SPECIAL DIFFERENTIAL EQUATIONS AND FUNCTIONS

In modelling problems in engineering and physics, several special functions are developed which are solutions of certain differential equations in power series. Bessel and Legendre differential equations are the two most common of them that are discussed in this section

4.1. Bessel Differential Equations and Bessel Functions

The family of Bessel differential equations

$x^2y''+xy'+(x^2 - p^2)y=0$ where $p \in R$

appears in many problems in engineering and physics. The simplest form of the Bessel differential equation's order is zero

Bessel D.E. of Order p=0

Let p=0 in the family of Bessel differential equation in the above. So

$xy''+y'+xy=0$

which is known as the zeroth order Bessel differential equation.

Let's first normalize this differential equation

$y''+(1/x)y'+y=0$

since $1/x$ is not analytic at $x_0 = 0$, and

$$\lim_{x\to 0} 1/x = 1; \lim_{x\to 0} x^2(1) = 0$$

$x_0 = 0$ is a regular singular point for the differential equation. According to the Frobenius theorem, the appropriate solution proposal is,

$$\sum_{n=0}^{\infty} c_n (n+r)(n+r1-1)x^{r+n-1} + \sum_{n=0}^{\infty} c_n (n+r)x^{r+n-1} + \sum_{n=0}^{\infty} c_n x^{r+n+1} = 0$$

$$\sum_{n=0}^{\infty} c_n (n+r)^2 x^{r+n-1} + \sum_{n=0}^{\infty} c_n x^{r+n+1} = 0$$

$n \to n-2$ *substitution in the last therm yields*

$$\sum_{n=0}^{\infty} c_n (n+r)^2 x^{r+n-1} + \sum_{n=2}^{\infty} c_{n-2} x^{r+n+1} = 0$$

$$c_0 r^2 x^{r-1} + c_1 (r+1)^2 x^r + \sum_{n=2}^{\infty} \left[c_n (n+r)^2 + c_{n-2} \right] x^{r+n-1} = 0$$

where the indicial equation with the two roots is

$r^2=0$ and $r_{1,2} = 0$, $c_0 \neq 0$

The condition equation is

$c_1(r+1)^2=0$ implies $c_1=0$.

The recurrence relation ; $c_n(n+r)^2+c_{n-2}=0$

Leads to

$$c_n = -\frac{c_{n-2}}{(r+n)^2}, n \geq 2$$

Therefore

$$n=2 \rightarrow c_2 = -\frac{c_0}{(r+2)^2}$$

$$n=3 \rightarrow c_3 = -\frac{c_1}{(r+3)^2} = 0 \rightarrow c_{2n+1} = 0, n = 1,2,3$$

while

$$n=4 \rightarrow c_4 = -\frac{c_2}{(r+4)^2} = \frac{c_0}{(r+2)^2(r+4)^2}$$

$$n=6 \rightarrow c_6 = -\frac{c_4}{(r+6)^2} = -\frac{c_0}{(r+2)^2(r+4)^2(r+6)^2} = -\frac{c_0\Gamma(r/2)^2}{2^6\left(\Gamma\left(\frac{r}{2}+3\right)\right)^2}$$

$$c_{2n} = (-1)^n \frac{c_0}{(r+2)^2(r+4)^2 \ldots (r+2n)^2} = (-1)^n \frac{c_0\left(\Gamma(r/2)\right)^2}{2^{2n}\left(\Gamma\left(\frac{r}{2}+n\right)\right)^2}$$

where

$$\Gamma\left(\frac{r}{2}+n\right) = \Gamma\left(\frac{r}{2}\right)\left(\frac{r}{2}+1\right)\left(\frac{r}{2}+2\right)\ldots\left(\frac{r}{2}+n\right)$$

is the Gamma function that we will discuss in the forthcoming section. Further substitution of these yields that;

$$y(x,r) = x^r c_0 \sum_{n=0}^{\infty} (-1)^n \frac{c_0\left(\Gamma(r/2)\right)^2}{2^{2n}\left(\Gamma\left(\frac{r}{2}+n\right)\right)^2}$$

