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(1) $\{X_1, \dots, X_n\} \overset{iid}{\sim} N(0, \theta)$, $E(X) = 0$

$V(X) = \theta > 0$

$F(X; \theta) = \dfrac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta}$

$E(X^2) = \theta$

$E(X^4) = 3\theta^2$

(a) $L(\theta) = \prod_{i=1}^{m} \dfrac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta}$

$L(\theta) = \dfrac{1}{(\sqrt{2\pi\theta})^m} e^{-\sum_{i=1}^{m} x_i^2 / 2\theta}$

$\ell(\theta) = -m \ln \sqrt{2\pi\theta} - \dfrac{\sum_{i=1}^{m} x_i^2}{2\theta}$

CPO:

$$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{-m}{\sqrt{2\pi\theta}} \cdot \sqrt{2\pi} \frac{1}{2\sqrt{\theta}} - \left(- \frac{\sum x_i^2}{2\theta^2} \right)$$

$$= \frac{-m}{2\theta} + \frac{\sum_{i=1}^{m} x_i^2}{2\theta^2} = 0$$

$$\frac{\sum_{i=1}^{m} x_i^2}{2\theta^2} = \frac{m}{2\theta}$$

$$\frac{\cancel{2}\theta^{\cancel{2}}}{\cancel{2\theta}} = \frac{\sum_{i=1}^{m} x_i^2}{m}$$

$$\hat{\theta}_m = \frac{1}{m} \sum_{i=1}^{m} x_i^2 .$$

CSO:

$$\frac{\partial^2 \ell(\theta)}{\partial \theta^2} = \frac{m}{2\theta^2} - 2 \frac{\sum_{i=1}^{m} x_i^2}{2\theta^3} = \frac{m}{2\theta^2} - \frac{\overbrace{\sum_{i=1}^{m} x_i^2}^{\hat{\theta}_m m}}{\theta^3} = \frac{m}{2\hat{\theta}_m^2} - \frac{\hat{\theta}_m m}{\hat{\theta}_m^3}$$

$$= \frac{m - 2m}{2\hat{\theta}_m^2} = \frac{-m}{2\hat{\theta}_m^2} < 0$$

Por lo tanto, dado que $\frac{\partial^2 \ell(\theta)}{\partial^2 \theta}$ corresponde al EMV.

b) $\bullet$ $E(\hat{\theta}_m) = E\left(\frac{1}{m}\sum_{i=1}^{m} x_i^2\right)$

$= \frac{1}{m} E\left(\sum_{i=1}^{m} x_i^2\right)$

$= \frac{1}{m}\sum_{i=1}^{m} E(x_i^2)$

$= \frac{1}{m} m\theta$

$E(\hat{\theta}_m) = \theta.$ $\therefore$ $\hat{\theta}_m$ es insesgado.

$\bullet$ Por LGN $\frac{\sum_{i=1}^{m} x_i^2}{m} \xrightarrow{P} E(X^2)$, entonces:

$\hat{\theta}_m = \frac{\sum_{i=1}^{n} x_i^2}{m} \xrightarrow{P} E(X^2) = \theta$ $\therefore$ $\hat{\theta}_m$ es consistente para θ.

c) $I_1(\theta) = -E\left[\frac{\partial^2 \ln f(X;\theta)}{\partial^2 \theta}\right]$

$= -E\left[\frac{1}{2\theta^2} - \frac{X^2}{\theta^3}\right]$

$= -\left[E\left(\frac{1}{2\theta^2}\right) - E\left(\frac{X^2}{\theta^3}\right)\right]$

$= -\left[\frac{1}{2\theta^2} - \frac{1}{\theta^3}E(X)\right]$

$= -\left(\frac{1}{2\theta^2} - \frac{1}{\theta^3}\theta\right)$

$= -\left(\frac{1}{2\theta^2} - \frac{1}{\theta^2}\right)$

$= -\left(\frac{-1}{2\theta^2}\right)$

$i(\theta) = I_1(\theta) = \frac{1}{2\theta^2}.$

$\Rightarrow CR = \frac{1}{m\, i(\theta)} = \frac{1}{m\frac{1}{2\theta^2}} = \frac{2\theta^2}{m}.$

