

Trabajo Práctico N° 6: Estimación Puntual.

Ejercicio 1.

Suponer que se tiene una muestra aleatoria de tamaño n tomada de una población X , que $E(X) = \mu$ y $V(X) = \sigma^2$. Sean

$$\bar{X}_1 = \frac{1}{n-1} \sum_{i=1}^{n-1} X_i \text{ y } \bar{X}_2 = \frac{1}{n} \sum_{i=1}^n X_i$$

dos estimadores de μ . ¿Cuál es el mejor estimador de μ ? Explicar la elección.

$$E(\bar{X}_1) = E\left(\frac{1}{n-1} \sum_{i=1}^{n-1} X_i\right)$$

$$E(\bar{X}_1) = \frac{1}{n-1} E\left(\sum_{i=1}^{n-1} X_i\right)$$

$$E(\bar{X}_1) = \frac{1}{n-1} \sum_{i=1}^{n-1} E(X_i)$$

por (*)

$$E(\bar{X}_1) = \frac{1}{n-1} \sum_{i=1}^{n-1} \mu$$

$$E(\bar{X}_1) = \frac{1}{n-1} (n - 1) \mu$$

$$E(\bar{X}_1) = \mu.$$

$$V(\bar{X}_1) = V\left(\frac{1}{n-1} \sum_{i=1}^{n-1} X_i\right)$$

$$V(\bar{X}_1) = \left(\frac{1}{n-1}\right)^2 V\left(\sum_{i=1}^{n-1} X_i\right)$$

$$V(\bar{X}_1) = \frac{1}{(n-1)^2} \sum_{i=1}^{n-1} V(X_i)$$

por (**)

$$V(\bar{X}_1) = \frac{1}{(n-1)^2} \sum_{i=1}^{n-1} \sigma^2$$

$$V(\bar{X}_1) = \frac{1}{(n-1)^2} (n - 1) \sigma^2$$

$$V(\bar{X}_1) = \frac{\sigma^2}{n-1}.$$

$$ECM(\bar{X}_1) = \{E[\bar{X}_1 - E(\bar{X}_1)]\}^2 + V(\bar{X}_1)$$

$$ECM(\bar{X}_1) = [E(\bar{X}_1) - \mu]^2 + \frac{\sigma^2}{n-1}$$

$$ECM(\bar{X}_1) = [E(\bar{X}_1) - E(\mu)]^2 + \frac{\sigma^2}{n-1}$$

$$ECM(\bar{X}_1) = (\mu - \mu)^2 + \frac{\sigma^2}{n-1}$$

$$ECM(\bar{X}_1) = 0^2 + \frac{\sigma^2}{n-1}$$

$$ECM(\bar{X}_1) = 0 + \frac{\sigma^2}{n-1}$$

$$ECM(\bar{X}_1) = \frac{\sigma^2}{n-1}.$$

$$E(\bar{X}_2) = E\left(\frac{1}{n} \sum_{i=1}^n X_i\right)$$

$$E(\bar{X}_2) = \frac{1}{n} E\left(\sum_{i=1}^n X_i\right)$$

$$E(\bar{X}_2) = \frac{1}{n} \sum_{i=1}^n E(X_i)$$

por (*)

$$E(\bar{X}_2) = \frac{1}{n} \sum_{i=1}^n \mu$$

$$E(\bar{X}_2) = \frac{1}{n} n\mu$$

$$E(\bar{X}_2) = \mu.$$

$$V(\bar{X}_2) = V\left(\frac{1}{n} \sum_{i=1}^n X_i\right)$$

$$V(\bar{X}_2) = \left(\frac{1}{n}\right)^2 V\left(\sum_{i=1}^n X_i\right)$$

$$V(\bar{X}_2) = \frac{1}{n^2} \sum_{i=1}^n V(X_i)$$

por (**)

$$V(\bar{X}_2) = \frac{1}{n^2} \sum_{i=1}^n \sigma^2$$

$$V(\bar{X}_2) = \frac{1}{n^2} n\sigma^2$$

$$V(\bar{X}_2) = \frac{\sigma^2}{n}.$$

(*) propiedad de linealidad de la esperanza.

(**) propiedad de la varianza e independencia.

$$ECM(\bar{X}_2) = \{E[\bar{X}_2 - E(\bar{X}_2)]\}^2 + V(\bar{X}_2)$$

$$ECM(\bar{X}_2) = [E(\bar{X}_2 - \mu)]^2 + \frac{\sigma^2}{n}$$

$$ECM(\bar{X}_2) = [E(\bar{X}_2) - E(\mu)]^2 + \frac{\sigma^2}{n}$$

$$ECM(\bar{X}_2) = (\mu - \mu)^2 + \frac{\sigma^2}{n}$$

$$ECM(\bar{X}_2) = 0^2 + \frac{\sigma^2}{n}$$

$$ECM(\bar{X}_2) = 0 + \frac{\sigma^2}{n}$$

$$ECM(\bar{X}_2) = \frac{\sigma^2}{n-1}.$$

$$ECM(\bar{X}_1) = \frac{\sigma^2}{n-1} < ECM(\bar{X}_2) = \frac{\sigma^2}{n}.$$

Por lo tanto, el mejor estimador de μ es \bar{X}_2 , ya que tiene menor error cuadrático medio.

Ejercicio 2.