Let c_0=1. Then recalling

$$\Gamma(0) = 1, and\, \Gamma(n) = n!\, one\, has$$

$$y_1 = y(x,1)|_{r=0} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(n!)^2\, 2^{2n}} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^4}{2^2 4^2 6^2} + \ldots + (-1)^n \frac{x^{2n}}{(n!)^2\, 2^{2n}} + \ldots$$

So, one of the solution is

$$y_1 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(n!)^2\, 2^{2n}}$$

This special series is known as Bessel function of first kind (standing 1st solution $y|_{r=0}$) and order zero (p=0), and denoted by J_0; so the first of the two solutions is found to be

$y_1 = J_0(x)$

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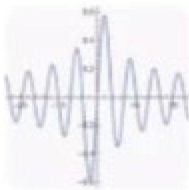

What is the difference between Bessel function of the first ...

Apr 18, 2018 — When changing the n subscript value (i.e. determining the order of Bessel function), I know that when graphing order zero, **Bessel function peaks** ...

2 answers · Top answer: Let's start with the true definition of Bessel type 1 functions:

Since the indicial equation has a double root $r_{1,2}$=0, the other independent solution is found through the differentiation of y(x,r) with respect to r,

To simplify the computation let us recall

$$y(x,r) = x^r c_0 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(r+2)^2 (r+4)^2 \ldots (r+2n)^2} x^{2n}$$

Let

$$F(r) = (r+2)(r+4)\ldots(r+2n)$$

Then for $C_0 = 1$

$$y(x,r) = x^r \sum_{n=0}^{\infty} (-1)^n F(r)^{-2} x^{2n}$$

The one has

$$y_2 = \frac{\partial y(x,r)}{\partial r}|_{r=0} = \left[x^r \ln|x|\, y_1 - x^r \sum_{n=1}^{\infty} (-1)^n \frac{\partial}{\partial r} F(r)^{-2} x^{2n} \right]_{r=0}$$

$$= J_0(x)\ln|x| + 2\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{\left(F(r+2n)\right)^3}\left(\frac{1}{r+2}F(r+2n)+\ldots+\frac{1}{r+2n}F(r+2n)\right)\Big|_{r=0}$$

$$= J_0(x)\ln|x| - 2\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{\left(F(r+2n)\right)^2}\left(\frac{1}{r+2}+\ldots+\frac{1}{r+2n}\right)\Big|_{r=0}$$

$$= J_0(x)\ln|x| - 2\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{2^{2n}(n!)^2}\left(\frac{1}{2}+\ldots+\frac{1}{2n}\right)$$

Hence

$$y_2 = J_0(x)\ln|x| + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{2^{2n}(n!)^2}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)$$

This solution y_2 is traditionally given in Weber type of expression

$$y_2 = \frac{2}{\pi}\left[\left(\ln\left|\frac{x}{2}\right|+\gamma\right)J_0(x)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n}}{2^{2n}(n!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)\right]$$

where γ is the Euler constant

$$\gamma = \lim_{n\to 0}\left[\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)-\ln n\right] \cong 0.5772$$

This solution with a slight modification, is known as Bessel function of the 2nd kind and order zero and denoted by $Y_0(x)$,

$$Y_0(x) = \frac{2}{\pi}\left[\left(\ln\left|\frac{x}{2}\right|+\gamma\right)J_0(x)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n}}{2^{2n}(n!)^2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)\right]$$

Then the second of the two linearly independent solutions is

$$y_2 = Y_0(x)$$

The general solution of the Bessel differential equation of order zero is then obtained as;

$$y_{gen} = c_1 J_0(x) + c_2 Y_0(x)$$

Bessel D.E. of Order $p\neq 0$