Chapter 1

AN OVERVIEW OF DIFFERENTIAL EQUATIONS

In this chapter, an overview of differential equations will be presented. A special attention is paid to classifications and solution of linear differential equations.

1.1 Differential Equations and Their Classifications

In science and engineering, mathematical models are developed to aid in understanding of physical phenomena. These models often yield an equation that contains some derivatives of an unknown function. Such equations involving derivatives or differentials of one dependent variable with respect to one or more independent variables are called differential equations. Differential equations can be classified according to number of independent variables and that to input output-output relationship. According to number of independent variables, differential equations are grouped as ordinary differential equations and partial differential equations.

Ordinary Differential Equations:

A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation. Some examples for ordinary differential equations may be as follows.

1) $$\frac{d^2y}{dx^2} + xy(\frac{dy}{dx})^2 = 0$$

2) $$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} = y^2 \tan t$$

Partial Differential Equations:

A differential equation involving partial derivatives of one or more dependent variables with respect to two or more dependent variables is called a partial differential equation. Some examples for partial differential equations may be as follows.

1) $$\frac{\partial v}{\partial s}+\frac{\partial v}{\partial t}=v$$

2) $$\frac{\partial z}{\partial x}+z=\frac{\partial^2 z}{\partial y^2}$$

Linearity of Differential Equations:

Differential equations can also be classified according to input-output relations as linear and nonlinear differential equations. A differential equation is called linear if;

a) Every dependent variable and every derivative involved occur to the first degree only, and
b) No products of dependent variables and/or derivatives occur.

A differential equation which is not linear is called a non-linear differential equation. Some examples for linearity and non-linearity of differential equations may be as follows. In all of the ordinary differential equations in the below, ***y*** **is the dependent variable**, and ***x*** **is the independent variable.**

1) $$\frac{d^2y}{\mathrm{d}x^2}+5\frac{\mathrm{dy}}{\mathrm{dx}}+6y=0$$ **Linear**

2) $$\frac{d^4y}{\mathrm{d}x^4}+x^2\frac{d^3y}{dx^3}+x^3\frac{dy}{dx}=xe^x$$ **Linear**

3) $$x\frac{d^2y}{dx^2}+y\frac{dy}{dx}=e^x$$ **Non-linear**

4) $\frac{d^2y}{dx^2} + 5\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^3 + 6y = \boldsymbol{x^2}$ **Non-linear**

5) $\frac{d^2y}{dx^2} + 5\frac{\mathrm{dy}}{\mathrm{dx}} + 6y^2 = \mathrm{Sinx}$ **Non-linear**

<u>Order of Differential Equations:</u>

The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation as shown below.

1) $\frac{d^2y}{dx^2} + xy\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^2 = 0$ O=2, O.D.E, Dep= ***y***, Ind= ***x***

2) $\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t$ O=4, O.D.E, Dep= ***x***, Ind= ***t***

3) $\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v$ O=1, P.D.E., Dep= ***v***, Ind= ***s***, ***t***

<u>Degree of a Differential Equation:</u>

If a differential equation can be rationalized and cleared from fractions with regard to all derivatives present, the exponent of the highest order derivative is called the degree of the differential equation as shown below.

1) $(\frac{d^2y}{dt^2})^2 + (1 + \frac{dy}{dt})^3 = 0$ O=2, O.D.E., D=3, Dep=***y***, Ind=***x***

2) $\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} = y^2 \, tan\, t$ O=3, O.D.E., D=2, Dep=***y***, Ind=***t***

Exercises:

Classify each of the following differential equations as:PDE,ODE, Linear, Non-Linear. Determine the order, and degree of them;

1) $\frac{dy}{dx} + xy = xe^x$ **ODE, Linear, 1st order, Degree=1**

2) $\frac{d^3y}{\mathrm{d}t^3} + x\sqrt{y} = \sin \mathrm{x}$ **ODE, non-Linear, 3rd order, Degree =1**

3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **PDE, Linear, 2nd order, Degree =1**

4) $\frac{d^2y}{\mathrm{d}x^2} + x \sin y = 0$ **ODE, non-Linear, 2nd order**

5) $\frac{d^2y}{\mathrm{d}x^2} + y \sin x = 0$ **ODE, Linear, 2nd order, Degree =1**

1.2 First Order Differential Equations

The most general first order differential equation is of the form

$$f(x, y, y') = 0.$$

Under certain analyticity conditions it can be reduced to the form;

$$y' = f(x, y).$$

Or considering that $y' = dy/dx$, and $f(x, y) = -M(x, y)/N(x, y)$, the most general first order differential equation can be written as

$$M(x, y)dx + N(x, y)dy = 0,$$

which is called the differential form of the first order differential equations.

Separable Differential Equations

Some of the first order differential equations can be reduced to the form; $g(y)y' = f(x)$ and substitution of $y' = \frac{dy}{dx}$ yields; $g(y)dy = f(x)dx$ in which the terms with variables x and y are separated. Therefore, these equations are called **Separable Differential Equations**. In separable differential equations, integrations of both sides yields an implicit solution straightforward, as follows;

$$\int g(y)dy = \int f(x)dx + c.$$

Example 1.1

Solve the differential equation $9yy' + 4x = 0$

Solution: After substitution that $y' = dy/dx$, it is possible to rewrite the equation in a form that the two variables are separated.

$$9ydy = -4xdx.$$

Then integrations of both sides yields an implicit solution

$$\int 9ydy = \int -4xdx$$

$$\frac{9}{2}y^2 = -2x^2 + C_1$$

$$\frac{x^2}{9} + \frac{y^2}{4} = \mathrm{C}.$$

This solution is an equation of an ellipse family for C > 0.

Exact Differential Equations:

A first order differential equation of the form $\mathrm{M}(x, y)dx + \mathrm{N}(x, y)dy = 0$ is called exact if there is a function $u(x, y)$ such that

$\frac{\partial u}{\partial x} = M$ and $\frac{\partial u}{\partial y} = N$.

In such a case, the left hand side of the differential equation can be simplified to an exact differential

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

and the equation is reduced to $du = 0$. That is why the equation is called exact. An implicit solution is simply $u(x, y) = C$.

A sufficient condition for the exactness of a first order ODE is obtained by the observation

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

For twice differentiable functions

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Hence a sufficient condition for a first order differential equation $\mathrm{M}(x, y)dx + \mathrm{N}(x, y)dy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Then the solutions are obtained using the following steps;

i. $\frac{\partial u}{\partial x} = M \rightarrow u = \int M dx + k(y)$
ii. Use the equation $\frac{\partial u}{\partial y} = \mathrm{N}$ to obtaindk/dy
iii. integrate $\frac{dk}{dy}$ to get k
iv. The solution of the exact equation is $u(x, y) = C$

Example 1.2

Show that the following equation is exact and find the solution.

$$(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$$

$$M = x^3 + 3xy^2 \Longrightarrow \frac{\partial M}{\partial y} = 6xy$$

$N = 3x^2y + y^3 \Longrightarrow \dfrac{\partial N}{\partial x} = 6xy$

Hence

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 6xy$$

Therefore the equation is exact.

$u = \int M dx + k(y) \; = \; \int (x^3 + 3xy^2)dx + k(y) = \dfrac{x^4}{4} + \; \dfrac{3x^2y^2}{2} + \; k(y)$

From this proposal

$\dfrac{\partial u}{\partial y} = 3x^2y + \dfrac{dk}{dy}$

From differential equation

$\dfrac{\partial u}{\partial y} = N = \; 3x^2y + y^3$

Comparison gives

$\dfrac{dk}{dy} = y^3 \Longrightarrow \; k = \dfrac{y^4}{4} + c$

Hence

$$u(x,y) = \frac{y^4}{4} + \frac{3x^2y^2}{2} + \frac{y^4}{4} + c$$

An implicit solution of the exact equation is $u(x,y) = C$. That is

$$\frac{y^4}{4} + \frac{3x^2y^2}{2} + \frac{y^4}{4} = C$$

a family of plane curves for C > 0.

Linear First Order Differential Equations:

First order equations that are linear in y and y' are called linear first order equations. The most general for of this kind of differential equations is

$$y' + p(x)\,y = q(x)\,.$$

To solve this equation let us multiply both sides by the function $e^{\int p(x)dx}$:

$$e^{\int p(x)dx}y' + p(x)e^{\int p(x)dx}\,y = q(x)e^{\int p(x)dx}\,.$$

It is clear that the left hand side is a derivative:

$$\left(e^{\int p(x)dx}y\right)' = q(x)e^{\int p(x)dx}\,.$$

which yields the solution

$$y = e^{-\int p(x)dx}\int q(x)e^{\int p(x)dx}dx\,.$$

Example 1.3

Find the solution of the linear equation

$$y' + \frac{4}{x}y = 8x^3\,.$$

For this equation

$$\int p(x)dx = \int \frac{4}{x}dx = 4\ln x = \ln x^4\,.$$

Hence by (2.35) one has

$$y = e^{-\ln x^4}\int 8x^3 e^{\ln x^4}dx = \frac{1}{x^4}\int 8x^7 dx = \frac{1}{x^4}\left(x^8 + C\right).$$

1.3 2nd Order Linear DifferentialEquations

In this section, solutions of second order differential equations of different kinds are discussed.

A second order linear differential equation is linear if it has the form

$$y'' + p(x)y' + q(x)y = r(x)$$

where $p(x),\ q(x),\ r(x)$ are smooth functions.

If $r(x) = 0$, the equation is called homogeneous.

Examples

1) $y'' + 4y = e^{-x}\sin x$ non-homogeneous, linear
2) $(1 - x^2)y'' - 2xy' + 6y = 0$ homogeneous, linear

Initial and Boundary Value Problems for Second Order Linear Differential Equations

If a second order linear differential equation

$$y'' + p(x)y' + q(x)y = r(x)$$

is accompanied by the conditions

$y'(x_0) = a$ and $y(x_0) = b$, it is called an **Initial Value Problem (IVP)**

$y'(x_0) = a$ and $y(x_1) = b$, or

$y(x_0) = a$ and $y(x_1) = b$, it is called a **Boundary Value Problem (BVP)**

Second Order Linear Homogeneous Differential Equations With Constant Coefficients:

Consider a 2nd order linear homogenous differential equation.

$$y'' + ay' + by = 0$$

Try a solution $y = e^{rx}$ then,

$$r^2 e^{rx} + a.\mathrm{r}e^{rx} + \mathrm{b}.e^{rx} = 0\,, (r^2 + ra + b)e^{rx} = 0$$

$e^{rx} \neq 0$, hence

$$y = e^{rx} \Longrightarrow r^2 + \text{ ar } + \text{ b} = 0$$

Three cases may arise in this case;

	Case	**Roots**	**General Solution**
I	Distinct real roots	$r_1 \neq r_2 \in R$	$y \;=\; C_1 e^{r_1 x} + C_2 e^{r_2 x}$
II	Double real roots	$r_1 = r_2 \in R$	$y \;=\; (C_1 + C_2 x)e^{r_1 x}$
III	Complex conjugate coots	$r_{1,2} = m \mp in$	$\mathrm{y} \;=\; C_1 e^{r_1 \mathrm{x}} + C_2 e^{r_2 x}$ or $y = e^{mx}[A\cos nx + B\sin nx]$

<u>Example 1.4</u>

Solve the boundary value problem

$y'' + y = 0, \quad \mathrm{y}(0) = 3, y(\pi/2) = -3$

$y = e^{rx} \Longrightarrow (r^2 + 1)e^{rx} = 0 \;\Longrightarrow r_{1,2} = \mp i$

$y \;=\; C_1 e^{ix} + C_2 e^{-ix} = A\,Sinx + BCosx$

$\mathrm{y}(0) = 3 \;\rightarrow 3 = \mathrm{B}, \qquad \mathrm{y}\left(\frac{\pi}{2}\right) = -3 \;\rightarrow -3 = \mathrm{A} \;\rightarrow\; \mathrm{A} = -3$

$$y = -3\;sinx + 3cosx$$

is the solution of the given boundary value problem.

<u>Example 1.5</u>

Find the general solution of $\quad y'' + 6y' + 9y = 0.$

$y = e^{rx} \Longrightarrow (r^2 + 6r + 9)e^{rx} = 0 \;\Longrightarrow (r + 3)^2 = 0 \;\Longrightarrow r_{1,2} = -3 \qquad$ double root

$y \;=\; (C_1 + C_2 x)e^{-3x}$

Euler-Cauchy Equations

Euler-Cauchy Equations are second order linear equations with variable coefficients of the special form

$$x^2y'' + axy' + by = r(x)$$

where a, b are constants, and $r(x)$ is a smooth function.

The change of the independent variable

$$x = e^z \rightarrow z = \ln x$$

leads to the transformations of the derivatives

$$\frac{d}{dx} = \frac{1}{x}\frac{d}{dz} \rightarrow x\frac{dy}{dx} = \frac{dy}{dz}$$

and

$$\frac{dy}{dx} + x\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2y}{dz^2} \rightarrow x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} \rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

Substituting these in Euler-Cauchy equation, one has

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + a\frac{dy}{dz} + by = r(e^z) \rightarrow \frac{d^2y}{dz^2} + (a-1)\frac{dy}{dz} + by = r(e^z)$$

The resulting equation is a second order linear equation with constant coefficients.

Example 1.6

Solve $x^2y'' + 7xy' + 13y = 0$ Let $x = e^z$

Then

$$x\frac{dy}{dx} = \frac{dy}{dz}, \qquad \text{and } x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

Substituting these in the given Euler-Cauchy equation, one has

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + 7\frac{dy}{dz} + 13y = 0 \rightarrow y" + 6y' + 13y = 0, \qquad ' = \frac{d}{dz}$$

$m^2 + 6m + 13 = 0 \implies m_{1,2} = -3 \mp 2i$

$y = e^{-3z}[A\cos(2z) + B\sin(2z)]$

Transforming back to the original independent variable x one has

$y = x^{-3}[ACos(2lnx) + BSin(2lnx)]$

Non-homogeneous Second Order Linear Differential Equations with Constant Coefficients:

These equations are in the form of

$$y'' + p(x)y' + q(x)y = r(x) \quad , \quad r(x) \neq 0 \qquad (2)$$

General solution of non-homogeneous differential equations are of the form

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the general solution of the corresponding homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0$$

and $y_p(x)$ is any solution of the non-homogeneous differential equation

$$y'' + p(x)y' + q(x)y = r(x) \quad , \quad r(x) \neq 0$$

In previous section we have seen how to find the general solution of a homogeneous second order linear differential equation. Now we will elaborate methods to find a particular solution of the non-homogeneous differential equation.

The Method of Undetermined Coefficients:

If the non-homogeneity function $\boldsymbol{r(x)}$ is from a special kind of functions that create only a finite number of root functions upon successive differentiations, it is called a function of finite derivatives.

Example 1.7

1) $\boldsymbol{r(x)} = \mathbf{Sin2x}$ is a function of finite derivatives since upon successive differentiations create only two root functions;

$$\mathrm{D} = \{\mathrm{Sin2x}, \mathrm{Cos\ 2x}\}$$

2) $r(x) = \mathbf{x^5}$ is also a function of finite derivatives

$$\mathrm{D} = \{\mathbf{x^5}, \mathrm{x}^4, \mathrm{x}^3, \mathrm{x}^2, \mathrm{x}, 1\}$$

3) $r(x) = \ln x$ is not a function of finite derivatives

$$\mathbf{D} = \{\ln x, \frac{1}{\mathrm{x}}, \frac{1}{\mathrm{x}^2}, \frac{1}{\mathrm{x}^3}, \dots\}$$

Now let the set of the two linearly independent solutions of homogeneous second order linear differential equation is

$$\mathrm{H} = \{\varphi_1(\mathrm{x}), \varphi_2(\mathrm{x})\}$$

and the D set of $r(x)$ is

$$\mathrm{D} = \{\mathrm{f}_1(\mathrm{x}), \mathrm{f}_2(\mathrm{x}), \dots, \mathrm{f}_\mathrm{n}(\mathrm{x})\}$$

1) If the sets H and D do not have any common function, then we propose a particular solution for the non-homogeneous differential equation as a linear combination of functions in D with coefficients $\{\mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_\mathrm{n}\}$ to be determined

$$\mathrm{y_p} = \mathrm{a}_1\mathrm{f}_1(\mathrm{x}) + \mathrm{a}_2\mathrm{f}_2(\mathrm{x}) + \cdots + \mathrm{a}_\mathrm{n}\mathrm{f}_\mathrm{n}(\mathrm{x})\}$$

Then this proposed solution is substituted in the non-homogeneous differential equation to determine $\{\mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_\mathrm{n}\}$.

Example 1.8

Solve $\quad y'' - 3y' + 2y = e^{3x}$

First homogenous solution is found. The characteristic equation has roots

$r^2 - 3r + 2 = 0 \Longrightarrow r_{1,2} = 1,2$

$y_h = C_1 e^x + C_2 e^{2x}; \quad H = \{e^x, e^{2x}\}$

where

$$D = \{e^{3x}\}, \qquad \text{and } H \cap D = \emptyset$$

Hence particular solution proposal is

$$y_p = Ce^{3x}$$

Upon substitution into the non-homogeneous differential equation one has

$y_p' = 3Ce^{3x} \Longrightarrow y_p'' = 9Ce^{3x}$

$9Ce^{3x} - 9Ce^{3x} + 2Ce^{3x} = e^{3x}$

$2Ce^{3x} = e^{3x} \Longrightarrow c = 1/2$

Therefore

$$y_p = \frac{1}{2}e^{3x}$$

and the general solution is

$y = C_1 e^x + C_2 e^{2x} + \frac{1}{2}e^{3x}$

2) If the sets H and D are not disjoint, we multiply the set D till we get

$$H \cap x^m D = \emptyset$$

then we propose a particular solution for the non-homogeneous differential equation as a linear combination of functions in $x^m D$ with coefficients $\{a_1, a_2, \dots, a_n\}$ to be determined

$$y_p = a_1 x^m f_1(x) + a_2 x^m f_2(x) + \cdots + a_n x^m f_n(x)\}$$

Example 1.9

Solve $\quad y'' - 3y' + 2y = e^x$

First homogenous solution is found. The characteristic equation has roots

$r^2 - 3\mathrm{r} + 2 = 0 \implies r_{1,2} = 1{,}2$

$y_h = C_1 e^x + C_2 e^{2x}; \quad H = \{e^x, e^{2x}\}$

where

$$\mathrm{D} = \{\mathrm{e^x}\}, \text{and } \mathrm{H} \cap \mathrm{D} \neq \emptyset \text{ but } \mathrm{xD} = \{\mathrm{xe^x}\}, \text{and } \mathrm{H} \cap \mathrm{xD} = \emptyset$$

Hence particular solution proposal is

$y_p = Cxe^x \implies y_p{}' = C(e^x + xe^x) \implies y_p{}'' = C(2e^x + xe^x)$

$C(2e^x + xe^x) - 3C(e^x + xe^x) + 2Cxe^x = e^x$

$-Ce^x = e^x \implies C = -1$

Therefore

$$y_p = -xe^x$$

and the general solution is

$y = C_1 e^x + C_2 e^{2x} - xe^x$

<u>Example 1.10</u>

Solve $\qquad y'' + 2y' + 5y = 16e^x + \sin 2x.$

The roots of the characteristic equation, and the homogenous solution are;

$r^2 + 2\mathrm{r} + 5 = 0 \implies r_{1,2} = -1 \mp 2i$

$y_h = e^{-x}(A\cos 2x + B\sin 2x)$

$H = \{e^{-x}\cos 2x, e^{-x}\sin 2x\},$

where the D sets are

$$\mathrm{D_1} = \{\mathrm{e^x}\}, \text{and } \mathrm{D_2} = \{\cos 2x, \sin 2x\},$$

$$\mathrm{H} \cap \mathrm{D_1} = \emptyset, \qquad \mathrm{H} \cap \mathrm{D_2} = \emptyset$$

Therefore

$y_p = Ce^x + K \cos 2x + M \sin 2x$

$y_p' = Ce^x - 2K \sin 2x + 2M \cos 2x$

$y_p'' = Ce^x - 4K \cos 2x - 4M \sin 2x$

$y'' + 2y' + 5y = 16e^x + \sin 2x \rightarrow$

$$Ce^x - 4K \cos 2x - 4M \sin 2x + 2(Ce^x - 2K \sin 2x + 2M \cos 2x) + 5(Ce^x + K \cos 2x + M \sin 2x) = 16e^x + \sin 2x \rightarrow$$

$8C = 16, \rightarrow C = 2$

$\cos 2x: -4K + 4M + 5K = 0 \rightarrow K + 4M = 0$

$\sin 2x: -4M - 4K + 5M = 1 \rightarrow -4K + M = 1$

$K = -\frac{4}{17};\ M = \frac{1}{17}$

Therefore

$$y_p = -\frac{4}{17}\cos 2x + \frac{1}{17}\sin 2x$$

and the general solution is

$$y = e^{-x}(A \cos 2x + B \sin 2x) + 2e^x - \frac{4}{17}\cos 2x + \frac{1}{17}\sin 2x$$

Wronskian Determinant and Solution by Variation of Parameters:

The previous solution technique can be applied only to constant-coefficient equations with special *r(x)*. For the more general equations like

$y'' + p(x)y' + q(x)y = r(x)$

with functions p , q and r which are continuous in the given interval, the method of variation of parameters is used to find a particular solution y_p.

***Claim*:** For equations like

$y'' + p(x)y' + q(x)y = r(x)$

with functions p , q and r which are continuous in the given interval, the method of variation of parameters yields a particular solution y_p in the form

$$y_p(x) = -y_1 \int \frac{y_2\, r}{|W|} dx + y_2 \int \frac{y_1\, r}{|W|} dx$$

where y_1 , y_2 form a basis of solutions of the homogeneous equation

$y'' + p(x)y' + q(x)y = 0$

corresponding to original equation, and

$$W = y_1\, y_2' - y_1' y_2 \quad \rightarrow \quad |W| = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

That is the Wronskian determinant ofy_1 , y_2.

Note: The Wronskian determinant is not zero when y_1 , y_2 are linearly independent and solution is feasible.

Proof:

Let the homogeneous solution of the given equation is

$$\mathrm{y_h(x) = C_1 y_1(x) + C_2 y_2(x).}$$

Let us propose a particular solution y_p in the form

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$

where u and v are functions to be determined.

$$y_p' = u'y_1 + uy_1' + v'y_2 + vy_2' \qquad (*)$$

Assume

$u'y_1 + v'y_2 = 0.$ (1)

Then the equation (*) reduces to $y_p{}' = uy_1{}' + vy_2{}'$

Second derivative $y_p{}''$ is now

$$y_p{}'' = u'y_1{}' + uy_1{}'' + v'y_2{}' + vy_2{}''$$

Insert $y_p, y_p{}'$ and $y_p{}''$ in the original non homogeneous equation,

$$u(y_1{}'' + py_1{}' + qy_1) + v(y_2{}'' + py_2{}' + qy_2) + u'y_1{}' + v'y_2{}' = r(x)$$

since y_1 and y_2 are solutions of the differential equation, the above equation reduces to

$u'y_1{}' + v'y_2{}' = r(x)$ (2)

and combining equation (2), with equation (1), a linear algebraic system of two linear equations for $u',\ v'$ is obtained.

$$\begin{bmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{bmatrix}\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}.$$

Using Cramer's method, the solution of this linear algebraic system is

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{|W|} = -\frac{y_2\, r}{W}, \qquad v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{|W|} = \frac{y_1\, r}{|W|}.$$

By integration, we find the unknown functions u and v

$$u = -\int \frac{y_2\, r}{|W|}\, dx, \qquad v = \int \frac{y_1\, r}{|W|}\, dx.$$

Hence it is proved that a particular solution of the non homogeneous equation is

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x) = -y_1 \int \frac{y_2\, r}{W}\, dx + y_2 \int \frac{y_1\, r}{W}\, dx.$$

Example 1.11

Solve $y'' + y = \sec x$

The roots of the characteristic equation, and the two linearly independent solutions are;

$r^2 + 1 = 0 \implies r_{1,2} = \mp i \implies y_1 = \cos x, \quad y_2 = \sin x$

and the homogenous solution is;

$$y_h = A\cos \mathrm{x} + B\sin \mathrm{x}$$

Therefore particular solution is;

$$W(\mathrm{y}_1, \mathrm{y}_2) = \begin{vmatrix} \cos \mathrm{x} & \sin \mathrm{x} \\ -\sin \mathrm{x} & \cos \mathrm{x} \end{vmatrix} = \cos^2 \mathrm{x} + \sin^2 \mathrm{x} = 1$$

$$y_p(x) = -\cos \mathrm{x} \int \sin \mathrm{x} \sec \mathrm{x}\, \mathrm{dx} + \sin \mathrm{x} \int \underbrace{\cos \mathrm{x} \sec \mathrm{x}}_{1} \mathrm{dx}$$

$$y_p(x) = \cos \mathrm{x}\ ln(\cos \mathrm{x}) + \mathrm{x} \sin \mathrm{x}$$

The general solution is obtained as;

$$y = y_h + y_p = (C_1 + ln \cos \mathrm{x}) \cos \mathrm{x} + (C_2 + x) \sin \mathrm{x}$$

Example 1.12

Solve $\quad y'' + 4\mathrm{y}' + 4y = \frac{e^{-2x}}{x^2}$

The roots of the characteristic equation, and the two linearly independent solutions are;

$$\mathrm{r}^2 + 4\mathrm{r} + 4 = 0 \Rightarrow \mathrm{r}_{1,2} = -2 \Rightarrow \mathrm{y}_1 = \mathrm{e}^{-2\mathrm{x}} \text{ and } \mathrm{y}_2 = \mathrm{xe}^{-2\mathrm{x}}$$

and the homogenous solution is;

$$\mathrm{y_h} = \mathrm{C}_1\mathrm{e}^{-2\mathrm{x}} + \mathrm{C}_2\mathrm{xe}^{-2\mathrm{x}}$$

Therefore particular solution is;

$$\mathrm{y_p}(\mathrm{x}) = -\mathrm{y}_1 \int \frac{\mathrm{y}_2\, \mathrm{r}}{\mathrm{W}} \mathrm{dx} + \mathrm{y}_2 \int \frac{\mathrm{y}_1\, \mathrm{r}}{\mathrm{W}} \mathrm{dx}$$

$$\mathrm{W} = \begin{vmatrix} \mathrm{y}_1 & \mathrm{y}_2 \\ \mathrm{y}_1' & \mathrm{y}_2' \end{vmatrix} = \begin{vmatrix} \mathrm{e}^{-2\mathrm{x}} & \mathrm{xe}^{-2\mathrm{x}} \\ -2\mathrm{e}^{-2\mathrm{x}} & \mathrm{e}^{-2\mathrm{x}} - 2\mathrm{xe}^{-2\mathrm{x}} \end{vmatrix} = \mathrm{e}^{-4\mathrm{x}} - 2\mathrm{xe}^{-4\mathrm{x}} + 2\mathrm{xe}^{-4\mathrm{x}} = \mathrm{e}^{-4\mathrm{x}}$$

$$\mathrm{y_p}(\mathrm{x}) = -\mathrm{e}^{-2\mathrm{x}} \int \frac{\mathrm{xe}^{-2\mathrm{x}}\mathrm{e}^{-2\mathrm{x}}}{\mathrm{e}^{-4\mathrm{x}}\mathrm{x}^2} \mathrm{dx} + \mathrm{xe}^{-2\mathrm{x}} \int \frac{\mathrm{e}^{-2\mathrm{x}}\mathrm{e}^{-2\mathrm{x}}}{\mathrm{e}^{-4\mathrm{x}}\mathrm{x}^2} \mathrm{dx} = -\mathrm{e}^{-2\mathrm{x}} \int \frac{\mathrm{dx}}{\mathrm{x}} + \mathrm{xe}^{-2\mathrm{x}} \int \frac{\mathrm{dx}}{\mathrm{x}^2}$$

$$y_p(x) = -e^{-2x}\ln x - \frac{x(\ e^{-2x})}{x} = -e^{-2x}\ln x - e^{-2x}$$

Then the general solution is

$$y = y_h + y_p = C_1e^{-2x} + C_2xe^{-2x} - e^{-2x}\ln x$$

Example 1.13

Solve

$$y'' - 4y' + 5y = \frac{2e^{2x}}{\sin(x)}$$

The roots of the characteristic equation, and the two linearly independent solutions are;

$$r^2 - 4r + 5 = 0 \Rightarrow r_{1,2} = 2 \pm i \ \Longrightarrow y_1 = e^{2x}\sin x \text{ and } y_2 = e^{2x}\cos x$$

and the homogenous solution is;

$$y_h = e^{2x}(C_1\sin x + C_2\cos x).$$

Therefore a particular solution is found as;

$$W = \begin{vmatrix} e^{2x}\sin x & e^{2x}\cos x \\ 2e^{2x}\sin x + e^{2x}\cos x & 2e^{2x}\cos x - e^{2x}\sin x \end{vmatrix}$$

$$= 2e^{4x}\cos x\sin x - e^{4x}\sin^2 x - 2e^{4x}\cos x\sin x - e^{4x}\cos^2 x \quad = -e^{4x}$$

$$y_p = -e^{2x}\sin x\int\frac{e^{2x}\cos x\frac{2e^{2x}}{\sin x}}{-e^{4x}}dx + e^{2x}\cos x\int\frac{e^{2x}\sin x\frac{2e^{2x}}{\sin x}}{-e^{4x}}dx$$

$$= 2e^{2x}\sin x\int\cot x\,dx - 2e^{2x}\cos x\int dx = 2e^{2x}(\ln|\sin x|\sin x - x\cos x)$$

Then the general solution is

$$y_{gen} = y_h + y_p = e^{2x}(C_1\sin x + C_2\cos x) + 2e^{2x}(\ln|\sin x|\sin x - x\cos x).$$

Example 1.14

Solve $y'' - 4y' + 4y = 6x^{-4}e^{2x}$

The roots of the characteristic equation, and the two linearly independent solutions are;

$r^2 - 4r + 4 = 0 \Rightarrow r_{1,2} = 2 \implies y_1 = e^{2x} y_2 = xe^{2x}$

and the homogenous solution is;

$y_h = C_1 e^{2x} + C_2 x e^{2x}$

Therefore a particular solution is found as;

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}$$

$$y_p = -e^{2x}\int \frac{xe^{2x}6x^{-4}e^{2x}}{e^{4x}}dx + xe^{2x}\int \frac{e^{2x}6x^{-4}e^{2x}}{e^{4x}}dx$$

$$= -e^{2x}\int 6x^{-3}dx + xe^{2x}\int 6x^{-4}dx\, 3e^{2x}x^{-2} - 2e^{2x}x^{-2} = e^{2x}x^{-2}.$$

Then the general solution is

$y_{gen} = y_h + y_p = C_1 e^{2x} + C_2 x e^{2x} + e^{2x}x^{-2}.$

1.4 Exercises

Find the solutions of the following ODE's

1. $(x^2 - 4)y' = 2xy + 6x$
2. $3xy' - y = \ln x + 1,\ x > 0,\ y(1) = -4$
3. $1 + \left(\frac{t}{y} - \cos y\right) y' = 0$
4. $y' = e^{\ln x - y + 2}$
5. $y'(e^{-y} + 2x) - e^{y} = 0,\ y(0) = 0$

Answers: (Check)

1. $y = \frac{C}{x^2-4} - 3$
2. $y = -(\ln x + 4)$
3. $\mathrm{t} = \sin \mathrm{y} + \mathrm{y}^{-1} \cos \mathrm{y} + \mathrm{C}\mathrm{y}^{-1}$
4. $\mathrm{y} = \ln\left(\mathrm{C} + \frac{\mathrm{e}^2\mathrm{x}^2}{2}\right)$
5. $x = -2e^{-2(1+e^{-y})} - \frac{1}{2}e^{-y} + \frac{1}{4}$

Chapter2

SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

Linear differential equations are obtained in modeling various applications such as electrical circuits, vibration systems mixture of substances etc. Consider for example a mass and spring system in the Figure 2.1.

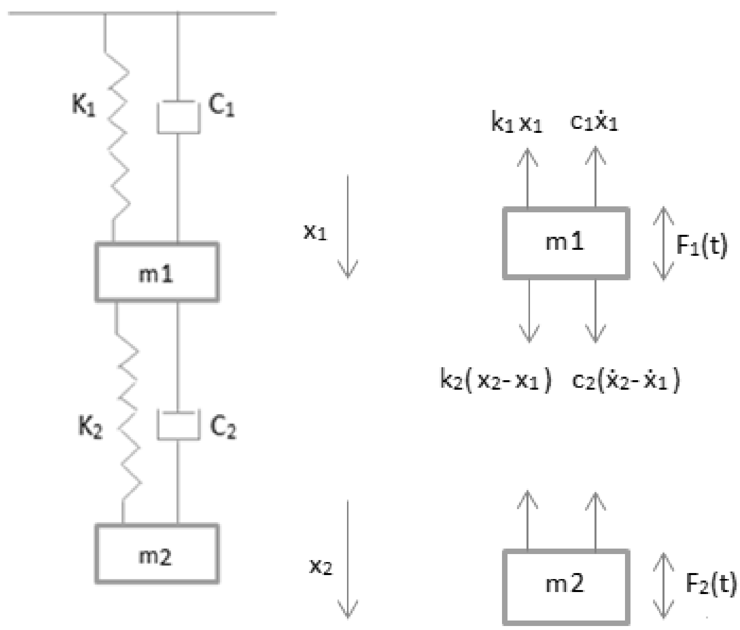

Figure 2.1 Modelling a spring and mass system.

Newton's 2nd law states that; $\sum \mathrm{F} = \mathrm{m.a}$ which implies for two bodies respectively

$$m_1\ddot{x}_1 = k_2(x_2 - x_1) + C_2(\dot{x}_2 - \dot{x}_1) - k_1x_1 - C_1x_1 + F_1(t),$$

and

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - C_2(\dot{x}_2 - \dot{x}_1) + F_2(t).$$

Manipulations yield that;

$$m_1\ddot{x}_1 + (C_1 + C_2)\dot{x}_1 - C_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1(t)$$

$$m_2\ddot{x}_2 - C_2\dot{x}_1 + C_2\dot{x}_2 - k_2x_1 + k_2x_2 = F_2(t)$$

This is a system of two second order linear differential equations. There have been many other physical applications that produce similar systems of linear equations.

There are various solution techniques for such linear systems of differential equations. One approach is to decouple the system, and obtain a number of differential equations with only one dependent variable. This approach inevitably increases the order of differential equations. Another approach is to decrease the order of differential equations to first order which increases the number of dependent variables, and equations in the set. Some of these techniques will be explained in this section.

2.1 Decoupling Method

This is probably one of the most popular techniques that decouple the system of coupled equations into several higher order differential equations in each of the dependent variables. These higher order linear differential equations are then solved individually.

Example 2.1

Solve the system of the two differential equations

$$\frac{dy}{dt} + z = e^t;$$

$$y + \frac{dz}{dt} = e^{-t}$$

$$\frac{dy}{dt} + z = e^t \rightarrow \frac{dy}{dt} = -z + e^t \rightarrow \frac{d^2y}{dt^2} = -\frac{dz}{dt} + e^t$$

$$y + \frac{dz}{dt} = e^{-t} \rightarrow \frac{dz}{dt} = -y + e^{-t} \rightarrow \frac{d^2z}{dt^2} = -\frac{dy}{dt} - e^{-t}$$

then,

$$\frac{d^2y}{dt^2} = \mathrm{y} - e^{-t} + e^t \rightarrow \frac{d^2y}{dt^2} - \mathrm{y} = -e^{-t} + e^t$$
$$\frac{d^2z}{dt^2} = \mathrm{z} - e^t - e^{-t} \rightarrow \frac{d^2z}{dt^2} - \mathrm{z} = -e^t - e^{-t}$$

These two 2nd order linear differential equations are to be solved individually. Let's solve homogeneous equations first;

$$\begin{array}{lclll} y'' - y = 0 & \multirow{2}{*}{Let} & Y_h = e^{mt} & Y_h'' = m^2 e^{mt} & (m^2-1)e^{mt} = 0 \\ z'' - z = 0 & & Z_h = e^{mt} & Z_h'' = m^2 e^{mt} & m_{1,2} = \pm 1 \end{array}$$

$$Y_h = C_1 e^t + C_2 e^{-t}$$
$$Z_h = C_3 e^t + C_4 e^{-t}$$

The general solution of a system of the two first order linear differential equations contains exactly two arbitrary constants. The relations between the four arbitrary constants in the homogeneous solution are obtained from homogeneous form of the given set of equations

$$\frac{dy}{dt} + \mathrm{z} = 0;$$
$$y + \frac{dz}{dt} = 0$$

Substitution yields that

$$C_1 e^t - C_2 e^{-t} + C_3 e^t + C_4 e^{-t} = 0 \xrightarrow{yields} (C_1 + C_3)e^t + (-C_2 + C_4)e^{-t} = 0$$
$$C_1 e^t + C_2 e^{-t} + C_3 e^t - C_4 e^{-t} = 0 \xrightarrow{yields} (C_1 + C_3)e^t + (C_2 - C_4)e^{-t} = 0$$

$C_3 = -C_1 \& C_4 = C_2$.

So homogeneous solutions with two arbitrary constants is obtained;

$$y_h = C_1 e^t + C_2 e^{-t}$$
$$z_h = -C_1 e^t + C_2 e^{-t}$$

Particular solution can be proposed based on RHS of the equations as seen in Chapter 1.

$$y_p = t(A_1 e^t + A_2 e^{-t})$$

$$y_p' = A_1 e^t + A_2 e^{-t} + t(A_1 e^t - A_2 e^{-t})$$

$$y_p'' = A_1 e^t - A_2 e^{-t} + A_1 e^t - A_2 e^{-t} + t(A_1 e^t + A_2 e^{-t})$$

Substitution of these into $y'' - y = e^t - e^{-t}$ yields that

$(A_1 + A_1 + A_1 t - A_1 t - 1)e^t + (-A_2 - A_2 + A_2 t - A_2 t + 1)e^{-t} = 0$

$A_1 = \frac{1}{2} \& A_2 = \frac{1}{2}$

$y_p = \frac{1}{2} t(e^t + e^{-t})$ Since

$y_{gen} = y_h + y_p = C_1 e^t + C_2 e^{-t} + \frac{1}{2} t(e^t + e^{-t})$

Similarly, particular solution for *z* is proposed as

$z_p = t(B_1 e^t + B_2 e^{-t})$

$z_p'' = B_1 e^t - B_2 e^{-t} + B_1 e^t - B_2 e^{-t} + t(B_1 e^t + B_2 e^{-t})$

Substitution of these into $z'' - z = -e^t - e^{-t}$ yields that

$(B_1 + B_1 + B_1 t - B_1 t + 1)e^t + (-B_2 - B_2 + B_2 t - B_2 t + 1)e^{-t} = 0$

$B_1 = -\frac{1}{2} \& B_2 = \frac{1}{2}$ then $z_p = \frac{1}{2} t(e^{-t} - e^t)$

$z_{gen} = -C_1 e^t + C_2 e^{-t} + \frac{1}{2} t(e^{-t} - e^t)$

Another strategy to find z_{gen} is to use y_{gen} in

$$\frac{dy_{gen}}{dt} + z_{gen} = e^t.$$

$$z_{gen} = e^t - \frac{dy_{gen}}{dt} = -C_1 e^t + C_2 e^{-t} + \frac{1}{2} t(e^{-t} - e^t).$$

Example 2.2

Solve the system of the two first order linear differential equations

$$\frac{dx}{dt} + \ y = 2$$
$$x + \frac{dy}{dt} = \cos t$$

$$\frac{dx}{dt} + \ y = 2, \qquad x' = -\,y + 2 \qquad x'' = -y'\,, \qquad x'' - x = -\cos t$$

$$x + \frac{dy}{dt} = \cos t, \quad y' = -x + \cos t \quad y'' = -x' - \sin t\,, \qquad y'' - y = -2 - \sin t$$

Let's solve them individually.

Let's start with the two corresponding homogeneous equations;

$$\begin{matrix} x'' - x = 0 \\ y'' - y = 0 \end{matrix} \qquad \text{Let} \qquad \begin{matrix} x_h = e^{mt} \Longrightarrow x_h'' = m^2 e^{mt} & \Longrightarrow & m_{1,2} & = \pm 1 \\ y_h = e^{mt} \Longrightarrow y_h'' = m^2 e^{mt} & \Longrightarrow & m_{1,2} & = \pm 1 \end{matrix}$$

$$x_h = C_1 e^t + C_2 e^{-t}$$
$$y_h = C_3 e^t + C_4 e^{-t}$$

For the first second order equation

$$x'' - x = -\cos t$$

since the right side is {-cost} let a particular solution be;

$$x_p = A\cos t + B\sin t$$

Then

$$x_p' = -A\sin t + B\cos t$$

$$x_p'' = -A\cos t - B\sin t$$

Substitution of these into $\qquad x'' - x = -\cos t \qquad$ yields that

$$-A\cos t - B\sin t - A\cos t - B\sin t = -\cos t$$

$$-2A\cos t - 2B\sin t = -\cos t \xrightarrow{yields} \begin{matrix} A = 1/2 \\ B = 0 \end{matrix} \xrightarrow{yields} x_p = \frac{1}{2}\cos t$$

Hence

$x_{gen} = x_h + x_p = C_1 e^t + C_2 e^{-t} + \frac{1}{2}\cos t.$

Consider the other equation; $\quad y'' - y = -\sin t - 2,$

let a particular solution be;

$y_p = C\cos t + D\sin t + E.$

Then

$y_p' = -C\sin t + D\cos t$

$y_p'' = -C\cos t - D\sin t$

Substitution of these into $\quad y'' - y = -\sin t - 2 \quad$ yields that

$-C\cos t - D\sin t - Ccost - D\sin t - E + \sin t + 2 = 0$

$-2C\cos t + (-2D + 1)\sin t + (-E + 2) = 0$

$C = 0, \qquad D = \frac{1}{2} \;\&\; E = 2 \xrightarrow{yields} y_p = \frac{1}{2}\sin t + 2$

Hence

$y_{gen} = y_h + y_p = C_3 e^t + C_4 e^{-t} + \frac{1}{2}\sin t + 2$

The general solution of a linear system of two first order differential equations contains exactly two arbitrary constants. Now we will substitute the two solutions into one of the equations

$$x + \frac{dy}{dt} = \cos t$$

in the given system to eliminate extra arbitrary constants.

Recall

$$x_{gen} = C_1 e^t + C_2 e^{-t} + \frac{1}{2}\cos t$$

$$\frac{dy_{gen}}{dt} = C_3 e^t - C_4 e^{-t} + \frac{1}{2}\cos t$$

Substitution yields that

$$C_1e^t + C_2e^{-t} + \frac{1}{2}\cos t + C_3e^t - C_4e^{-t} + \frac{1}{2}\cos t - \cos t = 0$$

$$(C_1 + C_3)e^t + (C_2 - C_4)e^{-t} + \left(\frac{1}{2} + \frac{1}{2} - 1\right)cost = 0 \xrightarrow{yields} \begin{array}{l} C_3 = -C_1 \\ C_4 = C_2 \end{array}$$

Therefore general solution of the two unknowns are:

$$x_{gen} = C_1e^t + C_2e^{-t} + \frac{1}{2}\cos t$$

$$y_{gen} = -C_1e^t + C_2e^{-t} + \frac{1}{2}\sin t + 2$$

This result may be verified using the other differential equation $\frac{dy}{dt} + x = 2$. The same result will be obtained.

