3. SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS

3.2 Series Solutions About Regular Singular Points

For the differential equation

$$y'' + p(x)y' + q(x)y = r(x).$$

let $p(x)$ or $q(x)$ is not analytic at the point x_0, but both of $(x - x_0)p(x)$, and $(x - x_0)^2 q(x)$ are analytic at the point x_0. Then x_0 is called a regular singular point for the differential equation. For the existence of a series solution to the equation one has the following theorem.

Let x_0 be a regular singular point for the differential equation

$$y'' + p(x)y' + q(x)y = r(x).$$

Then, the differential equation has at least one series solution of the form

$$y = |x - x_0|^r \sum_{n=0}^{\infty} C_n (x - x_0)^n.$$

If the solution is valid in some interval $0 < x - x_0 < R$,

$$y = |x - x_0|^r \sum_{n=0}^{\infty} C_n (x - x_0)^n = \sum_{n=0}^{\infty} C_n (x - x_0)^{n+r}$$

The rest is performed as before. This technique is also known as ***Frobenius Method.***

A brief outline of the Frobenius Method is as follows: a proposal is first made as,

I)

$$y = \sum_{n=0}^{\infty} C_n (x-x_0)^{n+r} = C_0(x-x_0)^r + C_1 (x-x_0)^{r+1} + C_2 (x-x_0)^{r+2} + \cdots$$

$$\begin{aligned} y' &= rC_0(x-x_0)^{r-1} + (r+1)C_1 (x-x_0)^r + (r+2)C_2 (x-x_0)^{r+1} + \cdots \\ &= \sum_{n=0}^{\infty} (n+r)C_n (x-x_0)^{n+r-1} \end{aligned}$$

$$\begin{aligned} y'' &= r(r-1)C_0(x-x_0)^{r-2} + r(r+1)C_1 (x-x_0)^{r-1} + (r+1)(r+2)C_2 (x-x_0)^r + \cdots \\ &= \sum_{n=0}^{\infty} (n+r)(n+r-1)C_n (x-x_0)^{n+r-2} \end{aligned}$$

II) Substitution and manipulations yield a polynomial in $(x-x_0)$;

$$K_0(x-x_0)^{r+k} + K_1(x-x_0)^{r+k-1} + K_2(x-x_0)^{r+k-2} + \cdots = 0$$

For this to be satisfied, the coefficients must be equal to zero;

$$K_0 = K_1 = K_2 = \cdots = 0$$

III) The lowest power of $(x-x_0)^{r+k} \xrightarrow{yields} K_0 = 0$ is a quadratic equation in r, called the *indicial equation*;

The roots of this is called exponents of differential equation, let them be $r_1$and r_2, where $Re(r_1) \geq Re(r_2)$, where $Re(r_i)$ is the real part of the exponent r_i. If r_1, r_2 are real and unequal, then r_1 is the larger root.

IV) Equate $K_0 = K_1 = K_2 = \cdots = 0$; substitute $r = r_1$ and obtain condition C_n 's

V) If $r_2 \neq r_1$, repeat the previous procedure for $r = r_2$, to obtain a second solution (for smaller root r_2) which is linearly independent to the previous one. According to different types of r_1- r_2, different solution styles are obtained.

VI) Let $\mathrm{r}_1 - \mathrm{r}_2 \neq \mathrm{N}$, N non-negative integer ($\mathrm{r}_1 - \mathrm{r}_2 \neq 0, 1, 2, 3, \ldots$) then two independent solutions do exist

$$y_1(x) = |x-x_0|^{r_1} \sum_{n=0}^{\infty} C_n (x-x_0)^n , \quad C_0 \neq 0$$

$$y_2(x) = |x-x_0|^{r_2} \sum_{n=0}^{\infty} C_n^{*} (x-x_0)^n , \quad C_0^{*} \neq 0$$

