MA1512

AY24/25 sem 1

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01. Intro to Differential Equations

First Principles

1. (**The method of separation of variables**) A first-order differential equation of the form $\frac{dy}{dx} = F(x, y)$ is **separable** if it can be written as $M(x)dx = N(y)dy$. To solve this, directly integrate both sides of the equation, we will get $\int N(y)dy = \int M(x)dx + C$, where C is an arbitrary constant.
   - (**The position of arbitrary constant** C) The arbitrary constant must be added immediately when you integrate w.r.t independent variable.
   - (**Some useful substitution**)
     - If $y' = f(ax + by + c)$, we employ a *linear change of variable*. Let $u = ax + by + c \rightarrow u' = a + by'$
     - If $y' = f(y/x)$, we let $y = xv$, and $y' = xv' + v$ (Product Rule) (This is same as let $v = \frac{y}{x}$ first.)

     Note that in both substitutions, we assume the function at the right side can be written as $f(\cdots)$, which is the soul in substitution.
2. (**Classic Examples**)
   - (**Half-Life**) The typical scenario for half-life is *exponential decay*, in which we have $\frac{dx}{dt} = -kx, k > 0$ and since its solution is $x = x_0e^{-kt}$. We can get the formula to determine the *decay rate* k given that we know the half-life τ, so, it will be $\frac{x_0}{2} = x_0e^{-kt} \rightarrow k = \frac{ln2}{\tau}$
   - Remember that $\int \frac{2u+4}{u^2+4u+5}\, du = \ln(u^2 + 4u + 5) + c$.
   - To solve $y' \cdot y'' = 2$, to use separation of variables, use substitution $u = y' \rightarrow u \cdot u' = 2$. Solve u first, then substitute u with y.
3. (**Tips**)
   - Note that the solution to a differential equation can be in **implicit form**.

The Geometry of Differential Equations

1. (**Methods to find equilibrium solution**)
   - Judge the order of the diff eq and let all the **dependent variables' derivatives** to be 0. e.g. First order $\rightarrow y' = 0$, Second order $\rightarrow y' = 0$ and $y'' = 0$. e.g. $y' = -10y(20 - y)(1 - \frac{1}{30}y) \rightarrow 0 = -10y(20 - y)(1 - \frac{1}{30}y)$
   - Use basic inequality technique to solve and sketch out a sign diagram for dy/dt. The **y-axis is dependent variable** and the **x-axis is the independent variable**.
   - Draw the corresponding direction/slope field to judge the stability.
2. **Tips**
   - (**Stable Equilibrium Solution**) An equilibrium solution $y(t) = \beta$ is said to be **stable** if solutions about/near this equilibrium approach β as $t \rightarrow \infty$.
   - (**Unstable Equilibrium Solution**) Otherwise, the equilibrium point is said to be **unstable**.

Population Dynamics

1. (**Malthusian model**) It assumes that the rate of change of a population is proportional to its present value. That is $dy/dt = ky \rightarrow y(t) = y_0e^{kt}$, where $y_0 = y(0)$. This model suggests that a population would grow exponentially with **growth rate** k. (**Note that in Malthusian model, it is** $y = y_0 \cdot e^{kt}$**, however, in half-life model, it is** $x = x_0 \cdot e^{-kt}$)
2. (**Verhulst model**) In Verhulst, our **growth rate** varies according to the present value y of the population. The formula is given by $\frac{dy}{dt} = [k(1 - \frac{y}{y_\infty})]y$, where dy/dt is **rate of change** and $k(1 - y/y_\infty)$ is **growth rate**. This model assumes that a population grows **logistically**, such that given any initial population, $lim_{t\rightarrow\infty}y(t) = y_\infty$, and y_∞ is called the *carrying capacity*, y_∞ is also a stable equilibrium solution for the original Verhulst model.
3. (**Hunt rate**) Usually we directly minus the hunt rate (E) at the right side of our equation. e.g. In Malthusian Model, if the hunt rate is 100, $dy/dt = ky - 100$.
4. (**Some useful tips regarding Verhulst model**)
   - Regard the R.H.S as a quadratic equation, we can see that when $y = y_\infty/2$, the rate of change dy/dt will be **maximum**.
   - Given the initial condition $y(0) = y_0$, the solution for Verhulst Model is $y(t) = \frac{y_\infty}{1+(\frac{y_\infty}{y_0}-1)e^{-kt}}$
   - In problems regarding Verhulst model, we are often interested in **finding the diff eq's equilibrium solutions**. Use the equilibrium solutions and slope field, we can determine whether the population will disappear or towards a constant. And when Hunt rate is included, we are interested in E and $ky_\infty/4$

