Chapter 1

AN OVERVIEW OF DIFFERENTIAL EQUATIONS

In this chapter, an overview of differential equations will be presented. A special attention is paid to classifications and solution of linear differential equations.

1.1 Differential Equations and Their Classifications

In science and engineering, mathematical models are developed to aid in understanding of physical phenomena. These models often yield an equation that contains some derivatives of an unknown function. Such equations involving derivatives or differentials of one dependent variable with respect to one or more independent variables are called differential equations. Differential equations can be classified according to number of independent variables and that to input output-output relationship. According to number of independent variables, differential equations are grouped as ordinary differential equations and partial differential equations.

Ordinary Differential Equations:

A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation. Some examples for ordinary differential equations may be as follows.

1) $$\frac{d^2y}{dx^2} + xy(\frac{dy}{dx})^2 = 0$$

2) $$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} = y^2 \tan t$$

Partial Differential Equations:

A differential equation involving partial derivatives of one or more dependent variables with respect to two or more dependent variables is called a partial differential equation. Some examples for partial differential equations may be as follows.

1) $$\frac{\partial v}{\partial s}+\frac{\partial v}{\partial t}=v$$

2) $$\frac{\partial z}{\partial x}+z=\frac{\partial^2 z}{\partial y^2}$$

Linearity of Differential Equations:

Differential equations can also be classified according to input-output relations as linear and nonlinear differential equations. A differential equation is called linear if;

a) Every dependent variable and every derivative involved occur to the first degree only, and
b) No products of dependent variables and/or derivatives occur.

A differential equation which is not linear is called a non-linear differential equation. Some examples for linearity and non-linearity of differential equations may be as follows. In all of the ordinary differential equations in the below, ***y* is the dependent variable**, and ***x* is the independent variable.**

1) $$\frac{d^2y}{\mathrm{d}x^2}+5\frac{\mathrm{dy}}{\mathrm{dx}}+6y=0$$ **Linear**

2) $$\frac{d^4y}{\mathrm{d}x^4}+x^2\frac{d^3y}{dx^3}+x^3\frac{dy}{dx}=xe^x$$ **Linear**

3) $$x\frac{d^2y}{dx^2}+y\frac{dy}{dx}=e^x$$ **Non-linear**

4) $\frac{d^2y}{dx^2} + 5\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^3 + 6y = \boldsymbol{x^2}$ **Non-linear**

5) $\frac{d^2y}{dx^2} + 5\frac{\mathrm{dy}}{\mathrm{dx}} + 6y^2 = \mathrm{Sinx}$ **Non-linear**

Order of Differential Equations:

The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation as shown below.

1) $\frac{d^2y}{dx^2} + xy\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right)^2 = 0$ O=2, O.D.E, Dep= ***y***, Ind= ***x***

2) $\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t$ O=4, O.D.E, Dep= ***x***, Ind= ***t***

3) $\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v$ O=1, P.D.E., Dep= ***v***, Ind= ***s***, ***t***

Degree of a Differential Equation:

If a differential equation can be rationalized and cleared from fractions with regard to all derivatives present, the exponent of the highest order derivative is called the degree of the differential equation as shown below.

1) $(\frac{d^2y}{dt^2})^2 + (1 + \frac{dy}{dt})^3 = 0$ O=2, O.D.E., D=3, Dep=***y***, Ind=***x***

2) $\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} = y^2 \tan t$ O=3, O.D.E., D=2, Dep=***y***, Ind=***t***

Exercises:

Classify each of the following differential equations as:PDE,ODE, Linear, Non-Linear. Determine the order, and degree of them;

1) $\frac{dy}{dx} + xy = xe^x$ **ODE, Linear, 1st order, Degree=1**

2) $\frac{d^3y}{\mathrm{d}t^3} + x\sqrt{y} = \sin \mathrm{x}$ **ODE, non-Linear, 3rd order, Degree =1**

3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **PDE, Linear, 2nd order, Degree =1**

4) $\frac{d^2y}{\mathrm{d}x^2} + x \sin y = 0$ **ODE, non-Linear, 2nd order**

5) $\frac{d^2y}{\mathrm{d}x^2} + y \sin x = 0$ **ODE, Linear, 2nd order, Degree =1**

1.2 First Order Differential Equations

The most general first order differential equation is of the form

$$f(x, y, y') = 0.$$

Under certain analyticity conditions it can be reduced to the form;

$$y' = f(x, y).$$

Or considering that $y' = dy/dx$, and $f(x, y) = -M(x, y)/N(x, y)$, the most general first order differential equation can be written as

$$M(x, y)dx + N(x, y)dy = 0,$$

which is called the differential form of the first order differential equations.