VII) Let $\mathrm{r}_1 - \mathrm{r}_2 = \mathrm{N} \neq 0$, N a positive integer; then two independent solutions are expressed as;

$$y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} C_n\,(x - x_0)^n\ , \qquad\qquad C_0 \neq 0$$

$$y_2(x) = |x - x_0|^{r_2} \sum_{n=0}^{\infty} C_n^*(x - x_0)^n + C y_1(x) \ln|x - x_0|$$

VIII) Let $\mathrm{r}_1 - \mathrm{r}_2 = 0$, then two independent solutions are;

$$y_1(x) = \ |x - x_0|^{r_1} \sum_{n=0}^{\infty} C_n\,(x - x_0)^n\ , \qquad\qquad C_0 \neq 0$$

$$y_2(x) = |x - x_0|^{r_1+1} \sum_{n=0}^{\infty} C_n^*(x - x_0)^n + C y_1(x) \ln|x - x_0|$$

Example 3.1

Given the differential equation

$$2x^2 y'' - xy' + (x - 5)y = 0,$$

find a series solution about $x_0 = 0$.

The normalized differential equation is $y'' - \frac{x}{2x^2} y' + \frac{x-5}{2x^2} y = 0$

where

$$p(x) = \frac{x}{2x^2}, q(x) = \frac{x-5}{2x^2}$$

are not analytic at $x_0 = 0$, but

$$\lim_{x \to x_0 = 0} x\,(-\tfrac{x}{2x^2}) = -\tfrac{1}{2}$$
$$\lim_{x \to x_0 = 0} x^2 \left(\tfrac{x-5}{2x^2}\right) = -\tfrac{5}{2} \quad \Rightarrow \qquad \text{so } x_0 = 0 \text{ is a regular singular point}$$

According to the Frobenius theorem we propose a solution

$$y(x) \quad = C_0\,\mathrm{x}^r + C_1\,\mathrm{x}^{r+1} + C_2\,\mathrm{x}^{r+2} + \cdots = \sum_{n=0}^{\infty} C_n\,\mathrm{x}^{n+r} \ \rightarrow$$

$$y'(x) \quad = C_0 r\mathrm{x}^{r-1} + C_1\,(\mathrm{r}+1)\,\mathrm{x}^r + C_2\,(\mathrm{r}+2)\,\mathrm{x}^{r+1} + \cdots = \sum_{n=0}^{\infty}(n+r)C_n\,\mathrm{x}^{n+r-1}$$

$$y''(x) \quad = C_0\,r(r-1)\mathrm{x}^{r-2} + C_1\,(\mathrm{r}+1)\mathrm{r}\,\mathrm{x}^{r-1} + C_2\,(\mathrm{r}+1)(\mathrm{r}+2)\,\mathrm{x}^r + \cdots$$

$$= \sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\,\mathrm{x}^{n+r-2}$$

Substitution of these into the differential equation yields

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1} + (x-5)\sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\,\mathrm{x}^{n+r} - \sum_{n=0}^{\infty}(n+r)C_n\,\mathrm{x}^{n+r} - 5\sum_{n=0}^{\infty} C_n\,\mathrm{x}^{n+r} + \sum_{n=0}^{\infty} C_n\,\mathrm{x}^{n+r+1} = 0$$

for the last term we let $n \rightarrow n-1$

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\,\mathrm{x}^{n+r} - \sum_{n=0}^{\infty}(n+r)C_n\,\mathrm{x}^{n+r} - 5\sum_{n=0}^{\infty} C_n\,\mathrm{x}^{n+r} + \sum_{n=1}^{\infty} C_{n-1}\,\mathrm{x}^{n+r} = 0$$

Since the common range is $1 < n < \infty$

$$(2r(r-1)C_0 - rC_0 - 5C_0)x^r + \sum_{n=1}^{\infty}[2(n+r)(n+r-1)C_n - (n+r)C_n - 5C_n + C_{n-1}]x^{n+r} = 0$$

$$(2r(r-1) - r - 5)C_0 x^r + \sum_{n=1}^{\infty}\{[2(n+r)(n+r-1) - (n+r) - 5]C_n + C_{n-1}\}x^{n+r} = 0$$

First and last terms are indicial equation and recurrence relations, respectively.

