Juan I. Mendiviño 37.102.205	Maestría en Econometría Econometría	HOJA N° 1/3 FECHA 29/05/24

① $\{y_i, x_i\}_{i=1}^{n}$ tal que:

$$y_i = x_i\beta + e_i$$
$$E(e_i|x_i) = 0$$
$$E(e_i^2|x_i) = \sigma^2(x_i)$$

$$\tilde{\beta} = \frac{\sum_{i=1}^{n} h(x_i)y_i}{\sum_{i=1}^{n} h(x_i)x_i} \qquad E[h(x_i)x_i] \neq 0$$

a) $\tilde{\beta} = \left[n^{-1}\left(\sum_{i=1}^{n} h(x_i)x_i\right)\right]^{-1}\left[n^{-1}\sum_{i=1}^{n} h(x_i)y_i\right]$

$$\tilde{\beta} = \left[n^{-1}\left(\sum_{i=1}^{n} h(x_i)x_i\right)\right]^{-1}\left[n^{-1}\sum_{i=1}^{n} h(x_i)(x_i\beta + e_i)\right]$$

$$\tilde{\beta} = \left[n^{-1}\sum_{i=1}^{n} h(x_i)x_i\right]^{-1}\left[n^{-1}\beta\sum_{i=1}^{n} h(x_i)x_i + n^{-1}\sum_{i=1}^{n} h(x_i)e_i\right]$$

$$\tilde{\beta} = \beta + \left[n^{-1}\sum_{i=1}^{n} h(x_i)x_i\right]\left[n^{-1}\sum_{i=1}^{n} h(x_i)e_i\right]$$

$$\text{plim}\,\tilde{\beta} = \beta + [E[h(x_i)x_i]]^{-1} E[h(x_i)e_i]$$
$$'' \quad = \beta + [E[h(x_i)x_i]]^{-1} E(h(x_i))\, E(e_i)$$
$$'' \quad = \beta + [E[h(x_i)x_i]]^{-1} E[h(x_i)]\, E[E(e_i|x_i)]$$
$$'' \quad = \beta + [E[h(x_i)x_i]]^{-1} E[h(x_i)]\, E(\underbrace{0}_{=0})$$

$$\text{plim}\,\tilde{\beta} = \beta$$

$$\tilde{\beta} - \beta = \left[n^{-1}\sum_{i=1}^{n} h(x_i)x_i\right]^{-1}\left[n^{-1}\sum_{i=1}^{n} h(x_i)e_i\right]$$

$$\sqrt{n}(\tilde{\beta} - \beta) = \left[n^{-1}\sum_{i=1}^{n} h(x_i)x_i\right]^{-1}\left[n^{-1/2}\sum_{i=1}^{n} h(x_i)e_i\right]$$

Se sabe que:

$$n^{-1/2}\sum_{i=1}^{n} h(x_i)e_i \xrightarrow{d} \text{Normal}\left(0, \sigma^2_{(x_i)} E[h(x_i)x_i]\right)$$

NOTA

De la demostración de consistencia, se sabe que:

$$\left[n^{-1}\sum_{i=1}^{n} h(x_i)\,x_i\right]^{-1} - \left[E\left[h(x_i)\,x_i\right]\right]^{-1} = o_p(1)$$

Por lo tanto:

$$\sqrt{n}\,(\tilde{\beta}-\beta) = \left[E\left[h(x_i)x_i\right]\right]^{-1}\left[n^{-1}\sum_{i=1}^{n} h(x_i)\,e_i\right] + o_p(1)$$

$$\sqrt{n}\,(\tilde{\beta}-\beta) \xrightarrow{d} \text{Normal}\left(0, \underbrace{\sigma^2_{(x_i)}\left[E\left[h(x_i)x_i\right]\right]^{-1}}_{\text{varianza asintótica}}\right).$$

b) $h(x_i) = \dfrac{x_i}{\sigma^2(x_i)}$

$$\sigma^2_{(x_i)}\left[E\left[h(x_i)x_i\right]\right]^{-1} = \sigma^2(x_i)\left[E\left(\frac{x_i}{\sigma^2(x_i)}x_i\right)\right]^{-1}$$

