Numerical approximation of DSGE models

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This note discusses the numerical approximation of DSGE (dynamic, stochastic, general equilibrium) models. The equilibrium conditions of a model can be cast in the form of a system of non-linear stochastic difference equations. The solution to this system determines the solution of the model. In this note we discuss how to approximate the nonlinear system using a linear approximation around the non-stochastic steady state. To build intution, we start with the example of a simple RBC model. The method, however, is general and can be applied to a wide range of models.

A REAL BUSINESS CYCLE (RBC) MODEL

This is a model of a closed economy with a large number of identical households with size normalized to one. We focus on the behavior of the "representative" household. There are no market failures. The First Welfare Theorem holds and the competivite equilibrium is Pareto efficient. We thus use the Second Welfare Theorem to find the equilibrium allocation by solving the planner's problem of maximizing the utility of the representative household subject to feasibility constraints. Prices, if needed, can be computed from the appropriate conditions of the associated competitive equilibrium.

A representative household has preferences over consumption $\{c_t\}$ and leisure $\{h_t\}$ represented by the utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, h_t) \tag{1}$$

where $U(c,h)$ is increasing in each argument and concave, $\beta \in (0,1)$ is a discount factor, and E_0 is the expectation operator conditional on information at time $t=0$.

The feasibility constraint of the economy is

$$c_t + i_t = y_t,$$

where i_t is investment and y_t is output. Output is produced with capital, k_t, and labor, l_t, according to the production function

$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha},$$

where A_t is the level of technology. Technology evolves according to

$$\log(A_{t+1}) = \rho \log(A_t) + \varepsilon_{t+1},$$

where $|\rho| < 1$ and $\varepsilon_t \sim N\left(0, \sigma_{\varepsilon}^2\right)$.

The stock of capital evolves according to

$$k_{t+1} = (1-\delta)\, k_t + i_t.$$

With this notation, k_t is the stock of capital installed at the beginning of period t that was chosen in period $t-1$. At time t, agents choose k_{t+1}.

If we normalize the total amount of time for consuming leisure or working to one, labor market feasibility requires

$$l_t + h_t = 1.$$

The initial stock of capital k_0 and level of technology A_0 are given.

Using these constraints, the planner's problem can be written as

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U\left(c_t, 1-l_t\right)$$

subject to

$$\begin{aligned} c_t + k_{t+1} &= A_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)\, k_t && (2)\\ \log A_{t+1} &= \rho \log A_t + \varepsilon_{t+1} && (3)\\ & \quad k_0 \text{ and } A_0 \text{ given.} \end{aligned}$$

The Lagrangian of this problem is

$$\max_{\{c_t, l_t, k_{t+1}, \lambda_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \left[U\left(c_t, 1-l_t\right) - \lambda_t \left(c_t + k_{t+1} - A_t k_t^{\alpha} l_t^{1-\alpha} - (1-\delta)\, k_t\right)\right].$$

where $\beta^t \lambda_t$ are the (stochastic) Lagrange multipliers, and k_0 and A_0 are given.

The first order conditions with respect to c_t, l_t, k_{t+1}, and λ_t are given by

$$U_c\left(c_t, 1-l_t\right) - \lambda_t = 0 \tag{4}$$

$$-U_h\left(c_t, 1-l_t\right) - \lambda_t (1-\alpha) A_t k_t^{\alpha} l_t^{-\alpha} = 0 \tag{5}$$

$$\beta E_t \left[\lambda_{t+1} \left(\alpha A_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta\right)\right] - \lambda_t = 0 \tag{6}$$

$$c_t + k_{t+1} - A_t k_t^{\alpha} l_t^{1-\alpha} - (1-\delta)\, k_t = 0, \tag{7}$$

where U_c and U_h denote the partial derivative of U with respect to consumption and leisure, respectively.

There is another necessary condition of the planner that is called the "transversality" condition, which

is given by

$$\lim_{T\to\infty} E_0\left[\beta^T \lambda_T k_{T+1}\right] = 0.$$

This condition imposes a constraint on the rate of growth of the stock of capital and is analogous to the requirement that households do not end their lives with a positive stock of capital in a finite horizon version of this model.

We look for solutions in the form of time-invariant "policy functions." To do this, we need to categorize all variables either as state or control variables. The set of state variables (also called predetermined variables) is the minimum set of variables that completely characterizes the state of the economy at the beginning of time t. These are the variables that are taken as given at the beginning of time t and that suffice to determine the current and future evolution of the economy. In some cases it is not obvious how to identify the state variables. Here it is simple: the state variables are the stock of capital k_t and the level of technology A_t; we denote by $x_t = (k_t, A_t)'$ the vector of state variables. The "jump," "control," or "non-predetermined" variables are all those variables that are chosen by the planner and are not predetermined at time t. In this example, the jump variables are c_t, l_t, and λ_t. We denote the set of control variables as $y_t = (c_t, l_t, \lambda_t)'$. It is possible to simplify the system and, for example, get rid of the multiplier λ_t from the above first order conditions. How we choose to solve the model is a matter of convenience. In this note, however, we will keep all the first order conditions derived above because it will be somewhat easier when we perform the linearization of the equilibrium conditions.

Thus, with some abuse of notation, we look for policy functions of the form

$$\begin{aligned} c_t &= \mathsf{c}(k_t, A_t) \\ l_t &= \mathsf{l}(k_t, A_t) \\ \lambda_t &= \lambda(k_t, A_t) \\ k_{t+1} &= \mathsf{k}(k_t, A_t). \end{aligned}$$

where $\mathsf{c}(k_t, A_t)$, $\mathsf{l}(k_t, A_t)$, $\lambda(k_t, A_t)$, and $\mathsf{k}(k_t, A_t)$ are time invariant functions of the state.

Sometimes it is useful to separate the set of state variables into exogenous and endogenous state variable. The former evolve exogenously and cannot be changed by anyone. In contrast, the latter can be influenced by the agents' choices. In the RBC example A_t is the exogenous state variable and k_t is the endogenous state variable.

Note that we can write the equilibrium conditions of the model as a system of equations of the form

$$E_t\left[f\left(x_{t+1}, y_{t+1}, x_t, y_t\right)\right] = \bar{0}, \tag{8}$$

where $\bar{0}$ is a vector of zeros of dimension 5 (one for each equation) and $f : R^{2\times2+2\times3} \to R^5$ is given by

$$f\left(x_{t+1}, y_{t+1}, x_t, y_t\right) = \begin{bmatrix} U_c\left(c_t, 1-l_t\right) - \lambda_t \\ -U_h\left(c_t, 1-l_t\right) - \lambda_t\left(1-\alpha\right) A_t k_t^{\alpha} l_t^{-\alpha} \\ \beta\lambda_{t+1}\left(\alpha A_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta\right) - \lambda_t \\ c_t + k_{t+1} - A_t k_t^{\alpha} l_t^{1-\alpha} - \left(1-\delta\right) k_t \\ \log\left(A_{t+1}\right) - \rho\log\left(A_t\right) - \varepsilon_{t+1} \end{bmatrix}$$

In general, the equilibrium conditions of a wide range of models can be written as (8), which shows that the equilibrium allocation (and prices if we were solving the competitive equilibrium directly without using the Second Welfare Theorem) solves a nonlinear system of stochastic difference equations.

