MA1512
AY24/25 sem 1
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01. Introduction to Differential Equations

First Principles

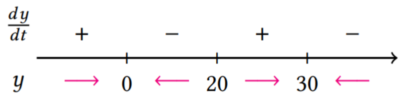
1. (**Differential Equation**) Let x be an independent variable and y be a dependent variable. An equation that involves x, y and **various derivatives of** y is called a **differential equation**. e.g. $(\frac{dy}{dx})^3 + e^x + 2 = \frac{d^2y}{dx^2}$
2. (**Ordinary Differential Equation**) In general, an equation of the form $F(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}) = 0$ is an **ordinary differential equation**. It is called so because there is only **one** independent variable and only **ordinary derivatives (not partial derivatives)** are involved.
3. (**Order of a Differential Equation**) The **order** of a differential equation is the order of the **highest derivative** appearing in the differential equation. e.g. dy/dx is first order derivative, d^2y/dx^2 is second order derivative.
4. (**Solution of a Differential Equation**)
   - (**General Solution**) A **general solution** to a differential equation is a family of infinitely many possible solutions, often involving **arbitrary constants** and they satisfy the differential equation when they are substituted into the differential equation.
   - (**Particular Solution**) With additional information such as *initial condition* (where a differential equation is required to satisfy conditions on the dependent variable and its derivatives specified at one value of the independent variable), we can determine a **particular solution** that no longer involves arbitrary constants.

   Note that the solution can be in **implicit form**.
5. (**The method of separation of variables**) A first-order differential equation of the form $\frac{dy}{dx} = F(x, y)$ is **separable** if it can be written as $M(x)dx = N(y)dy$. To solve this, directly integrate both sides of the equation, we will get $\int N(y)dy = \int M(x)dx + C$, where C is an arbitrary constant.
   - (**The position of arbitrary constant** C) The arbitrary constant must be added immediately when you integrate w.r.t independent variable.
   - (**Notation**) Sometimes dy/dx is written simply as y'.
   - (**Some useful substitution**)
     - If $y' = f(ax + by + c)$, we employ a *linear change of variable*. Let $u = ax + by + c \to u' = a + by'$
     - If $y' = f(y/x)$, we let $y = xv$, and $y' = xv' + v$ (Chain Rule) (This is same as let $v = \frac{y}{x}$ first.

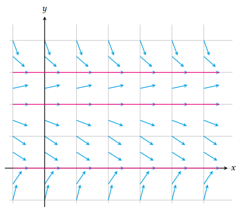
   Note that in both substitutions, we assume the function at the right side can be written as $f(\cdots)$, which is the soul in substitution.
6. (**Classic Examples**)
   - (**Half-Life**) The typical scenario for half-life is *exponential decay*, in which we have $\frac{dx}{dt} = -kx, k > 0$ and since its solution is $x = x_0e^{-kt}$. We can get the formula to determine the *decay rate* k given that we know the half-life τ, so, it will be $\frac{x_0}{2} = x_0e^{-kt} \to k = \frac{ln2}{\tau}$
   - Remember that $\int \frac{2u+4}{u^2+4u+5} = \ln(u^2 + 4u + 5)$.
   - For $y' \cdot y'' = 2$, to use separation of variables, use substitution $u = y' \to u \cdot u' = 2$ Then solve it accordingly.

