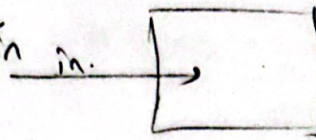


1

Mehmet Emre ÖZ

$$V = 20000 \text{ m}^3$$

$$600 \frac{\text{m}^3}{\text{min}}$$



$$y(0) = 0 \rightarrow \text{constraint.}$$

$$\left. \begin{array}{l} y \rightarrow \text{Clean air in room} \\ V \rightarrow \text{Volume of room} = 20000 \text{ m}^3 \\ R \rightarrow \text{Rate of transport air} = 600 \frac{\text{m}^3}{\text{min}} \end{array} \right\} \frac{dy}{dt} = R - \frac{R}{V} \cdot y \quad *$$

$$\frac{dy}{dt} = 600 - \frac{600}{20000} y \quad *$$

$$\frac{dy}{dt} = 600 - \frac{6}{200} y$$

$$y' + 0.03y - 600 = 0 \quad \begin{array}{l} p(t) = 0.03 \\ q(t) = 600 \end{array}$$

$$e^{\int p(t) dt} = e^{\int 0.03 dt} = e^{0.03t}$$

$$y \cdot e^{0.03t} = \int q(t) e^{0.03t} dt$$

$$y e^{0.03t} = 600 \cdot \frac{1}{0.03} e^{0.03t} + C_1$$

$$y = 20000 + C_1 e^{-0.03t}$$

$$y(0) = 20000 + C_1 e^{-0.03 \cdot 0} = 0$$

$$C_1 = -20000$$

$$y = 20000 - 20000 \cdot e^{-0.03 \cdot t}$$

$$V = 20000 \cdot (0.9) = 18000 \text{ m}^3$$

$$18000 = 20000 \cdot (1 - e^{-0.03t})$$

$$0.9 = 1 - e^{-0.03 \cdot t} \rightarrow e^{-0.03 \cdot t} = 0.1$$

$$t = 76.7 \text{ min.}$$

2

Newton's cooling Law

$$\frac{dT}{dt} = k \cdot (T - T_s)$$

Separable DE.

$$dT = k \cdot (T - T_s) dt$$

$$T_i = 90^\circ\text{C}$$

$$T_s = 0^\circ\text{C}$$

$$T(60) = 40^\circ\text{C}$$

$$\int \frac{1}{T - T_s} dT = \int k dt = \ln|T - T_s| = kt + C$$

$$T - T_s = e^{kt+C}$$

$$T = T_s + e^{kt+C} \Rightarrow T(t)$$

init.

$$t = 0$$

$$T_i = T(0) = T_s + e^C \rightarrow C = \ln|T_i - T_s|$$

$$T = T_s + e^{kt} \cdot e^{\ln|T_i - T_s|}$$

$$T = T_s + (T_i - T_s) \cdot e^{kt}$$

for k.

$$t = 60s$$

$$T = 0 + (90 - 0) \cdot e^{k \cdot 60} = 40$$

$$90 \cdot e^{k \cdot 60} = 40$$

$$e^{k \cdot 60} = 0.66$$

$$k \cdot 60 = \ln(0.66)$$

$$k = -0.0135$$

$$T = T_s + (T_i - T_s) \cdot e^{-0.0135 \cdot t}$$

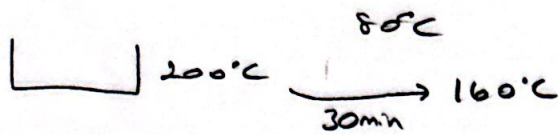
$$T = 90 \cdot e^{-0.0135 \cdot t}$$

$$\text{at } t = 5 \text{ min} = 300s.$$

$$T = 90 \cdot e^{-0.0135 \cdot 300} = 1.57^\circ\text{C}$$

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Mehmet Emre ÖZ



$$\frac{dT}{dt} = k \cdot (T_i - T_s)$$

$$T = T_s + (T_i - T_s) \cdot e^{k \cdot t}$$

$$160 = 80 + (200 - 80) \cdot e^{k \cdot 30}$$

$$160 = 80 + 120 \cdot e^{k \cdot 30}$$

$$120 \cdot e^{k \cdot 30} = 80$$

$$e^{k \cdot 30} = 0.666$$

$$k \cdot 30 = \ln(0.666) = -0.4055$$

$$k = -0.01351$$

$$T = 80 + (200 - 80) \cdot e^{-0.01351 \cdot t}$$

$t = 60 \text{ min.}$

$$T = 80 + (200 - 80) \cdot e^{-0.01351 \cdot 60} = \underline{\underline{133.35^\circ\text{C}}}$$

