a combination of terms (1.1) and (1.2) representing inflow and outflow with respect to unit volume

Example 1: Conservation law of chemical species

$$\frac{\partial(\rho m_l)}{\partial t} + div(\rho \mathbf{U} m_l + \mathbf{J}_l) = R_l \qquad (1.3)$$

where:

m_l - mass fraction of the species l

$\frac{\partial(\rho m_l)}{\partial t}$ - rate of change of the volume fraction of species l

$\rho \mathbf{U} m_l$ - convective flux of species l

J_l - diffusive flux of spciese l the most often caused by gradient of its mass fraction m_l

$$\Phi = a_0 + a_1 x + a_2 x^2 + \quad + a_m x^m \qquad (2.2)$$

The substitution of the approximation (2.2) into Eq.(2.1) gives non zero residual:

$$L(\Phi) = R \qquad (2.3)$$

minimization of the residual R in sense of certain norm leads to:

$$\int WRdx = 0 \qquad (2.4)$$

Where : W - weighting function

Outflow with respect to volume unit:

$$\frac{\partial J_x}{\partial x}+\frac{\partial J_y}{\partial y}+\frac{\partial J_z}{\partial z}=div\mathbf{J} \qquad (1.1)$$

Rate of change of physical property Φ with respect to volume unit

$$\frac{\partial(\rho\Phi)}{\partial t} \qquad (1.2)$$

Differential equations of fluid mechanics and heat transfer are a combination of terms (1.1) and (1.2) representing inflow and outflow with respect to unit volume

Example 1: Conservation law of chemical species

$$\frac{\partial(\rho m_l)}{\partial t}+div(\rho\mathbf{U}m_l+\mathbf{J}_l)=R_l \qquad (1.3)$$

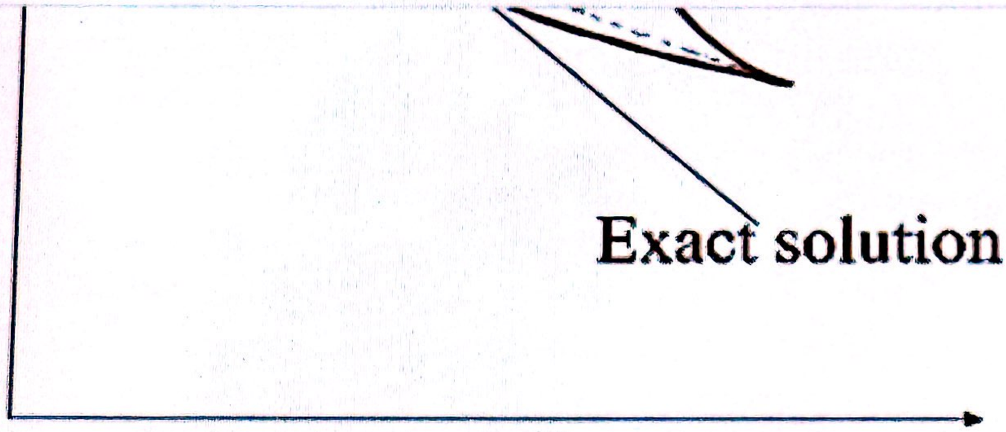


Fig.2.3. Example of physically realistic and unrealistic behaviour

Source term approximation:

$$\bar{S} = S_C + S_P T_P \qquad (2.14)$$

$$a_P = a_E + a_W - S_P \Delta x \qquad (2.15)$$

$$b = S_C \Delta x \qquad (2.16)$$

The four basic rules of discretization formulation

1. Consistency at control-volume faces:

Flux of the dependent variable transmitted through the common face of two control volumes must be approximated by the use of the same formula for both adjacent control volumes

