Bessel Diff. Eqn. of order $p=1/2$

20 Aralık 2023 Çarşamba 14:01

The characteristic eqn.

$$C_n = \frac{-C_{n-2}}{(n+p)^2-p^2} = \frac{-C_{n-2}}{n^2+n} = -\frac{C_{n-2}}{n(n+1)} \; ; \; n \geq 2$$

$$n=2 \Rightarrow C_2 = \frac{-C_0}{2.3}$$

$$n=3 \Rightarrow C_3 = \frac{-C_1}{3.4} = 0$$

$$n=4 \Rightarrow C_4 = \frac{-C_2}{4.5} = \frac{C_0}{5!}$$

$$n=5 \Rightarrow C_5 = 0$$

$$n=6 \Rightarrow C_6 = \frac{-C_4}{6.7} = -\frac{C_0}{7!}$$

$$C_{2n} = \frac{(-1)^n C_0}{(2n+1)!} \; ; \; n \geq 1$$

and $C_{2n-1} = 0 \; ; \; n \geq 2$

$$y = C_0 x^{1/2}\left(1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \cdots\right)$$

For $C_0 = 1$ the first solution is

$$y_1 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

The Case $r=-p<0$

For any value $p \neq 1/2$ the second solution of the Bessel eqn. of order p is straightforward. For $p=1/2$ the solution corresponding to $r=-p=-1/2$ has a difference. In this case the condition equation $C_1[(r+1)^2-p^2]=0$ vanishes for $p=-1/2$, for all $C_1 \in R$. then one has $C_0 \neq 0$, $C_1 \neq 0$ are arbitrary. Hence

$$C_n = \frac{-C_{n-2}}{(n+p)^2-p^2} = \frac{-C_{n-2}}{n^2-n} = -\frac{C_{n-2}}{n(n-1)} \; ; \; n \geq 2$$

$$n=2 \Rightarrow C_2 = \frac{-C_0}{1.2}$$

$$n=3 \Rightarrow C_3 = \frac{-C_1}{2.3}$$

$$n=4 \Rightarrow C_4 = \frac{-C_2}{3.4} = \frac{C_0}{4!}$$

$$n=5 \Rightarrow C_5 = \frac{-C_3}{4.5} = \frac{C_1}{5!}$$

. . . . . .

Summing up

$$y = C_0 x^{-1/2}\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) + \cdots$$

$$\cdots + C_1 x^{-1/2}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right)$$

$$y = C_0 x^{-1/2}\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) + \cdots$$

$$\cdots + C_1 x^{1/2}\left(1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \cdots\right)$$

In general

$$C_{2n} = \frac{(-1)^n C_0}{(2n)!} \; ; \; n \geq 1 \quad \text{and}$$

$$C_{2n+1} = \frac{(-1)^n C_1}{(2n+1)!} \; ; \; n \geq 1$$

Therefore the general solution is

$$y(x) = C_0 x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + C_1 x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

For $C_0=1$, $C_1=0$ the first solution is

$$y_1 = x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

For $C_0=0$, $C_1=1$ the second solution is

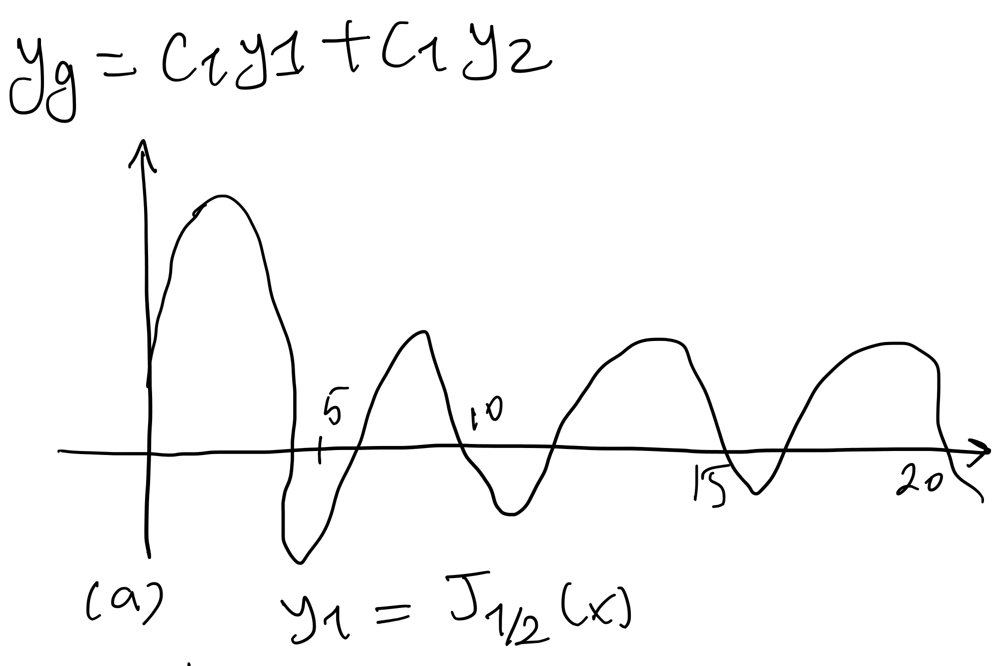
$$y_2 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