02. Linear Differential Equation

First-Order Linear Equations

1. (**First-order Linear Equation**) A **first-order linear** differential equation is an equation of the form $a(x)y' + b(x)y = c(x)$, with $a(x) \neq 0$.
2. (**Method of Integrating factor**)
   - Rewrite the entire equation in **standard linear form** $y' + p(x)y = q(x)$ (The coefficient of y' must be 1)
   - Calculate the integrating factor $u = e^{\int p(x)dx}$ (No need to add arbitrary constant in this step)
   - Multiply both sides of the equation by u: $u(y' + py) = uq \rightarrow (uy)' = uq$ (When doing calculation, for the L.H.S, just substitute u in and leave it as it is)
   - Integrate both sides of the equation. (Remember to add the arbitrary constant C at the R.H.S of the equation at this step! And sometimes **Integration by parts** is needed! Don't be scared!)
3. (**Bernoulli differential equation**) It is a diff eq of the form $y' + p(x)y = q(x)y^n \equiv y^{-n}y' + y^{1-n}p(x) = q(x)$. To solve it, use substitution. Substitute $v = y^{1-n}$, so $v' = (1 - n)y^{-n}y'$. Then, the Bernoulli equation is simply $v' + v(1 - n)p(x) = q(x)(1 - n)$. Then use integrating factor to solve it. And now, the integrating factor is $e^{\int (1-n)p(x)\, dx}$ (Always remember to fit in the exact form of Bernoulli equation, then try to find the suitable substitution)
4. (**Some useful substitution**)
   - $v = siny, v' = cosy \cdot y'$
5. (**Newton's Law of Cooling Down**) It states that the rate of change of the temperature of an object is proportional to the difference between its temperature $T(t)$ and that of its environment T_{env}. So, the differential equation we can get is $T' = -k(T - T_{\text{env}}), k > 0$

Higher Order Differential Equation

In this part, we mainly focus on how to solve **homogeneous** linear differential equation with **constant coefficients**.

1. (**Homogeneous/Non-homogeneous**) A **linear differential equation with constant coefficients** is an equation of the form $a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = f(x)$, where $a_i \in R$. When $f(x) = 0$, the equation is said to be **homogeneous**; otherwise, it is said to be **non-homogeneous**.
2. (**The method of characteristic equation**)
   - Make a guess $y(x) = e^{\lambda x}$
   - Plug the guess into the diff eq and form a **characteristic equation**: $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$. Solve for λ
     - If $\lambda \in R$ is a real, distinct root, then a solution is given by $e^{\lambda x}$
     - If $\lambda \in R$ is a repeated root with multiplicity γ (repeats γ times), then solutions are obtained by modifying our trial solution by a factor of x: $e^{\lambda x}, xe^{\lambda x}, x^2e^{\lambda x}, \cdots, x^{r-1}e^{\lambda x}$
     - If $\lambda, \bar{\lambda} \in \mathbb{C}$ are conjugate roots $\alpha \pm i\beta$, the solutions $e^{\alpha x}\cos\beta x$, $e^{\alpha x}\sin\beta x$. (Obtained by Euler's formula)
     - If $\lambda, \bar{\lambda} \in \mathbb{C}$ are *repeated roots*, the solutions are $e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, x^2e^{\alpha x}\cos\beta x, \cdots$ and $e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, x^2e^{\alpha x}\sin\beta x, \cdots$

   Note that for an real number order-N equation, the sum of the multiplicity of all its roots must be equal to N.
3. (**Superposition Principle**) Let $y_1(x)$ and $y_2(x)$ be solutions to a **homogeneous linear differential equation** $a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$. Then a solution to this diff eq is also given by $y(x) = c_1y_1(x) + c_2y_2(x)$, for all $c_1, c_2 \in R$. (This is the general solution)

03. The Harmonic Oscillator

Non-Homogeneous Linear Diff Eq

1. (**Rule of Thumb**) Consider the non-homogeneous second-order linear equation $y'' + py' + qy = f(x)$, where $f(x) \neq 0$ and p, q must be **constant** (cannot be function of x). The general solution to this differential equation is given by $y(x) = y_h(x) + y_p(x)$, where $y_h(x)$ is the **general solution** to the *complementary homogeneous equation* $y'' + py' + qy = 0$, and $y_p(x)$ is any **particular solution**.
2. (**Methods to obtain the particular solution**) Usually, y_p is a determined/exact function.