2. Positive coefficients

3. Negative-slope source-term linearization

4. Sum of the neighbour coefficients in the discretization equation should fulfil the following condition:

$$a_P = \sum a_{nb} \tag{2.17}$$

1.2. Fundamental idea of numerical analysis

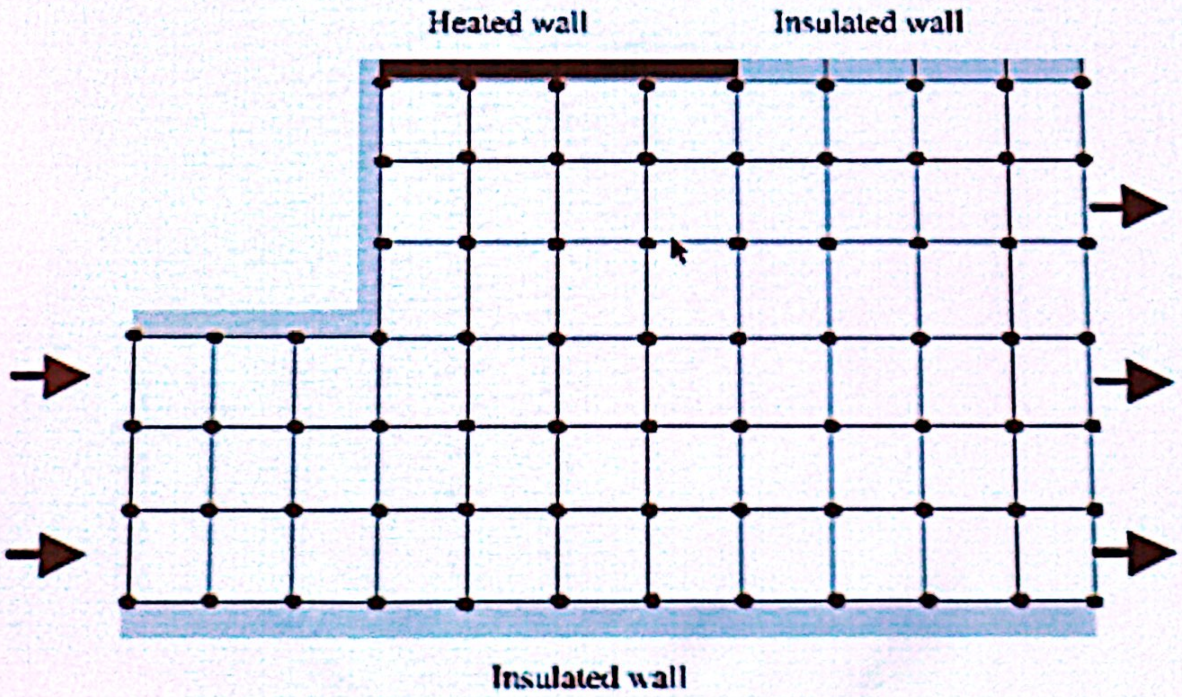


Fig.1.1. Example of numerical grid

➤ Discretization of the computational domain

Conservation of Mass (Differential CV & Taylor series)

- Dividing through by volume *dxdydz*

$$\boxed{\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial x}+\frac{\partial(\rho w)}{\partial x}=0}$$

Or, if we apply the definition of the divergence of a vector

$$\boxed{\frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot\left(\rho\vec{V}\right)=0}$$

- Use product rule on divergence term

$$\frac{\partial \rho}{\partial t}+\vec{\nabla}\cdot\left(\rho\vec{V}\right)=\frac{\partial \rho}{\partial t}+\vec{V}\cdot\nabla\rho+\rho\nabla\cdot\vec{V}=0$$

Where, *D/Dt* is material or total derivative

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\vec{V}=0 \quad \text{or} \quad \boxed{\frac{1}{\rho}\frac{D\rho}{Dt}+\nabla\cdot\vec{V}=0}$$

