Classic Examples for Differential Equations

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1 Introduction to Differential Equations

1.1 The method of Separation of variables

1. (**Linear change of variable**) Solve the differential equation

$$\frac{dy}{dx} = \frac{1 - 2y - 4x}{1 + y + 2x} \tag{1}$$

Solution. Observe that we may rewrite the differential equation as

$$\frac{dy}{dx} = \frac{1 - 2(y + 2x)}{1 + (y + 2x)}$$

We employ a *linear change of variable*: let

$$u = y + 2x \Rightarrow \frac{du}{dx} = \frac{dy}{dx} + 2 \text{ or, } u' = y' + 2$$

Thus, our differential equation becomes

$$\frac{du}{dx} = \frac{1 - 2u}{1 + u} + 2 \Rightarrow \frac{du}{dx} = \frac{3}{1 + u} \text{ or, } u' = \frac{3}{1 + u}$$

This is now a separable equation!

$$\int (1 + u)\ du = \int 3\ dx \Rightarrow u + \frac{u^2}{2} = 3x + c$$

Since $u = y + 2x$, we thus have the general solution[1]

$$y + 2x + \frac{(y + 2x)^2}{2} = 3x + c$$

[1] This is in *implicit form*. The general solution can be in implicit form.

2. (**Fraction change of variable**) Solve the differential equation

$$2xy\frac{dy}{dx} - y^2 + x^2 = 0 \tag{2}$$

Solution. Observe that we may rewrite the differential equation as

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{1}{2}(\frac{y}{x} - \frac{x}{y}).$$

Consider the substitution $u = \frac{y}{x}$, or $y = xu$, such that $y' = u + xu'$[2]. Substituting this into the differential equation yields

$$u + x\frac{du}{dx} = \frac{1}{2}(u - \frac{1}{u}) \Rightarrow x\frac{du}{dx} = -\frac{1}{2}(u + \frac{1}{u})$$

Observe that this is now a separable differential equation!

$$\int \frac{2u}{u^2+1}\ du = \int -\frac{1}{x}\ dx \Rightarrow \ln(u^2+1) = -\ln x + c$$

Since $u = \frac{y}{x}$, we thus obtain the general solution[3]

$$ln(\frac{y^2}{x} + x) = c \Rightarrow \frac{y^2}{x} + x = A.$$

3. (**Special substitution**) Solve the following differential equation with given initial conditions

$$y'y'' = 2, \text{ with } y(0) = 1 \text{ and } y'(0) = 2. \tag{3}$$

Solution. To reduce the order of the differential equation, let $u = y'$ and we will have $u' = y''$. Substituting this into the differential equation yields

$$u\frac{du}{dx} = 2$$

Observe that this is now a separable differential equation!

$$\int u\ du = \int 2\ dx \Rightarrow \frac{u^2}{2} = 2x + c$$

Since $u = y'$, the substitution yields

$$(y')^2 = 4x + 2c$$

Since $y'(0) = 2$, then $2c = 4$. We can now directly integrate our expression for y' to find y

$$y' = (4x+4)^{\frac{1}{2}} = 2(x+1)^{\frac{1}{2}} \Rightarrow y = \frac{4}{3}(x+1)^{\frac{3}{2}} + D$$

Since $y(0) = 1$, after substitution, we can get

$$1 = \frac{4}{3}(0+1)^{\frac{3}{2}} + D \Rightarrow D = -\frac{1}{3}$$

Therefore, the solution to the initial value problem is given by

$$y = \frac{4}{3}(x+1)^{\frac{3}{2}} - \frac{1}{3}$$

[2] This is done by *product rule*

[3] Here the constant $A = e^c$, which will be determined by the initial condition

1.2 Population Model

1. (**Application of Equilibrium Solutions**) The population of a certain species of bugs behaves according to the Verhulst (logistic) model, and the formula is given below:

$$\frac{dy}{dt} = [k(1 - \frac{y}{y_\infty})]y \tag{4}$$

where $k = -15, y_\infty \approx 375.75$. Now, what is the maximum number of bugs you can put to death per day without causing the population to die out?
Solution. We may now modify the Verhulst equation to account for the harvesting of bugs. Suppose E bugs are put to death per day. Then

$$\frac{dy}{dt} = [k(1 - \frac{y}{y_\infty})]y - E = -\frac{k}{y_\infty}y^2 + ky - E$$

where $k = 1.5$ and $y_\infty \approx 375.75$. Now, the equilibrium solutions are

$$y = \frac{-k \pm \sqrt{k^2 - \frac{4kE}{y_\infty}}}{-\frac{2k}{y_\infty}} = \frac{1.5 \mp \sqrt{2.25 - \frac{6E}{375.75}}}{\frac{3}{375.75}}$$