2.1. (**Method of undetermined coefficients**) When $f(x)$ involves simple functions, we can attempt to guess $y_p(x)$ using the rule below and **leave** $f(x)$ **as it is at R.H.S (except when it involves complex numbers)**
   - If $f(x) = x^n$, let $y_p = A_1x^n + A_2x^{n-1} + \cdots + A_nx^1 + C$, where $A_1, A_2, \cdots, A_n, C \in R$ are undetermined coefficients.
   - If $f(x) = e^{kx}$, let $y_p = Ae^{kx}$, where $A \in R$ is an undetermined coefficients. k can be real/complex number.
   - If $f(x) = x \pm e^{kx}$, separate according to $\pm$, find y_{p1} for x (we first try $Ax + B$ for x, if any term appears in the y_h (constant term also counts!), multiply by x and try $Ax^2 + Bx$ now) and y_{p2} for e^{kx}. Combine them, $y_p = y_{p1} + y_{p2}$.
   - If $f(x) = x \cdot e^{kx}$, make the guess as $y_p = (Ax + B) \cdot e^{kx}$
   - If $f(x)$ is either $coskx = \Re\mathfrak{e}(e^{ikx})$ or $sinkx = \Im\mathfrak{m}(e^{ikx})$, let $y_p = Ae^{ikx}$, where $A \in C$ is an undetermined coefficient.
   - If $f(x) = x \pm sin(kx)$, separate according to $\pm$, find y_{p1} for x and y_{p2} for $sin(kx)$. Combine them, $y_p = y_{p1} + y_{p2}$.
   - If $f(x) = x \cdot sin(kx)$, Make the guess as $y_p = (Ax + B) \cdot e^{ikx}$.
   - If $f(x) = e^x\cos 2x$, guess the solution as $Ae^{x(1+2i)}$
   - If $f(x) = C$, where C is a constant, let $y_p = C$, where $C \in R$ is the undetermined coefficient.
   - (**Important**) If any term of your guess y_p appears in y_h (notice the constant term counts also!) or you cannot solve for the undetermined coefficients, **multiply** y_p **by** x. If still cannot, multiply it by $x^2 \cdots$.

   The steps to solve for y_p **after making the guess of** y_p
   - If $f(x)$ **does not contain trigonometric functions**.
     i. Leave R.H.S as it is
     ii. Calculate the derivative of your guess y_p if necessary.
     iii. Substitute $y, y', \cdots$ with $y_p, y_p' \cdots$.
     iv. Solve for the undetermined coefficients.
   - If $f(x)$ **contains trigonometric functions**.
     i. Change the trigonometric function part at R.H.S ($sinkx, coskx$) to e^{ikx} (without undetermined coefficients). Leave the remaining R.H.S as it is (Don't forget the coefficients).
     ii. Calculate the derivative of your guess $y_p = Ae^{ikx}$ if necessary.
     iii. Substitute $y, y', \cdots$ with $y_p, y_p', \cdots$.
     iv. Solve for the undetermined coefficient.
     v. Then substitute the undetermined coefficient (usually it is a complex number) in, change e^{ikx} to $\cos kx + i\sin kx$.
     vi. If the original $f(x)$ contains $sin(kx)$ only, then find the $\Im\mathfrak{m}$ part of y_p. If the original $f(x)$ contains $cos(kx)$ only, then find the $\Re\mathfrak{e}$

part of y_p.

Tips

- When calculating y_p', y_p'' after making the guess of y_p, extract the factor and combine together, then do the further derivation.
- When encounter higher degree e.g. $\sin^2 x, \cos^2 x$, use the double angle formula to decrease the degree to 1. Then make the guess.

2.2. (**Method of variation of parameters**) Given a solution $y_h(x)$ to the complementary equation, we can perform a **variation of parameters** to obtain a particular solution. $y_h = c_1y_1(x)+c_2y_2(x) \to y_p(x) = u(x)y_1(x)+v(x)y_2(x)$, where the functions $u(x) = -\int \frac{y_2 f}{y_1y_2'-y_1'y_2}dx$, $v(x) = \int \frac{y_1 f}{y_1y_2'-y_1'y_2}dx$, where f is the $f(x)$ at the R.H.S of the initial diff eq.

