M(\{q\})] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5\kappa^2\left(1+(\cos q_4)^2\right) & 0 & \kappa^2\cos q_4 \\ 0 & 0 & 0 & \left(0.5\kappa^2 + R^2(\cos q_4)^2\right) & 0 \\ 0 & 0 & \kappa^2\cos q_4 & 0 & \kappa^2 \end{bmatrix}. \tag{2}$$

The excitation matrix is

$$\{F(\{q\},\{\dot q\})\} = \begin{Bmatrix} 0 \\ 0 \\ \kappa^2\left[\frac{1}{2}\dot q_3\dot q_4\sin(2q_4) + \dot q_4\dot q_5\sin(q_4)\right] \\ \frac{1}{2}\left(R^2\dot q_4^2 - \frac{1}{2}\kappa^2\dot q_3^2\right)\sin(2q_4) - \kappa^2\dot q_3\dot q_5\sin(q_4) - gR\cos(q_4) \\ \kappa^2\dot q_3\dot q_4\sin(q_4) \end{Bmatrix}. \tag{3}$$

The Jacobian constraint matrix and its time derivative are

$$[a(\{q\},\{\dot q\})] = R\begin{bmatrix} 1/R & 0 & -\cos(q_3)\cos(q_4) & \sin(q_3)\sin(q_4) & -\cos(q_3) \\ 0 & 1/R & -\sin(q_3)\cos(q_4) & -\cos(q_3)\sin(q_4) & -\sin(q_3) \end{bmatrix}, \tag{4}$$

$$\left[\frac{da}{dt}(\{q\},\{\dot q\})\right] = R\begin{bmatrix} 0 & 0 & (\dot q_3\sin(q_3)\cos(q_4) + \dot q_4\cos(q_3)\sin(q_4)) \\ 0 & 0 & (-\dot q_3\cos(q_3)\cos(q_4) + \dot q_4\sin(q_3)\sin(q_4)) \end{bmatrix}$$
$$\begin{matrix} (\dot q_3\cos(q_3)\sin(q_4) + \dot q_4\sin(q_3)\cos(q_4)) & \dot q_3\sin(q_3) \\ (\dot q_3\sin(q_3)\sin(q_4) - \dot q_4\cos(q_3)\cos(q_4)) & -\dot q_3\cos(q_3) \end{matrix}\Bigg]. \tag{5}$$

Also $\{b\} = \{0\}$. Note that the mass m has been factored out of all equations, so the computed Lagrange multipliers will be the true λ_n values divided by m. Regardless of which algorithm is implemented, functions to evaluate each of these matrices corresponding to current values of $\{q\}$ and $\{\dot q\}$ need to be provided. The discussion that follows is framed in terms of Matlab, so it is assumed that function m-files named `M.m`, `F.m`, `a.m`, and `a_dot.m` exist in the search path.

The values $\dot X = \dot q_1$ and $\dot Y = \dot q_2$ are not specified in the problem statement, so we know that, at the given instant,

$$\{q\} = [2 \quad -1 \quad -0.6109 \quad \theta \quad 3.6652]^{\mathrm{T}},$$

$$\{\dot q\} = [? \quad ? \quad 0.4000 \quad -0.2000 \quad 80.0000]^{\mathrm{T}}.$$

To determine $\dot{X}$ and $\dot{Y}$ we observe that $[a]\{\dot{q}\} = \{0\}$ constitutes two equations for the five generalized coordinates. Because X and Y are the first two variables in $\{q\}$, the appropriate partitioning of these equations is

$$\begin{bmatrix}[a_L]_{2\times 2} & [a_R]_{2\times 3}\end{bmatrix}\begin{bmatrix}[\dot{q}_1 \;\; \dot{q}_2] & [\dot{q}_3 \;\; \dot{q}_4 \;\; \dot{q}_5]\end{bmatrix}^{\mathrm{T}} = \{0\}.$$

Thus we find that

$$\begin{Bmatrix}\dot{q}_1 \\ \dot{q}_2\end{Bmatrix} = -[a_L]^{-1}[a_R]\begin{Bmatrix}\dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5\end{Bmatrix}.$$

To extract $[a_L]$ and $[a_R]$ from $[a]$ we use the given values of ψ and θ to evaluate Eq. (4):

$$[a] = \begin{bmatrix}1 & 0 & -0.1856 & -0.0606 & -0.2048 \\ 0 & 1 & 0.1300 & -0.0865 & 0.1434\end{bmatrix},$$

Correspondingly, we find from the preceding algebraic solution that

$$\dot{X} = \dot{q}_1 = 16.4452, \quad \dot{Y} = \dot{q}_2 = -11.5408. \qquad \triangleleft$$

Now that we have the initial values of $\{q\}$ and $\{\dot{q}\}$, we may evaluate the other system matrices. The results corresponding to $\kappa = R/\sqrt{2}$ for a solid disk are

$$[M] = \begin{bmatrix}1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.0285 & 0 & 0.0283 \\ 0 & 0 & 0 & 0.0670 & 0 \\ 0 & 0 & 0.0283 & 0 & 0.0312\end{bmatrix},$$

$$\{F\} = \begin{bmatrix}0 & 0 & -0.2123 & -2.6447 & -0.0011\end{bmatrix}^{\mathrm{T}},$$

$$\left[\frac{da}{dt}\right] = \begin{bmatrix}0 & 0 & -0.0693 & 0.0606 & -0.0574 \\ 0 & 0 & -0.0621 & 0.0129 & -0.0819\end{bmatrix}.$$

We are now ready to follow each algorithm. Each will be described in terms of Matlab operations, based on knowing the current value of the state-space vector $\{z\}$ for that formulation. In the integrated multiplier method the state-space vector is

$$\{z\} = \begin{Bmatrix}\{q\} \\ \{\dot{q}\} \\ \{\mu\}\end{Bmatrix} = \begin{Bmatrix}[X \;\; Y \;\; \psi \;\; \theta \;\; \phi]^{\mathrm{T}} \\ [\dot{X} \;\; \dot{Y} \;\; \dot{\psi} \;\; \dot{\theta} \;\; \dot{\phi}]^{\mathrm{T}} \\ [\mu_1 \;\; \mu_2]^{\mathrm{T}}\end{Bmatrix}.$$

The actual values of μ_1 and μ_2 do not arise in the assembled equations, so we may set them to zero. Therefore the state-space vector at the given instant is

$$\{z\} = \left\{ \begin{matrix} [2 \quad -1 \quad -0.6109 \quad 0.4363 \quad 3.6652]^{\mathrm{T}} \\ [16.4452 \quad -11.5408 \;\; 0.40 \quad -0.20 \quad 80.00]^{\mathrm{T}} \\ [0 \quad 0]^{\mathrm{T}} \end{matrix} \right\}. \qquad \triangleleft$$

The manner in which $\{z\}$ is displayed is intended to facilitate identifying which parts are $\{q\}$, $\{\dot{q}\}$, or $\{\mu\}$. Because $\{z\}$ is the variable that would be returned by the differential equation solver, we extract the values of $\{q\}$ and $\{\dot{q}\}$ from $\{z\}$ as submatrices. The following Matlab fragment defines `LHS` and `RHS` as the left coefficient matrix and right coefficient vector appearing in Eqs. (8.2.9):

```
q = z(1:5);  q_dot = z(6:10);
LHS = [[eye(5)  zeros(5,7)];  [zeros(5)  M(q)  -a(q)'];...
       [zeros(2,5)  -a(q)  zeros(2)]];
RHS = [q_dot;  F(q,q_dot);  a_dot(q,q_dot) * q_dot];
z_dot = LHS\RHS;
```

where `M(q)`, `a(q)`, `a_dot(q,q_dot)`, and `F(q,q_dot)`are defined in function m-files. The result is

$$\{\dot{z}\} = \left\{ \begin{matrix} [16.4452 \quad -11.5408 \;\; 0.40 \quad -0.20 \;\; 80.00]^{\mathrm{T}} \\ [1.9061 \;\; 2.7173 \quad -75.7185 \quad -44.7319 \;\; 68.5679]^{\mathrm{T}} \\ [1.906 \;\; 2.717]^{\mathrm{T}} \end{matrix} \right\}. \qquad \triangleleft$$

The state-space vector in the augmented method is

$$\{z\} = \left\{ \begin{matrix} \{q\} \\ \{\dot{q}\} \end{matrix} \right\} = \left\{ \begin{matrix} [X \;\; Y \;\; \psi \;\; \theta \;\; \phi]^{\mathrm{T}} \\ [\dot{X} \;\; \dot{Y} \;\; \dot{\psi} \;\; \dot{\theta} \;\; \dot{\phi}]^{\mathrm{T}} \end{matrix} \right\},$$

whose value at the given instant is known from the analysis of initial conditions to be

$$\{z\} = \left\{ \begin{matrix} [2 \quad -1 \quad -0.6109 \;\; 0.4363 \;\; 3.6652]^{\mathrm{T}} \\ [16.4452 \quad -11.5408 \;\; 0.40 \quad -0.20 \;\; 80.00]^{\mathrm{T}} \end{matrix} \right\}. \qquad \triangleleft$$

The Matlab operations to compute $\{\dot{z}\}$ are like those for the integrated multiplier method, but there are fewer partitions to define. As before, it is assumed that function m-files have been written to evaluate the basic coefficient matrices, so the following program fragments implement the operations associated with Eqs. (8.2.11):

```
q = z(1:5);  q_dot = z(6:10);
LHS = [[M(q)  -a(q)'];.[-a(q)  zeros(2)]];
RHS = [F(q,q_dot);  a_dot(q,q_dot) * q_dot];
sol = LHS\RHS; q2_dot = sol(1:5);
z_dot=[q_dot;  q_2dot]];
```

Note that we could extract the instantaneous Lagrange multiplier values as the lower part of the solution vector `sol` if their behavior was of interest. The value of $\{\dot{z}\}$ that results is

$$\{\dot{z}\} = \begin{Bmatrix} [16.4452 \quad -11.5408 \;\; 0.40 \quad -0.20 \;\; 80.00]^{\mathrm{T}} \\ [1.9061 \;\; 2.7173 \quad -75.7185 \quad -44.7319 \;\; 68.5679]^{\mathrm{T}} \end{Bmatrix}. \qquad \triangleleft$$

The state-space vector $\{z\}$ in the elimination method is the same as the one for the augmented method, so its value at the present instant is as stated previously. There are two Lagrange multipliers to eliminate, so the first partition $[a_1]$ in Eq. (8.2.13) will be 2×2. Inspection of $[a]$ for this problem shows that its left 2×2 partition is the identity matrix, which is invertible, so there is no need to rearrange any matrices. Thus we partition the computed values of $[M]$, $[a]$, and $\{F\}$ as prescribed in Eq. (8.2.12), with $J = 2$. After we perform these operations, we may proceed directly to evaluate $\{\ddot{q}\}$ according to Eqs. (8.2.16). The Matlab program fragment implementing these operations is

```
q = z(1:5);  q_dot = z(6:10);
a_mat = a(q);   M_mat=M(q);  F_mat = F(q,q_dot);
a_1 = a_mat(1:2,1:2);  a_2 = a_mat(1:2,3:5);
M_1 = M_mat(1:2,1:5);  M_2 = M_mat(3:5,1:5);
F_1 = F_mat(1:2);  F_2 = F_mat(3:5);
LHS = [(M_2-a_2' * inv(a_1)' * M_1;  -a_mat ];
RHS = [F_2 - a_2' * inv(a_1)' * F_1;  a_dot(q,q_dot) * q_dot];
q_2dot = LHS\RHS;    z_dot = [q_dot;  q_2dot]
```

The value of `z_dot` produced by these operations is identical to the result of the augmented method.

We use SVD to implement the orthogonal complement method. The state-space vector here is the same as that of the augmented and elimination methods. Determination of the orthogonal complement $[C]$ corresponding to the present state must be done before $\{\ddot{q}\}$ can be evaluated according to Eqs. (8.2.26). Toward that end we define $[A]$ according to Eq. (8.2.34). After SVD has been performed the singular values w_j are sorted in ascending order, with the same sorting sequence also applied to $[U]$ and $[R]$. Matlab steps carrying out these operations are

```
q = z(1:5);  q_dot = z(6:10);
A = [a(q);  zeros(3,5)];  [L_svd,w_svd,R_svd] = svd(A);
w_diag = diag(w_svd);  [w,n_sort] = sort(w_diag);
for j=1:5;
        L(:,j) = L_svd(:,n_sort(j));
        R(:,j) = R_svd(:,n_sort(j));
end
```

The first three singular values `w_diag` should be zero because $[A]$ has three rows of zeros. This is confirmed by the computed results, which give

$$w = [0.0000 \quad 0.0000 \quad 0.0000 \quad 1.0056 \quad 1.0554]^{\mathrm{T}}.$$

We correspondingly form the rows of $[C]$ from the first three columns of $[R]$, then use that value to implement Eqs. (8.2.26),

```
C = R(:,1:3)';   LHS = [C*M(q);  -a(q)];
RHS = [C * F(q,q_dot);  a_dot(q,q_dot) * q_dot]
q_2dot = LHS\RHS;    z_dot = [q_dot;  q_2dot]
```

The value of $\{\dot{z}\}$ that results is identical to that obtained from the augmented and elimination methods. An interesting feature of Matlab is that it contains the null function, which gives the orthogonal complement without all of the intermediate programming steps. Specifically, we could have written `C = null(A)'`. The solution did not focus on this alternative because most mathematical software contains a SVD routine, whereas the null routine is less common. In the event that one wishes to compare these alternative functions, it is important to realize that $[C]$ obtained from the null function is likely to differ from the result obtained from SVD.

Implementation of the embedding method version begins by selecting the generalized coordinates to place in the constrained set and defining the corresponding sorting matrix $[P]$. In the present context it is logical to select X and Y as the constrained variables, consistent with our analysis of initial conditions, so that

$$\{q_c\} = [X \;\; Y]^{\mathrm{T}}, \quad \{q_u\} = [\psi \;\; \theta \;\; \phi]^{\mathrm{T}}.$$

Satisfaction of Eq. (8.2.37) for these definitions of $\{q_u\}$ and $\{q_c\}$ requires that

$$[q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5]^{\mathrm{T}} = [P]\,[q_3 \quad q_4 \quad q_5 \quad q_1 \quad q_2]^{\mathrm{T}}.$$

The nonzero element in row j of $[P]$ occurs in column n, where n is the row where q_j occurs in the right side vector multiplying $[P]$. A Matlab fragment implementing this definition is

```
P = zeros(5);  P(1,4) = 1;  P(2,5) = 1;
P(3,1) = 1;   P(4,2) = 1;  P(5,3) = 1;
```

The corresponding definition of the state vector is

$$\{z\} = \begin{Bmatrix} \{q_u\} \\ \{\dot{q}_u\} \\ \{q_c\} \end{Bmatrix} = \begin{Bmatrix} [\psi \quad \theta \quad \phi]^{\mathrm{T}} \\ [\dot{\psi} \quad \dot{\theta} \quad \dot{\phi}]^{\mathrm{T}} \\ [X \quad Y]^{\mathrm{T}} \end{Bmatrix}.$$

The current value of $\{z\}$ is composed of the given parameters. (There is no need to determine the values of $\dot{X}$ and $\dot{Y}$ as in the other algorithms because those variables have been eliminated as part of the embedding process.) Thus we set

$$\{z\} = \left\{ \begin{matrix} [-0.6109 \quad 0.4363 \quad 3.6652]^{\mathrm{T}} \\ [0.40 \quad -0.20 \quad 80.00]^{\mathrm{T}} \\ [2 \quad -1]^{\mathrm{T}} \end{matrix} \right\}. \qquad \triangleleft$$

The sequence of operations by which we proceed from a known value of $\{z\}$ at some instant is to form the full $\{q\}$ from that $\{z\}$, which we use to evaluate $[a_c]$ and $[a_u]$. We then use these submatrices to evaluate the full $\{\dot{q}\}$, at which point we have the data required for solving for $\{\ddot{q}_u^*\}$. The following fragment from a Matlab script file calls the same functions as previously described to evaluate $[M]$, $[a]$, $[da/dt]$, and $\{F\}$:

```
q_u = z(1:3);  q_dot_u = z(4:6);  q_c = z(7:8);
q = P * [q_u;  q_c];   a_sort = a(q) * P;
a_u = a_sort(1:2,1:3);  a_c = a_sort(1:2,4:5);
a_c_inv = inv(a_c);   D = P * [eye(3);  -a_c_inv * a_u];
B = P * [zeros(3,2);  -a_c_inv];   q_dot = D * q_dot_u;
LHS = D' * M(q) * D;
RHS = D' * (F(q,q_dot) - M(q) * B * a_dot(q,q_dot) ...
        * D * q_dot)
q_2dot_u = LHS\RHS;  z_dot = [q_dot_u;  q_2dot_u;  q_dot_c];
```

Note that there is no need to define $[E]$ because $\{b\} = \{0\}$. The computation yields

$$\{\dot{z}\} = \left\{ \begin{matrix} [0.40 \quad -0.20 \quad 80.00]^{\mathrm{T}} \\ [-75.7185 \quad -44.7319 \quad 68.5679]^{\mathrm{T}} \\ [16.4452 \quad -11.5408 \quad]^{\mathrm{T}} \end{matrix} \right\}. \qquad \triangleleft$$

The preceding presentation illustrates the computations required to apply each algorithm in conjunction with a differential equation solver that marches forward in time. The value of $\{\dot{z}\}$ that was computed would be $\{G(z_i, t)\}$ in a differential equation solver associated with Eq. (8.2.3). The output of that solver would be an updated value of $\{z\}$. The program fragments presented here could be incorporated into programs for each algorithm. These programs can be made quite general by allowing for an arbitrary number of generalized coordinates. Of course, the functions/subroutines that evaluate $[M]$, $\{F\}$, $[a]$, and $\{b\}$ would need to be modified to describe a specific system.

EXAMPLE 8.6 A thin disk whose radius is 250 mm rolls without slipping after being set into motion. Two sets of initial conditions are of interest:

1. The precession and spin rates are set to the values required for a steady precession at $\theta = 60^o$, such that the center would follow a circular path having radius $\rho = 5R$. However, the initial nutation angle is $\theta = 30^o$ and the initial nutation rate is $\dot{\theta} = -\dot{\psi}/2$, rather than zero. Other initial conditions are $X = 0$, $Y = 5R$, $\psi = 0$, $\phi = 0$.
2. The disk is set into planar motion with its center's velocity being $0.1155\,(gR)^{1/2}$ in the X direction, and $\dot{\psi} = \dot{\theta} = 0$. The disk is leaning slightly, with a nutation angle of $\pi/2 - 0.01$ rad, and the other initial conditions are $X = Y = 0$, $\psi = 0$, $\phi = 0$.

For each case use numerical methods to solve the equations of motion. Graph the response in a manner that describes the motion in each case.

SOLUTION This example builds on the expertise developed in the previous example, while also leading to responses that have numerous interesting features. The computation is based on the augmented algorithm because it is relatively easy to program, and reasonably efficient. For either set of initial conditions the main program performs a number of fundamental tasks. It begins with an evaluation of the full set of initial generalized velocities corresponding to the given unconstrained values. It then executes a loop that saves the current set of $\{q\}$ and $\{\dot{q}\}$ values, increments time, and calls the differential equation solver. As noted in Example 7.19, some subprograms for solving differential equations are adaptive. Such a routine will automatically select time steps and perform multiple corresponding substeps between the specified beginning and termination times. The present work uses such a solver with the time interval subdivided into many subintervals. The state vector computed at each step initializes the next step. Specifically, the results were obtained with the Matlab `ODE45` function, which employs fourth- and fifth-order Runge–Kutta formulas. (The alternative `ODE23`, which implements lower-order formulas, was found to require approximately 30% more time to execute, with no difference in the output.) The programmatic steps that were executed to call `ODE45` were constructed as follows. The state-space vector formed from the values of $\{q\}$ and $\{\dot{q}\}$ at the beginning of the integration step is contained in the variable `z_0` and the corresponding time is `t0`. The time for the desired response is `tf`. The program fragment is

```
z_dot_anon = @(t,z) z_dot_aug(t,z,radius,kappa)
for i = 1:i_max;  t0 = (i - 1) * dt;  tf = i * dt;
        < Data storage and error checking operations >
        [t,z] = ode45(z_dot_anon, [t0,tf], z_0, options);
        n_steps = size(z,1);   z_0 = z(n_steps,1:10)';
end
```

The first argument in the call to the `ode45` function is the handle of the anonymous function `z_dot_anon`, which passes the requisite parameter values to a function m-file `z_dot_aug.m` that evaluates $\{\dot{z}\} = [\{\dot{q}\}^{\mathrm{T}} \quad \{\ddot{q}\}^{\mathrm{T}}]^{\mathrm{T}}$ corresponding to the current $\{z\} = [\{q\}^{\mathrm{T}} \quad \{\dot{q}\}^{\mathrm{T}}]^{\mathrm{T}}$. (See Example 7.19 for a more detailed discussion of anonymous functions in Matlab.) These operations parallel those described in Example 8.5 for a single time step using the augmented algorithm. The `options` value contains various parameters for `ode45` that are set with the `odeset` function. All of these parameters were left at their defaults, except that the relative and absolute errors were set to 10^{-8}, both of which are substantially smaller than the defaults. The output variable t is a vector of time steps at which `ode45` computed a response. The output `z` has the same number of rows, with each row containing the state vector corresponding to each row of t.

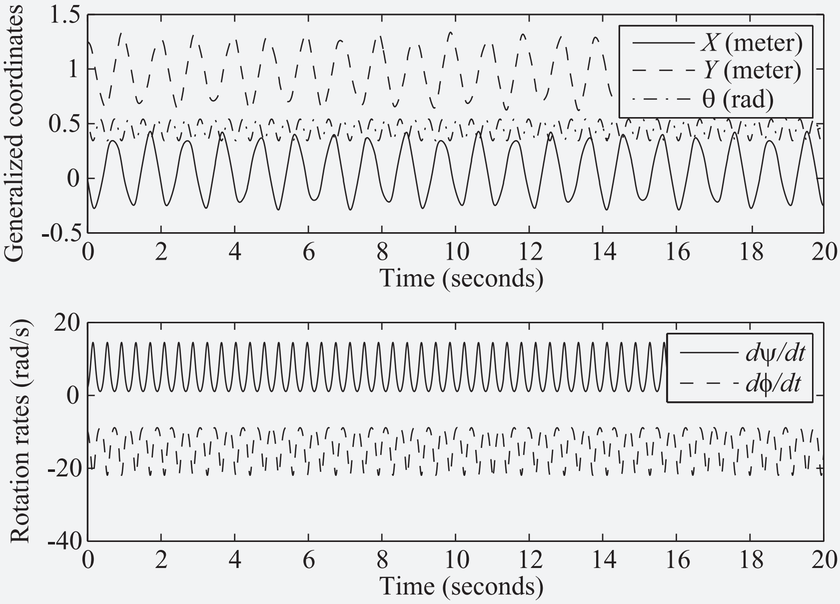
The storage and checking operations mentioned in the listing evaluate the total mechanical energy $E = T + V$ corresponding to the current $\{z\}$. They also evaluate $[a]\{\dot{q}\}$, which would grow if the velocity constraints drift even though the acceleration constraints are satisfied. The values of E and norm$([a]\{\dot{q}\})$ are appended to `z_0`, which is saved in additional operations before the next integration step is begun.

In general, it is a good idea to verify that any numerical simulation gives the correct results in situations in which a known solution exists. Such is the case here. The conditions for steady precession were derived in Example 8.5. If we modify the initial conditions in Case 1 such that $\theta = 60^{\mathrm{o}}$ and $\dot{\theta} = 0$, the steady precession formulas indicate that the response should be given by

$$X = 1.25\sin(1.7234t), \quad X = 1.25\cos(1.7234t) \text{ m},$$
$$\psi = 1.7234t, \quad \theta = \pi/3, \quad \phi = -9.479t \text{ rad}.$$

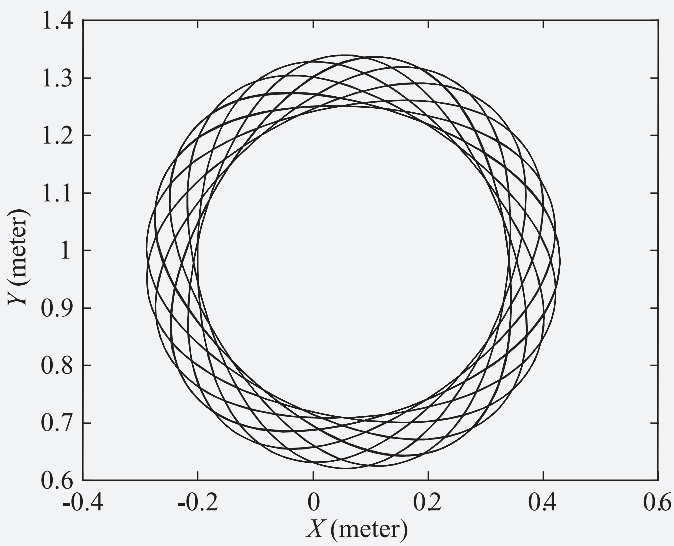
The computed result closely matched these expressions. Throughout the integration interval the values of $\dot{\psi}$ and $\dot{\phi}$ matched the formula values to seven significant figures, and $\left|\dot{\theta}\right|$ remained smaller than 10^{-15} rad/s.

The responses corresponding to the Case 1 initial conditions are displayed in the first two graphs as plots of X, Y, θ $\dot{\psi}$, and $\dot{\phi}$ as functions of time. Throughout the time interval of the plots, the mechanical energy remained at $E/m = 4.250498$ and the norm of the velocity constraint error remained below $2\left(10^{-13}\right)$. The first set of plots suggests that the motion is like a steady precession, in that the time dependence of X and Y is oscillatory and nearly sinusoidal. Also, θ, which represents the wobble, executes a relatively small oscillation about the mean value at which it was set. In contrast, the plots of $\dot{\psi}$ and $\dot{\phi}$ differ substantially from the steady precession case, in which they would be constant. These rotation rates now execute a synchronized oscillation that is a large fraction of their respective mean values.



Generalized coordinates and velocities as functions of time when the disk is released with initial conditions resembling a steady precession

A clearer picture emerges from the third graph, which plots (X, Y) pairs at successive values of t; this is the path of the center of the disk as viewed looking down the Z axis. Careful inspection would show that the center follows a nearly elliptical path relative to the XY plane, with the major and minor axes of this ellipse rotating a little in successive cycles. At any instant the distance to the disk's center from



Path of the center of the disk that is released with initial conditions resembling a steady precession the origin is between the length of the semiminor and semimajor axes. The average distance from this center is approximately 0.32 m, which is much less than the value $\rho = 5R$ that was intended with the initial conditions.

One might be surprised that a nearly steady precession is obtained, despite the difference between the given initial conditions and those for the intended motion. This is a manifestation of the overall stability resulting from the gyroscopic moment, which induces a rotation about an axis perpendicular to the axis of an applied moment. Stability is the motivation for considering the second set of initial conditions. A classical result states that, if one attempts to cause a disk to roll freely in the vertical plane, that motion will be unstable if the forward speed is below a certain value. Before we examine the computational result, let us see how such a stability analysis is carried out. The reference free-rolling motion corresponds to $\dot{X} = v$, $\dot{Y} = 0$, $\theta = \pi/2$, $\dot{\psi} = 0$, and $\dot{\phi} = v/R$. A stability analysis entails considering a motion in which the generalized coordinates differ by a very small amount from this reference solution. To highlight this difference we let ε designate a small parameter, and define the deviations as $\varepsilon\xi_j$, where the ξ_j variables are taken to have a unit order of magnitude, specifically,

$$X = vt + \varepsilon\xi_1, \quad Y = \varepsilon\xi_2, \quad \psi = \varepsilon\xi_3, \quad \theta = \frac{\pi}{2} - \varepsilon\xi_4, \quad \phi = \frac{vt}{R} + \varepsilon\xi_5. \tag{1}$$

The idea is to substitute these expressions into the equations of motion and drop all terms that contain ε in quadratic and higher powers, on the grounds that higher-order terms will be negligible if ε is sufficiently small. The trigonometric terms are linearized with the aid of identities, such that

$$\begin{gathered} \sin\psi = \sin(\varepsilon\xi_3) \approx \varepsilon\xi_3, \quad \cos(\psi) = \cos(\varepsilon\xi_3) \approx 1, \\ \sin(\theta) = \sin\left(\frac{\pi}{2} - \varepsilon\xi_4\right) = \cos(\varepsilon\xi_4) \approx 1, \\ \cos(\theta) = \cos\left(\frac{\pi}{2} - \varepsilon\xi_4\right) = \sin(\varepsilon\xi_4) \approx \varepsilon\xi_4. \end{gathered} \tag{2}$$

The equations of motion and constraint equations were derived in Example 8.4. Substitution of Eqs. (1) and (2) into the constraint equations gives

$$\begin{gathered} v + \varepsilon\dot{\xi}_1 - R\left[(\varepsilon\dot{\xi}_3)(\varepsilon\xi_4) + \left(\frac{v}{R} + \varepsilon\dot{\xi}_5\right)\right] + R\left(-\varepsilon\dot{\xi}_4\right)(\varepsilon\xi_3) = 0, \\ \varepsilon\dot{\xi}_2 - R\left[(\varepsilon\dot{\xi}_3)(\varepsilon\xi_4) + \left(\frac{v}{R} + \varepsilon\dot{\xi}_5\right)\right](\varepsilon\xi_3) - R\left(-\varepsilon\dot{\xi}_4\right) = 0. \end{gathered}$$

The terms in the preceding equations that are independent of ε cancel, and linearization eliminates all higher-order terms in ε. Thus the constraint equations reduce to

$$\dot{\xi}_1 = R\dot{\xi}_5, \quad \dot{\xi}_2 = -R\dot{\xi}_4 + v\xi_3. \tag{3}$$

To eliminate as many generalized coordinates as possible from the equations of motion, we substitute Eqs. (1) and (3) into the Lagrange equations for X and Y, which leads to

$$m\varepsilon\left(R\ddot{\xi}_5\right)=\lambda_1,\quad m\varepsilon\left(-R\ddot{\xi}_4+v\dot{\xi}_3\right)=\lambda_2. \tag{4}$$

Substitution of these expressions into the remaining Lagrange equations eliminates the Lagrange multipliers. We simplify the algebraic operations by observing that both λ_1 and λ_2 are of order ε, which makes it possible to ignore *a priori* some small terms containing λ_1 and λ_2. The result is

$$\begin{gathered}m\kappa^2\left(\frac{1}{2}\varepsilon\ddot{\xi}_3+\varepsilon\dot{\xi}_4\frac{v}{R}\right)=0,\\ m\varepsilon\left[\frac{1}{2}\kappa^2\left(-\varepsilon\ddot{\xi}_4\right)+\kappa^2\left(\varepsilon\dot{\xi}_3\right)\frac{v}{R}+gR\left(\varepsilon\xi_4\right)\right]=-\lambda_2R,\\ m\kappa^2\left(\varepsilon\ddot{\xi}_5\right)=-\lambda_1R.\end{gathered} \tag{5}$$

Satisfaction of the first of Eqs. (4) and the last of Eqs. (5) requires that $\xi_5=0$. This enables us to obtain a single differential equation for one variable. The first of Eqs. (5) indicates that $\dot{\xi}_3=-2\left(v/R\right)\xi_4$, which we substitute along with λ_2 from Eqs. (4), into the second of Eqs. (5). We thereby obtain

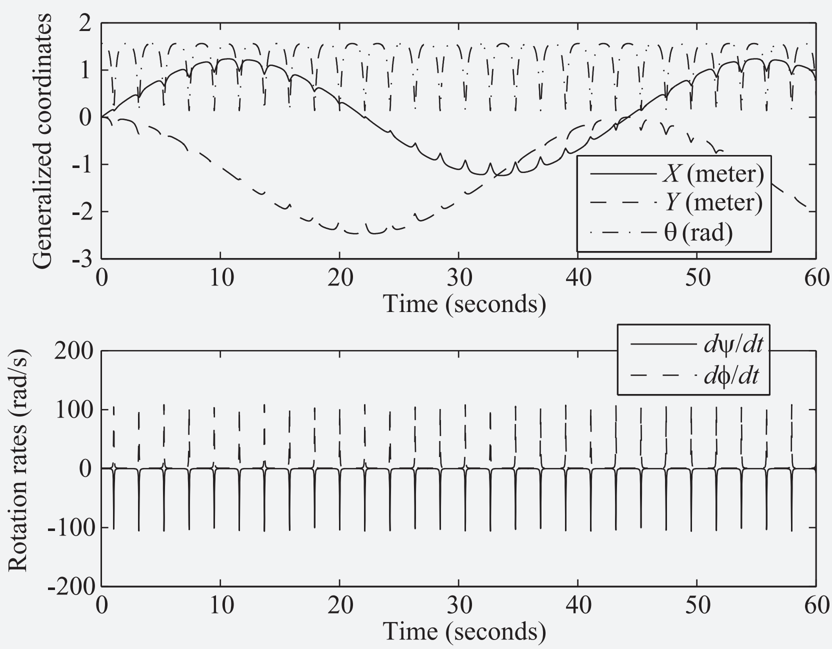
$$\left(\frac{1}{2}\kappa^2+R^2\right)\ddot{\xi}_4+\left[2\left(\frac{\kappa^2}{R^2}+1\right)v^2-GR\right]\xi_4=0. \tag{6}$$

This is the differential equation for a linear undamped spring–mass system. The coefficient of the acceleration term is always positive, so the solution for ζ_4 will be sinusoidal if the coefficient of the springlike terms is also positive. A sinusoidal term does not grow as time evolves, so it corresponds to a stable situation. In contrast, if the coefficient is negative, the solutions are exponential, with one positive characteristic exponent. Because this means that ξ_4 will grow as t increases, corresponding to a divergence from the nominal value $\theta=\pi/2$, the response is said to be unstable. The radius of gyration for a thin disk is $\kappa=R/\sqrt{2}$, so we have found that

$$v>\left(\frac{gR}{3}\right)^{1/2}\implies\text{stability.}$$

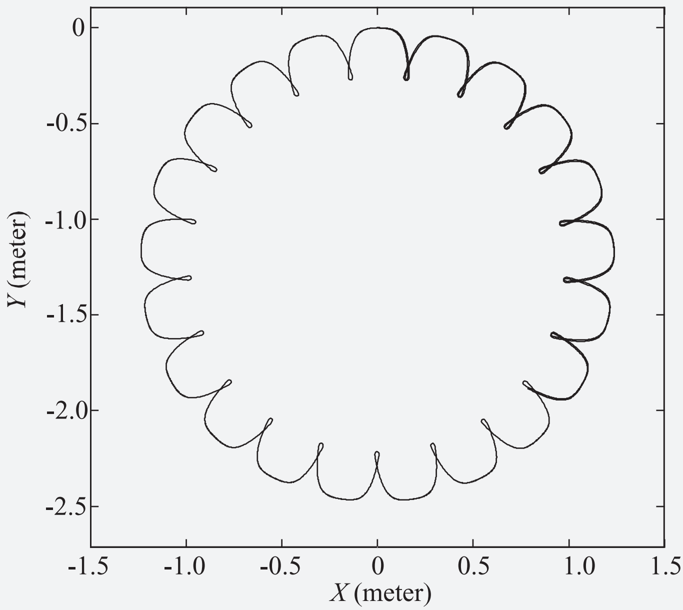
This result is often misinterpreted. The analysis is valid only if ξ_4 remains at unit order, so we may conclude that a disk will not roll upright along a straight line if the speed does not satisfy the stability criterion. However, the analysis does not tell us what will happen in the unstable case, just as the fact that a ball at the apex of a hill is unstable does not tell us much. Finding out what actually happens requires a numerical solution. The given forward speed in the Case 2 initial conditions is 20% of this stability limit. The full set of initial conditions, which is obtained by satisfying the constraint equations with the given initial parameters, is $\{q\}=[0\ \ 0\ \ 0\ \ \pi/2-0.1\ \ 0]^{\mathrm{T}}$, $\{\dot{q}\}=\left[0.1155\,(gR)^{1/2}\ \ 0\ \ 0\ \ 0\ \ 0.1155\,(g/R)^{1/2}\right]^{\mathrm{T}}$. This is the only alteration required to invoke the same computer routine as that used to obtain the Case 1 response. As before, conservation of energy and the norm of the velocity constraints are monitored as error checks. It was found that $E = 2.476145$ throughout and the norm of the velocity constraint equations never exceeded 10^{-9}.

Inspection of the plots of X and Y as functions of time shows that the overall motion is a slow sinusoidal variation with Y leading X by $\pi/2$, which is suggestive of a steady precession. However, additional oscillations are superimposed on these responses. The plot of $\theta(t)$ shows the extraordinary behavior of oscillating between a minimum that is nearly zero and a maximum that is close to $\pi/2$. (The actual minimum $\theta = 0.1308$ and the maximum is the initial value $\pi/2 - 0.10$.) We also see that the values of $\dot{\psi}$ and $\dot{\phi}$ are very large when θ is at the minima, and almost zero at maximum θ, and the higher-frequency aspect of the oscillation in X and Y is also synchronized with the oscillation of θ.



Generalized coordinates and velocities when the disk is released with a slow forward motion in a nearly upright orientation

The plot of Y as a function of X in the fourth graph helps us to interpret this response. It shows that the overall motion resembles a steady precession following a circular path. However, this overall motion is composed of many loops. The outer part of the loop consists of the disk being nearly upright, $\theta \approx \pi/2$. When θ begins to decrease, the center moves inward to the center of the overall path. The disk's center executes a small loop at the bottom of its descent, and then rises back to a nearly upright position. In the course of executing this looping motion, the overall precession angle has changed, thereby setting up the overall circular pattern. The computed result for the center's path is almost closed. Another run with a larger initial deviation from an upright orientation, $\theta = \pi/2 - 0.1$, showed a similar looping pattern, but it was visibly open. This suggests that setting the initial value of θ extremely close to $\pi/2$ would indeed produce a closed path corresponding to a truly periodic response.



Path of the center of the disk when it is released with a slow forward motion in a nearly upright orientation

Although this result is extremely interesting, it also points out the limits of computer modeling that do not truly capture reality. An actual disk could not sustain this motion because of rolling friction, which dissipates energy, whereas energy is conserved in the present idealized model. The only condition in our analysis that would indicate cessation of the motion would be that $\theta < 0$ or $\theta > \pi$, both of which correspond to the center falling to the ground. It would also be appropriate to terminate our ideal model if the instantaneous friction force required to prevent slippage exceeds the maximum that can be developed for the current normal force. However, checking this condition would require that we redo the entire formulation, for we then would need to track the actual friction and normal forces, rather than employing Lagrange multipliers to represent those force. We also would need to evaluate the normal force at each instant. We can implement such modifications by following techniques developed in the the next section.

In closing, it is useful to consider using an algorithm other than the augmented method. Matlab's `null` function makes it particularly easy to program the orthogonal complement algorithm, as described in the previous example. Other than altering how $\{\ddot{q}\}$ is evaluated at each time step, all other aspects of the solution obtained with the augmented method were retained. Because the orthogonal complement eliminates the Lagrange multipliers, the instantaneous generalized accelerations are obtained by solving five simultaneous equations, whereas the augmented method entails solving seven equations. Nevertheless, the orthogonal complement method required 41% more cpu time. This apparently is due to the computational overhead required to perform the SVD.

8.2.4 Analysis of Coulomb Friction

When it is necessary to account for Coulomb friction forces, it is mandatory that the constraint forces be evaluated, rather than represented by Lagrange multipliers. Let us partition the virtual work into three parts: δW_a is the contribution of known applied forces, δW_C denotes the contribution of the constraint forces that act perpendicularly to contact surfaces, and δW_f is the contribution of the friction forces. The first part is described in the usual manner as

$$\delta W^{(a)} = \sum_{j=1}^{N} Q_j^{(a)} \delta q_j. \tag{8.2.62}$$

The explicit contribution of a single constraint force C_i to the virtual work is described by Eq. (7.4.13). We adapt that description to a friction analysis by using N_i to denote the normal force, and allow for the possibility that there are several such forces by adding the individual contributions. Thus we have

$$\delta W_C = \sum_{i=1}^{J} N_i \sum_{j=1}^{N} c_{ij} \delta q_j. \tag{8.2.63}$$

The form of δW_f is a generalization of the typical situation described by Eq. (8.1.7). Each friction force may be written as

$$\bar{f}_i = -\mu \, |N_i| \operatorname{sgn}\left((\bar{v}_{\text{rel}})_i \cdot \bar{e}_t\right) \bar{e}_t, \tag{8.2.64}$$

where $(\bar{v}_{\text{rel}})_i$ is the velocity of the surface on which $\bar{f}_i$ acts relative to the other surface and $\bar{e}_t$ is the direction that is tangent to the plane of contact. Furthermore, velocities may be expressed as a linear sum of generalized velocities. It follows that the virtual work done by all friction forces is described in general by

$$\delta W_f = \sum_{i=1}^{J} \sum_{j=1}^{N} u_{ij}\left(q_i, \dot{q}_i, t\right) |N_i| \, \delta q_j, \tag{8.2.65}$$

where the $u_{ij}\left(q_i, \dot{q}_i, t\right)$ will contain signum functions of the generalized velocities.

The generalized forces obtained from these three contributions to the virtual work are

$$Q_j = Q_j^{(a)} + \sum_{i=1}^{J} c_{ij} N_i + \sum_{i=1}^{J} u_{ij} |N_i| . \tag{8.2.66}$$

The analog of Eqs. (8.1.4) when the generalized forces are described in this manner is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j^{(a)} + \sum_{i=1}^{J} c_{ij} N_i + \sum_{i=1}^{J} u_{ij} |N_i| , \quad j = 1, \dots, N. \tag{8.2.67}$$

The matrix version of these equations is

$$[M]\{\ddot{q}\} = \{F\} + [c]^{\mathrm{T}}\{N\} + [u]\{|N|\} , \tag{8.2.68}$$

where $\{N\}$ is a vector of normal force values and $\{|N|\}$ is the corresponding vector of magnitudes.

If the contacting surfaces are such that the normal force can act in only one direction, as in the case of a block on a plane, then the corresponding $\bar{N}_i$ can act only to press the surfaces together. In that case, N_i can only be positive, because negative N_i would mean that the surfaces are about to separate. If only compressive contact forces are permitted, $|N_i|$ can be replaced with N_i, so the equations of motion are linear in the normal forces. Any of the algorithms that entail evaluating the Lagrange multipliers are readily modified to address this situation. An important aspect of the one-sided nature of the contact force in this case is that the solution ceases to be valid whenever any N_i becomes negative.

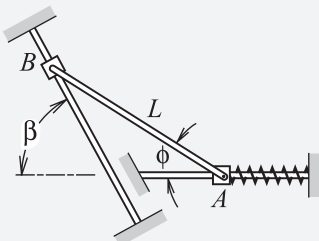
The problematic case is that in which the sense of the normal force is not known in advance, as would be the case for a collar sliding on a guide bar. In planar motion, this means that the sign of the normal force is unknown, whereas the magnitude of the normal force in spatial motion is the resultant of the two force components perpendicular to $\bar{e}_t$. In either case, the normal force does not occur linearly in the equations of motion, so there is no simple modification of the solution algorithms. One approach is implemented in the next example. The idea is to use the fact that one can obtain the magnitude of a scalar by multiplying the value by its sign. Thus

$$|N_i| = N_i \operatorname{sgn}(N_i) . \tag{8.2.69}$$

The value of N_i is taken to be unknown, and $\operatorname{sgn}(N_i)$ is set by the value of N_i that was most recently determined.

A separate issue for Coulomb friction is whether slippage actually occurs. The static friction law states that, at any instant when there is no relative velocity between the contacting surfaces, sliding will resume only if the magnitude of the friction force exceeds $\mu_s |N_i|$, where μ_s is the coefficient of static friction. Thus a simulation of the Coulomb friction force should implement a separate test for this condition, which would be called for whenever the contacting surfaces are sensed to have reversed the sense of their sliding motion.

EXAMPLE 8.7 The coefficient of sliding friction between collar A and its horizontal guide is μ, but friction between collar B and the inclined guide is negligible. The spring, whose stiffness is k, is unstretched when $\phi = 0$. The bar and each collar have identical mass m. Determine the equations of motion of the system. Then consider a specific case in which $kL/mg = 4$, $\beta = 60^\circ$, and the coefficient of sliding friction is $\mu = 0.25$. The bar is released from rest at $\phi = 30^\circ$. Determine the elapsed time until the bar comes to rest and the mechanical energy that is dissipated by friction in that interval.



Example 8.7

SOLUTION This example is representative of how sliding friction can be incorporated into the Lagrange equations of motion, and the numerical solution will clarify some of the issues regarding how to handle friction forces. Also, its similarity with Example 6.9, which was analyzed with the Newton–Euler equations of motion, provides an opportunity to assess the merits of each approach.

The position of the bar is fully specified by the angle ϕ, so this is a holonomic system with one degree of freedom. We use two generalized coordinates in order to violate the constraint that collar A must slide along its guide, because doing so will cause the normal force at that location to appear in the virtual work. Because friction at collar B is negligible, we do not wish to violate the constraint condition for that collar. Thus the generalized coordinates we use are the angle of orientation and the distance from the intersection of the guide bars to collar B, so $q_1 = \phi$, $q_2 = s$. We place the origin O of a fixed coordinate system XYZ at this intersection, so the vectors corresponding to arbitrary values of the generalized coordinates are

$$\begin{aligned}\bar{r}_{B/O} &= s\left(-\cos\beta\bar{I} + \sin\beta\bar{J}\right),\\ \bar{r}_{A/O} &= \bar{r}_{B/O} + \bar{r}_{A/B} = s\left(-\cos\beta\bar{I} + \sin\beta\bar{J}\right) + L\left(\cos\phi\bar{I} - \sin\phi\bar{J}\right),\\ \bar{r}_{G/O} &= \bar{r}_{B/O} + \bar{r}_{G/B} = s\left(-\cos\beta\bar{I} + \sin\beta\bar{J}\right) + \frac{L}{2}\left(\cos\phi\bar{I} - \sin\phi\bar{J}\right).\end{aligned} \tag{1}$$

We readily obtain the configuration constraint by applying the law of sines with the collar A situated on the horizontal guide bar. The result is written as

$$s\sin\beta - L\sin\phi = 0. \tag{2}$$

Differentiating this gives the velocity constraint equation:

$$\dot{s}\sin\beta - L\dot{\phi}\cos\phi = 0. \tag{3}$$

We describe the virtual work done by the normal force at point A explicitly, as called for by the general procedure for handling friction forces. Arbitrary increments in the generalized coordinates cause collar B to merely displace parallel to its guide, so the normal force acting on collar B will not contribute to the virtual work. For collar A we take the normal force $\bar{N}_A$ to be positive upward, and the friction force $\bar{f}_A$ is taken to be positive when it acts to the left, corresponding to $\bar{v}_A$ being to the right. Applying the analytical method for virtual displacement to the second of Eqs. (1) gives

$$\delta \bar{r}_A = \delta s \left(-\cos\beta \bar{I} + \sin\beta \bar{J}\right) + L\delta\phi \left(-\sin\phi \bar{I} - \cos\phi \bar{J}\right).$$

The gravity and spring forces are conservative, so the virtual work is

$$\delta W = \left(-f_A \bar{I} + N_A \bar{J}\right) \cdot \delta \bar{r}_A = L(-N_A \cos\phi + f_A \sin\phi)\,\delta\phi + (N_A \sin\beta + f_A \cos\beta)\,\delta s,$$

from which it follows that the generalized forces are

$$Q_1 = -N_A L \cos\phi + f_A L \sin\phi, \quad Q_2 = N_A \sin\beta + f_A \cos\beta. \tag{4}$$

We will introduce Coulomb's law later, so we proceed to formulating the energy expressions. The bar is in general motion, with $-\dot{\phi}\bar{k}$ being its angular velocity. The required velocities may be described by by direct differentiation of the positions in Eqs. (1):

$$\begin{aligned}
\bar{v}_B &= \frac{d}{dt}\bar{r}_{G/O} = \dot{s}\left(-\cos\beta \bar{I} + \sin\beta \bar{J}\right), \\
\bar{v}_A &= \frac{d}{dt}\bar{r}_{A/O} = \left(-\dot{s}\cos\beta - L\dot{\phi}\sin\phi\right)\bar{I} + \left(\dot{s}\sin\beta - L\dot{\phi}\cos\phi\right)\bar{J}, \\
\bar{v}_G &= \frac{d}{dt}\bar{r}_{G/O} = \left(-\dot{s}\cos\beta - \frac{L}{2}\dot{\phi}\sin\phi\right)\bar{I} + \left(\dot{s}\sin\beta - \frac{L}{2}\dot{\phi}\cos\phi\right)\bar{J}.
\end{aligned}$$

The kinetic energy corresponding to these quantities is

$$\begin{aligned}
T &= \frac{1}{2}mv_G^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\phi}^2 + \frac{1}{2}mv_A^2 + \frac{1}{2}mv_B^2 \\
&= \frac{1}{2}m\Bigg[\left(-\dot{s}\cos\beta - \frac{L}{2}\dot{\phi}\sin\phi\right)^2 + \left(\dot{s}\sin\beta - \frac{L}{2}\dot{\phi}\cos\phi\right)^2 + \frac{1}{12}L^2\dot{\phi}^2 \\
&\quad + \left(-\dot{s}\cos\beta - L\dot{\phi}\sin\phi\right)^2 + \left(\dot{s}\sin\beta - L\dot{\phi}\cos\phi\right)^2 + \dot{s}^2\Bigg] \qquad (5) \\
&= \frac{1}{2}m\left[\frac{4}{3}L^2\dot{\phi}^2 + 3\dot{s}^2 - 3L\dot{\phi}\dot{s}\sin(\beta - \phi)\right].
\end{aligned}$$

The potential energy is readily constructed in terms of the constrained generalized coordinates. The distance from the intersection O of the guide bars to the collar is $\bar{r}_{A/O} \cdot \bar{I}$. At $\phi = 0$, where the spring is undeformed, this distance is L. Thus the spring elongation is $\Delta = L - \bar{r}_{A/O} \cdot \bar{I}$. Let the X axis be the reference elevation for gravity potential energy. Then the elevation of the collars and of the center of mass are the vertical components of the respective position vectors given in Eqs. (1). These considerations lead to

$$V = \frac{1}{2}k\left[L(1-\cos\phi)+s\cos\beta\right]^2 + mg\left(3s\sin\beta - \frac{3}{2}L\sin\phi\right).$$

The Lagrange equations corresponding to these energy expressions and the generalized forces in Eq. (2) are

$$\begin{aligned}&\frac{4}{3}mL^2\ddot{\phi} - \frac{3}{2}mL\ddot{s}\sin(\beta-\phi) + kL[L(1-\cos\phi)+s\cos\beta]\sin\phi \\ &\qquad - \frac{3}{2}mgL\cos\phi = -N_A L\cos\phi + f_A L\sin\phi, \\ &3m\ddot{s} - \frac{3}{2}mL\ddot{\phi}\sin(\beta-\phi) + \frac{3}{2}mL\dot{\phi}^2\cos(\beta-\phi) + k\left[[L(1-\cos\phi)]\right. \\ &\qquad \left.+ s\cos\beta\right]\cos\beta + 3mg\sin\beta = N_A\sin\beta + f_A\cos\beta.\end{aligned} \tag{6} \triangleleft$$

The last step in the derivation is to characterize the friction force. The generalized forces were derived with positive N_A corresponding to the normal force acting upward. However, the constraint imposed by the guide bar is still effective if the collar is tending to move upward, which would lead to N_A being negative. We therefore describe the magnitude of the sliding friction force as $|f_A| = \mu|N_A|$. Furthermore, $\bar{f}_A$ was taken to be directed to the left, which corresponds to collar A moving to the right. Such a movement means that $\bar{v}_A \cdot \bar{I} > 0$, so we may say that $f_A = \mu|N_A|\text{sgn}(\bar{v}_A \cdot \bar{I})$. Because $\bar{v}_A \cdot \bar{I} = -\dot{s}\cos\beta - L\dot{\phi}\sin\phi$, substituting f_A into Eqs. (6) shows that we have three unknowns, ϕ, s, and N_A, that are governed by the two Lagrange differential equations and the velocity constraint equation.

These equations are highly nonlinear, so we solve them numerically. Even if dimensional system parameters were given, it is good practice to begin by nondimensionalizing the equations. Let $\xi = s/L$, and let nondimensional time be $\tau = t(g/L)^{1/2}$, so that each time derivative is replaced with a derivative with respect to τ according to $d/dt = (g/L)^{1/2}\,d/d\tau$. Changing the variables in this manner converts Lagrange equations (6), and velocity constraint equation (3) to

$$\begin{aligned}&\frac{4}{3}\ddot{\phi} - \frac{3}{2}\ddot{\xi}\sin(\beta-\phi) + \Omega^2[(1-\cos\phi)+\xi\cos\beta]\sin\phi - \frac{3}{2}\cos\phi \\ &\qquad = -\hat{N}\cos\phi + \mu\cos\beta\left|\hat{N}\right|\text{sgn}(v_A)\sin\phi, \\ &3\ddot{\xi} - \frac{3}{2}\ddot{\phi}\sin(\beta-\phi) + \frac{3}{2}\dot{\phi}^2\cos(\phi-\beta) + \Omega^2[(1-\cos\phi)+\xi\cos\beta]\cos\beta + 3\sin\beta \\ &\qquad = \hat{N}\sin\beta + \mu\left|\hat{N}\right|\text{sgn}(v_A)\cos\beta, \\ &\dot{\xi}\sin\beta - \dot{\phi}\cos\phi = 0,\end{aligned}$$

where an overdot now denotes differentiation with respect to τ, and

$$v_A = -\dot{\xi}\cos\beta - \dot{\phi}\sin\phi, \quad \Omega^2 = \frac{kL}{mg}, \quad \hat{N} = \frac{N_A}{mg}.$$

As suggested by Eq. (8.2.69), we handle the appearance of $|N|$ in the friction force by replacing it with $N\,\mathrm{sgn}(N)$. The resulting differential equations to solve may be written as

$$\begin{gathered}[M]\{\ddot{q}\} = \{F\} + \{H\}\,\hat{N},\\ -[a]\{\ddot{q}\} = [\dot{a}]\{\dot{q}\},\end{gathered} \tag{7}$$

where $\{q\}$ now is formed from ϕ and ξ, and

$$[M] = \begin{bmatrix} \dfrac{4}{3} & -\dfrac{3}{2}\sin(\beta - q_1) \\ -\dfrac{3}{2}\sin(\beta - q_1) & 3 \end{bmatrix},$$

$$\{F\} = \begin{Bmatrix} -\Omega^2[(1-\cos q_1) + q_2\cos\beta]\sin q_1 + \dfrac{3}{2}\cos q_1 \\ -\dfrac{3}{2}\dot{q}_1^2\cos(\beta - q_1) - \Omega^2[(1-\cos q_1) + q_2\cos\beta]\cos\beta - 3\sin\beta \end{Bmatrix}, \tag{8}$$

$$\{H\} = \begin{Bmatrix} -\cos q_1 + \mu\cos\beta\,\mathrm{sgn}\left(\hat{N}\right)\mathrm{sgn}(-\dot{q}_2\cos\beta - \dot{q}_1\sin q_1)\sin q_1 \\ \sin\beta + \mu\,\mathrm{sgn}\left(\hat{N}\right)\mathrm{sgn}(-\dot{q}_2\cos\beta - \dot{q}_1\sin q_1)\cos\beta \end{Bmatrix},$$

$$[a] = [-\cos q_1 \quad \sin\beta], \quad \left[\frac{da}{dt}\right] = [\dot{q}_1\sin q_1 \quad 0].$$

We employ the augmented method to solve these equations, which requires that the values of $\{\ddot{q}\}$ and $\hat{N}$ be evaluated at each instant. Toward that end we write Eqs. (7) as

$$\begin{bmatrix} [M] & -\{H\} \\ -[\dot{a}] & 0 \end{bmatrix} \begin{Bmatrix} \{\ddot{q}\} \\ \hat{N} \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ \left[\dfrac{da}{dt}\right]\{\dot{q}\} \end{Bmatrix}. \tag{9}$$

This is the standard form for the augmented algorithm, the third of Eqs. (8.2.11), except that the coefficient matrix is not symmetric. The state vector formed by stacking the current $\{\dot{q}\}$ value above the $\{\ddot{q}\}$ value obtained from Eq. (9) constitutes the input for the differential equation solver. Solving these equations linearly requires that $\{H\}$, which depends on $\mathrm{sgn}\left(\hat{N}\right)$, be evaluated before solving the equations. The results presented in the following discussion were computed by setting the sign of $\hat{N}$ at each time step to the sign of the result for $\hat{N}$ computed at the previous time step.

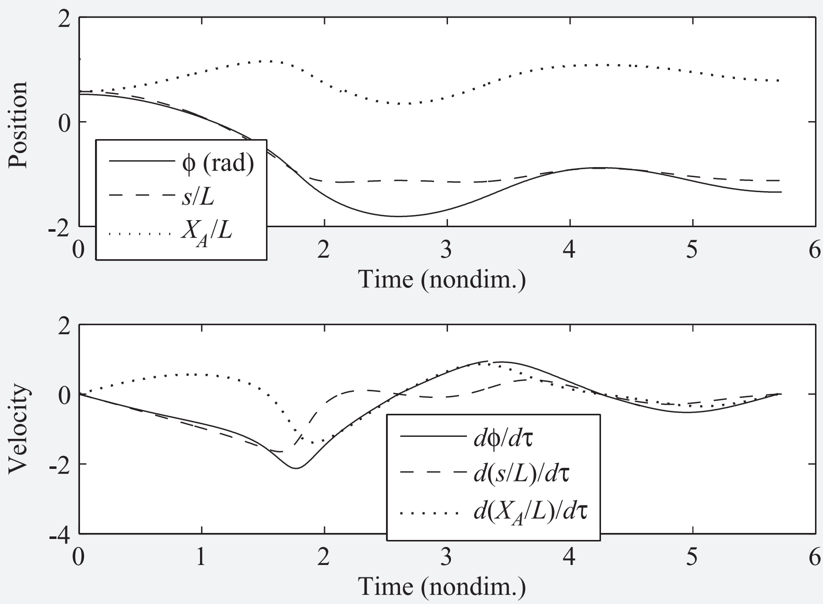
Starting the differential equation solver requires initial values of $\{q\}$ and $\{\dot{q}\}$, but this is a holonomic system, which means that only the initial values ϕ_0 and $\dot{\phi}_0$ may be defined freely. The constraint conditions then give

$$s_0 = \frac{L\sin\phi_0}{\sin\beta}, \quad \dot{s}_0 = \frac{L\dot{\phi}_0\cos\phi_0}{\sin\beta}.$$

The results presented here were obtained with the Matlab `ODE45` function, which uses Runge–Kutta integration. The time step was set at $\Delta\tau = 0.004$. Before we examine the results, it is appropriate to consider some general issues. Error checking should be done to verify that the configuration constraint equation actually is satisfied. This was done by computing at each time instant the nondimensional configuration constraint $\xi \sin\beta - \sin\phi$, which should be zero. The maximum value of this quantity was found to be less than $5\left(10^{-11}\right)$. Another check is to set $\mu = 0$ in the computer program, in which case the system is conservative. We can monitor the mechanical energy in a simulation with $\mu = 0$ to verify that $E = T + V$ is indeed constant. If we seek further verification, we can use the principle of conservation of energy when $\mu = 0$ to evaluate the angular velocity when $\phi = 0$, and compare that result with the computed response. In the same vein, we know that the sliding friction force always opposes the movement, so it must be that E decreases monotonically whenever $\mu > 0$.

The last consideration is the decision to halt computation. Properly done, whenever the velocity of collar A reverses sign, meaning that collar A has momentarily come to rest, we should launch a separate static analysis of the friction and normal forces required to hold the system in equilibrium at the current location. If $|f_{\text{static}}| < \mu_{\text{static}} |N_{\text{static}}|$, then there is adequate friction to sustain the equilibrium state, which would indicate that the simulation should be halted. A simpler alternative, which was employed here, is to step forward in time until it is observed that v_A becomes extremely small for several time steps. One weakness of this approach is that it does not account for the difference between the static and sliding friction coefficients, but it is easier to program.

The first set of graphs depicts the position and velocity variables. Of course, only one such variable is actually required for this one-degree-of-freedom system, whereas the others could be computed from the kinematical equations, but it is convenient to see them assembled.



Position and velocity as functions of time for the falling bar.

The motion ceased at $\tau = 5.696$, when the value of $v_A/(gL)^{1/2} = -\dot{\phi}\sin\phi - \dot{\xi}\sin\beta$ reversed sign and remained below 10^{-5}. At that time the bar was at $\phi = -77.035^{\circ}$. The graphed velocity shows that this was the third instant after release at which collar A came to rest. ◁

The next set of graphs displays the value of $\hat{N}$ obtained from the function that solves Eq. (9) when called by the differential equation solver. The graph also displays the mechanical energy E computed from the generalized coordinates and velocities at each instant. The nondimensional form of $E = T + V$ is scaled by mgL, such that

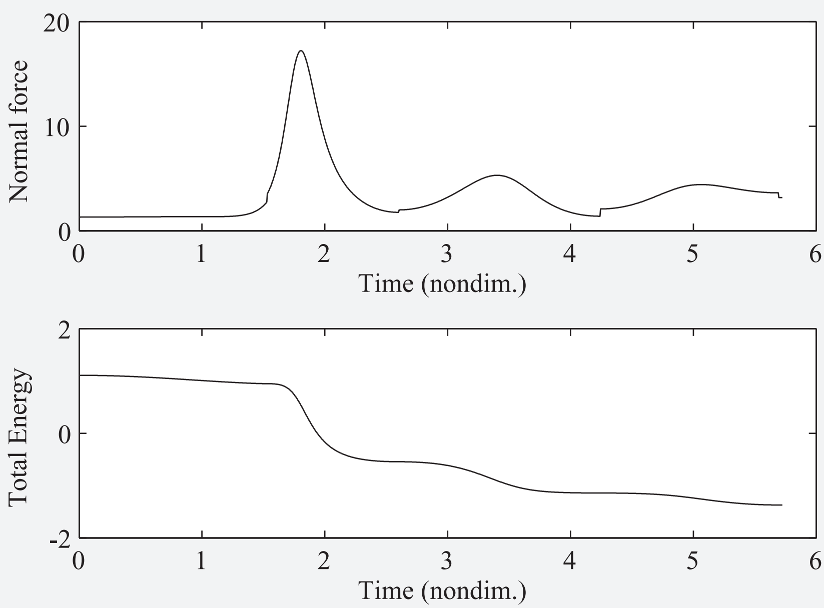
$$\frac{T}{mgL} = \frac{1}{2}\left[\frac{4}{3}\dot{\phi}^2 + 3\dot{\xi}^2 + 3\dot{\phi}\dot{\xi}\sin(\phi - \beta)\right],$$

$$\frac{V}{mgL} = \frac{1}{2}\Omega^2\left[(1 - \cos\phi) + \xi\cos\beta\right]^2 + \left(\xi\sin\beta - \frac{1}{2}\sin\phi\right).$$

The energy that is dissipated by friction is the difference between the initial and final values of E, which leads to

$$\Delta E = (T + V)\big|_{\tau=0} - (T + V)\big|_{\tau=5.696} = 2.478mgL. \quad ◁$$

The graph of the normal force is interesting for the fact that it shows small discontinuities in $\hat{N}$. The instants for this occurrence are those at which v_A changes sign, causing the friction force to reverse.



Time dependence of the normal force and total mechanical energy as the bar descends.

In a strict sense, the present derivation of equations of motion cannot be compared with the solution of Example 6.9 because the present situation featured a spring, and the collars had mass, whereas friction at both collars was important for the previous case. Furthermore, the previous analysis dealt with a situation in which collar A had a constant velocity. Nevertheless, we can see that the kinematical aspects of the current solution were simpler by virtue of its use of unconstrained generalized coordinates. In fact, if it were necessary to account for friction at both collars, as was done previously, we would use three constrained generalized coordinates, which would need to satisfy constraint conditions associated with both collars. Another noteworthy aspect is that accounting for each collar's mass entailed little additional effort for the Lagrange equation formulation, whereas the previous approach would require that Newton's Second Law be applied to each collar to account for the internal forces exerted between each collar and the bar.

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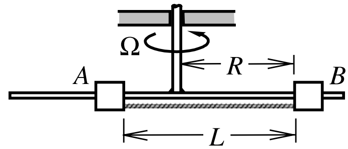
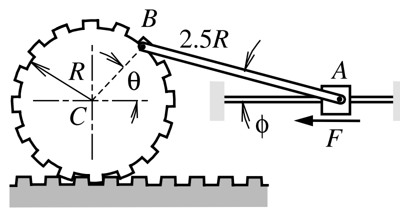
HOMEWORK PROBLEMS



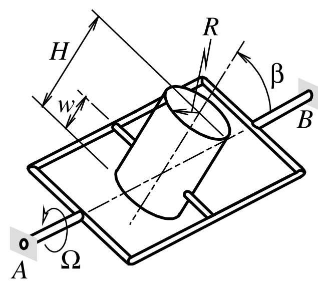
Exercise 8.1

EXERCISE 8.1 A torque Γ acting about the vertical shaft is such that the precession rate is $\Omega = \Omega_0 \sin(\beta t)$. The sliders, whose masses are m_A and m_B, are tied together by an inextensible cable. The moment of inertia of the T-bar about the precession axis is I_p. Derive the equation of motion for the radial distance R assuming that frictional resistance is negligible, and also obtain an expression for Γ.



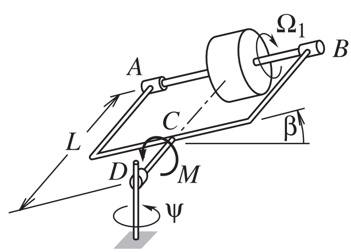
Exercise 8.2

EXERCISE 8.2 Force $\bar{F}(t)$ pushes collar A, whose mass is m, to the left. This causes the gear to roll over the horizontal rack. The mass of the gear is $2m$, and its radius of gyration about center point C is κ; the mass of bar AB is m. Use the angles θ and ϕ as constrained generalized coordinates to derive the equations of motion for the system.



Exercise 8.3

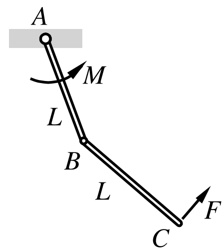
EXERCISE 8.3 The cylinder of mass m is free to rotate by angle β relative to the gimbal, which rotates about the horizontal axis. The precessional rate Ω is held constant by varying an unknown torque Γ that acts about the horizontal axis AB of the gimbal. Use Lagrange's equations to derive the equation of motion governing β, as well as an expression for Γ.



Exercise 8.4

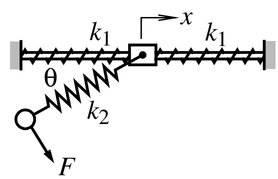
EXERCISE 8.4 A torque Γ that is a known function of time acts about the vertical shaft, whereas M is a servotorque that controls the gimbal's precession, such that $\dot{\beta} = c_1\dot{\psi} + c_2\beta + c_3$, where the c_n are constants. The spin rate Ω_1 is maintained at a constant value by a servomotor. The mass of this motor and the gimbal are negligible. The mass of the flywheel is m, and its principal radii of gyration for centroidal axes are κ_1 about its spin axis and κ_2 normal to that axis. Derive Lagrange equations of motion, then assemble a set of state-space equations suitable for implementing a computer solution.

EXERCISE 8.5 A known couple $M(t)$ is applied to the upper bar. Force F, which is applied perpendicularly to the lower bar, acts to make the velocity of end C always be parallel to the line from joint A to end B. The bars have equal mass m, and the system lies in the vertical plane. Use the method of Lagrange multipliers to derive the equations of motion.



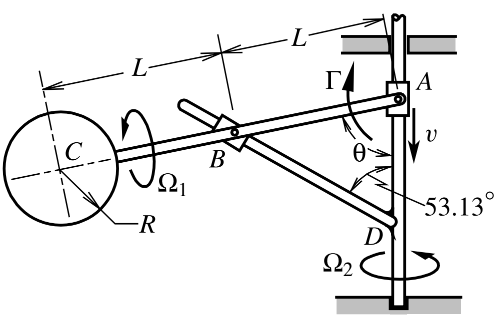
Exercise 8.5

EXERCISE 8.6 The force $\bar{F}$ applied to the small suspended sphere m_2 acts perpendicularly to spring k_2, which represents an elastic cable. This force is such that the velocity of the sphere is always parallel to the cable. Springs k_1 are compressed by the same amount at the position where the displacement x of the collar is zero. The mass of the collar is m_1. Friction between the collar and the horizontal guide bar is negligible. Derive the differential equations governing the motion of this system.



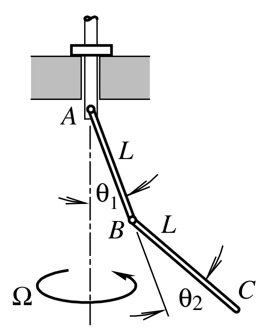
Exercise 8.6

EXERCISE 8.7 Collars A and B are pinned to the bar that is welded to sphere C. A servomotor acting about the pin in collar A applies torque Γ to bar AC with the result that collar A moves downward at a constant specified speed v. The system rotates freely about the vertical axis, so the precession rate Ω_2 is unknown, but the spin rate Ω_1 is held at a constant value by another servomotor. The mass m of the sphere is sufficiently large to neglect the inertia of the other parts. Derive a set of equations whose solution would give Ω_2 as a function of time and the corresponding value of Γ.



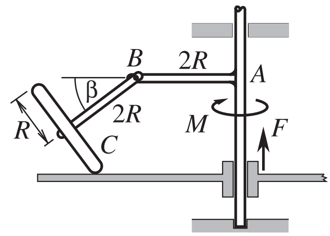
Exercise 8.7

EXERCISE 8.8 A servoactuator at joint B applies a torque that maintains $\theta_2 = 1.5\theta_1$. The rotation rate Ω is held constant by a torque applied to the vertical shaft. Derive a set of equations of motion whose solution would yield θ_1, as well as the controlling torques applied to the vertical shaft and between the bars at pin B.



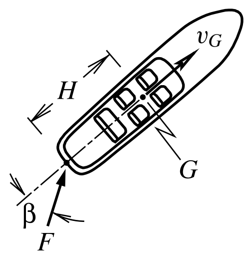
Exercise 8.8

EXERCISE 8.9 The horizontal platform translates upward, and the T-bar rotates about the vertical axis. Both the force $\bar{F}$ and torque $\bar{M}$ are known functions of time. The rolling motion of disk C, which spins freely about shaft BC, is such that there is no slippage in the direction perpendicular to the sketch, but there is slippage in the radial direction. Derive the Lagrange equation(s) of motion governing this system in the situation where only the disk's mass is significant.



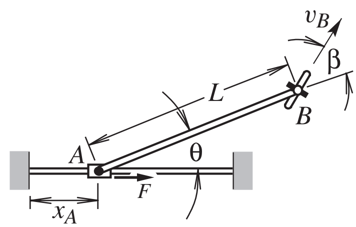
Exercise 8.9

EXERCISE 8.10 The thrust of an outboard motor on a boat may be represented as a force $\bar{F}$ acting at an angle β relative to the axis of the boat. The hydrodynamic properties of the boat are such that the velocity of the center of mass G is constrained to be parallel to the longitudinal axis of the boat. The component of the hydrodynamic force parallel to the axis of the boat is the drag f_d. Derive the equations of motion for the boat by using Lagrange multipliers. The mass of the boat is m, and its centroidal moment of inertia is I.



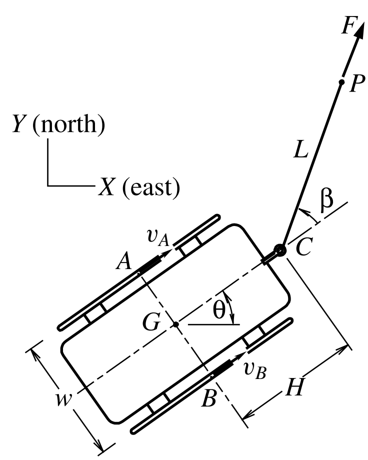
Exercise 8.10

EXERCISE 8.11 The diagram depicts a linkage that is situated in the horizontal plane. Collar A slides along the stationary guide bar, and the steerable wheel B at the other end of rigid bar AB rolls over the ground. A servomotor controls the steering angle β. There is no slippage in the rolling motion, so the velocity of end B must be in the sense shown in the diagram. Movement is actuated by the known force $F(t)$ applied to the collar. Generalized coordinates are the horizontal position x_A and the rotation angle θ. Use the method of Lagrange multipliers to derive the equations of motion governing these variables.



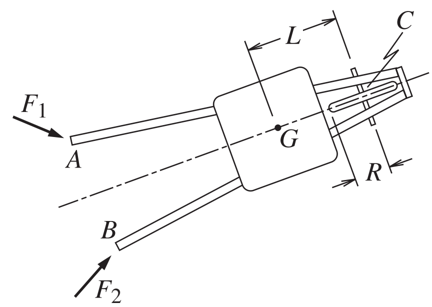
Exercise 8.11

EXERCISE 8.12 The sketch shows the top view of a sled that is towed over the ice by fastening a cable to hook C. Force F is the tensile force applied to the tow cable at its free end P; its magnitude and direction angle β relative to the sled's center line are known functions of time. The edge contact constrains the velocity of points A and B to be parallel to the rails, but the speed of each point may be different. Generalized coordinates have been selected as the east and north coordinates of the center of mass G and the heading angle θ. The radius of gyration about the vertical centroidal axis is κ_G. Use Lagrange's equations to derive the equations governing the motion for this system.



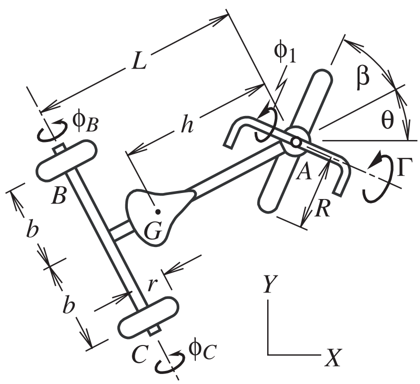
Exercise 8.12

EXERCISE 8.13 The wheelbarrow is pushed in the horizontal plane by forces $\bar{F}_1$ and $\bar{F}_2$ acting at the ends of the handles. The chassis has mass m, with its center of mass situated at point G on the centerline. The centroidal moment of inertia of the chassis about a vertical axis is I. The wheel, which may be approximated as a thin disk, rolls without slipping. Derive the Lagrange equations of motion.



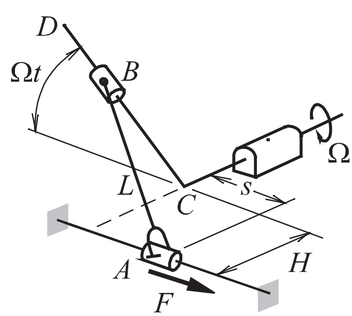
Exercise 8.13

EXERCISE 8.14 Beth applies torque Γ about the front axle, thereby causing the front wheel to spin by angle ϕ_1. She also applies a torque to the handlebars, with the result that $\beta(t)$ is is a known time function. Beth may be considered to be immobile relative to the chassis, so she and the chassis represent a rigid body having mass m_c, center of mass G, and moment of inertia I_c about a vertical axis through point G. All wheels may be approximated as thin disks with mass m_1 and radius R for the front wheel and mass m_2 and radius r for those in the rear, and they may be assumed to roll without slipping. Generalized coordinates are the coordinates X_A and Y_A of the steering fork, the chassis angle θ, and the spin angles ϕ_1, ϕ_2, and ϕ_3 of the wheels. Derive the corresponding differential equations of motion.