The Geometry of Differential Equations

1. (**Geometry of First-Order Differential Equation**) Note that y' is the slope of curve $y = y(x)$ on the $x - y$ plane. Hence, solving differential equation $y' = f(x, y)$ means **finding curves whose slope at any given point** (x, y) **is equal to** $f(x, y)$. If adding initial value condition $y(x_0) = y_0$, that means the curve must pass through (x_0, y_0).
2. (**Using Direction/Slope Field to understand**) $\cdots$ **finding curves that are tangent to the short straight line at each point** (x, y). If adding initial value condition $y(x_0) = y_0$, that means the curve must pass through (x_0, y_0).
3. (**Equilibrium Solution**) An **equilibrium solution** of a differential equation is a solution that is **constant** ($y(t) = \beta$); these correspond to **horizontal lines** on a direction field (can have multiple equilibrium solutions).
   - (**Stable Equilibrium Solution**) An equilibrium solution $y(t) = \beta$ is said to be **stable** if solutions about/near this equilibrium approach β as $t \to \infty$.
   - (**Unstable Equilibrium Solution)** Otherwise, the equilibrium point is said to be **unstable**.
4. (**Methods to find equilibrium solution**)
   - Judge the order of the diff eq and let all the dependent variables' derivatives to be 0. e.g. First order $\to y' = 0$, Second order $\to y' = 0$ and $y'' = 0$. e.g. $y' = -10y(20 - y)(1 - \frac{1}{30}y) \to 0 = -10y(20 - y)(1 - \frac{1}{30}y)$
   - Use basic inequality technique to solve and sketch out a sign diagram for dy/dt with y For example,



   - Draw the corresponding direction/slope field to judge the stability. For example,



Population Dynamics

1. (**Malthusian model**) It assumes that the rate of change of a population is proportional to its present value. That is $dy/dt = ky \to y(t) = y_0e^{kt}$, where $y_0 = y(0)$. This model suggests that a population would grow exponentially with **growth rate** k. (**Note that in Malthusan model, it is** $y = y_0 \cdot e^{kt}$**, however, in half-life model, it is** $x = x_0 \cdot e^{-kt}$)
2. (**Verhulst model**) In Verhulst, our **growth rate** varies according to the present value y of the population. The formula is given by $\frac{dy}{dt} = [k(1 - \frac{y}{y_\infty})]y$, where dy/dt is **rate of change** and $k(1 - y/y_\infty)$ is **growth rate**. This model assumes that a population grows **logistically**, such that given any initial population, $lim_{t\to\infty}y(t) = y_\infty$, and y_∞ is called the *carrying capacity*.
3. (**Hunt rate**) Since hunt rate(E) is usually given in constant number per period, and in both models we have rate of change dy/dt, so usually we just minus the hunt rate (E) at the right side of our equation. e.g. In Malthusian Model, if the hunt rate is 100, $dy/dt = ky - 100$
4. (**Some useful tips regarding Verhulst model**)
   - Regard the R.H.S as a quadratic equation, we can see that when $y = y_\infty/2$, the rate of change dy/dt will be maximum.
   - Given the initial condition $y(0) = y_0$, the solution for Verhulst Model is $y(t) = \frac{y_\infty}{1+(\frac{y_\infty}{y_0}-1)e^{-kt}}$
   - In problems regarding Verhulst model, we are often interested in **finding the diff eq's equilibrium solutions**. Use the equilibrium solutions and slope field, we can determine whether the population will disappear or towards a constant.