$$2r(r-1) - r - 5 = 0 \ \Rightarrow\ 2r^2 - 3r - 5 = 0 \ \Rightarrow\ \begin{array}{l} r_1 = \dfrac{5}{2} \\ r_2 = -1 \end{array}$$

Since $r_1 - r_2 = \frac{5}{2} - (-1) = \frac{7}{2} \neq 0,1,2,3\,\ldots$; two independent solutions are simply obtained by the substitutions of r_1 and r_2 into the proposal

$$y(x) = |x - x_0|^r \sum_{n=0}^{\infty} C_n\,(x - x_0)^n\ , \qquad C_0 \neq 0$$

Let’s obtain the recurrence relation;

$$[2(n+r)(n+r-1)-(n+r)-5]C_n \ +C_{n-1}=0$$

$C_n=-\frac{C_{n-1}}{2(\mathrm{n+r})(\mathrm{n+r}-1)-(\mathrm{n+r})-5}$; Recurrence relation

One of the solutions for $r=\frac{5}{2}$

$$C_n=-\frac{C_{n-1}}{2\left(\mathrm{n}+\frac{5}{2}\right)\left(\mathrm{n}+\frac{3}{2}\right)-\left(\mathrm{n}+\frac{5}{2}\right)-5} \quad with \ \ n\geq 1$$

$$C_1=-\frac{C_0}{2\left(1+\frac{5}{2}\right)\left(1+\frac{3}{2}\right)-\left(1+\frac{5}{2}\right)-5}=-\frac{C_0}{2\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)-\left(\frac{7}{2}\right)-\frac{10}{2}}=-\frac{C_0}{9}$$

$$C_2=-\frac{C_1}{2\left(2+\frac{5}{2}\right)\left(2+\frac{3}{2}\right)-\left(2+\frac{5}{2}\right)-5}=-\frac{C_1}{2\left(\frac{9}{2}\right)\left(\frac{7}{2}\right)-\left(\frac{9}{2}\right)-\frac{10}{2}}=-\frac{C_1}{22}=\frac{C_0}{198}$$

$$y_1(x)=C_0x^{\frac{5}{2}}\left(1-\frac{1}{9}x+\frac{1}{198}x^2-\cdots\right)$$

The other solution for $r=-1$

$$C_n=-\frac{C_{n-1}}{2(\mathrm{n}-1)(\mathrm{n}-2)-(\mathrm{n}-1)-5}$$

$$C_1=-\frac{C_0}{-5}=\frac{C_0}{5}, \qquad C_2=\frac{C_1}{6}=\frac{C_0}{30}, \qquad C_3=-\frac{C_2}{-3}=\frac{C_0}{90}$$

$$y_2(x)=C_0x^{-1}\left(1+\frac{1}{5}x+\frac{1}{30}x^2+\frac{1}{90}x^3\ldots\right)$$

Assuming $\quad C_0=1,$

$$y_{gen}=C_1y_1+C_2y_2$$

$$y_{gen}=C_1x^{\frac{5}{2}}\left(1-\frac{1}{9}x+\frac{1}{198}x^2-\cdots\right)+C_2x^{-1}\left(1+\frac{1}{5}x+\frac{1}{30}x^2+\frac{1}{90}x^3\ldots\right)$$

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Determination of solutions according to the roots of indicial equation may be trivial in some cases, so sharp classifications are not very straight forward. A closer look may be performed as follows. Let the roots of indicial equation are; $\quad r_1=\alpha\,;\, r_2=\beta$

1) $\quad \alpha\neq\beta \quad$ and $\quad \alpha-\beta \quad$ is not an integer.