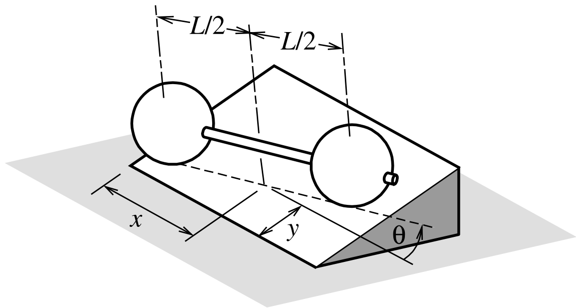
Exercise 8.14

EXERCISE 8.15 Force F is applied to collar A, and arm CD rotates at the constant rate Ω. Each collar has mass m, and the mass of the connecting bar AB is $2m$. Friction is negligible, as is the effect of gravity. Constrained generalized coordinates have been selected to be the horizontal position s of collar A, the precession of bar AB relative to the horizontal guide, and the angle between bar AB and the horizontal guide. Derive the corresponding Lagrange equations of motion.



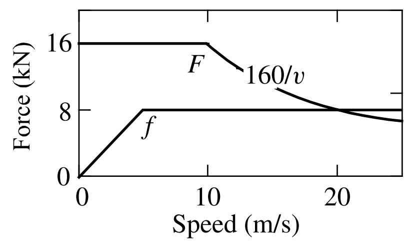
Exercise 8.15

EXERCISE 8.16 The distance L between the spheres remains constant as the spheres spin freely about the connecting shaft. The spheres are identical, with mass m and radius R, and the mass of the shaft also is m. Derive the Lagrange equations of motion governing the x, y coordinates of the center of the shaft and the inclination angle θ in the case in which the spheres roll without slippage over the inclined surface.



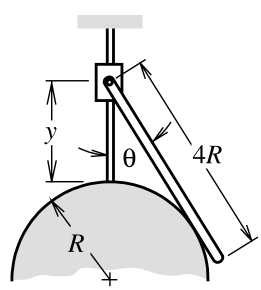
Exercise 8.16

EXERCISE 8.17 Consider the motorboat in Exercise 8.10. The magnitude of the drag force depends on the speed of the center of mass. In addition, when the motor is set to full throttle, the thrust is speed dependent because of cavitation at the propeller. Thus, if $\bar{v}_m$ and $\bar{v}_G$ are the instantaneous velocities at the motor and the center of mass, then the respective forces are described by the functions $f\,(|\bar{v}_G|)$ and $F\,(u)$ described in the accompanying graphs, where $u \equiv \bar{v}_m \cdot \left[\cos(\theta+\beta)\,\bar{I} + \sin(\theta+\beta)\,\bar{J}\right]$. The boat's mass is 4000 kg, and its centroidal radius of gyration about the vertical axis is 4 m. Also, the distance $H = 6$ m. Suppose the boat is at rest pointing northward when the motor is suddenly moved to full throttle and held there. The steering angle is set and held at 1° for 30 s, then reversed to −4° for the next 30 s. Use numerical methods to determine the path of the center of mass during this 1-min interval. Also, determine the speed and heading of the boat at $t = 60$ s.



Exercise 8.17

EXERCISE 8.18 The collar and bar have equal mass m, and the effect of friction is negligible. The system is released from rest at $\theta = 90°$. Determine the inclination angle θ as a function of t for the case where $(g/R)^{1/2} = 1.6$ rad/s by following alternative formulations: (a) θ is a single unconstrained generalized coordinate, (b) θ and y are constrained generalized coordinates.



Exercise 8.18

EXERCISE 8.19 The sled in Exercise 8.12 is at rest pointed eastward $(\theta = 0)$, with its hook C at $X = -L$, $Y = 0$. At $t = 0$, Tundra, the great Alaskan Malamute, begins to pull it by applying a constant tensile force of 300 N as she walks northward along the Y axis. Parameters for the system are $m = 500$ kg, $\kappa_G = 600$ mm, $H = 1.5$ m, $w = 1$ m, and $L = 10$ m. Solve the equations of motion to determine how the position coordinates of point C and the heading angle θ vary as a function of time. How long does it take for the heading angle to be $\theta = 89°$?

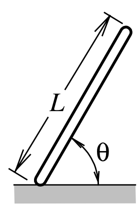
EXERCISE 8.20 The linkage in Exercise 8.2 was at rest at $t = 0$ with bar AB inclined at $\phi = 10°$. The actuating force is $F = 50\sin(4t)$ N. The system parameters are $m = 800$ g, $R = 200$ mm, and $\kappa = 150$ mm. Use numerical methods to solve the state-space equations governing the constrained generalized coordinates ϕ and θ. What is the largest value of θ attained in the ensuing motion?

EXERCISE 8.21 The torque in Exercise 8.5 is constant at 60 N-m. The system is at rest at $t = 0$ with bar AB horizontal to the right, and bar BC vertically downward. Determine the angle of inclination of each bar. Use those results to graph the path followed by end C. The mass of each bar is $m = 4$ kg, and $L = 500$ mm.

EXERCISE 8.22 The force F in Exercise 8.11 is constant at 50 N. The masses are 2 kg for bar AB and 0.5 kg for collar A, the mass of the wheel is negligible, and $L = 800$ mm. At $t = 0$, the system was at rest with $\theta = 0$. The steering angle is $\beta = 0.5\pi\,[2\exp(-2t) - 1]$ rad, where t is in units of seconds. Determine x_A and θ as functions of time. What are the extreme values of θ attained in the motion?

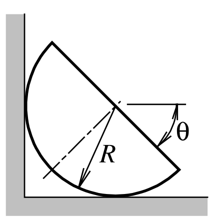
EXERCISE 8.23 A sphere of radius R rolls without slipping over a horizontal turntable that rotates about its center at constant angular speed Ω. Two generalized coordinates are selected to be the radial distance r from the turntable axis to the center of the sphere and the angle ψ of that radial line. Rotation of the sphere about its vertical diameter does not affect the velocity of the center relative to the turntable, so the nutation and spin angles of the sphere suffice as the other generalized coordinates. Derive the equations of motion for this system. Then solve them numerically for the case where the initial conditions are $r = 3R$, $\dot{r} = \psi = 0$, $\dot{\psi} = \Omega$ at $t = 0$. *Hint*: Nondimensionalize the equations of motion using R as the length scale and $1/\Omega$ as the time scale.

EXERCISE 8.24 The bar is slipping relative to the ground as it falls. The coefficient of kinetic friction is μ. Use Lagrange's equations to derive the equations of motion for the bar.



Exercise 8.24

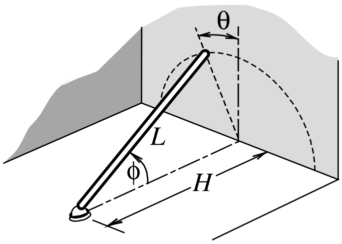
EXERCISE 8.25 The semicylinder, whose mass is m, is released from rest at an initial orientation $\theta > 0$. The coefficient of kinetic friction μ between the cylinder and the ground is not adequate to prevent sliding, and friction with the wall is negligible. Use Lagrange's equations to derive the equations of motion for the bar.



Exercises 8.25 and 8.26

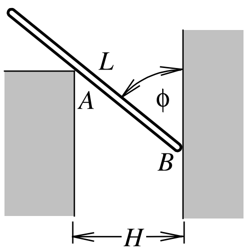
EXERCISE 8.26 The semicylinder, whose mass is m, is released from rest at an initial orientation $\theta > 0$. The coefficient of kinetic friction μ between the cylinder and both surfaces it is in contact with is μ. The system is released from rest at $\theta = 90^\circ$, and the value of μ is not adequate to prevent sliding. Derive Lagrange's equations of motion.

EXERCISE 8.27 The bar is supported by a ball-and-socket joint A. The coefficient of sliding friction between end B of the bar and the wall is μ. Use Lagrange's equations to derive the equation of motion governing the angle of inclination θ.



Exercise 8.27

EXERCISE 8.28 The coefficient of sliding friction between the bar and corner A is μ, but friction between end B of the bar and the vertical wall is negligible. Derive the Lagrange equations of motion for this system.



Exercise 8.28

EXERCISE 8.29 The bar in 8.24 is released from rest at $\theta = 75^\circ$. The coefficient of sliding friction is 0.1, and $(g/L)^{1/2} = 0.5$ rad/s. Determine the time required for the bar to fall to the ground, and compare that time with the case in which friction is negligible.

EXERCISE 8.30 In Exercise 8.18 the coefficient of sliding friction between the bar and the cylinder is μ, but friction between the collar and the vertical post is negligible. (a) Derive the differential equations of motion. (b) Consider the case in which the coefficient of friction is $\mu = 0.1$ and $(g/R)^{1/2} = 1.6$ rad/s. The system is released from rest at $\theta = 20^\circ$. Solve the Lagrange equations of motion to determine the inclination angle θ as a function of the elapsed time t.

EXERCISE 8.31 In Exercise 8.28 the coefficient of sliding friction between the bar and both surfaces it is in contact with is $\mu = 0.1$. (a) Derive the Lagrange equations of motion. (b) Consider the situation in which the bar is released from rest at $\phi = 85^\circ$. The dimensions are $L = 600$ mm and $H = 500$ mm. Determine the angle of elevation as a function of time for the interval during which it remains in contact with both surfaces. Does the bar lose contact with the wall or with the corner?

CHAPTER 9

Alternative Formulations

The preceding developments suffice to treat systems that are described by a finite number of degrees of freedom. They are not directly applicable if a system is best modeled as a flexible continuum, in which bodies deform and also have mass. One cannot compartmentalize kinetic and potential energy in such systems, because a mass element also stores strain energy. Consequently, concepts like generalized coordinates become problematic. The derivation of principles that can be used to model continuous media is the first priority for this chapter.

Another focus here is exploration of alternative formulations for deriving equations of motion for discrete systems. Derivation of these formulations has received considerable attention for more than a century and a half. Those efforts were motivated by a desire to seek simpler equation forms, either from the perspective of ease of implementation or ease of solution. We consider a few formulations, but extensive discussions may be found in the works by Greenwood (2003) or Papastavridis (1998, 2002).

One of the outcomes of these alternative formulations are conservation principles that sometimes can be used when the standard momentum and energy principles cannot be implemented. Such principles enable us to determine features of a system's response without solving equations of motion and also provide checks for computation solutions. Overall, the developments that follow are intended to enhance understanding of the basic concepts of analytical mechanics and to provide increased versatility to carry an analysis to completion.

9.1 HAMILTON'S PRINCIPLE

Lagrange's equations are restricted to systems composed of rigid bodies that have mass, and therefore store kinetic energy, whereas potential energy is stored in massless springs and other sources of conservative forces. This is an idealization, as is evident when one considers that no material can sustain stresses without deforming, and all springs have mass. Any model of a system needs to be tested by examination of the assumptions on which the model is founded. This leads to the need to consider models in which the relative position of mass points is not kinematically constrained. Each body's displacement then is a function of location within the body. This function is the *displacement field*. It represents the spatial distribution of displacement, which causes some individuals to refer to such models as *distributed parameter systems.*

One approach for deriving equations of motion of continua relies on the concepts in mechanics of materials. Specifically, a differential element of the system is isolated in a free-body diagram, and the internal stress distribution is described by stress-strain and strain-displacement relations. The primary difference from the static case is the usage of Newton–Euler equations of motion, rather than the laws of statics, to describe the balance of forces acting on the isolated mass element. Such an approach works well when applied to simple systems, such as a vibrating elastic bar, but it becomes cumbersome and prone to error in more complicated situations. Fortunately, there is an alternative formulation that relies on *Hamilton's Principle*, which was presented by Sir William Rowan Hamilton in 1834. From one perspective, the relationship of continuum formulations using Hamilton's Principle and mechanics of materials approaches is analogous to the relationship of Lagrange's equations and the Newton–Euler equations for discrete systems. However, the remarkable aspect of Hamilton's Principle is that it is far more general. It provides an alternative derivation of Lagrange's equations, yet, with suitable adjustments in the definition of energy, it can be applied to relativistic systems. It also provides the basis for many approximation techniques, including finite element analysis, that are used to derive discrete models of continua.

9.1.1 Derivation

The steps leading to Hamilton's Principle parallel the derivation of Lagrange's equations in Chapter 7, with the important difference that generalized coordinates are not used to describe position. We begin with a single particle. The principle of dynamic virtual work states that

$$\left(\Sigma \bar{F} - m\bar{a}\right) \cdot \delta\bar{r} = \bar{0}. \tag{9.1.1}$$

The virtual work done by the actual forces is δW, so we have

$$\delta W - m\bar{a} \cdot \delta\bar{r} = 0. \tag{9.1.2}$$

The explicit occurrence of the acceleration may be removed by introducing the rule for differentiating a product, which leads to

$$\delta W - \frac{d}{dt}\left(m\bar{v} \cdot \delta\bar{r}\right) + m\bar{v} \cdot \frac{d}{dt}\left(\delta\bar{r}\right) = 0. \tag{9.1.3}$$

Interchanging the virtual increment and the time derivative in the last term enables us to rewrite the preceding as

$$\delta W - \frac{d}{dt}\left(m\bar{v} \cdot \delta\bar{r}\right) + m\bar{v} \cdot \delta\bar{v} = 0. \tag{9.1.4}$$

Aside from holding t constant, the rules for a virtual increment are like those for a differential, so that

$$m\bar{v} \cdot \delta\bar{v} = \delta\left(\frac{1}{2}m\bar{v} \cdot \bar{v}\right) \equiv \delta T, \tag{9.1.5}$$

where T is the kinetic energy of the particle. It follows that

$$\delta T + \delta W - \frac{d}{dt}\left(m\bar{v}\cdot\delta\bar{r}\right) = 0. \tag{9.1.6}$$

We now extend this relation to treat any system. All systems are composed of a collection of particles,* and Eq. (9.1.6) is descriptive of any particle. Let index n denote which particle it represents. Because the preceding is a scalar equation, each term becomes a simple sum when we add the equations for all particles. This leads to redefinition of T as the kinetic energy of all particles in the system and δW as the virtual work done by all forces. The latter was decomposed in Eq. (7.4.34), which means that the virtual work done by the conservative forces is the negative of the virtual change in the potential energy. The symbol δW shall henceforth represent the virtual work done by all forces that have not been described by a potential-energy function. In that case, addition of Eq. (9.1.6) for each particle in the system yields

$$\delta T - \delta V + \delta W - \sum_n \frac{d}{dt}\left(m_n\bar{v}_n\cdot\delta\bar{r}_n\right) = 0. \tag{9.1.7}$$

The first two quantities are the increments of the kinetic and potential energies if the system were to follow the variational path, rather than the true one associated with the actual forces. The third is the virtual work done by those forces. Each quantity is evaluated for points on the true and variational points corresponding to the same time instant. The fourth term has no significance relative to the standard kinetics principles, although we recognize $m_n\bar{v}_n$ as a particle's momentum. To handle this term, we observe that it is differentiated with respect to time. Integrating Eq. (9.1.7) over time enables us to consider this problematic term solely at the limits of the integration. These limits are the initial time t_0 at which we have initial conditions and an arbitrary subsequent time t_1. Thus we have

$$\int_{t_0}^{t_1}\left(\delta T - \delta V + \delta W\right)dt - \sum_n \left(m_n\bar{v}_n\cdot\delta\bar{r}_n\right)\Big|_{t=t_0}^{t=t_1} = 0. \tag{9.1.8}$$

This relation is best understood by considering the motion of the system through the configuration space. We consider generalized coordinates for the system to be the set of position coordinates for all particles contained in that system. Then the configuration space for a system of N particles is the $3N$-dimensional Cartesian space whose coordinates are the set of position coordinates. (Letting the dimensionality be infinite for a continuous distribution of mass does not lessen the validity of the discussion.) Figure 9.1 shows the true path and a variational path. Recall that the concept of a variational path is that it represents a different evolution of a system's state, corresponding to a set of forces that is altered from the true one. However, regardless of what forces are applied, the system must start from the specified initial state, which we take to occur at t_0. In addition, we wish to arrive at the true state of the system at the instant corresponding

* The rigid-body and continuum models are mathematical abstractions, which are included in the discussion by considering the particles to be differential elements of mass. A summation accounting for each particle then becomes an integration over the domain.

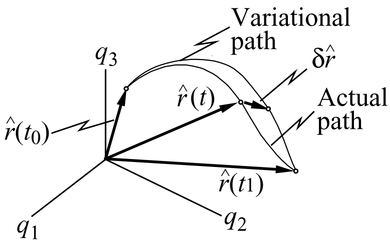


Figure 9.1. Actual and variational paths in the configuration space starting from a specified initial position and arriving at the same ultimate state.

to the end of the variational path. Thus the true and variational paths should intersect at both instants, as shown in Fig. 9.1, so that $\delta\hat{r} = \hat{0}$ at t_0 and t_1. Correspondingly, we set $\delta\bar{r}_n = \bar{0}$ at these instants. Under these conditions, Eq. (9.1.8) reduces to *Hamilton's Principle*:

$$\boxed{\int_{t_0}^{t_1} (\delta T - \delta V + \delta W)\, dt = 0.} \tag{9.1.9}$$

According to this relation the true path is distinguishable from all possible variational paths by the fact that the time dependence of the generalized coordinates and forces yields a zero mean value for $\delta T - \delta V + \delta W$. Introducing the Lagrangian $\mathcal{L} = T - V$ gives a more compact form of Hamilton's Principle:

$$\int_{t_0}^{t_1} (\delta\mathcal{L} + \delta W)\, dt = 0. \tag{9.1.10}$$

Note that δT and δV represent the difference between quantities associated with the variational and true paths at an arbitrary instant. In contrast, there is no "work" quantity W from which the virtual work of a nonconservative force may be derived. Suppose we consider the restricted class of systems for which $\delta W = 0$. Clearly, this can be the case only if a system is conservative, but it also is necessary that the system be holonomic. Otherwise, the constraint forces will do work when the generalized coordinates are given arbitrary virtual increments. In this special case of a conservative holonomic system, we find that, among all variational paths connecting the initial and final position, the true one is the one for which the *action integral* $\mathcal{I}$ has a stationary value, which is stated as

$$\mathcal{I} = \int_{t_0}^{t_1} (T - V)\, dt, \quad \delta\mathcal{I} = 0. \tag{9.1.11}$$

In other words, the motion of a conservative holonomic system evolves in a manner that gives a stationary value of the action integral (maximum, minimum, or inflection point).

It is logical at this juncture to question the significance of these results, as Hamilton's Principle seems to represent only one relation. For example, we know that the work–energy principle $\Delta T + \Delta V = W_{1\to 2}$ is not adequate by itself to solve problems involving several generalized coordinates. The difference is that Eq. (9.1.9) leads to many relations, because the virtual movement is arbitrary except at the initial and final instants. An infinite number of variational curves can be constructed, and Hamilton's Principle must be satisfied for each. If the system consists of particles and rigid bodies, we can construct different variational paths by holding all generalized coordinates except one constant, and imparting a virtual increment to the remaining generalized coordinate that is arbitrary, other than being zero at instants t_0 and t_1.

Hamilton's Principle can be used in a variety of ways. One application is based on the recognition that the various kinetics principles that have been formulated in this text rely on a set of axioms, specifically Newton's Laws. We could instead take Hamilton's Principle to be the fundamental axiom on which other principles are founded. Indeed, as we will soon see, Lagrange's equations are readily derived from Hamilton's Principle. If one is merely concerned with classical mechanics, this capability might seem to be unimportant. However, generalizing the definition of kinetic energy to include relativistic effects would enable us to greatly expand the scope of our investigations.

In the present context Hamilton's Principle was introduced to derive equations of motion for continua. The mathematical tool by which those equations can be obtained is the calculus of variations. This entails treating the displacement of each particle in the system as a function of its starting position $\bar{r}$ and time, denoted as $\bar{u}(\bar{r}, t)$. The concept is to evaluate the increment of the kinetic and potential energies associated with a virtual increment of the displacement field $\delta\bar{u}$, and then use the calculus to replace dependences on derivatives of $\delta\bar{u}$ with explicit dependence on $\delta\bar{u}$. Arbitrariness of the virtual displacement field will lead to the governing equations. This application is addressed in Subsection 9.1.2.

Hamilton's Principle can also be used as the foundation for approximate solutions of the response of continuous media. The notion here is to introduce a simplified representation of the displacement field, for example, a series expansion. This leads to energy expressions corresponding to the coefficients of this series. Satisfaction of Hamilton's Principle yields the best possible equations governing these coefficients. This is the essence of the Ritz series method, which is introduced in a subsequent section.

To illustrate the use of Hamilton's Principle, let us use it to derive Lagrange's equations. Independently of the previous derivation, it is a simple matter to identify that, if position is uniquely defined by a set of generalized coordinates q_j, then the kinetic energy is a function of the q_j, $\dot{q}_j$, and t, whereas the potential energy is a function of the q_j and t, that is, $T(q_j, \dot{q}_j, t)$ and $V(q_j, t)$. The definitions of δT and δV are that they are the change of the respective values when the generalized coordinates and velocities are given infinitesimal increments at a specified instant, so we have

$$\begin{aligned} \delta T &= T(q_j + \delta q_j, \dot{q}_j + \delta\dot{q}_j, t) - T(q_j, \dot{q}_j, t), \\ \delta V &= V(q_j + \delta q_j, t) - T(q_j, t). \end{aligned} \tag{9.1.12}$$

Because the δq_j and $\delta\dot{q}_j$ values are infinitesimal, we may use the rules of differential calculus, specifically, the chain rule for differentiation, to describe the difference. This leads to

$$\delta T = \sum_{j=1}^{N} \left(\frac{\partial T}{\partial \dot{q}_j}\delta\dot{q}_j + \frac{\partial T}{\partial q_j}\delta q_j \right), \quad \delta V = \sum_{j=1}^{N} \frac{\partial V}{\partial q_j}\delta q_j. \tag{9.1.13}$$

To apply Hamilton's Principle we consider the δq_j values to be a selected set of time functions, because we need to apply the variation at every instant of time along the configuration path. The values of the $\delta\dot{q}_j$ are defined by the choice for the $\delta q_j(t)$ functions, because

$$\delta\dot{q}_j \equiv \frac{d}{dt}\delta q_j. \tag{9.1.14}$$

Accordingly, we account for this dependence by manipulating the first term in Eq. (9.1.13):

$$\frac{\partial T}{\partial \dot{q}_j}\delta\dot{q}_j = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\delta q_j\right) - \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\right]\delta q_j. \tag{9.1.15}$$

The next step is to substitute into Hamilton's Principle this expression, Eq. (9.1.13) for δV, and the standard description of δW in terms of generalized forces. The result is

$$\int_{t_0}^{t_1}\sum_{j=1}^{N}\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\delta q_j\right) + \left[-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j}\delta q_j + Q_j\right]\delta q_j\right\}dt = 0. \tag{9.1.16}$$

The first term in the integrand is a perfect differential; its integration yields

$$\int_{t_0}^{t_1}\sum_{j=1}^{N}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\delta q_j\right)dt = \sum_{j=1}^{N}\left(\frac{\partial T}{\partial \dot{q}_j}\delta q_j\right)\Bigg|_{t_0}^{t_1}. \tag{9.1.17}$$

In the derivation of Hamilton's principle we asserted that the variational path must originate and end at the true points in the configuration space. This means that $\delta\bar{r} = \hat{0}$ at t_0 and t_1, so it must be that all of the $\delta q_j = 0$ at those instants. Consequently, the preceding term vanishes, which reduces Eq. (9.1.16) to

$$\int_{t_0}^{t_1}\left\{\sum_{j=1}^{N}\left[-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j}\delta q_j + Q_j\right]\delta q_j\right\}dt = 0. \tag{9.1.18}$$

Hamilton's Principle must apply for any variational path, which means that the δq_j may be arbitrary time functions, subject only to the condition that $\delta q_j(t_0) = \delta q_j(t_1) = 0$. For a specific set of δq_j functions, the integral could vanish by merely having the negative values over portions of the time interval cancel the positive values over the remaining time portions. However, the same result must be obtained for any set of δq_j functions, whose value at any instant may be positive or negative. This leads to the *fundamental lemma of the calculus of variations:*

> If an integral over any dependent variable ξ is such that $\mathcal{I} = \int f(\xi)\, g(\xi)\, d\xi$, and $\mathcal{I}$ must be zero for arbitrary $g(\xi)$, then it must be that $f(\xi) = 0$.

Application of this reasoning to Eq. (9.1.18) leads to the conclusion that the factor of δq_j must be zero for each value of j, which is the same relation as the basic form of Lagrange's equations.

9.1.2 Calculus of Variations

The motion of continua is characterized by displacements that are functions of the initial position of the moving point, as well as t. One advantage of using the calculus of variations to derive field equations is that it will definitively identify what boundary conditions are appropriate. Furthermore, although the procedure is more intricate in its mathematical steps, it requires less physical insight. This expedites the derivation of field equations for complicated continuum models. The calculus of variations is a mathematical procedure that exists independently of Hamilton's Principle. It can be applied to a variety of other areas. For example, one can use it address a number of problems in geometry, such as the identification of a geodesic curve, which is the shortest path connecting two points on a specified curved surface. Weinstock (1974) offers a fundamental treatise on this important concept.

In the most general situation, the displacement of any continuum is defined by a vector function of position and time. For simplicity, we restrict the development to situations in which the displacement of any point may be described in terms of a scalar function $u(x,t)$, where x is a spatial coordinate spanning $0 \le x \le L$. Many systems of engineering significance fit this specification, including vibrating cables and beams. In a continuum model, particles are replaced with differential mass elements, so the energy expressions will consist of integrals over all mass elements. To understand how the calculus of variations is implemented, we must recognize that the kinetic energy will depend on the time derivative of the displacement, and the potential energy will depend on the material strain, which in turn depends on spatial gradients of displacement. Thus let us consider the Lagrangian $\mathcal{L}$, which is the primary part of the integrand in Hamilton's principle, to depend on $\dot{u}$, u', u'', and $\dot{u}'$, where an overdot and prime respectively denote temporal and spatial partial differentiation. (This dependence is sufficiently general to enable us to identify the appropriate manner in which to proceed if higher derivatives of u or several displacement components appear in $\mathcal{L}$.) A convenient notation is to use brackets to indicate the dependent variables that appear in the integrand of $\mathcal{L}$, so we are considering $\mathcal{L}\left[\dot{u},\ u', u'', \dot{u}'\right]$. Mathematically, the use of square brackets is intended to convey the fact that $\mathcal{L}$ is a functional of the bracketed variables, meaning that a variety of operations must be performed on these variables in order to obtain $\mathcal{L}$.

The virtual displacement we impart is a differential increment $\delta u(x,t)$ to the actual displacement. It is important to recognize that this quantity is an arbitrary function we select, subject to some restrictions that will evolve in the course of the development. Selection of a specified $\delta u(x,t)$ in turn sets the derivatives of δu. Furthermore, the outcome is the same regardless of whether one increments u and then differentiates it, or takes the virtual increment of the derivative, so that

$$\boxed{\delta \dot{u} = \frac{\partial}{\partial t}\delta u, \quad \delta u' = \frac{\partial}{\partial x}\delta u.} \tag{9.1.19}$$

The definition of $\delta\mathcal{L}$ is that it is the difference in the value of the integral when its integrand is formed with $u + \delta u$, rather than u. Thus,

$$\delta\mathcal{L} = \mathcal{L}\left[\dot{u} + \delta\dot{u}, u + \delta u', u'' + \delta u'', \dot{u}' + \delta\dot{u}'\right] - \mathcal{L}\left[\dot{u},\ u', u'', \dot{u}'\right]. \tag{9.1.20}$$

Let $g(\dot{u},\ u', u'', \dot{u}')$ denote the integrand of $\mathcal{L}$, so that

$$\delta\mathcal{L} = \int_0^L [g(\dot{u} + \delta\dot{u}, u + \delta u', u'' + \delta u'', \dot{u}' + \delta\dot{u}') - g(\dot{u},\ u', u'', \dot{u}')]\,dx. \tag{9.1.21}$$

Because u and its derivatives are infinitesimal, a Taylor series expansion of the first integrand may be truncated at first-order terms. Examination of the expression that results shows that it is the same as what one would obtain if δ were considered to be a differential operator that is applied to the integrand. We call this the *variational derivative*, the result of which is written as

$$\boxed{\delta\mathcal{L} = \int_0^L \left[\left(\frac{\partial g}{\partial(\dot{u})}\right)\delta\dot{u} + \left(\frac{\partial g}{\partial(u')}\right)\delta u' + \left(\frac{\partial g}{\partial(u'')}\right)\delta u'' + \left(\frac{\partial g}{\partial(\dot{u}')}\right)\delta\dot{u}'\right]dx.} \tag{9.1.22}$$

Note that if the integrand contains other derivatives of u, the variational derivative would account for increments of g corresponding to each derivative.

The form of $\delta\mathcal{L}$ in Eq. (9.1.22) does not recognize that δu represents a function that we have selected, so that derivatives of δu are known in terms of δu. Integration by parts enables us to account for such relationships. Spatial derivatives of δu are handled by integrating over x, whereas time derivatives require an integration of $\delta\mathcal{L}$ over time. Thus we form

$$\begin{aligned}
\int_{t_0}^{t_1} \delta\mathcal{L}dt &= \int_{t_0}^{t_1}\int_0^L \left[\left(\frac{\partial g}{\partial(\dot{u})}\right)\delta\dot{u} + \left(\frac{\partial g}{\partial(u')}\right)\delta u' + \left(\frac{\partial g}{\partial(u'')}\right)\delta u'' + \left(\frac{\partial g}{\partial(\dot{u}')}\right)\delta u'\right]dxdt \\
&= \int_0^L \left(\frac{\partial g}{\partial(\dot{u})}\right)\delta u\,dx\Bigg|_{t=t_0}^{t=t_1} - \int_{t_0}^{t_1}\int_0^L \left[\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial(\dot{u})}\right)\right]\delta u\,dxdt \\
&\quad + \int_{t_0}^{t_1} \left(\frac{\partial g}{\partial(u')}\right)\delta u\,dt\Bigg|_{x=0}^{x=L} - \int_{t_0}^{t_1}\int_0^L \left[\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial(u')}\right)\right]\delta u\,dxdt \\
&\quad + \int_{t_0}^{t_1} \left(\frac{\partial g}{\partial(u'')}\right)\delta u'\,dt\Bigg|_{x=0}^{x=L} - \int_{t_0}^{t_1}\int_0^L \left[\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial(u'')}\right)\right]\delta u'\,dxdt \\
&\quad + \int_0^L \left(\frac{\partial g}{\partial(\dot{u}')}\right)\delta u'\,dx\Bigg|_{t=t_0}^{t=t_1} - \int_{t_0}^{t_1}\int_0^L \left[\frac{\partial}{\partial t}\left(\frac{\partial g}{\partial(\dot{u}')}\right)\right]\delta u'\,dxdt.
\end{aligned} \tag{9.1.23}$$

The last three integral terms still contain $\delta u'\ dx$, which gives δu when integrated. Thus we apply an integration by parts over x to these terms, with the result that

$$\begin{aligned}
\int_{t_0}^{t_1} \delta\mathcal{L}dt &= \int_{t_0}^{t_1}\int_0^L \left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial g}{\partial(u'')}\right) + \frac{\partial^2}{\partial x\partial t}\left(\frac{\partial g}{\partial(\dot{u}')}\right) - \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial(\dot{u})}\right)\right. \\
&\quad \left. - \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial(u')}\right)\right]\delta u\,dxdt + \int_0^L \left[\left(\frac{\partial g}{\partial(\dot{u})}\right) - \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial(\dot{u}')}\right)\right]\delta u\,dx\Bigg|_{t=t_0}^{t=t_1} \\
&\quad + \int_{t_0}^{t_1} \left[\frac{\partial g}{\partial(u')} - \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial(\dot{u}')}\right) - \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial(u'')}\right)\right]\delta u\Bigg|_{x=0}^{x=L} dt \\
&\quad + \int_{t_0}^{t_1} \left(\frac{\partial g}{\partial(u'')}\right)\delta u'\Bigg|_{x=0}^{x=L} dt + \left(\frac{\partial g}{\partial(\dot{u}')}\right)\delta u\Bigg|_{x=0}^{x=L}\Bigg|_{t=t_0}^{t=t_1}.
\end{aligned} \tag{9.1.24}$$

In addition to a description of $\delta\mathcal{L}$, Hamilton's Principle requires specification of the virtual work. For this we consider the external force system to consist of a distributed force $f(x,t)$ (units of force per unit length) acting in the same sense as that of the displacement. Thus, if u is a displacement in a certain direction, then f acts in that direction. Similarly, if u is a torsional rotation, then f is distributed torsional loading. The force resultant acting on a differential segment dx is $f\,dx$, and δu is the virtual displacement. Hence the virtual work is

$$\delta W = \int_0^L f\,\delta u\,dx. \tag{9.1.25}$$

The result of adding Eqs. (9.1.24) and (9.1.25) to form Hamilton's Principle, as prescribed by Eq. (9.1.10), is called the *variational equation.* It is obvious that it contains a variety of terms, all of which must sum to zero. However, the function δu associated with the variation is selected arbitrarily, which means that the fundamental lemma of the calculus of variations applies. We begin by applying this theorem to the terms in the variational derivative that are integrated over both x and t, which must be zero. If we knew the actual response function $u(x,t)$ associated with the distributed force $fx,t)$, we certainly could find a function δu that would cause this double integral to vanish. However, a different $f(x,t)$ would lead to a different $u(x,t)$, yet the integral must still vanish in that circumstance. In order that Hamilton's principle be satisfied for any set of applied forces, it must be that the factor of δu in the integrand vanishes, so we have

$$\boxed{\frac{\partial^2}{\partial x^2}\left(\frac{\partial g}{\partial(u'')}\right) + \frac{\partial^2}{\partial x\partial t}\left(\frac{\partial g}{\partial(\dot{u}')}\right) - \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial(\dot{u})}\right) - \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial(u')}\right) = 0.} \tag{9.1.26}$$

This is the *Euler–Lagrange equation*. When we form it for the function $g(\dot{u},\ u', u'', \dot{u}')$ appropriate to the system of interest, we obtain the partial differential equation governing the displacement field.

When the Euler–Lagrange equation is satisfied, the terms remaining in the variational derivative are those that are evaluated at $x = 0$ and $x = L$, and/or $t = t_0$ and $t = t_1$. Recall that the latter correspond to the beginning and end points of the path through the configuration space of each particle in the system. The fact that there are an infinite number of such points, so that the configuration space has infinite dimension, does not alter the requirement that a variational path should begin and end at the true state of the system, so that $\delta u = 0$ at $t = t_0$ and $t = t_1$ for every x. This condition causes all terms evaluated at the time limits to be zero, so the terms that remain in the variational derivative are

$$\begin{aligned}\int_{t_0}^{t_1} \delta\mathcal{L}dt = \int_{t_0}^{t_1} \left[\frac{\partial g}{\partial(u')} - \frac{\partial}{\partial t}\left(\frac{\partial g}{\partial(\dot{u}')}\right) - \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial(u'')}\right)\right]\delta u\Big|_{x=0}^{x=L} dt \\ + \int_{t_0}^{t_1}\left(\frac{\partial g}{\partial(u'')}\right)\delta u'\Big|_{x=0}^{x=L} dt.\end{aligned} \tag{9.1.27}$$

The arbitrariness of δu allows us to consider the value of $\delta u'$ at $x = 0$ or $x = L$ to be a separately selectable parameter, independent of the value of δu at these locations. Thus the time integral associated with evaluation of the integrand at each boundary must be zero. Furthermore, $\delta u(L,t)$ is independent of $\delta u(0,t)$. This leads to the requirement that each integrand evaluated at each end must vanish. There are several possible ways in which this condition might be satisfied, because each integrand is a product of a term that depends on g and a term that depends on δu, either of which being zero will lead to a zero value for the integral. Let us address each possibility individually. The condition that $\delta u = 0$ at $x = 0$ or $x = L$ for all t means that $u(0,t)$ or $u(L,t)$ is fully specified, and therefore not changeable. The most common condition of this type is that in which the displacement is made to be zero, as is the case for the stationary end of a cable. We must examine the system of interest to ascertain whether such a condition applies. If not, then it must be that the coefficient of δu in the first integral is identically zero. The condition that u is known at a boundary constitutes a kinematical condition, so it is referred to as a *geometric boundary condition*. In contrast, the factor of δu depends on the function g, which means that it depends on the nature of the kinetic and potential energies. These quantities correspond to inertial effects and internal forces, so the factor of δu could said to be said to be a force boundary condition. However, the common terminology is to refer to it as a *natural boundary condition.*

A similar argument applies to the second integral in Eq. (9.1.27). This integral will be zero if $\delta u' = 0$ at the ends. Such a condition results if u' is specified at $x = 0$ or $x = L$. An example of such a situation is the condition on the transverse displacement of a beam, where $u' = 0$ for an end that is clamped or welded to a wall. Once again this is a kinematical condition, so a situation in which $\delta u' = 0$ at $x = 0$ or $x = L$ is a geometric boundary condition. If $\delta u'$ is not known to be zero at either end, then there is a natural boundary condition to satisfy.

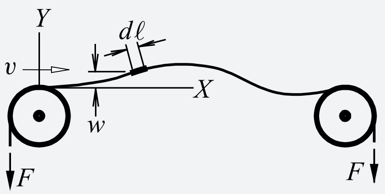
In summary, there are alternative geometrical or natural boundary conditions at each end. In the case in which g depends on u', u'', $\dot{u}$, and $\dot{u}'$, there are two boundary conditions at each end. The possibilities are

$$\boxed{\begin{aligned} &x = 0 \text{ and } x = L: \quad \begin{cases} \text{specify } u \text{ as a function of time } \textit{or} \\ \text{set } \dfrac{\partial g}{\partial (u')} - \dfrac{\partial}{\partial t}\left(\dfrac{\partial g}{\partial (\dot{u}')}\right) - \dfrac{\partial}{\partial x}\left(\dfrac{\partial g}{\partial (u'')}\right) = 0; \end{cases} \\ &x = 0 \text{ and } x = L: \quad \begin{cases} \text{specify } u' \text{ as a function of time } \textit{or} \\ \text{set } \dfrac{\partial g}{\partial (u'')} = 0. \end{cases} \end{aligned}} \tag{9.1.28}$$

Thus the calculus of variations has led us to the differential equation governing the displacement field, as well as alternative boundary conditions that might apply. Selecting from these alternatives the set of boundary conditions appropriate to a specific system is our responsibility.

EXAMPLE 9.1 Transverse vibration of axially moving cables is important for a variety of applications, such as conveyor belts, cable–pulley systems, and computer tape drives. Consider the situation in which a cable is stretched to a very high tensile force and wrapped around two rollers, whose rotation imparts a horizontal translational velocity v to the unsupported segment of the cable. The mass per unit length of the cable is μ, and the cable tension is F. Use the calculus of variations to derive the differential equation and boundary conditions governing the transverse displacement w. It is permissible to consider w to be very small compared with the distance between the rollers, such that the slope of the cable in the displaced position also is small.

SOLUTION This example is an application of the general calculus of variations formulation to a system whose equations will be found to have some interesting features. In the sketch XYZ is a fixed reference frame with its origin at the left end of the cable. If the cable did not displace transversely, the position of a point along the span would be defined solely by the horizontal distance X, so this system fits the prescription of a system having a displacement that depends on a single position coordinate.



Coordinates and displacement variables for a translating cable.

We begin by considering the kinetic energy of the cable segment $d\ell$ highlighted in the sketch. The position of $d\ell$ is $\bar{r} = X\bar{I} + w(X,t)\bar{J}$. The stated smallness of w means that the cable primarily moves horizontally, so $\dot{X} \approx v$. The vertical velocity results from explicit time dependence of w, as well as the fact that different points on the cable, having different w, arrive at a specified X as time elapses. Both effects are captured by taking a total derivative of $\bar{r}$, so the velocity of the cable segment is

$$\bar{v}_s = \frac{d\bar{r}}{dt} = \dot{X}\bar{I} + \left(\dot{X}\frac{\partial w}{\partial X} + \frac{\partial w}{\partial t}\right)\bar{J} = v\bar{I} + [\dot{w} + vw']\,\bar{j}.$$

The segment $d\ell$ would be situated on the X axis if the cable did not deflect, so it must be that the mass of $d\ell$ is the same as that of a segment of length dX in a situation in which w is zero, which is μdX. The kinetic energy of this segment is $\frac{1}{2}(\mu dx)\bar{v}_s \cdot \bar{v}_s$, so the kinetic energy functional is

$$T = \frac{1}{2}\mu\int_0^L \left[\dot{w}^2 + 2vw'\dot{w} + v^2(w')^2\right] dX + \frac{1}{2}mLv^2.$$

The last term is the translational kinetic energy the cable would have if it did not displace transversely. It is independent of w, so it will be unimportant to the derivation of the field equation.

Potential energy is stored as a result of stretching the cable. The stipulation that the cable tension F is very high in the initial state means that there is a substantial strain within the cable, even if $w = 0$. Because w is assumed to be small, the additional strain associated with w will also be small relative to the initial strain. It follows that it is reasonable to assume that F remains constant when w is nonzero. Thus the work done by F, which is stored as potential energy, is the product of F and the amount $\Delta\ell$ by which the cable length is increased from the undeflected to deflected conditions. By the Pythagorean theorem, the differential arc length is

$$d\ell = \left[(dX)^2 + (w' dX)^2\right]^{1/2}.$$

Integrating the preceding equation to obtain the deflected arc length ℓ and then subtracting the original length L yields $\Delta\ell$. To $F\Delta\ell$ we add the potential energy of gravity. Let the X axis be the datum, so that w is the elevation of segment $d\ell$, whose weight is $\mu d\ell$. Hence the potential-energy functional is

$$V = F\left[\int_0^L \left(1 + (w')^2\right)^{1/2} dX - L\right] + \int_0^L w\,\mu g dX.$$

The virtual work δW is zero because all forces doing work when w is incremented have been included in the potential energy.

Rather than merely forming Eqs. (9.1.26) and (9.1.28), let us explicitly perform the basic operations on the present functionals. We begin with the kinetic-energy term in Hamilton's Principle:

$$\begin{aligned}\int_{t_0}^{t_1} \delta T dt &= \int_{t_0}^{t_1} \delta\left\{\frac{1}{2}\mu \int_0^L \left[\dot{w}^2 + 2vw'\dot{w} + v^2 (w')^2\right] dX + \frac{1}{2} m L v^2\right\} dt \\ &= \int_{t_0}^{t_1} \mu \int_0^L \left[\dot{w}\delta(\dot{w}) + v\dot{w}\,\delta w' + vw'\delta\dot{w} + v^2 w'\delta w'\right] dXdt.\end{aligned}$$

Note how the variational derivative is implemented as a differential operator applied to all terms containing w. We interchange the virtual increments and derivatives of w in accord with Eqs. (9.1.19). Terms containing time derivatives of δw are integrated by parts with respect to t, whereas terms containing spatial derivatives of δw are integrated by parts with respect to x. These operations replace terms containing derivatives of δw with equivalent representations in terms of δw:

$$\begin{aligned}\int_{t_0}^{t_1} \delta T dt = \mu \int_0^L \left[(\dot{w} + vw')\,\delta w\right]\Big|_{t=t_0}^{t=t_1} + \int_{t_0}^{t_1} \mu\left\{\left[v\dot{w} + v^2 w'\right]\delta w\Big|_{x=0}^{x=L}\right\} dt \\ - \int_{t_0}^{t_1} \mu \int_0^L \left(\ddot{w} + 2v\dot{w}' + v^2 w''\right)\delta w\, dXdt.\end{aligned}$$

It is interesting to observe that the slope w' is essentially the rotation of the tangent line because of the smallness of the displacement. Consequently, the middle term in the integrand is the product of the angular velocity of the segment $d\ell$ and the translational speed v. The two factor confirms that this term represents the effects of Coriolis acceleration.

Similar operations are followed to express the contribution of the strain-energy term to Hamilton's Principle. Thus,

$$
\begin{aligned}
\int_{t_0}^{t_1} \delta V dt &= \int_{t_0}^{t_1} \delta \left\{ F \left[\int_0^L \left(1+(w')^2\right)^{1/2} dX - L \right] + \int_0^L w\,\mu g dX \right\} dt \\
&= \int_{t_0}^{t_1} \left\{ F \int_0^L \left(\frac{1}{2}\right) \left(1+(w')^2\right)^{-1/2} (2w'\delta w')\, dX + \int_0^L \mu g \delta w dX \right\} dt \\
&= \int_{t_0}^{t_1} \left\{ F \left(\frac{w'}{\left(1+(w')^2\right)^{1/2}} \right) \delta w \Bigg|_{X=0}^{X=L} \right\} dt \\
&\quad - \int_{t_0}^{t_1} \int_0^L \left[F \frac{\partial}{\partial X} \left(\frac{w'}{\left(1+(w')^2\right)^{1/2}} \right) - \mu g \right] \delta w dX dt.
\end{aligned}
$$

The variational equation is the difference of the preceding equations. The variational path is such that $\delta w = 0$ at $t = t_0$ and $t = t_1$, so

$$
\begin{aligned}
\int_{t_0}^{t_1} \delta T dt - \int_{t_0}^{t_1} \delta V dt &= \int_{t_0}^{t_1} \left\{ \left[\mu \left[v\,\dot{w} + v^2 w' \right] - F \left(\frac{w'}{\left(1+(w')^2\right)^{1/2}} \right) \right] \delta w \Bigg|_{x=0}^{x=L} \right\} dt \\
&\quad - \int_{t_0}^{t_1} \int_0^L \left\{ \left[\mu \left(\ddot{w} + 2v\,\dot{w}' + v^2 w'' \right) \right] - F \frac{\partial}{\partial X} \left(\frac{w'}{\left(1+(w')^2\right)^{1/2}} \right) \right. \\
&\quad \left. - \mu g \right\} \delta w dX dt = 0.
\end{aligned}
$$

We obtain the Euler–Lagrange equation by equating to zero the coefficient of δw in the integrand of the double integral over X and t:

$$
F \frac{\partial}{\partial X} \left(\frac{w'}{\left(1+(w')^2\right)^{1/2}} \right) - \mu \left(\ddot{w} + 2v\dot{w}' + v^2 w'' \right) = -\mu g. \qquad \triangleleft
$$

When the Euler–Lagrange equation is satisfied, the remaining terms in the variational derivative are

$$
\int_{t_0}^{t_1} \left\{ \left[\mu \left(v\,\dot{w} + v^2 w' \right) - F \left(\frac{w'}{\left(1+(w')^2\right)^{1/2}} \right) \right] \delta w \Bigg|_{x=0}^{x=L} \right\} dt = 0.
$$

Satisfaction of this condition requires that the integrand evaluated at each end be zero, because the value of δw at $x = L$ is independent of its value at $x = 0$. Thus we find that one boundary condition must be satisfied at each end. This boundary condition is an alternative between the geometric condition, which requires that $\delta w = 0$ at an end, and the natural condition we obtain by equating to zero the coefficient of δw at that end. The latter is a condition on the vertical component of the tensile force, which does not apply to the system under condition. The rollers prevent the ends from displacing, so we set

$$w(0,t) = w(L,t) = 0. \qquad \triangleleft$$

The Euler–Lagrange equation is a nonlinear partial differential equation. We may simplify it by considering $|w'| \ll 1$, which should be valid because $|w|$ at all locations is small compared with L.† When we use this to simplify the denominator of the X derivative term, we obtain

$$Fw'' - \mu\left(\ddot{w} + 2v\,\dot{w}' + v^2 w''\right) = -g.$$

An interesting aspect of this differential equation is that is not amenable to direct solution by the standard method of separation of variables. The solution of this equation and an exploration of numerous interesting phenomena may be found in the work of Wickert and Mote (1990), and many others after that.

9.1.3 Ritz Series Method

The Euler–Lagrange equation that is obtained from Hamilton's Principle can be quite challenging to solve. For that reason approximate analysis techniques are required. One approach begins by assuming that the displacement field can be represented as a series having a finite number of terms. Walther Ritz (1878–1909) proposed the use of a specific form of such a series, in which each term is a product of a selected function of position and a time function whose dependence is to be determined. The spatial functions are referred to as *basis functions.* The general form of a *Ritz series* for the vectorial displacement of an arbitrary continuous system therefore has the form

$$\bar{u}(\bar{r}) = \sum_{n=1}^{N} \bar{\psi}_n(\bar{r})\, q_n(t). \qquad (9.1.29)$$

The basis functions are $\bar{\psi}_n(\bar{r})$. They are indicated to be vectors because, in the most general situation, $\bar{u}$ might have independently variable components. It will be noted that the symbol q_n is used to denote the time functions. This is done to emphasize that, as a consequence of selecting the basis functions as a first step in formulating the analysis, the sole variables required to evaluate the displacement $\bar{u}$ are the $q_n(t)$ variables, so they are effectively generalized coordinates.

† As the frequency of vibration increases, the length scale over which w changes might become significantly smaller than the span L. The consequence is that the approximation of small $|\partial w/\partial x|$ loses validity.

A Ritz series may be considered to be a generalization of a Fourier series. [In fact, some texts have referred to Eq. (9.1.29) as "generalized Fourier series."] To understand why, suppose a displacement field is a function of a single spatial coordinate and that it is periodic in a distance L measured along that coordinate direction. Any periodic function may be expanded in a Fourier series, so the displacement in this scenario may be expressed as

$$\bar{u}\,(x) = C_0 + \sum_{n=1}^{N} \left[C_n \cos\left(\frac{2n\pi x}{L}\right) + S_n \sin\left(\frac{2n\pi x}{L}\right) \right]. \tag{9.1.30}$$

If we consider the coefficients C_n and S_n to be time dependent, we extend the Fourier series to situations in which the displacement field has an arbitrary time dependence, while still maintaining its spatial periodicity. Thus, in this scenario, the cosine and sine functions are the basis functions of a Ritz series, and the C_n and S_n coefficients are the generalized coordinates q_n. In a linear algebra perspective, the $\bar{\psi}_n\,(\bar{x})$ define favored directions in a functional space, like the unit vectors $\bar{e}_j$ for a vector space, which is why they are said to be basis functions. The q_n values may then be considered to be the displacement components parallel to the basis functions.

The most problematic aspect of formulating a Ritz series analysis is selection of the basis functions. There are a few inviolable requirements. The first is that the functions must be linearly independent, which is equivalent to requiring that the Ritz series yield a zero value of $\bar{u}$ at all $\bar{r}$ only if all q_n are zero. The second requirement is that the basis functions must satisfy all *geometric boundary conditions* that pertain to the system being analyzed. (This requirement, of course, assumes that one has identified what the geometric boundary conditions are. In extraordinary circumstances, it might be necessary to apply the calculus of variations, but we usually can identify these conditions by inspection.) It is important to realize that the natural boundary conditions need not be considered when the basis functions are selected. Because they constitute conditions on force resultants at the boundary, natural boundary conditions are accounted for in the virtual work.‡

Some individuals consider the Ritz series to be a guessing procedure, in that one has no other guidelines as to how to select the basis functions. However, in many cases it is possible to identify basis functions that satisfy the geometric boundary conditions and also consitute a parameterized family of functions. For example, the sine and cosine functions in Eq. (9.1.30) are parameterized by the harmonic number n. The terms of a power series, $(x/L)^n$, are another parameterized set, as are the basis functions used in finite element analysis. In fact, one can consider the finite element technique for elastic systems to be a special case of a Ritz series analysis, as discussed by Ginsberg (2001). The significance of increasing the length N of the Ritz series is that doing so will bring the solution closer to what one would obtain by solving the differential equations of

‡ In some situations elastic springs might be situated at the boundaries. In that case the aspects of a natural boundary condition associated with the spring force are addressed by including in the V functional the spring's potential energy.

motion, if such a solution were possible. It can be proven that the limit $N \to \infty$ would yield an infinite dimensional representation that is the true response.

Once we have selected the basis functions, generation of the equations of motion for the generalized coordinates is fairly straightforward. It is assumed that prior to initiating the analysis we have derived the appropriate functionals describing how the kinetic and potential energies of the continuous system depend on the displacement. These functionals feature integrals over the spatial domain of the system. When we substitute the Ritz series into these functionals, all spatial dependencies have been set, so the integrals may be evaluated. This process converts the functionals of displacement into functions of the generalized coordinates. As before, we limit our attention here to cases in which a displacement field consists of a single component that depends on a single spatial coordinate and time, $u(x,t)$. Correspondingly, we take all ψ_n to be scalar functions of x and t. The mathematical statement of the operation of substituting the Ritz series is

$$\boxed{\mathcal{L}\left[\dot{u},\ u',u'',\dot{u}',t\right] = \mathcal{L}\left[\sum_{n=1}^{N}\psi_n\dot{q}_n, \sum_{n=1}^{N}\psi'_n q_n, \sum_{n=1}^{N}\psi''_n q_n, \sum_{n=1}^{N}\psi'_n\dot{q}_n, t\right] = \mathcal{L}\left(\dot{q}_n, q_n, t\right).} \tag{9.1.31}$$

In other words, substitution of the Ritz series enables us to reduce the Lagrangian from a functional featuring spatial integration to a discrete function of the generalized coordinates and velocities. In practice, such a reduction will entail integrating a variety of terms containing the basis functions and their derivatives. These integrations may be carried out analytically or by means of computer techniques using symbolic or numerical software.

Formulation of Hamilton's Principle also requires description of the virtual work. Let $f(x,t)$ denote the distributed force per unit length acting in the direction of the displacement u. The external force acting on an interval dx is $f\,dx$, and the virtual displacement at this location is $\delta u(x,t)$, so the virtual work is

$$\delta W = \int_0^L f(x,t)\,\delta u dx = \int_0^L f(x,t)\,\delta u dx. \tag{9.1.32}$$

When we form the Ritz series, Eq. (9.1.29), the basis functions have been set. Thus a virtual displacement entails imparting virtual increments solely to the series coefficients:

$$\delta u = \sum_{n=1}^{N}\psi_n\delta q_n. \tag{9.1.33}$$

We substitute this representation into the preceding expression for δW and bring the integral inside the summation. The result is a representation of δW that matches the standard form containing generalized forces, specifically,

$$\boxed{\delta W = \sum_{n=1}^{N} Q_n\delta q_n, \quad Q_n = \int_0^L f(x,t)\,\psi_n(x)\,dx.} \tag{9.1.34}$$

This completes the basic operations of a Ritz series analysis. The system model has been reduced to one in which the Lagrangian is a function of the generalized coordinates and generalized velocities, and the virtual work is characterized by a set of generalized forces. As we saw earlier, Hamilton's Principle for such a system reduces to Lagrange's equations. Thus the formulation of Ritz series in essence has reduced the continuous system from one having an infinite number of degrees of freedom to an approximate one described by a finite number of degrees of freedom. An important aspect of this procedure, which is often forgotten, is that consideration should be given to whether a convergent representation of the phenomena of interest results from the analysis. If one has normalized the basis functions such that they have unit order of magnitude everywhere, then it is possible to ascertain that convergence has been attained by merely examining the convergence properties of the series coefficients q_n.

EXAMPLE 9.2 Consider the analysis of transverse displacement of a cable that translates at a constant speed, which was described in Example 9.1. Derive the equations of motion governing a set of Ritz series coefficients.

SOLUTION Beyond merely illustrating how to implement a Ritz series analysis, this example is intended to interest the reader in some of the more modern issues encountered in vibration research. A suitable Ritz series is readily identified. As shown in the previous example, both ends are constrained by the rollers to not displace, so the geometric boundary condition applies at each end. We therefore seek a set of basis functions that is linearly independent and has zero value at $X = 0$ and $X = L$. The simplest such function is a set of sines, specifically,

$$\psi_n = \sin\left(\frac{n\pi X}{L}\right),$$

so our Ritz series for this system is

$$w(X,t) = \sum_{n=1}^{N} q_n(t) \sin\left(\frac{n\pi X}{L}\right).$$

We substitute this series into the kinetic-energy functional derived previously, which leads to

$$\begin{aligned} T &= \frac{1}{2}\mu \int_0^L \left[\dot{w}^2 + 2vw'\dot{w} + v^2 (w')^2\right] dX + \frac{1}{2} m L v^2 \\ &= \frac{1}{2}\mu \int_0^L \left\{\left[\sum_{j=1}^{N} \dot{q}_j \sin\left(\frac{j\pi X}{L}\right)\right]\left[\sum_{n=1}^{N} \dot{q}_n \sin\left(\frac{n\pi X}{L}\right)\right]\right\} dX \\ &\quad + \mu v \int_0^L \left\{\left[\sum_{j=1}^{N} \left(\frac{j\pi}{L}\right) q_j \cos\left(\frac{j\pi X}{L}\right)\right]\left[\sum_{n=1}^{N} \dot{q}_n \sin\left(\frac{n\pi X}{L}\right)\right]\right\} dX \end{aligned}$$

$$+\frac{1}{2}\mu v^2\int_0^L\left\{\left[\sum_{j=1}^N q_j\left(\frac{j\pi}{L}\right)\cos\left(\frac{j\pi X}{L}\right)\right]\left[\sum_{n=1}^N q_n\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi X}{L}\right)\right]\right\}dX$$
$$+\frac{1}{2}\mu L v^2.$$

The most important aspect of this substitution is that a different index is used to describe the summation for each factor in products of w and its derivatives. This is done to ensure that we account for each possible combination of terms. We focus on integrating over x by bringing the integration operator inside the summations. The first and third integrals simplify as a consequence of the orthogonality of the sine function, with the result that

$$T=\frac{1}{4}\mu L\sum_{j=1}^N\left[\dot{q}_j^2+\left(\frac{j\pi v}{L}\right)^2 q_j^2\right]+\mu v\sum_{j=1}^N\sum_{n=1}^N B_{jn}q_j\dot{q}_n+\frac{1}{2}mLv^2,$$

where

$$B_{jn}=\begin{cases}\dfrac{2jn}{n^2-j^2}\text{ if } j+n \text{ is odd}\\[2ex] 0 \text{ if } j+n \text{ is odd}\end{cases}.$$

The reduction of V from a functional of w to a function of the Ritz series coefficients follows similar steps. However, the presence of the square root substantially complicates the task of integrating over x. We address this by recalling that $|\partial w/\partial x|\ll 1$. When this condition is valid, the square root term may be approximated by a binomial series that is truncated. Thus we write

$$\begin{aligned}V&\approx F\left[\int_0^L\left(1+\frac{1}{2}(w')^2\right)dX-L\right]+\int_0^L w\,\mu g dX+O\left(w^4\right)\\&=\frac{1}{2}F\int_0^L(w')^2\,dX+\int_0^L w\,\mu g dX.\end{aligned}\tag{9.1.35}$$

Note that the series expansion of the square root term has been truncated at the lowest order possible. This level of approximation is equivalent in every respect to linearizing the Euler–Lagrange equation in Example 9.1. Substitution of the Ritz series into this functional yields

$$V=\frac{1}{2}F\int_0^L\left(\sum_{j=1}^N\left(\frac{j\pi}{L}\right)q_j\cos\left(\frac{j\pi X}{L}\right)\right)\left(\sum_{n=1}^N\left(\frac{n\pi}{L}\right)q_n\cos\left(\frac{n\pi X}{L}\right)\right)^{1/2}dX$$
$$+\mu g\int_0^L\sum_{n=1}^N q_n\sin\left(\frac{n\pi X}{L}\right)dX$$
$$=\frac{1}{4}FL\sum_{j=1}^N\left(\frac{j\pi}{L}\right)^2 q_j^2+\mu g\sum_{n=1}^N\left(\frac{L}{n\pi}\right)[1-(-1)^n]\,q_n.$$

As in the previous example the virtual work is zero. Thus we proceed to evaluate Lagrange's equations for this system. For this, we observe that partial differentiation of a summation with respect to a specific q_n or $\dot{q}_n$ will yield a nonzero contribution only when the index of the summation matches n. To avoid confusion in this process the symbol denoting the summation index should be changed to differ from n. The operations are carried out as follows:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) &= \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_n}\left(\frac{1}{4}\mu L\sum_{j=1}^{N}\left[\dot{q}_j^2 + \left(\frac{j\pi v}{L}\right)^2 q_j^2\right] + \mu v\sum_{j=1}^{N}\sum_{k=1}^{N} B_{jk} q_j \dot{q}_k\right)\right\}\\ &= \frac{d}{dt}\left(\frac{1}{2}\mu L\dot{q}_n + \mu v\sum_{j=1}^{N} B_{jn} q_j\right) = \frac{1}{2}\mu L\ddot{q}_n + \mu v\sum_{j=1}^{N} B_{jn}\dot{q}_j,\end{aligned}$$

$$\begin{aligned}\frac{\partial T}{\partial q_n} &= \frac{\partial}{\partial q_n}\left(\frac{1}{4}\mu L\sum_{j=1}^{N}\left[\dot{q}_j^2 + \left(\frac{j\pi v}{L}\right)^2 q_j^2\right] + \mu v\sum_{j=1}^{N}\sum_{k=1}^{N} B_{jk} q_j \dot{q}_k\right)\\ &= \frac{\mu v^2}{L}\left(\frac{n^2\pi^2}{2}\right) q_n + \mu v\sum_{k=1}^{N} B_{nk}\dot{q}_k,\end{aligned}$$

$$\begin{aligned}\frac{\partial V}{\partial q_n} &= \frac{\partial}{\partial q_n}\left(\frac{1}{4}FL\sum_{j=1}^{N}\left(\frac{j\pi}{L}\right)^2 q_j^2 + \mu g\sum_{j=1}^{N}\left(\frac{L}{j\pi}\right)\left[1-(-1)^j\right] q_j\right)\\ &= \frac{F}{L}\left(\frac{n^2\pi^2}{2}\right) q_n + \mu g L\frac{[1-(-1)^n]}{n\pi}.\end{aligned}$$

The assembled Lagrange equations are thereby found to be

$$\begin{aligned}\frac{\mu L}{2}\ddot{q}_n &+ \left(\frac{F}{L} - \mu v^2\right)\left(\frac{n^2\pi^2}{2}\right) q_n + \mu v\sum_{j=1}^{N} B_{jn}\dot{q}_j - \mu v\sum_{k=1}^{N} B_{nk}\dot{q}_k\\ &+ \mu g L\frac{[1-(-1)^n]}{n\pi} = 0.\end{aligned}$$

We may simplify this equation by recognizing that any symbol may be used as the index for a summation. Changing j to k in the first summation leads to

$$\frac{\mu L}{2}\ddot{q}_n - \mu v\sum_{k=1}^{N} G_{nk}\dot{q}_k + \left(\frac{F}{L} - \mu v^2\right)\left(\frac{n^2\pi^2}{2}\right) q_n = -\mu g L\frac{[1-(-1)^n]}{n\pi};\ \ j = 1, 2, \ldots, N,$$

◁

where

$$G_{nk} = -G_{kn} = B_{nk} - B_{kn} = \begin{cases}\dfrac{4kn}{k^2-n^2} \text{ if } j+n \text{ is odd}\\ 0 \text{ if } j+n \text{ is odd}\end{cases}.$$

This set of Lagrange's equations constitutes a set of coupled linear ordinary differential equations. They have an interesting feature when written in matrix form as

$$\mu L\{\ddot{q}\} - 2\mu v\,[G]\{\dot{q}\} + \left(\frac{F}{L} - \mu v^2\right)[K]\{q\} = \{F\},$$

where $[K]$ is a diagonal array whose elements are $K_{jj} = j^2\pi^2$. In the study of vibratory response of structural and mechanical systems the coefficient of $\{\ddot{q}\}$ in equations of motion is referred to as the inertia matrix, and the coefficient of $\{q\}$ is called the stiffness matrix. In the present context, the inertia matrix is proportional to the identity matrix, and the stiffness matrix is diagonal. Fundamental vibration studies typically do not encounter terms like $[G]$, which is skew symmetric. This term is the Ritz series manifestation of the Coriolis acceleration encountered in the Euler–Lagrange equation derived in Example 9.1. Similar Coriolis acceleration terms are encountered in vibrating pipes with high-speed internal flows. Terms having a similar appearance also are encountered in studies of flexible rotordynamic systems, where they are associated with gyroscopic effects. The presence of Coriolis and gyroscopic effects has a profound effect on the nature of the vibratory response. The solutions of the Ritz series equations of motion were addressed by the author (Ginsberg, 2001).

9.2 GENERALIZED MOMENTUM PRINCIPLES

Hamilton's enunciation of the principle bearing his name is arguably his most important contribution to the dynamics of systems, but not his only one. He used Lagrange's equations to derive a set of first-order equations of motion that are quite different from the state-space formulations encountered thus far. This formulation leads to additional conservation principles that would be more difficult to derive by other approaches.

9.2.1 Hamilton's Equations

Hamilton's equations of motion, which govern systems that have a finite number of degrees of freedom, are founded on a relation between momentum and kinetic energy. A suggestion that there is such a relation in general comes from considering a single particle whose motion is described by its Cartesian coordinates in a fixed reference frame, $q_1 = x$, $q_2 = y$, and $q_3 = z$. The momentum components are $m\dot{x}$, $m\dot{y}$, and $m\dot{z}$, and the kinetic energy is $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$, so we have

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x}, \quad \frac{\partial T}{\partial \dot{y}} = m\dot{y}, \quad \frac{\partial T}{\partial \dot{z}} = m\dot{z}. \tag{9.2.1}$$

These simple relations between momentum and kinetic energy actually are manifestations of a more general perspective. Consider the kinetic energy of a single particle.

Differentiation of T for this particle with respect to a selected generalized velocity $\dot{q}_n$ gives

$$\frac{\partial T}{\partial \dot{q}_n} = \frac{\partial}{\partial \dot{q}_n}\left(\frac{1}{2}m\bar{v}\cdot\bar{v}\right) = m\bar{v}\cdot\frac{\partial \bar{v}}{\partial \dot{q}_n}. \tag{9.2.2}$$

The particle's velocity is related to the generalized coordinates by

$$\bar{v}\left(\dot{q}_i, q_i, t\right) = \sum_{j=1}^{N}\left(\frac{\partial}{\partial q_j}\bar{r}\left(q_i, t\right)\right)\dot{q}_j + \frac{\partial}{\partial t}\bar{r}\left(q_i, t\right). \tag{9.2.3}$$

Because $\dot{q}_n$ represents a specific variable, it occurs within the summation only when the index j matches n, with the result that

$$\frac{\partial}{\partial \dot{q}_n}\bar{v}\left(\dot{q}_i, q_i, t\right) \equiv \frac{\partial}{\partial q_j}\bar{r}\left(q_i, t\right), \tag{9.2.4}$$

which is an identity we first encountered in the derivation of Lagrange's equations. With this, Eq. (9.2.2) becomes

$$\frac{\partial T}{\partial \dot{q}_n} = m\bar{v}\cdot\frac{\partial \bar{r}}{\partial q_n}. \tag{9.2.5}$$

One way of interpreting this relation is that $\partial\bar{r}/\partial q_n$ is a vector in the direction of increasing q_n, like the unit vectors of a curvilinear coordinate system. In this interpretation $\partial T/\partial \dot{q}_n$ is the component of momentum in the direction of increasing q_n, which leads to its being referred to as the *generalized momentum.* The kinetic energy of any system may be decomposed into contributions of individual particles, so it follows that $\partial T/\partial \dot{q}_n$ is the total momentum of a system in the sense of q_n. The symbol p_n is typically reserved for a generalized momenta, so we define

$$\boxed{p_n = \frac{\partial T}{\partial \dot{q}_n} = \frac{\partial \mathcal{L}}{\partial \dot{q}_n},} \tag{9.2.6}$$

where the form containing the Lagrangian is a direct consequence of the fact that V cannot depend on the generalized velocities.

The essence of Hamilton's formulation is that the generalized momenta should replace the generalized velocities as the rate variables in the equations of motion. This is accompanied by the introduction of a new energylike quantity called the *Hamiltonian* and denoted by the symbol $\mathcal{H}$. Motivation for the definition may be found in comparing the mechanical energy $E = T + V$ and the Lagrangian $\mathcal{L}$ for a particle whose position is described by its Cartesian coordinates. If we express T in terms of the momentum components in this case, we have

$$E = 2T - \mathcal{L} = \left(m\dot{x}\right)\dot{x} + \left(m\dot{y}\right)\dot{y} + \left(m\dot{z}\right)\dot{z} - \mathcal{L}. \tag{9.2.7}$$

This suggests a generalization in which the momentum components are replaced with p_n and the velocity components are replaced with $\dot{q}_n$. The result is the definition of $\mathcal{H}$:

$$\boxed{\mathcal{H} = \sum_{j=1}^{N} p_j \dot{q}_j - \mathcal{L}.} \tag{9.2.8}$$

An alternative definition of the Hamiltonian results when we recall Eqs. (7.6.4), which decomposed the kinetic energy into a term T_2 representing contributions that are quadratic in the generalized velocities, a term T_1 that is linear in those variables, and a term T_0 that is independent of the $\dot{q}_j$. The result of using Eqs. (7.6.4) to form p_j is

$$\begin{aligned} p_j &= \frac{\partial}{\partial \dot{q}_j} \left[\frac{1}{2} \sum_{k=1}^{N} \sum_{n=1}^{N} M_{kn}(q_i, t)\, \dot{q}_k \dot{q}_n + \sum_{k=1}^{N} N_k(q_i, t)\, \dot{q}_k + T_0(q_i, t) \right] \\ &= \sum_{k=1}^{N} M_{kj}(q_i, t)\, \dot{q}_k + N_j(q_i, t), \quad j = 1, 2, \dots, N. \end{aligned} \tag{9.2.9}$$

This expression shows that the generalized momenta are linearly related to the generalized velocities. We will soon have another use for this relation, but currently we merely substitute it into Eq. (9.2.8), which leads to

$$\begin{aligned} \mathcal{H} &= \sum_{j=1}^{N} \left[\sum_{k=1}^{N} M_{kj}(q_i, t)\, \dot{q}_k + N_j(q_i, t) \right] \dot{q}_j - (T - V) \\ &= \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} M_{jk}(q_i, t)\, \dot{q}_k \dot{q}_j - T_0(q_i, t) + V(q_i, t) \\ &= T_2(\dot{q}_i, q_i, t) - T_0(q_i, t) + V(q_i, t). \end{aligned} \tag{9.2.10}$$

Thus, if T_1 and T_0 happen to be zero for a system, so that T_2 is the total kinetic energy, then $\mathcal{H}$ is indeed the mechanical energy E.

We have not yet faced the most troublesome aspect of the Hamiltonian formulation, which is removal of the generalized velocities from the dependence. With this as our objective, we return to Eq. (9.2.9), which we now view as a set of N simultaneous equations for the generalized velocities corresponding to a specified set of q_i and p_i. Recall that $[M]$ is always invertible because the kinetic energy is positive definite. Hence it is always possible to solve Eq. (9.2.9) for the generalized velocities. The matrix form of that solution is

$$\boxed{\{\dot{q}\} = [M(q_i, t)]^{-1} \left\{ \{p\} - \{N(q_i, t)\} \right\}.} \tag{9.2.11}$$

These constitute one part of Hamilton's equations. It is imperative to recognize that, although matrix notation is employed here, these operations must be done algebraically. Symbolic mathematical software can be quite useful for this task.

Substitution of Eq. (9.2.11) into the alternative form of $\mathcal{H}$, Eq. (9.2.10), eliminates the generalized velocities. The matrix form of the result is

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\{\dot{q}\}^{\mathrm{T}}[M]\{\dot{q}\} - T_0(q_i,t) + V(q_i,t) \\ &= \frac{1}{2}\left\{\{p\}^{\mathrm{T}} - \{N(q_i,t)\}^{\mathrm{T}}\right\}[M(q_i,t)]^{-1}\left\{\{p\} - \{N(q_i,t)\}\right\} - T_0 + V.\end{aligned} \tag{9.2.12}$$

When we collect like terms in this expression and recognize that $[M]$ is symmetric, so that $\left([M]^{-1}\right)^{\mathrm{T}} = [M]^{-1}$, we obtain a definition of the Hamiltonian that contains only the p_n and q_n variables, specifically,

$$\boxed{\begin{aligned}\mathcal{H} = &\frac{1}{2}\{p\}^{\mathrm{T}}[M(q_i,t)]^{-1}\{p\} - \{N(q_i,t)\}^{\mathrm{T}}[M(q_i,t)]^{-1}\{p\} \\ &- \frac{1}{2}\{N(q_i,t)\}^{\mathrm{T}}[M(q_i,t)]^{-1}\{N(q_i,t)\} - T_0(q_i,t) + V(q_i,t).\end{aligned}} \tag{9.2.13}$$

Now that we know how to construct $\mathcal{H}$, we turn our attention to identifying a corresponding set of equations of motion. The most direct route is to return to the original definition, Eq. (9.2.8). To ascertain some identities obeyed by derivatives of $\mathcal{H}$, we form the total time derivative $d\mathcal{H}/dt$ by using two perspectives. From one view $\mathcal{H}$ is a general function of the p_j, q_j, and t, so that

$$\frac{d\mathcal{H}}{dt} = \sum_{j=1}^{N}\frac{\partial\mathcal{H}}{\partial p_j}\dot{p}_j + \sum_{j=1}^{N}\frac{\partial\mathcal{H}}{\partial q_j}\dot{q}_j + \frac{\partial\mathcal{H}}{\partial t}. \tag{9.2.14}$$

In the other view Eq. (9.2.8) explicitly defines $\mathcal{H}$ in terms of these variables, so differentiating that expression yields

$$\begin{aligned}\frac{d\mathcal{H}}{dt} &= \sum_{j=1}^{N}(\dot{p}_j\dot{q}_j + p_j\ddot{q}_j) - \left[\sum_{j=1}^{N}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\ddot{q}_j + \frac{\partial\mathcal{L}}{\partial q_j}\dot{q}_j\right) + \frac{\partial\mathcal{L}}{\partial t}\right] \\ &= \sum_{j=1}^{N}(\dot{p}_j\dot{q}_j + p_j\ddot{q}_j) - \sum_{j=1}^{N}\left(p_j\ddot{q}_j + \frac{\partial\mathcal{L}}{\partial q_j}\dot{q}_j\right) - \frac{\partial\mathcal{L}}{\partial t} \\ &= \sum_{j=1}^{N}\left(\dot{p}_j\dot{q}_j - \frac{\partial\mathcal{L}}{\partial q_j}\dot{q}_j\right) - \frac{\partial\mathcal{L}}{\partial t}.\end{aligned} \tag{9.2.15}$$

Both descriptions of the time derivative must be the same, regardless of the details of the system's response. To enforce this assertion we observe that, although p_j and $\dot{q}_j$ are related by Eq. (9.2.9), one set of variables does not define the other because the relation depends on the current values of the generalized coordinates. Thus we may consider the p_j variables at any instant to be independent of the $\dot{q}_j$ variables. This means that the coefficient of each $\dot{p}_j$ in Eq. (9.2.14) must match the corresponding coefficient in Eq. (9.2.15). When this equality holds, then the remaining terms that are coefficients of the $\dot{q}_j$ must also match, which in turn requires that the remaining partial time derivatives also match. The result is that

$$\dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j}, \quad \frac{\partial \mathcal{L}}{\partial q_j} = -\frac{\partial \mathcal{H}}{\partial q_j}, \quad \frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial \mathcal{H}}{\partial t}. \tag{9.2.16}$$

We substitute these relations and the definition of the p_j variables as derivatives $\mathcal{L}$ into Lagrange's equations, which yields

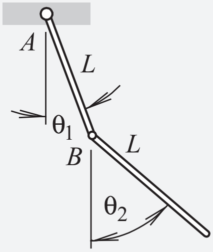
$$\boxed{\begin{aligned} \dot{q}_j &= \frac{\partial \mathcal{H}}{\partial p_j}, \\ \dot{p}_j + \frac{\partial \mathcal{H}}{\partial q_j} &= Q_j. \end{aligned}} \tag{9.2.17}$$

These are *Hamilton's equations of motion*.

The only occurrence of generalized velocities is in the first set of equations, because the $\dot{q}_j$ have been removed from $\mathcal{H}$. These equations, which describe how the generalized coordinates are influenced by the other effects, are the scalar equivalent of Eq. (9.2.11). Some individuals prefer to use that relation to obtain the $\dot{q}_j$ variables, rather than differentiating $\mathcal{H}$. In any event, Hamilton's equations constitute a set of $2N$ coupled, first-order differential equations governing the N generalized coordinates q_j and N generalized momenta p_j. If the generalized coordinates are constrained, these equations must be supplemented by the constraint equations, in which any occurrence of generalized velocity must be replaced with its relation to the generalized momenta, as described by Eq. (9.2.11). The constraint forces would then appear in the generalized forces, or alternatively, the Lagrange multiplier terms may be added to the generalized force array.