IMPORTANT RESULTS FROM LINEAR ALGEBRA

Here we compile some useful results from linear algebra. Let A be a matrix. We use a_{ij} to denote element (i, j) of A. If A is a square matrix of complex numbers, we denote by A^H the Hermitian transpose of A. The Hermitian transpose is the generalization of the transpose of a real matrix and means that we first transpose A and next take the complex conjugate of the resulting numbers.

Definition 1: A square matrix A of complex number is said to be unitary if $A^H A = AA^H = I$, where I is the identity matrix.

Comment: Clearly, the inverse of a unitary matrix A exists and equals A^H.

Result 1: A square matrix A is invertible if and only if all its eigenvalues are different from zero.

Definition 2: A square matriz A is upper triangular if its entries below the main diagonal are zero.

Result 2: If A is upper triangular with non-zero entries on its main diagonal, then A is invertible.

The proof follows from the observation that the eigenvalues of a triangular matrix are its diagonal entries and by Result 1.

Result 3: If A is upper triangular and invertible, then A^{-1} is upper triangular. Moreover, the diagonal elements of A^{-1} are the reciprocal of the diagonal elements of A, that is, element (i, i) of A^{-1} is $1/a_{ii}$.

Result 4: If A and B are $n \times n$ upper triangular matrices, then AB is also upper triangular.

The following is a key theorem that we use in the rest of this note.

Theorem 1: *(QZ decomposition): Let A and B be $n \times n$ matrices. If there is a complex number z such that $\det\left(B - Az\right) \neq 0$, then there are matrices Q, Z, S, and T such that:*

1. *Q and Z are unitary, i.e.* $Q^H Q = QQ^H = I$ and $Z^H Z = ZZ^H = I$

2. *T and S are upper triangular*

3. *The matrices Q, Z, S, and T satisfy*

$$\begin{aligned} QAZ &= S \\ QBZ &= T, \end{aligned}$$

4. *There is no index i such that $s_{ii} = t_{ii} = 0$, and*

5. *The matrices Q, Z, S, and T can be chosen in such a way as to make the diagonal entries s_{ii} and t_{ii} appear in any desired order.*

Remark 1: in Theorem 1, the ratios t_{ii}/s_{ii} are called the *generalized eigenvalues* of the matrix pair (A, B) because they can be shown to solve the following eigenvalue problem: find the λ that satisfies $\det(B - A\lambda) = 0$. By convention, we say that $s_{ii} = 0$ corresponds to an infinite generalized eigenvalue.

Remark 2: Theorem 1 is also called the *Generalized Schur Decomposition.*

SOLUTION TO FIRST ORDER APPROXIMATIONS OF DSGE MODELS

Let $x_t \in R^n$ denote a vector of state variables (predetermined variables) and $y_t \in R^m$ a vector of non-state variables (control variables).[1] The vector of state variables includes endogenous state variables, like the stock of capital, and exogenous state variables, like the evolution of technology shocks.

The equilibrium conditions of a wide variety of dynamic economic models can be expressed as the following system of nonlinear stochastic difference equations

$$E_t \left[f\left(x_{t+1}, y_{t+1}, x_t, y_t \right) \right] = \bar{0}, \tag{9}$$

where E_t denotes the mathematical expectation conditional on the information available at time t, the function $f : R^{2n+2m} \to R^{n+m}$, contains all the equilibrium conditions of the model, like Euler equations, feasibiliy conditions, state evolution equations, and so forth, and $\bar{0}$ denotes a vector of zeros of dimension $n + m$. The function f has image on R^{n+m}. The reason for this is that there is one equation (such as a first order condition or a law of motion) for each (control or state) variable: since there are $n + m$ variables, there must be $n + m$ equations.

Please note that there are other ways of writing the equilibrium conditions of the model, as you will notice if you read different papers in the literature. For example, one can also write equation (9) as a

[1] This part of the note follows Schmitt-Grohé and Uribe (2017), Appendix to chapter 4. See also Klein (2000) and Gomme and Klein (2011).

second order stochastic difference equation in the state variables alone and no control variables; that is, as a nonlinear function of x_t, x_{t+1}, and x_{t+2} only. This means that we need two sets of conditions to pin down the solution to (9). The first set of conditions are the initial conditions of the state variables at time $t = 0$, x_0 (like the initial stock of capital). We need to come up with another condition, for otherwise the second order difference equation does not uniquely pin down the solution. This additional condition is usually a "transversality" condition that requires the state variables to be bounded in an appropriate sense. By focusing on bounded solutions, the method that we develop below automatically imposes the required transversality condition.

We further assume that the state vector x_t can be partitioned as $x_t = \left(x'_{1,t}\ x'_{2,t}\right)'$, where $x_{1,t}$ contains all endogenous state variables and $x_{2,t}$ contains all exogenos state variables.[2] We assume that the dimensions of $x_{1,t}$ and $x_{2,t}$ are, respectively n_1 and n_2, where $n_1 + n_2 = n$. Furthermore, we will focus on cases where the exogenous state variables evolve according to the following stochastic process

$$x_{2,t+1} = \Lambda x_{2,t} + \tilde{\eta}\varepsilon_{t+1}, \tag{10}$$

where Λ is an $n_2 \times n_2$ matrix, ε_{t+1} is an n_2 vector of i.i.d. shocks with zero mean and identity covariance matrix, and $\tilde{\eta}$ is an $n_2 \times n_2$ matrix that is introduced to allow for correlated structural shocks. We also assume that the eigenvalues of Λ are all less than one in absolute value, which guarantees that the exogenous stochastic process is stationary. The process (10) can be generalized to a nonlinear stochastic process of the form $x_{2,t+1} = H\left(x_{2,t}\right) + \tilde{\eta}\varepsilon_{t+1}$ for some nonlinear function H. But for our purposes, (10) is enough. The methods developed here can also be applied, with some modifications, when the exogenous stochastic process has a unit root.

We will solve the model using a first order perturbation (i.e. local linear approximation) around the non-stochastic steady state. To do this, it is convenient to add an "auxiliary" parameter (positive real number) σ that controls the "amount of uncertainty" in the model, and replace (10) by

$$x_{2,t+1} = \Lambda x_{2,t} + \sigma\tilde{\eta}\varepsilon_{t+1}. \tag{11}$$

When $\sigma = 0$ the model becomes deterministic while setting $\sigma = 1$ we recover the original model. The perturbation technique involves approximating the model around the deterministic model $\sigma = 0$.

Equation (9) together with the appropriate stability condition imply solutions in the form of policy functions $g : R^n \times R^+ \to R^m$ and $h : R^n \times R^+ \to R^n$ such that

$$y_t = g\left(x_t, \sigma\right) \tag{12}$$

$$x_{t+1} = h\left(x_t, \sigma\right) + \sigma\eta\varepsilon_{t+1} \tag{13}$$

[2] In the RBC example, x_t^1 is the stock of capital (a state variable that can be affected by agent choices) and x_t^2 is the level of technology (an exogenous state variable that cannot be affected by any agent).

where

$$\eta_{n\times n_2} = \begin{bmatrix} 0_{n_1\times n_2} \\ \tilde{\eta}_{n_2\times n_2} \end{bmatrix}.$$

Equations (12) and (13) are the policy functions of the model. In particular, (12) determines the control variables y_t as a function of the state variables x_t and (13) determines the evolution of the state variables. Under regularity conditions that will depend on the particular case being analized, the policy functions (12) and (13) are unique—more on this below.