This case is relatively straightforward; the two independent solutions y_1, y_2 and the general solution y_{gen} is obtained as

$$y_1 = y(x,r)|_{r=\alpha} \;\; ; \;\; y_2 = y(x,r)|_{r=\beta} \;\Rightarrow\; y_{gen} = C_1 y_1 + C_2 y_2$$

2) $\alpha = \beta$ This is also a relatively straightforward case

One of solution is $y_1 = y(x,r)|_{r=\alpha}$

The other independent solution is $y_2 = \left.\frac{\partial y}{\partial r}\right|_{r=\alpha}$

Note that substitution of one of the solutions in r into the differential equation yields that if say

$$y(x,r) = a_0 x^r \left[1 + \frac{1}{(r+1)^2} + \cdots\right]$$

then

$$[\text{differential equation}]|_{y(x,r)} = a_0(\mathrm{r} - \alpha)(\mathrm{r} - \beta)x^{r-1}$$

3) $\alpha \neq \beta$ and $\alpha - \beta$ is an integer.

In this case, some of the coefficients C may become infinite for either α or β. Obviously for the root without trouble, say β, the solution is obtained as before,

$$y_1 = y(x,r)|_{r=\beta}$$

For the other troublesome root α some of the coefficients may involve terms like, $\frac{a_0}{r-\alpha}$. In such cases, arbitrary constants may be selected in such a way that $a_0 = b_0(r-\alpha)$ to eliminate $(r-\alpha)$ term. Substitution of α into $y(x,r)$ as

$y_2 = y(x,r)|_{r=\alpha}$ may result in a dependent solution, in this case

$y_2 = \left.\frac{\partial y(x,r)}{\partial r}\right|_{r=\alpha}$ is used

Notice that since $y(x,r) = x^r(\ldots)$ then $\frac{\partial y}{\partial r} = x^r \ln x\,(\ldots) + x^r \frac{\partial y_p}{\partial r}$ where $(\ldots)$ is y_p

Example 3.2

Find series solution of

$xy'' + y' - y = 0$

about $x_0 = 0$

The normalized differential equation is

$$y'' - \frac{1}{x}y' + \frac{1}{x}y = 0$$

since $\frac{1}{x}$ is not analytic at $x_0 = 0$, but

$\lim_{x\to 0} x\,(\frac{1}{x}) = 1;$ $\lim_{x\to 0} x^2\,(-\frac{1}{x}) = 0$, therefore $x_0 = 0$is a regular singular point.

The solution proposal is,

$$y = \sum_{n=0}^{\infty} C_n\, \mathrm{x}^{n+r} \to$$

$y' = \sum_{n=0}^{\infty} (n+r)C_n\, \mathrm{x}^{n+r-1}$ and

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)C_n\, \mathrm{x}^{n+r-2}$$

Substitution of these into differential equation yields that

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\, \mathrm{x}^{n+r-2} + \sum_{n=0}^{\infty}(n+r)C_n\, \mathrm{x}^{n+r-1} - \sum_{n=0}^{\infty} C_n\, \mathrm{x}^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\, \mathrm{x}^{n+r-1} + \sum_{n=0}^{\infty}(n+r)C_n\, \mathrm{x}^{n+r-1} - \sum_{n=0}^{\infty} C_n\, \mathrm{x}^{n+r} = 0$$

for the last term $n \to n-1$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\, \mathrm{x}^{n+r-1} + \sum_{n=0}^{\infty}(n+r)C_n\, \mathrm{x}^{n+r-1} - \sum_{n=1}^{\infty} C_{n-1}\, \mathrm{x}^{n+r-1} = 0$$

Since the common range is $1 \to \infty$

$$r(r-1)C_0x^{r-1} + rC_0x^{r-1} + \sum_{n=1}^{\infty}\{[(n+r) + (n+r)(n+r-1)]C_n - C_{n-1}\}x^{r+n-1} = 0 \to$$

$$C_0r^2x^{r-1} + \sum_{n=1}^{\infty}\{(n+r)^2\, C_n - C_{n-1}\}\, x^{r+n-1} = 0$$

Equating the coefficients of x^i to zero one has the indicial equation;