After finding the solution x_{gen}, a simpler way to find y_{gen} is to substitute x_{gen} in the first equation $\frac{dx}{dt} + \mathrm{y} = 2$ which leads

$$y_{gen} = 2 - \frac{dx_{gen}}{dt} = -C_1e^t + C_2e^{-t} + \frac{1}{2}\sin t + 2.$$

Example 2.3

Solve the system of the two first order linear differential equations

$$\frac{dx}{dt} - \mathrm{y} = \frac{1}{\sin t}$$
$$x + \frac{dy}{dt} = \frac{1}{\cos t}$$

$$x' = y + \frac{1}{\sin t}, \quad x'' = y' - \frac{\cos t}{\sin^2 t}, \quad x'' = -x + \frac{1}{\cos t} - \frac{\cos t}{\sin^2 t}$$

$$y' = -x + \frac{1}{\cos t}, \quad y'' = -x' + \frac{\sin t}{cos^2 t}, \quad y'' = -y' - \frac{1}{\sin t} + \frac{\sin t}{cos^2 t}$$

The two uncoupled equations are

$$x'' + x = \frac{1}{\cos t} - \frac{\cos t}{\sin^2 t}$$

and

$$y'' + y' = -\frac{1}{\sin t} + \frac{\sin t}{cos^2 t}.$$

Homogeneous solutions;

$$\begin{aligned} x'' + x &= 0 \\ y'' + y &= 0 \end{aligned} \qquad \text{Let} \qquad \begin{aligned} x_h &= e^{mt} \Longrightarrow x_h'' = m^2 e^{mt} \quad \Rightarrow \quad m_{1,2} &= \pm i \\ y_h &= e^{mt} \Longrightarrow y_h'' = m^2 e^{mt} \quad \Rightarrow \quad m_{1,2} &= \pm i \end{aligned}$$

$$\begin{aligned} x_h &= C_1 \cos t + C_2 \sin t \\ y_h &= C_3 \cos t + C_4 \sin t \end{aligned}$$

These should satisfy homogeneous equations. Therefore one has

$$\begin{aligned} C_4 &= -C_1 \\ C_3 &= C_2 \end{aligned}$$

2.2 **Operator Method**:

This is probably one of the most popular techniques that eliminate the number of dependent parameters to a single variable of higher order differential equations. In operator method each differentiation is denoted by a linear operator, D, and the resulting algebraic equation set is solved in terms of D using linear algebra rules. The unknowns are solved in terms of D. Since D is a differential operator, higher order independent linear differential equations are then obtained and solved individually.

Example

Solve the differential equation system
$$\begin{aligned} \frac{dy}{dt} + z &= e^t \\ y + \frac{dz}{dt} &= e^{-t} \end{aligned}$$

Let $D = \frac{d}{dt}$ then,

$$\begin{matrix} Dy + z = e^t \\ y + Dz = e^{-t} \end{matrix} \quad \Longrightarrow \quad \Delta = \begin{vmatrix} D & 1 \\ 1 & D \end{vmatrix} = \mathrm{D}^2 - 1$$

$$\Delta y = \begin{vmatrix} e^t & 1 \\ e^{-t} & D \end{vmatrix} = De^t - e^{-t} = e^t - e^t$$

$$\Delta z = \begin{vmatrix} D & e^t \\ 1 & e^{-t} \end{vmatrix} = De^{-t} - e^{-t} = -e^{-t} - e^t$$

$$y = \frac{\Delta y}{\Delta} = \frac{e^t - e^t}{D^2 - 1} \xrightarrow{yields} (D^2 - 1)y = e^t - e^{-t} \xrightarrow{yields} y'' - y = e^t - e^{-t}$$

$$z = \frac{\Delta z}{\Delta} = \frac{-e^{-t} - e^t}{D^2 - 1} \xrightarrow{yields} (D^2 - 1)z = -e^{-t} - e^t \xrightarrow{yields} z'' - z = -e^t - e^{-t}$$

These two 2nd order differential equations are to be solved individually. Let's solve homogeneous equations first;

$$\begin{matrix} y'' - y = 0 \\ z'' - z = 0 \end{matrix} \qquad \text{Let} \qquad \begin{matrix} Y_h = e^{mt} & Y_h'' = m^2 e^{mt} & (m^2 - 1)e^{mt} = 0 \\ Z_h = e^{mt} & Z_h'' = m^2 e^{mt} & m_{1,2} = \pm 1 \end{matrix}$$

$$\begin{matrix} Y_h = C_1 e^t + C_2 e^{-t} \\ Z_h = C_3 e^t + C_4 e^{-t} \end{matrix}$$

The relation between coefficients are obtained from homogeneous form of the given equation set

$$\begin{matrix} Dy_h = C_1 e^t - C_2 e^{-t} \\ Dz_h = C_3 e^t - C_4 e^{-t} \end{matrix} \qquad \text{Substitution yields that}$$

$$\begin{matrix} Dy + z = e^t & \xrightarrow{yields} & C_1 e^t - C_2 e^{-t} + C_3 e^t + C_4 e^{-t} = 0 & \xrightarrow{yields} & (C_1 + C_3)e^t + (-C_2 + C_4)e^{-t} = 0 \\ y + Dz = e^{-t} & \xrightarrow{yields} & C_1 e^t + C_2 e^{-t} + C_3 e^t - C_4 e^{-t} = 0 & \xrightarrow{yields} & (C_1 + C_3)e^t + (C_2 - C_4)e^{-t} = 0 \end{matrix}$$

$C_3 = -C_1$ & $C_4 = C_3$.

So homogeneous solutions;

$$\begin{matrix} Y_h = C_1 e^t + C_2 e^{-t} \\ Z_h = -C_1 e^t + C_2 e^{-t} \end{matrix}$$

Particular solution can be proposed based on RHS of the equation;

$$Y_p = t(A_1 e^t + A_2 e^{-t})$$

$$Y_p' = A_1 e^t + A_2 e^{-t} + t(A_1 e^t - A_2 e^{-t})$$

$Y_p'' = A_1e^t - A_2e^{-t} + A_1e^t - A_2e^{-t} + t(A_1e^t + A_2e^{-t})$

Substitution of these into $y'' - y = e^t - e^{-t}$ yields that

$(A_1 + A_1 + A_1t - A_1t - 1)e^t + (-A_2 - A_2 + A_2t - A_2t + 1)e^{-t} = 0$

$A_1 = \frac{1}{2}$ & $A_2 = \frac{1}{2}$

$Y_p = \frac{1}{2}t(e^t + e^{-t})$ Since

$Y_{gen} = Y_h + Y_p = C_1e^t + C_2e^{-t} + \frac{1}{2}t(e^t + e^{-t})$

Similarly, particular solution for Z is proposed to be

$Z_p = t(B_1e^t + B_2e^{-t})$

$Z_p'' = B_1e^t - B_2e^{-t} + B_1e^t - B_2e^{-t} + t(B_1e^t + B_2e^{-t})$

Substitution of these into $z'' - z = -e^t - e^{-t}$ yields that

$(B_1 + B_1 + B_1t - B_1t + 1)e^t + (-B_2 - B_2 + B_2t - B_2t + 1)e^{-t} = 0$

$B_1 = -\frac{1}{2}$ & $B_2 = \frac{1}{2}$ then $Z_p = \frac{1}{2}t(e^{-t} - e^t)$

$Z_{gen} = -C_1e^t + C_2e^{-t} + \frac{1}{2}t(e^{-t} - e^t)$

Example

Solve the given differential equations
$$\begin{aligned} \frac{dx}{dt} + y &= 2 \\ x + \frac{dy}{dt} &= \cos t \end{aligned}$$

$D = \frac{d}{dt}$, substituting D into the system;

$$\begin{aligned} Dx + y &= 2 \\ x + Dy &= \cos t \end{aligned} \Longrightarrow \Delta = \begin{vmatrix} D & 1 \\ 1 & D \end{vmatrix} = D^2 - 1$$

$$\Delta x = \begin{vmatrix} 2 & 1 \\ cost & D \end{vmatrix} = D^2 - \cos t = -\cos t$$

$$\Delta y = \begin{vmatrix} D & 2 \\ 1 & \cos t \end{vmatrix} = D\cos t - \ 2 = -\sin t - 2$$

$$\mathrm{x} = \frac{\Delta x}{\Delta} = \frac{-\cos t}{\mathrm{D}^2 \ - 1} \xrightarrow{yields} (\mathrm{D}^2 - 1)\mathrm{x} = -\cos t \xrightarrow{yields} x'' - x = -\cos t$$

$$\mathrm{y} = \frac{\Delta y}{\Delta} = \frac{-\sin t - 2}{\mathrm{D}^2 \ - 1} \xrightarrow{yields} (\mathrm{D}^2 - 1)\mathrm{y} = -\sin t - 2 \xrightarrow{yields} y'' - y = -\sin t - 2$$

Let's solve them individually. Let's start with the corresponding homogeneous system;

$$\begin{matrix} x'' - x = 0 \\ y'' - y = 0 \end{matrix} \qquad \text{Let} \qquad \begin{matrix} x_h = e^{mt} \implies x_h'' = m^2 e^{mt} & \implies & m^2 - 1 & = 0 \\ y_h = e^{mt} \implies y_h'' = m^2 e^{mt} & \implies & m_{1,2} & = \pm 1 \end{matrix}$$

$$x_h = C_1 e^t + C_2 e^{-t}$$
$$y_h = C_3 e^t + C_4 e^{-t}$$

Since the right side is {-cost} let a particular solution be;

$$x_p = A\cos t + B\sin t$$

$$x_p' = -A\sin t + B\cos t$$

$$x_p'' = -A\cos t - B\sin t$$

Substitution of these into $x'' - x = -\cos t$ yields that

$$-A\cos t - B\sin t - A\cos t + B\sin t = -\cos t$$

$$-2A\cos t - 2B\sin t = -\cos t \xrightarrow{yields} \begin{matrix} A = 1/2 \\ B = 0 \end{matrix} \xrightarrow{yields} x_p = \frac{1}{2}\cos t$$

$$x_{gen} = x_h + x_p = C_1 e^t + C_2 e^{-t} + \frac{1}{2}\cos t$$

Consider the other equation; $y'' - y = -\sin t - 2,$ let a particular solution be;

$$y_p = C\cos t + D\sin t + E$$

$$y_p' = -C\sin t + D\cos t$$

$$y_p'' = -C\cos t - D\sin t$$

Substitution of these into $y'' - y = -\sin t - 2$ yields that

$$-C\cos t - D\sin t - Ccost + D\sin t + E + \sin t + 2 = 0$$

$$-2C\cos t + (-2D+1)\sin t + (-E+2) = 0$$

$$C = 0, \qquad D = \frac{1}{2} \;\;\&\;\; E = 2 \xrightarrow{yields} y_p = \frac{1}{2}\sin t + 2$$

$$y_{gen} = y_h + y_p = C_3 e^t + C_4 e^{-t} + \frac{1}{2}\sin t + 2$$

Now we should substitute the solution into the given set to eliminate the dependent parameters.

$$x + \frac{dy}{dt} = \cos t \qquad \text{where} \qquad \begin{aligned} x &= C_1 e^t + C_2 e^{-t} + \tfrac{1}{2}\cos t \\ \frac{dy}{dt} &= C_3 e^t - C_4 e^{-t} + \tfrac{1}{2}\cos t \end{aligned}$$

Substitution yields that

$$C_1 e^t + C_2 e^{-t} + \frac{1}{2}\cos t + C_3 e^t - C_4 e^{-t} + \frac{1}{2}\cos t - \cos t = 0$$

$$(C_1 + C_3)e^t + (C_2 - C_4)e^{-t} + \left(\frac{1}{2} + \frac{1}{2} - 1\right) cost = 0 \xrightarrow{yields} \begin{aligned} C_3 &= -C_1 \\ C_4 &= C_2 \end{aligned}$$

$$x_{gen} = C_1 e^t + C_2 e^{-t} + \frac{1}{2}\cos t$$

$$y_{gen} = -C_1 e^t + C_2 e^{-t} + \frac{1}{2}\sin t + 2$$

This result may be verified using the other differential equation $\frac{dy}{dt} + x = 2$. The same result will be obtained.

Example

Solve the differential equations
$$\begin{aligned} \frac{dx}{dt} - \mathrm{y} &= \frac{1}{\sin t} \\ x + \frac{dy}{dt} &= \frac{1}{\cos t} \end{aligned}$$

$$\begin{aligned} Dx - \mathrm{y} &= \frac{1}{\sin t} \\ x + Dy &= \frac{1}{\cos t} \end{aligned} \Longrightarrow \Delta = \begin{vmatrix} D & -1 \\ 1 & D \end{vmatrix} = D^2 + 1$$

$$\Delta x = \begin{vmatrix} \frac{1}{\sin t} & -1 \\ \frac{1}{\cos t} & D \end{vmatrix} = D\frac{1}{\sin t} + \frac{1}{\cos t} = -\frac{\cos t}{sin^2 t} + \frac{1}{\cos t}$$

$$\Delta y = \begin{vmatrix} \mathrm{D} & \frac{1}{\sin t} \\ 1 & \frac{1}{\cos t} \end{vmatrix} = D\frac{1}{\cos t} + \frac{1}{\sin t} = -\frac{\sin t}{cos^2 t} - \frac{1}{\sin t}$$

$$\mathrm{x} = \frac{\Delta x}{\Delta} = \frac{\Delta x}{\mathrm{D}^2 + 1} \xrightarrow{yields} (\mathrm{D}^2 + 1)\mathrm{x} = \Delta x \xrightarrow{yields} x'' + x = -\frac{\cos t}{sin^2 t} + \frac{1}{\cos t}$$

$$\mathrm{y} = \frac{\Delta y}{\Delta} = \frac{\Delta y}{\mathrm{D}^2 + 1} \xrightarrow{yields} (\mathrm{D}^2 + 1)\mathrm{y} = \Delta y \xrightarrow{yields} y'' + y = -\frac{\sin t}{cos^2 t} - \frac{1}{\sin t}$$

Homogeneous solutions;

$$\begin{matrix} x'' + x = 0 \\ y'' + y = 0 \end{matrix} \quad \text{Let} \quad \begin{matrix} x_h = e^{mt} \Rightarrow x_h'' = m^2 e^{mt} & \Rightarrow & m^2 + 1 & = 0 \\ y_h = e^{mt} \Rightarrow y_h'' = m^2 e^{mt} & \Rightarrow & m_{1,2} & = \pm i \end{matrix}$$

$$x_h = C_1 \cos t + C_2 \sin t$$
$$y_h = C_3 \cos t + C_4 \sin t$$

<u>Example 2.4</u> Consider two masses connected to the walls by springs, and they are connected to each other by massless springs. Suppose that $m_1 = m_2 = k_1 = k_2 = k_3 = 1$. Find $x_1(t)$ and $x_2(t)$ are the deviation from equilibrium for both masses.

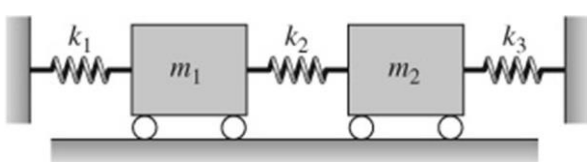

Figure.2.2 Spring and mass system.

Newton's Second Law of Motion yields

Example 2.5

Solve the differential equations

$$\begin{aligned} \dot{x} - x - y \quad &= 3t \\ \dot{x} + \dot{y} - 5x - 2y \quad &= 5 \end{aligned}$$

$\dot{x} - x = y + 3t$ (1)

$\dot{x} - 5x = 2y - \dot{y} + 5$ (2)

From (1) solve y, compute $\dot{y}$

$y = \dot{x} - x - 3t \rightarrow \dot{y} = \ddot{x} - \dot{x} - 3$

Substitute in (2) get the decoupled differential equation for x

$\dot{x} - 5x = 2y - \dot{y} + 5 = 2(\dot{x} - x - 3t) - (\ddot{x} - \dot{x} - 3) + 5$

$\rightarrow \ddot{x} - 2\dot{x} - 3x = 8 - 6t$

From these equations (1) and (2) we solve $x, \dot{x}$

$4x = \dot{y} - y + 3t - 5 \rightarrow 4\dot{x} = \ddot{y} - \dot{y} + 3$

$4\dot{x} = 3y + \dot{y} + 15t - 5$

Equating the two equations, we get the decoupled differential equation for y

$\ddot{y} - \dot{y} + 3 = \dot{y} + 3y - 5 + 15t$

$\rightarrow \ddot{y} - 2\dot{y} - 3y = -8 + 15t$

Solution of x-equation

$$\ddot{x} - 2\dot{x} - 3x = 8 - 6t$$

Homogeneous solution;

$r^2 - 2r - 3 = 0 \Longrightarrow \begin{matrix} r_1 = -1 \\ r_2 = 3 \end{matrix} \Longrightarrow x_h = c_1 e^{-t} + c_2 e^{3t}$

Particular solution;

$x_p = A + Bt \rightarrow \dot{x}_p = B \rightarrow \ddot{x}_p = 0$

$\ddot{x} - 2\dot{x} - 3x = 8 - 6t \rightarrow -2B - 3(A + Bt) = 8 - 6t$

$-3B = -6 \rightarrow \mathrm{B} = 2$

$-2B - 3A = 8 \rightarrow \mathrm{A} = -4$

$x_p = 2t - 4$

$x_{gen} = 2t - 4 + c_1 e^{-t} + c_2 e^{3t}$

Solution of y-equation

$\ddot{\mathrm{y}} - 2\dot{y} - 3\mathrm{y} = -8 + 15\mathrm{t}$

Homogeneous solution;

$$r^2 - 2r - 3 = 0 \Longrightarrow \begin{matrix} r_1 = -1 \\ r_2 = 3 \end{matrix} \Longrightarrow y_h = c_3 e^{-t} + c_4 e^{3t}$$

Particular solution;

$y_p = \mathrm{A} + \mathrm{Bt} \rightarrow \dot{y}_p = \mathrm{B} \rightarrow \ddot{\mathrm{y}}_p = 0$

$\ddot{\mathrm{y}} - 2\dot{y} - 3\mathrm{y} = -8 + 15\mathrm{t} \rightarrow -2B - 3(\mathrm{A} + \mathrm{Bt}) = -8 + 15\mathrm{t}$

$-3B = 15 \rightarrow \mathrm{B} = -5$

$-2B - 3A = -8 \rightarrow \mathrm{A} = 6$

$y_p = 6 - 5\mathrm{t}$

$y_{gen} = 6 - 5\mathrm{t} + c_3 e^{-t} + c_4 e^{3t}$

To remove extra arbitrary constants we use one of the equations in the system

$\dot{x} - x = y + 3t$

which leads

$2 - e^{-t} c_1 + 3e^{3t} c_2 - (2t - 4 + c_1 e^{-t} + c_2 e^{3t}) = 6 - 5\mathrm{t} + c_3 e^{-t} + c_4 e^{3t} + 3\mathrm{t}$

Therefore $C_3 = -2C_1$, $C_4 = 2C_2$, and hence

$y_{gen} = 6 - 5\mathrm{t} - 2c_1 e^{-t} - 2c_2 e^{3t}$

Homeworks

1. $$\begin{aligned} x' + 3x &= \frac{2e^{-t}-3t}{\sqrt{2}} \\ y' + y &= \frac{2e^{-t}+3t}{\sqrt{2}} \end{aligned}$$

2. $$\begin{aligned} x' - 2x - y &= -1, \quad x(0) = 1 \\ y' + x - 2y &= 8, \quad y(0) = 1 \end{aligned}$$

3. $$\begin{aligned} x' - 3x &= -4\sin 2t, \quad x(0) = 2 \\ y' - 5x + 2y &= \cos 2t, \quad y(0) = -1 \end{aligned}$$

4. $$\begin{aligned} \dot{x} - x - 2y &= 2t \\ \dot{y} - 3x - 2y &= -4t \end{aligned}$$

5. $$\begin{aligned} x' + 5x - y &= 6e^{2t} \\ y' - 4x + 2y &= -e^{2t} - 4t \end{aligned}$$

2.3 Canonical Forms and Eigen Values of Linear Differential Equation Systems

Systems of any order coupled differential equations can be expressed in term of a first order set of differential equations which can be expressed in matrix form. Consider such a set of homogeneous set of linear systems with constant coefficients.

$$\begin{gathered} x_1' = a_{11}x_1 + a_{12}\ x_2 + \cdots + a_{1n}x_n \\ x_2' = a_{21}x_1 + a_{22}\ x_2 + \cdots + a_{2n}x_n \\ \vdots \\ x_n' = a_{n1}x_1 + a_{n2}\ x_2 + \cdots + a_{nn}x_n \end{gathered}$$

or in matrix form; $x' = Ax$

where

$$x' = \begin{bmatrix} x_1{}' \\ x_2' \\ . \\ x_n{}' \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_n \end{bmatrix}$$

Let possible solutions be;

$$\begin{gathered} x_1 = \alpha_1 e^{\lambda t} \\ x_2 = \alpha_2 e^{\lambda t} \\ \vdots \\ x_n = \alpha_n e^{\lambda t} \end{gathered} \qquad \text{in matrix form;} \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} e^{\lambda t}, \qquad \text{where } x \text{ and } \alpha \text{ are vectors.}$$

Substitution of this solution proposal matrix expression $x = \alpha e^{\lambda t}$ into the given set in matrix form; $x' = Ax$, yields that

$$x = \alpha e^{\lambda t}$$

$$A\alpha - \lambda\alpha = 0$$

$$(A - \lambda \mathrm{I})\alpha = 0$$

As expected, for α to have a nontrivial solution, the condition $|A - \lambda \mathrm{I}| = 0$ must be satisfied. This condition is known as "*characteristic equation*". Notice that $|\lambda \mathrm{I} - A| = 0$, is also the same expression. The solutions of this expression are called *characteristic values* or more commonly, *eigenvalues* of the system, which are;

$$\lambda = \{\lambda_1\ ;\ \lambda_2\ ;\ ...\ ;\lambda_n\}\ \text{eigenvalues,}$$

Substitution of each λ_i produces a corresponding α_i which is known as "*characteristic vector*" or "*eigenvector*". Each eigenvector corresponds to a solution proposal $x_i = \propto_i e^{\lambda_i t}$.

General solution is the linear combination of these solution proposals, provided that each of them is linearly independent. Otherwise some more manipulations are needed to be carried out to obtain corresponding independent solutions. As known, a popular measure for linear dependency of solution proposals (or any other functions) is that if the Wronskian determinant is not zero, $W(x_1, x_2, \ldots, x_n) \neq 0$, then the proposals are *linearly independent*, otherwise *dependent*. According to the eigenvalues, three cases can arise.

<u>Case 1</u>: All eigenvalues are distinct; that is

$$\lambda_1 \neq \lambda_2 \neq\ \ldots\ \neq \lambda_n$$

In this case, all of the proposals are linearly independent which can be verified using Wronskian determinant. Therefore, the general solution is obtained as the linear combination of the independent parameters;

$$x_{gen} = C_1 x_1 + C_2 x_2 + \cdots + C_n x_n$$

where $C_1, C_2, \ldots \ldots, C_n$ are arbitrary constants.

<u>Example 2.6</u>

Consider the homogeneous linear system, and solve them using matrix method.

$$\begin{aligned} x_1' &= 7x_1 - x_2 + 6x_3 \\ x_2' &= -10x_1 + 4x_2 - 12x_3 \\ x_3' &= -2x_1 + x_2 - x_3 \end{aligned}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{if} \quad x = \alpha e^{\lambda t} \quad \text{then} \quad \begin{aligned} x_1 &= \alpha_1 e^{\lambda t} \\ x_2 &= \alpha_2 e^{\lambda t} \\ x_3 &= \alpha_3 e^{\lambda t} \end{aligned}$$

Characteristic equation;

$$|\lambda I - A| = \begin{vmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)\begin{vmatrix} 4-\lambda & -12 \\ 1 & -1-\lambda \end{vmatrix} + \begin{vmatrix} -10 & -12 \\ -2 & -1-\lambda \end{vmatrix} + 6\begin{vmatrix} -10 & 4-\lambda \\ -2 & 1 \end{vmatrix} = 0$$

$$(7-\lambda)[(4-\lambda)(-1-\lambda)+12] + [-10(-1-\lambda)-24] + [-10+2(4-\lambda)] = 0$$

after some manipulations;

$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$;or $\quad (\lambda-2)(\lambda-3)(\lambda-5) = 0$, so eigenvalues are

$\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 5$; are all distinct. Let's obtain the corresponding eigenvectors;

For $\lambda = \lambda_1 = 2$;

$$A\alpha = \lambda\alpha \xrightarrow{\textbf{yields}} \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 2\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\begin{matrix} 7\alpha_1 & -\alpha_2 & +6\alpha_3 & = 2\alpha_1 \\ -10\alpha_1 & +4\alpha_2 & -12\alpha_3 & = 2\alpha_2 \\ -2\alpha_1 & +\alpha_2 & -\alpha_3 & = 2\alpha_3 \end{matrix} \xrightarrow{\textbf{yields}} \begin{matrix} 5\alpha_1 & -\alpha_2 & +6\alpha_3 & = 0 \\ -10\alpha_1 & +2\alpha_2 & -12\alpha_3 & = 0 \\ -2\alpha_1 & +\alpha_2 & -3\alpha_3 & = 0 \end{matrix}$$

First two are linearly dependent. So two of the independent expressions;

$$\begin{matrix} 5\alpha_1 & -\alpha_2 & +6\alpha_3 & = 0 \\ -2\alpha_1 & +\alpha_2 & -3\alpha_3 & = 0 \end{matrix} \xrightarrow{\textbf{yields}} \begin{matrix} -\alpha_2 & +6\alpha_3 & = -5\alpha_1 \\ +\alpha_2 & -3\alpha_3 & = \ \ 2\alpha_1 \end{matrix} \xrightarrow{\textbf{yields}} \begin{matrix} \alpha_3 & = -\alpha_1 \\ \alpha_2 & = -\alpha_1 \end{matrix}$$

Let $\alpha_1 = 1$ *then* $\alpha_2 = -1,\ \alpha_3 = -1$ so the corresponding eigenvector

For $\lambda_1 = 2 \Longrightarrow \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ then the corresponding solution;

$$x_1 = \alpha e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{bmatrix}$$

For $\lambda = \lambda_2 = 3$;

$$A\alpha = \lambda\alpha \xrightarrow{\textbf{yields}} \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 3\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\begin{matrix} 7\alpha_1 & -\alpha_2 & +6\alpha_3 & = 3\alpha_1 \\ -10\alpha_1 & +4\alpha_2 & -12\alpha_3 & = 3\alpha_2 \\ -2\alpha_1 & +\alpha_2 & -\alpha_3 & = 3\alpha_3 \end{matrix} \xrightarrow{\textbf{yields}} \begin{matrix} 4\alpha_1 & -\alpha_2 & +6\alpha_3 & = 0 \\ -10\alpha_1 & +\alpha_2 & -12\alpha_3 & = 0 \\ -2\alpha_1 & +\alpha_2 & -4\alpha_3 & = 0 \end{matrix}$$

Let $\alpha_1 = 1$ then

$$\begin{aligned} -\alpha_2 \quad +6\alpha_3 &= -4 \\ +\alpha_2 \quad -12\alpha_3 &= 10 \\ +\alpha_2 \quad -4\alpha_3 &= 2 \end{aligned}$$ so the corresponding eigenvectors $\alpha_2 = -2,\ \alpha_3 = -1$

For $\lambda_2 = 3 \rightarrow \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ then the corresponding solution;

$$x_2 = \alpha e^{\lambda_2 t} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix}$$

For $\lambda = \lambda_3 = 5$;

$$A\alpha = \lambda\alpha \xrightarrow{\textbf{yields}} \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 5 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\begin{matrix} 7\alpha_1 & -\alpha_2 & +6\alpha_3 & = 5\alpha_1 \\ -10\alpha_1 & +4\alpha_2 & -12\alpha_3 & = 5\alpha_2 \\ -2\alpha_1 & +\alpha_2 & -\alpha_3 & = 5\alpha_3 \end{matrix} \xrightarrow{\textbf{yields}} \begin{matrix} 2\alpha_1 & -\alpha_2 & +6\alpha_3 & = 0 \\ -10\alpha_1 & -\alpha_2 & -12\alpha_3 & = 0 \\ -2\alpha_1 & +\alpha_2 & -6\alpha_3 & = 0 \end{matrix}$$

First and last are linearly dependent. Let $\alpha_1 = 3$ then two of the independent expressions;

$$\begin{matrix} 2\alpha_1 & -\alpha_2 & +6\alpha_3 & = 0 \\ -10\alpha_1 & -\alpha_2 & -12\alpha_3 & = 0 \end{matrix} \xrightarrow{\textbf{yields}} \begin{matrix} -\alpha_2 & +6\alpha_3 & = -2\alpha_1 \\ -\alpha_2 & -12\alpha_3 & = 10\alpha_1 \end{matrix} \xrightarrow{\textbf{yields}} \begin{matrix} \alpha_3 & = -2 \\ \alpha_2 & = -6 \end{matrix}$$

So the corresponding eigenvector

For $\lambda_3 = 5 \rightarrow \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix}$ then the corresponding solution;

$$x_3 = \alpha e^{\lambda_3 t} = \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix} e^{5t} = \begin{bmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{bmatrix}$$

Since general solution is expressed in terms of their linear combinations

$$x_{gen} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C_1 \begin{bmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{bmatrix} + C_2 \begin{bmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix} + C_3 \begin{bmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{bmatrix} \quad \text{then;}$$

$$x_1 = C_1 e^{2t} + C_2 e^{3t} + 3C_3 e^{5t}$$

$$x_2 = -C_1 e^{2t} - 2C_2 e^{3t} - 6C_3 e^{5t}$$

$$x_3 = -C_1 e^{2t} - C_2 e^{3t} - 2C_3 e^{5t}$$

Case 2: If a pair of the distinct eigenvalues is complex conjugate numbers, the corresponding eigenvectors are also complex conjugate vectors. Let

$$\lambda_1 = \alpha + i\beta, \text{and } \lambda_2 = \alpha - i\beta$$

Are complex conjugate eigenvalues, then

$v^1 = u^1 + iu^2$ and $v^2 = u^1 - iu^2$

are complex conjugate eigenvectors. Then the two linearly independent real solutions are

$$x^1 = \frac{\left[(u^1 + iu^2)e^{(\alpha+i\beta)t} + (u^1 - iu^2)e^{(\alpha-i\beta)t}\right]}{2} = \text{RealPart}\left((u^1 + iu^2)e^{(\alpha+i\beta)t}\right)$$

$$= e^{\alpha t}\text{RealPart}\big((u^1 + iu^2)(\cos\beta t + i\sin\beta t)\big)$$

Hence

$$x^1 = e^{\alpha t}(u^1\cos\beta t - u^2\sin\beta t)$$

And

$$x^2 = \frac{\left[(u^1 + iu^2)e^{(\alpha+i\beta)t} - (u^1 - iu^2)e^{(\alpha-i\beta)t}\right]}{2\mathrm{i}} = \text{ImPart}\left((u^1 + iu^2)e^{(\alpha+i\beta)t}\right)$$

$$= e^{\alpha t}\text{ImPart}\big((u^1 + iu^2)(\cos\beta t + i\sin\beta t)\big)$$

Hence

$$x^2 = e^{\alpha t}(u^1\sin\beta t + u^2\cos\beta t).$$

Case 3: In the case of repeated eigenvalues, determinant of a different solution corresponding to the all eigenvalues may require a different procedure. In the state equation, $x' = Ax$, let the state matrix A be an $n_x n$ real constant matrix; and λ_1 has a multiplicity m where $1 < m \leq n$; and the other eigenvalues λ_{m+1}, $\lambda_{m+2}, \ldots, \lambda_n$ are distinct. Under these considerations two possible sub cases may arise;

a. Repeated eigenvalue λ_1 with the multiplicity m may have equal number of independent solution ($p = m$, where p is the number of independent solution). In this sub case the number of independent solutions are complete, so the general solution are obtained in terms of the linear combination of them similar to before

b. Repeated eigenvalue λ, with the multiplicity *m* may have smaller number independent solution (*p*<*m*) where *p* is a number of independent solutions. For the sake of simplicity let *m=2* and *p=1*. As known, in the case of distinct eigenvalues, a solution proposal is made as

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} e^{\lambda t} \Longrightarrow x = \alpha e^{\lambda t} \quad \text{where } x \text{ and } \alpha \text{ are vectors.}$$

This is equally valid for the repeated eigenvalue. For the missing solutions, however, a different proposal should be made as

$$x = (\gamma t + \beta) e^{\lambda t}$$

Substitution of this proposal into the state equation, $\dot{x} = Ax$, yields;

$$(\gamma t + \beta)\lambda e^{\lambda t} + \gamma e^{\lambda t} = A(\gamma t + \beta) e^{\lambda t}$$

After grouping and cancellations,

$$(\lambda\gamma - A\gamma)t + (\lambda\beta + \gamma - A\beta) = 0, \quad \text{yields two equations}$$

$$\lambda\gamma - A\gamma = 0 \implies (A - \lambda I)\gamma = 0 \; and \implies (A - \lambda I) = 0$$
$$\lambda\beta + \gamma - A\beta = 0$$

The first expression implies that γ in fact is an eigenvector a corresponding to the eigenvalue λ. Substitution of this feature ($\gamma = \alpha$) into the second equation;

$$\lambda\beta + \alpha - A\beta = 0 \xrightarrow{yields} (A\beta - \lambda\beta) = \alpha \quad \text{so} \quad (A - \lambda I)\beta = \alpha$$

As the equation for the determination of β' It can be shown that the solutions $x = \alpha\, e^{\lambda t}$ and $x = (\alpha t + \beta)e^{\lambda t}$ are linearly independent. Also, notice that in the case of further missing independent solutions; the next solution can be proposed as:

$$x = (\theta t^2 + \tau t + \emptyset)e^{\lambda t} \qquad \text{and so on.}$$

Example 2.7

Solve the given homogeneous linear system;

$$\begin{aligned} x_1' &= 3x_1 + x_2 - x_3 \\ x_2' &= x_1 + 3x_2 - x_3 \\ x_3' &= 3x_1 + 3x_2 - x_3 \end{aligned}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad x' = Ax \quad \text{where} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Eigen values are obtained using $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ 3 & 3 & -1-\lambda \end{vmatrix} = 0 \xrightarrow{yields} \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0 \implies (\lambda - 1)(\lambda - 2)^2 = 0$$

So the eigenvalues are $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 2$ and the corresponding eigenvector is obtain using

$$A\alpha = \lambda\alpha \quad \text{where} \quad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\text{For } \lambda = 1 \implies \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 1 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

This corresponds to the equation set

$$\begin{aligned} 2\alpha_1 + \alpha_2 - \alpha_3 &= 0 \\ \alpha_1 + 2\alpha_2 - \alpha_3 &= 0 \\ 3\alpha_1 + 3\alpha_2 - 2\alpha_3 &= 0 \end{aligned}$$

Let's check the rank

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 0 & -3 & 1 \\ 1 & 2 & -1 \\ 0 & -3 & -1 \end{vmatrix} = -\begin{vmatrix} -3 & 1 \\ -3 & 1 \end{vmatrix} = 0$$

$$\Delta_{\text{sub}} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 13 \neq 0 \implies \text{rank} = 2$$

So, select 3-2 =1 unknown as known (say $\alpha_1 = 1$) and solve the others using any two equation, solution; $\alpha_1 = 1,\ \alpha_2 = 1,\ \alpha_3 = 3$

Therefore, the eigenvector corresponding to λ=1 is; $\alpha = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

Since the solution is $x = \alpha e^{\lambda t}$ $\quad x = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} e^{t}$ $\quad x = \begin{bmatrix} e^{t} \\ e^{t} \\ 3e^{t} \end{bmatrix}$

Let's obtain the eigenvector for the repeated eigenvalue λ=2;

$$\mathrm{A}\alpha = \lambda\alpha \Longrightarrow \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

The corresponding equation set;

$$\begin{array}{cccc} \alpha_1 & +\alpha_2 & -\alpha_3 & = 0 \\ \alpha_1 & +\alpha_2 & -\alpha_3 & = 0 \\ 3\alpha_1 & +3\alpha_2 & -3\alpha_3 & = 0 \end{array}$$

The first one is redundant equation.

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{vmatrix} \Longrightarrow$$ **Rank=1** So 3-2 two of them must be selected arbitrary.

Let's select $\alpha_1 = 1,\ \alpha_3 = 0 \ \Longrightarrow\ \alpha_2 = -1$ or $\alpha_1 = 1,\ \alpha_2 = 0 \ \Longrightarrow\ \alpha_3 = 1$

$$\alpha = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Longrightarrow x = \alpha e^{\lambda t} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ -e^{2t} \\ 0 \end{bmatrix}$$ and

$$\alpha = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow x = \alpha e^{\lambda t} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}$$

Therefore, the general solution is the linear combination of these three solution as follows;

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C_1 \begin{bmatrix} e^{t} \\ e^{t} \\ 3e^{t} \end{bmatrix} + C_2 \begin{bmatrix} e^{2t} \\ -e^{2t} \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}$$

Then the solution of the given homogeneous set is;

$$x_1 = C_1 e^{t} + (C_2 + C_3) e^{2t}$$

$$x_2 = C_1 e^{t} - C_2 e^{2t}$$

$$x_3 = 3C_1 e^{t} + C_3 e^{2t}$$

Example 2.8

Solve the homogeneous linear system

$$
\begin{aligned}
x_1' &= 4x_1 + 3x_2 + x_3 \\
x_2' &= -4x_1 - 4x_2 - 2x_3 \\
x_3' &= 8x_1 + 12x_2 + 6x_3
\end{aligned}
$$

The matrix representation of this set,

$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ Assuming a solution of the form, $x = \alpha e^{\lambda t}$ where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$;

$|A - \lambda I| = \begin{bmatrix} 4-\lambda & 3 & 1 \\ -4 & -4-\lambda & -2 \\ 8 & 12 & 6-\lambda \end{bmatrix} = 0$ Expanding the characteristic equation;

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0 \Longrightarrow (\lambda - 2)^3 = 0 \Longrightarrow \lambda_1 = \lambda_2 = \lambda_3 = 2$$

Let's obtain the eigenvector of the eigenvalue $\lambda=2$;

$$A\alpha = \lambda\alpha \Longrightarrow \begin{bmatrix} 4 & 3 & 1 \\ -4 & -4 & -2 \\ 8 & 12 & 6 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

Some manipulations yield the homogenous set,

$$
\begin{aligned}
2\alpha_1 + 3\alpha_2 + \alpha_3 &= 0 \\
-4\alpha_1 - 6\alpha_2 - 2\alpha_3 &= 0 \\
8\alpha_1 + 12\alpha_2 + 4\alpha_3 &= 0
\end{aligned}
$$

The rank of the coefficient matrix $\begin{bmatrix} 2 & 3 & 1 \\ -4 & -6 & -2 \\ 8 & 12 & 4 \end{bmatrix}$ is 1

because all of the 2x2 matrices are all zero, too, so, two arbitrary selections are made

Let's select $\alpha_1 = 1,\ \alpha_2 = 0$ then $\alpha_3 = -2$ or selected; $\alpha_1 = 0,\ \alpha_2 = 1$ then $\alpha_3 = -3$

Therefore, two of the eigenvectors and independent solutions corresponding to $\lambda=2$

$$\alpha = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ 0 \\ -2e^{2t} \end{bmatrix} \quad \& \quad \alpha = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} e^{2t} = \begin{bmatrix} 0 \\ e^{2t} \\ -3e^{2t} \end{bmatrix}$$

Since the state matrix 3x3 then the total number of independent solutions is 3; so, one of the solutions is missing.

The third solution proposal for λ=2 is made as $(\alpha t+\beta)e^{2t}$ where

α satisfies $(A-2I)\alpha=0$ and β satisfies $(A-2I)\beta=\alpha$.

Notice that α above is the linear combinations of the eigenvector obtained before

$$\alpha = k_1\begin{bmatrix}1\\0\\-2\end{bmatrix}+k_2\begin{bmatrix}0\\1\\-3\end{bmatrix}=\begin{bmatrix}k_1\\k_2\\-2k_1-3k_2\end{bmatrix}$$

and $\beta=\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}$ under these considerations, since $(A-2I)\beta=\alpha$ then,

$$\left[\begin{bmatrix}4&3&1\\-4&-4&-2\\8&12&6\end{bmatrix}-2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right]\begin{bmatrix}\beta_1\\\beta_2\\\beta_3\end{bmatrix}=\begin{bmatrix}k_1\\k_2\\-2k_1-3k_2\end{bmatrix}$$

Manipulations yields that

$$\begin{array}{llll}2\beta_1 & +3\beta_2 & +\beta_3 & =k_1\\ -4\beta_1 & -6\beta_2 & -2\beta_3 & =k_2\\ 8\beta_1 & 12\beta_2 & +4\beta_3 & =-2k_1-3k_2\end{array}\quad\Longrightarrow k_2=-2k_1$$

Since the rank of the coefficient matrix is 2, 3-2=1 free selection is made; say

$k_1=1\Longrightarrow\ k_2=-2$; substitution yields,

$$\begin{array}{lllll}2\beta_1 & +3\beta_2 & +\beta_3 & = & 1\\ -4\beta_1 & -6\beta_2 & -2\beta_3 & = & -2\\ 8\beta_1 & 12\beta_2 & +4\beta_3 & = & 4\end{array}$$ Rank=1, so proposal should be made

Let $\beta_1=\beta_2=0\ \Longrightarrow\ \beta_3=1\ \Longrightarrow\ \beta=\begin{bmatrix}0\\0\\1\end{bmatrix}$

Since $\alpha=\begin{bmatrix}k_1\\k_2\\-2k_1-3k_2\end{bmatrix}=\begin{bmatrix}1\\-2\\4\end{bmatrix}$ and $\beta=\begin{bmatrix}0\\0\\1\end{bmatrix}$

Substitution of these into $x=[\alpha t+\beta]e^{\lambda t}$

$$x = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} = \begin{bmatrix} te^{2t} \\ -2te^{2t} \\ (4t+1)e^{2t} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C_1 \begin{bmatrix} e^{2t} \\ 0 \\ -2e^{2t} \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ e^{2t} \\ -3e^{2t} \end{bmatrix} + C_3 \begin{bmatrix} te^{2t} \\ -2te^{2t} \\ (4t+1)e^{2t} \end{bmatrix}$$

Therefore;

$x_1 = C_1 e^{2t} + C_3 t e^{2t}$

$x_2 = C_2 e^{2t} - 2C_3 t e^{2t}$

$x_3 = -2C_1 e^{2t} - 3C_2 e^{2t} + C_3 (4t+1) e^{2t}$

Non-homogeneous Systems of Linear Differential Equations:

Let $f(t) \neq 0$, and consider the system,

$$x' = Ax + f, \qquad f \neq 0.$$

Let solutions of the homogeneous system

$$x' = Ax$$

be;

$$x_h = C_1 x^1 + C_2 x^2 + \cdots + C_n x^n$$

If a particular solution of the non-homogeneous system is x_p, then the general solution of the non-homogeneous system will be

$$x_{gen} = x_h + x_p$$

As in the case of scalar equations we'll distinguish two cases, and apply techniques similar to the method of undetermined coefficients, and variation of parameters accordingly.

