3. SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS

Let the power series

$$C_0 + C_1(x - x_0) + C_2(x - x_0)^2 = \sum_{n=0}^{\infty} C_n (x - x_0)^n$$

converges uniformly to a function $f(x)$ in an interval for which x_0 is an inner point. Then $f(x)$ is analytical around the point x_0, and Taylor's theorem says that

$$C_n = \frac{f^{(n)}(x_0)}{n!}, \qquad n = 0,1,2,\dots$$

3.1 Second Order Linear Differential Equations with Variable Coefficients

In Chapter 1. the techniques to solve second order linear differential equations with constant coefficients

$$ay'' + by' + cy = r(x)$$

are studied. The solution of equations with variable coefficients

$$y'' + p(x)y' + q(x)y = r(x)$$

are more complicated.

Under certain conditions on $p(x)$, and $q(x)$, the equation may have series solutions in appropriate forms.

Classification of Points With respect to a Differential Equation

An Ordinary Point: If $p(x)$, and $q(x)$, are both analytical around a point x_0, then x_0 is an ordinary point for the differential equation

$$y'' + p(x)y' + q(x)y = r(x).$$

A Regular Singular Point: If $p(x)$, or $q(x)$, is not analytical around a point x_0, but $(x-x_0)p(x)$, and $(x-x_0)^2q(x)$, are both analytical around the point x_0, then x_0 is a regular singular point for the differential equation

Irregular Singular Point: If $(x-x_0)p(x)$, or $(x-x_0)^2q(x)$, is not analytical around the point x_0, then x_0 is an irregular singular point of the differential equation.

3.2 Series Solution About an Ordinary Point

Frobenius Theorem I: Let x_0 be an ordinary point for the differential equation

$$y'' + p(x)y' + q(x)y = r(x).$$

Then, the differential equation has at least one series solution of the form

$$y = \sum_{n=0}^{\infty} C_n\,(x-x_0)^n.$$

Example 3.1

Find the power series solution of the initial value problem in the below around the point $x_0 = 0$.

$(x^2-1)y'' + 3xy' + xy = 0\,;\ y(0) = 4, \qquad y'(0) = 6$

The normalized differential equation is;

$$y'' + \frac{3x}{x^2-1}y' + \frac{x}{x^2-1}y = 0$$

Obviously, coefficient functions are not analytic only at $x = \pm 1$, so $x_0 = 0$ is an ordinary point, therefore the solution proposal be;

$$y = \sum_{n=0}^{\infty} C_n\,x^n$$

$$y = \sum_{n=0}^{\infty} C_n\,x^n = C_0 + C_1\,x + C_2\,x^2 + C_3\,x^3 + \cdots \rightarrow$$

$y' = C_1 + 2C_2\,x + \cdots + 3C_3\,x^2 + \cdots = \sum_{n=1}^{\infty} nC_n\,x^{n-1}$, and

$$y'' = 2C_2 + 6C_3x + \cdots = \sum_{n=2}^{\infty} n(n-1)C_n\, x^{n-2}$$

Substitution of these into the differential equation yields that;

$$(x^2 - 1)\sum_{n=2}^{\infty} n(n-1)C_n\, x^{n-2} + 3x\sum_{n=1}^{\infty} nC_n\, x^{n-1} + x\sum_{n=0}^{\infty} C_n\, x^n \qquad = 0 \;\rightarrow$$

$$\sum_{n=2}^{\infty} n(n-1)C_n\, x^n - \sum_{n=2}^{\infty} n(n-1)C_n\, x^{n-2} + 3\sum_{n=1}^{\infty} nC_n\, x^n + \sum_{n=0}^{\infty} C_n\, x^{n+1} \qquad = 0$$

In order to add the above series, it is necessary that both summation indices start with the same number and the powers of $x$terms in each series should be such that if one series starts with a multiple of x to the first power, then the other series should also have the same power. Next, to make the exponent of all x terms same n, the same as the first and the third terms. So, the second term is modified by replacing $n \rightarrow n + 2$

$$\sum_{n=2}^{\infty} n(n-1)C_n\, x^{n-2} \;\; for\;\; n \rightarrow n+2 \;\Rightarrow\; \sum_{n+2=2}^{\infty} (n+2)(n+1)C_{n+2}\, x^n$$

Similarly, for the last term $\quad n \rightarrow n - 1$

$$\sum_{n=0}^{\infty} C_n\, x^{n+1} \;\; for\, n \rightarrow n-1 \;\Rightarrow\; \sum_{n-1=0}^{\infty} C_{n-1}\, x^n$$

Substitution of these above yields that;

$$\sum_{n=2}^{\infty} n(n-1)C_n\, x^n - \sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}\, x^n + 3\sum_{n=1}^{\infty} nC_n\, x^n + \sum_{n=1}^{\infty} C_{n-1}\, x^n = 0$$

Since the exponents are now common, the next step is to make the range of Σ ‘s common as well. Clearly the common range is $n = 2 \ldots \infty$; so expand the Σ ‘s to makethe index as n=2;

$$\sum_{n=2}^{\infty} n(n-1)C_n\, x^n$$

already n=2;

$$\sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}\, x^n = 2C_2 + 6C_3\, x + \sum_{n=2}^{\infty}(n+2)(n+1)C_{n+2}\, x^n$$

$$\sum_{n=1}^{\infty} nC_n \, \mathrm{x}^n = C_1 \, x + \sum_{n=2}^{\infty} nC_n \, \mathrm{x}^n$$

$$\sum_{n=1}^{\infty} C_{n-1} \, \mathrm{x}^n = C_0 x + \sum_{n=2}^{\infty} C_{n-1} \, \mathrm{x}^n$$