$r^2 = 0 \implies r_1 = r_2 = 0$ for $C_0 \neq 0$

Recurrence relation is

$$C_n = \frac{C_{n-1}}{(n+r)^2}, \quad n \geq 1$$

Hence

$$C_1 = \frac{C_0}{(r+1)^2}$$

$$C_2 = \frac{C_1}{(r+2)^2} = \frac{C_0}{(r+2)^2(r+1)^2} = \frac{C_0}{[(r+1)!]^2}$$

$$C_3 = \frac{C_2}{(r+3)^2} = \frac{C_0}{(r+3)^2(r+2)^2(r+1)^2} = \frac{C_0}{[(r+3)!]^2}$$

$$C_n = \frac{C_{n-1}}{(n+r)^2} = \frac{C_0}{[(r+n)!]^2}$$

Substitution of these into the proposal

$$y(x,r) = \sum_{n=0}^{\infty} C_n \mathrm{x}^{n+r} = C_0\mathrm{x}^r + C_1\mathrm{x}^r\mathrm{x} + C_2\mathrm{x}^r\mathrm{x}^2 + C_3\mathrm{x}^r\mathrm{x}^3 + \cdots$$

so

$$y(x,r) = C_0\mathrm{x}^r\left[1 + \frac{x}{(r+1)^2} + \frac{x^2}{[(r+2)!]^2} + \frac{x^3}{[(r+3)!]^2} + \cdots + \frac{x^n}{[(r+n)!]^2} + \cdots\right]$$

Assuming $C_0 = 1$

$$y_1 = y(x,r)|_{r=0} = 1 + \frac{x}{(1!)^2} + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \cdots + \frac{x^n}{(n!)^2} + \cdots$$

One of the solutions is obtained as mentioned. A second independent solution is obtained by;

$$y_2 = \frac{\partial y(x,r)}{\partial r}\Big|_{r=0}$$

Hence for, $C_0 = 1$,

$$\frac{\partial y(x,r)}{\partial r} = x^r \ln|x|\left[1 + \frac{x}{(r+1)^2} + \frac{x^2}{[(r+2)!]^2} + \cdots + \frac{x^n}{[(r+n)!]^2}\right] - 2x^r\left[\frac{x}{(r+1)^3} + \frac{x^2}{(2!)^2}\left(1+\frac{1}{2}\right) + \cdots\right]$$

$$y_2 = \frac{\partial y(x,r)}{\partial r}\Big|_{r=0} = \ln|x|\, y_1 - 2\left[\frac{x}{1^3} + \frac{x^2}{(2!)^2}\left(1+\frac{1}{2}\right) + \cdots + \frac{x^n}{(n!)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) + \cdots\right]$$

So, the general solution is;

$$y_{gen} = C_1y_1 + C_2y_2$$

$$y_{gen} = C_1 y_1 + C_2 \ln|x|\, y_1 - 2C_2 \left[x + \frac{x^2}{(2!)^2}\left(1+\frac{1}{2}\right) + \cdots + \frac{x^n}{(n!)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right]$$

so,

$$y_{gen} = y_1(C_1 + C_2 \ln|x|) - 2C_2 \left[x + \frac{3}{2}\frac{x^2}{(2!)^2} + \cdots + \frac{x^n}{(n!)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right]$$

Example 3.4

Find series solution of the differential equation

$xy'' - 3y' + xy = 0$

about $x_0 = 0$.

The normalized equation; $y'' - \frac{3}{x}y' + \frac{x}{x}y = 0$

since $\frac{1}{x}$ is not analytic at $x_0 = 0$, but

$\lim_{x\to 0} x\,(-\frac{3}{x}) = -3;$ $\lim_{x\to 0} x^2\, \frac{x}{x} = 0$, $x_0 = 0$ is a regular singular point for the given differential equation. By the Frobenius theorem, the solution proposal is;

$y = \sum_{n=0}^{\infty} C_n\, x^{n+r}$ as in the previous examples

$$y' = \sum_{n=0}^{\infty} (n+r)C_n\, x^{n+r-1}$$

and

$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\, x^{n+r-2}$.