02. Linear Differential Equation

First-Order Linear Equations

1. (**First-order Linear Equation**) A **first-order linear** differential equation is an equation of the form $a(x)y' + b(x)y = c(x)$, with $a(x) \neq 0$. **First-order** means only have y', cannot have $y'' \cdots$. **Linear** means the highest order of y must be one, cannot have $y^2 \cdots$. A tip is to treat y' also as a function of y, so terms like $y'y$ is also not allowed.
2. (**Method of Integrating factor**)
   - Rewrite the entire equation in standard linear form $y' + p(x)y = q(x)$
   - Calculate the integrating factor $u = e^{\int p(x)dx}$ (No need to add arbitrary constant in this step)
   - Multiply both sides of the equation by u: $u(y' + py) = uq \to (uy)' = uq$ (When doing calculation, for the L.H.S, just substitute u in and leave it as it is)
   - Integrate both sides of the equation. (Remember to add the arbitrary constant C at the right side of the equation at this step! And sometimes **Integration by parts** is needed! Don't be scared!)
3. (**Bernoulli differential equation**) It is a diff eq of the form $y' + p(x)y = q(x)y^n \equiv y^{-n}y' + y^{1-n}p(x) = q(x)$. To solve it, use substitution. Substitute $v = y^{1-n}$, so $v' = (1 - n)y^{-n}y'$. Then, the Bernouli equation is simply $v' + v(1 - n)p(x) = q(x)(1 - n)$. Then use integrating factor to solve it. (Always remember to fit in the exact form of Bernoulli equation, but the coefficient of y' can include x, then try to find the suitable substitution)
4. (**Some useful substitution**) The sole of substitution is to reduce the equation for first-order linear equation. So, the whole idea is to try some substitution v and find whether v' can help me achieve the goal.
   - $v = siny, v' = cosy \cdot y'$
5. (**Newton's Law of Cooling Down**) It states that the rate of change of the temperature of an object is proportional to the difference between its temperature $T(t)$ and that of its environment T_{env}. So, the differential equation we can get is $T' = -k(T - T_{\text{env}}), k > 0$

Higher Order Differential Equation

In this part, we mainly focus on how to solve **homogeneous** linear differential equation with **constant coefficients**.

1. (**Homogeneous/Non-homogeneous**) A **linear differential equation with constant coefficients** is an equation of the form $a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(x)$, where $a_i \in R$. When $f(x) = 0$, the equation is said to be **homogeneous**; otherwise, it is said to be **non-homogeneous**.
2. (**Methods to solve homogeneous diff eq with constant coefficients**)
   - Make a guess $y(x) = e^{\lambda x}$
   - Plug the guess into the diff eq and form a **characteristic equation**: $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$. Solve for λ
     - If $\lambda \in R$ is a real, distinct root, then a solution is given by $e^{\lambda x}$
     - If $\lambda \in R$ is a repeated root with multiplicity γ (repeats γ times), then solutions are obtained by modifying our trial solution by a factor of x: $e^{\lambda x}, xe^{\lambda x}, x^2e^{\lambda x}, \cdots, x^{r-1}e^{\lambda x}$
     - If $\lambda, \bar{\lambda} \in \mathbb{C}$ are conjugate roots $\alpha \pm i\beta$, the solutions $e^{\alpha x} \cos \beta x$, $e^{\alpha x} \sin \beta x$. (Obtained by Euler's formula)
     - If $\lambda, \bar{\lambda} \in \mathbb{C}$ are *repeated roots*, the solutions are $e^{\alpha x} \cos \beta x, xe^{\alpha x} \cos \beta x, x^2e^{\alpha x} \cos \beta x, \cdots$ and $e^{\alpha x} \sin \beta x, xe^{\alpha x} \sin \beta x, x^2e^{\alpha x} \sin \beta x, \cdots$

   Note that for an real number order-N equation, the sum of the multiplicity of all its roots must be equal to N.
3. (**Superposition Principle**) Let $y_1(x)$ and $y_2(x)$ be solutions to a **homogeneous linear differential equation** $a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$. Then a solution to this diff eq is also given by $y(x) = c_1y_1(x) + c_2y_2(x)$, for all $c_1, c_2 \in R$. (This is the general solution)

03. The Harmonic Oscillator

Non-Homogeneous Linear Differential Equations

1. (**Rule of Thumb**) Consider the non-homogeneous second-order linear equation $y'' + py' + qy = f(x)$, where $f(x) \neq 0$ and p, q must be **constant** (cannot be function of x). The general solution to this differential equation is given by $y(x) = y_h(x) + y_p(x)$, where $y_h(x)$ is the **general solution** to the *complementary homogeneous equation* $y'' + py' + qy = 0$, and $y_p(x)$ is any **particular solution**.

2. (**Methods to obtain the particular solution**) Usually, the y_p is a determined/exact function.