$Var(\hat{\theta}_m) = Var\left(\frac{1}{m}\sum_{i=1}^{m} x_i^2\right)$

$= \frac{1}{m^2} Var\left(\sum_{i=1}^{m} x_i^2\right)$

$\overset{iid}{=} \frac{1}{m^2}\sum_{i=1}^{m} Var(x_i^2)$

$= \frac{1}{m^2}\sum_{i=1}^{m}\left\{E(x_i^4) - [E(x_i^2)]^2\right\}$

$= \frac{1}{m^2}\sum_{i=1}^{m}(3\theta^2 - \theta^2)$

$= \frac{1}{m^2}\sum_{i=1}^{m} 2\theta^2$

$= \frac{m}{m^2} 2\theta^2$

$Var(\hat{\theta}_m) = \frac{2\theta^2}{m}.$

$\therefore$ $\hat{\theta}_m$ es el UMVUE de θ, ya que alcanza la CR

d) $n = 100$ $\quad \sum_{i=1}^{100} x_i^2 = 500$ $\quad \dfrac{\sqrt{n}\,(\hat{\theta}_n - \theta)}{\sqrt{\mathrm{Var}(\hat{\theta}_n)}} \xrightarrow{D} N(0,1)$

$$IC_{0,95}(\theta) = \left[\hat{\theta}_n - 1{,}96\sqrt{\frac{2\hat{\theta}_n^2}{n}}\ ;\ \hat{\theta}_n + 1{,}96\sqrt{\frac{2\hat{\theta}_n^2}{n}}\right]$$

$$= \left[\hat{\theta}_n - 1{,}96 \cdot \sqrt{\frac{2}{n}}\ \hat{\theta}_n\ ;\ \hat{\theta}_n + 1{,}96\sqrt{\frac{2}{n}}\ \hat{\theta}_n^2\right]$$

$$= \left[\hat{\theta}_n\left(1 - 1{,}96\sqrt{\frac{2}{n}}\right);\ \hat{\theta}_n\left(1 + 1{,}96\sqrt{\frac{2}{n}}\right)\right]$$

$$= \left[\frac{500}{100}\left(1 - 1{,}96\sqrt{\frac{2}{100}}\right);\ \frac{500}{100}\left(1 + 1{,}96\sqrt{\frac{2}{100}}\right)\right]$$

$$= \left[5\left(1 - 1{,}96\frac{\sqrt{2}}{10}\right);\ 5\left(1 + 1{,}96\frac{\sqrt{2}}{10}\right)\right]$$

$$IC_{0,95}(\theta) = [-0{,}886\ ;\ 6{,}386].$$

② $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, con $\sigma^2 = 4$, $\sigma = 2$

$H_0: \mu = 1$ versus $H_1: \mu \neq 1$

$W_n \equiv \sqrt{n}\,(\bar{X}_n - \mu)$

ⓐ $\frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

$\Rightarrow \frac{(\bar{X}_n - \mu)}{\sqrt{\frac{\sigma^2}{n}}} \sim N(0,1)$

$\Rightarrow \sqrt{n}\,\frac{(\bar{X}_n - \mu)}{\sigma} \sim N(0,1)$

$\therefore \quad W_n \equiv \sqrt{n}\,(\bar{X}_n - \mu) \sim N(0, \sigma^2)$

y bajo H_0 : $W_n^{H_0} \equiv \sqrt{n}\,(\bar{X}_n - 1) \sim (0, \sigma^2)$.

ⓑ $R = \{\bar{X}_n : |\sqrt{n}\,(\bar{X}_n - 1)| \geq 3\}$

$\alpha = P(\text{rechazar } H_0 \mid \mu = 1)$

$\alpha = P(|W_n| \geq 3 \mid \mu = 1)$

$\alpha = 1 - P(-3 \leq W_n^{H_0} \leq 3)$

$\alpha = 1 - P\left(\frac{-3}{2} \leq \frac{\sqrt{n}\,(\bar{X}_n - 1)}{\sigma} \leq \frac{3}{2}\right)$

