PDE:

Cryzieg es Advanced Math.

Differential equations can be classified according to number of independent variables and that to input-output relations.

PDE

ODE

A DE involving ordinary derivatives of one or more dependent variable with respect to a single independent variable is called ODE.

$$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$$

$$\frac{d^2y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} = y^2$$

A DE involving partial derivatives of one or more dependent variables with respect to two or more dependent variables is PDE.

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial v} = v \quad , \quad \frac{\partial z}{\partial x} + z = \frac{\partial^2 z}{\partial y^2}$$

Linear Differential Equations

A DE is called linear if;

1- Every dependent variable and every derivative involved occur to the first degree only and.

2- No products or dependent variable and/or derivatives occur.

$y \rightarrow$ dependent variable

$x \rightarrow$ independent variable

2. der

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \rightarrow \text{ODE, Linear}$$

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$$y\frac{d^2y}{dx} + \frac{dy}{dx} = e^x \rightarrow \text{ODE, Non-linear}$$

$$\frac{d^4y}{dx^4} + x^4\frac{d^3y}{dx^3} + x^3\frac{dy}{dx} = x \cdot e^x \rightarrow \text{ODE, Linear.}$$

Order of Differential Equation

The highest order of DE.

First order	$\frac{dy}{dx}$, y'
Second order	$\frac{d^2y}{dx^2}$, y''

$$\frac{d^2y}{dx} + xy\left(\frac{dy}{dx}\right)^2 = 0 \rightarrow \text{ODE, Order} = 2, \text{ dep} = y \text{ ind} = x$$

$$\frac{d^4x}{dt} + 5\frac{d^2x}{dt} + 3x = \sin t \rightarrow \text{ODE}, O = 4, \text{ dep} = x, \text{ ind} = t$$

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \rightarrow \text{PDE}, O = 1, \text{ dep} = v, \text{ ind} = s, t$$

Degree of DE.

En yüksek Order lı turevin Exponenti

If a DE can be rationalized and cleared from fractions with regard to all derivatives present, the exponent of the highest order derivative is called the degree of the DE.

$$\left(\frac{d^2y}{dt^2}\right)^2 + \left(1 + \frac{dy}{dt}\right)^3 = 0 \rightarrow \text{ODE}, O = 2, D = 3, \text{ Dep} = y \text{ ind} = t$$

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} = y^2 \tan t \rightarrow \text{ODE}, O = 3, D = 2, \text{ dep} = y \text{ ind} = t$$

$f(x,y,y')=0$

$y'=f(x,y)$ or $y'=\frac{dy}{dx}$ and $f(x,y)=\frac{-M(x,y)}{N(xy)}$

$M(x,y)dx + N(x,y)\,dy = 0$

Seperable DE.

The DE that can be reduced to the form $g(y)\cdot y' = f(x)$

$y'=dy/dx$ substitution would yield $g(y)\,dy = f(x)\,dx$

$$\int g(y)\,dx = \int f(x)\,dx + K$$

Let's solve the DE $9yy'+4x=0$

$$9y\frac{dy}{dx} = -4x$$

$$\int 9y\,dy = \int -4x\,dx$$

$$\frac{9y^2}{2} = -2x^2+C$$

$$\frac{x^2}{9}+\frac{y^2}{4}=C$$

Exact DE.

A first order DE of the form $M(x,y)dx + N(x,y)\,dy = 0$ is called exact if there is a function $u(x,y)$ such that

$$\frac{\partial u}{\partial x} = M \quad \text{and} \quad \frac{\partial u}{\partial y} = N$$

In this case,

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ and the equation is reduced to $\partial u = 0$

An implicit soln. is simplify $u(x,y)=C$

A sufficient condition for the exactness of a first order ODE is.

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Hence a sufficient condition for a 1^{st} order DE. $M(x,y)\,dx + N(x,y)dy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Apply following steps for solution

1) $\frac{\partial u}{\partial x} = M \rightarrow u = \int M\,dx + k(y)$

2) Use the equation $\frac{\partial u}{\partial y} = N$ to obtain dk/dy

3) Integrate $\frac{dk}{dy}$ to get k

4) The solution of the exact eqn. is $u(x,y) = 0$

Show that the following eqn. is exact and find the soln.

$$(x^3 + 3xy^2)\,dx + (3x^2y + y^3)\,dy = 0$$

$M = x^3 + 3xy^2 \rightarrow \frac{\partial M}{\partial y} = 6xy$

$N = 3x^2y + y^3 \rightarrow \frac{\partial N}{\partial x} = 6xy$

its Exact DE. ✓

$$u = \int M\,dx + k(y) = \int (x^3 + 3xy^2)\,dx + ky = \frac{x^4}{4} + \frac{3x^2}{2}y^2 + k(y)$$

From this proposal.

$$\frac{\partial u}{\partial y} = 3x^2y + \frac{dk}{dy}$$

From DE

$$\frac{\partial u}{\partial y} = N = 3x^2y + y^3$$

→ Lets Compare

$$\frac{dk}{dy} = y^3 \rightarrow k = \frac{y^4}{4} + c$$

Hence,

$$u(x,y) = \frac{x^4}{4} + \frac{3x^2y^2}{2} + \frac{y^4}{4} + c$$

An implicit soln. of the exact eqn. is $u(x,y) = 0$

That's

$$\frac{x^4}{4} + \frac{3x^2y^2}{2} + \frac{y^4}{4} = c$$

Linear 1^{st} Order DE

$y' + P(x)\cdot y = q(x)$

To solve this eqn, we multiply both side by the function $e^{\int P(x)dx}$

$$e^{\int P(x)\,dx}\cdot y' + P(x)\cdot e^{\int P(x)dx}\cdot y = e^{\int P(x)\,dx}\cdot q(x)$$

$$\left(e^{\int P(x)\,dx}\cdot y\right)' = q(x)\cdot e^{\int P(x)\,dx} \qquad \text{çarpım türevi}$$

$$y = e^{-\int P(x)dx}\cdot \int q(x)\cdot e^{\int P(x)\,dx}\cdot dx$$

find the soln. of the linear eqn! Example

$$y' + \frac{4}{x}\cdot y = 8x^3 \qquad e^{\ln x^4} = x^4$$

$$\int P(x)\,dx = \int \frac{4}{x}\,dx = 4\ln x = \ln x^4 \qquad e^{-\ln x^4} = \frac{1}{x^4}$$

$$y = e^{-\ln x^4}\cdot\int 8\cancel{x}^3\cdot e^{\ln x^4}dx = \frac{1}{x^4}\cdot\int 8x^7dx = \frac{1}{x^4}\cdot(x^8+c) \checkmark$$

2nd Order Linear DE

$y'' + P(x)\cdot y' + q(x)\cdot y = r(x)$

If $r(x)=0 \rightarrow$ homogeneous eqn.

Example

$y'' + 4y = e^{-x}\sin x$ - non-homogeneous, linear.

$(1-x^2)\cdot y'' - 2xy' + 6y = 0$ - homogeneous, linear.

If a 2^{nd} order linear DE.

$y'' + P(x)\,y' + q(x)\,y = r(x)$

is accompanied by the conditions.

$y'(x_0) = a$ and $y(x_0) = b$ its called IVP

$y'(x_0) = a$ and $y'(x_1) = b$ — boundary value problem

$y'(x_0) = a$ and $y(x_1) = b$ or its called BVP

Consider 2^{nd} order linear homogeneous DE.

$$y'' + ay' + by = 0$$

Try a solution. $y = e^{rx}$

$$r^2 \cdot e^{rx} + a \cdot re^{rx} + b \cdot e^{rx} = 0$$

$$(r^2 + ra + b) \cdot e^{rx} = 0$$

$e^{rx} \neq 0$, hence

$y = e^{rx} \rightarrow r^2 + ar + b = 0$

3 cases may arise in this situation

1) Distinct real roots	$r_1 \neq r_2 \in R$	$y = c_1 \cdot e^{r_1 x} + c_2 \cdot e^{r_2 x}$
2) Double real roots	$r_1 = r_2 \in R$	$y = (c_1 + c_2 x) \cdot e^{r_1 x}$
3) Complex conjugate roots	$r_{1,2} = m \mp in$	$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ or $y = e^{mx} \cdot [A \cdot \cos nx + B \cdot \sin nx]$

Solve the B.V.P

$y'' + y = 0$, $y(0) = 3$, $y(\pi/2) = -3$

$y = e^{rx} \rightarrow (r^2 + 1) e^{rx} = 0 \rightarrow r_{1,2} = \mp i$

$y = c_1 \cdot e^{ix} + c_2 \cdot e^{-ix} = A \sin x + B \cos x$

$y(0) = 3 \rightarrow B = 3$

$y(\pi/2) = -3 \rightarrow A = -3$

$$y = -3 \sin x + 3 \cos x$$

Find the general Soln of $y'' + 6y' + 9y = 0$

$y = e^{rx} \rightarrow (r^2 + 6r + 9) \cdot e^{rx} = 0$ ✓

$(r+3)^2 = 0 \rightarrow r_{1,2} = -3$ ✓ OK

$y = (c_1 + c_2 \cdot x) \cdot e^{-3 \cdot x}$ — ? Nasıl $c_1\, c_2$?

Euler - Cauchy Equations

These equations are second order linear eqn.s with variable coefficient.s of the special form.

$$x^2 y'' + a x y' + b y = r(x)$$

The change of the independent variable

$$x = e^z \rightarrow z = \ln x$$

$$\frac{d}{dx} = \frac{1}{x} \frac{d}{dz} \rightarrow x \cdot \frac{dy}{dx} = \frac{dy}{dz}$$

and

$$\frac{dy}{dx} + x \frac{d^2y}{dx^2} = \frac{1}{x} \cdot \frac{d^2y}{dz^2}$$

$$x \frac{dy}{dx} + x^2 \cdot \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}$$

$$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

Substituting these in Euler - Cauchy eqn. one has.