Separable Differential Equations

Some of the first order differential equations can be reduced to the form; $g(y)y' = f(x)$ and substitution of $y' = \frac{dy}{dx}$ yields; $g(y)dy = f(x)dx$ in which the terms with variables x and y are separated. Therefore, these equations are called **Separable Differential Equations**. In separable differential equations, integrations of both sides yields an implicit solution straightforward, as follows;

$$\int g(y)dy = \int f(x)dx + c.$$

Example 1.1

Solve the differential equation $9yy' + 4x = 0$

Solution: After substitution that $y' = dy/dx$, it is possible to rewrite the equation in a form that the two variables are separated.

$$9ydy = -4xdx.$$

Then integrations of both sides yields an implicit solution

$$\int 9ydy = \int -4xdx$$

$$\frac{9}{2}y^2 = -2x^2 + C_1$$

$$\frac{x^2}{9} + \frac{y^2}{4} = \text{C}.$$

This solution is an equation of an ellipse family for C > 0.

Exact Differential Equations:

A first order differential equation of the form $\text{M}(x, y)dx + \text{N}(x, y)dy = 0$ is called exact if there is a function $u(x, y)$ such that

$\frac{\partial u}{\partial x} = M$ and $\frac{\partial u}{\partial y} = N$.

In such a case, the left hand side of the differential equation can be simplified to an exact differential

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

and the equation is reduced to $du = 0$. That is why the equation is called exact. An implicit solution is simply $u(x, y) = C$.

A sufficient condition for the exactness of a first order ODE is obtained by the observation

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

For twice differentiable functions

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Hence a sufficient condition for a first order differential equation $\mathrm{M}(x, y)dx + \mathrm{N}(x, y)dy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Then the solutions are obtained using the following steps;

i. $\frac{\partial u}{\partial x} = M \rightarrow u = \int M dx + k(y)$
ii. Use the equation $\frac{\partial u}{\partial y} = \mathrm{N}$ to obtaindk/dy
iii. integrate $\frac{dk}{dy}$ to get k
iv. The solution of the exact equation is $u(x, y) = C$

Example 1.2

Show that the following equation is exact and find the solution.

$$(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$$

$$M = x^3 + 3xy^2 \Longrightarrow \frac{\partial M}{\partial y} = 6xy$$

$N = 3x^2y + y^3 \Longrightarrow \dfrac{\partial N}{\partial x} = 6xy$

Hence

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 6xy$$

Therefore the equation is exact.

$u = \int M dx + k(y) \;=\; \int (x^3 + 3xy^2) dx + k(y) = \dfrac{x^4}{4} + \dfrac{3x^2y^2}{2} + \; k(y)$

From this proposal

$\dfrac{\partial u}{\partial y} = 3x^2y + \dfrac{dk}{dy}$

From differential equation

$\dfrac{\partial u}{\partial y} = N = \; 3x^2y + y^3$

Comparison gives

$\dfrac{dk}{dy} = y^3 \Longrightarrow \; k = \dfrac{y^4}{4} + c$

Hence

$$u(x,y) = \frac{y^4}{4} + \frac{3x^2y^2}{2} + \frac{y^4}{4} + c$$

An implicit solution of the exact equation is $u(x,y) = C$. That is

$$\frac{y^4}{4} + \frac{3x^2y^2}{2} + \frac{y^4}{4} = C$$

a family of plane curves for C > 0.