$\alpha = 1 - P(-1{,}5 \leq Z \leq 1{,}5)$

$\alpha = 1 - [F_Z(1{,}5) - F_Z(-1{,}5)]$

$\alpha = 1 - (0{,}933 - 0{,}067)$

$\alpha = 1 - 0{,}866$

$\alpha = 0{,}134$.

c) $n=16$; $\overline{X}_n = 2$

i) $|\sqrt{n}(\overline{X}_n - 1)| = |\sqrt{16}(2-1)|$
$= |4 \cdot 1|$
$= |4|$

$|\sqrt{n}(\overline{X}_n - 1)| = 4 \geq 3$ ∴ se rechaza Ho

ii) P-valor $= 2P(W_n \geq w_n)$
$= 2P(W_n \geq 4)$
$= 2P\left(\frac{W_n}{\sigma} \geq \frac{4}{2}\right)$
$= 2P(Z \geq 2)$
$= 2P(Z \leq -2)$ → por simetría de la normal
$= 2 \cdot 0{,}023$

P-valor $= 0{,}046$.

iii) En base a lo anterior, si el nivel de significatividad para el test fuera $\alpha = 0{,}01$, no existe evidencia suficiente para rechazar Ho, ya que P-valor $= 0{,}046 > \alpha = 0{,}01$.

d) $\mu = 1{,}5$; $n = 16$

$\beta = P(\text{no rechazar Ho} \mid \mu = 1{,}5)$

$\beta = P(|W_n| \leq 3 \mid \mu = 1{,}5)$

$\beta = P(-3 \leq W_{n=16}^{\mu=1{,}5} \leq 3)$

$\beta = P(-3 \leq \sqrt{16}(\overline{X}_n - 1{,}5) \leq 3)$

$\beta = P\left(1{,}5 - \frac{3}{4} \leq \overline{X}_n \leq 1{,}5 + \frac{3}{4}\right)$

$\beta = P\left(\frac{3}{4} \leq \overline{X}_n \leq \frac{9}{4}\right)$

$\beta = P\left(\frac{\sqrt{16}\left(\frac{3}{4} - \frac{3}{2}\right)}{2} \leq \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} \leq \frac{\sqrt{16}\left(\frac{9}{4} - \frac{3}{2}\right)}{2}\right)$ ↑

$\beta = P\left(\frac{4}{2} \cdot \left(\frac{-3}{2}\right) \leq Z \leq \frac{4}{2} \frac{3}{4}\right)$

$\beta = P(-6 \leq Z \leq 1{,}5)$

$\beta = P(Z \leq 1{,}5) - P(Z \leq -6)$

$\beta = F_Z(1{,}5) - F_Z(-6)$

$\beta = 0{,}933 - 0$

$\beta = 0{,}933$.

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3) $\{X_1, \ldots, X_n\} \overset{iid}{\sim} Unif(\theta, 1)$, donde $0 < \theta < 1$

$$f(x;\theta) = \frac{1}{1-\theta} \mathbb{1}_{\{[\theta,1]\}}(x)$$

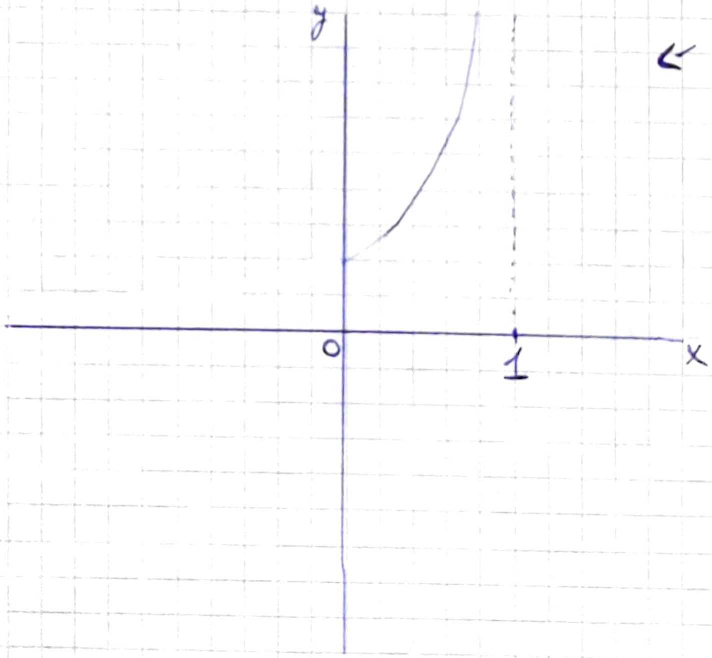
$$E(X) = \frac{1+\theta}{2} \quad ; \quad Var(X) = \frac{(1-\theta)^2}{12}$$