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + a \frac{dy}{dz} + by = r(e^z)$$

$$\frac{d^2y}{dz^2} + (a-1) \frac{dy}{dz} + by = r(e^z)$$

Solve $x^2 y'' + 7xy' + 13y = 0$

Let: $x = e^z$ Then.

$$x\frac{dy}{dx} = \frac{dy}{dz} \quad \text{and} \quad x^2 \cdot \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

Substituting these in the Euler-Cauchy eqn

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + 7\frac{dy}{dz} + 13y = 0$$

$$y'' + 6y' + 13y = 0$$

$$m^2 + 6m + 13 = 0 \rightarrow m_{1,2} = -3 \mp 2\hat{i}$$

$$y = e^{-3z} \cdot [A\cos(2z) + B\sin(2z)]$$

Transforming back to the original independent variable

$$y = x^{-3}[A\cos(2\ln x) + B\sin(2\ln x)]$$

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Non-Homogeneous 2nd Order Linear DE with Constant Coefficients

$y'' + p(x)\, y' + q(x)\, y = r(x)$, $r(x) \neq 0$

$y(x) = y_h(x) + y_p(x)$ → any soln. of the non-homogeneous DE

The Method of Undetermined Coefficients

If RHS function $r(x)$ is from a special kind of functions that create only a finite number of root functions upon successive differentiations, it's called a function of finite derivatives.

If $r(x) = \sin 2x$ is a function of finite derivatives since upon successive differentiations create only two root function.

$$D = \{\sin 2x,\ \cos 2x\}$$

$r(x) = x^5$ is also a function of finite derivatives $D = \{x^5, x^4, x^3, x^2, x, 1\}$

$r(x) = \ln(x)$ is not a function of finite derivatives.

$$D = \left\{\ln x,\ \frac{1}{x},\ \frac{1}{x^2},\ \frac{1}{x^3},\ \ldots\right\}$$

Now, let the set of the two linearly independent solns. of homogeneous 2nd order. DE. is

$$H = \{\varphi_1(x),\ \varphi_2(x)\}$$

and the D set of $r(x)$ is

$$D = \{f_1(x),\ f_2(x), \ldots, f_n(x)\}$$

① If the sets H and D do not have any commen function, the we prepose a particular soln. fer the nen-hemogeneous DE. as a lineer cambinitien. of functions in D with coefficient $\{a_1, a_2, a_3 ... a_n\}$

Solve $y'' - 3y' + 2y = e^{3x}$

chactaristics roots

$r^2 - 3r + 2 = 0 \rightarrow r_{1,2} = 1, 2$

$y_h = c_1 \cdot e^x + c_2 e^{2x}$, $H = \{e^x, e^{2x}\}$

where $D = \{e^{3x}\}$ and $H \cap D = \emptyset$

Hence particular soln. preposal is $y_p = c e^{3x}$

Upon substition inte the non-homogeneous DE, one has

$y_p' = 3 \cdot c e^{3x} \rightarrow y_p'' = 9 c e^{3x}$

$9 c e^{3x} - 9 c e^{3x} + 2 c e^{3x} = e^{3x}$

$2 c e^{3x} = e^{3x}$. $c = 1/2 \rightarrow y_p = \frac{1}{2} e^{3x}$

$y = c_1 e^x + c_2 e^{2x} + \frac{1}{2} e^{3x}$

② If the sets H and D are not disjoint we multiply the set D till we

$H \cap x^m \cdot D = \emptyset$

then we prepose a particular soln. fer the nen-homogenous DE as a lineor combination of functions in $x^m D$ with coefficients $\{a_1, a_2, a_3 .. a_n\}$ to be detemined.

$y_p = a_1 \cdot x^m f_1(x) + a_2 x^m \cdot f_2 \cdot (x) + .. + a_n x^m f_n(x)$

Solve $\cdot\ y'' - 3y' + 2y = e^x$

$r^2 - 3r + 2 = 0 \rightarrow r_{1,2} = 1, 2$

$y_h = c_1 \cdot e^x + c_2 e^{2x}$: $H = \{e^x, e^{2x}\}$, $D = \{e^x\}$ and $H \cap D \neq \emptyset$ but

$xD = \{xe^x\}$ and $H \cap xD = \emptyset$

Hence

$y_p = Cx\,e^x \rightarrow y_p' = C(e^x + x\,e^x)$

$y_p'' = C(2e^x + xe^x)$

$C(2e^x + xe^x) - 3C(e^x + xe^x) + 2Cxe^x = e^x$

$-Ce^x = e^x \rightarrow C = -1$

$y_p = -xe^x$

$y = c_1 e^x + c_2 e^{2x} - xe^x$

HW! $y'' + 2y' + 5y = 16e^x + \sin \cdot 2x$

$y'' + 2y' + 5y = 16e^{x} + \sin 2x$

The roots of the characteristics eqn and the homogeneous solution are

$r^2 + 2r + 5 = 0 \quad \rightarrow \quad r_{1,2} = -1 \mp 2i$

$y_h = e^{-x}(A\cos 2x + B\sin 2x)$

$H = \{e^{-x}\cos 2x, e^{-x}\sin 2x)$ where D sets are

$D_1 = \{e^{x}\}$ and $D_2 = \{\cos 2x, \sin 2x\}$

$H \cap D_1 = \emptyset \quad , H \cap D_2 = \emptyset$

Therefore

$y_p = Ce^{x} + K\cos 2x + M\sin 2x$

$y_p' = Ce^{x} - 2K\sin 2x + 2M\cos 2x$

$y_p'' = Ce^{x} - 4K\cos 2x - 4M\sin 2x$

$y'' + 2y' + 5y = 16e^{x} + \sin 2x \Rightarrow$

$Ce^{x} - 4K\cos 2x - 4M\sin 2x + 2(Ce^{x} - 2K\sin 2x + 2M\cos 2x) + 5(Ce^{x} + K\cos 2x + M\sin 2x)$
$= 16e^{x} + \sin 2x$

$8C = 16 \rightarrow C = 2$

$\cos 2x: -4K + 4M + 5K = 0 \rightarrow K + 4M = 0$

$\sin 2x: -4M - 4K + 5M = 1 \rightarrow -4K + M = 1$

$K = -4/17 \quad , M = 1/17$

$y_p = \frac{4}{17}\cos 2x + \frac{1}{17}\sin 2x$

The general solution is

$y = e^{-x}(A\cos 2x + B\sin 2x) + 2e^{x} - \frac{4}{17}\cos 2x + \frac{1}{17}\sin 2x$

Wronskian Determinant and Variation of Parameters

The previous solution technique can be applied only the constant-coefficient eqns. with special $r(x)$. For the more general soln. is

$$y'' + P(x)\, y' + q(x) \cdot y = r(x)$$

with function P, q and r which are continuous in the given interval, the method of variation of parameters is used to find particular soln. y_p

Claim: For equns like

$y'' + P(x) \cdot y' + q(x)\, y = r(x)$ with functions P, q and r which are continous in the given interval, the method of variation of parameters yields a particular soln. y_p in the form,

$$y_p(x) = -y_1 \int \frac{y_2 r}{|W|} dx + y_2 \int \frac{y_1 \cdot r}{|W|} dx$$

where y_1, y_2 form a basis of solns of the homogeneous eqn.

$$y'' + P(x) \cdot y' + q(x)\, y = 0$$

corresponding to original eqn. and

$$W = y_1 y' - y' y_2 \rightarrow |W| = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

When y_1 and y_2 are linearly independent the wronskian determinant $|W|$ is not zero and the soln. is feasible.

Proof: Let the homogeneous soln of the given eqn. is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

Let also propose a particular soln. y_p in the form

$$y_p(x) = u(x)\, y_1(x) + v(x)\, y_2(x)$$

where u and v are functions to be determined

$$y_p' = u' y_1 + u y_1' + v' y_2 + v y_2' \quad (*)$$

Assume: $u' y_1 + v' y_2 = 0 \quad (1)$

Then the eqn (*) reduces to $y_p' = u \cdot y_1' + v y_2'$

Second derivative y_p'' is now

$$y_p'' = u' y_1' + u y_1'' + v' y_2' + v y_2''$$

Insert y_p, y_p', y_p'' in the original non-homogeneous eqn.

$$u(y_1'' + p y_1' + q y_1) + v(y_2'' + p y_2' + q y_2) + u' y_1' + v' y_2' = r(x)$$

Since y_1 and y_2 are solns of the DE, the above eqn. reduce to

$$u' y_1' + v' y_2' = r(x) \quad (2)$$

and combining eqn 2 with eqn 1 a linear algebratic systems of two linear.

eqn. for u' v' is obtained

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}$$

using Cramer's method the soln. is

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{|W|} = -\frac{y_2 r}{|W|}$$

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{|W|} = \frac{y_1 \cdot r}{|W|}$$

Integrating

$$u = -\int \frac{y_2 r}{|W|} dx \qquad v = \int \frac{y_1 \cdot r}{|W|} dx$$

$$y_p(x) = -y_1 \cdot \int \frac{y_2 \cdot r}{W} \cdot dx + y_2 \cdot \int \frac{y_1 r}{W} dx$$