2.3. (**Tips**)

- Use **the method of variation of parameter**s to solve question when it is hard/impossible to guess the solution. e.g. $y''+y=\tan x$

Simple Harmonic Motion

1. (**Simple Harmonic Motion (SHM)**) Its differential equation is $m\ddot{x} = -kx \equiv \ddot{x} = -\frac{k}{m}x$ or $\ddot{x} = -\omega^2 x \equiv \ddot{x}+\omega^2 x = 0$, where $\omega = \sqrt{\frac{k}{m}}$ and ω denotes the **angular frequency** of the oscillation, which is to differ from f, which is equal to $\omega/2\pi$. And its solution is $x = c_1\cos\omega t + c_2\sin\omega t$ or $R\cos(\omega t-\varphi)$, where c_1, c_2, R, φ are two arbitrary constants that can be determined by **initial conditions**. (It is recommended to use c_1, c_2 version in the test)
2. (**Some useful tips**)
   - (Speed) $v = -\omega R\sin(\omega t-\varphi)$
   - (Acceleration) $a = -\omega^2 R\cos(\omega t-\varphi)$
   - (Use initial conditions) The soul is to read the given conditions, and determine when $t=0$, whether it is $x=0/v=0/a=0$, then substitute t into the equation to solve for φ. e.g. if the oscillator starts at rest at equilibrium, that means at $t=0, x=0, v=0$. Usually R is given directly.
3. (**Damped Harmonic Motion (DHM)**) The damping force is $(F=-bv)$, where b is a constant.
   - (**Its differential equation**) $m\ddot{x}+b\dot{x}+kx=0 \equiv \ddot{x}+\frac{b}{m}\dot{x}+\frac{k}{m}x=0$. Or if we denote $\omega = \sqrt{\frac{k}{m}}, \gamma = \frac{b}{2m}$, our differential equation becomes $\ddot{x} = -\omega^2 x - 2\gamma x \equiv \ddot{x}+2\gamma\dot{x}+\omega^2 = 0$
   - (**Its solution**) It depends on the constant b. When b is small (underdamped), $x = Ae^{-\gamma t}\cos(\omega' t-\varphi)$. Use the initial condition $t=0, x=A$, our solution becomes $x = Ae^{-\gamma t}\cos\omega' t$, where $\gamma = \frac{b}{2m}, \omega' = \sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}$
   - (**Different amounts of damping**) By using the characteristic equation method to solve the differential equation $\lambda^2+2\gamma\lambda+\omega^2=0$, with solutions $\lambda = -\gamma\pm\sqrt{\gamma^2-\omega^2} < 0$. The nature of the object's motion now depends on the value of the discriminant $\Delta = \gamma^2-\omega^2$
     - (**Overdamped**) It occurs when $\gamma^2 > \omega^2$ or, $b^2 \gg 4mk$, when our ω' becomes imaginary. It means the damping is so large and it takes a long time to reach equilibrium. It has two real roots $\lambda_1, \lambda_2 \in R$ and $x = c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}$.
     - (**Underdamped**) It occurs when $\gamma^2 < \omega^2$ or, $b^2 < 4mk$. It means the system makes several swings before coming to rest. It has two complex conjugate roots $\lambda = \alpha+i\beta \in C$ and $x = c_1e^{\alpha t}\cos\beta t + c_2e^{\alpha t}\sin\beta t$ (Still oscillate)
     - (**Critical damping**) It occurs when $\gamma^2 = \omega^2$ or, $b^2 = 4mk$. In this case, the equilibrium is reached in the shortest time. It has a **repeated** real root $\lambda \in R$ and $x = c_1e^{\lambda t}+c_2te^{\lambda t}$.
4. (**Forced Oscillation (FO): Resonance**) When an oscillating system has an external force applied to it that has its own **particular frequency**, we have a **forced oscillation**.
   - (**Its differential equation**) Suppose our $F_{\text{ext}} = F\cos\omega t$ and consider the damping force also, we have $\ddot{x}+\frac{b}{m}\dot{x}+\omega_0^2x = F\cos\omega t$.
   - (**Its solution**) Given the initial condition $t=0, x=A$, our general solution is $x = A\cdot e^{-\gamma t}\cdot\cos\omega_0 t + A_0\cdot\sin(\omega t+\varphi)$, where $\gamma = \frac{b}{2m}, A_0 = \frac{F}{m\sqrt{(\omega^2-\omega_0^2)^2+b^2\omega^2/m^2}}, \varphi = \arctan\frac{\omega_0^2-\omega^2}{b\omega/m}$.
   - (**Some tips**)
     - When solving oscillation problems, the first thing is to decide whether it is **SHM**, or **DHM** or **FO**. Then find the suitable equation to plug in.
     - When $\omega = \omega_0$, **resonance** occurs. **If damping force is not considered** and **initial condition is not given**, only when $\omega = \omega_0$, our solution becomes $x = c_1\cos\omega t + c_2\sin\omega t + \frac{F}{2\omega}t\sin\omega t$ (if $F_{\text{ext}} = F\cos\omega t$). If given that the initial condition is *the oscillator starts at rest at equilibrium*, then $A=0$ $(x(0)=0, \dot{x}(0)=0)$, a.k.a we can kick out of the first term and now $x = \frac{F}{2\omega}t\sin\omega t$. If $F_{ext} = F\sin\omega t$, then solve it manually.
       Otherwise $(\omega \neq \omega_0)$, the oscillator will be **stable** (means never tend to infinity).
     - Notice that in our general solution, as our t increases, the first term approaches 0 because of $Ae^{-\gamma t}$.
     - The natural frequency of the system ω_0 is called the **resonance frequency**. And if the external force has several angular frequency, when either one of them is equal to ω_0, the resonace will occur.

