Individual Project

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1. INTRODUCTION

2. STOCHASTIC MODEL FOR MEDIA ATTENTION

2.1. MEDIA ATTENTION INDEX

To begin with, lets introduce the media attention index

$$M_t = \exp\left\{\mu t + \sigma W_t + \sum_{i=1}^{N_t} \xi i\right\}, \quad t \geq 0,$$

where W_t - a Wiener process, $\sum\limits_{i=1}^{N_t} \xi_i$ - a compound Poisson process such that N_t is a homogeneous Poisson process with intensity λ and ξ_1, ξ_2,... are i.i.d. r.v. with absolutely continuous distribution having the density p.

2.2. WHY EXPONENT?

First of all, I would like to notice that, as I understand, the main reason for using exponent is the exclusion of negative values of M_t that don't make sense while W_t and $\sum\limits_{i=1}^{N_t} \xi_i$ can be negative.

2.3. WHAT IS THE SENSE OF D_k?

We have the discrete grid $0, \Delta, 2\Delta, ...$ with fixed $\Delta > 0$ on which M_t is considered. Now lets introduce

$$D_k := \log M_{k\Delta} - \log M_{(k-1)\Delta}, \quad k = 1, 2, ...$$

with $\log M_0 = 0$ what means that $M_0 = 1$. Thus due to properties of the logarithm D_k can be submitted as

$$D_k = \log \frac{M_{k\Delta}}{M_{(k-1)\Delta}}, \quad k = 1, 2, ...,$$

where $\frac{M_{k\Delta}}{M_{(k-1)\Delta}}$ is the growth of M_t during period k. So in this case we are considering the change of M_t and it would be great, if we could somehow model it.

2.4. WHAT PROCESS IS $\log M_{k\Delta}$?

The first obvious observation is that

$$\log M_{k\Delta} = \mu k\Delta + \sigma W_{k\Delta} + \sum_{i=1}^{N_{k\Delta}} \xi_i \Rightarrow D_k = \mu\Delta + \sigma(W_{k\Delta} - W_{(k-1)\Delta}) + \sum_{i=1}^{N_{k\Delta}} \xi_i - \sum_{i=1}^{N_{(k-1)\Delta}} \xi_i,$$

where $W_{k\Delta} - W_{(k-1)\Delta} \sim N(0, \Delta)$ and the characteristic function of $\sum\limits_{i=1}^{N_{k\Delta}} \xi_i - \sum\limits_{i=1}^{N_{(k-1)\Delta}} \xi_i$ is $\exp\{\lambda\Delta(\phi_\xi(u) - 1)\}$.

Also it makes sense to understand what process $\log M_{k\Delta}$ is. The most promising suggestion is that $\log M_{k\Delta}$ is a Lévy process. So then D_k is an increment of a Lévy process, therefore, $\forall i, j = 0, 1, 2, ... : i \neq j :\ D_{i\Delta}$ and $D_{j\Delta}$ are independent.

As we saw before $\log M_{k\Delta} = \mu k\Delta + \sigma W_{k\Delta} + \sum\limits_{i=1}^{N_{k\Delta}} \xi_i$ - a sum of a Brownian motion, a compound Poisson process which are known to be Lévy processes and a non-stochastic part. Thus, taking into an account that $\log M_0 = 0$ we also get that $\log M_{k\Delta}$ has independent increments and it is stochastic continious.

Therefore, $\log M_{k\Delta}$ is a Lévy process and D_k represents its increments.

NB :

1. Since we can get D_k from the real data, thus it is a sample

2. A jump in $\log M_\Delta$ is caused only by an increment in the CPP beacause other parts are continious

2.5. THE LÉVY-KHINCHINE FORMULA

Let us write the Lévy-Khinchine formula in general terms. Let X_t be a Lévy process. Then

$$\phi_t(u) = \mathbb{E}[e^{iuX_t}] = \exp\{t(iub - \frac{1}{2}c^2u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{I}\{|x| < 1\})\nu(dx))\},$$

where $b \in \mathbb{R}$, $c \in \mathbb{R}_+$ and ν is a Lévy measure of X_t.

So if we take into account that $\nu(dx) = \lambda p_\xi(x)$, consideration the integral in this formula gives us

$$\int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{I}\{|x| < 1\})\nu(dx) = \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx) - iu \int_{\mathbb{R}} x\mathbb{I}\{|x| < 1\}\nu(dx)$$

because

$$1)\ \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx) = \int_{|x|<1} (e^{iux} - 1)\nu(dx) + \int_{|x|\geq 1} (e^{iux} - 1)\nu(dx) \leq$$

$$\leq ||\ e^{iux} - 1 \leq |e^{iux} - 1| \leq |e^{iux}| + 1 = 2 \text{ and } \int_{|x|<1} (e^{iux} - 1)\nu(dx) \text{ is finite } || \leq 2 \int_{|x|\geq 1} \nu(dx)$$

$$2)\ \int_{\mathbb{R}} x\mathbb{I}\{|x| < 1\}\nu(dx) = \int_{|x|<1} x\nu(dx) \leq \int_{|x|<1} |x|\nu(dx) \leq \int_{|x|<1} \nu(dx)$$

thus, both parts of the integral are finite. Therefore

$$\phi_t(u) = \exp\{t(iub - \frac{1}{2}c^2u^2 + \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx) - iu \int_{\mathbb{R}} x\mathbb{I}\{|x| < 1\})\nu(dx))\} =$$

$$= \exp\{t(iu(b - \int_{\mathbb{R}} x\mathbb{I}\{|x| < 1\}\nu(dx)) - \frac{1}{2}c^2u^2 + \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx))\}$$

So if

$$\mu = b - \int_{\mathbb{R}} x\mathbb{I}\{|x| < 1\}\nu(dx) \text{ and } \sigma = c,$$

then in our case the characteristic function of $\log M_\Delta$ is

$$\phi_\Delta(u) = \mathbb{E}[e^{iu\log M_\Delta}] = \exp\{\Delta\big(i\mu u - \frac{1}{2}\sigma^2u^2 + \int_{\mathbb{R}} (e^{iux} - 1)\nu(dx)\big)\} = \exp\{\Delta\big(i\mu u - \frac{1}{2}\sigma^2u^2 + \lambda\int_{\mathbb{R}} e^{iux}p_\xi(x)dx - \lambda\int_{\mathbb{R}} p_\xi(x)dx\big)\} =$$

$$= \exp\{\Delta\big(i\mu u - \frac{1}{2}\sigma^2u^2 - \lambda + \lambda\mathcal{F}[p](u)\big)\},\ u \in \mathbb{R},$$

where $\nu(dx) = \lambda p_\xi(x)dx$ and $\mathcal{F}[p](u) = \int_{\mathbb{R}} e^{iux}p_\xi(x)dx$