Solve $y''+y=\sec x$

$r^2+1=0 \quad r_{1,2}=\mp i \rightarrow y_1=\cos x \;,\; y_2=\sin x$

The homogeneous soln is $y_h=A\cos x+B\sin x$

Therefore particular soln is

$$W(y_1,y_2)=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1$$

$$y_p(x)=-\cos x\int \sin x\sec x\,dx+\sin x\int \cos x\sec x\,dx$$

$$=\cos x\ln\cos x+x\sin x$$

$$y=y_p+y_p=(C_1\ln\cos x)\cdot\cos x+(C_2+x)\sin x$$

Solve $y''+4y'+4y=\dfrac{e^{-2x}}{x^2}$

$r^2+4r+4=0 \rightarrow r_{1,2}=-2$

$y_1=e^{-2x}$ and $y_2=x\,e^{-2x}$ $\qquad \rightarrow$ Tablodan gelen $y_1\, y_2$ Tipine göre

and the homogeneous soln is

$$y_h=C_1e^{-2x}+C_2x\,e^{-2x}$$

Therefore particular soln is

$$y_p(x)=-y_1\int\frac{y_2 r}{W}dx+y_2\int\frac{y_1 r}{W}dx$$

$$W=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}=\begin{vmatrix}e^{-2x} & xe^{-2x}\\ -2e^{-2x} & e^{-2x}-2xe^{-2x}\end{vmatrix}=e^{-4x}$$

Particular soln

$$y_p(x)=-e^{-2x}\cdot\int\frac{xe^{-2x}\cdot e^{-2x}}{e^{-4x}\,x^2}dx+x\,e^{-2x}\cdot\int\frac{e^{-2x}\cdot e^{-2x}}{e^{-4x}\,x^2}dx$$

$$=-e^{-2x}\int\frac{dx}{x}+x\,e^{-2x}\cdot\int\frac{dx}{x}=-e^{-2x}\cdot\ln x-e^{-2x}$$

$$y=y_h+y_p=C_1e^{-2x}+C_2x\,e^{-2x}-e^{-2x}\cdot\ln x-e^{-2x}$$

Solve $y''-4y'+5y=\dfrac{2e^{2x}}{\sin(x)}$

$r^2-4r+5=0 \to r_{1,2}=2\mp i$

$y_1=e^{2x}\sin x$ and $y_2=e^{2x}\cos x$

$y_h=e^{2x}\cdot(c_1\sin x+c_2\cos x)$

$$W=\begin{vmatrix} e^{2x}\sin x & e^{2x}\cos x \\ 2e^{2x}\sin x+e^{2x}\cdot\cos x & 2e^{2x}\cos x-e^{2x}\sin x \end{vmatrix} = -e^{-4x}$$

$$y_p=-e^{2x}\sin x\int\frac{e^{2x}\cos x\,\frac{2e^{2x}}{\sin x}}{-e^{4x}}dx+e^{2x}\cos x\cdot\int\frac{e^{2x}\sin x\,\frac{2e^{2x}}{\sin x}}{-e^{4x}}dx$$

$$=2e^{2x}\sin x\int\cot x\,dx-2e^{2x}\cos x\int dx$$

$y_p=2e^{2x}\cdot\left(\ln|\sin x|\sin x-x\cos x\right)$

$y=y_h+y_p=e^{2x}(c_1\sin x+c_2\cos x)+2e^{2x}(\ln|\sin x|\sin x-x\cos x)$

Solve $y''-4y'+4y=6x^{-4}e^{2x}$

$r^2-4r+4=0 \to r_{1,2}=2 \to y_1=e^{2x}, y_2=xe^{2x}$

$y_h=c_1e^{2x}+c_2xe^{2x}$

$$W=\begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x}+2xe^{2x}\end{vmatrix}=e^{4x}$$

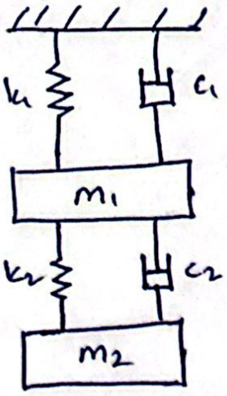
$$y_p=-e^{2x}\int\frac{xe^{2x}6x^{-4}e^{2x}}{e^{4x}}dx+xe^{2x}\int\frac{e^{2x}6x^{-4}e^{2x}}{e^{4x}}dx$$

$$=-e^{2x}\int 6x^{-3}dx+xe^{2x}\int 6x^{-4}dx \quad 3e^{2x}x^{-2}-2e^{2x}x^{-2}$$

$$=e^{2x}x^{-2}$$

$y=y_h+y_p=c_1e^{2x}+c_2xe^{2x}+e^{2x}x^{-2}$

mass spring damper



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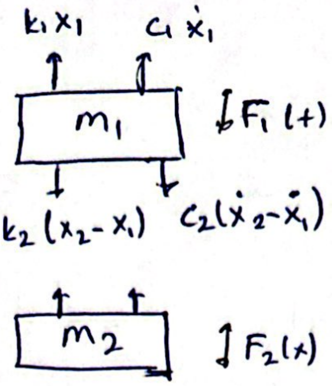


Newton 2nd law

$$m_1 \ddot{x}_1 = k_2(x_2-x_1) + c_2(\dot{x}_2-\dot{x}_1) - k_1 x_1 - c_1 \dot{x}_1 + F_1(t)$$

$$m_2 \ddot{x}_2 = -k_2(x_2-x_1) - c_2(\dot{x}_2-\dot{x}_1) + F_2(t)$$

Manipulation yield

$$m_1 \ddot{x}_1 + (c_1+c_2)\dot{x}_1 - c_2 \dot{x}_2 + (k_1+k_2)x_1 - k_2 x_2 = F_1(t)$$

$$m_2 \ddot{x}_2 - c_2 \dot{x}_1 + c_2 \dot{x}_2 - k_2 x_1 + k_2 x_2 = F_2(t)$$ linear eqn.

Solution Techniques

Decoupling Method

Solve the system of the two differential eqns.

$$\frac{dy}{dt} + z = e^{t}$$ türev al içine koy. birbirine eşitle.

$$y + \frac{dz}{dt} = e^{-t}$$

$$\frac{dy}{dx} = -z + e^{t} \rightarrow \frac{d^2y}{dt^2} = -\frac{dz}{dt} + e^{t}$$

$$\frac{dz}{dt} = -y + e^{-t} \rightarrow \frac{d^2z}{dt^2} = -\frac{dy}{dt} - e^{-t}$$

$$\frac{d^2y}{dt^2} = y - e^{-t} + e^{t} \rightarrow \frac{d^2y}{dt^2} - y = -e^{-t} + e^{t}$$

$$\frac{d^2z}{dt^2} = z - e^{t} - e^{-t} \rightarrow \frac{d^2z}{dt^2} - z = -e^{t} - e^{-t}$$

$y'' - y = 0 \qquad y_h = e^{mt} \quad , \; y_h'' = m^2 e^{mt}$

$z'' - z = 0 \qquad z_h = e^{mt} \quad , \; z_h'' = m^2 e^{mt}$

$(m^2 - 1) e^{mt} = 0 \qquad m_{1,2} = \mp 1$

$y_h = c_1 e^t + c_2 e^{-t}$

$z_h = c_3 e^t + c_4 e^{-t}$

$c_1 e^t - c_2 e^{-t} + c_3 e^t + c_4 e^{-t} = 0 \rightarrow e^t (c_1 + c_3) + e^{-t} (-c_2 + c_4) = 0$

$c_1 e^t + c_2 e^{-t} + c_3 e^t - c_4 e^{-t} = 0 \rightarrow e^t (c_1 + c_3) + e^{-t} (c_2 - c_4) = 0$

$c_3 = -c_1$ and $c_4 = c_2$

$y_h = c_1 e^t + c_2 e^{-t}$

$z_h = -c_1 e^t + c_2 e^{-t}$

$y_p = t (A_1 e^t + A_2 e^{-t})$

$y_p' = A_1 e^t + A_2 e^{-t} + t (A_1 e^t - A_2 e^{-t})$

$y_p'' = A_1 e^t - A_2 e^{-t} + A_1 e^t - A_2 e^{-t} + t (A_1 e^t + A_2 e^{-t})$

Substitution ~~of~~ these into $y'' - y = e^t - e^{-t}$ yield that

$(A_1 + A_1 + A_1 t - A_1 t - 1) e^t + (-A_2 - A_2 + A_2 t - A_2 t + 1) e^{-t} = 0$

$A_1 = 1/2 \; , \; A_2 = 1/2$

$y_p = \frac{1}{2} t (e^t + e^{-t})$

$y_{general} = y_h + y_p = c_1 e^{t} + c_2 e^{-t} + \frac{1}{2} + t(e^{t} + e^{-t})$

$z_p = t(B_1 e^{t} + B_2 e^{-t})$

$z_p'' = B_1 e^{t} - B_2 e^{-t} + B_1 e^{t} - B_2 e^{-t} + t(B_1 e^{t} + B_2 \cdot e^{-t})$

Substitution into $z'' - z = -e^{t} - e^{-t}$ yields that

$(B_1 + B_1 + B_1 t - B_1 t + 1) e^{t} + (-B_2 - B_2 + B_2 t - B_2 t + 1) e^{-t} = 0$

$B_1 = -1/2 , \quad B_2 = 1/2$

$z_p = \frac{1}{2} + t(e^{-t} - e^{t})$

$z_{general} = -c_1 e^{t} + c_2 e^{-t} + \frac{1}{2} + t(e^{-t} - e^{t})$

Ex 1:

$\frac{dx}{dt} + y = 2$ $\qquad x_{general} = c_1 e^{t} + c_2 e^{-t} + \frac{1}{2} \cos t$

$x + \frac{dy}{dt} = \cos t$ $\qquad y_{general} = c_3 e^{t} + c_4 e^{-t} + \frac{1}{2} \sin t + 2$