$$100 = 80 + (200 - 80) \cdot e^{-0.01351 \cdot t}$$

$$20 = 120 \cdot e^{-0.01351 \cdot t}$$

$$e^{-0.01351 \cdot t} = \frac{1}{6}$$

$$-0.01351 \cdot t = \ln\left(\frac{1}{6}\right) = -1.7917$$

$$\underline{\underline{t = 132.62 \text{ min}}}$$

4

$$120 \text{ kg} \downarrow \quad V_1 = 200 \text{ km/h} = 55.5 \text{ m/s} \quad V = 2.5 \text{ m/s} \Rightarrow 0.25 \text{ m/s}$$

$$\bullet V_2 = 0.25 \text{ m/s}$$

$$m \cdot \frac{dv}{dt} = mg - bv \rightarrow \text{drag force}$$

$$\int \frac{1}{m} \cdot (mg - bv) dt = \int dv$$

$$\frac{1}{b} \cdot \ln \left(\frac{mg - bv}{mg} \right) = t + c$$

$$\frac{mg - bv}{mg} = e^{b(t+c)}$$

max or limit velocity is Terminal velocity

$$t \text{ goes } \rightarrow \infty, \quad e^{b(t+c)} = 0$$

$$V_{\text{terminal}} = \frac{mg}{b}, \text{ as known.}$$

↑ Force equilibrium
↓

Person reach 10% of $V_{\text{terminal}} = V = 0.1 \cdot V_{\text{terminal}}$

$$V = \frac{mg}{b} \cdot (1 - e^{b(t+c)})$$

$$e^{b(t+c)} = 1 - \frac{10b}{mg} \Rightarrow b(t+c) = \ln \left(1 - \frac{10b}{mg} \right)$$

$$t = -\frac{c}{b} + \frac{1}{b} \cdot \ln \left(1 - \frac{10b}{mg} \right)$$

Solve the following homogeneous linear system using matrix form.

5

$$x_1' = 7x_1 - x_2 + 6x_3 + \cos 2t$$

$$x_2' = -10x_1 + 4x_2 - 12x_3$$

$$x_3' = -2x_1 + x_2 - x_3$$

Convert into matrix form

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos 2t$$

$$\text{if } x = \alpha e^{\lambda t} \rightarrow \begin{aligned} x_1 &= \alpha_1 e^{\lambda t} \\ x_2 &= \alpha_2 e^{\lambda t} \\ x_3 &= \alpha_3 e^{\lambda t} \end{aligned}$$

$$-(\lambda - 7) = 7 - \lambda$$

$$|\lambda I - A| = \begin{vmatrix} 7-\lambda & -1 & -6 \\ +10 & 4-\lambda & +12 \\ +2 & -1 & -1-\lambda \end{vmatrix} = 0$$

$$(7-\lambda) \cdot \begin{vmatrix} 4-\lambda & +12 \\ -1 & -1-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} +10 & +12 \\ +2 & -1-\lambda \end{vmatrix} - (-6) \begin{vmatrix} 10 & 4-\lambda \\ 2 & -1 \end{vmatrix} = 0$$

$$(7-\lambda) [(4-\lambda)(-1-\lambda)+12] + [-10(-1-\lambda)-24] + 1[-10+2(4-\lambda)] = 0$$

$$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$$

$$(\lambda-2)(\lambda-3)(\lambda-5) = 0$$

$$\lambda_1 = 2 \quad \lambda_2 = 3 \quad \lambda_3 = 5$$

for $\lambda = 2$

$$A\alpha = \lambda\alpha \rightarrow \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$7\alpha_1 - \alpha_2 + 6\alpha_3 = 2\alpha_1$$

$$-10\alpha_1 + 4\alpha_2 - 12\alpha_3 = 2\alpha_2$$

$$-2\alpha_1 + \alpha_2 - \alpha_3 = 2\alpha_3$$

$$5\alpha_1 - \alpha_2 + 6\alpha_3 = 0$$

$$-10\alpha_1 + 2\alpha_2 - 12\alpha_3 = 0$$

$$-2\alpha_1 + \alpha_2 - 3\alpha_3 = 0$$

$$5\alpha_1 - \alpha_2 + 6\alpha_3 = 0$$

$$\rightarrow \alpha_3 = -\alpha_1$$

$$-2\alpha_1 + \alpha_2 - 3\alpha_3 = 0$$

$$\alpha_2 = -\alpha_1$$

Let $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = -1$

$$\text{So that for } \lambda = 2 \rightarrow \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$x_1 = \alpha e^{\lambda_1 t} \rightarrow$ so the corresponding soln is