3. ESTIMATION OF MEDIA ATTENTION

3.1. APPROACH TO ESTIMATION

In what follows, we use function

$$\varphi_\Delta(u) := \frac{1}{\Delta}\log(\phi_\Delta(u)) = iu\mu - \frac{1}{2}\sigma^2u^2 - \lambda + \lambda\mathcal{F}[p](u), \quad u \in \mathbb{R}.$$

Due to the Riemann-Lebesque lemma, $\mathcal{F}[p](u) \to 0$ as $|u| \to \infty$. Therefore,

$$Re(\varphi_\Delta(u)) = -\frac{1}{2}\sigma^2u^2 - \lambda + o(1)$$
$$Im(\varphi_\Delta(u)) = u\mu + o(1),$$

as $|u| \to \infty$. So in estimation we should use only large u.

Complex logarithm

One should consider $z = x + iy = re^{i\varphi}$, where $r = \sqrt{x^2 + y^2}$ is an absolute value of z, and φ is its argument.

Obviously $\tan\varphi = \frac{y}{x} \Rightarrow \varphi = \arctan\frac{y}{x}$.

Then $\log z = \log r + i\varphi = \log\sqrt{x^2 + y^2} + i\arctan\frac{y}{x}$

It is known that mathematical expectation can be estimated by mean value then a natural estimation of $\varphi_\Delta(u)$ is **(don't know why D_k)**

$$\hat{\varphi}_\Delta(u) := \frac{1}{\Delta}\log(\frac{1}{n}\sum_{k=1}^{n} e^{iuD_k}) = -\frac{1}{\Delta}\log n + \frac{1}{\Delta}\log\sum_{k=1}^{n} e^{iuD_k} = -\frac{1}{\Delta}\log n + \frac{1}{\Delta}\log\big(\sum_{k=1}^{n}\cos(uD_k) + i\sum_{k=1}^{n}\sin(uD_k)\big) =$$

$$= -\frac{1}{\Delta}\log n + \frac{1}{\Delta}\left(\frac{1}{2}\log\left[\left(\sum_{k=1}^{n}\cos(uD_k)\right)^2 + \left(\sum_{k=1}^{n}\sin(uD_k)\right)^2\right] + i\arctan\left(\frac{\sum_{k=1}^{n}\sin(uD_k)}{\sum_{k=1}^{n}\cos(uD_k)}\right)\right) =$$

$$= -\frac{1}{\Delta}\log n + \frac{1}{2\Delta}\log\left[\left(\sum_{k=1}^{n}\cos(uD_k)\right)^2 + \left(\sum_{k=1}^{n}\sin(uD_k)\right)^2\right] + i\frac{1}{\Delta}\arctan\left(\frac{\sum_{k=1}^{n}\sin(uD_k)}{\sum_{k=1}^{n}\cos(uD_k)}\right), \quad u \in \mathbb{R}.$$

Thus, for large u it makes sense to consider **(don't know the sense of $w^{U_n}(u)$)**

1. $\int_{\mathbb{R}_+} w^{U_n}(u)\left[Re(\hat{\varphi}_\Delta(u)) - (-\frac{1}{2}\sigma^2u^2 - \lambda)\right]^2 du = \int_{\mathbb{R}_+} w^{U_n}(u)\left[Re(\hat{\varphi}_\Delta(u)) + \frac{1}{2}\sigma^2u^2 + \lambda\right]^2 du$
2. $\int_{\mathbb{R}_+} w^{V_n}(u)\left[Im(\hat{\varphi}_\Delta(u)) - \mu u\right]^2 du$

Therefore,

1. $(\hat{\sigma}^2,\ \hat{\lambda}^2) = arg\min_{\sigma^2,\,\lambda}\int_{\mathbb{R}_+} w^{U_n}(u)\left[Re(\hat{\varphi}_\Delta(u)) + \frac{1}{2}\sigma^2u^2 + \lambda\right]^2 du \stackrel{?}{=} arg\min_{\sigma^2,\,\lambda}\int_{\varepsilon}^{1} w(u)\left[Re(\hat{\varphi}_\Delta(uU_n)) + \frac{1}{2}\sigma^2u^2U_n^2 + \lambda\right]^2 du$
2. $\hat{\mu} = arg\min_{\mu}\int_{\mathbb{R}_+} w^{V_n}(u)\left[Im(\hat{\varphi}_\Delta(u)) - \mu u\right]^2 du \stackrel{?}{=} arg\min_{\mu}\int_{\varepsilon}^{1} w(u)\left[Im(\hat{\varphi}_\Delta(uV_n)) - \mu uV_n\right]^2 du$

3.2. ALGORITHM 1

$$Q_1(\sigma^2,\lambda) = \sum_{j=1}^{N} w(\tilde{u}_j)\left[Re(\hat{\varphi}_\Delta(\tilde{u}_jU_n)) + \frac{1}{2}\sigma^2\tilde{u}_j^2U_n^2 + \lambda\right]^2 \longrightarrow \min_{\sigma^2,\lambda}$$

So one needs to consider

$$\begin{cases} \frac{\partial Q_1(\sigma^2,\lambda)}{\partial\sigma^2} = \sum\limits_{j=1}^{N} w(\tilde{u}_j)\tilde{u}_j^2U_n^2\left[Re(\hat{\varphi}_\Delta(\tilde{u}_jU_n)) + \frac{1}{2}\sigma^2\tilde{u}_j^2U_n^2 + \lambda\right] = 0 \\ \frac{\partial Q_1(\sigma^2,\lambda)}{\partial\lambda} = \sum\limits_{j=1}^{N} w(\tilde{u}_j)\left[Re(\hat{\varphi}_\Delta(\tilde{u}_jU_n)) + \frac{1}{2}\sigma^2\tilde{u}_j^2U_n^2 + \lambda\right] = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \sum\limits_{j=1}^{N} w(\tilde{u}_j)Re(\hat{\varphi}_\Delta(\tilde{u}_jU_n))(\tilde{u}_jU_n)^2 + \frac{\sigma^2}{2}U_n^4\sum\limits_{j=1}^{N} w(\tilde{u}_j)\tilde{u}_j^{2\cdot2} + \lambda U_n^2\sum\limits_{j=1}^{N} w(\tilde{u}_j)\tilde{u}_j^2 = 0 \\ \sum\limits_{j=1}^{N} w(\tilde{u}_j)Re(\hat{\varphi}_\Delta(\tilde{u}_jU_n)) + \frac{\sigma^2}{2}U_n^2\sum\limits_{j=1}^{N} w(\tilde{u}_j)\tilde{u}_j^2 + \lambda\sum\limits_{j=1}^{N} w(u_j) = 0. \end{cases}$$