When both sets of Hamilton's equations are assembled, the result has an attractive state-space form, in that only the state vector $\{x\} = [\{q\}^{\mathrm{T}} \ \ \{p\}^{\mathrm{T}}]^{\mathrm{T}}$ appears on the left side. This feature might seem to make Hamilton's equations more efficient for numerical analysis because evaluation of the right side of the assembled equations gives the value of $\{\dot{x}\}$ to be sent to a differential equation solver. In comparison, numerical solution of the Lagrange equations for a holonomic system require evaluation of the current generalized accelerations by solving $[M]\{\ddot{q}\} = \{F\}$; see Section 7.6. The fact that Hamilon's equations of motion directly give the $\{\dot{x}\}$ vector, whereas the state-space formulation of Lagrange's equations requires intermediate calculations, might seem to make Hamilton's equations more efficient for a numerical solution. However, this is an incomplete view because it fails to recognize that $[M]$ must be inverted algebraically to form $\mathcal{H}$ according to Eq. (9.2.13), whereas evaluation of $\{\ddot{q}\}$ for the state-space equations can be done by generating a formal inverse of $[M]$ or by numerically solving $[M]\{\ddot{q}\} = \{F\}$ at each instant.

EXAMPLE 9.3 The angles of rotation are selected as generalized coordinates for the double pendulum, whose bars have equal mass m. Derive Hamiltonian equations of motion.



Example 9.3

SOLUTION This system is sufficiently simple that we could skip some steps, but we will follow the formal procedures in order to get a good picture of what operations are required for formulating Hamilton's equations for a complicated system. We begin the analysis by expressing the Lagrangian in terms of the generalized coordinates and velocities. Bar AB is in pure rotation about end A, and bar BC is in general motion. We obtain the velocity of the center of mass G of bar BC by differentiating its position, which gives

$$\bar{v}_G = \left(L\dot{\theta}_1 \cos\theta_1 + \frac{L}{2}\dot{\theta}_2 \cos\theta_2\right)\bar{i} + \left(L\dot{\theta}_1 \sin\theta_1 + \frac{L}{2}\dot{\theta}_2 \sin\theta_2\right)\bar{j},$$

where $\bar{i}$ is to the right and $\bar{j}$ is upward. The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}m\bar{v}_G\cdot\bar{v}_G + \frac{1}{2}I_G\dot{\theta}_2^2 \\ &= \frac{1}{2}\left(\frac{1}{3}mL^2\right)\dot{\theta}_1^2 + \frac{1}{2}m\Big[L^2\dot{\theta}_1^2 + \frac{L^2}{4}\dot{\theta}_2^2 \\ &\quad + L^2\dot{\theta}_1\dot{\theta}_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)\Big] + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}_2^2 \\ &= \frac{1}{2}mL^2\left[\frac{4}{3}\dot{\theta}_1^2 + \frac{1}{3}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)\right]. \end{aligned}$$

The elevation of pin A is a convenient datum for gravitational potential energy. The corresponding Lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}mL^2\left[\frac{4}{3}\dot{\theta}_1^2 + \frac{1}{3}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1)\right] \\ &\quad + mg\frac{L}{2}\cos\theta_1 + mg\left(L\cos\theta_1 + \frac{L}{2}\cos\theta_2\right). \end{aligned} \tag{1}$$

The next step is to form the generalized momenta. The kinetic energy has only quadratic terms in the generalized velocities, so $\{N\} = \{0\}$ and $T_0 = 0$. The easiest way to identify $[M]$ is to form the p_j variables by differentiating $\mathcal{L}$ and then compare the result with Eq. (9.2.9). Differentiation yields

$$\begin{aligned} p_1 &= \frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = mL^2\left[\frac{4}{3}\dot{\theta}_1 + \frac{1}{2}\dot{\theta}_2\cos(\theta_2 - \theta_1)\right], \\ p_2 &= \frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} = mL^2\left[\frac{1}{3}\dot{\theta}_2 + \frac{1}{2}\dot{\theta}_1\cos(\theta_2 - \theta_1)\right]. \end{aligned}$$

Matching this to $\{p\} = [M]\{\dot{q}\} + \{N\}$ leads to

$$[M] = mL^2 \begin{bmatrix} \frac{4}{3} & \frac{1}{2}\cos(\theta_2 - \theta_1) \\ \frac{1}{2}\cos(\theta_2 - \theta_1) & \frac{1}{3} \end{bmatrix}. \tag{2}$$

The inverse of the inertia matrix is

$$[M]^{-1} = \frac{1}{mL^2\Delta} \begin{bmatrix} 12 & -18\cos(\theta_2 - \theta_1) \\ -18\cos(\theta_2 - \theta_1) & 48 \end{bmatrix}, \tag{3}$$

where

$$\Delta = 16 - 9\cos(\theta_2 - \theta_1)^2. \tag{4}$$

From this, we form $\mathcal{H}$ by following Eq. (9.2.13), which yields

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}[p_1 \;\; p_2][M]^{-1}\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} + V \\ &= \frac{1}{2mL^2\Delta}\left[12p_1^2 + 48p_2^2 - 36p_1p_2\cos(\theta_2 - \theta_1)\right] \\ &\quad - mg\frac{L}{2}(3\cos\theta_1 + \cos\theta_2). \end{aligned} \tag{5}$$

All applied forces are conservative forces, and the generalized coordinates are unconstrained, so the generalized forces are zero.

We obtain the first set of Hamilton's equations by differentiating $\mathcal{H}$ with respect to the generalized momenta:

$$\begin{aligned} \dot{\theta}_1 &= \frac{\partial\mathcal{H}}{\partial p_1} = \frac{1}{mL^2\Delta}[12p_1 - 18p_2\cos(\theta_1 - \theta_2)], \\ \dot{\theta}_2 &= \frac{\partial\mathcal{H}}{\partial p_2} = \frac{1}{mL^2\Delta}[48p_2 - 18p_1\cos(\theta_1 - \theta_2)]. \end{aligned} \tag{6 $\triangleleft$}$$

Differentiation of $\mathcal{H}$ with respect to the generalized coordinates leads to the second portion of Hamilton's equations. The generalized coordinates appear in the inertial terms only as $\theta_2 - \theta_1$, which simplifies the operations. Thus,

$$\begin{aligned} \dot{p}_1 = -\frac{\partial\mathcal{H}}{\partial\theta_1} &= \frac{1}{2mL^2}\left\{\frac{1}{\Delta^2}\frac{d\Delta}{d\theta_1}\left[12p_1^2 + 48p_2^2 - 36p_1p_2\cos(\theta_2 - \theta_1)\right]\right\} \\ &\quad + \frac{1}{2mL^2\Delta}[-36\sin(\theta_2 - \theta_1)] - \frac{3}{2}mgL\sin\theta_1 \\ &= -\frac{18\sin(\theta_2 - \theta_1)}{mL^2\Delta^2}\left[(6p_1^2 + 24p_2^2)\cos(\theta_2 - \theta_1)\right. \\ &\quad \left. - p_1p_2\left(16 + 9\cos(\theta_2 - \theta_1)^2\right)\right] - \frac{3}{2}mgL\sin\theta_1, \end{aligned} \tag{7 $\triangleleft$}$$

$$\dot{p}_2 = -\frac{\partial \mathcal{H}}{\partial \theta_2} = \frac{18\sin(\theta_2-\theta_1)}{mL^2\Delta^2}\left[\left(6p_1^2+24p_2^2\right)\cos(\theta_2-\theta_1)\right.$$
$$\left.-\,p_1p_2\left(16+9\cos(\theta_2-\theta_1)^2\right)\right]-\frac{1}{2}mgL\sin\theta_2.$$

Given that this system is relatively uncomplicated, it is clear that forming Hamilton's equations in more engineering-oriented systems is likely to be a tedious process, unless one uses symbolic mathematical software.

9.2.2 Conservation of the Hamiltonian

One of the features of Hamilton's equations that has attracted attention is that they permit canonical transformations. The result of such transformations are new variables other than generalized coordinates and momenta that nevertheless satisfy Hamilton's equations. Goldstein (1980) and Desloges (1982) give good explanations of this application. For us, the primary value of the Hamiltonian formulation lies in an associated conservation theorem. We return to Eq. (9.2.15), into which we substitute the definition of a generalized momentum, Eq. (9.2.6), in order to eliminate p_j. The altered form is

$$\frac{d\mathcal{H}}{dt} = \sum_{j=1}^{N}\left[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_j}\right)-\frac{\partial\mathcal{L}}{\partial q_j}\right]\dot{q}_j-\frac{\partial\mathcal{L}}{\partial t}. \tag{9.2.18}$$

The term in the brackets equals the generalized force in Lagrange's equations, so we find that

$$\frac{d\mathcal{H}}{dt} = \sum_{j=1}^{N} Q_j\dot{q}_j-\frac{\partial\mathcal{L}}{\partial t}. \tag{9.2.19}$$

The first case we consider is a time-invariant system, which means that the position of all points at any instant depends on the generalized coordinates in a functional relationship that does not explicitly display the elapsed time. This has a number of consequences regarding Eq. (9.2.19) because physical velocities are related homogeneously to the generalized velocities according to

$$\bar{v}\left(\dot{q}_i, q_i, t\right) = \sum_{j=1}^{N}\frac{\partial\bar{r}}{\partial q_j}\dot{q}_j. \tag{9.2.20}$$

Correspondingly, the power input by a set of nonconservative forces is

$$\dot{W} = \sum_n \bar{F}_n\cdot\bar{v}_n = \sum_n \bar{F}_n\cdot\sum_{j=1}^{N}\left(\frac{\partial\bar{r}_n}{\partial q_j}\right)\dot{q}_j. \tag{9.2.21}$$

According to Eq. (7.4.4) the generalized forces are describable as

$$Q_j = \sum_n \bar{F}_n\cdot\frac{\partial\bar{r}_n}{\partial q_j}. \tag{9.2.22}$$

A rearrangement of the sequence in which the sums in Eq. (9.2.21) are performed reveals that the power input $\dot{W}$ for a time-invariant system attributable to nonconservative forces is given by

$$\dot{W} = \sum_{j=1}^{N} Q_j \dot{q}_j. \tag{9.2.23}$$

Another corollary of the time-invariant feature is that the kinetic energy consists solely of terms in which the generalized velocities appear quadratically, $T = T_2(\dot{q}_i, q_i)$. According to Eq. (9.2.10), $\mathcal{H}$ is identical to the mechanical energy in this case. In addition to T not depending explicitly on the elapsed time, V is also independent of t, because the position is known if the q_j values are known, regardless of the value of t. It follows that $\partial\mathcal{L}/\partial t = 0$. Hence, we find that Eq. (9.2.19) reduces to $\dot{E} = \dot{W}$. This is the power balance law, that is, the time derivative of the basic work–energy relation. Thus no new insights will result from evaluating the Hamiltonian for a time-invariant system.

Consider now an important special class of time-dependent systems in which velocities contain a term that is independent of generalized velocities, but they do not explicitly depend on t. In other words, suppose that positions have the form $\bar{r}(q_i, t)$, but $\partial\bar{r}/\partial t$ is not an explicit function of t, so that all velocities are described by

$$\bar{v}(\dot{q}_i, q_i, t) = \sum_{j=1}^{N} \bar{v}_j(q_i)\dot{q}_j + \bar{v}_0(q_j), \quad \bar{v}_j = \frac{\partial\bar{r}}{\partial q_j}. \tag{9.2.24}$$

A body that precesses at a constant rate typifies this situation. The precession angle in that case is proportional to t, so we do not take it to be a generalized coordinate. All positions depend on this angle, which means that they are explicitly time dependent. In contrast, the angular velocity components relative to body-fixed coordinate axes are independent of the precession angle. Consequently, the product of the precession rate and a unit vector parallel to the precession axis will represent a contribution to $\bar{\omega}$ that does not contain a generalized velocity; this is the type of quantity that $\bar{v}_0$ represents.

When Eq. (9.2.24) applies for all relevant velocities, both T_0 and T_1 are likely to be nonzero. In that case, $\mathcal{H}$ will be different from E, even though T does not depend explicitly on t. If it happens that V also is not an explicit function of t, then $\partial\mathcal{L}/\partial t = 0$. The consequence is that Eq. (9.2.19) indicates that

$$\frac{d\mathcal{H}}{dt} = \sum_{j=1}^{N} Q_j \dot{q}_j. \tag{9.2.25}$$

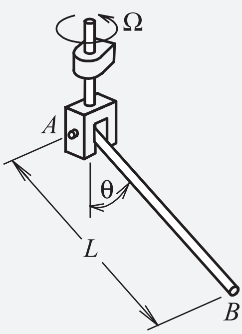
Because $\mathcal{H}$ is different from E, the left side is not the rate at which mechanical energy changes. Correspondingly, the summation on the right side does not represent the power input by nonconservative forces.

To understand the difference between Eq. (9.2.25) and the work–energy principle, suppose the generalized coordinates are unconstrained. Constraint forces then do not appear in any of the generalized forces, so the left side of Eq. (9.2.25) is the instantaneous power of the applied forces. However, making a system move in the specified time-dependent manner requires that there be constraint forces to sustain the motion.

The work done by these forces is not described by Eq. (9.2.25), but it does contribute to the power balance law. For example, consider the situation discussed previously, in which a body precesses at a constant rate. A torque must be applied to maintain that rate, regardless of how the nutation angle changes. The product of this torque and the precession rate is the power input from a constraint force. If the precession angle is used as a generalized coordinate, this power input does not contribute to the right side of Eq. (9.2.25).

It follows that Eq. (9.2.19) for time-dependent systems is quite different from the power balance form of the basic work–energy principle. If it happens that $\partial\mathcal{L}/\partial t = 0$, then we may employ Eq. (9.2.25) in the same way that we would use the power balance law for a time-invariant system. A further specialization pertains to a conservative system, in which case all Q_j are zero. Equation (9.2.25) indicates that, if $\partial\mathcal{L}/\partial t = 0$ for a conservative system, then $d\mathcal{H}/dt = 0$, which is equivalent to stating that $\mathcal{H}$ is a constant quantity. This is *Jacobi's integral*. Knowledge of constancy of $\mathcal{H}$ can be used in the same way that one would apply conservation of energy to a conservative system. Thus we may obtain a relation for the rate variables at some position by equating the value of $\mathcal{H}$ at that position to the value at the initial position, where the system's state presumably is known.

EXAMPLE 9.4 Application of an unspecified torque Γ causes the system to rotate about the vertical axis at the known constant rate Ω. Pin A has ideal properties, which allows the angle θ to change freely. Compare the rate of change of the Hamiltonian of the system with the result of applying the power balance principle.



Example 9.4

SOLUTION This example clarifies the difference between Jacobi's integral and energy principles. The central quantity is the kinetic energy as a function of the generalized coordinates. The only generalized coordinate required here is the nutation angle θ because the precession rate is specified as $\dot{\psi} = \Omega$. Let pin A be the origin of xyz, which is attached to bar AB such that its x axis is aligned with the bar and the z axis is situated in the vertical plane. Then the angular velocity of the bar is

$$\bar{\omega} = -\Omega\cos\theta\bar{i} - \dot{\theta}\,j + \Omega\sin\theta\bar{k}.$$

With the pin defined to be the datum for gravitational potential energy, the corresponding energy functions are

$$T = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\left[\dot{\theta}^2 + \Omega^2 (\sin\theta)^2\right], \quad V = -mg\frac{L}{2}\cos\theta. \tag{1}$$

Time does not occur explicitly in these expressions, so Eq. (9.2.25) is applicable. The generalized momentum corresponding to the sole generalized coordinate is

$$p_1 = \frac{\partial T}{\partial \dot{\theta}} = \frac{1}{3}mL^2\dot{\theta}, \tag{2}$$

so $M_{1,1} = (1/3)\,mL^2$. Inspection of T shows that it does not contain a term that is linear in $\dot{\theta}$, so the kinetic energy matches the standard form, Eqs. (7.6.4), with

$$N_1 = 0, \quad T_0 = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\Omega^2 (\sin\theta)^2 .$$

These expressions enable us to employ Eq. (9.2.13) to derive the Hamiltonian, which leads to

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\frac{1}{M_{1,1}}p_1^2 - T_0 + V \\ &= \frac{3}{2mL^2}p_1^2 - \frac{1}{6}mL^2\Omega^2 (\sin\theta)^2 - mg\frac{L}{2}\cos\theta. \end{aligned} \tag{3}$$

A virtual displacement increments θ by $\delta\theta$, but the only force that does work in such a movement is gravity, which is included in V. Therefore $Q_1 = 0$, from which it follows that Jacobi's integral, which is the conservation form of Eq. (9.2.25), applies. We may establish the constant value $\mathcal{H}_0$ of the Hamiltonian by evaluating it at the initial state. Hence any motion of the system must be such that the Hamiltonian in Eq. (3) remains constant:

$$\frac{3}{2mL^2}p_1^2 - \frac{1}{6}mL^2\Omega^2 (\sin\theta)^2 - mg\frac{L}{2}\cos\theta = \mathcal{H}_0.$$

We may express this conservation equation in terms of θ by substituting Eq. (2), which yields

$$\frac{1}{6}mL^2\left[\dot{\theta}^2 - \Omega^2 (\sin\theta)^2\right] - mg\frac{L}{2}\cos\theta = \mathcal{H}_0. \tag{4} \ \triangleleft$$

This relation is an integral of the equation of motion, just like the principle of conservation of energy for a conservative, time-independent system. Knowledge of the constancy of $\mathcal{H}$ can be used to determine a value of $\dot{\theta}$ at a specified θ. Another use is to check the results of a computer simulation.

Now let us examine the power balance principle for this system. The mechanical energy is $E = T + V$, so

$$E = \frac{1}{6}mL^2\left[\dot{\theta}^2 + \Omega^2 (\sin\theta)^2\right] - mg\frac{L}{2}\cos\theta.$$

The only nonconservative force that does work when the system moves is the couple Γ that sustains a constant precession rate Ω, so the power input is

$$\dot{W} = \Gamma\Omega.$$

The work–energy principle states that $\dot{E} = \dot{W}$, which leads to

$$\frac{1}{3}mL^2\left[\dot{\theta}\ddot{\theta} + \Omega^2\dot{\theta}\left(\sin\theta\right)\left(\cos\theta\right)\right] + mg\frac{L}{2}\dot{\theta}\sin\theta = \Gamma\Omega. \qquad (5) \lhd$$

We may derive another differential equation for θ from Jacobi's integral by observing that $d\mathcal{H}/dt = 0$. Differentiation of Eq. (4) shows that

$$\frac{1}{3}mL^2\left[\dot{\theta}\ddot{\theta} - \Omega^2\dot{\theta}\left(\sin\theta\right)\left(\cos\theta\right)\right] + mg\frac{L}{2}\dot{\theta}\sin\theta = 0. \qquad 6\ (\lhd)$$

This is identical to Lagrange's equation for θ multiplied by an additional θ factor. Taking the difference between Eqs. (5) and (6) yields an expression for Γ:

$$\Gamma = \frac{2}{3}mL^2\Omega\dot{\theta}\left(\sin\theta\right)\left(\cos\theta\right). \qquad (7)$$

Equations (5) and (6) can be obtained by an alternative formulation that uses as constrained generalized coordinates $q_1 = \theta$ and $q_2 = \psi$, which must satisfy the constraint equation that $\dot{\psi} = \Omega$. The Lagrange equation for θ would be the same as Eq. (6) divided by the common factor $\dot{\theta}$. Substitution of the condition that $\dot{\psi} = \Omega$, $\ddot{\psi} = 0$, into the Lagrange equation for ψ would yield Eq. (7).

9.2.3 Ignorable Coordinates and Routh's Method

Consideration of generalized momenta in some circumstances can lead to additional conservation theorems, as well as simplification of the equations of motion. The first situation arises when the Lagrangian does not depend explicitly on a specific generalized coordinate q_n, even though the corresponding $\dot{q}_n$ does appear explicitly. Because $\partial\mathcal{L}/\partial q_n = 0$ in this case, Lagrange's equation for this generalized coordinate is

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_n}\right) \equiv \frac{dp_n}{dt} = Q_n. \qquad (9.2.26)$$

The solution of this differential equation is

$$p_n|_{t_2} = p_n|_{t_2} + \int_{t_1}^{t_2} Q_n dt. \qquad (9.2.27)$$

This is a *generalized impulse–momentum principle*. Both linear and angular momenta are described by it, depending on the type of geometric quantity associated with q_n. It should be noted that, if Q_n depends on any of the generalized coordinates, this momentum principle could not be used independently to relate the states at t_1 and t_2, because the impulse could not be evaluated until the time dependence of the generalized coordinates has been determined. Such a situation corresponds to the Newtonian formulation of position-dependent forces, in which momentum principles are not used because the impulse cannot be evaluated.

Now consider the more restrictive situation, in which $\mathcal{L}$ does not depend on a specific q_n and the corresponding generalized force Q_n vanishes, $Q_n = 0$. We find from the foregoing that p_n is constant, which corresponds to *conservation of a generalized momentum*. When such a situation occurs, the value of p_n may be computed from the system's initial conditions. On the other hand, when we return to the definition of p_n we obtain an expression that depends on the full set of generalized coordinates and velocities. Thus, if we form

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_n} = p_n = \text{ constant,} \tag{9.2.28}$$

we may algebraically solve this relation for $\dot{q}_n$ in terms of the other generalized coordinates and generalized velocities. This solution may be substituted into the remaining Lagrange's equations to obtain equations of motion in which neither q_n nor $\dot{q}_n$ appears. For this reason q_n is said to be an *ignorable coordinate*.[SS]

We used this procedure to simplify the equations of motion in Example 7.14 in Chapter 7. Although the resulting equations will have a more complicated form in comparison with those that feature the ignorable coordinate, there will be fewer equations to solve. Thus one should assess whether the effort to remove the ignorable coordinate is warranted in view of the potential improvement for the solution. When several generalized coordinates are ignorable, the set of momentum conservation equations, Eq. (9.2.28), comprises simultaneous algebraic equations that may be solved for each ignorable coordinate's generalized velocity in terms of the other variables.

An important aspect of the elimination process is that the coordinates are removed by operating on the Lagrange's equations that are derived by use of the full set of equations. It might seem that a shortcut would be to substitute the $\dot{q}_n$ expressions for the ignorable coordinates derived from Eq. (9.2.28) directly into the Lagrangian and use that reduced Lagrangian to obtain the equations for the nonignorable coordinates. Doing so would be incorrect because a fundamental step leading to Lagrange's equations required that the virtual increments in the generalized coordinates be completely arbitrary.

Routh's method for the ignoration of coordinates is an alternative elimination procedure that enables us to remove the ignorable coordinates before the equations for the nonignorable coordinates are derived. In essence, it derives the Lagrangian for an equivalent system having a reduced number of degrees of freedom. To develop the procedure, let K denote the number of generalized coordinates, sequenced as q_1 to q_K, that appear explicitly in the Lagrangian, and therefore are not ignorable. Then there are $N - K$ ignorable coordinates, sequenced as q_{K+1} to q_N. The first step in following Routh's method is to form Eq. (9.2.28) for each ignorable coordinate and to solve those equations algebraically. These operations will yield the ignorable generalized velocities

[SS] Rather than being said to be ignorable, such variables are often said to be *cyclic coordinates*. This name stems from the observation that most cases in which a generalized coordinate is ignorable involve rotation about an axis. Occasionally, these variables are said to be *kinosthenic coordinates*.

in terms of the other variables, that is,

$$\frac{\partial \mathcal{L}}{\partial q_n} = p_n \text{ for } n = K+1, \ldots, N \text{ yields}$$
$$\dot{q}_n = f_n\left(\dot{q}_1, \ldots, \dot{q}_K, q_1, \ldots, q_J, p_{K+1}, \ldots, p_N, t\right) \text{ for } n = K+1, \ldots, N. \tag{9.2.29}$$

The *Routhian function* is defined in terms of the Lagrangian for the system according to

$$\Re = \mathcal{L} - \sum_{n=K+1}^{N} p_n \dot{q}_n. \tag{9.2.30}$$

The result of substituting the expressions for the ignorable generalized velocities is that the functional dependence of the Routhian is $\Re(\dot{q}_1, \ldots, \dot{q}_K, q_1, \ldots, q_K, p_{K+1}, \ldots, p_N, t)$. The definition of $\Re$ is reminiscent of $\mathcal{H}$. Thus it should not be surprising that the Routhian equations of motion are derived by a procedure that is like the route we took to derive Hamilton's equations. As was true for the direct elimination procedure discussed previously, here we must recognize that, although $p_{N_q+1}, \ldots p_N$, are constants in the actual motion, they must be treated as variables when we derive equations of motion. Rather than a time derivative, we consider the virtual increment of the Routhian. In view of the general functional dependence of $\Re$, its variation is described by

$$\delta\Re = \sum_{j=1}^{K} \left(\frac{\partial \Re}{\partial \dot{q}_j} \delta \dot{q}_j + \frac{\partial \Re}{\partial q_j} \delta q_j \right) + \sum_{n=K+1}^{N} \frac{\partial \Re}{\partial p_n} \delta p_n. \tag{9.2.31}$$

Alternatively, we may form $\delta\Re$ based on the definition in Eq. (9.2.30). Because the Lagrangian does not depend on ignorable coordinates, the variation of $\Re$ is

$$\begin{aligned} \delta\Re &= \sum_{j=1}^{K} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j} \delta \dot{q}_j + \frac{\partial \mathcal{L}}{\partial q_j} \delta q_j \right) + \sum_{n=K+1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{q}_n} \delta \dot{q}_n - \sum_{n=K+1}^{N} \left(p_n \delta \dot{q}_n + \delta p_n \dot{q}_n \right) \\ &= \sum_{j=1}^{K} \left(p_j \delta \dot{q}_j + \frac{\partial \mathcal{L}}{\partial q_j} \delta q_j \right) - \sum_{n=K+1}^{N} \delta p_n \dot{q}_n. \end{aligned} \tag{9.2.32}$$

We are free to select the time dependence of each δq_j arbitrarily. Hence, like coefficients of the virtual increments in the alternative representations of $\delta\Re$ in Eqs. (9.2.31) and (9.2.32) must match, so that

$$\begin{gathered} \frac{\partial \Re}{\partial \dot{q}_j} = \frac{\partial \mathcal{L}}{\partial \dot{q}_j}, \quad \frac{\partial \Re}{\partial q_j} = \frac{\partial \mathcal{L}}{\partial q_j}, \quad j = 1, \ldots, K, \\ \frac{\partial \Re}{\partial p_n} = -\dot{q}_n, \quad n = K+1, \ldots, N \end{gathered} \tag{9.2.33}$$

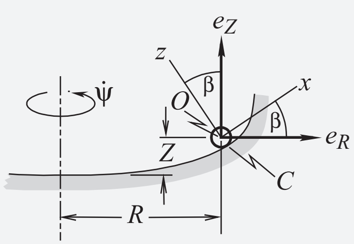
When we substitute the first two identities into Lagrange's equations we obtain the Routhian equations of motion:

$$\boxed{\frac{d}{dt}\left(\frac{\partial \Re}{\partial q_j}\right) - \frac{\partial \Re}{\partial q_j} = Q_j, \quad j = 1, \ldots, K.} \tag{9.2.34}$$

In other words, the Routhian may be considered to be the Lagrangian for a system described by K generalized coordinates. If the generalized coordinates are an unconstrained set, then the Routhian represents an equivalent system having K degrees of freedom.

EXAMPLE 9.5 Assume that the ball within a roulette wheel rolls without slipping and that rolling resistance is negligible. The position of the center of the ball is defined by two cylindrical coordinates, the azimuthal angle θ measured about the wheel's rotation axis and the transverse distance R measured from that axis. The elevation is a function $Z(R)$ that specifies the profile of the roulette wheel, and σ is the radius of the ball. The mass of the ball is m, the moment of inertia of the wheel is I, and it is permissible to assume that the roulette wheel rolls without slipping relative to the roulette wheel. Determine the Routhian corresponding to the minimum number of differential equations of motion governing this system.

SOLUTION This example is an interesting application of Routh's method, but the omission of numerous effects, such as bearing and rolling friction, makes the model incapable of predicting the actual motion of a ball in a roulette wheel. The absence of slippage means that it is sufficient to select generalized coordinates that locate the center of the ball, and another generalized coordinate is required to locating the roulette wheel. Thus we select $q_1 = R$, $q_2 = \theta$, and $q_3 = \psi$, which is the angle the wheel rotates about the vertical.



Example 9.5

We begin with a kinematical analysis in terms of the generalized coordinates. The sketch shows a vertical cross section. This plane has rotated by angle θ relative to some reference orientation, and R is the perpendicular distance to the center of the ball from the wheel's axis, which corresponds to a set of cylindrical coordinates having horizontal and vertical unit vectors, as shown. The sketch also shows an xyz coordinate system that is defined such that the z axis is normal to the surface of the wheel at contact point C with the xz plane always coincident with the plane of the sketch. Because the x axis is tangent to the surface in the vertical plane, its angle of elevation β is related to the slope $z' \equiv dz/dR$, so that

$$\beta = \tan^{-1}(Z') \implies \sin\beta = \frac{Z'}{\left[1+(Z')^2\right]^{1/2}}, \quad \cos\beta = \frac{1}{\left[1+(Z')^2\right]^{1/2}}, \\ \bar{e}_R = \cos\beta\bar{i} - \sin\beta\bar{k}, \quad \bar{e}_Z = \sin\beta\bar{i} + \cos\beta\bar{k}. \tag{1}$$

The angular velocity of the ball consists of the precession $\dot{\theta}$ at which the plane of the sketch rotates about the vertical axis, onto which is superposed unknown rotations Ω_x about the x axis and Ω_z about the z axis. Therefore the angular velocity of the ball is given by

$$\bar{\omega} = \dot{\theta}\left(\sin\beta\bar{i} + \cos\beta\bar{k}\right) + \Omega_x\bar{i} + \Omega_y\bar{j}. \tag{2}$$

The no-slip condition requires that $\bar{v}_O = \bar{v}_C + \bar{\omega} \times \bar{r}_{O/C}$, where the position of the center O relative to the contact point C is $\bar{r}_{O/C} = \sigma\bar{k}$. The transverse distance from the axis of rotation to the contact point is $R_C = R + \sigma\sin\beta$, and the velocity of this point on the roulette wheel is $R_C\dot{\psi}\bar{j}$. The condition that there is no slippage at point C therefore leads to

$$\begin{aligned}\bar{v}_O &= R_C\dot{\psi}\bar{j} + \left[\left(\dot{\theta}\sin\beta + \Omega_x\right)\bar{i} + \Omega_y\bar{j} + \dot{\theta}\cos\beta\bar{k}\right] \times \sigma\bar{k} \\ &= \Omega_y\sigma\bar{i} + \left[R_C\dot{\psi} - \left(\dot{\theta}\sin\beta + \Omega_x\right)\sigma\right]\bar{j}.\end{aligned} \tag{3}$$

We also can describe the velocity of the ball's center in terms of the cylindrical coordinates. Here we use the fact that Z is a known function of R, so that $\dot{Z} = \dot{R}(dZ/dR) \equiv \dot{R}Z'$. In view of the representation of the cylindrical coordinate unit vectors in Eqs. (1), it follows that

$$\begin{aligned}\bar{v}_O &= \dot{R}\bar{e}_R + R\dot{\theta}\bar{e}_\theta + \dot{Z}\bar{k} \\ &= \left(\dot{R}\cos\beta + \dot{R}Z'\sin\beta\right)\bar{i} + R\dot{\theta}\bar{j} + \left(-\dot{R}\sin\beta + \dot{R}Z'\cos\beta\right)\bar{k} \\ &\equiv \frac{\dot{R}}{\cos\beta}\bar{i} + R\dot{\theta}\bar{j},\end{aligned} \tag{4}$$

where the last form is a consequence of the definition of β in Eqs. (1). Each component of this description of $\bar{v}_O$ must match the corresponding component in Eq. (3), which leads to

$$\begin{aligned}\Omega_y &= \frac{\dot{R}}{\sigma}\dot{R}\left[1+(Z')^2\right]^{1/2}, \\ \Omega_x &= -\left(\frac{R}{\sigma} + \sin\beta\right)\dot{\theta} + \frac{R_C}{\sigma}\dot{\psi} \equiv \left(\frac{R}{\sigma} + \sin\beta\right)\left(\dot{\psi} - \dot{\theta}\right).\end{aligned}$$

The angular velocity in Eq. (2) correspondingly is

$$\bar{\omega} = \left[\left(\frac{R}{\sigma} + \sin\beta\right)\dot{\psi} - \frac{R}{\sigma}\dot{\theta}\right]\bar{i} + \frac{\dot{R}}{\sigma\cos\beta}\bar{j} + \dot{\theta}\cos\beta\bar{k}. \tag{5}$$

Because β is a known function of R, Eqs. (4) and (5) are suitable for describing the energy functionals in terms of the generalized coordinates. The moment of inertia of the sphere about any centroidal axis is $(2/5)\,m\sigma^2$, so the kinetic energy is

$$\begin{aligned}T &= \frac{1}{2}m\bar{v}_O\cdot\bar{v}_O+\frac{1}{2}\left(\frac{2}{5}m\sigma^2\right)\bar{\omega}\cdot\bar{\omega}\\ &= \frac{1}{2}m\left[\left(\frac{\dot{R}}{\cos\beta}\right)^2+\left(R\dot{\theta}\right)^2\right]\\ &\quad+\frac{1}{2}\left(\frac{2}{5}m\sigma^2\right)\left\{\left[\left(\frac{R}{\sigma}+\sin\beta\right)\dot{\psi}-\frac{R}{\sigma}\dot{\theta}\right]^2\right.\\ &\quad\left.+\left(\frac{\dot{R}}{\sigma\cos\beta}\right)^2+\left(\dot{\theta}\cos\beta\right)^2\right\}+\frac{1}{2}I\dot{\psi}^2.\end{aligned}$$

Collecting like coefficients in this expression yields

$$\begin{aligned}2T &= \frac{7}{5}m\left(\frac{\dot{R}}{\cos\beta}\right)^2+m\left[\frac{7}{5}R^2+\frac{2}{5}\sigma^2\left(\cos\beta\right)^2\right]\dot{\theta}^2\\ &\quad+\left[\frac{2}{5}m\left(R+\sigma\sin\beta\right)^2+I\right]\dot{\psi}^2-\frac{4}{5}\left(R^2+\sigma^2\sin\beta\right)\dot{\theta}\dot{\psi}.\end{aligned}$$

This expression for T has the standard quadratic form of kinetic energy. The inertia coefficients are

$$M_{1,1}=\frac{7}{5}\frac{m}{\left(\cos\beta\right)^2},\qquad M_{2,2}=m\left[\frac{7}{5}R^2+\frac{2}{5}\sigma^2\left(\cos\beta\right)^2\right],$$
$$M_{3,3}=\frac{2}{5}m\left(R+\sigma\sin\beta\right)^2+I,$$
$$M_{2,3}=M_{3,2}=-\frac{2}{5}m\left(R^2+\sigma R\sin\beta\right),\qquad M_{i,j}=0\text{ otherwise},\ \ N_j=T_0=0.$$

Gravity is conservative, with $Z(R)$ defining the elevation of the sphere, and no other force contributes to the virtual work. Thus the Lagrangian is

$$\mathcal{L}=\frac{1}{2}\left(M_{1,1}\dot{R}^2+M_{2,2}\dot{\theta}^2+M_{3,3}\dot{\psi}^2+2M_{2,3}\dot{\theta}\dot{\psi}\right)-mgZ(R).$$

The only generalized coordinate appearing explicitly in this expression is R, so both θ and ψ are ignorable coordinates. The corresponding generalized momenta are

$$\begin{aligned}p_2 &= \frac{\partial\mathcal{L}}{\partial\dot{\theta}}=M_{2,2}\dot{\theta}+M_{2,3}\dot{\psi},\\ p_3 &= \frac{\partial\mathcal{L}}{\partial\dot{\psi}}=M_{3,2}\dot{\theta}+M_{3,3}\dot{\psi}.\end{aligned}$$

The values of p_2 and p_3 are constants that may be evaluated from the initial conditions. The $M_{j,n}$ coefficients are known functions of R, so solution of these generalized momentum equations yields expressions for the values of $\dot{\theta}$ and $\dot{\psi}$ at a specified R. This solution is most conveniently described in matrix form. Let $[M_I]$ denote the submatrix of $[M]$ associated with the ignorable coordinates,

$$[M_I] = \begin{bmatrix} M_{2,2} & M_{2,3} \\ M_{3,2} & M_{3,3} \end{bmatrix},$$

which leads to

$$\begin{bmatrix} \dot{\theta} & \dot{\psi} \end{bmatrix}^{\mathrm{T}} = [M_I]^{-1} [p_2 \quad p_3]^{\mathrm{T}}.$$

Forming the Routhian with this expression is assisted by the fact that R is inertially uncoupled from the ignorable coordinates, so $\mathcal{L}$ may be rewritten as

$$\mathcal{L} = \frac{1}{2} M_{1,1} \dot{R}^2 + \frac{1}{2} \begin{bmatrix} \dot{\theta} & \dot{\psi} \end{bmatrix} [M_I] \begin{bmatrix} \dot{\theta} & \dot{\psi} \end{bmatrix}^{\mathrm{T}} - mgZ.$$

The summation in Eq. (9.2.30) defining the Routhian is equivalent to $\{p_I\}^{\mathrm{T}} \{\dot{q}_I\}$, where each vector contains the variables for the ignorable coordinates. Thus substitution of the solution for the ignorable generalized coordinates into the Routhian for the present system yields

$$\begin{aligned}
\Re &= \mathcal{L} - \{p_I\}^{\mathrm{T}} \{\dot{q}_I\} \\
&= \frac{1}{2} M_{1,1} \dot{R}^2 + \frac{1}{2} \begin{bmatrix} \dot{\theta} & \dot{\psi} \end{bmatrix} [M_I] \begin{bmatrix} \dot{\theta} & \dot{\psi} \end{bmatrix}^{\mathrm{T}} - [p_2 \quad p_3] \begin{bmatrix} \dot{\theta} & \dot{\psi} \end{bmatrix}^{\mathrm{T}} - mgZ \\
&= \frac{1}{2} M_{1,1} \dot{R}^2 + \frac{1}{2} [p_2 \quad p_3] [M_I]^{-1} [M_I] [M_I]^{-1} [p_2 \quad p_3]^{\mathrm{T}} \\
&\quad - [p_2 \quad p_3] [M_I]^{-1} [p_2 \quad p_3]^{\mathrm{T}} - mgZ \\
&= \frac{1}{2} M_{1,1} \dot{R}^2 - \frac{1}{2} [p_2 \quad p_3] [M_I]^{-1} [p_2 \quad p_3]^{\mathrm{T}} - mgZ.
\end{aligned} \qquad \triangleleft$$

Because Z and $[M_I]$ are known functions of R, the Routhian $\Re$ depends on only R and $\dot{R}$, so it represents the Lagrangian of an equivalent one-degree-of-freedom system. The first term, which contains the generalized velocity, is the equivalent kinetic energy, whereas the equivalent potential energy is the negative of the sum of the remaining terms. The single differential equation of motion is Lagrange's equation formulated in terms of $\Re$ rather than the Lagrangian, with $q_1 = R$ and $Q_1 = 0$. After this equation has been solved for R, the time dependence of θ and ϕ may be found by treating the definitions of p_2 and p_3 as first-order differantial equations. One advantage of having to analyze only one acceleration equation is that fundamental properties are readily established. For example, we may identify the equilibrium positions by seeking the values of R at which $\partial \Re / \partial R = 0$.

9.3 FORMULATIONS WITH QUASI-COORDINATES

The dynamics principles associated with Lagrange and Hamilton share the attribute that the sole kinematical variables to be determined are the generalized coordinates. They also share the feature of being based on formulating the kinetic and potential energies exclusively in terms of those variables. A different line of development has led to formulations in which the variables used to describe the velocity of a system are not the time derivatives of the generalized coordinates. We consider two such formulations here. The Gibbs–Appell equations bear some similarity to Lagrange's equations, in that their equations of motion are obtained by differentiation of a function of the kinematical variables, whereas Kane's equations are derived as a generalization of the dynamical virtual work principle.

Despite the difference in the way they are derived and their ultimate appearance, the Gibbs–Appell and Kane's equation formulations yield the same differential equations of motion. A source of confusion that has arisen is that some quantities are common to both formulations, but are referred to differently in each. We first develop the Gibbs–Appell approach, which chronologically preceded Kane's approach, using the terminology associated with it. The similarities will be pointed out when we address Kane's equations by using its associated terminology.

Implementing the Gibbs–Appell or Kane's equations requires that we describe the acceleration of a system. This is a distinct disadvantage relative to Lagrange's equations, which require only a velocity analysis. On the other hand, much freedom is gained by allowing us to select velocity variables independently of the position variables. The merits of these formulations will be discussed after each has been developed. In any event, our study of the Gibbs–Appell equations and Kane's equations will give a picture of the variety of possible approaches that we might encounter.

9.3.1 Quasi-Velocities and Quasi-Coordinates

By definition, we may uniquely describe the instantaneous position of a system in terms of the generalized coordinates q_j. Up to now, we have correspondingly described the instantaneous velocity of a system in terms of generalized velocities $\dot{q}_j$. However, this is not the only way in which we could describe velocity. For example, consider a general angular motion. The Eulerian angles could be the generalized coordinates, and the angular velocity components ω_x, ω_y, and ω_z could be the variables we use to describe velocity and acceleration in Euler's equations. Except for planar motion, an angular velocity component seldom is the rate at which a rotation angle changes. However, the definition of angular velocity is that it is $\overline{d\theta}/dt$, so the infinitesimal angle of rotation about the x axis during an interval dt would be $\omega_x dt$. Thus this is a situation in which a rate variable corresponds to a differential change, but there is no corresponding finite change. Rate variables like ω_x are said to be *quasi-velocities*, which are denoted as $\dot{\gamma}_j$. Correspondingly, the symbol γ_j is called a *quasi-coordinate*. The prefix "quasi" denotes that γ_j need not have any meaning as a position coordinate, whereas the $\dot{\gamma}_j$ parameters and the associated differential increments $d\gamma_j = \dot{\gamma}_j dt$ will have physical significance. It is acceptable, however, to employ a position variable as a quasi-coordinate, in which case we would have $\gamma_j = q_j$.

To get an idea of how quasi-coordinates arise, consider the familiar relation between the velocity of two points in a rigid body, $\bar{v}_B = \bar{v}_A + \bar{\omega} \times \bar{r}_{B/A}$, when we describe the position, and angular velocity vectors in terms of components relative to a body-fixed reference frame. The result is

$$\bar{v}_B = \bar{v}_A + \left(-r_z \bar{j} + r_y \bar{k}\right) \omega_x + \left(r_z \bar{i} - r_x \bar{k}\right) \omega_y + \left(r_x \bar{j} - r_y \bar{i}\right) \omega_z. \tag{9.3.1}$$

The components of $\bar{\omega}$ can be considered quasi-velocities. They occur linearly in the preceding description of velocity, with vector coefficients that may depend on generalized coordinates and time. There also is a term in the preceding equation that does not contain these quasi-velocities. This form represents a generalization of Eq. (9.2.3), with derivatives of generalized velocities $\dot{q}_j$ replaced with quasi-velocities.

To develop this concept we begin by considering a situation in which the number of quasi-coordinates we select is N, the number of generalized coordinates. Then the mathematical form of a physical velocity will be

$$\bar{v}_P \left(\dot{\gamma}_i,\ q_i, t\right) = \sum_{j=1}^{N} \bar{v}_{Pj} \left(q_i, t\right) \dot{\gamma}_j + \bar{v}_{P0} \left(q_i, t\right). \tag{9.3.2}$$

In view of the fact that we have the option of describing the velocity of any point in terms of the $\dot{q}_j$ according to Eq. (9.2.3) or in terms of the $\dot{\gamma}_j$ variables according to Eq. (9.3.2), it must be that the two sets of variables are kinematically related. Indeed, when we contemplate matching like components of the alternative descriptions of velocity, it becomes apparent that there is a linear relationship between the two sets of variables, with the coefficients of that relationship being dependent on the generalized coordinates. In other words,

$$\dot{q}_j = \sum_{n=1}^{N} C_{jn} \left(q_i, t\right) \dot{\gamma}_n + D_j \left(q_i, t\right); \quad j = 1, \ldots, N. \tag{9.3.3}$$

These are the *kinematical equations*, which will form one part of the differential equations of motion.

Suppose that there are J velocity constraint equations relating the generalized coordinates. We can formulate these equations in terms of quasi-velocities by using descriptions of linear and angular velocities having the form of Eq. (9.3.2). The result will be a set of constraint equations that are linear in the quasi-velocities. Thus, when there are J constraint equations to be satisfied, it is necessary that the quasi-velocities at any instant satisfy

$$\sum_{n=1}^{N} A_{jn} \left(q_i, t\right) \dot{\gamma}_n + B_j \left(q_i, t\right) = 0; \quad j = 1, \ldots, J. \tag{9.3.4}$$

[In some situations it might be that linear velocity constraint equations have already been described in terms of the generalized velocities. In that case, substitution of Eq. (9.3.3) will lead to the preceding form.]

We now come to one of the beneficial features of quasi-coordinates. Velocity constraint equations that are expressed in terms of generalized velocities also depend on the values of the generalized coordinates. Consequently, in the nonholonomic case, they are differential equations that can be solved only in conjunction with the kinetics equations. In contrast, the $\dot{\gamma}_i$ appear in the preceding constraint equations only linearly, with coefficients that are independent of the quasi-coordinates. Thus we may consider Eq. (9.3.4) to be a set of J algebraic equations for the instantaneous $\dot{\gamma}_i$ values at a specified position and time. We could solve those equations for J values in terms of the remaining $N - J$ values. The latter set constitutes *unconstrained quasi-velocities*. This name conveys the fact that any choice of the $N - J$ remaining quasi-velocities will inherently lead to a motion that is kinematically admissible. On the other hand, there are situations in which constrained quasi-velocities are useful, such as systems in which Coulomb friction is important.

We can accommodate all possibilities by letting K denote the number of quasi-coordinates, while we retain N as the number of generalized coordinates and J as the number of constraint equations. In the initial discussion of nonholonomic constraint equations, the number of degrees of freedom was defined to be the number of generalized velocities whose value at any instant can be selected arbitrarily and still obtain a kinematically admissible velocity for all parts of the system. Thus $K - J$ is the number of degrees of freedom. This generalization requires an adjustment of the summation range in the kinematical equations, such that

$$\boxed{\dot{q}_j = \sum_{n=1}^{K} C_{jn}(q_i, t)\,\dot{\gamma}_n + D_j(q_i, t); \quad j = 1, \ldots, N.} \tag{9.3.5}$$

If $J > 0$, then the constraint equations will also be like the earlier set, except for the altered summation range. Thus the quasi-velocities must satisfy constraint equations whose form is

$$\boxed{\sum_{n=1}^{K} A_{jn}(q_i, t)\,\dot{\gamma}_n + B_j(q_i, t) = 0; \quad j = 1, \ldots, J.} \tag{9.3.6}$$

Quasi-coordinates are also used to describe virtual movements. The true displacement of point P is $\bar{v}_P dt$, which Eq. (9.3.2) shows to be

$$d\bar{r}_P = \sum_{j=1}^{K} \bar{v}_{Pj}(q_i, t)\,\dot{\gamma}_j dt + \bar{v}_{P0}(q_i, t)\,dt. \tag{9.3.7}$$

To convert this to a virtual displacement, for which time is held constant, we drop the last term and replace each $\dot{\gamma}_j dt$ with an infinitesimal value $\delta\gamma_j$. (Just as the $\dot{\gamma}_j$ parameters need not directly describe the rates at which position coordinates change, the $\delta\gamma_j$ values might have an abstract meaning in terms of change of position.) Thus the virtual displacement of a point is described by

$$\delta\bar{r}_P = \sum_{j=1}^{K} \bar{v}_{Pj}(q_i, t)\,\delta\gamma_j. \tag{9.3.8}$$

Similarly, the virtual increments of the generalized coordinates are found from Eq. (9.3.5) to be

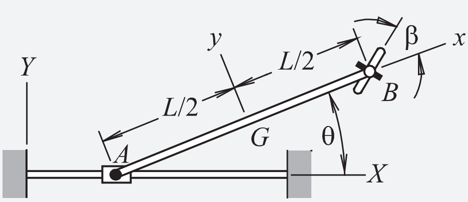
$$\delta q_n = \sum_{j=1}^{K} C_{nj}\,(q_i, t)\,\delta\gamma_j. \tag{9.3.9}$$

We further observe that, if the quasi-coordinates are constrained, then their virtual increments must be selected to be kinematically admissible. We multiply the constraint equations, Eq. (9.3.6), by dt to obtain the Pfaffian form, then convert $\dot{\gamma}_j dt$ to $\delta\gamma_j$, and drop the term containing dt, which leads to the condition that the virtual displacement will be kinematically admissible if

$$\sum_{j=1}^{N} A_{nj}\,(q_i, t)\,\delta\gamma_j = 0; \quad j = 1, \ldots, J. \tag{9.3.10}$$

An overview of the kinematical concepts associated with quasi-coordinates shows that generalized coordinates solely describe position dependence, whereas quasi-coordinates describe velocity and virtual displacement. We begin a kinematical description by selecting the geometrical variables that constitute the set of q_j and the rate variables that constitute the $\dot{\gamma}_j$ parameters. It is permissible that some or all of the latter are the rate of change of a generalized coordinate. The number K of quasi-velocities must be sufficient to describe every possible kinematical movement of the system.

EXAMPLE 9.6 End A of the bar is constrained to follow the guide, whereas end B has a steerable wheel that rolls in the horizontal XY plane. The steering angle β is controlled by a servomotor, so it is a known function of time. Let the position coordinates of pivot B and angle θ be generalized coordinates, and let the quasi-velocities be $\dot{\theta}$ and the velocity components of point G relative to the body-fixed xyz coordinate system. (a) Determine the constraint equations relating the quasi-velocities. (b) Determine the kinematical equations relating the generalized coordinates to the quasi-velocities.



Example 9.6

SOLUTION This example demonstrates that the basic procedure for formulating velocity constraint equations by use of quasi-velocities differs little from the procedure for generalized velocities. However, to highlight this attribute, the generalized coordinates and quasi-velocities that have been selected are not the parameters an experienced practitioner would use. It is specified that the generalized coordinates are $q_1 = X_A$, $q_2 = Y_A$, and $q_3 = \theta$, and that the quasi-velocities are $\dot{\gamma}_1 = (v_G)_x$, $\dot{\gamma}_2 = (v_G)_y$, and $\dot{\gamma}_3 = \dot{\theta}$. Collar A requires that $\bar{v}_A$ be parallel to the X axis, whereas wheel B requires that $\bar{v}_B$ be at angle β relative to the x axis. Thus we begin by relating the velocity of each end to the velocity of the center of mass and the angular velocity, which are known in terms of the quasi-velocities:

$$\begin{aligned}\bar{v}_A &= \bar{v}_G + \bar{\omega} \times \bar{r}_{A/G} = (v_G)_x \bar{i} + (v_G)_y \bar{j} + \dot{\theta}\bar{k} \times \left(-\frac{L}{2}\bar{i}\right), \\ \bar{v}_B &= \bar{v}_G + \bar{\omega} \times \bar{r}_{B/G} = (v_G)_x \bar{i} + (v_G)_y \bar{j} + \dot{\theta}\bar{k} \times \left(\frac{L}{2}\bar{i}\right).\end{aligned} \tag{1}$$

The constraints require that $\bar{v}_A$ be parallel to the X axis and $\bar{v}_B$ be at angle β above the x axis. Both conditions can be imposed with cross products and unit vectors in the respective directions, specifically,

$$\begin{aligned}\bar{v}_A \times \left(\cos\theta \bar{i} - \sin\theta \bar{j}\right) &= \bar{0}, \\ \bar{v}_B \times \left(\cos\beta \bar{i} + \sin\beta \bar{j}\right) &= \bar{0}.\end{aligned} \tag{2}$$

We substitute Eqs. (1) to eliminate v_A and v_B, neither of which is a quasi-velocity. The cross products have only $\bar{k}$ components, so evaluation of Eqs. (2) leads to

$$\begin{aligned}-(v_G)_x \sin\theta - \left[(v_G)_y - \frac{L}{2}\dot{\theta}\right]\cos\theta &= 0, \\ (v_G)_x \sin\beta - \left[(v_G)_y + \frac{L}{2}\dot{\theta}\right]\cos\beta &= 0.\end{aligned}$$

When each quasi-velocity is designated by the appropriate $\dot{\gamma}$ symbol, these constraint equations become

$$\begin{aligned}\dot{\gamma}_1 \sin\theta + \dot{\gamma}_2 \cos\theta - \frac{L}{2}\dot{\gamma}_3 \cos\theta &= 0, \\ \dot{\gamma}_1 \sin\beta - \dot{\gamma}_2 \cos\beta - \frac{L}{2}\dot{\gamma}_3 \cos\beta &= 0.\end{aligned} \qquad \triangleleft$$

Both of these have the standard linear velocity constraint form in Eq. (9.3.4), with

$$\begin{aligned}&A_{1,1} = \sin\theta, \quad A_{1,2} = \cos\theta, \quad A_{1,3} = -\frac{L}{2}\cos\theta, \quad B_1 = 0, \\ &A_{2,1} = \sin\beta, \quad A_{2,2} = -\cos\beta, \quad A_{2,3} = -\frac{L}{2}\cos\beta, \quad B_2 = 0.\end{aligned}$$

There are three quasi-velocities that must satisfy two velocity constraints, so there is one degree of freedom.

The kinematical equations express the rates at which the generalized coordinates X_B, Y_B, and θ change in terms of the quasi-velocities. We observe that $\dot{\theta}$ is defined to be $\dot{\gamma}_3$, so one kinematical equation is an identity. To obtain the other kinematical equations associated with X_B and Y_B we observe that these variables describe the position of point B. Thus, instead of describing the velocity of point B according to Eqs. (1), we could have expressed $\bar{v}_B$ in terms of the generalized velocities as

$$\bar{v}_B = \dot{X}_B \bar{I} + \dot{Y}_B \bar{J} = (v_G)_x \,\bar{i} + \left[(v_G)_y \,\bar{j} + \dot{\theta} \frac{L}{2} \right] \bar{j}.$$

Taking components of each expression and replacing the various symbols with their quasi-velocity designations yields

$$\dot{X}_B = \dot{\gamma}_1 \cos\theta - \left(\dot{\gamma}_2 + \frac{L}{2} \dot{\gamma}_3 \right) \sin\theta,$$

$$\dot{Y}_B = \dot{\gamma}_1 \sin\theta + \left(\dot{\gamma}_2 + \frac{L}{2} \dot{\gamma}_3 \right) \cos\theta, \qquad \triangleleft$$

$$\dot{\theta} = \dot{\gamma}_3,$$

which matches Eq. (9.3.5) with

$$C_{1,1} = \cos\theta, \quad C_{1,2} = -\sin\theta, \quad C_{1,3} = -\frac{L}{2}\sin\theta, \quad D_1 = 0,$$

$$C_{2,1} = \sin\theta, \quad C_{2,2} = \cos\theta, \quad C_{2,3} = \frac{L}{2}\cos\theta, \quad D_2 = 0,$$

$$C_{3,1} = C_{3,2} = 0, \quad C_{3,3} = 1, \quad D_3 = 0.$$

9.3.2 Gibbs–Appell Equations

It is possible to modify Lagrange's equations such that the term that contains a derivative with respect to a generalized velocities is replaced with terms that depend on the quasi-velocities, see Desloge (1982). However, the transformation may be conveniently carried out only for a holonomic system. In contrast, the Gibbs–Appell equations place no restrictions on the conditions to which they apply. The derivation begins at the fundamental level of a particle P. We describe the position of this particle in terms of its coordinates with respect to a fixed XYZ reference frame, which we denote indicially as x_1, x_2, x_3. Let $(F_P)_n$ denote the corresponding components of the resultant force acting on this particle. In view of Newton's Second Law, the virtual work done on this particle must be such that

$$\delta W_P = \sum_{n=1}^{3} (F_P)_n \,\delta x_n = \sum_{n=1}^{3} m_P \ddot{x}_n \delta x_n. \tag{9.3.11}$$

Now consider using quasi-velocities to describe the motion. The relation between the $\dot{x}_n$ and the $\dot{\gamma}_n$ variables has the form of Eq. (9.3.5), and the relation between the virtual increments must have the associated form in Eq. (9.3.10). We therefore have

$$\dot{x}_n = \sum_{j=1}^{K} C_{nj}(x_i, t)\,\dot{\gamma}_j + D_n(x_i, t),$$
$$\delta x_n = \sum_{j=1}^{K} C_{nj}(x_i, t)\,\delta\gamma_j. \tag{9.3.12}$$

The result of substituting δx_n into the left side of Eq. (9.3.11) is the usual description of generalized forces, with the difference that δq_j variables are replaced with $\delta\gamma_j$ parameters. We accordingly use the symbol Γ to denote generalized forces associated with quasi-coordinates, so that

$$\delta W_P = \sum_{j=1}^{3} (\Gamma_P)_j\,\delta\gamma_j, \quad (\Gamma_P)_j = \sum_{n=1}^{3} (F_P)_n\,C_{nj}(x_i, t). \tag{9.3.13}$$

Substitution of δx_n into the right side of Eq. (9.3.11) gives

$$\delta W_P = \sum_{n=1}^{3} \left(m_P \ddot{x}_n \sum_{j=1}^{K} C_{nj}(x_i, t)\,\delta\gamma_j \right). \tag{9.3.14}$$

To handle the acceleration components we differentiate the generalized velocities in Eq. (9.3.12) to find

$$\ddot{x}_n = \sum_{k=1}^{K} \left(C_{nk}(q_i, t)\,\ddot{\gamma}_k + \dot{\gamma}_k \frac{d}{dt} C_{nk}(q_i, t) \right) + \frac{d}{dt} D_n(q_i, t). \tag{9.3.15}$$

There is no need to actually differentiate the coefficients. Rather, we observe that such derivatives cannot lead to $\ddot{q}_j$ factors, so no additional occurrences of quasi-accelerations $\ddot{\gamma}_j$ will be found in the preceding equation. In other words each $\ddot{x}_n$ depends linearly on the $\ddot{\gamma}_j$. It follows that

$$\frac{\partial}{\partial \ddot{\gamma}_j}(\ddot{x}_n) = C_{nj}. \tag{9.3.16}$$

We use this relation to eliminate the C_{nj} coefficients in Eq. (9.3.14). When we rearrange the sequence in which the summations are performed, we find that

$$\delta W_P = \sum_{j=1}^{K} \sum_{n=1}^{3} m_P \ddot{x}_n \frac{\partial}{\partial \ddot{\gamma}_j}(\ddot{x}_n)\,\delta\gamma_j \equiv \sum_{j=1}^{K} \left\{ \frac{\partial}{\partial \ddot{\gamma}_j} \sum_{n=1}^{3} \left[\frac{1}{2} m (\ddot{x}_n)^2 \right] \right\} \delta\gamma_j. \tag{9.3.17}$$

The inner summation can be rewritten vectorially in terms of the particle's acceleration $\bar{a}_P$, so equating the alternative descriptions of virtual work in Eqs. (9.3.13) and (9.3.17) for this particle leads to

$$\delta W_P = \sum_{j=1}^{K} \left[\frac{\partial}{\partial \ddot{\gamma}_j} \left(\frac{1}{2} m_P \bar{a}_P \cdot \bar{a}_P \right) \right] \delta\gamma_j = \sum_{n=1}^{K} (\Gamma_P)_j\,\delta\gamma_j. \tag{9.3.18}$$

If particle P is part of a large set, the virtual work done on this particle will still be as previously described. The virtual work for all particles will be the sum of these scalar equations. The result of making the sum over all particles interior to the sum over the quasi-coordinates results is

$$\delta W = \sum_P \delta W_P = \sum_{j=1}^{K} \left\{ \frac{\partial}{\partial \ddot{\gamma}_j} \left(\sum_P \frac{1}{2} m_P \bar{a}_P \cdot \bar{a}_P \right) \right\} \delta\gamma_j = \sum_{n=1}^{K} \left[\sum_P (\Gamma_P)_j \right] \delta\gamma_j. \quad (9.3.19)$$

We now invoke the arguments that enabled us to extract Lagrange's equations from the description of virtual work. Thus we observe that, in the case in which the generalized coordinates are unconstrained, the $\delta\gamma_j$ values may be selected arbitrarily at any instant. Furthermore, even if there are constraint equations to satisfy, which subset of $\delta\gamma_j$ is taken to be unconstrained is arbitrary. Consequently, corresponding coefficients of $\delta\gamma_j$ in these alternative descriptions of virtual work must match, which leads us to the *Gibbs–Appell equations of motion*,

$$\boxed{\frac{\partial S}{\partial \ddot{\gamma}_j} = \Gamma_j, \quad k = 1, \ldots, K,} \quad (9.3.20)$$

where

$$\boxed{S = \sum_P \frac{1}{2} m_P \bar{a}_P \cdot \bar{a}_P} \quad (9.3.21)$$

and

$$\boxed{\delta W = \sum_{j1}^{K} \Gamma_j \delta\gamma_j. \ \ \Gamma_j = \sum_P \bar{F}_P \cdot \bar{v}_{Pj} \left(q_i, t \right) \delta\gamma_j.} \quad (9.3.22)$$

The quantity S is the *Gibbs–Appell function.* Its similarity to the definition of kinetic energy causes some to refer to S as an acceleration energy, but that is obviously a misnomer.

Although they are associated with the $\delta\gamma_j$ quantities, rather than the δq_j, the Γ_j are generalized forces. Their definition in terms of virtual work, Eqs. (9.3.22), is similar to the definition of the Q_j terms for Lagrange's equations. Thus the basic procedure for characterizing the Γ_j entails matching the standard form of δW to the virtual work done by the actual forces.

After the generalized coordinates and quasi-velocities have been selected for a specific system, formulation of the Gibbs–Appell equations of motion requires that we describe $S(\ddot{\gamma}_i, \dot{\gamma}_i, q_j, t)$ and the associated generalized forces. Application of this formulation to describe the motion of rigid bodies requires an expression for S in terms of angular motion variables and inertia properties. Toward that end, we define a convenient reference location as point A and consider a differential element of mass that is situated at position P. Let $\bar{r}$ denote the position of point P relative to point A, so the acceleration of this mass element is

$$\bar{a} = \bar{a}_A + \bar{\alpha} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r}) . \tag{9.3.23}$$

Before we use this expression to form S, it is useful to observe that only those terms in S that contain a quasi-acceleration $\ddot{\gamma}_j$ will give a nonzero derivative in regard to Eq. (9.3.20). However, the quasi-accelerations occur only in linear and angular accelerations, as evidenced by Eq. (9.3.15), which leads to the realization that any term in S that does not contain a linear or angular acceleration may be ignored. It follows that the contribution of the mass element to S for the rigid body may be written as

$$\begin{aligned} dS &= \frac{1}{2} dm \, [\bar{a}_A + \bar{\alpha} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})] \cdot [\bar{a}_A + \bar{\alpha} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})] \\ &= \frac{1}{2} dm \, (\bar{a}_A \cdot \bar{a}_A) + \frac{1}{2} dm \, [(\bar{\alpha} \times \bar{r}) \cdot (\bar{\alpha} \times \bar{r})] + dm \, \bar{a}_A \cdot (\bar{\alpha} \times \bar{r}) \\ &\quad + dm \, \bar{a}_A \cdot [\bar{\omega} \times (\bar{\omega} \times \bar{r})] + dm \, (\bar{\alpha} \times \bar{r}) \cdot [\bar{\omega} \times (\bar{\omega} \times \bar{r})] + \cdots , \end{aligned} \tag{9.3.24}$$

where "$\cdots$" is our standard notation indicating terms that depend solely on the angular velocity components, and therefore are irrelevant to the equations of motion.

It is necessary to integrate this description of dS over all mass elements. The velocity and acceleration variables are properties of the body, whereas $\bar{r}$ and dm are variables for such integrals. To isolate the latter we call on basic identities for the scalar and vector triple products:

$$\bar{a} \cdot (\bar{b} \times \bar{c}) \equiv \bar{b} \cdot (\bar{c} \times \bar{a}) \equiv \bar{c} \cdot (\bar{a} \times \bar{b}) , \tag{9.3.25a}$$

$$\bar{a} \times (\bar{b} \times \bar{c}) \equiv \bar{b} \, (\bar{a} \cdot \bar{c}) - \bar{c} \, (\bar{a} \cdot \bar{b}) . \tag{9.3.25b}$$

The first identity yields

$$\begin{aligned} (\bar{\alpha} \times \bar{r}) \cdot (\bar{\alpha} \times \bar{r}) &\equiv \bar{\alpha} \cdot [\bar{r} \times (\bar{\alpha} \times \bar{r})] , \\ (\bar{\alpha} \times \bar{r}) \cdot [\bar{\omega} \times (\bar{\omega} \times \bar{r})] &\equiv \bar{\alpha} \cdot \{\bar{r} \times [\bar{\omega} \times (\bar{\omega} \times \bar{r})]\} . \end{aligned} \tag{9.3.26}$$

Now consider the term within the braces, $\bar{r} \times [\bar{\omega} \times (\bar{\omega} \times \bar{r})]$. Because $\bar{\omega} \times \bar{r}$ is perpendicular to both $\bar{\omega}$ and $\bar{r}$, invoking Eq. (9.3.25b) with $\bar{a} = \bar{r}$ and $\bar{b} = \bar{\omega}$ gives

$$\bar{r} \times [\bar{\omega} \times (\bar{\omega} \times \bar{r})] \equiv -(\bar{\omega} \times \bar{r}) \, (\bar{r} \cdot \bar{\omega}) . \tag{9.3.27}$$

It is readily verified that this is the same quantity as the result of applying Eq. (9.3.25b) to $\bar{\omega} \times [\bar{r} \times (\bar{\omega} \times \bar{r})]$ with $\bar{a} = \bar{\omega}$ and $\bar{b} = \bar{r}$. It follows that

$$\bar{\omega} \times [\bar{r} \times (\bar{\omega} \times \bar{r})] \equiv \bar{r} \times [\bar{\omega} \times (\bar{\omega} \times \bar{r})] . \tag{9.3.28}$$

These identities convert the expression for dS to

$$\begin{aligned} dS = {} & \frac{1}{2} dm \, (\bar{a}_A \cdot \bar{a}_A) + \frac{1}{2} dm \bar{\alpha} \cdot [\bar{r} \times (\bar{\alpha} \times \bar{r})] + dm \, \bar{a}_A \cdot (\bar{\alpha} \times \bar{r}) \\ & + dm \, \bar{a}_A \cdot [\bar{\omega} \times (\bar{\omega} \times \bar{r})] + dm \bar{\alpha} \cdot \{\bar{\omega} \times [\bar{r} \times (\bar{\omega} \times \bar{r})]\} + \cdots . \end{aligned} \tag{9.3.29}$$

Recall that the integral of $\bar{r}\,dm$ is the first moment of mass with respect to the point from which $\bar{r}$ is measured, which is point A, so that

$$\iiint \bar{r}\;dm = m\bar{r}_{G/A}.$$

Thus we find that

$$S = \iiint dS = \frac{1}{2}m\,(\bar{a}_A\cdot\bar{a}_A) + \frac{1}{2}\bar{\alpha}\cdot\iiint[\bar{r}\times(\bar{\alpha}\times\bar{r})]\,dm + m\bar{a}_A\cdot(\bar{\alpha}\times\bar{r}_{G/A})$$
$$+ m\bar{a}_A\cdot[\bar{\omega}\times(\bar{\omega}\times\bar{r}_{G/A})] + \bar{\alpha}\cdot\left\{\bar{\omega}\times\iiint[\bar{r}\times(\bar{\omega}\times\bar{r})]\,dm\right\} + \cdots. \tag{9.3.30}$$

The final step in the derivation comes from recognition that the last integral is the angular momentum $\bar{H}_A$ of the body relative to point A. The integral contained in the second term has the same form, except that $\bar{\alpha}$ replaces $\bar{\omega}$. Recall that, when we employ body-fixed axes to represent the components of $\bar{\omega}$, the components of $\bar{\alpha}$ are the same as the rates at which the respective components of $\bar{\omega}$ change. Hence the integral in the second term is $\partial H_A/\partial t$, which reduces the Gibbs–Appell function to

$$\begin{aligned} S = \frac{1}{2}m\,(\bar{a}_A\cdot\bar{a}_A) + \frac{1}{2}\bar{\alpha}\cdot\frac{\partial\bar{H}_A}{\partial t} + \bar{\alpha}\cdot\left(\bar{\omega}\times\bar{H}_A\right) \\ + m\bar{a}_A\cdot(\bar{\alpha}\times\bar{r}_{G/A}) + m\bar{a}_A\cdot[\bar{\omega}\times(\bar{\omega}\times\bar{r}_{G/A})] + \cdots. \end{aligned} \tag{9.3.31}$$

In our previous analyses of rigid bodies, we selected the reference point A for angular momentum as the center of mass, or the fixed point of a body in pure rotation. The former gives $\bar{r}_{G/A} = \bar{0}$, whereas the latter leads to $\bar{a}_A$. Each causes the last two terms in the preceding description of S to vanish, thereby reducing S to three terms, none of which couples rotational motion with the translational motion of the reference point. Thus, we maintain the earlier restriction that *point A should be the center of mass for a translating body or a body in general motion, or the fixed pivot point for a body in pure rotation*. In either case, the Gibbs–Appell function for the body is given by

$$\boxed{S = \frac{1}{2}m\,(\bar{a}_A\cdot\bar{a}_A) + \frac{1}{2}\bar{\alpha}\cdot\frac{\partial\bar{H}_A}{\partial t} + \bar{\alpha}\cdot\left(\bar{\omega}\times\bar{H}_A\right).} \tag{9.3.32}$$

The significant aspect of this formula is that it allows us to evaluate the Gibbs–Appell function in terms of the same quantities as those we encountered for the Newton–Euler equations of motion. Note that S is a scalar, so we obtain its value for a system by summing the contribution of each constituent body.

Planar motion is an important special case. Without loss of generality, let $\bar{\omega} = \omega\bar{k}$ and $\bar{\alpha} = \dot{\omega}\bar{k}$ for motion parallel to the xy plane. In this case only the $\bar{k}$ component of $\partial\bar{H}_A/\partial t$ will lead to terms in S. Regardless of the inertia matrix properties, this component is $\left(\partial\bar{H}_A/\partial t\right)\cdot\bar{k} = I_{zz}\dot{\omega}$. Also, $\bar{\omega}\times\bar{H}_A$ is perpendicular to $\bar{\alpha}$ in this case, so it has no role in regard to forming S. It follows that

$$\text{planar motion: } S = \frac{1}{2}m\,(\bar{a}_A\cdot\bar{a}_A) + \frac{1}{2}I_{zz}\dot{\omega}^2. \tag{9.3.33}$$

As was true for Lagrange's equations, shortcuts are available to assist the evaluation of the Γ_j terms. Regardless of the nature of a physical force, it is always permissible to identify its contribution to the generalized forces by explicitly evaluating its virtual work, as described by Eq. (9.3.22). For conservative forces we can alternatively use the associated potential-energy function, V, which is a function solely of the generalized coordinates and, possibly, time. The virtual work such forces do is

$$\delta W_{\text{cons}} = -\delta V = -\sum_{n=1}^{N} \frac{\partial V}{\partial q_n} \delta q_n. \tag{9.3.34}$$

In the Gibbs–Appell formulation, the virtual increments are the $\delta\gamma_j$ quantities, which are kinematically related to the corresponding δq_j by Eq. (9.3.9). Thus,

$$\delta W_{\text{cons}} = -\sum_{n=1}^{N} \frac{\partial V}{\partial q_n} \sum_{j=1}^{K} C_{nj} \delta\gamma_j. \tag{9.3.35}$$

We obtain a useful identity for extracting the C_{nj} coefficients by differentiating the kinematical equations, Eq. (9.3.5), with respect to a specific quasi-velocity, which yields

$$C_{nj} = \frac{\partial \dot{q}_n}{\partial \dot{\gamma}_j}.$$

We substitute this relation into δW_{cons} and recognize that the coefficient of $\delta\gamma_j$ is the contribution of conservative forces to the jth generalized force. It follows that

$$(\Gamma_j)_{\text{conservative}} = -\sum_{n=1}^{N} \frac{\partial V}{\partial q_n} \frac{\partial \dot{q}_n}{\partial \dot{\gamma}_j}. \tag{9.3.36}$$

Now consider the constraint forces. The generalized forces here are like those for Lagrange's equations, so any constraint force whose kinematical constraint condition is satisfied regardless of the q_j and $\dot{\gamma}_j$ values will not appear in any generalized force. As a corollary, using unconstrained quasi-velocities leads to equations of motion that do not contain constraint forces. In contrast, if the quasi-velocities are a constrained set that must satisfy kinematical constraint equations like Eq. (9.3.6), then an arbitrary set of $\delta\gamma_j$ values will correspond to a kinematically inadmissible movement, which means that the constraint forces will contribute to the virtual work. As was true for constrained generalized coordinates, the contribution of the ith constraint force to Γ_j may be described by a Lagrange multiplier λ_i defined in conjunction with the Jacobian constraint coefficients A_{ij}. We thereby find that

$$(\Gamma_j)_{\text{reactions}} = \sum_{i=1}^{J} A_{ij} \lambda_i. \tag{9.3.37}$$

Thus we find that a generalized force in the Gibbs–Appell formulation may consist of three types of terms,

$$\Gamma_j = \sum_P \bar{F}_P \cdot \bar{v}_{Pj}(q_i, t) - \sum_{n=1}^{N} \frac{\partial V}{\partial q_n} \frac{\partial \dot{q}_n}{\partial \dot{\gamma}_j} + \sum_{i=1}^{J} A_{ij} \lambda_i. \tag{9.3.38}$$

This leads us to the form of the Gibbs–Appell equations that we will implement:

$$\boxed{\frac{\partial S}{\partial \ddot{\gamma}_j} = \sum_P \bar{F}_P \cdot \bar{v}_{Pj}(q_i, t) - \sum_{n=1}^{N} \frac{\partial V}{\partial q_n}\frac{\partial \dot{q}_n}{\partial \dot{\gamma}_j} + \sum_{i=1}^{J} A_{ij}\lambda_i, \quad j = 1, \ldots, K.} \tag{9.3.39}$$

Many texts use the symbol $\tilde{\gamma}$ to denote unconstrained quasi-coordinates, corresponding to the absence of constraint equations. However, from our viewpoint there is nothing special about these variables. We merely set $J = 0$ in the preceding equation, which removes the term with Lagrange multipliers from the preceding Gibbs–Appell equations.

A summary of the procedure required for formulating the Gibbs–Appell equations of motion is appropriate at this junction. The first step is to define a set of N generalized coordinates q_i to describe the instantaneous position and a set of K quasi-velocities $\dot{\gamma}_i$. A kinematical analysis is performed to derive an expression for each $\dot{q}_i$ in terms of the instantaneous values of the $\dot{\gamma}_i$ and q_i parameters. These relations constitute the kinematical equations, whose form is described by Eq. (9.3.5). A kinematical analysis is also required to ascertain whether assigning an arbitrary value to each $\dot{\gamma}_i$ at an arbitrary position will lead to a kinematically admissible motion for the system. If not, then velocity constraint equations like Eq. (9.3.6) must be derived. The number of such equations is J.