We consider approximations around the non-stochastic steady state. The steady state is defined as the values of x and y that satisfy $x_{t+1} = x_t = \bar{x}$, $y_{t+1} = y_t = \bar{y}$, and that solve the system of equations

$$f(\bar{x}, \bar{y}, \bar{x}, \bar{y}) = 0. \tag{14}$$

This problem, in principle, has a solution as there are $n + m$ equations to find the $n + m$ unknowns.

The way we impose the appropriate stability conditions is to restrict attention to bounded equilibria in which the economy, absent any shock, converges to the steady state. As we will see below, this involves computing some eigenvalues and counting the number of stable versus unstable eigenvalues.

We now compute a first order Taylor approximation of the policy functions g and h around the point $(x, \sigma) = (\bar{x}, 0)$. Here, $\bar{x}$ is the steady state of the state variables of the deterministic economy and the scalar $\sigma = 0$ implies that we are perturbing the solution around an economy with no shocks. The Taylor expansions of the functions g and h around $(\bar{x}, 0)$ are

$$g(x, \sigma) \approx g(\bar{x}, 0) + g_x(\bar{x}, 0)(x - \bar{x}) + g_\sigma(\bar{x}, 0)\sigma \tag{15}$$

$$h(x, \sigma) \approx h(\bar{x}, 0) + h_x(\bar{x}, 0)(x - \bar{x}) + h_\sigma(\bar{x}, 0)\sigma \tag{16}$$

where g_x is the $(m\times n)$ Jacobian matrix of partial derivatives $\partial g_i/\partial x_j$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$; h_x is the $(n \times n)$ Jacobian with component $\partial h_i/\partial x_j$ for $i, j = 1, 2, ..., n$; g_σ is a $(m \times 1)$ vector with component $\partial g_i/\partial \sigma$ for $i = 1, 2, ..., m$; and h_σ is a $(n \times 1)$ vector with entries $\partial h_i/\partial \sigma$ for $i = 1, 2, ..., n$. All of these matrices are evaluated at the point $(x, \sigma) = (\bar{x}, 0)$.

By definition of the steady state, we know that

$$\begin{aligned} g(\bar{x}, 0) &= \bar{y} \\ h(\bar{x}, 0) &= \bar{x}. \end{aligned}$$

Therefore, the approximate policy functions are given by

$$y_t \approx \bar{y} + g_x(\bar{x}, 0)(x_t - \bar{x}) + g_\sigma(\bar{x}, 0)\sigma \tag{17}$$

$$x_{t+1} \approx \bar{x} + h_x(\bar{x}, 0)(x_t - \bar{x}) + h_\sigma(\bar{x}, 0)\sigma + \sigma\eta\varepsilon_{t+1} \tag{18}$$

Our task is to obtain the matrices $g_x(\bar{x},0)$, $g_\sigma(\bar{x},0)$, $h_x(\bar{x},0)$, and $h_\sigma(\bar{x},0)$. To that end, we start by substituting the true (unknown) policy functions (12) and (13) into (9), and define a new function $F: R^n \times R^+ \to R^{n+m}$ as

$$\begin{aligned} F(x_t,\sigma) &\equiv E_t\left[f(x_{t+1},y_{t+1},x_t,y_t)\right] \\ &= E_t\left[f\left(h(x_t,\sigma)+\sigma\eta\varepsilon_{t+1}, g\left(h(x_t,\sigma)+\sigma\eta\varepsilon_{t+1},\sigma\right), x_t, g(x_t,\sigma)\right)\right] = 0. \end{aligned} \tag{19}$$

Under the true solution, $F(x,\sigma)$ equals zero for all x and σ. This implies that all the derivatives of $F(x,\sigma)$ are also zero, or

$$F_\sigma(x,\sigma) = 0 \tag{20}$$

$$F_x(x,\sigma) = 0 \tag{21}$$

for all x and σ. Here, $F_\sigma(x,\sigma)$ is a vector of dimension $n+m$, and $F_x(x,\sigma)$ is a Jacobian matrix of dimension $(n+m)\times n$.

To find the matrices associated to the linear approximation—that is, $g_x(\bar{x},0)$, $g_\sigma(\bar{x},0)$, $h_x(\bar{x},0)$, and $h_\sigma(\bar{x},0)$—we note that (20) and (21) must also hold at the steady state $(\bar{x},0)$. We now use (19) to find the required derivatives. Differentiating (19) with respect to the scalar σ—and keeping the argument of the functions implicit to save on notation—gives

$$\begin{aligned} F_\sigma(\bar{x},0) &= E_t\left[f_{x'}(h_\sigma+\eta\varepsilon_{t+1}) + f_{y'}(g_x(h_\sigma+\eta\varepsilon_{t+1})+g_\sigma) + f_y g_\sigma\right] \\ &= f_{x'}(h_\sigma+\eta E_t[\varepsilon_{t+1}]) + f_{y'}(g_x(h_\sigma+\eta E_t[\varepsilon_{t+1}])+g_\sigma) + f_y g_\sigma \\ &= f_{x'}h_\sigma + f_{y'}(g_x h_\sigma + g_\sigma) + f_y g_\sigma \\ &= \left[f_{x'}+f_{y'}g_x\right]h_\sigma + \left[f_{y'}+f_y\right]g_\sigma, \end{aligned}$$

where the second equality uses that we are evaluating all expressions at the steady state, so that the only random object is the mean zero shock ε_{t+1}, and the third equality uses $E_t[\varepsilon_{t+1}]=0$. At this point we are after the derivatives $h_\sigma(\bar{x},0)$ and $g_\sigma(\bar{x},0)$. The expressions f_y, $f_{y'}$, and $f_{x'}$ are evaluated at the steady state and, thus, are known. As mentioned above, the previous expression has to be equal to zero, so that $F_\sigma(\bar{x},0)=0$ implies

$$\left[f_{x'}+f_{y'}g_x\right]h_\sigma + \left[f_{y'}+f_y\right]g_\sigma = \bar{0}_{(n+m)\times 1},$$

or

$$\begin{bmatrix} f_{x'}+f_{y'}g_x & f_{y'}+f_y \end{bmatrix}\begin{bmatrix} h_\sigma \\ g_\sigma \end{bmatrix} = \bar{0}.$$

This is a homogeneous linear sytem of equations in the unknowns (h_σ, g_σ). Clearly, $h_\sigma=0$ and $g_\sigma=0$ is *one solution* of this system. If there is another solution with $\tilde{h}_\sigma \neq 0$ or $\tilde{g}_\sigma \neq 0$, then $\alpha\tilde{h}_\sigma$ and $\alpha\tilde{g}_\sigma$

is also a solution of the system for any number α. But since we are looking for a pair of *unique* policy functions, it then must be the case that $h_\sigma = 0$ and $g_\sigma = 0$.