Let

$$\Lambda_d = \sum_{j=1}^{N} w(\tilde{u}_j)\tilde{u}_j^{2d},\ d = 0,1,2$$

$$\Psi_d = \sum_{j=1}^{N} w(\tilde{u}_j)Re(\hat{\varphi}_\Delta(\tilde{u}_jU_n))(\tilde{u}_jU_n)^{2d},\ d = 0,1$$

Therefore,

$$\begin{cases} \Psi_1 + \frac{\sigma^2}{2}U_n^4\Lambda_2 + \lambda U_n^2\Lambda_1 = 0 \\ \Psi_0 + \frac{\sigma^2}{2}U_n^2\Lambda_1 + \lambda\Lambda_0 = 0 \end{cases} \Leftrightarrow \begin{cases} \sigma_n^2 = 2\frac{\Psi_0\Lambda_1U_n^2 - \Psi_1\Lambda_0}{(\Lambda_2\Lambda_0 - \Lambda_1^2)U_n^4} \\ \lambda_n = \frac{\Psi_1\Lambda_1 - \Psi_0\Lambda_2U_n^2}{(\Lambda_2\Lambda_0 - \Lambda_1^2)U_n^2} \end{cases}$$

3.3. ALGORITHM 2

Considering the function

$$\varphi_\Delta(u) := \frac{1}{\Delta}\log(\phi_\Delta(u)) = iu\mu - \frac{1}{2}\sigma^2u^2 - \lambda + \lambda\mathcal{F}[p](u), \quad u \in \mathbb{R}$$

provides for us a representation of the density p as follows

$$\lambda\mathcal{F}[p](u) = \varphi_\Delta(u) - iu\mu + \frac{1}{2}\sigma^2u^2 + \lambda \Rightarrow \hat{p}_n(x) = \hat{\mathcal{F}}^{-1}\left(\frac{1}{\hat{\lambda}}(\hat{\varphi}_\Delta(u) - iu\hat{\mu} + \frac{1}{2}\hat{\sigma}^2u^2 + \hat{\lambda})K(u/T_n)\right) =$$

$$= \frac{1}{2\pi\hat{\lambda}}\int_{\mathbb{R}} e^{-iux}\left(\hat{\varphi}_\Delta(u) - iu\hat{\mu} + \frac{1}{2}\hat{\sigma}^2u^2 + \hat{\lambda}\right)K(u/T_n)du$$

as an inverse Fourier transform, where $K : \mathbb{R} \to \mathbb{R}_+$ is a smoothing kernel

$$K(x) = \begin{cases} 1, & |x| < 0.05 \\ exp\left\{-\frac{e^{-1/(|x|-0.05)}}{1-|x|}\right\}, & 0.05 < |x| < 1 \\ 0, & |x| > 1. \end{cases}$$

Then according to *algorithm 2* one should take points $u_j \in [-1, 1],\ j = 1, ..., N$. Therefore if u in the theoretical case is an analog of u_j

$$\hat{p}_n(x) = \frac{1}{2\pi\hat{\lambda}}\int_{-1}^{1} e^{-i(uT_n)x}\left(\hat{\varphi}_\Delta(uT_n) - i(uT_n)\hat{\mu} + \frac{1}{2}\hat{\sigma}^2(uT_n)^2 + \hat{\lambda}\right)K(u)d(uT_n)$$

can be represented in discrete form as

$$\hat{p}_n(x_s) = \frac{T_n\delta}{2\pi\hat{\lambda}}\sum_{j=1}^{N} e^{-i\breve{u}_jx_s}\left(\hat{\varphi}_\Delta(\breve{u}_j) - i\breve{u}_j\hat{\mu} + \frac{1}{2}\hat{\sigma}^2\breve{u}_j^2 + \hat{\lambda}\right)K(\breve{u}_j/T_n).$$

3.4. SIMULATION STUDY

Let us consider the Merton jump-diffusion model

$$M_t = \exp\left\{\mu t + \sigma W_t + \sum_{i=1}^{N_t}\xi_i\right\},$$

where $W_t \sim N(0,t), N_t \sim Pois(\lambda t)$ and $\xi_i \sim N(0,1)$.

For further calculations I have taken 25 samples and the parameters have been chosen to be $\mu = 0, \sigma = 1, \lambda = 10$ and $\Delta = 0.1$. Therefore, if n is the size of a sample we will use $\{0,\, 0.1,\, 0.2, ..., 0.1\, n\}$ grid.

3.4.1. BROWNIAN MOTION

Note that $B_{k\Delta} = \sum_{j=1}^{k} \left[B_{j\Delta} - B_{(j-1)\Delta}\right]$, where $B_{j\Delta} - B_{(j-1)} \sim N(0,\ \Delta)$. So we can calculate the value of $B_{k\Delta}$ in each point.