04. The Laplace Transform

1. (**Definition**) The **Laplace transform of** $f(t)$ is defined by $\mathcal{L}[f(t)] = F(s) = \lim_{h\to\infty}\int_0^h e^{-st}\cdot f(t)dt$, and $f(t)$ is the **inverse** Laplace transform of $F(s)$: $f(t) = \mathcal{L}^{-1}[F(s)]$. (Note that s is a new intermediate variable).

$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}[f(t)]$
$f(t)\delta(t-c)$	$e^{-cs}f(c)$
$e^{at}u(t-c)$	$\frac{e^{-c(s-a)}}{s-a}$

2. (**Linearity**) Given functions $f(t)$ and $g(t)$, $\mathcal{L}[af(t)+bg(t)] = a\mathcal{L}[f(t)]+b\mathcal{L}[g(t)]$ for all $a, b \in R$. Note that $\mathcal{L}^{-1}$ also has **Linearity**.
3. (**Differentiation Property**) $\mathcal{L}[t\cdot f(t)] = -\frac{d}{ds}F(s)$ or $-F'(s)$. General form $\mathcal{L}[t^n\cdot f(t)] = (-1)^nF^{(n)}(s)$, where $F(s) = \mathcal{L}[f(t)]$
4. (**First Shifting Theorem**) $\mathcal{L}[e^{at}\cdot f(t)] = F(s-a)$, where $F(s) = \mathcal{L}[f(t)]$
5. (**Derivatives**) $\mathcal{L}[y'] = s\mathcal{L}[y]-y(0)$, $\mathcal{L}[y''] = s^2\mathcal{L}[y]-sy(0)-y'(0)$. General form is $\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)]-s^{n-1}f(0)-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$
6. (**Second Shifting Theorem**) $\mathcal{L}[f(t-c)\cdot u(t-c)] = e^{-sc}\mathcal{L}[f(t)]$, or $\mathcal{L}[f(t)u(t-c)] = e^{-sc}\mathcal{L}[f(t+c)]$. (Sometimes the later one will be faster!)
   - (**Forward**)
     6.1. Find $u(t-c)$, then use $\mathcal{L}[f(t)u(t-c)] = e^{-sc}\mathcal{L}[f(t+c)]$.
     6.2. Then $t \to t+c$.
     6.3. Lastly, use linearity and other properties to find the corresponding Laplace Transform $\mathcal{L}[f(t+c)]$.
   - (**Reverse**)
     6.1. Use $\mathcal{L}^{-1}[e^{-cs}F(s)] = f(t-c)u(t-c)$. Firstly, use the term e^{-cs} to find c.
     6.2. Then do the **Inverse Laplace Transform** on every term of $F(s)$ to find $f(t)$.
     6.3. Lastly, change t to $t-c$ to form $f(t-c)$.
7. (**The method of partial fraction decomposition**)
   - If the denominator is **not repeated**, the degree of the **numerator** should be **1 less than** the degree of the **denominator**. e.g. $\frac{1}{s(ms^2+k)} = \frac{A}{s}+\frac{Bs+C}{ms^2+k}$, $\frac{2s^3+4}{s^4+2s^3} = \frac{As^2+Bs+C}{s^3}+\frac{D}{s}$
   - If the denominator is **repeated** (a.k.a degree is bigger than 1), start from degree of 1, sum to the current degree. e.g. $\frac{1}{x^2(x^2+1)^2} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}+\frac{Ex+F}{(x^2+1)^2}$.
8. (**Tips**)
   - To use the above properties, the first step is to find your y or $f(t)$, this is your base function. Then find the Laplace Transform of your base function. Then use the properties.
   - (**The Inverse Laplace Transform**) e.g. Evaluate $\mathcal{L}^{-1}[\frac{1+e^{-3s}}{s^4}]$. $F(s) = \frac{1}{s^4}+e^{-3s}\frac{1}{s^4}$. Note that $\mathcal{L}^{-1}[\frac{1}{s^n}] = \frac{t^{n-1}}{(n-1)!}, \mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)u(t-a)$. Hence, $f(t) = \mathcal{L}^{-1}[F(s)] = \frac{t^3}{3!}+\frac{(t-3)^3}{3!}u(t-3)$
   - (**Variation of the Laplace Transform Definition**) To get $\mathcal{L}[f(ct)]$, let $u = ct, du = c\,dt$. Thus $\mathcal{L}[f(ct)] = \frac{1}{c}\int_0^\infty e^{-(s/c)u}f(u)\,du = \frac{1}{c}F(\frac{s}{c})$. What matters is the exponential term $e^{\cdots s}$.
   - (**Differentiate the unit step function**) For a unit step function $u(t-c)$, if differentiate w.r.t c, we treat $u(t-c)$ as a constant.