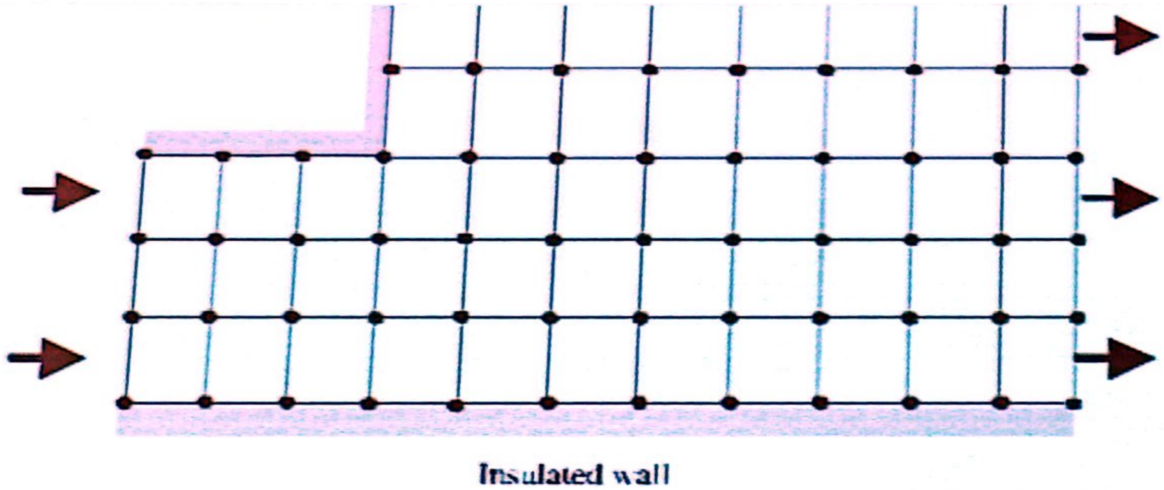


Fig.1.1. Example of numerical grid

- Discretization of the computational domain
- The dependent variables like velocity, temperature, pressure or density are determined at grid points of computational domain which represent their neighbourhood of finite volume
- Transformation of the problem described by the system of differential equations into the problem described by the system of algebraic equations
- Application of the linear algebra methods to solve the system of algebraic equations

$\rho U m_l$ - convective flux of species l

J_l - diffusive flux of spciese l the most often caused by gradient of its mass fraction m_l

R_l - rate of generation of destruction of species l by chemical reaction

Diffusion Fick's law

$$J_l = -\Gamma_l \, grad(m_l) \qquad (1.4)$$

where: Γ_l – diffusion coefficient two-dimensional one in a cylindrical coordinate system

- Two dimensional boundary layer on flat plate becomes the one-dimensional problem after the following transformation :

$$\eta = \frac{cy}{\sqrt{x}}$$

- Plane free jet is a two-diemsional flow. However it can be described by:

$$\tilde{U} = \tilde{U}(\eta)$$

$$gdzie:$$

$$\tilde{U} = \frac{U}{U_c} \eta = \frac{y}{\delta}$$

U_c – center - line veolocity

y – cross - stream coordinate

δ - characteristic jet width

× + Oluştur

nza

1.3.2. One-way and two-way coordinates

Definition:

The coordinate is called two-way if the conditions at a chosen point are influenced by a change of the conditions on either side of that point

The coordinate is called on-way if a change of conditions in a chosen point is influenced by the change of the conditions on only one side of that point

Examples :

- One-dimensional heat transfer – two-way coordinate
- Time is always one-way coordinate
- Convective transport – one-way process
- Diffusive transport – two-way process

Spatial coordinate along which convective flux dominates has properties of one-way coordinate .
Spatial coordinate along which diffusive flux dominates has properties of two-way coordinate

Example 3: Momentum equation

$$\frac{\partial(\rho U_x)}{\partial t} + div(\rho \mathbf{U} U_x) = div(\mu \ grad\, U_x) - \frac{\partial p}{\partial x} + F_x + V_x \quad (1.9)$$

where:

μ - dynamic viscosity coefficient
p – pressure
F_x – component of unitary mass force
V_x- additional component caused by viscosity

Example 4: Kinetic energy of turbulence transport equation

$$\frac{\partial(\rho k)}{\partial t} + div(\rho \mathbf{U} k) = div(\Gamma_k \ grad\ k) + P - \rho \in \quad (1.10)$$

where:

k – turbulence kinetic energy
Γ_k- diffusion coefficient
P – turbulence energy production from mean flow

General form of differential equation describing fluid flow and heat transfer processes

$$\underset{1}{\frac{\partial(\rho\Phi)}{\partial t}} + \underset{2}{div(\rho \mathbf{U}\Phi)} = \underset{3}{div(\Gamma\ grad\ \Phi)} + \underset{4}{S} \qquad (1.11)$$

where:

