Sturm-Liouville Boundry Value Problem

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Boundary value problems for second order linear ordinary differential equations consist of a differential eqn.

$$p(x)y''+q(x)y'+r(x)y=0$$

And one set of two boundry conditions which the

$y(x_0)=\alpha_1,\ y(x_1)=\beta_1$
$y'(x_0)=\alpha_2,\ y'(x_1)=\beta_2$
$y'(x_0)=\alpha_3,\ y(x_1)=\beta_3$
$y(x_0)=\alpha_4,\ y'(x_1)=\beta_4$

solution of the differential eqn. is going to satisfy. In this section we'll discuss a special kind of boundary value problems; Sturm-Liouville BVP. Characteristic functions, orthogonality of functions, and Fourier series expansions are some of the aspects we will encounter.

Sturm-Liouville BVP

Definition: A boundary value problem that consists of a second order linear ordinary differential eqn.

$$(p(x)y')'+[q(x)+\lambda r(x)]y=0 \quad (1)$$

and boundry conditions

$$\alpha_1 y(x_0)+\alpha_2 y'(x_0)=0 \quad (2)$$
$$\beta_1 y(x_1)+\beta_2 y'(x_1)=0$$

is a sturm-liouvielle BVP.

where p, q, r, p' are real continous functions and $p(x)>0$, $r(x)>0$, in an interval $a\le x\le b$, and λ is a parameter independent of x.

In a boundry conditions $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real constants such that

$$\alpha_1\cdot\alpha_2\neq 0,\quad \beta_1\beta_2\neq 0$$

Legendre, Bessel and many other ODE's can be written in the form of a Sturm-Liouville eqn.

Example!

i. Legendre's eqn.

$$(1-x^2)y''-2xy'+n(n+1)y=0$$

may be written as

$[py']'+\lambda y=0$ with

$p=1-x^2,\ q=0,\ r=1,\ \lambda=n(n+1)$

ii. Bessel's Eqn.

$$x^2y''+xy'+(x^2-v^2)y^2=0$$

divided by x

$$xy''+y'+\left(x-\frac{v^2}{x}\right)y=0$$

$$[xy']'+\left(x-\frac{v^2}{x}\right)y=0 \quad (2)$$

$p=x$

$q=-\frac{v^2}{x},\ r=x,\ \lambda=1$

$q=x,\ r=-\frac{1}{x},\ \lambda=v^2$

It can be seen that $y(x)=0$ is the solution for (1) and (2). It's trivial soln, but we want to find non-trivial solns. in terms or Eigenfunctions for particular value of λ (eigenvalues).

Example: $y''+\lambda y=0,\ y(0)=0,\ y(\pi)=0$

$$[y']'+\lambda y=0 \quad \underbrace{\begin{matrix}p=1\\ q=0\\ r=1\end{matrix}}_{\text{in (1)}} \quad \underbrace{\begin{matrix}\alpha_1=1 & \alpha_2=0\\ \beta_1=1 & \beta_2=0\\ x_1=0 & x_2=\pi\end{matrix}}_{\text{in (2)}}$$

a) λ is negative, $\lambda=-v^2$

$y''-v^2y=0 \Rightarrow y(x)=c_1e^{-vx}+c_2e^{vx}$

$y(0)=c_1+c_2=0$
$y(\pi)=c_1e^{-v\pi}+c_2e^{v\pi}=0$ $\Rightarrow c_1=c_2=0$

$y(x)=0,\ \forall x\in[0,\pi]$

b) $\lambda=0 \Rightarrow y''=0 \Rightarrow y=c_3x+c_4$

$y(0)=0=c_4$
$y(\pi)=0=c_3\pi$ $\Rightarrow c_3=0 \Rightarrow y(x)=0\ \forall x\in[0,\pi]$

c) λ is positive, $\lambda=v^2$

$y''+v^2y=0 \Rightarrow r_{1,2}=\mp iv \Rightarrow y=c_5\cos vx+c_6\sin vx$

$y(0)=c_5\cos 0=0 \Rightarrow c_5=0$
$y(\pi)=c_6\sin v\pi=0 \Rightarrow \sin v\pi=0 \quad c_6\neq 0$

for $n=0$, $v=0$, trivial soln. $\left(v=\frac{n\pi}{\pi}=n\right)$
$y(x)=0,\ \forall x\in[0,\pi]$ $\quad n=\mp1,\mp2$

$\lambda=v^2=\left(\frac{n\pi}{\pi}\right)^2=n^2$ are eigenvalues

$\sin nx$, $n=1,2$ are eigenfunctions

In homogenous linear ODE's, summation of individual solns. is also a solution. So a series soln. can be obtained by summation of eigenfunctions.

Orthogonality of Functions

If p, q, r and p' are real and continous in $a\le x\le b$ and $p(x), r(x)$ is positive then the eigenvalues are real.

Consider the functions $y_1(x), y_2(x)$ continous on $a\le x\le b$. They are called orthogonal on this interval w.r.t. to the $r(x)$ (weight function) $r(x)>0$ if

$$\int_a^b r(x)y_1(x)y_2(x)\,dx=0$$

The norm $y_m(x)$ is

$$\|y_m(x)\|=\sqrt{\int_a^b r(x)y_m^2\,dx}$$

The functions y_1 and y_2 are called "orthonormal" on $a\le x\le b$ if $r(x)=1$

$$\int_a^b y_m(x)y_n(x)\,dx=0,\ (m\neq n)\Rightarrow \|y_m\|=1,$$

$\|y_n\|=1$