Bessel Function of Order p; $2p \notin \mathbb{Z}^+$

13 Aralık 2023 Çarşamba 14:33

Let $2p$ is not positive integer. And $r=p$

$$C_n = \frac{-C_{n-2}}{(n+p)^2-p^2} = \frac{-C_{n-2}}{n^2+2np} = -\frac{C_{n-2}}{n(n+2p)} \; ; \; n \geq 2$$

$$n=2 \Rightarrow C_2 = \frac{-C_0}{2(2+2p)} = \frac{-C_0}{2^2 \cdot 1 \cdot (1+p)}$$

$$n=3 \Rightarrow C_3 = \frac{-C_1}{3(3+2p)} = 0, \; C_5 = 0, \dots, C_{2n+1} = 0 \quad n \geq 1$$

$$n=4 \Rightarrow C_4 = \frac{-C_2}{4(4+2p)} = \frac{C_0}{[2 \cdot 4][(2+2p)(4+2p)]}$$

$$= \frac{C_0}{2^4(1 \cdot 2)[(1+p)(2+p)]}$$

$$n=6 \Rightarrow C_6 = \frac{-C_4}{6(6+2p)} = -\frac{C_0}{[2 \cdot 4 \cdot 6][(2+2p)(4+2p)(6+2p)]}$$

$$= -\frac{C_0}{2^6(1 \cdot 2 \cdot 3)[(1+p)(2+p)(3+p)]}$$

$$C_{2n} = \frac{(-1)^n C_0}{[2 \cdot 4 \cdot 6 \cdots (2n)][(1+p)(2+p)\cdots(n+p)]} \; ; \; n \geq 1$$

or

$$C_{2n} = \frac{(-1)^n C_0}{2^{2n}(n!)[(1+p)(2+p)\cdots(n+p)]} \; ; \; n \geq 1$$

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$$y_1 = y(x,r)\Big|_{r=p} = C_0 x^r + C_1 x^{r+1} + C_2 x^{r+2} + C_3 x^{r+3} + \cdots \Big|_{r=p}$$

$$y_1 = C_0 x^p - \frac{C_0}{2^2 \cdot 1 \cdot (1+p)} x^{p+2} + \frac{C_0}{2^4(1 \cdot 2)[(1+p)(2+p)]} x^{p+4}$$

so,

$$y_1 = C_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p}}{2^{2n}(n!)[(1+p)(2+p)\cdots(n+p)]}$$

After some manipulations, for $C_0 = 1$

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p} \Gamma(p)}{2^{2n} n! [\Gamma(p)(1+p)(2+p)\cdots(n+p)]}$$

$$= \Gamma(p) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, 2^{2n} \Gamma(n+p)} x^{2n+p}$$

where Γ in

$$\Gamma(n+p) = \Gamma(p)(1+p)(2+p)\cdots(n+p)$$

is the Γ gamma function. This expression is, obviously reduced to J_0 for $p=0$.
y_1 is called Bessel function of 1st kind and order p, and denoted by $J_p(x)$

$$J_p(x) = \Gamma(p) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, 2^{2n} \Gamma(n+p)} x^{2n+p}$$