Linear First Order Differential Equations:

First order equations that are linear in y and y' are called linear first order equations. The most general for of this kind of differential equations is

$$y' + p(x)\,y = q(x)\,.$$

To solve this equation let us multiply both sides by the function $e^{\int p(x)dx}$:

$$e^{\int p(x)dx}y' + p(x)e^{\int p(x)dx}\,y = q(x)e^{\int p(x)dx}\,.$$

It is clear that the left hand side is a derivative:

$$\left(e^{\int p(x)dx}y\right)' = q(x)e^{\int p(x)dx}\,.$$

which yields the solution

$$y = e^{-\int p(x)dx}\int q(x)e^{\int p(x)dx}dx\,.$$

Example 1.3

Find the solution of the linear equation

$$y' + \frac{4}{x}y = 8x^3.$$

For this equation

$$\int p(x)dx = \int \frac{4}{x}dx = 4\ln x = \ln x^4\,.$$

Hence by (2.35) one has

$$y = e^{-\ln x^4}\int 8x^3 e^{\ln x^4}dx = \frac{1}{x^4}\int 8x^7 dx = \frac{1}{x^4}\left(x^8 + C\right).$$

1.3 2nd Order Linear DifferentialEquations

In this section, solutions of second order differential equations of different kinds are discussed.

A second order linear differential equation is linear if it has the form

$$y'' + p(x)y' + q(x)y = r(x)$$

where $p(x)$, $q(x)$, $r(x)$ are smooth functions.

If $r(x) = 0$, the equation is called homogeneous.

Examples

1) $y'' + 4y = e^{-x}\sin x$ non-homogeneous, linear
2) $(1-x^2)y'' - 2xy' + 6y = 0$ homogeneous, linear

Initial and Boundary Value Problems for Second Order Linear Differential Equations

If a second order linear differential equation

$$y'' + p(x)y' + q(x)y = r(x)$$

is accompanied by the conditions

$y'(x_0) = a$ and $y(x_0) = b$, it is called an **Initial Value Problem (IVP)**

$y'(x_0) = a$ and $y(x_1) = b$, or

$y(x_0) = a$ and $y(x_1) = b$, it is called a **Boundary Value Problem (BVP)**

Second Order Linear Homogeneous Differential Equations With Constant Coefficients:

Consider a 2nd order linear homogenous differential equation.

$$y'' + ay' + by = 0$$

Try a solution $y = e^{rx}$ then,

$$r^2 e^{rx} + a.\mathrm{r}e^{rx} + \mathrm{b}.e^{rx} = 0\,,\, (r^2 + ra + b)e^{rx} = 0$$

$e^{rx} \neq 0$, hence

$$y = e^{rx} \Longrightarrow r^2 + \mathrm{ar} + \mathrm{b} = 0$$

Three cases may arise in this case;

	Case	**Roots**	**General Solution**
I	Distinct real roots	$r_1 \neq r_2 \in R$	$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
II	Double real roots	$r_1 = r_2 \in R$	$y = (C_1 + C_2 x)e^{r_1 x}$
III	Complex conjugate coots	$r_{1,2} = m \mp in$	$\mathrm{y} = C_1 e^{r_1 \mathrm{x}} + C_2 e^{r_2 x}$ or $y = e^{mx}[A\cos nx + B\sin nx]$

<u>Example 1.4</u>

Solve the boundary value problem

$y'' + y = 0, \quad \mathrm{y}(0) = 3, y(\pi/2) = -3$

$y = e^{rx} \Longrightarrow (r^2 + 1)e^{rx} = 0 \;\Longrightarrow r_{1,2} = \mp i$

$y = C_1 e^{ix} + C_2 e^{-ix} = A\,Sinx + BCosx$

$\mathrm{y}(0) = 3 \;\rightarrow 3 = \mathrm{B}, \qquad \mathrm{y}\left(\frac{\pi}{2}\right) = -3 \;\rightarrow -3 = \mathrm{A} \;\rightarrow\; \mathrm{A} = -3$

$$y = -3\; sinx + 3cosx$$

is the solution of the given boundary value problem.