Substitution of these into the differential equation yields that;

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\, x^{n+r-2} - 3\sum_{n=0}^{\infty}(n+r)C_n\, x^{n+r-1} + x\sum_{n=0}^{\infty} C_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\, x^{n+r-1} - 3\sum_{n=0}^{\infty}(n+r)C_n\, x^{n+r-1} + \sum_{n=0}^{\infty} C_n\, x^{n+r+1} = 0$$

for the last term $n \to n-2$ substitution gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\, \mathrm{x}^{n+r-1} - 3\sum_{n=0}^{\infty}(n+r)C_n\, \mathrm{x}^{n+r-1} + \sum_{n=2}^{\infty} C_{n-2}\, \mathrm{x}^{n+r-1} = 0$$

Since the exponents are the same, next we make the ranges the same $n{:}\, 0 \to \infty$.

$$r(r-1)C_0x^{r-1} + r(r+1)C_1x^r + \sum_{n=2}^{\infty}(n+r)(n+r-1)C_n\, \mathrm{x}^{n+r+1} - 3rC_0x^{r-1} - 3(r+1)C_1x^r$$

$$-3\sum_{n=2}^{\infty}(n+r)C_n\, \mathrm{x}^{n+r+1} + \sum_{n=2}^{\infty} C_{n-2}\, \mathrm{x}^{n+r+1} = 0$$

$$[\mathrm{r}(\mathrm{r}-1) - 3\mathrm{r}]C_0x^{r-1} + [\mathrm{r}(\mathrm{r}+1) - 3(\mathrm{r}+1)]C_1x^r$$

$$+\sum_{n=0}^{\infty}[C_n[(n+r)(n+r-1) - 3(n+r)] + C_{n-2}\,]\mathrm{x}^{n+r+1} = 0$$

Equating the coefficients of x^i to zero, one gets the indicial equation

$$\mathrm{r}(\mathrm{r}-1) - 3\mathrm{r} = 0 \implies \mathrm{r}(\mathrm{r}-4) = 0 \implies r_1 = 4, \qquad r_2 = 0$$

Considering the second coefficient

$$C_1[\mathrm{r}(\mathrm{r}+1) - 3(\mathrm{r}+1)] = 0$$

one obtains

$r_2 = 0 \Longrightarrow -3C_1 = 0$
$r_1 = 4 \Longrightarrow 5C_1 = 0$

Hence $C_1 = 0$.

Recurrence relation is;

$$C_n(n+r)((n+r-1) - 3) + C_{n-2} = 0$$

which yields

$$C_n = -\frac{C_{n-2}}{(\mathrm{n}+\mathrm{r})(\mathrm{n}+\mathrm{r}-4)}, \quad n \geq 2$$

$$n = 2 \Longrightarrow \; C_2 = -\frac{C_0}{(\mathrm{r}+2)(\mathrm{r}-2)}$$

$n = 3 \Longrightarrow \; C_3 = -\frac{C_1}{(\mathrm{r}+5)(\mathrm{r}+1)} = 0$ Similarly, $C_1 = C_3 = C_5 = C_7 = \cdots = 0$

$$n = 4 \Longrightarrow\ C_4 = -\frac{C_2}{(\mathrm{r}+4)\mathrm{r}} = \frac{C_0}{(\mathrm{r}+2)(\mathrm{r}-2)(\mathrm{r}+4)\mathrm{r}}$$

Clearly, for $r = 0 \Longrightarrow C_4 = \infty$, for which the series solution fails. To avoid this, we let $C_0 = b_0\mathrm{r}$, with $b_0 = 1$. Hence

$$C_0 = b_0 r = r$$

$$C_2 = -\frac{b_0.\mathrm{r}}{(\mathrm{r}+2)(\mathrm{r}-2)} = -\frac{\mathrm{r}}{(\mathrm{r}+2)(\mathrm{r}-2)}$$

$$C_4 = \frac{b_0.\mathrm{r}}{(\mathrm{r}+2)(\mathrm{r}-2)(\mathrm{r}+4)\mathrm{r}} = \frac{1}{(\mathrm{r}+2)(\mathrm{r}-2)(\mathrm{r}+4)}$$

$$C_6 = -\frac{C_4}{(\mathrm{r}+2)(\mathrm{r}+6)} = -\frac{1}{(\mathrm{r}+2)^2(\mathrm{r}-2)(\mathrm{r}+4)(\mathrm{r}+6)} \quad \dots \dots$$