The Method of Undetermined Coefficients

Let

$$f(t) = \begin{bmatrix} a_1 \\ a_2 \\ . \\ a_n \end{bmatrix} \varphi(t),$$

and $\varphi(t)$ is a function of finite derivatives, where the set

$$D = \{\varphi_1(t), \varphi_2(t), \ldots, \varphi_p(t)\}$$

is finite. And let functions in the solutions

$$x^i = \propto_i e^{\lambda_i t}, i = 1,2,..,n$$

of the homogeneous system be

$$H = \{f_1(t), f_2(t), \ldots, f_n(t)\}$$

If $H \cap D = \emptyset$ empty set, a particular solution proposal be

$$x_p = c^1\varphi_1(t) + c^2\varphi_2(t) + \cdots + c^p\varphi_p(t)$$

Where $\{c^1, c^2, \ldots, c^p\}$ are n dimensional vectors to be specified.

If $H \cap D \neq \emptyset$ a non-empty set, D is multiplied by t till $H \cap t^n D = \emptyset$ is obtained. Then a particular solution proposal be a linear combination of contents of $t^n D$ with n dimensional vectors $\{c^0, c^1, c^2, \ldots, c^p\}$ that are to be specified.

<u>Example 2.10 (Non-Homogeneous Version)</u>

Consider the homogenous linear system, and solve it using matrix method.

$$\begin{aligned} x_1' &= 7x_1 - x_2 + 6x_3 + \cos 2t \\ x_2' &= -10x_1 + 4x_2 - 12x_3 \\ x_3' &= -2x_1 + x_2 - x_3 \end{aligned}$$

In matrix notation

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos 2t \quad (1)$$

Along the previous Examples the homogeneous solution is found to be

$$x_h = C_1 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} e^{3t} + C_3 \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix} e^{5t} \quad \text{then;}$$

$$H = \{e^{2t}, e^{3t}, e^{5t}\}$$

and here

$$D = \{\cos 2t\,, \sin 2t\}$$

Since $H \cap D = \emptyset$ empty set, a particular solution proposal be

$x_p = c^1 \cos 2t + c^2 \sin 2t, \quad x_p{}' = -2c^1 \sin 2t + 2c^2 \cos 2t.$

Then Equation (1) implies

$$-2c^1 \sin 2t + 2c^2 \cos 2t = Ac^1 \cos 2t + Ac^2 \sin 2t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos 2t$$

Equating the coefficients of $\{\cos 2t\,, \sin 2t\}$ one has

$$\sin 2t\colon -2c^1 = Ac^2$$

$$\cos 2t\colon\ 2c^2 = Ac^1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to 2Ac^2 = A^2c^1 + \mathrm{A} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to -4c^1 = A^2c^1 + \mathrm{A} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\to (A^2 + 4I)c^1 = \mathrm{A} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \to c^1 = (A^2 + 4I)^{-1} \,\mathrm{x}\, \mathrm{A} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$c^1 = \{-0.25, -0.19, -0.05\}$

$c^2 = -2A^{-1}c^1$

$c^2 = \{0.24, 0.21, -0.17\}$

$$x_p = \{-0.25, -0.19, -0.05\} \cos 2t + \{0.24, 0.21, -0.17\} \sin 2t$$

$$x_{gen} = x_h + x_p$$

$x_1 = C_1 e^{2t} + C_2 e^{3t} + 3C_3 e^{5t} - 0.25 \cos 2t + 0.24 \sin 2t$

$$x_2 = -C_1 e^{2t} - 2C_2 e^{3t} - 6C_3 e^{5t} - 0.19 \cos 2t + 0.21 \sin 2t$$

$$x_3 = -C_1 e^{2t} - C_2 e^{3t} - 2C_3 e^{5t} - 0.05 \cos 2t - 0.17 \sin 2t$$

■

Exercises

1. $\begin{aligned} x' - 2x - y &= -1, \quad x(0) = 1 \\ y' + x - 2y &= 8, \quad\ \ y(0) = 1 \end{aligned}$

2. $\begin{aligned} x' - 3x \quad &= -4 \sin 2t\,, \quad x(0) = 2 \\ y' - 5x + 2y &= \cos 2t\,, \qquad y(0) = -1 \end{aligned}$

3. SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS

Let the power series

$$C_0 + C_1(x - x_0) + C_2(x - x_0)^2 = \sum_{n=0}^{\infty} C_n(x - x_0)^n$$

converges uniformly to a function $f(x)$ in an interval for which x_0 is an inner point. Then $f(x)$ is analytical around the point x_0, and Taylor's theorem says that

$$C_n = \frac{f^{(n)}(x_0)}{n!}, \qquad n = 0,1,2, \ldots$$

3.1 Second Order Linear Differential Equations with Variable Coefficients

In Chapter 1. the techniques to solve second order linear differential equations with constant coefficients

$$ay'' + by' + cy = r(x)$$

are studied. The solution of equations with variable coefficients

$$y'' + p(x)y' + q(x)y = r(x)$$

are more complicated.

Under certain conditions on $p(x)$, and $q(x)$, the equation may have series solutions in appropriate forms.

Classification of Points With respect to a Differential Equation

An Ordinary Point: If $p(x)$, and $q(x)$, are both analytical around a point x_0, then x_0 is an ordinary point for the differential equation

$$y'' + p(x)y' + q(x)y = r(x).$$

A Regular Singular Point: If $p(x)$, or $q(x)$, is not analytical around a point x_0, but $(x-x_0)p(x)$, and $(x-x_0)^2q(x)$, are both analytical around the point x_0, then x_0 is a regular singular point for the differential equation

Irregular Singular Point: If $(x-x_0)p(x)$, or $(x-x_0)^2q(x)$, is not analytical around the point x_0, then x_0 is an irregular singular point of the differential equation.

3.2 Series Solution About an Ordinary Point

Frobenius Theorem I: Let x_0 be an ordinary point for the differential equation

$$y'' + p(x)y' + q(x)y = r(x).$$

Then, the differential equation has at least one series solution of the form

$$y = \sum_{n=0}^{\infty} C_n\,(x-x_0)^n.$$

Example 3.1

Find the power series solution of the initial value problem in the below around the point $x_0 = 0$.

$(x^2-1)y'' + 3xy' + xy = 0\,;\;\; y(0) = 4, \qquad y'(0) = 6$

The normalized differential equation is;

$$y'' + \frac{3x}{x^2-1}y' + \frac{x}{x^2-1}y = 0$$

Obviously, coefficient functions are not analytic only at $x = \pm 1$, so $x_0 = 0$ is an ordinary point, therefore the solution proposal be;

$$y = \sum_{n=0}^{\infty} C_n\,x^n$$

$$y = \sum_{n=0}^{\infty} C_n\,x^n = C_0 + C_1\,x + C_2\,x^2 + C_3\,x^3 + \cdots \to$$

$y' = C_1 + 2C_2\,x + \cdots + 3C_3\,x^2 + \cdots = \sum_{n=1}^{\infty} nC_n\,x^{n-1}$, and

$$y'' = 2C_2 + 6C_3x + \cdots = \sum_{n=2}^{\infty} n(n-1)C_n\, x^{n-2}$$

Substitution of these into the differential equation yields that;

$$(x^2 - 1)\sum_{n=2}^{\infty} n(n-1)C_n\, x^{n-2} + 3x\sum_{n=1}^{\infty} nC_n\, x^{n-1} + x\sum_{n=0}^{\infty} C_n\, x^n \qquad = 0 \;\rightarrow$$

$$\sum_{n=2}^{\infty} n(n-1)C_n\, x^n - \sum_{n=2}^{\infty} n(n-1)C_n\, x^{n-2} + 3\sum_{n=1}^{\infty} nC_n\, x^n + \sum_{n=0}^{\infty} C_n\, x^{n+1} \qquad = 0$$

In order to add the above series, it is necessary that both summation indices start with the same number and the powers of $x$terms in each series should be such that if one series starts with a multiple of x to the first power, then the other series should also have the same power. Next, to make the exponent of all x terms same n, the same as the first and the third terms. So, the second term is modified by replacing $n \rightarrow n + 2$

$$\sum_{n=2}^{\infty} n(n-1)C_n\, x^{n-2} \;\; for\;\; n \rightarrow n+2 \;\Rightarrow\; \sum_{n+2=2}^{\infty} (n+2)(n+1)C_{n+2}\, x^n$$

Similarly, for the last term $\quad n \rightarrow n - 1$

$$\sum_{n=0}^{\infty} C_n\, x^{n+1} \;\; for\, n \rightarrow n-1 \;\Rightarrow\; \sum_{n-1=0}^{\infty} C_{n-1}\, x^n$$

Substitution of these above yields that;

$$\sum_{n=2}^{\infty} n(n-1)C_n\, x^n - \sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}\, x^n + 3\sum_{n=1}^{\infty} nC_n\, x^n + \sum_{n=1}^{\infty} C_{n-1}\, x^n = 0$$

Since the exponents are now common, the next step is to make the range of Σ ‘s common as well. Clearly the common range is $n = 2 \ldots \infty$; so expand the Σ ‘s to makethe index as n=2;

$$\sum_{n=2}^{\infty} n(n-1)C_n\, x^n$$

already n=2;

$$\sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}\, x^n = 2C_2 + 6C_3\, x + \sum_{n=2}^{\infty}(n+2)(n+1)C_{n+2}\, x^n$$

$$\sum_{n=1}^{\infty} nC_n\,\mathrm{x}^n = C_1\,x + \sum_{n=2}^{\infty} nC_n\,\mathrm{x}^n$$

$$\sum_{n=1}^{\infty} C_{n-1}\,\mathrm{x}^n = C_0 x + \sum_{n=2}^{\infty} C_{n-1}\,\mathrm{x}^n$$

Substitution of these back yields that;

$$\sum_{n=2}^{\infty} n(n-1)C_n\,\mathrm{x}^n - 2C_2 - 6C_3\,x - \sum_{n=2}^{\infty}(n+2)(n+1)C_{n+2}\,\mathrm{x}^n + 3C_1\,x + 3\sum_{n=2}^{\infty} nC_n\,\mathrm{x}^n + C_0 x + \sum_{n=2}^{\infty} C_{n-1}\,\mathrm{x}^n = 0$$

Let's group this according to the powers of x;

$$-2C_2 + (C_0 + 3C_1 - 6C_3)x + \sum_{n=2}^{\infty}[-(n+2)(n+1)C_{n+2} + [n(n-1) + 3n]C_n + C_{n-1}]\mathrm{x}^n = 0$$

Equating the coefficients of x^i to zero one has

$$-2C_2 = 0 \quad \Rightarrow \quad C_2 = 0$$

$$C_0 + 3C_1 - 6C_3 = 0 \quad \Rightarrow \quad C_3 = \frac{1}{6}C_0 + \frac{1}{2}C_1$$

$$[-(n+2)(n+1)C_{n+2} + [n(n+2)]C_n + C_{n-1} = 0$$

$$C_{n+2} = \frac{n(n+2)C_n + C_{n-1}}{(n+1)(n+2)}\;; \qquad n \geq 2$$

This last expression is called the *RECURRENCE formula*. Apparently there is no restriction on C_0, and C_1. Therefore they remain as arbitrary. If we use the recurrence formula to compute coefficients with higher indices, we have

$$n = 2 \Rightarrow C_4 = \frac{8C_2 + C_1}{12} = \frac{1}{12}C_1$$

$$\mathrm{n} = 3 \Rightarrow C_5 = \frac{15C_3 + C_2}{20} = \frac{3}{4}\left(\frac{1}{6}C_0 + \frac{1}{2}C_1\right) + \frac{C_2}{20} = \frac{1}{8}C_0 + \frac{3}{8}C_1$$

Substitution of these C_0, C_1, C_2 ... into the proposal

$$y = C_0 + C_1 x + C_2\mathrm{x}^2 + \cdots$$

we have

$$y = C_0 + C_1 x + \left(\frac{1}{6}C_0 + \frac{1}{2}C_1\right)x^3 + \frac{1}{2}C_1 x^4 + \left(\frac{1}{8}C_0 + \frac{3}{8}C_1\right)x^5 + \cdots$$

Let's group C_o and C_1 terms separately,

$$y = C_0\left(1 + \frac{1}{6}x^3 + \frac{1}{8}x^5 + \cdots\right) + \ C_1\left(x + \frac{1}{2}x^3 + \frac{1}{12}x^4 + \frac{3}{8}x^5 + \cdots\right)$$

This is the series solution of the differential equation. As expected, there appeared two arbitrary constants C_0, C_1 to be obtained using initial conditions

for $y(0) = 4 \ \Rightarrow C_0 = 4$

for $y'(0) = 6 \Rightarrow y' = C_0\left(\frac{1}{2}x^2 + \frac{5}{8}x^4 + \cdots\right) + C_1\left(1 + \frac{3}{2}x^2 + \frac{1}{3}x^3 + \frac{15}{8}x^4 + \cdots\right) \Rightarrow C_1 = 6$

substitution of this into above,

$$y = 4\left(1 + \frac{1}{6}x^3 + \frac{1}{8}x^5 + \cdots\right) + \ 6\left(x + \frac{1}{2}x^3 + \frac{1}{12}x^4 + \frac{3}{8}x^5 + \cdots\right)$$

manipulations yield that

$$y = 4 + 6x + \frac{11}{3}x^3 + \frac{1}{2}x^4 + \frac{11}{4}x^5 + \cdots$$

This is the solution of the initial value problem.

Notice that if initial condition would be $y(5) = 4, y'(5) = 6$; it is a good idea to shift the differential equation using independent variable transformation $t = x - 5$ and obtain a series solution about $t = 0$, instead of $x = 5$.

More precisely instead of solving the boundary value problem, for example;

$(x^2 - 1)y'' + 3xy' + xy = 0\,;\ \ y(x = 5) = 4, \qquad y'(x = 5) = 6$

using proposal

$$y = \sum_{n=0}^{\infty} C_n\,(\mathrm{x} - \ 5)^n$$

A change of variable $t = x - 5$ replaces this initial value problem by the equivalent problem;

$$t = x - 5 \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} \Rightarrow y' = \frac{dy}{dt};\ y'' = \frac{d^2y}{dt^2}$$

$x = t + 5 \Rightarrow (x^2 - 1) = (t^2 + 10t + 24),$ so

$(t^2 + 10t + 24)\frac{d^2y}{dt^2} + (3t + 15)\frac{dy}{dt} + 3y = 0$ and initial conditions; $y(0) = 4, y'(0) = 6$. And solution proposal to be made is;

$$y = \sum_{n=0}^{\infty} C_n\, t^n$$

Having obtained the solution in *t*; the substitution $t = x - 5$, replaces the solution in terms of x . This technique may be suggested for the sake of simplicity.■

3. SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS

3.2 Series Solutions About Regular Singular Points

For the differential equation

$$y'' + p(x)y' + q(x)y = r(x).$$

let $p(x)$ or $q(x)$ is not analytic at the point x_0, but both of $(x - x_0)p(x)$, and $(x - x_0)^2 q(x)$ are analytic at the point x_0. Then x_0 is called a regular singular point for the differential equation. For the existence of a series solution to the equation one has the following theorem.

Let x_0 be a regular singular point for the differential equation

$$y'' + p(x)y' + q(x)y = r(x).$$

Then, the differential equation has at least one series solution of the form

$$y = |x - x_0|^r \sum_{n=0}^{\infty} C_n (x - x_0)^n.$$

If the solution is valid in some interval $0 < x - x_0 < R$,

$$y = |x - x_0|^r \sum_{n=0}^{\infty} C_n (x - x_0)^n = \sum_{n=0}^{\infty} C_n (x - x_0)^{n+r}$$

The rest is performed as before. This technique is also known as ***Frobenius Method.***

A brief outline of the Frobenius Method is as follows: a proposal is first made as,

I)

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} C_n\,(x-x_0)^{n+r} = C_0(x-x_0)^r + C_1\,(x-x_0)^{r+1} + C_2\,(x-x_0)^{r+2} + \cdots \\
y' &= rC_0(x-x_0)^{r-1} + (r+1)C_1\,(x-x_0)^r + (r+2)C_2\,(x-x_0)^{r+1} + \cdots \\
&= \sum_{n=0}^{\infty} (n+r)C_n\,(x-x_0)^{n+r-1} \\
y'' &= r(r-1)C_0(x-x_0)^{r-2} + r(r+1)C_1\,(x-x_0)^{r-1} + (r+1)(r+2)C_2\,(x-x_0)^r + \cdots \\
&= \sum_{n=0}^{\infty} (n+r)(n+r-1)C_n\,(x-x_0)^{n+r-2}
\end{aligned}
$$

II) Substitution and manipulations yield a polynomial in $(x-x_0)$;

$$K_0(x-x_0)^{r+k} + K_1(x-x_0)^{r+k-1} + K_2(x-x_0)^{r+k-2} + \cdots = 0$$

For this to be satisfied, the coefficients must be equal to zero;

$$K_0 = K_1 = K_2 = \cdots = 0$$

III) The lowest power of $(x-x_0)^{r+k} \xrightarrow{yields} K_0 = 0$ is a quadratic equation in r, called the *indicial equation*;

The roots of this is called exponents of differential equation, let them be $r_1$and r_2, where $Re(r_1) \geq Re(r_2)$, where $Re(r_i)$ is the real part of the exponent r_i. If r_1, r_2 are real and unequal, then r_1 is the larger root.

IV) Equate $K_0 = K_1 = K_2 = \cdots = 0$; substitute $r = r_1$ and obtain condition C_n 's

V) If $r_2 \neq r_1$, repeat the previous procedure for $r = r_2$, to obtain a second solution (for smaller root r_2) which is linearly independent to the previous one. According to different types of r_1- r_2, different solution styles are obtained.

VI) Let $\mathrm{r}_1 - \mathrm{r}_2 \neq \mathrm{N}$, N non-negative integer $(\mathrm{r}_1 - \mathrm{r}_2 \neq 0, 1, 2,3, \dots)$ then two independent solutions do exist

$$y_1(x) = |x-x_0|^{r_1} \sum_{n=0}^{\infty} C_n\,(x-x_0)^n\ , \qquad C_0 \neq 0$$

$$y_2(x) = |x-x_0|^{r_2} \sum_{n=0}^{\infty} C_n^{\,*}(x-x_0)^n\ , \quad C_0^{\,*} \neq 0$$

VII) Let $r_1 - r_2 = N \neq 0$, N a positive integer; then two independent solutions are expressed as;

$$y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} C_n\,(x - x_0)^n\ , \qquad\qquad C_0 \neq 0$$
$$y_2(x) = |x - x_0|^{r_2} \sum_{n=0}^{\infty} C_n^*(x - x_0)^n + Cy_1(x)\ln|x - x_0|$$

VIII) Let $r_1 - r_2 = 0$, then two independent solutions are;

$$y_1(x) = |x - x_0|^{r_1} \sum_{n=0}^{\infty} C_n\,(x - x_0)^n\ , \qquad\qquad C_0 \neq 0$$
$$y_2(x) = |x - x_0|^{r_1+1} \sum_{n=0}^{\infty} C_n^*(x - x_0)^n + Cy_1(x)\ln|x - x_0|$$

Example 3.1

Given the differential equation

$$2x^2y'' - xy' + (x - 5)y = 0,$$

find a series solution about $x_0 = 0$.

The normalized differential equation is $y'' - \frac{x}{2x^2}y' + \frac{x-5}{2x^2}y = 0$

where

$$p(x) = \frac{x}{2x^2}, q(x) = \frac{x-5}{2x^2}$$

are not analytic at $x_0 = 0$, but

$$\left.\begin{array}{l} \lim\limits_{x\to x_0=0} x\left(-\frac{x}{2x^2}\right) = -\frac{1}{2} \\ \lim\limits_{x\to x_0=0} x^2\left(\frac{x-5}{2x^2}\right) = -\frac{5}{2} \end{array}\right\} \Rightarrow \qquad \text{so } x_0 = 0 \text{ is a regular singular point}$$

According to the Frobenius theorem we propose a solution

$$y(x) \quad = C_0\,\mathrm{x}^r + C_1\,\mathrm{x}^{r+1} + C_2\,\mathrm{x}^{r+2} + \cdots = \sum_{n=0}^{\infty} C_n\,\mathrm{x}^{n+r} \;\rightarrow$$

$$y'(x) \quad = C_0 r\mathrm{x}^{r-1} + C_1\,(\mathrm{r}+1)\,\mathrm{x}^r + C_2\,(\mathrm{r}+2)\,\mathrm{x}^{r+1} + \cdots = \sum_{n=0}^{\infty}(n+r)C_n\,\mathrm{x}^{n+r-1}$$

$$y''(x) \quad = C_0\,r(r-1)\mathrm{x}^{r-2} + C_1\,(\mathrm{r}+1)\mathrm{r}\,\mathrm{x}^{r-1} + C_2\,(\mathrm{r}+1)(\mathrm{r}+2)\,\mathrm{x}^r + \cdots$$

$$= \sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\,\mathrm{x}^{n+r-2}$$

Substitution of these into the differential equation yields

$$2x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n x^{n+r-2} - x\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1} + (x-5)\sum_{n=0}^{\infty} C_n x^{n+r} = 0$$

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\,\mathrm{x}^{n+r} - \sum_{n=0}^{\infty}(n+r)C_n\,\mathrm{x}^{n+r} - 5\sum_{n=0}^{\infty} C_n\,\mathrm{x}^{n+r} + \sum_{n=0}^{\infty} C_n\,\mathrm{x}^{n+r+1} = 0$$

for the last term we let $n \rightarrow n-1$

$$2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\,\mathrm{x}^{n+r} - \sum_{n=0}^{\infty}(n+r)C_n\,\mathrm{x}^{n+r} - 5\sum_{n=0}^{\infty} C_n\,\mathrm{x}^{n+r} + \sum_{n=1}^{\infty} C_{n-1}\,\mathrm{x}^{n+r} = 0$$

Since the common range is $1 < n < \infty$

$$(2r(r-1)C_0 - rC_0 - 5C_0)x^r + \sum_{n=1}^{\infty}[2(n+r)(n+r-1)C_n - (n+r)C_n - 5C_n + C_{n-1}]x^{n+r} = 0$$

$$(2r(r-1) - r - 5)C_0 x^r + \sum_{n=1}^{\infty}\{[2(n+r)(n+r-1) - (n+r) - 5]C_n + C_{n-1}\}x^{n+r} = 0$$

First and last terms are indicial equation and recurrence relations, respectively.

$$2r(r-1) - r - 5 = 0 \;\Rightarrow\; 2r^2 - 3r - 5 = 0 \;\Rightarrow\; \begin{array}{l} r_1 = \dfrac{5}{2} \\ r_2 = -1 \end{array}$$

Since $r_1 - r_2 = \frac{5}{2} - (-1) = \frac{7}{2} \neq 0,1,2,3\ldots$; two independent solutions are simply obtained by the substitutions of r_1 and r_2 into the proposal

$$y(x) = |x - x_0|^r \sum_{n=0}^{\infty} C_n\,(x - x_0)^n\;, \qquad C_0 \neq 0$$

Let’s obtain the recurrence relation;

$[2(n+r)(n+r-1)-(n+r)-5]C_n \ +C_{n-1}=0$

$C_n=-\frac{C_{n-1}}{2(\mathrm{n+r})(\mathrm{n+r}-1)-(\mathrm{n+r})-5}$; Recurrence relation

One of the solutions for $r=\frac{5}{2}$

$C_n=-\frac{C_{n-1}}{2\left(\mathrm{n}+\frac{5}{2}\right)\left(\mathrm{n}+\frac{3}{2}\right)-\left(\mathrm{n}+\frac{5}{2}\right)-5} \quad with \ \ n\geq 1$

$$C_1=-\frac{C_0}{2\left(1+\frac{5}{2}\right)\left(1+\frac{3}{2}\right)-\left(1+\frac{5}{2}\right)-5}=-\frac{C_0}{2\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)-\left(\frac{7}{2}\right)-\frac{10}{2}}=-\frac{C_0}{9}$$

$$C_2=-\frac{C_1}{2\left(2+\frac{5}{2}\right)\left(2+\frac{3}{2}\right)-\left(2+\frac{5}{2}\right)-5}=-\frac{C_1}{2\left(\frac{9}{2}\right)\left(\frac{7}{2}\right)-\left(\frac{9}{2}\right)-\frac{10}{2}}=-\frac{C_1}{22}=\frac{C_0}{198}$$

$$y_1(x)=C_0x^{\frac{5}{2}}\left(1-\frac{1}{9}x+\frac{1}{198}x^2-\cdots\right)$$

The other solution for $r=-1$

$$C_n=-\frac{C_{n-1}}{2(\mathrm{n}-1)(\mathrm{n}-2)-(\mathrm{n}-1)-5}$$

$$C_1=-\frac{C_0}{-5}=\frac{C_0}{5}, \qquad C_2=\frac{C_1}{6}=\frac{C_0}{30}, \qquad C_3=-\frac{C_2}{-3}=\frac{C_0}{90}$$

$$y_2(x)=C_0x^{-1}\left(1+\frac{1}{5}x+\frac{1}{30}x^2+\frac{1}{90}x^3\ldots\right)$$

Assuming $\quad C_0=1,$

$y_{gen}=C_1y_1+C_2y_2$

$$y_{gen}=C_1x^{\frac{5}{2}}\left(1-\frac{1}{9}x+\frac{1}{198}x^2-\cdots\right)+C_2x^{-1}\left(1+\frac{1}{5}x+\frac{1}{30}x^2+\frac{1}{90}x^3\ldots\right)$$

■

Determination of solutions according to the roots of indicial equation may be trivial in some cases, so sharp classifications are not very straight forward. A closer look may be performed as follows. Let the roots of indicial equation are; $\quad r_1=\alpha\,;\,r_2=\beta$

1) $\quad \alpha\neq\beta \quad$ and $\quad \alpha-\beta \quad$ is not an integer.

This case is relatively straightforward; the two independent solutions y_1, y_2 and the general solution y_{gen} is obtained as

$$y_1 = y(x,r)|_{r=\alpha} \quad ; \quad y_2 = y(x,r)|_{r=\beta} \;\Rightarrow\; y_{gen} = C_1 y_1 + C_2 y_2$$

2) $\alpha = \beta$ This is also a relatively straightforward case

One of solution is $y_1 = y(x,r)|_{r=\alpha}$

The other independent solution is $y_2 = \left.\frac{\partial y}{\partial r}\right|_{r=\alpha}$

Note that substitution of one of the solutions in r into the differential equation yields that if say

$$y(x,r) = a_0 x^r \left[1 + \frac{1}{(r+1)^2} + \cdots\right]$$

then

$$[\text{differential equation}]|_{y(x,r)} = a_0(\mathrm{r} - \alpha)(\mathrm{r} - \beta)x^{r-1}$$

3) $\alpha \neq \beta$ and $\alpha - \beta$ is an integer.

In this case, some of the coefficients C may become infinite for either α or β. Obviously for the root without trouble, say β, the solution is obtained as before,

$$y_1 = y(x,r)|_{r=\beta}$$

For the other troublesome root α some of the coefficients may involve terms like, $\frac{a_0}{r-\alpha}$. In such cases, arbitrary constants may be selected in such a way that $a_0 = b_0(r-\alpha)$ to eliminate $(r-\alpha)$ term. Substitution of α into $y(x,r)$ as

$y_2 = y(x,r)|_{r=\alpha}$ may result in a dependent solution, in this case

$y_2 = \left.\frac{\partial y(x,r)}{\partial r}\right|_{r=\alpha}$ is used

Notice that since $y(x,r) = x^r(\ldots)$ then $\frac{\partial y}{\partial r} = x^r \ln x\,(\ldots) + x^r \frac{\partial y_p}{\partial r}$ where $(\ldots)$ is y_p

Example 3.2

Find series solution of

$xy'' + y' - y = 0$

about $x_0 = 0$

The normalized differential equation is

$$y'' - \frac{1}{x}y' + \frac{1}{x}y = 0$$

since $\frac{1}{x}$ is not analytic at $x_0 = 0$, but

$\lim_{x\to 0} x\,(\frac{1}{x}) = 1;$ $\lim_{x\to 0} x^2\,(-\frac{1}{x}) = 0$, therefore $x_0 = 0$is a regular singular point.

The solution proposal is,

$$y = \sum_{n=0}^{\infty} C_n\,\mathrm{x}^{n+r} \rightarrow$$

$y' = \sum_{n=0}^{\infty}(n+r)C_n\,\mathrm{x}^{n+r-1}$ and

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\,\mathrm{x}^{n+r-2}$$

Substitution of these into differential equation yields that

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\,\mathrm{x}^{n+r-2} + \sum_{n=0}^{\infty}(n+r)C_n\,\mathrm{x}^{n+r-1} - \sum_{n=0}^{\infty}C_n\,\mathrm{x}^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\,\mathrm{x}^{n+r-1} + \sum_{n=0}^{\infty}(n+r)C_n\,\mathrm{x}^{n+r-1} - \sum_{n=0}^{\infty}C_n\,\mathrm{x}^{n+r} = 0$$

for the last term $n \rightarrow n-1$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\,\mathrm{x}^{n+r-1} + \sum_{n=0}^{\infty}(n+r)C_n\,\mathrm{x}^{n+r-1} - \sum_{n=1}^{\infty}C_{n-1}\,\mathrm{x}^{n+r-1} = 0$$

Since the common range is $1 \rightarrow \infty$

$$r(r-1)C_0x^{r-1} + rC_0x^{r-1} + \sum_{n=1}^{\infty}\{[(n+r) + (n+r)(n+r-1)]C_n - C_{n-1}\}x^{r+n-1} = 0 \rightarrow$$

$$C_0r^2x^{r-1} + \sum_{n=1}^{\infty}\{(n+r)^2\,C_n - C_{n-1}\}\,x^{r+n-1} = 0$$

Equating the coefficients of x^i to zero one has the indicial equation;

$r^2 = 0 \implies r_1 = r_2 = 0$ for $C_0 \neq 0$

Recurrence relation is

$$C_n = \frac{C_{n-1}}{(n+r)^2}, \quad \mathrm{n} \geq 1$$

Hence

$$C_1 = \frac{C_0}{(r+1)^2}$$

$$C_2 = \frac{C_1}{(r+2)^2} = \frac{C_0}{(r+2)^2(r+1)^2} = \frac{C_0}{[(r+1)!]^2}$$

$$C_3 = \frac{C_2}{(r+3)^2} = \frac{C_0}{(r+3)^2(r+2)^2(r+1)^2} = \frac{C_0}{[(r+3)!]^2}$$

$$C_n = \frac{C_{n-1}}{(n+r)^2} = \frac{C_0}{[(r+n)!]^2}$$

Substitution of these into the proposal

$$y(x,r) = \sum_{n=0}^{\infty} C_n \, \mathrm{x}^{n+r} = C_0 \mathrm{x}^r + C_1 \mathrm{x}^r \mathrm{x} + C_2 \mathrm{x}^r \mathrm{x}^2 + C_3 \mathrm{x}^r \mathrm{x}^3 + \cdots$$

so

$$y(x,r) = C_0 \mathrm{x}^r \left[1 + \frac{x}{(r+1)^2} + \frac{x^2}{[(r+2)!]^2} + \frac{x^3}{[(r+3)!]^2} + \cdots + \frac{x^n}{[(r+n)!]^2} + \cdots \right]$$

Assuming $C_0 = 1$

$$y_1 = y(x,r)\big|_{r=0} = 1 + \frac{x}{(1!)^2} + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \cdots + \frac{x^n}{(n!)^2} + \cdots$$

One of the solutions is obtained as mentioned. A second independent solution is obtained by;

$$y_2 = \frac{\partial y(x,r)}{\partial r}\bigg|_{r=0}$$

Hence for, $C_0 = 1$,

$$\frac{\partial y(x,r)}{\partial r} = x^r \ln|x| \left[1 + \frac{x}{(r+1)^2} + \frac{x^2}{[(r+2)!]^2} + \cdots + \frac{x^n}{[(r+n)!]^2} \right] - 2x^r \left[\frac{x}{(r+1)^3} + \frac{x^2}{(2!)^2}\left(1 + \frac{1}{2}\right) + \cdots \right]$$

$$y_2 = \frac{\partial y(x,r)}{\partial r}\bigg|_{r=0} = \ln|x| \, y_1 - 2\left[\frac{x}{1^3} + \frac{x^2}{(2!)^2}\left(1 + \frac{1}{2}\right) + \cdots + \frac{x^n}{(n!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) + \cdots \right]$$

So, the general solution is;

$$y_{gen} = C_1 y_1 + C_2 y_2$$

$$y_{gen} = C_1 y_1 + C_2 \ln|x|\, y_1 - 2C_2 \left[x + \frac{x^2}{(2!)^2}\left(1+\frac{1}{2}\right) + \cdots + \frac{x^n}{(n!)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right]$$

so,

$$y_{gen} = y_1(C_1 + C_2 \ln|x|) - 2C_2 \left[x + \frac{3}{2}\frac{x^2}{(2!)^2} + \cdots + \frac{x^n}{(n!)^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right]$$

Example 3.4

Find series solution of the differential equation

$xy'' - 3y' + xy = 0$

about $x_0 = 0$.

The normalized equation; $y'' - \frac{3}{x}y' + \frac{x}{x}y = 0$

since $\frac{1}{x}$ is not analytic at $x_0 = 0$, but

$\lim_{x\to 0} x\,(-\frac{3}{x}) = -3;$ $\lim_{x\to 0} x^2\ \frac{x}{x} = 0$, $x_0 = 0$ is a regular singular point for the given differential equation. By the Frobenius theorem, the solution proposal is;

$y = \sum_{n=0}^{\infty} C_n\, x^{n+r}$ as in the previous examples

$$y' = \sum_{n=0}^{\infty} (n+r) C_n\, x^{n+r-1}$$

and

$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n\, x^{n+r-2}$.

Substitution of these into the differential equation yields that;

$$x \sum_{n=0}^{\infty} (n+r)(n+r-1) C_n\, x^{n+r-2} - 3\sum_{n=0}^{\infty} (n+r) C_n\, x^{n+r-1} + x \sum_{n=0}^{\infty} C_n\, x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) C_n\, x^{n+r-1} - 3\sum_{n=0}^{\infty} (n+r) C_n\, x^{n+r-1} + \sum_{n=0}^{\infty} C_n\, x^{n+r+1} = 0$$

for the last term $n \to n-2$ substitution gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_n\,\mathrm{x}^{n+r-1} - 3\sum_{n=0}^{\infty}(n+r)C_n\,\mathrm{x}^{n+r-1} + \sum_{n=2}^{\infty}C_{n-2}\,\mathrm{x}^{n+r-1} = 0$$

Since the exponents are the same, next we make the ranges the same $n: 0 \to \infty$.

$$r(r-1)C_0x^{r-1} + r(r+1)C_1x^r + \sum_{n=2}^{\infty}(n+r)(n+r-1)C_n\,\mathrm{x}^{n+r+1} - 3rC_0x^{r-1} - 3(r+1)C_1x^r$$

$$-3\sum_{n=2}^{\infty}(n+r)C_n\,\mathrm{x}^{n+r+1} + \sum_{n=2}^{\infty}C_{n-2}\,\mathrm{x}^{n+r+1} = 0$$

$$[\mathrm{r}(\mathrm{r}-1) - 3\mathrm{r}]C_0x^{r-1} + [\mathrm{r}(\mathrm{r}+1) - 3(\mathrm{r}+1)]C_1x^r$$

$$+\sum_{n=0}^{\infty}[C_n[(n+r)(n+r-1) - 3(n+r)] + C_{n-2}\,]\mathrm{x}^{n+r+1} = 0$$

Equating the coefficients of x^i to zero, one gets the indicial equation

$$\mathrm{r}(\mathrm{r}-1) - 3\mathrm{r} = 0 \implies \mathrm{r}(\mathrm{r}-4) = 0 \implies r_1 = 4, \qquad r_2 = 0$$

Considering the second coefficient

$$C_1[\mathrm{r}(\mathrm{r}+1) - 3(\mathrm{r}+1)] = 0$$

one obtains

$r_2 = 0 \Longrightarrow -3C_1 = 0$
$r_1 = 4 \Longrightarrow 5C_1 = 0$

Hence $C_1 = 0$.

Recurrence relation is;

$$C_n(n+r)((n+r-1)-3) + C_{n-2} = 0$$

which yields

$$C_n = -\frac{C_{n-2}}{(\mathrm{n}+\mathrm{r})(\mathrm{n}+\mathrm{r}-4)}, \quad n \geq 2$$

$$n = 2 \Longrightarrow \; C_2 = -\frac{C_0}{(\mathrm{r}+2)(\mathrm{r}-2)}$$

$n = 3 \Longrightarrow \; C_3 = -\frac{C_1}{(\mathrm{r}+5)(\mathrm{r}+1)} = 0$ Similarly, $C_1 = C_3 = C_5 = C_7 = \cdots = 0$

$$n = 4 \Longrightarrow \ C_4 = -\frac{C_2}{(\mathrm{r}+4)\mathrm{r}} = \frac{C_0}{(\mathrm{r}+2)(\mathrm{r}-2)(\mathrm{r}+4)\mathrm{r}}$$

Clearly, for $r = 0 \Longrightarrow C_4 = \infty$, for which the series solution fails. To avoid this, we let $C_0 = b_0\mathrm{r}$, with $b_0 = 1$. Hence

$$C_0 = b_0 r = r$$

$$C_2 = -\frac{b_0.\mathrm{r}}{(\mathrm{r}+2)(\mathrm{r}-2)} = -\frac{\mathrm{r}}{(\mathrm{r}+2)(\mathrm{r}-2)}$$

$$C_4 = \frac{b_0.\mathrm{r}}{(\mathrm{r}+2)(\mathrm{r}-2)(\mathrm{r}+4)\mathrm{r}} = \frac{1}{(\mathrm{r}+2)(\mathrm{r}-2)(\mathrm{r}+4)}$$

$$C_6 = -\frac{C_4}{(\mathrm{r}+2)(\mathrm{r}+6)} = -\frac{1}{(\mathrm{r}+2)^2(\mathrm{r}-2)(\mathrm{r}+4)(\mathrm{r}+6)} \quad \dots \dots$$

In general for even coefficients

$$C_{2k} = (-1)^k \frac{1}{(\mathrm{r}-2)\underbrace{(\mathrm{r}+2)^2(\mathrm{r}+4)^2 \dots (\mathrm{r}+2\mathrm{k}-2)}_{\text{All square}}(\mathrm{r}+2\mathrm{k})}$$

The solution proposal would then be;

$$y(x,r) = x^r \sum_{n=0}^{\infty} C_n \, \mathrm{x}^n = x^r [C_0 + C_1 x + C_2 x^2 + \cdots + C_{2k} \, \mathrm{x}^{2k} + \cdots]$$

Substitution to the differential equation yields that

$$y(x,r) = x^r \Big[r - \frac{r}{(r+2)(r-2)} x^2 + \frac{1}{(r+2)(r-2)(r+4)} x^4 - \frac{1}{(\mathrm{r}+2)^2(\mathrm{r}-2)(\mathrm{r}+4)(\mathrm{r}+6)} x^6 + \cdots + (-1)^k \frac{1}{(\mathrm{r}+2)^2(\mathrm{r}-2)(\mathrm{r}+4)^2 \dots (\mathrm{r}+2\mathrm{k}-2)(\mathrm{r}+2\mathrm{k})} x^{2k} + \cdots \Big]$$

For $r = 0$ we get

$$y_1 = y(x,r)|_{r=0} = -\frac{1}{16} x^4 + \frac{1}{16 * 12} x^6 + \cdots + (-1)^k \frac{1}{(-2)2^2 4^2 \dots (2\mathrm{k}-2)2\mathrm{k}} x^{2k} + \cdots$$

Since if $r = 4$ some terms become indefinite

$$y_2 = \left.\frac{\partial y(x,r)}{\partial r}\right|_{r=0} = x^r \ln|x| \, y_1 + x^r \left.\frac{\partial y(x,r)}{\partial r}\right|_{r=0}$$

so $\quad y_{gen} = C_1 y_1 + C_2 y_2$

Example3.5

Find series solution of the differential equation

$6x^2y'' + 7xy' - (1 + x^2)y = 0$

about $x_0 = 0$.

The normalized equation is

$$y'' + \frac{7x}{6x^2}y' + \frac{-(1 + x^2)}{6x^2}y = 0$$

since $\frac{1}{x^2}$ is not analytic at $x_0 = 0$, but

$$p(x) = \frac{7x}{6x^2} \Longrightarrow \lim_{x\to 0} x\left(\frac{7x}{6x^2}\right) = \frac{7}{6},$$

$$q(x) = \frac{-(1 + x^2)}{6x^2} \Longrightarrow \lim_{x\to 0} x^2\left(\frac{-(1 + x^2)}{6x^2}\right) = -\frac{1}{6}$$

Hence $x_0 = 0$ is a regular singular point for the differential equation. From Frobenius theorem the solution proposal will then be;

$$y = \sum_{n=0}^{\infty} C_n\, x^{n+r}$$

similar to the previous examples

$$y' = \sum_{n=0}^{\infty} (n + r)C_n\, x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)C_n\, x^{n+r-2}$$

Substitution of these into the differential equation yields that;

$$6x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)C_nx^{n+r-2}+7x\sum_{n=0}^{\infty}(n+r)C_nx^{n+r-1}-(1+x^2)\sum_{n=0}^{\infty}C_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)C_nx^{n+r}+7\sum_{n=0}^{\infty}(n+r)C_nx^{n+r}-\sum_{n=0}^{\infty}C_nx^{n+r}-\sum_{n=2}^{\infty}C_{n-2}x^{n+r}=0$$

Hence

$$6(r)(r-1)C_0x^r+7(r)C_0x^r-C_0x^r+6(1+r)(r)C_1x^r+7(1+r)C_1x^{1+r}$$
$$-C_1x^{1+r}-\sum_{n=2}^{\infty}(6(n+r)(n+r-1)C_n+7(n+r)C_n-C_n-C_{n-2})x^{n+r}=0$$

$$(6(r)(r-1)+7r-1)C_0x^r+(6(1+r)(r)+7(1+r)-1)C_1x^r$$
$$-\sum_{n=2}^{\infty}((6(n+r)(n+r-1)+7(n+r)-1)C_n-C_{n-2})x^{n+r}=0$$

$$(6r^2+r-1)C_0x^r+(6r^2+13r+6)C_1x^r$$
$$-\sum_{n=2}^{\infty}[(6(n+r)(n+r-1)+7(n+r)-1)C_n-C_{n-2}]x^{n+r}=0$$

which yields

$$(6r^2+r-1)C_0x^r=0 \quad\Longrightarrow\quad r_1=\frac{1}{3},\qquad r_2=-\frac{1}{2},\quad r_1-r_2\neq N$$
$$(6r^2+13r+6)C_1x^r=0 \quad\Longrightarrow\quad r_3=-\frac{2}{3},\quad r_4=-\frac{3}{2},\quad r_3-r_4\neq N$$

For r_1, r_2 there are two independent solutions.

$$[6(n+r)(n+r-1)+7(n+r)-1]C_n-C_{n-2}=0$$

Hence the recurrence relation is;

$$C_n=\frac{C_{n-2}}{6(n+r)(n+r-1)+7(n+r)-1}$$

for $r_1={}^1\!/_3$

$$n=2\Longrightarrow C_2=\frac{C_0}{\frac{56}{3}+\frac{49}{3}-1}=\frac{C_0}{34}$$

$$n=4\Longrightarrow C_4=\frac{C_0}{\frac{260}{3}+\frac{91}{3}-1}=\frac{C_2}{116}=\frac{C_0}{3944}$$

$$y_1 = C_0 x^{1/3}\left(1 + \frac{x^2}{34} + \frac{x^4}{3944} + \cdots\right)$$

for $r_2 = -{}^1\!/_2$

$$n = 2 \Longrightarrow C_2 = \frac{C_0}{\frac{9}{2} + \frac{21}{2} - 1} = \frac{C_0}{14}$$

$$n = 2 \Longrightarrow C_4 = \frac{C_0}{\frac{105}{2} + \frac{49}{2} - 1} = \frac{C_2}{76} = \frac{C_0}{1064}$$

$$y_2 = C_0 x^{-1/2}\left(1 + \frac{x^2}{14} + \frac{x^4}{1064} + \cdots\right)$$

Assume $C_0 = 1$

$$y_{gen} = C_1 y_1 + C_2 y_2$$

$$y_{gen} = C_1 x^{1/3}\left(1 + \frac{x^2}{34} + \frac{x^4}{3944} + \cdots\right) + C_2 x^{-1/2}\left(1 + \frac{x^2}{14} + \frac{x^4}{1064} + \cdots\right)$$

■

Solution Around an Irregular Singular Point

If the series solution is required about an irregular singular point x_0; then the change of variable $z = \frac{1}{x}$ may sometimes make the solution possible about a regular singular point $z = z_0$. More precisely;

$$z = \frac{1}{x} \implies x = \frac{1}{z} \implies \frac{dz}{dx} = -\frac{1}{x^2} = -z^2$$

$$y' = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = -z^2\frac{dy}{dz}$$

$$y'' = \frac{d}{dx}y' = \frac{d(y')}{dz}\frac{dz}{dx} = -z^2\frac{d}{dz}\left[-z^2\frac{dy}{dz}\right]$$

$$y'' = -z^2\left[-2z\frac{dy}{dz} - z^2\frac{d^2y}{dz^2}\right] = 2z^3\frac{dy}{dz} + z^4\frac{d^2y}{dz^2}$$

Substitution of y' and y'' into differential equation changes independent variable from x to z. Irregular singular point x_0 is then replaced by a regular singular point $z_0 = \frac{1}{x_0}$; and the new differential equation is solved as before. Finally, the solution in z is translated back to x by changing the variable $z = \frac{1}{x}$.

