Bessel Functions of Order p; $2p \in \mathbb{Z}^+$, $m=2p$ odd

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In the family of Bessel D.E.

$$x^2y''+xy'+(x^2-p^2)y^2=0, \quad p\in\mathbb{R}$$

Equating the coefficients of x^i to zero one gets the indicial eqn. with two roots

$$r^2-p^2=0, \quad C_0\neq 0 \Rightarrow r_1=p>0, \; r_2=-p$$

The condition eqn.

$$C_1[(r+1)^2-p^2]=0 \Rightarrow C_1=0 \quad \text{unless,}$$

$$(r+1)^2-p^2=0$$

For $r=p>0$ $\quad (r+1)^2-p^2=2p+1\neq 0$

For $r=-p$ $\quad (r+1)^2-p^2=-2p+1=0$

$$C_0\neq 0 \Rightarrow r_1=p>0, \; r_2=-p$$

$C_0\neq 0$, arbitrary, $C_1=0$ unless $p\neq 1/2$

The recurrence relation is

$$[(n+r)^2-p^2]C_n+C_{n-2}=0$$

$$\Rightarrow C_n=-\frac{C_{n-2}}{(n+r)^2-p^2}, \quad n\geqslant 2$$

For $r=p$

The first soln. is found as before

$$y_1=\Gamma(p)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,2^{2n}\Gamma(n+p)}x^{2n+p}$$

For $r=-p$

The recurrence relation is

$$[(n+r)^2-p^2]C_n+C_{n-2}=0$$

$$\Rightarrow (n^2-2np)C_n+C_{n-2}=0$$

$$\Rightarrow n(n-2p)C_n+C_{n-2}=0$$

$$\Rightarrow n(n-m)C_n+C_{n-2}=0$$

$m=2p$, odd integer

$$C_n=\frac{-C_{n-2}}{(n-p)^2-p^2}=\frac{-C_{n-2}}{n^2-2np}=-\frac{C_{n-2}}{n(n-m)}; \quad n\geqslant 2$$

C_0 is arbitrary

$$n=2 \Rightarrow \quad C_2=\frac{-C_0}{2(2-m)}=\frac{-C_0}{2^2.1.(1+p)}$$

$$n=4 \Rightarrow \quad C_4=\frac{-C_2}{4(4-m)}=\frac{C_0}{[2.4][(2-m)(4-m)]}$$

$$y_3=\Gamma(-p)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,2^{2n}\Gamma(n-p)}x^{2n-p}$$