Sea X_1, X_2, \dots, X_7 una muestra aleatoria de una población que tiene media μ y varianza σ^2 . Considerar los siguientes estimadores de μ :

$$\widehat{\Theta}_1 = \frac{X_1 + X_2 + \dots + X_7}{7}; \quad \widehat{\Theta}_2 = \frac{2X_1 - X_6 + X_4}{2}; \quad \widehat{\Theta}_3 = \frac{2X_1 - X_7 + X_3}{3}.$$

(a) ¿Alguno de estos estimadores es insesgado?

$$\begin{aligned} E(\widehat{\Theta}_1) &= E\left(\frac{X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7}{7}\right) \\ E(\widehat{\Theta}_1) &= \frac{1}{7} E(X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7) \\ E(\widehat{\Theta}_1) &= \frac{1}{7} [E(X_1) + E(X_2) + E(X_3) + E(X_4) + E(X_5) + E(X_6) + E(X_7)] \quad \text{por (*)} \\ E(\widehat{\Theta}_1) &= \frac{1}{7} 7 E(X_1) \\ E(\widehat{\Theta}_1) &= \mu. \end{aligned}$$

$$\begin{aligned} E(\widehat{\Theta}_2) &= E\left(\frac{2X_1 - X_6 + X_4}{2}\right) \\ E(\widehat{\Theta}_2) &= \frac{1}{2} E(2X_1 - X_6 + X_4) \\ E(\widehat{\Theta}_2) &= \frac{1}{2} [E(2X_1) - E(X_6) + E(X_4)] \quad \text{por (*)} \\ E(\widehat{\Theta}_2) &= \frac{1}{2} [2 E(X_1) - \mu + \mu] \\ E(\widehat{\Theta}_2) &= \frac{1}{2} (2\mu - \mu + \mu) \\ E(\widehat{\Theta}_2) &= \frac{1}{2} 2\mu \\ E(\widehat{\Theta}_2) &= \mu. \end{aligned}$$

$$\begin{aligned} E(\widehat{\Theta}_3) &= E\left(\frac{2X_1 - X_7 + X_3}{3}\right) \\ E(\widehat{\Theta}_3) &= \frac{1}{3} E(2X_1 - X_7 + X_3) \\ E(\widehat{\Theta}_3) &= \frac{1}{3} [E(2X_1) - E(X_7) + E(X_3)] \quad \text{por (*)} \\ E(\widehat{\Theta}_3) &= \frac{1}{3} [2 E(X_1) - \mu + \mu] \\ E(\widehat{\Theta}_3) &= \frac{1}{3} (2\mu - \mu + \mu) \\ E(\widehat{\Theta}_3) &= \frac{2}{3} \mu. \end{aligned}$$

(*) propiedad de linealidad de la esperanza.

Por lo tanto, $\widehat{\Theta}_1$ y $\widehat{\Theta}_2$ son insesgados.

(b) Hallar el error cuadrático medio de los estimadores.

$$\begin{aligned} V(\widehat{\Theta}_1) &= V\left(\frac{X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7}{7}\right) \\ V(\widehat{\Theta}_1) &= \frac{1}{49} V(X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7) \end{aligned}$$

$$V(\widehat{\Theta}_1) = \frac{1}{49} [V(X_1) + V(X_2) + V(X_3) + V(X_4) + V(X_5) + V(X_6) + V(X_7)] \quad \text{por}$$

(**)

$$V(\widehat{\Theta}_1) = \frac{1}{49} 7 V(X_1)$$

$$V(\widehat{\Theta}_1) = \frac{1}{7} \sigma^2.$$

$$V(\widehat{\Theta}_2) = V\left(\frac{2X_1 - X_6 + X_4}{2}\right)$$

$$V(\widehat{\Theta}_2) = \frac{1}{4} V(2X_1 - X_6 + X_4)$$

$$V(\widehat{\Theta}_2) = \frac{1}{4} [V(2X_1) + V(X_6) + V(X_4)] \quad \text{por (**)}$$

$$V(\widehat{\Theta}_2) = \frac{1}{4} [4 V(X_1) + \sigma^2 + \sigma^2]$$

$$V(\widehat{\Theta}_2) = \frac{1}{4} (4\sigma^2 + \sigma^2 + \sigma^2)$$

$$V(\widehat{\Theta}_2) = \frac{1}{4} 6\sigma^2$$

$$V(\widehat{\Theta}_2) = \frac{3}{2} \sigma^2.$$

$$V(\widehat{\Theta}_3) = V\left(\frac{2X_1 - X_7 + X_3}{3}\right)$$

$$V(\widehat{\Theta}_3) = \frac{1}{9} V(2X_1 + X_7 + X_3)$$

$$V(\widehat{\Theta}_3) = \frac{1}{9} [V(2X_1) + V(X_7) + V(X_3)] \quad \text{por (**)}$$

$$V(\widehat{\Theta}_3) = \frac{1}{9} [4 V(X_1) + \sigma^2 + \sigma^2]$$

$$V(\widehat{\Theta}_3) = \frac{1}{9} (4\sigma^2 + \sigma^2 + \sigma^2)$$

$$V(\widehat{\Theta}_3) = \frac{1}{9} 6\sigma^2$$

$$V(\widehat{\Theta}_3) = \frac{2}{3} \sigma^2.$$

(**) propiedad de la varianza e independencia.

$$ECM(\widehat{\Theta}_1) = \{E[\widehat{\Theta}_1 - E(\widehat{\Theta}_1)]\}^2 + V(\widehat{\Theta}_1)$$

$$ECM(\widehat{\Theta}_1) = [E(\widehat{\Theta}_1) - \mu]^2 + \frac{1}{7} \sigma^2$$

$$ECM(\widehat{\Theta}_1) = [E(\widehat{\Theta}_1) - E(\mu)]^2 + \frac{1}{7} \sigma^2$$

$$ECM(\widehat{\Theta}_1) = (\mu - \mu)^2 + \frac{1}{7} \sigma^2$$

$$ECM(\widehat{\Theta}_1) = 0^2 + \frac{1}{7} \sigma^2$$

$$ECM(\widehat{\Theta}_1) = 0 + \frac{1}{7} \sigma^2$$

$$ECM(\widehat{\Theta}_1) = \frac{1}{7} \sigma^2.$$

$$ECM(\widehat{\Theta}_2) = \{E[\widehat{\Theta}_2 - E(\widehat{\Theta}_2)]\}^2 + V(\widehat{\Theta}_2)$$

