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Newton's cooling Law

$$\frac{dT}{dt} = k \cdot (T - T_s)$$

$T_i = 90°C$

$T_s = 0°C$

$T(60) = 40°C$

Seperable DE.

$$dT = k \cdot (T - T_s)\, dt$$

$$\int \frac{1}{T - T_s}\, dT = \int k\, dt = \ln|T - T_s| = kt + c$$

$$T - T_s = e^{kt+c}$$

$$T = T_s + e^{kt+c} \Rightarrow T(t)$$

init.

$t = 0$

$$T_i = T(0) = T_s + e^{c} \rightarrow c = \ln|T_i - T_s|$$

$$T = T_s + e^{kt} \cdot e^{\ln|T_i - T_0|}$$

$$\boxed{T = T_s + (T_i - T_s) \cdot e^{kt}}$$

for k.

$t = 60s$

$$T = 0 + (90 - 0) \cdot e^{k \cdot 60} = 40$$

$$90 \cdot e^{k \cdot 60} = 40$$

$$e^{k \cdot 60} = 0.66$$

$$k \cdot 60 = \ln(0.66)$$

$$k = -0.0135$$

$$T = T_s + (T_i - T_s) \cdot e^{-0.0135 \cdot t}$$

$$T = 90 \cdot e^{-0.0135 \cdot t}$$

at $t = 5min = 300s$.

$$T = 90 \cdot e^{-0.0135 \cdot 300} = 1.57°C$$

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80°C

200°C $\xrightarrow[30\,min]{}$ 160°C

$$\frac{dT}{dt} = k \cdot (T_i - T_s)$$

$$T = T_s + (T_i - T_s) \cdot e^{k \cdot t}$$

$$160 = 80 + (200 - 80)\, e^{k \cdot 30}$$

$$160 = 80 + 120 \cdot e^{k \cdot 30}$$

$$120 \cdot e^{k \cdot 30} = 80$$

$$e^{k \cdot 30} = 0.666$$

$$k \cdot 30 = \ln(0.666) = -0.4055$$

$$k = -0.01351$$

$$T = 80 + (200 - 80) \cdot e^{-0.01351 \cdot t}$$

$t = 60$ min.

$$T = 80 + (200 - 80) \cdot e^{-0.01351 \cdot 60} = \underline{\underline{133.35\,°C}}$$

$$100 = 80 + (200 - 80) \cdot e^{-0.01351 \cdot t}$$

$$20 = 120 \cdot e^{-0.01351 \cdot t}$$

$$e^{-0.01351 \cdot t} = \frac{1}{6}$$

$$-0.01351 \cdot t = \ln\left(\frac{1}{6}\right) = -1.7917$$

$$\underline{\underline{t = 132.62 \text{ min}}}$$

Solve the following homogeneous linear system using matrix form.

(5)

$$x_1' = 7x_1 - x_2 + 6x_3 + \cos 2t$$
$$x_2' = -10x_1 + 4x_2 - 12x_3$$
$$x_3' = -2x_1 + x_2 - x_3 .$$

Convert into matrix form

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos 2t$$

if $x = \alpha e^{\lambda t} \rightarrow$
$$x_1 = \alpha_1 e^{\lambda t}$$
$$x_2 = \alpha_2 e^{\lambda t}$$
$$x_2 = \alpha_3 e^{\lambda t}$$

$\boxed{-(\lambda-7) = 7-\lambda}$

$$|\lambda I - A| = \begin{bmatrix} 7-\lambda & -1 & -6 \\ +10 & 4-\lambda & +12 \\ +2 & -1 & -1-\lambda \end{bmatrix} = 0$$

$$7-\lambda \cdot \begin{bmatrix} 4-\lambda & +12 \\ -1 & -1-\lambda \end{bmatrix} + 1 \cdot \begin{bmatrix} +10 & +12 \\ +2 & -1-\lambda \end{bmatrix} \overset{-1(-6)}{+} \begin{bmatrix} 10 & 4-\lambda \\ 2 & -1 \end{bmatrix} = 0$$

$$(7-\lambda)\left[(4-\lambda)(-1-\lambda)+12\right] + \left[-10(-1-\lambda)-24\right] + 1\left[-10+2(4-\lambda)\right] = 0$$

$$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$$

$$(\lambda-2)(\lambda-3)(\lambda-5) = 0$$

$$\lambda_1 = 2 \quad \lambda_2 = 3 \quad \lambda_3 = 5$$

for $\lambda = 2$

$$A\alpha = \lambda \alpha \rightarrow \begin{bmatrix} 7 & -1 & 6 \\ -10 & 4 & -12 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 2 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$7\alpha_1 - \alpha_2 + 6\alpha_3 = 2\alpha_1$$
$$-10\alpha_1 + 4\alpha_2 - 12\alpha_3 = 2\alpha_2$$
$$-2\alpha_1 + \alpha_2 - \alpha_3 = 2\alpha_3$$

$$5\alpha_1 - \alpha_2 + 6\alpha_3 = 0$$
$$-10\alpha_1 + 2\alpha_2 - 12\alpha_3 = 0$$
$$-2\alpha_1 + \alpha_2 - 3\alpha_3 = 0$$

$$\begin{matrix} 5\alpha_1 - \alpha_2 + 6\alpha_3 = 0 \\ -2\alpha_1 + \alpha_2 - 3\alpha_3 = 0 \end{matrix} \rightarrow \begin{matrix} \alpha_3 = -\alpha_1 \\ \alpha_2 = -\alpha_1 \end{matrix}$$