In general for even coefficients

$$C_{2k} = (-1)^k \frac{1}{(\mathrm{r}-2)\underbrace{(\mathrm{r}+2)^2(\mathrm{r}+4)^2 \dots}_{\text{All square}}(\mathrm{r}+2\mathrm{k}-2)(\mathrm{r}+2\mathrm{k})}$$

The solution proposal would then be;

$$y(x,r) = x^r \sum_{n=0}^{\infty} C_n\, \mathrm{x}^n = x^r [C_0 + C_1 x + C_2 x^2 + \cdots + C_{2k}\, \mathrm{x}^{2k} + \cdots]$$

Substitution to the differential equation yields that

$$y(x,r) = x^r \left[r - \frac{r}{(r+2)(r-2)}x^2 + \frac{1}{(r+2)(r-2)(r+4)}x^4 - \frac{1}{(\mathrm{r}+2)^2(\mathrm{r}-2)(\mathrm{r}+4)(\mathrm{r}+6)}x^6 \right.$$
$$\left. + \cdots + (-1)^k \frac{1}{(\mathrm{r}+2)^2(\mathrm{r}-2)(\mathrm{r}+4)^2 \dots (\mathrm{r}+2\mathrm{k}-2)(\mathrm{r}+2\mathrm{k})} x^{2k} + \cdots \right]$$

For $r = 0$ we get

$$y_1 = y(x,r)|_{r=0} = -\frac{1}{16}x^4 + \frac{1}{16 * 12}x^6 + \cdots + (-1)^k \frac{1}{(-2)2^2 4^2 \dots (2\mathrm{k}-2)2\mathrm{k}} x^{2k} + \cdots$$

Since if $r = 4$ some terms become indefinite

$$y_2 = \left.\frac{\partial y(x,r)}{\partial r}\right|_{r=0} = x^r \ln|x|\, y_1 + x^r \left.\frac{\partial y(x,r)}{\partial r}\right|_{r=0}$$

so $\qquad y_{gen} = C_1 y_1 + C_2 y_2$

Example3.5

Find series solution of the differential equation

$$6x^2y'' + 7xy' - (1 + x^2)y = 0$$

about $x_0 = 0$.

The normalized equation is

$$y'' + \frac{7x}{6x^2}y' + \frac{-(1 + x^2)}{6x^2}y = 0$$

since $\frac{1}{x^2}$ is not analytic at $x_0 = 0$, but

$$p(x) = \frac{7x}{6x^2} \Longrightarrow \lim_{x\to 0} x\left(\frac{7x}{6x^2}\right) = \frac{7}{6},$$

$$q(x) = \frac{-(1 + x^2)}{6x^2} \Longrightarrow \lim_{x\to 0} x^2\left(\frac{-(1 + x^2)}{6x^2}\right) = -\frac{1}{6}$$

Hence $x_0 = 0$ is a regular singular point for the differential equation. From Frobenius theorem the solution proposal will then be;

$$y = \sum_{n=0}^{\infty} C_n\, x^{n+r}$$

similar to the previous examples

$$y' = \sum_{n=0}^{\infty} (n + r)C_n\, x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)C_n\, x^{n+r-2}$$

Substitution of these into the differential equation yields that;

$$6x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_nx^{n+r-2}+7x\sum_{n=0}^{\infty}(n+r)C_nx^{n+r-1}-(1+x^2)\sum_{n=0}^{\infty}C_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_nx^{n+r}+7\sum_{n=0}^{\infty}(n+r)C_nx^{n+r}-\sum_{n=0}^{\infty}C_nx^{n+r}-\sum_{n=2}^{\infty}C_{n-2}x^{n+r}=0$$