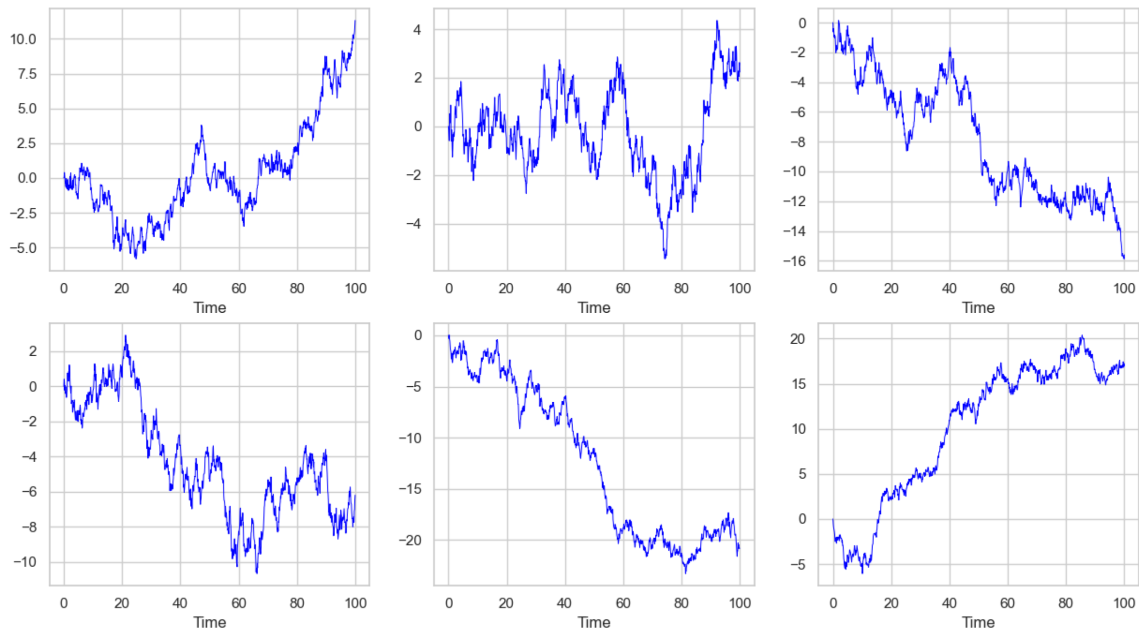


Рис. 1: Trajectory of $B_{k\Delta}$

3.4.2. POISSON PROCESS

Similarly $N_{k\Delta} = \sum_{j=1}^{k} \left[N_{j\Delta} - N_{(j-1)\Delta}\right]$, where $N_{j\Delta} - N_{(j-1)\Delta} \sim Pois(\lambda\Delta)$.

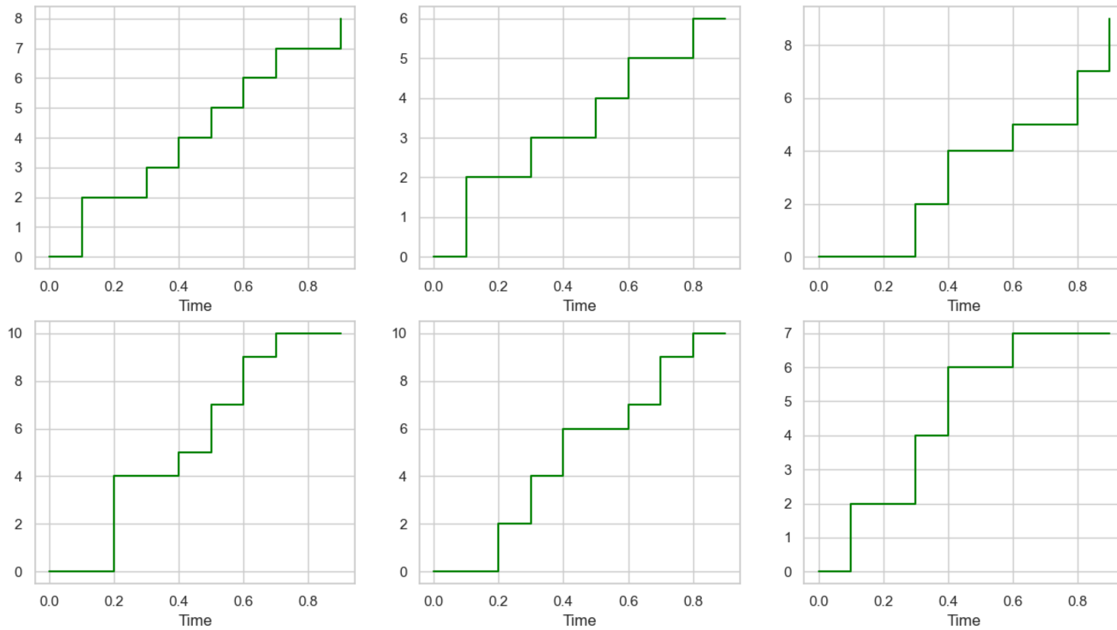


Рис. 2: Trajectories of $N_{k\Delta}$

3.4.3. COMPOUND POISSON PROCESS

Now we have everything for counting values of Compound Poisson process $\sum\limits_{i=1}^{N_{k\Delta}} \xi_i$.

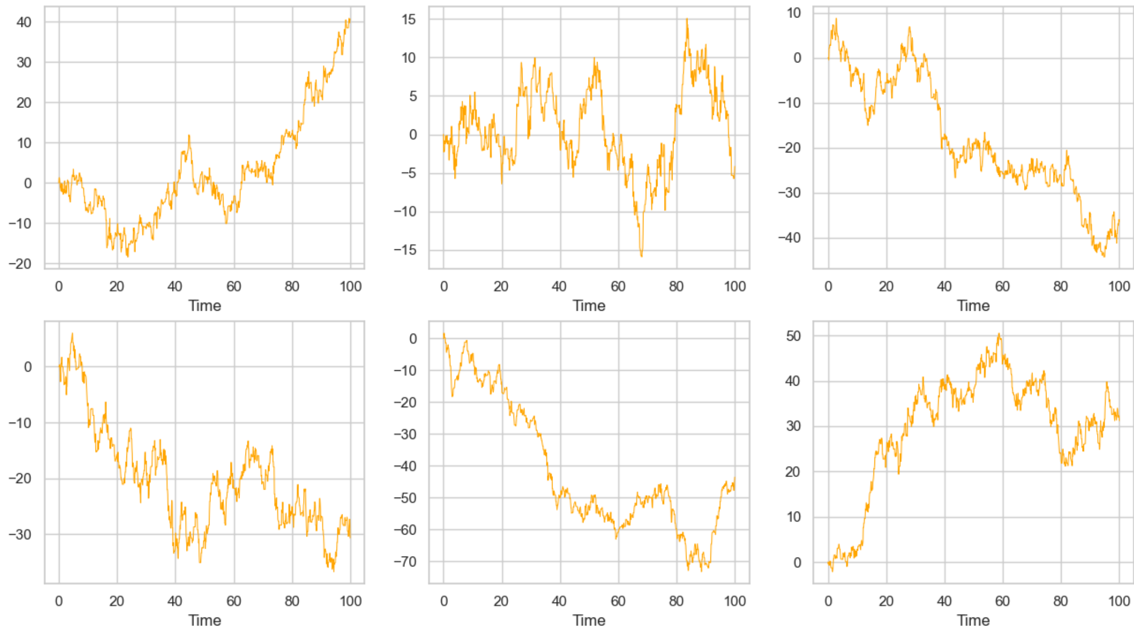
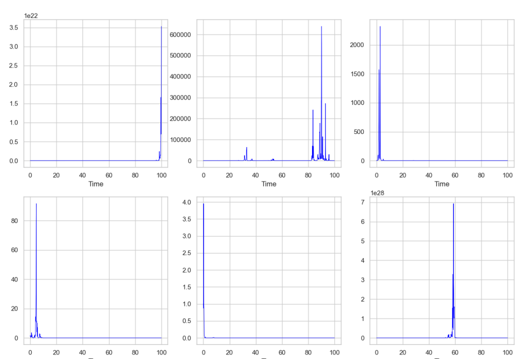
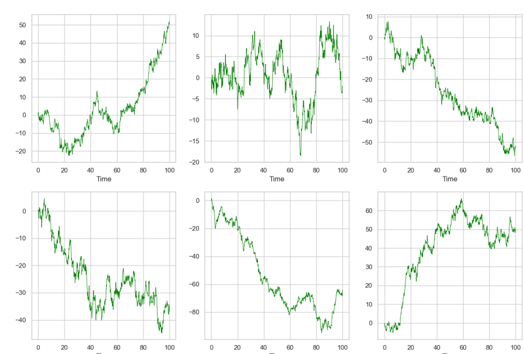