This has the important implication that the "amount of uncertainty", as reflected in the parameter σ, is irrelevant in a first order approximation to the policy functions. This result is usually called the "certainty equivalence principle," a property that no longer holds in higher order approximations of the model or using other solution methods. This also means that, to a first order approximation, the unconditional expected values of x_t and y_t are equal to their non-stochastic steady state values $\bar{x}$ and $\bar{y}$. In effect, to see that $E[x_t] = \bar{x}$ and $E[y_t] = \bar{y}$, note that using the linear approximation to the policy functions we obtain

$$\begin{aligned} E[x_{t+1}] &= E[h(x_t, \sigma)] \\ &\cong h(\bar{x}, 0) + h_x(\bar{x}, 0)(E[x_t] - \bar{x}) + h_\sigma(\bar{x}, 0)\sigma \\ &= \bar{x} + h_x(\bar{x}, 0)(E[x_t] - \bar{x}), \end{aligned}$$

where the third equality uses $h(\bar{x}, 0) = \bar{x}$ and $h_\sigma = 0$. Furthermore, using $E[x_t] = E[x_{t+1}]$ and rearranging gives

$$(I - h_x(\bar{x}, 0))(E[x_t] - \bar{x}) = 0.$$

If no eigenvalue of $h_x(\bar{x}, 0)$ is one in absolute value—as will be the case in the set of solutions we are looking for—then the only solution to this equation is $E[x_t] = \bar{x}$ which proves the claim.

We now discuss how to find the perturbation matrices $g_x(\bar{x}, 0)$ and $h_x(\bar{x}, 0)$. To do this, differentiate expression (19) with respect to x and evaluate the resulting expressions at the steady state $(\bar{x}, 0)$,

$$\begin{aligned} \bar{0}_{(n+m)\times n} &= F_x(\bar{x}, 0) \\ &= f_{x'} h_x + f_{y'} g_x h_x + f_x + f_y g_x \\ &= (f_{x'} + f_{y'} g_x) h_x + f_x + f_y g_x \\ &= \begin{bmatrix} f_{x'} & f_{y'} \end{bmatrix} \begin{bmatrix} I \\ g_x \end{bmatrix} h_x + \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} I \\ g_x \end{bmatrix}. \end{aligned}$$

Rearranging gives

$$\begin{bmatrix} f_{x'} & f_{y'} \end{bmatrix} \begin{bmatrix} I \\ g_x \end{bmatrix} h_x = - \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} I \\ g_x \end{bmatrix}. \tag{22}$$

This is a system of $(n+m) \times n$ quadratic equations in $(n+m) \times n$ unknowns given by the elements of g_x and h_x.

Let $A = \begin{bmatrix} f_{x'} & f_{y'} \end{bmatrix}$ and $B = - \begin{bmatrix} f_x & f_y \end{bmatrix}$ be $(n+m) \times (n+m)$ square matrices and note that, because $f_{x'}$, f_x, $f_{y'}$, and f_y are evaluated at the known steady state, A and B are known.

Let's define $\hat{x}_{t+j} = E_t[x_{t+j}] - \bar{x}$ and $\hat{y}_{t+j} = E_t[y_{t+j}] - \bar{y}$. When $j = 0$, we have $\hat{x}_t = x_t - \bar{x}$ and

$\hat{y}_t = y_t - \bar{y}$. Now post-multiply both sides of the system of equations (22) by $\hat{x}_t$ to obtain

$$A \begin{bmatrix} I \\ g_x \end{bmatrix} h_x \hat{x}_t = B \begin{bmatrix} I \\ g_x \end{bmatrix} \hat{x}_t$$

or

$$A \begin{bmatrix} h_x \hat{x}_t \\ g_x h_x \hat{x}_t \end{bmatrix} = B \begin{bmatrix} \hat{x}_t \\ g_x \hat{x}_t \end{bmatrix} \tag{23}$$

We now prove that $\hat{x}_{t+1} \approx h_x \hat{x}_t$, $\hat{y}_t \approx g_x \hat{x}_t$ and $\hat{y}_{t+1} \approx g_x h_x \hat{x}_t$. To see this, consider first

$$h_x \hat{x}_t = h_x \left(\bar{x}, 0\right) \left(x_t - \bar{x}\right).$$

Now, the Taylor expansion (18) and the result $h_\sigma = 0$, gives

$$x_{t+1} - \bar{x} \approx h_x \hat{x}_t + \eta \varepsilon_{t+1}. \tag{24}$$

Taking the conditional expectations at time t on both sides of the last equation then gives

$$E_t \left[x_{t+1}\right] - \bar{x} \equiv \hat{x}_{t+1} \approx h_x \hat{x}_t.$$

Likewise, using (17) and $g_\sigma = 0$ implies

$$y_t - \bar{y} \equiv \hat{y}_t \approx g_x \hat{x}_t$$

and

$$\hat{y}_{t+1} \approx g_x \hat{x}_{t+1} \approx g_x h_x \hat{x}_t.$$

Using these results into (23) gives

$$A \begin{bmatrix} \hat{x}_{t+1} \\ \hat{y}_{t+1} \end{bmatrix} = B \begin{bmatrix} \hat{x}_t \\ \hat{y}_t \end{bmatrix}. \tag{25}$$

which is a usual representation of the equilibrium conditions of a linearized rational expectations model. Equivalently, this condition can be written as

$$A E_t \begin{bmatrix} x_{t+1} - \bar{x} \\ y_{t+1} - \bar{y} \end{bmatrix} = B \begin{bmatrix} x_t - \bar{x} \\ y_t - \bar{y} \end{bmatrix}, \tag{26}$$

We now use the QZ decomposition of Theorem 1 to solve this linear system of difference equations.

Define the vector

$$\hat{w}_t = \begin{bmatrix} \hat{x}_t \\ \hat{y}_t \end{bmatrix} \tag{27}$$

and write the linearized system as

$$A\hat{w}_{t+1} = B\hat{w}_t \tag{28}$$

We look for non-explosive solutions that satisfy

$$\lim_{j\to\infty} \hat{w}_{t+j} = 0 \tag{29}$$

which means that, at any point in time, the system is expected to converge to the steady state.

By Theorem 1, there are unitary matrices Q and Z, and upper triangular matrices S and T such that

$$\begin{aligned} QAZ &= S \\ QBZ &= T. \end{aligned}$$

Moreover, the matrices can be arranged so that the diagonal elements s_{ii} and t_{ii} appear in any order. We choose the following ordering of the pairs (s_{ii}, t_{ii}) : the ones satisfying $|s_{ii}| > |t_{ii}|$ appear in the first block of diagonal elements of S and T. We call these pairs of elements the *stable generalized eigenvalues* for reasons that will become apparent below. We also impose the following assumption,

Assumption 1: There is no i such that $|s_{ii}| = |t_{ii}|$. In other words, the system (28) does not have a unit root.