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{bmatrix}$$

$$\text{for } \lambda = 3 \rightarrow x_2 = \alpha e^{\lambda_2 t} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix}$$

$$\text{for } \lambda = 5 \rightarrow x_3 = \alpha e^{\lambda_3 t} = \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix} e^{5t} = \begin{bmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{bmatrix}$$

general soln

$$x(t) = c_1 \begin{bmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix} + c_3 \begin{bmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{bmatrix}$$

$$H = \{e^{2t}, e^{3t}, e^{5t}\}$$

$$D = \{\cos 2t, \sin 2t\}$$

Since $H \cap D = \emptyset$ empty set, a particular soln. proposed be

$$x_p = C^1 \cos 2t + C^2 \sin 2t$$

$$x_{p'} = -2 \cdot C^1 \sin 2t + 2 C^2 \cos 2t$$

Then eqn. 1 implies

$$-2 C^1 \sin 2t + 2 C^2 \cos 2t = A C^1 \cos 2t + A C^2 \sin 2t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos 2t$$

Equating the coefficients of $\{\cos 2t, \sin 2t\}$

$$\sin 2t: -2 C^1 = A C^2$$

$$\cos 2t: 2 C^2 = A \cdot C^1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow 2 A C^2 = A^2 C^1 + A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$-4 C^1 = A^2 C^1 + A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow (A^2 + 4I) C^1 = A \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow C^1 = (A^2 + 4I)^{-1} \times A \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$C^1 = \{-0.25, -0.19, -0.05\}$$

$$C^2 = -2A^{-1}C^1$$

$$C^2 = \{0.24, 0.21, -0.17\}$$

$$X_p = \{-0.25, -0.19, -0.05\} \cdot \cos 2t + \{0.24, 0.21, -0.17\} \sin 2t$$

$$X_{gen} = X_n + X_p$$

$$X_1 = C_1 e^{2t} + C_2 e^{3t} + 3C_3 e^{5t} - 0.25 \cos 2t + 0.24 \sin 2t$$

$$X_2 = -C_1 e^{2t} - 2C_2 e^{3t} - 6C_3 e^{5t} - 0.19 \cos 2t + 0.21 \sin 2t$$

$$X_3 = -C_1 e^{2t} - 1C_2 e^{3t} - 2C_3 e^{5t} - 0.05 \cos 2t - 0.17 \sin 2t$$

$$6x^2 y'' + 7xy' - (1+x^2)y = 0 \quad \text{about } x_0 = 0$$

(1)

$$y'' + \frac{7x}{6x^2} y' - \frac{1+x^2}{6x^2} y = 0$$

$$p(x) = \frac{7x}{6x^2} \quad q(x) = -\frac{1+x^2}{6x^2}$$

$$\lim_{x \rightarrow x_0=0} x \cdot \left(\frac{7x}{6x^2} \right) = \frac{7}{6}$$

$$\lim_{x \rightarrow x_0=0} x^2 \left(-\frac{1+x^2}{6x^2} \right) = -\frac{1}{6}$$

$$y(x) = \sum_{n=0}^{\infty} c_n \cdot x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} c_n \cdot (n+r) x^{n-1+r}$$

$$y'' = \sum_{n=0}^{\infty} c_n (n+r) \cdot (n-1+r) \cdot x^{n-2+r}$$

$$6x^2 \cdot \sum_{n=0}^{\infty} c_n (n+r) \cdot (n-1+r) \cdot x^{n-2+r} + 7x \cdot \sum_{n=0}^{\infty} c_n (n+r) \cdot x^{n-1+r} - (1+x^2) \cdot \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$6 \cdot \sum_{n=0}^{\infty} c_n (n+r) (n-1+r) \cdot x^{n+r} + 7 \cdot \sum_{n=0}^{\infty} c_n (n+r) x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r} -$$

$$- \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0$$

$$- \sum_{n=2}^{\infty} c_{n-2} x^{n+r} = 0$$

(2)

$$6 \cdot \sum_{n=0}^{\infty} C_n (n+r)(n-1+r) x^{n+r} + 7 \cdot \sum_{n=2}^{\infty} C_n \cdot (n+r) x^{n+r} - \sum_{n=2}^{\infty} C_n \cdot x^{n+r} - \underbrace{\sum_{n=2}^{\infty} C_{n-2} x^{n+r}}_*$$

