Ejemplo #2: Quiero modelar $E(Y|X) = \mu_{Y|X}$

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

$$\varepsilon \sim N(0, \sigma^2)$$

$$Y|X \sim N(\beta_0 + \beta_1 X; \sigma^2)$$

$$\theta = (\beta_0, \beta_1, \sigma^2).$$

Identificabilidad:

$$X \sim f(x;\theta) \Rightarrow \text{si } f(x,\theta_1) = f(x;\theta_2) \Leftrightarrow \theta_1 = \theta_2$$

$$\hat{\psi}_1(\theta) = \hat{E}(X) = \int x\, f(x;\hat{\theta})\, dx$$

$$\psi_2(\theta) = Var(x) = \int (x - E(x))^2 f(x;\theta)\, dx$$

$$\psi_3(\theta) = \left\{ m : \int_{-\infty}^{m} f(x;\theta)\,dx = 1/2 \right\}$$

$$\psi_4(\theta) = P(X \geq 0) = \int_0^{\infty} f(x,\theta)\,dx$$

$X \sim Bern(\theta)$ $\quad \Theta = (0,1)$

$\boxed{\text{Familia Exponenc.}}$

$\mathcal{X} = \{0,1\}$

$$f(x;\theta) = \begin{cases} \theta^x (1-\theta)^{1-x} & \text{si } x \in \{0,1\} \\ 0 & \text{en otro caso.} \end{cases}$$

$$\mathbb{1}_{\{0,1\}}(x) = \begin{cases} 1 & \text{si } x \in \{0,1\} \\ 0 & \text{en otro caso} \end{cases}$$

$$f(x,\theta) = \overbrace{\mathbb{1}_{\{0,1\}}(x)}^{h(x) \geq 0} \underline{\theta^x (1-\theta)^{1-x}}$$

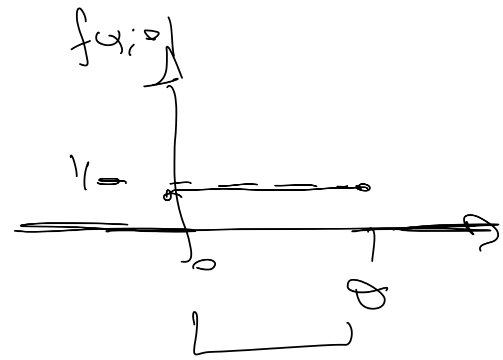
$$= \mathbb{1}_{\{0,1\}}(x) (1-\theta)^1 \underline{\left(\frac{\theta}{1-\theta}\right)^x} \qquad \exp(\ln(a)) = a$$

$$= \mathbb{1}_{\{0,1\}}(x) (1-\theta) \exp\left\{\ln\left(\frac{\theta}{1-\theta}\right)^x\right\}$$

$$= \underbrace{\mathbb{1}_{\{0,1\}}(x)}_{h(x)\geq 0} \underbrace{(1-\theta)}_{c(\theta)\geq 0} \exp\Big\{ \underbrace{x}_{t(x)} \underbrace{\ln\left(\frac{\theta}{1-\theta}\right)}_{w(\theta)} \Big\}$$

- **Contra-ejemplo:**

$X \sim \text{Unif}\,[0,\theta) \qquad \mathcal{X} = [0,\theta]$



$$f(x;\theta) = \frac{1}{\theta}\, \mathbb{1}_{[0,\theta]}(x)$$

$$= \underbrace{\mathbb{1}_{[0,\theta]}(x)} \exp\Big\{ \overset{t(x)=1}{1} \cdot \overbrace{\ln\left(\frac{1}{\theta}\right)}^{w(\theta)} \Big\}$$

¿ $h(x)$ y $c(\theta)$?

Caso multiparámetro ($k=2$)

$X \sim N(\mu; \sigma^2)$

$$f(x;\mu,\sigma^2) = \underbrace{1}_{h(x)} \times \underbrace{\frac{e^{-\mu^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}}_{C(\theta)} \exp\left\{\underbrace{\left(-\frac{1}{2\sigma^2}\right)}_{w_1(\theta)} \overset{t_1(x)}{x^2} + \underbrace{\frac{\mu}{\sigma^2}}_{w_2(\theta)} \overset{t_2(x)}{x}\right\}$$

$\theta = (\mu; \sigma^2)$

Familia loc y Escala

$$\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

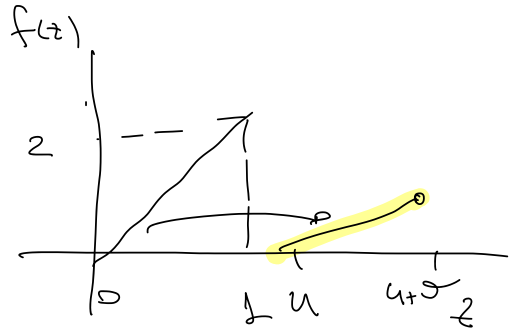
$$= N(\mu; \sigma^2)$$

Ejemplo loc y Escala:

$$Z \sim f(z) = 2z \quad \text{si } z \in (0,1)$$

$$X \sim f(x; \mu, \sigma) = \frac{1}{\sigma}\, 2 \overbrace{\left(\frac{x-\mu}{\sigma}\right)}^{z}$$

$$= \frac{2}{\sigma^2}(x-\mu)$$



$$0 \le \frac{x-\mu}{\sigma} \le 1 \Rightarrow \mu \le x \le \mu+\sigma$$

$$E(Z) = \frac{2}{3} \quad ; \quad V(Z) = \frac{1}{18}$$

$$\boxed{E(X) = \sigma E(Z) + \mu = \sigma \frac{2}{3} + \mu}$$

$$\boxed{V(X) = V(Z)\,\sigma^2 = \frac{\sigma^2}{18}}$$

Muestreo Aleatorio

$X \sim Exp(\theta)$ $\quad \underline{f(x;\theta)} = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & \text{si } x \geq 0 \\ 0 & \text{en otro caso} \end{cases}$