Because Z is unitary (so that $ZZ^H = I$), we can premultiply (28) by Q to obtain

$$\begin{aligned} QA\hat{w}_{t+1} &= QB\hat{w}_t \\ QAZZ^H\hat{w}_{t+1} &= QBZZ^H\hat{w}_t \\ SZ^H\hat{w}_{t+1} &= TZ^H\hat{w}_t. \end{aligned}$$

where the second line uses the QZ decomposition.

Definining the new variable

$$z_t \equiv Z^H\hat{w}_t, \tag{30}$$

the system can be written as

$$Sz_{t+1} = Tz_t.$$

This is progress because we wrote the general first order difference equation as another first order difference equation but where the relevant matrices (S and T) are triangular.

Using that S and T are upper triangular matrices, we write the system in block form as

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} z_{t+1}^s \\ z_{t+1}^u \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} z_t^s \\ z_t^u \end{bmatrix}. \tag{31}$$

where the partition of the matrices is constructed so that S_{11} and T_{11} contain the pairs of elements s_{ii}, t_{ii} such that $|s_{ii}| > |t_{ii}|$, and z_t is partitioned accordingly—that is, we put the stable eigenvalues first in the system. This inequality together with Assumption 1 means that the diagonal elements of S_{22} and T_{22} satisfy $|s_{ii}| < |t_{ii}|$.

Now focus on the lower block of the system (31),

$$S_{22} z_{t+1}^u = T_{22} z_t^u,$$

Theorem 1 (Point 4) implies that there is no index i such that $s_{ii} = t_{ii} = 0$. Therefore, the inequality $|s_{ii}| < |t_{ii}|$ for the submatrices S_{22} and T_{22} implies that all the diagonal elements of T_{22} are non-zero. Furthermore, because T_{22} is upper triangular, Result 2 above implies that T_{22} is invertible. Therefore, premultiplying both sides of the previous expression by T_{22}^{-1} we obtain

$$T_{22}^{-1} S_{22} z_{t+1}^u = z_t^u.$$

Results 3 and 4 above imply that $T_{22}^{-1} S_{22}$ is also upper triangular with diagonal elements s_{ii}/t_{ii} (because the product of triangular matrices is triangular). Since the eigenvalues of an upper triangular matrix are its diagonal entries, it follows that the matrix $T_{22}^{-1} S_{22}$ *has all its eigenvalues smaller than one in absolute value*. It then follows that at least one element of z_t^u has to explode to infinity in absolute value *unless* $z_t^u = 0$ for all t.[3] In other words, the only stable solution of the lower block of the system (31) is $z_t^u = 0$ for all t.

We now proceed to find the solution of the first block of equations in the system (31) (that corresponding to z_t^s). Using that $z_t^u = 0$ for all t, the first block of (31) implies

$$S_{11} z_{t+1}^s = T_{11} z_t^s.$$

Now, because S_{11} and T_{11} are such that $|s_{ii}| > |t_{ii}|$, it follows that all the diagonal elements of S_{11} are different from zero and, therefore, Result 2 implies that S_{11} is invertible. We thus have

$$z_{t+1}^s = S_{11}^{-1} T_{11} z_t^s. \tag{32}$$

By Results 3 and 4, $S_{11}^{-1} T_{11}$ is upper triangular with diagonal elements $|t_{ii}/s_{ii}| < 1$ for all i (this follows

[3] To obtain intuition, note that if S_{22} were invertible (which may not be), then $\left(T_{22}^{-1} S_{22}\right)^{-1} = S_{22}^{-1} T_{22}$ would be an upper triangular matrix with all its eigenvalues greater than 1 in absolute value and $z_{t+1}^u = S_{22}^{-1} T_{22} z_t^u$. This means that, unless $z_t^u = 0$, z_t^u would explode towards (plus or minus) infinity.

from $|s_{ii}| > |t_{ii}|$ on S_{11} and T_{11}). Therefore, (32) is a stable first order difference equation that converges to zero as $t \to \infty$ for any value of z_0^s. In other words, (32) is the solution to the first block of the system (31).

However, we are interested in the solution in terms of the variables $\hat{w}_t$, not in terms of z_t. To recover the solution in terms of $\hat{w}_t$, we use that Z^H is invertible (with inverse Z) and, from (30), it follows that

$$\hat{w}_t \equiv \begin{bmatrix} \hat{w}_t^s \\ \hat{w}_t^u \end{bmatrix} = Z \begin{bmatrix} z_t^s \\ z_t^u \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} z_t^s \\ z_t^u \end{bmatrix} \tag{33}$$

where $\hat{w}_t$ and Z were partitioned in terms of the stable and unstable eigenvalues as we did above with S and T. That is, $\hat{w}_t$ is a linear combination of z_t^s and z_t^u. But because $z_t^u = 0$ for all t and $z_t^s \to 0$ as $t \to \infty$ for any initial condition, it follows that $\hat{w}_t$ also converges to zero as $t \to \infty$. This proves that the stability condition (29) is satisfied.

Consider now the first block of (33), where we use $z_t^u = 0$ for all t,

$$\hat{w}_t^s = Z_{11} z_t^s.$$

We now impose the following assumption:

Assumption 2: Z_{11} is invertible.

With this assumption,

$$z_t^s = Z_{11}^{-1} \hat{w}_t^s.$$

Combining this expression with (32) gives

$$Z_{11}^{-1} \hat{w}_{t+1}^s = S_{11}^{-1} T_{11} Z_{11}^{-1} \hat{w}_t^s.$$

Now, pre-multiplying both sides of this expression by Z_{11} gives

$$\hat{w}_{t+1}^s = H \hat{w}_t^s. \tag{34}$$

where

$$H = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}.$$

Under Assumption 2, the eigenvalues of H are the same as the eigenvalues of $S_{11}^{-1} T_{11}$,[4] which is an upper triangular matrix with eigenvalues $|t_{ii}/s_{ii}| < 1$. This is the solution to the first block of equations $\hat{w}_t^s$.

[4] This follow because of the following result: Let A by a square $n \times n$ matrix and G an $n \times n$ non-singular matrix. Then, A and GAG^{-1} have the same eigenvalues. Proof:

$$\det\left(GAG^{-1} - \lambda I\right) = \det\left(GAG^{-1} - \lambda GG^{-1}\right) = \det\left(G\left(A - \lambda I\right)G^{-1}\right) = \det\left(G\right)\det\left(A - \lambda I\right)\det\left(G^{-1}\right) = \det\left(A - \lambda I\right)$$

because $\det\left(G^{-1}\right) = 1/\det\left(G\right)$.

We now obtain the solution of $\hat{w}_t^u$. The second block of (33) together with $z_t^s = Z_{11}^{-1}\hat{w}_t^s$ implies

$$\hat{w}_t^u = Z_{21} z_t^s = Z_{21} Z_{11}^{-1} \hat{w}_t^s, \tag{35}$$

which gives the solution of $\hat{w}_t^u$ as a function of $\hat{w}_t^s$.

Summarizing the results so far, given an initial condition for w_t^s we have the solution for the entire vector $\hat{w}_t$. As we will show below, in a locally unique rational expectation equilibrium w_t^s coincides with the vector of state variables.