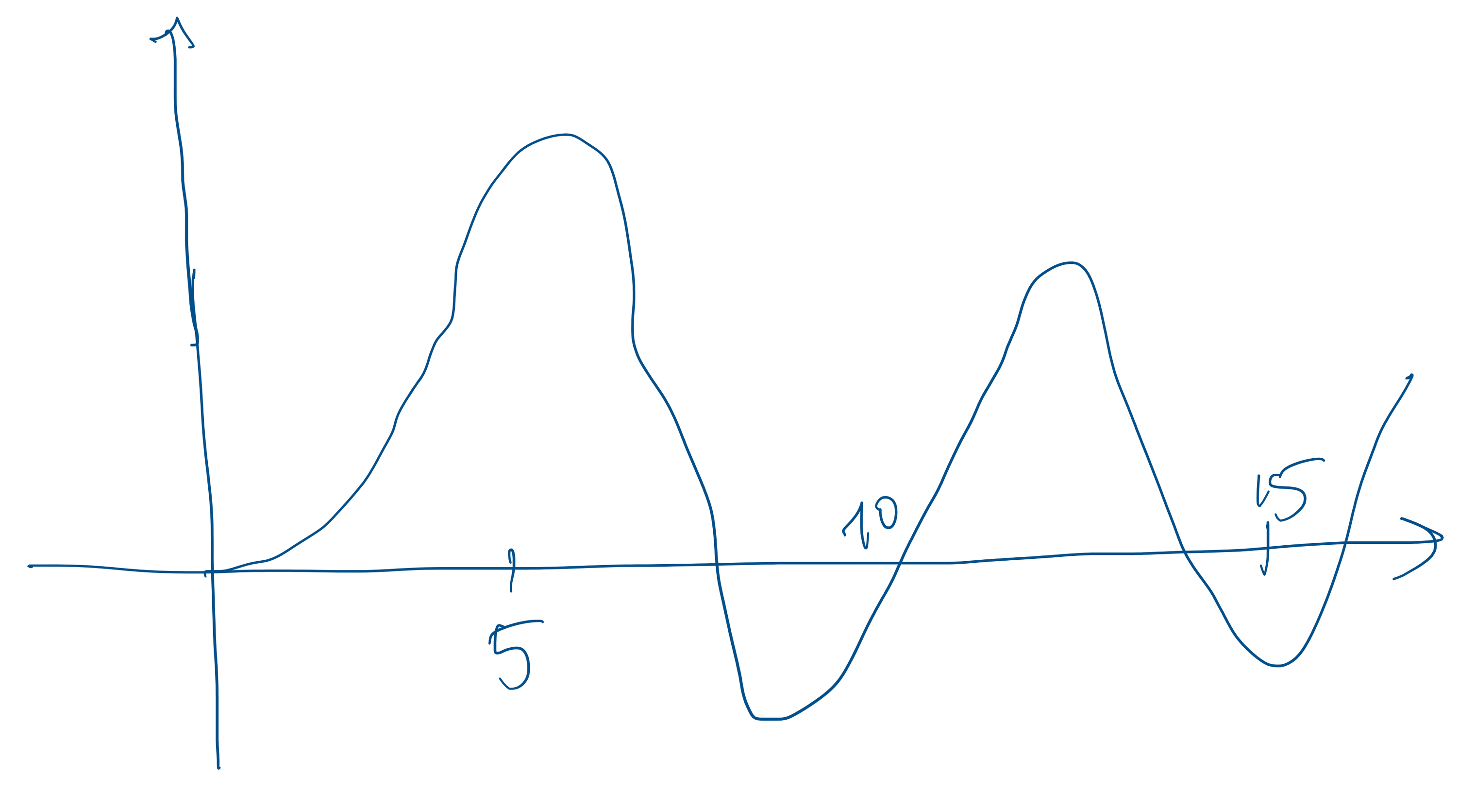
$C_1=0$

$$n=3 \Rightarrow C_3=\frac{-C_1}{3(3+2p)}=0, \; C_5=0, \ldots, C_{2n+1}=0, \; n\geqslant 1$$

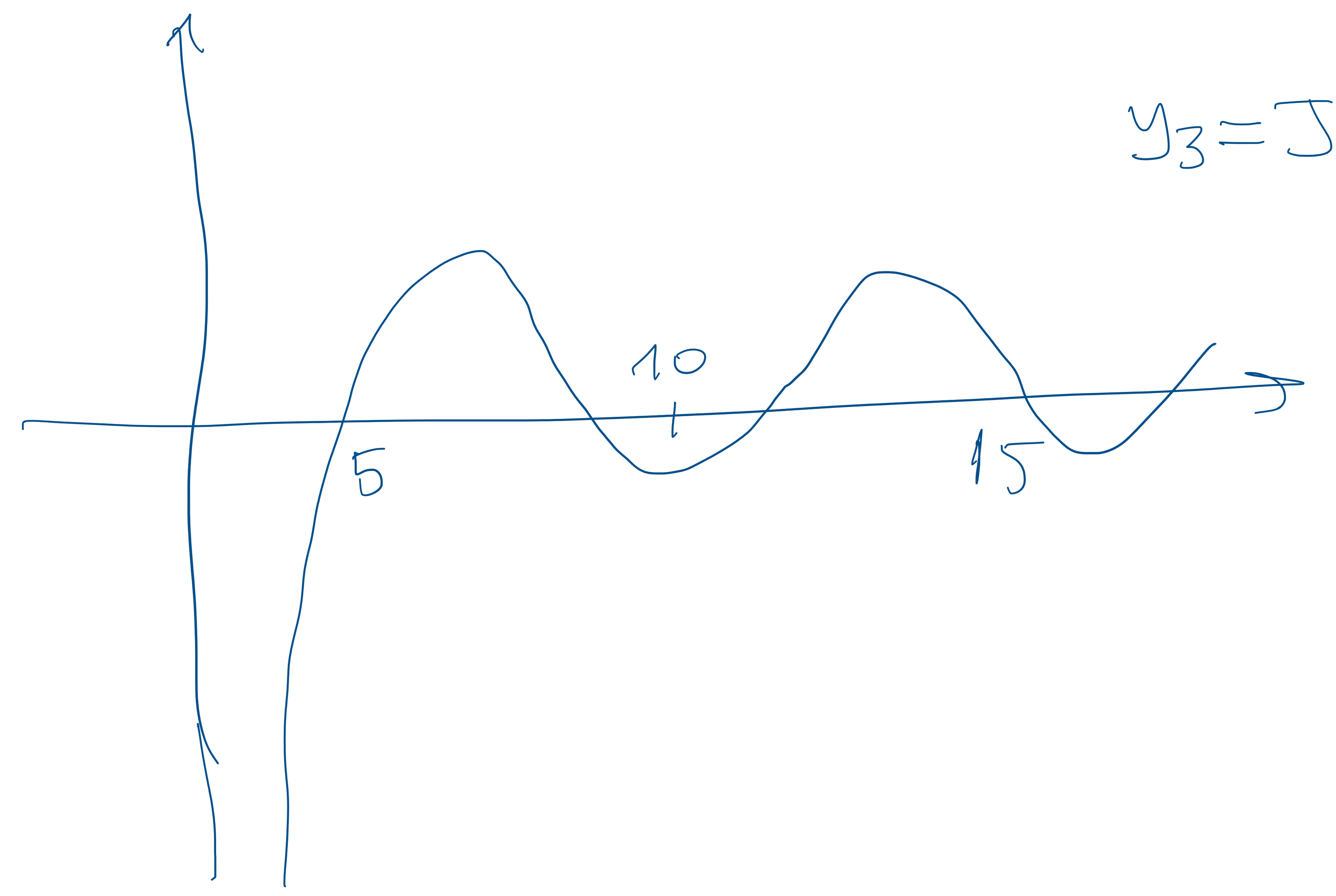
Therefore the eqn. $n(n-m)C_n+C_{n-2}=0$ does not bring any restriction since $C_{2n+1}=0$, $n\geqslant 1$

$y_4=0$

y_3 is the second linear independent soln. of the Bessel eqn.



$y_1=J_{7/2}(x)$



$y_3=J_{-7/2}(x)$