Рис. 3: Trajectories of $\sum_{i=1}^{N_{k\Delta}} \xi_i$

3.4.4. $M_{k\Delta}$ and $\log M_{k\Delta}$



(a) $M_{k\Delta}$



(b) $\log M_{k\Delta}$

Рис. 4: Graphs of $M_{k\Delta}$ and $\log M_{k\Delta}$

3.4.5. CALCULATIONS

As it has been said before, we consider

$$M_t = \exp\left\{\mu t + \sigma W_t + \sum_{i=1}^{N_t} \xi_i\right\}$$

and for each point of grid $\{k\Delta\}_{k=0}^{n} = \{0,\, 0.1,\, 0.2, ...,\, 0.1\, n\}$ the values of $W_{k\Delta}$ and $\sum\limits_{i=1}^{N_{k\Delta}} \xi_i$ then $M_{k\Delta}$ can be counted. To be more precise,

$$M_{k\Delta} = \exp\left\{\mu k\Delta + \sigma W_{k\Delta} + \sum_{i=1}^{N_{k\Delta}} \xi_i\right\}$$

Thus, we can also get $D_k = \log M_{k\Delta} - \log M_{(k-1)\Delta}$.

Approximate $Re(\varphi_\Delta(u))$

As $|u| \to \infty$ approximately

$$\varphi_\Delta(u) \approx i\mu u - \frac{1}{2}\sigma^2 u^2 - \lambda \Rightarrow Re(\varphi_\Delta(u)) \approx -\frac{1}{2}\sigma^2 u^2 - \lambda \text{ (denoted as } re\_approx)$$

Estimated $Re(\varphi_\Delta(u))$

We found before that

$$\hat{\varphi}_\Delta(u) = -\frac{1}{\Delta}\log n + \frac{1}{2\Delta}\log\left[\left(\sum_{k=1}^{n}\cos(uD_k)\right)^2 + \left(\sum_{k=1}^{n}\sin(uD_k)\right)^2\right] + i\frac{1}{\Delta}\arctan\left(\frac{\sum_{k=1}^{n}\sin(uD_k))}{\sum_{k=1}^{n}\cos(uD_k)}\right) \Rightarrow$$

$$\Rightarrow Re(\hat{\varphi}_\Delta(u)) = -\frac{1}{\Delta}\log n + \frac{1}{2\Delta}\log\left[\left(\sum_{k=1}^{n}\cos(uD_k)\right)^2 + \left(\sum_{k=1}^{n}\sin(uD_k)\right)^2\right] \text{ (denoted as } re\_hat)$$

Factual $Re(\varphi_\Delta(u))$

To find factual real part of $\varphi_\Delta(u) = \frac{1}{\Delta}\log\phi_\Delta(u)$ one should remember that in our case $\xi_i \sim \mathcal{N}(0,1)$. Thus, $p_\xi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. Then

$$\varphi_\Delta(u) = i\mu u - \frac{1}{2}\sigma^2u^2 + \int_{\mathbb{R}}(e^{iux}-1)\nu(x) = i\mu u - \frac{1}{2}\sigma^2u^2 + \lambda\int_{\mathbb{R}}e^{iux}p_\xi(x)dx - \lambda\int_{\mathbb{R}}p_\xi(x)dx = i\mu u - \frac{1}{2}\sigma^2u^2 - \lambda + \frac{\lambda}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-\frac{x^2}{2}+iux}dx =$$

$$= i\mu u - \frac{1}{2}\sigma^2u^2 - \lambda + \lambda e^{-\frac{u^2}{2}}.$$

Therefore

$$Re(\varphi_\Delta(u)) = -\frac{1}{2}\sigma^2u^2 - \lambda + \lambda e^{-\frac{u^2}{2}} \text{ (denoted as } re\_fact).$$