Step Functions and the Unit Impulse

1. (**Use step function to represent the range**)
   - $1-u(t-1)$ can represent $t<1$, sometimes it will be $0<t<1$ (depends on the question)
   - $u(t-1)-u(t-\frac{\pi}{2})$ can represent $1<t<\pi/2$
   - $u(t-\frac{\pi}{2})$ can represent $t>\pi/2$
2. (**Dirac Delta/Unit Impulse Function**) Defined when Impulse $I=1$ and as $\epsilon\to\infty$, the dirac delta function is often used to represent a **sudden** change in the question and the magnitude/Impulse is 1.
3. (**Tips**)
   - The dirac delta function are defined to be an instantaneous amount of change, but in problems, it should be considered as **a spike rate of change**!

05. Partial Differential Equations

1. (**Solve PDE - Method of separation of variables**)
   1.1. Suppose that a solution is given by $u(x,y) = A(x)B(y)$
   1.2. Rewrite the equation in A and B, e.g. $u = AB, u_x = A'B, u_y = AB', u_{xx} = A''B\cdots$
   1.3. Separate the variables: $f(x, A, A', \cdots) = g(y, B, B', \cdots)$ and let both sides equal to a *separation constant* k. Thus, we have two ODEs.
   1.4. Solve these two ODEs using separation of variables in ODE. Get $A = \cdots x, B = \cdots y$. Then combine these two solutions by $u(x,y) = AB$, which will be a general solution.
2. (**Superposition Principle**) Let $u_1(x,y)$ and $u_2(x,y)$ be solutions of a **homogeneous linear PDE**. Then, a solution is also given by $u(x,y) = c_1u_1(x,y)+c_2u_2(x,y)$, for any $c_1, c_2 \in R$
3. (**Solving Tips**)
   - When using the method of separation of variables, if the PDE becomes an ”ODE” modify your constants c_1, c_2 to the functions contain the constant variables $f(y), g(y)$
   - $\int\frac{1}{A}dA = \int(k+1)\frac{1}{x}dx \to \ln|A| = (k+1)\ln|x|+c$, thus we have $A(x) = c_1x^{k+1}$

The Heat Equation

1. (**Definition**) The dispersion of heat on a metal rod of length l is described by the **heat equation** $u_t = c^2u_{xx}$, $0<x<l, t>0$. c^2 is the *thermal diffusivity of the metal*, and the **solution** $u(x,t)$ describes the temperature of the rod at a given point x and time t. Assuming that at $x=0$ and $x=l$, the rod is insulated, so we have *boundary conditions* $u(0,t) = 0, u(l,t) = 0$. If the initial distribution of heat is given by the function $f(x)$, then we have the initial condition $u(x,0) = f(x)$
2. (**Solution**) The general solution to the heat equation is $u(x,t) = \beta_ne^{-c^2n^2\pi^2t/l^2}sin(\frac{n\pi}{l}x)$, where c^2, l are usually given by question. n, β_n are constants that can be derived using the *initial condition*. And by superposition principle, we can divide the initial condition into each $\sin$ function, find its corresponding n, β_n and combine them together into one particular solution using superposition principle again.