$$ECM(\widehat{\Theta}_2) = [E(\widehat{\Theta}_2) - \mu]^2 + \frac{3}{2} \sigma^2$$

$$ECM(\widehat{\Theta}_2) = [E(\widehat{\Theta}_2) - E(\mu)]^2 + \frac{3}{2} \sigma^2$$

$$ECM(\widehat{\Theta}_2) = (\mu - \mu)^2 + \frac{3}{2} \sigma^2$$

$$ECM(\widehat{\Theta}_2) = 0^2 + \frac{3}{2} \sigma^2$$

$$ECM(\widehat{\Theta}_2) = 0 + \frac{3}{2} \sigma^2$$

$$\text{ECM}(\widehat{\Theta}_2) = \frac{3}{2}\sigma^2.$$

$$\text{ECM}(\widehat{\Theta}_3) = \{E[\widehat{\Theta}_3] - E(\widehat{\Theta}_3)\}^2 + V(\widehat{\Theta}_3)$$

$$\text{ECM}(\widehat{\Theta}_3) = [E(\widehat{\Theta}_3) - \mu]^2 + \frac{2}{3}\sigma^2$$

$$\text{ECM}(\widehat{\Theta}_3) = [E(\widehat{\Theta}_3) - E(\mu)]^2 + \frac{2}{3}\sigma^2$$

$$\text{ECM}(\widehat{\Theta}_3) = (\frac{2}{3}\mu - \mu)^2 + \frac{2}{3}\sigma^2$$

$$\text{ECM}(\widehat{\Theta}_3) = (\frac{-1}{3}\mu)^2 + \frac{2}{3}\sigma^2$$

$$\text{ECM}(\widehat{\Theta}_3) = \frac{1}{9}\mu + \frac{2}{3}\sigma^2$$

$$\text{ECM}(\widehat{\Theta}_3) = \frac{1}{9}\mu + \frac{2}{3}\sigma^2.$$

(c) ¿Cuál estimador es el “mejor”? ¿En qué sentido es mejor?

$$\text{ECM}(\widehat{\Theta}_1) = \frac{1}{7}\sigma^2 < \text{ECM}(\widehat{\Theta}_2) = \frac{3}{2}\sigma^2.$$

$$\text{ECM}(\widehat{\Theta}_1) = \frac{1}{7}\sigma^2 < \text{ECM}(\widehat{\Theta}_3) = \frac{1}{9}\mu + \frac{2}{3}\sigma^2.$$

El “mejor” estimador es $\widehat{\Theta}_1$, ya que tiene menor error cuadrático medio.

Ejercicio 3.

Sea X_1, X_2, \dots, X_n una muestra aleatoria de tamaño n .

(a) Demostrar que \bar{X}^2 es un estimador sesgado de μ^2 .

Media poblacional de X_i , $i=1, 2, \dots, n$: μ .

Varianza poblacional de X_i , $i=1, 2, \dots, n$: σ^2 .

$$E(\bar{X}^2) = V(\bar{X}) + [E(\bar{X})]^2$$

$$E(\bar{X}^2) = V\left(\frac{\sum_{i=1}^n X_i}{n}\right) + [E\left(\frac{\sum_{i=1}^n X_i}{n}\right)]^2$$

$$E(\bar{X}^2) = \frac{1}{n^2} V(\sum_{i=1}^n X_i) + [\frac{1}{n} E(\sum_{i=1}^n X_i)]^2$$

$$E(\bar{X}^2) = \frac{1}{n^2} \sum_{i=1}^n V(X_i) + [\frac{1}{n} \sum_{i=1}^n E(X_i)]^2$$

por (*)

$$E(\bar{X}^2) = \frac{1}{n^2} \sum_{i=1}^n \sigma^2 + [\frac{1}{n} \sum_{i=1}^n \mu]^2$$

$$E(\bar{X}^2) = \frac{1}{n^2} n \sigma^2 + (\frac{1}{n} n \mu)^2$$

$$E(\bar{X}^2) = \frac{\sigma^2}{n} + \mu^2.$$

(*) propiedad de linealidad de la esperanza.

Por lo tanto, \bar{X}^2 es un estimador sesgado de μ^2 .

(b) Determinar la magnitud del sesgo de este estimador.

$$\text{Sesgo } (\bar{X}^2) = E(\bar{X}^2) - \mu^2$$

$$\text{Sesgo } (\bar{X}^2) = \frac{\sigma^2}{n} + \mu^2 - \mu^2$$

$$\text{Sesgo } (\bar{X}^2) = \frac{\sigma^2}{n}.$$

(c) ¿Qué sucede con el sesgo a medida que aumenta el tamaño de n de la muestra?

$$\lim_{n \rightarrow +\infty} \text{Sesgo } (\bar{X}^2) = \lim_{n \rightarrow +\infty} \frac{\sigma^2}{n}$$

$$\lim_{n \rightarrow +\infty} \text{Sesgo } (\bar{X}^2) = 0.$$

A medida que aumenta el tamaño de n de la muestra, el sesgo tiende a cero.

Ejercicio 4.

El número diario de desconexiones accidentales de un servidor sigue una distribución de Poisson. En cinco días, se observan: 2, 5, 3, 3, 7 desconexiones accidentales.