4. SPECIAL DIFFERENTIAL EQUATIONS AND FUNCTIONS

In modeling problems in engineering and physics, several special functions are developed which are solutions of certain differential equations in power series. Bessel and Legendre differential equations are the two most common of them that are discussed in this section.

4.1 Bessel Differential Equations and Bessel Functions

The family of Bessel differential equations

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \qquad p \in \mathbb{R}$$

appears in many problems in engineering and physics. The simplest of them is Bessel differential equation of order zero.

Bessel Differential Equation of Order $p = 0$

Let $p = 0$ in the family of Bessel differential equations in the above. Then one has

$$xy'' + y' + xy = 0$$

which is known as the zeroth order Bessel differential equation.

Let's normalize this differential equation;

$$y'' + \frac{1}{x}y' + y = 0$$

Since $\frac{1}{x}$ is not analytic at $x_0 = 0$, and

$$\lim_{x\to 0} x\left(\frac{1}{x}\right) = 1\,; \quad \lim_{x\to 0} x^2\,(1) = 0\,,$$

$x_0 = 0$ is a regular singular point for the differential equation. According to the Frobenius Theorem , the appropriate solution proposal is,

$$y = \sum_{n=0}^{\infty} C_n x^{r+n}$$

which implies

$$y' = \sum_{n=0}^{\infty} C_n \,(n+r) x^{r+n-1}$$

and

$$y'' = \sum_{n=0}^{\infty} C_n \,(n+r)(n+r-1) x^{r+n-2}$$

Substitution of them into the equation yields that;

$$x \sum_{n=0}^{\infty} C_n \,(n+r)(n+r-1) x^{r+n-2} + \sum_{n=0}^{\infty} C_n \,(n+r) x^{r+n-1} + x \sum_{n=0}^{\infty} C_n x^{r+n} = 0 \;\rightarrow$$

$$\sum_{n=0}^{\infty} C_n \,(n+r)(r+n-1) x^{r+n-1} + \sum_{n=0}^{\infty} C_n \,(n+r) x^{r+n-1} + \sum_{n=0}^{\infty} C_n x^{r+n+1} = 0 \;\rightarrow$$

$$\sum_{n=0}^{\infty} C_n \,(n+r)^2 x^{r+n-1} + \sum_{n=0}^{\infty} C_n x^{r+n+1} = 0$$

$n \rightarrow n-2$ substitution in the last term yields

$$\sum_{n=0}^{\infty} C_n \,(n+r)^2 x^{r+n-1} + \sum_{n=2}^{\infty} C_{n-2} \, x^{r+n-1} = 0 \;\rightarrow$$

$$C_0 r^2 x^{r-1} + C_1 (r+1)^2 x^r + \sum_{n=2}^{\infty} [C_n (r+n)^2 + C_{n-2}] \, x^{r+n-1} = 0$$

where the indicial equation with the two roots is

$r^2 = 0 \;\Longrightarrow\; r_{1,2} = 0,\; C_0 \neq 0,$

The condition equation

$$C_1 (r+1)^2 = 0$$

Implies $C_1 = 0$.

The recurrence relation; $\quad C_n (n+r)^2 + C_{n-2} = 0$

Leads to

$C_n = -\frac{C_{n-2}}{(r+n)^2}$, n ≥ 2.

Therefore

$$n = 2 \Longrightarrow \quad C_2 = -\frac{C_0}{(r+2)^2}$$

$$n = 3 \Longrightarrow \quad C_3 = \frac{C_1}{(r+3)^2} = 0 \Longrightarrow \; C_{2n+1} = 0, \qquad n = 1,2,3 \ldots$$

while

$$n = 4 \Longrightarrow \quad C_4 = -\frac{C_2}{(r+4)^2} = \frac{C_o}{(r+2)^2(r+4)^2}$$

$$n = 6 \Longrightarrow \quad C_6 = -\frac{C_4}{(r+6)^2} = -\frac{C_o}{(r+2)^2(r+4)^2(r+6)^2}$$

$$= -\frac{C_o\Gamma(r/2)^2}{2^6\Gamma(r/2)^2(r/2+1)^2(r/2+2)^2(r/2+3)^2} == -\frac{C_o\Gamma(r/2)^2}{2^6\left(\Gamma\left(\frac{r}{2}+3\right)\right)^2}$$

$$C_{2n} = (-1)^n \, \frac{C_o}{(r+2)^2(r+4)^2 \ldots \, (r+2n)^2} = (-1)^n \, \frac{C_o(\Gamma(r/2))^2}{2^{2n}\left(\Gamma\left(\frac{r}{2}+n\right)\right)^2}$$

Where

$$\Gamma(r/2+n) = \Gamma(r/2)\left(\frac{r}{2}+1\right)\left(\frac{r}{2}+2\right) \ldots (r/2+n)$$

is the Gamma function (see section 4.2). Further substitution of these yields that;

$$y(x,r) = x^r C_0 \sum_{n=0}^{\infty}(-1)^n \frac{C_o(\Gamma(r/2))^2}{2^{2n}\left(\Gamma\left(\frac{r}{2}+n\right)\right)^2}$$

Let $C_0 = 1$. Then recalling $\Gamma(0) = 1$, and $\Gamma(n) = n!$ one has

$$y_1 = y(x,r)|_{r=0} = \sum_{n=0}^{\infty}(-1)^n \, \frac{x^{2n}}{(n!)^2 \, 2^{2n}} = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 \, 6^2} + \cdots + \; (-1)^n \, \frac{x^{2n}}{(n!)^2 \, 2^{2n}} + \cdots$$

So, one of the two solutions is;

$$y_1 = \sum_{n=0}^{\infty}(-1)^n \, \frac{x^{2n}}{(n!)^2 \, 2^{2n}}$$

This special series is known as Bessel function of first kind (standing 1st solution $y|_{r=0}$) and order zero ($p = 0$), and denoted by J_0; so, the first of the two solutions is found to be

$$y_1 = J_0(x)$$

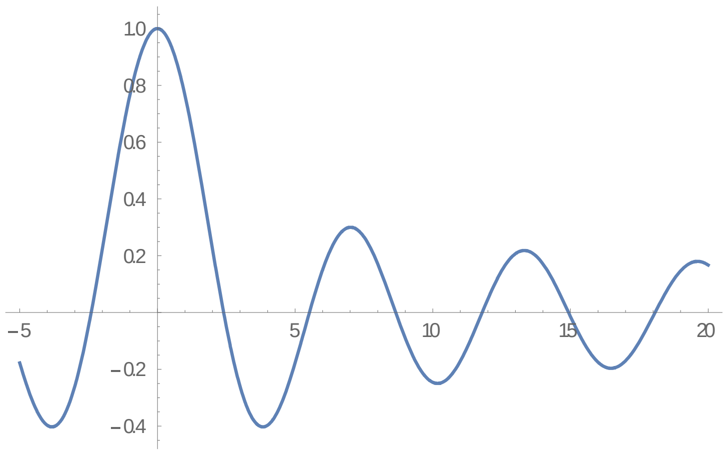

Figure 4.1. The plot of $J_0(x)$. It is seen that the function is symmetrical w.r.t. the y-axis, and diminishes as x gets larger.

Since the indicial equation has a double root $r_{1,2} = 0$, the other independent solution is found through the differentiation of $y(x,r)$ with respect to r,

To simplify the computation let us recall

$$y(x,r) = x^r C_0 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(r+2)^2 (r+4)^2 \ldots (r+2n)^2} x^{2n}$$

Let

$$F(r) = (r+2)(r+4)\ldots(r+2n)$$

Then for $C_0 = 1$,

$$y(x,r) = x^r \sum_{n=0}^{\infty} (-1)^n F(r)^{-2} x^{2n}$$

Then one has

$$y_2 = \left.\frac{\partial y(x,r)}{\partial r}\right|_{r=0} = \left[x^r \ln|x|\, y_1 - x^r \sum_{n=1}^{\infty} (-1)^n \frac{\partial}{\partial r} F(r)^{-2} x^{2n}\right]_{r=0}$$

$$= J_0(x)\ln|x| + 2\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{\big(\mathrm{F}(r+2n)\big)^3}\left(\frac{1}{r+2}\mathrm{F}(r+2n) + \cdots + \frac{1}{r+2n}\mathrm{F}(r+2n)\right)\Bigg|_{r=0}$$

$$= J_0(x)\ln|x| - 2\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{\big(\mathrm{F}(r+2n)\big)^2}\left(\frac{1}{r+2} + \cdots + \frac{1}{r+2n}\right)\Bigg|_{r=0}$$

$$= J_0(x)\ln|x| - 2\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^{2n}(n!)^2}\left(\frac{1}{2} + \cdots + \frac{1}{2n}\right)$$

Hence

$$y_2 = J_0(x)\ln|x| + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{2^{2n}(n!)^2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

This solution y_2 is traditionally given in Weber type of expression

$$y_2 = \frac{2}{\pi}\left[\left(\ln\left|\frac{x}{2}\right| + \gamma\right)J_0(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{2^{2n}(n!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)\right]$$

Where γ is the Euler constant

$$\gamma = \lim_{n\to 0}\left[\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \ln n\right] \cong 0.5772.$$

This solution with a slight modification, is known as ***Bessel function of the 2nd kind and order zero*** and, denoted by $Y_0(x)$,

$$Y_0(x) = \frac{2}{\pi}\left[\left(\ln\left|\frac{x}{2}\right| + \gamma\right)J_0(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^{2n}}{2^{2n}(n!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)\right]$$

Then the second of the two linearly independent solutions is

$$y_2 = Y_0(x)$$

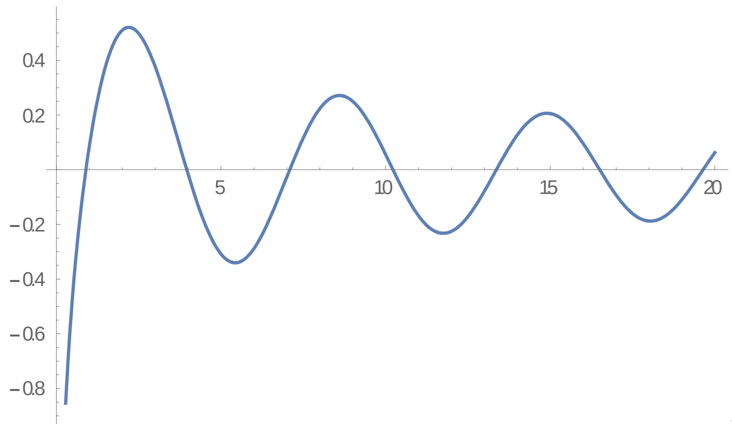

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Figure 4.2. The plot of $Y_0(x)$. It is seen that the function is defined only for positive values of x because of the logarithmic term in its definition, and diminishes as x gets larger.

The general solution of the Bessel differential equation of order zero is then obtained as;

$$y_{gen} = C_1 J_0(x) + C_2 Y_0(x)$$

4. SPECIAL DIFFERENTIAL EQUATIONS AND FUNCTIONS

Bessel Differential Equations of Order $p \neq 0$:

In a more general context, when$p \neq 0$, one has

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \;\; p \in \mathbb{R}.$$

The normalized differential equation is now

$$y'' + \frac{1}{x}y' + \frac{x^2 - p^2}{x^2}y = 0$$

Since

$$\frac{1}{x}, \text{and } \frac{x^2 - p^2}{x^2}$$

are not analytic at $x_0 = 0$, and

$$\lim_{x \to 0} x\left(\frac{1}{x}\right) = 1\,, \quad \lim_{x \to 0} x^2\left(\frac{x^2 - p^2}{x^2}\right) = -p^2\,,$$

are analytic every where, $x_0 = 0$ is a regular singular point for the differential equation. According to the Frobenius Theorem II, the appropriate solution proposal is,

$$y = \sum_{n=0}^{\infty} C_n x^{n+r}$$

which implies

$$y' = \sum_{n=0}^{\infty} C_n\,(n+r)x^{n+r-1}, \;\text{ and }\; y'' = \sum_{n=0}^{\infty} C_n\,(n+r)(n+r-1)x^{n+r-2}$$

Substitution of these into the differential equation

$$x^2 \sum_{n=0}^{\infty} C_n\,(n+r)(n+r-1)x^{n+r-2}$$
$$+x \sum_{n=0}^{\infty} C_n\,(n+r)x^{n+r-1} + x^2 \sum_{n=0}^{\infty} C_n x^{n+r} - p^2 \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} C_n\,(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} C_n\,(n+r)x^{n+r} + \sum_{n=0}^{\infty} C_n x^{n+r+2} - p^2 \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

in the third term, shifting $n \to n-2$ one has e has

$$\sum_{n=0}^{\infty} C_n\,[(n+r)^2 - p^2]x^{r+n} + \sum_{n=2}^{\infty} C_{n-2}\,x^{r+n} = 0$$

which gives

$$C_0(r^2 - p^2)x^r + C_1[(r+1)^2 - p^2]x^{r+1} + \sum_{n=2}^{\infty}\{[(n+r)^2 - p^2]c_n + c_{n-2}\}\,x^{n+r} = 0$$

Equating the coefficients of x^i to zero one gets the indicial equation with two roots

$$r^2 - p^2 = 0\,, \qquad C_0 \neq 0 \Longrightarrow r_1 = p > 0,\;\; r_2 = -p$$

The condition equation

$C_1[(r+1)^2 - p^2] = 0$

Further implies $C_1 = 0$, unless $r \neq -1+p, -1-p,$

Since we have

$$C_0 \neq 0 \Longrightarrow r_1 = p > 0,\;\; r_2 = -p$$

$C_0 \neq 0,\; C_1 = 0$, unless $p \neq 1/2$.

The recurrence relation is;

$$[(n+r)^2 - p^2]C_n + C_{n-2} = 0 \;\Longrightarrow\; C_n = -\frac{C_{n-2}}{(n+r)^2 - p^2}\,, n \geq 2.$$

The two cases are distinguished

1. $r_1 - r_2 = 2p$ is not a positive integer, then two roots lead to two independent solutions

2. $r_1 - r_2 = 2p$ is a positive integer, then the second solution is going to be found by a special proposal.

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Bessel Functions of Order p; $2p \notin Z^+$

Let $2p$ is not positive integer. And $r = p$ then;

$$C_n = \frac{-C_{n-2}}{(n+p)^2 - p^2} = \frac{-C_{n-2}}{n^2 + 2np} = -\frac{C_{n-2}}{n(n+2p)}\ ; n \geq 2$$

$$\text{n} = 2 \Longrightarrow \quad C_2 = \frac{-C_0}{2(2+2p)} = \frac{-C_0}{2^2.1.(1+p)}$$

$$n = 3 \Longrightarrow \quad C_3 = \frac{-C_1}{3(3+2p)} = 0, C_5 = 0, \dots, C_{2n+1} = 0,\ \ \text{n} \geq 1.$$

$$n = 4 \Longrightarrow \quad C_4 = \frac{-C_2}{4(4+2p)} = \frac{C_0}{[2.4].[(2+2p)(4+2p)]} = \frac{C_0}{2^4(1.2)[(1+p)(2+p)]}$$

$$n = 6 \Longrightarrow \quad C_6 = \frac{-C_4}{6(6+2p)} = -\frac{C_0}{[2.4.6][(2+2p)(4+2p)(6+2p)]}$$

$$= -\frac{C_0}{2^6(1.2.3)[(1+p)(2+p)(3+p)]}$$

$$C_{2n} = \frac{(-1)^n C_0}{[2.4.6 \dots .(2n)][(1+p)(2+p)\dots(n+p)]}\ ;\ \ \text{n} \geq 1$$

or

$$C_{2n} = \frac{(-1)^n C_0}{2^{2n}(n!)[(1+p)(2+p)\dots(n+p)]};\ \ \text{n} \geq 1$$

$$y_1 = y(x,r)|_{r=p} = C_0 x^r + C_1 x^{r+1} + C_2 x^{r+2} + C_3 x^{r+3} + \cdots |_{r=p}$$

$$y_1 = \quad C_0 x^p - \frac{C_0}{2^2.1.(1+p)} x^{p+2} + \frac{C_0}{2^4(1.2)[(1+p)(2+p)]} x^{p+4} + \cdots + \frac{(-1)^n C_0 x^{p+2n}}{2^{2n}(n!)[(1+p)(2+p)\dots(n+p)]}$$

so,

$$y_1 = C_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p}}{2^{2n}(n!)[(1+p)(2+p)\dots(n+p)]}$$

After some manipulations, for $C_0 = 1$,

$$y_1 = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+p} \Gamma(p)}{2^{2n} n!\,[\Gamma(p)(1+p)(2+p)\ldots(n+p)]} = \Gamma(p) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,2^{2n}\Gamma(n+p)} x^{2n+p}$$

Where Γ in

$$\Gamma(n+p) = \Gamma(p)(1+p)(2+p)\ldots(n+p)$$

is the Γ function (See Section 4.2). This expression is, obviously, reduced to J_0 for p=0.

y_1 is called Bessel function of 1st kind and order p, and denoted by $J_p(x)$

$$J_p(x) = \Gamma(p) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,2^{2n}\Gamma(n+p)} x^{2n+p}$$

and the second linearly independent solutions is obviously,

$y_2 = J_{-p}(x)$.

For example, for p=1/4;

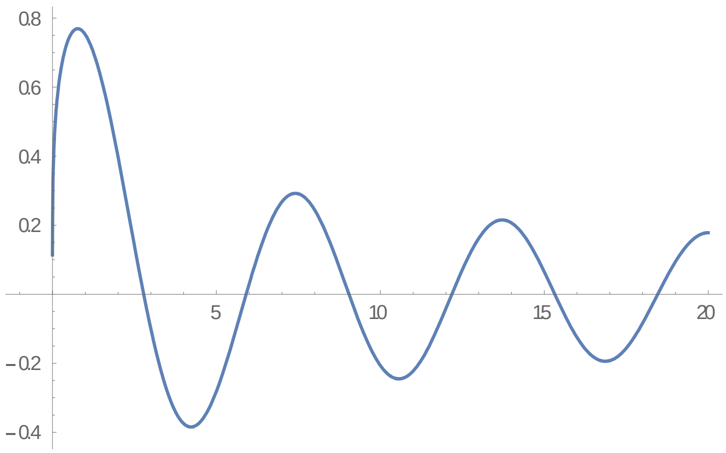
$$y_1 = \Gamma(1/4) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,2^{2n}\Gamma(n+1/4)} x^{2n+1/4}$$

$$y_2 = \Gamma(-1/4) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,2^{2n}\Gamma(n-1/4)} x^{2n-1/4}$$

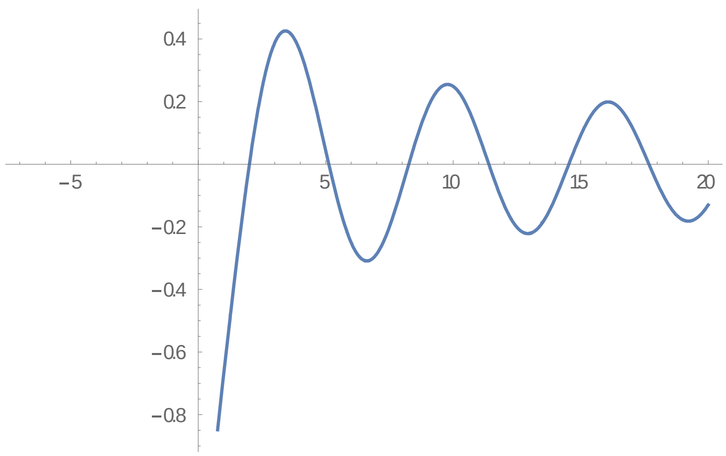
After some scaling

$y_1 = J_{1/4}(x)$

$y_2 = J_{-1/4}(x)$



(a)



(b)

Figure 4.3. The plot of (a) $y_1 = J_{1/4}(x)$, and (b) $y_2 = J_{-1/4}(x)$

<u>Bessel Functions of Order p</u>; $p \in Z^+$

In the family of Bessel differential equations

$$x^2 y'' + xy' + (x^2 - p^2)y = 0, \qquad \mathrm{p} \in \mathbb{R}$$

Equating the coefficients of x^i to zero one gets the indicial equation with two roots

$$r^2 - p^2 = 0 , \qquad C_0 \neq 0 \Longrightarrow r_1 = p > 0, \;\; r_2 = -p$$

The condition equation

$C_1[(r+1)^2 - p^2] = 0$

Further implies $C_1 = 0$, unless

$$(r+1)^2 - p^2 = 0$$

For $r = p > 0$

$$(r+1)^2 - p^2 = 2p + 1 \neq 0$$

For $r = -p$

$$(r+1)^2 - p^2 = -2p + 1 = 0$$

$$C_0 \neq 0 \Longrightarrow r_1 = p > 0, \;\; r_2 = -p$$

$C_0 \neq 0,$ arbitrary, $C_1 = 0$, unless $\qquad p \neq 1/2.$

The recurrence relation is;

$$[(n+r)^2 - p^2]C_n + C_{n-2} = 0 \Longrightarrow C_n = -\frac{C_{n-2}}{(n+r)^2 - p^2} \text{ , n} \geq 2.$$

<u>For $r = p$</u>

ıhe first solution is found as before

$$y_1 = \Gamma(p) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 2^{2n} \Gamma(n+p)} x^{2n+p}$$

<u>**For $\boldsymbol{r = -p}$**</u>

The recurrence relation is

$[(n+r)^2 - p^2]C_n + C_{n-2} = 0 \Longrightarrow (n^2 - 2np)C_n + C_{n-2} = 0$

$\Longrightarrow n(n-2p)C_n + C_{n-2} = 0 \Longrightarrow n(n-m)C_n + C_{n-2} = 0$,

$m = 2p,$ even integer

$C_n = \dfrac{-C_{n-2}}{(n-p)^2 - p^2} = \dfrac{-C_{n-2}}{n^2 - 2np} = -\dfrac{C_{n-2}}{n(n-m)}$; $n < m$, n even

C_0 is arbitrary

when $2p = m$, an even integer.

$$\Longrightarrow n(n-2p)C_n + C_{n-2} = 0 \Longrightarrow n(n-m)C_n + C_{n-2} = 0$$

C_m is arbitrary

$C_{m+2} = -\dfrac{C_m}{2(m+2)}$; $n \geq m+2$

$y_3 = x^{-p}(C_0 + C_2x^2 + C_4x^4 + \cdots + C_{m-2}x^{m-2})$

$y_4 = x^{-p}(C_m x^m + \cdots)$

n $= 2 \Longrightarrow \quad C_2 = \dfrac{-C_0}{2(2-m)} = \dfrac{-C_0}{2^2.1.(1+p)}$

$n = 3 \Longrightarrow \quad C_3 = \frac{-C_1}{3(3+2p)} = 0, C_5 = 0, \ldots, C_{2n+1} = 0,$ n $\geq 1.$

$n = 4 \Longrightarrow \quad C_4 = \dfrac{-C_2}{4(4-m)} = \dfrac{C_0}{[2.4].[(2-m)(4-m)]}$

… … ….

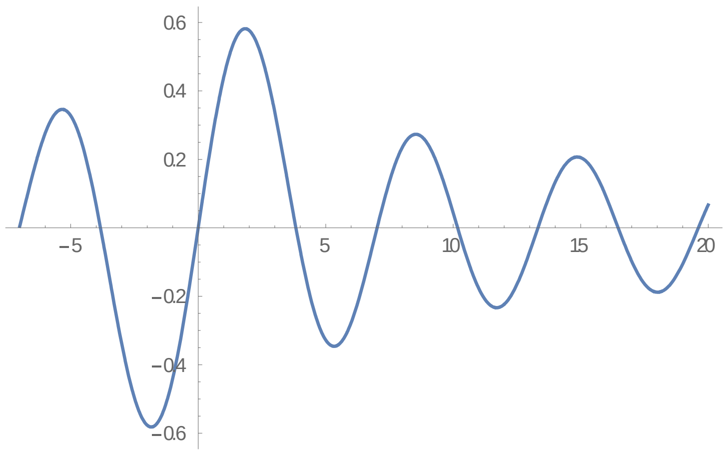
n ≥ m

$C_{m+2} = -\dfrac{C_m}{2(m+2)}$; $n \geq m+2$

$y_3 = x^{-p}(C_0 + C_2x^2 + C_4x^4 + \cdots + C_{m-2}x^{m-2})$

$y_4 = x^{-p}(C_m x^m + \cdots)$

y_3 is the second linear independent solution of the Bessel equation, $y_4 = \alpha y_1$ for some α.



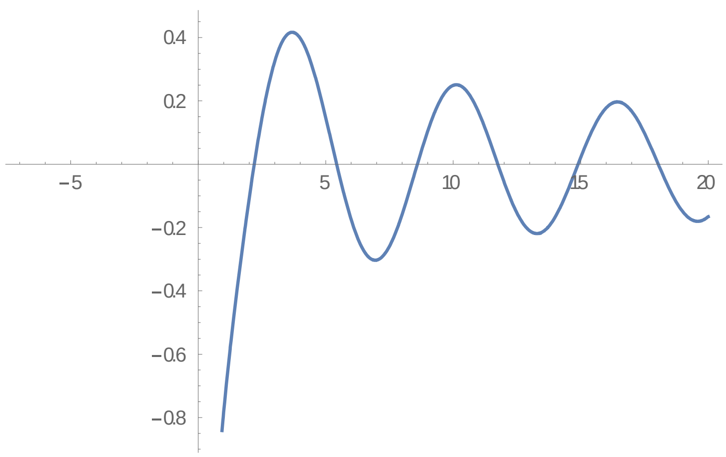


Figure 4.4. The plot of (a) $y_1 = J_1(x)$, and (b) $y_3 = Y_1(x)$

<u>Bessel Functions of Order p</u>; $2p \in Z^+, m = 2\text{p odd}$

In the family of Bessel differential equations

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \qquad \text{p} \in \mathbb{R}$$

Equating the coefficients of x^i to zero one gets the indicial equation with two roots

$$r^2 - p^2 = 0\,, \qquad C_0 \neq 0 \Longrightarrow r_1 = p > 0,\;\; r_2 = -p$$

The condition equation

$C_1[(r+1)^2 - p^2] = 0$

Further implies $C_1 = 0$, unless

$$(r+1)^2 - p^2 = 0$$

For $r = p > 0$

$$(r+1)^2 - p^2 = 2p + 1 \neq 0$$

For $r = -p$

$$(r+1)^2 - p^2 = -2p + 1 = 0$$

$$C_0 \neq 0 \Longrightarrow r_1 = p > 0, \;\; r_2 = -p$$

$C_0 \neq 0$, arbitrary, $C_1 = 0$, unless $\quad p \neq 1/2$.

The recurrence relation is;

$$[(n+r)^2 - p^2]C_n + C_{n-2} = 0 \implies C_n = -\frac{C_{n-2}}{(n+r)^2 - p^2}, \text{n} \geq 2.$$

<u>For $r = p$</u>

ıhe first solution is found as before

$$y_1 = \Gamma(p) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 2^{2n} \Gamma(n+p)} x^{2n+p}$$

<u>**For $r = -p$**</u>

The recurrence relation is

$[(n+r)^2 - p^2]C_n + C_{n-2} = 0 \implies (n^2 - 2np)C_n + C_{n-2} = 0$

$\implies n(n-2p)C_n + C_{n-2} = 0 \implies n(n-m)C_n + C_{n-2} = 0$,

$m = 2p$, odd integer

$$C_n = \frac{-C_{n-2}}{(n-p)^2 - p^2} = \frac{-C_{n-2}}{n^2 - 2np} = -\frac{C_{n-2}}{n(n-m)}\ ; n > 2$$

C_0 is arbitrary

$$\text{n} = 2 \implies \quad C_2 = \frac{-C_0}{2(2-m)} = \frac{-C_0}{2^2.1.(1+p)}$$

$$n = 4 \implies C_4 = \frac{-C_2}{4(4-m)} = \frac{C_0}{[2.4].[(2-m)(4-m)]}$$

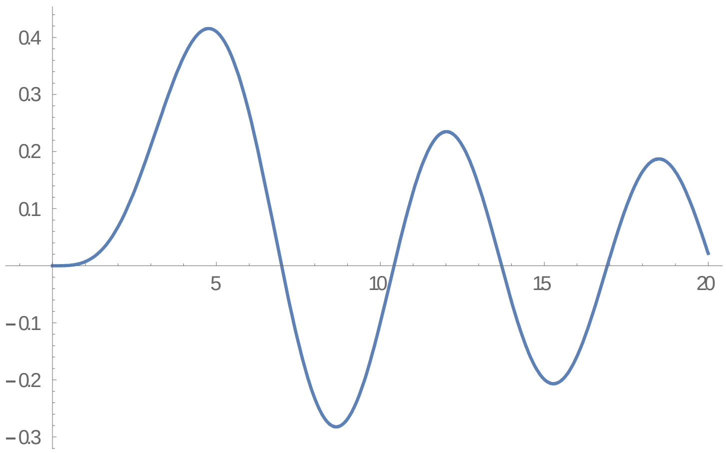
$$y_3 = \Gamma(-p) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, 2^{2n} \Gamma(n-p)} x^{2n-p}$$

$$C_1 = 0$$

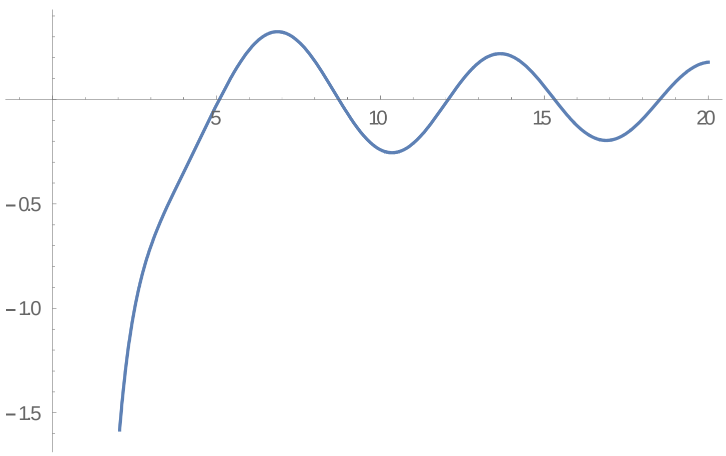
$$n = 3 \implies C_3 = \frac{-C_1}{3(3+2p)} = 0, C_5 = 0, \dots, C_{2n+1} = 0, \text{ n} \geq 1.$$

Therefore the equation $n(n-m)C_n + C_{n-2} = 0$ does not bring qny restriction since $C_{2n+1} = 0$, $\text{n} \geq 1$
$y_4 = 0$

y_3 is the second linear independent solution of the Bessel equation.



(a)



(b)

Figure4.5. The graph of (a) $y_1 = J_{7/2}(x)$, and (b) $y_3 = Y_{7/2}(x) y_3 = J_{-7/2}(x)$

Bessel Differential Equations of Order $p = 1/2$:

The characteristic equation

$$C_n = \frac{-C_{n-2}}{(n+p)^2 - p^2} = \frac{-C_{n-2}}{n^2+n} = -\frac{C_{n-2}}{n(n+1)} \; ; n \geq 2$$

$$\text{n} = 2 \Longrightarrow \quad C_2 = \frac{-C_0}{2.3}$$

$$n = 3 \Longrightarrow \quad C_3 = \frac{-C_1}{3.4} = 0$$

$$n = 4 \Longrightarrow \quad C_4 = \frac{-C_2}{4.5} = \frac{C_0}{5!}$$

$$n = 5 \Longrightarrow \quad C_5 = \; 0$$

$$n = 6 \Longrightarrow \quad C_6 = \frac{-C_4}{6.7} = -\frac{C_0}{7!}$$

$$C_{2n} = \frac{(-1)^n C_0}{(2n+1)!} \; ; \; \text{n} \geq 1$$

and

$$C_{2n-1} = 0; \;\; \text{n} \geq 2$$

$$y = C_0 x^{1/2}\left(1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \cdots\right)$$

For $C_0 = 1$ the first solution is

$$y_1 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

<u>The Case r$= -p < 0$:</u>

For any value $p \neq 1/2$ the second solution of the Bessel equation of order p is straightforward. For $p = 1/2$ the solution corresponding to $r = -p = -1/2$ has a difference. In this case the condition equation

$$C_1[(r+1)^2 - p^2] = 0$$

vanishes for $p = -1/2$, for all $C_1 \in R$. then one has $C_0 \neq 0, C_1 \neq 0$ are arbitrary. Hence

$$C_n = \frac{-C_{n-2}}{(n+p)^2 - p^2} = \frac{-C_{n-2}}{n^2 - n} = -\frac{C_{n-2}}{n(n-1)} \; ; n \geq 2$$

$$\text{n} = 2 \Longrightarrow \quad C_2 = \frac{-C_0}{1.2}$$

$$n = 3 \Longrightarrow \quad C_3 = \frac{-C_1}{2.3.}$$

$$\text{n} = 4 \Longrightarrow \quad C_4 = \frac{-C_2}{3.4} = \frac{C_0}{4!}$$

$$n = 5 \Longrightarrow \quad C_5 = \frac{-C_3}{4.5} = \frac{C_1}{5!}$$

....

Summing up

$$y = C_0 x^{-1/2}\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) + C_1 x^{-1/2}\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots\right)$$

$$y = C_0 x^{-1/2}\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots\right) + C_1 x^{1/2}\left(1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \cdots\right)$$

In general

$$C_{2n} = \frac{(-1)^n C_0}{(2n)!} \; ; \; \text{n} \geq 1$$

and

$$C_{2n+1} = \frac{(-1)^n C_1}{(2n+1)!} ; \; \text{n} \geq 1$$

Therefore the general solution is

$$y(x) = C_0 x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + C_1 x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

For $C_0 = 1, C_1 = 0$ the first solution is

$$y_1 = x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

For $C_1 = 0, C_1 = 1$ the second solution is

$$y_2 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

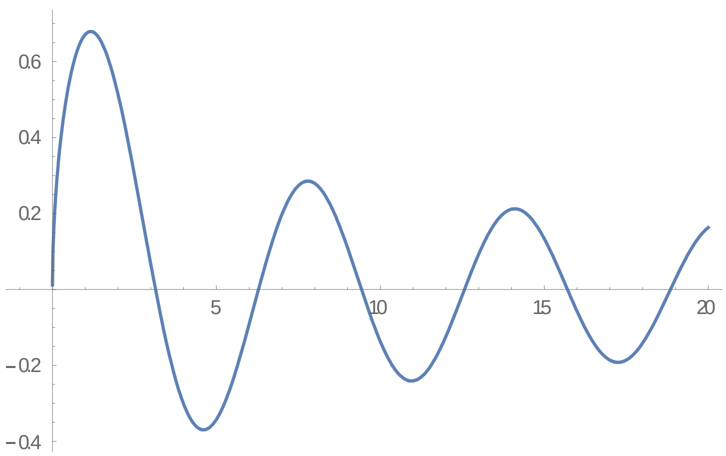
But this solution is same as the one of the case $r = p = 1/2$. Therefore for $p = 1/2$ the linearly independent solutions for $r = \frac{1}{2}, r = -\frac{1}{2}$ are

$$y_1 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

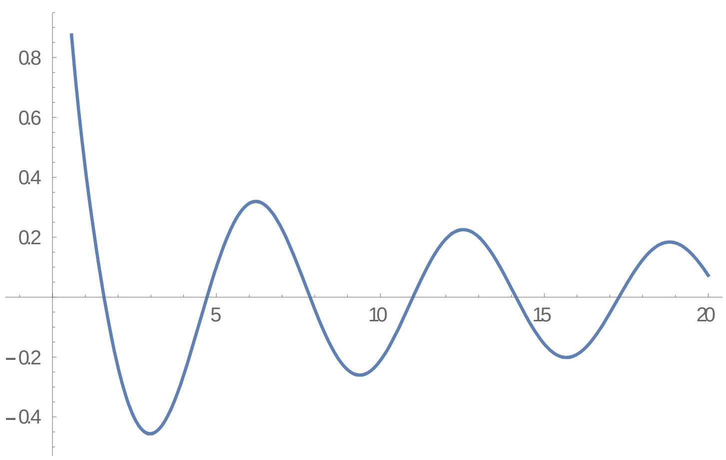
$$y_2 = x^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

and general solution is

$$y_g = C_1 y_1 + C_1 y_2$$



(a)



(b)

Figure 4.6. The graph of (a) $y_1 = J_{1/2}(x)$, and (b) $y_3 = -Y_{1/2}(x) y_3 = J_{-1/2}(x)$

Let $r = -p$ then, substitution of this $J_p(x)$;

$$p!\,2^{-p} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n-p)!} \left(\frac{x}{2}\right)^{2n-p}$$

Since, $(-p)!$ is undefined and 2^{-p} is a multiplication of the Σ function, they will be excluded and $J_{-p}(x)$ is defined as;

$$J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n-p)!} \left(\frac{x}{2}\right)^{2n-p}$$

Since, $J_p(x)$ and $J_{-p}(x)$ are independent of each other, then the general solution for the case that p is not positive integer;

$$y_{gen} = C_1 J_p(x) + C_2 J_{-p}(x)$$

<u>The case that 'p is positive integer'</u>

$$y_1 = y(x,r)|_{r=p}$$

y_1 is obtained as

$$J_p(x) = \int_n^{\infty} \frac{(-1)^n}{n!\,(n+p)!} \left(\frac{x}{2}\right)^{2n+p}$$

for $r = -p$, y_1 is simply obtained by a substitution $-p \xrightarrow{for} +p$ for the other solution;

$$y_1 = y(x,r)|_{r=-p} = J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n-p)!} \left(\frac{x}{2}\right)^{2n-p}$$

But it can be shown that this is linearly dependent to the former. Another independent solution y_2 is obtained as

$$y_2 = \left.\frac{\partial y(x,r)}{\partial r}\right|_{r=p}$$ as stated before.

Having jumped over cumbersome manipulations, the result is below,

$$y_2 = Y_{-p}(x) = \frac{2}{\pi}\left\{\left(\ln\left|\frac{x}{2}\right| + \gamma\right)J_p(x) - \frac{1}{2}\sum_0^{\infty}\frac{(p-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n-p}\right.$$

$$\left. + \frac{1}{2}\sum_0^{\infty}(-1)^{n+1}\left(\sum_{k=1}^{\mathrm{n}}\frac{1}{k} + \sum_{k=1}^{\mathrm{n+p}}\frac{1}{k}\right)\left[\frac{1}{n!\,(n+p)!}\left(\frac{x}{2}\right)^{2n+p}\right]\right\}$$

So the general solution;

$y_{gen} = C_1 J_{-p}(x) + C_2 Y_{-p}(x)$

■

$-\int x^{-1} J_2 = x^{-1} J_1$

<u>Some Properties of the Bessel Functions</u>

$x^{0} J_1 = \int x^1 J_0 \, dx)$

The derivative of $J_v(x)$ with respect to x can be expressed by $J_{v-1}(x)$ and $J_{v+1}(x)$ such as;

i) $[x^v J_v(x)]' = x^v J_{v-1}(x)$

ii) $[x^{-v} J_v(x)]' = -x^{-v} J_{v+1}(x)$ $= x^{-3} J_3 = -\int x^{-3} J_4 \, dx$

Moreover, $-J_0 = \int x^0 J_1$

$x^3 J_3 = \int x^3 J_2 \, dx$

iii) $J_{v-1}(x) + J_{v+1}(x) = \frac{2v}{x} J_v(x)$

iv) $J_{v-1}(x) - J_{v+1}(x) = 2J_v'(x)$

$\int J_5 = \int J_3 - 2 J_4$

Elementary $J_v(x)$ functions

Bessel function $J_v(x)$ of orders $\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}$ ... are called Elementary Bessel functions. They can be expressed in terms of trigonometric functions.

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin(x) \;\; and \; J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos(x)$$

<u>Proof:</u>

For $v = \frac{1}{2}$ from definition of Bessel function

$(\; J_3 = \int J_1 - 2 J_2$

$$J_{\frac{1}{2}}(x) = \sqrt{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^{2n+\frac{1}{2}}n!\,\Gamma(n+\frac{3}{2})} = \sqrt{\frac{2}{x}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2^{2n+1}n!\,\Gamma(n+\frac{3}{2})}$$

Let

$A = 2^n n! = 2n.(2n-2)(2n-4) \dots \dots 4.2$ and

$$B = 2^{n+1}\Gamma\left(n+\frac{3}{2}\right) = 2^{n+1}\frac{1}{2}\Gamma\left(n+\frac{1}{2}\right) = 2^{n+1}\frac{1}{2}.\frac{1}{2}\Gamma\left(n-\frac{1}{2}\right) = \cdots\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

it can be rewritten with $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ as

$B = 2^{n+1}\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\dots\dots.\frac{3}{2}.\frac{1}{2}\sqrt{\pi} = (2n+1)(2n-1)\dots.3.1\sqrt{\pi}$

Then $A.B = (2n+1)!\sqrt{\pi}$

Replacing this result into $J_{\frac{1}{2}}(x)$

$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sqrt{\frac{2}{\pi x}}\sin(x)$

and using above mentioned properties of Bessel functions

$$\left[\sqrt{x}J_{\frac{1}{2}}(x)\right]' = x^{\frac{1}{2}}J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\cos(x) \Longrightarrow J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos(x)$$

Other elementary functions can be also obtained using the properties of Bessel functions.