2.1. (**Method of undetermined coefficients**) When $f(x)$ involves simple functions, we can attempt to guess $y_p(x)$ using the rule below and leave $f(x)$ as it is at R.H.S (except when it involves complex numbers)
- If $f(x)=x^n$, let $y_p=A_1x^n+A_2x^{n-1}+\cdots+A_nx^1+C$, where $A_1,A_2,\cdots,A_n,C\in R$ are undetermined coefficients.
- If $f(x)=e^{kx}$, let $y_p=Ae^{kx}$, where $A\in R$ is an undetermined coefficients. k can be real/complex number.
- If $f(x)=x\pm e^{kx}$, separate according to $\pm$, find y_{p1} for x (we first try $Ax+B$ for x, if it doesn't work, multiply by x and try Ax^2+Bx now) and y_{p2} for e^{kx}. Combine them, $y_p=y_{p1}+y_{p2}$.
- If $f(x)=x\cdot e^{kx}$, suppose y_{p1} is the guess for x and y_{p2} is the guess for e^{kx}. Time them, $y_p=(Ax+B)\cdot e^{kx}$
- If $f(x)$ is either $coskx=\Re\mathfrak{e}(e^{ikx})$ or $sinkx=\Im\mathfrak{m}(e^{ikx})$, let $y_p=Ae^{ikx}$, where $A\in C$ is an undetermined coefficient.
- If $f(x)=x\pm sin(kx)$, separate according to $\pm$, find y_{p1} for x and y_{p2} for $sin(kx)$. Combine them, $y_p=y_{p1}+y_{p2}$.
- If $f(x)=x\cdot sin(kx)$, suppose y_{p1} is the guess for x and y_{p2} is the guess for $sin(kx)$. Time them, $y_p=(Ax+B)\cdot e^{ikx}$. After solving for A,B, get the $\Im\mathfrak{m}$ part as the particular solution.
- If $f(x)=C$, where C is a constant, let $y_p=C$, where $C\in R$ is the undetermined coefficient.
- (**Important**) If any term of the trial solution is already a solution of the complementary equation (notice the constant term counts also!) or you cannot solve for the undetermined coefficients, **multiply the trial solution by** x. If still cannot, multiply it by $x^2\cdots$.

The steps to solve for y_p after making the guess of y_p
- If $f(x)$ **does not contain trigonometric functions**. Leave R.H.S as it is, calculate the derivative of your guess y_p if necessary. Substitute $y,y',\cdots$ with $y_p,y_p',\cdots$. Solve for the undetermined coefficients.
- If $f(x)$ **contains trigonometric functions**. Change the trigonometric function part at R.H.S ($sinkx, coskx$) to e^{ikx}. Leave the remaining R.H.S as it is (Don't forget the coefficients). Calculate the derivative of your guess $y_p=Ae^{ikx}$ if necessary. Substitute $y,y',\cdots$ with $y_p,y_p',\cdots$. Solve for the undetermined coefficient. Then substitute the undetermined coefficient (usually it is a complex number) in, change e^{ikx} to $\cos kx+i\sin kx$. If the original $f(x)$ contains $sin(kx)$ only, then find the $\Im\mathfrak{m}$ part of y_p. If the original $f(x)$ contains $cos(kx)$ only, then find the $\Re\mathfrak{e}$ part of y_p.

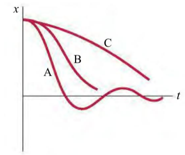
Tips
- When calculating y_p', y_p'' after making the guess of y_p, extract the factor and combine together, then do the further derivation.
- When encounter higher degree e.g. $\sin^2 x, \cos^2 x$, use the double angle formula to decrease the degree to 1. Then make the guess.

2.2. (**Method of variation of parameters**) Given a solution $y_h(x)$ to the complementary equation, we can perform a **variation of parameters** to obtain a particular solution. $y_h=c_1y_1(x)+c_2y_2(x)\to y_p(x)=u(x)y_1(x)+v(x)y_2(x)$, where the functions $u(x)=-\int\frac{y_2f}{y_1y_2'-y_1'y_2}dx$, $v(x)=\int\frac{y_1f}{y_1y_2'-y_1'y_2}dx$, where f is the $f(x)$ at the R.H.S of the initial diff eq.