A kinematics analysis also must characterize the velocity and acceleration parameters required to form the Gibbs–Appell function for each body according to Eq. (9.3.32). Thus expressions for the body-fixed angular velocity components as functions of the $\dot{\gamma}_j$ and q_j parameters, and for the body-fixed angular acceleration components as functions of the $\ddot{\gamma}_j$, $\dot{\gamma}_j$, and q_j parameters, need to be derived. If the reference point for the angular motion is the center of mass, then the acceleration of the center of mass of that body must also be derived as a function of $\ddot{\gamma}_j$, $\dot{\gamma}_j$, and q_j. We obtain the Gibbs–Appell function for the system by summing the contribution of each of its bodies. The last task prior to forming the Gibbs–Appell equations is to determine the generalized force associated with each quasi-coordinate. We address the contribution of conservative forces by describing their potential energy as a function of the generalized coordinates. The contribution of forces that are not associated with constraints is described by evaluation of their virtual work. This entails further kinematical analysis in order to characterize the associated virtual displacement (or rotation) resulting from arbitrary increments $\delta\gamma_j$ when the system is at an arbitrary location. Any constraint force that we wish to appear explicitly in the equations of motion, especially the normal force associated with Coulomb friction, may be described in this manner. The portion of the generalized forces associated with the remaining constraint forces is described by the Lagrange multipliers. In the case of unconstrained quasi-coordinates, for which all physical constraint conditions are implicitly satisfied, the Lagrange multipliers are omitted from the formulation.

After we have carried out these preliminaries, we differentiate S as called for in Eq. (9.3.39). This yields a set of K Gibbs–Appell equations of motion in which the sole variables are $\ddot{\gamma}_j$, $\dot{\gamma}_j$, q_j, possibly t, and the Lagrange multipliers. These may be considered to be set of ordinary differential equations governing the $\dot{\gamma}_j$, which will occur only linearly. In other words, the Gibbs–Appell equations may be written in matrix notation as

$$\boxed{[M(q_i,t)]_{K\times K}\frac{d}{dt}\left\{\dot{\gamma}_j\right\}_{K\times 1}=\left\{F(\dot{\gamma}_i,q_i,t)\right\}_{K\times 1}+\left([A]_{J\times K}\right)^{\mathrm{T}}\{\lambda\}_{J\times 1}.} \tag{9.3.40}$$

Obviously, $[M]$ here is not the same as the inertia matrix for Lagrange's equations. Also, if constraint forces R_i are used in lieu of Lagrange multipliers, the last term would be replaced with $[A']\{R\}$, where the $[A']$ coefficients would be identified from the virtual work.

By themselves these equations cannot be sent to a differential equation solver because they feature the current values of the q_j, as well as the Lagrange multipliers if $J\neq 0$. A numerical solution requires the aforementioned kinematical equations, whose matrix form is

$$\boxed{\frac{d}{dt}\{q_j\}_{N\times 1}=[C(q_i,t)]_{N\times K}\left\{\dot{\gamma}_j\right\}_{K\times 1}+\left\{D(q_i,t)\right\}_{N\times 1}.} \tag{9.3.41}$$

In addition, if $J>0$, so that there are Lagrange multipliers to determine, then there are J velocity constraint equations. To satisfy the constraint equations simultaneously with the Gibbs–Appell and kinematical equations, we differentiate the velocity constraints to obtain acceleration constraint equations, whose matrix form is

$$\boxed{[A(q_i,t)]_{J\times K}\frac{d}{dt}\left\{\dot{\gamma}_j(q_i,t)\right\}_{K\times 1}=-\left[\frac{d}{dt}A(q_i,t)\right]_{J\times K}\left\{\dot{\gamma}_j(q_i,t)\right\}_{K\times 1}-\left\{\frac{d}{dt}B\right\}.} \tag{9.3.42}$$

Thus we obtain $K+N+J$ coupled ordinary differential equations for $\dot{\gamma}_j$ for $j=1,\ldots,K$, q_j for $j=1,\ldots,N$, and λ_j for $j=1,\ldots,J$. These equations have the form required to employ the numerical solution techniques in Subsection 8.2.1. As always, care must be taken to select initial conditions that are consistent with all constraints. This issue was addressed in Subsection 8.2.3.

The ability to use quasi-coordinates is the primary merit of the Gibbs–Appell approach. It affords us more flexibility than Lagrange's equations in deciding how to formulate an analysis. In many cases this freedom simplifies the equations of motion because the variables naturally fit the fundamental system features. Also, the number of quasi-coordinates can be less than the number of generalized coordinates, in which case there will be fewer differential equations of motion than there would be with Lagrange's equations. We will examine the merits of these alternative approaches after several examples applying the Gibbs–Appell equations have been presented. These examples focus on nonholonomic systems, because the primary advantage of the Gibbs–Appell equations is claimed to lie in its application to such systems.

EXAMPLE 9.7 Use the Gibbs–Appell equations to derive Euler's equations of motion for a rigid body.

SOLUTION The fact that Euler's equations can be derived from a formulation of the Gibbs–Appell equations, but not from Lagrange's equations, has been argued to be a philosophical advantage. The velocity variables in Euler's equations are the angular velocity components relative to a set of body-fixed axes, so we define the quasi-velocities to be these components, $\dot{\gamma}_1 = \omega_x,\ \dot{\gamma}_2 = \omega_y,\ \dot{\gamma}_3 = \omega_z$. Because $\bar{\alpha} \equiv \partial\bar{\omega}/\partial t$, the angular velocity and angular acceleration are

$$\bar{\omega} = \dot{\gamma}_1\bar{i} + \dot{\gamma}_2\bar{j} + \dot{\gamma}_3\bar{k},\ \ \bar{\alpha} = \ddot{\gamma}_1\bar{i} + \ddot{\gamma}_2\bar{j} + \ddot{\gamma}_3\bar{k}.$$

Euler's equations pertain to principal axes, in which case we have

$$\bar{H}_A = I_{xx}\dot{\gamma}_1\bar{i} + I_{yy}\dot{\gamma}_2\bar{j} + I_{zz}\dot{\gamma}_3\bar{k},$$
$$\frac{\partial\bar{H}_A}{\partial t} = I_{xx}\ddot{\gamma}_1\bar{i} + I_{yy}\ddot{\gamma}_2\bar{j} + I_{zz}\ddot{\gamma}_3\bar{k}.$$

At this juncture, we could use the preceding expressions to form the full expression for S. However, it is sufficient to demonstrate the equivalence between a Gibbs–Appell equation for one quasi-velocity and the corresponding Euler equation because the components of all of the preceding vectors are symbolic permutations. Hence we evaluate the equation for $\partial S/\partial\ddot{\gamma}_1$. This reduces the analytical effort because we may ignore any terms that do not contain $\ddot{\gamma}_1$. We also may ignore the translational acceleration term in Eq. (9.3.32) because the motion of the center of mass is independent of the angular motion until we introduce constraint conditions. The terms we need to form S are

$$\bar{\alpha}\cdot\frac{\partial\bar{H}_A}{\partial t} = I_{xx}\ddot{\gamma}_1^2 + \cdots,$$
$$\begin{aligned}\bar{\alpha}\cdot\left(\bar{\omega}\times\bar{H}_A\right) &= \left(\ddot{\gamma}_1\bar{i} + \cdots\right)\cdot\left[\left(\cdots + \dot{\gamma}_2\bar{j} + \dot{\gamma}_3\bar{k}\right)\times\left(\cdots + I_{yy}\dot{\gamma}_2\bar{j} + I_{zz}\dot{\gamma}_3\bar{k}\right)\right]\\ &= \ddot{\gamma}_1\left[\left(I_{zz} - I_{yy}\right)\dot{\gamma}_2\dot{\gamma}_3\right] + \cdots.\end{aligned}$$

Correspondingly, the Gibbs–Appell function is

$$S = \frac{1}{2}I_{xx}\ddot{\gamma}_1^2 + \ddot{\gamma}_1\left[\left(I_{zz} - I_{yy}\right)\dot{\gamma}_2\dot{\gamma}_3\right] + \cdots.$$

The force system acting on any rigid body is equivalent to a resultant force $\Sigma\bar{F}$ applied at point A and a couple $\Sigma\bar{M}_A$ that is the moment of the force system about point A. In the absence of constraint conditions the virtual displacement of point A does not depend on the angular motion variables. To obtain the virtual rotation we observe that the true rotation in an infinitesimal time interval is

$$\overline{d\theta} = \bar{\omega}dt = \left(\dot{\gamma}_1 dt\right)\bar{i} + \left(\dot{\gamma}_2 dt\right)\bar{j} + \left(\dot{\gamma}_3 dt\right)\bar{k}.$$

We obtain the virtual rotation by replacing each $\dot{\gamma}\,dt$ with $\delta\gamma$, so

$$\overline{\delta\theta} = \delta\gamma_1 \bar{i} + \delta\gamma_2 \bar{j} + \delta\gamma_3 \bar{k}.$$

The virtual work is

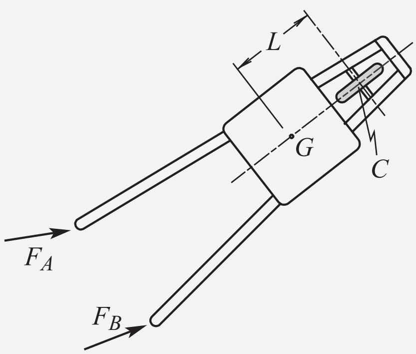
$$\delta W = \Sigma \bar{M}_A \cdot \overline{\delta\theta} = (\Sigma M_A)_x \,\delta\gamma_1 + \cdots = \Gamma_1 \delta\gamma_1 + \cdots .$$

The Gibbs–Appell equation for $\dot{\gamma}_1$ is

$$\frac{\partial S}{\partial \ddot{\gamma}_1} = I_{xx}\ddot{\gamma}_1 + (I_{zz} - I_{yy})\,\dot{\gamma}_2\dot{\gamma}_3 = \Gamma_1 = (\Sigma M_A)_x \,. \qquad \triangleleft$$

Replacing each quasi-velocity with the corresponding component of $\bar{\omega}$ makes it evident that this is the same as the Euler equation for moment about the x axis.

EXAMPLE 9.8 The wheelbarrow remains upright as it is pushed in the horizontal plane by forces $\bar{F}_A$ and $\bar{F}_B$ acting at each handle. The body of the wheelbarrow has mass m_1 with its center of mass at point G; the corresponding centroidal moment of inertia about a vertical axis is I. The wheel, which rolls without slipping, has mass m_2, moment of inertia J about its axle, and radius R. Derive the Gibbs–Appell equations of motion.



Example 9.8

SOLUTION This example of a nonholonomic system illustrates how one may employ a quasi-velocity that is not a generalized velocity. Because the wheel rolls without slipping and the wheelbarrow remains upright, the angular velocity component of the wheel about its bearing axis is $\omega_1 = v/R$, where v is the speed of the center C of the axle. Also, because there is no slippage, the velocity of point C must be along the longitudinal centerline, which we define to be the x axis of a coordinate system that is attached to the wheelbarrow's body, that is, $\bar{i} = \bar{e}_{C/G}$. Convenient generalized coordinates for the wheelbarrow are the absolute position coordinates of point C, $q_1 = X_C$, $q_2 = Y_C$, and the angle θ locating the x axis relative to a fixed XYZ reference frame, which we define such that X is to the right and Z is vertical.

We could define a fourth generalized coordinate to be the spin angle of the wheel. This can be avoided here because no forces or couples depend on this angle, and we can account for the spin rate through the no-slip condition.

We may identify a suitable set of unconstrained quasi-velocities by visualizing the motion. At an arbitrary position, we know that point C must move in the direction indicated by the longitudinal axis, but we do not know its speed. This suggests that one quasi-velocity should be $\dot{\gamma}_1 = v$. Knowledge of the current values of v and θ sets the velocity of point C, so only the rotation of the wheelbarrow is undefined. Thus we take $\dot{\gamma}_2 = \dot{\theta}$. Note that the condition that the velocity of point C be parallel to the x axis is a velocity constraint that is identically satisfied for arbitrary values of these two quasi-velocities, which shows that the system has two degrees of freedom. A formulation using Lagrange's equations would require explicit enforcement of a constraint equation relating the three generalized coordinates, whereas the quasi-velocities we have defined constitute an unconstrained set.

Two kinematical equations come from the rolling constraint, which requires that $\dot{X}_C$ and $\dot{Y}_C$ be the components in the fixed directions of the velocity of point C. The third kinematical equation comes from the definition of $\dot{\gamma}_2$. We use $\dot{\gamma}_1$ rather than v in order to remember that v is a quasi-velocity. Hence the kinematical equations are

$$\dot{X}_C = \dot{\gamma}_1 \cos\theta, \quad \dot{Y}_C = \dot{\gamma}_1 \sin\theta, \quad \dot{\theta} = \dot{\gamma}_2. \tag{1} \triangleleft$$

It is good practice at this juncture to describe S for the system in terms of the physical parameters in order to highlight which accelerations should be described in terms of the quasi-velocities. We must account for the inertia of the wheel in addition to that of the wheelbarrow's chassis. The latter body is in planar motion, so we add Eq. (9.3.32) for the wheel to Eq. (9.3.33) for the chassis:

$$S = \frac{1}{2} m_1 a_G^2 + \frac{1}{2} I \dot{\omega}_1^2 + \frac{1}{2} m_2 a_C^2 + \frac{1}{2} \bar{\alpha}_2 \cdot \frac{\partial \bar{H}_C}{\partial t} + \bar{\alpha}_2 \cdot \left(\bar{\omega}_2 \times \bar{H}_G\right). \tag{2}$$

We may obtain an expression for $\bar{a}_C$ in terms of the quasi-velocities by differentiating a general description of $\bar{v}_C$. We know that $\bar{v}_C = v\bar{i}$ and note that $\bar{i}$ is fixed to the chassis, whose angular velocity is $\dot{\gamma}_2 \bar{k}$. Consequently, it must be that

$$\bar{a}_C = \frac{d}{dt}\left(\dot{\gamma}_1 \bar{i}\right) = \ddot{\gamma}_1 \bar{i} + \dot{\gamma}_1 \left(\dot{\gamma}_2 \bar{k} \times \bar{i}\right) = \ddot{\gamma}_1 \bar{i} + \dot{\gamma}_1 \dot{\gamma}_2 \bar{j}. \tag{3}$$

The fact that points C and G belong to the chassis then leads to

$$\bar{a}_G = \bar{a}_C + \left(\ddot{\gamma}_2 \bar{k}\right) \times \bar{r}_{G/C} - \dot{\gamma}_1^2 \bar{r}_{G/C} = \left(\ddot{\gamma}_1 + \dot{\gamma}_2^2 L\right) \bar{i} + \left(\dot{\gamma}_1 \dot{\gamma}_2 - \ddot{\gamma}_2 L\right) \bar{j}. \tag{4}$$

To describe the angular motion for the wheel, we observe that it precesses about a vertical axis at the angular speed $\dot{\theta}$ of the wheelbarrow while it simultaneously spins about its own axis. As noted earlier, the spin rate is v/R. We defined $\bar{i} = \bar{e}_{C/B}$, and

$\bar{k}$ has been taken to be the upward vertical, so the sense of the spin axis is $\bar{j}$. Thus the angular velocity and angular acceleration of the wheel are

$$\bar{\omega}_2 = \frac{\dot{\gamma}_1}{R}\bar{j} + \dot{\gamma}_2\bar{k},$$
$$\bar{\alpha}_2 = \frac{\ddot{\gamma}_1}{R}\bar{j} + \frac{\dot{\gamma}_1}{R}\left(\dot{\gamma}_2\bar{k} \times \bar{j}\right) + \ddot{\gamma}_2\bar{k} = -\frac{\dot{\gamma}_1\dot{\gamma}_2}{R}\bar{i} + \frac{\ddot{\gamma}_1}{R}\bar{j} + \ddot{\gamma}_2\bar{k}. \tag{5}$$

Based on the assumption that the wheel is thin, so that the transverse centroidal moment of inertia is $J/2$, the angular momentum and its body-fixed derivative are

$$\bar{H}_C = \frac{J}{R}\dot{\gamma}_1\bar{j} + \frac{J}{2}\dot{\gamma}_2\bar{k},$$
$$\frac{\partial \bar{H}_C}{\partial t} = -\frac{J}{2R}\dot{\gamma}_1\dot{\gamma}_2\bar{i} + \frac{J}{R}\ddot{\gamma}_1\bar{j} + \frac{J}{2}\ddot{\gamma}_2\bar{k}. \tag{6}$$

When we substitute Eqs. (3)–(6) into Eq. (2) for S, we omit any terms that do not contain either $\ddot{\gamma}_1$ or $\ddot{\gamma}_2$. The specific operations are

$$\begin{aligned} S = {} & \frac{1}{2}m_1\left[\left(\ddot{\gamma}_1 + \dot{\gamma}_2^2 L\right)^2 + \left(\dot{\gamma}_1\dot{\gamma}_2 - \ddot{\gamma}_2 L\right)^2\right] + \frac{1}{2}I\,\ddot{\gamma}_2^2 \\ & + \frac{1}{2}m_2\left[\left(\ddot{\gamma}_1\right)^2 + \cdots\right] + \frac{1}{2}\left[\frac{J}{R^2}\ddot{\gamma}_1^2 + \frac{J}{2}\ddot{\gamma}_2^2\right] \\ & + \left(-\frac{\dot{\gamma}_1\dot{\gamma}_2}{R}\bar{i} + \frac{\ddot{\gamma}_1}{R}\bar{j} + \ddot{\gamma}_2\bar{k}\right)\cdot\left[\left(\frac{\dot{\gamma}_1}{R}\bar{j} + \dot{\gamma}_2\bar{k}\right) \times \left(\frac{J}{R}\dot{\gamma}_1\bar{j} + \frac{J}{2}\dot{\gamma}_2\bar{k}\right)\right] \\ = {} & \frac{1}{2}m_1\left[\left(\ddot{\gamma}_1 + \dot{\gamma}_2^2 L\right)^2 + \left(\dot{\gamma}_1\dot{\gamma}_2 - \ddot{\gamma}_2 L\right)^2\right] \\ & + \frac{1}{2}\left(I + \frac{J}{2}\right)\ddot{\gamma}_2^2 + \frac{1}{2}\left(m_2 + \frac{J}{R^2}\right)\ddot{\gamma}_1^2 + \cdots . \end{aligned} \tag{7}$$

Note that the terms associated with a gyroscopic moment for the wheel, which are the $\bar{i}$ terms in $\partial\bar{H}_C/\partial t$ and $\bar{\omega}_2 \times \bar{H}_C$, do not appear in S because the system has been constrained to move in the horizontal plane. Momentum effects tending to cause the wheelbarrow to tilt are balanced by reactions, which would be out-of-plane components of the forces $\bar{F}_A$ and $\bar{F}_B$.

The next step is to evaluate the generalized forces. The friction and normal forces exerted on the wheel by the ground are constraint forces that respectively prevent slipping and prevent the wheel from penetrating the ground. Both constraint conditions are satisfied for any set of quasi-velocities, so these forces do not contribute to the virtual work. Gravity acts perpendicularly to the plane of motion, so it too does not contribute to δW. To characterize the effect of the applied forces $\bar{F}_A$ and $\bar{F}_B$, we recall that we have already described the velocity of point C. We therefore replace these forces with a force–couple system, $\bar{R}$ and $\bar{M}$, acting at point C:

$$\bar{R} = \bar{F}_A + \bar{F}_B, \quad \bar{M} = \bar{r}_{A/C} \times \bar{F}_A + \bar{r}_{B/C} \times \bar{F}_B.$$

An expression for $\delta\bar{r}_C$ follows directly from the velocity of that point,

$$\bar{v}_C = \dot{\gamma}_1\bar{i} \implies \delta\bar{r}_C = \delta\gamma_1\bar{i},$$

so the virtual work associated with the forces applied to the handles is

$$\delta W = \bar{R} \cdot \delta \bar{r}_C + \bar{M} \cdot \delta \gamma_2 \bar{k} = \left(\bar{R} \cdot \bar{i} \right) \delta \gamma_1 + \left(\bar{M} \cdot \bar{k} \right) \delta \gamma_2 .$$

Matching the two descriptions of δW yields the generalized forces:

$$\Gamma_1 = \bar{R} \cdot \bar{i}, \quad \Gamma_2 = \bar{M} \cdot \bar{k}.$$

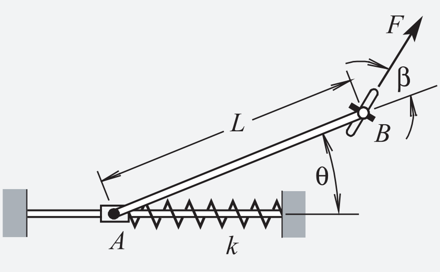
All that remains to be done is to differentiate the Gibbs–Appell function, Eq. (7), with respect to each of the quasi-acceleration variables. This yields

$$\frac{\partial S}{\partial \ddot{\gamma}_1} = \left(m_1 + m_2 + \frac{J}{R^2} \right) \ddot{\gamma}_1 + m_1 L \dot{\gamma}_2^2 = \bar{R} \cdot \bar{i},$$

$$\frac{\partial S}{\partial \ddot{\gamma}_2} = \left(m_1 L^2 + I + \frac{1}{2} J \right) \ddot{\gamma}_2 - m_1 L \dot{\gamma}_1 \dot{\gamma}_2 = \bar{M} \cdot \bar{k}. \qquad \triangleleft$$

These are two coupled first-order differential equations governing $\dot{\gamma}_1$ and $\dot{\gamma}_2$. They have the interesting feature that the generalized coordinates do not appear in them. This means that the equations can be solved numerically for $\dot{\gamma}_1 = v$ and $\dot{\gamma}_2 = \dot{\theta}$ as functions of time. If we wish to know where the wheelbarrow goes, we would substitute these solutions into the kinematical equations, Eqs. (1)–(3), which would give X_C, Y_C, and θ as functions of time. This is an exceptional situation stemming from the fact that the motion is unaffected by the wheelbarrow's position. In most cases, such as the one in the next example, it is necessary to treat the Gibbs–Appell and kinematical equations as a coupled set of differential equations.

EXAMPLE 9.9 The system in the sketch is like the one in Example 9.6. A motor mounted internally applies a known torque to the wheel's axle, thereby generating the known traction force $F(t)$. This force acts tangentially to the wheel. The wheel rolls without slipping, and the steering angle β changes in a prescribed manner. The collar is attached to a linearly elastic spring k. The mass of the bar, the collar, and the wheel are m_1, m_2, and m_3, respectively, and the centroidal moments of inertia of the wheel are I_1 and I_2, respectively parallel and perpendicular to the wheel's axle. Determine the Gibbs–Appell equations of motion governing a set of constrained quasi-coordinates.



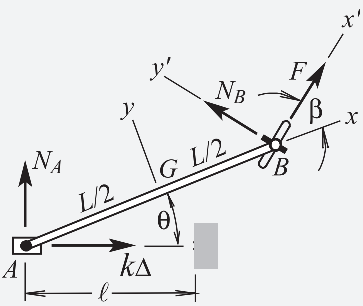
Example 9.9

SOLUTION Most of the basic operations involved in formulating the Gibbs–Appell equations of motion in terms of constrained quasi-velocities are demonstrated here. Example 9.6 established that this system has one degree of freedom. The position of all parts of the system is readily described if we know the angle θ and the length ℓ of the spring, so we use these variables as generalized coordinates, $q_1 = \theta$, $q_2 = \ell$. Similarly, all velocities and accelerations are readily described in terms of $\dot{\theta}$ and $\dot{\ell}$ and their time derivatives, so we select these variables as the quasi-velocities:

$$\dot{\theta} = \dot{\gamma}_1, \quad \dot{\ell} = \dot{\gamma}_2. \tag{1, 2 $\triangleleft$}$$

These are the kinematical equations relating the generalized coordinates to the quasi-velocities.

We consider the bar, the collar, and the wheel as a system. The forces acting on this system at the wheeel are the traction force and a constraint force perpendicular to $\bar{v}_B$. The only other force of concern is the normal force exerted on the collar. We show these forces in a free-body diagram that also defines the generalized coordinates. The xyz coordinate system in this diagram is attached to the bar. The $x'y'z'$ system is attached to the wheel, but the fact that this body is axisymmetric enables us to take the x' axis to be horizontal without loss of generality.



Free-body diagram showing generalized coordinates

The constraint that collar A must follow the guide bar is identically satisfied by these variables, because ℓ is the sole variable affecting the position of the collar, so that $\bar{v}_A = -\dot{\ell}\bar{I} = -\dot{\gamma}_2\bar{I}$. The constraint that $\bar{v}_B$ be parallel to the orientation of the wheel must be enforced explicitly. Because points A and B belong to bar AB, it must be that $\bar{v}_B = \bar{v}_A + \dot{\theta}\bar{k} \times \bar{r}_{B/A}$. To be sure that the appropriate rate variables are recognized as quasi-velocities, we denote them as $\dot{\gamma}$ rather than use their symbolic name. Hence the velocity of point B is described as

$$\begin{aligned}\bar{v}_B &= v_B\left(\cos\beta\bar{i} + \sin\beta\,\bar{j}\right) \\ &= \dot{\gamma}_2\left(-\cos\theta\bar{i} + \sin\theta\,\bar{j}\right) + \dot{\gamma}_1\bar{k} \times L\bar{i}.\end{aligned}$$

Matching like components in the preceding equation gives two simultaneous equations for v_B in terms of $\dot{\gamma}_1$ and $\dot{\gamma}_2$. Eliminating v_B gives the constraint equation, and the resulting expression for v_B will soon be needed. The constraint equation is

$$\dot{\gamma}_1 L\cos\beta + \dot{\gamma}_2\sin(\theta + \beta) = 0, \tag{3 $\triangleleft$}$$

and the speed of end B is

$$v_B = -\dot{\gamma}_2 \frac{\cos\theta}{\cos\beta}. \tag{4}$$

Collar A translates, and bar AB is in general planar motion with angular acceleration $\ddot{\theta}$ about the vertical axis. In contrast, the wheel executes a general three-dimensional motion. Hence the Gibbs–Appell function for this system is given by

$$\begin{aligned} S = {} & \frac{1}{2} m_1 a_G^2 + \frac{1}{2} I_G \ddot{\gamma}_1^2 + \frac{1}{2} m_2 a_A^2 + \frac{1}{2} m_3 a_B^2 \\ & + \frac{1}{2}\left(\bar{\alpha} \cdot \frac{\partial \bar{H}_B}{\partial t}\right)_{\text{wheel}} + \left[\bar{\alpha} \cdot \left(\bar{\omega} \times \bar{H}_B\right)\right]_{\text{wheel}}. \end{aligned} \tag{5}$$

The only rate variables allowed in S are the quasi-velocities. Expressions for the point accelerations are readily obtained from the facts that all points reside on bar AB and that collar A translates to the left at $\dot{\gamma}_2$. Thus,

$$\begin{aligned} \bar{a}_A &= \ddot{\gamma}_2 \left(-\cos\theta \bar{i} + \sin\theta \bar{j}\right), \\ \bar{a}_G &= \bar{a}_A + \ddot{\gamma}_1 \bar{k} \times \frac{L}{2}\bar{i} - \frac{L}{2}\dot{\gamma}_1^2 \bar{i} = \left(-\ddot{\gamma}_2 \cos\theta - \frac{L}{2}\dot{\gamma}_1^2\right)\bar{i} + \left(\frac{L}{2}\ddot{\gamma}_1 + \ddot{\gamma}_2 \sin\theta\right)\bar{j}, \\ \bar{a}_B &= \bar{a}_A + \ddot{\gamma}_1 \bar{k} \times L\bar{i} - L\dot{\gamma}_1^2 \bar{i} = \left(-\ddot{\gamma}_2 \cos\theta - L\dot{\gamma}_1^2\right)\bar{i} + \left(L\ddot{\gamma}_1 + \ddot{\gamma}_2 \sin\theta\right)\bar{j}. \end{aligned} \tag{6}$$

Because the wheel rolls without slipping, its rotational motion consists of a spin about its axle at angular speed v_B/R, where R is the wheel's radius, accompanied by a precession about the vertical axis at angular speed $\dot{\theta} + \dot{\beta}$. Therefore

$$\bar{\omega}_{\text{wheel}} = \frac{v_B}{R}\bar{j}' + \left(\dot{\gamma}_1 + \dot{\beta}\right)\bar{k}'.$$

The parameter v_B is not a quasi-coordinate, so we use Eq. (4) to eliminate it:

$$\bar{\omega}_{\text{wheel}} = -\frac{\dot{\gamma}_2}{R}\frac{\cos\theta}{\cos\beta}\bar{j}' + \left(\dot{\gamma}_1 + \dot{\beta}\right)\bar{k}'. \tag{7}$$

This expression is generally valid, so it may be differentiated. In doing so, it is important to recognize that β is time dependent, so that

$$\begin{aligned} \bar{\alpha}_{\text{wheel}} = {} & \left(-\frac{\ddot{\gamma}_2}{R}\frac{\cos\theta}{\cos\beta} + \frac{\dot{\gamma}_1\dot{\gamma}_2}{R}\frac{\sin\theta}{\cos\beta} - \frac{\dot{\gamma}_2\dot{\beta}}{R}\frac{\cos\theta\sin\beta}{(\cos\beta)^2}\right)\bar{j} \\ & - \frac{\dot{\gamma}_2}{R}\frac{\cos\theta}{\cos\beta}\left[\left(\dot{\gamma}_1 + \dot{\beta}\right)\bar{k}' \times \bar{j}'\right] + \left(\ddot{\gamma}_1 + \ddot{\beta}\right)\bar{k}' \\ = {} & \left(-\frac{\ddot{\gamma}_2}{R}\frac{\cos\theta}{\cos\beta} + \frac{\dot{\gamma}_1\dot{\gamma}_2}{R}\frac{\sin\theta}{\cos\beta} - \frac{\dot{\gamma}_2\dot{\beta}}{R}\frac{\cos\theta\sin\beta}{(\cos\beta)^2}\right)\bar{j}' \\ & + \frac{\dot{\gamma}_2\left(\dot{\gamma}_1 + \dot{\beta}\right)}{R}\frac{\cos\theta}{\cos\beta}\bar{i}' + \left(\ddot{\gamma}_1 + \ddot{\beta}\right)\bar{k}'. \end{aligned} \tag{8}$$

When the components of $\bar{\omega}$ and $\bar{\alpha}$ are denoted symbolically, with $\omega_x = 0$ in the present case, the portion of S attributable to the wheel's rotation is

$$\begin{aligned}S_{\mathrm{rot}} &= \frac{1}{2}\left(\alpha_x \bar{i}' + \alpha_y \bar{j}' + \alpha_z \bar{k}'\right)\cdot\left(I_2\alpha_x \bar{i}' + I_1\alpha_y \bar{j}' + I_2\alpha_z \bar{k}'\right)\\&\quad + \left(\alpha_x \bar{i}' + \alpha_y \bar{j}' + \alpha_z \bar{k}'\right)\cdot\left[\left(\omega_y \bar{j}' + \omega_z \bar{k}'\right)\times\left(I_1\omega_y \bar{j}' + I_2\omega_z \bar{k}'\right)\right]\\&= \frac{1}{2}\left(I_2\alpha_x^2 + I_1\alpha_y^2 + I_2\alpha_z^2\right) + \left(I_2 - I_1\right)\alpha_x\omega_y\omega_z.\end{aligned}$$

Writing S_{rot} in this symbolic form enables us to recognize that, because neither the angular velocity components nor α_x contains quasi-accelerations, the first and last terms in the preceding equation may be ignored. Furthermore, terms that do not contain either quasi-acceleration may be ignored when α_y and α_z are squared. We extract the $\bar{\alpha}$ components from Eq. (8), and substitute them into S_{rot}, with the result that

$$\begin{aligned}S_{\mathrm{rot}} = \frac{1}{2}I_1\Bigg[&\left(\frac{\ddot{\gamma}_2}{R}\frac{\cos\theta}{\cos\beta}\right)^2 - 2\frac{\ddot{\gamma}_2}{R}\frac{\cos\theta}{\cos\beta}\left(\frac{\dot{\gamma}_1\dot{\gamma}_2}{R}\frac{\sin\theta}{\cos\beta}\right.\\&\left. - \frac{\dot{\gamma}_2\dot{\beta}}{R}\frac{\cos\theta\sin\beta}{(\cos\beta)^2}\right) + \cdots\Bigg] + \frac{1}{2}I_2\left(\ddot{\gamma}_1 + \ddot{\beta}\right)^2.\end{aligned}$$

Substitution of the acceleration terms, Eqs. (6), into Eq. (5) yields the other terms of S. Once again, terms that do not contain quasi-accelerations are ignored when the other contributions to S are formed. The evaluation proceeds as follows:

$$\begin{aligned}S &= \frac{1}{2}m_1\left[\left(-\ddot{\gamma}_2\cos\theta - \frac{L}{2}\dot{\gamma}_1^2\right)^2 + \left(\frac{L}{2}\ddot{\gamma}_1 + \ddot{\gamma}_2\sin\theta\right)^2\right]\\&\quad + \frac{1}{2}\left(\frac{1}{12}m_1L^2\right)\ddot{\gamma}_1^2 + \frac{1}{2}m_2\ddot{\gamma}_2^2 + \frac{1}{2}m_3\Big[\left(-\ddot{\gamma}_2\cos\theta - L\dot{\gamma}_1^2\right)^2\\&\quad + \left(L\ddot{\gamma}_1 + \ddot{\gamma}_2\sin\theta\right)^2\Big] + S_{\mathrm{rot}}\\&= \frac{1}{2}m_1\left[\ddot{\gamma}_2^2 + \ddot{\gamma}_2\dot{\gamma}_1^2L\cos\theta + \frac{1}{4}L^2\ddot{\gamma}_1^2 + \ddot{\gamma}_1\ddot{\gamma}_2L\sin\theta + \cdots\right]\\&\quad + \frac{1}{2}\left(\frac{1}{12}m_1L^2\right)\ddot{\gamma}_1^2 + \frac{1}{2}m_2\ddot{\gamma}_2^2 + \frac{1}{2}m_3\big[\ddot{\gamma}_2^2 + 2\ddot{\gamma}_2\dot{\gamma}_1^2L\cos\theta\\&\quad + L^2\ddot{\gamma}_1^2 + 2\ddot{\gamma}_1\ddot{\gamma}_2L\sin\theta + \cdots\big] + S_{\mathrm{rot}}.\end{aligned}$$

We collect like terms and substitute S_{rot} to find

$$\begin{aligned}S &= \frac{1}{2}\left(\frac{1}{3}m_1 + m_3\right)L^2\ddot{\gamma}_1^2 + \frac{1}{2}\left(m_2 + m_1 + m_3\right)\ddot{\gamma}_2^2\\&\quad + \frac{1}{2}\left(m_1 + 2m_3\right)L\left(\ddot{\gamma}_1\ddot{\gamma}_2\sin\theta + \ddot{\gamma}_2\dot{\gamma}_1^2\cos\theta\right)\end{aligned}$$

$$+\frac{1}{2}\frac{I_1}{R^2}\left[\left(\ddot{\gamma}_2\frac{\cos\theta}{\cos\beta}\right)^2-2\ddot{\gamma}_2\frac{\cos\theta}{\cos\beta}\left(\dot{\gamma}_1\dot{\gamma}_2\frac{\sin\theta}{\cos\beta}\right.\right.$$
$$\left.\left.-\dot{\gamma}_2\dot{\beta}\frac{\cos\theta\sin\beta}{(\cos\beta)^2}\right)+\cdots\right]+\frac{1}{2}I_2\left(\ddot{\gamma}_1+\ddot{\beta}\right)^2. \tag{9}$$

The last aspect to be considered prior to evaluating the equations of motion is the role of the forces. The spring is conservative, and its deformation is the difference between the current and original length of the spring, so that

$$V=\frac{1}{2}k\left(\ell-\ell_0\right)^2.$$

The coefficients required convert derivatives of V to generalized coordinates are found from kinematical equations (1) and (2) to be

$$\frac{\partial\dot{\ell}}{\partial\dot{\gamma}_1}=0,\quad \frac{\partial\dot{\ell}}{\partial\dot{\gamma}_2}=1.$$

Correspondingly the contribution of the spring force to the generalized forces is

$$(\Gamma_1)_k=-\frac{\partial V}{\partial\ell}\frac{\partial\dot{\ell}}{\partial\dot{\gamma}_1}=0,\quad (\Gamma_2)_k=-\frac{\partial V}{\partial\ell}\frac{\partial\dot{\ell}}{\partial\dot{\gamma}_2}=-k\left(\ell-\ell_0\right).$$

The thrust $\bar{F}$ is not conservative. The virtual displacement of point B, where it is applied, may be obtained from the initial expression for $\bar{v}_B$, which was

$$\bar{v}_B=-\dot{\gamma}_2\cos\theta\bar{i}+\left(\dot{\gamma}_1L+\dot{\gamma}_2\sin\theta\right)\bar{j}.$$

The infinitesimal displacement of this point is $\bar{v}_B dt$, and changing each $\dot{\gamma}dt$ to $\delta\gamma_j$ gives the virtual displacement:

$$\delta\bar{r}_B=-\delta\gamma_2\cos\theta\bar{i}+\left(\delta\gamma_1L+\delta\gamma_2\sin\theta\right)\bar{j}.$$

The corresponding virtual work done by $\bar{F}$ is

$$\begin{aligned}\delta W_F&=F\left(\cos\beta\bar{i}+\sin\beta\bar{j}\right)\cdot\delta\bar{r}_B\\&=F\delta\gamma_2\left(-\cos\beta\cos\theta+\sin\beta\sin\theta\right)+FL\sin\beta\delta\gamma_1.\end{aligned}$$

The generalized forces are the coefficients of each $\delta\gamma$, so

$$(\Gamma_1)_F=FL\sin\beta,\quad (\Gamma_2)_F=-F\cos\left(\theta+\beta\right).$$

The constraint of collar A is implicitly satisfied for all values of the generalized coordinates and quasi-velocities, so the normal constraint force acting on the collar does not contribute to the virtual work. The same is not true for $\bar{N}_B$, which makes the wheel move in the intended direction. This force is associated with constraint equation (3), so we may describe its contribution to the generalized force with a Lagrange multiplier by using the Jacobian coefficients of Eq. (3), which are

$$A_{1,1}=L\cos\beta,\quad A_{1,2}=\sin\left(\theta+\beta\right).$$

Thus the generalized forces corresponding to the quasi-coordinates are

$$\Gamma_1 = FL\sin\beta + \lambda_1 L\cos\beta,$$
$$\Gamma_2 = -F\cos(\theta+\beta) - k(\ell-\ell_0) + \lambda_1\sin(\theta+\beta).$$

The least difficult aspect of forming the Gibbs–Appell equations is evaluating the derivatives. The first equation is $\partial S/\partial\ddot{\gamma}_1 = \Gamma_1$, which evaluates to

$$\begin{aligned}&\left(\frac{1}{3}m_1+m_3\right)L^2\ddot{\gamma}_1+\frac{1}{2}(m_1+2m_3)L\ddot{\gamma}_2\sin\theta\\&+I_2\left(\ddot{\gamma}_1+\ddot{\beta}\right)=FL\sin\beta+\lambda_1 L\cos\beta.\end{aligned} \qquad (10) \lhd$$

The second Gibbs–Appell equation, $\partial S/\partial\ddot{\gamma}_1 = \Gamma_1$, is

$$\begin{aligned}&(m_1+m_2+m_3)\ddot{\gamma}_2+\frac{1}{2}(m_1+2m_3)L\left(\ddot{\gamma}_1\sin\theta+\dot{\gamma}_1^2\cos\theta\right)\\&+\frac{I_1}{R^2}\left[\ddot{\gamma}_2\left(\frac{\cos\theta}{\cos\beta}\right)^2-\frac{\cos\theta}{\cos\beta}\left(\dot{\gamma}_1\dot{\gamma}_2\frac{\sin\theta}{\cos\beta}-\dot{\gamma}_2\dot{\beta}\frac{\cos\theta\sin\beta}{(\cos\beta)^2}\right)\right]\\&=-F\cos(\theta+\beta)-k(\ell-\ell_0)+\lambda_1\sin(\theta+\beta).\end{aligned} \qquad (11) \lhd$$

Thus we have obtained five equations of the differential-algebraic type, consisting of these two Gibbs–Appell equations, the two kinematical equations, Eqs. (1) and (2), and the constraint equation, Eq. (3). The variables for these equations are $\dot{\gamma}_1$, $\dot{\gamma}_2$, θ, ℓ, and λ_1. The highest order of the derivatives of the variables is one, and those derivatives occur linearly. Hence, any of the techniques in Subsection 8.2.1 may be employed directly to obtain the response for specified initial conditions.

EXAMPLE 9.10 Consider the system in Example 9.9. Determine the Gibbs–Appell equations of motion using unconstrained quasi-coordinates.

SOLUTION A comparison of this solution with the preceding one will enable us to judge the merits of eliminating the constraint equations through the use of constrained generalized coordinates. Another feature of this example is its usage of a quasi-velocity that is not a derivative of a generalized coordinate. This system has one degree of freedom. The single unconstrained quasi-velocity we use is the speed of the wheel's center:

$$\dot{\gamma}_1 = v_B.$$

As we did in the preceding example, we let the generalized coordinates be $q_1 = \theta$, $q_2 = \ell$.

Enforcing the condition that the collar must move along its guide leads to the kinematical equations. It is necessary that $\bar{v}_A = \bar{v}_B + \bar{\omega}_{AB} \times \bar{r}_{A/B}$, and we know the directions in which both end points move. It actually is preferable for this system to use as the global coordinate system stationary XYZ, with X to the right and Y upward in the previous sketch, rather than the body-fixed xyz coordinate system used previously. Thus we have

$$-\dot{\ell}\bar{I} = v_B\left[\cos(\theta+\beta)\,\bar{I} + \sin(\theta+\beta)\,\bar{J}\right] + \dot{\theta}_1\bar{K} \times L\left(-\cos\theta\,\bar{I} - \sin\theta\,\bar{J}\right).$$

Matching like components gives

$$\begin{aligned} -\dot{\ell} &= v_B\cos(\theta+\beta) + L\dot{\theta}\sin\theta, \\ 0 &= v_B\sin(\theta+\beta) - L\dot{\theta}\cos\theta. \end{aligned}$$

The solution of these equations for $\dot{\theta}$ and $\dot{\ell}$ in terms of v_B are the kinematical equations:

$$\begin{aligned} \dot{\theta} &= \frac{v_B}{L}\frac{\sin(\theta+\beta)}{\cos\theta}, \\ \dot{\ell} &= -v_B\left[\cos(\theta+\beta) + \frac{\sin(\theta+\beta)}{\cos\theta}\sin\theta\right] \equiv -v_B\frac{\cos\beta}{\cos\theta}. \end{aligned} \tag{1}$$

All parts have significant inertia, and the wheel is executing a rotation about two axes, so the Gibbs–Appell function is

$$\begin{aligned} S = {} & \frac{1}{2}m_1 a_G^2 + \frac{1}{2}I_G\ddot{\theta}^2 + \frac{1}{2}m_2 a_A^2 + \frac{1}{2}m_3 a_B^2 \\ & + \frac{1}{2}\left(\bar{\alpha}\cdot\frac{\partial \bar{H}_B}{\partial t}\right)_{\text{w}} + \left[\bar{\alpha}\cdot\left(\bar{\omega}\times\bar{H}_B\right)\right]_{\text{w}}. \end{aligned} \tag{2}$$

The only rate variables that may appear in S are v_B and $\dot{v}_B$. This requires that we eliminate $\ddot{\theta}$ and $\ddot{\ell}$, so we differentitate Eqs. (1) to find that

$$\begin{aligned} \ddot{\theta} &= \frac{\dot{v}_B}{L}\frac{\sin(\theta+\beta)}{\cos\theta} + \frac{v_B\left(\dot{\theta}+\dot{\beta}\right)}{L}\frac{\cos(\theta+\beta)}{\cos\theta} + \frac{v_B\dot{\theta}}{L}\frac{\sin(\theta+\beta)\sin\theta}{(\cos\theta)^2} \\ &= \frac{\dot{v}_B}{L}\frac{\sin(\theta+\beta)}{\cos\theta} + \frac{v_B^2}{L^2}\frac{\sin(\theta+\beta)\cos\beta}{(\cos\theta)^3} + \frac{v_B\dot{\beta}}{L}\frac{\cos(\theta+\beta)}{\cos\theta}, \\ \ddot{\ell} &= -\dot{v}_B\frac{\cos\beta}{\cos\theta} - v_B\dot{\theta}\frac{\sin\theta\cos\beta}{(\cos\theta)^2} + v_B\dot{\beta}\frac{\sin\beta}{\cos\theta} \\ &= -\dot{v}_B\frac{\cos\beta}{\cos\theta} - \frac{v_B^2}{L}\frac{\sin(\theta+\beta)\sin\theta\cos\beta}{(\cos\theta)^3} + v_B\dot{\beta}\frac{\sin\beta}{\cos\theta}. \end{aligned} \tag{3}$$

Note that these forms were obtained by use of the first of Eqs. (1) to eliminate $\dot{\theta}$ because that variable is not a quasi-velocity.

We know that $|\bar{a}_A| = \ddot{\ell}$, but describing the other variables appearing in S requires a kinematical analysis. To describe $\bar{a}_B$ we differentiate the component representation of $\bar{v}_B$ and again use Eqs. (1) to eliminate $\dot{\theta}$. This gives

$$\begin{aligned}
\bar{a}_B &= \frac{d}{dt}\left[v_B \cos(\theta+\beta)\,\bar{I} + v_B \sin(\theta+\beta)\,\bar{J}\right] \\
&= \dot{v}_B\left[\cos(\theta+\beta)\,\bar{I} + \sin(\theta+\beta)\,\bar{J}\right] + v_B\left(\dot{\theta}+\dot{\beta}\right)\left[-\sin(\theta+\beta)\,\bar{I} + \cos(\theta+\beta)\,\bar{J}\right] \\
&= \left[\dot{v}_B \cos(\theta+\beta) - \frac{v_B^2}{L}\frac{(\sin(\theta+\beta))^2}{\cos\theta} - v_B\dot{\beta}\sin(\theta+\beta)\right]\bar{I} \\
&\quad + \left[\dot{v}_B \sin(\theta+\beta) + \frac{v_B^2}{L}\frac{\sin(\theta+\beta)\cos(\theta+\beta)}{\cos\theta} + v_B\dot{\beta}\cos(\theta+\beta)\right]\bar{J}.
\end{aligned} \tag{4}$$

It is not necessary to describe the motion of the center of mass explicity, because the consequence of its being the midpoint is that

$$\bar{v}_G = \frac{1}{2}\left(\bar{v}_A + \bar{v}_B\right), \quad \bar{a}_G = \frac{1}{2}\left(\bar{a}_A + \bar{a}_B\right). \tag{5}$$

We use the $x'y'z'$ coordinate system defined in the preceding example to describe the contribution of the wheel to S. The corresponding angular velocity and angular acceleration are found by the standard method to be

$$\begin{aligned}
\bar{\omega}_{\mathrm{w}} &= \frac{v_B}{R}\bar{j}' + \left(\dot{\theta}+\dot{\beta}\right)\bar{k}', \\
\bar{\alpha}_{\mathrm{w}} &= \frac{\dot{v}_B}{R}\bar{j}' + \frac{v_B}{R}\left[\left(\dot{\theta}+\dot{\beta}\right)\bar{k}' \times \bar{j}'\right] + \left(\ddot{\theta}+\ddot{\beta}\right)\bar{k}'.
\end{aligned}$$

Substitution of Eqs. (1) and (3) to eliminate $\dot{\theta}$ and $\ddot{\theta}$ yields

$$\begin{aligned}
\bar{\omega}_{\mathrm{w}} &= \frac{v_B}{R}\bar{j}' + \left(\frac{v_B}{L}\frac{\sin(\theta+\beta)}{\cos\theta} + \dot{\beta}\right)\bar{k}', \\
\bar{\alpha}_{\mathrm{w}} &= -\left[\frac{v_B^2}{LR}\frac{\sin(\theta+\beta)}{\cos\theta} + \frac{v_B\dot{\beta}}{R}\right]\bar{i}' + \frac{\dot{v}_B}{R}\bar{j}' + \left[\frac{\dot{v}_B}{L}\frac{\sin(\theta+\beta)}{\cos\theta}\right. \\
&\quad \left. + \frac{v_B^2}{L^2}\frac{\sin(\theta+\beta)\cos\beta}{(\cos\theta)^3} + \frac{v_B\dot{\beta}}{L}\frac{\cos(\theta+\beta)}{\cos\theta} + \ddot{\beta}\right]\bar{k}'.
\end{aligned} \tag{6}$$

The next step is to use Eqs. (3)–(6) to form the individual terms in S. As a start, we use Eqs. (5) to replace $\bar{a}_G$ in S, and also account for the facts that $\bar{a}_A = -\ddot{\ell}\bar{I}$ and $\bar{\omega}_{\mathrm{w}}$ has no component in the x' direction, from which it follows that

$$\begin{aligned}
S &= \frac{1}{8}m_1\left(\bar{a}_A\cdot\bar{a}_A + 2\bar{a}_A\cdot\bar{a}_B + \bar{a}_B\cdot\bar{a}_B\right) + \frac{1}{2}I_G\ddot{\theta}^2 + \frac{1}{2}m_2a_A^2 + \frac{1}{2}m_3a_B^2 \\
&\quad + \frac{1}{2}\left(I_{xx}a_x^2 + I_{yy}a_y^2 + I_{zz}a_z^2\right)_{\mathrm{wheel}} + \left(\alpha_x\bar{i} + \alpha_y\bar{j} + \alpha_z\bar{k}\right)\cdot\left[\left(\omega_y j + \omega_z\bar{k}\right)\right. \\
&\quad \left.\times\left(I_{yy}\omega_y j + I_{zz}\omega_z\bar{k}\right)\right]_{\mathrm{wheel}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\left(m_1 + 4m_2\right)\ddot{\ell}^2 + \frac{1}{8}\left(m_1 + 4m_3\right)\bar{a}_B \cdot \bar{a}_B - \frac{1}{4} m_1 \ddot{\ell}\bar{I} \cdot \bar{a}_B \\
&\quad + \frac{1}{2} I_G \ddot{\theta}^2 + \frac{1}{2}\left(I_{yy} a_y^2 + I_{zz} a_z^2\right)_{\mathrm{w}} + \cdots .
\end{aligned}
$$

The terms that are omitted from the preceding equation are associated with the wheel's rotation, but do not contain $\dot{v}_B$. We substitute Eqs. (3), (4), and (6) into this description of S to find that

$$
\begin{aligned}
S = {} & \frac{1}{8}\left(m_1 + 4m_2\right)\left[-\dot{v}_B \frac{\cos\beta}{\cos\theta} - \frac{v_B^2}{L}\frac{\sin(\theta+\beta)\sin\theta\cos\beta}{(\cos\theta)^3} + v_B\dot{\beta}\frac{\sin\beta}{\cos\theta}\right]^2 \\
& + \frac{1}{8}\left(m_1 + 4m_3\right)\left[\dot{v}_B\cos(\theta+\beta) - \frac{v_B^2}{L}\frac{(\sin(\theta+\beta))^2}{\cos\theta} - v_B\dot{\beta}\sin(\theta+\beta)\right]^2 \\
& + \frac{1}{8}\left(m_1 + 4m_3\right)\Bigg[\dot{v}_B\sin(\theta+\beta) + \frac{v_B^2}{L}\frac{\sin(\theta+\beta)\cos(\theta+\beta)}{\cos\theta} \\
& + v_B\dot{\beta}\cos(\theta+\beta)\Bigg]^2 - \frac{1}{4} m_1 \Bigg[-\dot{v}_B\frac{\cos\beta}{\cos\theta} - \frac{v_B^2}{L}\frac{\sin(\theta+\beta)\sin\theta\cos\beta}{(\cos\theta)^3} \\
& + v_B\dot{\beta}\frac{\sin\beta}{\cos\theta}\Bigg]\left[\dot{v}_B\cos(\theta+\beta) \frac{v_B^2}{L}\frac{(\sin(\theta+\beta))^2}{\cos\theta} - v_B\dot{\beta}\sin(\theta+\beta)\right] \\
& + \frac{1}{2} I_G\left[\frac{\dot{v}_B}{L}\frac{\sin(\theta+\beta)}{\cos\theta} + \frac{v_B^2}{L^2}\frac{\sin(\theta+\beta)\cos\beta}{(\cos\theta)^3} + \frac{v_B\dot{\beta}}{L}\frac{\cos(\theta+\beta)}{\cos\theta}\right]^2 \\
& + \frac{1}{2} I_1\left(\frac{\dot{v}_B}{R}\right)^2 + \frac{1}{2}\Bigg[\frac{\dot{v}_B}{L}\frac{\sin(\theta+\beta)}{\cos\theta} + \frac{v_B^2}{L^2}\frac{\sin(\theta+\beta)\cos\beta}{(\cos\theta)^3} \\
& + \frac{v_B\dot{\beta}}{L}\frac{\cos(\theta+\beta)}{\cos\theta} + \ddot{\beta}\Bigg]^2 .
\end{aligned}
$$

When the products appearing in this expression are evaluated, the quasi-acceleration $\dot{v}_B$ will appear in S in only three combinations: $\dot{v}_B^2$, $\dot{v}_B v_B^2/L$, or $\dot{v}_B v_B\dot{\beta}$. All other terms are unimportant, so the final form of S may be written as

$$
S = \frac{1}{2} S_1\left(v_B, \theta, \beta\right)\dot{v}_B^2 + S_2\left(v_B, \theta, \beta\right)\dot{v}_B v_B^2/L + S_3\left(v_B, \theta, \beta\right)\dot{v}_B v_B\dot{\beta}. \tag{7}
$$

It still remains to describe the effect of the forces. The spring force is conservative, so its influence is described by the potential energy. The current length of the spring is ℓ, and its unstretched length is ℓ_0, from which it follows that

$$
V = \frac{1}{2} k\left(\ell - \ell_0\right)^2 .
$$

Equation (9.3.36) describes the contribution of potential energy to the generalized forces. The coefficients $\partial \dot{q}_j / \partial \dot{\gamma}_1$ are obtained from Eqs. (1), with the result that

$$(\Gamma_1)_{\text{conservative}} = -\left(\frac{\partial V}{\partial \theta} \frac{\partial \dot{\theta}}{\partial v_B} + \frac{\partial V}{\partial \ell} \frac{\partial \dot{\ell}}{\partial v_B} \right) = -k(\ell - \ell_0) \left[-\frac{\cos\beta}{\cos\theta} \right].$$

There is no need to consider the reaction forces $\bar{N}_A$ or $\bar{N}_B$ because they are the constraint forces that prevent movement in the direction in which they act, and both constraints were satisfied to derive the kinematical equations. The thrust $\bar{F}$ is not conservative, so we form the virtual work it does. It acts parallel to $\bar{v}_B$, and $\dot{\gamma}_1 = v_B$, which leads to

$$\bar{F} = F\bar{e}_t, \quad \delta\bar{r}_B = \delta\gamma_1\bar{e}_t, \quad \delta W = F\delta\gamma_1,$$

so that

$$\Gamma_1 = (\Gamma_1)_{\text{conservative}} + (\Gamma_1)_F = k(\ell - \ell_0)\frac{\cos\beta}{\cos\theta} + F.$$

The single Gibbs–Appell equation obtained from Eq. (7) is

$$\frac{\partial S}{\partial \dot{v}_B} = S_1\dot{v}_B + S_2 v_B^2/L + S_3 v_B\dot{\beta} = k(\ell - \ell_0)\frac{\cos\beta}{\cos\theta} + F. \qquad (8) \triangleleft$$

The S_j terms are lengthy and are not listed here for the sake of brevity. This omission does not detract from our ability to compare the present analysis with the one in the previous example, which used constrained quasi-coordinates. In the present case there are three first-order differential equations, consisting of one Gibbs–Appell equation, Eq. (8), and two kinematical equations, Eqs. (1), which govern v_B, θ, and ℓ. In the previous analysis there were five first-order DAEs, consisting of two kinematical equations, one constraint equation, and two Gibbs–Appell equations. The unknowns there were two quasi-velocities, two generalized coordinates, and one Lagrange multiplier. Thus the present set of equations is easier to solve, both because they are fewer in number and also because they are purely differential equations. Countering this positive aspect of the present approach is the observation that the process of obtaining an expression for S was much more difficult. The primary reason for the greater intricacy of the manipulations here lies in the fact that the kinematical equations, which express the role of the constraint conditions, become embedded into all aspects of the kinematical relations. In this respect, the merits of using constrained quasi-velocities are like those of using constrained generalized coordinates to analyze a holonomic system with Lagrange's equations.

It also is instructive to compare both analyses of this system using quasi-coordinates with an analysis using generalized coordinates in conjunction with Lagrange's equations. This system is nonholonomic, so the generalized coordinates for such an analysis would of necessity be constrained. If we were to select $q_1 = \theta$ and $q_2 = \ell$, there would be one velocity constraint equation. The kinetic energy of this system in terms of these generalized coordinates is significantly easier to construct than was S in either analysis, primarily because neither linear nor angular accelerations need to be described. The derivation of the generalized forces would involve comparable effort with the analyses for the Gibbs–Appell equations. However, the derivatives for Lagrange's equations are more intricate. There would be two Lagrange equations that are second-order differential equations in regard to the generalized coordinates, which would be supplemented by the constraint equation. The unknowns would be the two generalized coordinates and one Lagrange multiplier, whose presence would be purely algebraic. Converting the Lagrange equations to state-space form would therefore result in a set of five first-order DAEs, which is the same as the situation for the equations derived in Example 9.9.

For some individuals, the derivation of equations of motion that are of a purely differential type is of primary importance, in which case an analysis using unconstrained quasi-coordinates to form the Gibbs-Appell equations (or Kane's equations, which are derived in the next section) would be the preferred route. However, in situations featuring friction, it is often is necessary to examine the associated forces. Doing so would require the use of constrained quasi-coordinates, in order to violate the sliding constraint condition. The author's perspective is that there are numerous readily implemented techniques for solving DAEs, such as those in Subsection 8.2.1, which, when coupled with the greater ease in deriving the required expressions for T and V, makes Lagrange's equations the preferred method.

9.3.3 Kane's Equations

Kane's equations may be considered to be a generalization of the dynamic virtual work principle, which we examined in Subsection 7.1.2. The differential equations of motion that are derived from Kane's formulation for a specified set of generalized coordinates and quasi-velocities are the same as those derived from the Gibbs–Appell equations. However, the operations leading to those equations are quite different. We will examine the similarities and differences after Kane's equations are derived, at which time we will address some of the aspects that have caused Kane's equations to be somewhat controversial.

Recall that the basic concept is to formulate a virtual work of the actual forces and of the equivalent inertial forces. It is convenient to restate the principle here:

$$\begin{aligned}\sum_k \left[-m_k \bar{a}_{Gk} \cdot \delta\bar{r}_{Gk} + \left(-\frac{d\bar{H}_{Gk}}{dt} \right) \cdot \overline{\delta\theta}_k \right] \\ + \sum_n \bar{F}_n \cdot \delta\bar{r}_n + \sum_k \bar{M}_k \cdot \overline{\delta\theta}_k = 0.\end{aligned} \tag{9.3.43}$$

In this expression $\bar{r}_n$ is the point at which force $\bar{F}_n$ is applied, and each body in the system is assigned an index k, with $\bar{M}_k$ being the applied torsional couples. There are a variety of virtual displacements that we can consider. Some choices for these virtual displacements might not be kinematically admissible, in which case the constraint forces associated with violated kinematical restrictions will need to be included.

The difficulty with applying the dynamic virtual work principle directly is the need to identify suitable virtual movements for various parts of a system. This issue was addressed earlier by considered virtual displacements that are produced by differential increments of the generalized coordinates, which led to Lagrange's equations. Now we consider the implication of describing virtual displacements in terms of differentials $\delta\gamma_j$ associated with the quasi-coordinates. The virtual displacement of points in this case is given by Eq. (9.3.8), and a similar relation applies for virtual rotation. The inertial terms in Eq. (9.3.43) already require that we express the velocity of the center of mass and the angular velocity of each body in terms of the $\dot{\gamma}_j$. Hence this means that for each body we can derive the required virtual displacements from the corresponding velocity variables:

$$\begin{aligned}
\bar{v}_{Gk} &= \sum_{j=1}^{N} (\bar{v}_{Gk})_j \, \dot{\gamma}_j + (\bar{v}_{Gk})_0 \implies \delta\bar{r}_{Gk} = \sum_{j=1}^{K} (\bar{v}_{Gk})_j \, \delta\gamma_j, \\
\bar{\omega}_k &= \sum_{j=1}^{N} (\bar{\omega}_k)_j \, \dot{\gamma}_j + (\bar{v}_{Gk})_0 \implies \overline{\delta\theta}_k = \sum_{j=1}^{N} (\bar{\omega}_k)_j \, \delta\gamma_j.
\end{aligned} \tag{9.3.44}$$

It is necessary that we also derive similar expressions for the point where each force is applied and the body to which each couple is applied. When we substitute these representations of virtual displacements and rotations into Eq. (9.3.43), and collect like coefficients of $\delta\gamma_j$, the result is

$$\begin{aligned}
&\sum_{j=1}^{N} \left\{ \sum_{k} \left[-m_k \bar{a}_{Gk} \cdot (\bar{v}_{Gk})_j + \left(-\frac{d\bar{H}_{Gk}}{dt} \right) \cdot (\bar{\omega}_k)_j \right] \right. \\
&\qquad \left. + \sum_{n} \bar{F}_n \cdot (\bar{v}_n)_j + \sum_{n} \bar{M}_n \cdot (\bar{\omega}_n)_j \right\} \delta\gamma_j = 0.
\end{aligned} \tag{9.3.45}$$

We now invoke the usual argument that independence of the $\delta\gamma_j$ quantities requires that the coefficient of each virtual increment vanish. Thus we obtain *Kane's equations:*

$$\boxed{\begin{aligned}
&\sum_{k} \left[m_k \bar{a}_{Gk} \cdot (\bar{v}_{Gk})_j + \left(\frac{d\bar{H}_{Gk}}{dt} \right) \cdot (\bar{\omega}_k)_j \right] \\
&\quad = \sum_{n} \bar{F}_n \cdot (\bar{v}_n)_j + \sum_{n} \bar{M}_n \cdot (\bar{\omega}_n)_j ; \quad j = 1, \ldots, K.
\end{aligned}} \tag{9.3.46}$$

There is one Kane's equation for each quasi-coordinate. In addition to the quasi-velocities, the variables in these equations will be the N generalized coordinates and the J constraint forces associated with the kinematical constraints conditions that are not automatically satisfied. As is true for the Gibbs–Appell equations, N kinematical equations in the form of Eq. (9.3.5), which give the generalized velocities $\dot{q}_j$ in terms of the current $\dot{\gamma}_j$ and q_j values, and J velocity constraint equations supplement Kane's equations.

One interpretation of Kane's equations comes from considering a physical velocity to consist of quasi-velocities measured parallel to corresponding directions vectors $(\bar{v})_j$

or $(\bar{\omega})_j$, which are not unit vectors. It is evident from Eqs. (9.3.44) that

$$(\bar{v}_{Gk})_j \equiv \frac{\partial}{\partial \dot{\gamma}_j}\bar{v}_{Gk}, \quad (\bar{\omega}_k)_j \equiv \frac{\partial}{\partial \dot{\gamma}_j}\bar{\omega}_k. \tag{9.3.47}$$

For this reason the $(\bar{v}_{Gk})_j$ and $(\bar{\omega}_k)_j$ are referred to as *partial velocities*. Thus Kane's equations constitute components, in a generalized sense, of the combined dynamic force and moment equilibrium equations for a system in the directions of each partial velocity. Kane's choice to call the $\dot{\gamma}_j$ values *generalized speeds* does not acknowledge the prior reference to them as quasi-velocities in other formulations. More profoundly, Kane's equations are typically stated in texts devoted to Kane's approach [see, for example, Kane and Levinson (1985)] directly as a new concept, without placing them in the context of prior development. From the author's perspective, it seems unlikely that one would arrive at this concept without considering the principle of dynamic virtual work.

9.3.4 Relationship of the Formulations

The relationship of Kane's equations to the Gibbs–Appell equations has been the topic of much discussions. Although the alternative formulations are derived quite differently, they lead to the same differential equations of motion governing a specific set of generalized coordinates and quasi-velocities. The first step in recognizing this equivalence comes from returning to the virtual work expressed in terms of quasi-coordinates, Eq. (9.3.45). The terms containing the force resultants $\bar{F}_n$ and moment resultants $\bar{M}_n$ represent the virtual work done by all forces. A comparison with Eq. (9.3.38) shows that the Kane equation terms containing the actual forces are merely the generalized forces Γ_j associated with quasi-coordinates when one ignores whether a force is conservative or imposes a constraint. Thus we can rewrite Eq. (9.3.46) as

$$\sum_k \left[m_k \bar{a}_{Gk} \cdot (\bar{v}_{Gk})_j + \left(\frac{d\bar{H}_{Gk}}{dt}\right) \cdot (\bar{\omega}_k)_j \right] = \Gamma_j, \quad j = 1, \ldots, K. \tag{9.3.48}$$

Now let us consider the left side of the Gibbs–Appell equations when S is the sum of the contributions of several bodies, with each term defined by Eq. (9.3.32). In any case we may take point A for each body to be the center of mass G. Both $\bar{\omega}$ and $\bar{H}_G$ depend on velocity variables, but not accelerations. Hence the derivative of S_k for body k with respect to $\ddot{\gamma}_j$ is

$$\begin{aligned}\frac{\partial S_k}{\partial \ddot{\gamma}_j} &= m\left(\bar{a}_{Gk} \cdot \frac{\partial \bar{a}_{Gk}}{\partial \ddot{\gamma}_j}\right) + \frac{1}{2}\frac{\partial \bar{\alpha}_k}{\partial \ddot{\gamma}_j} \cdot \frac{\partial \bar{H}_{Gk}}{\partial t} + \frac{1}{2}\bar{\alpha}_k \cdot \frac{\partial}{\partial \ddot{\gamma}_j}\left(\frac{\partial \bar{H}_{Gk}}{\partial t}\right) \\ &\quad + \frac{\partial \bar{\alpha}_k}{\partial \ddot{\gamma}_j} \cdot (\bar{\omega}_k \times \bar{H}_{Gk}) \\ &= m\left(\bar{a}_{Gk} \cdot \frac{\partial \bar{a}_{Gk}}{\partial \ddot{\gamma}_j}\right) + \frac{\partial \bar{\alpha}_k}{\partial \ddot{\gamma}_j} \cdot \frac{d\bar{H}_{Gk}}{dt} - \frac{1}{2}\frac{\partial \bar{\alpha}_k}{\partial \ddot{\gamma}_j} \cdot \frac{\partial \bar{H}_{Gk}}{\partial t} \\ &\quad + \frac{1}{2}\bar{\alpha}_k \cdot \frac{\partial}{\partial \ddot{\gamma}_j}\left(\frac{\partial \bar{H}_{Gk}}{\partial t}\right).\end{aligned} \tag{9.3.49}$$

To progress further it is convenient to recall the matrix representation of $\partial \bar{H}_{Gk}/\partial t$, when $\bar{\omega}_k$ is represented in terms of components relative to body-fixed axes:

$$\left\{ \frac{\partial H_{Gk}}{\partial t} \right\} = [I_k]\{\alpha_k\}. \tag{9.3.50}$$

In this representation $[I_k]$ is constant, so we have

$$\bar{\alpha}_k \cdot \frac{\partial}{\partial \ddot{\gamma}_j} \left(\frac{\partial \bar{H}_{Gk}}{\partial t} \right) = \{\alpha_k\}^{\mathrm{T}} \frac{\partial}{\partial \ddot{\gamma}_j} \left\{ \frac{\partial H_{Gk}}{\partial t} \right\} = \{\alpha_k\}^{\mathrm{T}} [I_k] \left\{ \frac{\partial \alpha_k}{\partial \ddot{\gamma}_j} \right\}. \tag{9.3.51}$$

Because $[I_k]$ is symmetric, we also have

$$\begin{aligned} \frac{\partial \bar{\alpha}_k}{\partial \ddot{\gamma}_j} \cdot \frac{\partial \bar{H}_{Gk}}{\partial t} &= \left\{ \frac{\partial \alpha_k}{\partial \ddot{\gamma}_j} \right\}^{\mathrm{T}} \left\{ \frac{\partial H_{Gk}}{\partial t} \right\} = \left\{ \frac{\partial \alpha_k}{\partial \ddot{\gamma}_j} \right\}^{\mathrm{T}} [I_k]\{\alpha_k\} \\ &= \{\alpha_k\}^{\mathrm{T}} [I_k] \left\{ \frac{\partial \alpha_k}{\partial \ddot{\gamma}_j} \right\} = \bar{\alpha}_k \cdot \frac{\partial}{\partial \ddot{\gamma}_j} \left(\frac{\partial \bar{H}_{Gk}}{\partial t} \right). \end{aligned} \tag{9.3.52}$$

This reduces Eq. (9.3.49) to

$$\frac{\partial S_k}{\partial \ddot{\gamma}_j} = m \bar{a}_{Gk} \cdot \frac{\partial \bar{a}_{Gk}}{\partial \ddot{\gamma}_j} + \frac{\partial \bar{\alpha}_k}{\partial \ddot{\gamma}_j} \cdot \frac{d \bar{H}_{Gk}}{dt}. \tag{9.3.53}$$

The final step is to recall the discussion centered around Eq. (9.3.15). As we did there, we consider the time derivative of a velocity that is expressed in terms of quasi-coordinates. The variables of interest now are the derivatives of the velocity of the center of mass and the angular velocity, which are described by Eqs. (9.3.44). The result may be written as

$$\begin{aligned} \bar{a}_{Gk} &= \sum_{j=1}^{N} (\bar{v}_{Gk})_j \ddot{\gamma}_j + \sum_{j=1}^{N} \left[\frac{d}{dt} (\bar{v}_{Gk})_j \right] \dot{\gamma}_j + \frac{d}{dt} (\bar{v}_{Gk})_0, \\ \bar{\alpha}_k &= \sum_{j=1}^{N} (\bar{\omega}_k)_j \ddot{\gamma}_j + \sum_{j=1}^{N} \left[\frac{d}{dt} (\bar{\omega}_k)_j \right] \dot{\gamma}_j + \sum_{j=1}^{N} \left[\frac{d}{dt} (\bar{\omega}_k)_j \right] \dot{\gamma}_j + \frac{d}{dt} (\bar{\omega}_k)_0. \end{aligned} \tag{9.3.54}$$

Our focus here is the dependence on the $\ddot{\gamma}_j$ parameters. Such dependence cannot result from differentiating the partial velocities, which depend on only the generalized coordinates. Thus we find that

$$\frac{\partial \bar{a}_{Gk}}{\partial \ddot{\gamma}_j} \equiv (\bar{v}_{Gk})_j, \quad \frac{\partial \bar{\alpha}_k}{\partial \ddot{\gamma}_j} \equiv (\bar{\omega}_k)_j. \tag{9.3.55}$$

Substitution of these relations into Eq. (9.3.53) yields

$$\frac{\partial S_k}{\partial \ddot{\gamma}_j} = m \bar{a}_{Gk} \cdot (\bar{v}_{Gk})_j + \frac{d \bar{H}_{Gn}}{dt} \cdot (\bar{\omega}_k)_j. \tag{9.3.56}$$

This term is the quantity within the summation in Kane's equations, Eq. (9.3.48). Because S for a system is the sum of the S_k of each body, we conclude that the Gibbs–Appell equations and Kane's equations will yield identical differential equations of motion governing a specified set of generalized coordinates and quasi-coordinates.

Although the equations are derived from the two formulations, the operations required for implementing each are substantially different. The Gibbs–Appell formulation is more automated in terms of the operations, in that one has a single function S to describe. In the Kane equation approach, one shows the equivalent inertial forces in a free-body diagram and then evaluates the partial velocities for the centroidal motion and the rotation of the body. The former approach can be cumbersome, in that numerous terms usually arise in S, some of which might not even be relevant to the equations of motion. Also, by coercing us to carefully consider free-body diagrams, Kane's equation tends to lead us to a greater understanding of the way the parts of a system interact.

Although Kane's equations and the Gibbs–Appell equations seem to be quite different from Lagrange's equations, these differences actually only stem from the use of quasi-velocities that are not identically generalized velocities. If one defines the quasi-velocities to be the derivatives of the generalized coordinates, then implementing Kane's equations or the Gibbs–Appell equations will lead to essentially the same differential equations as those derived from Lagrange's equations. The only difference is that Lagrange's equations are second order, but converting the resulting differential equations to state-space form will yield a set of first-order differential equations in which the dynamical equations are the same and the derivative identity is the same as the Kane's and Gibbs–Appell kinematical equations.

From one perspective this equivalence is obvious. Our derivation of Kane's equations began by representing the virtual displacement in terms of quasi-coordinates to formulate the dynamical virtual work principle. In contrast, we derived Lagrange's equations by satisfying the same principle by using a representation of virtual displacements in terms of generalized coordinates. If one has $\dot{\gamma}_j \equiv \dot{q}_j$, then the two approaches are the same, with the Lagrange equation derivation proceeding further in order to introduce kinetic energy.

The aforementioned equivalence can be proven mathematically. Consider the translational acceleration term in Eq. (9.3.48) for body k in this case. In view of Eqs. (9.3.47) with $\dot{\gamma}_j = \dot{q}_j$, this term may be rewritten as

$$\begin{aligned} m_k \bar{a}_{Gk} \cdot (\bar{v}_{Gk})_j &= m_k \frac{d\bar{v}_{Gk}}{dt} \cdot \frac{\partial}{\partial \dot{q}_j} \bar{v}_{Gk} \\ &= \frac{d}{dt}\left(m_k \bar{v}_{Gk} \cdot \frac{\partial}{\partial \dot{q}_j} \bar{v}_{Gk} \right) - m_k \bar{v}_{Gk} \cdot \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j} \bar{v}_{Gk} \right). \end{aligned} \tag{9.3.57}$$

To simplify the second term, recall the identity in Eq. (7.5.4), which was used to derive Lagrange's equations. Applying it in the present situation leads to

$$\frac{\partial}{\partial \dot{q}_j} \bar{v}_{Gk} \equiv \frac{\partial}{\partial q_j} \bar{r}_{Gk}, \tag{9.3.58}$$

from which we find that

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j} \bar{v}_{Gk} \right) = \frac{\partial}{\partial q_j}\left(\frac{d}{dt} \bar{r}_{Gk} \right) = \frac{\partial}{\partial q_j}\left(\bar{v}_{Gk} \right). \tag{9.3.59}$$

This converts the Kane equation term to

$$\begin{aligned} m_k \bar{a}_{Gk} \cdot (\bar{v}_{Gk})_j &= \frac{d}{dt}\left(m_k \bar{v}_{Gk} \cdot \frac{\partial}{\partial \dot{q}_j}\bar{v}_{Gk}\right) - m_k \bar{v}_{Gk} \cdot \frac{\partial}{\partial q_j}(\bar{v}_{Gk}) \\ &= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m_k \bar{v}_{Gk} \cdot \bar{v}_{Gk}\right)\right] - \frac{\partial}{\partial q_j}\left(\frac{1}{2}m_k \bar{v}_{Gk} \cdot \bar{v}_{Gk}\right). \end{aligned} \tag{9.3.60}$$

The term $\frac{1}{2}m_k \bar{v}_{Gk} \cdot \bar{v}_{Gk}$ is the translational kinetic energy of body k relative to its center of mass, and the overall expression is like the kinetic-energy terms in Lagrange's equations.