Ex. 2:

homojen: sağ 0. =0

$\frac{dx}{dt} - y = \frac{1}{\sin t}$ $\qquad x_h = c_1 \cos t + c_2 \sin t$

$y_h = c_3 \cos t + c_4 \sin t$

$x + \frac{dy}{dt} = \frac{1}{\cos t}$

$c_4 = -c_1 \qquad c_3 = c_2$

Operator Method

In Operator method each differentiation is denoted by a linear operator D and the resulting algebraic eqn.
Set is solved in terms of D using algebraic rules.
The unknowns are solved in terms of D. Since D is a differential operator higher order linear differential eqns are then obtained and solved individually

$$\frac{dy}{dt}+z=e^{t}$$

$$y+\frac{dz}{dt}=e^{-t}$$

Let $D=\frac{d}{dt}$

$$Dy+z=e^{t}$$
$$y+Dz=e^{-t}$$

$$\rightarrow \Delta=\begin{vmatrix} D & 1 \\ 1 & D \end{vmatrix}=D^2-1$$

$$\Delta y=\begin{vmatrix} e^{t} & 1 \\ e^{-t} & D \end{vmatrix}=De^{t}-e^{-t}=e^{t}-e^{-t}$$

$$\Delta z=\begin{vmatrix} D & e^{t} \\ 1 & e^{-t} \end{vmatrix}=De^{-t}-e^{t}=-e^{-t}-e^{t}$$

$$y=\frac{\Delta y}{\Delta}=\frac{e^{t}-e^{t}}{D^2-1}$$

$$(D^2-1)\,y=e^{t}-e^{-t}\rightarrow y''-y=e^{t}-e^{-t}$$

$$z=\frac{\Delta z}{\Delta}=\frac{-e^{-t}-e^{t}}{D^2-1}\rightarrow (D^2-1)\cdot z=-e^{t}-e^{t}$$

$$z''-z=-e^{t}-e^{-t}$$

geri kalan aynı

$$\frac{dx}{dt} + y = 2$$

$$x + \frac{dy}{dt} = \cos t$$

$$\begin{aligned} Dx + y &= 2 \\ x + Dy &= \cos t \end{aligned} \rightarrow \Delta = \begin{vmatrix} D & 1 \\ 1 & D \end{vmatrix} = D^2 - 1$$

$$\Delta x = \begin{vmatrix} 2 & 1 \\ \cos t & D \end{vmatrix} = D^2 - \cos t = -\cos t$$

$$\Delta y = \begin{vmatrix} D & 2 \\ 1 & \cos t \end{vmatrix} = D\cos t - 2 = -\sin t - 2$$

$$x = \frac{\Delta x}{\Delta} = \frac{-\cos t}{D^2 - 1} \xrightarrow{\text{yields}} (D^2 - 1)\, x = -\cos t$$

$$\xrightarrow{\text{yields}} \quad x'' - x = -\cos t$$

$$y = \frac{\Delta y}{\Delta} = \frac{-\cos t}{D^2 - 1} \xrightarrow{\text{yields}} (D^2 - 1)\, y = -\sin t - 2$$

$$y'' - y = -\sin t - 2 .$$

[illegible]

$$\frac{dx}{dt} - y = \frac{1}{\sin t} \qquad Dx - y = \frac{1}{\sin t}$$

$$x + \frac{dy}{dt} = \frac{1}{\cos t} \qquad x - Dy = \frac{1}{\cos t}$$

$$\Delta = \begin{vmatrix} D & -1 \\ 1 & D \end{vmatrix} = D^2+1$$

$$\Delta x = \begin{vmatrix} 1/\sin t & -1 \\ 1/\cos t & D \end{vmatrix} = D\frac{1}{\sin t} + \frac{1}{\cos t} = \frac{-\cos t}{\sin^2 t} + \frac{1}{\cos t}$$

$$\Delta y = \begin{vmatrix} D & 1/\sin t \\ 1 & 1/\cos t \end{vmatrix} = D\frac{1}{\cos t} + \frac{1}{\sin t} = \frac{-\sin t}{\cos^2 t} - \frac{1}{\sin t}$$

$$x = \frac{\Delta x}{\Delta} = \frac{\Delta x}{D^2+1} \xrightarrow{\text{yields}} (D^2+1)\,x = \Delta x$$

$$\xrightarrow{\text{yields}} \quad x'' + x = -\frac{\cos t}{\sin^2 t} + \frac{1}{\cos t}$$

$$y = \frac{\Delta y}{\Delta} = \frac{\Delta y}{D^2+1} \xrightarrow{\text{yield}} (D^2+1)\,y = \Delta y$$

$$\xrightarrow{\text{yields}} \quad y'' + y = \frac{-\sin t}{\cos^2 t} - \frac{1}{\sin t}$$

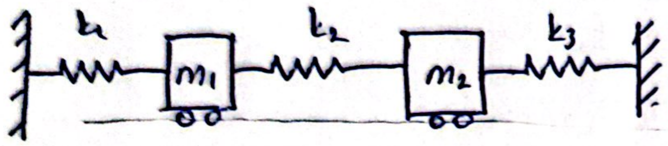
Problem 7

Consider two masses connected to the walls by springs and they're connected to each other by massles springs.

Suppose that $m_1 = m_2 = k_1 = k_2 = k_3 = 1$

Find $x_1(t)$ and $x_2(t)$ are the deviation from equilibrium for both masses.

Newton 2nd l..



Kasim [redacted]

Math

(1)

Solve: $\dot{x} - x - y = 3t$

$\dot{x} + \dot{y} - 5x - 2y = 5$

$\dot{x} - x = y + 3t \quad (1)$

$\dot{x} - 5x = 2y - \dot{y} + 5 \quad (2)$

substitute

$y = \dot{x} - x - 3t \rightarrow \dot{y} = \ddot{x} - \dot{x} - 3$

$\dot{x} - 5x = 2y - \dot{y} + 5 = 2(\dot{x} - x - 3t) - (\ddot{x} - \dot{x} - 3) + 5$

$\ddot{x} - 2\dot{x} - 3x = 8 - 6t$

$\ddot{y} - \dot{y} + 3 = \dot{y} + 3y - 5 + 15t$

$\ddot{y} - 2\dot{y} - 3y = -8 + 15t$

soln. of x-eqn

$\ddot{x} - 2\dot{x} - 3x = 8 - 6t$

Homogenous Soln

$r^2 - 2r - 3 = 0 \quad r_1 = -1 \quad r_2 = 3 \quad x_h = C_1 e^{-t} + C_2 e^{3t}$

Particular Soln

$x_p = A + Bt \rightarrow \dot{x}_p = B \rightarrow \ddot{x}_p = 0$

$\ddot{x} - 2\dot{x} - 3x = 8 - 6t$

$-2B - 3(A + Bt) = 8 - 6t$

$-3B = -6 \rightarrow B = 2$

$-2B - 3A = 8 \rightarrow A = -4 \quad x_p = 2t - 4$

$x_{gen} = 2t - 4 + C_1 e^{-t} + C_2 e^{3t}$

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Soln. of y-eqn.

$\ddot{y}-2\dot{y}-3y=-8+15t$

$r^2-2r-3=0 \quad \begin{array}{l} r_1=-1 \\ r_2=3 \end{array} \rightarrow y_h=C_3e^{-t}+C_4e^{3t}$

Particular Soln.

$y_p=A+Bt \rightarrow \dot{y}_p=B \rightarrow \ddot{y}_p=0$

$\ddot{y}-2y-3y=-8+15t$

$-2B-3(A+Bt)=-8+15t$

$-3B=15 \rightarrow B=-5$

$-2B-3A=-8 \rightarrow A=6$

$y_p=6-5t$

$y_{gen}=6-5t+C_3e^{-t}+C_4 \cdot e^{3t}$

$2-e^{-t}C_1+3e^{3t}C_2-(2t-4+C_1e^{-t}+C_2e^{3t})=6-5t+C_3 \cdot e^{-t}+C_4 \cdot e^{3t}+3t$

$C_3=-2C_1 \quad , C_4=2C_2$

$y_{gen}=6-5t-2C_1e^{-t}-2C_2e^{3t}$

Canonical Forms and Eigen Values of LDE Systems

Let's consider a set of homogeneous set of linear systems with constant coefficient

$$x_1' = a_{11} \cdot x_1 + a_{12} \cdot x_2 + \cdots + a_{1n} x_n$$

$$x_2' = a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n$$

$$\vdots$$

$$x_n' = a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n$$

$$x' = A x$$

$$x' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ \vdots \\ x_n' \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Let possible solutions be;

$$x_1 = \alpha_1 e^{\lambda t}$$

$$x_2 = \alpha_2 e^{\lambda t}$$

$$\vdots$$

$$x_n = \alpha_n e^{\lambda t}$$

in matrix form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix} \cdot e^{\lambda t}$$

$$x = \alpha e^{\lambda t}$$

$$A\alpha - \lambda \alpha = 0$$

$$(A - \lambda I)\alpha = 0 \rightarrow |A - \lambda I| = 0$$

must be satisfied for non-trivial soln.

$$|\lambda I - A| = 0$$

Characteristic eqn.

$$\lambda = \{\lambda_1 ; \lambda_2 ; \ldots \lambda_n\}$$

Case 1: $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_n$

$$X_{gen} = C_1 X_1 + C_2 X_2 \cdots + C_n X_n$$

Solve the following homogeneous linear system using matrix form.

$$x_1' = 7x_1 - x_2 + 6x_3$$
$$x_2' = -10x_1 + 4x_2 - 12x_3$$
$$x_3' = -2x_1 + x_2 - x_3.$$

Convert into matrix form

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

if $x = \alpha e^{\lambda t} \rightarrow$ $x_1 = \alpha_1 e^{\lambda t}$, $x_2 = \alpha_2 e^{\lambda t}$, $x_2 = \alpha_3 e^{\lambda t}$