The population will die out when E is chosen such that there are no equilibria. In this case, $\frac{dy}{dt}$ will be **negative** and any solution $y(t)$ will always be decreasing. This occurs when,

$$k^2 - \frac{4kE}{y_\infty} < 0 \Rightarrow E > \frac{375.75}{6} \times 2.25 = 140.91\ldots$$

Hence, we can only kill as much as 140 bugs per day.

2 Linear Differential Equations

2.1 The method of Integrating Factors

1. Solve the following first-order linear differential equation

$$y' - (1 + 3x^{-1})y = x + 2 \tag{5}$$

Solution. The equation is in standard form. The integrating factor is $e^{\int p(x)\ dx}$, where $p(x) = -1 - 3x^{-1}$. Thus,

$$\int p(x)\ dx = -\int 1 + 3x^{-1}\ dx = -x - 3\ln x$$

and we have the integrating factor

$$e^{\int p(x)\ dx} = e^{-x-3\ln x} = x^{-3}e^{-x}$$

Multiplying this factor to both sides of the equation,

$$(x^{-3}e^{-x})[y' - (1 - \frac{3}{x})y] = (x^{-3}e^{-x})[x + 2]$$
$$(x^{-3}e^{-x}y)' = x^{-2}e^{-x} + 2x^{-3}e^{-x}$$

Integrating both sides of the equation yields[4]

$$x^{-3}e^{-x}y + c = \int x^{-2}e^{-x}\ dx + 2\int x^{-3}e^{-x}\ dx$$

To solve the integral in the second term, we integrate by parts: let $u = e^{-x}, du = -e^{-x}\ dx$, and $dv = x^{-3}\ dx, v = \frac{-x^{-2}}{2}$:[5]

$$2\int x^{-3}e^{-x}\ dx = -e^{-x}x^{-2} - \int x^{-2}e^{-x}\ dx$$

Thus, we have[6]

$$x^{-3}e^{-x}y + c = -e^{-x}x^{-2}$$

2.2 Bernoulli differential equation

1. (**Normal Substitution**) Find the general solution to the following differential equation

$$2xyy' + (x - 1)y^2 = x^2e^x \tag{6}$$

Solution. Expressing the differential equation into Bernoulli form to find our v, we have

$$2xy' + (x - 1)y = x^2e^xy^{-1}$$

Now, we can let $v = y^2$, and $v' = 2yy'$, substituting them into (6) yields[7]

$$xv' + (x - 1)v = x^2e^x \Rightarrow v' + (1 - x^{-1})v = xe^x$$

We have the integrating factor $e^{\int 1-\frac{1}{x}\ dx} = e^{x-\ln x} = e^xx^{-1}$. Multiplying this to both sides of the equation,

$$(e^xx^{-1})[v' + (1 - x^{-1})v] = (\frac{e^x}{x})xe^x$$
$$(e^xx^{-1}v)' = e^{2x}$$

Integrating both sides, we have

$$\frac{e^x}{x}y^2 = \frac{1}{2}e^{2x} + c$$

[4] Here, integrate R.H.S w.r.t x. L.H.S not sure

[5] Note that $\int e^{-x}x^{-2}\ dx$ is unsolvable using my limited knowledge.

[6] Here we cancel off the term $\int x^{-2}e^{-x}\ dx$

[7] Afterwards, we should write the equation in v in the linear form, which is the coefficient of v' must be 1

2. (**Special Substitution**) Find the general solution to the following differential equation

$$x^2 + \sin y = y' \cos y \tag{7}$$

Solution. This is a Bernoulli equation: letting $v = \sin y, v' = y' \cos y$, the differential equation becomes

$$x^2 + v = v' \Rightarrow v' - v = x^2$$

We have the integrating factor $e^{\int -1 \; dx} = e^{-x}$. Multiplying this to both sides of the equation,

$$e^{-x}[v' - v] = e^{-x}[x^2] \Rightarrow (e^{-x}v)' = e^{-x}x^2$$

We integrate the right-hand side by parts:

$$\int e^{-x}x^2 \; dx = -x^2e^{-x} + 2\int e^{-x}x \; dx = -x^2e^{-x} + 2(-xe^{-x} - e^{-x}) + c$$

