# Bessel Function of order p; $p \in Z^+$

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In the family of Bessel D.E.

$x^2y'' + xy' + (x^2 - p^2)y = 0$, $p \in R$

Equating coefficients of $x^i$ to zero onefets the indicial eqn. with two roots

$r^2 - p^2 = 0$, $C_0 \neq 0 \Rightarrow r_1 = p > 0, r_2 = -p$

The condition eqn. is

$C_1[(r+1)^2 - p^2] = 0 \Rightarrow C_1 = 0$ unless,

For $r = p > 0$ $\quad (r+1)^2 - p^2 = 0$

$(r+1)^2 - p^2 = 2p + 1 \neq 0$

For $r = -p$ $\quad (r+1)^2 - p^2 = -2p + 1 = 0$

$C_0 \neq 0 \Rightarrow r_1 = p > 0, r_2 = -p$

$C_0 \neq 0$, arbitrary, $C_1 = 0$, unless $p \neq 1/2$

The recurrence relation is;

$[(n+r)^2 - p^2]C_n + C_{n-2} = 0 \Rightarrow C_n = -\frac{C_{n-2}}{(n+r)^2 - p^2}$, $n \geq 2$

For $r = p$

the first soln. is found as before

$y_1 = \Gamma(p) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! 2^{2n} \Gamma(n+p)} x^{2n+p}$

For $r = -p$

The recurrence relation is

$[(n+r)^2 - p^2]C_n + C_{n-2} = 0 \Rightarrow (n^2 - 2np)C_n + C_{n-2} = 0$

$\Rightarrow n(n-2p)C_n + C_{n-2} = 0$

$\Rightarrow n(n-m)C_n + C_{n-2} = 0$

$m = 2p$, even integer

$C_n = \frac{-C_{n-2}}{(n-p)^2 - p^2} = \frac{-C_{n-2}}{n^2 - 2np} = -\frac{C_{n-2}}{n(n-m)}$; $n < m$, $n$ even

$C_0$ is arbitrary

when $2p = m$, an even integer

$\Rightarrow n(n-2p)C_n + C_{n-2} = 0$

$\Rightarrow n(n-m)C_n + C_{n-2} = 0$

$C_m$ is arbitrary

$C_{m+2} = -\frac{C_m}{2(m+2)}$ ; $n \geq m+2$

$y_3 = x^{-p}(C_0 + C_2x^2 + C_4x^4 + \cdots + C_{m-2}x^{m-2})$

$y_4 = x^{-p}(C_m x^m + \cdots)$

$n = 2 \Rightarrow C_2 = \frac{-C_0}{2(2-m)} = \frac{-C_0}{2^2 \cdot 1 \cdot (1+p)}$

$n = 3 \Rightarrow C_3 = \frac{-C_1}{3(3+2p)} = 0$, $C_5 = 0, \ldots, C_{2n+1} = 0$, $n \geq 1$

$n = 4 \Rightarrow C_4 = \frac{-C_2}{4(4-m)} = \frac{C_0}{[2 \cdot 4][(2-m)(4-m)]}$

... ....

$n \geq m$

$C_{m+2} = -\frac{C_m}{2(m+2)}$ ; $n \geq m+2$

$y_3 = x^{-p}(C_0 + C_2x^2 + C_4x^4 + \cdots + C_{m-2}x^{m-2})$

$y_4 = x^{-p}(C_m x^m + \cdots)$

$y_3$ is the second linear independent solution of the Bessel eqn., $y_4 = \alpha y_1$ for some $\alpha$