1 – unsteady term
2 – convective term
3 – diffusive term
4 – source term
Γ- general diffusion coefficient

Gözden Geçir Görünüm Kayıt Yardım

Paragraf

Yerleştir

Çizim

Conservation of Mass *(Cylindrical coordinates)*

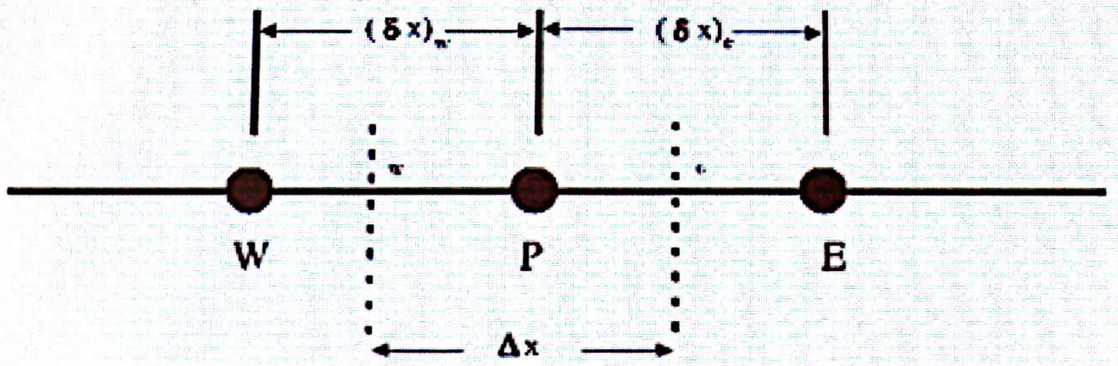
$$\vec{\nabla} = \frac{1}{r}\frac{\partial(r)}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z \qquad \vec{V} = U_r\hat{e}_r + U_\theta\hat{e}_\theta + U_z\hat{e}_z$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right) = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho U_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho U_\theta)}{\partial \theta} + \frac{\partial(\rho U_z)}{\partial z} = 0}$$

- In general, continuity equation cannot be used by itself to solve for flow field, however it can be used to:

1. Determine if velocity field is incompressible.
2. Find missing velocity component.



Rys.2.1. Grid-point distribution for the one-dimensional problem

Integration of the Eq. (2.5) over the control volume leads to:

$$\left(k\frac{dT}{dx}\right)_e - \left(k\frac{dT}{dx}\right)_w + \int_w^e S\,dx = 0 \qquad (2.6)$$

The simplest profiles of dependent variable over the control volume are as follows

(a) Stepwise profile

The proper choice of a sequence of weighting (test) functions allows to formulate a number of equations necessary to determine the parameters of the approximation (2.2)

There are various methods originating from the formulation presented above:

- ✓ Finite element methods
- ✓ Spectral methods

➢ Control volume method

(main subject of the current lecture)

2.1. Basic idea of the control volume formulation

The differential equation governing steady one-dimensional heat conduction can be presented as follows:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + S = 0 \qquad (2.5)$$

Total or Material derivative

- The total derivative operator (d/dt) is also given special notation, D/Dt.

  Material derivative: $\frac{D}{Dt} = \frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{V} \cdot \vec{\nabla})$

- Remember D/Dt (or d/dt) and ∂/∂t are physically and numerically different quantities, The former is the time rate of change following a moving fluid particle while later is the time rate of change at fixed location.
- Provides ``transformation" between Lagrangian and Eulerian frames.
- Other names for the material derivative include: **total, particle, Lagrangian, Eulerian,** and **substantial** derivative.
- It can be applied to any fluid properties, both scalars and vectors (e.g. V, p, p etc.). For example, the material derivative of velocity & pressure can be written as:

$$\frac{D\vec{V}}{Dt} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V} \qquad \frac{DP}{Dt} = \frac{dP}{dt} = \frac{\partial P}{\partial t} + (\vec{V} \cdot \vec{\nabla})P$$