Substituting back $v = \sin y$, we find that

$$e^{-x} \sin y = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + c$$

3 The Harmonic Oscillator

3.1 The method of undetermined coefficients

1. (**Don't have Trigonometric Terms**) Find a particular solution to

$$y''' - 2y'' + y' = 2x \tag{8}$$

Solution. Firstly, the characteristic equation is $\lambda^3 - 2\lambda^2 + \lambda = \lambda(\lambda - 1)^2 = 0$, so

$$y_h = c_1 + c_2 e^x + c_3 x e^x$$

To find the particular solution, try $Ax + B$. Since a constant term appears in y_h (c_1), we modify this trial solution: instead, try $y = Ax^2 + Bx$. Then,

$$y' = 2Ax + b, y'' = 2A, y''' = 0$$

Plugging this into the equation yields

$$0 - 4A + 2Ax + B = 2x \Rightarrow 2Ax + (-4A + B) = 2x + 0$$

Comparing coefficients, we have $A = 1$ and $B = 4A = 4$. Thus,

$$y_p = x^2 + 4x$$

2. (**Have Trigonometric Terms**) Find a particular solution to

$$y'' + 2y' + 3y = 34e^x \cos 2x \tag{9}$$

Solution. First, we observe that $f(x) = 34e^x \cos 2x$ is equal to

$$\mathfrak{Re}(34e^x e^{i2x}) = \mathfrak{Re}(34e^{x(1+2i)})$$

Thus, we can equivalently solve for

$$y'' + 2y' + 3y = 34e^{x(1+2i)}$$

noting that we only want the real part of this solution. We try

$$\begin{aligned} y_p &= Ae^{x(1+2i)}, \\ y_p' &= A(1+2i)e^{x(1+2i)}, \\ y_p'' &= A(1+2i)^2 e^{x(1+2i)} = A(-3+4i)e^{x(1+2i)}, \end{aligned}$$

Substituting this into the equation, we have

$$A(-3+4i)e^{x(1+2i)} + 2A(1+2i)e^{x(1+2i)} + 3Ae^{x(1+2i)} = 34e^{x(1+2i)}$$

and $A(2 + 8i) = 34$. Simplifying this, we find

$$A = \frac{34}{2+8i} = \frac{17}{1+4i} \cdot \frac{1-4i}{1-4i} = \frac{17-68i}{17} = 1 - 4i \tag{10}$$

So,

$$\begin{aligned} y_p = (1-4i)e^{x(1+2i)} &= e^x(1-4i)(\cos 2x + i \sin 2x) \\ &= e^x(\cos 2x + 4\sin 2x + i(-4\cos 2x + \sin 2x)) \end{aligned}$$

Thus, a particular solution is given by

$$\mathfrak{Re}(y_p) = e^x(\cos 2x + 4\sin 2x)$$

4 The Laplace Transform

4.1 Unit Step Functions and Dirac Delta Functions

1. (**Application of Laplace transform on Unit Step Functions**) Do the Laplace Transform on the following unit step function

$$\frac{t^2}{2} \cdot u(t-1) \tag{11}$$

Solution. We apply the Second Shifting Theorem[8]

$$\begin{aligned} \mathcal{L}[\frac{t^2}{2} \cdot u(t-1)] &= \frac{1}{2} e^{-s} \mathcal{L}[(t+1)^2] \\ &= \frac{e^{-s}}{2} \mathcal{L}[t^2 + 2t + 1] \\ &= \frac{e^{-s}}{2} (\frac{2!}{s^3} + \frac{2}{s^2} + \frac{1}{s}) \end{aligned}$$

2. (**Application of the Inverse Laplace Transform**) Solve the following initial value problem

$$y' = tu(t-2), y(0) = 4 \tag{12}$$

Solution. Take the Laplace Transform of both sides yields

$$s\mathcal{L}[y] - y(0) = \mathcal{L}[tu(t-2)]$$

To compute the right-hand side, we apply the Second-Shifting Theorem to $f(t) = t$

$$\mathcal{L}[tu(t-2)] = e^{-2s} \mathcal{L}[t+2] = e^{-2s} (\frac{1}{s^2} + \frac{2}{s})$$

Thus, we have

$$s\mathcal{L}[y] - 4 = e^{-2s} (\frac{1}{s^2} + \frac{2}{s}) \Rightarrow \mathcal{L}[y] = e^{-2s} (\frac{1}{s^3} + \frac{2}{s^2}) + \frac{4}{s}$$