$\boxed{-(\lambda - 7) = 7 - \lambda}$

$$|\lambda I - A| = \begin{bmatrix} 7-\lambda & -1 & -6 \\ +10 & 4-\lambda & +12 \\ +2 & -1 & -1-\lambda \end{bmatrix} = 0$$

$$7-\lambda \cdot \begin{bmatrix} 4-\lambda & +12 \\ -1 & -1-\lambda \end{bmatrix} + 1 \cdot \begin{bmatrix} +10 & +12 \\ +2 & -1-\lambda \end{bmatrix} \overset{-(-6)}{+} \begin{bmatrix} 10 & 4-\lambda \\ 2 & -1 \end{bmatrix} = 0$$

$$(7-\lambda)\left[(4-\lambda)(-1-\lambda)+12\right] + \left[-10(-1-\lambda) - 24\right] + 1\left[-10 + 2(4-\lambda)\right] = 0$$

$$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$$

$$(\lambda-2)(\lambda-3)(\lambda-5) = 0$$

$$\lambda_1 = 2 \quad \lambda_2 = 3 \quad \lambda_3 = 5$$

For $\lambda = 2$

$$A\alpha = \lambda\alpha \rightarrow \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$7\alpha_1 - \alpha_2 + 6\alpha_3 = 2\alpha_1$$
$$-10\alpha_1 + 4\alpha_2 - 12\alpha_3 = 2\alpha_2$$
$$-2\alpha_1 + \alpha_2 - \alpha_3 = 2\alpha_3$$

$$5\alpha_1 - \alpha_2 + 6\alpha_3 = 0$$
$$-10\alpha_1 + 2\alpha_2 - 12\alpha_3 = 0$$
$$-2\alpha_1 + \alpha_2 - 3\alpha_3 = 0$$

$$\begin{array}{l} 5\alpha_1 - \alpha_2 + 6\alpha_3 = 0 \\ -2\alpha_1 + \alpha_2 - 3\alpha_3 = 0 \end{array} \rightarrow \begin{array}{l} \alpha_3 = -\alpha_1 \\ \alpha_2 = -\alpha_1 \end{array}$$

Let $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = -1$

So that for $\lambda = 2 \rightarrow \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$x_1 = \alpha e^{\lambda_1 t}$ → so the corresponding soln is

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{bmatrix}$$

for $\lambda = 3 \rightarrow x_2 = \alpha e^{\lambda_2 t} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix}$

for $\lambda = 5 \rightarrow x_3 = \alpha e^{\lambda_3 t} = \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix} e^{5t} = \begin{bmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{bmatrix}$

general soln.

$$X_{gen} = c_1 \begin{bmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix} + c_3 \begin{bmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{bmatrix}$$

Case 2: if a pair of the distinct eigenvalues is complex conjugate numbers the corresponding eigenvectors. are also complex conjugate.

$$\lambda_1 = \alpha + i\beta \quad \text{and} \quad \lambda_2 = \alpha - i\beta$$

$$x^1 = e^{\alpha t}\left(u^1 \cos\beta t - u^2 \sin\beta t\right)$$
$$x^2 = e^{\alpha t}\left(u^1 \sin\beta t + u^2 \cos\beta t\right)$$

Here. $v^1 = u^1 + i u^2$
$v^2 = u^1 - i u^2$

Case 3: If eigenvalues are repeated

a) Repeated eigenvalue λ_1 with the multiplicity m have have equal number of independent soln. Then you will behave as it's case 1.

b) Repeated eigenvalue λ, with the multiplicity m may be have smaller independent soln.

For instance $m=2$ and $\underbrace{p=1}$ → number of independent soln.

Let solve linear DE.

$x_1' = 3x_1 + x_2 - x_3$

$x_2' = x_1 + 3x_2 - x_3$

$x_3' = 3x_1 + 3x_2 - x_3$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad x' = Ax$$

$|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3 & 3 & -1-\lambda \end{vmatrix} = 0$$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$(\lambda - 1)(\lambda - 2)^2 = 0$

$\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 2$

For $\lambda_1 = 1$

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 1 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

→ Düzeltme

$2\alpha_1 + \alpha_2 - \alpha_3 = 0$

$\alpha_1 + 2\alpha_2 - \alpha_3 = 0$

$3\alpha_1 + 3\alpha_2 - 2\alpha_3 = 0$

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Lets Check the rank

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 0 & -3 & 1 \\ 1 & 2 & -1 \\ 0 & -3 & -1 \end{vmatrix} = - \begin{vmatrix} -3 & 1 \\ -3 & 1 \end{vmatrix} = 0$$

Determinant

$$\Delta_{sub} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 \neq 0 \qquad \text{rank} = 2$$

So select $3-2=1$ unknown as known

lets say $\alpha_1 = 1 \rightarrow \alpha_2 = 1$, $\alpha_3 = 3$

for $\lambda = 1 \rightarrow \alpha = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

$$x = \alpha e^{\lambda t} \rightarrow x = \begin{bmatrix} e^t \\ e^t \\ 3e^t \end{bmatrix}$$

$\lambda = 2$ repeated

$$A\alpha = \lambda \alpha = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} = 2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\alpha_1 + \alpha_2 - \alpha_3 = 0$$
$$\alpha_1 + \alpha_2 - \alpha_3 = 0$$
$$3\alpha_1 + 3\alpha_2 - 3\alpha_3 = 0$$

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{vmatrix} \rightarrow$$ Rank $=1$ $\quad 3-1$ two of them must be select arbitrary

Let's select $\alpha_1 = 1$ $\alpha_3 = 0 \rightarrow \alpha_2 = -1$

$$\alpha = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \rightarrow x = \alpha e^{\lambda t} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ -e^{2t} \\ 0 \end{bmatrix}$$

$$X_{gen} = c_1 \begin{bmatrix} e^t \\ e^t \\ 3e^t \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ -e^{2t} \\ 0 \end{bmatrix} + c_3 \cdot \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}$$

Ex:

$$x_1' = 4x_1 + 3x_2 + x_3$$

$$x_2' = -4x_1 - 4x_2 - 2x_3$$

$$x_3' = 8x_1 + 12x_2 + 6x_3$$

Example 2.8

Solve the homogeneous linear system

$$\begin{aligned} x_1' &= 4x_1 + 3x_2 + x_3 \\ x_2' &= -4x_1 - 4x_2 - 2x_3 \\ x_3' &= 8x_1 + 12x_2 + 6x_3 \end{aligned}$$

The matrix representation of this set,

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ Assuming a solution of the form, $x = \alpha e^{\lambda t}$ where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$;

$$|A - \lambda I| = \begin{bmatrix} 4-\lambda & 3 & 1 \\ -4 & -4-\lambda & -2 \\ 8 & 12 & 6-\lambda \end{bmatrix} = 0$$ Expanding the characteristic equation;

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0 \Rightarrow (\lambda - 2)^3 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 2$$

Let's obtain the eigenvector of the eigenvalue λ=2;

$$A\alpha = \lambda\alpha \Rightarrow \begin{bmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

Some manipulations yield the homogenous set,

$$\begin{aligned} 2\alpha_1 + 3\alpha_2 + \alpha_3 &= 0 \\ -4\alpha_1 - 6\alpha_2 - 2\alpha_3 &= 0 \\ 8\alpha_1 + 12\alpha_2 + 4\alpha_3 &= 0 \end{aligned}$$

The rank of the coefficient matrix $\begin{bmatrix} 2 & 3 & 1 \\ -4 & -6 & -2 \\ 8 & 12 & 4 \end{bmatrix}$ is 1

because all of the 2x2 matrices are all zero, too, so, two arbitrary selections are made

Let's select $\alpha_1 = 1,\ \alpha_2 = 0$ then $\alpha_3 = -2$ or selected; $\alpha_1 = 0,\ \alpha_2 = 1$ then $\alpha_3 = -3$

Therefore, two of the eigenvectors and independent solutions corresponding to λ=2

$$\alpha = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ 0 \\ -2e^{2t} \end{bmatrix} \quad \& \quad \alpha = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} e^{2t} = \begin{bmatrix} 0 \\ e^{2t} \\ -3e^{2t} \end{bmatrix}$$

Since the state matrix 3x3 then the total number of independent solutions is 3; so, one of the solutions is missing.

The third solution proposal for λ=2 is made as $(\alpha t + \beta)e^{2t}$ where

α satisfies $(A - 2I)\alpha = 0$ and β satisfies $(A - 2I)\beta = \alpha$.

Notice that α above is the linear combinations of the eigenvector obtained before

$$\alpha = k_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{bmatrix}$$

and $\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$ under these considerations, since $(A - 2I)\beta = \alpha$ then,

$$\left[\begin{bmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right] \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{bmatrix}$$

Manipulations yields that

$$\begin{array}{llll} 2\beta_1 & +3\beta_2 & +\beta_3 & = k_1 \\ -4\beta_1 & -6\beta_2 & -2\beta_3 & = k_2 \\ 8\beta_1 & 12\beta_2 & +4\beta_3 & = -2k_1 - 3k_2 \end{array} \quad \Rightarrow k_2 = -2k_1$$

Since the rank of the coefficient matrix is 2, 3-2=1 free selection is made; say

$k_1 = 1 \Rightarrow k_2 = -2$; substitution yields,

$$\begin{array}{lllll} 2\beta_1 & +3\beta_2 & +\beta_3 & = & 1 \\ -4\beta_1 & -6\beta_2 & -2\beta_3 & = & -2 \\ 8\beta_1 & 12\beta_2 & +4\beta_3 & = & 4 \end{array}$$

Rank=1, so proposal should be made

Let $\beta_1 = \beta_2 = 0 \Rightarrow \beta_3 = 1 \Rightarrow \beta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Since $\alpha = \begin{bmatrix} k_1 \\ k_2 \\ -2k_1 - 3k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Substitution of these into $x = [\alpha t + \beta]e^{\lambda t}$

$$x = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} te^{2t} \\ -2te^{2t} \\ (4t+1)e^{2t} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C_1 \begin{bmatrix} e^{2t} \\ 0 \\ -2e^{2t} \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ e^{2t} \\ -3e^{2t} \end{bmatrix} + C_3 \begin{bmatrix} te^{2t} \\ -2te^{2t} \\ (4t+1)e^{2t} \end{bmatrix}$$

Therefore;

$$x_1 = C_1 e^{2t} + C_3 te^{2t}$$

$$x_2 = C_2 e^{2t} - 2C_3 te^{2t}$$

$$x_3 = -2C_1 e^{2t} - 3C_2 e^{2t} + C_3(4t+1)e^{2t}$$

Non-Homogenous Systems of Linear Differential Equ.

