Bayesian inference and data assimilation, SS2019
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Please send questions about the exercise sheet to hanlie@uni-potsdam.de. E-mails will be replied to during working hours.

You may use results from the previous Exercise Sheets. Justify each step in your proofs. When giving the formula of a function $f : \mathbb{R} \to \mathbb{R}$, you must specify the value of f over all of $\mathbb{R}$.

Exercise Sheet 2 (Due date: Thursday, 09.05.2019)

Exercise 1. (Expectations for discrete random variables; **4 points**).

(a) Given $p \in (0,1)$ and $X \sim \text{Bernoulli}(p)$, prove that $\mathbb{E}[X] = p$ and $\mathbb{V}[X] = p(1-p)$. (**1 point**)

(b) Given $p \in (0,1)$ and $n \in \mathbb{N}$ and $X \sim \text{Binomial}(n,p)$, prove that $\mathbb{E}[X] = np$ and $\mathbb{V}[X] = np(1-p)$. (**1 point**)

(c) Given $\lambda > 0$ and $X \sim \text{Poisson}(\lambda)$, prove that $\mathbb{E}[X] = \lambda$ and $\mathbb{V}[X] = \lambda$. Hint: use the series expansion for the exponential. (**2 points**)

Exercise 2. (Expectations for continuous random variables; **4.5 points**).

For the following random variables, (i) derive the formula for the probability density function (PDF) f_X of X as a function on $\mathbb{R}$ from the formula for the cumulative distribution function (CDF) F_X, and (ii) use the formula for the PDF to prove the given formulas for the expectation $\mathbb{E}[X]$ and variance $\mathbb{V}[X]$.

(a) Fix $a < b$ for $a, b \in \mathbb{R}$ and let $X \sim \text{Uniform}([a,b])$. Then $\mathbb{E}[X] = \frac{b+a}{2}$ and $\mathbb{V}[X] = \frac{(b-a)^2}{12}$. (**1.5 points**)

(b) Let $\lambda > 0$ and $X \sim \text{Exponential}(\lambda)$. Then $\mathbb{E}[X] = \frac{1}{\lambda}$ and $\mathbb{V}[X] = \frac{1}{\lambda^2}$. (**1.5 points**)

(c) Let $m \in \mathbb{R}$, $\sigma > 0$, and $X \sim \text{Normal}(m, \sigma^2)$. Then $\mathbb{E}[X] = m$ and $\mathbb{V}[X] = \sigma^2$. (**1.5 points**)

Exercise 3. (Functions of random variables; **5 points**)

(a) Let $a < b$ for $a, b \in \mathbb{R}$, $X \sim \text{Uniform}([a,b])$, and $\alpha > 0$, $\beta \in \mathbb{R}$ be fixed but arbitrary deterministic numbers. Let $Y := \alpha X + \beta$.

  (i) Compute $\mathbb{E}[Y]$ and $\mathbb{V}[Y]$. (**1 point**)

  (ii) Give the formula for the CDF F_Y as a function on $\mathbb{R}$. (**1 point**)

  (iii) Use your answer to part (ii) to give the formula for the PDF f_Y as a function on $\mathbb{R}$. (**0.5 point**)

(b) Let $a > 0$ be an arbitrary, fixed, nondeterministic number, $X \sim \text{Uniform}([-a, +a])$, and $Y := X^2$. Repeat parts (i), (ii), and (iii) for this Y. (**2.5 points**)

Exercise 4. (Conditioning; **4.5 points**)
You are given two random variables X, Y on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $X : \Omega \to \{0, 1, 2\}$ and $Y : \Omega \to \{0, 1\}$, and the following probabilities:

$$\begin{aligned} &\mathbb{P}(X = 0) = 0.07, && \mathbb{P}(X = 1) = 0.03, && \mathbb{P}(X = 2) = 0.9, \\ &\mathbb{P}(Y = 1|X = 0) = 0.95, && \mathbb{P}(Y = 1|X = 1) = 1.0, && \mathbb{P}(Y = 1|X = 2) = 0.1. \end{aligned}$$

(For the following, round your answers to four decimal places, e.g. if some quantity $a = 0.93215...$ then write '$a \approx 0.9322$', or if $a = 0.12424...$ write '$a \approx 0.1242$'.)

(a) Compute $\mathbb{P}(Y = 0)$ and $\mathbb{P}(Y = 1)$. (**1 point**)

(b) Compute the posterior probabilities of X assuming each of the values $0, 1, 2$ given that $Y = 1$, i.e. compute $\mathbb{P}(X = x|Y = 1)$ for $x = 0, 1, 2$. (**1.5 points**)

(c) Compute $\mathbb{P}(X = x|Y = 0)$ for $x = 0, 1, 2$. (**1 point**)

(d) Compute the conditional expectations $\mathbb{E}[X|Y = 0]$ and $\mathbb{E}[X|Y = 1]$, using the fact that the conditional probability measures $\mathbb{P}(\cdot|Y = 0)$ and $\mathbb{P}(\cdot|Y = 1)$ are probability measures. Compare $\mathbb{E}[X|Y = 0]$ and $\mathbb{E}[X|Y = 1]$ with $\mathbb{E}[X]$. (**1 point**)

Exercise 5. (Practice with independence; **2 points**)

(a) Let $X, Y : \Omega \to \mathbb{R}$ be independent, discrete random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{V}[X], \mathbb{V}[Y] < \infty$. Prove that

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]. \tag{1}$$

Note: You must use the formula for the expectation of discrete random variables and the definition of independence of discrete random variables in terms of the joint probability distribution (**1 point**).

(b) Let $X, Y : \Omega \to \{0, +1, -1\}$ be discrete random variables such that

$$\begin{aligned} \mathbb{P}(X = +1, Y = 0) = \mathbb{P}(X = -1, Y = 0) = \frac{1}{4}, \\ \mathbb{P}(X = 0, Y = +1) = \mathbb{P}(X = 0, Y = -1) = \frac{1}{4}, \end{aligned}$$

and all other combinations of X- and Y-values have probability zero. In other words, the random variable $(X, Y)^\top : \Omega \to \mathbb{R}^2$ satisfies

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \text{Uniform}\left(\left\{\begin{pmatrix} +1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ +1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}\right).$$

Show that (1) is satisfied (**0.5 point**). Show that X and Y are not independent (**0.5 point**). What do you conclude from this example?