2.3. (**Tips**)
- This is used to solve question when it is hard/impossible to guess the solution. e.g. $y''+y=\tan x$

Simple Harmonic Motion

1. (**Simple Harmonic Motion**) Any oscillating system for which the **net restoring force** is directly proportional to the **negative** of the **displacement** (e.g. $F=-kx$) is said to exhibit **Simple Harmonic Motion (SHM)**, such a system is called a **Simple Harmonic Oscillator (SHO)**.
   - (**Its differential equation**) $m\ddot{x}=-kx\equiv\ddot{x}=-\frac{k}{m}x$ or $\ddot{x}=-\omega^2x\equiv\ddot{x}+\omega^2x=0$, where $\omega=\sqrt{\frac{k}{m}}$ and ω denotes the **angular frequency** of the oscillation, which is to differ from f, which is equal to $\omega/2\pi$.
   - (**Its solution**) $x=R\cos(\omega t-\varphi)$, where R and φ are two arbitrary constants that can be determined by **initial conditions**.
   - (**Some useful tips**)
     - (Speed) $v=-\omega R\sin(\omega t-\varphi)$
     - (Acceleration) $a=-\omega^2R\cos(\omega t-\varphi)$
     - (Use initial conditions) The soul is to read the given conditions, and determine when $t=0$, whether it is $x=0/v=0/a=0$, then substitute t into the equation to solve for φ. e.g. if the oscillator starts at rest at equilibrium, that means at $t=0$, $x=0,v=0$. Usually R is given directly.
2. (**Damped Harmonic Motion**) To damp means to diminish, restrain or extinguish. Consider a **damped oscillator**, in which the damping force can be simply approximated to be proportional to the speed ($F=-bv$), where b is a constant.
   - (**Its differential equation**) $m\ddot{x}+b\dot{x}+kx=0\equiv\ddot{x}+\frac{b}{m}\dot{x}+\frac{k}{m}x=0$. Or if we denote $\omega=\sqrt{\frac{k}{m}},\gamma=\frac{b}{2m}$, our differential equation becomes $\ddot{x}=-\omega^2x-2\gamma\dot{x}\equiv\ddot{x}+2\gamma\dot{x}+\omega^2x$
   - (**Its solution**) It depends on the constant b. When b is small (underdamped), $x=Ae^{-\gamma t}\cos(\omega' t-\varphi)$. Use the initial condition $t=0,x=A$, our solution becomes $x=Ae^{-\gamma t}\cos\omega' t$, where $\gamma=\frac{b}{2m},\omega'=\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$ (This solution is used to give us a intuitive feeling about how the amplitude and angular frequency changes when b changes)
   - (**Different amounts of damping**) By using the characteristic equation method to solve the differential equation $\lambda^2+2\gamma\lambda+\omega^2=0$, with solutions $\lambda=-\gamma\pm\sqrt{\gamma^2-\omega^2}$. The nature of the object's motion now depends on the value of the discriminant $\Delta=\gamma^2-\omega^2$
     - (**Overdamped**) (Shown as Curve C) It occurs when $\gamma^2>\omega^2$ or, $b^2\gg 4mk$, when our ω' becomes imaginary. It means the damping is so large and it takes a long time to reach equilibrium. It has two real roots $\lambda_1,\lambda_2\in R$ and $x=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}$.
     - (**Underdamped**) (Shown as Curve A) It occurs when $\gamma^2<\omega^2$ or, $b^2<4mk$. It means the system makes several swings before coming to rest. It has two complex conjugate roots $\lambda=\alpha+i\beta\in C$ and $x=c_1e^{\alpha t}\cos\beta t+c_2e^{\alpha t}\sin\beta t$ (Still oscillate)
     - (**Critical damping**) (Shown as Curve B) It occurs when $\gamma^2=\omega^2$ or, $b^2=4mk$. In this case, the equilibrium is reached in the shortest time. It has a **repeated** real root $\lambda\in R$ and $x=c_1e^{\lambda t}+c_2te^{\lambda t}$.