Substitution of these back yields that;

$$\sum_{n=2}^{\infty} n(n-1)C_n \, \mathrm{x}^n - 2C_2 \; - 6C_3 \, x - \sum_{n=2}^{\infty} (n+2)(n+1)C_{n+2} \, \mathrm{x}^n + 3C_1 \, x + 3\sum_{n=2}^{\infty} nC_n \, \mathrm{x}^n + C_0 x + \sum_{n=2}^{\infty} C_{n-1} \, \mathrm{x}^n = 0$$

Let’s group this according to the powers of x;

$$-2C_2 + (C_0 + 3C_1 - 6C_3)x + \sum_{n=2}^{\infty} [-(n+2)(n+1)C_{n+2} \; + [n(n-1) + 3n]C_n + C_{n-1} \,]\mathrm{x}^n = 0$$

Equating the coefficients of x^i to zero one has

$$-2C_2 = 0 \quad \Rightarrow \quad C_2 = 0$$

$$C_0 + 3C_1 - 6C_3 = 0 \quad \Rightarrow \quad C_3 = \frac{1}{6}C_0 + \frac{1}{2}C_1$$

$$[-(n+2)(n+1)C_{n+2} + [n(n+2)]C_n + C_{n-1} = 0$$

$$C_{n+2} = \frac{n(n+2)C_n + C_{n-1}}{(n+1)(n+2)} \; ; \qquad\qquad n \geq 2$$

This last expression is called the *RECURRENCE formula*. Apparently there is no restriction on C_0, and C_1. Therefore they remain as arbitrary. If we use the recurrence formula to compute coefficients with higher indices, we have

$$n = 2 \Rightarrow C_4 = \frac{8C_2 + C_1}{12} = \frac{1}{12}C_1$$

$$\mathrm{n} = 3 \Rightarrow C_5 = \frac{15C_3 + C_2}{20} = \frac{3}{4}\left(\frac{1}{6}C_0 + \frac{1}{2}C_1\right) + \frac{C_2}{20} = \frac{1}{8}C_0 + \frac{3}{8}C_1$$

Substitution of these C_0, C_1, C_2 ... into the proposal

$$y = C_0 + C_1 x + C_2 \mathrm{x}^2 + \cdots$$

we have

$$y = C_0 + C_1 x + \left(\frac{1}{6}C_0 + \frac{1}{2}C_1\right)x^3 + \frac{1}{2}C_1 x^4 + \left(\frac{1}{8}C_0 + \frac{3}{8}C_1\right)x^5 + \cdots$$

Let's group C_o and C_1 terms separately,

$$y = C_0\left(1 + \frac{1}{6}x^3 + \frac{1}{8}x^5 + \cdots\right) + \ C_1\left(x + \frac{1}{2}x^3 + \frac{1}{12}x^4 + \frac{3}{8}x^5 + \cdots\right)$$

This is the series solution of the differential equation. As expected, there appeared two arbitrary constants C_0, C_1 to be obtained using initial conditions

for $y(0) = 4 \ \Rightarrow C_0 = 4$

for $y'(0) = 6 \Rightarrow y' = C_0\left(\frac{1}{2}x^2 + \frac{5}{8}x^4 + \cdots\right) + C_1\left(1 + \frac{3}{2}x^2 + \frac{1}{3}x^3 + \frac{15}{8}x^4 + \cdots\right) \Rightarrow C_1 = 6$

substitution of this into above,

$$y = 4\left(1 + \frac{1}{6}x^3 + \frac{1}{8}x^5 + \cdots\right) + \ 6\left(x + \frac{1}{2}x^3 + \frac{1}{12}x^4 + \frac{3}{8}x^5 + \cdots\right)$$

manipulations yield that

$$y = 4 + 6x + \frac{11}{3}x^3 + \frac{1}{2}x^4 + \frac{11}{4}x^5 + \cdots$$

This is the solution of the initial value problem.

Notice that if initial condition would be $y(5) = 4, y'(5) = 6$; it is a good idea to shift the differential equation using independent variable transformation $t = x - 5$ and obtain a series solution about $t = 0$, instead of $x = 5$.

More precisely instead of solving the boundary value problem, for example;

$(x^2 - 1)y'' + 3xy' + xy = 0\,; \ \ y(x = 5) = 4, \qquad y'(x = 5) = 6$

using proposal

$$y = \sum_{n=0}^{\infty} C_n\,(\mathrm{x} - \ 5)^n$$

A change of variable $t = x - 5$ replaces this initial value problem by the equivalent problem;

$$t = x - 5 \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} \Rightarrow y' = \frac{dy}{dt};\ y'' = \frac{d^2y}{dt^2}$$

$x = t + 5 \Rightarrow (x^2 - 1) = (t^2 + 10t + 24),$ so

$(t^2 + 10t + 24)\frac{d^2y}{dt^2} + (3t + 15)\frac{dy}{dt} + 3y = 0$ and initial conditions; $y(0) = 4, y'(0) = 6$. And solution proposal to be made is;

$$y = \sum_{n=0}^{\infty} C_n \, t^n$$

Having obtained the solution in *t*; the substitution $t = x - 5$, replaces the solution in terms of x . This technique may be suggested for the sake of simplicity.■