Now consider the rotational inertia term in Kane's equations. Following similar steps to the preceding equation gives

$$\left(\frac{d\bar{H}_{Gk}}{dt}\right) \cdot (\bar{\omega}_k)_j = \frac{d}{dt}\left(\bar{H}_{Gk} \cdot \frac{\partial}{\partial \dot{q}_j}\bar{\omega}_k\right) - \bar{H}_{Gk} \cdot \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\bar{\omega}_k\right). \tag{9.3.61}$$

The rotational equivalent of Eq. (9.3.59) is

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}\bar{\omega}_k\right) \equiv \frac{\partial}{\partial q_j}\bar{\omega}_k, \tag{9.3.62}$$

which leads to

$$\begin{aligned} \left(\frac{d\bar{H}_{Gk}}{dt}\right) \cdot (\bar{\omega}_k)_j &= \frac{d}{dt}\left(\bar{H}_{Gk} \cdot \frac{\partial}{\partial \dot{q}_j}\bar{\omega}_k\right) - \bar{H}_{Gk} \cdot \frac{\partial}{\partial q_j}\bar{\omega}_k \\ &= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\bar{H}_{Gk} \cdot \bar{\omega}_k\right)\right] - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\bar{H}_{Gk} \cdot \bar{\omega}_k\right). \end{aligned} \tag{9.3.63}$$

The term $\frac{1}{2}\bar{H}_{Gk} \cdot \bar{\omega}_k$ is the rotational kinetic energy of body k. The total kinetic energy of this body is the sum of its translational and rotational parts. Hence, if quasi-velocity $\dot{\gamma}_j$ is the same as generalized velocity $\dot{q}_j$, Eq. (9.3.48) is equivalent to

$$\begin{aligned} &\sum\nolimits_k \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m_k \bar{v}_{Gk} \cdot \bar{v}_{Gk}\right) + \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\bar{H}_{Gk} \cdot \bar{\omega}_k\right)\right] \\ &\quad - \sum\nolimits_k \left[\frac{\partial}{\partial q_j}\left(\frac{1}{2}m_k \bar{v}_{Gk} \cdot \bar{v}_{Gk}\right) + \frac{\partial}{\partial q_j}\left(\frac{1}{2}\bar{H}_{Gk} \cdot \bar{\omega}_k\right)\right] \\ &\quad \equiv \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\sum\nolimits_k \left(\frac{1}{2}m_k \bar{v}_{Gk} \cdot \bar{v}_{Gk} + \frac{1}{2}\bar{H}_{Gk} \cdot \bar{\omega}_k\right)\right] \\ &\quad - \frac{\partial}{\partial q_j}\sum\nolimits_k \left(\frac{1}{2}m_k \bar{v}_{Gk} \cdot \bar{v}_{Gk} + \frac{1}{2}\bar{H}_{Gk} \cdot \bar{\omega}_k\right) \\ &\quad \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \Gamma_j. \end{aligned} \tag{9.3.64}$$

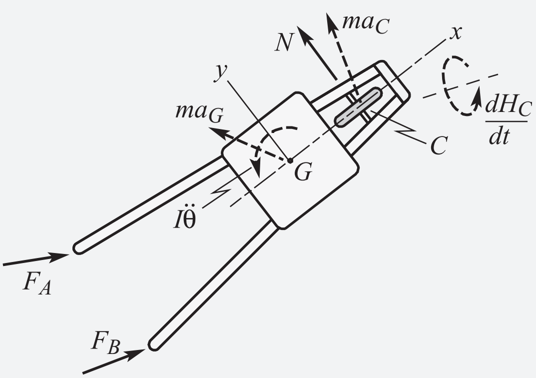
This of course is the same as the raw form of Lagrange's equations prior to finding alternative ways for describing the generalized force.

An overview of the developments here shows that, for a given set of generalized coordinates and quasi-velocities, the equations of motion derived by following the

Gibbs–Appell or Kane's equation formulations will be the same. Furthermore, if the quasi-velocities actually are generalized velocities, then both yield the same equations of motion as those obtained by Lagrange's equations. Both the Gibbs–Appell and Kane's formulations have strong advocates. There also are other formulations of the equations of motion. All are obviously related by the fact that they describe the same system subject to the same set of axioms, Newton's Laws. The mathematical relationship between the various formulations has been extensively explored by Papastavridis (2002). In special circumstances one approach might prove to be advantageous. The author's experience has been that Lagrange's equations, supplemented by the Newton–Euler equations to provide physical insight, serve well as an overall tool for a broad range of engineering problems.

EXAMPLE 9.11 Use Kane's equations to derive the differential equations of motion of the wheelbarrow in Example 9.8.

SOLUTION The objective here is to compare the operations required to implement Kane's equations and the Gibbs–Appell equations. We proceed as though we had not previously solved this problem using the Gibbs–Appell equations, except that we call on the previous results of the kinematical analysis. We begin with a free-body diagram of the system. In this diagram the $m\bar{a}$ vector for each center of mass and the centroidal $d\bar{H}/dt$ for each body are shown as dashed lines in order to identify them as the inertial effects. (In situations in which there are many bodies and/or many forces, it might be advantageous to draw separate diagrams for the actual forces and their inertial equivalents.) Note that, because the wheelbarrow is in planar motion, $d\bar{H}_G/dt$ for it reduces to $I\ddot{\theta}$ about the vertical axis.



Free-body diagram of the wheelbarrow showing the equivalent inertial effects

We use the same generalized coordinates and quasi-velocities as in the previous analysis, so $q_1 = X_C$, $q_2 = Y_C$, $q_3 = \theta$, $\dot{\gamma}_1 = v$, $\dot{\gamma}_2 = \dot{\theta}$. It is helpful to use Eqs. (9.3.47) to describe the partial velocities explicitly when we write Kane's equation. Thus Kane's equations for this system are

$$\begin{aligned}
&(m_1\bar{a}_G)\cdot\frac{\partial\bar{v}_G}{\partial\dot{\gamma}_j}+(I\ddot{\gamma}_2)\frac{\partial\dot{\theta}}{\partial\dot{\gamma}_j}+(m_2\bar{a}_C)\cdot\frac{\partial\bar{v}_C}{\partial\dot{\gamma}_j}+\left(\frac{d\bar{H}_C}{dt}\right)\cdot\frac{\partial\bar{\omega}_2}{\partial\dot{\gamma}_j}\\
&\quad=\left(\bar{F}_A+\bar{F}_C+N\bar{j}\right)\cdot\frac{\partial\bar{v}_C}{\partial\dot{\gamma}_j}+\left(\bar{r}_{A/C}\times\bar{F}_A+\bar{r}_{A/C}\times\bar{F}_C\right)\cdot\frac{\partial\dot{\theta}}{\partial\dot{\gamma}_j}\bar{k},\quad j=1,2,
\end{aligned}\tag{1}$$

where we have used the previous approach to replace the forces acting on the handles with an equivalent force–couple system acting at point C. Each term must be described solely in terms of the q_i and $\dot{\gamma}_i$ variables, which would require a kinematical analysis if we had not already done so in Example 9.8. From that development we have

$$\begin{aligned}
&\bar{a}_C=\ddot{\gamma}_1\bar{i}+\dot{\gamma}_1\dot{\gamma}_2\bar{j},\ \ \bar{a}_G=\left(\ddot{\gamma}_1+\dot{\gamma}_2^2L\right)\bar{i}+\left(\dot{\gamma}_1\dot{\gamma}_2-\ddot{\gamma}_2L\right)\bar{j}\\
&\bar{\omega}_2=\frac{\dot{\gamma}_1}{R}\bar{j}+\dot{\gamma}_2\bar{k},\ \ \bar{\alpha}_2=-\frac{\dot{\gamma}_1\dot{\gamma}_2}{R}\bar{i}+\frac{\ddot{\gamma}_1}{R}\bar{j}+\ddot{\gamma}_2\bar{k}.
\end{aligned}\tag{2}$$

We had not explicitly described the velocity of point G, but it is readily obtained. Thus the quantities required for evaluating the partial velocities are

$$\bar{v}_C=v\bar{i}=\dot{\gamma}_1\bar{i},\quad\bar{v}_G=\bar{v}_C+\dot{\theta}\bar{k}\times\left(-L\bar{i}\right)=\dot{\gamma}_1\bar{i}-L\dot{\gamma}_2\bar{j}.\tag{3}$$

The angular momentum terms corresponding to Eqs. (2) for $\bar{\omega}_2$ and $\bar{\alpha}_2$ are

$$\begin{aligned}
&\bar{H}_C=J\left(\frac{\dot{\gamma}_1}{R}\right)\bar{j}+\frac{J}{2}(\dot{\gamma}_2)\bar{k},\ \ \frac{\partial\bar{H}_C}{\partial t}=\frac{J}{2}\left(-\frac{\dot{\gamma}_1\dot{\gamma}_2}{R}\right)\bar{i}+J\left(\frac{\ddot{\gamma}_1}{R}\right)\bar{j}+\frac{J}{2}\ddot{\gamma}_2\bar{k},\\
&\frac{d\bar{H}_C}{dt}=\frac{\partial\bar{H}_C}{\partial t}+\bar{\omega}_2\times\bar{H}_C=-\frac{J}{R}\dot{\gamma}_1\dot{\gamma}_2\bar{i}+\frac{J}{R}\ddot{\gamma}_1\bar{j}+\frac{J}{2}\ddot{\gamma}_2\bar{k}.
\end{aligned}\tag{4}$$

Because $\dot{\theta}=\dot{\gamma}_1$, substitution of Eqs. (2) and (3) into Eq. (1) for each j yields

$$j=1:\ \ m_1\bar{a}_G\cdot\bar{i}+I\ddot{\theta}\,(0)+m_2\bar{a}_C\cdot\bar{i}+\frac{d\bar{H}_C}{dt}\cdot\left(\frac{1}{R}\bar{j}\right)=\bar{R}\cdot\bar{i}+\bar{M}\cdot\bar{0},$$

$$j=2:\ \ m_1\bar{a}_G\cdot\left(-L\dot{\gamma}_2\bar{j}\right)+I\ddot{\theta}\,(1)+m_2\bar{a}_C\cdot\left(\bar{0}\right)+\frac{d\bar{H}_C}{dt}\cdot\left(1\bar{k}\right)=\bar{R}\cdot\bar{0}+\bar{M}\cdot\left(1\bar{k}\right),$$

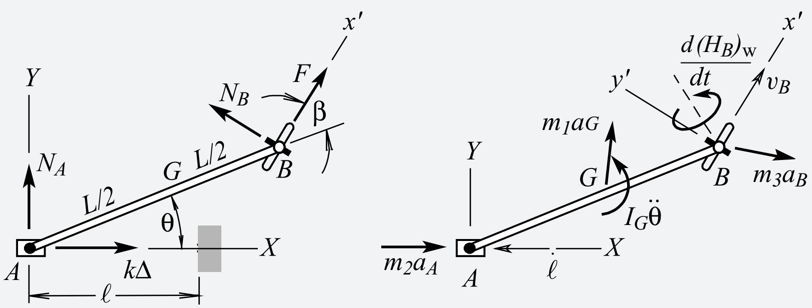
where $\bar{R}$ and $\bar{M}$ are the resultant force and moment about point C of the forces $\bar{F}_A$ and $\bar{F}_B$. The result of substituting Eqs. (2) and (4) into the preceding equations is

$$\begin{aligned}
&m_1\left(\ddot{\gamma}_1+\dot{\gamma}_2^2L\right)+m_2\ddot{\gamma}_1+\frac{J}{R^2}\ddot{\gamma}_1=\bar{R}\cdot\bar{i},\\
&m_1\left(\dot{\gamma}_1\dot{\gamma}_2-\ddot{\gamma}_2L\right)\left(-L\dot{\gamma}_2\right)+I\ddot{\gamma}_2+\frac{J}{2}\ddot{\gamma}_2=\bar{M}\cdot\bar{k}.
\end{aligned}\qquad\triangleleft$$

These equations of motion are the same as the result of Example 9.8. The kinematical quantities that must be described in each approach are essentially the same. The primary difference is that S is generated in a more automatic fashion, but the manipulations are more tedious. The operations in Kane's equations are broken into smaller steps, and one can gain some physical insights by examining the forcing effects in the directions associated with the partial velocities.

EXAMPLE 9.12 Use Kane's equations with $\dot{\gamma}_1 = v_B$ and $q_1 = \theta$, $q_2 = \ell$ to derive equations of motion for the steerable linkage in Example 9.10.

SOLUTION As a closure to our exploration of alternative techniques for deriving equations of motion, we apply Kane's equations to a system in which the derivation of the Gibbs–Appell function encountered considerable complexity. Whether the same level of complexity is encountered here is a key question to be addressed. We begin with a free-body diagram. For the sake of clarity, we use separate diagrams to describe the actual forces and the inertial equivalents. (The diagram on the left is like the one in Example 9.9, except that it now shows the fixed XYZ coordinate system that is selected as the global system.) Bar AB is in planar motion, so the forces acting on it are equivalent to a force $m_1\bar{a}_G$ at the center of mass and a couple $I_G\ddot{\theta}$ acting about the vertical axis. Collar A follows its stationary guide in any movement because the quasi-velocity is unconstrained. Hence the inertial effect for this body is merely $m_2\bar{a}_A$ along the guide. In contrast, the wheel executes a spatial rotation, so the equivalent system is a force $m_3\bar{a}_B$ and a couple $\left(d\bar{H}_B/dt\right)_{\text{w}}$.



Actual forces and their inertial equivalents

There is no need to carry out a kinematical analysis here, because the present motion variables are the same as those previously used to describe S. The only quasi-velocity is v_B, so Kane's equation for this system is

$$m_1\bar{a}_G \cdot \frac{\partial \bar{v}_G}{\partial v_B} + I_G\ddot{\theta}\frac{\partial \dot{\theta}}{\partial v_B} + \left(-m_2\ddot{\ell}\bar{I}\right) \cdot \frac{\partial \bar{v}_A}{\partial v_B} + m_3\bar{a}_B \cdot \frac{\partial \bar{v}_B}{\partial v_B} + \left(\frac{d\bar{H}_B}{dt}\right)_{\text{w}} \cdot \frac{\partial \bar{\omega}_{\text{w}}}{\partial v_B}$$
$$= k\Delta\bar{I} \cdot \frac{\partial \bar{v}_A}{\partial v_B} + F\left[\cos(\theta+\beta)\,\bar{I} + \sin(\theta+\beta)\,\bar{J}\right] \cdot \frac{\partial \bar{v}_B}{\partial v_B}.$$

It should be noted that the reaction forces $\bar{N}_A$ and $\bar{N}_B$ do not appear here because the respective partial velocities will be perpendicular to these constraint forces as a result of using unconstrained quasi-coordinates. Before we perform the main substitutions, it is useful to simplify the preceding by using Eqs. (5) in Example 9.10 to describe the motion of the center of mass. We also account for the facts that $\bar{v}_A = -\dot{\ell}\bar{I}$, that $\bar{F}$ acts parallel to $\bar{v}_B$, and that $\bar{\omega}_{\text{w}} \cdot \bar{i}' = 0$. Thus, we have

$$\begin{aligned}
&\frac{1}{4}m_1\left(-\ddot{\ell}\bar{I}+\bar{a}_B\right)\cdot\left(-\frac{\partial\dot{\ell}}{\partial v_B}\bar{I}+\frac{\partial\bar{v}_B}{\partial v_B}\right)+I_G\ddot{\theta}\frac{\partial\dot{\theta}}{\partial v_B}+m_2\ddot{\ell}\frac{\partial\dot{\ell}}{\partial v_B}\\
&\quad+m_3\bar{a}_B\cdot\frac{\partial\bar{v}_B}{\partial v_B}+\left(I_1\alpha_{y'}\bar{j}'+I_2\alpha_{z'}\bar{k}'\right)\cdot\frac{\partial\left(\omega'_y\bar{j}'+\omega_{z'}\bar{k}'\right)}{\partial v_B}\\
&\quad+\left[\left(\omega'_y\bar{j}'+\omega_{z'}\bar{k}'\right)\times\left(I_1\omega'_y\bar{j}'+I_2\omega_{z'}\bar{k}'\right)\right]_{\text{wheel}}\cdot\frac{\partial\left(\omega'_y\bar{j}'+\omega_{z'}\bar{k}'\right)}{\partial v_B}\\
&=-k\Delta\frac{\partial\dot{\ell}}{\partial v_B}+F\left(\bar{e}_t\right)_B\cdot\frac{\partial}{\partial v_B}\left[\left(v_B\right)\left(\bar{e}_t\right)_B\right].
\end{aligned}$$

It is evident that the last of the inertial terms in the preceding equation vanishes. Combining like terms in the remainder reduces Kane's equation to

$$\begin{aligned}
&\left(\frac{1}{4}m_1+m_2\right)\ddot{\ell}\frac{\partial\dot{\ell}}{\partial v_B}+\left(\frac{1}{4}m_1+m_3\right)\bar{a}_B\cdot\frac{\partial\bar{v}_B}{\partial v_B}\\
&\quad-\frac{1}{4}m_1\left(\ddot{\ell}\bar{I}\cdot\frac{\partial\bar{v}_B}{\partial v_B}+\bar{a}_B\cdot\frac{\partial\dot{\ell}}{\partial v_B}\bar{I}\right)+I_G\ddot{\theta}\frac{\partial\dot{\theta}}{\partial v_B}\\
&\quad+I_1\alpha_{y'}\frac{\partial\omega'_y}{\partial v_B}+I_2\alpha_{z'}\frac{\partial\omega_{z'}}{\partial v_B}\\
&=-k\Delta\frac{\partial\dot{\ell}}{\partial v_B}+F.
\end{aligned}$$

The acceleration terms were derived in Example 9.10. We obtain the partial velocities by differentiating the kinematical equations and velocity expressions that were derived there, which gives

$$\begin{aligned}
&\frac{\partial\dot{\ell}}{\partial v_B}=-\frac{\cos\beta}{\cos\theta},\quad\frac{\partial\dot{\theta}}{\partial v_B}=\frac{1}{L}\frac{\sin\left(\theta+\beta\right)}{\cos\theta},\\
&\frac{\partial\bar{v}_B}{\partial v_B}=\cos\left(\theta+\beta\right)\bar{I}+\sin\left(\theta+\beta\right)\bar{J},\\
&\frac{\partial\omega_{y'}}{\partial v_B}=\frac{1}{R},\quad\frac{\partial\omega_{z'}}{\partial v_B}=\frac{\partial\dot{\theta}}{\partial v_B}.
\end{aligned}$$

Substitution of these expressions and the previously derived acceleration terms converts Kane's equation for this system to

$$\begin{aligned}
&\left(\frac{1}{4}m_1+m_2\right)\left[-\dot{v}_B\frac{\cos\beta}{\cos\theta}-\frac{v_B^2}{L}\frac{\sin\left(\theta+\beta\right)\sin\theta\cos\beta}{\left(\cos\theta\right)^3}+v_B\dot{\beta}\frac{\sin\beta}{\cos\theta}\right]\left(-\frac{\cos\beta}{\cos\theta}\right)\\
&\quad+\left(\frac{1}{4}m_1+m_3\right)\left[\dot{v}_B\cos\left(\theta+\beta\right)-\frac{v_B^2}{L}\frac{\left(\sin\left(\theta+\beta\right)\right)^2}{\cos\theta}\right.
\end{aligned}$$

$$\begin{aligned}
&- v_B\dot{\beta}\sin(\theta+\beta)\big]\cos(\theta+\beta) + \left(\frac{1}{4}m_1 + m_3\right)\Big[\dot{v}_B\sin(\theta+\beta) \\
&+ \frac{v_B^2}{L}\frac{\sin(\theta+\beta)\cos(\theta+\beta)}{\cos\theta} + v_B\dot{\beta}\cos(\theta+\beta)\Big]\sin(\theta+\beta) \\
&- \frac{1}{4}m_1\Bigg[-\dot{v}_B\frac{\cos\beta}{\cos\theta} - \frac{v_B^2}{L}\frac{\sin(\theta+\beta)\sin\theta\cos\beta}{(\cos\theta)^3} \\
&+ v_B\dot{\beta}\frac{\sin\beta}{\cos\theta}\Bigg]\cos(\theta+\beta) - \frac{1}{4}m_1\big[\dot{v}_B\cos(\theta+\beta) \\
&- \frac{v_B^2}{L}\frac{(\sin(\theta+\beta))^2}{\cos\theta} - v_B\dot{\beta}\sin(\theta+\beta)\Big]\left(-\frac{\cos\beta}{\cos\theta}\right) \\
&+ I_G\Bigg[\frac{\dot{v}_B}{L}\frac{\sin(\theta+\beta)}{\cos\theta} + \frac{v_B^2}{L^2}\frac{\sin(\theta+\beta)\cos\beta}{(\cos\theta)^3} \\
&+ \frac{v_B\dot{\beta}}{L}\frac{\cos(\theta+\beta)}{\cos\theta}\Bigg]\frac{1}{L}\frac{\sin(\theta+\beta)}{\cos\theta} + I_1\left(\frac{\dot{v}_B}{R}\right)\frac{1}{R} \\
&+ \Bigg[\frac{\dot{v}_B}{L}\frac{\sin(\theta+\beta)}{\cos\theta} + \frac{v_B^2}{L^2}\frac{\sin(\theta+\beta)\cos\beta}{(\cos\theta)^3} \\
&+ \frac{v_B\dot{\beta}}{L}\frac{\cos(\theta+\beta)}{\cos\theta} + \ddot{\beta}\Bigg]\frac{1}{L}\frac{\sin(\theta+\beta)}{\cos\theta} \\
= {}& k(\ell - \ell_0)\frac{\cos\beta}{\cos\theta} + F.
\end{aligned}$$

We could go on to simplify this equation of motion by collecting the coefficients of $\dot{v}_B$, v_B^2/L, and $v_B\dot{\beta}$. There is no need to do so in the present context, because the main features are evident. We see that, for a given set of generalized coordinates and quasi-velocities, the effort to analyze the kinematical features of a system is the same for the Gibbs–Appell equations and Kane's equations. From that common starting point, the steps leading to the final equations are considerably less intricate for Kane's equations. The comparison of the two examples here suggests that this aspect is more important than the ability to construct the Gibbs–Appell function in a standard procedure.

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HOMEWORK PROBLEMS

EXERCISE 9.1 The kinetic and potential energies of a stretched cable are

$$T = \frac{1}{2}\mu \int_0^L \dot{w}^2 dx,$$

$$V = F\left[\int_0^L \left(1 + (w')^2\right)^{1/2} dx - L\right] + \int_0^L w\,\mu g dx,$$

where $w(x, t)$ is the transverse displacement. Only unforced motion is of interest, so $\delta W = 0$. Use Hamilton's Principle to derive the nonlinear field equation and the associated boundary conditions at $x = 0$ and $x = L$. The square root term in V may be approximated by a three-term truncation of its series expansion for $|w'| \ll 1$.

EXERCISE 9.2 The transverse vibration of a straight beam according to linear theory is described by a function $w(x, t)$ that is the displacement of the beam's centerline. The energy functionals and virtual work are

$$T = \frac{1}{2}\int_0^L \mu \dot{w}^2 dx,$$

$$V = \frac{1}{2}\int_0^L EI\,(w'')^2\,dx + \int_0^L w\,\mu g dx,$$

$$\delta W = \int_0^L f(x, t)\,\delta w dx,$$

where μ is the mass per unit length and EI is the bending rigidity, both of which can depend on x. Also, $f(x, t)$ is the transverse load per unit length. Use Hamilton's Principle to derive the field equation governing w, as well as the possible boundary conditions. Recall that δw is an arbitrary, but selected, function of x. Consequently, all derivatives of δw are defined by δw in the interior region $0 < x < L$, whereas w and its derivatives are mutually independent at $x = 0$ and $x = L$.

EXERCISE 9.3 Torsion of a short bar whose cross section is not circular features warping of the cross section. This means that, in addition to exhibiting an overall torsional rotation $\theta(x, t)$, a cross section at distance x along the beam will also undergo a displacement in the axial direction that varies with position on the cross section. Approximate potential- and kinetic-energy functionals capturing this effect are

$$T = \frac{1}{2}\int_0^L \rho J \dot{\theta}^2 dx,$$

$$V = \frac{1}{2}\int_0^L \left[GJ\left(\frac{\partial\theta}{\partial x}\right) + E\Gamma\left(\frac{\partial^2\theta}{\partial x^2}\right)\right]^2 dx,$$

where GJ and $E\Gamma$ are cross-section properties that may depend on x. Use Hamilton's Principle to derive the field equation governing w, as well as the possible boundary conditions corresponding to unforced motions.

EXERCISE 9.4 When an axial force P acts on a beam that undergoes transverse displacement $w(x,t)$, the force does work unless both ends are immobile. This combines with the strain energy associated with flexure and leads to the possibility of dynamic buckling. The corresponding energy functionals are

$$T = \frac{1}{2}\int_0^L \mu \dot{w}^2 dx,$$

$$V = \frac{1}{2}\int_0^L \left[EI\left(w''\right)^2 - P\left(w'\right)^2 \right] dx.$$

The coefficient EI is the bending rigidity, which might depend on x. Use Hamilton's Principle to derive the differential equation and possible boundary conditions governing w.

EXERCISE 9.5 Under certain circumstances, a horizontally stretched cable that is excited in the vertical plane can exhibit a whirling response in which the displacement has nonzero horizontal and vertical components, labeled $w_y(x,t)$ and $w_z(x,t)$. The energy functionals and virtual work in this case are

$$T = \frac{1}{2}\int_0^L \mu\left[\dot{w}_y^2 + \dot{w}_z^2\right] dx,$$

$$V = F\left[\int_0^L \left[1 + \left(\frac{\partial w_y}{\partial x}\right)^2 + \left(\frac{\partial w_z}{\partial x}\right)^2\right]^{1/2} dx - L\right] + \int_0^L w_y\,\mu g dx,$$

$$\delta W = \int_0^L \left[f_y(x,t)\,\delta w_y + f_z(x,t)\,\delta w_z\right] dx,$$

where f_y and f_z are distributed transverse forces in the respective directions. Both ends of the cable are stationary. Use Hamilton's Principle to derive the differential equations governing w_y and w_z. It is permissible to expand the integrand of V in a Taylor series of the displacement variables, in which terms higher than the fourth power of displacement variables are discarded.

EXERCISE 9.6 When a bar having an arbitrary cross-sectional shape is loaded by a transverse force, the cross section will undergo displacements w_y and w_z in two orthogonal directions transverse to the longitudinal x axis. The kinetic and potential energies in this case are

$$T = \frac{1}{2}\int_0^L \left(\dot{w}_y^2 + \dot{w}_z^2\right) \rho A\, dx,$$

$$V = \frac{1}{2}\int_0^L E\left[I_{zz}\left(\frac{\partial^2 w_y}{\partial x^2}\right) + 2I_{yz}\left(\frac{\partial^2 w_y}{\partial x^2}\right)\left(\frac{\partial^2 w_z}{\partial x^2}\right) + I_{yy}\left(\frac{\partial^2 w_z}{\partial x^2}\right)^2\right] dx,$$

$$\delta W = \int_0^L \left[f_y(x,t)\,\delta w_y + f_z(x,t)\,\delta w_z\right] dx,$$

where I_{yy}, I_{zz}, and I_{yz}, which are second moments of the cross-sectional area, might depend on x. Also, f_y and f_z are distributed transverse forces in the respective directions.

The virtual displacement field in this case consists of independent fields $\delta w_y(x,t)$ and $\delta w_z(x,t)$, both of which must be continuously differentiable in order to be kinematically admissible. Use Hamilton's Principle to derive the pair of differential equations governing w_y and w_z, as well as the possible boundary conditions pertaining to each displacement component.

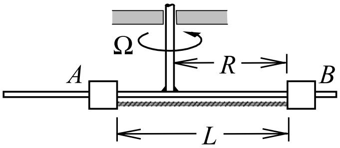
EXERCISE 9.7 It is desired to use the Ritz series method to derive approximate equations describing the effects of nonlinearity in the vibration of the stretched cable in Exercise 9.1. Both ends are stationary, so that $w = 0$ at $x = 0$ and $x = L$, which means that suitable basis functions are $\psi_j = \sin(j\pi x/L)$. Derive the differential equations governing the two-term Ritz series that uses these basis functions.

EXERCISE 9.8 The energy functionals for flexural vibration of a beam are given in Exercise 9.2. A cantilever beam is one in which one end is not permitted to displace or rotate, corresponding to $w = \partial w/\partial x = 0$ at $x = 0$, while the other end, $x = L$, is free to move. Suitable basis functions have the form $\psi_j = (x/L)^{j+p}$, where p is an integer. (a) Identify the lowest value of p that is appropriate to the cantilever beam. (b) Derive the equations for the Ritz series coefficients when the series is truncated at two terms and p has the value in Part (a). The cross-sectional properties are constant, and the transverse loading is a uniformly distributed harmonic excitation, $f(x,t) = F_0 \sin(\Omega t)$.

EXERCISE 9.9 The energy functionals for an axially loaded beam are given in Exercise 9.4. A cantilever beam is one in which one end is not permitted to displace or rotate, corresponding to $w = \partial w/\partial x = 0$ at $x = 0$, while the other end, $x = L$, is free to move. Use a three-term Ritz series for the transverse displacement to derive approximate equations for this beam's vibration in the presence of the axial force.

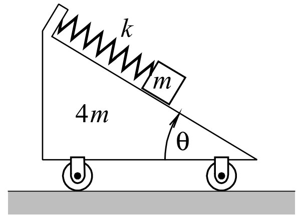
EXERCISE 9.10 Both ends of a cable like the one described in Exercise 9.5 are fixed, so suitable basis functions for both displacement components are $\sin(j\pi x/L)$. The cable is excited by a uniformly distributed transverse load in the xy plane, $f_y(x,t) = F_0 \sin(\Omega t)$, $f_z = 0$. Use a two-term Ritz series for each transverse displacement component to derive approximate equations describing the whirling response. All terms in the differential equations may be truncated at cubic powers of the Ritz series coefficients, which means that Taylor series expansions of the energy functionals may be truncated at quartic powers of w_y and w_z.

EXERCISE 9.11 Torsional excitation Γ acting about the vertical shaft causes the T-bar to rotate at angular speed Ω. Collars A and B are connected by an inextensible cable. Derive Hamilton's equations of motion governing Ω and the radial distance R to a collar based on the assumption that the cable remains taut.



Exercise 9.11

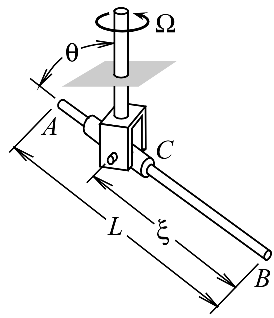
EXERCISE 9.12 The small mass m is supported by a spring as it moves along the smooth incline on the cart, whose mass is M. The spring has stiffness k and its unstretched length is ℓ. Derive Hamilton's canonical equations for the system.



Exercise 9.12

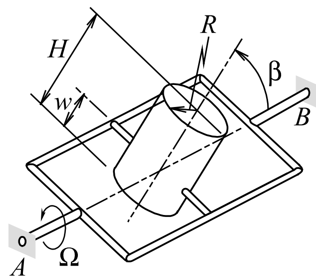
EXERCISE 9.13 Derive Hamilton's equations of motion for the system in Exercise 7.35.

EXERCISE 9.14 Collar C is attached to the vertical shaft by a fork-and-clevis, so the angle of inclination θ of bar AB is arbitrary. Because this bar slides through the collar, the distance ξ from the pivot point to the end of the bar is variable, but it may be assumed that the bar does not spin about its own axis. The vertical shaft rotates at the constant rate Ω. Derive Hamilton's equations governing ξ and θ.



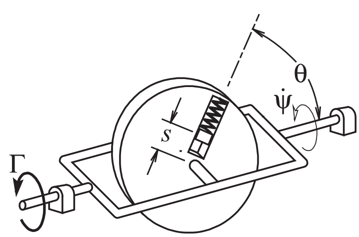
Exercise 9.14

EXERCISE 9.15 The cylinder of mass m is free to rotate by angle β relative to the gimbal, which rotates about the horizontal axis. The precessional rotation is unconstrained, so $\dot{\psi} = \Omega$ is unknown. Derive Hamilton's equations governing the gimbal's precession angle ψ and the relative rotation angle β.



Exercise 9.15

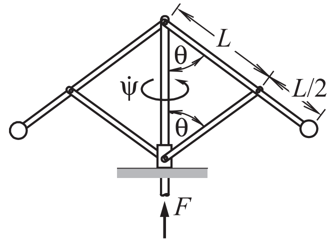
EXERCISE 9.16 The disk rotates by angle θ relative to the gimbal, which precesses at a constant angular speed $\dot{\psi}$ because of the constraining torque Γ. A block of mass m slides inside a slot within the disk. The spring has stiffness is k, and $s = 0$ corresponds to the unstretched position of the spring. The disk has been balanced to be axisymmetric about the gimbal shaft, with centroidal moments of inertia I about this shaft and I' perpendicular to it. Derive Hamilton's equations of motion governing θ and s.



Exercise 9.16

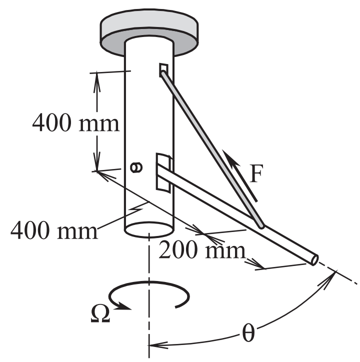
EXERCISE 9.17 Consider the system in Exercise 9.11 in the situation in which Ω is maintained at a constant value by the torque Γ. Use conservation of the Hamiltonian and the work–energy principle to find the differential equation governing R and an expression for the associated value of Γ.

EXERCISE 9.18 The precession rate $\dot{\psi}$ of the flyball governor is held constant by a torque Γ that acts about the vertical shaft, while that shaft is pushed upward by a constant force F. Each ball has mass m, and the inertia of each bar is negligible. Formulate the angular momentum about the rotation axis, the mechanical energy, and the Hamiltonian of this system. Are any of these quantities conserved? What is the differential equation of motion associated with the time derivative of each quantity?



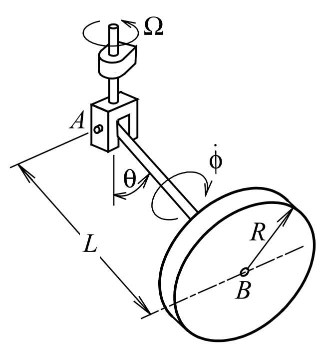
Exercise 9.18

EXERCISE 9.19 The angle θ is adjusted by pulling the cable inward, thereby causing the bar to pivot about the horizontal pin in the vertical shaft. The mass of the bar is 20 kg, and the moment of inertia of the vertical shaft about its axis of rotation is 0.2 kg-m^2. The cable tension is constant at 2 kN. At the initial position $\theta = 20^\circ$, $\dot{\theta} = 0$, and $\Omega = 5$ rad/s. Determine the value of $\dot{\theta}$ when $\theta = 90^\circ$ in the following circumstances: (a) A torque is applied to the vertical shaft, with the result that Ω is constant. (b) There is no torque acting on the vertical shaft, with the result that the system precesses freely.



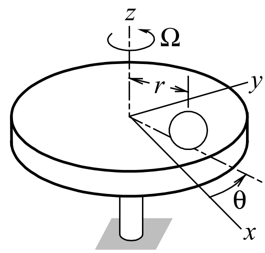
Exercise 9.19

EXERCISE 9.20 Application of an unspecified torque Γ to the vertical shaft causes the system to rotate at constant rate Ω. The mass of the disk is m_1, its centroidal radii of gyration are κ_1 and κ_2 respectively about axis AB and transverse to that axis, and the mass of shaft AB is m_2. Pin A has ideal properties, which allows the angle θ to change freely. Consider two situations: (a) The disk spins freely relative to shaft AB at angular speed $\dot{\phi}$ that is an unknown function of time; (b) a servomotor maintains the spin rate $\dot{\phi}$ at a known constant. For each case, identify whether any angular momenta are conserved, and compare the rate of change of the Hamiltonian of the system with the result of applying the power balance principle.



Exercise 9.20

EXERCISE 9.21 The ball bearing rolls without slipping on the horizontal turntable, whose rotation rate Ω is constant. The xyz coordinate system is attached to the turntable, so the position of the ball bearing relative to the turntable is defined by the radial distance r and azimuthal angle θ. It may be assumed that the ball bearing does not spin about its vertical centerline relative to the turntable. The radius of the ball bearing is R. Determine whether the differential equations of motion governing r and θ can be derived solely as time derivatives of conservation equations.



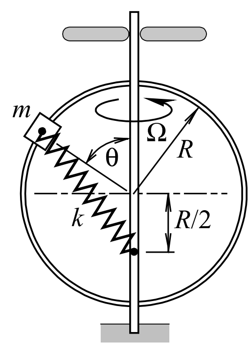
Exercise 9.21

EXERCISE 9.22 Angle θ for the flyball governor in Exercise 9.18 is controlled by applying force F, whose value at any instant is known. The system rotates freely about the vertical axis. The mass of each sphere is m, and the mass of the linkage is negligible. Use Routh's method for the ignoration of coordinates to derive a single differential equation governing θ.

EXERCISE 9.23 Consider the cart with a small attached mass in Exercise 9.12. Derive a single differential equation of motion for the relative distance x, by using Routh's method for the ignoration of coordinates.

EXERCISE 9.24 Consider the system in Exercise 9.15. Let $w = H/2$, so that the center of mass is coincident with the horizontal axis. Use Routh's method for the ignoration of coordinates to derive a single differential equation governing β. Can such a formulation be used when $w \neq H/2$? Explain your answer.

EXERCISE 9.25 Collar m slides along the circular guide, which spins freely about the vertical axis at angular speed Ω. The spring's stiffness is $k = mg/R$ and its unstretched length is $R/2$. The collar was at rest relative to the guide at its initial position $\theta = 60°$, at which location the rotation rate was $\Omega = (g/R)^{1/2}$. Use Routh's method to derive a single differential equation of motion for θ.



Exercise 9.25

EXERCISE 9.26 Use the Routhian to derive differential equations of motion governing the system in Exercise 9.14 in the case where the precession rate Ω is constant.

EXERCISE 9.27 Consider the system in Exercise 7.66 in the case where the torque $\Gamma = 0$ and the flywheel disk spins freely without servocontrol. The rotation angles β, ψ, and ϕ are suitable generalized coordinates, but the latter two are ignorable. Thus there is a Routhian function from which a single differential equation governing β can be extracted. Derive this function.

EXERCISE 9.28 The absolute velocity of a particle may be represented by the components v_x, v_y, and v_z relative to the axes of a moving reference system xyz. Suppose that the angular velocity $\bar{\omega}$ of xyz and the velocity $\bar{v}_O$ of the origin of xyz are known as functions of time. Derive the Gibbs–Appell equations of motion relating the quasi-velocities $\dot{\gamma}_1 = v_x$, $\dot{\gamma}_2 = v_y$, and $\dot{\gamma}_3 = v_z$ to the resultant force acting on the particle.

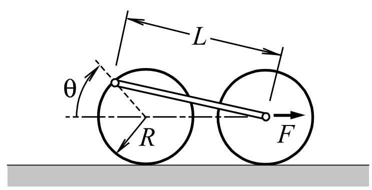
EXERCISE 9.29 Derive the Gibbs–Appell equations of motion for the system in Exercise 9.12.

EXERCISE 9.30 Derive the Gibbs–Appell equations of motion for the flyball governor in Exercise 9.18.

EXERCISE 9.31 Derive the Gibbs–Appell equations of motion for the semicylinder in Exercise 7.43.

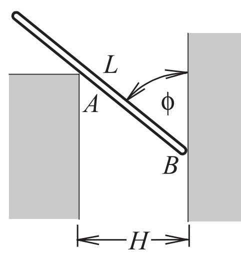
EXERCISE 9.32 Derive the Gibbs–Appell equations of motion for the motorboat in Exercise 8.10

EXERCISE 9.33 Two cylinders, each having mass m, are linked by a connecting rod whose mass is negligible. A known horizontal force $F(t)$ is applied to the right cylinder, and neither cylinder slips in its rolling motion. In the initial position, the angle θ locating the connecting pin is zero. Derive the Gibbs–Appell equation(s) of motion.



Exercise 9.33

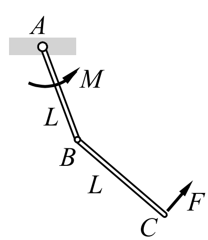
EXERCISE 9.34 Friction between the rod and the surfaces it contacts is negligible. Use the Gibbs–Appell formulation to derive differential equations describing the motion while the rod remains in contact with the wall.



Exercise 9.34

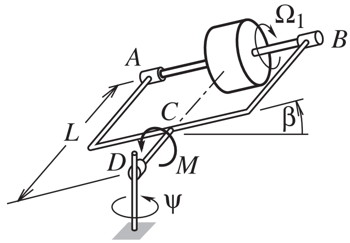
EXERCISE 9.35 Derive the Gibbs–Appell equations of motion for the system in Exercise 7.58.

EXERCISE 9.36 A known couple $\bar{M}(t)$ is applied to the upper bar. Force $\bar{F}$, which is applied perpendicularly to the lower bar, acts to make the velocity of end C always be collinear with the line from joint A to end B. The bars have equal mass m, and the system lies in the vertical plane. Derive the Gibbs–Appell equations of motion governing a set of generalized coordinates and unconstrained quasi-velocities.



Exercise 9.36

EXERCISE 9.37 Torque M, which induces the gimbal rotation β, is a specified function of time, while the torque Γ causes the precession rate to vary according to $\dot{\psi} = c_1\dot{\beta} + c_2\beta + c_3$, where the c_n are constants. The spin rate Ω_1 is maintained at a constant value by a servomotor. The mass of this motor and the gimbal are negligible. The mass of the flywheel is m, and its principal radii of gyration for centroidal axes are κ_1 about its spin axis and κ_2 normal to that axis. Derive the Gibbs–Appell equations of motion.

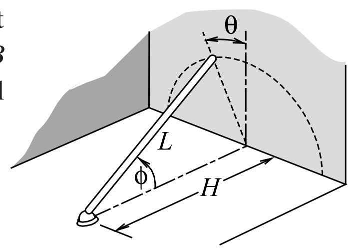


Exercise 9.37

EXERCISE 9.38 Consider the system in Exercise 9.15 in the case in which $D \neq L/2$. Derive the Gibbs–Appell equations governing the precession angle ψ and nutation angle θ.

EXERCISE 9.39 Derive the Gibbs–Appell equations of motion for the sphere on a rotating turntable in Exercise 8.23 when unconstrained quasi-velocities are used to describe the motion.

EXERCISE 9.40 The bar is supported by a ball-and-socket joint A. The coefficient of sliding friction between end B of the bar and the wall is μ. Derive a set of Gibbs–Appell equations of motion.



Exercise 9.40

EXERCISE 9.41 Use Kane's formulation to derive the differential equations of motion describing the system in Exercise 9.12.

EXERCISE 9.42 Derive Kane's equations of motion for the flyball governor in Exercise 9.18.

EXERCISE 9.43 Consider the system in Exercise 9.15 in the case in which $D \neq L/2$. Derive Kane's equations for the precession angle ψ and nutation angle θ.

EXERCISE 9.44 Derive Kane's equations of motion for the power boat in Exercise 9.32.

EXERCISE 9.45 Derive Kane's equations of motion for the gyroscope in Exercise 9.37 when unconstrained quasi-velocities are used to describe the motion.

EXERCISE 9.46 Friction between the rod in Exercise 9.34 and the surfaces it contacts is negligible. Use Kane's formulation to derive the differential equations of motion. Consider only the interval during which the rod remains in contact with the wall.

EXERCISE 9.47 The coefficient of kinetic friction between the rod and corner A in Exercise 9.34 is μ, while frictional resistance at the wall is negligible. Use Kane's formulation to derive the equations of motion for this system corresponding to the interval during which the rod remains in contact with the wall.

EXERCISE 9.48 Use Kane's formulation to derive the equations of motion of the steerable linkage in Example 9.9 when $\dot{\theta}$ and $\dot{\ell}$ are selected as the quasi-velocities.

EXERCISE 9.49 Derive Kane's equations governing the steerable linkage in Example 9.9 when the speed v of the wheel is used as the sole quasi-velocity.

EXERCISE 9.50 Beth, the tricycle rider in Exercise 8.14, steers by applying a known torque M to the handlebars. Derive Kane's equations corresponding to using the speed v_A of the center of the front wheel, the chassis rotation rate $\dot{\theta}$, and the steering rate $\dot{\beta}$ as unconstrained quasi-velocities. The inertia of the rear wheels may be ignored.

EXERCISE 9.51 Derive Kane' equations of motion governing a set of unconstrained quasi-velocities for the sphere on a rotating turntable in Exercise 8.23.

CHAPTER 10

Gyroscopic Effects

The focus thus far has been on the derivation of equations of motion for rigid-body motion. Sometimes the goal was to characterize the force system required to produce a specified motion, while other situations entailed where the response was not known, which led to differential equations of motion. Both situations will be encountered in this chapter, where the common thread is the prominent role of gyroscopic action. Such phenomena are exploited in gyroscopes, whose theory will be introduced here. However, much can be learned about the nature of dynamical responses by beginning with studies of simpler, yet more common, systems that display similar effects.

10.1 FREE MOTION

One of the first types of spatial motion treated in basic physics and engineering courses on mechanics is projectile motion, whose study is devoted to the determination of the motion of the center of mass. In contrast, the manner in which the body rotates about its center of mass is seldom discussed in a fundamental course. Our study of free rotation will be based on the assumption that the only external force is gravitational attraction acting at the center of mass. The corollary is that the resultant moment about the center of mass is zero, from which it follows that

$$\boxed{\bar{H}_G \text{ is constant.}} \tag{10.1.1}$$

A further corollary of taking the resultant moment to be zero is that the rotational motion occurs independently of the motion of the center of mass. In reality, aerodynamic forces acting on a body may be represented as a force–couple system acting at the mass center. Such forces depend on the orientation (angle of attack) as well as on the overall velocity. Hence, accurate models of the motion of objects through the air might require consideration of coupling between the translational and rotational motions.

10.1.1 Axisymmetric Bodies

An axisymmetric body, whose shape is obtained by rotating a curve about the axis of symmetry, and whose mass is independent of the azimuthal angle relative to that axis, has special inertial properties. Specifically, if the axis of symmetry is one of the coordinate axes, then the coordinate system is principal with the moments of inertia about the other two axes having equal values. Without loss of generality, we select the z axis to be the axis of symmetry, and correspondingly let $I_{zz} = I$ and $I_{xx} = I_{yy} = I'$. (Actually, any body having two equal principal moments of inertia behaves inertially as though it were axisymmetric. The analysis that follows is valid for the free motion of any such object, provided that the z axis is aligned with the axis that has the distinct moment of inertia.)

The constant value of $\bar{H}_G$ in corollary (10.1.1) can be determined if the angular velocity is known at the instant the body is released. The direction of this constant vector defines a reference for studying the response. Let us define this direction as the Z axis for a translating reference frame XYZ, so we have

$$\bar{K} = \frac{\bar{H}_G}{|\bar{H}_G|}. \tag{10.1.2}$$

We define Z to be the precession axis for a set of Eulerian angles and let the body's axis of symmetry be the spin axis. The nutation angle is measured between the precession and spin axes, so we have the situation in Fig. 10.1. The body-fixed x axis depicted there is the centroidal transverse axis that lies in the plane containing the space-fixed Z axis and the body-fixed z axis at the instant of interest.

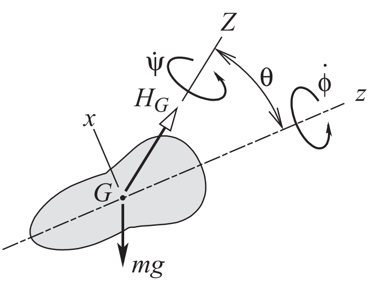


Figure 10.1. Free-body diagram and Eulerian angles for free rotation of an axisymmetric body.

The angular velocity corresponding to an arbitrary set of Eulerian angles is the precessional, spinning, and nutational rotations. The latter occurs about the line of nodes, which is perpendicular to both the precession and spin axes. This direction is opposite the sense of the y axis associated with Fig. 10.1, so

$$\bar{\omega} = \dot{\psi}\bar{K} - \dot{\theta}\bar{j} + \dot{\phi}\bar{k} = \dot{\psi}\sin\theta\bar{i} - \dot{\theta}\bar{j} + \left(\dot{\psi}\cos\theta + \dot{\phi}\right)\bar{k}. \tag{10.1.3}$$

We can use this description of $\bar{\omega}$ to construct $\bar{H}_G$. A different representation comes from the observation that $\bar{H}_G$ lies in the Zz plane at an angle θ from the z axis, so we have

$$\begin{aligned}\bar{H}_G &= I'\dot{\psi}\sin\theta\bar{i} - I'\dot{\theta}\bar{j} + I\left(\dot{\psi}\cos\theta + \dot{\phi}\right)\bar{k} \\ &= H_G\sin\theta\bar{i} + H_G\cos\theta\bar{k}.\end{aligned} \tag{10.1.4}$$

These alternative representations must be consistent, which requires that

$$I'\dot{\psi}\sin\theta = H_G\sin\theta, \quad I'\dot{\theta} = 0, \quad I\left(\dot{\psi}\cos\theta + \dot{\phi}\right) = H_G\cos\theta. \tag{10.1.5}$$

Solving these relations for the rotation rates yields

$$\dot{\psi} = \frac{H_G}{I'}, \quad \dot{\theta} = 0, \quad \dot{\phi} = \left(\frac{1}{I} - \frac{1}{I'}\right)H_G\cos\theta. \tag{10.1.6}$$

This shows that the free rotation of an axisymmetric body is characterized by a steady spinning rotation about the axis of symmetry, accompanied by a steady precession about an axis that is parallel to the angular momentum, with an invariant nutation angle between these axes.

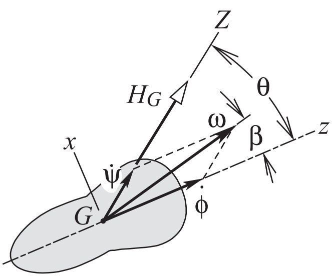


Figure 10.2. Evaluation of the Eulerian angle properties and angular momentum when the angular velocity of an axisymmetric body is known.

Presumably, the body's orientation and angular velocity are known when it was released. As noted earlier, this allows us to construct $\bar{H}_G$, from which we may determine θ. Let us formalize this construction. Because $\dot{\theta} = 0$, it must be that $\bar{\omega}$, $\bar{H}_G$, and the z axis are coplanar. Therefore the initial condition must be as depicted in Fig. 10.2, where the vector $\bar{\omega}$ and its angle β from the z axis presumably are known as initial conditions. We may construct the components of $\bar{\omega}$ from this angle. When we use these components to construct $\bar{H}_G$, we obtain a representation that must match either form in Eqs. (10.1.4), so it must be that

$$\begin{aligned} \bar{H}_G \cdot \bar{k} &= I\,(\omega \cos\beta) = I\left(\dot{\psi}\cos\theta + \dot{\phi}\right) = H_G \cos\theta, \\ \bar{H}_G \cdot \bar{i} &= I'\,(\omega \sin\beta) = I'\dot{\psi}\sin\theta = H_G \sin\theta. \end{aligned} \tag{10.1.7}$$

Eliminating all kinematical parameters except ω and β from these relations yields

$$\boxed{\begin{aligned} \tan\theta &= \frac{I'}{I}\tan\beta, \\ \dot{\psi} &= \frac{H_G}{I'} = \left[(\sin\beta)^2 + \left(\frac{I}{I'}\cos\beta\right)^2\right]^{1/2}\omega, \\ \dot{\phi} &= \left(1 - \frac{I}{I'}\right)\omega\cos\beta. \end{aligned}} \tag{10.1.8}$$

It is evident that there are two cases to consider, depending in whether I'/I is less than or greater than unity. If the body is slender, like a javelin, then $I' > I$, whereas $I' < I$ for an oblate body, such as a disk. (In the case of a sphere, for which $I' = I$, $\bar{H}_G$ is always parallel to $\bar{\omega}$, so there is no need to distinguish between $\dot{\psi}$ and $\dot{\phi}$.) The rotation of a slender body is such $\theta > \beta$ and $\dot{\phi}$ has the same sign as ω if $\beta < 90^\circ$. (Cases in which $\beta > 90^\circ$ are best treated by reversing the sense of the z axis.) The property that $\dot{\phi}$ and ω have the same sign indicates that the spin is in the same sense as the overall rotation; this is said to be a *regular precession.* Conversely, an oblate body executes a *retrograde precession,* in which the spin is in the opposite sense from the overall rotation ($\dot{\phi}/\omega < 0$ when $\beta < 90^\circ$). In this case $\theta < \beta$.

The picture provided by Fig. 10.2 is descriptive of any instant, provided we recognize that, because of the precession, the plane depicted by this figure rotates about the Z axis. Thus the angular velocity is situated on a cone whose axis coincides with Z and whose semivertex angle is $\theta - \beta$. The significance of this observation is that $\bar{\omega}$ determines the velocity of points relative to the center of mass, according to $\bar{v}_{P/G} = \bar{\omega} \times \bar{r}_{P/G}$. Thus every point on a line parallel to $\bar{\omega}$ and intersecting point G is not moving relative to point G. This leads us to a conceptual model formed from (right circular) space and body cones. The space cone consists of the locus of the vector $\bar{\omega}$ from the viewpoint of an observer translating with the center of mass when the tail of $\bar{\omega}$ is placed at that point. Thus $\bar{H}_G$ defines the axis of the space cone, and $|\theta - \beta|$ is the semivertex angle. The body cone represents the locus of the same $\bar{\omega}$ vector from the perspective of an observer who moves in unison with the body. Hence the body cone's semivertex angle is β, and its axis is the body's axis of symmetry. The free rotation of the body may be visualized by letting the body cone, which is attached to the body, roll without slipping over the stationary space cone. The line of contact for this rolling motion is parallel to $\bar{\omega}$ at the instant of interest.

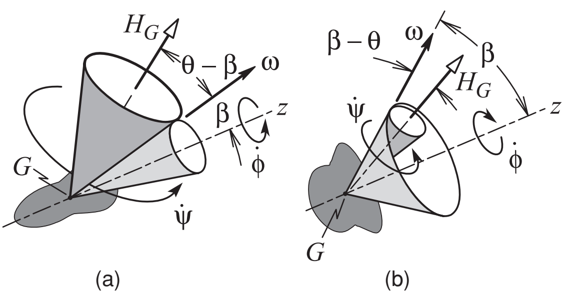
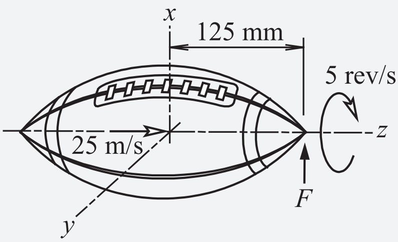


Figure 10.3. Body and space cones for free motion. The body cone and the body are shaded. (a) Regular precession, (b) retrograde precession.

Because $\theta > \beta$ for a regular precession such as the one in Fig. 10.3(a), the exterior of the body cone in this case rolls over the exterior of the space cone. The precession describes the rotation of the body cone's axis about the axis marked by $\bar{H}_G$, and the spin rate is such that points on the body cone that contact the space cone have zero velocity relative to the center of mass. Retrograde precession, which is depicted in Fig. 10.3(b), features $\beta > \theta$, so the space cone is interior to the body cone. In this case the sense of ω indicates the sense of the precession, but the spin takes place in the opposite sense. The term "retrograde" stems from the perception that the body is spinning oppositely to the sense of its overall rotation.

EXAMPLE 10.1 A football has an instantaneous velocity of 25 m/s parallel to its longitudinal axis, z, and it is spinning about that axis at 5 rev/s. At that instant, the ball is deflected by a transverse force $\bar{F}$ at the forward tip. As a result of the action of $\bar{F}$, whose duration is very short, the ensuing motion relative to the center of mass is such that the longitudinal axis always lies on the surface of a cone whose apex angle is 60°. The radii of gyration about centroidal axes are 40 mm and 70 mm along and transversely to the longitudinal axis, respectively. Determine (a) the angular velocity and the velocity of the center of mass immediately after the application of $\bar{F}$, (b) the orientation of the precession axis for the subsequent rotation relative to the orientation of the longitudinal axis prior to the application of $\bar{F}$, (c) the precession and spin rates for the rotational motion.



Example 10.1

SOLUTION This example applies the basic relations to a system that is rather familiar to some sports fans. The force $\bar{F}$ fits the impulsive model, which enables us to consider the orientation of the body-fixed xyz reference frame to be unaltered during the interval of the impulse. The moments of inertia are

$$I = I_{zz} = m\,(0.040)^2 = 0.0016m \text{ kg-m}^2,$$
$$I' = I_{xx} = I_{yy} = m\,(0.070)^2 = 0.0049m \text{ kg-m}^2,$$

where m is the mass (in units of kilograms).

The initial linear and angular momenta are

$$\bar{P}_1 = m\,(\bar{v}_G)_1 = 25m\bar{k} \text{ kg-m/s},$$
$$\left(\bar{H}_G\right)_1 = I\omega_z\bar{k} = I\,(-5)\,(2\pi)\,\bar{k} \text{ kg-m}^2\text{/s}.$$

Because the change in the position of any point on the football is negligible during the impulse interval, the point of application of $\bar{F}$ is essentially constant at $\bar{r}_{P/G} = 0.125\bar{k}$ m. The corresponding impulse–momentum principles are

$$\begin{aligned} m(\bar{v}_G)_2 &= 25m\bar{k} + F\,(\Delta t)\,\bar{i}, \\ (\bar{H}_G)_2 &= (\bar{H}_G)_1 + \bar{r}_{P/G} \times F\,(\Delta t)\,\bar{i} = -0.05027m\bar{k} + 0.125F\,(\Delta t)\,\bar{j}. \end{aligned} \tag{1}$$

Because $I' > I$, the free rotation of the football is a regular precession. According to Fig. 10.3, the given information that the z axis sweeps out a 60° cone in the subsequent rotation means that the nutation angle is $\theta = 30°$, with the precession axis coincident with the axis of that cone. The corresponding angle β between the angular velocity and the axis of symmetry is found from Eqs. (10.1.8) to be

$$\beta = \tan^{-1}\left(\frac{I}{I'}\tan\theta\right) = 10.08°. \tag{2}$$

The expression for $(\bar{H}_G)_2$ in Eqs. (1) indicates that $\omega_x = 0$ at the instant when $\bar{F}$ terminates and $\omega_y > 0$ and $\omega_z < 0$. Hence the angular velocity at that instant must be

$$\bar{\omega} = \omega\left(\sin\beta\,\bar{j} - \cos\beta\bar{k}\right) = \omega\left(0.18526\bar{j} - 0.9027\bar{k}\right).$$

The corresponding angular momentum is

$$(\bar{H}_G)_2 = I'\omega_y\bar{i} + I\omega_z\bar{k} = m\omega(0.9078\bar{j} - 1.5723\bar{k})(10^{-3}).$$

Matching this to $(\bar{H}_G)_2$ in Eqs. (1) yields

$$\begin{aligned} m\omega(0.9078)(10^{-3}) &= 0.125F\,(\Delta t)\,, \\ m\omega(-1.5723)(10^{-3}) &= -0.05027m, \end{aligned} \tag{3}$$

from which we find that

$$\omega = 31.97 \text{ rad/s}, \quad \frac{F\,(\Delta t)}{m} = 0.2322 \text{ m/s}.$$

This defines the final value of $\bar{\omega}$, and the corresponding value of $\bar{v}_G$ is given by the first of Eqs. (1),

$$\bar{\omega} = 5.923\bar{j} - 31.42\bar{k} \text{ rad/s}, \quad (\bar{v}_G)_2 = 0.2322\bar{j} + 25k \text{ m/s}. \qquad \triangleleft$$

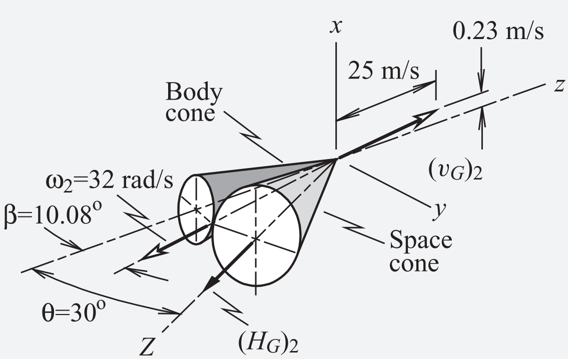
The precession axis is parallel to $(\bar{H}_G)_2$, so we have

$$\bar{K} = \frac{(\bar{H}_G)_2}{\left|(\bar{H}_G)_2\right|} = 0.500\bar{j} - 0.8660\bar{k}. \qquad \triangleleft$$

Note that the angle between $\bar{K}$ and $\bar{k}$, that is, between the symmetry and precession axes, is $\cos^{-1}(-0.8660) = 150°$, in agreement with the stated conditions.

From these results we may draw a sketch of the position of the body cone relative to the space cone at the initiation of the free motion. We also show $(\bar{v}_G)_2$ in that sketch. The corresponding precession and spin rates are found from Eqs. (10.1.8):

$$\dot{\psi} = 11.85 \text{ rad/s}, \quad \dot{\theta} = 21.16 \text{ rad/s}. \qquad \triangleleft$$



Movement of the football subsequent to application of the impulsive force.

10.1.2 Arbitrary Bodies

Analysis of the free motion of an arbitrary body proceeds similarly to the treatment of axisymmetric bodies. The analysis is substantially simplified by defining xyz to be the body's principal centroidal axes. The principal moments of inertia are denoted as $I_{xx} = I_1$, $I_{yy} = I_2$, and $I_{zz} = I_3$. Even though the body has arbitrary inertia properties, $\bar{H}_G$ must still be constant, so the fixed Z axis for precession ψ is again defined again to coincide with this vector. The other Eulerian angles are defined consistently with the conventions established in Section 4.2. Thus the spin ϕ occurs about the principal axis denoted as z, the nutation angle θ is measured from Z to z, and the line of nodes is perpendicular to the plane formed by the Z and z axes.

The intermediate $x'y'z'$ coordinate system, which only precesses and nutates, facilitates the description of $\bar{\omega}$. The z' axis is defined to coincide with the z axis, and the y' axis is parallel to the line of nodes in the sense of the right-hand rule from Z to z. The angular velocity is then

$$\bar{\omega} = \dot{\psi}\bar{K} + \dot{\theta}\bar{j}' + \dot{\phi}\bar{k}. \tag{10.1.9}$$

We can use projections or rotation transformations to resolve $\bar{K}$ and $\bar{j}'$ into xyz components, with the result that

$$\begin{aligned} \bar{K} &= -\sin\theta\cos\phi\bar{i} + \sin\theta\sin\phi\,\bar{j} + \cos\phi\bar{k}, \\ \bar{j}' &= \sin\phi\bar{i} + \cos\phi\,\bar{j}. \end{aligned} \tag{10.1.10}$$

The corresponding representation of the angular velocity is

$$\begin{aligned} \bar{\omega} = &\left(-\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi\right)\bar{i} + \left(\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi\right)\bar{j} \\ &+ \left(\dot{\psi}\cos\phi + \dot{\phi}\right)\bar{k}. \end{aligned} \tag{10.1.11}$$

As was true for axisymmetric bodies, the angular momentum can be constructed from this description of $\bar{\omega}$, as well as from the fact that it coincides with the Z axis. Thus it must be that

$$\begin{aligned} \bar{H}_G &= H_G\bar{K} = H_G\left(-\sin\theta\cos\phi\bar{i} + \sin\theta\sin\phi\,\bar{j} + \cos\phi\bar{k}\right) \\ &= I_1\left(-\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi\right)\bar{i} + I_2\left(\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi\right)\bar{j} \\ &\quad + I_3\left(\dot{\psi}\cos\phi + \dot{\phi}\right)\bar{k}. \end{aligned} \tag{10.1.12}$$

The constant value of $\bar{H}_G$ is set by the initial conditions, in the form of a known value of $\bar{\omega}$. If the components of the initial angular velocity are denoted as $(\omega_x)_0$, $(\omega_y)_0$, and $(\omega_z)_0$, the corresponding angular momentum is

$$\bar{H}_G = H_G K = I_1(\omega_x)_0\,\bar{i} + I_2(\omega_y)_0\,\bar{j} + I_3(\omega_z)_0\,\bar{k}. \tag{10.1.13}$$

It follows that

$$H_G = \left[I_1^2(\omega_x)_0^2\,\bar{i} + I_2^2(\omega_y)_0^2 + I_3^2(\omega_z)_0^2\right]^{1/2}.$$

Thus, matching like components of $\bar{H}_G$ in Eq. (10.1.12) leads to a set of three ordinary differential equations governing the Eulerian angles:

$$\boxed{\begin{aligned} I_1\left(\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi\right) &= H_G\sin\theta\cos\phi, \\ I_2\left(\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi\right) &= H_G\sin\theta\sin\phi, \\ I_3\left(\dot{\psi}\cos\phi + \dot{\phi}\right) &= H_G\cos\theta. \end{aligned}} \tag{10.1.14}$$

These equations are alternatives to the Euler or Lagrange equations of motion for the case in which all resultant moments are zero. The latter would be second order, whereas Eqs. (10.1.14) are first order. This difference is a consequence of the fact that conservation of angular momentum is an integral of the standard equations of motion. Solution of Eqs. (10.1.14) as differential equations governing the Eulerian angles or as algebraic equations for $\dot{\psi}$, $\dot{\theta}$, and $\dot{\phi}$ at specified Eulerian angles requires knowledge of the value of H_G, which can be obtained from the initial conditions.

Equations (10.1.14) are highly nonlinear. Solution schemes for specified initial conditions are easier to implement if the equations are not coupled in the derivatives. We obtain such a form by solving the first two of Eqs. (10.1.14) for $\dot{\psi}$ and $\dot{\theta}$, after which $\dot{\phi}$ is found from the last equation. Because $\sin\theta$ is factored out in those operations, the cases in which $\theta = 0$ or π are special, and shall be treated separately. If $\sin\theta \neq 0$, it must be that

$$\begin{aligned} \dot{\psi} &= H_G\left(\frac{(\cos\phi)^2}{I_1} + \frac{(\sin\phi)^2}{I_2}\right), \\ \dot{\theta} &= H_G\left(\frac{1}{I_2} - \frac{1}{I_1}\right)\sin\theta\sin\phi\cos\phi, \\ \dot{\phi} &= H_G\left(\frac{1}{I_3} - \frac{(\cos\phi)^2}{I_1} - \frac{(\sin\phi)^2}{I_2}\right)\cos\theta. \end{aligned} \tag{10.1.15}$$

We obtain a quick verification of this result by noting that, if the body is axisymmetric and z is defined to be the axis of symmetry, then $I_1 = I_2 = I'$ and $I_3 = I$. These expressions for the Eulerian rotation rates then reduce to Eqs. (10.1.6).

Analytical solutions for the Eulerian angles as functions of time in the form of elliptic functions can be obtained (Synge and Griffith, 1959). Also, because Eqs. (10.1.15) already are in first-order form, the differential equations are readily integrated numerically. Rather than following either line of analysis, the next section will develop the Poinsot construction, which is a graphical way of understanding the rotation. Before we continue, there are a few general observations that we can make regarding the solutions of these differential equations.

First, note that $\dot{\psi}$ is always positive, which means that a body in free motion never changes the direction in which it precesses. However, the signs of $\dot{\theta}$ and $\dot{\phi}$ depend on the relative magnitudes of the moments of inertia, and on the current quadrant in which θ and ϕ reside. This suggests that there might be free motions in which the nutation and spin rates, and therefore the corresponding angles, oscillate. In turn, this leads to questions regarding the stability of a rotational motion that has been established.

Specific results regarding stability can be obtained in the case in which a body is released with an initial rotation rate Ω about a principal axis. Without loss of generality, we define this axis to be the z axis for the Eulerian angles. We may take the initial spin rate to be Ω, corresponding to $\theta = 0$, whereas the initial precession rate and nutation angle are both zero. The case in which $\sin\theta = 0$ was excluded from Eqs. (10.1.15), so we return to Eqs. (10.1.14). These equations are indentically satisfied by constant $\theta = 0$ or π, $\dot{\psi} = 0$, and $\dot{\phi} = H_G/\Omega$. Thus a pure spinning motion about any of the principal axes is a possible type of rotation. However, we are not likely to impart an initial rotation to a body in which the axis of rotation is exactly aligned with one of the principal axes. Even if we could, any temporary disturbance, such as a mild gust of wind, would alter the initial rotation. A more realistic expectation is that, because of a small error, the initial motion will feature nutation and precession rates that are much smaller than Ω, with the initial nutation angle being small. To study the ensuing rotation we perform a linearized perturbation analysis. If an evaluation of the response confirms that these small initial perturbations remain small for all t, then we may conclude that the rotation is stable. In contrast, a result indicating that the the disturbances grow merely means that the ensuing rotation will differ much from the initial motion.

The disturbed response is represented as

$$\dot{\phi} = \Omega + \varepsilon\dot{\xi}_3, \quad \theta = \varepsilon\xi_2, \quad \psi = \varepsilon\xi_1, \tag{10.1.16}$$

where the ξ_n parameters are time functions that are taken to be unit order and the perturbation parameter ε, which scales the magnitude of the disturbance, is a positive number much less than unity. Note that this representation considers the precession and spin nutation angles to be small values, whereas the spin angle is approximately Ωt. We obtain the easiest derivation of solvable equations for these parameters by substituting this representation into Eqs. (10.1.14). After we do so, we apply small-angle approximations to the trigonometric terms, which leads to

$$\begin{aligned}
I_1\left[\left(\varepsilon\dot{\xi}_1\right)\left(\varepsilon\xi_2\right)\cos\phi - \varepsilon\dot{\xi}_2\sin\phi\right] &= H_G\left(\varepsilon\xi_2\right)\cos\phi, \\
I_2\left[\left(\varepsilon\dot{\xi}_1\right)\left(\varepsilon\xi_2\right)\sin\phi + \varepsilon\dot{\xi}_2\cos\phi\right] &= H_G\left(\varepsilon\xi_2\right)\sin\phi, \\
I_3\left[\left(\varepsilon\dot{\xi}_1\right)\cos\phi + \left(\Omega + \varepsilon\dot{\xi}_3\right)\right] &= H_G\cos\theta.
\end{aligned} \tag{10.1.17}$$

We retain only the leading-order term in each equation, with the result that

$$\begin{aligned}
-I_1\dot{\xi}_2\sin\phi &= H_G\xi_2\cos\phi, \\
I_2\dot{\xi}_2\cos\phi &= H_G\xi_2\sin\phi, \\
I_3\Omega &= H_G\cos\phi.
\end{aligned} \tag{10.1.18}$$

A change of variables that expedites solution of these equations is suggested by the observation that $\sin\phi$ and $\cos\phi$ are the direction cosines of $\bar{j}'$, which is the line of nodes, relative to the body-fixed x and y coordinate axes. Wherever ξ_2 occurs in the perturbation equations, it is multiplied by one of these direction cosines. Because ξ_2 represents the disturbance of the nutation, which is about the line of nodes, let us define a vector $\bar{\xi}_2$ parallel to the line of nodes, whose components are u and v, such that

$$\begin{gathered}\bar{\xi}_2 = \xi_2 \bar{j}' = u\bar{i} + v\bar{j}, \\ u = \xi_2 \sin\phi, \quad v = \xi_2 \cos\phi.\end{gathered} \tag{10.1.19}$$

According to Eqs. (10.1.16), the first approximation of $\dot{\phi}$ is Ω, so differentiation of the preceding equations shows that

$$\begin{gathered}\dot{u} \approx \dot{\xi}_2 \sin\phi + \Omega\xi_2 \cos\phi = \dot{\xi}_2 \sin\phi + \Omega v, \\ \dot{v} \approx \dot{\xi}_2 \cos\phi - \Omega\xi_2 \sin\phi = \dot{\xi}_2 \cos\phi - \Omega u.\end{gathered} \tag{10.1.20}$$

We use these relations to eliminate ξ_2 and ϕ from Eqs. (10.1.18), which yields

$$\begin{gathered}I_1 \left(\dot{u} - \Omega v\right) = -I_3 \Omega v, \\ I_2 \left(\dot{v} + \Omega u\right) = I_3 \Omega u.\end{gathered} \tag{10.1.21}$$

These are a pair of coupled, homogeneous, linear differential equations with constant coefficients. Their solution must be exponential in time, so we set

$$u = A \exp\left(\lambda t\right), \quad v = B \exp\left(\lambda t\right). \tag{10.1.22}$$

Substitution of these forms into Eqs. (10.1.21) gives

$$\begin{bmatrix} I_1\lambda & \left(I_3 - I_1\right)\Omega \\ \left(I_2 - I_3\right)\Omega & I_2\lambda \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{10.1.23}$$

This is an eigenvalue problem because the only solution is the trivial one, $A = B = 0$, unless the value of λ is such that the determinant of the coefficient matrix vanishes. The characteristic equation obtained from setting this determinant to zero is

$$I_1 I_2 \lambda^2 + \left(I_3 - I_1\right)\left(I_3 - I_2\right) = 0. \tag{10.1.24}$$

Because the moments of inertia are positive values, the roots of this quadratic equation occur either as a pair of conjugate imaginary values or as real values with opposite signs, depending on the sign of the last term. A positive value of λ corresponds to exponential growth of u and v, and therefore of ξ_2. This is the instability condition, which indicates that the spin axis will not remain close to the precession axis. In contrast, purely imaginary values of λ represent an oscillatory ξ_2. This is the stable case in which the spin axis remains close to the precession axis.

Recall that I_3 is the moment of inertia about the principal axis for the nominal spin. The stability condition requires $\left(I_3 - I_1\right)\left(I_3 - I_2\right) > 0$, which is satisfied either if $I_3 > I_1$ and $I_3 > I_2$, or else, $I_3 < I_1$ and $I_3 < I_2$. In other words, a body that is released with an initial angular velocity that is essentially a spin about the principal axis having either the largest or smallest moment of inertia will continue to have that type of rotation. An initial spin about the principal axis for which the moment of inertia is the intermediate value will show a growth in the nutation angle, such that the eventual rotation does not resemble the attempted initial state. Note in this regard that the present analysis merely describes the tendency to deviate from the initial state. It provides no information regarding the ultimate response, because the assumption of a small nutation angle on which it is based would not be valid.

If you wish, you may test these stability properties by throwing a homogeneous rectangular object, such as a wooden block or a board eraser. Try to impart to it an initial spin about an axis parallel to one of its edges. It is fairly easy to obtain a motion in which the object spins about an axis parallel to the shortest or longest edge. However, a comparable attempt for rotation about the intermediate edge does not produce the desired steady spin.

These results are consistent with the previous investigation of an axisymmetric body. To use the stability analysis in the case in which the initial angular velocity is close to the symmetry axis, we set $I_{zz} = I = I_3$ and $I_{xx} = I_{yy} = I' = I_1 = I_2$. In the space and body cone construction we set the apex angle β for the body cone to a very small value. Equations (10.1.8) indicate that the apex angle θ for the body cone will also be small, from which it follows that the angular velocity will be close to the symmetry axis.* The stability analysis leads to the same conclusion because either $I_3 < I_2$ and $I_3 < I_1$ for a slender body or $I_3 > I_2$ and $I_3 > I_1$ for an oblate body. Interestingly, the stability analysis does not yield a definitive result for the case of an axisymmetric body to which one imparts an initial spinning motion about the transverse axis. That situation is covered by setting $I_3 = I'$, in which case $I_1 = I$ and $I_2 = I'$ or vice versa. The characteristic equation yields $\lambda = 0$, which means that it is a degenerate condition in which the linearized analysis is inadequate. We will examine this situation in the next subsection, which does not rely on simplifications.

10.1.3 Poinsot's Construction for Arbitrary Bodies

As was mentioned earlier, one approach for determining how a body with arbitrary inertia properties rotates is to seek analytical or numerical solutions of the first-order equations of motion, Eqs. (10.1.14) or (10.1.15). Here we develop a pictorial representation of the motion that considerably enhances our qualitative understanding of free rotation. We derive this representation by considering the kinetic energy and angular momentum jointly.

The only force acting on a body in our model of free motion is gravitational attraction, which acts through the center of mass. It was shown in Subsection 6.4.2 that the rotational kinetic energy can be altered only by the application of a moment about the center of mass. Thus the rotational kinetic energy is constant. The ellipsoid of inertia, which was developed in Subsection 5.2.3, describes how the angular velocity must vary relative to the body if the rotational kinetic energy is to remain constant.

* An extremely slender body, for which $I' \gg I$, represents an exceptional case, because a small nonzero β will correspond to a large θ. Although the spin axis of this type of body cannot remain close to the precession axis unless the angular velocity is extremely close to the spin axis, it is still stable in the sense that θ remains below 90°.