3. (**Forced Oscillation: Resonance**) When an oscillating system has an external force applied to it that has its own **particular frequency**, we have a **forced oscillation**.
   - (**Its differential equation**) Suppose our $F_{\text{ext}}=F_0\cos\omega t$ and consider the damping force also, we have $\ddot{x}+\frac{b}{m}\dot{x}+\omega_0^2x=F_0\cos\omega t$.
   - (**Its solution**) Given the initial condition $t=0,x=A$, our general solution is $x=A\cdot e^{-\gamma t}\cdot\cos\omega_0 t+A_0\cdot\sin(\omega t+\varphi)$, where $\gamma=\frac{b}{2m},A_0=\frac{F_0}{m\sqrt{(\omega^2-\omega_0^2)^2+b^2\omega^2/m^2}},\varphi=\arctan\frac{\omega_0^2-\omega^2}{b\omega/m}$.
   - (**Some tips**)
     - When solving oscillation problems, the first thing is to decide whether it is **Simple Harmonic Motion**, or **Damped Harmonic Motion** or **Forced Oscillation**. Then find the suitable equation to plug in.
     - When $\omega=\omega_0$, **resonance** occurs and if damping force is not considered and initial condition is not given, when $\omega=\omega_0$, then our solution becomes $x=A\cos(\omega t+\varphi)+\frac{F}{2\omega}t\sin\omega t$. Otherwise, the oscillator will be **stable** (means never tend to infinity). Given that the initial condition is *the oscillator starts at rest at equilibrium*, then $A=0$ ($x(0)=0),\dot{x}(0)=0)$, a.k.a we can kick out of the first term and now $x=\frac{F}{2\omega}t\sin\omega t$
     - Notice that in our general solution, as our t increases, the first term approaches 0 because of $Ae^{-\gamma t}$.
     - The natural frequency of the system ω_0 is called the **resonance frequency**.

04. The Laplace Transform

1. (**Definition**) The **Laplace transform of** $f(t)$ is defined by $\mathcal{L}[f(t)]=F(s)=\lim_{h\to\infty}\int_0^h e^{-st}\cdot f(t)dt$, and $f(t)$ is the **inverse** Laplace transform of $F(s)$: $f(t)=\mathcal{L}^{-1}[F(s)]$. (Note that s is a new intermediate variable).

$f(t)=\mathcal{L}^{-1}[F(s)]$	$F(s)=\mathcal{L}[f(t)]$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$t^n, n\in N$	$\frac{n!}{s^{n+1}}$
$\delta(t-c)$	e^{-cs}
$f(t)\delta(t-c)$	$e^{-cs}f(c)$
$u(t-c)$	$\frac{1}{s}e^{-cs}$
$e^{at}u(t-c)$	$\frac{e^{-c(s-a)}}{s-a}$