Local existence and uniqueness of the equilibrium

We were able to find a solution to the linear stochastic difference equation (28). However, we didn't find yet the coefficient matrices $h_x(\bar{x}, 0)$ and $g_x(\bar{x}, 0)$ of the linear approximation to the policy functions. This is what we do now. In the process, we analyze the important issues of local existence and uniqueness of the equilibrium.

Local uniqueness of equilibrium and the Blanchard-Kahn condition.—

The **Blanchard and Kahn condition** is the requirement that *the number of stable generalized eigenvalues of the matrix pair* (A, B) *(that is, the number of elements* i *such that* $|t_{ii}/s_{ii}| < 1$*) is exactly equal to the number of state variables* n*.*

If the Blanchard and Kahn condition is satisfied, then the equilibrium of the DSGE model exists and is locally unique. Indeed, if the Blanchard and Kahn condition is satisfied, then Z_{11} is of size $n \times n$ and $\hat{w}_t^s$ is of size $n \times 1$. The definition of $\hat{w}_t$ in equation (27) and the last observation imply that

$$\begin{aligned} \hat{w}_{t+1}^s &= \hat{x}_{t+1} = E_t[x_{t+1}] - \bar{x} \\ \hat{w}_t^s &= \hat{x}_t = x_t - \bar{x}. \end{aligned}$$

Then, (34) implies

$$E_t[x_{t+1}] - \bar{x} = H(x_t - \bar{x}),$$

where $H = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$. Dropping the expectation operator and using (24) leads to

$$x_{t+1} = \bar{x} + H(x_t - \bar{x}) + \eta \varepsilon_{t+1}. \tag{36}$$

This implies that $h_x(\bar{x}, 0) = H = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$ is the matrix of elements that we were looking for. Since x_t is uniquely determined from $\hat{w}_t^s$, the linearized policy function is also unique.

Next, using that $\hat{w}_t^u = y_t - \bar{y}$ (because $\hat{w}_t^s = \hat{x}_t$), equation (35) implies

$$y_t = \bar{y} + Z_{21} Z_{11}^{-1}(x_t - \bar{x}), \tag{37}$$

so that $g_x(\bar{x}, 0) = G = Z_{21}Z_{11}^{-1}$. Again, since y_t is uniquely determined from x_t, the policy function for control variables is also locally unique.

Summarizing, we showed that if the number of stable generalized eigenvalues with absolute value less than one is equal to the number of predetermined variables x_t (the so-called Blanchard and Kahn condition), the equilibrium exists locally around the steady state and is unique. In some models, like the RBC model described above, it can be shown that the Blanchard and Kahn condition must hold and that the equilibrium is locally unique.

No local existence of the equilibrium.—

Suppose now that the number of generalized eigenvalues of (A, B) with absolute value less than unity ($|t_{ii}/s_{ii}| < 1$) is smaller than the number of state variables n. Specifically, suppose that Z_{11} is of size $(n-q) \times (n-q)$ for $0 < q < n$. In this case, $\hat{w}_t^s$ has less elements than the state vector $\hat{x}_t$. The vectors $\hat{w}_t^s$ and $\hat{w}_t^u$ take the form

$$\hat{w}_t^s = \hat{x}_t^a \text{ and } \hat{w}_t^u = \begin{bmatrix} \hat{x}_t^b \\ \hat{y}_t \end{bmatrix},$$

where

$$\hat{x}_t = \begin{bmatrix} \hat{x}_t^a \\ \hat{x}_t^b \end{bmatrix}.$$

In this case, the solution of the difference equation (34) and (35) imply

$$\hat{x}_{t+1}^a = H\hat{x}_t^a$$

$$\begin{bmatrix} \hat{x}_t^b \\ \hat{y}_t \end{bmatrix} = G\hat{x}_t^a.$$

The last expression states that $\hat{x}_t^b$ is determined by $\hat{x}_t^a$. But this is impossible because $\hat{x}_t^b$ is a predetermined variable independent of $\hat{x}_t^a$. Therefore, there is no local equilibrium when the number of generalized eigenvalues of (A, B) with absolute value less than unity is smaller than the number of state variables n.

Local indeterminacy of the equilibrium.—

Finally, suppose that the number of generalized eigenvalues of (A, B) with absolute value less than one is greater than the number of state variables, n. Specifically, suppose that there are $n + q$, for $0 < q \leq m$ generalized eigenvalues with absolute value less than one. Then Z_{11} is size $(n+q) \times (n+q)$. Thus, $\hat{w}_t^s$ has more elements than $\hat{x}_t$ and $\hat{w}_t^u$ has less elements than $\hat{y}_t$. In particular,

$$\hat{w}_t^s = \begin{bmatrix} \hat{x}_t \\ \hat{y}_t^a \end{bmatrix}; \ \hat{w}_t^u = \hat{y}_t^b$$

where $\hat{y}_t^a$ is a vector with the first q elements of the vector $\hat{y}_t$, and $\hat{y}_t^b$ is a vector with the remaning $m-q$ elements of $\hat{y}_t$, so that

$$\hat{y}_t = \begin{bmatrix} \hat{y}_t^a \\ \hat{y}_t^b \end{bmatrix}.$$

According to (34) and (35), the laws of motion for $\hat{x}_t$ and $\hat{y}_t$ are as follows

$$\begin{bmatrix} \hat{x}_{t+1} \\ \hat{y}_{t+1}^a \end{bmatrix} = H \begin{bmatrix} \hat{x}_t \\ \hat{y}_t^a \end{bmatrix} \tag{38}$$

$$\hat{y}_t^b = G \begin{bmatrix} \hat{x}_t \\ \hat{y}_t^a \end{bmatrix}. \tag{39}$$

Now, since $\hat{y}_t^a$ is a jump variable, it is not predetermined at time t. Therefore, we can freely pick $\hat{y}_t^a$ at an arbitrary period, say $t=0$, and then solve the above system in the variables $\hat{x}_t$ and $\hat{y}_t$. But since $\hat{y}_0^a$ was arbitrary, this means that the equilibrium is indeterminate.

Furthermore, there is even a deeper sense in which the equilibrium is indeterminate. The above equations are written in terms of variables with expected values. Since there is nothing that ties $\hat{y}_t^a$ to previous decisions, we can always drop the expectation operator and write the above system as

$$\begin{bmatrix} x_{t+1} \\ y_{t+1}^a \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{y}^a \end{bmatrix} + H \begin{bmatrix} x_t - \bar{x} \\ y_t^a - \bar{y}^a \end{bmatrix} + \begin{bmatrix} \eta & 0 \\ \nu_\varepsilon & \nu_\mu \end{bmatrix} \begin{bmatrix} \varepsilon_{t+1} \\ \mu_{t+1} \end{bmatrix}$$

$$y_t^b - \bar{y}^b = G \begin{bmatrix} x_t - \bar{x} \\ y_t^a - \bar{y}^a \end{bmatrix},$$

where μ_{t+1} is an arbitrary mean zero stochastic process of size $q \times 1$ and variance covariance matrix equal to the identity matrix. The matrices ν_ε and ν_μ are also arbitrary. To see that this solves the difference equation, take conditional expectations and we return to the system (38) and (39).