In terms of body-fixed principal axes, the shape of the ellipsoid of inertia is described by

$$\boxed{I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 = 1.} \tag{10.1.25}$$

The corresponding expression for the rotational kinetic energy is

$$T_{\text{rot}} = \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right). \tag{10.1.26}$$

Let us depict the angular velocity as a vector whose tail is always situated at the origin of xyz. If the head of this vector is always situated on the inertia ellipsoid, then we would have $\bar{\omega} = x\bar{i} + y\bar{j} + z\bar{k}$, where x, y, and z satisfy Eq. (10.1.25). In that case the rotational kinetic energy would maintain the constant value at $T_{\text{rot}} = 1/2$. The actual rotational kinetic energy will be obtained if this vector is scaled by $(2T_{\text{rot}})^{1/2}$. In other words, if $\bar{\rho}$ denotes the position vector extending from the origin to a point on the inertia ellipsoid, then constancy of the rotational kinetic energy requires that the angular velocity be such that

$$\bar{\omega} = (2T_{\text{rot}})^{1/2}\,\bar{\rho}. \tag{10.1.27}$$

The component representation of this relation is

$$x = \frac{\omega_x}{(2T_{\text{rot}})^{1/2}}, \quad y = \frac{\omega_y}{(2T_{\text{rot}})^{1/2}}, \quad z = \frac{\omega_x}{(2T_{\text{rot}})^{1/2}}. \tag{10.1.28}$$

To exploit this property we let the inertia ellipsoid serve as a fictitious proxy that moves in unison with the body. The inertia ellipsoid's properties give us some idea of how the angular velocity is viewed from the perspective of the body. The next feature comes from the constancy of the angular momentum about the center of mass. The component of $\bar{\rho}$ parallel to $\bar{H}_G$, which is denoted as ρ_H, may be evaluated from a dot product. In view of the preceding relation this distance is

$$\rho_H = \bar{\rho}\cdot\frac{\bar{H}_G}{|\bar{H}_G|} = \left(\frac{1}{2T_{\text{rot}}}\right)^{1/2}\frac{\bar{\omega}\cdot\bar{H}_G}{|\bar{H}_G|}. \tag{10.1.29}$$

Recall that $\bar{\omega}\cdot\bar{H}_G = 2T_{\text{rot}}$. In view of the fact that both $|\bar{H}_G|$ and T_{rot} are constant in a free motion, the angular velocity must always be oriented relative to the body such that ρ_H maintains a constant value given by

$$\boxed{\rho_H = \frac{(2T_{\text{rot}})^{1/2}}{|\bar{H}_G|}.} \tag{10.1.30}$$

One definition of a plane states that it is the locus of points whose distance to a specified point, measured in the direction normal to the plane, is constant. It follows that the point P on the inertia ellipsoidal to which $\bar{\rho}$ extends always lies on a plane that is at the constant distance ρ_H from the center of mass and that $\bar{H}_G$ is the normal to that plane. If we ignore the movement of the center of mass, this plane appears to be stationary; it is the *invariable plane*.

This property is important, but it does not tell us how point P moves along the invariable plane, nor does it tell us how the ellipsoid is oriented relative to the plane. These questions are resolved by another property of the ellipsoid of inertia. We saw that, if $\bar{\omega}$ equals $\bar{\rho}$, then the motion corresponds to $T_{\text{rot}} = 1/2$. Setting $\rho' = c\rho$ defines a family of concurrent ellipsoids having the same proportions as the inertia ellipsoid. It follows from Eq. (10.1.25) that the coordinates of a point on one of these concurrent ellipsoids is defined by

$$F(x, y, z) = I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 = c^2, \tag{10.1.31}$$

where $c = 1$ corresponds to the ellipsoid of inertia. In general, the gradient applied to a family of functions defined in this manner indicates the direction in which the value of c changes most rapidly in going from one surface to another. Therefore the gradient of F, which is

$$\nabla F = 2\left(I_{xx}x\bar{i} + I_{yy}y\bar{j} + I_{zz}z\bar{k}\right), \tag{10.1.32}$$

defines the normal to the ellipsoid at the point on the surface whose coordinates are (x, y, z). To make this point lie on the ellipsoid of inertia, we require that these coordinates satisfy Eq. (10.1.31) with $c = 1$. However, Eq. (10.1.28) indicates that these coordinates are the respective components of $\bar{\omega}$ divided by $(2T_{\text{rot}})^{1/2}$. It follows that the normal direction at any point on the inertia ellipsoid is defined by

$$\nabla F = \frac{2}{(2T_{\text{rot}})^{1/2}}\left(I_{xx}\omega_x\bar{i} + I_{yy}\omega_h\bar{j} + I_{zz}\omega_z\bar{k}\right) = \left(\frac{2}{T_{\text{rot}}}\right)^{1/2}\bar{H}_G. \tag{10.1.33}$$

We conclude from this that the normal to the inertia ellipsoid at any point where it intersects the invariable plane is parallel to $\bar{H}_G$. However, the normal to the invariable plane is also parallel to $\bar{H}_G$. These two conditions can be satisfied simultaneously only if the ellipsoid of inertia tangentially contacts the invariable plane. Furthermore, the velocity of this point of contact relative to the center of mass is zero, because $\bar{\omega} \times \bar{\rho} = \bar{0}$. These observations lead us to the *Poinsot construction*, first disclosed in 1857:

> *The ellipsoid of inertia of a body in free motion rotates about the center of mass such that it rolls without slipping over the invariable plane. The normal to the invariable plane is parallel to the constant angular momentum of the body. The line extending from the center of mass to the point where the ellipsoid tangentially comes into contact with the invariable plane is parallel to the instantaneous axis of rotation. The rolling motion is such that the perpendicular distance from the center of mass to the invariable plane is constant at a value that depends on the angular momentum and constant rotational kinetic energy,.*

Figure 10.4 provides a pictorial representation of the Poinsot construction. It is important to realize that this construction is solely for the purpose of understanding the orientation of the ellipsoid of inertia, and therefore of the body. Application of Chasle's theorem relative to the center of mass in combination with the Poinsot construction would allow us to fully describe the motion.

The initial conditions at the instant the body was released define $\bar{H}_G$ and T_{rot}, which, in turn, define the invariable plane and the distance from the center of mass to the

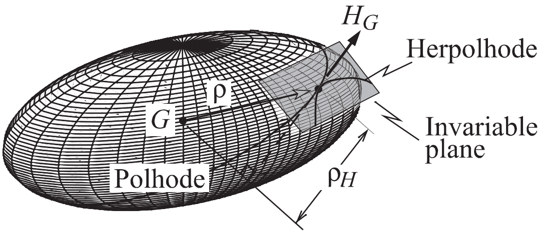


Figure 10.4. Poinsot's construction showing the inertia ellipsoid rolling over the invariable plane.

invariable plane. At each instant, a different point on the inertia ellipsoid contacts the invariable plane. The locus of contact points on the inertia ellipsoid is a curve called the *polhode*, whereas the locus on the invariable plane is the *herpolhode*. The herpolhode is generally an open curve, which means that the rotation does not repeat, but the polhode is a closed curve. We may establish the closure of the polhode, as well as the overall nature of these curves, by noting that the inertia ellipsoid represents the constancy of the kinetic energy in the rotational motion. However, the angular momentum is also constant, which means that

$$\left|\bar{H}_G\right|^2 = I_{xx}^2\omega_x^2 + I_{yy}^2\omega_y^2 + I_{zz}^2\omega_z^2. \tag{10.1.34}$$

When Eqs. (10.1.28) are used to represent the angular velocity components, the preceding equation becomes

$$\boxed{I_{xx}^2 x^2 + I_{yy}^2 y^2 + I_{zz}^2 z^2 = \frac{\left|\bar{H}_G\right|^2}{2T_{\text{rot}}} = \frac{1}{\rho_H^2} \equiv D.} \tag{10.1.35}$$

The usage of D rather than $1/\rho_H^2$ is done for convenience. Clearly, D is constant, so the preceding equation represents another ellipsoid whose origin is at the center of mass and whose principal axes are x, y, and z. In other words, constancy of the angular momentum's magnitude leads to the conclusion that the point where the inertia ellipsoid contacts the invariable plane is situated on another ellipsoid that is stationary with respect to the body-fixed xyz coordinate system. The intersection of this ellipsoid with the ellipsoid of inertia, given by Eq. (10.1.25), is the polhode. The closure of the polhode is a direct consequence of the fact that both ellipsoids, and therefore their intersection, rotate with the body.

The value of the constant D is determined by the initial motion. If this initial state is a rotation about the x axis, then $\bar{H}_G = I_{xx}\omega\bar{i}$, so $|H_G|^2 = 2T_{\text{rot}}/I_{xx}$. Similar statements apply to initial rotations about the other principal axes. This leads to the observation that initial rotations about each of the principal axes correspond to $D = I_1$, I_2, and I_3, respectively. Without loss of generality, we now specify the labeling of the xyz axes to be such that $I_1 = I_{xx}$ is the smallest value and $I_3 = I_{zz}$ is the largest. Then, $I_1 \le I_2 \le I_3$. We may construct the polhode for a specified value of D by picking a value of one coordinate, and then solving Eqs. (10.1.25) and (10.1.35) simultaneously for the other two. A visualization that is easier to obtain depicts polhode curves in terms of their projections onto the principal coordinate planes. We derive equations for these projections by eliminating the coordinate normal to that plane from the two ellipsoid equations. These projection equations are

$$\begin{aligned} x-y\text{ plane: } & I_1(I_3-I_1)x^2+I_2(I_3-I_2)y^2=I_3-D, \\ y-z\text{ plane: } & I_2(I_2-I_1)y^2+I_3(I_3-I_1)z^2=D-I_2, \\ x-z\text{ plane: } & I_1(I_2-I_1)x^2-I_3(I_3-I_2)z^2=I_2-D. \end{aligned} \tag{10.1.36}$$

The coefficients in these equations are positive for the assigned sequence $I_1 \le I_3 \le I_3$. For this ordering, the projections onto the $x-y$ and $y-z$ planes are ellipses, whereas the projections onto the $x-z$ plane are hyperbolas. These projections and the outline of the inertia ellipsoid are illustrated in Fig. 10.5 for the positive quadrants.

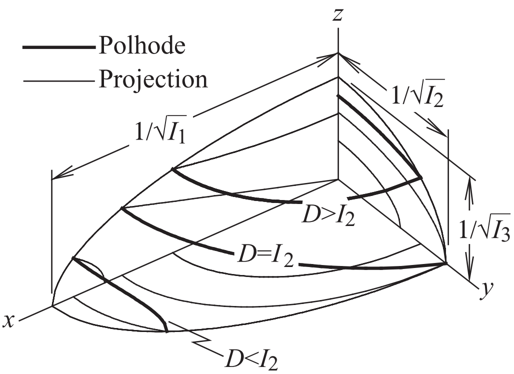


Figure 10.5. Typical polhode curves and their projections onto the principal axis coordinate planes for $I_1=1$, $I_2=4$, $I_3=8$.

The curve corresponding to $D=I_2$ is the separatrix between the hyperbolas in the $x-z$ plane, but it appears as an ellipse in the other coordinate planes. The corresponding polhode curves on the ellipsoid of inertia are shown in Fig. 10.6.

We concluded in the previous section that an attempt to impart a rotation about the principal axis of smallest or largest moment of inertia would produce a stable rotation. This is further demonstrated here. Recall that the instantaneous angular velocity $\bar{\omega}$ is parallel to a line from the origin to the point where the polhode curve contacts the invariable plane. If the initial rotation is approximately about the z axis, then D is slightly smaller than I_3. In that case the projection of the polhode curve onto the $x-y$ plane is a small ellipse, corresponding to an angular velocity that always is nearly parallel to the z axis. Similarly, an initial rotation approximately about the x axis, which gives a value

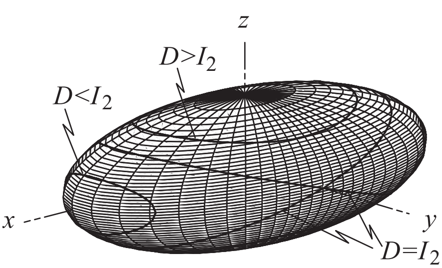
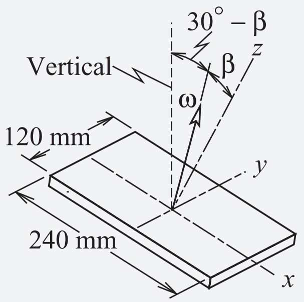


Figure 10.6. Typical polhode curves and the inertia ellipsoid for $I_1=1$, $I_2=4$, $I_3=8$.

of D slightly larger than I_1, leads to a polhode curve projection on the $y - z$ plane that is a small ellipse. This corresponds to an angular velocity that is always nearly parallel to the x axis. In contrast, if the initial motion is approximately about the y axis, then $D \approx I_2$. Then the polhode curves are close to the separatrices. Depending on whether D is greater than or less than I_2, the closed polhode curve is centered about either the z axis or the x axis, respectively. In either case, the angle between ω and any of the coordinate axes varies greatly in the motion. This explains why the rotation of an arbitrary body is often difficult to observe.

An axisymmetric body is a special case in which two principal moments of inertia are equal. To match the earlier analysis of axisymmetric bodies, let us place the tail of $\bar{\omega}$ at the apex of the body cone and divide $\bar{\omega}$ by $(2T_{\mathrm{rot}})^{1/2}$. The vector formed in this manner is the position vector $\bar{\rho}$ in Poinsot's construction. Because the magnitude of $\bar{\omega}$ for free motion of an axisymmetric body is constant, it must be that the tip of the scaled vector will always be situated on a circular cross section of the body cone. This is the polhode curve. Poinsot's construction leads to the same conclusion. The ellipsoid of inertia for an axisymmetric body is spheroidal. Constancy of $|\bar{H}_G|$ corresponds to another spheroid having the same axis of symmetry. The polhode curve is the intersection of these two spheroids, which is a circular cross section they have in common. Recall that $\bar{H}_G$ is perpendicular to the cross sections of the space cone. Thus the invariable plane for an axisymmetric body is the cross section of the space cone that contacts the cross section of the body cone that is the polhode curve. In regard to the previous stability analysis, we now see that an initial motion that is almost a pure spin about the axis of symmetry corresponds to a very small radius polhode, which means that the angular velocity does not deviate much from the initial condition. This confirms the earlier analysis. We also can now answer the degenerate case for the previous stability analysis, in which an axisymmetric body is set into motion with a rotation that is almost a pure rotation about a transverse axis. The polhode curve for such a motion is a circle close to the equator of the spheroid. This proximity leads to the conclusion that an axisymmetric body also is stable if it is launched with an angular velocity that is perpendicular to the axis of symmetry. Followers of American football will recognize this as the case of an "end-over-end" kick.

EXAMPLE 10.2 The xyz coordinate system in the sketch is a set of body-fixed principal centroidal axes for the 10-kg rectangular plate. At the instant the plate is dropped, the y axis is horizontal and the z axis is inclined at 30° from vertical. The angular velocity at that instant lies in the xz plane at an unspecified angle β from the z axis. (a) Determine the value of β for which the precession axis is vertical. (b) Determine the maximum value of β for which the angle between the z axis and the precession axis will not exceed 90° in the rotation after release. (c) For the case in which β is one half the critical value in Part (b), determine the minimum and maximum angles between the plate's normal and the precession axis during the rotation. Evaluate the corresponding angular velocity at these limits.



Example 10.2

SOLUTION We see in this example that much information regarding the free motion of a body can be extracted without solving the differential equations of motion. The sequence in which the coordinate axes have been labeled matches the derivation, where x is the principal axis having the smallest moment of inertia, $I_{xx} = I_1$, and z is the principal axis having the largest moment of inertia, $I_{zz} = I_3$. Here, these values are

$$I_1 = \frac{1}{12}(10)(0.12)^2 = 0.012, \quad I_2 = \frac{1}{12}(10)(0.24)^2 = 0.048,$$
$$I_3 = I_1 + I_2 = 0.060 \text{ kg-m}^2.$$

The orientation of the initial angular velocity relative to xyz is described by the angle β, so we have

$$\bar{\omega} = \omega\left(-\sin\beta\bar{i} + \cos\beta\bar{k}\right). \tag{1}$$

The corresponding constant angular momentum is

$$\bar{H}_G = \omega\left(-I_1\sin\beta\bar{i} + I_3\cos\beta\bar{k}\right). \tag{2}$$

To answer the first question, we require that the angular momentum be parallel to the vertical axis, which is stated to be situated in the xz plane at an angle of 30° from the z axis. Resolving this vector into xyz components gives

$$\bar{H}_G = H_G\left(-0.5\bar{i} + 0.8660\bar{k}\right).$$

This description must be consistent with Eq. (2), so like components must match, which leads to

$$-I_1\omega\sin\beta = -0.5H_G, \quad I_3\omega\cos\beta = 0.8660H_G,$$

so that

$$\frac{I_1}{I_3}\tan\beta = 0.5774 \implies \beta = 70.89^\circ \begin{cases}\text{for precession about} \\ \text{the vertical axis}\end{cases}. \qquad \triangleleft$$

To address the second question, we turn to Fig. 10.6. In the Poinsot construction the invariable plane is tangent to the inertia ellipsoid and the normal to the invariable plane is parallel to the precession axis. Consider the polhode curves corresponding to $D > I_2$, which surround the z axis. The tangent plane at any point on such a curve has a normal that forms an acute angle with the z axis. The limiting case is the separatrix, for which half of the curve lies on a portion of the ellipsoid where the normal is more than 90° from the z axis. Thus the maximum value of β satisfying the specification in Part (b) is that which gives $D = I_2$. The initial rotational kinetic energy for an arbitrary β is found from Eqs. (1) and (2) to be

$$2T_{\text{rot}} = \bar{\omega} \cdot \bar{H}_G = \omega^2 \left[I_1 (\sin\beta)^2 + I_3 (\cos\beta)^2 \right]. \tag{3}$$

Using this expression to form the critical condition, $D = I_2$, leads to

$$D = \frac{\left|\bar{H}_G\right|^2}{2T_{\text{rot}}} = \frac{I_1^2 (\sin\beta)^2 + I_3^2 (\cos\beta)^2}{I_1 (\sin\beta)^2 + I_3 (\cos\beta)^2} = I_2. \tag{4}$$

In other words,

$$I_1 (I_2 - I_1) (\sin\beta)^2 = I_3 (I_3 - I_2) (\cos\beta)^2,$$

$$\beta = \tan^{-1} \left[\frac{I_3 (I_3 - I_2)}{I_1 (I_2 - I_1)} \right]^{1/2} = 52.239° \begin{cases} \text{for rotation about} \\ \text{the } z \text{ axis} \end{cases}. \qquad \triangleleft$$

For Part (c), we set $\beta = 26.119°$. The value of D obtained from Eq. (4) for this angle is

$$D = 0.057798.$$

Because $D > I_2$, the polhode curve surrounds the z axis. Now examine the polhode curve for this case in Fig. 10.6. The minimum and maximum angles between the z axis and the normal to the tangent plane, which is the fixed precession axis, occur when the polhode curve intersects the xz plane and yz plane, respectively. Thus the task of identifying the minimum and maximum angle conditions reduces to establishing the conditions for which the angular velocity has either a zero y or x component, respectively. This observation leads us to conclude that the initial motion must represent the minimum angle condition, because it is specified that $\omega_y = 0$ at that instant. Thus we obtain the minimum angle condition by setting $\beta = 26.119°$ in Eq. (2), from which we can determine the direction of $\bar{H}_G$, which is $\bar{K}$:

$$\bar{H}_G = \omega(-0.005283\bar{i} + 0.053873\bar{k}) = H_G \bar{K}.$$

The angle from $\bar{K}$ to the z axis is θ, so we find that

$$\theta_{\min} = \cos^{-1} \left(\frac{\left|\bar{H}_G \cdot \bar{k}\right|}{\left|\bar{H}_G\right|} \right) = 5.601°. \qquad \triangleleft$$

To determine the maximum angle we seek the solution of the polhode equations corresponding to the preceding value of D subject to the condition that $x = 0$. The polhode curves correspond to values of (x, y, z) that simultaneously satisfy

Eq. (10.1.25) for the inertia ellipsoid and Eq. (10.1.35) describing constancy of $\left|\bar{H}_G\right|$. For the present values of the parameters, these equations are

$$0.012(x^2 + 4y^2 + 5z^2) = 1,$$
$$(0.012)^2\left(x^2 + 16y^2 + 25z^2\right) = 0.057798.$$

The polhode we are interested corresponds to $z > 0$, because ω_z was initially positive. We therefore seek the root of these equations for which $x = 0$ and $z > 0$. The roots are

$$y = \pm 1.9553, \quad z = 3.6889.$$

The alternative sign results from the symmetry of the polhode curves. We use the positive value of y; either sign will yield the same angle. According to Eqs. (10.1.28), the corresponding angular velocity components are

$$\omega_x = 0, \quad \omega_y = 1.9553\,(2T_{\text{rot}})^{1/2}, \quad \omega_z = 3.6889\,(2T_{\text{rot}})^{1/2}, \tag{5}$$

which leads to the angular momentum being

$$\bar{H}_G = (2T_{\text{rot}})^{1/2}\left(0.093854\,\bar{j} + 0.221334\bar{k}\right) = H_G\bar{K}.$$

From this we find the angle β between $\bar{\omega}$ and the z axis and the angle θ between the z axis and the precession axis to be

$$\beta = \tan^{-1}\left(\frac{\omega_y}{\omega_z}\right) = 27.926^\circ, \quad \theta_{\max} = \cos^{-1}\left(\frac{\bar{H}_G\cdot\bar{k}}{\left|\bar{H}_G\right|}\right) = 22.979^\circ.$$

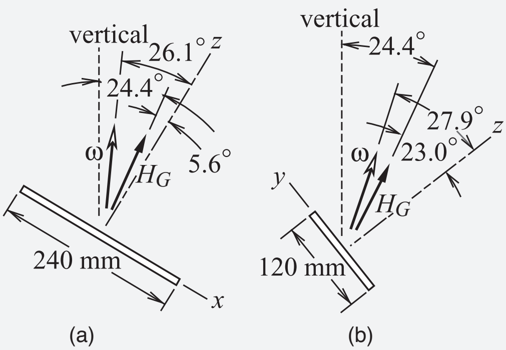
The rotational kinetic energy corresponding to Eq. (5) must be the same as the value when the plate was released. We obtain this value by substituting $\beta = 26.119^\circ$ into Eq. (3), which gives

$$2T_{\text{rot}} = 0.050697\omega_0^2,$$

where ω_0 is the magnitude of the angular velocity when the plate was released. Substitution of this value into Eq. (5) results in

$$\bar{\omega} = \omega_0\left(0.4403\,j + 0.8306\bar{k}\right). \qquad \triangleleft$$

The orientation of the principal axes and of the angular velocity at this instant are depicted in the accompanying sketch. To construct the picture, the fixed orientation of $\bar{H}_G$ relative to vertical is established in the release position as the difference of the 30° angle to the z axis and the 5.6° angle from the z axis to $\bar{H}_G$. We find the orientation of all vectors at the other extreme orientation by copying the $\bar{H}_G$ vector to that sketch, then using the computed $\theta_{\max}$ to locate the z axis. Then the z axis provides the reference from which β is measured to locate $\bar{\omega}$. Note that the planes for each sketch are not the same; to locate the plane for the second sketch we would need to know the precession angle. Also, each configuration in the sketch is mirrored by another, not shown, in which all features are reflected to the other side of $\bar{H}_G$, corresponding to a 180° rotation about the precession axis. At an arbitrary instant, the angles between the z axis and the precession axis, and between the angular velocity and the precession axis, will be intermediate to the illustrated conditions and its mirror image.



Orientation of the plate at extreme locations on the polhode curve: (a) release at $\beta = 26.1°$, (b) largest β.

10.2 SPINNING TOP

The toy known as a spinning top consists of an axially symmetric body that executes a pure rotation about an apex situated on the axis of symmetry. (We shall not worry here about the drift that occurs when the apex is not anchored, primarily because such effects are complicated by minor irregularities in the surface over which the apex would move.) The study of a spinning top leads to many insights regarding the interplay among rotation, angular momentum, and the moment exerted by forces. The results for its motion may be extended to other bodies that rotate about a reference point due to the moment of the gravitational force; such systems include certain types of gyroscopes.

Point O in Fig. 10.7 is stationary because of a reaction force having three components F_j in orthogonal directions. The gravity force acts through the center of mass G. Its moment about point O is $mgL\sin\theta$ in the sense of the horizontal axis through point O that is perpendicular to the axis of symmetry. Because such an axis is the line of nodes for a set of Eulerian angles, it is natural to use those parameters as generalized coordinates. The reactions exert no moments about the precession, spin, and nutation axes, and gravity is conservative, so the generalized force associated with each angle is identically zero. Thus the principal difference between a spinning top and an axisymmetric body in free motion is the presence of a moment about the reference point for the rotation. This moment depends on the nutation angle and must be balanced by an angular momentum that varies with time.

We employ Lagrange's equations to formulate the equations of motion. Let I be the moment of inertia about the axis of symmetry and let I' be the moment of inertia about any axis perpendicular to the axis of symmetry and intersecting point O. In terms of

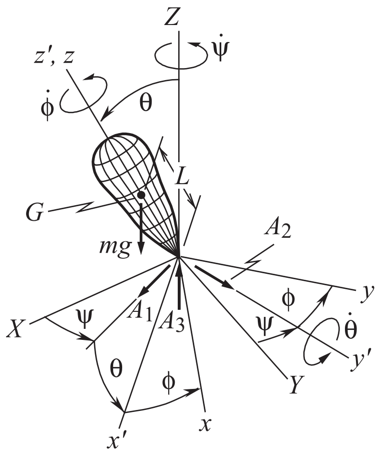


Figure 10.7. Free-body diagram and coordinate systems for a spinning top.

components relative to the $x'y'z'$, which only precesses and nutates, the angular velocity of the body is

$$\bar{\omega} = -\dot{\psi}\sin\theta\bar{i}' + \dot{\theta}\,\bar{j}' + \left(\dot{\psi}\cos\theta + \dot{\phi}\right)\bar{k}'. \tag{10.2.1}$$

The moment of inertia about any body-fixed axis that is aligned with $\bar{j}'$ at the instant of interest is I', so the kinetic energy corresponding to this expression for $\bar{\omega}$ is

$$T = \frac{1}{2}\left[I'\dot{\psi}^2(\sin\theta)^2 + I'\dot{\theta}^2 + I\left(\dot{\psi}\cos\theta + \dot{\phi}\right)^2\right]. \tag{10.2.2}$$

The elevation of the apex is a convenient reference for the gravitational potential energy, so

$$V = mgL\cos\theta. \tag{10.2.3}$$

We noted earlier that the generalized forces are all zero in our idealized model. Furthermore, the Lagrangian, $\mathcal{L} = T - V$, does not depend explicitly on either the precession or spin angles. As a result the precession and spin angles are ignorable coordinates, corresponding to conservation of the generalized momenta associated with these variables. These momenta are

$$\begin{aligned} p_\psi &= \frac{\partial T}{\partial\dot{\psi}} = I\left(\dot{\psi}\cos\theta + \dot{\phi}\right)\cos\theta + I'\dot{\psi}(\sin\theta)^2 \equiv I'\beta_\psi, \\ p_\phi &= \frac{\partial T}{\partial\dot{\phi}} = I\left(\dot{\psi}\cos\theta + \dot{\phi}\right) \equiv I'\beta_\phi, \end{aligned} \tag{10.2.4}$$

where β_ψ and β_ϕ are constants having the units of angular speed. These parameters may be evaluated from the initial conditions, so these conservation equations may be solved for the rotation rates:

$$\boxed{\begin{aligned} \dot{\psi} &= \frac{\beta_\psi - \beta_\phi\cos\theta}{(\sin\theta)^2}, \\ \dot{\phi} &= \frac{\beta_\phi\left[I'(\sin\theta)^2 + I(\cos\theta)^2\right] - \beta_\psi I\cos\theta}{I(\sin\theta)^2}. \end{aligned}} \tag{10.2.5}$$

An important distinction is that a constant value of β_ϕ corresponds to constancy of the total rotation rate about the axis of symmetry, $\omega_z = \dot{\psi}\cos\theta + \dot{\phi}$. Thus the precession and spin rates are individually constant only when the nutation angle is constant. If θ is known as function of time, then these equations could be integrated to determine the precession and spin angles.

Constancy of p_ψ and p_ϕ are derived from the Lagrange equations associated with ψ and ϕ. Proper evaluation of the Lagrange equation associated with θ requires that the derivatives of the energy expressions be evaluated before the conserved momenta are used to eliminate the ignorable coordinates. The resulting equation of motion is

$$I'\ddot{\theta} - \left(\dot{\psi}\sin\theta\right)\left[I'\dot{\psi}\cos\theta - I\left(\dot{\psi}\cos\theta + \dot{\phi}\right)\right] - mgL\sin\theta = 0. \tag{10.2.6}$$

Substitution of Eqs. (10.2.5) eliminates the precession and spin rates, with the result that

$$\ddot{\theta} + \frac{1}{(\sin\theta)^3}\left(\beta_\psi - \beta_\phi\cos\theta\right)\left(\beta_\phi - \beta_\psi\cos\theta\right) - \frac{mgL}{I'}\sin\theta = 0. \tag{10.2.7}$$

The only variable in this second-order differential equation is θ. We shall return to it later, but for now we use conservation of energy to obtain a first integral describing the nutation rate. We require that $T + V = E$ be constant, and we use Eqs. (10.2.5) to eliminate $\dot{\psi}$ and $\dot{\phi}$, which leads to

$$\frac{1}{2}I'\dot{\theta}^2 + \frac{1}{2}I'\left(\frac{\beta_\psi - \beta_\phi\cos\theta}{\sin\theta}\right)^2 + \frac{(I')^2}{2I}\beta_\phi^2 + mgL\cos\theta = E. \tag{10.2.8}$$

As is true for the generalized momenta, the value of E is specified by the initial conditions.

Multiplication of this equation by $(\sin\theta)^2$ leads to a form in which the derivative occurs only in the combination $\dot{\theta}\sin\theta$. Also, because $(\sin\theta)^2 = 1 - (\cos\theta)^2$, the variable terms that do not contain derivatives may be considered to depend on $\cos\theta$. This suggests that it would be useful to define a new variable such that

$$u = \cos\theta, \quad \dot{u} = -\dot{\theta}\sin\theta. \tag{10.2.9}$$

Also, it is convenient to define the following combination of parameters:

$$\varepsilon = \frac{2E}{I'} - \frac{I'}{I}\beta_\phi^2, \quad \gamma = \frac{2mgL}{I'}. \tag{10.2.10}$$

Substitution of these expressions into the energy conservation equation, Eq. (10.2.8), gives

$$\boxed{\dot{u}^2 = (\varepsilon - \gamma u)\left(1 - u^2\right) - \left(\beta_\psi - \beta_\phi u\right)^2.} \tag{10.2.11}$$

This differential equation may be solved by separating variables, which would yield an expression for t as a function of u in the form of an elliptic integral; see Gradsteyn and Rhyzek (2000). Numerical methods provide an alternative solution procedure. However, we may determine much qualitative information about the motion by merely studying the roots of the cubic polynomial on the right side of Eq. (10.2.11). These roots describe the conditions for which $\dot{u}$ is zero, corresponding to either an extreme or a

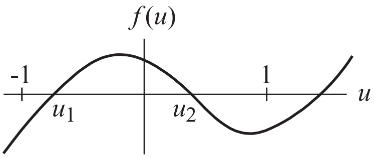


Figure 10.8. Roots of the function $f(u)$ describing the nutation rate of a spinning top.

constant value of the nutation angle. The polynomial is

$$f(u) = (\varepsilon - \gamma u)\left(1 - u^2\right) - \left(\beta_\psi - \beta_\phi u\right)^2. \tag{10.2.12}$$

Because $u = \cos\theta$, the physically meaningful values of u must lie in the range $-1 \leq u \leq 1$, subject to the requirement that $f(u) \geq 0$ in order that $\dot{\theta}$ be real.

Let us investigate the nature of the roots of $f(u)$. If $u \gg 1$, then $f(u) \approx \gamma u^3 > 0$ because γ is a positive parameter. Similarly, $u \ll -1$ corresponds to $f(u) < 0$. Furthermore, $u = \pm 1$ corresponds to $f(u) < 0$. Thus one root must be situated in $u > 1$, which is extraneous. We reason that there must be some range of precession angles for which $\dot{\theta}^2 > 0$, which means that $f(u)$ must be positive somewhere between $u = -1$ and $u = +1$. One possible situation is depicted in Fig. 10.8, where the region of positive $f(u)$ straddles $u = 0$. In that case the significant roots are $u_1 < 0$ and $u_2 > 0$. However, it might be that both roots are positive or negative.

The variable u may be interpreted geometrically as being the elevation above the apex of a point P on the (z) axis of symmetry at a unit distance from the apex. In this interpretation the precession angle ψ and the nutation angle θ are spherical coordinates for point P, whose path lies on a sphere of unit radius. Because $f(u) = 0$ corresponds to $\dot{\theta} = 0$, the larger root of $f(u)$, that is, $u = u_2$, represents the highest elevation of point P, or equivalently the smallest nutation angle is the smallest. Similarly, the lowest elevation attained in the motion is $u = u_1$, corresponding to the largest nutation angle. The nutational motion is such that the symmetry axis oscillates between high and low positions, $u_1 \leq u \leq u_2$. The exceptional situation in which the roots are repeated, $u_1 = u_2$, corresponds to a constant nutation angle. This is an important possibility, because we saw in Eqs. (10.2.5) that the precession and spin rates are constant when θ is constant. Thus the case of repeated roots corresponds to steady precession, which we shall treat later.

If we have determined that a certain value of u occurs at some instant, the corresponding precession rate given by Eqs. (10.2.5) is found to be

$$\dot{\psi} = \frac{\beta_\phi (u_0 - u)}{1 - u^2}, \quad u_0 = \frac{\beta_\psi}{\beta_\phi}. \tag{10.2.13}$$

Because $|u| \leq 1$, we observe from this relation that the sense of the precession, which is defined by the sign of $\dot{\psi}$, is determined by the value of $u - u_0$. Whether $\dot{\psi}$ actually vanishes in a specific response depends on whether the value of u_0 lies in the range $u_1 \leq u \leq u_2$.

There are three ways in which the value of u_0 may be situated relative to u_1 and u_2. Understanding each requires recognition of the interplay between the alteration in the rotational motions necessary to conserve angular momentum and mechanical energy.

Conservation of angular momentum about the axis of symmetry requires that the total rate of rotation about that axis, $\omega_z = \dot{\phi} + \dot{\psi}\cos\theta$, remain constant. This means that a decrease in the precession rate or an increase in the nutation angle will be compensated by an increase in the spin rate. The effect of the nutation on the precession rate may be seen from the expression for p_ψ, Eqs. (10.2.4). This term originates from two sources: the projection of the spin momentum p_ϕ onto the precession axis, and the angular momentum associated with the precession itself. The equivalent moment of inertia for the latter effect is $I'(\sin\theta)^2$. Increasing the nutation angle increases this moment of inertia, while it simultaneously decreases the projection of p_ϕ. Hence an increase in the nutation angle has competing effects on the precession rate, depending on the value of p_ϕ relative to p_ψ.

Equation (10.2.8) indicates that changing the nutation angle also has competing effects on the mechanical energy. The portion of the kinetic energy attributable to the precession and spin rates might increase or decrease when θ increases, depending on the values of β_ϕ and β_ψ. This is accompanied by a decrease in the potential energy with increasing θ. The nutational portion of the mechanical energy, that is, $I'\dot{\theta}^2/2$, must maintain the balance between kinetic and potential energy. At the extremes of the nutational motion, the change in potential energy is exactly compensated by the change in the precession and spin kinetic energy, so the nutational energy vanishes at those locations.

- *Unidirectional Precession*
  This is the case in which the precession rate $\dot{\psi}$ never changes sign. According to Eq. (10.2.13), such a situation is encountered if $u_0 \le u_1$ or $u_0 \ge u_2$. The sense of the precessional motion that is imparted at the initial instant continues throughout the motion. The nutation angle has its maximum and minimum values at $u = u_1$ and u_2, but the precession continues at those locations. As shown in Fig. 10.9, the path of point P at its highest and lowest elevations is tangent to horizontal circles on the unit sphere. One way in which we may initiate a unidirectional precession is to release the top at the highest elevation of point P, where $u = u_2$, with the precession rate set in the desired sense, and the initial nutation rate set to zero, corresponding to $\dot{u} = 0$. The initial precession rate should be relatively large, sufficient to make $u_0 = \beta_\psi/\beta_\phi$ exceed u_2.
- *Looping Precession*
  This response is characterized by a precession that reverses direction, although the overall motion is in one direction. Such a motion corresponds to $u_1 \le u_0 \le u_2$. The reversal of direction is indicated by $\dot{\psi} = 0$, whose occurrence is indicated by Eq.

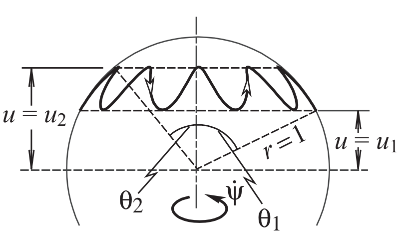


Figure 10.9. Path of the spin axis of a top in unidirectional precession.

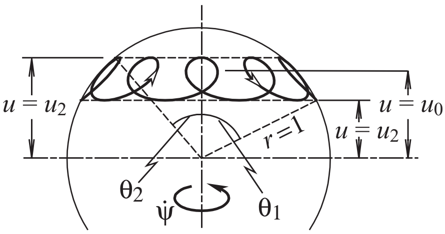


Figure 10.10. Path of the spin axis of a top in looping precession.

(10.2.13) to correspond to $u = u_0$. The tangent to the path at these locations coincides with a meridian of the unit circle. In contrast, the nutation rate vanishes at the lowest and highest elevations. At those locations, the tangent to the path is horizontal, with movement at the low point indicating the direction of the overall precession. As shown in Fig. 10.10, the movement at the upper limit is opposite the overall motion. (The overall precession might proceed oppositely to the motion depicted in the figure.)

A looping precession may be attained by releasing a top at the highest elevation, $u = u_2$, with a comparatively small precession rate. The nutation rate at release must be zero in order for u_2 to be the maximum elevation. As the top falls, the portion of the precession associated with β_ψ is eventually overwhelmed by the counter effect associated with β_ϕ. The elevation u_0 marks the location where these effects balance.

- *Cuspidial Motion*

This case is a transition between the unidirectional and looping precessions previously discussed. Like looping precession, the precession comes to rest, but this occurs at the highest elevation, $u_0 = u_2$. Because the tangent to the path at the highest elevation coincides with the meridian, the path at that location has a cusp. As shown in Fig. 10.11, the path of point P resembles a cycloidal path that is wrapped around the unit sphere.

Cuspidial motion may be attained by releasing the top at the highest elevation, $u = u_2$, with no initial precessional or nutational motion. The precessional motion that arises as the top falls is therefore due to only the spin momentum β_ϕ. As the top falls, it gains kinetic energy and loses potential energy. Conservation of β_ψ and β_ϕ causes the the precession rate to increase, until at the low point the contributions of the precession and spin rates to the kinetic energy equal the decrease in the potential energy, so $\dot{\theta}$ vanishes. Incidentally, we may prove by this reasoning that the cusps cannot arise at the lowest elevation, where $u = u_1$. Such a motion would lead to

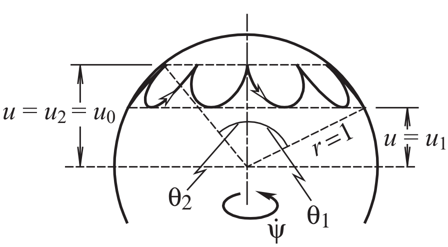


Figure 10.11. Path of the spin axis of a top in cuspidial precession.

kinetic and potential energies that are both maximum values at the lowest elevation, in violation of energy conservation.

The coincidence of the values of u_0 and u_2 in this case simplifies the governing equations sufficiently to permit solving them for the Eulerian angles as functions of time. We have seen that suitable initial conditions leading to cuspidial motion are $\dot{\psi} = \dot{\theta} = 0$, with the initial spin rate $\dot{\phi}_0$ nonzero. The elevation at the instant of release is $\cos\theta = u_2 = u_0$. The corresponding momentum parameters are given by Eqs. (10.2.4) to be

$$\beta_\phi = \frac{I}{I'}\acute{\phi}_0, \quad \beta_\psi = \beta_\phi u_0. \tag{10.2.14}$$

The energy level parameter in Eq. (10.2.8) is

$$E = mglu_0 + \frac{(I')^2}{2I}\beta_\phi^2, \tag{10.2.15}$$

whose substitution into Eqs. (10.2.10) gives

$$\varepsilon = \gamma u_0. \tag{10.2.16}$$

On substitution of these parameters, the energy function $f(u)$ defined in Eq. (10.2.12) becomes

$$f(u) = (u_0 - u)\left[\gamma\left(1 - u^2\right) - \beta_\phi^2(u_0 - u)\right], \tag{10.2.17}$$

whose roots in the acceptable range $|u| \le 1$ are

$$u_1 = U - \left(U^2 - 2u_0U + 1\right)^{1/2}, \quad u_2 = u_0, \tag{10.2.18}$$

where

$$U = \frac{\beta_\phi^2}{2\gamma} = \frac{I^2\acute{\phi}_0^2}{4I'mgL}. \tag{10.2.19}$$

These expressions for the limits of the nutation in cuspidial motion as a function of the initial conditions may be simplified further if we limit our attention to the situation that usually arises: a *fast top*, in which the spin rate imparted in the initial motion is large. We quantify this restriction by specifying that

$$U \gg 1 \iff \beta_\phi^2 \gg 2\gamma. \tag{10.2.20}$$

The corresponding minimum elevation obtained from the leading terms in a series expansion of the first of Eqs. (10.2.18) is

$$u_1 = u_0 - \frac{1 - u_0^2}{2U}. \tag{10.2.21}$$

In view of the definition of U, this equation states that the difference between the maximum and minimum elevations for a fast top decreases as the inverse square of the initial spin rate.

The smallness of the cuspidial motion at high spin rates allows us to evaluate the precessional and nutational rotations as explicit functions of time. The technique for such an investigation is perturbation analysis, based on the observation that the difference between u_0 and u must be small throughout the motion, and that $1/U$ is a small quantity that scales this difference. This suggests a series expansion for u:

$$u = u_0 - \frac{1}{U}v_1(t) - \frac{1}{U^2}v_2(t) + \cdots, \tag{10.2.22}$$

where the $v_j(t)$ are unknown functions of time that are independent of the parameter U. Many terms would be required to obtain a convergent series when the value of U is arbitrary. In contrast, the error that arises from truncating the series becomes smaller and smaller as the value of U increases. We say that Eq. (10.2.22) is an *asymptotic series* for the variable u in terms of the perturbation parameter $1/U \ll 1$.

We obtain differential equations for the unknown functions v_j by requiring that the asymptotic series satisfy the equation of motion at each level of approximation, associated with increasing powers of $1/U$. The most direct approach to deriving these equations uses the second-order differential equation that is the time derivative of the energy conservation relation, Eq. (10.2.11), with the right side represented by Eq. (10.2.17). These operations lead to

$$2\dot{u}\ddot{u} = \frac{dF}{du}\dot{u} = \left[-\gamma\left(1-u^2\right) - 2\gamma u\left(u_0-u\right) + 2\beta_\phi^2\left(u_0-u\right)\right]\dot{u}. \tag{10.2.23}$$

We cancel the common factor $\dot{u}$, then substitute the asymptotic series. Concurrently, we use Eq. (10.2.19) to eliminate γ, with the result that

$$\begin{aligned} -\frac{2}{U}\ddot{v}_1 - \frac{2}{U^2}\ddot{v}_2 = {} & -\frac{\beta_\phi^2}{2U}\left[1-\left(u_0-\frac{1}{U}v_1-\frac{1}{U^2}v_2\right)^2\right] \\ & -\frac{\beta_\phi^2}{U}\left(u_0-\frac{1}{U}v_1-\frac{1}{U^2}v_2\right)\left(\frac{1}{U}v_1+\frac{1}{U^2}v_2\right) \\ & +2\beta_\phi^2\left(\frac{1}{U}v_1+\frac{1}{U^2}v_2\right). \end{aligned} \tag{10.2.24}$$

This equation must be satisfied for arbitrary (large) values of U, which means that the coefficients of like powers of $1/U$ must match. The leading order is $1/U$, which gives results that are sufficiently descriptive. Thus we expand the preceding and retain only the coefficients of $1/U$, which leads to

$$\ddot{v}_1 + \beta_\phi^2 v_1 = \frac{1}{4}\beta_\phi^2\left(1-u_0^2\right). \tag{10.2.25}$$

The solution of this differential equation must satisfy the initial conditions for cuspidial motion, which we have taken to be that $u = u_0$ and $\dot{u} = 0$ at the instant of release. The leading term in Eq. (10.2.22) satisfies these conditions, so the next order of approximation must satisfy rest conditions, $v_1 = \dot{v}_1 = 0$ at $t = 0$. The sum of the complementary and particular solutions satisfying the initial conditions is

$$v_1 = \frac{1}{4}\left(1-u_0^2\right)\left[1-\cos\left(\beta_\phi t\right)\right], \tag{10.2.26}$$

which corresponds to an elevation given by

$$u \approx u_0 - \frac{v_1}{U} = u_0 - \frac{\gamma}{2\beta_\phi^2}\left(1 - u_0^2\right)\left[1 - \cos\left(\beta_\phi t\right)\right]. \tag{10.2.27}$$

Recall that $\dot{u} = -\dot{\theta}\sin\theta$. In the present context it is sufficient to apply this relation with θ approximated be the angle θ_2 at which the motion was initiated. Because $1 - u_0^2 = (\sin\theta_2)^2$, we find that

$$\dot{\theta} = -\frac{\dot{u}}{\sin\theta_2} \approx \frac{\gamma}{2\beta_\phi}\sin\theta_2 \sin\left(\beta_\phi t\right). \tag{10.2.28}$$

We obtain a simplified expression for the precession rate by using $u = u_0$ to simplify the denominator of Eq. (10.2.13). Substitution of β_ϕ from Eqs. (10.2.14) and u from Eq. (10.2.27) then yields

$$\dot{\psi} = \frac{\gamma}{2\beta_\phi}\left[1 - \cos\left(\beta_\phi t\right)\right]. \tag{10.2.29}$$

The interpretation of these results is that the average precession rate of a fast top varies harmonically about the mean value $\gamma/2\beta_\phi$, with an amplitude equal to the mean value. When the precession rate is zero, [$\cos\left(\beta_\phi t\right) = 1$], the nutation rate is zero and the top is at its highest elevation. At the instant when the precession rate is maximum [$\cos\left(\beta_\phi t\right) = -1$], the nutation rate is zero once again, corresponding to the lowest elevation.

- *Steady precession*

  If the appropriate initial motion is imparted to the top, it is possible to obtain a rotation in which the nutation angle and precession rate are constant. The corresponding spin rate in that case also will not vary from its initial value. Attaining such a motion requires that the top be released with $\dot{\theta} = 0$. It also is necessary that the initial values of θ, $\dot{\psi}$, and $\dot{\phi}$ be such that the coefficient of θ in the original differential equation, Eq. (10.2.6), vanishes. Then $\ddot{\theta}$ will be zero at the instant of release, which in combination with $\dot{\theta}$ being zero initially, leads to $\dot{\theta}$ being constant for all t. Thus we seek the combination of θ, $\dot{\psi}$, and $\dot{\phi}$ for which

$$\left(\dot{\psi}\sin\theta\right)\left[I'\dot{\psi}\cos\theta - I\left(\dot{\psi}\cos\theta + \dot{\phi}\right)\right] + mgL\sin\theta = 0. \tag{10.2.30}$$

In most cases the spin rate is set before the top is set into motion. Thus the preceding condition can be considered to govern the initial precession rate required to obtain a steady precession with specified values $\dot{\phi}$ and θ. One possibility is $\sin\theta = 0$, which we will address separately. Factoring out $\sin\theta$ from the equation leads to a quadratic equation, whose roots are

$$\dot{\psi} = \frac{I\dot{\phi} \pm \left[I^2\dot{\phi}^2 - 4mgL(I' - I)\cos\theta\right]^{1/2}}{2(I' - I)\cos\theta}. \tag{10.2.31}$$

An important aspect is that no real root exists unless the spin rate exceeds a critical value given by

$$\dot{\phi} > \omega_{\text{cr}} = \left[\frac{4mgL(I' - I)\cos\theta}{I^2}\right]^{1/2}. \tag{10.2.32}$$

The roots in Eq. (10.2.31) entail no approximations, but we usually are interested in the case of a fast top. In that case $\dot{\phi}$ is large, which leads to simplification of the formula based on a binomial series expansion of the square root, which gives

$$\left[I^2\dot{\phi}^2 - 4mgL(I' - I)\cos\theta\right]^{1/2} \approx I\dot{\phi} - 2mgL\frac{(I' - I)}{I\dot{\phi}}\cos\theta. \tag{10.2.33}$$

This reduces the roots to

$$\dot{\psi}_1 = \frac{mgL}{I\dot{\phi}}, \quad \dot{\psi}_2 = \frac{I}{(I' - I)\cos\theta}\dot{\phi}. \tag{10.2.34}$$

The first root, $\dot{\psi}_1$, is inversely proportional to $\dot{\phi}$, so it is comparatively low. The fast precession rate $\dot{\psi}_2$ matches the value obtained from Eqs. (10.1.6) for a symmetric body in free motion. In essence, the spin and precession rates in the fast case are so high that the gravitational moment is negligible in comparison with the moments required to alter the angular momentum of the top. Steady precession of a top usually occurs at the slow precession rate, because the kinetic energy required to attain $\dot{\psi}_2$ is prohibitive.

A special case of steady rotation is the *sleeping top*, which is the term used when the axis of symmetry of the top is vertical, $\sin\theta = 0$. The precession and spin are indistinguishable in a sleeping top, because the rotations are about concurrent axes. (The name "sleeping top" stems from the merger of spin and precession, which causes a polished body of revolution without markings to appear to be stationary.) Because of the similarity of precession and spin in such a rotation, some of the relations for steady precession become trivial. For example, $\sin\theta = 0$ identically satisfies Eq. (10.2.30), which is the condition for $\ddot{\theta}$ being identically zero. However, all relations for steady precession remain valid in the limit as $\theta \to 0$. We treat this degenerate case by noting that the angular velocity of a sleeping top is merely

$$\omega = \dot{\psi} + \dot{\phi}. \tag{10.2.35}$$

Hence, we find from Eq. (10.2.32) that the minimum rotation rate required for a top to sleep is

$$\omega > \omega_{\text{cr}} = \left[\frac{4mgL(I' - I)}{I^2}\right]^{1/2}. \tag{10.2.36}$$

Our analysis suggests that the axis of symmetry cannot remain vertical if ω is below this value. However, we saw earlier that $\ddot{\theta} = 0$ whenever $\sin\theta = 0$, which suggests that the top will sleep for any spin rate if $\theta = \dot{\theta} = 0$ when the top is released. In that condition the angular momentum is vertical, and therefore constant, and the moment of the gravity force about the pivot point vanishes. In essence, by obtaining the sleeping top as a special case, we have demonstrated that $\theta = 0$ is unstable if $\omega < \omega_{\text{cr}}$.

In actuality, the effect of friction at the apex O is to slow the rate of rotation. When the value of ω for a sleeping top falls below ω_{cr}, the top begins to nutate. Because the nutational velocity is zero at the instant when the rotation rate falls below critical, the ensuing motion is a cuspidial precession. If the spin rate decreases slowly, the amplitude of the nutation will slowly increase until the top hits the ground or falls from its support.

EXAMPLE 10.3 A 2-kg top is in a state of steady slow precession at a spin rate of 500 rev/min with its axis at $\theta = 120^\circ$. A vertical impulsive force acting through the axis of symmetry suddenly induces an upward nutation, such that the ensuing motion is observed to be cuspidial. The radii of gyration of the top about its pivot are 360 mm and 480 mm parallel and transversely to the axis of symmetry, respectively, and the distance from the center of mass to the pivot is 200 mm. Determine (a) the nutation rate induced by the impulsive force, (b) the largest and smallest values of the nutation angle in the cuspidial precession, (c) the number of cusps in the path of the axis of symmetry for one revolution of the top about the vertical axis, and (d) the maximum, minimum, and average precession rates in the cuspidial motion.

SOLUTION Many of the basic formulas for a spinning top are implemented in the solution of this example. We begin by evaluating the steady precession preceding the application of the impulse force. The first of Eqs. (10.2.34) gives an approximate value for the slow precession rate. We could use that relation provided that we verify that the associated value of β_ϕ is sufficiently high to warrant using that expression. Instead, we use Eq. (10.2.31) to find the slow precession rate. The parameters for the present system are $m = 2$ kg, $I = m\kappa^2 = 0.2592$ kg-m^2, $I' = m(\kappa')^2 = 0.4608$ kg-m^2, $L = 0.2$ m, and $\dot{\phi} = 52.36$ rad/s, which leads to the two roots

$$\dot{\psi}_1 = 0.28843 \text{ rad/s}, \quad \dot{\psi}_2 = -134.93 \text{ rad/s}.$$

Both values are extremely close to the approximations obtained from Eqs. (10.2.34). Because it is stated that the initial precession is slow, we use $\dot{\psi}_1$ as the initial precession rate.

The impulsive force induces an unknown nutation rate $\dot{\theta}$, because it exerts a moment about the horizontal axis through the pivot. However, the spin and precession rates are not altered during the impulse interval because the force exerts no moment about either axis. We find $\dot{\theta}$ from the fact that the subsequent precession is cuspidial. We need the values of the precession and spin momentum parameters to evaluate cuspidial motion, so we substitute the values of $\dot{\psi}_1$ and $\dot{\phi}$ into Eqs. (10.2.4), which gives

$$\beta_\phi = 29.371 \text{ rad/s}, \quad \beta_\psi = -14.469 \text{ rad/s}.$$

From this we find that the highest elevation in the cuspidial motion is

$$u_0 = \frac{\beta_\psi}{\beta_\phi} = -0.49264. \tag{1}$$

The perturbation analysis of cuspidial motion may be employed if the parameter U is large. The imbalance parameter is found from Eqs. (10.2.10) to be

$$\gamma = \frac{2mgL}{I'} = 17.026, \tag{10.2.37}$$

whose substitution into Eq. (10.2.19) gives

$$U = \frac{\beta_\phi^2}{2\gamma} = 25.334.$$

This value is sufficiently large to warrant using Eq. (10.2.27) to describe the nutation angle. Because $u = \cos\theta$, we find that

$$\begin{aligned}\theta &= \cos^{-1}\left[u_0 - \frac{1-u_0^2}{4U}\left[1-\cos\left(\beta_\phi t\right)\right]\right] \\ &= \cos^{-1}\left[-0.49264 - 0.007473\left[1-\cos\left(29.371t\right)\right]\right] \text{ rad.}\end{aligned} \tag{2}$$

The extreme values of the nutation angle in Eq. (2) correspond to $\cos(29.371t) = \pm 1$, which leads to

$$\theta_2 = \theta_{\min} = 119.512^\circ, \quad \theta_1 = \theta_{\max} = 120.503.^\circ \qquad \triangleleft$$

Direct substitutions into Eqs. (10.2.28) and (10.2.29) yield the corresponding nutation and precession rates:

$$\dot\theta = 0.25223\sin(29.371t), \quad \dot\psi = 0.28984\left[1-\cos(29.371t)\right] \text{ rad/s,} \tag{3}$$

To determine the nutation rate imparted by the impulsive force we need to establish when the force was applied. (Recall that $t = 0$ is an instant when $\theta = \theta_{\min}$, at which the cusp occurs.) The instant $t = t_0$ at which the force ceased may be determined from the fact that $\theta = 2\pi/3$ initially. Setting θ in Eq. (2) to this value gives

$$-0.5 = -0.49264 - 0.007473\left[1-\cos(29.371t_0)\right] \implies t_0 = 0.05299 \text{ s.}$$

The first of Eqs. (3) indicates that the nutation rate at this instant is

$$\dot\theta = 0.25223 \text{ rad/s.} \qquad \triangleleft$$

To determine the average precession rate we observe that the average of a sinusoidal function is zero, so the second of Eqs. (3) leads to

$$\dot\psi_{\text{av}} = 0.2898, \quad \dot\psi_{\min} = 0, \quad \dot\psi_{\max} = 0.5796 \text{ rad/s.} \qquad \triangleleft$$

Finally, we note that cusps occur when $\theta = \theta_{\min}$, or alternatively when $\dot\psi = 0$. According to Eqs. (2) and (3), these conditions occur whenever $\cos(29.371t) = 1$. Therefore the time interval between two adjacent cusps is the period of the oscillation:

$$\Delta t = \frac{2\pi}{29.371} = 0.2139 \text{ s.}$$

At the average precession rate, the time interval for one revolution about the vertical axis is

$$T = \frac{2\pi}{\dot{\psi}_{\text{av}}} = 21.68 \text{ s}.$$

The number of cusps is the ratio of this period to the period Δt for the θ oscillation, so

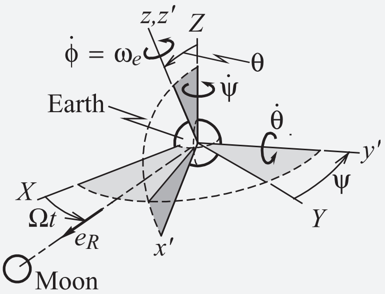
$$N = 101.34.$$

EXAMPLE 10.4 Example 5.4 examined the implications of the inverse square law for gravitation in the context of the oblateness of the Earth and the tilt of the Earth's polar action. It was shown there that the Moon and the Sun each exert a force–couple system acting at the Earth's center of mass. Show that the primary consequence of these couples is a nearly steady precession of the Earth's polar axis about an axis that is normal to the orbital plane, accompanied by a very minor wobble. Determine the precession rate associated with each attracting body.

SOLUTION Although the role of gravitational moments exerted on the Earth is not like the effect of gravity on a spinning top, the implications of the similarities of a top and the Earth's rotation is quite interesting. It was shown in Example 5.4 that the moment exerted by the Moon is more than twice as large as the Sun's effect, so we start by considering the Moon's effect. The previous analysis found that the Moon's gravitational attraction can be resolved into a force and couple acting at the Earth's center (of mass) given by

$$\begin{aligned} \bar{F} &= \frac{GM_{\text{moon}} M_e}{R_{\text{moon}}^2} \bar{e}_R, \\ \bar{M} &= \frac{3GM_{\text{moon}} (I - I')}{R_{\text{moon}}^3} \left(\bar{e}_R \cdot \bar{k}\right) \left(\bar{e}_R \times \bar{k}\right), \end{aligned} \tag{1}$$

where $\bar{k}$ is the unit vector from the South Pole to the North Pole, $\bar{e}_R$ is the unit vector from the center of the Earth to the center of the Moon, R is the distance between centers, and I and I' are the centroidal moments of inertia, with I being about the polar axis. To describe the unit vectors and angular motion of the Earth, we let xyz be the body-fixed coordinate system, and let XYZ be a translating reference frame whose origin is always situated at the Earth's center with the XY plane coincident with the orbital plane of the Moon. An auxiliary coordinate system $x'y'z'$ is defined such that it precesses about the Z axis by ψ and nutates about the y' axis by angle θ. The y' axis, which is the line of nodes, lies in the orbital plane. The spin rate of xyz relative to $x'y'z'$ is $\dot{\phi}$, which is the spin rate ω_e of the Earth. As shown in the sketch, the radial line to the Moon coincides with the Moon's orbital plane, and this line rotates at the orbital speed Ω of the Moon, which we consider to be constant. (By taking R_{moon} and Ω to be constant, we are ignoring the eccentricity of the orbit.)



Coordinate systems and Eulerian angles for analyzing the rotation of the Earth resulting from the gravitational attraction of the Moon.

Thus the angle between the X axis and $\bar{e}_R$ is Ωt. (The definition of the instant when $t = 0$ will not be relevant to the analysis.)

Because of the axisymmetry of the Earth about its polar axis, we can use $x'y'z'$ as the global coordinate system for the equations of motion. The transformation from XYZ to $x'y'z'$ components is the standard one for Eulerian angles:

$$[x' \quad y' \quad z']^{\mathrm{T}} = [R]\,[X \quad Y \quad Z]^{\mathrm{T}},$$
$$[R] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

The components of $\bar{e}_R$ relative to XYZ are found from the sketch to be

$$\bar{e}_R = \cos(\Omega t)\,\bar{I} + \sin(\Omega t)\,\bar{J}.$$

Using this representation in conjunction with $[R]$ gives the $x'y'z'$ components of $\bar{e}_R$. Trigonometric identities for the sine and cosine of the difference of two angles simplify the result to

$$\{e_R\}_{x'y'z'} = [R]\begin{Bmatrix} \cos(\Omega t) \\ \sin(\Omega t) \\ 0 \end{Bmatrix} = \begin{Bmatrix} \cos\theta\cos(\Omega t - \psi) \\ \sin(\Omega t - \psi) \\ \sin\theta\cos(\Omega t - \psi) \end{Bmatrix}. \tag{3}$$

This expression enables us to compute the dot and cross products of $\bar{e}_R$ and $\bar{k}' \equiv \bar{k}$:

$$\bar{e}_R \cdot \bar{k} = \sin\theta\cos(\Omega t - \psi),$$
$$\bar{e}_R \times \bar{k} = \sin(\Omega t - \psi)\,\bar{i} - \cos\theta\cos(\Omega t - \psi)\,\bar{j}.$$

Substituting these terms into $\bar{M}$, as described in Eqs. (1), leads to

$$\bar{M} = \Gamma\sin\theta\sin(2\Omega t - 2\psi)\,\bar{i} - G\sin\theta\cos\theta\,[1 + \cos(2\Omega t - 2\psi)]\,\bar{j},$$
$$\Gamma = \frac{3GM_{\mathrm{moon}}\,(I - I')}{2R_{\mathrm{moon}}^3}. \tag{4}$$

We are now ready to address the equations of motion. The angular velocity of the Earth is

$$\bar{\omega} = \dot{\psi}\bar{K} + \dot{\theta}\bar{j}' + \dot{\phi}\bar{k}' = -\dot{\psi}\sin\theta\bar{i}' + \dot{\theta}\bar{j}' + \left(\dot{\phi} + \dot{\psi}\cos\theta\right)\bar{k}'. \tag{5}$$

The translational and rotational portions of the motion are uncoupled, so the rotational kinetic energy is the same as in Eq. (10.2.2) for a spinning top. Rather than using potential energy to describe the role of gravity, it is easier to evaluate the virtual work done by the gravitational moment. The Eulerian angles are the generalized coordinates. The virtual rotation is found by analogy to the angular velocity to be

$$\overline{\delta\Theta} = -\delta\psi\sin\theta\bar{i}' + \delta\theta\bar{j}' + (\delta\phi + \delta\psi\cos\theta)\bar{k}'.$$

The virtual work is $\bar{M}\cdot\overline{\delta\Theta}$, which leads to the generalized forces being

$$Q_\psi = \Gamma(\sin\theta)^2\sin(2\Omega t - 2\psi), \quad Q_\theta = -\Gamma\sin\theta\cos\theta\left[1 + \cos(2\Omega t - 2\psi)\right].$$

The spin angle does not appear explicitly in T and $Q_\phi = 0$, so the associated generalized momentum is conserved. Thus the Lagrange equation for the spin angle is the same as for a spinning top:

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = I\left(\dot{\psi}\cos\theta + \dot{\phi}\right) \equiv I'\beta_\phi. \tag{6}$$

Unlike the precession angle of a spinning top, here the precession angle is not ignorable because the generalized forces depend on this variable. The Lagrange equation for the precession is

$$\begin{gathered}\left[I'(\sin\theta)^2 + I(\cos\theta)^2\right]\ddot{\psi} + I\ddot{\phi}\cos\theta + 2(I - I')\dot{\psi}\dot{\theta}\sin\theta\cos\theta \\ -I\dot{\theta}\dot{\phi}\sin\theta = \Gamma(\sin\theta)^2\sin(2\Omega t - 2\psi).\end{gathered} \tag{7}$$

The equation of motion for the nutation is the same as that for the spinning top, except that the potential-energy term is replaced with the generalized force Q_θ. Substitution of Eq. (6) into that equation gives

$$I'\ddot{\theta} - I'\left(\dot{\psi}\sin\theta\right)\left(\dot{\psi}\cos\theta - \beta_\phi\right) = -\Gamma\sin\theta\cos\theta\left[1 + \cos(2\Omega t - 2\psi)\right]. \tag{8}$$

We know from astronomical observation that the nutation angle is nearly constant. Furthermore, if the time-dependent term were not present in Eq. (6), a constant value of θ would lead to a constant value of $\dot{\psi}$. Thus we let

$$\theta = \theta_0 + \varepsilon\theta_1(t), \quad \dot{\psi} = \dot{\psi}_0 + \varepsilon\dot{\psi}_1(t), \tag{9}$$

where ε is a small parameter and the quantities with subscript one are not large. When the leading-order terms that arise in Eq. (8) are matched to the constant portion of the right side, the result is

$$I'\left(\dot{\psi}_0\sin\theta_0\right)\left(\dot{\psi}_0\cos\theta_0 - \beta_\phi\right) = \Gamma\sin\theta_0\cos\theta_0.$$

The nutation angle is not zero, so $\sin\theta_0$ can be factored out. The precession rate obtained from this quadratic equation is much less than the Earth's spin rate, so we may simplify the preceding relation by ignoring the contribution of $\dot{\psi}$ to Eq. (6), which gives $I'\beta_\phi \approx I\dot{\phi}_0$. Correspondingly, we find†

$$\dot{\psi}_0 \approx -\frac{\Gamma\cos\theta_0}{I\dot{\phi}_0} = \frac{3}{2}\frac{GM_{\text{moon}}}{R_{\text{moon}}^3\dot{\phi}_0}\left(\frac{I'-I}{I}\right)\cos\theta_0. \tag{10}$$

The spin rate of the Earth is $\dot{\phi}_0 = 7.292(10^{-5})$ rad/s. Example 5.4 lists $M_{\text{moon}} = 7.348\left(10^{22}\right)$ kg and $R_{\text{moon}} = 3.844\left(10^{8}\right)$ m. The quantity $(I'-I)/I$ is called the *dynamic elipticity*. The accepted value for this parameter is 1/305, which leads to

$$\left(\dot{\psi}_0\right)_{\text{moon}} = 5.356\left(10^{-12}\right) \text{ rad/s.} \qquad \lhd$$

The corresponding period is $2\pi/\left(\dot{\psi}_0\right)_{\text{moon}}$, which is 37 180 years. This value is 50% greater than the observed period of 26 000 years. One source for the discrepancy is that we have not accounted for the effect of the Sun. Changing the parameters in Eq. (10) to those for the Sun gives an expression for the precession rate if the Sun is the sole attracting body. The significant aspect is that the orbital plane of the Moon is approximately 5° different from the ecliptic plane, which is the term used to refer to the plane of the Earth's orbit. Thus it is a reasonable approximation to use the same precession axis to analyze the effects of the Sun and the Moon. Correspondingly, we may add the associated gravitational moment components associated with each attracting body. The oscillatory frequency 2Ω is much lower for the Sun than for the Moon, but the mean values, which are the nonoscillatory parts of the moments, add. The parameters for the Sun are $M_s = 1.98892\left(10^{30}\right)$ kg and $R_s = 1.4960\left(10^{11}\right)$ m, which leads to

$$\left(\dot{\psi}_0\right)_{\text{sun}} = 2.452\left(10^{-12}\right) \text{ rad/s.}$$

The period obtained from adding to the precession rate the contributions of the Moon and the Sun is

$$T = \frac{2\pi}{\left(\dot{\psi}_0\right)_{\text{moon}} + \left(\dot{\psi}_0\right)_{\text{sun}}} = 25\,500 \text{ years.} \qquad \lhd$$

This result is very close to the measured period of 26 000 years. The agreement is even more remarkable when one considers that the analysis treated the Earth as though it were a rigid body, which ignores such effects as tidal motions of the oceans. The average precessional motion is referred to in astronomy as the *precession of the equinoxes*. (At an equinox the polar axis is perpendicular to $\bar{e}_R$, corresponding to Ωt in the figure for this problem being an odd multiple of $\pi/2$.) The consequence

† Goldstein (1980) in Section 11-3 derived this expression from Lagrange's equations by evaluating the gravitational potential energy of a nonspherical axisymmetric body whose axis is tilted relative to the ecliptic plane. The derivation of the gravitational force–couple system in Example 5.4 is more general because it addresses an arbitrary body.

of the precession of the equinoxes is that 13 000 years from now the seasons will be reversed, so that the first day of summer in the Northern Hemisphere will come in December. Another consequence is that Polaris, which is popularly known as the North Star because it is almost concurrent with the Earth's polar axis, in the future will no longer be aligned in that manner.

One could proceed to analyze the second-order approximation associated with Eqs. (9). Because the first-order terms describe the effect of the mean value of the gravitational moment, the higher-order terms are driven by the oscillatory portion of the moment. The contributions of the Moon and the Sun add, but, as was noted previously, they have vastly different frequencies. In any event, the nutation and precession rates must exhibit minor fluctuations. The more noticeable one is the nutation, which represents a minor wobble. Earth-based astronomers must compensate for this wobble, as well as the precession, when they aim a telescope.

10.3 GYROSCOPES FOR INERTIAL GUIDANCE

The gyroscopic moment, which is associated with altering the direction of a rotation axis, is generally proportional to the product of rotation rates. We explore here a number of ways in which this effect has been employed in devices that guide moving vehicles autonomously. In essence, these devices provide an inertial reference system that moves with the vehicle, so they are referred to as *inertial guidance systems*. The conceptual pictures for our studies will be quite crude. In practice, the various pieces of equipment are manufactured with exceptionally high accuracy and with the finest bearings, in order to match as closely as possible the ideal conditions that we shall treat. Some of the concepts have been rendered obsolete with the availability of GPS navigation systems, but the virtue of the ones we examine is that they lead to self-contained systems. Also, modern electronic capabilities have made it possible to construct devices that achieve the same purpose with other technologies. For example, some systems exploit relativistic effects associated with the fact that light rays traveling in opposite directions through a rotating fiber-optic loop have differing phase speeds. However, many conventional gyroscopes are still in use, and their study will enhance our general understanding of fundamental phenomena.

10.3.1 Free Gyroscope

The gyroscope appearing in Fig. 10.12 is said to be free because the rotation of the rotor is unconstrained. The outer gimbal permits precessional rotation, the inner gimbal permits nutation, and the rotor shaft permits spin. For our introductory study we ignore the effect of the motion of the vehicle supporting the outer gimbal. In that case the center point O is stationary, because the three rotation axes are concurrent at that point.

If the center of mass G of the rotor does not coincide with the fixed point O, the gimbals must rotate. The excitation is the moment of the gravity force about the line

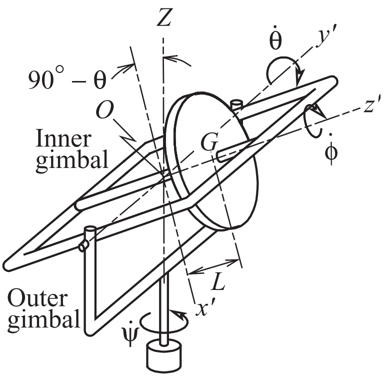


Figure 10.12. Rotations of a free gyro: precession of the outer gimbal, nutation of the inner gimbal, and spin of the rotor.

of nodes, which is the axis about which the inner gimbal rotates relative to the outer gimbal. The configuration of the system, with the center of mass situated on the spin axis relative to a fixed point that is also on that axis, is identical to that of a spinning top. It follows that the two systems behave in the same manner. In the special case in which the center of mass coincides with the fixed point O, the free gyroscope behaves like a body in free motion, Section 10.1, because there are no external moments. We shall employ the results of the previous sections, as necessary.

Suppose a steady precession, in which the nutation angle is constant, has been established. The relation among the precession rate, the spin rate, and the nutation angle is given by Eq. (10.2.31). In order for there to be a steady precession, the spin rate must exceed the minimum value given by Eq. (10.2.32). If the center of mass coincides with the fixed point O, then the steady slow precession rate is zero, which means that the axis of symmetry has a constant orientation.

An important question that must be addressed is whether the steady precession is a stable response. If it is not, then such motion would not be observed in reality. Our analysis of the dynamic stability of steady precession will follow the procedure used in Subsection 10.1.2 to study a pure spin of a body about either of its principal axes. We introduce a small disturbance of the nutation angle from the constant value it has when a steady precession has been established. Thus, let

$$\theta = \theta^* + \varepsilon\xi, \tag{10.3.1}$$

where θ^* denotes the constant value for steady precession, ε is a small parameter, and ξ is a time-dependent function that is assumed to have unit order of magnitude.

This description of the nutation angle must satisfy the basic equation of motion, Eq. (10.2.7), which we multiply by $(\sin\theta)^3$. We require only the terms that have the leading order of magnitude, so we employ Taylor series and retain only terms that are either independent of, or proportional to, ε. For example,

$$\cos\theta = \cos\theta^* - \varepsilon\xi\sin\theta^*, \quad (\sin\theta)^4 = (\sin\theta^*)^4 + 4\varepsilon\xi\,(\sin\theta^*)^3\cos\theta^*. \tag{10.3.2}$$

These series expansions are substituted into Eq. (10.2.7), and terms are grouped into those that are independent of ε and those that are proportional to ε. The former vanishes because it is the relation that must be satisfied for steady precession, that is,

$$\left(\beta_\psi - \beta_\phi \cos\theta^*\right)\left(\beta_\phi - \beta_\psi \cos\theta^*\right) - \frac{\gamma}{2}\left(\sin\theta^*\right)^4 = 0. \tag{10.3.3}$$

Canceling the ε factor in the remaining terms leads to

$$\begin{aligned}&\left(\sin\theta^*\right)^3 \ddot{\xi} + \{[\beta_\phi\left(\beta_\phi - \beta_\psi \cos\theta^*\right)\\ &\quad + \beta_\psi(\beta_\psi - \beta_\phi \cos\theta^*)]\sin\theta^* - 2\gamma\left(\sin\theta^*\right)^3 \cos\theta^*\}\xi = 0.\end{aligned} \tag{10.3.4}$$

The definitions of β_ψ and β_ϕ in Eqs. (10.2.4) simplifies this differential equation to

$$\begin{aligned}&\ddot{\xi} + K\xi = 0,\\ &K = \beta_\phi^2 + \left(\dot{\psi}^* \sin\theta^*\right)^2 - 2\gamma\cos\theta^*.\end{aligned} \tag{10.3.5}$$

The steady precession is stable to small disturbances if the value of ξ remains bounded. Such a condition is obtained if $K > 0$, in which case the solution for ξ is oscillatory. However, Eq. (10.2.32) states that a steady precession can exist only if the the spin rate, and therefore β_ϕ, is sufficiently large. It follows that $K > 0$ for any steady precession. In other words, if the spin momentum is sufficiently large to establish a steady precession at nutation angle θ^*, then an attempt to change the nutation angle by a small amount will result in an oscillatory nutational motion whose mean value is θ^*.

The balanced free gyroscope, for which $\gamma = 0$ corresponding to $L = 0$, is stable regardless of the spin momentum. It has been used to provide an *inertial platform* as part of an inertial navigation system that tracks vehicle motion in aircraft and missiles. The concept uses a free gyroscope and accelerometers. An accelerometer measures acceleration in the direction that it is oriented. [As explained in most fundamental texts on vibrations, such as Ginsberg (2001), an accelerometer features a very small mass that is suspended by a stiff spring. The displacement of the mass relative to the base on which the spring is supported is proportional to the base acceleration.] Three accelerometers mounted orthogonally therefore can measure an acceleration vector. An inertial navigation system using a free gyroscope mounts the gyroscope and orthogonal accelerometers on a freely suspended platform. If the supports had ideal properties, the platform's inertia would prevent its rotation. In that case, the platform would constitute a translating reference frame, so the accelerometer would measure the absolute acceleration of the platform. However, friction and other nonideal effects, regardless of how small they are, cause the platform to rotate. Sensors measure the rotation of the gimbals relative to the gyroscope, whose axis indicates a invariant direction. Servomotors apply counterrotations to the platform in order to hold its orientation constant. (Because only rotations about two axes can be sensed with a single free gyroscope, two or more are required.) The absolute acceleration is obtained from the accelerometer outputs. It is integrated twice in order to determine the absolute position at each time instant. The position relative to the Earth is readily determined because the absolute motion of any point on the Earth is a known function of time.