(a) Obtener el estimador de máxima verosimilitud de λ . ¿El estimador es insesgado? ¿Es consistente?

$$L(\lambda) = \prod_{i=1}^n \frac{e^{-\lambda} \lambda^{x_i}}{x_i!}$$

$$L(\lambda) = \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}.$$

$$\ln L(\lambda) = \ln \left(\frac{e^{-n\lambda} \lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} \right)$$

$$\ln L(\lambda) = \ln(e^{-n\lambda}) + \ln(\lambda^{\sum_{i=1}^n x_i}) - \ln(\prod_{i=1}^n x_i!)$$

$$\ln L(\lambda) = -n\lambda \ln e + \sum_{i=1}^n x_i \ln \lambda - \sum_{i=1}^n \ln(x_i!)$$

$$\ln L(\lambda) = -n\lambda * 1 + \sum_{i=1}^n x_i \ln \lambda - \sum_{i=1}^n \ln(x_i!)$$

$$\ln L(\lambda) = -n\lambda + \sum_{i=1}^n x_i \ln \lambda - \sum_{i=1}^n \ln(x_i!).$$

$$\frac{\partial \ln L(\lambda)}{\partial \lambda} = 0$$

$$-n + \frac{\sum_{i=1}^n x_i}{\lambda} = 0$$

$$\frac{\sum_{i=1}^n x_i}{\lambda} = n$$

$$\hat{\lambda}_{EMV} = \frac{\sum_{i=1}^n x_i}{n}$$

$$E(\hat{\lambda}_{EMV}) = E\left(\frac{\sum_{i=1}^n x_i}{n}\right)$$

$$E(\hat{\lambda}_{EMV}) = \frac{1}{n} E(\sum_{i=1}^n x_i)$$

$$E(\hat{\lambda}_{EMV}) = \frac{1}{n} \sum_{i=1}^n E(x_i) \quad \text{por (*)}$$

$$E(\hat{\lambda}_{EMV}) = \frac{1}{n} \sum_{i=1}^n \lambda$$

$$E(\hat{\lambda}_{EMV}) = \frac{1}{n} n\lambda$$

$$E(\hat{\lambda}_{EMV}) = \lambda.$$

$$V(\hat{\lambda}_{EMV}) = V\left(\frac{\sum_{i=1}^n x_i}{n}\right)$$

$$V(\hat{\lambda}_{EMV}) = \frac{1}{n^2} V(\sum_{i=1}^n x_i)$$

$$V(\hat{\lambda}_{EMV}) = \frac{1}{n^2} \sum_{i=1}^n V(x_i) \quad \text{por (**)}$$

$$V(\hat{\lambda}_{EMV}) = \frac{1}{n^2} \sum_{i=1}^n \lambda$$

$$V(\hat{\lambda}_{EMV}) = \frac{1}{n^2} n\lambda$$

$$V(\hat{\lambda}_{EMV}) = \frac{\lambda}{n}$$

(*) propiedad de linealidad de la esperanza.

(**) propiedad de la varianza e independencia.

$$\lim_{n \rightarrow +\infty} E(\hat{\lambda}_{EMV}) = \lambda.$$

$$\lim_{n \rightarrow +\infty} V(\hat{\lambda}_{EMV}) = 0.$$

Por lo tanto, el estimador es insesgado (ya que $E(\hat{\lambda}_{EMV}) = \lambda$) y consistente (ya que $\lim_{n \rightarrow +\infty} E(\hat{\lambda}_{EMV}) = \lambda$ y $\lim_{n \rightarrow +\infty} V(\hat{\lambda}_{EMV}) = 0$).

(b) Obtener la estimación de λ a partir de la muestra dada.

$$\hat{\lambda}_{EMV} = \frac{2+5+3+3+7}{5}$$

$$\hat{\lambda}_{EMV} = \frac{20}{5}$$

$$\hat{\lambda}_{EMV} = 4.$$

(c) Encontrar el estimador de máxima verosimilitud de la probabilidad de que ocurrirán 3 o más desconexiones accidentales y encontrar la estimación de dicha probabilidad a partir de los datos.

$$P(X \geq 3) = 1 - P(X < 3)$$

$$P(X \geq 3) = 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$P(X \geq 3) = 1 - \left(\frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!} \right)$$

$$P(X \geq 3) = 1 - \left(\frac{e^{-\lambda} * 1}{1} + \frac{e^{-\lambda}\lambda}{1} + \frac{e^{-\lambda}\lambda^2}{2} \right)$$

$$P(X \geq 3) = 1 - (e^{-\lambda} + e^{-\lambda}\lambda + \frac{e^{-\lambda}\lambda^2}{2})$$

$$P(X \geq 3) = 1 - e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2} \right).$$

$$\hat{P}_{EMV}(X \geq 3 | \hat{\lambda}_{EMV}) = 1 - e^{-\hat{\lambda}_{EMV}} \left(1 + \hat{\lambda}_{EMV} + \frac{\hat{\lambda}_{EMV}^2}{2} \right) \quad \text{por propiedad de invarianza}$$

$$\hat{P}_{EMV}(X \geq 3 | \hat{\lambda}_{EMV}) = 1 - e^{-4} \left(1 + 4 + \frac{4^2}{2} \right)$$

$$\hat{P}_{EMV}(X \geq 3 | \hat{\lambda}_{EMV}) = 1 - e^{-4} \left(1 + 4 + \frac{16}{2} \right)$$

$$\hat{P}_{EMV}(X \geq 3 | \hat{\lambda}_{EMV}) = 1 - e^{-4} (1 + 4 + 8)$$

$$\hat{P}_{EMV}(X \geq 3 | \hat{\lambda}_{EMV}) = 1 - 13e^{-4}$$

$$\hat{P}_{EMV}(X \geq 3 | \hat{\lambda}_{EMV}) = 1 - 13 * 0,018$$

$$\hat{P}_{EMV}(X \geq 3 | \hat{\lambda}_{EMV}) = 1 - 0,238$$

$$\hat{P}_{EMV}(X \geq 3 | \hat{\lambda}_{EMV}) = 0,762.$$

Ejercicio 5.