a) $$L(\theta) = \prod_{i=1}^{n} \frac{1}{1-\theta} \mathbb{1}_{\{[\theta,1]\}}(x)$$

$$L(\theta) = \frac{1}{(1-\theta)^n} \quad \text{para } x_i \in [\theta, 0]$$

La función de verosimilitud se construye como el productorio de la función de densidad uniforme.

$$\frac{\partial L(\theta)}{\partial \theta} = \frac{-n(-1)}{(1-\theta)^{n-1}} > 0 \quad , \forall n, 0<\theta<1$$



← aprox...

Por lo tanto, la función $L(\theta)$ es siempre creciente como función de θ, por lo que se querría elegir el valor de θ más alto posible como su estimador máximo verosímil. Sin embargo, esta función sólo está definida para $x_i > \theta$, por lo que el estimador máximo verosímil será $\hat{\theta}_n = X_{(1)} = \min\{X_1, \ldots, X_n\}$.

(b) $M_1 = E(X)$

$$\frac{1}{m}\sum_{i=1}^{m} x_i = \frac{1+\theta}{2}$$

$$\overline{X}_m = \frac{1+\theta}{2}$$

$$1+\theta = 2\overline{X}_m$$

$$\tilde{\theta}_m = 2\overline{X}_m - 1.$$

$$ECM(\tilde{\theta}_m,\theta) = sesgo^2(\tilde{\theta}_m) + Var(\tilde{\theta}_m)$$
$$= [E(\tilde{\theta}_m - \theta)]^2 + Var(\tilde{\theta}_m)$$
$$= [E(\tilde{\theta}_m) - E(\theta)]^2 + \frac{(1-\theta)^2}{4m^2} \quad (*) \text{ y } (**)$$
$$= (\theta-\theta)^2 + \left(\frac{1-\theta}{2m}\right)^2$$
$$= 0^2 + \left(\frac{1-\theta}{2m}\right)^2$$

$$ECM(\tilde{\theta}_m,\theta) = \left(\frac{1-\theta}{2m}\right).$$

(*) $E(\tilde{\theta}_m) = E(2\overline{X}_m - 1)$
$$= 2E\left(\frac{\sum_{i=1}^{m} X_i}{m}\right) - E(1)$$
$$= \frac{2}{m}\sum_{i=1}^{m} E(X_i) - 1$$
$$= \frac{2m}{m}\,\frac{(1+\theta)}{2} - 1$$
$$= 1 + \theta - 1$$
$$E(\tilde{\theta}_m) = \theta.$$

(**) $Var(\tilde{\theta}_m) = Var(2\overline{X}_m - 1)$
$$= 4Var\left(\frac{\sum_{i=1}^{m} X_i}{m}\right) + Var(1)$$
$$\overset{iid}{=} \frac{4}{m^2}\sum_{i=1}^{m} Var(X_i) + 0$$
$$= \frac{4}{m^2}\,\frac{(1-\theta)^2}{12}$$
$$Var(\tilde{\theta}_m) = \frac{(1-\theta)^2}{4m^2}.$$

(c) $m = 5$; $x_1 = 0{,}52$; $x_2 = 0{,}78$; $x_3 = 0{,}1$; $x_4 = 0{,}33$; $x_5 = 0{,}89$

$$\mu \equiv E(X) = \frac{1+0{,}1}{2} = \frac{1{,}1}{2}$$
$$\mu = 0{,}55.$$

$$\sigma^2 \equiv Var(X) = \frac{(1-0{,}1)^2}{12} = \frac{0{,}9^2}{12} = \frac{0{,}81}{12}$$
$$\sigma^2 = 0{,}0675.$$