<u>Example 1.5</u>

Find the general solution of $\quad y'' + 6y' + 9y = 0.$

$y = e^{rx} \Longrightarrow (r^2 + 6r + 9)e^{rx} = 0 \;\Longrightarrow (r+3)^2 = 0 \;\Longrightarrow r_{1,2} = -3 \qquad$ double root

$y = (C_1 + C_2 x)e^{-3x}$

Euler-Cauchy Equations

Euler-Cauchy Equations are second order linear equations with variable coefficients of the special form

$$x^2y'' + axy' + by = r(x)$$

where a, b are constants, and $r(x)$ is a smooth function.

The change of the independent variable

$$x = e^z \ \rightarrow z = \ln x$$

leads to the transformations of the derivatives

$$\frac{d}{dx} = \frac{1}{x}\frac{d}{dz} \rightarrow \ x\frac{dy}{dx} = \ \frac{dy}{dz}$$

and

$$\frac{dy}{dx} + x\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2y}{dz^2} \rightarrow \ x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} = \ \frac{d^2y}{dz^2} \rightarrow \ x^2\frac{d^2y}{dx^2} = \ \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

Substituting these in Euler-Cauchy equation, one has

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + a\frac{dy}{dz} + by = r(e^z) \rightarrow \frac{d^2y}{dz^2} + (a-1)\frac{dy}{dz} + by = r(e^z)$$

The resulting equation is a second order linear equation with constant coefficients.

Example 1.6

Solve $x^2y'' + 7xy' + 13y = 0$ Let $x = e^z$

Then

$$x\frac{dy}{dx} = \ \frac{dy}{dz}, \qquad \text{and } x^2\frac{d^2y}{dx^2} = \ \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

Substituting these in the given Euler-Cauchy equation, one has

$$\frac{d^2y}{dz^2} - \frac{dy}{dz} + 7\ \frac{dy}{dz} + 13y = 0 \rightarrow y'' + 6y' + 13y = 0, \qquad ' = \frac{d}{dz}$$

$m^2 + 6m + 13 = 0 \implies m_{1,2} = -3 \mp 2i$

$y = e^{-3z}[A\cos(2z) + B\sin(2z)]$

Transforming back to the original independent variable x one has

$y = x^{-3}[ACos(2lnx) + BSin(2lnx)]$

Non-homogeneous Second Order Linear Differential Equations with Constant Coefficients:

These equations are in the form of

$$y'' + p(x)y' + q(x)y = r(x) \quad , \quad r(x) \neq 0 \qquad (2)$$

General solution of non-homogeneous differential equations are of the form

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the general solution of the corresponding homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0$$

and $y_p(x)$ is any solution of the non-homogeneous differential equation

$$y'' + p(x)y' + q(x)y = r(x) \quad , \quad r(x) \neq 0$$

In previous section we have seen how to find the general solution of a homogeneous second order linear differential equation. Now we will elaborate methods to find a particular solution of the non-homogeneous differential equation.

The Method of Undetermined Coefficients:

If the non-homogeneity function $\boldsymbol{r(x)}$ is from a special kind of functions that create only a finite number of root functions upon successive differentiations, it is called a function of finite derivatives.

Example 1.7

1) $\boldsymbol{r(x)} = \mathbf{Sin2x}$ is a function of finite derivatives since upon successive differentiations create only two root functions;

$$D = \{\mathrm{Sin2x}, \mathrm{Cos\ 2x}\}$$

2) $r(x) = \mathbf{x^5}$ is also a function of finite derivatives

$$D = \{\mathbf{x^5}, x^4, x^3, x^2, x, 1\}$$

3) $r(x) = \ln x$ is not a function of finite derivatives

$$\mathbf{D} = \{\ln x, \frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \dots\}$$

Now let the set of the two linearly independent solutions of homogeneous second order linear differential equation is

$$H = \{\varphi_1(x), \varphi_2(x)\}$$

and the D set of $r(x)$ is

$$D = \{f_1(x), f_2(x), \dots, f_n(x)\}$$

1) If the sets H and D do not have any common function, then we propose a particular solution for the non-homogeneous differential equation as a linear combination of functions in D with coefficients $\{a_1, a_2, \dots, a_n\}$ to be determined

$$y_p = a_1 f_1(x) + a_2 f_2(x) + \cdots + a_n f_n(x)\}$$

Then this proposed solution is substituted in the non-homogeneous differential equation to determine $\{a_1, a_2, \dots, a_n\}$.