(a) Sea X_1, X_2, \dots, X_n una muestra aleatoria de una v.a. $B(1, p)$. Hallar un estimador de máxima verosimilitud (EMV) de p .

$$L(p) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i}$$

$$L(p) = p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i}$$

$$\ln L(p) = \ln [p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i}]$$

$$\ln L(p) = \ln(p^{\sum_{i=1}^n x_i}) + \ln(1-p)^{n-\sum_{i=1}^n x_i}$$

$$\ln L(p) = \sum_{i=1}^n x_i \ln p + (n - \sum_{i=1}^n x_i) \ln(1-p).$$

$$\frac{\partial \ln L(p)}{\partial p} = 0$$

$$\frac{\sum_{i=1}^n x_i}{p} + \frac{n - \sum_{i=1}^n x_i}{1-p} (-1) = 0$$

$$\frac{\sum_{i=1}^n x_i}{p} - \frac{n - \sum_{i=1}^n x_i}{1-p} = 0$$

$$\frac{\sum_{i=1}^n x_i}{p} = \frac{n - \sum_{i=1}^n x_i}{1-p}$$

$$\frac{1-p}{p} = \frac{n - \sum_{i=1}^n x_i}{\sum_{i=1}^n x_i}$$

$$\frac{1}{p} - 1 = \frac{n}{\sum_{i=1}^n x_i} - 1$$

$$\frac{1}{p} = \frac{n}{\sum_{i=1}^n x_i}$$

$$\hat{p}_{EMV} = \frac{\sum_{i=1}^n x_i}{n}$$

(b) Se selecciona una muestra aleatoria de n chips fabricados por cierta compañía. Sea $X =$ el número entre los n que tienen defectos y $p = P$ (el chip tiene defecto). Se supone que sólo se observa X (el número de chips con defectos).

(i) Si $n = 100$ y $x = 5$, ¿cuál es la estimación de p ?

$$\hat{p}_{EMV} = \frac{5}{100}$$

$$\hat{p}_{EMV} = 0,05.$$

Por lo tanto, si $n = 100$ y $x = 5$, la estimación de p es 0,05.

(ii) Si $n = 100$ y $x = 5$, ¿cuál es el EMV de la probabilidad $(1-p)^6$, de que ninguno de los siguientes 6 chips que se examinen tenga defectos?

$$\hat{P}_{EMV} = (1 - \hat{p}_{EMV})^6$$

$$\hat{P}_{EMV} = (1 - 0,05)^6$$

$$\hat{P}_{EMV} = 0,95^6$$

por propiedad de invarianza

$\hat{P}_{EMV} = 0,735$.

Por lo tanto, si $n=100$ y $x=5$, el EMV de la probabilidad $(1-p)^6$ es $0,735$.

Ejercicio 6.

Se denota por X la proporción de tiempo asignado que un estudiante seleccionado al azar emplea trabajando en cierta prueba de actitud, y se supone que la f.d.p. de X es:

$$f(x) = \begin{cases} (2\theta + 1)x^{2\theta}, & 0 \leq x \leq 1 \\ 0, & \text{c. c.} \end{cases} \text{ donde } \theta > \frac{-1}{2}.$$

Una muestra aleatoria de diez estudiantes produce la siguiente información: 0.92, 0.79, 0.90, 0.65, 0.86, 0.47, 0.73, 0.97, 0.94, 0.77.

(a) Utilizar el método de los momentos para obtener un estimador de θ y, luego, calcular la estimación para esta información.

$$\mu = \int_{-\infty}^{+\infty} x(2\theta + 1)x^{2\theta} dx$$

$$\mu = \int_0^1 (2\theta + 1)x^{2\theta+1} dx$$

$$\mu = (2\theta + 1) \int_0^1 x^{2\theta+1} dx$$

$$\mu = (2\theta + 1) \frac{x^{2\theta+2}}{2\theta+2} \Big|_0^1$$

$$\mu = \frac{2\theta+1}{2\theta+2} (1^{2\theta+2} - 0^{2\theta+2})$$

$$\mu = \frac{2\theta+1}{2\theta+2} (1 - 0)$$

$$\mu = \frac{2\theta+1}{2\theta+2} * 1$$

$$\mu = \frac{2\theta+1}{2\theta+2}$$

$$M_1 = \frac{\sum_{i=1}^n X_i}{n}.$$

$$\mu = M_1$$

$$\frac{2\theta+1}{2\theta+2} = \frac{\sum_{i=1}^n X_i}{n}$$

$$2\theta + 1 = (2\theta + 2) \frac{\sum_{i=1}^n X_i}{n}$$

$$2\theta + 1 = 2\theta \frac{\sum_{i=1}^n X_i}{n} + 2 \frac{\sum_{i=1}^n X_i}{n}$$

$$2\theta - 2\theta \frac{\sum_{i=1}^n X_i}{n} = 2 \frac{\sum_{i=1}^n X_i}{n} - 1$$

$$\theta (2 - 2 \frac{\sum_{i=1}^n X_i}{n}) = \frac{2 \sum_{i=1}^n X_i - n}{n}$$

$$\frac{2n - 2 \sum_{i=1}^n X_i}{n} \theta = \frac{2 \sum_{i=1}^n X_i - n}{n}$$

$$\theta = \frac{\frac{n}{2 \sum_{i=1}^n X_i - n}}{\frac{n}{2n - 2 \sum_{i=1}^n X_i}}$$

$$\hat{\theta}_{MM} = \frac{2 \sum_{i=1}^n X_i - n}{2(n - \sum_{i=1}^n X_i)}.$$

$$\hat{\theta}_{MM} = \frac{2*8-10}{2(10-8)}$$

$$\hat{\theta}_{MM} = \frac{16-10}{2*2}$$

$$\begin{aligned}\hat{\theta}_{MM} &= \frac{6}{4} \\ \hat{\theta}_{MM} &= \frac{3}{2} \\ \hat{\theta}_{MM} &= 1,5.\end{aligned}$$

(b) Obtener el EMV de θ y, luego, calcular la estimación para la información dada.