2. (**Linearity**) Given functions $f(t)$ and $g(t)$, $\mathcal{L}[af(t)+bg(t)]=a\mathcal{L}[f(t)]+b\mathcal{L}[g(t)]$ for all $a,b\in R$. Note that $\mathcal{L}^{-1}$ also has **Linearity**.
3. (**Differentiation Property**) $\mathcal{L}[t\cdot f(t)]=-\frac{d}{ds}F(s)$ or $-F'(s)$. General form $\mathcal{L}[t^n\cdot f(t)]=(-1)^nF^{(n)}(s)$, where $F(s)=\mathcal{L}[f(t)]$
4. (**First Shifting Theorem**) $\mathcal{L}[e^{at}\cdot f(t)]=F(s-a)$, where $F(s)=\mathcal{L}[f(t)]$
5. (**Derivatives**) $\mathcal{L}[y']=s\mathcal{L}[y]-y(0)$, $\mathcal{L}[y'']=s^2\mathcal{L}[y]-sy(0)-y'(0)$. General form is $\mathcal{L}[f^{(n)}(t)]=s^n\mathcal{L}[f(t)]-s^{n-1}f(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$
6. (**Second Shifting Theorem**) $\mathcal{L}[f(t-c)\cdot u(t-c)]=e^{-sc}\mathcal{L}[f(t)]$, or $\mathcal{L}[f(t)u(t-c)]=e^{-sc}\mathcal{L}[f(t+c)]$. (Sometimes the later one will be faster!)
   - (**Forward**) Find $u(t-c)$, then use $\mathcal{L}[f(t)u(t-c)]=e^{-sc}\mathcal{L}[f(t+c)]$, then use linearity and other properties to find the corresponding Laplace Transform $\mathcal{L}[\{(\sqcup)]$, then $t\to t+c$.
   - (**Reverse**) Use $\mathcal{L}^{-1}[e^{-cs}F(s)]=f(t-c)u(t-c)$ Firstly, use the term e^{-cs} to find c. Then do the **Inverse Laplace Transform** on every term of $F(s)$ to find $f(t)$. Then, change t to $t-c$.
7. (**The method of partial fraction decomposition**)
   - If the denominator is **not repeated**, the degree of the **numerator** should be **1 less than** the degree of the **denominator**. e.g. $\frac{1}{s(ms^2+k)}=\frac{A}{s}+\frac{Bs+C}{ms^2+k}$, $\frac{2s^3+4}{s^4+2s^3}=\frac{As^2+Bs+C}{s^3}+\frac{D}{s}$
   - If the denominator is **repeated** (a.k.a degree is bigger than 1), start from degree of 1, sum to the current degree. e.g. $\frac{1}{x^2(x^2+1)^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}+\frac{Ex+F}{(x^2+1)^2}$.
8. (**Tips**)
   - To use the above properties, the first step is to find your y or $f(t)$, this is your base function. Then find the Laplace Transform of your base function. Then use the properties.
   - (**The Inverse Laplace Transform**) e.g. Evaluate $\mathcal{L}^{-1}[\frac{1+e^{-3s}}{s^4}]$. $F(s)=\frac{1}{s^4}+e^{-3s}\frac{1}{s^4}$. Note that $\mathcal{L}^{-1}[\frac{1}{s^n}]=\frac{t^{n-1}}{(n-1)!},\mathcal{L}^{-1}[e^{-as}F(s)]=$

$f(t-a)u(t-a)$. Hence, $f(t)=\mathcal{L}^{-1}[F(s)]=\frac{t^3}{3!}+\frac{(t-3)^3}{3!}u(t-3)$

- (**Convolution Integral**) If $\mathcal{L}^{-1}[F(s)]=f(t)$ and $\mathcal{L}^{-1}[G(s)]=g(t)$, then $\mathcal{L}^{-1}[F(s)G(s)]=\int_0^t f(u)g(t-u)du=\int_0^t g(u)f(t-u)du=(f*g)(t)$
- (**Differentiate the unit step function**) For a unit step function $u(t-c)$, if differentiate w.r.t c, we treat $u(t-c)$ as a constant.