4.2 Method of Reducing Equations to Bessel Equations

There are many differential equations that can be reduced to Bessel or modified Bessel equations by transformations of variables. Then the reduced equation can be solved in terms of Bessel functions.

If a differential equation can be put into the form

$$\frac{d}{dx}\left(x^a\frac{dy}{dx}\right) + bx^c y = 0$$

where a,b,c are real constants, It can be expressed into a Bessel equation by transforming both independent and dependent variables. The transformation can be obtained by defining independent and dependent variables as follows;

$$t = \alpha\sqrt{b}x^{1/\alpha} \text{ and } u = x^{-v/\alpha}y$$

Moreover, α and v are calculated as

$$\alpha = \frac{2}{c-a+2} \text{ and } v = \frac{1-a}{c-a+2}$$

Thus, using the transformation parameters the differential equation will be transformed to

$$t^2\frac{d^2u}{dt^2} + t\frac{du}{dt} + (t^2 - v^2)u = 0$$

Then the solution becomes $u(t) = Z_v$ where Z_v is the Bessel function solution of the transformed equation. If $b > 0$, Z_v denotes Bessel functions of first kind; J_v and Y_v, and if $b < 0$ Z_v denotes the Bessel functions of second kind, I_v and K_v.

Finally, one can write the solution for y as;

$$y(x) = x^{v/\alpha}Z_v(\alpha\sqrt{b}x^{1/\alpha})$$

Example 4.3

Solve the differential equation

$$y'' + \left(e^{2x} - \frac{1}{9}\right)y = 0$$

$$\begin{aligned} z &= e^x \\ dz &= e^x dx \end{aligned} \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}e^x = \frac{dy}{dz}z$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d}{dz}\frac{dz}{dx}\left(z\frac{dy}{dz}\right) = z\frac{dy}{dz} + z^2\frac{d^2y}{dz^2}$$

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + \left(z^2 - \frac{1}{9}\right)y = 0$$

$$y = C_1J_{1/3}(z) + C_2J_{-1/3}(z)$$

$$y = C_1J_{1/3}(e^x) + C_2J_{-1/3}(e^x)$$

Example 4.4

Solve the differential equation $xy'' + 2y' + xy = 0$

$$y = zx^{-1/2}$$

$$y' = -\frac{zx^{-3/2}}{2} + \frac{dz}{dx}x^{-1/2}$$

$$y'' = \frac{3zx^{-\frac{5}{2}}}{4} - \frac{x^{-\frac{3}{2}}}{2}\frac{dz}{dx} - \frac{x^{-\frac{3}{2}}}{2}\frac{dz}{dx} + x^{-\frac{1}{2}}\frac{d^2z}{dx^2} = \frac{3zx^{-\frac{5}{2}}}{4} - x^{-\frac{3}{2}}\frac{dz}{dx} + x^{-\frac{1}{2}}\frac{d^2z}{dx^2}$$

Substituting into the differential equation, yields that

$$\frac{3}{4}zx^{-\frac{3}{2}} - x^{-\frac{1}{2}}\frac{dz}{dx} + x^{\frac{1}{2}}\frac{d^2z}{dx^2} - zx^{-\frac{3}{2}} + 2x^{-\frac{1}{2}}\frac{dz}{dx} + zx^{\frac{1}{2}} = 0$$

$$x^{\frac{1}{2}}\frac{d^2z}{dx^2} + x^{-\frac{1}{2}}\frac{dz}{dx} - \frac{1}{4}zx^{-\frac{3}{2}} + zx^{\frac{1}{2}} = 0$$

Scale by $x^{3/2}$

$$x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + \left(x^2 - \frac{1}{4}\right)z = 0$$

$$\begin{aligned} z &= C_1J_{1/2}(x) + C_2Y_{1/2}(x) \\ yx^{\frac{1}{2}} &= C_1J_{1/2}(x) + C_2Y_{1/2}(x) \\ y &= x^{-\frac{1}{2}}\left[C_1J_{1/2}(x) + C_2Y_{1/2}(x)\right] \end{aligned}$$

Example 4.5

Evaluate $\int x^{-1}J_4$

$$\begin{aligned} J_4(x) &= \frac{2(3)}{x}J_3 - J_2 \\ \int x^{-1}J_4 &= \int x^{-1}\left(\frac{6}{x}J_3 - J_2\right) \\ \int x^{-1}J_4 &= \int 6x^{-2}J_3 - \int x^{-1}J_2 = -\,6x^{-2}J_2 + x^{-1}J_1 \end{aligned}$$

Example 4.6

Solve the differential equation in terms of Bessel fuctions

$$y'' + xy = 0$$

$$a = 0, \qquad b = 1, \quad c = 1$$

$$\alpha = \frac{2}{c-a+2} = \frac{2}{3}, \qquad n = \frac{1-a}{c-a+2} = \frac{1}{3}$$

$$t = \alpha\sqrt{b}x^{1/\alpha} = \frac{2}{3}x^{3/2}, \qquad u = x^{-n/\alpha}y = x^{-1/2}y$$

$$t^2\frac{d^2u}{dt^2} + t\frac{du}{dt} + (t^2 + n^2)u = 0$$

$$y(x) = x^{\frac{n}{\alpha}}J_n\left(\alpha\sqrt{b}x^{\frac{1}{\alpha}}\right)$$

$$y(x) = C_1x^{\frac{1}{2}}J_{1/3}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + C_2x^{\frac{1}{2}}J_{-1/3}\left(\frac{2}{3}x^{\frac{3}{2}}\right)$$

4.3 The Gamma Function

Definition for Gamma function is;

$$\Gamma(x) = \int_0^\infty x^{p-1}\, e^{-x} dx; \ \mathrm{p} \in \mathbb{R} \quad \& \quad p > 0$$

Let's verify some of its features;

1) $p = 1 \Longrightarrow \quad \Gamma(1) = \int_0^\infty x^0\, e^{-x} dx = -e^x|_0^\infty = -(0-1) = 1$

so; $\quad \Gamma(1) = 1$

2)

$\Gamma(p+1) = p\Gamma(p)$

$; \quad \mathrm{p} > 0$

$\Gamma(n+1) = n!$

$$\Gamma(p+1)=\int_0^\infty x^p\, e^{-x}dx \qquad \begin{matrix} u= & x^p & dv= & e^{-x}dx \\ du= & px^{p-1}dx & v\;= & -e^{-x}\end{matrix}$$

$$\Gamma(p+1)=-x^pe^{-x}|_{x=0}^{\infty}+\int_0^\infty e^{-x}p\,x^{p-1}dx=p\int_0^\infty x^{p-1}e^{-x}dx$$

so; $\qquad \Gamma(p+1)=p\Gamma(p)$

$p=1 \Longrightarrow \quad \Gamma(2)=1\Gamma(1)=1=1!$

$p=2 \Longrightarrow \quad \Gamma(3)=2\Gamma(2)=2.1=2!$

$p=3 \Longrightarrow \quad \Gamma(4)=3\Gamma(3)=3.2.1=3!$

$p=\mathrm{n} \Longrightarrow \quad \Gamma(n)=(n-1)!$

Verify that;

$$\Gamma\left(\frac{1}{2}\right)=\left(-\frac{1}{2}\right)!=\sqrt{\pi}$$

$$\Gamma\left(\frac{1}{2}\right)=\int_0^\infty x^{\frac{1}{2}-1}\,e^{-x}dx=\int_0^\infty x^{-\frac{1}{2}}e^{-x}dx \qquad (*)$$

Substitute

$$x=u^2 \Longrightarrow dx=2u\,du \Longrightarrow \frac{dx}{\sqrt{\mathrm{x}}}=2\,du$$

Insert in (*)

$$\Gamma\left(\frac{1}{2}\right)=2\int_0^\infty e^{-u^2}du$$

Taking square of both sides

$$\Gamma^2\left(\frac{1}{2}\right)=4\int_0^\infty e^{-u^2}du\int_0^\infty e^{-v^2}dv=4\iint\limits_0^\infty e^{-(u^2+v^2)}du\,dv$$

the double integral is evaluated by transforming to polar coordinates

$u=\mathrm{r}\cos\theta \quad \& \quad v=\mathrm{r}\sin\theta \quad \Longrightarrow du\,dv=r\,dr\,d\theta$

$$\Gamma^2\left(\frac{1}{2}\right)=4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\;r\,dr\,d\theta=4\int_0^{\frac{\pi}{2}}\left(-\frac{e^{-r^2}}{2}\right)\Bigg|_0^\infty d\theta=4\int_0^{\frac{\pi}{2}}\left(0+\frac{1}{2}\right)d\theta$$

$$\Gamma^2\left(\frac{1}{2}\right) = 2\,\frac{\pi}{2} = \pi \Longrightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \left(\frac{1}{2}\right)! = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)! = \frac{(3\sqrt{\pi})}{2^2}$$

4.4 Legendre Differential Equation and Legendre Polynomials

Second family of differential equation that appears in many problems in engineering and physics is the Legendre differential equations

$$(1-x^2)y'' + 2xy' + p(p+1)y = 0, \quad \mathrm{p} \in \mathbb{R}$$

The normalized equation is

$$y'' - \frac{2x}{1-x^2}y' + \frac{p(p+1)}{1-x^2}y = 0$$

Since

$$\frac{2x}{1-x^2}, \text{and } \frac{p(p+1)}{1-x^2}$$

are all analytic at $x_0 = 0$, in fact for $|x| < 1$, $x = 0$ so, is an *ordinary point* for the differential equation. Let $\lambda = p(p+1)$, then an appropriate solution proposal is

$$y = \sum_{n=0}^{\infty} C_n x^n$$

Upon successive differentiation one has

$$y' = \sum_{n=1}^{\infty} C_n\, n\, x^{n-1} \text{ and } y'' = \sum_{n=2}^{\infty} C_n\, n(n-1)\, x^{n-2}.$$

Substitution of these into the differential equation one obtains

$$(1-x^2)\sum_{n=2}^{\infty} C_n\, n(n-1)\, x^{n-2} + 2x\sum_{n=1}^{\infty} C_n n x^{n-1} + \lambda \sum_{n=0}^{\infty} C_n x^n = 0 \rightarrow$$

$$\sum_{n=2}^{\infty} C_n\, n(n-1)\, x^{n-2} - \sum_{n=2}^{\infty} C_n\, n(n-1)\, x^n + 2\sum_{n=1}^{\infty} C_n n x^n + \lambda \sum_{n=0}^{\infty} C_n\, x^n = 0$$

Shifting in the first term $n \rightarrow n+2$ one has

$$\sum_{n=0}^{\infty} C_{n+2}\,(n+2)(n+1)x^n - \sum_{n=2}^{\infty} C_n\, n(n-1)x^n - 2\sum_{n=1}^{\infty} C_n n x^n + \lambda \sum_{n=0}^{\infty} C_n x^n = 0$$

Since the common range is $2 \to \infty$, we cut extra terms to

$$2C_2 + 6C_3 x + \sum_{n=2}^{\infty} C_{n+2}(n+2)(n+1)x^n$$
$$-\sum_{n=2}^{\infty} C_n\, n(n-1)x^n - 2C_1 x - 2\sum_{n=2}^{\infty} C_n n x^n + \lambda C_0 + \lambda C_1 x + \lambda \sum_{n=2}^{\infty} C_n x^n = 0 \ \rightarrow$$

The grouping of the terms yields

$$(\lambda C_0 + 2C_2) + (\lambda C_1 - 2C_1 + 6C_3)x + \sum_{n=2}^{\infty} \{[C_n\,[\lambda - n(n-1) - 2n] + C_{n+2}(n+2)(n+1)]x^n\}$$

There is no restriction on C_0. Equating the coefficients of x^i $i = 0, 1, \ldots$ to zero one has

$$C_2 = -\frac{\lambda}{2} C_0 = -\frac{p(p+1)}{2} C_0$$

$$C_3 = -\frac{1}{6}(\lambda - 2)C_1 = \frac{2 - \ p(p+1)}{6} C_1 = -\frac{(p-1)(p+2)}{3!} C_1$$

The recurrence relation;

$$C_{n+2} \ = \frac{p(p+1) - n(n-1) - 2n}{(n+2)(n+1)} C_n$$

$$C_{n+2} \ = -\frac{(p-n)(p+n+1)}{(n+2)(n+1)} \mathrm{C}_n, n \geq 2$$

Therefore

$$n = 2 \Longrightarrow \ C_4 = -\frac{(p-2)(p+3)}{4 \cdot 3} C_2 = \frac{(p-2)p(p+1)(p+3)}{4!} C_0$$

$$n = 3 \Longrightarrow \ C_5 = -\frac{(p-3)(p+4)}{5 \cdot 4} C_3 = \frac{(p-3)(p-1)(p+2)(p+4)}{5!} C_1$$

Substitution of these into the solution proposal;

$$y = \sum_{n=0}^{\infty} C_n\, \mathrm{x}^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots$$

This yields that;

$$y_{gen} = \quad C_0 + C_1 x - \frac{p(p+1)}{2!} C_0 x^2 - \frac{(p-1)(p+2)}{3!} C_1 x^3$$

$$+ \frac{(p-2)p(p+1)(p+3)}{4!} C_0 x^4 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!} C_1 x^5 + \cdots$$

Grouping C_0, C_1 terms

$$y_{gen} = \quad C_0 \left[1 - \frac{p(p+1)}{2!} x^2 + \frac{(p-2)p(p+1)(p+3)}{4!} x^4 - \cdots \right]$$

$$+ C_1 \left[x - \frac{(p-1)(p+2)}{3!} x^3 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!} x^5 + \cdots \right]$$

Simplifying by summation symbols

$$y_{gen} = \quad C_0 \left[1 - \frac{p(p+1)}{2!} x^2 + \frac{\prod_{k=0}^{2-1} (p-2k)(p+2k+1)}{(2.2)!} x^{2.2} - \cdots \right]$$

$$+ C_1 \left[x - \frac{(p-1)(p+2)}{3!} x^3 + \frac{\prod_{k=0}^{2-1} (p-2k-1)(p+2k+2))}{(2.2+1)!} x^{2.2+1} + \cdots \right]$$

With more simplifications

$$y_{gen} = \quad C_0 \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{k=0}^{n-1} (p-2k)(p+2k+1)}{(2n)!} x^{2n} \right]$$

$$+ C_1 \left[x + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{k=0}^{n-1} (p-2k-1)(p+2k+2))}{(2n+1)!} x^{2n+1} \right]$$

Therefore the general solution is

$y_{gen} = C_0 y_1 + C_1 y_2$;

where

$$y_1 = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{k=0}^{n-1} (p-2k)(p+2k+1)}{(2n)!} x^{2n}$$

$$y_2 = x + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{k=0}^{n-1} (p-2k-1)(p+2k+2))}{(2n+1)!} x^{2n+1}$$

$$y_1 = \quad 1 - \frac{p(p+1)}{2!}x^2 + \frac{(p-2)p(p+1)(p+3)}{4!}x^4$$
$$-\sum_{p=3}^{\infty} \frac{\alpha \ldots (\alpha - 2p + 2)(\alpha + 1) \cdots (\alpha + 2p - 1)}{2p!} x^{2p}$$

$$y_2 = \quad x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!}x^5 +$$
$$-\sum_{p=3}^{\infty} (-1)^p \frac{(\alpha - 1) \ldots (\alpha - 2p + 1)(\alpha + 2) \cdots (\alpha + 2p)}{(2p+1)!} x^{2p+1}$$

hence

$y_{gen} = Y_1 + Y_2$, where $Y_1 = C_0 y_1$, and $Y_2 = C_1 y_2$

It can be observed that for *odd values of* p's, the solution y_1 (and Y_1) reduces to a polynomial with order p $(n = p)$. Similarly for *even values of* p's , the solution y_2 (and Y_2) reduces to a polynomial with order p $(n = p)$. Let's obtain these polynomials using Y_1 and Y_2 ;

$$p = 0 \quad \Rightarrow \quad P_0(x) = Y_1|_{p=0} \quad = C_0 \cdot 1$$

$$p = 2 \quad \Rightarrow \quad P_2(x) = Y_1|_{p=2} \quad = C_0[1 - 3x^2]$$

$$p = 4 \quad \Rightarrow \quad P_4(x) = Y_1|_{p=4} \quad = C_0\left[1 - 10x^2 + \frac{35}{3}x^4\right]$$
$$\vdots$$
$$p = 1 \quad \Rightarrow \quad P_1(x) = Y_2|_{p=1} \quad = C_1 x$$

$$p = 3 \quad \Rightarrow \quad P_3(x) = Y_2|_{p=3} \quad = C_1\left[x - \frac{5}{3}x^3\right]$$

$$p = 5 \quad \Rightarrow \quad P_5(x) = Y_2|_{p=5} \quad = C_1\left[x - \frac{14}{3}x^3 + \frac{21}{5}x^5\right]$$
$$\vdots$$

If C_0 and C_1 are chosen in such a way that $P_i(x) = 1$ for $x = 1 \Rightarrow P_i(1) = 1$ then the resulting polynomials will be simplified and still be the solutions of the Legendre equations as follows;

$$\Rightarrow C_0 = 1 \qquad \Rightarrow P_0(x) = 1$$

$$\Rightarrow C_0 = -\frac{1}{2} \quad \Rightarrow P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$= C_0 = \frac{1}{8} \qquad \Rightarrow P_4(x) = \frac{1}{8}(35\,x^4 - 30x^2 + 3)$$

$$\Rightarrow C_1 = 1 \qquad \Rightarrow P_1(x) = x$$

$$\Rightarrow C_1 = -\frac{3}{2} \quad \Rightarrow P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\Rightarrow C_1 = \frac{15}{8} \qquad \Rightarrow P_3(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

Note that these polynomials are the solutions of Legendre differential equations only for the interval $-1 < x < +1.$

A generalized expression for Legendre polynomials is also given as follows

$$P_p(x) = \frac{1}{2^p p!}\frac{d^p}{dx^p}(x^2 - 1)\ ; \quad p \geq 0, \text{integer}$$

for example, for $p = 3$;

$$P_3(x) = \frac{1}{2^3.3!}\frac{d^3}{dx^3}(x^2 - 1)^3 = \frac{1}{48}\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1) = \frac{1}{48}(120x^3 - 72x)$$

then

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{2^4 \cdot 4!}\frac{d^4}{dx^4}(x^2 - 1)^4$$

$$P_4(x) = \frac{1}{16 \cdot 4 \cdot 3 \cdot 2 \cdot 1}\frac{d^4}{dx^4}[1(x^2)^4 1^0 - 4(x^2)^3 1^1 + 6(x^2)^2 1^2 - 4(x^2)^1 1^2 + 1(x^2)^0 1^4]$$

$$\frac{1}{16\cdot4\cdot3\cdot2}\frac{d^4}{dx^4}[x^8-4x^6+6x^4-4x^2+1] \quad = \frac{1}{16\cdot4\cdot3\cdot2}\frac{d^3}{dx^3}[8x^7-24x^5+24x^3-8x]$$

$$\frac{1}{16\cdot4\cdot3\cdot2}\frac{d^2}{dx^2}[56x^6-120x^4+72x^2-8] \quad = \frac{1}{16\cdot4\cdot3\cdot2}\frac{d}{dx}[336x^5-480x^3+144x]$$

$$\frac{1}{16\cdot4\cdot3\cdot2}[1680x^4-1440x^2+144] \quad = \frac{1}{16}[70x^4-60x^2+3]$$

so

$$P_4(x) = \frac{1}{16}[70x^4-60x^2+3]$$

4.5 Sturm-Liouville Boundary Value Problems

Boundary value problems for second order linear ordinary differential equations consist of a differential equation

$$p(x)y" + q(x)y' + r(x)y = 0$$

And one set of two boundary conditions which the

$$y(x_0) = \alpha_1 , y(x_1) = \beta_1$$
$$y'(x_0) = \alpha_2 , y'(x_1) = \beta_2$$
$$y'(x_0) = \alpha_3 , y(x_1) = \beta_3$$
$$y(x_0) = \alpha_4 , y'(x_1) = \beta_4$$

solution of the differential equation is going to satisfy. In this section we will discuss a special kind of boundary value problems; Sturm-Liouville Boundary Value Problems. Characteristic functions, orthogonality of functions, and Fourier series expansions are some of the aspects we will encounter. These concepts will be very useful in modeling problems from engineering and physics.

4.5.1 Sturm-Liouville Boundary Value Problems

Definition

A boundary value problem that consists of a second order linear ordinary differential equation

$$(p(x)y')' + [q(x) + \lambda r(x)]\, y = 0 \qquad (1)$$

and boundary conditions

$$\alpha_1 y(x_0) + \alpha_2\, y'(x_0) = 0 \qquad (2)$$
$$\beta_1 y(x_1) + \beta_2 y'(x_1) = 0.$$

is a Sturm-Liouville Boundary Value Problem.

where p, q, r, p' are real continuous functions and $p(x) > 0$, $r(x) > 0$, in an interval $a \le x \le b$, and λ is a parameter independent of x.

In boundary conditions $\alpha_{1,}\, \alpha_2,\ \beta_{1,}\, \beta_2$ are real constants such that

$$\alpha_1 . \alpha_2 \neq 0,\ \beta_{1.}\beta_2 \neq 0$$

Legendre, Bessel and many other ODE's can be written in the form of a Sturm-Liouville equation.

EXAMPLE;

i. Legendre's Equation:

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$

may be written as

$[py']' + \lambda y = 0$

with

$p = 1 - x^2, \qquad q = 0, \qquad r = 1, \qquad \lambda = n(n+1)$

ii.Bessel's Equation:

$x^2y'' + xy' + (x^2 - v^2)y = 0$

divide by *x*

$$xy'' + y' + \left(x - \frac{v^2}{x}\right)y = 0$$

$$[xy']' + \left(x - \frac{v^2}{x}\right)y = 0 \qquad (2)$$

$p = x,$

$$q = -\frac{v^2}{x}, \qquad r = x, \qquad \lambda = 1$$

$$or$$

$$q = x, \qquad r = -\frac{1}{x}, \quad \lambda = v^2$$

It can be seen that $y(x) = 0$ is the solution for (1) and (2). It is trivial solution, but we want to find non-trivial solutions in terms of Eigenfunctions for particular value of λ (eigenvalues).

Example 4.7

$y'' + \lambda y = 0, \; y(0) = 0, \; y(\pi) = 0$

$$[y']' + \lambda y = 0 \quad \Rightarrow \underbrace{\begin{matrix} p = 1 \\ q = 0 \\ r = 1 \end{matrix}}_{\text{in (1)}} \quad \underbrace{\begin{matrix} \alpha_1 = 1 & \alpha_2 = 0 \\ \beta_1 = 1 & \beta_2 = 0 \\ x_1 = 0 & x_2 = \pi \end{matrix}}_{\text{in (2)}}$$

a) λ is negative, $\lambda = -v^2$

$y'' - v^2 y = 0 \quad \Rightarrow \quad y(x) = C_1 e^{-vx} + C_2 e^{vx}$

$\begin{matrix} y(0) = C_1 + C_2 & = 0 \\ y(\pi) = C_1 e^{-v\pi} + C_2 e^{v\pi} & = 0 \end{matrix} \Rightarrow C_1 = C_2 = 0$ no nonzero eigenfunctions, then

$y(x) = 0, \forall x \in [0, \pi]$.

b) $\lambda = 0$

$y'' = 0 \quad \Rightarrow \quad y = C_3 x + C_4 \quad \Rightarrow \begin{matrix} y(0) = 0 = C_4 \\ y(\pi) = 0 = C_3 \pi \end{matrix} \quad \Rightarrow C_3 = 0$ no nonzero eigenfunctions $y(x) = 0, \forall x \in [0, \pi]$

c) λ is positive, $\lambda = v^2$

$y'' + v^2 y = 0 \quad \Rightarrow \quad r_{1,2} = \pm iv \quad \Rightarrow \quad y = C_5 \cos vx + C_6 \sin vx$

$\begin{matrix} y(0) = C_5 \cos 0 & = 0 & \Rightarrow & C_5 = 0 & \\ y(\pi) = C_6 \sin v\pi & = 0 & \Rightarrow & \sin v\pi = 0 & \therefore \quad C_6 \neq 0 \end{matrix}$ for non-trivial solution

for $\begin{matrix} n = 0 \\ v = 0 \\ y(x) = 0, \forall x \in [0, \pi] \end{matrix}$, trivial solution $\left(v = \frac{n\pi}{\pi} = n\right)$, $n = \pm 1, \pm 2$

$\lambda = v^2 = \left(\frac{n\pi}{\pi}\right)^2 = n^2$ are eigenvalues

$\sin nx$, $n = 1,2$ are eigenfunctions

In **homogeneous** linear ODE's, summation of individual solutions is also a solution. So a series solution can be obtained by summation of eigenfunctions.

<u>Orthogonality of Functions</u>:

If p, q, r and p' are real and continuous in $a \leq x \leq b$ and $p(x), r(x)$ is positive then the eigenvalues are real.

Consider the functions $y_1(x), y_2(x)$ continuous on $a \le x \le b$. They are called orthogonal on this internal with respect to the $r(x)$ (weight function) $r(x) > 0$ if

$$\int_a^b r(x) y_1(x) y_2(x)\, dx = 0,$$

The norm $y_m(x)$ is

$$||y_m(x)|| = \sqrt{\int_a^b r(x) y_m^2(x)\, dx}$$

The functions y_1 and y_2 are called "orthonormal" on $a \le x \le b$ if $r(x) = 1$

$$\int_a^b y_m(x) y_n(x)\, dx = 0, \qquad (\mathrm{m} \ne \mathrm{n}) \Longrightarrow ||y_m|| = 1, \qquad ||y_n|| = 1$$

Example 4.8

Consider the set of functions $\{\sin nx\,, n = 1,2, \ldots\},\ \ x \in [-\pi, \pi]$

$$\begin{aligned}\int_{-\pi}^{\pi} y_m(x) y_n(x)\, dx &= \int_{-\pi}^{\pi} \sin mx \cdot \sin nx\, dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)x\, dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x\, dx \\ &= \frac{1}{2(m-n)} \sin(m-n)x\Big|_{-\pi}^{\pi} - \frac{1}{2(m+n)} \sin(m-n)x\Big|_{-\pi}^{\pi} = 0\end{aligned}$$

They are orthogonal with $\quad r(x) = 1$

The norm of $\quad y_m$

$$||y_m||^2 = \int_{-\pi}^{\pi} \sin^2 mx\, dx = \int_{-\pi}^{\pi} \frac{1-\cos mx}{2} dx = \frac{1}{2}x - \frac{\sin(2mx)}{4m}\Big|_{-\pi}^{\pi} \Longrightarrow ||y_m|| = \sqrt{\pi}$$

Then

$$\left\{\frac{\sin x}{\sqrt{\pi}}\ ,\ \frac{\sin 2x}{\sqrt{\pi}}\ ,\frac{\sin 3x}{\sqrt{\pi}}, \ldots \ldots\right\}$$

Is an orthonormal set

Example 4.9

Orthogonality of Bessel functions consider

The Bessel's Equation

$\tilde{x}^2\ddot{y} \quad + \tilde{x}\dot{y} \quad + (\tilde{x}^2 - n^2)y = 0$ $\qquad y_n(\tilde{x})$ $\qquad$ is the solution

Let $\quad \tilde{x} = kx$

$$\dot{y} = \frac{dy}{d\tilde{x}} = \frac{dy}{dx}\frac{dx}{d\tilde{x}} = \frac{y'}{k} \quad \Longrightarrow \quad \ddot{y} = \frac{y''}{k^2} \quad , \quad \dot{y}' = \frac{dy}{dx}$$

$$k^2x^2\frac{y''}{k^2} + kx\frac{y'}{k} + (k^2x^2 - n^2)y = 0$$

$$x^2y'' + xy' + (k^2x^2 - n^2)y = 0$$

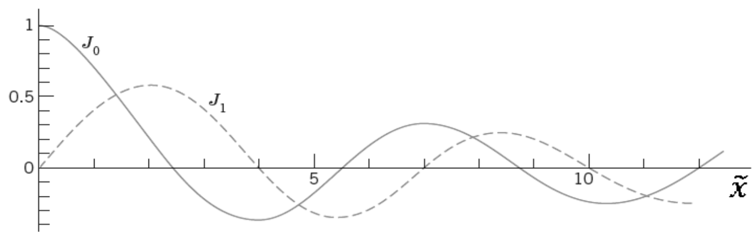
$$xy'' + y' + \left(k^2x - \frac{n^2}{x}\right)y = 0$$

$$[xy']' + \left[-\frac{n^2}{x} + k^2x\right]y = 0 \quad \Longrightarrow \quad p(x) = x, \quad q = -\frac{n^2}{x}, \quad r = x, \quad \lambda = k^2$$

$J_n(kx)$ $\quad$ is a solution

Let $\quad 0 \le x \le n, y(0) = 0, \; y(R) = 0$

$J_n(kR) = 0 \Longrightarrow$ $\qquad$ Bessel Functions have zero than…



$\tilde{x} = \alpha_{n,1}, \alpha_{n,2}, a_{n,3}$ …. (zero location)

$$kR = a_{n,m} \;\Longrightarrow\; k_{n,m} = \frac{\alpha_{n,m}}{R}$$

$\lambda = \; k^2$ $\qquad$ eigenvalues

$$\int_0^R x\,J_n\left(k_{n,m}x\right)J_n\left(k_{n,j}x\right)dx = 0$$

for $m \neq j$ orthogonality of Bessel's function.

Orthogonal Eigenfunctions Expansion:

Inner product of two functions

$$(y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x) dx\,, \quad a \leq x \leq b$$

If y_m and y_n are orthogonal an a ≤ λ ≤ b

$(y_m, y_n) = \delta_{m,n}\, ||y_m||^2$ $\delta_{m,n}$ is Kronecker delta

$$\delta_{m,n} = \begin{Bmatrix} 0 \;\; if \; m \neq n \\ 1 \;\; if \;\; m = n \end{Bmatrix}$$

Orthogonal series:

Let $y_1, y_2, \ldots$ be an orthogonal set of functions with respect to $R(x)$ an $a \leq x \leq b$. Let $f(x)$ be a function, it can be represented

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$$

orthogonal expansion. We need to find the coefficient's a_m's

$$(f, y_n) = \sum_{m=0}^{\infty} a_m \big(y_m(x), y_n(x)\big) = \int_a^b rf\;\, y_n dx = \int_a^b r(\sum_{m=0}^{\infty} a_m y_m) y_n dx$$

For m = n $\Longrightarrow a_n(y_n, y_n) = a_n \|y_n\|^2$

Then

$$a_m = \frac{(f, y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b rf\, y_m dx\,, \qquad \mathrm{m} = 0,1,\ldots$$

Example 4.10

$$y'' + xy = 0, \qquad \begin{matrix} y(\pi) = y(-\pi) \\ y'(\pi) = y'(-\pi) \end{matrix}, \qquad \begin{matrix} p = 1 \\ q = 0\,, \\ r = 1 \end{matrix} \qquad -\pi \le x \le \pi$$

Linear solution $\qquad y(x) = Acoskx + Bsinkx, \quad k^2 = \lambda$

$$\begin{matrix} A\cos k\pi + B\sin k\pi & = & A\cos(-k\pi) + B\sin(-k\pi) & \because & \cos(-\propto) = \cos(\propto) \\ -kA\sin k\pi + kB\cos k\pi & = & -kA\sin(-k\pi) + kB\cos(-k\pi) & \because & \sin(-\propto) = -\sin(\propto) \end{matrix}$$

$$\begin{matrix} 2B\sin k\pi = 0 & \Longrightarrow & \sin k\pi = 0 & \Longrightarrow & k\pi & = m\pi \\ -2kA\sin k\pi = 0 & & k\pi = 0 & & m & = 0,1,2,\ldots \end{matrix}$$

	Eigenvalues		Eigenfunctions
$k = 0$	$\lambda = 0$	$\Longrightarrow$	$\cos(0) = 1, \sin(0) = 0\ (trivial)$
$k = 1$	$\lambda = 1$	$\Longrightarrow$	$\cos x\,, \sin x$
$k = 2$	$\lambda = 4$	$\Longrightarrow$	$\cos 2x\,, \sin 2x$
$k = 3$	$\lambda = 9$	$\Longrightarrow$	$\cos 3x\,, \sin 3x$

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = 0 \quad \text{for} \quad m \ne n$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = 0 \quad \text{for} \quad m \ne n$$

Norm

$$\int_{-\pi}^{\pi} 1^2 dx = 2\pi \qquad \sqrt{2\pi}$$

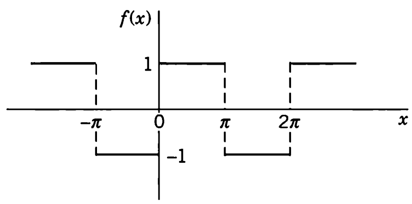
$$\int_{-\pi}^{\pi} \cos^2 nx\, dx = \int_{-\pi}^{\pi} \left(\frac{\cos 2nx + 1}{2}\right) dx = \pi \qquad \sqrt{\pi}$$

$$\int_{-\pi}^{\pi} \sin^2 nx\, dx = \int_{-\pi}^{\pi} \left(\frac{1 - \cos 2nx}{2}\right) dx = \pi \qquad \sqrt{\pi}$$

Fourier series: $\quad f(x) \quad = \quad a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$

Euler Formula: $\begin{cases} a_0 &= \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)dx \\ a_m &= \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos mx\, dx \\ b_m &= \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin mx\, dx \end{cases}$

Example 4.11



$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$
$$f(x) = f(x + 2\pi)$$

$$a_m = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-1)\cos mx\, dx + \int_{0}^{\pi} 1 \cos mx\, dx\right] = 0$$

$$b_m = \frac{1}{\pi}\left[\int_{-\pi}^{0} -1 \sin mx\, dx + \int_{0}^{\pi} 1 \sin mx\, dx\right]$$

$$= \frac{1}{\pi m}[1 - 2\cos mx + 1]$$

$$= \begin{cases} \dfrac{4}{\pi m} & m = 1,3,\ldots \\ 0 & m = 2,4,\ldots \end{cases}$$

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right) \ldots.$$

Example 4.12

Find the eigenvalues and eigenfunctions of SL problem

$$\frac{d}{dx}\left(\frac{1}{3x^2 + 1}\frac{dy}{dx}\right) + \lambda(3x^2 + 1)y = 0, \quad y(0) = 0, \quad y(\pi) = 0$$

Let

$$x^3 + x = t \Longrightarrow (3x^2 + 1) = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}(3x^2+1)$$

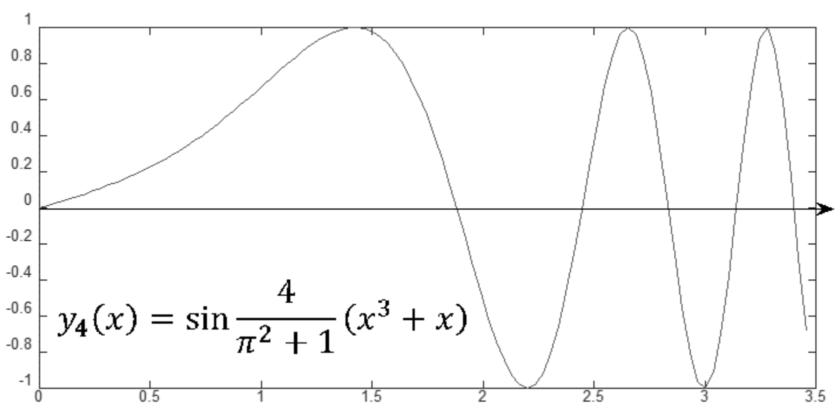
$$\frac{d}{dx} = \frac{d}{dt}\frac{dt}{dx} = (3x^2+1)\frac{d}{dt}$$

$$(3x^2+1)\frac{d}{dt}\left(\frac{dy}{dt}\right) + \lambda(3x^2+1)y = 0 \quad \Longrightarrow \quad \frac{d^2y}{dt^2} + \lambda y = 0\,, \quad \lambda = k^2$$

$$y = C_1\cos kt + C_2 \sin kt = C_1 \cos k(x^3+x) + C_2 \sin k(x^3+x)$$

$$y(0) = 0 \quad \Longrightarrow \quad C_1 = 0$$
$$y(\pi) = 0 \quad \Longrightarrow \quad C_2 \sin k(\pi^3+\pi) = 0 \quad \Longrightarrow \quad k(\pi^3+\pi) = n\pi$$
$$k = \frac{n}{\pi^2+1} \qquad n = 1,2,\ldots$$
$$\therefore \qquad \lambda_n = \left(\frac{n}{\pi^2+1}\right)^2$$

$$y_n(x) = \sin\frac{n}{\pi^2+1}(x^3+x) \qquad n = 1,2\ldots$$



■

4.5 Some Special Equations and Summary of Their Solutions

$x^2y'' + xy' + (x^2 - p^2)y = 0,$ Bessel equation of order *p*

$x^2y'' + y' + xy = 0,$ Bessel equation of order zero

In solution:

$$J_0(x) = \frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}$$ Bessel function of the 1st kind of order zero

$$v = \lim_{x\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right) \cong 0.577$$ Euler constant

$$Y_0(x) = \frac{2}{\pi}\left\{\left(\ln\left|\frac{x}{2}\right| + v\right)J_0(x) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n}}{2^{2n}(n!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)\right\}$$ Bessel function of the 2nd kind of order zero (Weber form)

Then the solution is: $y = C_1J_0(x) + C_2Y_0(x)$

$x^2y'' + xy' + (x^2 - p^2)y = 0,$ Bessel equation of order *p*

In solution:

$$\Gamma(N) = \int_0^{\infty} e^{-x}x^{N-1}dx$$ Gamma function

$$N! = N\Gamma(N+1) = N\Gamma(N)$$ Generalized factorial for real number

$$J_p(x) = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!\,(n+p!)}\left(\frac{x}{2}\right)^{2n+p}$$ Bessel function of the 1st kind or order zero

$$Y_p(x) = \frac{2}{\pi}\left\{\left(\ln\left|\frac{x}{2}\right| + v\right)J_0(x) - \frac{1}{2}\sum_{n=1}^{\infty}\frac{(p-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n+p}\right.$$
$$\left. + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+1}\left(\sum_{k=1}^{\mathrm{n}}\frac{1}{k} + \sum_{k=1}^{\mathrm{n+p}}\frac{1}{k}\right)\left[\frac{1}{n!\,(n+p)!}\left(\frac{x}{2}\right)^{2n+p}\right]\right\}$$ Bessel function of 2nd kind of order p (Webber's form)

Then the solution is: $y = C_1 J_p(x) + C_2 Y_p(x)$

$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$ α: constant — Legendre equation

In solution:

Legendre polynomials are obtained as;

$P_p(x) = \frac{1}{2^p p!} \frac{d^p}{dx^p} (x^2 - 1)^p; \quad p \geq 0$

$P_0(x) = 1$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

These are the solutions of Legendre equations for $-1 < x < +1$; so $x = 0$, too??

Some Other Special Equations:

$y'' - 2xy' + \lambda y = 0, \quad -\infty < x < \infty$ — Hermite equation (Hermite polynomials are obtained from Hermit equation)

$(1 - x^2)y'' - xy' + \alpha 2y = 0$ α: constant — Chebyshev equation (Chebyshev polynomials are obtained from Chebyshev equation)

$x^2 y'' + \alpha x y' + \beta y = 0$ α & β real — Euler equation

$xy'' + (1 - x)y' + \lambda y = 0$ — Laguerre equation

$(x - x^2)y'' + [\gamma - (\alpha + \beta + 1)x]y' - \alpha\beta y = 0$ — Gauss equations or Hypergeometric equation

Solution of hyper geometric equations

$$y = C_1 y_1 + C_2 y_2$$

where $y_{1,2}(\alpha, \beta, \gamma, x)$ are known as hyper geometric functions.

5. FOURIER SERIES AND FOURIER INTEGRALS

Mathematical models for physical and engineering problems often contains periodic forcing terms. For models expressed with partial differential equations there are boundary conditions defined on finite boundaries. After the separation of the variables, these boundary value problems lead us to boundary value problems in ordinary differential equations which are mostly nonhomogeneous. The corresponding homogeneous problems are mostly Sturm-Liouville problems. In such a case to propose a particular solution one needs to expand inhomogenities in Fourier series with the eigenfunctions of the Sturm-Liouville problem taken as base functions. If the eigenfunctions are trigonometric cosine and sine functions, then the corresponding Fourier series are called Trigonometric Fourier series.

Periodic Functions

A function $f(x)$ is called periodic if it is defined for all $x \in R$, reals, and if there is some positive number p such that

$$f(x+p) = f(x) \qquad \text{for all } x \in R \tag{6.1}$$

this number p is called a period of $f(x)$.

Periodic phenomena and functions occur in many applications. For example, sine, cos functions. The function $f = c = constant$ is also a periodic function since it satisfies the equation (5.1) for every p.

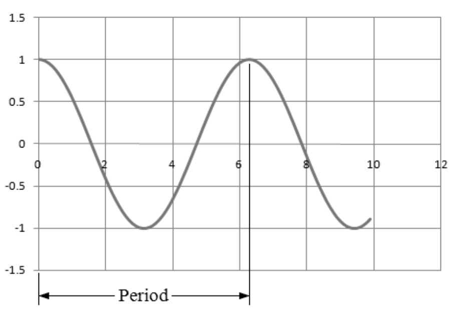


Figure. The graph of the periodic function $f(x) = \cos x$, which is periodic with period 2π.

$x, x^2, e^x, \cosh x, \ln x$ are examples of non-periodic functions**.**

If $f(x)$ and $g(x)$ have period p, then the functions, then

$h(x) = af(x) + bg(x), \;\; a, b \in R$

also a periodic function with period p.

Since functions

$\{\cos nx, \sin mx\}, \qquad n, m \in I$ (nonnegative integers)

All have period of 2π. Then the series

$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$

also have the same period 2π.

The series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots.$ are real constants, is called a trigonometric series known as Fourier series.