Let $f(t) \neq 0$ and consider the system.

$$x' = Ax + f \quad , \quad f \neq 0$$

Let the solutions of homogenous system is

$$x' = Ax$$

be

$$x_h = c_1 x^1 + c_2 x^2 + \cdots + c_n x^n$$

If a particular soln. of the non-homogenous system is x_p, then the general soln. of the non-homogenous system will be

$$x_{gen} = x_h + x_p$$

The method of Undetermined Coefficients

$$f(t) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \psi(t)$$ and $\psi(t)$ is a function of finite derivatives, where

the set $D = \{\psi_1(t), \psi_2(t) + \cdots + \psi_p(t)\}$ is finite.

And let the function in the solutions

$x^i = \alpha_i e^{\lambda_i t}$, $i = 1, 2, \ldots, n$ of the homogenous system be

$$H = \{f_1(t), f_2(t), \ldots, f_n(t)\}$$

if $H \cap D \neq \emptyset$ empty set, a particular soln. proposal would be

$$x_p = c^1 \psi_1(t) + c^2 \psi_2(t) + \cdots + c^p \psi_p(t)$$

where $\{c^1, c^2, \ldots c^p\}$ are n-dimensional vector to be specified.

Consider the homogeneous linear system: and Solve it using matrix method ②

$x_1' = 7x_1 - x_2 + 6x_3 + \cos 2t$

$x_2' = -10x_1 + 4x_2 - 12x_3$

$x_3 = -2x_1 + x_2 - x_3$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos 2t$$

Recall the homogenous soln. is

$$X_n = c_1 \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \cdot e^{2t} + c_2 \cdot \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \cdot e^{3t} + c_3 \cdot \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix} \cdot e^{5t}$$ then

$H = \{e^{2t}, e^{3t}, e^{5t}\}$

$D = \{\cos 2t, \sin 2t\}$

Since $H \cap D = \emptyset$ empty set. a particular soln. proposed be

$X_p = c^1 \cos 2t + c^2 \sin 2t$

$X_p' = -2 \cdot c^1 \sin 2t + 2c^2 \cos 2t$

Then eqn. 1 implies

$$-2c^1 \sin 2t + 2c^2 \cos 2t = Ac^1 \cos 2t + Ac^2 \sin 2t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos 2t$$

Equating the coefficients of $\{\cos 2t, \sin 2t\}$

$\sin 2t : -2c^1 = Ac^2$

$\cos 2t : 2c^2 = A \cdot c^1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow 2Ac^2 = A^2 c^1 + A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$-4c^1 = A^2 c^1 + A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\rightarrow (A^2 + 4I)\, c^1 = A \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$\rightarrow c^1 = (A^2 + 4I)^{-1} \times A \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$c^1 = \{-0.25, -0.19, -0.05\}$

$c^2 = -2A^{-1}c^1$

$c^2 = \{0.24, 0.21, -0.17\}$

$x_p = \{-0.25, -0.19, -0.05\} \cdot \cos 2t + \{0.24, 0.21, -0.17\} \sin 2t$

$x_{gen} = x_n + x_p$

$x_1 = c_1 e^{2t} + c_2 e^{3t} + 3c_3 e^{5t} - 0.25 \cos 2t + 0.24 \cdot \sin 2t$

$x_2 =$

$x_3 =$

Series Solutions of D.E.

$$c_0 + c_1(x-x_0) + c_2(x-x_0)^2 = \sum_{n=0}^{\infty} c_n (x-x_0)^n$$

converges uniformly to a function $f(x)$ in an interval for which x_0 is an inner point. Then $f(x)$ is analytical around the point x_0, and Taylor's theorem says that

$$c_n = \frac{f^{(n)}(x_0)}{n!}, \quad n = 0, 1, 2 \ldots$$

2nd Order LDE with Variable Coefficients

In Chapter 1, the techniques to solve 2nd order LDE's with constant coefficient are satisfied

$$ay'' + by' + cy = r(x)$$

If you'd like to solve variable coefficients:

$$y'' + p(x) \cdot y' + q(x) \cdot y = r(x)$$ are more complex.

Under certain conditions on $p(x)$ and $q(x)$ the eqn. may have series solns. in appropriate forms

An ordinary point: if $p(x)$ and $q(x)$ are both analytical around a point x_0, then x_0 is an ordinary point for the DE.

$$y'' + p(x) \cdot y' + q(x)\, y = r(x)$$

A regular singular point; if $p(x)$ or $q(x)$ is not analytical around a point x_0, but $(x-x_0)\,p(x)$ and $(x-x_0)^2\, q(x)$ are both analytical around the point x_0, then x_0 is a regular singular point for DE.

Irregular Singular Point: if $(x-x_0)\,p(x)$ or $(x-x_0)^2\, q(x)$ is not analytical around the point x_0. then x_0 is an irregular singular point of the DE.

Series Soln. about an Ordinary Point

Frobenius Theorem1: Let x_0 be an ordinary point for the DE.

$$y'' + p(x)y' + q(x)y = r(x)$$

Then the DE has at least one series soln of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

Find the power series soln. of the initial value problem in the below around the point $x_0 = 0$

$$(x^2-1)y'' + 3xy' + xy = 0; \quad y(0) = 4, \quad y'(0) = 6$$

$$y'' + \frac{3x}{x^2-1} y' + \frac{x}{x^2-1} y = 0$$

obviously . coefficient functions are not analytic only at $x = \mp 1$, So $x_0 = 0$ is an ordinary point. Therefore, the soln. proposal

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = 2c_2 + 6c_3 x + \cdots = \sum_{n=1}^{\infty} n(n-1) \cdot c_n x^{n-2}$$

Substitution of these into DE

$$(x^2-1) \sum_{n=1}^{\infty} n(n-1) \cdot c_n x^{n-2} + 3x \sum_{n=1}^{\infty} n c_n x^{n-1} + x \sum_{n=1}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^n - \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + 3\sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

The second term is modified by replacing

$n \rightarrow n+2$

$$\sum_{n=2}^{\infty} n(n-1)\, c_n\, x^{n-2} \quad \cdot \text{for } n \rightarrow n+2 \qquad \sum_{n+2=2}^{\infty} (n+2)(n+1)\, c_{n+2}\, x^n$$

Similarly for the last term $n \rightarrow n-1$

$$\sum_{n=0}^{\infty} c_n\, x^{n+1} \text{ for } n \rightarrow n-1 \Rightarrow \sum_{n-1=0}^{\infty} c_{n-1}\, x^n$$

Substitution of above yields that:

$$\sum_{n=2}^{\infty} n(n-1)\, c_n \cdot x^n - \sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2}\, x^n + 3\sum_{n=1}^{\infty} n\, c_n\, x^n + \sum_{n=1}^{\infty} c_{n-1}\, x^n = 0$$

The common range is $n=2 \ldots \infty$ lets make the index $n=2$

$$\sum_{n=2}^{\infty} n(n-1)\, c_n\, x^n$$

already $n=2$

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2}\, x^n = 2c_2 + 6c_3 x + \sum_{n=2}^{\infty} (n+2)(n+1)\, c_{n+2}\, x^n$$

$$\sum_{n=1}^{\infty} n\, c_n\, x^n = c_1 x + \sum_{n=2}^{\infty} n\, c_n\, x^n$$

$$\sum_{n=1}^{\infty} c_{n-1}\, x^n = c_0 x + \sum_{n=2}^{\infty} c_{n-1}\, x^n$$

Substitution these back yields that

$$\sum_{n=2}^{\infty} n(n-1)\, c_n x^n - 2c_2 - 6c_3 x - \sum_{n=2}^{\infty} (n+2)(n+1)\, c_{n+2}\, x^n + 3c_1 x + 3\sum_{n=2}^{\infty} n\, c_n x^n +$$

$$+ c_0 x + \sum_{n=2}^{\infty} c_{n-1}\, x^n = 0$$

$$-2c_2 + (c_0 + 3c_1 - 6c_3)\, x + \sum_{n=2}^{\infty} \left[-(n+2)(n+1)\, c_{n+2} + [n(n-1) + 3n]\, c_n + c_{n-1}\right] x^n = 0$$

$$-2c_2 = 0 \rightarrow c_2 = 0$$

$$c_0 + 3c_1 - 6c_3 = 0 \rightarrow c_3 = \frac{1}{6} c_0 + \frac{1}{2} c_1$$

$$[-(n+2)(n+1)\, c_{n+2} + [n(n+2)]\, c_n + c_{n-1} = 0$$

$$c_{n+2} = \frac{n\cdot(n+2)\, c_n + c_{n-1}}{(n+1)(n+2)} \; ; \quad n \geq 2$$

$$n=2 \rightarrow c_4 = \frac{8c_2 + c_1}{12} = \frac{1}{12} c_1$$

$$n=3 \rightarrow c_5 = \frac{15c_3 + c_2}{20} = \frac{3}{4}\left(\frac{1}{6} c_0 + \frac{1}{2} c_1\right) + \frac{c_2}{20} = \frac{c_0}{8} + \frac{3c_1}{8}$$