Example 1.8

Solve $\quad y'' - 3y' + 2y = e^{3x}$

First homogenous solution is found. The characteristic equation has roots

$r^2 - 3\mathrm{r} + 2 = 0 \implies r_{1,2} = 1{,}2$

$y_h = C_1 e^x + C_2 e^{2x}; \quad H = \{e^x, e^{2x}\}$

where

$$\mathrm{D} = \{\mathrm{e}^{3\mathrm{x}}\}, \qquad \text{and } \mathrm{H} \cap \mathrm{D} = \emptyset$$

Hence particular solution proposal is

$$y_p = Ce^{3x}$$

Upon substitution into the non-homogeneous differential equation one has

$y_p{}' = 3Ce^{3x} \implies y_p{}'' = 9Ce^{3x}$

$9Ce^{3x} - 9Ce^{3x} + 2Ce^{3x} = e^{3x}$

$2Ce^{3x} = e^{3x} \implies c = 1/2$

Therefore

$$y_p = \frac{1}{2} e^{3x}$$

and the general solution is

$y = C_1 e^x + C_2 e^{2x} + \frac{1}{2} e^{3x}$

2) If the sets H and D are not disjoint, we multiply the set D till we get

$$\mathrm{H} \cap \mathrm{x}^{\mathrm{m}}\mathrm{D} = \emptyset$$

then we propose a particular solution for the non-homogeneous differential equation as a linear combination of functions in $\mathrm{x}^{\mathrm{m}}\mathrm{D}$ with coefficients $\{\mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_\mathrm{n}\}$ to be determined

$$\mathrm{y}_\mathrm{p} = \mathrm{a}_1\mathrm{x}^\mathrm{m}\mathrm{f}_1(\mathrm{x}) + \mathrm{a}_2\mathrm{x}^\mathrm{m}\mathrm{f}_2(\mathrm{x}) + \cdots + \mathrm{a}_\mathrm{n}\mathrm{x}^\mathrm{m}\mathrm{f}_\mathrm{n}(\mathrm{x})\}$$

<u>Example 1.9</u>

Solve $\quad y'' - 3y' + 2y = e^x$

First homogenous solution is found. The characteristic equation has roots

$r^2 - 3\mathrm{r} + 2 = 0 \implies r_{1,2} = 1{,}2$

$y_h = C_1 e^x + C_2 e^{2x}; \quad H = \{e^x, e^{2x}\}$

where

$$\mathrm{D} = \{\mathrm{e^x}\}, \text{and } \mathrm{H} \cap \mathrm{D} \neq \emptyset \text{ but } \mathrm{xD} = \{\mathrm{xe^x}\}, \text{and } \mathrm{H} \cap \mathrm{xD} = \emptyset$$

Hence particular solution proposal is

$y_p = Cxe^x \implies {y_p}' = C(e^x + xe^x) \implies {y_p}'' = C(2e^x + xe^x)$

$C(2e^x + xe^x) - 3C(e^x + xe^x) + 2Cxe^x = e^x$

$-Ce^x = e^x \implies C = -1$

Therefore

$$y_p = -xe^x$$

and the general solution is

$y = C_1 e^x + C_2 e^{2x} - xe^x$

Example 1.10

Solve $\quad y'' + 2y' + 5y = 16e^x + \sin 2x.$

The roots of the characteristic equation, and the homogenous solution are;

$r^2 + 2\mathrm{r} + 5 = 0 \implies r_{1,2} = -1 \mp 2i$

$y_h = e^{-x}(A\cos 2x + B\sin 2x)$

$H = \{e^{-x}\cos 2x, e^{-x}\sin 2x\},$

where the D sets are

$$\mathrm{D_1} = \{\mathrm{e^x}\}, \text{and } \mathrm{D_2} = \{\cos 2x, \sin 2x\},$$

$$\mathrm{H} \cap \mathrm{D_1} = \emptyset, \qquad \mathrm{H} \cap \mathrm{D_2} = \emptyset$$