$\underline{X} = \{X_1, \ldots, X_n\} \overset{iid}{\sim} X$

$$\underline{X} \sim f(x_1, \ldots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$$

$$= \prod_{i=1}^{n} \frac{1}{\theta} e^{-x_i/\theta}$$

$$= \underline{\underline{\frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i}}}$$

$$= P(X_1 \leq a_1, X_2 \leq a_2, \ldots, X_n \leq a_n)$$

$$= \int_0^{a_1} \int_0^{a_2} \cdots\cdots \int_0^{a_n} \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i} dx_n \ldots dx_1$$

$\underline{\underline{X \not\sim N(\mu, \sigma^2)}}$; $\underline{\underline{\bar{X}_n \sim\ ?}}$

TCL: $\underline{\sqrt{n}\,(\bar{X}_n - \underline{\underline{E}}(X))} \xrightarrow[F]{} N(0, Var(X))$

$n \gg 0$

$$\Longrightarrow \bar{X}_n \sim_a N\left(E(X);\ \frac{Var(X)}{n}\right)$$

Est. Orden

$X \sim Exp(\lambda)$ Flujo anual $\leq c$

X_i = "Cant. agua fluye al dique en el año i" $\sim Exp(\lambda)$

$$\{X_1, \ldots, X_n\} \overset{iid}{\sim} Exp(\lambda)$$

$f(x;\lambda) = \frac{1}{\lambda} e^{-x/\lambda}$

$F(x;\lambda) = 1 - e^{-x/\lambda}$

$$P(\underbrace{\min\{X_1, \ldots, X_n\}}_{X_{(1)}} \leq c)$$

$$f_{X_{(1)}}(x) = n\, f(x,\lambda)\left[1 - F(x;\lambda)\right]^{n-1}$$

$$= n \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \left(1 - \left(1 - e^{-x/\lambda}\right)\right)^{n-1}$$

$$= \frac{n}{\lambda} e^{-\frac{x}{\lambda} - (n-1)x/\lambda}$$

$$X_{(1)} \sim f_{X_{(1)}}(x;\lambda) = \frac{n}{\lambda} e^{-\frac{n}{\lambda}x} \sim Exp(\lambda/n)$$

$$E(X_{(1)}) = \lambda/n \qquad Var(X_{(1)}) = (\lambda/n)^2$$

$$P(X_{(1)} \leq c) = \int_0^c f_{X_{(1)}}(x;\theta)\,dx$$

$$= \int_0^c \frac{n}{\lambda} e^{-\frac{n}{\lambda}x} dx$$

$$\psi_1(\lambda) = P(X_{(1)} \leq c) = 1 - e^{-\frac{n}{\lambda}c}$$

$$\lambda = E(X) \Longrightarrow \hat{\lambda} = \bar{X}_n$$

$$\hat{\psi}_1 = 1 - e^{-\frac{n}{\bar{X}_n}c}$$

Método Delta

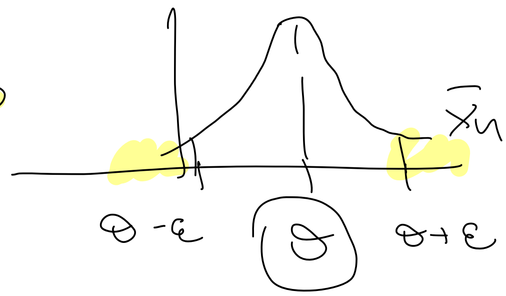
$X \sim Bern(\theta) \Rightarrow \bar{X}_n \Rightarrow$ "buen estim. de $\theta = E(X)$"

$$* \quad \sqrt{n}\,(\bar{X}_n - \underbrace{E(X)}_{\theta}) \xrightarrow[F]{d} N(0, \theta(1-\theta))$$

$$n >> 0 \qquad \bar{X}_n \sim_a N\left(\theta; \frac{\theta(1-\theta)}{n}\right)$$

Con $n >> 0$

$$P(|\bar{X}_n - \theta| > \varepsilon) \approx 0$$



$$g(\theta) = \frac{\theta}{1-\theta} = \frac{P(X=1)}{P(X=0)} = \psi$$

$$\rightarrow \boxed{\hat{\psi}} = g(\bar{X}_n) = \frac{\bar{X}_n}{1-\bar{X}_n} \xrightarrow[P]{} g(\theta)$$

$$g'(\theta) = \frac{1}{(1-\theta)^2} \neq 0 \quad \forall \theta \in (0,1).$$

$$\sqrt{n}\left(\underbrace{\frac{\bar{X}_n}{1-\bar{X}_n}}_{g(T)} - \underbrace{\frac{\theta}{1-\theta}}_{g(\theta)}\right) \xrightarrow[F]{} N\left(0; \frac{\theta\cancel{(1-\theta)}}{(1-\theta)^{\cancel{4}3}}\right)$$

$$T = \bar{X}_n$$

$$n \gg 0 \quad \hat{\psi} = \frac{\bar{X}_n}{1-\bar{X}_n} \sim_a N\left(\underbrace{\frac{\theta}{1-\theta}}_{\psi}; \frac{\theta}{n(1-\theta)^3}\right)$$