$n=0, 1.$

$$6 \left(C_0 (0+r)(0-1+r) \cdot x^{0+r} + C_1 \cdot (1+r)(1-1+r) x^{1+r} \right) + \sum_{n=2}^{\infty} C_n (n+r)(n-1+r) x^{n+r} +$$

$$7 \cdot \left(C_0 \cdot (0+r) x^{0+r} + C_1 \cdot (1+r) \cdot x^{1+r} \right) + \sum_{n=2}^{\infty} C_n (n+r) x^{n+r} -$$

$$C_0 \cdot x^{0+r} + C_1 \cdot x^{1+r} + \sum_{n=2}^{\infty} C_n \cdot x^{n+r} - \sum_{n=2}^{\infty} C_{n-2} x^{n+r}.$$

$$C_0 \cdot \left(6 \cdot r \cdot (-1+r) \cdot x^r + 7 \cdot r \cdot x^r - x^r \right) +$$

$$C_1 \cdot \left(6 \cdot (1+r)(1-1+r) \cdot x^{1+r} + 7 \cdot (1+r) \cdot x^{1+r} - x^{1+r} \right) +$$

$$\sum_{n=2}^{\infty} \left[C_n (n+r)(n-1+r) x^{n+r} + C_n \cdot (n+r) x^{n+r} + C_n x^{n+r} - C_{n-2} x^{n+r} \right] = 0$$

$$\sum_{n=2}^{\infty} x^{n+r} \left[C_n (1+(n+r) \cdot (n-1+r) + (n+r) + 1) - C_{n-2} \right] = 0$$

in order to be satisfied, every power of x must be 0. So,

$$C_0 \cdot (6 \cdot r \cdot (-1+r) + 7 \cdot r - 1) = 0$$

$$C_1 (6(1+r) \cdot (r) + 7 \cdot (1+r) - 1) = 0$$

$$\sum_{n=2}^{\infty} C_n ((n+r)(n-1+r) + (n+r) + 1) - C_{n-2} = 0$$

Because $C_0 \neq 0$ we solve,

$$6 \cdot r(-1+r) + 7r - 1 = 0$$

$$6r^2 - 6r + 7r - 1 = 0$$

$$6r^2 + r - 1 = 0$$

$$\begin{array}{cc} 3r & -1 \\ 2r & +1 \end{array}$$

$$(3r-1) \cdot (2r+1) = 0$$

$$r_1 = \frac{1}{3} \quad r_2 = -\frac{1}{2}$$

for $r_{22} = -\frac{1}{2}$

(4)

$$c_1 \left(6 \cdot \left(1 - \frac{1}{2}\right) \cdot \left(-\frac{1}{2}\right) + 7 \cdot \left(1 - \frac{1}{2}\right) - 1 \right) = 0$$

$$c_1 \cdot 1 = 0 \Rightarrow c_1 = 0$$

$$c_n \left(\left(n - \frac{1}{2}\right) \left(n - 1 - \frac{1}{2}\right) + \left(n - \frac{1}{2}\right) - 1 \right) = c_{n-2}$$

$$c_n \left(\left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) + \left(n - \frac{3}{2}\right) \right) = c_{n-2}$$

$$c_n = \frac{c_{n-2}}{\left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) + \left(n - \frac{3}{2}\right)} = \frac{4 \cdot c_{n-2}}{(2n-1)(2n-3) + (4n-6)}$$

$$c_1 = 0$$

$$c_2 = \frac{4c_0}{4 \cdot 3 \cdot 1 + 2} = \frac{4c_0}{5}$$

$$c_3 = \frac{c_1}{\dots} = 0$$

$$c_4 = \frac{c_2}{7 \cdot 5 + 10} = \frac{4c_0}{5} \cdot \frac{45}{45}$$

$$c_5 = 0$$

(5)

$$y_1(x) = C_0 \cdot x^r + C_1 \cdot x^{1+r} + C_2 \cdot x^{2+r} + C_3 \cdot x^{3+r} + \dots$$

from $C_1 = C_3 = C_5 = \dots$

$$y_1(x) = C_0 \cdot x^{\frac{1}{2}} \left(1 + \frac{9}{50} \cdot x^2 + \frac{9}{50} \cdot \frac{1}{175} x^4 + \dots \right)$$

$$y_2(x) = 4 \cdot C_0 \cdot x^{-\frac{1}{2}} \left(1 + \frac{1}{5} \cdot x^2 + \frac{1}{5} \cdot \frac{1}{45} x^4 + \dots \right)$$

$$\underline{y(x) = y_1(x) + y_2(x)}$$