and the second linearly independent soln. is obviously,

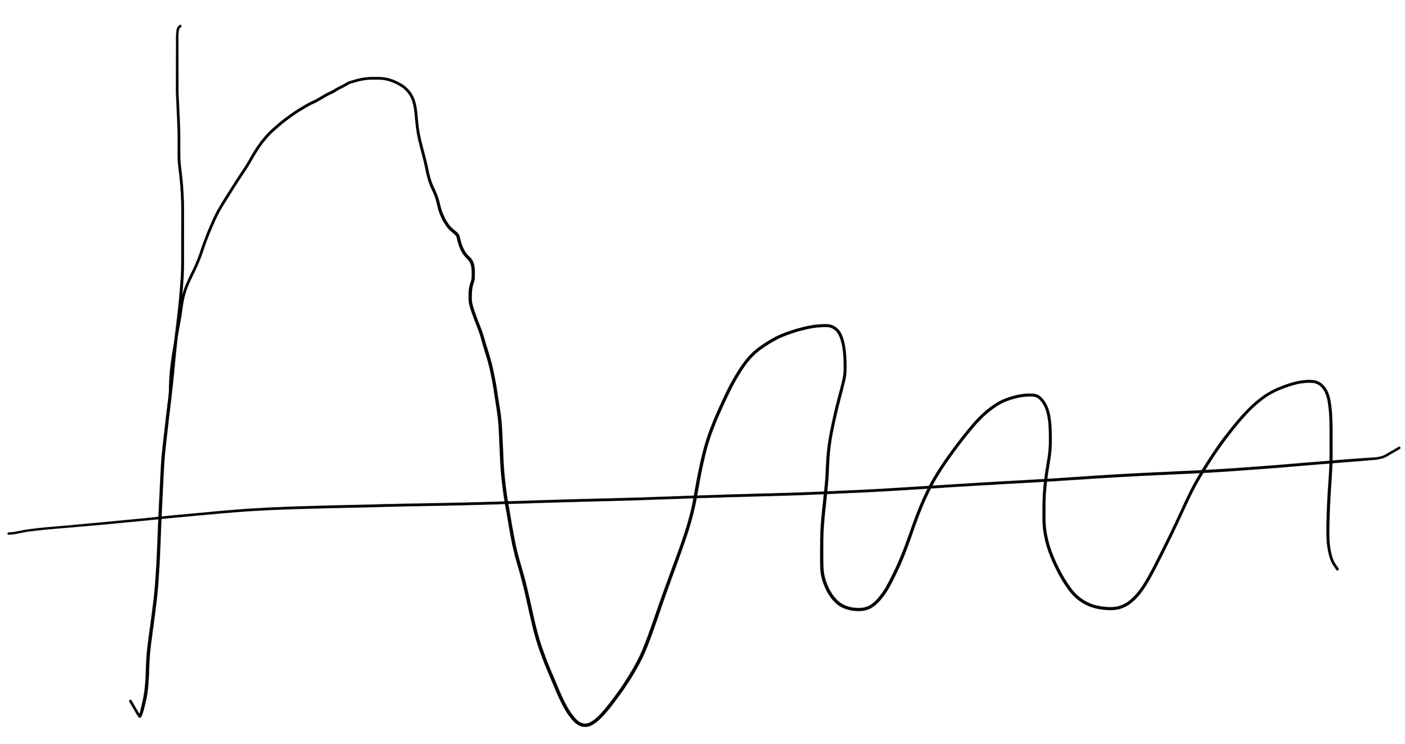
$$y_2 = J_{-p}(x)$$

For example, for $p = 1/4$

$$y_1 = \Gamma(1/4) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, 2^{2n} \Gamma(n+1/4)} x^{2n+1/4}$$

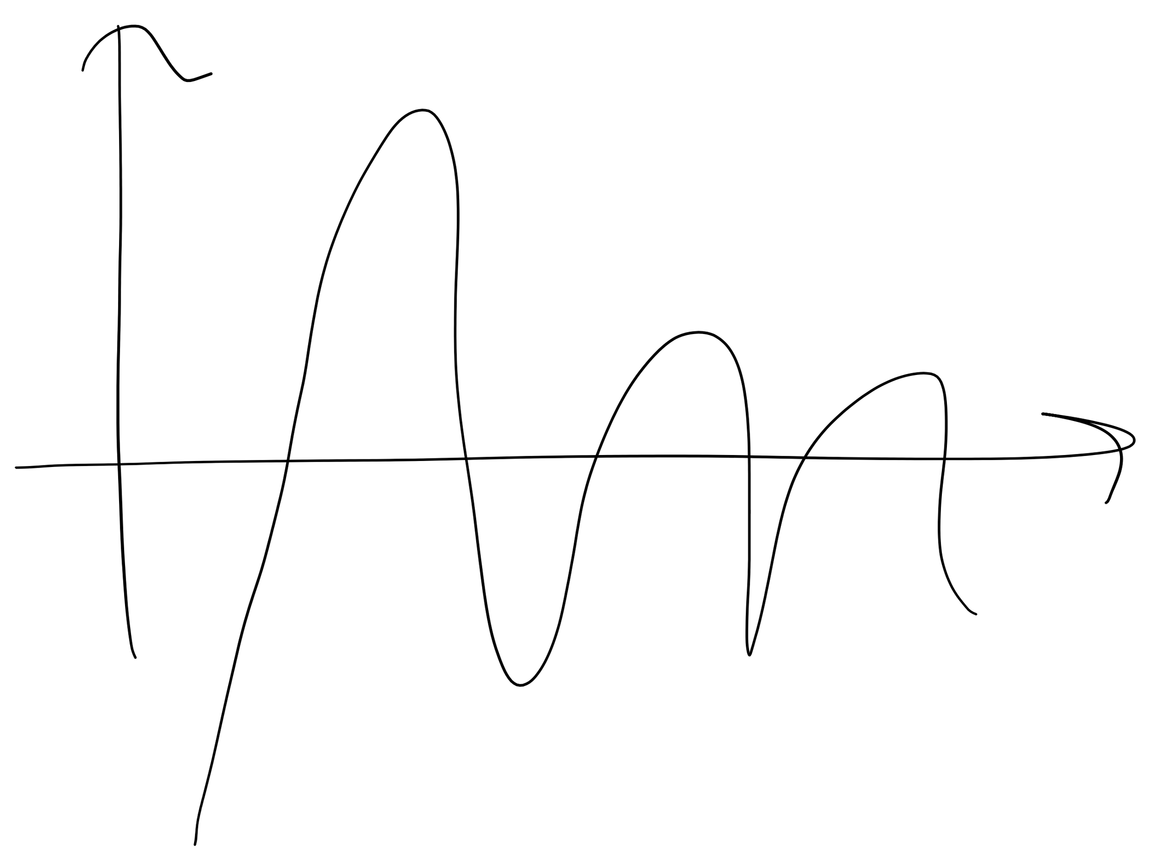
After some scaling

$y_1 = J_{1/4}(x)$, $y_2 = J_{-1/4}(x)$



$y_1 = J_{1/4}(x)$

(a)



$y_2 = J_{-1/4}(x)$