Therefore

$y_p = Ce^x + K\cos 2x + M\sin 2x$

$y_p' = Ce^x - 2K\sin 2x + 2M\cos 2x$

$y_p'' = Ce^x - 4K\cos 2x - 4M\sin 2x$

$y'' + 2y' + 5y = 16e^x + \sin 2x \rightarrow$

$$Ce^x - 4K\cos 2x - 4M\sin 2x + 2(Ce^x - 2K\sin 2x + 2M\cos 2x) + 5(Ce^x + K\cos 2x + M\sin 2x) = 16e^x + \sin 2x \rightarrow$$

$8C = 16, \rightarrow C = 2$

$\cos 2x: -4K + 4M + 5K = 0 \rightarrow K + 4M = 0$

$\sin 2x: -4M - 4K + 5M = 1 \rightarrow -4K + M = 1$

$K = -\frac{4}{17};\ M = \frac{1}{17}$

Therefore

$$y_p = -\frac{4}{17}\cos 2x + \frac{1}{17}\sin 2x$$

and the general solution is

$$y = e^{-x}(A\cos 2x + B\sin 2x) + 2e^x - \frac{4}{17}\cos 2x + \frac{1}{17}\sin 2x$$

Wronskian Determinant and Solution by Variation of Parameters:

The previous solution technique can be applied only to constant-coefficient equations with special *r(x)*. For the more general equations like

$y'' + p(x)y' + q(x)y = r(x)$

with functions p , q and r which are continuous in the given interval, the method of variation of parameters is used to find a particular solution y_p.

***Claim*:** For equations like

$y'' + p(x)y' + q(x)y = r(x)$

with functions p , q and r which are continuous in the given interval, the method of variation of parameters yields a particular solution y_p in the form

$$y_p(x) = -y_1 \int \frac{y_2\, r}{|W|} dx + y_2 \int \frac{y_1\, r}{|W|} dx$$

where y_1 , y_2 form a basis of solutions of the homogeneous equation

$y'' + p(x)y' + q(x)y = 0$

corresponding to original equation, and

$$W = y_1\, y_2' - y_1{}' y_2 \quad \rightarrow \quad |W| = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

That is the Wronskian determinant ofy_1 , y_2.

Note: The Wronskian determinant is not zero when y_1 , y_2 are linearly independent and solution is feasible.

Proof:

Let the homogeneous solution of the given equation is

$$\mathrm{y_h(x) = \ C_1 y_1(x) + C_2 y_2(x).}$$

Let us propose a particular solution y_p in the form

$$y_p(x) \ = u(x) y_1(x) + v(x) y_2(x)$$

where u and v are functions to be determined.

$$\mathrm{y}_p{}' \ = \mathrm{u' y_1 + u y_1{}' + v' y_2 + v y_2{}'} \qquad (*)$$

Assume

$u' y_1 + v' y_2 = 0.$ (1)

Then the equation (*) reduces to $\quad y_p{}' = \mathrm{u}y_1{}' + \mathrm{v}y_2{}'$

Second derivative $y_p{}''$ is now

$$y_p{}'' = u'y_1{}' + uy_1{}'' + v'y_2{}' + vy_2{}''$$

Insert $y_p, y_p{}'$ and $y_p{}''$ in the original non homogeneous equation,

$$u(y_1{}'' + py_1{}' + qy_1) + v(y_2{}'' + py_2{}' + qy_2) + u'y_1{}' + v'y_2{}' = r(x)$$

since y_1 and y_2 are solutions of the differential equation, the above equation reduces to

$u'y_1{}' + v'y_2{}' = r(x)$ (2)

and combining equation (2), with equation (1), a linear algebraic system of two linear equations for u', v' is obtained.

$$\begin{bmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{bmatrix}\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}.$$

Using Cramer's method, the solution of this linear algebraic system is

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{|W|} = -\frac{y_2\, r}{W}, \qquad v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{|W|} = \frac{y_1\, r}{|W|}.$$