So for $n = 1000$ we can observe

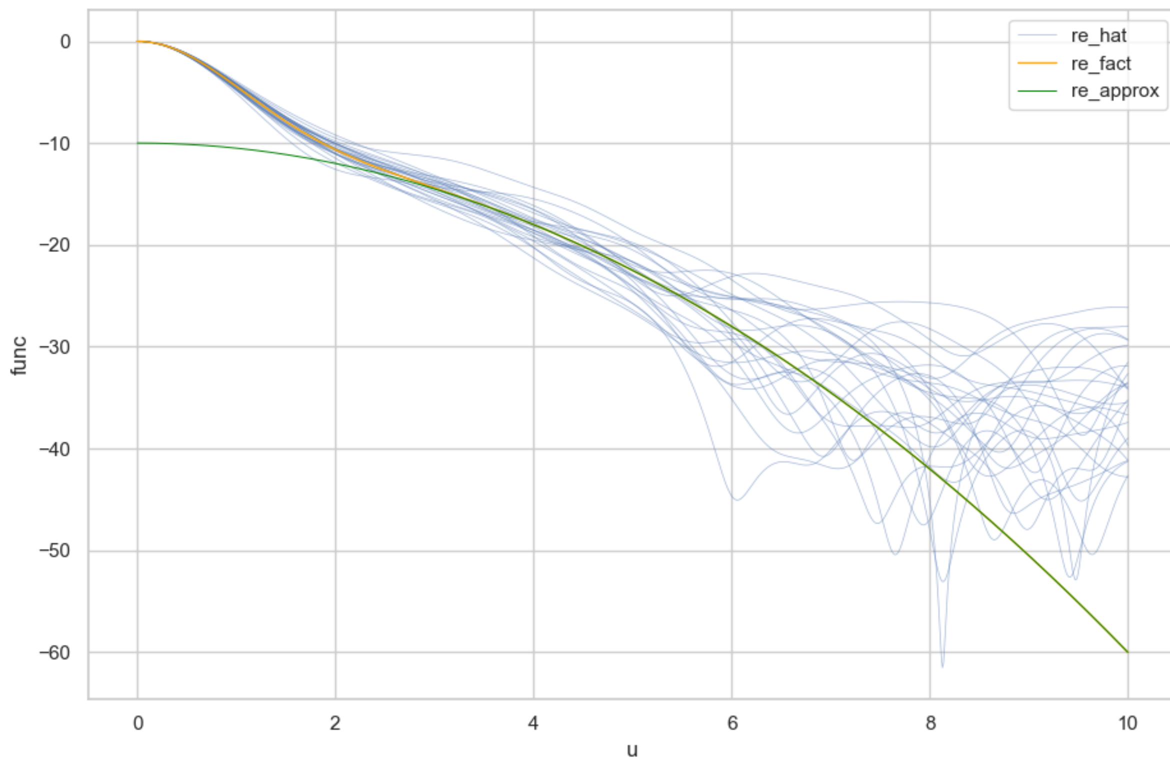


Рис. 5: Factual, approximate and estimated real parts of $\varphi_\Delta(u)$

And it can be seen that all three lines are close to each other when $u \in [3,6]$. Therefore, it makes sense to take $\varepsilon = 0.5, U_n = V_n = 6$.

After all calculations I received

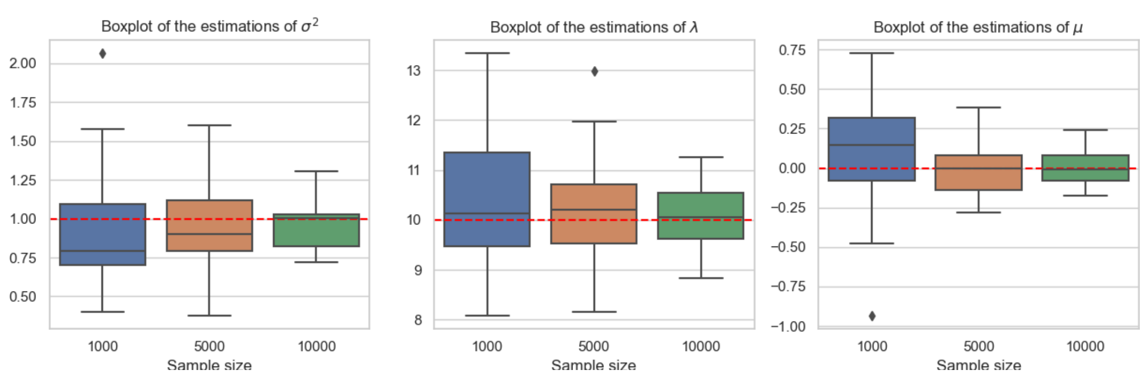


Рис. 6: Boxplots of the estimations of σ^2, λ and μ

As for the density's estimation I have taken $T_n = 3.3$ and $m = 100$ points between -5 and 5 as x_s. The result which

I have got according to algorithm 2 is presented in figure 7.

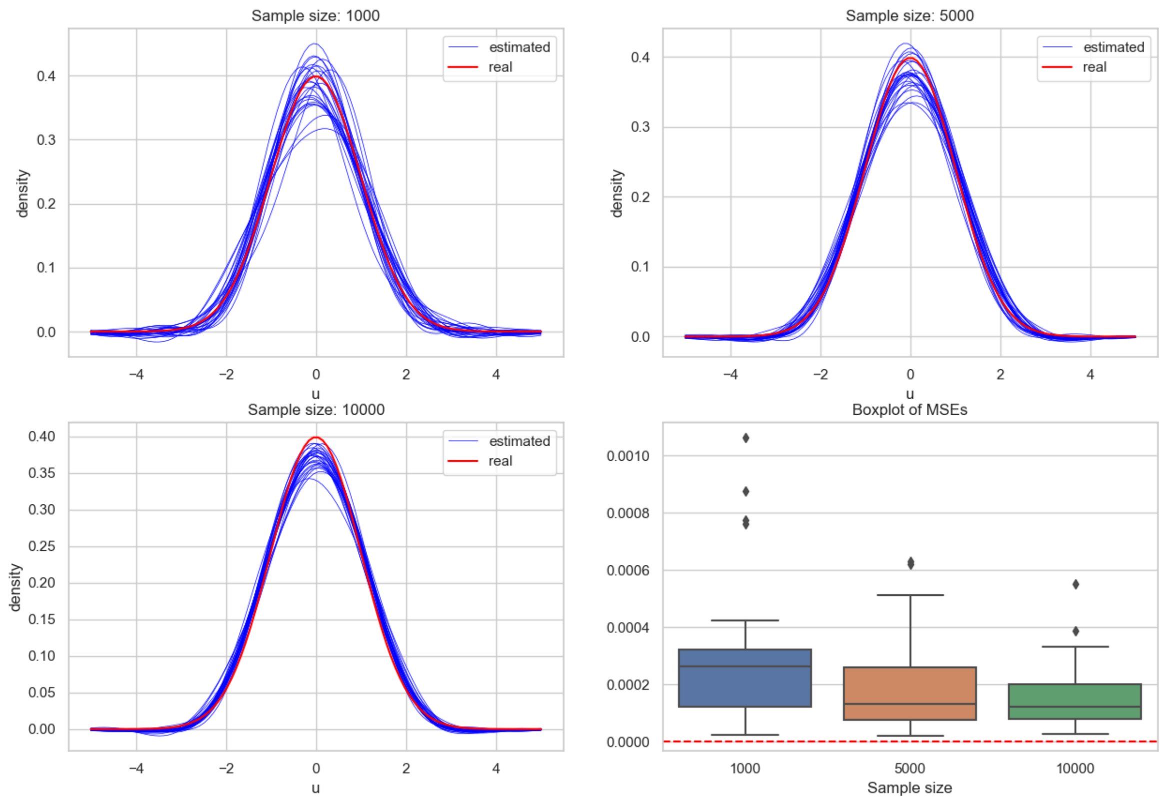


Рис. 7: Estimated density p

It is important to note that Figure 7 shows only real part of estimated points but ignoring imaginary part is not fatal because it is approximately zero. Also it is evident that this tendency is strengthening with a growth of a sample size.