$$\begin{aligned}L(\theta) &= \prod_{i=1}^n (2\theta + 1) x_i^{2\theta} \\ L(\theta) &= (2\theta + 1)^n \prod_{i=1}^n x_i^{2\theta}.\end{aligned}$$

$$\begin{aligned}\ln L(\theta) &= \ln [(2\theta + 1)^n \prod_{i=1}^n x_i^{2\theta}] \\ \ln L(\theta) &= \ln (2\theta + 1)^n + \ln (\prod_{i=1}^n x_i^{2\theta}) \\ \ln L(\theta) &= n \ln (2\theta + 1) + \sum_{i=1}^n \ln x_i^{2\theta} \\ \ln L(\theta) &= n \ln (2\theta + 1) + 2\theta \sum_{i=1}^n \ln x_i.\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln L(\theta)}{\partial \theta} &= 0 \\ \frac{n}{2\theta+1} * 2 + 2 \sum_{i=1}^n \ln x_i &= 0 \\ \frac{2n}{2\theta+1} &= -2 \sum_{i=1}^n \ln x_i \\ 2\theta + 1 &= \frac{2n}{-2 \sum_{i=1}^n \ln x_i} \\ 2\theta + 1 &= \frac{-n}{\sum_{i=1}^n \ln x_i} \\ 2\theta &= \frac{-n}{\sum_{i=1}^n \ln x_i} - 1 \\ \hat{\theta}_{EMV} &= \frac{-n}{2 \sum_{i=1}^n \ln x_i} - \frac{1}{2}.\end{aligned}$$

$$\begin{aligned}\hat{\theta}_{EMV} &= \frac{-10}{2(-2,43)} - \frac{1}{2} \\ \hat{\theta}_{EMV} &= \frac{-10}{-4,86} - \frac{1}{2} \\ \hat{\theta}_{EMV} &= 2,06 - \frac{1}{2} \\ \hat{\theta}_{EMV} &= 1,56.\end{aligned}$$

Ejercicio 7.

Sea X_1, X_2, \dots, X_n una muestra aleatoria de una v.a. $\mathcal{N}(\mu, \sigma^2)$.

(a) Hallar los estimadores de μ y σ^2 por el método de los momentos. ¿Los estimadores son insesgados?

$$\begin{aligned}\mu_1 &= M_1 \\ \hat{\mu}_{EMM} &= \frac{\sum_{i=1}^n X_i}{n}.\end{aligned}$$

$$\begin{aligned}\sigma^2 + \mu^2 &= M_2 \\ \sigma^2 + \mu^2 &= \frac{\sum_{i=1}^n X_i^2}{n} \\ \sigma^2 &= \frac{\sum_{i=1}^n X_i^2}{n} - \hat{\mu}_{EMM}^2 \\ \hat{\sigma}_{EMM}^2 &= \frac{\sum_{i=1}^n X_i^2}{n} - \left(\frac{\sum_{i=1}^n X_i}{n}\right)^2.\end{aligned}$$

$$\begin{aligned}E(\hat{\mu}_{EMM}) &= E\left(\frac{\sum_{i=1}^n X_i}{n}\right) \\ E(\hat{\mu}_{EMM}) &= \frac{1}{n} E(\sum_{i=1}^n X_i) \\ E(\hat{\mu}_{EMM}) &= \frac{1}{n} \sum_{i=1}^n E(X_i) && \text{por (*)} \\ E(\hat{\mu}_{EMM}) &= \frac{1}{n} \sum_{i=1}^n \mu \\ E(\hat{\mu}_{EMM}) &= \frac{1}{n} n\mu \\ E(\hat{\mu}_{EMM}) &= \mu.\end{aligned}$$

$$\begin{aligned}E(\hat{\sigma}_{EMM}^2) &= E\left[\frac{\sum_{i=1}^n X_i^2}{n} - \left(\frac{\sum_{i=1}^n X_i}{n}\right)^2\right] \\ E(\hat{\sigma}_{EMM}^2) &= E\left(\frac{\sum_{i=1}^n X_i^2}{n}\right) - E\left[\left(\frac{\sum_{i=1}^n X_i}{n}\right)^2\right] && \text{por (*)} \\ E(\hat{\sigma}_{EMM}^2) &= \frac{1}{n} E(\sum_{i=1}^n X_i^2) - \{V\left(\frac{\sum_{i=1}^n X_i}{n}\right) + [E(\frac{\sum_{i=1}^n X_i}{n})]^2\} \\ E(\hat{\sigma}_{EMM}^2) &= \frac{1}{n} \sum_{i=1}^n E(X_i^2) - [\frac{1}{n^2} V(\sum_{i=1}^n X_i) + \mu^2] && \text{por (*)} \\ E(\hat{\sigma}_{EMM}^2) &= \frac{1}{n} \sum_{i=1}^n (\sigma^2 + \mu^2) - [\frac{1}{n^2} \sum_{i=1}^n V(X_i) + \mu^2] && \text{por (**)} \\ E(\hat{\sigma}_{EMM}^2) &= \frac{1}{n} n (\sigma^2 + \mu^2) - (\frac{1}{n^2} n \sigma^2 + \mu^2) \\ E(\hat{\sigma}_{EMM}^2) &= \sigma^2 + \mu^2 - (\frac{\sigma^2}{n} + \mu^2) \\ E(\hat{\sigma}_{EMM}^2) &= \sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2 \\ E(\hat{\sigma}_{EMM}^2) &= \frac{n-1}{n} \sigma^2.\end{aligned}$$

(*) propiedad de linealidad de la esperanza.

(**) propiedad de la varianza e independencia.

Por lo tanto, el estimador EMM de μ es insesgado y el estimador EMM de σ^2 no es insesgado.