Orthogonality Properties of Trigonometric Sine and Cosine Functions

Let $L > 0$ be any number, then functions

$\left\{\cos \frac{n\pi x}{L}, \sin \frac{m\pi x}{L}\right\}, \qquad n, m \in I$ (nonnegative integers)

all have period of $2L$. It can be easily proved that integrals of these functions over a period have the following properties:

1) $\int_{-L}^{L} \cos m\pi x/L \, dx = 0 \, , \forall m \in Z^+$ the set of nonnegative integers.
2) $\int_{-L}^{L} \sin n\pi x/L \, dx = 0 \, , \forall n \in Z^+$
3) $\int_{-L}^{L} \cos m\pi x/L \cos n\pi x/L \, dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$
4) $\int_{-L}^{L} \sin m\pi x/L \sin n\pi x/L \, dx = \begin{cases} 0, & m \neq n \\ L, & m = n \end{cases}$
5) $\int_{-L}^{L} \cos m\pi x/L \sin n\pi x/L \, dx = 0, \forall m, n \in Z^+$

5.1 Fourier Series

Fourier series represents a given periodic function $f(x)$ in terms of sine and cosine functions. These series are trigonometric series whose coefficients are determined from $f(x)$ by "Euler Formulas", by Euler Formulas, by the use of orthogonality properties of the sine and cosine functions.

Let us assume that f(x) is a periodic function of period 2π. Assume the series

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx) \qquad (5.2)$$

converges uniformly to f(x). Let us determine coefficients a_n and b_n.

Integrating both sides of (5.2), using the integral properties in 1) and 2) one has

$$\int_{-\pi}^{\pi} f(x)dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos nx\, dx + b_n \int_{-\pi}^{\pi} \sin nx\, dx\right) = 2\pi a_0$$

And therefore one has

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$$

To determine a_1, a_2 .... We multiply (5.2) by $\cos mx$, where *m* is any fixed positive integer and integrate over the interval $[-\pi , \pi]$.

$$\int_{-\pi}^{\pi} f(x)\cos mx\, dx = a_0 \int_{-\pi}^{\pi} \cos mx\, dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos nx \cos mx\, dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx\, dx\right)$$

Using the orthogonality properties 1)-4) one has

$$\int_{-\pi}^{\pi} f(x)\cos mx\, dx = a_m\,\pi$$

Then one obtains

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, cos\, mx\, dx.$$

Similarly multiply multiply (5.2) by $\sin mx$, where *m* is any fixed positive integer and integrate over the interval $[-\pi , \pi]$.

$$\int_{-\pi}^{\pi} f(x)\sin mx\, dx = a_0 \int_{-\pi}^{\pi} \sin mx\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\left(\cos nx \sin mx\, dx + b_n \int_{-\pi}^{\pi} \sin nx \sin mx\right) dx$$

Using the orthogonality properties 1)-4) one has

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = b_m\,\pi$$

Then one obtains

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

Example 5.1

Find the Fourier coefficients of the periodic function $f(x)$ given in the figure which may be the external force acting on a mechanical system.

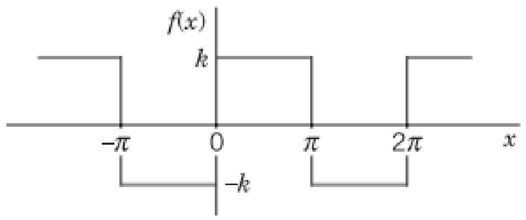


Figure. Periodic external force acting on a mechanical system

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}, \qquad f(x+2\pi) = f(x)$$

Let the Fourier series of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx)$$

To find a_0 use the formula derived in the above

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -kdx + \int_{0}^{\pi} kdx\right) = 0$$

To find $a_n, n > 0$ use the formula derived in the above again

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -k\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right) = 0, \forall n$$

To find $b_n, n > 0$ use the formula derived in the above again yields

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0} -k\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right)$$

$$= \frac{1}{\pi}\left(-k\frac{-cosnx}{n}\Big|_{-\pi}^{0} - k\frac{cosnx}{n}\Big|_{0}^{\pi}\right) = \frac{2k}{n\pi}[1-\cos n\pi]$$

Since

$$\cos n\pi = \begin{cases} -1, for\ odd\ n \\ 1, for\ even\ n \end{cases}$$

then

$$b_n = \begin{cases} 4k/n\pi, for\ odd\ n \\ 0, for\ even\ n \end{cases}$$

Therefore,

$$f(x) = \frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

in summation symbol

$$f(x) = \frac{4k}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)\,x$$

Partial sums are,

$$s_1 = \frac{4k}{\pi}\sin x$$

$s_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$ etc.

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\right)$$

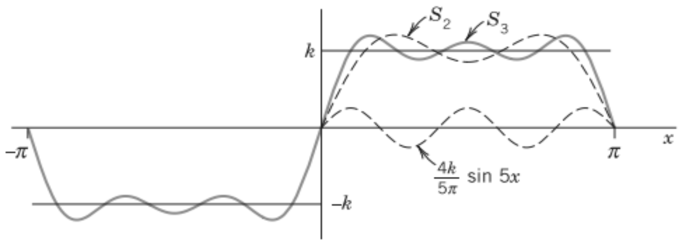


Figure. Partial sums s_1, s_2

Functions of any period p=2L:

For a function $f(x)$ with a period of $2L$, the Euler formulae for the Fourier coefficients are found as follows.

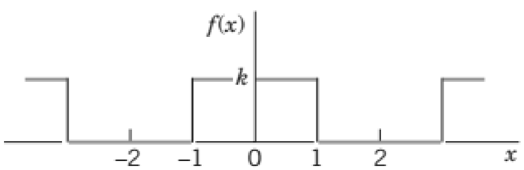
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx \qquad n = 1,2 \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx \qquad n = 1,2 \ldots$$

Example 5.2

Find the Fourier series of the function given the figure



$$f(x) = \begin{cases} 0, & -2 < x < -1 \\ k, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases} \quad \mathrm{P} = 2\mathrm{L} = 4 \Rightarrow \mathrm{L} = 2$$

Using the Euler formulae it is found that

$$a_0 = \frac{1}{4}\left(\int_{-2}^{2} f(x)dx = \frac{1}{4}\int_{-2}^{2} 0\,dx + \int_{-1}^{0} k\,dx + \int_{0}^{1} k\,dx + \int_{1}^{2} 0\,dx\right) = \frac{k}{2}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{4\pi x}{2}dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{4\pi x}{2}dx = \frac{k}{2}\frac{2}{4\pi}\sin\frac{4\pi x}{2}\Big|_{-1}^{1}$$

$$= \frac{k}{n\pi}\left[\sin\frac{n\pi}{2} - \sin\frac{(-n\pi)}{2}\right] = \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

Therefore

$$a_n = \begin{cases} 0, & \text{n is even} \\ \dfrac{2k}{n\pi}, & \text{n} = 1,5,9,\ldots \\ -\dfrac{2k}{n\pi}, & \text{n} = 3,7,11,\ldots \end{cases}$$

$$b_n \quad = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = -\frac{1}{2}\int_{-1}^{1} k\sin\frac{n\pi x}{2}\,dx = \frac{k}{2}\left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_{-1}^{1}\right]$$

$$= -\frac{k}{n\pi}\left[\cos\frac{n\pi}{2} - \cos\frac{n\pi}{2}\right] = 0$$

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x + \cdots\right)$$

in summation symbol

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\frac{(2n-1)\pi}{2}x$$

Even and odd functions

Definition: Let $g(x)$ is a function that is defined $\forall x \in R$.

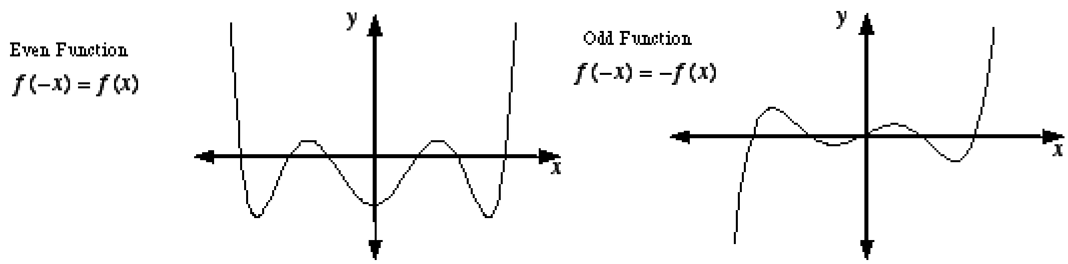
Function $y = g(x)$ is even, if $g(-x) = g(x),\ \forall x$

The graph of such a function is symmetric with respect to the y-axis.

The function $g(x)$ is odd, if $g(-x) = -g(x), \forall x \in R$

The graph of such a function is symmetric with respect to the origin.

The function $\cos nx$is even, while $\sin nx$ is odd



(a) (b)

Figure. (a) an even function, (b) an odd function,

Integrals of Even and odd functions over a Symmetrical Interval

If $g(x)$ is an even function then;

$$\int_{-2}^{2} g(x)dx = 2\int_{0}^{2} g(x)dx$$

If h(x) is an odd function then;

$$\int_{-2}^{2} h(x)dx = 0$$

The product $q = gh$ is odd, because;

$$q(-x) = g(-x)h(-x) = g(x)\big(-h(x)\big) = -g(x)h(x) = -q(x) \quad \text{odd}$$

Theorem 5.1: The Fourier series of an even function of period $2L$ is a Fourier cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

$a_0 = \frac{1}{L}\int_0^L f(x)\,dx,\ a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx,\ n = 1,2,\ldots$

The Fourier series of an odd function of period $2L$ is a Fourier sin series

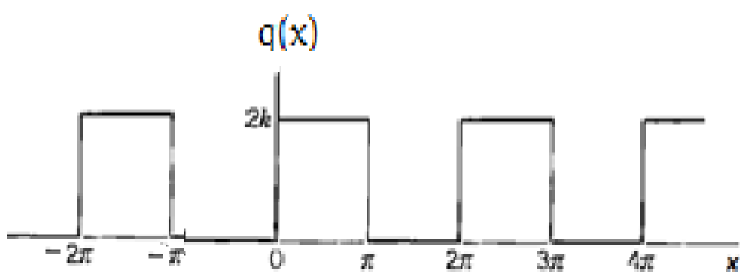
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{2\pi}{L}x$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx \quad n = 1,2,\ldots$$

Theorem 5.2: Let f_1, f_2 be two periodic functions with a common period p. Then Fourier coefficients of $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2. The Fourier coefficients of $c \cdot f$ are c times the corresponding Fourier coefficients of f.

Example 5.3

Find the Fourier series of the function q(x)



$$q(x) = \begin{cases} 0, & -\pi < x < 0 \\ 2k, & 0 < x < \pi \end{cases}, \qquad q(x + 2\pi) = q(x)$$

It has been seen that the Fourier series for $f(x)$

$$f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}, \qquad q(x + 2\pi) = q(x)$$

is

$$f(x) = \frac{4k}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x)$$

and

$$q(x) = k + f(x)$$

The Fourier series for the constant function $g(x) = k$ is the same. According to the superpositiğon theorem we have for the Fourier series for $q(x)$:

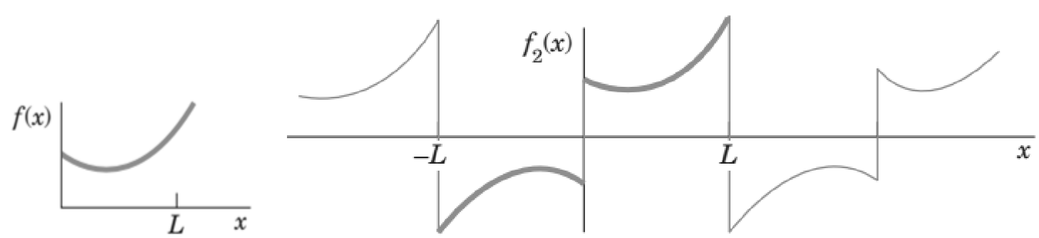
$$q(x) = k + \frac{4k}{\pi}(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x)$$

In general

$$f(x) = k + \frac{4k}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin(2n-1)\,x$$

Half range expansions:

In various applications, there may be a practical need to use Fourier series of a function $f(x)$ defined within the interval $0 \le x \le L$. We could extend this function to the range of $-L \le x \le L$ in two ways to make it periodical and expand it to a Fourier series. If we expand it to the right to have an odd function, then we have only sine terms in the Fourier series, and the series is called Fourier sine series.



. (a) (b)

Figure (a) $f(x)$ is given in the half interval $0 \le x \le L$, extended as an odd periodic function of period $2L$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) cos\, dx = 0, \qquad n = 1,2, \ldots$$

and

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi}{L}x\, dx$$

Hence

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x.$$

Alternatively if we expand it to the right to have an even function, then we have only cosine terms in the Fourier series, and the series is called Fourier cosine series. These two extension are called the ‘’two half range extensions’’ of the function $f(x)$.

If we extend $f(x)$ from $0 \le x \le L$ to $-L \le x \le L$ as an even function, then extend this new function as a periodic function of period $2L$, since it is odd, the Fourier series of it is a Fourier *cosine* series .

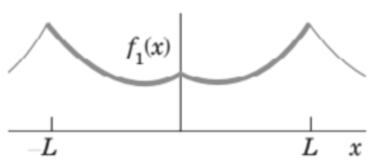


Figure. $f(x)$ extended as an even periodic function of period 2L

The following *cosine* half range expansion is (*even*)

where

$$a_0 = \frac{1}{L}\int_{0}^{L} f(x)\, dx, \quad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi}{L}x\, dx, \quad n = 1,2, \ldots$$

and

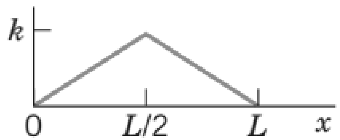
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx = 0, \qquad n = 1,2,\ldots$$

Hence

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

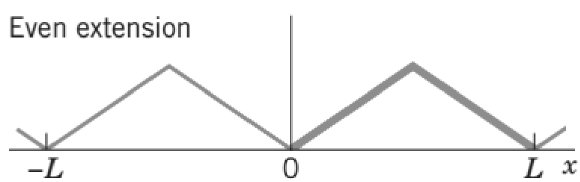
Example 5.4

Find the half range expansions of the function defined in $0 \leq x \leq L$.



$$f(x) = \begin{cases} \dfrac{2k}{L}x, & \text{if } \ 0 < x < \dfrac{L}{2} \\ \dfrac{2k}{L}(L-x), & \text{if } \ \dfrac{L}{2} < x < L \end{cases}$$

a) Let us extend $f(x)$ to the interval $-\mathrm{L} \leq x \leq L$ as an even periodic function with period $2L$.



Then

$$a_0 = \frac{k}{L^2}\left(\int_0^{L/2} x\,dx + \int_{L/2}^{L}(L-x)dx\right) = \frac{k}{L^2}\left(\frac{L^2}{8} + L^2 - \frac{L^2}{2} - \frac{L^2}{2} + \frac{L^2}{8}\right) = \frac{k}{4}$$

$$a_n = \frac{4}{L}\left(\frac{2k}{L}\int_0^{L/2} x\cos\frac{n\pi}{L}x\;dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)\cos\frac{n\pi}{L}x\,dx\right)$$

$$\int_{0}^{L/2} x\cos\frac{n\pi}{L}x\ dx = \frac{Lx}{n\pi}\sin\frac{n\pi}{L}x\Big|_{0}^{L/2} - \frac{L}{n\pi}\int_{0}^{L/2}\sin\frac{n\pi}{L}x\,dx = \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

$$\int_{L/2}^{L}(L-x)\cos\frac{n\pi}{L}x\,dx \quad = \frac{L}{n\pi}(L-x)\sin\frac{n\pi}{L}x\Big|_{0}^{L/2} + \frac{L}{n\pi}\int_{L/2}^{L}\sin\frac{n\pi}{L}x\,dx$$

$$= \left(0 - \frac{L}{n\pi}\left(L - \frac{L}{2}\right)\sin\frac{n\pi}{L}\right) - \frac{L^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right),\quad a_2 = -\frac{16k}{2^2\pi^2},\quad a_6 = -\frac{16k}{6^2\pi^2},\quad a_6 = -\frac{16k}{10^2\pi^2}$$

$a_n > 0$ if $n \neq 2,6,10,14,\ldots$

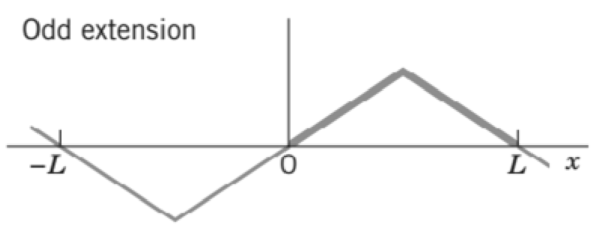
Hence even periodic expansion is

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right)$$

with the summation symbol

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{2^2(2n-1)^2}\cos\frac{2\pi(2n-1)}{L}x$$

b) Odd periodic expansion. Let us extend $f(x)$ to the interval $-\mathrm{L} \le x \le L$ as an odd periodic function with period $2L$. Then



$$b_n = \frac{4}{L}\left[\frac{2k}{L}\left[\int_{0}^{L/2} x\sin\frac{n\pi}{L}x\ dx + \int_{L/2}^{L}(L-x)\sin\frac{n\pi}{L}x\,dx\right]\right]$$

$$\int_0^{\frac{L}{2}} x\sin\frac{n\pi}{L}x\,dx \quad = \frac{-Lx}{n\pi}\cos\frac{n\pi}{L}x + \frac{L}{n\pi}\int_0^{\frac{L}{2}}\cos\frac{n\pi}{L}x\,dx = \frac{-Lx}{n\pi}\cos\frac{n\pi}{L}x\Big|_0^{\frac{L}{2}} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{L}x\Big|_0^{\frac{L}{2}}$$

$$= -\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\int_{L/2}^{L}(L-x)\sin\frac{n\pi}{L}x\,dx \quad = \left[-\frac{L^2}{2n\pi}\cos\frac{n\pi}{L}x + \frac{L}{n\pi}x\cos\frac{n\pi}{L}x - \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}x\right]\Bigg|_{\frac{L}{2}}^{L}$$

$$= \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\left(\sin n\pi - \sin\frac{n\pi}{2}\right) = \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$b_n = \frac{8k}{L^2}\left[\frac{2L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence odd periodic expansion is

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - \cdots\right)$$

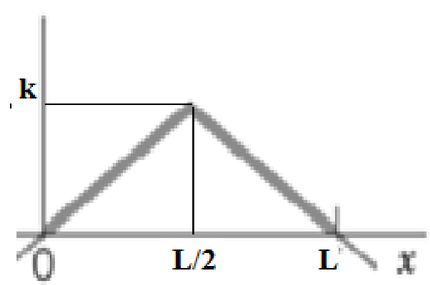
with the summation symbol

$$f(x) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin\frac{\pi(2n-1)}{L}x$$

Example 5.5

Find the Fourier coefficients of the periodic function $f(x)$ with period L

$$f(x) = \begin{cases} \frac{2k}{L}x, & 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$$



$$f(x) = \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

$$a_0 = \frac{\mathrm{k}}{2}$$

$$a_n = \frac{1}{L}\int_0^L f_{(x)} \cos\frac{2n\pi x}{L}\,dx$$

$$a_n = \frac{4\mathrm{k}}{L^2}\left(\int_0^{L/2} x\cos\frac{2n\pi x}{L}\,dx + \int_{L/2}^{L} (L-x)\cos\frac{2n\pi x}{L}\,dx\right) = \frac{2\mathrm{k}}{n^2\pi^2}(\cos n\pi - 1)$$

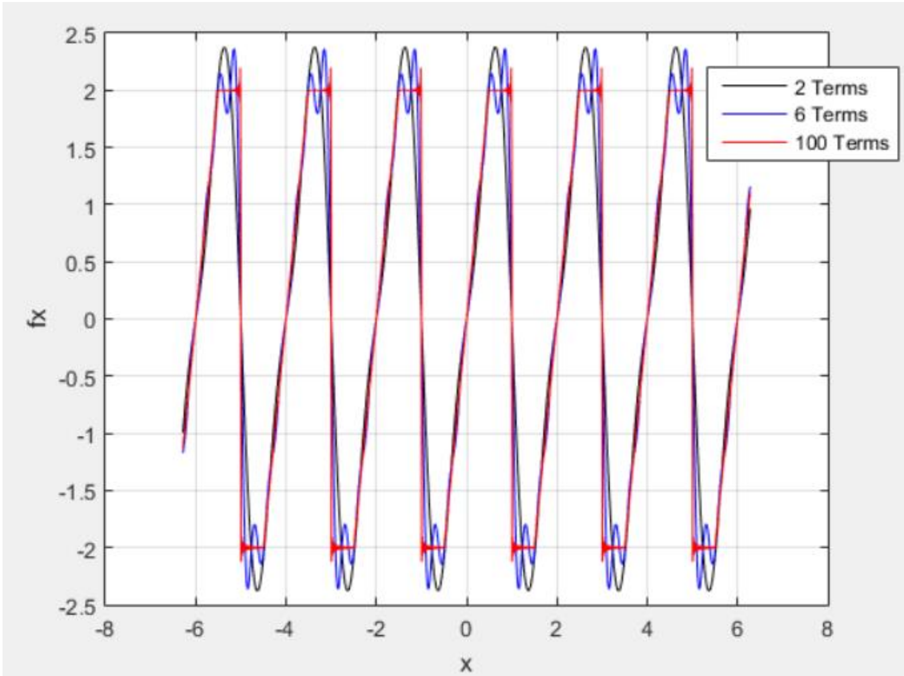
$$b_n = \frac{4\mathrm{k}}{L^2}\left(\int_0^{L/2} x\sin\frac{2n\pi x}{L}\,dx + \int_{L/2}^{L} (L-x)\sin\frac{2n\pi x}{L}\,dx\right) = 0$$

$$f(x) = \frac{\mathrm{k}}{2} + \sum_{n=1}^{\infty} \frac{2\mathrm{k}}{n^2\pi^2}(\cos n\pi - 1)\cos\frac{2n\pi x}{L}$$

Matlab Code:

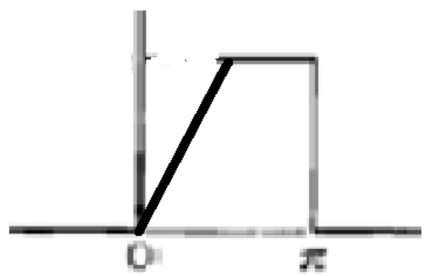
```
x=-2*pi:0.01:2*pi;
y1=0;y2=0;y3=0;
k=1;
L=1;
for n=1:2
    y1=y1+(8*k*sin(n*pi/2)/n^2/pi^2-4*k*cos(n*pi)/n/pi)*sin(n*pi*x/L);
end
for n=1:6
    y2=y2+(8*k*sin(n*pi/2)/n^2/pi^2-4*k*cos(n*pi)/n/pi)*sin(n*pi*x/L);
end
for n=1:100
    y3=y3+(8*k*sin(n*pi/2)/n^2/pi^2-4*k*cos(n*pi)/n/pi)*sin(n*pi*x/L);
end
plot(x,y1,'k',x,y2,'b',x,y3,'r')
xlabel('x')
ylabel('fx')
legend('2 Terms','6 Terms','100 Terms')
grid on
```

Figure:



Example 5.6

Find the Fourier series of the function $f(x)$



$$f(x) = \begin{cases} x & \text{if} \quad 0 < x < \pi/2 \\ \pi/2 & \text{if} \quad \pi/2 < x < \pi \end{cases}$$

a) Odd

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad (L = \pi)$$

$$b_n = \frac{2}{L}\left[\int_0^{\pi/2} x\sin\left(\frac{n\pi x}{L}\right) dx + \int_{\pi/2}^{\pi} \frac{\pi}{2}\sin\left(\frac{n\pi x}{L}\right) dx\right] = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin(nx)dx + \int_{\pi/2}^{\pi} \frac{\pi}{2}\sin(nx)dx\right]$$

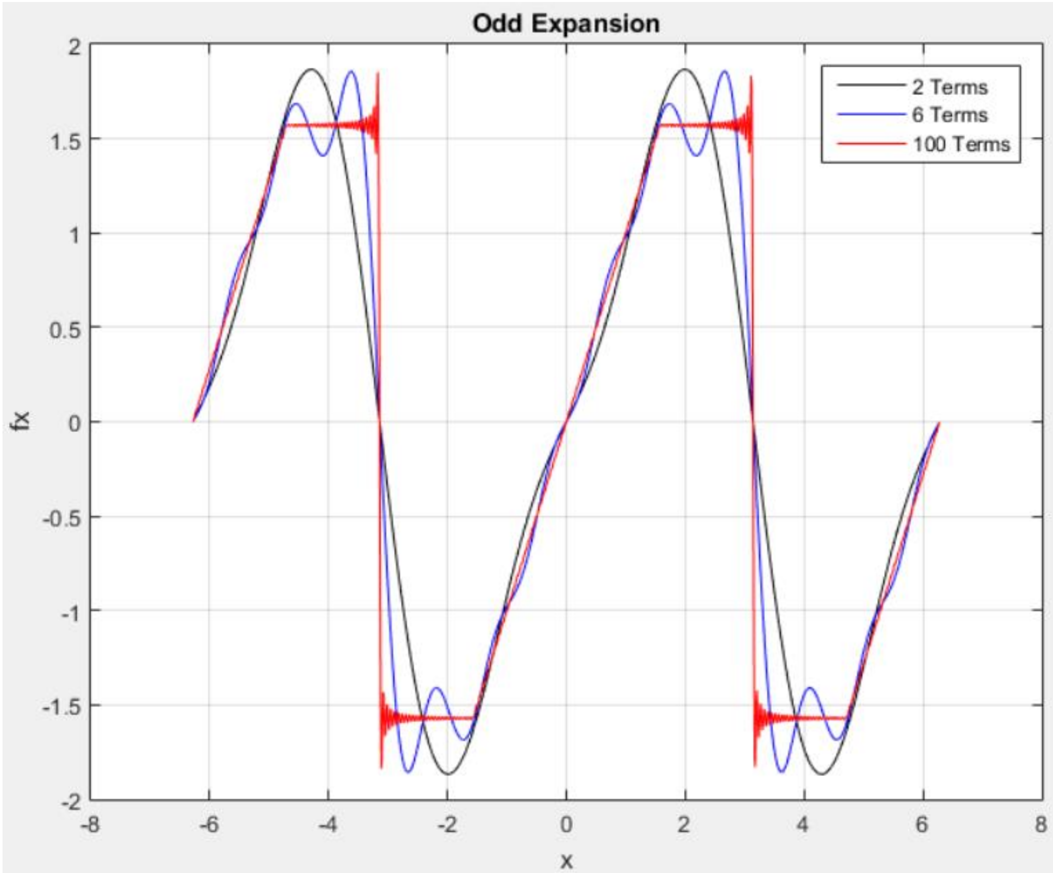
$$u = x \quad \Longrightarrow \quad du = 1$$
$$v = \frac{-\cos(nx)}{n} \quad \Longrightarrow \quad dv = \sin(nx)$$

$$b_n = \frac{2}{\pi}\left[\frac{-\pi\cos\left(\frac{n\pi}{2}\right)}{2n} - \int_0^{\pi/2} \frac{-\cos(nx)}{n}\,dx - \frac{\pi\cos(n\pi)}{2n} + \frac{\pi\cos\left(\frac{n\pi}{2}\right)}{2n}\right] = \frac{2}{\pi}\left[\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2} - \frac{\pi\cos(n\pi)}{2n}\right]$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left(\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2} - \frac{\pi}{2n}\cos(n\pi)\right)\sin(nx)$$

Matlab:

```
x=-2*pi:0.01:2*pi;
y1=0;y2=0;y3=0;
for n=1:2
    y1=y1+2/pi*(sin(pi*n/2)/n^2-pi*cos(pi*n)/2/n)*sin(n*x);
end
for n=1:6
    y2=y2+2/pi*(sin(pi*n/2)/n^2-pi*cos(pi*n)/2/n)*sin(n*x);
end
for n=1:100
    y3=y3+2/pi*(sin(pi*n/2)/n^2-pi*cos(pi*n)/2/n)*sin(n*x);
end
plot(x,y1,'k',x,y2,'b',x,y3,'r')
xlabel('x')
ylabel('fx')
title('Odd Expansion')
legend('2 Terms','6 Terms','100 Terms')
grid on
```



b) Even

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi/2} xdx + \int_{\pi/2}^{\pi} \frac{\pi}{2}dx\right] = \frac{1}{\pi}\left[\frac{\pi^2}{8} + \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right)\right] = \frac{3\pi}{8}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos(nx)dx + \frac{\pi}{2}\int_{\pi/2}^{\pi}\cos(nx)dx\right]$$

$$\begin{array}{llll} u = & x & \Longrightarrow & du = & 1 \\ v = & \dfrac{\sin(nx)}{n} & \Longrightarrow & dv = & \cos(nx) \end{array}$$

$$a_n \quad = \frac{2}{\pi}\left[\frac{\pi\sin\left(\frac{n\pi}{2}\right)}{2n} - \int_0^{\pi/2} \frac{\sin(nx)}{n}\,dx + \frac{\pi}{2n}\sin(\pi n) - \frac{\pi}{2n}\sin\left(\frac{\pi n}{2}\right)\right]$$

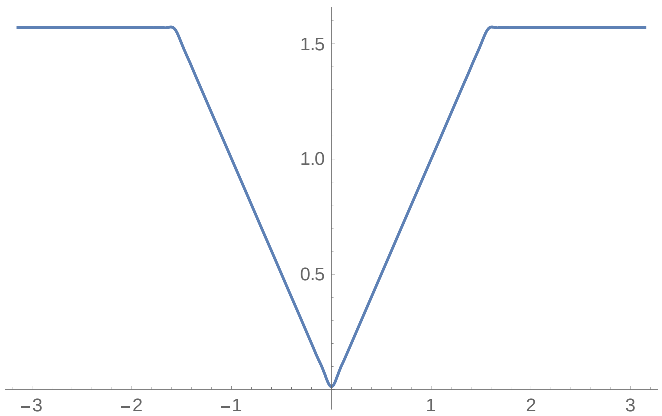
$$= \frac{2}{\pi}\left[\frac{\pi\sin\left(\frac{n\pi}{2}\right)}{2n} + \frac{\cos\left(\frac{n\pi}{2}\right)}{n^2} - \frac{1}{n^2} - \frac{\pi}{2n}\sin\left(\frac{\pi n}{2}\right)\right] = \frac{2}{\pi}\left(-\frac{1}{n^2}\right)$$

$$f(x) = \frac{3\pi}{8} + \sum_{n=1}^{\infty} \frac{2}{\pi}\left(\frac{\cos\left(\frac{n\pi}{2}\right) - 1}{n^2}\right)\cos(nx)$$

Matlab:

```
x=-2*pi:0.01:2*pi;
y1=0;y2=0;y3=0;
for n=1:2
    y1=y1+2/pi*((cos(n*pi/2)-1)/n^2)*cos(n*x);
end
for n=1:6
    y2=y2+2/pi*((cos(n*pi/2)-1)/n^2)*cos(n*x);
end
for n=1:100
    y3=y3+2/pi*((cos(n*pi/2)-1)/n^2)*cos(n*x);
end
plot(x,y1,'k',x,y2,'b',x,y3,'r')
xlabel('x')
ylabel('fx')
title('Even Expansion')
legend('2 Terms','6 Terms','100 Terms')
grid on
```

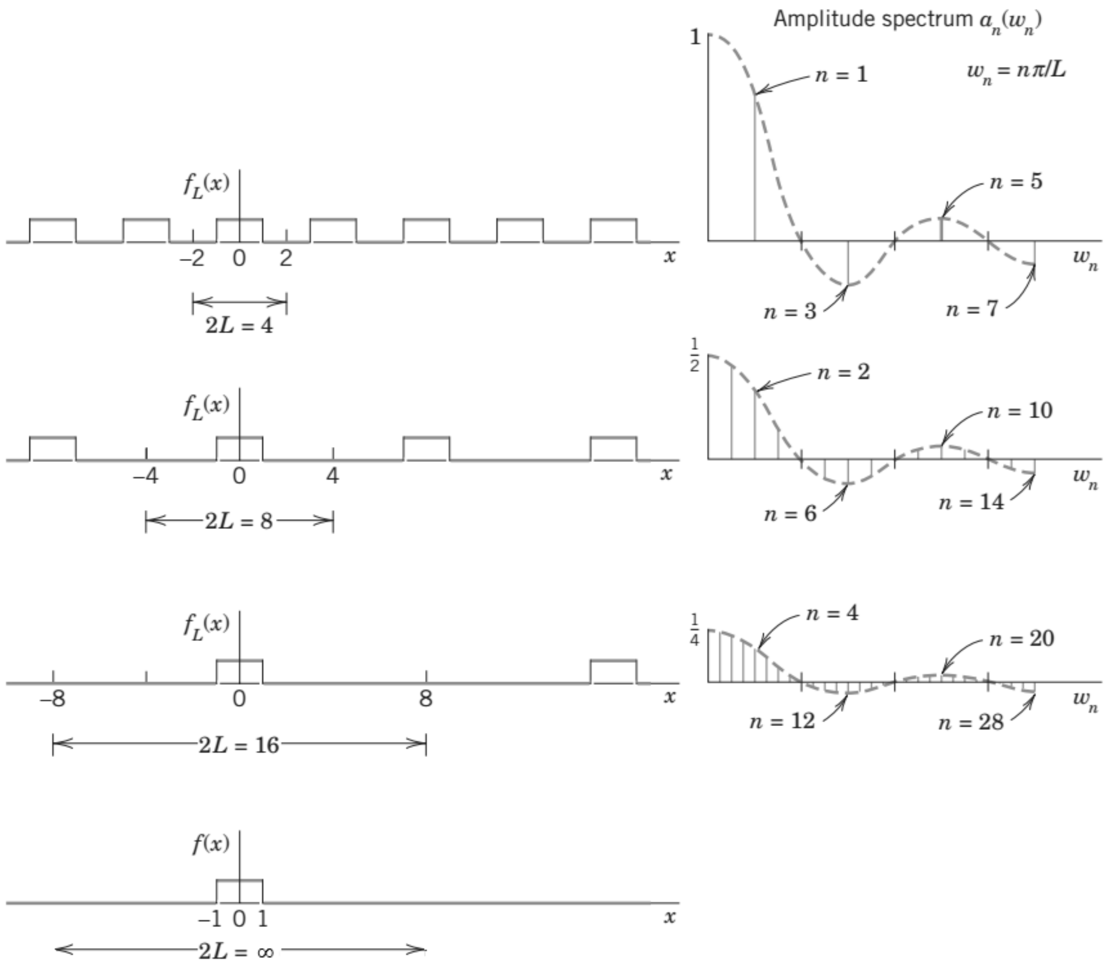
Type equation here.


1.5
1.0
0.5
−3
−2
−1
1
2
3

6.2 Fourier Integrals

Many practical problems involve non-periodic functions. So, it is desired to extend the method of Fourier series to such functions. Assume that we have the function $f_L(x)$ of period $2L$ and let $L \to \infty$. Then the resulting function is no longer periodic.

$$f_L(x) = \begin{cases} 0, & \text{if } -L < x < -1 \\ 1, & \text{if } -1 < x < 1 \\ 1, & \text{if } \ 1 < x < L \end{cases}$$



What happens to the Fourier Coefficients of f_L as L increases?

Since f_L is even $b_n = 0$ for all n

$$a_0 = \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L}, \quad a_n = \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{1} \cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\frac{\sin(n\pi/L)}{n\pi/L}$$

This sequence of Fourier Coefficients is called the amplitude spectrum of f_L.

Consider any periodic function $f_L(\mathrm{x})$ of period $2L$ that can be represented by Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty}(a_n Cos w_n x + \ b_n \, Sin \, w_n \, x) \, , \quad w_n \, = \frac{n\pi}{L}$$

Substitution of a_n and b_n and denoting the variable of integration by v , $f_L$yields that

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\, dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos w_n x \int_{-L}^{L} f_L(v) \cos w_n v \, dv + \sin w_n x \int_{-L}^{L} f_L(v) \sin w_n v \, dv\right] \quad (*)$$

Setting,

$$\Delta w \, = w_{n+1} - w_n \, = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$$

Hence we have

$$\frac{1}{L} = \frac{\Delta w}{\pi}$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\, dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\cos w_n x \, \Delta w \int_{-L}^{L} f_L(v) \cos w_n v \, dv + \sin w_n x \, \Delta w \int_{-L}^{L} f_L(v) \sin w_n v \, dv\right]$$

Valid for any fixed L, arbitrary large, but finite

We now let $L \to \infty$ and assume that the resulting nonperiodic function

$$f(x) \, = \lim_{L\to\infty} f_L \, (x)$$

is absolutely integrable on the x-axis; that is, the following limits exist

$$\lim_{a\to-\infty}\int_{a}^{0} |f(x)| dx + \lim_{b\to\infty}\int_{0}^{b} |f(x)| dx \, , \quad \left(= \int_{-\infty}^{\infty} |f(x)| dx \right)$$

Then, since

$$\Delta \mathrm{w} = \frac{\pi}{L} \longrightarrow 0, \ \ 1/L \longrightarrow 0 \, ,$$

first integral in (*) is zero. Infinite series turns into the integral form

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left[\cos wx \int_{-\infty}^{\infty} f(v) \cos wv \ dv + \sin wx \int_{-\infty}^{\infty} f(v) \sin wv \ dv\right] dw$$

Let

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv, \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\, dv$$

then

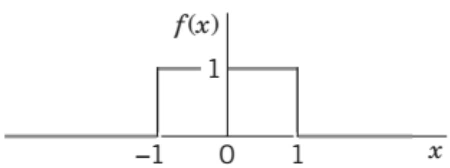
$$f(x) = \int_{0}^{\infty} [A(w)\cos wx + B(w)\sin wx]\, dw$$

This is the representation of f(x) by ***Fourier Integral***

<u>Theorem 5.3:</u> If f(x) is piecewise continuous in every finite interval, and has right- and left-hand side derivatives at every point and the integral $\int_{-\infty}^{\infty} f|x|dx$ exists, then f(x) can be represented by a Fourier integral. At a point where f(x) is discontinuous, the average of the right and left limits of the function f(x) is used in the Fourier integral.

<u>Example 5.7</u>

Single Pulse, sine integral



$$f(x) = \begin{cases} 1, & \text{if} |x| < 1 \\ 0, & \text{if} |x| > 1 \end{cases}$$

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv = \frac{1}{\pi}\int_{-1}^{1} \cos wv\, dv = \sin\frac{wv}{\pi w}\Big|_{-1}^{1} = 2\frac{\sin w}{\pi w}$$

$$B(w) = \frac{1}{\pi}\int_{-1}^{1} f(v)\sin wv\, dv = 0$$

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \frac{\cos wx \sin w}{w}\, dw$$

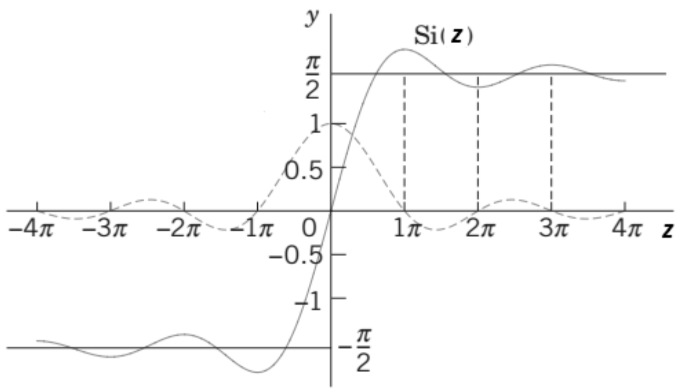
The average of the left- and right-hand side limits of $f(x)$ at $x = 1$is equal to (0+1)/2 = 0.5

$$f(x) = \int_0^\infty \frac{\cos wx \sin w}{w} dw \ = \begin{cases} \pi/2 & \text{if} \quad 0 < x < 1 \\ \pi/4 & \text{if} \qquad x = 1 \\ 0 & \text{if} \qquad x > 1 \end{cases} \qquad (**)$$

This integral is called Dirichlet's discontinuous factor. If $x = 0$

$\int_0^\infty \frac{\sin w}{w} dw = \frac{\pi}{2}$ the limit of sine integral

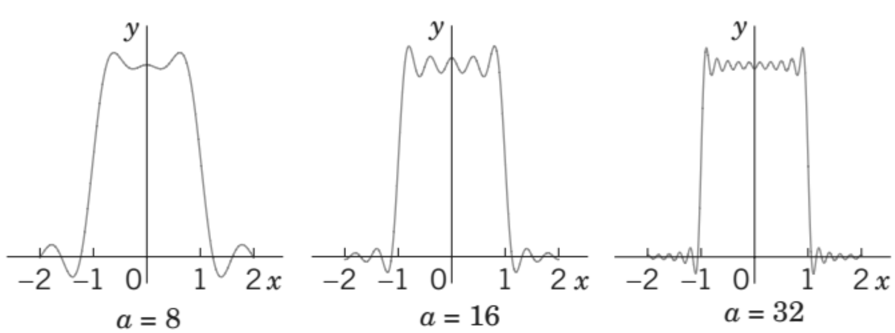
$\mathrm{Si}(z) = \int_0^z \frac{sinw}{w} \, dw$ as $z \longrightarrow \infty$



In the case of Fourier series, the graphs of the partial sums are approximation curves of the curve of the periodic function represented by the series. Similarly, for better approximation of Fourier integral, the upper limit a in the approximate integral

$\int_0^a \frac{\cos wx \sin w}{w} dw$

can be increased as follows.



Fourier Cosine and Sin integrals:

The Fourier integral becomes simpler for an even or odd function. If $f(x)$ is an **even function** then $B(w) = 0$, Then Fourier integral is the Fourier cosine integral.

$$f(x) = \int_0^\infty A(w) \cos wx \, dw \quad \text{where} \quad A(w) = \frac{2}{\pi} \int_0^\infty f(v) \cos wv \, dv$$

$$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$$

If the Fourier cosine transform of $f(x)$. $f(x)$ is retrieve from

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw$$

If $f(x)$ is an **odd function** then $A(w) = 0$

$$f(x) = \int_0^\infty A(w) \sin wx \, dw \quad \text{where} \quad B(w) = \frac{2}{\pi} \int_0^\infty f(v) \sin wv \, dv$$

Then the Fourier integral reduces to Fourier sine integral as follows.

$$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$$

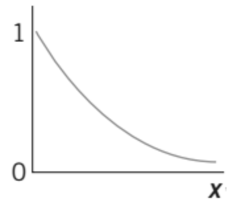
is the Fourier sine

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dw$$

is the inverse transform

$$F_c = \hat{f}_c(w), \quad F_s = \hat{f}_s(w)$$

Example 5.8



Find the Fourier cosine and sine integrals of

$f(x) = e^{-kx}$ where $x > 0, \quad k > 0$

$$A(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cos wv\, dv = \frac{2}{\pi}\left[-\frac{k}{k^2 + w^2}\left(-\frac{w}{k}\sin wv + \cos wv\right)\right]_0^\infty = \frac{k}{k^2 + w^2}$$

Thus;

$$A(w) = \frac{2k/\pi}{k^2 + w^2}$$

$$f(x) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos wx}{k^2 + w^2}dw \quad = \frac{2k}{\pi} \underbrace{\int_0^\infty \frac{\cos wx}{k^2 + w^2}dw}_{\text{Laplace integral}} = e^{-kx} \quad x > 0, \quad k > 0$$

similarly,

$$B(w) = \frac{2}{\pi}\int_0^\infty e^{-kv}\sin wv\, dv = -\frac{2w}{\pi(k^2 + w^2)}e^{-kv}\left(\frac{k}{w}\sin wv + \cos wv\right)\Big|_0^\infty = \frac{2w}{\pi(k^2 + w^2)}$$

$$f(x) = e^{-kx} = \frac{2}{\pi}\int_0^\infty \frac{w\, sin\, wx}{k^2 + w^2}dw \quad = \frac{2}{\pi} \underbrace{\int_0^\infty \frac{w\, sin\, wx}{k^2 + w^2}dw}_{\text{Laplace integral}} = e^{-kx} \quad x > 0, \quad k > 0$$

INTEGRAL TRANSFORMATIONS: FOURIER TRANSFORM

Integral transformations are powerful tools to simplify equations that transform them from time domain to a complex domain using a complex integral relation. The motivation behind integral transformation is that the mathematical model of a linear system in time domain may become much simpler to handle in a complex domain transformed; once the solution is obtained in complex domain, the solution of the problem in time domain can directly be obtained by the inverse integral transformation of this solution, as shown in Fig 6.1. Two common integral transformations will be explained in this section, namely, Fourier transformation and Laplace transformation.