Substitution of these c_0, c_1, c_2 into the proposal

$$y = c_0 + c_1 x + c_2 \cdot x^2 + \ldots$$

we have

$$y = c_0 + c_1 x + \left(\frac{1}{6} c_0 + \frac{1}{2} c_1\right) x^3 + \frac{1}{2} c_1 x^4 + \left(\frac{1}{8} c_0 + \frac{3}{8} c_1\right) x^5 + \ldots$$

$$y = c_0 \cdot \left(1 + \frac{x^3}{6} + \frac{x^5}{8} + \ldots\right) + c_1\left(x + \frac{x^3}{2} + \frac{x^4}{12} + \frac{3x^5}{8} + \ldots\right)$$

for $y(0) = 4 \rightarrow c_0 = 4$

for $y'(0) = 6 \Rightarrow y' = c_0\left(\frac{1}{2}x^2 + \frac{5}{8}x^4 + \dots\right) + c_1\left(1 + \frac{3}{2}x^2 + \frac{1}{3}x^3 + \frac{15}{8}x^4 \dots\right)$

$\Rightarrow c_1 = 6$

$$y = 4\left(1 + \frac{1}{6}x^3 + \frac{1}{8}x^5 + \dots\right) + 6\left(x + \frac{1}{2}x^3 + \frac{1}{12}x^4 + \dots + \frac{3}{8}x^5\right)$$

$$y = 4 + 6x + \frac{11}{3}x^3 + \frac{1}{2}x^4 + \frac{11}{4}x^5 + \dots$$

If initial condition would be $y(5) = 4$, $y'(5) = 6 \rightarrow$ then using transformation such as $t = x - 5$ and obtain a series soln. about $t = 0$ instead of $x = 5$

$(x^2 - 1)y'' + 3xy' + xy = 0$; $\quad y(x=5) = 4 \quad y'(x=5) = 6$

using proposal.

$y = \sum_{n=0}^{\infty} c_n (x-5)^n \rightarrow$ A change of variable $t = x - 5$ replaces this initial value problem by the equivalent problem

$$t = x - 5 \rightarrow \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \rightarrow y' = \frac{dy}{dt}; \quad y'' = \frac{d^2y}{dt^2}$$

$$x = t + 5 \rightarrow x^2 - 1 = t^2 + 10t + 24$$

$(t^2 + 10t + 24)\frac{d^2y}{dt^2} + (3t + 15)\frac{dy}{dt} + 3y = 0$ and initial conditions;

$y(0) = 4$. $y'(0) = 6$ and solution proposal to be made is!

$y = \sum_{n=0}^{\infty} c_n t^n \Rightarrow$ Having obtained the soln. into the substitution $t = x - 5$ replaces the soln in terms of x. This technique may be suggesed for the sake of simplicity.

Series Solutions About Regular Singular Points

15-11-2023
Math 1

For the differential equation

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

Let $p(x)$ or $q(x)$ is not analytic at the point x_0 but both of $(x-x_0)\,p(x)$ and $(x-x_0)^2 q(x)$ are analytic at point x_0.

Then x_0 is called regular singular point for the differential equatio.

For the existence of a series solution to the equation one has the following theorem.

Let x_0 be regular singular point for the differential equation.

$$y'' + p(x)\, y' + q(x)\, y = r(x)$$

Then, the differential equation has at least one series solution of the form.

$$y = |x-x_0|^r \cdot \sum_{n=0}^{\infty} c_n (x-x_0)^n = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$$

The rest is performed as before. This techique is also known as Frobenius Method. A brief outline of the Frobenius Method is as follows. a proposal is first made as

I) $$y = \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r} = c_0 (x-x_0)^r + c_1 (x-x_0)^{r+1} + c_2 (x-x_0)^{r+2} + \cdots$$

$$y' = r \cdot c_0 (x-x_0)^{r-1} + (r+1)\, c_1 (x-x_0)^r + (r+2)\, c_2 (x-x_0)^{r+1} + \cdots = \sum_{n=0}^{\infty} (n+r)\, c_n (x-x_0)^{n+r-1}$$

$$y'' = r(r-1)\, c_0 (x-x_0)^{r-2} + r(r+1)\, c_1 (x-x_0)^{r-1} + (r+2)(r+1)\, c_2 (x-x_0)^r = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n (x-x_0)^{n+r-2}$$

II) Substitution and manipulation yield a polynomial in $(x-x_0)$:

$$K_0 (x-x_0)^{r+k} + K_1 (x-x_0)^{r+k+1} + K_2 (x-x_0)^{r+k+2} + \cdots = 0$$

For this theorem satified the coefficients must be equal to zero

$$K_0 = K_1 = K_2 = \cdots = 0$$

III) The lowest power $(x-x_0)^{r+k}$ $\xrightarrow{\text{yields}}$ $K_0=0$ is a quadratic equation in r, called the indicial equation;

The roots of this is called exponents of of differential equation. let them r_1 and r_2, $Re(r_1) \geq Re(r_2)$, where $Re(r_i)$ is the real part of the exponent r_i, if r_1, r_2 are equal, then r_1 is the larger root

IV) Equate $K_0 = K_1 = K_2 = \cdots = 0$; substitute $r=r_1$ and obtain condition C_n's

V) If $r_2 \neq r_1$ repeat the previous procedure for $r=r_2$ to obtain a second Solution (for smaller root r_2) which is linearly independent to the previous one.

According to different types of $r_1 - r_2$, different solution styles are obtained.

VI) Let $r_1 - r_2 \neq N$, non-negative integer $(r_1 - r_2 \neq 0,1,2,3 \cdots)$ then two independent solutions do exist.

$$y_1 = |x-x_0|^{r_1} \cdot \sum_{n=0}^{\infty} c_n (x-x_0)^n, \quad c_0 \neq 0$$

$$y_2 = |x-x_0|^{r_2} \cdot \sum_{n=0}^{\infty} c_n^* \cdot (x-x_0)^n + c y_1(x) \ln|x-x_0|$$

VII) Let $r_1 - r_2 = N \neq 0$, N a positive integer then two independent solutions are expressed as.

$$y_1(x) = |x-x_0|^{r_1} \cdot \sum_{n=0}^{\infty} c_n \cdot (x-x_0)^n, \quad c_0 \neq 0$$

$$y_2(x) = |x-x_0|^{r_2} \sum_{n=0}^{\infty} c_n^* (x-x_0)^n + c y_1(x) \ln|x-x_0|$$

VIII) Let $r_1 - r_2 = 0$, then two independent solutions are

$$y_1(x) = |x-x_0|^{r_1} \sum_{n=0}^{\infty} c_n (x-x_0)^n, \quad c_0 \neq 0$$

$$y_2(x) = |x-x_0|^{r_1+1} \cdot \sum_{n=0}^{\infty} c_n^* (x-x_0)^n + c y_1(x) \ln|x-x_0|$$

15-11-2023

Math (3)

Given the differential equation

$2x^2y'' - xy' + (x-5)y = 0$

find a series solution about $x_0 = 0$

The normalized differential equation is $y'' - \frac{x}{2x^2}y' + \frac{x+5}{2x^2}\cdot y = 0$

$p(x) = \frac{x}{2x^2}$ and $q(x) = \frac{x-5}{2x^2}$ are not analytic at $x_0=0$ but

$$\left.\begin{aligned}&\lim_{x\to x_0=0} x\left(-\frac{x}{2x^2}\right) = -\frac{1}{2}\\ &\lim_{x\to x_0=0} x^2\left(\frac{x-5}{2x^2}\right) = -\frac{5}{2}\end{aligned}\right\}$$ So, $x_0=0$ is a regular singular point

According to the Frobenius theorem we propose a solution

$$y(x) = c_0 x^r + c_1 x^{r+1} + c_2 x^{r+2} + \dots = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y'(x) = c_0 r x^{r-1} + c_1(r+1)x^r + c_2(r+2)x^{r+1} + \dots = \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}$$

$$y''(x) = c_0 r(r-1)x^{r-2} + c_1(r+1)r\cdot x^{r-1} + c_2(r+1)(r+2)x^r + \dots = \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n \cdot x^{n+r-2}$$

dallemin Tahe kayulacak

Substitution of these into the differential equation yields

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)c_n\cdot x^{n+r-1} + (x-5)\sum_{n=0}^{\infty}c_n x^{n+r} = 0$$

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+1} - 5\sum_{n=0}^{\infty}c_n x^{n+r} + \sum_{n=0}^{\infty}c_n x^{n+r+1} = 0$$

for the last term we let $n \to n-1$

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} - 5\sum_{n=0}^{\infty}c_n x^{n+r} + \sum_{n=0}^{\infty}c_{n-1}x^{n+r} = 0$$

Since the common range is $1 < n < \infty$

$$(2r(r-1)\cdot c_0 - r\cdot c_0 - 5c_0)\, x^r + \sum_{n=1}^{\infty} \left[2(n+r)(n+r-1)c_n - (n+r)c_n - 5c_n + c_{n-1}\right] \cdot x^{n+r} = 0$$

$$(2r(r-1) - r - 5)\, c_0 \cdot x^r + \sum_{n=1}^{\infty} \{[2(n+r)(n+r-1) - (n+r) - 5]\, c_n + c_{n-1}\} \cdot x^{n+r} = 0$$