A much simpler concept is the *directional gyroscope*. It uses the free gyroscope in Fig. 10.12 in the balanced case, with the spin axis aligned along the longitudinal axis of the flight vehicle. The Z axis, which is the bearing axis of the outer gimbal, is stationary with respect to the vehicle; it is aligned perpendicularly to the longitudinal axis. This defines the yaw axis. The spin axis of the idealized balanced free gyroscope maintains a constant orientation regardless of how its gimbals rotate. Thus the yaw angle can be measured with a sensor that measures the rotation of the mount with respect to the outer gimbal. Similarly, the pitch angle is the rotation of the outer gimbal with respect to the inner one. A *vertical gyroscope* uses a similar arrangement, with the spin axis aligned perpendicularly relative to the vehicle's longitudinal axis. This is equivalent to the yaw axis for the directional gyroscope. (For an airplane this would be the vertical direction.) The axis of the outer gimbal is aligned along the longitudinal axis. Then the rotation of the mount relative to the outer gimbal is the roll, and the rotation of the outer gimbal with respect to the inner is the pitch. In practice, friction at the bearings would cause the rotor axis to drift for both the directional and rate gyros. This is compensated by force sensors/actuators that apply smoothing torques to counter such deviations.

EXAMPLE 10.5 To overcome the effects of friction, a servomotor applies a torque about the spin axis of an unbalanced gyroscope, with the result that the spin rate is constant. The initial conditions are such that the initial precession rate $\dot{\psi}^*$ and nutation angle θ^* correspond to steady precession. Determine whether the action of the servomotor can cause the gyroscope to be unstable to small disturbances.

SOLUTION The idea here is to develop the ability to analyze nonideal effects for gyroscopes. The primary difference between the present system and a free gyroscope is that there are only two degrees of freedom, because the spin rate $\dot{\phi}$ is constrained, rather than the spin momentum β_ϕ being constant. We commence to derive the equations of motion for the servogyroscope by using the energies in Eqs. (10.2.2) and (10.2.3):

$$T = \frac{1}{2}\left[I'\dot{\psi}^2(\sin\theta)^2 + I'\dot{\theta}^2 + I\left(\dot{\psi}\cos\theta + \dot{\phi}\right)^2\right], \quad V = mgL\cos\theta.$$

The precession angle is an ignorable generalized coordinate, because only its derivative appears in T. The corresponding conservation of momentum equation is identical to the first of Eqs. (10.2.4):

$$\beta_\psi = \dot{\psi}\left[\frac{I}{I'}(\cos\theta)^2 + (\sin\theta)^2\right] + \frac{I}{I'}\dot{\phi}\cos\theta. \tag{1}$$

We may find the actual value of β_ψ by substituting $\dot{\psi}^*$ and θ^*. Equation (10.2.6) is the Lagrange equation for θ corresponding to arbitrary dependence of ψ and ϕ. It may be written as

$$\begin{gathered}\ddot{\theta} + g(\psi,\theta) = 0,\\ g(\psi,\theta) = \left(\frac{I}{I'} - 1\right)\dot{\psi}^2\sin\theta\cos\theta + \left(\frac{I}{I'}\dot{\psi}\dot{\phi} - mgL\right)\sin\theta.\end{gathered} \tag{2}$$

Because the initial conditions are those appropriate to a steady precession, we have

$$g(\psi^*, \theta^*) = 0. \tag{3}$$

We assume that the configuration of interest is not a sleeping top, so $\sin\theta^* \neq 0$. Satisfaction of Eq. (3) therefore requires

$$\left(\frac{I}{I'} - 1\right)(\dot\psi^*)^2 \cos\theta^* + \frac{I}{I'}\dot\psi^*\dot\phi - mgL = 0, \tag{4}$$

which matches the expression established in Example 10.3 for the free gyroscope. Although the angular motion in steady precession is identical to that of a free gyroscope, the stability situation is different. Constancy of β_ψ now requires that any fluctuation in the nutation angle be compensated solely by a change in the precession rate. We consider the deviations from the nominal steady-state condition by setting $\dot\psi = \dot\psi^* + \varepsilon\dot\psi_1$ and $\theta = \theta^* + \varepsilon\theta_1$, where $\varepsilon \ll 1$ and $\dot\psi_1$ and θ_1 are taken to have unit magnitude. We substitute these expressions into the basic equations of motion, Eqs. (1) and (2), both of which are expanded in Taylor series in powers of ε. The results are

$$\begin{aligned}\beta_\psi = \dot\psi^*\left[\frac{I}{I'}(\cos\theta^*)^2 + (\sin\theta^*)^2\right] + \frac{I}{I'}\dot\phi\cos\theta^* + \varepsilon\left\{\dot\psi_1\left[\frac{I}{I'}(\cos\theta^*)^2 + (\sin\theta *)^2\right]\right.\\ \left.-\left[2\dot\psi^*\left(\frac{I}{I'} - 1\right)(\sin\theta^*\cos\theta^*) + \frac{I}{I'}\dot\phi\cos\theta^*\right]\theta_1\right\},\\ g(\psi^*, \theta^*) + \varepsilon\left[\ddot\theta_1 + \left(\frac{\partial g}{\partial\dot\psi}\right)^*\psi_1 + \left(\frac{\partial g}{\partial\theta}\right)^*\theta_1\right] = 0.\end{aligned}$$

The zero-order terms in each of the preceding equations convey no new information because $\dot\psi^*$ and θ^* satisfy Eq. (3) and also give the value of β_ψ in Eq. (1). The first-order terms in the preceding therefore must satisfy

$$\dot\psi_1\left[\frac{I}{I'}(\cos\theta)^2 + (\sin\theta)^2\right] - \left[\dot\psi^*\left(\frac{I}{I'} - 1\right)\sin(2\theta^*) + \frac{I}{I'}\dot\phi\sin\theta^*\right]\theta_1 = 0, \tag{5}$$

whereas the first-order terms obtained from the energy conservation equation (2) require that

$$\ddot\theta_1 + \left(\frac{\partial g}{\partial\dot\psi}\right)^*\psi_1 + \left(\frac{\partial g}{\partial\theta}\right)^*\theta_1 = 0. \tag{6}$$

Both derivatives are marked by an asterisk to remind us that they are evaluated at the steady precession state. We solve Eq. (5) for $\dot\psi_1$, and substitute that expression into Eq. (6). The result has the form

$$\ddot\theta_1 + K\theta_1 = 0. \tag{7}$$

The constant K, which is obtained by differentiating the expression for g in Eqs. (2) and then substituting the aforementioned expression for $\dot{\psi}_1$, is

$$K = \left[\left(\frac{I}{I'} - 1 \right) \left(\dot{\psi}^* \right)^2 \cos(2\theta^*) + \left(\frac{I}{I'} \dot{\phi} \dot{\psi}^* - \frac{\gamma}{2} \right) \cos\theta^* \right] + \frac{I' (\sin\theta^*)^2}{I (\cos\theta^*)^2 + I' (\sin\theta^*)^2} \left[2 \left(\frac{I}{I'} - 1 \right) \dot{\psi}^* \cos\theta^* + \frac{I}{I'} \dot{\phi} \right]^2 .$$

To simplify this further, we employ Eq. (4) to eliminate $\dot{\phi}$. The result after many manipulations is

$$K = \frac{I' (\sin\theta^*)^2}{\left(\dot{\psi}^* \right)^2 \left[I (\cos\theta^*)^2 + I' (\sin\theta^*)^2 \right]} \left\{ \frac{\gamma^2}{4} + \left(\frac{I}{I'} - 1 \right) \left(\dot{\psi}^* \right)^2 \left[\gamma \cos\theta^* - \left(\dot{\psi}^* \right)^2 \right] \right\} . \tag{7}$$

The solutions of Eq. (7) are sinusoidal when $K > 0$, whereas one grows exponentially if $K < 0$. Thus, as was true for the free gyroscope, whose stability was described by Eqs. (10.3.5), $K < 0$ indicates situations in which the servo-driven gyroscope is unstable to small disturbances of the nutation angle. Suppose we are interested in evaluating whether a specific design corresponding to specified values of $\dot{\phi}$, γ, and I/I' is unstable in some range of θ^*. To address this question we would consider Eq. (4) to be a quadratic equation for the steady precession rate $\dot{\psi}^*$. We would substitute the smaller root, corresponding to slow precession, into Eq. (7), which would yield an expression for K as a function of θ^*. Scanning this function over $0 < \theta^* < \pi$ would reveal whether K is negative in some range of θ^*. As an example of such a computation, consider $I/I' = 1.5$ and $\gamma = 20$ rad/s^2. Setting $\dot{\phi} > 3.366$ rad/s leads to stability for any θ^*, whereas $\dot{\phi} = 2$ rad/s leads to stability only for $0 < \theta^* < 97.09^\circ$.

The possibility that controlling the spin rate can lead to instability could have been anticipated on the basis of physical arguments. The free gyroscope is a conservative system. In contrast, the servo-driven gyroscope is not, because the servomotor does work to hold the spin rate constant. The energy provided to the system from this source can drive the nutational motion away from steady precession. However, most situations of practical interest are like the preceding numerical example, in that the spin rate below which the gyroscope would lose stability is sufficiently low to be of no concern.

10.3.2 Gyrocompass

A fundamental requirement for earthbound navigation is knowledge of the orientation of true north. The balanced free gyroscope maintains a fixed orientation as the Earth rotates; an observer on the Earth perceives the gyroscope to be rotating. The gyrocompass has the feature that its steady precession always matches the Earth's rotation, so an observer on the Earth perceives the rotor axis to always point in a constant direction.

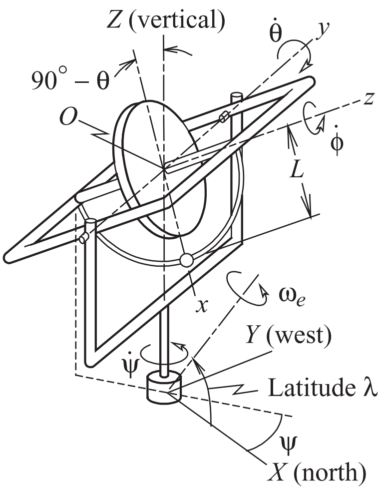


Figure 10.13. A gyrocompass on the surface of the Earth.

The gyrocompass bears much similarity to an unbalanced free gyroscope, except for the placement of the mass causing the imbalance. As shown in Fig. 10.13, the center of mass of the rotor is situated on the intersection of the precession and spin axes, but a small additional mass m is attached to the inner gimbal. This arrangement is selected in order that the gravitational moment be like that for a pendulum. The Earth-fixed coordinate system XYZ defined in Fig. 10.13 is introduced to describe the orientation relative to the Earth. The Z axis is oriented in the direction perceived as vertical to an observer on the Earth, and the X axis is situated in the northerly direction, which means that X is tangent to a meridian. In terms of these components the angular velocity of the Earth is

$$\bar{\omega}_e = \omega_e \left(\cos\lambda \bar{I} + \sin\lambda \bar{K}\right), \tag{10.3.6}$$

where $\omega_e = 7.292(10^{-5})$ rad/s $\approx 2\pi$ rad/24 h is the rotation rate.

The rotation of the gyrocompass' rotor is unconstrained. Eulerian angles are used to describe the orientation of the rotor, relative to XYZ. The xyz coordinate system, which is attached to the inner gimbal, facilitates the description. The z axis is aligned with the axis of symmetry of the rotor and the y axis is the line of nodes formed by the bearings of the inner gimbal. Our goal here is to determine whether there is any set of precession and nutation angles for which the axis of the rotor remains stationary relative to the Earth. For this reason we consider the values of ψ and θ to be constant, and we also assume that $\dot{\phi}$ remains constant. The corresponding angular velocity of the rotor relative to XYZ is $\dot{\phi}\bar{k}$. The absolute angular velocity $\bar{\omega}$ is the sum of this term and the angular velocity of the Earth-fixed reference frame. To express $\bar{\omega}$ in terms of component relative to xyz, we refer to Fig. 10.13 to find that

$$\begin{aligned}
\bar{I} &= \cos\psi\left[\cos\theta\bar{i} + \sin\theta\bar{k}\right] - \sin\psi\,\bar{j},\\
\bar{J} &= \sin\psi\left[\cos\theta\bar{i} + \sin\theta\bar{k}\right] + \cos\psi\,\bar{j},\\
\bar{K} &= -\sin\theta\bar{i} + \cos\theta\bar{k}.
\end{aligned} \tag{10.3.7}$$

Adding the Earth's rotation to the rotor spin then leads to

$$\begin{aligned}\bar{\omega} &= \omega_e (\cos\lambda \cos\psi \cos\theta - \sin\lambda \sin\theta)\,\bar{i} - \omega_e \cos\lambda \sin\psi\,\bar{j} \\ &\quad + \left(\omega_e \cos\lambda \cos\psi \sin\theta + \omega_e \sin\lambda \cos\theta + \dot{\phi}\right)\bar{k}.\end{aligned} \tag{10.3.8}$$

Terms containing ω_e have a very small value, so we may simplify the kinetic energy of the system by neglecting effects that are of the order of ω_e^2. The corresponding kinetic energy for the system is

$$\begin{aligned}T &= \frac{1}{2}\left(I'\omega_x^2 + I'\omega_y^2 + I\omega_z^2\right) \\ &= \frac{1}{2} I\left[\dot{\phi}^2 + 2\omega_e\dot{\phi}\,(\cos\lambda \cos\psi \sin\theta + \sin\lambda \cos\theta)\right].\end{aligned} \tag{10.3.9}$$

The potential energy is associated with the unbalanced mass on the inner gimbal. When the datum is set at the elevation of center O of the rotor, we find that

$$V = mgL\cos\left(\frac{\pi}{2} + \theta\right) = -mgL\sin\theta. \tag{10.3.10}$$

There are no nonconservative forces in this ideal model, so the Lagrange's equations for the generalized coordinates in this case of steady precession are

$$\begin{aligned}\omega_e\dot{\phi} \cos\lambda \sin\psi \sin\theta &= 0, \\ I\omega_e\dot{\phi}\,(-\cos\lambda \cos\psi \cos\theta + \sin\lambda \sin\theta) - mgL\cos\theta &= 0, \\ \dot{\phi} + \omega_e (\cos\lambda \cos\psi \sin\theta + \sin\lambda \cos\theta) &= \beta_\phi,\end{aligned} \tag{10.3.11}$$

where $\beta_\phi = p_\phi / I$ is the spin momentum parameter associated with the ignorable coordinate ϕ.

Because of the smallness of ω_e, the last of Eqs. (10.3.11) indicates that the spin momentum is primarily associated with the spin itself. The first equation is satisfied when $\sin\theta = 0$ or $\sin\psi = 0$. The first possibility is not useful, because then the rotor does not provide any directional information. The second case is the one we seek, because it is satisfied when $\psi = 0$ or π, which corresponds to the spin axis being aligned along the north–south meridian. Setting $\cos\psi = \pm 1$ in the second of Eqs. (10.3.11) yields

$$\left(mgL \pm I\omega_e\dot{\phi} \cos\lambda\right)\cos\theta = I\omega_e\dot{\phi} \sin\lambda \sin\theta, \tag{10.3.12}$$

from which we find that

$$\cot\theta = \frac{I\omega_e\dot{\phi} \sin\lambda}{mgL \pm I\omega_e\dot{\phi} \cos\lambda}. \tag{10.3.13}$$

The smallness of ω_e simplifies this result further. For any likely value of $\dot{\phi}$, the denominator term containing ω_e can be ignored. Furthermore, because $\cot\theta$ is very small, θ is close to $\pi/2$, so that $\cot\theta \equiv \tan(\pi/2 - \theta) \approx \pi/2 - \theta$. Consequently, the nutation angle for steady motion is

$$\theta = \frac{\pi}{2} - \frac{I\omega_e\dot{\phi} \sin\lambda}{mgL}, \quad \psi = 0 \text{ or } \pi. \tag{10.3.14}$$

In summary, the analysis shows that attaching a pendulous mass to an otherwise balanced gyroscope will result in a steady precession in which the spin axis is always situated in the vertical plane containing the local northerly direction. This behavior is obtained if the rotor is released with its spin axis situated in the vertical plane containing the northerly direction and slightly off horizontal, as described by the nutation angle in Eqs. (10.3.14). The only adjustment that is necessary is that this angle must be adjusted as the vehicle on which the gyrocompass moves in order to compensate for its dependence on the latitude λ.

The analysis identified the conditions for dynamic equilibrium at a specified latitude λ. The next example will show that the gyrocompass is stable to small disturbances. The primary limitation on its use is loss of accuracy that is due to rapid movement of the vehicle in which it is mounted. To learn why this is so, we observe that movement of the vehicle relative to the Earth may be approximated locally as following a great circle that is the intersection with the Earth's surface of the plane formed by the velocity and the center of the Earth. Such movement corresponds to rotation of the radial line from the center of the Earth to the vehicle, which is the vertical Z axis in Fig. 10.13. In other words, the vehicle's forward velocity $\bar{v}$ is accompanied by an angular velocity ω_{v}. The true angular velocity of the gyrocompass' base is the sum of this angular velocity and the angular velocity of the Earth. The consequence is that the northerly direction indicated by the gyrocompass is based on the angular velocity of the base, rather than the angular velocity of the Earth.

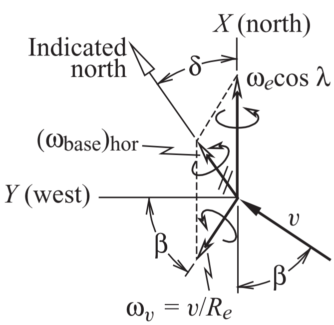


Figure 10.14. Directional error of a gyrocompass that is due to movement of its base at velocity $\bar{v}$. The angular velocity $\bar{\omega}_{\mathrm{rel}}$ represents the additional angular velocity associated with the motion being along a great circle.

We may quantify this effect by describing the vehicle's velocity along the local great circle as $\bar{v} = \bar{\omega}_{\mathrm{v}} \times \bar{r}_{O'/O}$, with $\bar{r}_{O'/O} = R_e \bar{K}$. If the vehicle is traveling at angle β west of north, then $\bar{v} = v\left(\cos\beta\bar{I} + \sin\beta\bar{J}\right)$, from which it follows that

$$\bar{\omega}_{\mathrm{v}} = \frac{v}{R_e}\left(-\sin\beta\bar{I} + \cos\beta\bar{J}\right). \tag{10.3.15}$$

An inspection of the analysis of the gyrocompass would reveal that the direction indicated as being north is the direction of the horizontal component of the base angular velocity. The addition of the horizontal angular velocity components is depicted in Fig. 10.14, where δ locates the true angular velocity of the base in the horizontal plane. The sum of the Earth's rotation, Eq. (10.3.6), and the preceding rotation of the base relative to the Earth is

$$\bar{\omega}_{\text{base}} = \left(\omega_e \cos\lambda - \frac{v}{R_e}\sin\beta\right)\bar{I} + \frac{v}{R_e}\cos\beta\,\bar{J} + \omega_e \sin\lambda\,\bar{K}. \tag{10.3.16}$$

Matching this vector to one that is oriented at angle δ west of north shows that

$$\delta = \tan^{-1}\left(\frac{v\cos\beta}{\omega_e R_e \cos\lambda - v\sin\beta}\right). \tag{10.3.17}$$

The mean radius of the Earth is $R_e = 6370$ km, which corresponds to $\omega_e R_e = 465$ m/s. The value of δ will be small if $v \ll \omega_e R_e$, which excludes aircraft and spacecraft. However, even for a slow-moving vehicle, like a ship, the error will be quite substantial near either the North or South Pole, where $|\lambda| = \pi/2$. In practice, λ is known, so one can use the preceding expression for δ to compensate for the gyrocompass readings. However, this does not entirely remove the difficulty near the Poles, because the manner in which a gyrocompass responds to disturbances at the Poles introduces additional errors, as is discussed in Example 10.6.

Other sources of error arise from motion of the vehicle on which the gyrocompass is mounted. These motions add to the base rotation and add accelerations to the pendulous mass. The former effect contributes additional deviations of the indicated northerly direction, whereas the latter alters the apparent magnitude and direction of the gravitational force. Both effects can be averaged out if the motion of the vehicle is not violent, just as one can average the rotation of a swinging pendulum to identify the vertical direction. For all of these reasons, the gyrocompass has been used primarily as a navigational aid for slowly moving vehicles, especially ships.

EXAMPLE 10.6 A gyrocompass tracking the northerly direction in a steady precession is given a small initial nutational disturbance $\Delta\theta$, causing it to deviate from its proper direction. Determine the response to this initial disturbance. Then, from that result, assess the stability of the gyrocompass.

SOLUTION In addition to enhancing one's ability to analyze the stability of a gyroscope, the result will disclose another limitation of a gyrocompass. We begin by assuming that the response is stable, because doing so will allow us to linearize the equations of motion, and thereby analytically solve for the response. We then will be able to assess the stability limits by seeking conditions for which the linearized response grows with increasing time. We assume that the rotor spins freely, so the orientation of the rotor relative to the base is described by the three Eulerian angles, none of which are constant. The previous equations of motion, Eqs. (10.3.11), are limited to constant values of ψ and θ, so we must derive a more general set of equations of motion.

The angular velocity of the flywheel is a superposition of the rotation of the Earth, given by Eq. (10.3.6), and the full set of Eulerian angles defined relative to the Earth-based XYZ coordinate system. Equations (10.3.7) give the transformation to body-fixed components, from which we find that

$$\begin{aligned}\bar{\omega} &= \bar{\omega}_e + \dot{\psi}\bar{K} + \dot{\theta}\bar{j} + \dot{\phi}\bar{k} \\ &= \left[-\dot{\psi}\sin\theta + \omega_e\left(\cos\lambda\cos\psi\cos\theta - \sin\lambda\sin\theta\right)\right]\bar{i} + \left(\dot{\theta} - \omega_e\cos\lambda\sin\psi\right)\bar{j} \\ &\quad + \left[\dot{\psi}\cos\theta + \dot{\phi} + \omega_e\left(\cos\lambda\cos\psi\sin\theta + \sin\lambda\cos\theta\right)\right]\bar{k}.\end{aligned}$$

Because the disturbance is small, the nutation angle θ will remain close to $\pi/2$, provided that the system is stable. We therefore define a new variable η that measures the deviation of θ from its nominal value, such that

$$\eta = \frac{\pi}{2} - \theta, \quad \dot{\eta} = -\dot{\theta}.$$

The corresponding mechanical energies are

$$\begin{aligned}T &= \frac{1}{2}I'\left[-\dot{\psi}\cos\eta + \omega_e\left(\cos\lambda\cos\psi\sin\eta - \sin\lambda\cos\eta\right)\right]^2 \\ &\quad + \frac{1}{2}I'\left(-\dot{\eta} - \omega_e\cos\lambda\sin\psi\right)^2 + \frac{1}{2}I\left[\dot{\psi}\sin\eta + \dot{\phi}\right. \\ &\quad \left. + \omega_e\left(\cos\lambda\cos\psi\cos\eta + \sin\lambda\sin\eta\right)\right]^2, \\ V &= -mgL\cos\eta.\end{aligned} \tag{1}$$

Neither T nor V depends explicitly on ϕ, and the generalized force associated with ϕ is zero. Hence ϕ is ignorable, as it was in the case of steady precession. The derivation of the equations of motion for η and ψ is considerably shortened if we follow Routh's method, rather than retaining ϕ as a generalized coordinate. We begin by evaluating the generalized momentum associated with ϕ, which is conserved,

$$\frac{\partial T}{\partial\dot{\phi}} = p_\phi = I\left[\dot{\phi} + \dot{\psi}\sin\eta + \omega_e\left(\cos\lambda\cos\psi\cos\eta + \sin\lambda\sin\eta\right)\right] = I\beta_\phi. \tag{2}$$

The expression for β_ϕ is the same as the one defined in Eqs. (10.3.11), but we now use it to form the Routhian, which is

$$\Re = T - V - p_\phi\dot{\phi}.$$

We solve Eq. (2) for $\dot{\phi}$, and use that expression to eliminate all occurrences of $\dot{\phi}$ in $\Re$. The result is

$$\begin{aligned}\Re &= \frac{1}{2}I'\left[-\dot{\psi}\cos\eta + \omega_e\left(\cos\lambda\cos\psi\sin\eta - \sin\lambda\cos\eta\right)\right]^2 \\ &\quad + \frac{1}{2}I'\left(\dot{\eta} + \omega_e\cos\lambda\sin\psi\right)^2 + \frac{1}{2}I\beta_\phi^2 - I\beta_\phi\left[\beta_\phi - \dot{\psi}\sin\eta\right. \\ &\quad \left. - \omega_e\left(\cos\lambda\cos\psi\cos\eta + \sin\lambda\sin\eta\right)\right] + mgL\cos\eta.\end{aligned} \tag{3}$$

We obtain the equations of motion for η and ψ by using $\Re$, rather than $\mathcal{L}$, to evaluate Lagrange's equations. Trigonometric identities for the sine and cosine of a double angle simplify the equations. The result for η is

$$\begin{aligned} & I'\ddot{\eta} + \frac{1}{2} I' \dot{\psi}^2 \sin 2\eta + I' \dot{\psi} \omega_e \left[2 \cos\lambda \cos\psi \, (\cos\eta)^2 + \sin\lambda \sin 2\eta \right] \\ & \quad - I\beta_\phi [\dot{\psi} \cos\eta - \omega_e (\cos\lambda \cos\psi \sin\eta - \sin\lambda \cos\eta)] \\ & \quad - I' \omega_e^2 (\cos\lambda \cos\psi \sin\eta - \sin\lambda \cos\eta) (\cos\lambda \cos\psi \cos\eta \\ & \quad + \sin\lambda \sin\eta) + mgL \sin\eta = 0, \end{aligned} \tag{4}$$

and the equation for ψ is

$$\begin{aligned} & I' \ddot{\psi} (\cos\eta)^2 - I' \dot{\psi} \dot{\eta} \sin 2\eta + I \dot{\eta} \beta_\phi \cos\eta - I' \dot{\eta} \omega_e [\cos\lambda \cos\psi \cos 2\eta \\ & + \sin\lambda \sin 2\eta] - I' \omega_e^2 \left[\frac{1}{2} (\cos\lambda)^2 \sin 2\psi + \frac{1}{4} \sin 2\lambda \cos\eta \sin\psi \sin 2\eta \right] \\ & + I\beta_\phi \omega_e \cos\lambda \sin\psi \cos\eta = 0. \end{aligned} \tag{5}$$

If the system is stable, then η and ψ will not grow relative to their initial magnitude. The smallness condition enables us to linearize Eqs. (4) and (5) by setting $\cos\eta \approx 1$, $\sin\eta \approx \eta$, $\cos\psi \approx 1$, and $\sin\psi \approx \psi$, and then dropping any terms that are quadratic or higher degree in the generalized coordinates. Concurrently, we also simplify the coefficients of the resulting equations by dropping any term containing ω_e if it is added to a term that does not contain ω_e. Another consistent approximation entails ignoring terms that are quadratic in ω_e. The simplified linearized equations of motion are

$$\begin{aligned} I'\ddot{\eta} - I\beta_\phi \dot{\psi} + mgL\eta &= I\beta_\phi \omega_e \sin\lambda, \\ I'\ddot{\psi} + I\beta_\phi \dot{\eta} + \left(I\beta_\phi \omega_e \cos\lambda \right) \psi &= 0. \end{aligned} \tag{6}$$

These are a coupled pair of linear ordinary differential equations, in which the inhomogeneous term is a constant. We form the solution of these differential equations by adding a complementary solution to the constant particular solution. The latter are

$$\eta_p = \frac{I\beta_\phi \omega_e}{mgL} \sin\lambda, \quad \psi_p = 0.$$

The value of β_ϕ is essentially the same as $\dot{\phi}$ because the spin rate is high, so the preceding is the same as the gyrocompass' nominal steady angles, as given by Eqs. (10.3.14). To derive the homogeneous solution of Eqs. (6), we reason that a stable response will be marked by oscillations about the nominal state, which means that the homogeneous solution of the linear equations should be sinusoidal. Furthermore, the homogeneous terms in Eqs. (6) relate a generalized coordinate and its second derivative to the first derivative of the other generalized coordinate. Consequently, one generalized coordinate must be 90° out of phase relative to the other. A suitable trial form for the complementary solution therefore is

$$\eta_c = A \sin(\sigma t - \gamma), \quad \psi_c = B \cos(\sigma t - \gamma). \tag{7}$$

Requiring that these expressions be solutions of the homogeneous portions of Eqs. (4) and (5) leads to

$$\begin{gathered}\left(mgL - I'\sigma^2\right) A + \left(I\beta_\phi\sigma\right) B = 0,\\ \left(I\beta_\phi\sigma\right) A + \left(I\beta_\phi\omega_e\cos\lambda - I'\sigma^2\right) B = 0.\end{gathered} \tag{8}$$

In order for there to be a nontrivial solution, the determinant of the coefficients of A and B must vanish, which leads to the characteristic equation,

$$\left(I'\right)^2\sigma^4 - \left(I^2\beta_\phi^2 + I'mgL + I'I\beta_\phi\omega_e\cos\lambda\right)\sigma^2 + mgL\left(I\beta_\phi\omega_e\cos\lambda\right) = 0.$$

The constant term in this biquadratic polynomial is very small, which means that one root is large and the other is small. For practical applications, the spin rate is sufficiently large that the middle coefficient may be simplified to $-\left(I\beta_\phi\right)^2$. Then the roots σ^2 of the quadratic equation are well approximated as

$$\sigma_1^2 = \frac{mgL\omega_e\cos\lambda}{I\beta_\phi}, \quad \sigma_2^2 = \left(\frac{I}{I'}\beta_\phi\right)^2. \tag{9}$$

Only the positive square roots of σ_1^2 and σ_2^2 are required. For each frequency σ_j, there is a corresponding ratio B/A. The first of Eqs. (8) indicates that

$$B_j = \mu_j A_j, \quad \mu_j = \frac{I'\sigma_j^2 - mgL}{I\beta_\phi\sigma_j}.$$

For the assumed orders of magnitude of β_ϕ, mgL/I, and ω_e, substitution of the roots in Eqs. (9) leads to

$$\mu_1 \approx -\left(\frac{mgL}{I\beta_\phi\omega_e\cos\lambda}\right)^{1/2}, \quad \mu_2 \approx 1$$

We conclude from the foregoing that the complementary solution, which is a free vibration, occurs as either of two modes. The frequency of the first mode is σ_1, which is very low. The magnitude of μ_1 is very large, which means that the amplitude of the nutation is much smaller than that of the precession. In contrast, the frequency σ_2 of the second mode is high, and the amplitudes of the nutation and the precession are approximately equal. The most general solution is a sum of the two modes, and of the particular solution. Thus we find that the response to the disturbance is

$$\begin{aligned}\eta &= \frac{I\omega_e\dot\phi\sin\lambda}{mgL} + A_1\sin\left(\sigma_1 t - \gamma_1\right) + A_2\sin\left(\sigma_2 t - \gamma_2\right),\\ \psi &= -\left(\frac{mgL}{I\beta_\phi\omega_e\cos\lambda}\right)^{1/2} A_1\cos\left(\sigma_1 t - \gamma_1\right) + A_2\cos\left(\sigma_2 t - \gamma_2\right).\end{aligned} \quad \triangleleft$$

The actual values of the amplitudes A_j and phase angles γ_j depend on the initial conditions, which are not stated. In most actual situations, dissipation effects damp the high-frequency mode much more than the low-frequency mode, in which case the oscillation at frequency σ_1 is more persistent and therefore more likely to be observed.

In regard to the question of stability, we note that the values of σ_1 and σ_2 are always real. Hence, disturbances of the gyrocompass always result in bounded oscillations, corresponding to a stable steady motion. However, the value of σ_1 becomes very small if $\cos\lambda = 0$, corresponding to locations near the North or South Pole. Hence, at those locations the gyrocompass executes very slow oscillations when disturbed. The northerly direction is indicated by the average value of ψ, that is, zero. The low oscillation frequency at the Earth's poles requires that ψ be measured over a long interval in order to see the average direction in which the rotor is aimed.

10.3.3 Single-Axis Gyroscope

Safe operation of aircraft requires that the vehicle's angular velocity be monitored. The single-axis gyroscope, which has only an inner gimbal, provides such information because its nutation is essentially proportional to the precession rate. A conceptual model of a single-axis gyro appears in Fig. 10.15, where the platform is assumed to undergo arbitrary rotations about axes ξ, η, and ζ that are attached to the platform. These axes are defined such that the $\xi\zeta$ plane is parallel to the platform, with ξ aligned with the bearing axis of the gimbal. The xyz reference frame is attached to the gimbal. Rotation of the gimbal relative to the platform is resisted by linear torsional springs whose total stiffness is K. A torsional damper whose constant is C acts in parallel with the springs. The springs are mounted such that they are undeformed in the position where the rotor axis is parallel to the platform.

The orientation of the rotor relative to the platform is defined by the spin angle ϕ and the rotation θ of the gimbal, which we consider to be a nutation about the x axis, with the precession set to zero. Thus θ is the angle from the ζ axis to the rotor axis. According to this definition, $\theta = 0$ represents the undeformed position of the spring.

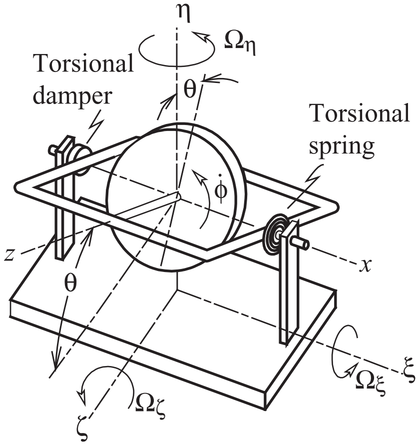


Figure 10.15. Conceptual model of a single-axis gyrosope that is mounted on a base whose angular velocity is $\bar{\Omega}$.

The angular motion of the platform is described in terms of rotation rates Ω_ξ, Ω_η, Ω_ζ about the body-fixed axes. The angular velocity of the rotor is the sum of this rotation and that of the platform, so

$$\bar{\omega} = \Omega_\xi \bar{e}_\xi + \Omega_\eta \bar{e}_\nu + \Omega_\zeta \bar{e}_\zeta + \dot{\theta}\bar{i} + \dot{\phi}\bar{k} \tag{10.3.18}$$

Let I denote the moment of inertia of the rotor about the z axis. Because of the axial symmetry, the moments of inertia of the rotor about the x and y axes are both I', regardless of the angle of spin of the rotor. Correspondingly, expressing $\bar{\omega}$ in terms of components relative to xyz will yield an expression for the kinetic energy that is descriptive of any instant. The result is

$$\bar{\omega} = \left(\Omega_\xi - \dot{\theta}\right)\bar{i} + \left(\Omega_\eta \cos\theta - \Omega_\zeta \sin\theta\right)\bar{j} + \left(\dot{\phi} + \Omega_\eta \sin\theta + \Omega_\zeta \cos\theta\right)\bar{k}. \tag{10.3.19}$$

We ignore gimbal inertia for this idealized analysis, so the corresponding kinetic-energy expression is

$$T = \frac{1}{2} I' \left[\left(\Omega_\xi - \dot{\theta}\right)^2 + \left(\Omega_\eta \cos\theta - \Omega_\zeta \sin\theta\right)^2\right] + \frac{1}{2} I \left(\dot{\phi} + \Omega_\eta \sin\theta + \Omega_\zeta \cos\theta\right)^2. \tag{10.3.20}$$

Our interest is how the platform's rotation affects θ, so we consider $\dot{\phi}$ to be a specified constant value. (Considering ϕ to be another generalized coordinate would yield an equation of motion that would either describe the torque that a servomotor must apply to hold $\dot{\phi}$ constant, or else an equation describing conservation of momentum. We will see that the spin rate should be much higher than the rotation rates of the platform, which makes the distinction between the two spin cases unimportant.) The position where the nutation angle is zero corresponds to the unstretched position of the torsional spring, so the potential energy is

$$V = \frac{1}{2} K\theta^2. \tag{10.3.21}$$

The dashpot exerts a torque $C\dot{\theta}$ in opposition to the rotation, so the virtual work is

$$\delta W = -c\dot{\theta}\delta\theta. \tag{10.3.22}$$

The corresponding Lagrange equation for θ is

$$\begin{aligned} &I'\left(\ddot{\theta} - \dot{\Omega}_\xi\right) + I'\left(\Omega_\eta \cos\theta - \Omega_\zeta \sin\theta\right)\left(\Omega_\eta \sin\theta + \Omega_\zeta \cos\theta\right) \\ &\quad - I\left(\dot{\phi} + \Omega_\eta \sin\theta + \Omega_\zeta \cos\theta\right)\left(\Omega_\eta \cos\theta - \Omega_\zeta \sin\theta\right) + K\theta = -C\dot{\theta}. \end{aligned} \tag{10.3.23}$$

The rotor in a practical single-axis gyroscope is made to spin much more rapidly than the highest anticipated rate of rotation of the platform. Also, the stiffness and damping parameters are usually selected to restrict the nutation angle to a small magnitude. Under these assumptions, trigonometric terms may be simplified by the small-angle approximation, and also by neglecting terms that are products of the platform's rotation rates. The equation of motion then reduces to

$$I'\ddot{\theta} + C\dot{\theta} + K\theta \approx I\dot{\phi}\Omega_\eta \quad \text{if} \quad \dot{\phi} \gg \Omega_\eta, \Omega_\zeta, \dot{\Omega}_\xi / \Omega_\eta. \tag{10.3.24}$$

This is the equation of motion for a damped, one-degree-of-freedom linear oscillator. Its natural frequency ω_{nat} and ratio of critical damping σ are

$$\omega_{\text{nat}} = \left(\frac{K}{I'}\right)^{1/2}, \quad \sigma = \frac{C}{2\,(KI')^{1/2}}. \tag{10.3.25}$$

Let us begin by evaluating the nutation when Ω_η is a constant, nonzero value. We may obtain the corresponding response by adding the complementary and particular solutions. In the absence of rotations of the platform, the gimbal will be at rest at its equilibrium position $\theta = 0$, so we set $\theta = \dot{\theta} = 0$ when $t = 0$ as initial conditions. If the system is underdamped, $\sigma < 1$, the corresponding response is

$$\theta = \frac{I\dot{\phi}\Omega_\eta}{K}\left\{1 - \exp\left(-\sigma\omega t\right)\left[\cos\left(\omega_d t\right) + \frac{\sigma}{\left(1-\sigma^2\right)^{1/2}}\sin\left(\omega_d t\right)\right]\right\}, \tag{10.3.26}$$

where $\omega_d = \omega\left(1-\sigma^2\right)^{1/2}$ is the damped natural frequency. This expression indicates that when $t > 4/\sigma\omega$ the nutation angle differs by less that 2% from a constant steady value that is proportional to the rate at which the platform is rotating about the η axis,

$$\theta \to \frac{I\Omega_\eta}{K}\dot{\phi}. \tag{10.3.27}$$

Thus the nutation angle may be measured and compared with a scale that is calibrated in units of the rotation rate Ω_η. Significantly, the foregoing steady-state response would also be obtained if Ω_η were time dependent, provided that the free-vibration response decays in a much smaller time than the interval required for observing substantial changes in Ω_η. This condition may be achieved by designing the system to have a high natural frequency and to be highly damped, subject to $\sigma < 1$. This calls for a single-axis gyro to be constructed with a spring and a dashpot that are both stiff. Such a device is called a *rate gyroscope*. Because a rate gyroscope indicates the rotation about only one axis, three, mounted about orthogonal axes, are employed to measure the angular velocity of flight vehicles.

There is an alternative configuration for a single-axis gyro that has been employed frequently. Suppose the torsional spring is not present. Setting $K = 0$ in Eq. (10.3.24) leads to

$$I'\ddot{\theta} + C\dot{\theta} \approx I\dot{\phi}\Omega_\eta. \tag{10.3.28}$$

A solution valid for arbitrary Ω_η, not necessarily constant, consists of a convolution integral that may be derived either from a Laplace transform or from a Duhamel integral by use of the impulse response of a second-order linear oscillator that has no spring. The result is

$$\theta = \frac{I\dot{\phi}}{C}\int_0^t \Omega_\eta\left(\tau\right)\left\{1 - \exp\left[-\frac{C}{I'}\left(t-\tau\right)\right]\right\}d\tau. \tag{10.3.29}$$

It is desirable that the damping rate be large, in order to make the exponential term in the integrand quickly decay. Then, after an initial start-up interval, the nutation angle will be well approximated by

$$\theta = \frac{I\dot{\phi}}{C}\int_0^t \Omega_\eta(\tau)\,d\tau. \tag{10.3.30}$$

The nutation angle in this case is proportional to the integral of the rotation rate about the η axis, which represents the cumulative rotation. For this reason, a well-damped single-axis gyro that is not restrained by a spring is called an *integrating gyroscope*. As is true for rate gyroscopes, a complete guidance system would require three integrating gyroscopes whose nutation axes are aligned with mutually orthogonal axes.

A common application is to use three rate gyros to measure the rotation rates about yaw, pitch, and roll axes of a flight vehicle. This arrangement is referred to as a *strapdown gyroscope*, because it is directly mounted on the vehicle. The rotation rates that are measured in this manner are about a set of body-fixed axes, so the system yields the instantaneous absolute angular velocity of the vehicle. Furthermore, integration of the rates, which can be done electronically, gives the angles of rotation about the body-fixed axes. Using these angles in conjunction with the rotation transformation concepts enables one to determine the orientation of the vehicle's axes relative to an inertial reference frame.

As a closing note, it is important that one recognize that this discussion of inertial guidance systems has been drastically simplified, both through the models that were created and the assumptions used to obtain responses. For example, we generally idealized systems by neglecting the inertia of the gimbals. In some cases this merely affects oscillation frequencies. However, the additional inertial resistance can lead to qualitative differences. Such is the case for a free gyroscope that is subjected to a small disturbance. The inertia of the outer gimbal can cause a precession that drifts away from the initial orientation, rather than merely oscillating about it. In regard to the analysis of responses, linearization often avoids some important questions, such as loss of dynamic stability through nonlinear mechanisms. Practical usage of the gyroscope as a tool for navigation over long ranges requires more sophisticated analyses, accounting for gimbal inertia and bearing friction, than those presented here. However, the features of such investigations would show many similarities to the steps we have pursued.

EXAMPLE 10.7 An airplane initially in level flight executes a body-fixed rotation about an axis that lies in the $\eta\zeta$ plane in Fig. 10.15, at angle γ from the ζ axis. The rotation rate Ω about this axis is a sinusoidal pulse over a time interval τ: $\Omega = \Omega_0 \sin(\pi t/\tau)$ for $0 \le t < \tau$, $\Omega = 0$ for $t > \tau$. The rotor of a rate gyro was spinning in its reference position $\theta = 0$ when the aircraft began its rotation. Determine the nutational response $\theta(t)$ of the gyroscope for the case in which damping is less than critical. From that solution determine the conditions for which the value of $K\theta/I\beta_\phi$ closely matches nominal response in Eq. (10.3.27).

SOLUTION This example is intended to shed light on some of the design issues that one must confront to make a practical rate gyro. Because the given rotation is specified as being about a body-fixed axis, the angular velocity components are found by projections onto the $\xi\eta\zeta$ axes, so that

$$\Omega_\xi = 0, \quad \Omega_\eta = \Omega \sin\gamma, \quad \Omega_\zeta = \Omega\cos\gamma.$$

The response we seek is the solution to Eq. (10.3.24) for this rotation of the base, subject to the initial conditions that $\theta = \dot{\theta} = 0$ when $t = 0$.

Several methods are available for determining this response. We exploit the similarity of the problem to that encountered in conventional transient vibrations. For the given angular velocity components, the conditions $\dot{\phi} \gg \Omega_\eta$, Ω_ζ, and $\dot{\Omega}_\xi/\Omega_\eta$ required for employing Eq. (10.3.24) are satisfied if $\beta_\phi \gg \Omega$ and $\beta_\phi \gg (\pi/\tau)\tan\gamma$. (When $\gamma = \pm\pi/2$, the rotation is essentially about the ζ axis. Such a rotation would presumably be measured by a rate gyro that is arranged orthogonally to the one under consideration.) We assume that β_ϕ meets these conditions. Substitution of the given functional form of Ω for $t < \tau$ then leads to

$$I'\ddot{\theta} + C\dot{\theta} + K\theta = I\beta_\phi \Omega_0 \sin\gamma \sin\left(\frac{\pi t}{\tau}\right).$$

This represents a one-degree-of-freedom system having natural frequency $\omega_{\text{nat}} = (K/I')^{1/2}$ and ratio of critical damping $\sigma = 0.5C/(I'K)^{1/2}$ that is subjected to a sinusoidal excitation at frequency π/τ. The response is the sum of the complementary solution and the particular solution. The latter, which is known as the steady-state response in vibration theory, consists of the quasi-static response, which is the amplitude that would be obtained if the system only had stiffness K, multiplied by a frequency response factor and delayed by a phase lag δ. For the parameters of the present system the steady-state response is

$$\theta_{\text{ss}} = \Theta \sin\left(\frac{\pi t}{\tau} - \delta\right), \quad \Theta = \frac{1}{D}\frac{I}{I'}\frac{\beta_\phi}{\omega_{\text{nat}}}\frac{\Omega_0}{\omega_{\text{nat}}}\sin\gamma,$$

$$D = \left\{\left[1 - \left(\frac{\pi}{\omega_{\text{nat}}\tau}\right)^2\right]^2 + 4\sigma^2\left(\frac{\pi}{\omega_{\text{nat}}\tau}\right)^2\right\}^{1/2}, \tag{1}$$

$$\delta = \tan^{-1}\left[\frac{2\sigma\pi/\omega_{\text{nat}}\tau}{1 - (\pi/\omega_{\text{nat}}\tau)^2}\right], \quad 0 \le \delta \le \pi.$$

The initial conditions must be satisfied by the combination of the particular and complementary solutions. Because the damping is stated to be less than critical, $\sigma < 1$, the complementary solution, whose form is the same as that of the free-vibration response, is oscillatory with an overall amplitude that decays exponentially in time, according to

$$\theta_c = \exp(-\sigma\omega_{\text{nat}}t)\left[A\sin(\omega_{\text{d}}t) + B\cos(\omega_{\text{d}}t)\right], \quad \omega_{\text{d}} = \left(1 - \sigma^2\right)^{1/2}\omega_{\text{nat}}. \tag{2}$$

Satisfaction of the initial conditions requires that $\theta_{\text{ss}} + \theta_c = 0$ and $\dot{\theta}_{\text{ss}} + \dot{\theta}_c = 0$ at $t = 0$. These represent two equations for A and B. The corresponding response is

$$\theta = \Theta \sin\left(\frac{\pi t}{\tau} - \delta\right) + \Theta \exp\left(-\sigma\omega_{\text{nat}} t\right)\left[\frac{(\sigma\omega_{\text{nat}}\tau)\sin\delta - \pi\cos\delta}{(1-\sigma^2)^{1/2}\,\omega_{\text{nat}}\tau}\sin\left(\omega_{\text{d}} t\right) + \sin\delta\cos\left(\omega_{\text{d}} t\right)\right], \quad 0 \le t \le \tau. \tag{3} \triangleleft$$

The airplane's rotation ceases at $t = \tau$, after which the rotor executes a free vibration. Equation (2) describes such a response if we change t to the elapsed time measured from the instant when the rotation ceases, which is $t' = t - \tau$. The initial conditions for this response are the values of θ and $\dot{\theta}$ obtained from Eq. (3) at the instant when the rotation ended. Let us denote these values as θ_τ and $\dot{\theta}_\tau$. Then the free response is given by

$$\theta = \exp\left[-\sigma\omega_{\text{nat}}\left(t-\tau\right)\right]\left\{\frac{\dot{\theta}_\tau + \sigma\omega_{\text{nat}}\theta_\tau}{\omega_{\text{d}}}\sin\left[\omega_{\text{d}}\left(t-\tau\right)\right] + \theta_\tau\cos\left[\omega_{\text{d}}\left(t-\tau\right)\right]\right\}, \quad t \ge \tau. \tag{4} \triangleleft$$

Equations (3) and (4) describe the response for any set of system parameters for which $\sigma < 1$. We desire that both expressions match Eq. (10.3.27). The rotation rate is given, and the stiffness parameter may be written as $K = I'\omega_{\text{nat}}^2$, so the desired nutation is described by

$$\theta_{\text{ideal}} = \frac{I\Omega_\eta}{K}\dot{\phi} = \begin{cases} \dfrac{I}{I'}\dfrac{\dot{\phi}}{\omega_{\text{nat}}}\dfrac{\Omega_0}{\omega_{\text{nat}}}\sin\gamma\sin\left(\dfrac{\pi t}{\tau}\right) \text{ if } 0 \le t \le \tau \\ 0 \text{ if } t > \tau \end{cases}. \tag{5}$$

In general, the spin rate will be much larger than the rate at which the aircraft rotates, so $\dot{\phi} \ggg \Omega_0$, which means that $\beta_\phi \approx \dot{\phi}$. When this condition applies, Θ, which is defined in Eqs. (1), is the coefficient appearing in θ_{ideal} multiplied by the frequency response factor $1/D$. Thus comparing Eq. (3) to θ_{ideal} in Eq. (5) for $t < \tau$ shows that the ideal response will be obtained if D is as close to unity as possible and δ is very small. In addition $\sigma\omega_{\text{nat}}$ should be very large, so that the exponential factor quickly decays. Furthermore, if $\sigma\omega_{\text{nat}}$ is very large, then Eq. (4) indicates that θ will decay to zero soon after $t = \tau$, which matches the ideal response in Eq. (5) for $t > \tau$.

Because $\sigma < 1$, the condition of large $\sigma\omega_{\text{nat}}$ requires that ω_{nat} itself be large. In addition, the analysis began with the requirements that $\dot{\phi} \gg \Omega_\eta$, Ω_ζ and $\dot{\Omega}_\xi/\Omega_\eta$. The first two conditions are met when the spin rate is sufficiently high to warrant approximating β_ϕ as $\dot{\phi}$. The third condition requires $\dot{\phi} \gg \pi/\tau$.

Fortunately, simultaneous satisfaction of each condition is attained if the natural frequency ω_{nat} is very high, the damping ratio σ is as large as possible, subject to the condition that $\sigma < 1$, and the spin rate is much higher than both the airplane's peak angular velocity and the fluctuation rate π/τ of that angular velocity. These requirements are not difficult to meet, because the spin, roll, and yaw motions of even a very high-performance aircraft are moderate from a mechanical standpoint. For example, a very violent maneuver might consist of several rolls in a few seconds, for which τ might be of the order of 0.2 s. In contrast, a natural frequency of 1000 rad/s and a spin rate of 25 000 rev/min are readily obtained.

It is evident from Eq. (5) that the rate gyro becomes less sensitive as the rotation axis becomes increasingly close to the spin axis, corresponding to γ decreasing. For this reason, three rate gyros arranged orthogonally are required for measuring an arbitrary spatial rotation.

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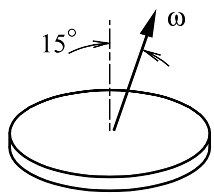
HOMEWORK PROBLEMS

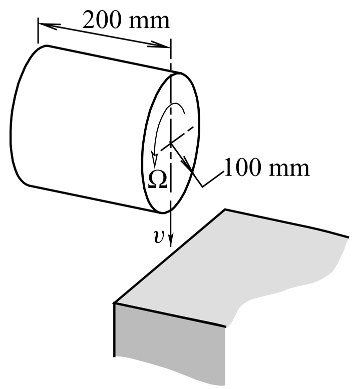
EXERCISE 10.1 An axially symmetric Earth satellite, whose ratio of principal moments of inertia is $I/I' = 1.6$, precesses about its axis once every 2 s. The spin rate in this state is 0.1 rad/s. Determine the overall rate of rotation and the angle from the axis of symmetry to the precession axis. Then determine the minimum angular impulse that a set of control rockets fastened to the satellite must exert in order to bring the precession axis into coincidence with the axis of symmetry. What is the rotation rate of the satellite at the conclusion of such a maneuver? (Assume that the rockets act impulsively.)

EXERCISE 10.2 A coin is tossed into the air with plane horizontal and its initial angular velocity at 15° off-vertical, as shown in the sketch. Construct the space and body cones at the instant of release, and also evaluate the precession and spin rates.



Exercise 10.2

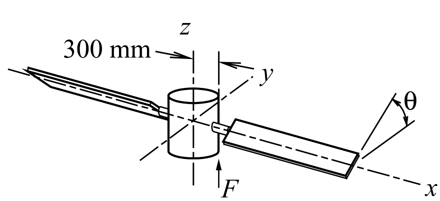
EXERCISE 10.3 The cylinder, whose mass is 20 kg, translates downward at $v = 40$ m/s with its axis of symmetry horizontal. The spin rate about that axis is $\Omega = 50$ rad/s. Its circular edge collides with the ledge, with the result that its center of mass has a downward velocity of 10 m/s immediately after the collision. Friction between the cylinder and the ledge is negligible. Describe the rotational motion of the cylinder after impact.



Exercise 10.3

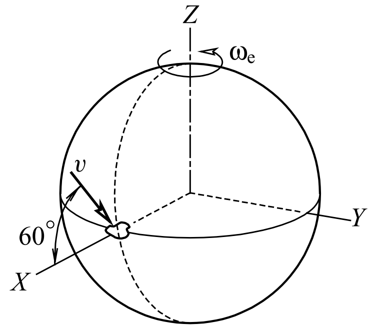
EXERCISE 10.4 Prove that the polhode description of free motion for an arbitrary body reduces to the space and body cone analogy when the body is axisymmetric.

EXERCISE 10.5 The communications satellite was precessing steadily about its z axis at 2 rad/s when the steering thruster was turned on, resulting in the application of an average thrust force of 20 kN for a 100 ms interval. The inertia properties relative to the principal xyz axes are $I_{xx} = 750$, $I_{yy} = I_{zz} = 1500$ kg-m^2. (From the viewpoint of inertia the satellite may be considered to be axisymmetric relative to the x axis.) Determine the orientation of the satellite's angular momentum when the thruster was turned off. Then determine the nutation angle, and the precession and spin rates for the subsequent free motion.



Exercise 10.5

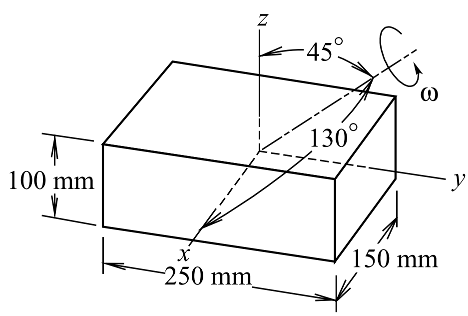
EXERCISE 10.6 Consider a model of the Earth in which its only motion is its daily spin relative to the inertial XYZ reference frame. The diagram depicts the impact of a meteorite at the Earth's equator. The velocity is $v = 21\,000$ km/h at 60° from the radial line to the center in the meridional plane. This meteorite becomes embedded impulsively in the Earth, which disturbs the Earth's steady rotation. Describe the rotational motion of the Earth that results from the collision. The meteorite's mass is 0.01% of the Earth's mass, and it may be treated as a particle.



Exercise 10.6

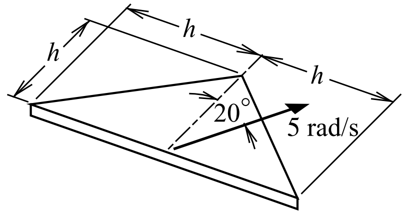
EXERCISE 10.7 Consider the coin in Exercise 10.2. Describe the location of the invariable plane relative to the coin. Graph the projections of the polhode curve onto the principal axis planes, and show in each graph the intersection of the coordinate plane with the inertia ellipsoid. Assess these projections in light of the space and body cone construction for axisymmetric bodies.

EXERCISE 10.8 The angular velocity of a wooden block at the instant it is released is as shown. Which body-fixed axis is surrounded by the polhode curve for the free rotation? What are the maximum and minimum angles between this axis and the constant direction of the angular momentum? What are the angular velocities of the block at these maximum and minimum conditions?



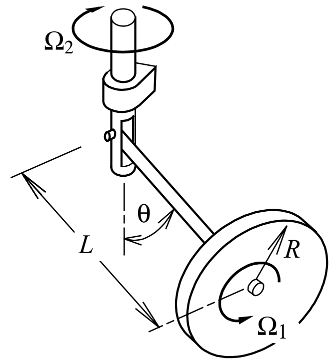
Exercise 10.8

EXERCISE 10.9 The sketch shows the initial angular velocity with which a plate in the shape of an isosceles triangle is thrown. Based on the assumption that air resistance is negligible, determine the location of the invariable plane relative to the plate. Then draw the projection of the polhode curve for this motion onto the principal axis planes. Also show in each graph the intersection of the inertia ellipsoid with the each plane.



Exercise 10.9

EXERCISE 10.10 The thin disk, whose mass is m_1, spins relative to shaft AB, whose mass is m_2. Both the spin rate Ω_1 and precession rate Ω_2 are held constant, and the pin connection A has ideal properties. The vertical orientation, $\theta = 0$, is like a sleeping top. Evaluate the stability of a steady precession in this position as a function of Ω_1, Ω_2, and the length ratio R/L.

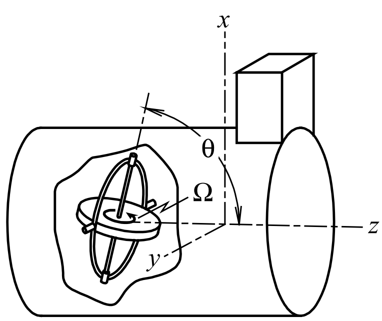


Exercise 10.10

EXERCISE 10.11 A free gyroscope is in a state of slow, steady precession at a nutation angle of 53.13° and a spin rate at 10 000 rev/min. The rotor's mass is 5 kg, its radii of gyration about its pivot are $\kappa = 100$ mm, $\kappa' = 180$ mm, and its center of mass is 120 mm from the pivot. A person accidentally touches the outer gimbal, thereby causing the precession rate to decrease suddenly by 0.6 rad/s. Determine whether the ensuing motion is unidirectional, looping, cuspidial, or steady precession. What are the maximum and minimum nutation angles in that motion?

EXERCISE 10.12 A free symmetric gyroscope initially in a state of steady slow precession is subjected to a small disturbing torque $\varepsilon mgL \sin(\Omega t)$ acting about the fixed vertical shaft supporting the outer gimbal. Use a perturbation analysis for $\varepsilon \ll 1$ to determine the frequency, if any, at which the system resonates.

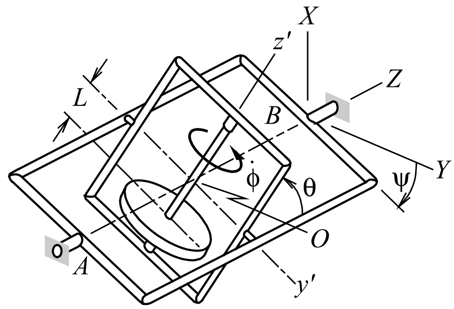
EXERCISE 10.13 The diagram shows a single-axis gyroscope that is used to aim a satellite. The gimbal's axis is coincident with the y axis of the satellite, and the flywheel spins at constant speed Ω relative to the gimbal. The principal moments of inertia of the satellite are arbitrary values I_{xx}, I_{yy}, and I_{zz} about the coordinate axes in the sketch, and the moments of inertia of the flywheel are I about the spin axis, and I' about the gimbal axis. The gimbal's mass is negligible, and the center of the flywheel is coincident with the satellite's center of mass. Because there are no external forces acting on this system and $\theta = 0$ initially, the angular momentum of the system is the constant at $H = I\Omega$ to the right in the sketch. A servomotor mounted to the satellite's body imparts a specified change to the angle θ of the spin axis, which causes the orientations of both the satellite and the gimbal to change. Derive equations of motion describing this process. *Hint:* Let $x'y'z'$ be a coordinate system attached to the gimbal.



Exercise 10.13

Then the rotation of the gimbal consists of precession ψ about the direction indicated by $\bar{H}$ and a nutation θ_g about the gimbal axis. Correspondingly, the rotation of the satellite relative to the gimbal is θ in the opposite sense from θ_g and the rotation of the flywheel relative to the gimbal is ϕ about the spin axis, with $\dot{\phi} = \Omega$.

EXERCISE 10.14 The device shown is a gyropendulum, which has been used in some applications to locate the vertical direction. The spin rate $\dot{\phi}$ is held constant by a servomotor. Let m be the mass, and let I and I' be the centroidal principal moments of inertia of the flywheel parallel and transverse to the spin axis; ignore the inertia of the gimbals. Evaluate the nutation response $\theta(t)$ and the precessional response $\psi(t)$ of the flywheel to a disturbance that causes it to rotate by very small angles away from the vertical reference position, at which $\theta = \pi/2$. Compare the frequency of these responses with that of a simple pendulum, and use that result to discuss an advantage of the gyropendulum.



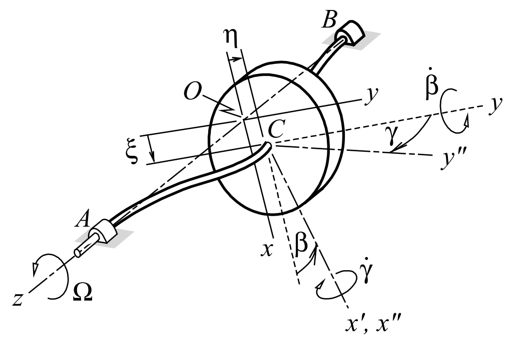
Exercise 10.14

EXERCISE 10.15 Consider the gyropendulum in Exercise 10.14. Because of movement of the vehicle, the center point O has a constant acceleration $\dot{v}$ directed parallel to the axis of the outer gimbal (that is, $\bar{a}_O = \dot{v}\bar{e}_{B/A}$). Let this acceleration be directed at angle β north of east. Derive equations of motion for the Eulerian angles including the effect of the Earth's rotation, and of the movement of the vehicle in a great circle at angular speed v/R_e.

EXERCISE 10.16 The platform of an integrating gyroscope rotates about the η axis in a time-dependent manner. Consider an angular speed that consists of an average value Ω_0, over which is superposed a harmonic fluctuation at amplitude Ω_1 and frequency λ, that is, $\Omega_\eta = \Omega_0 + \Omega_1 \sin(\lambda t)$. What conditions must be true if the nutational response $\theta(t)$ following the initial transient phase is to be proportional to the mean rotation Ω_0?

EXERCISE 10.17 A top is initially in a state of steady precession at a precession rate $\dot{\psi}^*$ and a nutation angle θ^*. Application of an impulsive force in the direction of the precession suddenly changes the precession rate by the amount $\varepsilon\dot{\psi}^*$, where $|\varepsilon| \ll 1$. It is desired to determine the precessional and nutational responses after cessation of the impulsive force. Toward that end, perform a perturbation analysis of the basic motion equations for a top, Eqs. (10.2.4) and (10.2.6), with $\dot{\psi} = \dot{\psi}^* + \varepsilon\dot{\psi}_1$, $\theta = \theta^* + \varepsilon\theta_1$, and $\dot{\phi} = \dot{\phi}^* + \varepsilon\dot{\phi}_1$, where the quantities having a subscript "1" are not large.

EXERCISE 10.18 Whirling is a phenomenon in turbomachinery in which a rotating shaft undergoes displacement as a beam, with the plane in which the displacement occurs rotating at an angular speed that might differ from the shaft speed. To study this effect, consider the shaft supporting the disk of mass m and radius R to be flexible in bending, rigid in extension and torsion, and massless. Thus the shaft acts like a set of springs that exert forces in opposition to displacement of the disk's center C, and couples in opposition to the disk's rotation. The movement of the disk is described by use of several moving coordinate systems. First, there is xyz, which rotates at the shaft speed Ω with its z axis coincident with the centerline between bearings A and B. The origin O of xyz is situated at the location where center C would reside if the shaft were rigid. Coordinate system $x'y'z'$ has center C as its origin, with its y' axis always parallel to the y axis. The y' axis is the line of nodes for the nutational rotation β of $x'y'z'$ relative to xyz. The third coordinate system, which is $x''y''z''$ in the sketch, is attached to the disk with its origin also at center C. The rotation of this coordinate system relative to $x'y'z'$ is γ about the x'' axis, which is defined to always coincide with the x' axis. (This is an alternative definition of Eulerian angles that also is used in aerospace applications.) The displacement of the center C is $\xi\bar{i} + \eta\bar{j}$ and the angular velocity is $\Omega\bar{k} + \dot{\beta}\bar{j}' + \dot{\gamma}\bar{j}''$. In a properly designed system the displacements will be very small compared with the span between bearings and the rotation angles will be sufficiently small to take them to equal their sine. For a shaft of symmetric cross section, each deformation is resisted solely by a corresponding proportional elastic force or moment. Thus the elastic effect of the shaft may be represented by a restoring force $-k_\xi\xi\bar{i} - k_\eta\eta\bar{j}$ applied to the disk at point C and a torque $-\kappa_\gamma\gamma\bar{i}'' - \kappa_\beta\beta\bar{j}''$, where the small-angle approximation makes it permissible to take $\bar{j}'' = j'$. Derive the corresponding linearized equations of motion.



Exercise 10.18

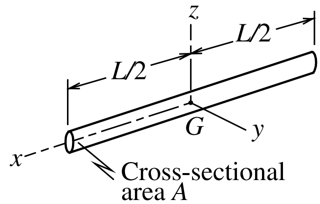
EXERCISE 10.19 Consider the effects of the inertia of the gimbals in a balanced free gyroscope. Let A, B, and C denote the (principal) moments of inertia of the inner gimbal about the $x'y'z'$ axes, where y' is the line of nodes and z' is the spin axis of the rotor. Also, let A' denote the moment of inertia of the outer gimbal about the precession axis. Derive the equations of motion for the gyroscope in this case.

EXERCISE 10.20 Suppose that the gyroscope in Exercise 10.19 is initially spinning at $\dot{\phi}$ and the nutation angle is constant at θ_0; there is no precession in this initial motion. At $t = 0$ a nutational velocity $\varepsilon \ll \dot{\phi}$ is imparted to the inner gimbal. Use a perturbation analysis in which $\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2$ and $\dot{\psi} = \varepsilon\dot{\psi}_1 + \varepsilon^2\dot{\psi}_2$ to determine the nutational and precessional fluctuations induced by the disturbance. Show that, because of gimbal inertia, the response exhibits *gimbal walk*, in which there is a nonzero average precessional rotation rate, even though the gyroscope is balanced.

Appendix

CENTROIDAL INERTIA PROPERTIES*

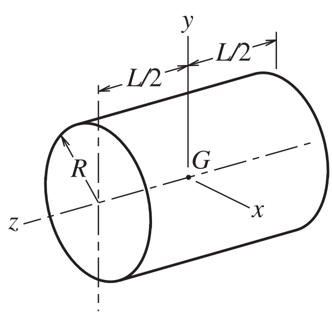
Slender bar



$$m = \rho A L,$$

$$I_{xx} \approx 0, \quad I_{yy} = I_{zz} = \frac{1}{12} m L^2.$$

Cylinder

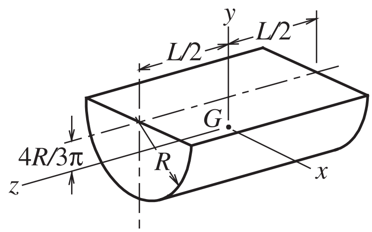


$$m = \pi \rho R^2 L,$$

$$I_{xx} = I_{yy} = \frac{1}{12} m \left(3R^2 + L^2\right), \quad I_{zz} = \frac{1}{2} m R^2.$$

* Products of inertia that are not listed are zero.

Semicylinder

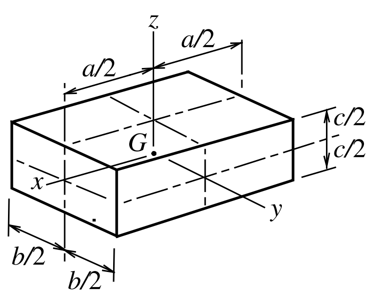


$$m = \frac{1}{2}\pi \rho R^2 L,$$

$$I_{xx} = \left(\frac{1}{4} - \frac{16}{9\pi^2}\right) m R^2 + \frac{1}{12} m L^2,$$

$$I_{yy} = \frac{m}{12}\left(3R^2 + L^2\right), \quad I_{zz} = \left(\frac{1}{2} - \frac{16}{9\pi^2}\right) m R^2.$$

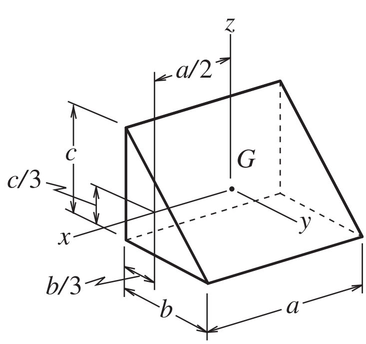
Rectangular parallelepiped



$$m = \rho abc,$$

$$I_{xx} = \frac{1}{12} m\left(b^2 + c^2\right), \quad I_{yy} = \frac{1}{12} m\left(a^2 + c^2\right), \quad I_{zz} = \frac{1}{12} m\left(a^2 + b^2\right).$$

Right prism

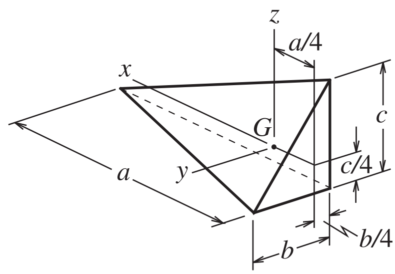


$$m = \frac{1}{2}\rho abc,$$

$$I_{xx} = \frac{1}{18}m\left(b^2 + c^2\right), \quad I_{yy} = \frac{1}{36}m\left(3a^2 + 2c^2\right), \quad I_{zz} = \frac{1}{36}m\left(3a^2 + 2b^2\right),$$

$$I_{yz} = -\frac{1}{36}mbc,$$

Right tetrahedron

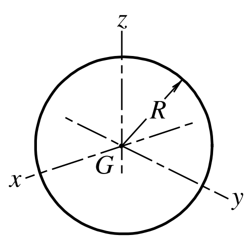


$$m = \frac{1}{6}\rho abc,$$

$$I_{xx} = \frac{3}{80}m\left(b^2 + c^2\right), \quad I_{yy} = \frac{3}{80}m\left(a^2 + c^2\right), \quad I_{zz} = \frac{3}{80}m\left(a^2 + b^2\right),$$

$$I_{xy} = -\frac{1}{80}mab, \quad I_{yz} = -\frac{1}{80}mbc, \quad I_{xz} = -\frac{1}{80}mac.$$

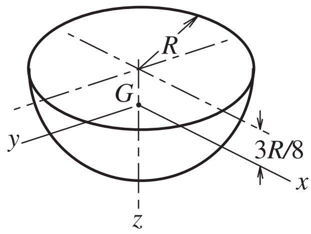
Sphere



$$m = \frac{4}{3}\pi\rho R^3,$$

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mR^2.$$

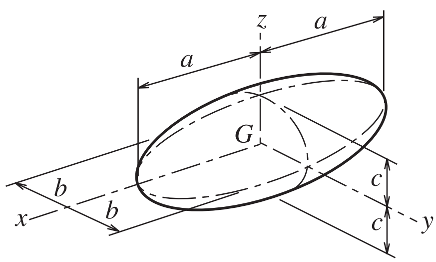
Hemisphere



$$m = \frac{2}{3}\pi \rho R^3,$$

$$I_{xx} = I_{yy} = \frac{83}{320} m R^2, \quad I_{zz} = \frac{2}{5} m R^2.$$

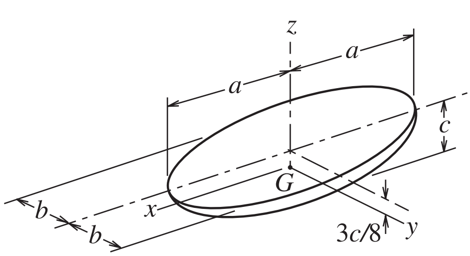
Ellipsoid



$$m = \frac{4}{3}\pi \rho abc,$$

$$I_{xx} = \frac{1}{5} m \left(b^2 + c^2\right), \quad I_{yy} = \frac{1}{5} m \left(a^2 + c^2\right), \quad I_{zz} = \frac{1}{5} m \left(a^2 + b^2\right).$$

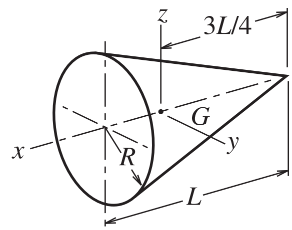
Semiellipsoid



$$m = \frac{2}{3}\pi \rho abc,$$

$$I_{xx} = m\left(\frac{1}{5} b^2 + \frac{19}{320} c^2\right), \quad I_{yy} = m\left(\frac{1}{5} a^2 + \frac{19}{320} c^2\right), \quad I_{zz} = \frac{1}{5} m \left(a^2 + b^2\right).$$

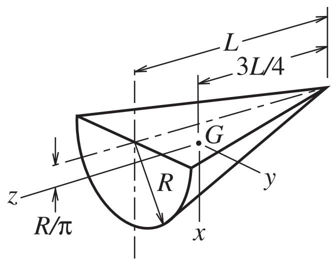
Right circular cone



$$m = \frac{1}{3}\pi\rho R^2 L,$$

$$I_{xx} = \frac{3}{10} m R^2, \quad I_{yy} = I_{zz} = \frac{3}{80} m\left(4R^2 + L^2\right).$$

Semicone

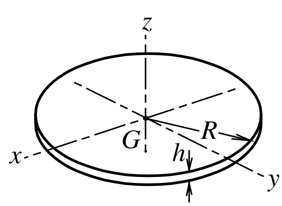


$$m = \frac{1}{6}\pi\rho R^2 L,$$

$$I_{xx} = \frac{3}{80} m\left(4R^2 + L^2\right), \quad I_{yy} = \left(\frac{3}{20} - \frac{1}{\pi^2}\right) m R^2 + \frac{3}{80} m L^2, \quad I_{zz} = \left(\frac{3}{10} - \frac{1}{\pi^2}\right) m R^2,$$

$$I_{xz} = \frac{1}{20\pi} m R L.$$

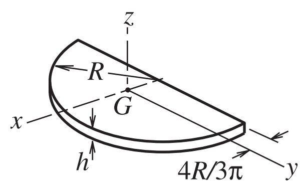
Thin disk ($h \ll R$)



$$m = \pi\rho R^2 h,$$

$$I_{xx} = I_{yy} = \frac{1}{4} m R^2, \quad I_{zz} = \frac{1}{2} m R^2.$$

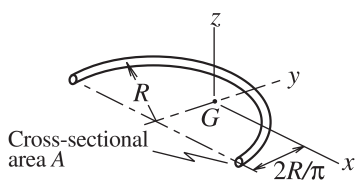
Semicircular plate ($h \ll R$)



$$m = \frac{1}{2}\pi\rho R^2 h,$$

$$I_{xx} = \frac{1}{4}mR^2, \quad I_{yy} = \left(\frac{1}{4} - \frac{16}{9\pi^2}\right)mR^2, \quad I_{zz} = I_{xx} + I_{yy}.$$

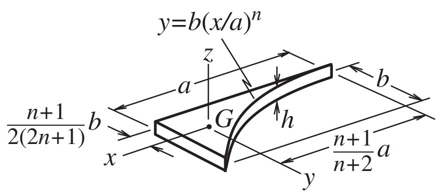
Semicircular wire arc



$$m = \pi\rho A R,$$

$$I_{xx} = \left(\frac{1}{2} - \frac{4}{\pi^2}\right)mR^2, \quad I_{yy} = \frac{1}{2}mR^2, \quad I_{zz} = I_{xx} + I_{yy}.$$

Monomial sector plate ($h \ll a, b$)



$$m = \frac{1}{n+1}\rho hab,$$

$$I_{xx} = \left[\frac{n+1}{3(3n+1)} - \frac{(n+1)^2}{4(2n+1)^2}\right]mb^2, \quad I_{yy} = \left[\frac{n+1}{n+3} - \frac{(n+1)^2}{(n+2)^2}\right]ma^2,$$

$$I_{zz} = I_{xx} + I_{yy}, \quad I_{xy} = \left[\frac{1}{4} - \frac{(n+1)^2}{(2n+1)(n+2)}\right]mab.$$

Answers to Selected Homework Problems

1.1 $\bar{F} = 3009\bar{i} - 3492\bar{j} + 1937\bar{k}$ N, $\bar{M}_A = 7749\bar{i} - 12034\bar{k}$, $M_{\text{shaft}} = -11624$ N-m.