(b) Hallar los estimadores de μ y σ^2 por el método de verosimilitud. ¿Los estimadores son insesgados?

$$\begin{aligned} L(\mu, \sigma^2) &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X_i-\mu)^2}{2\sigma^2}} \\ L(\mu, \sigma^2) &= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{\sum_{i=1}^n \frac{-(X_i-\mu)^2}{2\sigma^2}} \\ L(\mu, \sigma^2) &= \frac{1}{(\sqrt{2\pi\sigma^2})^n} e^{\frac{-1}{2\sigma^2} \sum_{i=1}^n (X_i-\mu)^2} \\ L(\mu, \sigma^2) &= (\sqrt{2\pi\sigma^2})^{-n} e^{\frac{-1}{2\sigma^2} \sum_{i=1}^n (X_i-\mu)^2} \\ L(\mu, \sigma^2) &= [(2\pi\sigma^2)^{\frac{1}{2}}]^{-n} e^{\frac{-1}{2\sigma^2} \sum_{i=1}^n (X_i-\mu)^2} \\ L(\mu, \sigma^2) &= (2\pi\sigma^2)^{\frac{-n}{2}} e^{\frac{-1}{2\sigma^2} \sum_{i=1}^n (X_i-\mu)^2}. \end{aligned}$$

$$\begin{aligned} \ln L(\mu, \sigma^2) &= \ln [(2\pi\sigma^2)^{\frac{-n}{2}} e^{\frac{-1}{2\sigma^2} \sum_{i=1}^n (X_i-\mu)^2}] \\ \ln L(\mu, \sigma^2) &= \ln (2\pi\sigma^2)^{\frac{-n}{2}} + \ln [e^{\frac{-1}{2\sigma^2} \sum_{i=1}^n (X_i-\mu)^2}] \\ \ln L(\mu, \sigma^2) &= \frac{-n}{2} \ln 2\pi\sigma^2 + [\frac{-1}{2\sigma^2} \sum_{i=1}^n (X_i - \mu)^2] \ln e \\ \ln L(\mu, \sigma^2) &= \frac{-n}{2} \ln 2\pi\sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^n (X_i - \mu)^2 * 1 \\ \ln L(\mu, \sigma^2) &= \frac{-n}{2} \ln 2\pi\sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^n (X_i - \mu)^2. \end{aligned}$$

$$\begin{aligned} \frac{\partial \ln L(\mu, \sigma^2)}{\partial \mu} &= 0 \\ \frac{-2}{2\sigma^2} \sum_{i=1}^n (X_i - \mu) (-1) &= 0 \\ \frac{1}{\sigma^2} \sum_{i=1}^n (X_i - \mu) &= 0 \\ \sum_{i=1}^n (X_i - \mu) &= 0 * \sigma^2 \\ \sum_{i=1}^n (X_i - \mu) &= 0 \\ \sum_{i=1}^n X_i - \sum_{i=1}^n \mu &= 0 \\ \sum_{i=1}^n X_i - n\mu &= 0 \\ n\mu &= \sum_{i=1}^n X_i \\ \hat{\mu}_{EMV} &= \frac{\sum_{i=1}^n X_i}{n}. \end{aligned}$$

$$\begin{aligned} \frac{\partial \ln L(\mu, \sigma^2)}{\partial \sigma^2} &= 0 \\ \frac{-n}{2} \frac{1}{2\pi\sigma^2} 2\pi - [\frac{-1}{2\sigma^4} \sum_{i=1}^n (X_i - \mu)^2] &= 0 \\ \frac{-n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (X_i - \mu)^2 &= 0 \\ \frac{n}{2\sigma^2} &= \frac{1}{2\sigma^4} \sum_{i=1}^n (X_i - \mu)^2 \\ \frac{2\sigma^4}{2\sigma^2} &= \frac{\sum_{i=1}^n (X_i - \mu)^2}{n} \\ \hat{\sigma}_{EMV}^2 &= \frac{\sum_{i=1}^n (X_i - \hat{\mu}_{EMV})^2}{n}. \end{aligned}$$

$$\begin{aligned} E(\hat{\mu}_{EMV}) &= E\left(\frac{\sum_{i=1}^n X_i}{n}\right) \\ E(\hat{\mu}_{EMV}) &= \frac{1}{n} E(\sum_{i=1}^n X_i) \\ E(\hat{\mu}_{EMV}) &= \frac{1}{n} \sum_{i=1}^n E(X_i) \quad \text{por (*)} \end{aligned}$$

$$E(\hat{\mu}_{EMV}) = \frac{1}{n} \sum_{i=1}^n \mu$$

$$E(\hat{\mu}_{EMV}) = \frac{1}{n} n\mu$$

$$E(\hat{\mu}_{EMV}) = \mu.$$

$$E(\hat{\sigma}_{EMV}^2) = E\left[\frac{\sum_{i=1}^n (X_i - \hat{\mu}_{EMV})^2}{n}\right]$$

$$E(\hat{\sigma}_{EMV}^2) = \frac{1}{n} E[\sum_{i=1}^n (X_i - \hat{\mu}_{EMV})^2]$$

$$E(\hat{\sigma}_{EMV}^2) = \frac{1}{n} E\{\sum_{i=1}^n [X_i^2 - 2X_i\hat{\mu}_{EMV} + \hat{\mu}_{EMV}^2]\}$$

$$E(\hat{\sigma}_{EMV}^2) = \frac{1}{n} E(\sum_{i=1}^n X_i^2 + \sum_{i=1}^n -2X_i\hat{\mu}_{EMV} + \sum_{i=1}^n \hat{\mu}_{EMV}^2)$$

$$E(\hat{\sigma}_{EMV}^2) = \frac{1}{n} E(\sum_{i=1}^n X_i^2 - 2\hat{\mu}_{EMV} \sum_{i=1}^n X_i + n\hat{\mu}_{EMV}^2)$$

$$E(\hat{\sigma}_{EMV}^2) = \frac{1}{n} E(\sum_{i=1}^n X_i^2 - 2\hat{\mu}_{EMV} n\hat{\mu}_{EMV} + n\hat{\mu}_{EMV}^2)$$