$$= \sigma^2(x_i)\left(\frac{x_i^2}{\sigma^2(x_i)}\right)^{-1}$$

$$= \frac{\sigma^4(x_i)}{x_i^2}.$$

Entonces:

$$\sqrt{n}\,(\tilde{\beta}-\beta) \xrightarrow{d} \text{Normal}\left(0, \frac{\sigma^4(x_i)}{x_i^2}\right)$$

$$\tilde{\beta}-\beta \xrightarrow{d} \text{Normal}\left(0, \frac{\sigma^4(x_i)}{x_i^2\, n}\right)$$

$$\tilde{\beta} \xrightarrow{d} \text{Normal}\left(\beta, \frac{\sigma^4(x_i)}{x_i^2\, n}\right).$$

NOTA

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② $f(y) = \frac{1}{\sqrt{2\pi\theta}} e\left[\frac{-(y-\theta)^2}{2\theta}\right]$, donde $\theta > 0$

$$L(y;\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} e\left[\frac{-(y-\theta)^2}{2\theta}\right]$$

$$L(y;\theta) = \left(\frac{1}{\sqrt{2\pi\theta}}\right)^n e\left[\frac{-1}{2\theta}\sum_{i=1}^{n}(y_i-\theta)^2\right]$$ ⇒ función de verosimilitud

$$\ln[L(y;\theta)] = \ln\left(\frac{1}{\sqrt{2\pi\theta}}\right)^n + \left[\frac{-1}{2\theta}\sum_{i=1}^{n}(y_i-\theta)^2\right]\ln e$$

$$\ln[L(y;\theta)] = n\ln\left(\frac{1}{\sqrt{2\pi\theta}}\right) - \frac{1}{2\theta}\sum_{i=1}^{n}(y_i-\theta)^2 . 1$$

$$\ln[L(y;\theta)] = n\ln 1 - n\ln\sqrt{2\pi\theta} - \frac{1}{2\theta}\sum_{i=1}^{n}(y_i-\theta)^2$$

$$\ln[L(y;\theta)] = n.0 - \frac{n}{2}\ln 2\pi\theta - \frac{1}{2\theta}\sum_{i=1}^{n}(y_i-\theta)^2$$

$$\ln[L(y;\theta)] = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\theta - \frac{1}{2\theta}\sum_{i=1}^{n}(y_i-\theta)^2$$

⇒ logaritmo de la función de verosimilitud

CPO:

$$\frac{\partial \ln[L(y;\theta)]}{\partial\theta} = 0$$

$$\frac{-n}{2\theta} - \frac{2}{2\theta}\sum_{i=1}^{n}(y_i-\theta)(-1) = 0$$

$$\frac{-n}{2\theta} + \frac{1}{\theta}\sum_{i=1}^{n}(y_i-\theta) = 0$$

$$\frac{1}{\theta}\left[\sum_{i=1}^{n}y_i - \sum_{i=1}^{n}\theta\right] = \frac{n}{2\theta}$$

$$\frac{2\theta}{\theta}\left[\sum_{i=1}^{n}y_i - n\theta\right] = n$$

$$2\left(\sum_{i=1}^{n}y_i - n\theta\right) = n$$

$$\sum_{i=1}^{n}y_i - n\theta = \frac{n}{2}$$

$$n\theta = \sum_{i=1}^{n}y_i - \frac{n}{2}$$

$$\theta = \frac{\sum_{i=1}^{n}y_i}{n} - \frac{n}{2n}$$

$$\hat{\theta}_{MLE} = \bar{y} - \frac{1}{2}$$ ⇒ estimador de máxima verosimilitud de θ.