By integration, we find the unknown functions u and v

$$u = -\int \frac{y_2\, r}{|W|}dx, \qquad v = \int \frac{y_1\, r}{|W|}dx.$$

Hence it is proved that a particular solution of the non homogeneous equation is

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x) = -y_1\int \frac{y_2\, r}{W}dx + y_2\int \frac{y_1\, r}{W}dx.$$

Example 1.11

Solve $\quad y'' + y = \sec x$

The roots of the characteristic equation, and the two linearly independent solutions are;

$r^2 + 1 = 0 \implies r_{1,2} = \mp i \implies y_1 = \cos x, \quad y_2 = \sin x$

and the homogenous solution is;

$$y_h = A\cos \mathrm{x} + B\sin \mathrm{x}$$

Therefore particular solution is;

$$W(\mathrm{y}_1, \mathrm{y}_2) = \begin{vmatrix} \cos \mathrm{x} & \sin \mathrm{x} \\ -\sin \mathrm{x} & \cos \mathrm{x} \end{vmatrix} = \cos^2 \mathrm{x} + \sin^2 \mathrm{x} = 1$$

$$y_p(x) = -\cos \mathrm{x} \int \sin \mathrm{x} \sec \mathrm{x}\, \mathrm{dx} + \sin \mathrm{x} \int \underbrace{\cos \mathrm{x} \sec \mathrm{x}}_{1} \mathrm{dx}$$

$$y_p(x) = \cos \mathrm{x}\ ln(\cos \mathrm{x}) + \mathrm{x} \sin \mathrm{x}$$

The general solution is obtained as;

$$y = y_h + y_p = (C_1 + ln \cos \mathrm{x}) \cos \mathrm{x} + (C_2 + x) \sin \mathrm{x}$$

Example 1.12

Solve $\quad y'' + 4\mathrm{y}' + 4y = \frac{e^{-2x}}{x^2}$

The roots of the characteristic equation, and the two linearly independent solutions are;

$\mathrm{r}^2 + 4\mathrm{r} + 4 = 0 \Rightarrow \mathrm{r}_{1,2} = -2 \Rightarrow \mathrm{y}_1 = \mathrm{e}^{-2\mathrm{x}}$ and $\mathrm{y}_2 = \mathrm{xe}^{-2\mathrm{x}}$

and the homogenous solution is;

$\mathrm{y_h} = \mathrm{C_1 e^{-2x}} + \mathrm{C_2 xe^{-2x}}$

Therefore particular solution is;

$$\mathrm{y_p(x)} = -\mathrm{y}_1 \int \frac{\mathrm{y}_2\, \mathrm{r}}{\mathrm{W}} \mathrm{dx} + \mathrm{y}_2 \int \frac{\mathrm{y}_1\, \mathrm{r}}{\mathrm{W}} \mathrm{dx}$$

$$\mathrm{W} = \begin{vmatrix} \mathrm{y}_1 & \mathrm{y}_2 \\ \mathrm{y}_1{}' & \mathrm{y}_2{}' \end{vmatrix} = \begin{vmatrix} \mathrm{e^{-2x}} & \mathrm{xe^{-2x}} \\ -2\mathrm{e^{-2x}} & \mathrm{e^{-2x}} - 2\mathrm{xe^{-2x}} \end{vmatrix} = \mathrm{e^{-4x}} - 2\mathrm{xe^{-4x}} + 2\mathrm{xe^{-4x}} = \mathrm{e^{-4x}}$$

$$\mathrm{y_p(x)} = -\mathrm{e^{-2x}} \int \frac{\mathrm{xe^{-2x}e^{-2x}}}{\mathrm{e^{-4x}x^2}} \mathrm{dx} + \mathrm{xe^{-2x}} \int \frac{\mathrm{e^{-2x}e^{-2x}}}{\mathrm{e^{-4x}x^2}} \mathrm{dx} = -\mathrm{e^{-2x}} \int \frac{\mathrm{dx}}{\mathrm{x}} + \mathrm{xe^{-2x}} \int \frac{\mathrm{dx}}{\mathrm{x^2}}$$