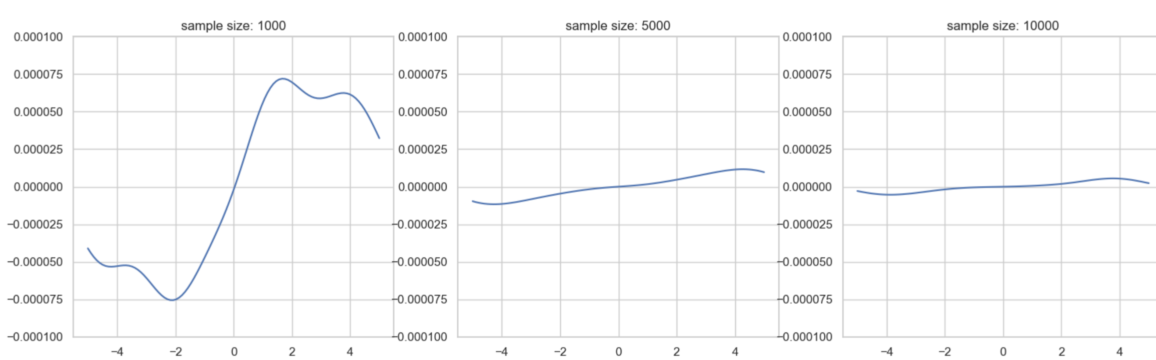


Рис. 8: Imaginary parts of $\hat{p}_n(x_s)$

3.5. REAL DATA

An estimation based on real data requires only a sample which I have obtained from Google Trends.