NOTA

CSO:

$$\frac{\partial^2 \ln[L(y;\theta)]}{\partial^2\theta} = \frac{n}{2\theta^2} - \frac{1}{\theta^2}\sum_{i=1}^{n}(y_i-\theta)(-1)$$

$$= \frac{n}{2\theta^2} + \frac{1}{\theta^2}\sum_{i=1}^{n}(y_i-\theta)$$

$$= \frac{n+2\sum_{i=1}^{n}(y_i-\theta)}{2\theta^2}$$

$$= \frac{n+2\sum_{i=1}^{n}y_i - 2n\theta}{2\theta^2}$$

$$= \frac{n+2\sum_{i=1}^{n}y_i - 2n\left(\bar{y}-\frac{1}{2}\right)}{2\left(\bar{y}-\frac{1}{2}\right)^2} \Rightarrow \text{evaluando en } \hat{\theta}_{MLE}$$

$$= \frac{n+2\sum_{i=1}^{n}y_i - 2\sum_{i=1}^{n}y_i + n}{2\left(\frac{\sum_{i=1}^{n}y_i}{n}-\frac{1}{2}\right)^2}$$

$$= \frac{n}{\left(\frac{\sum y_i}{n}-\frac{1}{2}\right)^2} < 0$$ (debería pero debo haber fallado en alguna cuenta)

(3) $y_i = \alpha_i + \beta_i x_i + \varepsilon_i$

$(y_i, x_i)'_{i=1,\dots,n}$ iid y $E(\varepsilon_i) = E(z_i \varepsilon_i) = 0$

$n = 3$

$$\begin{bmatrix} y_1 & x_1 & z_1 \\ y_2 & x_2 & z_2 \\ y_3 & x_3 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 0 & 0 \\ 1 & 1 & -2 \end{bmatrix}$$

a)

. $\sqrt{n}\,(\hat{\beta} - \beta) \xrightarrow{d} \text{Normal}\left(0,\ \sigma^2_{(x_i)} \{E(X'Z)[E(Z'Z)]^{-1} E(Z'X)\}^{-1}\right)$

. $X'Z(Z'Z)^{-1}Z'X = (-1\ 0\ 1)\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \left[(1\ 0\ -2)\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}\right]^{-1} (1\ 0\ -2)\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$= -3 \cdot 5^{-1} \cdot (-3)$

$= \dfrac{9}{5}.$

Entonces, el intervalo del 95% de confianza es:

$$I_{95\%} = \left[-z_{0,975} \cdot \sqrt{\tfrac{9}{5}\sigma^2_{(x_i)}} + \hat{\beta}_{VI}\ ;\ + z_{0,025} \sqrt{\tfrac{9}{5}\sigma^2_{(x_i)}} + \hat{\beta}_{VI} \right]$$

$\hat{\beta}_{VI} = (Z'X)^{-1} Z'y = (-3)^{-1} (1\ 0\ -2)\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = \dfrac{-1}{3}(-1) = \dfrac{1}{3}$

Por lo tanto, se tiene:

$$I_{95\%} = \left[-z_{0,975} \sqrt{\tfrac{9}{5}\sigma^2_{(x_i)}} + \tfrac{1}{3}\ ;\ + z_{0,025} \sqrt{\tfrac{9}{5}\sigma^2_{(x_i)}} + \tfrac{1}{3} \right].$$

NOTA

b) $y_i = \alpha + \beta x_i + \gamma e_i + \varepsilon_i$

$$\hat{\beta} = \frac{Cov(y, x)}{Var(X)}$$

$$\hat{\beta} = \frac{\sum_{i=1} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1} (x_i - \bar{x})}$$

- $\bar{X} = 0$
- $\bar{y} = 2$

$$\hat{\beta} = \frac{\sum_{i=1} x_i (y_i - 2)}{\sum_{i=1} x_i}$$

$$\hat{\beta} = \frac{\sum_{i=1} x_i y_i - 2 \sum_{i=1} x_i}{\sum_{i=1} x_i}$$

$$\hat{\beta} = \sum_{i=1} x_i y_i - 2$$

$$\hat{\beta} = -1.1 + 0.4 + 1.1 - 2$$

$$\hat{\beta} = -1 + 0 + 1 - 2$$

$$\hat{\beta} = -2.$$

Esta estimación comparada con la de variables instrumentales debería arrojar las mismas estimaciones debido a que fueron incorporados los residuos de X sobre Z, los cuales eran los causantes de la endogeneidad en la ecuación $y_i = \alpha + \beta x_i + \varepsilon_i$, debido a estar correlacionados con ε_i, es decir, $Cor(e_i, \varepsilon_i) \neq 0$, por lo cual al incorporar e_i ya no tenemos esa fuente de endogeneidad en la estimación.

NOTA