Hence

$$6(r)(r-1)C_0x^r+7(r)C_0x^r-C_0x^r+6(1+r)(r)C_1x^r+7(1+r)C_1x^{1+r}$$
$$-C_1x^{1+r}-\sum_{n=2}^{\infty}(6(n+r)(n+r-1)C_n+7(n+r)C_n-C_n-C_{n-2})x^{n+r}=0$$

$$(6(r)(r-1)+7r-1)C_0x^r+(6(1+r)(r)+7(1+r)-1)C_1x^r$$
$$-\sum_{n=2}^{\infty}((6(n+r)(n+r-1)+7(n+r)-1)C_n-C_{n-2})x^{n+r}=0$$

$$(6r^2+r-1)C_0x^r+(6r^2+13r+6)C_1x^r$$
$$-\sum_{n=2}^{\infty}[(6(n+r)(n+r-1)+7(n+r)-1)C_n-C_{n-2}]x^{n+r}=0$$

which yields

$$(6r^2+r-1)C_0x^r=0 \quad\Longrightarrow\quad r_1=\frac{1}{3},\qquad r_2=-\frac{1}{2},\quad r_1-r_2\neq N$$
$$(6r^2+13r+6)C_1x^r=0 \quad\Longrightarrow\quad r_3=-\frac{2}{3},\quad r_4=-\frac{3}{2},\quad r_3-r_4\neq N$$

For r_1, r_2 there are two independent solutions.

$$[6(n+r)(n+r-1)+7(n+r)-1]C_n-C_{n-2}=0$$

Hence the recurrence relation is;

$$C_n=\frac{C_{n-2}}{6(n+r)(n+r-1)+7(n+r)-1}$$

for $r_1={}^1\!/_3$

$$n=2\Longrightarrow C_2=\frac{C_0}{\frac{56}{3}+\frac{49}{3}-1}=\frac{C_0}{34}$$

$$n=4\Longrightarrow C_4=\frac{C_0}{\frac{260}{3}+\frac{91}{3}-1}=\frac{C_2}{116}=\frac{C_0}{3944}$$

$$y_1 = C_0 x^{1/3}\left(1 + \frac{x^2}{34} + \frac{x^4}{3944} + \cdots\right)$$

for $r_2 = -{}^{1}/_{2}$

$$n = 2 \Longrightarrow C_2 = \frac{C_0}{\frac{9}{2} + \frac{21}{2} - 1} = \frac{C_0}{14}$$

$$n = 2 \Longrightarrow C_4 = \frac{C_0}{\frac{105}{2} + \frac{49}{2} - 1} = \frac{C_2}{76} = \frac{C_0}{1064}$$

$$y_2 = C_0 x^{-1/2}\left(1 + \frac{x^2}{14} + \frac{x^4}{1064} + \cdots\right)$$

Assume $C_0 = 1$

$$y_{gen} = C_1 y_1 + C_2 y_2$$

$$y_{gen} = C_1 x^{1/3}\left(1 + \frac{x^2}{34} + \frac{x^4}{3944} + \cdots\right) + C_2 x^{-1/2}\left(1 + \frac{x^2}{14} + \frac{x^4}{1064} + \cdots\right)$$

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Solution Around an Irregular Singular Point

If the series solution is required about an irregular singular point x_0; then the change of variable $z = \frac{1}{x}$ may sometimes make the solution possible about a regular singular point $z = z_0$. More precisely;

$$z = \frac{1}{x} \implies x = \frac{1}{z} \implies \frac{dz}{dx} = -\frac{1}{x^2} = -z^2$$

$$y' = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = -z^2\frac{dy}{dz}$$

$$y'' = \frac{d}{dx}y' = \frac{d(y')}{dz}\frac{dz}{dx} = -z^2\frac{d}{dz}\left[-z^2\frac{dy}{dz}\right]$$

$$y'' = -z^2\left[-2z\frac{dy}{dz} - z^2\frac{d^2y}{dz^2}\right] = 2z^3\frac{dy}{dz} + z^4\frac{d^2y}{dz^2}$$

Substitution of y' and y'' into differential equation changes independent variable from x to z. Irregular singular point x_0 is then replaced by a regular singular point $z_0 = \frac{1}{x_0}$; and the new differential equation is solved as before. Finally, the solution in z is translated back to x by changing the variable $z = \frac{1}{x}$.