Let $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = -1$

So that for $\lambda = 2 \rightarrow \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$x_1 = \alpha e^{\lambda_1 t}$ → so the corresponding soln is

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} e^{2t} = \begin{bmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{bmatrix}$$

for $\lambda = 3 \rightarrow x_2 = \alpha e^{\lambda_2 t} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} e^{3t} = \begin{bmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix}$

for $\lambda = 5 \rightarrow x_3 = \alpha e^{\lambda_3 t} = \begin{bmatrix} 3 \\ -6 \\ -2 \end{bmatrix} e^{5t} = \begin{bmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{bmatrix}$

general soln

$$X_h = c_1 \begin{bmatrix} e^{2t} \\ -e^{2t} \\ -e^{2t} \end{bmatrix} + c_2 \cdot \begin{bmatrix} e^{3t} \\ -2e^{3t} \\ -e^{3t} \end{bmatrix} + c_3 \begin{bmatrix} 3e^{5t} \\ -6e^{5t} \\ -2e^{5t} \end{bmatrix}$$

$H = \{e^{2t}, e^{3t}, e^{5t}\}$

$D = \{\cos 2t, \sin 2t\}$

Since $H \cap D = \emptyset$ empty set. a particular soln. proposed be

$x_p = c^1 \cos 2t + c^2 \sin 2t$

$x_p' = -2 \cdot c^1 \sin 2t + 2 c^2 \cos 2t$

Then eqn. 1 implies

$-2 c^1 \sin 2t + 2 c^2 \cos 2t = A c^1 \cos 2t + A c^2 \sin 2t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos 2t$

Equating the coefficients of $\{\cos 2t, \sin 2t\}$

$\sin 2t : -2 c^1 = A c^2$

$\cos 2t : 2 c^2 = A \cdot c^1 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow 2 A c^2 = A^2 c^1 + A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$-4 c^1 = A^2 c^1 + A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\rightarrow (A^2 + 4I) c^1 = A \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$\rightarrow c^1 = (A^2 + 4I)^{-1} \times A \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$c^1 = \{-0.25, -0.19, -0.05\}$

$$c^2 = -2A^{-1}c^1$$

$$c^2 = \{0.24,\ 0.21,\ -0.17\}$$

$$x_p = \{-0.25,\ -0.19,\ -0.05\} \cdot \cos 2t + \{0.24,\ 0.21,\ -0.17\} \sin 2t$$

$$x_{gen} = x_n + x_p$$

$$x_1 = C_1 e^{2t} + C_2 e^{3t} + 3C_3 e^{5t} - 0.25 \cos 2t + 0.24 \cdot \sin 2t$$
$$x_2 = -C_1 e^{2t} - 2C_2 e^{3t} - 6C_3 e^{5t} - 0.19 \cos 2t + 0.21 \sin 2t$$
$$x_3 = -C_1 e^{2t} - 1C_2 e^{3t} - 2C_3 e^{5t} - 0.05 \cos 2t - 0.17 \cdot \sin 2t$$

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$$6x^2y'' + 7xy' - (1+x^2)y = 0 \quad \text{about } x_0 = 0$$

(1)

$$y'' + \frac{7x}{6x^2}y' - \frac{1+x^2}{6x^2}\cdot y = 0$$

$$p(x) = \frac{7x}{6x^2} \qquad q(x) = -\frac{1+x^2}{6x^2}$$

$$\lim_{x \to x_0 = 0} x\cdot\left(\frac{7x}{6x^2}\right) = \frac{7}{6}$$

$$\lim_{x \to x_0 = 0} x^2\left(-\frac{1+x^2}{6x^2}\right) = -\frac{1}{6}$$

$$y(x) = \sum_{n=0}^{\infty} c_n \cdot x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} c_n \cdot (n+r)\, x^{n-1+r}$$

$$y'' = \sum_{n=0}^{\infty} c_n\, (n+r)\cdot(n-1+r)\cdot x^{n-2+r}$$

$$6x^2\cdot\sum_{n=0}^{\infty} c_n(n+r)\cdot(n-1+r)\cdot x^{n-2+r} + 7x\cdot\sum_{n=0}^{\infty} c_n(n+r)\cdot x^{n-1+r} - (1+x^2)\cdot\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$6\cdot\sum_{n=0}^{\infty} c_n(n+r)(n-1+r)\cdot x^{n+r} + 7\cdot\sum_{n=0}^{\infty} c_n(n+r)\cdot x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r+2}}$$ $$= 0$$

$$-\sum_{n=2}^{\infty} c_{n-2}x^{n+r}.$$

$$6\cdot\sum_{n=0}^{\infty} c_n (n+r)(n-1+r)\, x^{n+r} + 7\cdot\sum_{n=0}^{\infty} c_n\cdot(n+r)\, x^{n+r} - \sum_{n=0}^{\infty} c_n\cdot x^{n+r} - \underbrace{\sum_{n=2}^{\infty} c_{n-2} x^{n+r}}_{*}$$