$$y_p(x) = -e^{-2x}\ln x - \frac{x(\ e^{-2x})}{x} = -e^{-2x}\ln x - e^{-2x}$$

Then the general solution is

$$y = y_h + y_p = C_1e^{-2x} + C_2xe^{-2x} - e^{-2x}\ln x$$

<u>Example 1.13</u>

Solve

$$y'' - 4y' + 5y = \frac{2e^{2x}}{\sin(x)}$$

The roots of the characteristic equation, and the two linearly independent solutions are;

$$r^2 - 4r + 5 = 0 \Rightarrow r_{1,2} = 2 \pm i \implies y_1 = e^{2x}\sin x \text{ and } y_2 = e^{2x}\cos x$$

and the homogenous solution is;

$y_h = e^{2x}(C_1\sin x + C_2\cos x).$

Therefore a particular solution is found as;

$$W = \begin{vmatrix} e^{2x}\sin x & e^{2x}\cos x \\ 2e^{2x}\sin x + e^{2x}\cos x & 2e^{2x}\cos x - e^{2x}\sin x \end{vmatrix}$$

$$= 2e^{4x}\cos x\sin x - e^{4x}\sin^2 x - 2e^{4x}\cos x\sin x - e^{4x}\cos^2 x = -e^{4x}$$

$$y_p = -e^{2x}\sin x\int\frac{e^{2x}\cos x\frac{2e^{2x}}{\sin x}}{-e^{4x}}dx + e^{2x}\cos x\int\frac{e^{2x}\sin x\frac{2e^{2x}}{\sin x}}{-e^{4x}}dx$$

$$= 2e^{2x}\sin x\int\cot x\,dx - 2e^{2x}\cos x\int dx = 2e^{2x}(\ln|\sin x|\sin x - x\cos x)$$

Then the general solution is

$y_{gen} = y_h + y_p = e^{2x}(C_1\sin x + C_2\cos x) + 2e^{2x}(\ln|\sin x|\sin x - x\cos x).$

Example 1.14

Solve $y'' - 4y' + 4y = 6x^{-4}e^{2x}$

The roots of the characteristic equation, and the two linearly independent solutions are;

$$r^2 - 4r + 4 = 0 \Rightarrow r_{1,2} = 2 \implies y_1 = e^{2x} y_2 = xe^{2x}$$

and the homogenous solution is;

$$y_h = C_1 e^{2x} + C_2 x e^{2x}$$

Therefore a particular solution is found as;

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}$$

$$y_p = -e^{2x} \int \frac{xe^{2x} 6x^{-4} e^{2x}}{e^{4x}} dx + xe^{2x} \int \frac{e^{2x} 6x^{-4} e^{2x}}{e^{4x}} dx$$

$$= -e^{2x} \int 6x^{-3} dx + xe^{2x} \int 6x^{-4} dx \, 3e^{2x} x^{-2} - 2e^{2x} x^{-2} = e^{2x} x^{-2}.$$

Then the general solution is

$$y_{gen} = y_h + y_p = C_1 e^{2x} + C_2 x e^{2x} + e^{2x} x^{-2}.$$

1.4 Exercises

Find the solutions of the following ODE's

1. $(x^2 - 4)y' = 2xy + 6x$
2. $3xy' - y = \ln x + 1,\ x > 0,\ y(1) = -4$
3. $1 + \left(\frac{t}{y} - \cos y\right) y' = 0$
4. $y' = e^{\ln x - y + 2}$
5. $y'(e^{-y} + 2x) - e^{y} = 0,\ y(0) = 0$

Answers: (Check)

1. $y = \frac{C}{x^2 - 4} - 3$
2. $y = -(\ln x + 4)$
3. $\mathrm{t} = \sin \mathrm{y} + \mathrm{y}^{-1} \cos \mathrm{y} + \mathrm{C}\mathrm{y}^{-1}$
4. $\mathrm{y} = \ln\left(\mathrm{C} + \frac{\mathrm{e}^2 \mathrm{x}^2}{2}\right)$
5. $x = -2e^{-2(1+e^{-y})} - \frac{1}{2}e^{-y} + \frac{1}{4}$