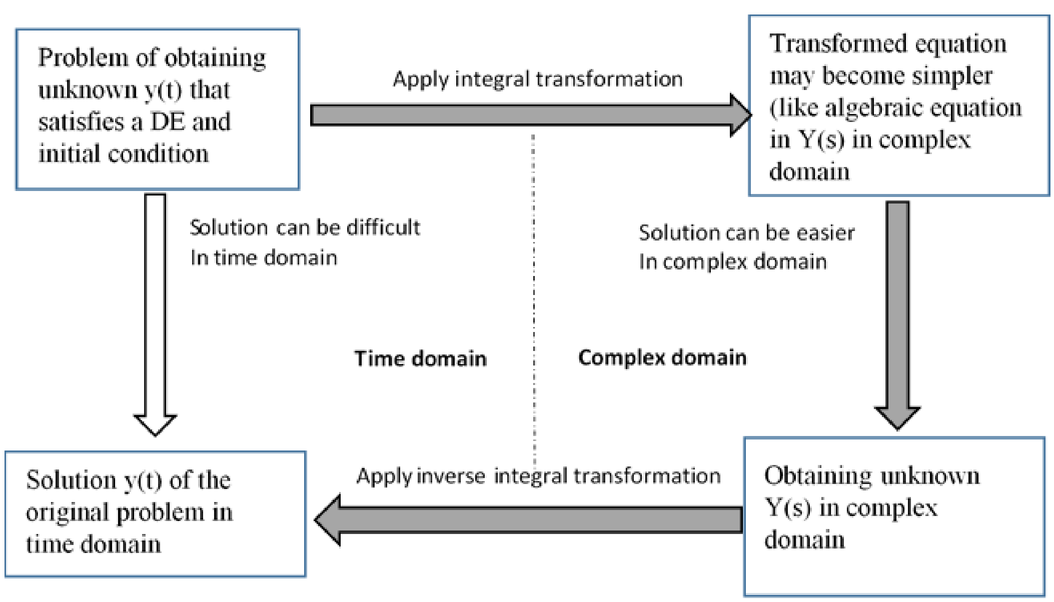


Figure 6.1 Relationship between time domain and complex domain in integral transformations.

6.1 Fourier Transform

Before explaining Fourier transformations, complex form of the Fourier Integral is required to be defined. The real Fourier integral is already known as below.

$$f(x) = \int_0^\infty [\mathrm{A(w)} \cos wx + B(w) \sin wx]\, dw$$

Where

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv\, dv$$

$$B(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv\, dv$$

Substituting A and B into the integral for f

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(v) [\cos wx \cos wv + \sin wv \sin wx]\, dv\, dw$$

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\underbrace{\int_{-\infty}^{\infty} f(v) \cos(wx - wv) dv}_{\text{Even function of } w, F(w)} \right] dw$$

But f is not function of $w \therefore f(v)$

1) $$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) \cos(wx - wv)\, dv \right] dw$$

2) $$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) \sin(wx - wv)\, dv \right] dw = 0$$ Since sin is an odd function

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} G(w)\, dw = 0, \quad G(w) \quad \text{is odd}$$

Using Euler formula $e^{it} = \cos t + i \sin t$, Adding $1 + 2$

Setting $t = wx - wv$ complex Fourier integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v)\, e^{iw\,(x-v)}\, dv\, dw$$

In order to obtain complex form of Fourier integral, the exponential function can be substituted.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)\, e^{iwv}\, dv}_{f(w),\text{Fourier transform of } f} \right] e^{iwx} dw$$

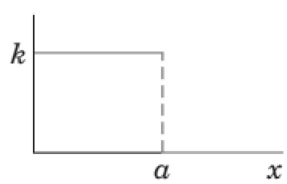
$$f(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)\, e^{iwv}\, dv$$

Then;
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(w)\, e^{iwx}\, dw$$

inverse Fourier transform of $f(w)$ or $F(f) = f(w)$ and F^{-1} for the inverse two conditions are sufficient for existence of the transform

1. $f(x)$ is piecewise continuous on every finite interval
2. $f(x)$ is absolutely integrable on the x-axis

Example 6.1



Find the Fourier transform of the function

$$f(x) = \begin{cases} k, & \text{if} 0 < x < a \\ 0, & \text{if} \quad x > a \end{cases}$$

$$f(w) = \frac{1}{\sqrt{2\pi}} \int_0^a k\, e^{-iwx}\, dx = \frac{k}{\sqrt{2\pi}} \frac{\left(e^{-iwa} - 1\right)}{-iw}$$

$$F_c = \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} k \int_0^a \cos wx \;\, dx = \sqrt{\frac{2}{\pi}} k \left(\frac{\sin aw}{w}\right)$$

$$F_s = \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} k \int_0^a \sin wx \;\, dx = \sqrt{\frac{2}{\pi}} k \left(\frac{1 - \cos aw}{w}\right)$$

Fourier Transform of Derivatives:

The Fourier transform is a *linear operation*; that is, for any functions $f(x)$and $g(x)$ whose Fourier transforms exist and any constants a and b

$$\mathcal{F}(af + by) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

Theorem 6.1:

Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $x \to \infty$, Furthermore, Let $f'(x)$ be integrable on the x-axis. Then

$$\mathcal{F}\{f'(x)\} = iw\mathcal{F}\{f(x)\}$$

Proof:

From the definition of the Fourier transform;

$$\mathcal{F}\{f'(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)\, e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \left[f(x) e^{-iwx} \Big|_{-\infty}^{\infty} - (-iw) \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx \right]$$

$$= iw\mathcal{F}\{f(x)\}$$

$$\mathcal{F}\{f''\} = iw\mathcal{F}\{f'\} = (iw)^2 \mathcal{F}\{f\} = -w^2 \mathcal{F}\{f(x)\}$$

If $f(x)$ is even, then the Fourier integral

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos wx\, dw \int_0^{\infty} f(x) \cos wu\, du$$

Fourier cCosine transform is $$f_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(u) \cos wx\, dx$$

Inverse Inverse transform $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_c(w) \cos wx\, dw$$

If $f(x)$ is odd,

Fourier sin transform $$f_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx\, dx$$

Inverse Fourier sin transform $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f_s(w) \sin wx\, dw$$

Example 6.2

Solve the integral equation

$$\int_0^\infty f(x)\cos wx\, dx = \begin{cases} 1-w, & 0 \le w \le 1 \\ 0, & w > 1 \end{cases}$$

Let $\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\, dx = f(w)$ and choose $f(w) = \sqrt{\frac{2}{\pi}}(1-w)$

if $0 \le w \le 1$ and 0 if $w > 1$

Then by Fourier cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\, dx = \sqrt{\frac{2}{\pi}}\int_0^1 \sqrt{\frac{2}{\pi}}(1-w)\cos wx\, dw = \frac{2}{\pi}\int_0^1 (1-w)\cos wx\, dw$$

$$= \frac{2}{\pi}\left[\frac{\sin wx}{x} - \left[\frac{w}{x}\sin wx - \frac{1}{x}\int_0^1 \sin wx\, dw\right]\right] = \frac{2}{\pi}\left[\frac{\sin wx}{x} - \left[\frac{w}{x} sin\ wx + \frac{1}{x}\frac{\cos wx}{x}\right]\right]\Bigg|_0^1$$

$$= \frac{2}{\pi}\left[\frac{\sin wx}{x} - \left[\frac{\sin x}{x} + \frac{1}{x^2}\cos x - \frac{1}{x^2}\right]\right] = \frac{2}{\pi}\left[\frac{\sin x}{x} - \frac{\sin x}{x} - \frac{1}{x^2}(\cos x - 1)\right]$$

$$= \frac{2}{\pi}\left[\frac{1-\cos \mathrm{x}}{x^2}\right]$$

<u>Example 6.3</u>

Find to transform of the function

$$f(x) = \begin{cases} 1-w, & x > 0 \\ 0, & x < 0 \end{cases}$$

$$f(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\ e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x}\, e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{(-iw-1)x}\, dx = \frac{1}{\sqrt{2\pi}}\frac{e^{(-iw-1)x}}{-iw-1}\Bigg|_0^1 = \frac{1}{\sqrt{2\pi}}\left[\frac{1}{iw+1}\right]$$

Example 6.4

Find to transform of the function xe^{-x^2}

$$\mathcal{F}\left(xe^{-x^2}\right) = \mathcal{F}\left(-\frac{1}{2}\left(e^{-x^2}\right)'\right) = -\frac{1}{2}\mathcal{F}\left(e^{-x^2}\right) = -\frac{1}{2}iw\mathcal{F}\left(e^{-x^2}\right)$$

$$= -\frac{1}{2}iw\frac{1}{\sqrt{2}}\left(e^{-w^2/4}\right) = -\frac{\mathrm{iw}}{2\sqrt{2}}\left(e^{-w^2/4}\right)$$

Chapter 7

PARTIAL DIFFERENTIAL EQUATIONS

In the previous chapters of this book several aspects and solution techniques of ordinary differential equations are discussed. Although it is a vast topic, in this chapter a brief introduction to partial differential equations is introduced with emphasize on a basic method of solution which is easily adapted to many applied physics, and engineering, problems.

7.1. Solution Concept of Partial Differential Equations

As it is said in Chapter 1. during classification of differential equations, a partial differential equation is a differential equation which involves partial derivatives of one or more dependent variables with respect to one or more independent variables. A solution of a partial differential equation is an explicit or implicit relation between the variables, free of any derivatives and identically satisfies the equation.

Example 7.1

Consider the first-order partial differential equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

in which $u(x, y)$ is the dependent variable and *x* and *y* are independent variables. It is easy to see that

$$u(x, y) = \frac{y}{x}$$

is a solution since,

$$\frac{\partial u}{\partial x} = -\frac{y}{x^2} \rightarrow x\frac{\partial u}{\partial x} = -\frac{y}{x}, \qquad \frac{\partial u}{\partial y} = \frac{1}{x}, \rightarrow y\frac{\partial u}{\partial y} = \frac{y}{x}$$

Indeed

$$u(x, y) = \emptyset\left(\frac{y}{x}\right)$$

Is more general solution of the given first-order partial differential equation, where $\emptyset$ is any twice differentiable function.

Example 7.2

Consider the second-order partial differential equation

$$x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} = 0$$

It is easy to see that

$$u(x, y) = \cos(xy)$$

is a solution since,

$$\frac{\partial u}{\partial x} = -y\ \sin(xy) \rightarrow \frac{\partial^2 u}{\partial x^2} = -y^2\cos(xy) \rightarrow x^2\ \frac{\partial^2 u}{\partial x^2} = -x^2y^2\cos(xy)$$

and

$$\frac{\partial u}{\partial y} = -x\ \sin(xy) \rightarrow \frac{\partial^2 u}{\partial y^2} = -x^2\cos(xy) \rightarrow y^2\ \frac{\partial^2 u}{\partial y^2} = -x^2y^2\cos(xy)$$

Indeed

$$v(x, y) = \sin(xy)$$

is also a solution of the given second-order partial differential equation.

The linear combination of these two solutions with the two arbitrary constants C_1, C_2

$$w(x,y) = C_1 \cos(xy) + C_2 \sin(xy),,$$

Is even more general solution of the given partial differential equation.

The mathematical formulation of some physical or engineering problem involves a partial differential equation and also certain supplementary conditions: boundary, and initial conditions or both. The solution of the problem must satisfy both the partial differential equation and these supplementary conditions. In that sense it is particular solution of the given partial differential equation.

In the following section, a basic method is developed which may be employed to obtain this particular solution in certain cases. First, however, we consider briefly the class of partial differential equations which most frequently occurs in such problems. This is the class of so-called linear partial differential equations of the second order.

7.2 Linear Partial Differential Equations of the Second Order

The general linear partial differential equation of the second order in two independent variables *x* and *y* is an equation of the form

$$R(x,y)U_{xx} + S(x,y)U_{xy} + T(x,y)U_{yy} + P(x,y)U_x + Q(x,y)U_y + W(x,y)u = G(x,y)$$

If $G(x,y) = 0, \forall(x,y)$, the above equation is called homogeneous.

7.3 Linear PDEs of the Second Order with Constant Coefficients

The general linear partial differential equation of the second order in two independent variables *x* and *y* with constant coefficients is an equation of the form

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + Fu = G(x,y)$$

where A, B, C, D, E, F are real constants. If $G(x,y) = 0, \forall(x,y)$, the above equation is called homogeneous.

Canonical Forms

The second-order homogeneous linear partial differential equation with constant coefficients in the above is

1. **Hyperbolic Type,** if $B^2 - 4AC > 0$.

Canonical Form:

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

$A = 1,\ B = O,\ C = -1,$ and $B^2 - 4AÇ = 4 > 0$. This equation is a special case of the *one-dimensional wave equation*, which is satisfied by the small transverse displacements of the points *x* of a vibrating string, at the time *t*.

2. **Parabolic Type,** if $B^2 - 4AC = 0$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0$$

$A = 1,\ B = O,\ C = 0,$ and $B^2 - 4AC = 0$. This equation is a special case of the *one-dimensional heath equation*, or the diffusion equation, which is satisfied by the temperature at a point *x* of a homogeneous rod, at the time *t*.

3. **Elliptic Type,** if $B^2 - 4AC < 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$A = 1,\ B = O,\ C = 1,$ and $B^2 - 4AC < 0$. This equation is a special case of the *two-dimensional Laplace equation*, which is satisfied by the steady-state temperature at points (x, y) of a thin rectangular plate.

The mathematical formulation of some physical or engineering problems involve second order linear partial differential equations with constant coefficients with certain supplementary conditions. In the sequel the three main types of these kind equations are studied.

7.4 Engineering Problems Modelled by Linear Second Order PDEs

The mathematical formulation of some physical or engineering problems involve a linear partial differential equation of the second order with constant coefficients. There are three canonical types of them as seen in the above. In following sections, mathematical models of some common engineering problems, related to these three canonical types, are studied.

Vibration Problems

Some examples of vibration problems are:

1) Vibration of a wire, one dimensional vibration problem,
2) Vibration of a membrane; two dimensional vibration problem,
3) Vibration of air like gases; three dimensional vibration problem.

One Dimensional Vibration Problems:

A one dimensional vibration problem is modeled as;

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}$$

where c is a physical constant of 1-D vibration equation related to the speed of propagation of vibrations.

This equation may represent vibrations of a finite string

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2}, \qquad a \le x \le b, \qquad t > 0.$$

If the string is clamped at the two ends, the two boundary conditions are;

$U(a, t) = 0$ and $U(b, t) = 0; \quad \forall\, t > 0$.

If the string is clamped at the end $= a$, and free at the end $x = b$, two boundary conditions are;

$U(a, t) = 0$ and $\frac{\partial U}{\partial x}(b, t) = 0; \quad \forall\, t > 0$.

If the string is infinitely long, then we do not impose any boundary conditions.

A second class of conditions imposed on the wave equations are initial conditions:

Initial displacement $U(x, 0) = f(x)$

Initial velocity $\left.\frac{\partial U}{\partial t}\right|_{t=0} = g(x)$

Two Dimensional Vibration Problems:

A two dimensional vibration problem for a rectangular plate is modeled as,

$$\frac{\partial^2 \mathrm{U}}{\partial \mathrm{t}^2} = \mathrm{c}^2 \left(\frac{\partial^2 \mathrm{U}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{U}}{\partial \mathrm{y}^2}\right), \qquad a \le x \le b, c \le y \le d, \qquad t > 0$$

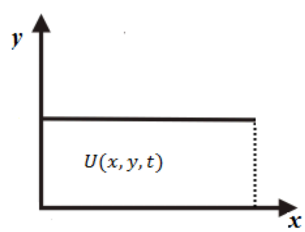


Figure 7.4 Two dimensional time varying domain.

If the plate is clamped on the frame, the four boundary conditions are;

$U(a, y, t) = U(b, y, t) = 0,\ c \leq y \leq d, \forall\, t > 0$ and

$$U(x, c, t) = U(x, d, t) = 0, \quad a \leq x \leq b, \forall\, t > 0$$

If there are free sides, then instead of (x, y, t) , $\partial U/\partial x$ is specified along these edges

A second class of conditions imposed on the wave equations are initial conditions.

Two possible initial conditions may be

$U(x, y, 0) = f(x, y);$ given initial displacement $f(x, y)$

$\left.\dfrac{\partial U}{\partial t}\right|_{t=0} = g(x);$ given initial velocity; $g(x, y)$

Three Dimensional Vibration Problems:

A three-dimensional vibration problem can be modeled as,

$$\frac{\partial^2 U}{\partial t^2} = c^2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right)$$

Operator ∇ (Greek letter nabla) is called the gradient operator and defined as

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

While the Laplace operator in three dimensions is defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Laplace operator is denoted sometimes by Δ, the Greek letter capital delta.

Hence the wave equation for three dimensional vibration problems is simplified as

$$\frac{\partial^2 U}{\partial t^2} = c^2 \nabla^2 U$$

To define the problem properly, boundary, as well as initial conditions must be imposed on the solutions.

Heat Dissipation and One-Dimensional Heat Flow Problems:

Similar to vibration problems, it is possible to define one, two and three dimensional heat dissipation problems. For a finite rod, a one-dimensional heat flow problem is modeled as,

$\frac{\partial U}{\partial t} = c^2 \frac{\partial^2 U}{\partial x^2}, \ a \leq x \leq b, \ t > 0$

Figure7.5 One dimensional heat flow.

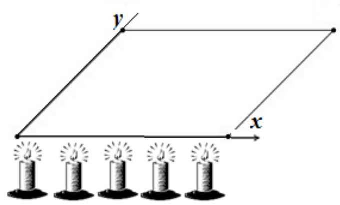
Possible boundary conditions for fixed end temperatures case are;

$$U(a,t) = U(b,t) = 0, \qquad \forall \ t > 0$$

and a possible initial condition $U(x, 0) = f(x)$ is imposed which specifies the initial temperature distribution on the rod.

Two-Dimensional Heat Flow Problem

A two-dimensional heat flow problem for a finite rectangular plate is modeled as



$$\frac{\partial U}{\partial t} = c^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \qquad a \le x \le b, c \le y \le d, \qquad t > 0$$

Figure7.6 Two dimensional heat flow.

If the frame of the plate is held at constant temperatures, T_a, T_b, T_c, T_d respectively, then possible boundary conditions may be

$U(a, y, t) = T_a,\ U(b, y, t) = T_a,\ c \le y \le d,\ \forall\, t > 0$ and

$U(x, c, t) = T_c, \qquad U(x, d, t) = T_d, \qquad a \le x \le b, \forall\, t > 0$

If some of the sides are left to cool to air, appropriate cooling conditions should be imposed as boundary conditions.

An initial condition $U(x, y, 0) = f(x, y)$ is also imposed which specifies the initial temperature distribution on the plate.

Three-Dimensional Heat Flow Problems

A three-dimensional heat flow problem on a rectangular box may be

$$\frac{\partial U}{\partial t} = c^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right),$$

$$a \le x \le b, c \le y \le d, \qquad e \le z \le f, \qquad t > 0$$

or briefly $\frac{\partial U}{\partial t} = c^2 \nabla^2 U.$

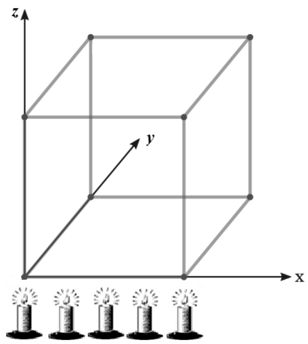


Figure7.7 Three dimensional heat flow.

To define the problem properly, on the solutions of the differential equation, certain appropriate boundary conditions is imposed. An initial condition $U(x, y, z, 0) = f(x, y, z)$ is also imposed which specifies the initial temperature distribution in the box.

The Laplace Equation: Steady State Temperature Distribution

Among the linear second order linear PDE's, elliptic equations are associated with steady state phenomena. The Laplace equation is the representative of this group.
Although the equation is appeared first in 1752 in a paper of Euler on hydrodynamics, Laplace's equation is named for Pierre-Simon de Laplace. Beginning from 1782, he studied the solution of this equation in gravitational field.

The Laplace equation in rectangular coordinates is of the form

$$\Delta_i u = 0, \quad \Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

in two and three dimensional spaces respectively.

Two-Dimensional Steady State Temperature Distribution Problems

A two dimensional steady state temperature distribution problem for a rectangular plate is modeled as,

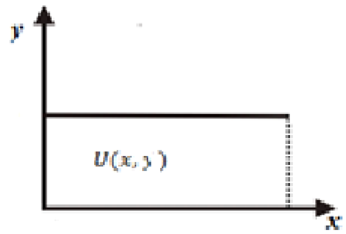


Figure 7.8 Two dimensional time invariant domain.

$$\frac{\partial^2 \mathrm{U}}{\partial \mathrm{x}^2} + \frac{\partial^2 \mathrm{U}}{\partial \mathrm{y}^2} = 0, \qquad a \leq x \leq b, c \leq y \leq d,$$

If the temperatures are specified on the frame, then the four boundary conditions are;

$U(a, y) = T_a,\ U(b, y) = T_a,\ c \leq y \leq d,$ and

$U(x, c) = T_c, \qquad U(x, d) = T_d, \qquad a \leq x \leq b.$

A three dimensional steady state temperature distribution problem for a box can also be modeled similarly.

7.5 Method of the Separation of Variables

In this section a basic method of solution, Method of the Separation of Variables is introduced with an emphasize on that it is easily adapted to many applied physics, and engineering, problems.

Method of the Separation of Variables for one Dimensional Vibrations

Consider the initial boundary value problem of one dimensional vibrations of a finite string

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L, \qquad t > 0.$$

If the string is clamped at the two ends, the two boundary conditions are;

$u(0,t) = 0$ and $u(L,t) = 0;$ $\forall\, t > 0$.

A second class of conditions imposed on the above wave equation are initial conditions:

Initial displacement $u(x,0) = f(x),\ \ 0 \le x \le L,$

Initial velocity $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x),\ \ 0 \le x \le L,$

Let us assume the one dimensional wave equation admits a solution as the product of two functions

$$u(x,t) = F(x)G(t)$$

Then

$$\frac{\partial u}{\partial t} = F\dot{G} \to \frac{\partial^2 u}{\partial t^2} = F\ddot{G}$$

where

$$\dot{G} = \frac{dG}{dt} \text{ and } \ddot{G} = \frac{d^2 G}{dt^2}$$

Similarly

$$\frac{\partial u}{\partial x} = F'G \to \frac{\partial^2 u}{\partial x^2} = F''G$$

where

$$F' = \frac{dF}{dx} \text{ and } F'' = \frac{d^2 F}{dx^2}$$

Substitute these into the PDE, then

$F\ddot{G} = c^2\, GF''$ dividing both sides by G(t). F(x) one has $\frac{\ddot{G}}{c^2\, G} = \frac{F''}{F}$

This equality is possible only when both sides are equal to the same constant

$$\frac{\ddot{G}}{c^2\, G} = \frac{F''}{F} = k \qquad k \text{ is a constant}$$

Then we have two ODE's

1) **$\boldsymbol{x}$ –Equation**

$F'' - kF = 0, \;\; 0 \le x \le L,$

and

2) **$\boldsymbol{t}$ –Equation**

$\ddot{G} - c^2 kG = 0, \qquad t > 0$

To have a solution $u(x,t) = F(x)G(t)$ periodic in time, $k = -\omega^2 < 0$

<u>Solution of the $\boldsymbol{x}$ – Equation</u>

$F'' + \omega^2 F = 0, \;\; 0 \le x \le L,$

A general solution is
$F(x) = A\ \mathrm{Cos}\ \omega x + B\ \mathrm{Sin}\ \omega x$

where A, B are arbitrary constants.

Applying BC's

$u(0,t) = F(0)G(t) = 0$ If $G(t) = 0, \; \forall\, t > 0$ then $u \equiv 0$ which is not interesting then $G(t) \neq 0$. Hence

a) $F(0) = 0 \Rightarrow A = 0$

Similarly

$u(L,t) = F(L)G(t) = 0$

b) $F(L) = 0 \Rightarrow B\text{Sin}\,\omega L = 0$

$B = 0$ leads to the trivial solution.

$\text{Sin}\,\omega L = 0 \Rightarrow \omega L = n\pi \Rightarrow \omega = n\pi/L,\ n \in Z$ an integer.

Therefore we get infinitely many solutions of the ODE which satisfy the boundary conditions

$F_n(x) = B_n\ \text{Sin}\frac{n\pi x}{L}$, $n \in Z$ an integer.

<u>Solution of the $t -$ Equation</u>

$\ddot{G} - c^2 kG = 0, \qquad t > 0$

Since $-k = \omega^2 = (n\pi/L)^2$, $n \in Z$ one has

$\ddot{G} + c^2(n\pi/L)^2 G = 0, \qquad t > 0$

Whose general solution is

$$G_n(t) = C_n \cos\frac{cn\pi}{\text{L}}t + D_n \sin\frac{cn\pi}{\text{L}}t,\ n \in Z$$

Then one has infinitely many solutions for the one dimensional wave equation which satisfy the boundary conditions:

$$u_n(x,t) = \left(C_n \cos\frac{cn\pi}{L}t + D_n \sin\frac{cn\pi}{L}t\right)\sin\frac{n\pi x}{L}, \qquad (n = 1,2,\ldots)$$

u_n's are called eigenfunctions

$\lambda_n = \frac{n\pi}{L}$'s are called eigenvalues (characteristic values) of the vibrating string.

Each u_n represents a harmonic motion having the frequency of

$\frac{\lambda_n}{2\pi} = \frac{n}{2L}$ cycles per unit time.

$u_n(x,t)$ is called the nth normal mode of the string, for $n = 1$ it is the fundamental mode. The others one known as overtones

$$\sin\frac{n\pi x}{L} = 0 \text{ at } x = \frac{L}{n}, \frac{2L}{n}, \dots, \frac{n-1}{n}L$$

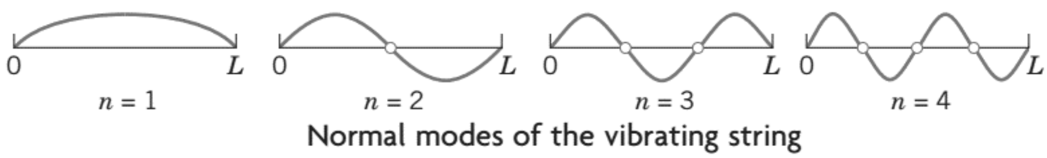


Figure 7.9 Mode shapes of a vibrating string.

Since the PDE is linear and homogeneous, sum of all possible solutions is also a solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos\frac{cn\pi}{\mathrm{L}}\, t + D_n \sin\frac{cn\pi}{\mathrm{L}} t\right) \sin\frac{n\pi x}{L}$$

With $C_n, D_n, n = 1,2,3, \dots$ not specified yet. To specify C_n, D_n we impose second class of conditions, which are the initial conditions:

1) Initial displacement $u(x,0) = f(x)\,,\ \ 0 \le x \le L,$

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{L} x = f(x)$$

Hence we must choose C_n's so that $u(x,0)$ becomes the Fourier series of the odd extension of $f(x)$ to the interval $-L \le x \le L,$

$$C_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx$$

2) Initial velocity $\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \left[\sum_{n=1}^{\infty}\left(-C_n \frac{cn\pi}{\mathrm{L}} \sin\frac{cn\pi}{\mathrm{L}} t + D_n \frac{cn\pi}{\mathrm{L}} \cos\frac{cn\pi}{\mathrm{L}}\ t\right) \sin\frac{n\pi}{L} x\right]_{t=0}$$

$$= \sum_{n=1}^{\infty} D_n \frac{cn\pi}{\mathrm{L}} \sin\frac{n\pi x}{L} = g(x)$$

Therefore, we must choose the D_n so that at $t = 0$, $\frac{\partial u}{\partial t}$ becomes Fourier sine series of the odd extension of $g(x)$, $0 \le x \le L$ to the interval $-L \le x \le L$,

$$D_n \frac{cn\pi}{\mathrm{L}} = \frac{2}{L}\int_0^L g(x) \sin\frac{n\pi x}{L} dx$$ And then

$$D_n = \frac{2}{cn\pi}\int_0^L g(x) \sin\frac{n\pi x}{L} dx, \qquad n = 1,2,\ldots$$ for simplicity,

If initial velocity is zero, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0, \quad D_n = 0$ then

$$u(x,t) = \sum_{n=1}^{\infty} C_n \cos\frac{cn\pi}{L}\ t \sin\frac{n\pi}{L} x$$

Recalling trigonometric identity

$$\cos\frac{cn\pi}{L} t \sin\frac{n\pi}{L} x = \frac{1}{2}\left[\sin\frac{n\pi}{L}(x - ct) + \sin\frac{n\pi}{L}(x + ct)\right]$$

one has

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{L}(x - ct) + \frac{1}{2}\sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{L}(x + ct)$$

$$u(x,t) = \frac{1}{2}[f^*(x - ct) + f^*(x + ct)]$$

where $f^*$is the odd periodic extension of f with period $2L$

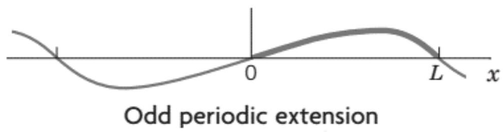


Figure 7.10 $f^*$is the odd periodic extension of f with period $2L$.

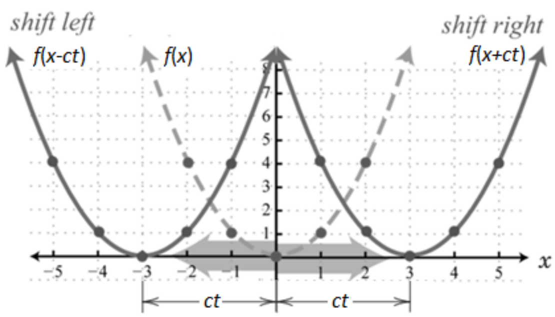


Figure 7.11 Wave is travelling to left and right. $u(x,t)$ is the superposition of both.

<u>Example 7.3</u>

Find the solution of the wave equation corresponding to the triangular initial defection and zero initial velocity

$$f(x) = \begin{cases} \dfrac{2kx}{L}, & 0 < x < \dfrac{L}{2} \\ \dfrac{2k(L-x)}{L}, & \dfrac{L}{2} < x < L \end{cases}$$

At various t wave is travelling left and right

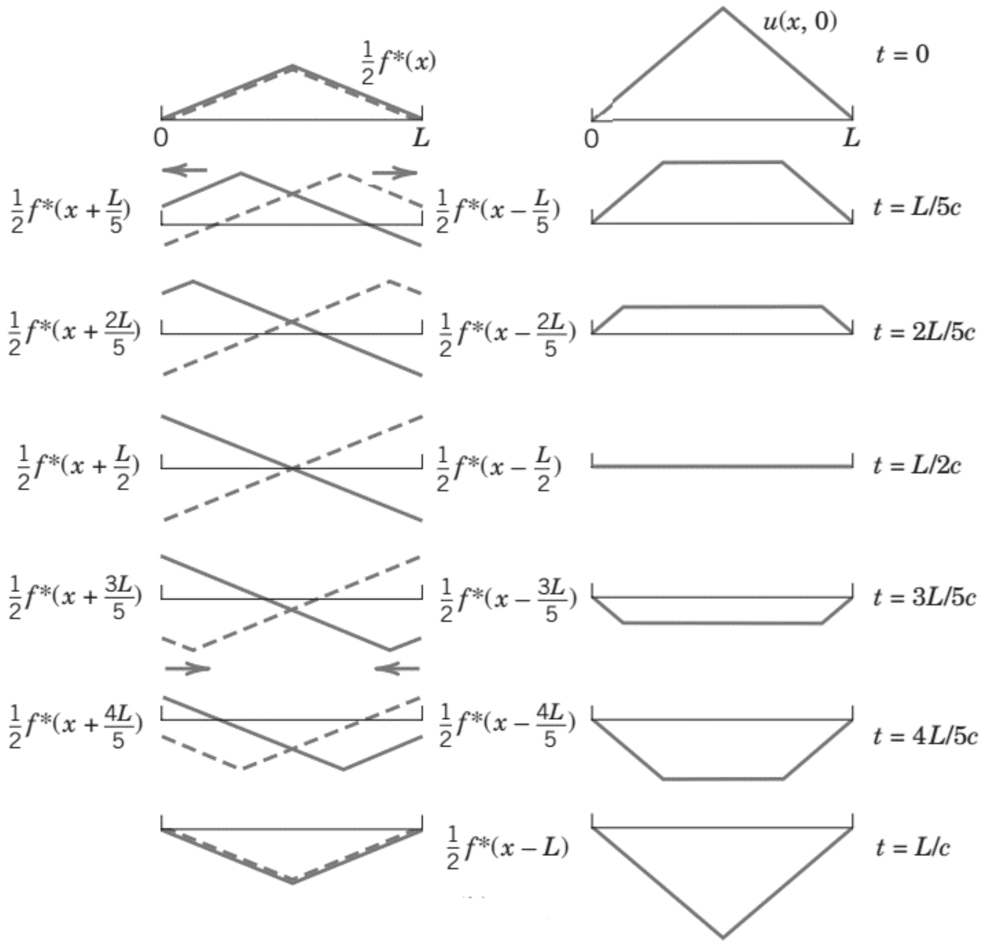


Figure 7.12 Various travelling wave forms.

since $g(x) \equiv 0$, $C_n = 0$ then $u = (x,t)$ becomes

$$u = (x,t) = \frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi}{L}x\cos\frac{\pi c}{L}t - \frac{1}{3^2}\sin\frac{3\pi}{L}x\cos\frac{3\pi c}{L}t + - \cdots\right]$$

7.6 Method of the Separation of Variables for One Dimensional Heat Equations

For a finite rod, a one-dimensional heat flow problem is modeled as,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L, \qquad t > 0$$

Where $c^2 = k/\rho C_p$ is the constant thermal diffusivity, $u(x,t)$ is the temperature, k is the thermal conductivity, C_p the specific heat, and ρ is the density of the material of the body.

Possible boundary conditions for fixed end temperatures case are;

$$u(0,t) = u(L,t) = 0, \qquad \forall\, t > 0$$

and a possible initial condition $u(x,0) = f(x)$ is imposed which specifies the initial temperature distribution on the body.

Let us assume the one dimensional heath equation admits a solution as the product of two functions

$$u(x,t) = F(x)G(t)$$

Then

$$\frac{\partial u}{\partial t} = F\dot{G},\ \dot{G} = \frac{dG}{dt}$$

Similarly

$$\frac{\partial u}{\partial x} = F'G \to \frac{\partial^2 u}{\partial x^2} = F"G$$

where

$$F' = \frac{dF}{dx} \ \text{ and } \ F" = \frac{d^2F}{dx^2}$$

Substitute these into the PDE, then

$F\dot{G} = c^2\, GF''$ dividing both sides by $G(t).F(x)$ one has $\frac{\dot{G}}{c^2\, G}$

$= \frac{F''}{F}$

This equality is possible only when both sides are equal to the same constant

$\frac{\dot{G}}{c^2\, G} = \frac{F''}{F} = k,$ k is a constant

Then we have two ODE's

3) x −Equation

$F'' - kF = 0, \;\; 0 \le x \le L,$

and

4) t −Equation

$\dot{G} - c^2 kG = 0, \qquad t > 0$

To have a solution $u(x,t) = F(x)G(t)$ which is diminishing in time, $k = -\omega^2 < 0$

<u>Solution of the x − Equation</u>

$F'' + \omega^2 F = 0, \;\; 0 \le x \le L,$

A general solution is
$F(x) = A$ Cos $\omega x + B$ Sin ωx

where A, B are arbitrary constants.

Applying BC's

$u(0,t) = F(0)G(t) = 0$ If $G(t) = 0, \; \forall\, t > 0$ then $u \equiv 0$ which is not interesting then $G(t) \neq 0$. Hence

b) $F(0) = 0 \Rightarrow A = 0$

Similarly

$u(L,t) = F(L)G(t) = 0$

b) $F(L) = 0 \Rightarrow B\text{Sin}\ \omega L = 0$

$B = 0$ leads to the trivial solution.

$\text{Sin}\ \omega L = 0 \ \Rightarrow \omega L = n\pi \Rightarrow \omega = n\pi/L,\ n \in Z$ an integer.

Therefore we get infinitely many solutions of the ODE which satisfy the boundary conditions

$F_n(x) = B_n\ \text{Sin}\frac{n\pi x}{L}\ ,\ n \in Z$ an integer.

Solution of the $t -$ Equation

$\dot{G} - c^2 kG = 0, \qquad t > 0$

Since $-k = \omega^2 = (n\pi/L)^2\ ,\ n \in Z$ one has

$\dot{G} + c^2(n\pi/L)^2 G = 0, \qquad t > 0$

Whose general solution is

$G_n(t) = C_n\ \ \mathrm{e}^{-(cn\pi/L)^2 \mathrm{t}}\ \ , n \in Z$

Then one has infinitely many solutions for the one dimensional heath equation which satisfy the boundary conditions:

$$u_n(x,t) = B_n \sin\frac{n\pi x}{L}\,\mathrm{e}^{-(cn\pi/L)^2\mathrm{t}}, \qquad (n = 1,2,\dots)$$

Since the PDE is linear and homogeneous, sum of all possible solutions is also a solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\,e^{-(cn\pi/L)^2 t}, \quad \lambda_n = \frac{c\,n\,\pi}{L}$$

With $B_n, n = 1,2,3,\dots$ is not specified yet. To specify B_n, we impose second class of conditions, which is the initial condition:

Initial temperature $u(x,0) = f(x)\ ,\ \ 0 \le x \le L$, which implies

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L} x = f(x)$$

Hence we must choose B_n's so that $u(x,0)$ becomes the Fourier series of the odd extension of $f(x)$ to the interval $-L \le x \le L$,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Example 7.4

Solve 1D heat transfer equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L, \qquad t > 0$$

with boundary conditions for fixed end temperatures case are;

$$u(0,t) = 0, \qquad u(L,t) = 100, \qquad \forall\, t > 0$$

and a possible initial condition $u(x,0) = f(x)$ is imposed which specifies the initial temperature distribution on the body.

$$f(x) = \begin{cases} x, & 0 < x < \frac{L}{2} \\ L - x, & \frac{L}{2} < x < L \end{cases}$$

Then

$$B_n = \frac{2}{L}\int_0^{L/2} x\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}\,dx$$

$B_n = 0$ n is even

$$B_n = \frac{4\,L}{n^2\pi^2}, \qquad n = 1,5,9\,\ldots$$

$$B_n = -\frac{4\,L}{n^2\pi^2}, \quad n = 3,7,11\ \ldots$$

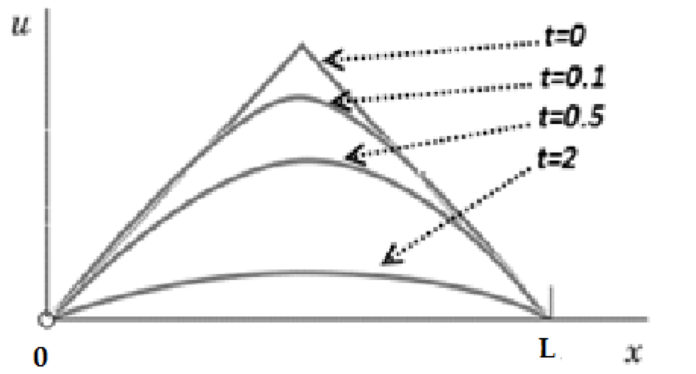


Figure 7.13 Various solutions of 1D heat transfer problem in (u,x) domain for different t values.

<u>Example 7.5</u>

Solve 1D heat transfer equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 \leq x \leq L, \qquad t > 0$$

with boundary conditions for fixed end temperatures case are;

$$\left.\frac{\partial u}{\partial t}\right|_{x=0} = 0, \qquad u(L,t) = 100, \qquad \forall\, t > 0$$

and a possible initial condition $u(x,0) = f(x)$ is imposed which specifies the initial temperature distribution on the body.

$$f(x) = \begin{cases} 60, & 0 < x < L/2 \\ 0, & L/2 < x < L \end{cases}$$

assume: $\qquad L = c = 1$

First problem is divided into two sub-problems

Sub-problem 1; Steady state solution $u_s = u_s(x)$,

$$\frac{\partial^2 u_s}{\partial x^2} = 0, \qquad 0 \le x \le \ 1,$$

$$\left.\frac{\partial u_s}{\partial x}\right|_{x=0} = 0, \qquad u_s(1) = 100$$

Sub-problem 2; Time dependent solution $u_T = u_T(x,t)$,

$$\frac{\partial u_T}{\partial t} = \frac{\partial^2 u_T}{\partial x^2}, \qquad 0 \le x \le \ 1, \qquad t > 0$$

$$\left.\frac{\partial u_T}{\partial x}\right|_{x=0} = 0, \qquad u_T(1,t) = 0, \qquad \forall\, t > 0$$

and a possible initial condition $u(x,0) = f(x)$ is imposed which specifies the initial temperature distribution on the body.

$$f(x) = -u_s + \begin{cases} 60, & 0 < x < 1/2 \\ 0, & 1/2 < x < 1 \end{cases}$$

It is clear that

$$u(x,t) = u_s(x) + u_T(x,t)$$

<u>Solution of the Steady State Problem:</u>

$$\frac{\partial^2 u_s}{\partial x^2} = 0, \qquad 0 \le x \le \ 1, \Longrightarrow \ u_s(x) = Ax + B$$

Imposing boundary conditions

$$\left.\frac{\partial u_s}{\partial x}\right|_{x=0} = 0, \qquad u_s(1) = 100$$

$\left.\frac{\partial u_s}{\partial x}\right|_{x=0} = A = 0$, and $u_s(1) = 100 \Longrightarrow B = 100$

Hence

$$u_s(x) = 100$$

Solution of the Time Dependent Problem:

Let us assume the time dependent problem for one dimensional heath equation admits a solution as the product of two functions

$$u_T(x,t) = F(x)G(t)$$

Then

$$\frac{\partial u_T}{\partial t} = F\dot{G},\ \dot{G} = \frac{dG}{dt}$$

Similarly

$$\frac{\partial u_T}{\partial x} = F'G \rightarrow \frac{\partial^2 u_T}{\partial x^2} = F"G$$

where

$$F' = \frac{dF}{dx} \text{ and } F" = \frac{d^2F}{dx^2}$$

Substitute these into the PDE, then

$F\dot{G} = GF''$ dividing both sides by G(t). F(x) one has $\frac{\dot{G}}{G} = \frac{F''}{F}$

This equality is possible only when both sides are equal to the same constant

$\frac{\dot{G}}{G} = \frac{F''}{F} = k,$ k is a constant

Then we have two ODE's

5) x −Equation

$F'' - kF = 0,\ 0 \leq x \leq L,$

and

6) t −Equation

$\dot{G} - kG = 0, \qquad t > 0$

To have a solution $u(x,t) = F(x)G(t)$ which is diminishing in time, $k = -\omega^2 < 0$

Solution of the $x -$ Equation

$F'' + \omega^2 F = 0, \;\; 0 \le x \le 1,$

A general solution is

$F(x) = A \text{ Cos } \omega x + B \text{ Sin } \omega x$

where A, B are arbitrary constants.