We now take the inverse Laplace transform of both sides. We evaluate the first term on the right: by the Second Shifting Theorem,

$$\mathcal{L}^{-1}[e^{-2s} (\frac{1}{s^3} + \frac{2}{s^2})] = f(t-2)u(t-2), \text{ where } \mathcal{L}[f(t)] = \frac{1}{s^3} + \frac{2}{s^2}$$

Since

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[\frac{1}{s^3}] + \mathcal{L}^{-1}[\frac{2}{s^2}] = \frac{t^2}{2} + 2t$$

we find that,

$$f(t-2) = \frac{(t-2)^2}{2} + 2(t-2) = \frac{(t-2)(t+2)}{2} = \frac{t^2 - 4}{2}$$

Thus,

$$\mathcal{L}^{-1}[e^{-2s} \cdot \frac{1+2s}{s^3}] = (\frac{t^2}{2} - 2) \cdot u(t-2)$$

Thus, the solution to the initial value problem is

$$y(t) = (\frac{t^2}{2} - 2) \cdot u(t-2) + 4$$

[8] $\mathcal{L}[f(t-c)u(t-c)] = e^{-sc} \mathcal{L}[f(t)]$ or $\mathcal{L}[f(t)u(t-c)] = e^{-sc} \mathcal{L}[f(t+c)]$

5 Partial Differential Equations

5.1 The method of Separation of variables

1. (PDE becomes fake ODE) Find a particular solution $u(x, y)$ to the partial differential equation

$$u_{xx} - 2u_x - 3u = 0$$

given the initial conditions $u(0, y) = 2y$ and $u_x = (0, y) = 2y - 4\sin y$ *Hint: The partial differential equation only involves derivatives in x*

Solution. We treat the partial differential equation as ordinary differential equation in x. Then, we have a second-order homogeneous linear differential equation:

$$u'' - 2u' - 3u = 0 \Rightarrow \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3) = 0$$

The partial differential equation thus has the general solution

$$u(x, y) = f(y)e^{-x} + g(y)e^{3x}$$

where f and g are functions of y (i.e., constants with respect to x). By the initial conditions, we have

$$u(0, y) = f(y) + g(y) = 2y$$

Likewise, taking the partial derivative of u with respect to x,

$$u_x = -fe^{-x} + 3ge^{3x} \Rightarrow u_x(0, y) = -f(y) + 3g(y) = 2y - 4\sin y$$

We thus require that

$$\begin{cases} f + g & = 2y \\ -f + 3g & = 2y - 4\sin y \end{cases}$$

Solve it, we have

$$\begin{cases} f(y) & = y + \sin y \\ g(y) & = y - \sin y \end{cases}$$

Hence, the desired solution to the partial differential equation is

$$u(x, y) = (y + \sin y)e^{-x} + (y - \sin y)e^{3x}$$

5.2 The Heat Equation

1. (**Superposition Principle**) Consider the following partial differential equation:

$$u_t = 2u_{xx}, 0 \leq x \leq 3, t > 0 \tag{13}$$

Solve the above equation with the given boundary and initial conditions

$$u(0, t) = u(3, t) = 0, u(x, 0) = \frac{5}{8}\sin \pi x - \frac{5}{16}\sin 3\pi x + \frac{1}{16}\sin 5\pi x$$

Solution. This is simply the heat equation with $c^2 = 2, l = 3$. Hence, our solutions will have the form

$$u(x, t) = \beta_n e^{-2n^2\pi^2 t/9} \sin(\frac{n\pi}{3}x)$$

By the initial condition $u(x, 0)$, we have at $t = 0$

$$\beta_n \sin(\frac{n\pi}{3}x) = \frac{5}{8}\sin \pi x - \frac{5}{16}\sin 3\pi x + \frac{1}{16}\sin 5\pi x$$

By the Fundamental Theorem of Superposition, we can simply combine the solutions that produces each term above. Comparing coefficients we have $n = 3, n = 9$, and $n = 15$, with $\beta_3 = 5/8, \beta_9 = -5/16$, and $\beta_{15} = 1/16$, respectively. Hence, we have the particular solution

$$u(x, t) = \frac{5}{8}e^{-2\pi^2 t}\sin(\pi x) - \frac{5}{16}e^{-18\pi^2 t}\sin(3\pi x) + \frac{1}{16}e^{-50\pi^2 t}\sin(5\pi x)$$