But this solution is same as the one of the case $r=p=1/2$. Therefore for $p=1/2$, the linearly independent solutions for $r=1/2$, $r=-1/2$ are

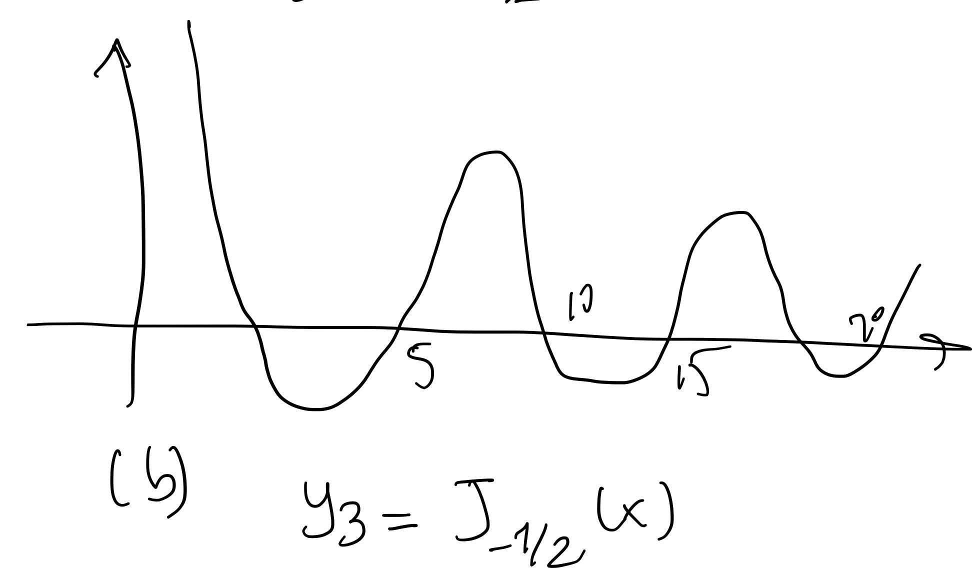
$$y_1 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}, \quad y_2 = x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

and general solution is

$$y_g = C_1 y_1 + C_1 y_2$$



(a) $y_1 = J_{1/2}(x)$



(b) $y_3 = J_{-1/2}(x)$

Let $r=-p$ then substitution of this $J_p(x)$

$$p!\, 2^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n-p)!} \left(\frac{x}{2}\right)^{2n-p}$$

Since, $(-p)!$ is undefined and 2^{-p} is a multiplication of the Σ function, they'll be excluded and $J_{-p}(x)$ is defined as;

$$J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n-p)!} \left(\frac{x}{2}\right)^{2n-p}$$

Since, $J_p(x)$ and $J_{-p}(x)$ are independent of each other, then the general solution for the case that p is not positive integer;

$$y_{gen} = C_1 J_p(x) + C_2 J_{-p}(x)$$

The case that "p" is positive integer

$$y_1 = y(x,r)\big|_{r=p}$$

y_1 is obtained as

$$J_p(x) = \int_n^{\infty} \frac{(-1)^n}{n!\,(n+p)!} \left(\frac{x}{2}\right)^{2n+p}$$

for $r=-p$, y_1 is simply obtained by a substitution $-p \xrightarrow{for} +p$ for other solution;

$$y_1 = y(x,r)\big|_{r=-p} = J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n-p)!} \left(\frac{x}{2}\right)^{2n-p}$$

But it can be shown that this is linearly dependent to the former. Another independent solution y_2 is obtained as

$$y_2 = \frac{\partial y(x,r)}{\partial r}\bigg|_{r=p} \quad \text{as stated before,}$$

Having jumped over cumbersome manipulations, the result is below

$$y_2 = Y_{-p}(x) = \frac{2}{\pi}\Bigg\{\left(\ln\left|\frac{x}{2}\right| + \gamma\right) J_p(x) - \frac{1}{2}\sum_{0}^{\infty} \frac{(p-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n-p} + \frac{1}{2}\sum_{0}^{\infty}(-1)^{n+1} \left(\sum_{k=1}^{n}\frac{1}{k} + \sum_{k=1}^{n+p}\frac{1}{k}\right)\left[\frac{1}{n!\,(n+p)!}\left(\frac{x}{2}\right)^{2n+p}\right]\Bigg\}$$

So the general solution;

$$y_{gen} = C_1 J_{-p}(x) + C_2 Y_{-p}(x)$$