Summarizing, this shows that if the number of generalized eigenvalues of (A, B) with absolute value less than one is greater than the number of state variables, we can construct many equilibria where non-fundamental uncertainty or sunspots, μ_{t+1}, affect the evolution of the equilibrium quantities of the model. There is a literature that exploits this indeterminacy to study sunspots and multiple equilibria.

UNCONDITIONAL SECOND MOMENTS

This part of the note discusses how to compute unconditional second moments using the first order approximation to the policy functions. Because the policy functions end up being linear, it is easy to compute unconditional (population) second moments for all the variables of interest.

We first compute the matrices $h_x(\bar{x}, 0) \equiv h_x$ and $g_x(\bar{x}, 0) \equiv g_x$ using the method described above. If

we let $\tilde{x}_t \equiv x_t - \bar{x}$ and $\tilde{y}_t = y_t - \bar{y}$ denote the deviation of the variables x_t and y_t relative to their steady state values, we can write the linearized policy functions as

$$\tilde{x}_{t+1} = h_x \tilde{x}_t + \eta \varepsilon_{t+1} \tag{40}$$

$$\tilde{y}_t = g_x \tilde{x}_t. \tag{41}$$

where we are evaluating the policy function at the original economy (that is, when $\sigma = 1$) Our objective in this section is to compute all the second moments of the variables $\tilde{x}_t$ and $\tilde{y}_t$.

Covariance matrix of x_t

Let

$$\Sigma_x \equiv E\left[(x_t - \bar{x})(x_t - \bar{x})'\right] = E\left[\tilde{x}_t \tilde{x}_t'\right]$$

Now take equation (40) and post-multiply both sides by its transpose:

$$\begin{aligned}
\tilde{x}_{t+1}\tilde{x}_{t+1}' &= (h_x \tilde{x}_t + \eta \varepsilon_{t+1})(h_x \tilde{x}_t + \eta \varepsilon_{t+1})' \\
&= (h_x \tilde{x}_t + \eta \varepsilon_{t+1})\left(\tilde{x}_t' h_x' + \varepsilon_{t+1}' \eta'\right) \\
&= h_x\left(\tilde{x}_t \tilde{x}_t'\right) h_x' + \eta\left(\varepsilon_{t+1}\tilde{x}_t'\right) h_x' + h_x\left(\tilde{x}_t \varepsilon_{t+1}'\right)\eta' + \eta \varepsilon_{t+1}\varepsilon_{t+1}'\eta'
\end{aligned}$$

Taking uncondicional expectations and using $E\left(\varepsilon_{t+1}\tilde{x}_t'\right) = E\left(\tilde{x}_t \varepsilon_{t+1}'\right) = 0$ because ε_{t+1} is an i.i.d. shock we obtain

$$E\left[\tilde{x}_{t+1}\tilde{x}_{t+1}'\right] = h_x E\left[\tilde{x}_t \tilde{x}_t'\right] h_x' + \eta E\left[\varepsilon_{t+1}\varepsilon_{t+1}'\right]\eta'$$

Now, using that $\tilde{x}_t$ is a covariance stationary process, so that $E\left[\tilde{x}_{t+1}\tilde{x}_{t+1}'\right] = E\left[\tilde{x}_t \tilde{x}_t'\right] = \Sigma_x$ and that $E\left[\varepsilon_{t+1}\varepsilon_{t+1}'\right] = I$, we have that Σ_x satisfies the following equation

$$\Sigma_x = h_x \Sigma_x h_x' + \Sigma_\varepsilon. \tag{42}$$

where we define $\Sigma_\varepsilon \equiv \eta\eta'$. This is an equation for Σ_x. The following are two methods to compute Σ_x.

Method 1: This method uses the following useful result. Let A, B, and C be matrices such that ABC is well defined. Then,

$$vec(ABC) = \left(C' \otimes A\right) vec(B),$$

where $vec(A)$ is the "vectorization" operator that transforms a matrix A into a column vector by stacking the column vectors of A, and $A \otimes B$ denotes the Kronecker product of the matrices A and B.

Now apply the *vec* operator to both sides of (42),

$$vec(\Sigma_x) = vec\left(h_x \Sigma_x h_x'\right) + vec(\Sigma_\varepsilon)$$

Using the result described above

$$vec\left(\Sigma_x\right) = h_x \otimes h_x vec\left(\Sigma_x\right) + vec(\Sigma_\varepsilon)$$

Therefore, the value of Σ_x solves

$$vec\left(\Sigma_x\right) = \left(I - h_x \otimes h_x\right)^{-1} vec(\Sigma_\varepsilon).$$

One possible drawback of this approach is that we have to invert a matrix of size $n^2 \times n^2$.

Method 2: The second method uses an iterative algorithm called *doubling algorithm.* We start with a guess of Σ_x, which we call Σ_x^0, and compute iteratively updated guesses of Σ_x according to the following algorithm: for $j = 1, 2, ...$ compute

$$\Sigma_x^j = h_x \Sigma_x^{j-1} h_x' + \Sigma_\varepsilon.$$

Stop the algorithm when $\Sigma_x^j \approx \Sigma_x^{j-1}$. We can start the algorithm with $\Sigma_x^0 = I$.

Autocovariances of x_t

We now can easily compute the autocovariances of x_t of any order. Consider the autocovariance of order $\tau > 0$, which we denote by $\Sigma_x(\tau)$. Then,

$$\Sigma_x(\tau) = E\left[\tilde{x}_t \tilde{x}_{t-\tau}'\right].$$

Note also that $\Sigma_x(0) = \Sigma_x$ is the covariance matrix of x_t.

To compute this matrix, let $\mu_t \equiv \eta \varepsilon_t$, and write

$$\begin{aligned}
\tilde{x}_t &= h_x \tilde{x}_{t-1} + \mu_t \\
&= h_x\left(h_x \tilde{x}_{t-2} + \mu_{t-1}\right) + \mu_t \\
&= h_x^2 \tilde{x}_{t-2} + h_x \mu_{t-1} + \mu_t \\
&= h_x^2\left(h_x \tilde{x}_{t-3} + \mu_{t-2}\right) + h_x \mu_{t-1} + \mu_t \\
&= h_x^3 \tilde{x}_{t-3} + h_x^2 \mu_{t-2} + h_x \mu_{t-1} + \mu_t \\
&\vdots \\
&= h_x^\tau \tilde{x}_{t-\tau} + h_x^{\tau-1} \mu_{t-(\tau-1)} + h_x^{\tau-2} \mu_{t-(\tau-2)} + ... + h_x \mu_{t-1} + \mu_t
\end{aligned}$$

or

$$\tilde{x}_t = h_x^\tau \tilde{x}_{t-\tau} + \sum_{j=0}^{\tau-1} h_x^j \mu_{t-j}$$

Therefore

$$\begin{aligned}
\Sigma_x(\tau) &= E\left[\tilde{x}_t\tilde{x}'_{t-\tau}\right] \\
&= E\left[\left(h_x^\tau\tilde{x}_{t-\tau}+\sum_{j=0}^{\tau-1}h_x^j\mu_{t-j}\right)\tilde{x}'_{t-\tau}\right] \\
&= E\left[h_x^\tau\tilde{x}_{t-\tau}\tilde{x}'_{t-\tau}+\sum_{j=0}^{\tau-1}h_x^j\mu_{t-j}\tilde{x}'_{t-\tau}\right] \\
&= h_x^\tau E\left[\tilde{x}_{t-\tau}\tilde{x}'_{t-\tau}\right]+\sum_{j=0}^{\tau-1}h_x^j E\left[\mu_{t-j}\tilde{x}'_{t-\tau}\right] \\
&= h_x^\tau\Sigma_x(0)
\end{aligned}$$

where we used that $E\left[\mu_{t-j}\tilde{x}'_{t-\tau}\right]=E\left[\eta\varepsilon_{t-j}\tilde{x}'_{t-\tau}\right]=0$ for all j and τ.