+ Oluştur

1.3. Nature of coordinate system

1.3.1. The proper choice of coordinate system

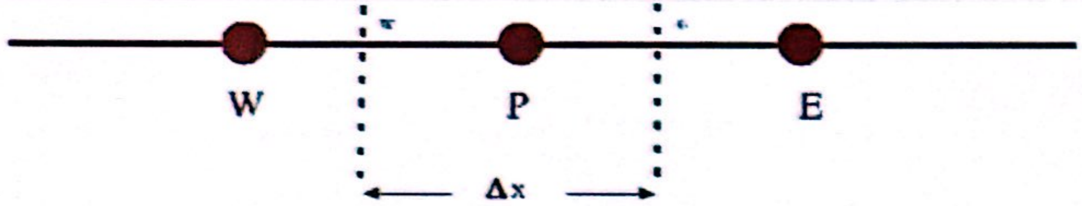
- Flow around the body moving at constant velocity is an unsteady problem from a point of view of motionless observer and steady one in the coordinate system moving with the body
- Fluid flow in a pipe of circular cross section is a three dimensional problem in Cartesian coordinate system and two-dimensional one in a cylindrical coordinate system
- Two dimensional boundary layer on flat plate becomes the one-dimensional problem after the following transformation :

$$\eta = \frac{cy}{\sqrt{x}}$$

- Plane free jet is a two-diemsional flow. However it can be described by:

$$\tilde{U} = \tilde{U}(\eta)$$

$$gdzie:$$



Rys.2.1. Grid-point distribution for the one-dimensional problem

Integration of the Eq. (2.5) over the control volume leads to:

$$\left(k\frac{dT}{dx}\right)_e - \left(k\frac{dT}{dx}\right)_w + \int_w^e S dx = 0 \qquad (2.6)$$

The simplest profiles of dependent variable over the control volume are as follows