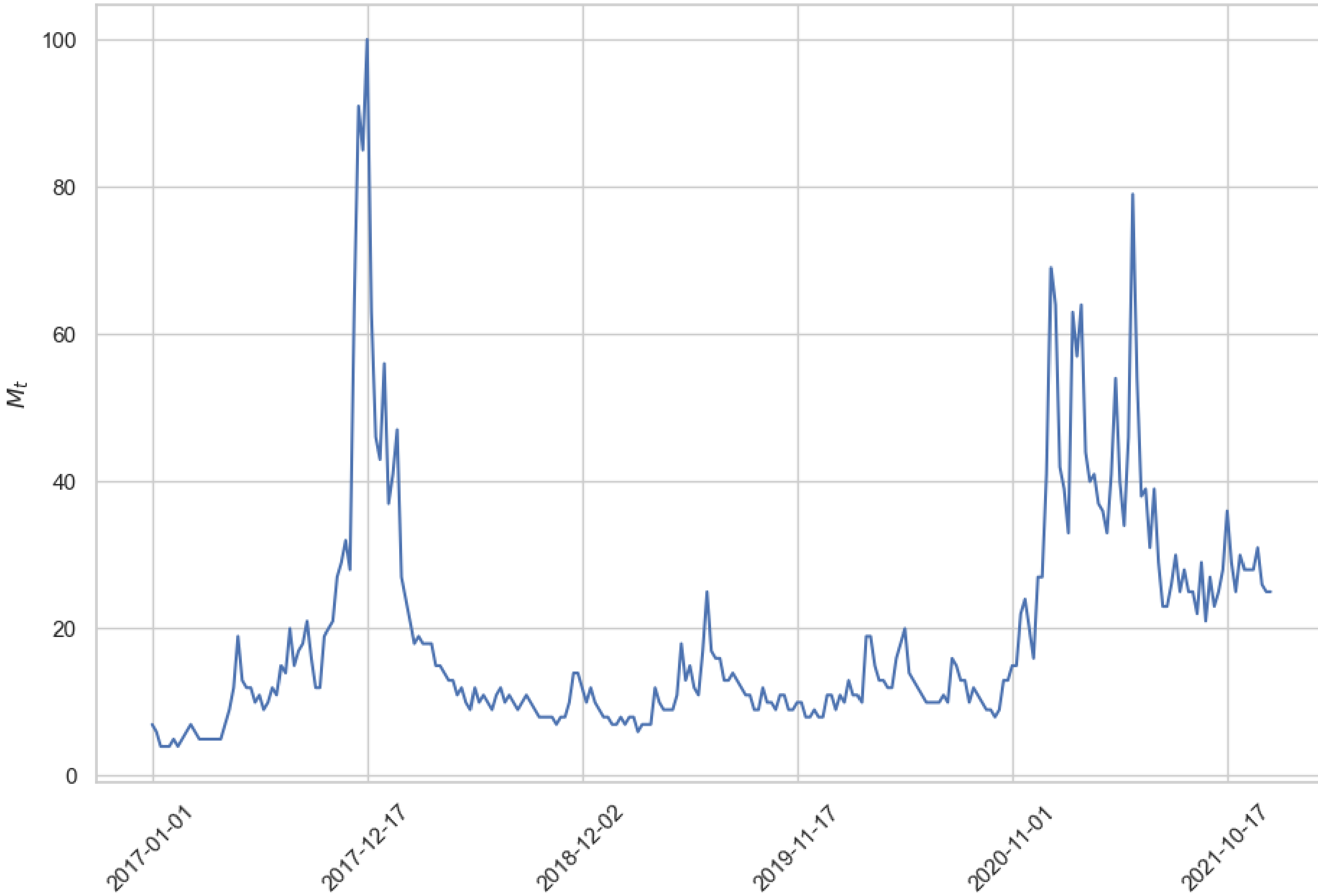


Рис. 9: Media attention index M_t based on real data

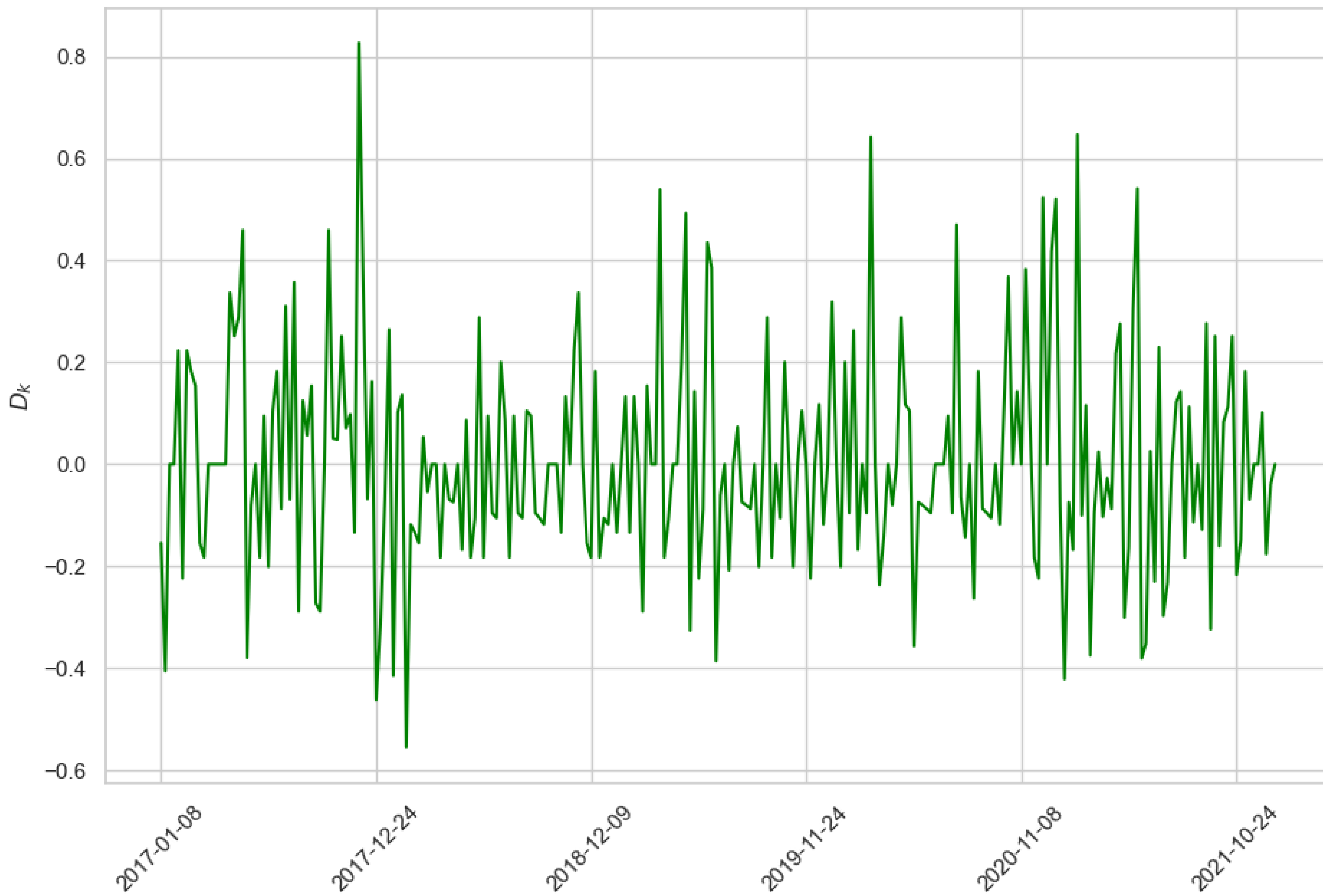


Рис. 10: Increments D_k based on real data

4. SOME SECTION

4.1. CHARACTERISTIC FUNCTION OF NORMAL DISTRIBUTION

Let $\xi \sim \mathcal{N}(0,1)$. Then characteristic function of ξ is

$$f(u) := \mathsf{E}\left[e^{iu\xi}\right] = \int_{\mathbb{R}} e^{iux} p_\xi(x)dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iux - \frac{x^2}{2}} dx.$$

Therefore

$$f'(u) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} x e^{iux - \frac{x^2}{2}} dx = -\frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iux} de^{-\frac{x^2}{2}} = -\frac{i}{\sqrt{2\pi}} \left(e^{iux - \frac{x^2}{2}} \Big|_{-\infty}^{+\infty} - \int_{\mathbb{R}} e^{-\frac{x^2}{2}} de^{iux} \right) =$$

$$= \left\| \left| e^{iux - \frac{x^2}{2}} \right| = e^{-\frac{x^2}{2}} \longrightarrow 0 \text{ as } x \to \pm\infty \right\| =$$

$$= -\frac{u}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iux - \frac{x^2}{2}} dx = -uf(u).$$

Thus, $f(u) = e^{-\frac{u^2}{2}} + c,\ c \in \mathbb{R}$. Using $f(0) = 1$ we get $c = 0$ and then $f(u) = e^{-\frac{u^2}{2}}$, $u \in \mathbb{R}$.

In case of $\xi \sim \mathcal{N}(\mu, \sigma^2)$ the characteristic function of ξ is

$$f(u) = \mathsf{E}\left[e^{iu\xi}\right] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iux} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

Also

$$f'(u) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} x e^{iux} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} (x - \mu + \mu) e^{iux} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx =$$

$$= \frac{i}{\sqrt{2\pi}} \left[\int_{\mathbb{R}} (x-\mu) e^{iux} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \mu \int_{\mathbb{R}} e^{iux} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \right] = \frac{i}{\sqrt{2\pi}} \left[-\sigma^2 \int_{\mathbb{R}} e^{iux} d e^{-\frac{(x-\mu)^2}{2\sigma^2}} + \mu \sqrt{2\pi} f(u) \right]$$

$$= \left\| \int_{\mathbb{R}} e^{iux} d e^{-\frac{(x-\mu)^2}{2\sigma^2}} = e^{iux - \frac{(x-\mu)^2}{2\sigma^2}} \Big|_{-\infty}^{+\infty} - iu \int_{\mathbb{R}} e^{iux - \frac{(x-\mu)^2}{2\sigma^2}} dx = -iu\sqrt{2\pi} f(u) \right\| =$$

$$= \frac{i}{\sqrt{2\pi}} \left[i\sigma^2 u \sqrt{2\pi} f(u) + \mu \sqrt{2\pi} f(u) \right] = \left(i\mu - \sigma^2 u \right) f(u).$$

Therefore

$$f(u) = e^{\int \left(i\mu - \sigma^2 u \right) du} = e^{i\mu u - \frac{1}{2}\sigma^2 u^2 + c},\ c \in \mathbb{R}.$$

Similarly to the previous case $f(0) = 1$. Thus, $c = 0$ and $f(u) = e^{i\mu u - \frac{1}{2}\sigma^2 u^2}$, $u \in \mathbb{R}$.