Applying BC's

$$\left.\frac{\partial u_T}{\partial x}\right|_{x=0} = 0, \qquad u_T(1,t) = 0, \qquad \forall\, t > 0$$

$\left.\frac{\partial u_T}{\partial x}\right|_{x=0} = 0 \Longrightarrow F'(0)G(t) = 0$

If $G(t) = 0, \; \forall\, t > 0$ then $u_T \equiv 0$ which is not interesting then $G(t) \neq 0$. Hence

c) $F'(0) = 0 \;\Rightarrow B = 0$

Similarly

$u(1,t) = F(1)G(t) = 0$

b) $F(1) = 0 \Rightarrow A\text{Cos}\,\omega = 0$

$A = 0$ leads to the trivial solution.

$\text{Cos}\,\omega = 0 \;\Rightarrow \omega = (2n+1)\pi/2, \; n \in Z$ an integer.

Therefore we get infinitely many solutions of the ODE which satisfy the boundary conditions

$F_n(x) = A_n \text{ Cos } (2n+1)\pi x/2 \; , \; n \in Z$ an integer.

Solution of the $t -$ Equation

$\dot{G} - kG = 0, \qquad t > 0$

Since $-k = \omega^2 = \left((2n+1)\pi/2\right)^2 \; , \; n \in Z$ one has

$\dot{G} + \left((2n+1)\pi/2\right)^2 G = 0, \qquad t > 0$

Whose general solution is

$G_n(t) = C_n \;\; \mathrm{e}^{-\left((2n+1)\pi/2\right)^2 \mathrm{t}} \;\; , n \in Z$

Then one has infinitely many solutions for the one dimensional heath equation which satisfy the boundary conditions:

$$u_n(x,t) = A_n \, \mathrm{Cos}(2n+1)\pi x/2 \, \mathrm{e}^{-\left((2n+1)\pi/2\right)^2 \mathrm{t}}, \qquad (n = 1,2, \dots)$$

Since the PDE is linear and homogeneous, sum of all possible solutions is also a solution

$$u_T(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n Cos(2n+1)\pi x/2 e^{-\left((2n+1)\pi/2\right)^2 t},$$

With $A_n, n = 1,2,3, \dots$ is not specified yet. To specify A_n, we impose the initial condition:

Initial temperature $u(x,0) = f(x)\,, \;\; 0 \le x \le 1, \Rightarrow u(x,0) = u_T(x,0) + u_s(x) = f(x)$ which implies

$$u_T(x,0) = g(x) = -u_s(x) + f(x) = -u_s + \begin{cases} 60, & 0 < x < 1/2 \\ 0, & 1/2 < x < 1 \end{cases}$$

$$g(x) = \begin{cases} -40, & 0 < x < 1/2 \\ -100, & 1/2 < x < 1 \end{cases}$$

$u(x,0) = f(x)\,, \;\; 0 \le x \le 1, \Rightarrow u(x,0) = u_T(x,0) + u_s(x) = f(x)$ which implies

$$g(x) = \sum_{n=1}^{\infty} A_n \operatorname{Cos}(2n+1)\pi x/2$$

$\operatorname{Cos} n\pi x$

Is a function of period 4. In a Fourier series if the function is even, we don't have sine terms, If we extend the function as a special even function, we don't have even cosine terms.

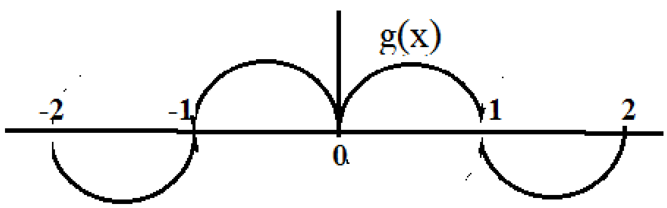


Figure 7.14 Function of g(x) with total period.

$$\int_{-2}^{2} g(x) \operatorname{Cos} n\pi x \;\; dx = A_n \int_{-2}^{2} (\operatorname{Cos} n\pi x)^2 \;\; dx$$

$$= \frac{A_n}{2} \int_{-2}^{2} (1 + \operatorname{Cos} 2n\pi x) \;\; dx = 2A_n$$

$$\int_{-2}^{2} g(x) \operatorname{Cos} n\pi x \;\; dx = 2 \int_{0}^{2} g(x) \operatorname{Cos} n\pi x \;\; dx$$

$$= 2 \int_{0}^{1} g(x) \operatorname{Cos} n\pi x \;\; dx + 2 \int_{1}^{2} g(x) \operatorname{Cos} n\pi x \;\; dx$$

$$= 2 \int_{0}^{1} g(x) \operatorname{Cos} n\pi x \;\; dx + 2 \int_{0}^{1} g(x+1) \operatorname{Cos} n\pi(x+1) \;\; dx$$

$$= 2 \int_{0}^{1} g(x) \operatorname{Cos} n\pi x \;\; dx - 2 \int_{0}^{1} g(x) \operatorname{Cos}(n\pi x + n\pi) \;\; dx$$

$$= \begin{cases} 4\int_0^1 g(x)\,\text{Cos}\, n\pi x \ \ dx\,, & n \text{ odd} \\ 0, \ \ n \text{ even} \end{cases}$$

while

$$\int_0^1 g(x)\,\text{Cos}\, n\pi x \ \ dx = -\,40\int_0^{1/2} \text{Cos}\, n\pi x \ \ dx - 100\int_{1/2}^{1} \text{Cos}\, n\pi x \ \ dx =$$

$$-40\ \text{Sin}\ n\pi x \Big|_0^{1/2} - 100\ \text{Sin}\ n\pi x \Big|_{1/2}^{1} = \begin{cases} 60\text{Sin}\ n\pi/2, & n \text{ odd} \\ 0, \ \ n \text{ even} \end{cases}$$

$$A_{2m-1} = 120\text{Sin}\ (2m+1)\pi/2$$

$$u_T(x,t) = \sum_{n=1}^{\infty} A_n Cos(2n+1)\pi x/2 e^{-((2n+1)\pi/2)^2 t}$$

$$= 120\sum_{n=1}^{\infty} \text{Sin}\ (2m+1)\pi/2 Cos(2n+1)\pi x/2 e^{-((2n+1)\pi/2)^2 t}$$

$$u(x,t) = u_s(x) + u_T(x,t)$$

$$u(x,t) = 100 + 120\sum_{n=1}^{\infty} \text{Sin}\ (2n+1)\pi/2 Cos(2n+1)\pi x/2 e^{-((2n+1)\pi/2)^2 t}$$

As time gets higher, f(x) goes to 100 at whole x.

7.7 Method of the Separation of Variables for Two Dimensional Laplace Equation

Among the linear second order linear PDE's, elliptic equations are associated with steady state phenomena. The Laplace equation is the representative of this group.

Two-Dimensional Steady State Temperature Distribution Problem

A two dimensional steady state temperature distribution problem for a rectangular plate is modeled as,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \le x \le a, 0 \le y \le b,$$

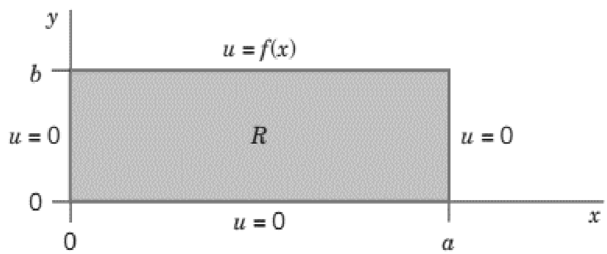


Figure 7.15 a rectangular plate for two dimensional heat distribution problem.

If the temperatures are specified on the frame as in the Figure, then the four boundary conditions are;

$u(0, y) = u(a, y) = 0,\ 0 \le y \le b,$ and

$u(x, 0) = 0, \quad u(x, b) = f(x), \quad 0 \le x \le a.$

Because of these special boundary conditions, this problem is called a Dirichlet problem:

Let us assume the one dimensional wave equation admits a solution as the product of two functions

$$u(x, y) = F(x)G(y)$$

Then

$$\frac{\partial u}{\partial y} = F\dot{G} \to \frac{\partial^2 u}{\partial y^2} = F\ddot{G}$$

where

$$\dot{G} = \frac{dG}{dy} \text{ and } \ddot{G} = \frac{d^2G}{dy^2}$$

Similarly

$$\frac{\partial u}{\partial x} = F'G \to \frac{\partial^2 u}{\partial x^2} = F''G$$

where

$$F' = \frac{dF}{dx} \text{ and } F'' = \frac{d^2F}{dx^2}$$

Substitute these into the PDE, then

$F\ddot{G} = -GF''$ dividing both sides by $G(t).F(x)$ one has $\frac{\ddot{G}}{G} = -\frac{F''}{F}$

This equality is possible only when both sides are equal to the same constant

$\frac{\ddot{G}}{G} + \frac{F''}{F} = 0 \Rightarrow \frac{F''}{F} = -\frac{\ddot{G}}{G} = k,$ k is a constant

Then we have two ODE's

1) x −Equation

$F'' - kF = 0, \;\; 0 \le x \le a,$

and

2) y −Equation

$\ddot{G} + kG = 0, \qquad t > 0$

To have a solution $u(x,t) = F(x)G(t)$ that satisfy the homogeneous boundary conditions on $x-$boundaries one must have $k = -\omega^2 < 0$ a negative constant.

Solution of the $x -$ Equation

$F'' + \omega^2 F = 0, \;\; 0 \le x \le a,$

A general solution is
$F(x) = A \text{ Cos } \omega x + B \text{ Sin } \omega x$

where A, B are arbitrary constants.

Applying BC's

$u(0,y) = F(0)G(t) = 0$ If $G(y) = 0, \;\; 0 \le y \le b$ then $u \equiv 0$ which is not interesting then $G(y) \neq 0$. Hence

d) $F(0) = 0 \Rightarrow A = 0$

Similarly

$u(a,t) = F(L)G(t) = 0$

b) $F(a) = 0 \Rightarrow B\text{Sin } \omega a = 0$

$B = 0$ leads to the trivial solution.

$\text{Sin } \omega a = 0 \Rightarrow \omega a = n\pi \Rightarrow \omega = n\pi/a, \; n \in Z$ an integer.

Therefore we get infinitely many solutions of the ODE which satisfy the boundary conditions

$F_n(x) = B_n \text{ Sin} \frac{n\pi x}{a}$, $n \in Z$ an integer.

Solution of the $y -$ Equation

$\ddot{G} - \omega^2 G = 0, \qquad t > 0$

Since $-k = \omega^2 = (n\pi/a)^2$, $n \in Z$ one has

$\ddot{G} + c^2(n\pi/a)^2 G = 0, \qquad t > 0$

Whose general solution is

$$G_n(y) = C_n \cosh\frac{cn\pi}{\mathrm{a}} y \ + D_n \sinh\frac{cn\pi}{\mathrm{a}} y, \ \ n \in Z$$

Imposing the boundary condition $u(x,0) = 0$, one has $G_n(0) = 0$, which leads $C_n = 0$.

Then one has infinitely many solutions for the one dimensional wave equation which satisfy the boundary conditions:

$$u_n(x,y) = C_n \cos\frac{n\pi}{a} x \sinh\frac{n\pi}{a} \mathrm{y}, \qquad (n = 1,2, \dots)$$

Since the PDE is linear and homogeneous, sum of all possible solutions is also a solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi}{\mathrm{a}} x \sinh\frac{n\pi}{a} y$$

With C_n, $n = 1,2,3, \dots$ not specified yet. To specify C_n ,we impose the the last boundary condition

$u(x,b) = f(x)$, $0 \leq x \leq a$,

Which leads

$$u(x,b) = \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi b}{a} \sin\frac{n\pi}{L} x = f(x)$$

Hence we must choose $C_n \sinh\frac{n\pi b}{a}$'s so that $u(x,b)$ becomes the Fourier series of the odd extension of $f(x)$ to the interval $-a \leq x \leq a$,

$$C_n \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi x}{a} dx \Rightarrow C_n = \frac{2}{a \sinh n\pi b/a} \int_0^a f(x) \sin\frac{n\pi x}{a} dx$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \ \cos\frac{cn\pi}{a} \ t \sinh\frac{n\pi}{a} x$$

is the solution of the given Dirichlet problem.

Rectangular Membrane. Double Fourier Series

Now we develop a solution for the PDE obtained in Sec. 12.8. Details are as follows.

The model of the vibrating membrane for obtaining the displacement $u(x, y, t)$ of a point (x, y) of the membrane from rest $(u = 0)$ at time t is

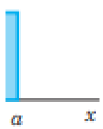
$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

$$u = 0 \text{ on the boundary} \tag{2}$$

$$u(x, y, 0) = f(x, y) \tag{3a}$$

$$u_t(x, y, 0) = g(x, y). \tag{3b}$$

Here (1) is the **two-dimensional wave equation** with $c^2 = T/\rho$ just derived, (2) is the **boundary condition** (membrane fixed along the boundary in the xy-plane for all times $t \geqq 0$), and (3) are the **initial conditions** at $t = 0$, consisting of the given *initial displacement* (initial shape) $f(x, y)$ and the given *initial velocity* $g(x, y)$, where $u_t = \partial u/\partial t$. We see that these conditions are quite similar to those for the string in Sec. 12.2.



2.
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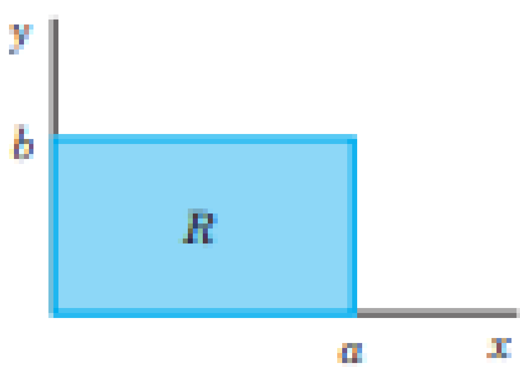
Let us consider the **rectangular membrane** R in Fig. 302. This is our first important model. It is much simpler than the circular drumhead, which will follow later. First we note that the boundary in equation (2) is the rectangle in Fig. 302. We shall solve this problem in three steps:



Step 1. By separating variables, first setting $u(x, y, t) = F(x, y)G(t)$ and later $F(x, y) = H(x)Q(y)$ we obtain from (1) an ODE (4) for G and later from a PDE (5) for F two ODEs (6) and (7) for H and Q.

Step 2. From the solutions of those ODEs we determine solutions (13) of (1) ("**eigenfunctions**" u_{mn}) that satisfy the boundary condition (2).

Step 3. We compose the u_{mn} into a double series (14) solving the whole model (1), (2), (3).

Step 1. Three ODEs From the Wave Equation (1)

To obtain ODEs from (1), we apply two successive separations of variables. In the first separation we set $u(x, y, t) = F(x, y)G(t)$. Substitution into (1) gives

$$F\ddot{G} = c^2(F_{xx}G + F_{yy}G)$$

where subscripts denote partial derivatives and dots denote derivatives with respect to t. To separate the variables, we divide both sides by c^2FG:

$$\frac{\ddot{G}}{c^2G} = \frac{1}{F}(F_{xx} + F_{yy}).$$

Since the left side depends only on t, whereas the right side is independent of t, both sides must equal a constant. By a simple investigation we see that only negative values of that constant will lead to solutions that satisfy (2) without being identically zero; this is similar to Sec. 12.3. Denoting that negative constant by $-\nu^2$, we have

$$\frac{\ddot{G}}{c^2G} = \frac{1}{F}(F_{xx} + F_{yy}) = -\nu^2.$$

This gives two equations: for the "**time function**" $G(t)$ we have the ODE

$$\ddot{G} + \lambda^2 G = 0 \qquad \text{where } \lambda = c\nu, \tag{4}$$

and for the "**amplitude function**" $F(x, y)$ a PDE, called the *two-dimensional* **Helmholtz**[3] **equation**

$$F_{xx} + F_{yy} + \nu^2 F = 0. \tag{5}$$

Separation of the Helmholtz equation is achieved if we set $F(x, y) = H(x)Q(y)$. By substitution of this into (5) we obtain

$$\frac{d^2H}{dx^2}Q = -\left(H\frac{d^2Q}{dy^2} + \nu^2 HQ\right).$$

To separate the variables, we divide both sides by HQ, finding

$$\frac{1}{H}\frac{d^2H}{dx^2} = -\frac{1}{Q}\left(\frac{d^2Q}{dy^2} + \nu^2 Q\right).$$

Both sides must equal a constant, by the usual argument. This constant must be negative, say, $-k^2$, because only negative values will lead to solutions that satisfy (2) without being identically zero. Thus

$$\frac{1}{H}\frac{d^2H}{dx^2} = -\frac{1}{Q}\left(\frac{d^2Q}{dy^2} + \nu^2 Q\right) = -k^2.$$

This yields two ODEs for H and Q, namely,

$$\textbf{(6)} \qquad \frac{d^2H}{dx^2} + k^2 H = 0$$

and

$$\textbf{(7)} \qquad \frac{d^2Q}{dy^2} + p^2 Q = 0 \qquad \text{where } p^2 = \nu^2 - k^2.$$

Step 2. Satisfying the Boundary Condition

General solutions of (6) and (7) are

$$H(x) = A\cos kx + B\sin kx \qquad \text{and} \qquad Q(y) = C\cos py + D\sin py$$

with constant A, B, C, D. From $u = FG$ and (2) it follows that $F = HQ$ must be zero on the boundary, that is, on the edges $x = 0, x = a, y = 0, y = b$; see Fig. 302. This gives the conditions

$$H(0) = 0, \qquad H(a) = 0, \qquad Q(0) = 0, \qquad Q(b) = 0.$$

Hence $H(0) = A = 0$ and then $H(a) = B\sin ka = 0$. Here we must take $B \neq 0$ since otherwise $H(x) \equiv 0$ and $F(x, y) \equiv 0$. Hence $\sin ka = 0$ or $ka = m\pi$, that is,

$$k = \frac{m\pi}{a} \quad (m \text{ integer}).$$

In precisely the same fashion we conclude that $C = 0$ and p must be restricted to the values $p = n\pi/b$ where n is an integer. We thus obtain the solutions $H = H_m$, $Q = Q_n$, where

$$H_m(x) = \sin\frac{m\pi x}{a} \qquad \text{and} \qquad Q_n(y) = \sin\frac{n\pi y}{b}, \qquad \begin{array}{l} m = 1, 2, \cdots, \\ n = 1, 2, \cdots. \end{array}$$

As in the case of the vibrating string, it is not necessary to consider $m, n = -1, -2, \cdots$ since the corresponding solutions are essentially the same as for positive m and n, expect for a factor -1. Hence the functions

$$F_{mn}(x, y) = H_m(x)Q_n(y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad \begin{array}{l} m = 1, 2, \cdots, \\ n = 1, 2, \cdots, \end{array} \tag{8}$$

are solutions of the Helmholtz equation (5) that are zero on the boundary of our membrane.

Eigenfunctions and Eigenvalues. Having taken care of (5), we turn to (4). Since $p^2 = \nu^2 - k^2$ in (7) and $\lambda = c\nu$ in (4), we have

$$\lambda = c\sqrt{k^2 + p^2}.$$

Hence to $k = m\pi/a$ and $p = n\pi/b$ there corresponds the value

$$\boxed{\lambda = \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}},} \qquad \begin{array}{l} m = 1, 2, \cdots, \\ n = 1, 2, \cdots, \end{array} \tag{9}$$

in the ODE (4). A corresponding general solution of (4) is

$$G_{mn}(t) = B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t.$$

It follows that the functions $u_{mn}(x, y, t) = F_{mn}(x, y)\,G_{mn}(t)$, written out

$$\boxed{u_{mn}(x, y, t) = (B_{mn}\cos\lambda_{mn}t + B^*_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}} \tag{10}$$

with λ_{mn} according to (9), are solutions of the wave equation (1) that are zero on the boundary of the rectangular membrane in Fig. 302. These functions are called the **eigenfunctions** or *characteristic functions,* and the numbers λ_{mn} are called the **eigenvalues** or *characteristic values* of the vibrating membrane. The frequency of u_{mn} is $\lambda_{mn}/2\pi$.

Step 3. Solution of the Model (1), (2), (3). Double Fourier Series

So far we have solutions (10) satisfying (1) and (2) only. To obtain the solutions that also satisfies (3), we proceed as in Sec. 12.3. We consider the double series

$$
\begin{aligned}
u(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}(x, y, t) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (B_{mn} \cos \lambda_{mn} t + B^*_{mn} \sin \lambda_{mn} t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\end{aligned}
\tag{14}
$$

(without discussing convergence and uniqueness). From (14) and (3a), setting $t = 0$, we have

$$
u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = f(x, y). \tag{15}
$$

Suppose that $f(x, y)$ can be represented by (15). (Sufficient for this is the continuity of f, $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x\, \partial y$ in R.) Then (15) is called the **double Fourier series** of $f(x, y)$. Its coefficients can be determined as follows. Setting

$$
K_m(y) = \sum_{n=1}^{\infty} B_{mn} \sin \frac{n\pi y}{b} \tag{16}
$$

we can write (15) in the form

$$
f(x, y) = \sum_{m=1}^{\infty} K_m(y) \sin \frac{m\pi x}{a}.
$$

For fixed y this is the Fourier sine series of $f(x, y)$, considered as a function of x. From (4) in Sec. 11.3 we see that the coefficients of this expansion are

$$
K_m(y) = \frac{2}{a} \int_0^a f(x, y) \sin \frac{m\pi x}{a}\, dx. \tag{17}
$$

Furthermore, (16) is the Fourier sine series of $K_m(y)$, and from (4) in Sec. 11.3 it follows that the coefficients are

$$
B_{mn} = \frac{2}{b} \int_0^b K_m(y) \sin \frac{n\pi y}{b}\, dy.
$$

From this and (17) we obtain the **generalized Euler formula**

$$
B_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy \qquad \begin{aligned} m &= 1, 2, \cdots \\ n &= 1, 2, \cdots \end{aligned} \tag{18}
$$

for the **Fourier coefficients** of $f(x, y)$ in the double Fourier series (15).

The B_{mn} in (14) are now determined in terms of $f(x, y)$. To determine the B^*_{mn}, we differentiate (14) termwise with respect to t; using (3b), we obtain

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B^*_{mn} \lambda_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = g(x, y).$$

Suppose that $g(x, y)$ can be developed in this double Fourier series. Then, proceeding as before, we find that the coefficients are

$$B^*_{mn} = \frac{4}{ab\lambda_{mn}} \int_0^b \int_0^a g(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \, dx \, dy \qquad \begin{aligned} m &= 1, 2, \cdots \\ n &= 1, 2, \cdots . \end{aligned} \tag{19}$$

Result. *If f and g in* (3) *are such that u can be represented by* (14), *then* (14) *with coefficients* (18) *and* (19) *is the solution of the model* (1), (2), (3).

Vibration of a Rectangular Membrane

Find the vibrations of a rectangular membrane of sides $a = 4$ ft and $b = 2$ ft (Fig. 305) if the tension is 12.5 lb/ft, the density is 2.5 slugs/ft^2 (as for light rubber), the initial velocity is 0, and the initial displacement is

$$f(x, y) = 0.1(4x - x^2)(2y - y^2) \text{ ft.} \tag{20}$$

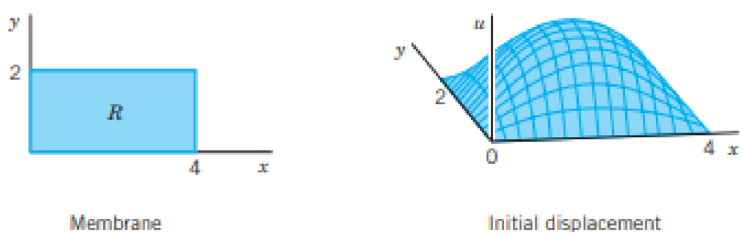


Fig. 305. Example 2

Solution. $c^2 = T/\rho = 12.5/2.5 = 5$ [ft^2/sec^2]. Also $B^*_{mn} = 0$ from (19). From (18) and (20),

$$\begin{aligned} B_{mn} &= \frac{4}{4 \cdot 2} \int_0^2 \int_0^4 0.1(4x - x^2)(2y - y^2) \sin \frac{m\pi x}{4} \sin \frac{n\pi y}{2} \, dx \, dy \\ &= \frac{1}{20} \int_0^4 (4x - x^2) \sin \frac{m\pi x}{4} dx \int_0^2 (2y - y^2) \sin \frac{n\pi y}{2} \, dy. \end{aligned}$$

Two integrations by parts give for the first integral on the right

$$\frac{128}{m^3\pi^3}[1 - (-1)^m] = \frac{256}{m^3\pi^3} \qquad (m \text{ odd})$$

and for the second integral

$$\frac{16}{n^3\pi^3}[1 - (-1)^n] = \frac{32}{n^3\pi^3} \qquad (n \text{ odd}).$$

For even m or n we get 0. Together with the factor $1/20$ we thus have $B_{mn} = 0$ if m or n is even and

$$B_{mn} = \frac{256 \cdot 32}{20m^3n^3\pi^6} \approx \frac{0.426050}{m^3n^3} \qquad (m \text{ and } n \text{ both odd}).$$

From this, (9), and (14) we obtain the answer

$$u(x, y, t) = 0.426050 \sum_{m,n \text{ odd}} \sum \frac{1}{m^3n^3} \cos\left(\frac{\sqrt{5}\pi}{4}\sqrt{m^2 + 4n^2}\right) t \sin\frac{m\pi x}{4} \sin\frac{n\pi y}{2}$$

$$(21) \qquad = 0.426050\left(\cos\frac{\sqrt{5}\pi\sqrt{5}}{4} t \sin\frac{\pi x}{4} \sin\frac{\pi y}{2} + \frac{1}{27}\cos\frac{\sqrt{5}\pi\sqrt{37}}{4} t \sin\frac{\pi x}{4} \sin\frac{3\pi y}{2}\right.$$
$$\left. + \frac{1}{27}\cos\frac{\sqrt{5}\pi\sqrt{13}}{4} t \sin\frac{3\pi x}{4} \sin\frac{\pi y}{2} + \frac{1}{729}\cos\frac{\sqrt{5}\pi\sqrt{45}}{4} t \sin\frac{3\pi x}{4} \sin\frac{3\pi y}{2} + \cdots\right).$$

To discuss this solution, we note that the first term is very similar to the initial shape of the membrane, has no nodal lines, and is by far the dominating term because the coefficients of the next terms are much smaller. The second term has two horizontal nodal lines ($y = \frac{2}{3}, \frac{4}{3}$), the third term two vertical ones ($x = \frac{4}{3}, \frac{8}{3}$), the fourth term two horizontal and two vertical ones, and so on.

Laplacian in Polar Coordinates. Circular Membrane. Fourier–Bessel Series

It is a ***general principle*** in boundary value problems for PDEs to *choose coordinates that make the formula for the boundary as simple as possible*. Here polar coordinates are used for this purpose as follows. Since we want to discuss circular membranes (drumheads), we first transform the Laplacian in the wave equation (1), Sec. 12.9,

$$(1) \qquad u_{tt} = c^2\nabla^2 u = c^2(u_{xx} + u_{yy})$$

(subscripts denoting partial derivatives) into **polar coordinates** r, θ defined by $x = r\cos\theta$, $y = r\sin\theta$; thus,

$$r = \sqrt{x^2 + y^2}, \qquad \tan\theta = \frac{y}{x}.$$

By the chain rule (Sec. 9.6) we obtain

$$u_x = u_r r_x + u_\theta \theta_x.$$

Differentiating once more with respect to x and using the product rule and then again the chain rule gives

$$\begin{aligned} u_{xx} &= (u_r r_x)_x + (u_\theta\theta_x)_x \\ (2) \qquad &= (u_r)_x r_x + u_r r_{xx} + (u_\theta)_x\theta_x + u_\theta\theta_{xx} \\ &= (u_{rr}r_x + u_{r\theta}\theta_x)r_x + u_r r_{xx} + (u_{\theta r}r_x + u_{\theta\theta}\theta_x)\theta_x + u_\theta\theta_{xx}. \end{aligned}$$

Also, by differentiation of r and θ we find

$$r_x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}, \qquad \theta_x = \frac{1}{1 + (y/x)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2}.$$

Differentiating these two formulas again, we obtain

$$r_{xx} = \frac{r - xr_x}{r^2} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{y^2}{r^3}, \qquad \theta_{xx} = -y\left(-\frac{2}{r^3}\right)r_x = \frac{2xy}{r^4}.$$

We substitute all these expressions into (2). Assuming continuity of the first and second partial derivatives, we have $u_{r\theta} = u_{\theta r}$, and by simplifying,

$$u_{xx} = \frac{x^2}{r^2}u_{rr} - 2\frac{xy}{r^3}u_{r\theta} + \frac{y^2}{r^4}u_{\theta\theta} + \frac{y^2}{r^3}u_r + 2\frac{xy}{r^4}u_\theta. \tag{3}$$

In a similar fashion it follows that

$$u_{yy} = \frac{y^2}{r^2}u_{rr} + 2\frac{xy}{r^3}u_{r\theta} + \frac{x^2}{r^4}u_{\theta\theta} + \frac{x^2}{r^3}u_r - 2\frac{xy}{r^4}u_\theta. \tag{4}$$

By adding (3) and (4) we see that the **Laplacian of u in polar coordinates** is

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{5}$$

Circular Membrane

Circular membranes are important parts of drums, pumps, microphones, telephones, and other devices. This accounts for their great importance in engineering. Whenever a circular membrane is plane and its material is elastic, but offers no resistance to bending (this excludes thin metallic membranes!), its vibrations are modeled by the **two-dimensional wave equation in polar coordinates** obtained from (1) with $\nabla^2 u$ given by (5), that is,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) \qquad c^2 = \frac{T}{\rho}. \tag{6}$$

We shall consider a membrane of radius R (Fig. 307) and determine solutions $u(r, t)$ that are radially symmetric. (Solutions also depending on the angle θ will be discussed in the problem set.) Then $u_{\theta\theta} = 0$ in (6) and the model of the problem (the analog of (1), (2), (3) in Sec. 12.9) is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{7}$$

$$u(R, t) = 0 \text{ for all } t \geqq 0 \tag{8}$$

$$u(r, 0) = f(r) \tag{9a}$$

$$u_t(r, 0) = g(r). \tag{9b}$$

Here (8) means that the membrane is fixed along the boundary circle $r = R$. The initial deflection $f(r)$ and the initial velocity $g(r)$ depend only on r, not on θ, so that we can expect radially symmetric solutions $u(r, t)$.

Step 1. Two ODEs From the Wave Equation (7). Bessel's Equation

Using the **method of separation of variables**, we first determine solutions $u(r, t) = W(r)G(t)$. (We write W, not F because W depends on r, whereas F, used before, depended on x.) Substituting $u = WG$ and its derivatives into (7) and dividing the result by c^2WG, we get

$$\frac{\ddot{G}}{c^2G} = \frac{1}{W}\left(W'' + \frac{1}{r}W'\right)$$

where dots denote derivatives with respect to t and primes denote derivatives with respect to r. The expressions on both sides must equal a constant. This constant must be negative, say, $-k^2$, in order to obtain solutions that satisfy the boundary condition without being identically zero. Thus,

$$\frac{\ddot{G}}{c^2G} = \frac{1}{W}\left(W'' + \frac{1}{r}W'\right) = -k^2.$$

This gives the two linear ODEs

$$\boxed{\ddot{G} + \lambda^2 G = 0} \qquad \text{where } \lambda = ck \tag{10}$$

and

$$\boxed{W'' + \frac{1}{r}W' + k^2W = 0.} \tag{11}$$

We can reduce (11) to Bessel's equation (Sec. 5.4) if we set $s = kr$. Then $1/r = k/s$ and, retaining the notation W for simplicity, we obtain by the chain rule

$$W' = \frac{dW}{dr} = \frac{dW}{ds}\frac{ds}{dr} = \frac{dW}{ds}k \qquad \text{and} \qquad W'' = \frac{d^2W}{ds^2}k^2.$$

By substituting this into (11) and omitting the common factor k^2 we have

$$\boxed{\frac{d^2W}{ds^2} + \frac{1}{s}\frac{dW}{ds} + W = 0.} \tag{12}$$

This is **Bessel's equation** (1), Sec. 5.4, with parameter $\nu = 0$.

Step 2. Satisfying the Boundary Condition (8)

Solutions of (12) are the Bessel functions J_0 and Y_0 of the first and second kind (see Secs. 5.4, 5.5). But Y_0 becomes infinite at 0, so that we cannot use it because the deflection of the membrane must always remain finite. This leaves us with

$$W(r) = J_0(s) = J_0(kr) \qquad (s = kr). \tag{13}$$

On the boundary $r = R$ we get $W(R) = J_0(kR) = 0$ from (8) (because $G = 0$ would imply $u = 0$). We can satisfy this condition because J_0 has (infinitely many) positive zeros, $s = \alpha_1, \alpha_2, \cdots$ (see Fig. 308), with numerical values

$$\alpha_1 = 2.4048, \quad \alpha_2 = 5.5201, \quad \alpha_3 = 8.6537, \quad \alpha_4 = 11.7915, \quad \alpha_5 = 14.9309$$

and so on. (For further values, consult your CAS or Ref. [GenRef1] in App. 1.) These zeros are slightly irregularly spaced, as we see. Equation (13) now implies

$$kR = \alpha_m \qquad \text{thus} \qquad k = k_m = \frac{\alpha_m}{R}, \qquad m = 1, 2, \cdots . \tag{14}$$

Hence the functions

$$W_m(r) = J_0(k_m r) = J_0\left(\frac{\alpha_m}{R} r\right), \qquad m = 1, 2, \cdots \tag{15}$$

are solutions of (11) that are zero on the boundary circle $r = R$.

Eigenfunctions and Eigenvalues. For W_m in (15), a corresponding general solution of (10) with $\lambda = \lambda_m = ck_m = c\alpha_m/R$ is

$$G_m(t) = A_m \cos \lambda_m t + B_m \sin \lambda_m t.$$

Hence the functions

$$u_m(r, t) = W_m(r)G_m(t) = (A_m \cos \lambda_m t + B_m \sin \lambda_m t)J_0(k_m r) \tag{16}$$

with $m = 1, 2, \cdots$ are solutions of the wave equation (7) satisfying the boundary condition (8). These are the **eigenfunctions** of our problem. The corresponding **eigenvalues** are λ_m.

The vibration of the membrane corresponding to u_m is called the mth **normal mode**; it has the frequency $\lambda_m/2\pi$ cycles per unit time. Since the zeros of the Bessel function J_0 are not regularly spaced on the axis (in contrast to the zeros of the sine functions appearing in the case of the vibrating string), the sound of a drum is entirely different from that of a violin. The forms of the normal modes can easily be obtained from Fig. 308 and are shown in Fig. 309. For $m = 1$, all the points of the membrane move up (or down) at the same time. For $m = 2$, the situation is as follows. The function $W_2(r) = J_0(\alpha_2 r/R)$ is zero for $\alpha_2 r/R = \alpha_1$, thus $r = \alpha_1 R/\alpha_2$. The circle $r = \alpha_1 R/\alpha_2$ is, therefore, **nodal line**, and when at some instant the central part of the membrane moves up, the outer part ($r > \alpha_1 R/\alpha_2$) moves down, and conversely. The solution $u_m(r, t)$ has $m - 1$ nodal lines, which are circles (Fig. 309).

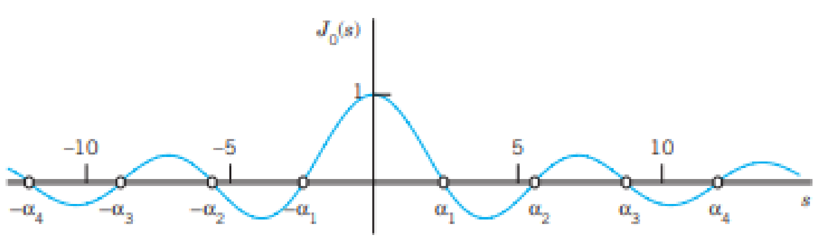


Fig. 308. Bessel function $J_0(s)$

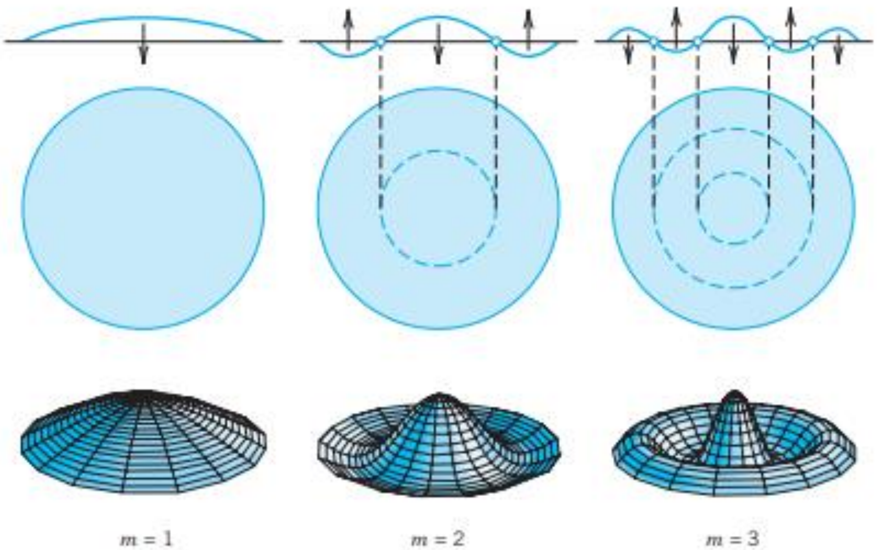


Fig. 309. Normal modes of the circular membrane in the case of vibrations independent of the angle

Step 3. Solution of the Entire Problem

To obtain a solution $u(r, t)$ that also satisfies the initial conditions (9), we may proceed as in the case of the string. That is, we consider the series

$$u(r, t) = \sum_{m=1}^{\infty} W_m(r)G_m(t) = \sum_{m=1}^{\infty} (A_m \cos \lambda_m t + B_m \sin \lambda_m t)J_0\left(\frac{\alpha_m}{R}r\right) \tag{17}$$

(leaving aside the problems of convergence and uniqueness). Setting $t = 0$ and using (9a), we obtain

$$u(r, 0) = \sum_{m=1}^{\infty} A_m J_0\left(\frac{\alpha_m}{R}r\right) = f(r). \tag{18}$$

Thus for the series (17) to satisfy the condition (9a), the constants A_m must be the coefficients of the **Fourier–Bessel series** (18) that represents $f(r)$ in terms of $J_0(\alpha_m r/R)$; that is [see (9) in Sec. 11.6 with $n = 0$, $\alpha_{0,m} = \alpha_m$, and $x = r$],

$$A_m = \frac{2}{R^2 J_1^2(\alpha_m)} \int_0^R r f(r) J_0\left(\frac{\alpha_m}{R}r\right) dr \qquad (m = 1, 2, \cdots). \tag{19}$$

Differentiability of $f(r)$ in the interval $0 \leqq r \leqq R$ is sufficient for the existence of the development (18); see Ref. [A13]. The coefficients B_m in (17) can be determined from (9b) in a similar fashion. Numeric values of A_m and B_m may be obtained from a CAS or by a numeric integration method, using tables of J_0 and J_1. However, numeric integration can sometimes be ***avoided***, as the following example shows.

Vibrations of a Circular Membrane

Find the vibrations of a circular drumhead of radius 1 ft and density 2 slugs/ft^2 if the tension is 8 lb/ft, the initial velocity is 0, and the initial displacement is

$$f(r) = 1 - r^2 \text{ [ft]}.$$

Solution. $c^2 = T/\rho = \frac{8}{2} = 4$ [ft^2/sec^2]. Also $B_m = 0$, since the initial velocity is 0. From (10) in Sec. 11.6, since $R = 1$, we obtain

$$\begin{aligned} A_m &= \frac{2}{J_1^2(\alpha_m)} \int_0^1 r(1 - r^2) J_0(\alpha_m r)\, dr \\ &= \frac{4J_2(\alpha_m)}{\alpha_m^2 J_1^2(\alpha_m)} \\ &= \frac{8}{\alpha_m^3 J_1(\alpha_m)} \end{aligned}$$

where the last equality follows from (21c), Sec. 5.4, with $\nu = 1$, that is,

$$J_2(\alpha_m) = \frac{2}{\alpha_m} J_1(\alpha_m) - J_0(\alpha_m) = \frac{2}{\alpha_m} J_1(\alpha_m).$$

Table 9.5 on p. 409 of [GenRef1] gives α_m and $J_0'(\alpha_m)$. From this we get $J_1(\alpha_m) = -J_0'(\alpha_m)$ by (21b), Sec. 5.4, with $\nu = 0$, and compute the coefficients A_m:

m	α_m	$J_1(\alpha_m)$	$J_2(\alpha_m)$	A_m
1	2.40483	0.51915	0.43176	1.10801
2	5.52008	−0.34026	−0.12328	−0.13978
3	8.65373	0.27145	0.06274	0.04548
4	11.79153	−0.23246	−0.03943	−0.02099
5	14.93092	0.20655	0.02767	0.01164
6	18.07106	−0.18773	−0.02078	−0.00722
7	21.21164	0.17327	0.01634	0.00484
8	24.35247	−0.16170	−0.01328	−0.00343
9	27.49348	0.15218	0.01107	0.00253
10	30.63461	−0.14417	−0.00941	−0.00193

Thus

$$f(r) = 1.108J_0(2.4048r) - 0.140J_0(5.5201r) + 0.045J_0(8.6537r) - \cdots.$$

We see that the coefficients decrease relatively slowly. The sum of the explicitly given coefficients in the table is 0.99915. The sum of *all* the coefficients should be 1. (Why?) Hence by the Leibniz test in App. A3.3 the partial sum of those terms gives about three correct decimals of the amplitude $f(r)$.

Since

$$\lambda_m = ck_m = c\alpha_m/R = 2\alpha_m,$$

from (17) we thus obtain the solution (with r measured in feet and t in seconds)

$$u(r, t) = 1.108J_0(2.4048r)\cos 4.8097t - 0.140J_0(5.5201r)\cos 11.0402t + 0.045J_0(8.6537r)\cos 17.3075t - \cdots.$$

In Fig. 309, $m = 1$ gives an idea of the motion of the first term of our series, $m = 2$ of the second term, and $m = 3$ of the third term, so that we can "see" our result about as well as for a violin string in Sec. 12.3. ■