Summarizing, the autocovariance matrices of x_t of order τ satisfy the equation

$$\Sigma_x(\tau)=h_x^\tau\Sigma_x. \tag{43}$$

Equivalently, we can compute iteratively this matrices as $\Sigma_x(\tau)=h_x\Sigma_x(\tau-1)$ starting with $\tau=1$.

Second moments of the control variables y_t

We now compute the second moments of y_t using the second moments of x_t. To that end, let

$$\Sigma_y=E\left[\tilde{y}_t\tilde{y}'_t\right]$$

$$\Sigma_y(\tau)=E\left[\tilde{y}_t\tilde{y}'_{t-\tau}\right]$$

Note that (41) implies

$$\begin{aligned}
\Sigma_y &= E\left[\tilde{y}_t\tilde{y}'_t\right] \\
&= E\left[\left(g_x\tilde{x}_t\right)\left(g_x\tilde{x}_t\right)'\right] \\
&= g_xE\left[\tilde{x}_t\tilde{x}'_t\right]g'_x
\end{aligned}$$

or

$$\Sigma_y=g_x\Sigma_xg'_x. \tag{44}$$

Likewise,

$$\begin{aligned}
\Sigma_y(\tau) &= E\left[\tilde{y}_t\tilde{y}'_{t-\tau}\right] \\
&= g_xE\left[\tilde{x}_t\tilde{x}'_{t-\tau}\right]g'_x
\end{aligned}$$

so that

$$\Sigma_y(\tau) = g_x \Sigma_x(\tau) g_x'. \tag{45}$$

Spectral density of x_t and y_t

Write the evolution of $\tilde{x}_t$ in terms of the lag operator L,

$$(1 - h_x L)\tilde{x}_t = \eta \varepsilon_t$$

If the model has no unit roots, then all the eigenvalues of the matrix $h_x(\bar{x}, 0)$ are less than one in absolute value. We can therefore obtain the infinite order moving average representation of $\tilde{x}_t$ inverting the lag-polynomial

$$\tilde{x}_t = (I - h_x L)^{-1} \eta \varepsilon_t,$$

where I an $n \times n$ identity matrix. Using results from our second lecture (or see formula 10.4.43 in page 277 of Hamilton, 1994), we can compute the spectral density of $\tilde{x}_t$ using the spectral density of $\eta \varepsilon_t$,

$$S_{\tilde{x}}(\omega) = \left(I - h_x e^{-i\omega}\right)^{-1} S_{\eta\varepsilon_t}(\omega) \left[\left(I - h_x e^{-i\omega}\right)^{-1}\right]^*$$

Using properties of the inverse and the conjugate transpose gives

$$S_{\tilde{x}}(\omega) = \left(I - h_x e^{-i\omega}\right)^{-1} S_{\eta\varepsilon_t}(\omega) \left(I - h_x' e^{i\omega}\right)^{-1}$$

But using that ε_t is uncorrelated with identity covariance matrix, we obtain

$$S_{\eta\varepsilon_t}(\omega) = \Sigma_\varepsilon (= \eta\eta').$$

Therefore, the spectral density of the state variables $\tilde{x}_t$ is

$$S_{\tilde{x}}(\omega) = \left(I - h_x e^{-i\omega}\right)^{-1} \Sigma_\varepsilon \left(I - h_x' e^{i\omega}\right)^{-1}. \tag{46}$$

To obtain the spectral density matrix of the control variables $\tilde{y}_t$, we note that

$$\tilde{y}_t = g_x \tilde{x}_t.$$

It then follows that

$$S_{\tilde{y}}(\omega) = g_x S_{\tilde{x}}(\omega) g_x'$$

or

$$S_{\tilde{y}}(\omega) = g_x \left(I - h_x e^{-i\omega}\right)^{-1} \Sigma_\varepsilon \left(I - h_x' e^{i\omega}\right)^{-1} g_x'. \tag{47}$$

IMPULSE RESPONSE FUNCTIONS (IR)

The easiest way to compute an impulse response function is to simulate the model for a particular history of shocks to the variable of interest in the shock vector ε_t. Suppose that we want to compute the impulse responses to a shock of one standard deviation in the first element of the vector ε_t, denoted by $\varepsilon_{1,t}$, at time $t = 0$. We simply simulate the model by setting $\varepsilon_{1,t} = 1$ for $t = 0$ and $\varepsilon_{1,t} = 0$ for all $t > 0$, and set all the other shocks to zero $\varepsilon_{j,t} = 0$ for all t and all $j \neq 1$.

But we can also compute the impulse response analytically using the policy functions of the model. We use the following definition

The impulse response to a variable z_t in period $t + j$ to an arbitrary impulse in period t (i.e. an arbitrary shock to the vector ε_t) is defined as

$$IR\left(z_{t+j}\right) = E_t\left[z_{t+j}\right] - E_{t-1}\left[z_{t+j}\right].$$

Notice the timing of the expectation operators. The impulse response tells us the new information that we acquire exactly at time t of the variable z_t in period $t + j$. New information means what we know at time t that we didn't know at time $t - 1$.

We start by computing the impulse response to the state variables x_t. Using the policy function $\tilde{x}_{t+1} = h_x\tilde{x}_t + \eta\varepsilon_{t+1}$ we have $E_t\left[\tilde{x}_{t+j}\right] = h_x^j\tilde{x}_t$. Suppose that at time $t = 0$ there is an initial impulse $\tilde{x}_0 = \eta\varepsilon_0$ (here we use $\tilde{x}_{-1} = 0$ so that the economy starts at the steady state). Applying the law of iterated expectations and $E_{-1}\varepsilon_0 = 0$ we have

$$\begin{aligned} E_0\left[\tilde{x}_t\right] &= h_x^t\tilde{x}_0 \\ E_{-1}\left[\tilde{x}_t\right] &= h_x^tE_{-1}\tilde{x}_0 = h_x^t\eta E_{-1}\varepsilon_0 = 0. \end{aligned}$$

Therefore, the impulse response to x_t at time t to an impulse ε_0 at time 0 is

$$IR\left(\tilde{x}_t\right) = E_0\left[\tilde{x}_t\right] - E_{-1}\left[\tilde{x}_t\right] = h_x^t\tilde{x}_0.$$

The impulse to the vector of control variables $\tilde{y}_t = g_x\tilde{x}_t$ is thus

$$IR\left(\tilde{y}_t\right) = g_xh_x^t\tilde{x}_0.$$

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