(a) Stepwise profile

(b) Piecewise-linear profile

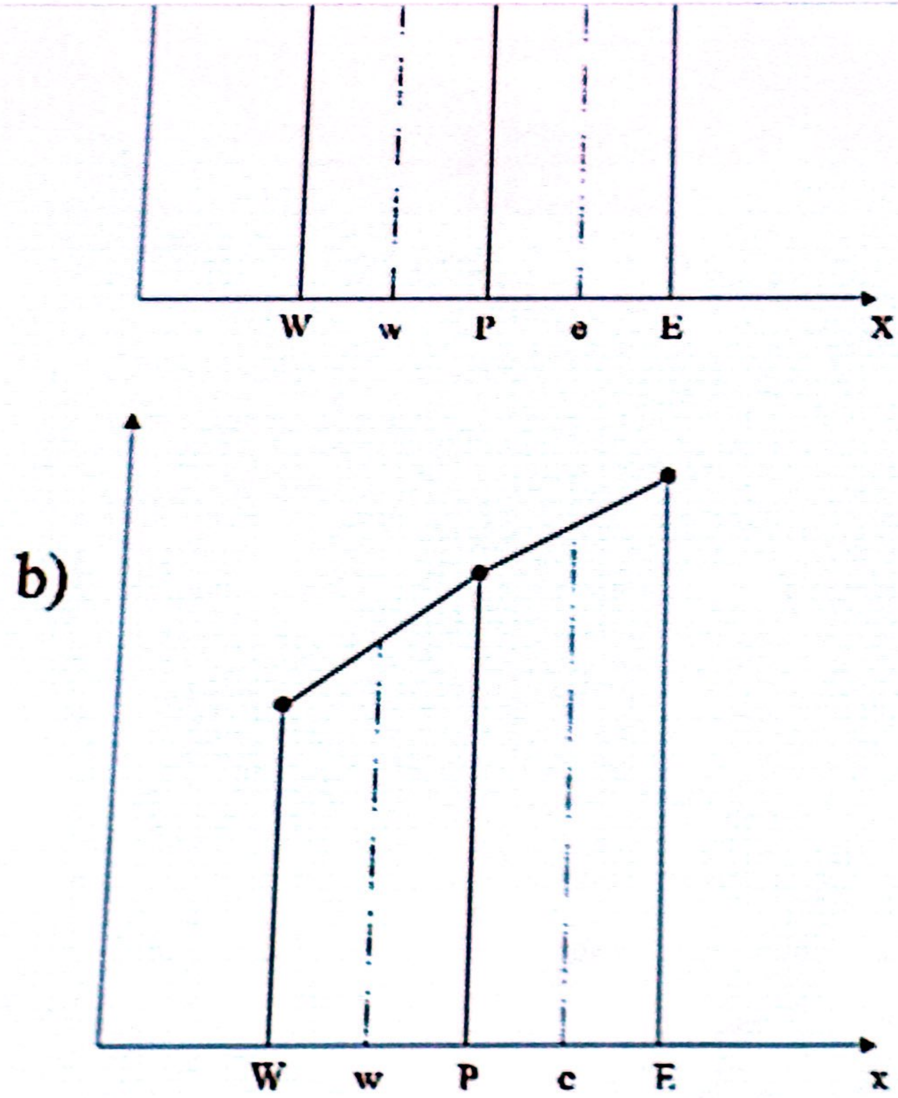


Fig. 2.2 The examples of dependent variable profiles over the control volume (a) Stepwise profile (b) Piecewise-linear profile

Conservation of Mass *(Special Cases)*

- Steady compressible flow: $\cancelto{0}{\frac{\partial \rho}{\partial t}} + \vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0$

$$\boxed{\vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0}$$

- Incompressible flow: ρ = constant *Also D□/Dt =0*

$$\frac{\partial \rho}{\partial t} = 0 \quad \square \quad \boxed{\vec{\nabla} \cdot \vec{V} = 0}$$

Cartesian $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} = 0$$

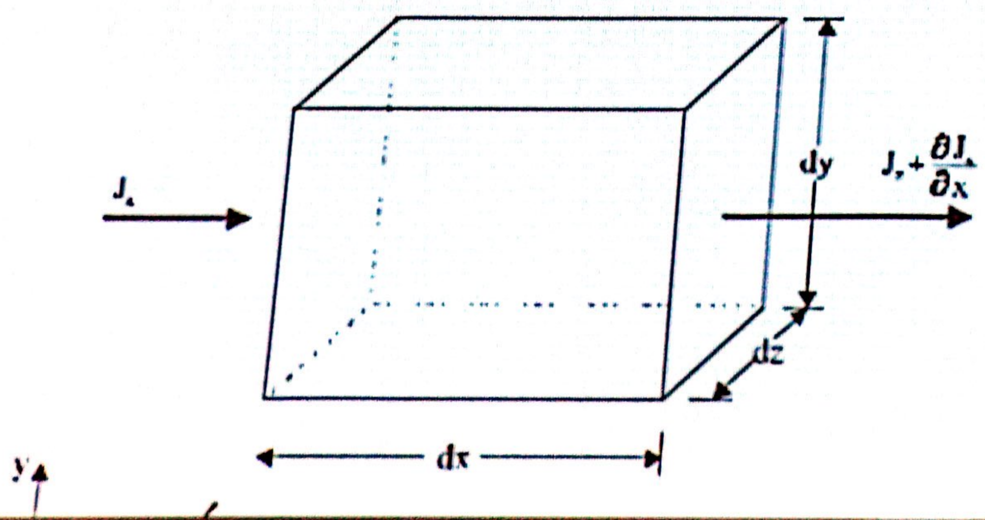
× + Oluştur

nza

1.3.Differential equations of heat transfer and fluid flow

- The dependent variables of these equations are usually specific properties for example:
  - ✓ Mass fraction of a chemical species,
  - ✓ Velocity (momentum per unit mass),
  - ✓ Specific enthalpy

- The terms in a differential equations describing a balance of the dependent varaible denote influences on a unit-volume



- Finite differences methods (expansion in Taylor series) (presented within previous lecture)

- Method of weighted residuals (to be presented within next lecture)