First and last terms are indicial equation are recurrence relation, respectively

$$2r(r-1) - r - 5 = 0 \to 2r^2 - 3r - 5 = 0 \to r_1 = 5/2 \quad \text{and} \quad r_2 = -1$$

Since $r_1 - r_2 = \frac{7}{2} \neq 0, 1, 2, 3 \ldots$; two independent solutions are simply obtained by the substitutions of r_1 and r_2 into the proposed

$$y(x) = |x - x_0|^r \cdot \sum_{n=0}^{\infty} c_n (x - x_0)^n, \quad c_0 \neq 0$$

Let's obtain the recurrence relation.

$$[2(n+r)(n+r-1) - (n+r) - 5]\, c_n + c_{n-1} = 0$$

$$c_n = \frac{c_{n-1}}{2(n+r)(n+r-1) - (n+r) - 5}$$; recurrence relation one of the solution for $r = 5/2$

$$c_n = -\frac{c_{n-1}}{2\left(n+\frac{5}{2}\right)\left(n+\frac{3}{2}\right) - \left(n+\frac{5}{2}\right) - 5}$$ with $n \geq 1$

$$c_1 = -\frac{c_0}{2\left(1+\frac{5}{2}\right)\left(1+\frac{3}{2}\right) - \left(1+\frac{5}{2}\right) - 5} = \frac{c_0}{2\left(\frac{7}{2}\right)\left(\frac{5}{2}\right) - \left(\frac{7}{2}\right) - \left(\frac{10}{2}\right)} = -\frac{c_0}{9}$$

$$c_2 = -\frac{c_1}{2\left(2+\frac{5}{2}\right)\left(2+\frac{3}{2}\right) - \left(2+\frac{5}{2}\right)} = \frac{c_1}{2\left(\frac{9}{2}\right)\left(\frac{7}{2}\right) - \left(\frac{9}{2}\right) - \frac{10}{2}} = \frac{c_0}{198}$$

$$y_1(x) = c_0 x^{5/2} \left(1 - \frac{1}{9}x + \frac{1}{198}x^2 - \ldots\right)$$

The other solution for $r=-1$

$$c_n = -\frac{c_{n-1}}{2(n-1)(n-2)-(n-1)-5} \quad , \quad c_1 =$$

$$c_1 = -\frac{c_0}{-5} \quad , \quad c_2 = \frac{c_1}{6} = \frac{c_0}{30} \quad , \quad c_3 = \frac{c_2}{-3} = \frac{c_0}{90}$$

$$y_2(x) = c_0 \cdot x^{-1}\left(1 + \frac{1}{5}x + \frac{1}{30}x^2 + \frac{1}{90}x^3 \dots\right)$$

Assuming $c_0 = 1$

$$y_{gen} = c_1 y_1 + c_2 y_2$$

$$y_{gen} = c_1 \cdot x^{5/2} \cdot \left(1 - \frac{1}{9}x + \frac{1}{198}x^2 - \dots\right) + c_2 \cdot x^{-1}\left(1 + \frac{1}{5}x + \frac{1}{30}x^2 + \frac{1}{90}x^3 \dots\right)$$

Determination of solutions according to roots of indicial equation may be trivial in some cases. So sharp classifications are not very straightforward. A closer look may be performed as follows. Let the roots of the indicial equation are $r_1 = \alpha$ $r_2 = \beta$

I) $\alpha \neq \beta$ and $\alpha - \beta$ is not an integer

This case is relatively straightforward; the independent solutions y_1, y_2 and the general solution y_{gen} is obtained as

$$y_1 = y(x,r)\big|_{r=\alpha} \; ; \quad y_2 = y(x,r)\big|_{r=\beta}$$

$$y_{gen} = c_1 y_1 + c_2 y_2$$

II) $\alpha = \beta$ This is also relatively straightforward case

One of solution is $y_1 = y(x,r)\big|_{r=\alpha}$

The other independent solution is $y_2 = \frac{\partial y}{\partial r}\Big|_{r=\alpha}$

Note that substitution of one of the solution in r into the differential equation yields that if we say

$$y(x,r) = a_0 x^r \left[1 + \frac{1}{(r+1)^2} + \cdots \right]$$

then

$$[\text{diff. equn}]\Big|_{y(x,r)} = \alpha_0 (r-\alpha)(r-\beta)\, x^{r-1}$$

III) $\alpha \neq \beta$ and $\alpha - \beta$ is an integer

in this case; some of the coefficients. C may become infinite for either α or β.. Obviously for the root without trouble β. the solution is obtained as before

$$y_1 = y(x,r)\Big|_{r=\beta}$$

For the other troublesome root α some of the coefficients may involve terms like, $\frac{a_0}{r-\alpha}$, In such cases, arbitrary constants may be selected such a way that $a_0 = b_0 (r-\alpha)$ to eliminate $(r-\alpha)$. Substitution of α into $y(x,r)$ as

$y_2 = y(x,r)\big|_{r=\alpha}$ may result in a dependent solution in this case

$y_2 = \frac{\partial y(x,r)}{\partial r}\Big|_{r=\alpha}$ is used

Please note that since $y(x,r) = x^r(\cdots)$ then $\frac{\partial y}{\partial r} = x^r \ln x(\cdots) + x^r \cdot \frac{\partial y_p}{\partial r}$

where $(\cdots)$ is y_p.

15-11-2023
Math (7)

Find the series solution of $xy''+y'-y=0$ about $x_0=0$

$$y''+\frac{1}{x}y'-\frac{1}{x}y=0$$

Since $\frac{1}{x}$ is not at $x_0=0$ but $\lim_{x\to0} x(1/x)=1$; $\lim_{x\to0} x^2(-1/x)=0$

therefore $x_0=0$ is a regular singular point.

The solution proposal is

$$y=\sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y'=\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}$$

$$x\cdot\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2}+\sum_{n=0}^{\infty}(n+r)c_n\cdot x^{n+r-1}-\sum_{n=0}^{\infty}c_n x^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n\cdot x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1}-\sum_{n=0}^{\infty}c_{n-1}\cdot x^{n+r-1}=0$$

Since the common range is $1\to\infty$

$$r(r-1)c_0\cdot x^{r-1}+rc_0\cdot x^{r-1}+\sum_{n=0}^{\infty}\{(n+r)+(n+r-1)]c_n-c_{n-1}\}\cdot x^{r+n-1}=0$$

$$c_0\cdot r^2x^{r-1}+\sum_{n=0}^{\infty}\{(n+r)^2\cdot c_n-c_{n-1}\}x^{r+n-1}=0$$

Equating the coefficients of x^1 to zero one has indicial equation

$r^2=0 \to r_1=r_2=0$ for $c_0\neq0$

Recurrence relations is

$$c_n=\frac{c_{n-1}}{(n+r)^2},\ n\geq1$$

Hence

$$c_1 = \frac{c_0}{(r+1)^2}, \quad c_2 = \frac{c_1}{(r+2)^2} = \frac{c_0}{(r+2)^2 (r+1)^2} = \frac{c_0}{[(r+1)!]^2}$$

$$c_3 = \frac{c_2}{(r+3)^2} = \frac{c_0}{(r+3)^2 \cdot (r+2)^2 \cdot (r+1)^2} = \frac{c_0}{[(r+3)!]^2}$$

$$c_n = \frac{c_{n-1}}{(n+1)^2} = \frac{c_0}{[(n+1)!]^2}$$

Substituting of these into the proposal

$$y(x,r) = \sum_{n=0}^{\infty} c_n x^{n+1} = c_0 x^r + c_1 x^r + c_2 x^2 + c_3 x^r x^3 + \cdots$$

so

$$y(x,r) = c_0 \cdot x^r \cdot \left[1 + \frac{x}{(r+1)^2} + \frac{x^2}{[(r+2)!]^2} + \frac{x^3}{[(r+3)!]^3} + \cdots + \frac{x^n}{[(r+n)!]^2} + \cdots \right]$$

$c_0 = 1$

$$y(x,r)\Big|_{r=0} = 1 + 1 + \frac{x}{(1)^2} + \frac{x^2}{[2!]^2} + \frac{x^3}{[3!]^2} + \cdots + \frac{x^n}{[n!]^2} + \cdots$$

A second independent solution is obtained by

$$y_2 = \frac{\partial y(x,r)}{\partial r}\Big|_{r=0} . \text{ Hence } c_0 = 1$$

$$\frac{\partial y(x,r)}{\partial r} = x^r \ln|x| \cdot \left[1 + \frac{x}{(r+1)^2} + \frac{x^2}{[(r+2)!]^2} + \cdots + \frac{x^n}{[(r+n)!]^2} \right] - 2x^r \cdot \left[\frac{x}{(r+1)^3} + \frac{x^2}{(2!)^2}\left(1 + \frac{1}{2}\right) + \cdots \right]$$

$$y_2 = \frac{\partial y(x,r)}{\partial r}\Big|_{r=0} = \ln|x| \cdot y_1 - 2\left[\frac{x}{1^3} + \frac{x^2}{(2!)^2}\left(1 + \frac{1}{2}\right) + \cdots + \frac{x^n}{[(n)!]^2} \cdot \left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right) + \cdots \right]$$

So the general solutions is

$$y_{gen} = c_1 y_1 + c_2 y_2 = c_1 y_1 + c_2 \cdot \ln|x|\, y_1 - 2c_2 \left[x + \frac{x^2}{(2!)^2}\left(1+\frac{1}{2}\right) + \cdots + \frac{x^n}{(n!)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right]$$

So

$$y_{gen} = y_1\left(c_1 + c_2 \cdot \ln|x|\right) - 2c_2\left[x + \frac{3}{2}\cdot\frac{x^2}{(2!)^2} + \cdots + \frac{x^n}{(n!)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right]$$