4 $X \sim f(x;\theta) = \begin{cases} (1-\theta)^x \theta, & \text{con } x \in \{0,1,2,\dots\}, \\ 0 & \text{, en otro caso} \end{cases}$

donde $0<\theta<1$, $\mu(\theta)=E(X)=\frac{1-\theta}{\theta}$

Prior: $\theta \sim Beta(\eta)$, donde $\eta=(\alpha,\beta)$

$$\theta \sim \pi(\theta;\eta)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, \text{ para } 0\le\theta\le1,$$

Con media y varianza apriori:

$$E(\theta)=\frac{\alpha}{\alpha+\beta} \quad ; \quad V(\theta)=\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

a) $$\pi(\theta\,|\,\underbrace{\text{datos}}_{D_m}) \propto L(\theta)\,\pi(\theta)$$

$$\propto \left[\prod_{i=1}^{m}(1-\theta)^{x_i}\theta\right]\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$\propto (1-\theta)^{\sum_{i=1}^{m}x_i}\theta^m\,\theta^{\alpha-1}(1-\theta)^{\beta-1}$$

$$\propto (1-\theta)^{\sum_{i=1}^{m}x_i+\beta-1}\,\theta^{m+\alpha-1}$$

$$\therefore\ \pi(\theta\,|\,\text{datos}) \sim Beta\left(m+\alpha;\ \sum_{i=1}^{m}x_i+\beta\right).$$

b) $$E(\theta\,|\,D_m)=\frac{m+\alpha}{m+\alpha+\sum_{i=1}^{m}x_i+\beta}$$

$$E(\theta\,|\,D_m)=\frac{1+\alpha/m}{1+\frac{\alpha+\beta}{m}+\bar{X}_m}.$$

$$Var(\theta\,|\,D_m)=\frac{(m+\alpha)\left(\sum_{i=1}^{m}x_i+\beta\right)}{\left(m+\alpha+\sum_{i=1}^{m}x_i+\beta\right)^2\left(m+\alpha+\sum_{i=1}^{m}x_i+\beta+1\right)}.$$

c) Si, es correcto afirmar que se trata de un modelo Bayesiano conjugado, ya que $\pi(\theta)$ y $\pi(\theta\,|\,\text{datos})$ siguen el mismo modelo estadístico.

NOTA

d) $H_0: E(X) \leq 1$ versus $H_1: E(X) > 1$

$\Rightarrow \frac{1-\theta}{\theta} \leq 1$ $\qquad$ $\Rightarrow \frac{1-\theta}{\theta} > 1$

$1-\theta \leq \theta$ $\qquad$ $1-\theta > \theta$

$1 \leq 2\theta$ $\qquad$ $1 > 2\theta$

$H_0: \theta \geq \frac{1}{2}$ $\qquad$ $H_1: \theta < \frac{1}{2}$

$$\begin{cases} P(\theta \in \Theta_1) = \int_0^{1/2} \pi(\theta; \eta_0)\, d\theta = 0{,}25 \Rightarrow \text{representa la prob. de } \theta \in \Theta_1 \\ P(\theta \in \Theta_0) = \int_{1/2}^{1} \pi(\theta; \eta_0)\, d\theta = 1 - 0{,}25 = 0{,}75 \end{cases}$$

$$\begin{cases} P(\theta \in \Theta_1 | \text{datos}) = \int_0^{1/2} \pi(\theta | D_m, \eta_0)\, d\theta = 0{,}95 \Rightarrow \text{representa la prob. de } \theta \in \Theta_1 \text{, dada la muestra} \\ P(\theta \in \Theta_0 | \text{datos}) = \int_{1/2}^{1} \pi(\theta | D_m, \eta_0)\, d\theta = 1 - 0{,}95 = 0{,}05. \end{cases}$$

De esta forma, la decisión sobre H_0 o H_1 se toma evaluando el Factor Bayesiano:

$$FB = \frac{P(\theta \in \Theta_1 | \text{datos})}{P(\theta \in \Theta_0 | \text{datos})} \; \frac{P(\theta \in \Theta_0)}{P(\theta \in \Theta_1)}$$

$$FB = \frac{0{,}95}{0{,}05} \cdot \frac{0{,}75}{0{,}25}$$

$FB = 57$ $\quad \therefore$ existe evidencia fuerte a favor de H_1.