Differential equation

$$L(\Phi) = 0 \qquad (2.1)$$

an approximation of a solution Φ:

$$\Phi = a_0 + a_1 x + a_2 x^2 + \quad + a_m x^m \qquad (2.2)$$

The substitution of the approximation (2.2) into Eq.(2.1) gives non zero residual:

$$L(\Phi) = R \qquad (2.3)$$

minimization of the residual R in sense of certain norm leads to:

$$\frac{\partial(\rho m_l)}{\partial t} + div(\rho \mathbf{U} m_l) = div\left[\Gamma_l \, grad(m_l)\right] + R_l \qquad (1.5)$$

Example 2: Energy equation of laminar steady flow with negligible viscous dissipation

$$div(\rho \mathbf{U} h) = div(k \;\, grad\, T) + S_h \qquad (1.6)$$

where:

h - unitary enthalpy

k - heat conduction coefficient

S_h – rate of heat creation per unit volume

$$c_p grad\, T = grad\, h \qquad (1.7)$$

$$div(\rho \mathbf{U} h) = div\left(\frac{k}{c_p}\, grad\, h\right) + S_h \qquad (1.8)$$

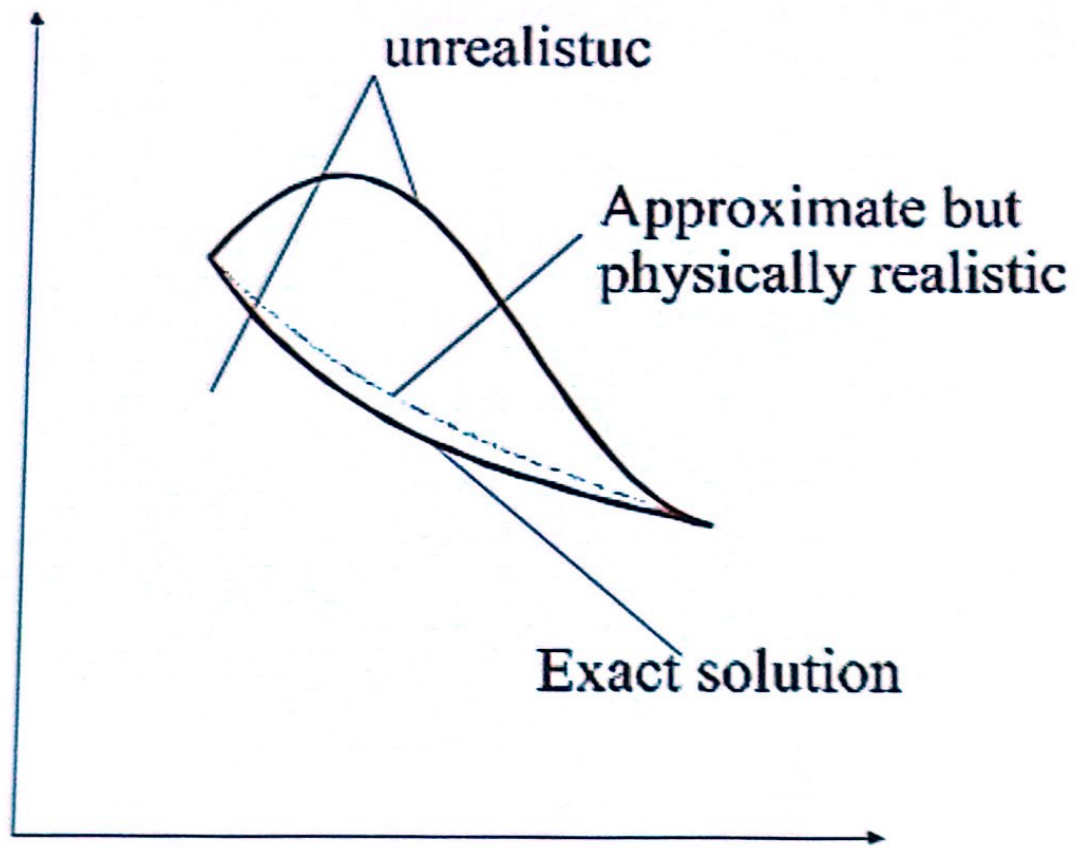


Fig.2.3. Example of physically realistic and unrealistic behaviour

Source term approximation:

Comments:

- Most of discretization equations considered during the lecture will have the form of equation (2.8), or in more general form corresponding to two- and three-dimensional problems:

$$a_P T_P = \sum a_{nb} T_{nb} + b \tag{2.13}$$

where subscript „nb" stand for neighbour points to point P

- In deriving Eq. (2.8) more sophisticated interpolation can be applied than the linear one shown above

- For each variable arising in equation a different interpolation method can be used

- Even for the same dependent variable arising in different terms in equation different interpolation schemes can be applied

Guide principles which ought to be taken into consideration in formulating the discretization equation:

of its mass fraction m_l

R_l - rate of generation of destruction of species l by chemical reaction

Diffusion Fick's law

$$J_l = -\Gamma_l grad(m_l) \qquad (1.4)$$

where: Γ_l – diffusion coefficient

$$\frac{\partial(\rho m_l)}{\partial t} + div(\rho U m_l) = div\left[\Gamma_l grad(m_l)\right] + R_l \qquad (1.5)$$

Example 2: Energy equation of laminar steady flow with negligible viscous dissipation

2. Discretization methods

- Finite differences methods (expansion in Taylor series) (presented within previous lecture)

- Method of weighted residuals (to be presented within next lecture)

Differential equation

$$L(\Phi) = 0 \tag{2.1}$$

an approximation of a solution Φ:

$$\Phi = a_0 + a_1 x + a_2 x^2 + \quad + a_m x^m \tag{2.2}$$

The substitution of the approximation (2.2) into Eq.(2.1) gives non zero residual:

The Navier-Stokes Equation

- Thus equation becomes ….. the Navier Stokes eqns:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\nabla^2=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$$

Laplacian Operator

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$\boxed{\underbrace{\rho\frac{D\vec{V}}{Dt}}_{\text{Advection}}=\underbrace{-\nabla p}_{\text{Pressure gradient}}+\underbrace{\rho\vec{g}}_{\text{Body force}}+\underbrace{\mu\nabla^2\vec{V}}_{\text{Diffusion}}}$$

$$\boxed{\nabla\cdot\vec{V}=0}$$

Incompressible NSE written in vector form

- This results in a *closed* system of equations!
  - 4 equations (continuity and momentum equations)
  - 4 unknowns (U, V, W, p)

Note: Above is unsteady, nonlinear, second order partial differential eqn.

The discretization equation

Evaluating of the slopes of the dependent variable at the volume faces from the piecewise-linear profile results in the following discretization equation

$$\frac{k_e(T_E - T_P)}{(\delta x)_e} - \frac{k_w(T_P - T_W)}{(\delta x)_w} + \overline{S}\Delta x = 0 \qquad (2.7)$$

where: $\overline{S}$ – average value of S over the control volume

It is useful to cast Eq. (2.7) in the following form:

$$a_P T_P = a_E T_E + a_W T_W + b \qquad (2.8)$$

where:

$$a_E = \frac{k_e}{(\delta x)_e} \qquad (2.9)$$

$$a_W = \frac{k_w}{(\delta x)_w} \qquad (2.10)$$

$$\frac{\cdots}{(\delta x)_e} - \frac{\cdots}{(\delta x)_w} + \bar{S}\Delta x = 0 \qquad (2.7)$$

where: $\bar{S}$ – average value of S over the control volume

It is useful to cast Eq. (2.7) in the following form:

$$a_P T_P = a_E T_E + a_W T_W + b \qquad (2.8)$$

where:

$$a_E = \frac{k_e}{(\delta x)_e} \qquad (2.9)$$

$$a_W = \frac{k_w}{(\delta x)_w} \qquad (2.10)$$

$$a_P = a_E + a_W \qquad (2.11)$$

$$b = \bar{S}\Delta x \qquad (2.12)$$

The Navier-Stokes Equation

Continuity

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0$$

X-momentum

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$

Y-momentum

$$\rho\left(\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right)$$

Z-momentum

$$\rho\left(\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2}\right)$$

$$\rho\frac{D\vec{V}}{Dt} = -\nabla p + \rho\vec{g} + \mu\nabla^2\vec{V}$$