1.4 $c_1 = -143.39$, $c_2 = -0.534501$, $c_3 = 440.334$ rad/s.

1.6 $F^1 = 336.8$, $F^2 = 386.2$ N.

1.8 $\bar{v}_P = [\varepsilon\alpha\cos(\alpha t)\cos\theta - R\alpha t\sin\theta]\,\bar{i} + [\varepsilon\alpha\cos(\beta t)\sin\theta + R\alpha t\cos\theta]\,\bar{j}$,

$$\theta = \frac{1}{2}\alpha t^2, \quad \bar{v}_{\parallel} = \varepsilon\alpha\cos(\alpha t), \quad \bar{v}_{\perp} = R\alpha t.$$

1.10
$$\bar{v} = \left[\dot{x} + L_1\dot{\theta}_1\cos\theta_1 + L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\cos(\theta_1 + \theta_2)\right]\bar{i} - \left[L_1\dot{\theta}_1\sin\theta_1 + L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\sin(\theta_1 + \theta_2)\right]\bar{j},$$

where $\dot{x} = 1000\cos(50t)$ mm/s, $\dot{\theta}_1 = -10\pi\sin(50t)$ rad/s,

$\dot{\theta}_2 = 10\pi\cos(50t - \pi/3)$ rad/s.

2.1 $\bar{v} = [2gR\sin(s/R)]^{1/2}\,\bar{e}_t$, $\bar{a} = g\cos(s/R)\,\bar{e}_t + 2g\sin(s/R)\,\bar{e}_n$.

2.3 (a) $s = s_0/4$; (b) $s = s_0/2$; (c) $R = 2s_0/\pi$; (d) $\sin(\pi s/s_0) = (2/3)^{1/2}$.

2.6 $\bar{v} = \left(0.1451\bar{i} - 0.4785\bar{j}\right)\beta$, $\bar{a} = \left(0.105280\bar{i} - 0.347154\bar{j}\right)\left(\beta^2/k\right)$.

2.9
$$\bar{v} = v_0(1 - k\beta\sinh\eta)\left(\frac{\bar{i} + \sinh\eta\,\bar{j}}{\cosh\eta}\right),$$

$$\bar{a} = -v_0^2 k(1 - k\beta\sinh\eta)\left(\frac{\bar{i} + \sinh\eta\,\bar{j}}{\cosh\eta}\right) + \frac{v_0^2}{\beta}(1 - k\beta\sinh\eta)^2\left[\frac{-\sinh\eta\,\bar{i} + \bar{j}}{(\cosh\eta)^3}\right].$$

2.11 $\bar{e}_t = -0.4804\bar{i} + 0.27740\bar{j} + 0.8321\bar{k}$, $\bar{e}_n = -0.5160\bar{i} + 0.6769\bar{j} - 0.5241\bar{k}$,

$\bar{e}_b = -0.7086\bar{i} - 0.6818\bar{j} - 0.1818\bar{k}$, $\rho = 6.392$ m, $\tau = 40.33$ m.

2.14
$$\bar{e}_b = \frac{\rho}{(s')^3}\bar{r}' \times \bar{r}'', \quad \tau = \frac{(s')^6}{\rho^2(\bar{r}' \times \bar{r}'')\cdot\bar{r}'''},$$

where
$$\rho = \frac{(s')^3}{\left[(\bar{r}''\cdot\bar{r}'')(s')^2 - (\bar{r}'\cdot\bar{r}'')^2\right]^{1/2}}.$$

2.16
$$\bar{v} = L\beta\left[-\sin(\beta t)\,\bar{i} + \cos(\beta t)\,\bar{j} - 2\sin(2\beta t)\,\bar{k}\right],$$
$$\bar{a} = L\beta^2\left[-\cos(\beta t)\,\bar{i} - \sin(\beta t)\,\bar{j} - 4\cos(2\beta t)\,\bar{k}\right].$$

2.19 (a) $x = 32.72$ m; (b) $\beta = 35.61^\circ$.

2.21 (a) $v_0 = 0.2887\left(\frac{\alpha H^3}{m}\right)^{1/2}$; (b) $t_f = 1.3095\left(\frac{m}{\alpha H}\right)^{1/2}$; $y_f = 0.8574\left(\frac{mg}{\alpha H}\right)$.

2.24 (a) $\bar{v} = u\bar{i} + \frac{\pi Hu}{L}\cos\left(\frac{\pi x}{L}\right)\bar{j}, \quad \bar{a} = -\frac{\pi^2 Hu^2}{L^2}\sin\left(\frac{\pi x}{L}\right)\bar{j};$

(b) $\max(v) = u\left[1 + \left(\frac{\pi H}{L}\right)^2\right]^{1/2}$ at $x = nL$, n is an integer;

(c) $\max(|\bar{a}|) = \frac{\pi^2 Hu^2}{L^2}$ at $x = \frac{2n-1}{2}L$; (d) $u < \left(\frac{gL^2}{\pi^2 H}\right)^{1/2}$.

2.26 $x = \frac{\dot{y}_0}{u}[1-\cos(\mu t)] + \frac{\dot{x}_0}{u}\sin(\mu t), \quad y = \frac{\dot{y}_0}{u}\sin(\mu t) - \frac{\dot{x}_0}{u}[1-\cos(\mu t)],$

$z = \dot{z}_0 t$, helical path.

2.28 $\bar{v} = (c\beta)^{1/2}\lambda\bar{e}_R - \frac{2\pi}{3}(c\beta)^{1/2}\gamma\lambda\bar{e}_\theta + \frac{8\pi}{3}\lambda\beta\bar{e}_z,$

$\bar{a} = -\frac{\pi}{3}(c\beta)^{1/2}\gamma^2\lambda^2\bar{e}_R + \left(\frac{2\pi}{\sqrt{3}} - 1\right)(c\beta)^{1/2}\gamma\lambda^2\bar{e}_\theta + 2\beta\lambda^2\bar{e}_z.$

2.31 $\bar{v} = u\cot\theta\bar{e}_R + L\Omega\sin\theta\bar{e}_\theta - u\bar{e}_z, \quad \bar{a} = -\left(\frac{u^2}{L(\sin\theta)^3} + L\Omega^2\sin\theta\right)\bar{e}_R + 2\Omega u\cot\theta\bar{e}_\theta.$

2.33 $N = 555.7$ N, $\mu = 1.788$.

2.36 $\bar{v} = ah\left(\dot{\alpha}\bar{e}_a + \dot{\beta}\bar{e}_\beta\right),$

$$\bar{a} = a\left[h\ddot{\alpha} + \frac{\dot{\alpha}^2 - \dot{\beta}^2}{h}(\sinh\alpha)(\cosh\alpha) - 2\frac{\dot{\alpha}\dot{\beta}}{h}(\sin\beta)(\cos\beta)\right]\bar{\varepsilon}_\alpha$$
$$+\left[h\ddot{\beta} + \frac{\dot{\alpha}^2 - \dot{\beta}^2}{h}(\sin\beta)(\cos\beta) + 2\frac{\dot{\alpha}\dot{\beta}}{h}(\sinh\alpha)(\cosh\alpha)\right]\bar{\varepsilon}_\beta.$$

2.39 $v = 12.329$ m/s, $\dot{v} = 8.922$ m/s^2, $\rho = 0.2689$ m.

2.41 $\bar{v} = \left(\dot{R}\cos\theta - R\dot{\theta}\sin\theta\right)\bar{i} + \left(\dot{R}\sin\theta + R\dot{\theta}\cos\theta\right)\bar{j},$

$$a = \left(\ddot{R}\cos\theta - 2\dot{R}\dot{\theta}\sin\theta - R\ddot{\theta}\sin\theta - R\dot{\theta}^2\cos\theta\right)\bar{i}$$
$$+\left(\ddot{R}\sin\theta + 2R\dot{\theta}\cos\theta + R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta\right)\bar{j}.$$

2.44 $v = A\omega\left[1 + 3(\sin\theta)^2\right]^{1/2}, \quad \dot{v} = 3\omega^2 A\frac{\sin\theta\cos\theta}{\left[1+3(\sin\theta)^2\right]^{1/2}},$

$$\bar{a}\cdot\bar{e}_n = \frac{2\omega^2 A}{\left[1+3(\sin\theta)^2\right]^{1/2}}.$$

2.46 $\dot{r} = -350.0$ m/s, $\dot{\lambda} = 0.06239$ rad/s, $\dot{\theta} = -0.0324$ rad/s,

$\ddot{r} = 40.40$ m/s, $\ddot{\lambda} = 0.02345$ rad/s, $\ddot{\theta} = -0.00516$ rad/s.

2.49 $\bar{v} = \dot{\theta}(R'\bar{e}_R + R\bar{e}_\theta), \quad \bar{a} = \dot{\theta}^2[(R'' - R)\bar{e}_R + 2R'\bar{e}_\theta], \quad \rho = \frac{\left[(R')^2 + R^2\right]^{3/2}}{R''R - R^2 - 2(R')^2}.$

2.51 $\bar{v} = 29.48$ m/s, $\bar{a} = -781.3$ m/s^2, $F = 280.4$ N.

2.54 $v = 30.05$ m/s, $\dot{v} = -520.8$ m/s^2, $F = 40.01$.N.

3.1 $[R] = \begin{bmatrix} 0.8321 & -0.5547 & 0 \\ 0.3714 & 0.5571 & 0.7428 \\ -0.412 & -0.618 & 0.6695 \end{bmatrix}$, $\bar{r}_{C/A} = 0.4828\bar{j} - 0.5356\bar{k}$ m.

3.4 $[R] = \begin{bmatrix} -0.9285 & 0.3714 & 0 \\ -0.1564 & -0.3910 & 0.9070 \\ 0.3369 & 0.8422 & 0.4211 \end{bmatrix}$, $\bar{r}_{O/A} = 46.42\bar{i} + 7.82\bar{j} - 16.84\bar{k}$ m.

3.6 (a) $[x_E \quad y_E \quad z_E] = [-75.09 \quad -48.32 \quad -42.73]$ mm;
(b) $[X_E \quad Y_E \quad Z_E] = [-1.62 \quad -98.80 \quad -6.06]$ mm.

3.9 $\beta = \cos^{-1}(0.7192\cos\theta - 0.6428\sin\theta)$.

3.11 $[X_C \quad Y_C \quad Z_C] = [0.1465 \quad 0.3357 \quad 0.0766]$ m.

3.14 $\phi = 77.14^\circ$ about $\bar{K}' = 0.9265\bar{I} - 0.3258\bar{J} - 0.1881\bar{K}$.

3.16 $[R] = \begin{bmatrix} 0.3185 & 0.8209 & -0.4740 \\ -0.8209 & 0.4889 & 0.29510 \\ 0.4740 & -0.29510 & 0.8296 \end{bmatrix}$, 50.58° between original and new y axes.

3.19 $\Delta\bar{r}_C = -406.9\bar{I} - 378.6\bar{J} - 505.1\bar{K}$ mm.

3.21 $\Delta\bar{r}_C = -171.88\bar{I} + 173.38\bar{J} - 13.76\bar{K}$ mm.

3.24 $\Delta\bar{r}_C = -49.65\bar{I} - 33.59\bar{J} + 5.30\bar{K}$ mm,
$\bar{v}_C(t=0)\,\Delta t = -56.12\bar{I} - 25.55\bar{J} + 5.80\bar{K}$ mm.

3.26 $\bar{\omega} = 1000\pi\bar{i} + 0.16667\bar{k}$ rad/s, $\bar{\alpha} = 166.7\pi\,\bar{j}$ rad/s^2,
where $\bar{i} = \bar{e}_t$ and $\bar{j} = \bar{e}_n$ for the airplane's path.

3.29 $\theta = 90^\circ$: $\bar{\omega} = 5236\bar{j} - 20\bar{k}$ rad/s, $\bar{\alpha} = 104\,720\bar{i} + 100\bar{k}$ rad/s^2.
$\theta = 60^\circ$: $\bar{\omega} = 5246\bar{j} - 17.32\bar{k}$ rad/s, $\bar{\alpha} = 90\,690\bar{i} - 50\bar{j} + 86.60\bar{k}$ rad/s^2.

3.32 $\bar{\omega} = -0.4330\left(\dot\theta + 2\dot\beta\right)\bar{i} + 0.5\left(\dot\theta + 2\dot\gamma\right)\bar{j} + 0.250\left(3\dot\theta - 2\dot\beta\right)\bar{k}$,
$\bar{\alpha} = -0.250\left(\dot\theta\dot\beta + 3\dot\theta\dot\gamma - 2\dot\beta\dot\gamma\right)\bar{i} - 0.8660\dot\theta\dot\beta\,\bar{j} + 0.4330\left(\dot\theta\dot\beta - \dot\theta\dot\gamma - 2\dot\beta\dot\gamma\right)\bar{k}$.

3.34 $\bar{a}_C = \left[L\ddot\theta\sin 2\theta - L\dot\theta^2(9 + \cos 2\theta)\right]\bar{i} + \left[L\ddot\theta\,(3 + \cos 2\theta) + L\dot\theta^2\sin 2\theta\right]\bar{j},\ \bar{i} = \bar{e}_{C/B}$.

3.37 $\bar{a}_{B/A} = \left(-\Omega^2 H + 2\Omega\dot\theta W\sin\theta\right)\bar{i} - \left(\Omega^2 + \dot\theta^2\right)W(\cos\theta)\,\bar{j} - \dot\theta^2 W\sin\theta\bar{k}$.

3.39 $v_E = -(226.2L + 102.6R)\,\bar{j} - 10\,(L + R)\,\bar{k}$,
$\bar{a}_E = -(2.285L + 1.064R)\,10^4\bar{i} + 1508\,(L + R)\,\bar{j} - (1.656L + 0.740)\,10^4\bar{k}$.

3.42 $\bar{v}_A = -32.83\bar{i} - 11.736\bar{j} + 35.68\bar{k}$ m/s, $\bar{a}_A = -1259.4\bar{i} - 811.4\bar{j} - 1586.7\bar{k}$ m/s^2,
where $\bar{j} = \bar{e}_{B/A}$ and $\bar{k} = \bar{e}_{B/A} \times \bar{e}_{C/B} / \left|\bar{e}_{B/A} \times \bar{e}_{C/B}\right|$.

3.45 $\bar{\omega} = \left(\frac{v}{\tau} + v\frac{d\beta}{ds}\right)\bar{e}_t + \frac{v}{\rho}\bar{e}_b,$

$$\bar{\alpha} = \left(\frac{\dot{v}}{\tau} - \frac{v^2}{\tau^2}\frac{d\tau}{ds} + \dot{v}\frac{d\beta}{ds} + v^2\frac{d^2\beta}{ds^2}\right)\bar{e}_t + \frac{v^2}{\rho}\frac{d\beta}{ds}\bar{e}_n + \left(\frac{\dot{v}}{\rho} - \frac{v^2}{\rho^2}\frac{d\rho}{ds}\right)\bar{e}_b.$$

3.47 $\dot{u} = (50 - \sin\theta)\, g + \dot{\theta}^2 L + \Omega^2 L(\cos\theta)^2,$

$$N_{\text{horizontal}} = 2m\Omega\left(u\cos\theta - \dot{\theta}L\sin\theta\right), \quad N_{\text{vertical}} = m\left(g\cos\theta + \Omega^2 L\sin\theta + 2\dot{\theta}u\right).$$

3.50 $\bar{v}_B = \dot{\xi}\,\bar{i} + \Omega\xi\sin\theta\,\bar{j} + \dot{\theta}\xi\bar{k},$

$$\bar{a}_B = \left[\ddot{\xi} - \dot{\theta}^2\xi - \Omega^2\xi\,(\sin\theta)^2\right]\bar{i} + \left[2\Omega\dot{\xi}\sin\theta + 2\Omega\dot{\theta}\xi\cos\theta\right]\bar{j}$$
$$+ \left[\ddot{\theta}\xi - \Omega^2\xi\sin\theta\cos\theta + 2\dot{\theta}\dot{\xi}\right]\bar{k}.$$

3.52 $\bar{\omega} = -0.9397\Omega\cos\phi\bar{i} + \left(0.3420\Omega + \dot{\phi}\right)\bar{j} - 0.9397\Omega\sin\theta\bar{k},$

$$\bar{\alpha}_D = 0.9397\Omega\dot{\phi}\sin\phi\bar{i} + \ddot{\phi}\bar{j} - 0.9397\Omega\sin\phi\bar{k}.$$
$$\bar{v}_D = 1.9397\Omega L\sin\phi\bar{i} - 0.9397\Omega L\sin\phi\,\bar{j}$$
$$+ \left(-1.9397\Omega L\cos\phi - 0.3420\Omega L - L\dot{\phi}\right)\bar{k}$$
$$\bar{a}_D = L\left[\left(-1 - 0.6634\cos\phi + 0.8830\,(\cos\phi)^2\right)\Omega^2 - \dot{\phi}^2 - 0.6480\Omega\dot{\phi}\right]\bar{i}$$
$$-L\left[(1.8227 + 0.3214\cos\phi)\,\Omega^2 + 1.8194\,(\cos\phi)\,\Omega\dot{\phi}\right]\bar{j}$$
$$+ \left[-\ddot{\phi} + (0.8830\cos\phi - 0.6634)\,(\sin\phi)\,\Omega^2\right]\bar{k}$$

3.55 $(\bar{v}_B)_{x_2y_2z_2} = 45.06\bar{i} - 41.30\bar{j}$ m/s, $\quad (\bar{a}_B)_{x_2y_2z_2} = -19\,767\bar{i} - 5072\bar{j}$ m/s^2.

3.57 $\bar{a}_P = 2.343\bar{i} - 9.145\bar{j} - 5.553\bar{k}$ m/s^2.

3.60 $s = \left[\left(\frac{u}{2\omega_e\sin\lambda}\right)^2 - d^2\right]^{1/2} + \left(\frac{u}{2\omega_e\sin\lambda}\right)$ to the right.

3.62 (a) $x = 0, \quad y = -\cos\lambda\left(\frac{2\omega_e H}{g}\right)^{1/2}\left(\frac{2H}{3}\right);$

(b) $x = 0, \quad y = \cos\lambda\left(\frac{2\omega_e H}{g}\right)^{1/2}\left(\frac{H}{3} + R_e\right).$

4.1 $\bar{v}_D = \left(R\omega_1\cos\beta - R\omega_2 - L\dot{\beta}\right)\bar{i} + L\omega_1\sin\beta\,\bar{j} - R\omega_1\sin\beta\bar{k},$

$$\bar{a}_D = \left(L\omega_1^2\sin\beta\cos\beta - L\ddot{\beta}\right)\bar{i} + \left[-R\left(\omega_1^2 + \omega_2^2\right) + 2\omega_1\left(L\dot{\beta} + R\omega_2\right)\cos\beta\right]\bar{j}$$
$$- \left[\omega_1^2 L(\sin\beta)^2 + L\dot{\beta}^2 + 2R\omega_2\dot{\beta}\right]\bar{k}.$$

4.3 $\dot{\phi} = 97.93$ rad/s, $\quad \dot{\theta} = -20$ rad/s, $\quad \dot{\psi} = -48.44$ rad/s,

$\bar{\omega} = 10.12\bar{i} - 19.19\bar{j} + 50.22\bar{k}$ rad/s.

4.6 $[R] = [R_x(\phi)]\,[R_y(\theta)]\,[R_z(\psi)],$

$$\bar{\omega} = \left(\dot{\phi} - \dot{\psi}\sin\theta\right)\bar{i} + \left(\dot{\theta}\cos\theta + \dot{\psi}\sin\phi\cos\theta\right)\bar{j} + \left(-\dot{\theta}\sin\theta + \dot{\psi}\cos\phi\cos\theta\right)\bar{k},$$

$$\bar{\alpha} = \left[\ddot{\phi} - \ddot{\psi}\sin\theta - \dot{\psi}\dot{\theta}\cos\theta\right]\bar{i} + \left[\ddot{\psi}\cos\theta\sin\phi + \ddot{\theta}\cos\phi - \dot{\psi}\dot{\theta}\sin\theta\sin\phi\right.$$
$$\left. + \dot{\psi}\dot{\phi}\cos\theta\cos\phi - \dot{\theta}\dot{\phi}\sin\phi\right]\bar{j} + \left[\ddot{\psi}\cos\theta\cos\phi - \ddot{\theta}\sin\phi - \dot{\psi}\dot{\theta}\sin\theta\cos\phi\right.$$
$$\left. - \dot{\psi}\dot{\phi}\cos\theta\sin\phi - \dot{\theta}\dot{\phi}\cos\phi\right]\bar{k}.$$

4.8 $v = 5.313$ m/s, $\dot{v} = 724.1$ m/s^2.

4.11 $\bar{v}_A = b\dot{\theta}\cos\theta\bar{J}, \quad \bar{a}_A = b\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\bar{J},$

$$\bar{v}_B = -b\dot{\theta}\sin\theta\bar{I}, \quad \bar{a}_B = -b\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\bar{I},$$

$$\bar{v}_G = \frac{\sqrt{2}}{2}b\dot{\theta}\cos\left(\theta + \frac{\pi}{4}\right)\left(\bar{I} + \bar{J}\right),$$

$$\bar{a}_G = \frac{\sqrt{2}}{2}b\left[\ddot{\theta}\cos\left(\theta + \frac{\pi}{4}\right) - \dot{\theta}^2\sin\left(\theta + \frac{\pi}{4}\right)\right]\left(\bar{I} + \bar{J}\right).$$

4.13 $\theta = 60^\circ: \ \bar{\omega}_{BC} = -0.1111\dot{\theta}\bar{k}, \quad \bar{\alpha}_{BC} = -0.5797\dot{\theta}^2\bar{k}$, $\bar{k}$ is outward.

$\theta = 120^\circ: \ \bar{\omega}_{BC} = -0.2727\dot{\theta}\bar{k}, \quad \bar{\alpha}_{BC} = -0.2586\dot{\theta}^2\bar{k}.$

4.16 $\bar{\omega}_{BC} = -0.8660\omega_{AB}\bar{k}, \quad \bar{\omega}_{CD} = 0.50\omega_{AB}\bar{k}, \ \bar{\alpha}_{BC} = -0.250\omega_{AB}^2\bar{k},$

$\bar{\alpha}_{CD} = -1.616\omega_{AB}^2\bar{k}$, $\bar{k}$ is outward.

4.18 $\bar{v}_B = 0.866v_A\bar{i} + 0.866L\Omega\bar{j} + 0.5v_A\bar{k}, \quad \bar{v}_G = 0.433v_A\bar{i} + 0.433L\Omega\bar{j} - 0.25v_A\bar{k},$

where $\bar{k}$ is upward and $\bar{i}$ is radial.

4.21 $\bar{v}_C = -R\Omega\sin\theta\bar{i} + u\bar{j} - u\tan\theta\bar{k}, \quad \bar{a}_C = -2\Omega u\bar{i} - R\Omega^2\sin\theta\bar{j} - \dfrac{u^2}{R}\dfrac{1}{(\cos\theta)^3}\bar{k}.$

4.23 $v_D = 18.138$ m/s, $\bar{\omega}_{CD} = 17.490\bar{i} + 141.22\bar{j} - 90.88\bar{k}$ rad/s,

$a_D = -13802$ m/s^2, $\alpha_{CD} = 950.7\bar{i} + 4718\bar{j} - 4940\bar{k}$ rad/s^2.

4.26 (a) No unique solution; (b) $\omega_{AB} = -\omega_{CD} = 1.20\bar{J} + 1.60\bar{k}$ rad/s, where $\bar{I} = \bar{e}_{A/C}$.

4.28 $\bar{v}_P = (R - r)\dot{\theta}\left[\left(1 + \dfrac{\varepsilon}{r}\cos\phi\right)\dot{\theta}\bar{i} - \dfrac{\varepsilon}{r}\sin\phi\bar{j}\right],$

$$\bar{a}_P = (R - r)\left[\ddot{\theta}\left(1 + \frac{\varepsilon}{r}\cos\phi\right) - \frac{\varepsilon}{r}\left(\frac{R}{r} - 1\right)\dot{\theta}^2\sin\phi\right]\bar{i}$$
$$+ (R - r)\left[-\ddot{\theta}\frac{\varepsilon}{r}\sin\phi + \dot{\theta}^2\left(1 + \frac{\varepsilon}{r} - \frac{\varepsilon R}{r^2}\right)\cos\phi\right]\bar{j}.$$

4.31 $\bar{\omega}_A = 0.7273\dfrac{v}{R}$ clockwise, $\bar{\alpha}_A = 0.1172\dfrac{v^2}{R^2}$ clockwise.

4.33 $\bar{\omega} = \dfrac{v(\cos\theta)^2}{R(\cos\theta)^2 + h}$ clockwise, $\bar{\alpha} = \dfrac{2v^2h(\cos\theta)^3\sin\theta}{\left[R(\cos\theta)^2 + h\right]^3}$ clockwise.

4.36 $\bar{\omega} = -\dfrac{v}{R}\bar{i} + \dfrac{v}{R}\cos\beta\bar{j}, \ \bar{\alpha} = \dfrac{v^2}{R^2}(1 + \cos\beta)\sin\beta\bar{k},$
$\bar{i}$ parallel to the cone generator and $\bar{j}$ upward.

4.39 $\bar{\omega} = \Omega_1\sin(\beta + \gamma)\bar{i} - \left[\Omega_1\cos(\beta + \gamma) + (\Omega_1 - \Omega_2)\dfrac{\sin\beta}{\sin\gamma}\right]\bar{k},$

$$\bar{\alpha} = \dot{\Omega}_1\sin(\beta + \gamma)\bar{i} + (\Omega_1 - \Omega_2)\Omega_1\frac{\sin\beta}{\sin\gamma}\sin(\beta + \gamma)\bar{j}$$
$$- \left[\dot{\Omega}_1\cos(\beta + \gamma) + \left(\dot{\Omega}_1 - \dot{\Omega}_2\right)\frac{\sin\beta}{\sin\gamma}\right]\bar{k}.$$

4.41 Precession: $\dot{\psi} = \Omega_1 + \dfrac{(\Omega_1 - \Omega_2)}{(\sin\beta)^2}\left[\left(\dfrac{R}{r} - 1\right)\cos\beta - 1\right]$,

Spin: $\dot{\phi} = \dfrac{(\Omega_1 - \Omega_2)}{(\sin\beta)^2}\left(\dfrac{R}{r} - 1 - \cos\beta\right)$,

$\bar{\omega} = \left(\dot{\psi}\cos\beta + \dot{\phi}\right)\bar{i} + \dot{\psi}\sin\beta\bar{k}, \quad \bar{\alpha} = \dot{\phi}\dot{\psi}\sin\beta\,\bar{j}$,

$\bar{i}$ parallel to the cone generator, $\bar{k}$ upward.

4.44 $\psi = 33.69^\circ, \quad \dot{\psi} = -15.428$ rad/s, $\quad \dot{\theta} = 0.6934$ rad/s, $\quad \dot{\phi} = 13.699$ rad/s.

4.46 $\bar{\omega} = 2\Omega\,(1 + \cos\beta)\,\bar{i} + \dfrac{u}{R\,(\sin\beta - 2\cos\beta)}\bar{j} - \Omega\cos\beta\bar{k}$,

$$\bar{\alpha} = -\frac{2\Omega u\sin\beta}{R\,(\sin\beta - 2\cos\beta)}\bar{i} - \left[\ddot{\beta} - \Omega^2\cos\beta\,(2 + 2\cos\beta - \sin\beta)\right]\bar{j}$$

$$+\frac{2\Omega u\,(1 + \cos\beta - \sin\beta)}{R\,(\sin\beta - 2\cos\beta)}\bar{k}, \quad \bar{i} = \bar{e}_{C/B}, \quad \bar{k} \text{ upward.}$$

5.1 Initial $\bar{H}_O = 2mh^2\Omega\sin\theta\left(-\sin\theta\,\bar{I} + \cos\theta\,\bar{J}\right)$,

$\Delta\bar{H}_O = -mh^2\Omega\sin 2\theta\left[1.523\left(10^{-4}\right)\bar{J} + 0.01745\bar{K}\right]$,

where XYZ is stationary, with X aligned with the shaft and

Z perpendicular to the initial plane of the bars.

5.6 $m = \dfrac{7}{12}\pi\rho R^2 L, \quad I_{xx} = \dfrac{31}{160}\pi\rho R^4 L, \quad I_{yy} = I_{zz} = \pi\rho R^2 L\left(\dfrac{31}{320}R^2 + \dfrac{2}{15}L^2\right)$.

5.9 $m = 6.369$ kg, $\quad x_G = 1.1078$ m, $\quad y_G = 0.5847$ m,

$I_{xx} = 3.544, \quad I_{yy} = 9.931, \quad I_{zz} = 13.475$ kg-m^2, $\quad I_{xy} = 5.776$ kg-m^2.

5.11 $m = 414.1$ kg, $\quad \bar{r}_{G/O} = 2.26\bar{j} + 200\bar{k}$ mm, centroidal $\hat{x}\hat{y}\hat{z}$ are principal axes with

$I_{\hat{x}\hat{x}} = 9.495, \quad I_{\hat{y}\hat{y}} = 9.498, \quad I_{\hat{z}\hat{z}} = 9.800$ kg-m^2.

5.13 $$[I] = mR^2\begin{bmatrix} 0.542 & 0.042 & 0 \\ 0.042 & 2.542 & 0 \\ 0 & 0 & 2.583 \end{bmatrix}.$$

5.16 $I_{xx} = 0.006667 - 0.004\,(\cos\theta)^2, \quad I_{yy} = 0.002667 + 0.004\,(\cos\theta)^2$,

$I_{xy} = -0.002\sin 2\theta$ kg-m^2.

5.19 $I_1 = 26.61\left(10^{-6}\right), \quad I_2 = 456.7\left(10^{-6}\right), \quad I_3 = 483.3\left(10^{-6}\right)$ kg-m^2.

5.21 $$[I] = \begin{bmatrix} 191.5 & 229.6 & -64.8 \\ 229.6 & 1687.5 & -11.1 \\ -64.8 & -11.1 & 1613.0 \end{bmatrix}\text{ kg-m}^2.$$

5.26 $\bar{H}_C = \dfrac{1}{9}mL^2\Omega\sin\theta\,\bar{j}, \quad d\bar{H}_C/dt = \dfrac{1}{9}mL^2\Omega^2\sin\theta\cos\theta\bar{k}$.

5.28 $I_{x'x'} = 0.125, \quad I_{y'y'} = 0.260, \quad I_{z'z'} = 0.225 \text{ kg-m}^2,$
$\bar{H}_C = 0.1320\omega\bar{i} + 0.054\omega\bar{j} = 0.1118\omega\bar{i}' + 0.1163\omega\bar{j}' \text{ kg-m}^2\text{/s}, \ T = 0.076\omega^2 \text{ J}.$

5.31 $\bar{F}_A = -\frac{1}{3}mb\omega^2\bar{j}, \quad \bar{M}_A = -\frac{17}{36}mab\omega^2\bar{i}, \ \bar{k} = \bar{e}_{B/A}.$

5.33 $$\bar{F}_O = -mL\left[\dot{\psi}^2(\sin\theta)^2 + \dot{\theta}^2\right]\bar{i} - mL\left[\ddot{\theta} - \dot{\psi}^2\sin\theta\cos\theta\right]\bar{j} - mL\left[\ddot{\psi}\sin\theta + 2\dot{\psi}\dot{\theta}\cos\theta\right]\bar{k},$$
$$\bar{M}_O = \left[I_1\ddot{\psi}\cos\theta - (I_1 - I_2 + I_3)\dot{\psi}\dot{\theta}\sin\theta\right]\bar{i} + \left[I_2\ddot{\psi}\sin\theta - (I_1 - I_2 - I_3)\dot{\psi}\dot{\theta}\cos\theta\right]\bar{i} - \left[I_3\ddot{\theta} + (I_1 - I_2)\dot{\psi}^2\sin\theta\cos\theta\right]\bar{k}, \quad \bar{i} = \bar{e}_{G/O}.$$

5.36 $\bar{H}_G = mR^2\omega_1\left[0.125\lambda\bar{i} + (0.433\lambda - 0.5)\bar{k}\right], \quad \frac{d\bar{H}_G}{dt} = mR^2\omega_1^2\left(0.25\lambda - 0.10825\lambda^2\right)\bar{j}$

$\lambda = 2.309$ for no dynamic reactions,

$\bar{i}$ is the axis of the disk, $\bar{j}$ is perpendicular to the diagram.

5.39 $\bar{F}_B = 3mR\Omega^2\left(\cos\theta\bar{i} - \sin\theta\bar{k}\right),$

$$\bar{M}_B = \frac{1}{4}mR^2\left[-2\Omega\dot{\phi}(\cos\theta)^2 + \left(\Omega^2 - \dot{\phi}^2\right)\sin\theta\cos\theta\right]\bar{j}$$

$\bar{i}$ is the axis of the disk, $\bar{j}$ is perpendicular to the diagram.

6.1 $\Omega = \left(\dfrac{3g}{2L\cos\theta}\right)^{1/2}.$

6.4 $$\bar{F}_A = -\left[mL\Omega^2\left(\frac{1}{3} + \frac{1}{9}\cos\theta\right) + \frac{1}{6}mg\sin\theta\right]\bar{i},$$
$$\bar{F}_B = -\left[mL\Omega^2\left(\frac{1}{3} + \frac{1}{9}\cos\theta\right) - \frac{1}{6}mg\sin\theta\right]\bar{i} + 2mg\bar{k},$$

$\bar{i}'$ is radial to the right, $\bar{k}$ is upward.

6.6 $\theta = 0: \quad \dot{v} = 0, \quad N_1 = N_2 = \frac{3}{2}mg, \quad f_1 = -f_2 = \frac{mv^2}{12R}.$

$\theta = 90^\circ: \dot{v} = 0, \quad N_1 = \frac{3}{2}mg + \frac{mv^2}{12R}, \quad N_2 = \frac{3}{2}mg - \frac{mv^2}{12R}, \quad f_1 = f_2 = 0.$

6.9 $\bar{F}_A = -0.7445\dot{\psi}^2\bar{i} + 1.143\dot{\psi}^2\bar{j} - 1.364\ddot{\psi}\bar{k}$ N,
$\bar{M}_A = 0.3352\ddot{\psi}\bar{i} + 0.4638\ddot{\psi}\bar{j} + \left(-41.16\dot{\psi} + 0.2058\dot{\psi}^2\right)\bar{k}$ N-m,
$\bar{i}$ along the axis of symmetry, $\bar{k}$ horizontal.

6.11 $\bar{F}_A = -\bar{F}_B = \dfrac{mR^2}{2L}\Omega_1\Omega_2\cos\theta$ parallel to the upward diameter of the disk,

$\bar{M}_1 = \frac{1}{8}mR^2\Omega_2^2\sin 2\theta\,\bar{e}_{A/B}.$

6.14 $\frac{1}{3}\ddot{\theta} + \Omega^2\left(\frac{1}{2} + \frac{1}{3}\cos\theta\right)\sin\theta = \frac{g}{2L}\cos\theta.$

6.16 $\ddot{\xi} + \dot{\psi}^2 \left[\left(\frac{L}{4} - \xi \right) (\sin\theta)^2 - \frac{L}{6} \sin\theta \cos\theta \cos\phi \right] + \frac{L}{3} \dot{\psi} \dot{\phi} \sin\theta \cos\phi = g \cos\theta,$

$$\frac{1}{27} L\ddot{\phi} - \dot{\psi}^2 \left[\frac{1}{27} L (\sin\theta)^2 \sin\phi \cos\phi + \frac{1}{24} (L - 4\xi) \sin\theta \cos\theta \sin\phi \right]$$
$$- \frac{1}{3} \dot{\psi} \dot{\xi} \sin\theta \cos\phi = \frac{1}{6} g \sin\theta \sin\phi.$$

6.19 $\omega_1^2 > \dfrac{2gR}{0.8 + 2.5 \sin 2\theta}.$

6.21 $N[L(1+\cos\gamma) - R\sin\gamma] = mgL(1+\cos\gamma) + mR^2\Omega^2 \left[\left(\frac{L}{2R} \cos\gamma - \frac{L^2}{R^2} \sin\gamma \right) \right.$

$$\left. (1+\cos\gamma) - \frac{1}{4} \sin\gamma \cos\gamma \right], \quad \gamma = \frac{\pi}{6}.$$

6.24 $N_B = mg \cot\beta + m(R-r)\Omega_2^2 - \frac{2}{5} mr\dot{\psi}\dot{\phi}\cos\beta,$

$f_B = -\frac{2}{5} mr\dot{\phi}\dot{\psi}, \quad N_A = \frac{mg}{\sin\beta} - \frac{2}{5} mr\dot{\psi}\dot{\phi},$

where $\dot{\phi} = \dfrac{R - r - r\cos\beta}{r(\sin\beta)^2} (\Omega_1 - \Omega_2)$ and $\dot{\psi} = \Omega_2 - \dot{\phi}\cos\beta.$

6.26 Front-wheel drive: $\dot{v} = \mu g \dfrac{L-b}{L+\mu h}$, rear-wheel drive: $\dot{v} = \mu g \dfrac{b}{L-\mu h}$,

all-wheel drive: $\dot{v} = \mu g.$

6.29 $\ddot{\theta} = 0.9262 \dfrac{g}{L}.$

6.31 $\ddot{\phi} = -13.845$ rad/s.

6.34 (a) $F_{\text{crit}} = \min \left(\dfrac{mg}{\sin\theta}, \dfrac{\mu mg\,(\kappa^2 + r_1^2)}{\kappa^2(\cos\theta + \mu\sin\theta) + \mu r_1^2 \sin\theta + r_1 r_2} \right),$

$\dot{v} = \dfrac{F_{\text{crit}}}{m} \dfrac{r_1^2 \cos\theta - r_1 r_2}{\kappa^2 + r_1^2}.$

6.36 $\dot{v} = \frac{1}{3} g, \quad \mu_{\min} = 0.2.$

6.39 $F = \sigma v.$

6.41 $\omega_A - \omega_B = \dfrac{gL}{R^2 \dot{\psi}}.$

6.44 $\Delta N = 128.8$ N (decrease at front wheels).

6.46 $v = 1.7194 \left(\dfrac{FR}{m} \right)^{1/2}.$

6.49 $\max\phi = 37.49^\circ, \quad \dot{\phi} = 38.62$ rad/s at $\phi = 37.49^\circ.$

6.51 $v^2 = \dfrac{2FR^3}{m(R^2 + \kappa^2)} \left[\theta + \sin\theta - \sqrt{5} + \left(8 - 2\cos\theta - (\cos\theta)^2 \right)^{1/2} \right].$

6.54 $\max\phi = 37.49^\circ.$

6.56 (a) $\dot{\theta} = 4.3503\left(\frac{g}{L}\right)^{1/2}$, (b) $\max(\theta) = 133.8^\circ$ above horizontal.

6.59 (a) $\beta = 46.56^\circ$, $\Omega_1 = 0.9494\left(\frac{g}{L}\right)^{1/2}$,

(b) $P\Delta t = 0.1148m\,(gL)^{1/2}$, $v_G = 1.1123\,(gL)^{1/2}$.

6.61 $\omega_2 = \dfrac{mvh}{2I_A}$, where point A is the corner where impact occurs.

6.64 $\omega_2 = 4.065$ rad/s, $\bar{v}_2 = 24.70$ rad/s at 59.17° above left horizontal.

6.66 $(\bar{v}_G)_2 = -0.2467v\sin\theta\,\bar{j}$, $\omega_2 = -0.2220v\sin\theta$,
$\bar{v}_{\text{ball}} = \left(-\cos\theta\,\bar{i} - 0.0333\sin\theta\,\bar{j}\right)v$.

6.69 $(\bar{v}_B)_2 = 0.8613v_1$ at 50.74° below the left direction.

6.71 $\bar{v}_G = 15.459$ m/s downward, $\omega_2 = 17.441$ rad/s counterclockwise.

6.74 $\bar{v}_2 = 19.20\bar{i} + 1.61\bar{j} + 6\bar{k}$ m/s, $\bar{\omega}_2 = 30.25\bar{i} + 170.12\bar{j} + 50\bar{k}$ rad/s.

7.1 $2\beta\,(X - ut)\,\dot{X} - \dot{Y} - 2\beta\,(X - ut)\,u = 0$, $Y = \beta\,(X - ut)^2$.

7.3 $R\dot{\theta} + \dot{s} - \dot{X}_C\cos\theta + \dot{Y}_C\sin\theta = 0$, $s\dot{\theta} + \dot{X}_C\sin\theta + \dot{Y}_C\cos\theta = 0$,
both constraints are holonomic.

7.6 $\dot{x}\cos\theta - \dot{\theta}x\sin\theta = 0$.

7.8 $\dot{\theta}_1\sin\theta_1 + \dot{\theta}_2\sin\theta_2 + \dot{\theta}_3\sin\theta_3 = 0$, $\dot{\theta}_1\cos\theta_1 - \dot{\theta}_2\cos\theta_2 - \dot{\theta}_3\cos\theta_3 = 0$.

7.11 $\dot{x}_A\sin(\theta + \beta) - L\dot{\theta}\cos\beta = 0$.

7.14 $(\sin\beta - 2\cos\beta)\,R\dot{\beta} - u = 0$, $[2\,(1 + \cos\beta) - \sin\beta]\,\dot{\psi} + \dot{\phi} = 0$.

7.16 $q_1 = y$: $Q_1^{\text{cons}} = -k\,(y - 1.5L) - 8\sigma L$, $Q_1^{\text{nc}} = -\dfrac{F\left(3y^2 + 4L^2\right) + 2ML}{2L(4L^2 - y^2)^{1/2}}$,

$q_1 = \theta$: $Q_1^{\text{cons}} = -2kL^2\,(2\sin\theta - 1.5)\cos\theta - 16\sigma L^2\cos\theta$,

$$Q_1^{\text{nc}} = -FL\left[2 + 6\,(\sin\theta)^2\right] - M.$$

7.19 $\delta W = -N_B\sin(\theta + \beta)\,\delta x_A + N_B L\delta\theta\cos\beta$,

$\delta x_A = L\delta\theta\dfrac{\cos\beta}{\sin(\theta\beta)}$ if kinematically admissible.

7.21 $V = -\left(x^2 - y^2\right)^{1/2}$.

7.24 $$m\left[\frac{1}{3}L^2 + \frac{H^2}{(\cos\theta)^4}\right]\ddot{\theta} + 2m\frac{H^2}{(\cos\theta)^5}\dot{\theta}^2 + \frac{1}{2}mgL\cos\theta$$

$$+kH^2\frac{\left(\tan\theta - \tan\dfrac{\pi}{9}\right)}{(\cos\theta)^2} + \frac{mgH}{(\cos\theta)^2} = -\Gamma,$$

$\theta = 16.662^\circ$ for static equilibrium.

7.26 $3m\ddot{X} + m\ddot{s}\cos\theta + 2kX = 0$, $m\ddot{s} + m\ddot{X}\cos\theta + ks - mg\sin\theta = 0$.

7.29 $$m\left[2\ddot{R}_1 + \ddot{R}_2 - (2R_1 + R_2)\,\dot{\theta}^2\right] + k\,(R_1 - L) + 2mgR_1\sin\theta = 0,$$

$$m\left[\ddot{R}_1 + \ddot{R}_2 - (2R_1 + R_2)\,\dot{\theta}^2\right] + k\,(R_2 - L) + mgR_2\sin\theta = 0.$$

7.31 $\frac{8}{3}mL^2\ddot{\theta} + kL^2\left(4\sin\theta - \frac{3}{2}L\right)\cos\theta = 2(F+mg)\,L\cos\theta.$

7.34 $\frac{1}{3}mL^2\ddot{\theta} + \frac{1}{2}mL(\ddot{u}+g)\sin\theta = 0.$

7.36 $\frac{1}{3}mL^2\ddot{\phi} + \frac{1}{2}mL\varepsilon\Omega^2\sin\phi + k\phi = 0.$

7.39 $4mR^2\left(\ddot{\theta} + \omega^2\sin\theta\cos\theta\right) + 2mgR\sin(2\theta+\omega t) = 2FR\sin\theta.$

7.41 $\left[1 + 8(\cos\theta)^2\right]\ddot{\theta} + \left(9\Omega^2 - 16\dot{\theta}^2\right)\sin\theta\cos\theta = \frac{(2mg-4F)}{mL}\sin\theta.$

7.44 $(m_1+m_2)\,\ddot{x} - m_2(R-r)\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) + kx = F,$
$\frac{3}{2}(R-r)\,\ddot{\theta} - \ddot{x}\cos\theta + g\sin\theta = 0.$

7.46 $\left(\frac{1}{12}L^2 + \frac{1}{3}h^2 + R^2\theta^2\right)\ddot{\theta} + mR^2\theta\dot{\theta}^2 + g\left(R\theta\cos\theta - \frac{h}{2}\sin\theta\right) = 0.$

7.48 $(m_1+m_2)\ddot{s} + m_1(R+\varepsilon\cos\theta)\ddot{\theta} - m_1\varepsilon\dot{\theta}^2\sin\theta = (m_1+m_2)g,$

$$m_1(R^2+\varepsilon^2+\kappa^2+2R\varepsilon\cos\theta)\ddot{\theta} + m_1(R+\varepsilon\cos\theta)\ddot{s}$$
$$- m_1R\varepsilon\dot{\theta}^2\sin\theta - m_1g\varepsilon\sin(\beta+\theta) = 0.$$

7.51 $$\frac{mL^2}{(\sin\beta)^2}\left\{\left[\left(\frac{4}{3} + \frac{\kappa^2}{R^2}\right)\left((\cos\theta)^2 + \cos(\beta-\theta)^2\right) + \frac{1}{2}\cos\theta\cos(\beta-\theta)\cos\beta\right]\ddot{\theta}\right.$$
$$\left. - \frac{1}{2}\left[\left(\frac{4}{3} + \frac{\kappa^2}{R^2}\right)(\sin 2\theta + \sin(\beta-2\theta)) + \frac{1}{2}\sin(2\theta-\beta)\cos\beta\right]\dot{\theta}^2\right\}$$
$$+ \frac{3}{2}mgL\cos\theta = FL\frac{\cos(\beta-\theta)}{\sin\beta}.$$

7.54 $m\ddot{s} - m\left[\dot{\theta}^2 + \dot{\psi}^2(\sin\theta)^2\right]s + ks + mg\sin\theta\cos\psi = 0,$
$\left(I_1 + ms^2\right)\ddot{\theta} + 2ms\dot{s}\dot{\theta} - m\dot{\psi}^2s^2\sin\theta\cos\theta + mgs\cos\theta\cos\psi = 0,$
$\left[I_2 + ms^2(\sin\theta)^2\right]\ddot{\psi} + 2m\dot{\psi}s\left[\dot{s}(\sin\theta)^2 + s\dot{\theta}\sin\theta\cos\theta\right] - mgs\sin\theta\sin\psi = M.$

7.56 $\frac{1}{4}mR^2\left[\left(1 + (\cos\theta)^2\right)\dot{\Omega}_1 + 2\dot{\Omega}_2\cos\theta\right] = C.$

7.58 $\frac{1}{12}mL^2\left[\left(1 + 3(\sin)^2\right)\ddot{\psi} + 6\dot{\psi}\dot{\theta}\sin\theta\cos\theta\right] = \Gamma,$
$\frac{1}{12}mL^2\left[11\ddot{\theta} + 3\dot{\psi}^2\sin\theta\cos\theta\right] + \frac{1}{2}mgL\sin\theta = 0.$

7.61 $\ddot{\xi} - \left[\Omega^2 (\sin\theta)^2 + \dot{\theta}^2\right]\xi + \frac{1}{2}L\dot{\theta}^2 + g\cos\theta = 0,$

$$\left(\frac{1}{3}L^2 - L\xi + \xi^2\right)\ddot{\theta} + 2\left(\xi - \frac{L}{2}\right)\dot{\xi}\dot{\theta} - \Omega^2\left(\xi^2 + \frac{1}{12}L^2\right)\sin\theta\cos\theta$$
$$- mg\left(\zeta - \frac{L}{2}\right)\sin\theta = 0.$$

7.63 $(m_1 + m_2)\ddot{z} + \frac{1}{2}m_2 L\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) + (m_1 + m_2)g + kz = F,$

$$\frac{1}{3}L\ddot{\theta} + \frac{1}{2}\ddot{z}\sin\theta - \frac{1}{3}L\dot{\psi}^2\sin\theta\cos\theta + \frac{1}{2}g\sin\theta = 0,$$

$$\frac{1}{3}m_2L^2\left[\ddot{\psi}(\sin\theta)^2 + 2\dot{\psi}\dot{\theta}\sin\theta\cos\theta\right] = M.$$

7.66 $\left[I_p + m\left(L^2 + \frac{1}{4}R^2\right)\left(1 + (\cos\beta)^2\right) + 2mL^2\cos\beta\right]\ddot{\psi}$

$$-2m\sin\beta\left[L^2 + \left(L^2 + \frac{1}{4}R^2\right)\cos\beta\right]\dot{\beta}\dot{\psi} + \frac{1}{2}mR^2\dot{\phi}\dot{\beta}\sin\beta = \Gamma,$$
$$m\left(L^2 + \frac{1}{4}R^2\right)\ddot{\beta} + m\sin\beta\left[L^2 + \left(L^2 + \frac{1}{4}R^2\right)\cos\beta\right]\dot{\psi}^2 - \frac{1}{2}mR^2\dot{\psi}\dot{\phi}\sin\beta$$
$$-mgL\cos\beta = 0.$$

8.2 $m\left(6R^2 + 2\kappa^2 + 4R^2\sin\theta\right)\ddot{\theta} + 3.75mR^2\left[\sin\phi - \cos(\theta+\phi)\right]\ddot{\phi} + 2mR^2\dot{\theta}^2\cos\theta$

$$+3.75mR^2\dot{\phi}^2\left[\cos\phi + \sin(\theta+\phi)\right] = FR(1 + \sin\theta) + \lambda_1\cos\theta,$$

$$\frac{25}{3}mR^2\ddot{\phi} + 3.75mR^2\left[\sin\phi - \cos(\theta+\phi)\right]\ddot{\theta} + 3.75mR^2\dot{\theta}^2\sin(\theta+\phi)$$

$$+1.25mgR\cos\phi = 2.5FR\sin\phi - 2.5\lambda_1\cos\phi.$$

8.4 $\{z\} = \begin{bmatrix}\psi & \beta & \dot{\psi} & \dot{\beta}\end{bmatrix}^{\mathrm{T}}, \quad I_1 = m\left[L^2 + \kappa_1^2(\sin\beta)^2 + \kappa_2^2(\cos\beta)^2\right],$

$$\begin{bmatrix} I_1 & 0 & -c_1 \\ 0 & m\kappa_2^2 & 1 \\ -c_1 & 1 & 0 \end{bmatrix}\begin{Bmatrix}\ddot{\psi} \\ \ddot{\beta} \\ \lambda_1\end{Bmatrix} = \begin{Bmatrix} -m\left(\kappa_1^2 - \kappa_2^2\right)\dot{\psi}\dot{\beta}\sin 2\beta + m\kappa_1^2\Omega_1\dot{\beta}\cos\beta + \Gamma \\ \frac{1}{2}m\left(\kappa_1^2 - \kappa_2^2\right)\dot{\psi}^2\sin 2\beta + m\kappa_1^2\Omega_1\dot{\psi}\cos\beta \\ c_2\dot{\psi}\end{Bmatrix}.$$

8.6 $(m_1 + m_2)\ddot{x} - m_2\left(\ddot{R} - R\dot{\theta}^2\right)\cos\theta + m_2\left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\sin\theta + 2k_1x = \lambda_1\sin\theta,$

$$m_2\left(\ddot{R} - \ddot{x}\cos\theta - R\dot{\theta}^2\right) + k_2(R - R_0) = 0,$$

$$m_2\left(R\ddot{\theta} + \ddot{x}\sin\theta + 2\dot{R}\dot{\theta}\right) - m_2g\cos\theta = \lambda_1, \quad \dot{x}\sin\theta + R\dot{\theta} = 0.$$

8.9 $m_1R^2\left\{\left[4.5 + 8\cos\theta + 3.75(\cos\theta)^2\right]\ddot{\psi} + \frac{1}{2}(\sin\phi)\ddot{\phi} - (8 + 7.5\cos\theta)(\sin\theta)\dot{\psi}\dot{\theta}\right.$

$$\left. + \frac{1}{2}\dot{\phi}\dot{\theta}\cos\theta\right\} = M + \lambda_1R(\sin\theta - 2 - 2\cos\theta),$$

$$\left\{4.25m_1R^2+m_2R^2\left[1-2\sin 2\theta+3(\cos\theta)^2\right]\right\}\ddot\theta-\frac{1}{2}m_2R^2(4\cos 2\theta+3\sin 2\theta)\dot\theta^2$$
$$+m_1R^2(4+3.75\cos\theta)(\sin\theta)\dot\psi^2-2(m_1+m_2)gR\cos\theta+m_2gR\sin\theta$$
$$=FR(\sin\theta-2\cos\theta),$$

$$\frac{1}{2}m_1R^2\left(\ddot\phi+\ddot\psi\sin\theta+\dot\psi\dot\theta\cos\theta\right)=\lambda_1R,\quad R(\sin\theta-2-2\cos\theta)\dot\psi+R\dot\phi=0.$$

8.11 $$(m_A+m_{AB})\ddot x_A-\frac{1}{2}m_{AB}L\left(\ddot\theta\sin\theta+\dot\theta^2\cos\theta\right)=F+\lambda_1\sin(\beta+\theta),$$

$$\frac{1}{3}m_{AB}L^2\ddot\theta-\frac{1}{2}m_{AB}L\ddot x_A\sin\theta=\lambda_1L\cos\beta,\quad \dot x_A\sin(\beta+\theta)+\dot\theta L\cos\beta=0.$$

8.13 $$q_1=X_C,\quad q_2=Y_C,\quad q_3=\theta,\quad \dot X_C\sin\theta-\dot Y_C\cos\theta=0$$
$$\left(m+\frac{3}{2}m_w\right)\ddot X_C+m\ell\left(\ddot\theta\sin\theta+\dot\theta^2\cos\theta\right)=\left(\bar F_1+\bar F_2\right)\cdot\bar I+\lambda_1\sin\theta,$$
$$\left(m+\frac{3}{2}m_w\right)\ddot Y_C-m\ell\left(\ddot\theta\cos\theta-\dot\theta^2\sin\theta\right)=\left(\bar F_1+\bar F_2\right)\cdot\bar J-\lambda_1\cos\theta,$$
$$\left(I+\frac{1}{4}m_wR^2\right)\ddot\theta+m\ell\left(\ddot X_C\sin\theta-\ddot Y_C\cos\theta\right)=\bar r_{A/C}\times\bar F_1+\bar r_{B/C}\times\bar F_2.$$

8.15 $$4m\ddot s+2mL\ddot\theta+2mL\dot\theta^2\cos\theta=F+\lambda_2\sin\Omega t,$$
$$\left[\frac{1}{6}+\frac{3}{2}(\sin\psi)^2(\sin\theta)^2\right]\ddot\psi-\frac{3}{8}\ddot\theta\sin 2\psi\sin 2\theta+\frac{3}{2}\dot\psi^2\sin\psi\cos\psi(\sin\theta)^2$$
$$+\frac{3}{4}\dot\theta^2\sin 2\psi+\frac{3}{2}\dot\psi\dot\theta(\sin\psi)^2\sin 2\theta=\frac{1}{mL}(\lambda_1\cos\psi\sin\theta-\lambda_2\cos\Omega t\sin\psi\sin\theta),$$

$$mL^2\left[\frac{1}{6}+\frac{3}{2}(\sin\theta)^2+\frac{3}{2}(\cos\psi)^2(\cos\theta)^2\right]\ddot\theta+2mL\ddot s\sin\theta$$
$$+\frac{3}{2}mL^2\dot\theta^2\sin 2\theta\left[1+(\cos\psi)^2\right]-\frac{1}{2}mL^2\dot\psi^2\left[\frac{1}{3}+3(\sin\psi)^2\right]\sin 2\theta$$
$$=\lambda_1L\sin\psi\cos\theta+\lambda_2L[\sin\Omega t\sin\theta+\cos\Omega t\cos\psi\cos\theta],$$

$$L\dot\psi\cos\psi\sin\theta+L\dot\theta\sin\psi\cos\theta=0,$$

$$\dot s\sin\Omega t-L\dot\psi\cos\Omega t\sin\psi\sin\theta+L\dot\theta[\sin\Omega t\sin\theta+\cos\Omega t\cos\psi\cos\theta]$$
$$+\Omega\left[s\cos\Omega t-L\sin\Omega t\cos\psi\sin\theta-L(\cos\Omega t)^2\right]=0.$$

8.17 $X_G=-111.2$ m, $Y_G=-70.0$ m, $\theta=56.2^\circ$ at $t=60$ s.

8.19 θ passes 89° when $t=9.28$ s.

8.22 $\max(\theta)=80.73^\circ$ when $t=1.995$ s.

8.25 $$\frac{1}{2}\ddot\theta+\frac{4}{3\pi}\frac{g}{R}\sin\theta+\mu\left[\frac{g}{R}-\frac{4}{3\pi}\left(\ddot\theta\sin\theta+\dot\theta^2\cos\theta\right)\right]\mathrm{sgn}(\dot\theta)=0.$$

8.28 $$\frac{1}{3}mL^2\ddot\phi+\frac{1}{2}ml\ddot Y_B\sin\phi-\frac{1}{2}mgL\sin\phi=\mu|N_A|(\cos\phi)\,\mathrm{sgn}\left(\dot\phi\right)-N_A\sin\phi,$$

$$m\ddot Y_B(\sin\phi)^2+\frac{1}{2}mL\ddot\phi\sin\phi+\frac{1}{2}mL\dot\phi^2\cos\phi=-\frac{H}{\sin\phi}N_A,\quad \dot Y_B(\sin\phi)^2+H\dot\phi=0.$$

8.31 $\{q\} = [\phi \;\; X_B \;\; Y_B]^{\mathrm{T}}, \;\; \{x\} = \left[\{q\}^{\mathrm{T}} \;\; \{\dot{q}\}^{\mathrm{T}}\right]^{\mathrm{T}}, \;\; \dfrac{d}{dt}\{x\} = \left[\{\dot{q}\}^{\mathrm{T}} \;\; \{\ddot{q}\}^{\mathrm{T}}\right]^{\mathrm{T}},$

$$\begin{bmatrix} [M] & -[B] \\ -[a] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{q}\} \\ \begin{Bmatrix} N_A \\ N_B \end{Bmatrix} \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ [\dot{a}]\{a\}\{\dot{q}\} \end{Bmatrix},$$

$$[M] = \begin{bmatrix} L^2/3 & (L/2)\cos\phi & (L/2)\sin\phi \\ (L/2)\cos\phi & 1 & 0 \\ (L/2)\sin\phi & 0 & 1 \end{bmatrix},$$

$$[B] = \begin{bmatrix} -H/\sin\phi & 0 \\ -\left(\mu\sin\phi\,\mathrm{sgn}\left(\dot{\phi}\right) + \cos\phi\right) & 1 \\ \left(\mu\cos\phi\,\mathrm{sgn}\left(\dot{\phi}\right) - \sin\phi\right) & 0 \end{bmatrix}, \quad \{F\} = \begin{Bmatrix} (L/2)\,g\sin\phi \\ (L/2)\,\dot{\phi}^2\sin\phi \\ g - (L/2)\,\dot{\phi}^2\cos\phi \end{Bmatrix},$$

$$[a] = \begin{bmatrix} 0 & 1 & 0 \\ H & 0 & (\sin\phi)^2 \end{bmatrix}, \quad [\dot{a}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dot{\phi}\sin 2\phi \end{bmatrix},$$

$N_B = 0$ at $t = 0.237$ s, $\phi = 45.54^\circ$.

9.2 $\mu\ddot{w} + (EIw'')'' + \mu g - f = 0$; either w is specified or $(EIw'')' = 0$ at $x = 0$ and $x = L$; either w' is specified or $EIw'' = 0$ at $x = 0$ and $x = L$.

9.5 $$-\mu\ddot{w}_y + F\left\{\left[1 - \frac{3}{2}\left(\frac{\partial w_y}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w_z}{\partial x}\right)^2\right]\frac{\partial^2 w_y}{\partial x^2} - \frac{\partial w_y}{\partial x}\frac{\partial w_z}{\partial x}\frac{\partial^2 w_z}{\partial x^2}\right\} - \mu g + f_y = 0,$$

$$-\mu\ddot{w}_z + F\left\{\left[1 - \frac{3}{2}\left(\frac{\partial w_z}{\partial x}\right)^2\frac{\partial^2 w_z}{\partial x^2} - \frac{1}{2}\left(\frac{\partial w_y}{\partial x}\right)^2\right]\frac{\partial^2 w_z}{\partial x^2} - \frac{\partial w_y}{\partial x}\frac{\partial w_z}{\partial x}\frac{\partial^2 w_y}{\partial x^2}\right\} + f_z = 0.$$

9.7 $$\frac{1}{2}\mu L\ddot{q}_1 + \frac{\pi^2 F}{2L}q_1 - \frac{\pi^4 F}{16L^3}\left(3q_1^3 + 24q_1q_2^2\right) + \frac{2}{\pi}\mu g L = 0,$$

$$\frac{1}{2}\mu L\ddot{q}_2 + \frac{4\pi^2 F}{L}q_2 - \frac{3\pi^4 F}{2L^3}\left(q_1^2q_2 + 2q_2^3\right) = 0.$$

9.10 $$\frac{1}{2}\mu L\ddot{p}_1 + \frac{\pi^2 F}{2L}p_1 - \frac{\pi^4 F}{L^3}\left[\frac{3}{16}p_1^3 + \frac{3}{16}p_1q_1^2 + \frac{3}{2}p_1p_2^2 + \frac{1}{2}p_1q_2^2 + q_1p_2q_2\right] + \frac{2}{\pi}\mu g L$$

$$= \frac{2}{\pi}f_0 L\sin(\Omega t),$$

$$\frac{1}{2}\mu L\ddot{q}_1 + \frac{\pi^2 F}{2L}q_1 - \frac{\pi^4 F}{L^3}\left[\frac{3}{16}q_1^3 + \frac{3}{16}p_1^2q_1 + \frac{1}{2}q_1p_2^2 + \frac{3}{2}q_1q_2^2 + p_1p_2q_1\right] = 0,$$

$$\frac{1}{2}\mu L\ddot{p}_2 + \frac{2\pi^2 F}{L}p_2 - \frac{\pi^4 F}{L^3}\left[\frac{3}{2}p_1^2p_2 + \frac{1}{2}q_1^2p_2 + p_1q_1q_2 + 2p_2^3 + 2p_2q_2^2\right] = 0,$$

$$\frac{1}{2}\mu L\ddot{q}_2 + \frac{2\pi^2 F}{L}q_2 - \frac{\pi^4 F}{L^3}\left[\frac{1}{2}p_1^2 q_2 + \frac{3}{2}q_1^2 q_2 + p_1 q_1 p_2 + 2q_2^3 + 2p_2^2 q_1\right] = 0.$$

9.12 $m\dot{x}_1 = \dfrac{p_1 - p_2\cos\theta}{5-(\cos\theta)^2}, \quad m\dot{x}_2 = \dfrac{-p_1\cos\theta + 5p_2}{5-(\cos\theta)^2}, \quad \dot{p}_1 = 0, \quad \dot{p}_2 = m_2 g\sin\theta - kx_2.$

9.15 $\dot{\psi} = \dfrac{p_1}{\Delta}, \quad \dot{\beta} = \dfrac{p_2}{I_2}, \quad \dot{p}_1 = -mg\left(\dfrac{H}{2} - w\right)\sin\beta\cos\psi,$

$$\dot{p}_2 = -\frac{p_1^2}{2\Delta^2}(I_1 - I_2)\sin 2.\beta - mg\left(\frac{H}{2} - w\right)\cos\beta\sin\psi,$$

$$\Delta = I_1(\cos\beta)^2 + I_2(\sin\beta)^2.$$

9.17 $\dot{\mathcal{H}} = 2m\dot{R}\ddot{R} - m(2R - L)\dot{R}\Omega^2 = 0,$

$$\dot{E} - \dot{\mathcal{H}} = 2m(2R - L)\dot{R}\Omega^2 = \Gamma\Omega.$$

9.20 Define $I_1 = m_1\kappa_1^2, \quad I_2 = m_1\kappa_2^2 + \left(m_1 + \frac{1}{3}m_2\right)L^2.$

Case (a): $\dot{\mathcal{H}} = I_1\dot{\phi}\ddot{\phi} + I_2\dot{\theta}\ddot{\theta} + \frac{1}{2}(I_1 - I_2)(\sin 2\theta)\Omega^2\dot{\theta} + \left(m_1 + \frac{1}{2}m_2\right)gL\dot{\theta}\sin\theta = 0,$

$$\dot{E} = I_1\dot{\phi}\ddot{\phi} + I_2\dot{\theta}\ddot{\theta} - \frac{1}{2}(I_1 - I_2)(\sin 2\theta)\Omega^2\dot{\theta} + \left(m_1 + \frac{1}{2}m_2\right)gL\dot{\theta}\sin\theta = \Gamma\Omega.$$

Case (b): $\dot{\mathcal{H}} = I_2\dot{\theta}\ddot{\theta} + \frac{1}{2}(I_1 - I_2)(\sin 2\theta)\Omega^2\dot{\theta} + \left(m_1 + \frac{1}{2}m_2\right)gL\dot{\theta}\sin\theta = 0,$

$$\dot{E} = I_2\dot{\theta}\ddot{\theta} - \frac{1}{2}(I_1 - I_2)(\sin 2\theta)\Omega^2\dot{\theta} + \left(m_1 + \frac{1}{2}m_2\right)gL\dot{\theta}\sin\theta = \Gamma\Omega + \Gamma_\phi\dot{\phi}.$$

9.22 $\frac{1}{2}mL^2\ddot{\theta}\left[1 + 8(\cos\theta)^2\right] - 4mL^2\dot{\theta}^2\sin\theta\cos\theta - \dfrac{2p_2^2}{9mL^2}\dfrac{\cos\theta}{(\sin\theta)^3} - mgL\sin\theta = 0,$

$$p_2 = \frac{9}{2}mL^2\dot{\psi}(\sin\theta)^2.$$

9.25 $mR^2\ddot{\theta} - \dfrac{p_2^2\cos\theta}{mR^2(\sin\theta)^3} + \dfrac{1}{2}mgR\sin\theta\left[1 + \dfrac{2\sin\theta}{(5 + 4\cos\theta)^{1/2}}\right] = 0,$

$$p_2 = mR^2\dot{\psi}(\sin\theta)^2 = \frac{3}{4}mgR.$$

9.27 $p_2 = \left[\tilde{I} + I_\phi(\cos\phi)^2\right]\dot{\psi} - I_\phi\dot{\phi}\cos\beta, \quad p_3 = I_\phi\left(\dot{\phi} - \dot{\psi}\cos\beta\right),$

$$\mathcal{R} = \frac{1}{2}\left(mL^2 + I_\phi\right)\dot{\beta}^2 - \frac{1}{2I_\phi\tilde{I}}\left[I_\phi(p_2 + p_3\cos\beta)^2 + \tilde{I}p_2^2\right] + mgL\sin\beta,$$

$$\tilde{I} = mL^2(1 + \cos\beta)^2 + \frac{1}{2}I_\phi(\sin\beta)^2 + I_P.$$

9.30 $\dot{\gamma}_1 = \dot{\psi}, \quad \dot{\gamma}_2 = \dot{\theta},$

$$mL^2\left[\frac{9}{2}\ddot{\gamma}_1(\sin\theta)^2 + 9\dot{\gamma}_1\dot{\gamma}_2\sin\theta\cos\theta\right] = 0,$$

$$mL^2\left[\frac{1}{2} + 4(\cos\theta)^2\right]\ddot{\gamma}_2 - mL^2\left(\frac{9}{2}\dot{\gamma}_1^2 + 4\dot{\gamma}_2^2\right)\sin\theta\cos\theta = (mg = 2F)L\sin\theta.$$

9.32 $\dot{\gamma}_1 = v, \quad \dot{\gamma}_2 = \dot{\theta}, \quad \dot{X}_G = \dot{\gamma}_1 \cos\theta, \quad \dot{Y}_G = \dot{\gamma}_1 \sin\theta, \quad m\ddot{\gamma}_1 = F\cos\beta, \quad I\ddot{\gamma}_2 = -FD\sin\beta.$

9.34 $$\left[\frac{1}{3}L^2(\cos\theta)^5 - DL(\cos\theta)^4 + D^2\cos\theta\right]\ddot{\gamma}_1 + \left[2D^2 - \frac{1}{2}DL(\cos\theta)^3(\sin\theta)\dot{\gamma}_1^2\right] = g\left[D - \frac{L}{2}(\cos\theta)^3\right](\cos\theta)^3, \quad \dot{\gamma}_1 = \dot{\theta}.$$

9.37 $\dot{\gamma}_1 = \dot{\psi}, \ \dot{\gamma}_2 = \dot{\beta}, \ \dot{\gamma}_1 - c_1\dot{\gamma}_2 - c_2\beta - c_3 = 0,$
$$m\left[k_1^2(\sin\beta)^2 + \kappa_2^2(\cos\beta)^2\right]\ddot{\gamma}_1 + m\left(\kappa_1^2 - \kappa_2^2\right)\left(\dot{\gamma}_1\dot{\gamma}_2 - \dot{\gamma}_1^2\right)\sin\beta\cos\beta + m\left(\kappa_1^2 - \kappa_2^2\right)\dot{\gamma}_1\Omega\cos\beta = \lambda_1,$$
$$m\kappa_2^2\ddot{\gamma}_1 + m\kappa_1^2\dot{\gamma}_1\Omega\cos\beta - m\left(\kappa_1^2 - \kappa_2^2\right)\gamma_1^2\sin\beta\cos\beta = -c_1\lambda_1.$$

9.39 $\dot{\gamma}_1 = r, \ \dot{\gamma}_2 = \dot{\theta}, \ \ \frac{7}{5}\ddot{\gamma}_1 - r\dot{\gamma}_2^2 = 0, \ \ \left(r^2 + \frac{2}{5}R^2\right)\ddot{\gamma}_2 + 2r\dot{\gamma}_1\dot{\gamma}_2 = 0.$

9.42 $\dot{\gamma}_1 = \dot{\theta}, \ \ \frac{1}{2}mL^2\left[1 + 8(\cos\theta)^2\right]\ddot{\gamma}_1 - \left(4mL^2\dot{\gamma}_1^2 + \frac{9}{2}\dot{\psi}^2\right)\sin\theta\cos\theta = (mgL - 2F)\sin\theta.$

9.45 $\dot{\gamma}_1 = \dot{\beta}, \ m\kappa_2^2\ddot{\gamma}_1 + m\kappa_1^2\Omega_1\dot{\psi}\cos\beta + m\left(\kappa_2^2 - \kappa_1^2\right)\dot{\psi}^2\sin\beta\cos\beta = M.$

9.48 $\dot{\gamma}_1 L\cos\beta + \dot{\gamma}_2\sin(\theta + \beta) = 0,$
$$\left(\frac{1}{3}m_1L^2 + m_2L^2 + I_2\right)\ddot{\gamma}_1 + \left(\frac{1}{2}m_1 + m_2\right)L\ddot{\gamma}_2\sin\theta + I_2\ddot{\beta} = F\sin\beta + N_B\cos\beta,$$
$$\left[m_1 + m_2 + m_3 + \frac{I_1}{R^2}\left(\frac{\cos\theta}{\cos\beta}\right)^2\right]\ddot{\gamma}_2 + \left(\frac{1}{2}m_1 + m_3\right)L\left(\ddot{\gamma}_1\sin\theta + \dot{\gamma}_1^2\cos\theta\right) - \frac{I_1}{R^2}\dot{\gamma}_1\dot{\gamma}_2\frac{\sin\theta\cos\theta}{(\cos\beta)^2} + \frac{I_1}{R^2}\dot{\gamma}_2\dot{\beta}\frac{(\cos\theta)^2\sin\theta}{(\cos\beta)^3} = -F\cos(\beta + \theta) - N_B\sin(\theta + \beta) + k(\ell - \ell_0), \ \ \dot{\gamma}_1 = \dot{\theta}, \ \dot{\gamma}_2 = \dot{\ell}.$$

9.50 $\dot{\gamma}_1 = v_A, \quad \dot{\gamma}_2 = \dot{\theta}, \ \dot{\gamma}_3 = \dot{\beta}, \ \ \dot{X}_A = \dot{\gamma}_1\cos(\theta + \beta), \quad \dot{Y}_A = \dot{\gamma}_1\sin(\theta + \beta),$
$$\left[m_1\left(1 + \frac{\kappa_1^2}{R^2}\right) + m_C\right]\dot{\gamma}_1 - m_C\ddot{\gamma}_2 h\sin\beta + m_C\dot{\gamma}_2^2 h\cos\beta = \frac{\Gamma}{R_1},$$
$$\left(\frac{1}{2}m\kappa_1^2 + m_Ch^2 + I_C\right)\ddot{\gamma}_2 + \frac{1}{2}m\kappa_1^2\ddot{\gamma}_3 - m_Ch\left[\ddot{\gamma}_1\sin\beta + \dot{\gamma}_1(\dot{\gamma}_2 + \dot{\gamma}_3)\cos\beta\right] = 0,$$
$$\frac{1}{2}m\kappa_1^2(\ddot{\gamma}_2 + \ddot{\gamma}_3) = \Gamma.$$

10.1 $|\bar{\omega}| = 3.135$ rad/s, $\theta = 94.87^\circ$, $\Delta\bar{H}_G = -3.130I'\bar{i}$, $|\bar{\omega}_2| = -0.1667$ rad/s.

10.3 $\omega_x = 50$ rad/s, $\omega_z = 450$ rad/s, $\beta = 83.66^\circ$, $\theta = 86.82^\circ$.

10.6 $\bar{\omega} = 7.2902\left(10^{-5}\right)$ rad/s about an axis through the center of the Earth at 0.1558° from the polar axis in the meridional plane at 90° from the meridian of impact.

10.8 Precession about the z azis; $\bar{\omega} = 7.594\bar{i} + 6.275\bar{k}$ rad/s @ $\max(\theta) = 45.91^\circ$; $\bar{\omega} = 5.534\bar{i} + 6.766\bar{k}$ rad/s @ $\min(\theta) = 13.57^\circ$.

10.11 Looping precession, $\min(\theta) = 53.130^\circ$, $\max(\theta) = 53.300^\circ$, $\dot{\psi} = 0$ at $\theta = 53.199^\circ$.

10.14 $\psi = A_1 \cos\left(\dfrac{\omega^2}{\sigma} t + \nu_1\right) + A_2 \cos(\sigma t + \nu_2)\,, \quad \omega^2 = \dfrac{mgL}{I'}, \quad \sigma = \dfrac{I}{I' + mL^2},$

$$\theta = \frac{\pi}{2} + A_1 \sin\left(\frac{\omega^2}{\sigma} t + \nu_1\right) - A_2 \sin(\sigma t + \nu_2)\,.$$

10.16 $\dfrac{I'\lambda}{c} \gg 1, \quad t \gg \dfrac{I'\Omega_1}{c\Omega_0}, \quad \dfrac{I'}{I} = O(1)\,.$

10.19 $\left[(I + C)(\cos\theta)^2 + (I' + A)\,\dot{\psi}\,(\sin\theta)^2 + A\right]\dot{\psi} + I\dot{\phi}\cos\theta = p_\psi,$

$$I\left(\dot{\psi}\cos\theta + \dot{\phi}\right) = p_\phi, \;\; (I' + B)\,\ddot{\theta} + (I + C - I' - A)\dot{\psi}^2 \sin\theta\cos\theta + I\dot{\phi}\dot{\psi}\sin\theta = 0.$$

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