$$E(\hat{\sigma}_{EMV}^2) = \frac{1}{n} E(\sum_{i=1}^n X_i^2 - 2n\hat{\mu}_{EMV}^2 + n\hat{\mu}_{EMV}^2)$$

$$E(\hat{\sigma}_{EMV}^2) = \frac{1}{n} E(\sum_{i=1}^n X_i^2 - n\hat{\mu}_{EMV}^2)$$

$$E(\hat{\sigma}_{EMV}^2) = E\left(\frac{\sum_{i=1}^n X_i^2}{n} - \hat{\mu}_{EMV}^2\right)$$

$$E(\hat{\sigma}_{EMM}^2) = E\left[\frac{\sum_{i=1}^n X_i^2}{n} - \left(\frac{\sum_{i=1}^n X_i}{n}\right)^2\right]$$

$$E(\hat{\sigma}_{EMM}^2) = E\left(\frac{\sum_{i=1}^n X_i^2}{n}\right) - E\left[\left(\frac{\sum_{i=1}^n X_i}{n}\right)^2\right] \quad \text{por (*)}$$

$$E(\hat{\sigma}_{EMM}^2) = \frac{1}{n} E(\sum_{i=1}^n X_i^2) - \{V\left(\frac{\sum_{i=1}^n X_i}{n}\right) + [E\left(\frac{\sum_{i=1}^n X_i}{n}\right)]^2\}$$

$$E(\hat{\sigma}_{EMM}^2) = \frac{1}{n} \sum_{i=1}^n E(X_i^2) - [\frac{1}{n^2} V(\sum_{i=1}^n X_i) + \mu^2] \quad \text{por (*)}$$

$$E(\hat{\sigma}_{EMM}^2) = \frac{1}{n} \sum_{i=1}^n (\sigma^2 + \mu^2) - [\frac{1}{n^2} \sum_{i=1}^n V(X_i) + \mu^2] \quad \text{por (**)}$$

$$E(\hat{\sigma}_{EMM}^2) = \frac{1}{n} n (\sigma^2 + \mu^2) - (\frac{1}{n^2} n \sigma^2 + \mu^2)$$

$$E(\hat{\sigma}_{EMM}^2) = \sigma^2 + \mu^2 - (\frac{\sigma^2}{n} + \mu^2)$$

$$E(\hat{\sigma}_{EMM}^2) = \sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2$$

$$E(\hat{\sigma}_{EMM}^2) = \frac{n-1}{n} \sigma^2.$$

(*) propiedad de linealidad de la esperanza.

(**) propiedad de la varianza e independencia.

Por lo tanto, el estimador EMV de μ es insesgado y el estimador EMV de σ^2 no es insesgado.

(c) Se determina la resistencia al corte de cada una de diez soldaduras eléctricas por puntos de prueba, dando los siguientes datos (lb/plg2): 392, 376, 401, 367, 389, 362, 409, 415, 358, 375. Si se supone que la resistencia al corte está normalmente distribuida, estimar la verdadera media de resistencia al corte y desviación estándar de resistencia al corte usando el método de máxima verosimilitud y el método de momentos.

X_i : “resistencia al corte de la i-ésima soldadura eléctrica”, $i = 1, 2, \dots, 10$.

$$X_i \sim \mathcal{N}(\mu, \sigma^2).$$

EMM:

$$\hat{\mu}_{EMM} = \frac{\sum_{i=1}^n X_i}{n}$$

$$\hat{\mu}_{EMM} = \frac{3844}{10}$$

$$\hat{\mu}_{EMM} = 384,4.$$

$$\hat{\sigma}_{EMM}^2 = \frac{\sum_{i=1}^n X_i^2}{n} - \left(\frac{\sum_{i=1}^n X_i}{n} \right)^2$$

$$\hat{\sigma}_{EMM}^2 = \frac{1481190}{10} - \left(\frac{3844}{10} \right)^2$$

$$\hat{\sigma}_{EMM}^2 = 148119 - 384,4^2$$

$$\hat{\sigma}_{EMM}^2 = 148119 - 147763,36$$

$$\hat{\sigma}_{EMM}^2 = 355,64.$$

$$\hat{\sigma}_{EMM} = \sqrt{\hat{\sigma}_{EMM}^2}$$

$$\hat{\sigma}_{EMM} = \sqrt{355,64}$$

$$\hat{\sigma}_{EMM} = 18,86.$$

EMV:

$$\hat{\mu}_{EMV} = \frac{\sum_{i=1}^n X_i}{n}$$

$$\hat{\mu}_{EMV} = \frac{3844}{10}$$

$$\hat{\mu}_{EMV} = 384,4.$$

$$\hat{\sigma}_{EMV}^2 = \frac{\sum_{i=1}^n (X_i - \hat{\mu}_{EMV})^2}{n}$$

$$\hat{\sigma}_{EMV}^2 = \frac{3556,4}{10}$$

$$\hat{\sigma}_{EMV}^2 = 355,64.$$

$$\hat{\sigma}_{EMV} = \sqrt{\hat{\sigma}_{EMV}^2}$$

$$\hat{\sigma}_{EMV} = \sqrt{355,64}$$

$$\hat{\sigma}_{EMV} = 18,86.$$

(d) Estimar la probabilidad de que la resistencia al corte de una soldadura al azar sea menor que 420.

$$\hat{P}_{EMV}(X < 420) = P\left(\frac{X - \hat{\mu}}{\hat{\sigma}} < \frac{420 - \hat{\mu}}{\hat{\sigma}}\right)$$

$$\hat{P}_{EMV}(X < 420) = P(Z < \frac{420 - 384,4}{18,86})$$

$$\hat{P}_{EMV}(X < 420) = P(Z < \frac{35,6}{18,86})$$

$$\hat{P}_{EMV}(X < 420) = P(Z < 1,89)$$

$$\hat{P}_{EMV}(X < 420) = F(1,89)$$

$$\hat{P}_{EMV}(X < 420) = 0,9706.$$