$n=0,1.$

$$6\left(c_0 (0+r)(0-1+r)\cdot x^{0+r} + c_1\cdot(1+r)(1-1+r)\, x^{1+r}\right) + \sum_{n=2}^{\infty} c_n (n+r)(n-1+r)\, x^{n+r} +$$

$$7\cdot\left(c_0\cdot(0+r)\, x^{0+r} + c_1\cdot(1+r)\cdot x^{1+r}\right) + \sum_{n=2}^{\infty} c_n (n+r)\, x^{n+r} -$$

$$c_0 : x^{0+r} + c_1\cdot x^{1+r} + \sum_{n=2}^{\infty} c_n\cdot x^{n+r} - \sum_{n=2}^{\infty} c_{n-2} x^{n+r}.$$

$$c_0\cdot\left(6\cdot r\cdot(-1+r)\cdot x^r + 7\cdot r\cdot x^r - x^r\right) +$$

$$c_1\left(6\cdot(1+r)(1-1+r)\cdot x^{1+r} + 7\cdot(1+r)\cdot x^{1+r} - x^{1+r}\right) +$$

$$\sum_{n=2}^{\infty}\left[c_n (n+r)(n-1+r)\, x^{n+r} + c_n\cdot(n+r)\, x^{n+r} + c_n x^{n+r} - c_{n-2} x^{n+r}\right] = 0$$

$$\sum_{n=2}^{\infty} x^{n+r}\left[c_n(1+(n+r)\cdot(n-1+r)+(n+r)+1) - c_{n-2}\right] = 0$$

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in order to satisfied, every power of x must be 0. So,

$$c_0\cdot\left(6\cdot r\cdot(-1+r) + 7\cdot r - 1\right) = 0$$

$$c_1\left(6(1+r)\cdot(r) + 7\cdot(1+r) - 1\right) = 0$$

$$\sum_{n=2}^{\infty} c_n\left((n+r)(n-1+r)+(n+r)+1\right) + c_{n-2} = 0$$

Because $c_0 \neq 0$ we. Solve,

$$6\cdot r(-1+r) + 7r - 1 = 0$$

$$6r^2 - 6r + 7r - 1 = 0$$

$$6r^2 + r - 1 = 0 \qquad (3r-1)\cdot(2r+1) = 0$$

$3r \quad -1$

$2r \quad +1$

$$r_1 = \frac{1}{3} \quad r_2 = -\frac{1}{2}$$

for $r_2 = -\frac{1}{2}$

$$c_1 \underbrace{\left(6 \cdot \left(1-\frac{1}{2}\right) \cdot \left(-\frac{1}{2}\right) + 7 \cdot \left(1-\frac{1}{2}\right) - 1\right)}_{} = 0$$

$$c_1 \cdot 1 = 0 \quad \Rightarrow \quad c_1 = 0$$

$$c_n \left(\left(n-\frac{1}{2}\right)\left(n-1-\frac{1}{2}\right) + \left(n-\frac{1}{2}\right) - 1\right) = c_{n-2}$$

$$c_n \left(\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) + \left(n-\frac{3}{2}\right)\right) = c_{n-2}$$

$$c_n = \frac{c_{n-2}}{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) + \left(n-\frac{3}{2}\right)} = \frac{4 \cdot c_{n-2}}{(2n-1)(2n-3) + (4n-6)}$$

$$c_1 = 0$$

$$c_2 = \frac{4c_0}{3 \cdot 1 + 2} = \frac{4c_0}{5}$$

$$c_3 = \frac{c_1}{\cdots} = 0$$

$$c_4 = \frac{c_2}{7 \cdot 5 + 10} = \frac{4c_0}{5} \cdot \frac{1}{45}$$

$$c_5 = 0$$

$$y_1(x) = C_0 \cdot x^{r} + C_1 \cdot x^{1+r} + C_2 \cdot x^{2+r} + C_3\, x^{3+r} + \dots$$

from $C_1 = C_3 = C_5 = 0$

$$y_1(x) = C_0 \cdot x^{\frac{1}{3}} \left(1 + \frac{9}{58} \cdot x^2 + \frac{9}{58} \cdot \frac{1}{178}\, x^4 + \dots \right)$$

$$y_2(x) = 4 \cdot C_0\, x^{-\frac{1}{2}} \left(1 + \frac{1}{5} \cdot x^2 + \frac{1}{5} \cdot \frac{1}{45}\, x^4 + \dots \right)$$

$$\underline{y(x) = y_1(x) + y_2(x)}$$