Step Functions and the Unit Impulse

1. (**Use step function to represent the range**)
   - $1-u(t-1)$ can represent $t<1$, sometimes it will be $0<t<1$ (depends on the question)
   - $u(t-1)-u(t-\frac{\pi}{2})$ can represent $1<t<\pi/2$
   - $u(t-\frac{\pi}{2})$ can represent $t>\pi/2$
2. (**Method to get the Laplace Transform of Unit Step Functions**) The idea is to separate the integrals e.g. $\mathcal{L}[u(t-c)]=\int_0^\infty e^{-st}u(t-c)dt=\int_0^c e^{-st}u(t-c)dt+\int_c^\infty e^{-st}u(t-c)dt=\int_0^c e^{-st}\cdot 0dt+\int_c^\infty e^{-st}\cdot 1dt=-\frac{1}{s}e^{-st}|_{t=c}^{\infty}=\frac{1}{s}e^{-cs}$
3. (**Dirac Delta/Unit Impulse Function**) Defined when Impulse $I=1$ and as $\epsilon\to\infty$, the dirac delta function is often used to represent a **sudden** change in the question and the magnitude/Impulse is 1.
4. (**Tips**)
   - The dirac delta function are defined to be an instantaneous amount of change, but in problems, it should be considered as **a spike rate of change**!

05. Partial Differential Equations

1. (**Definition**) A **partial differential equation** (PDE) is an equation involving one or more **partial derivatives** of a function that depends on **two or more variables**.
2. (**Linearity and Homogeneity**)
   - A PDE is **linear** if it is of the **first degree** in the **unknown function and its derivative**. But the **independent variables** can be of higher degrees. e.g. u is an unknown function of x,y, $u\cdot u'=0$ is **not linear**, $x^2u=0$ is $linear$
   - A PDE is **homogeneous** if each of the terms contains either or one of its partial derivatives. (Or, 0 is one of the solutions for the equation)
3. (**Solve PDE - Method of separation of variables**)

   3.1. Suppose that a solution is given by $u(x,y)=A(x)B(y)$

   3.2. Rewrite the equation in A and B, e.g. $u=AB, u_x=A'B, u_y=AB', u_{xx}=A''B\cdots$

   3.3. Separate the variables: $f(x,A,A',\cdots)=g(y,B,B',\cdots)$ and let both sides equal to a *separation constant* k. Thus, we have two ODEs.

   3.4. Solve these two ODEs using separation of variables in ODE. Get $A=\cdots x, B=\cdots y$. Then combine these two solutions by $u(x,y)=AB$
4. (**Superposition Principle**) Let $u_1(x,y)$ and $u_2(x,y)$ be solutions of a **homogeneous linear PDE**. Then, a solution is also given by $u(x,y)=c_1u_1(x,y)+c_2u_2(x,y)$, for any $c_1,c_2\in R$
5. (**Solving Tips**)
   - When using the method of separation of variables, if the PDE becomes an "ODE" (either A or B is eliminated), then after solving the remaining variable, change the constants c_1,c_2 to the one that includes the variable that is treated as constants in this PDE, e.g. $f(y),g(y)$
   - $\int\frac{1}{A}dA=\int(k+1)\frac{1}{x}dx\to\ln|A|=(k+1)\ln|x|+c$, thus we have $A(x)=c_1x^{k+1}$

The Heat Equation

1. (**Definition**) The dispersion of heat on a metal rod of length l is described by the **heat equation** $u_t=c^2u_{xx}$, $0<x<l, t>0$. c^2 is the *thermal diffusivity of the metal*, and the **solution** $u(x,t)$ describes the temperature of the rod at a given point x and time t. Assuming that at $x=0$ and $x=l$, the rod is insulated, so we have *boundary conditions* $u(0,t)=0, u(l,t)=0$. If the initial distribution of heat is given by the function $f(x)$, then we have the initial condition $u(x,0)=f(x)$
2. (**Solution**) The general solution to the heat equation is $u(x,t)=\beta_ne^{-c^2n^2\pi^2t/l^2}sin(\frac{n\pi}{l}x)$, where c^2, l are usually given by question. n,β_n are constants that can be derived using the *initial condition*,. And by superposition principle, we can divide the initial condition into each $\sin$ function, find its corresponding n,β_n and combine them together into one particular solution using superposition principle again.

Tips

1. Include integration by parts
2. Include the double angle, sum to product, product to sum formulas
3. Draw the curve diagram for damped harmonoic oscillation