An RBC model with supply and demand shocks

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Model with supply and demand shocks

- Chari, Kehoe, and McGrattan (2008)
- RBC model with technology and demand shocks
  - Technology shock has unit root
  - Demand shock modeled as a stochastic labor income tax.
- There are consumers, firms, and a government
- Because there are distorting taxes, cannot use the abstraction of the social planner to find the equilibrium
- Model so simple that it is easy to find the competitive equilibrium.

Consumers

Representative consumer maximizes the utility function

$$\max_{C_t, L_t, K_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$$

subject to the budget constraint

$$C_t + K_{t+1} - (1-\delta) K_t = (1-\tau_{lt}) W_t L_t + R_t K_t + T_t$$

- C_t : consumption
- L_t : labor (hours worked)
- K_t : stock capital with K_0 given
- τ_{lt} : a proportional labor income tax
- W_t : real wage
- R_t : real rental rate of capital
- T_t: is a lump-sum transfer
- $\beta \in (0,1)$: discount factor
- $\delta \in (0,1)$: depreciation rate of capital.

The utility function $U\left(C,L\right)$ is given by

$$U\left(C_t,L_t\right)=\frac{\left[C_t\left(1-L_t\right)^{\phi}\right]^{1-\gamma}}{1-\gamma}.$$

where $\gamma>0$ and $\phi>0$.

Lagrangian for the consumer's problem

$$E_0\sum_{t=0}^{\infty}\beta^t\frac{\left[C_t\left(1-L_t\right)^{\phi}\right]^{1-\gamma}}{1-\gamma}\ldots$$
$$-\beta^t\Lambda_t\left[C_t+K_{t+1}-\left(1-\delta\right)K_t-\left(1-\tau_{lt}\right)W_tL_t-R_tK_t-T_t\right]$$

where K_0 is given and $\beta^t\Lambda_t$ is the Lagrange multiplier on the budget constraint.

First order conditions with respect to C_t, L_t, and K_{t+1}:

$$C_t^{-\gamma} \left(1 - L_t\right)^{\phi(1-\gamma)} = \Lambda_t \tag{1}$$

$$\phi C_t^{1-\gamma} \left(1 - L_t\right)^{\phi(1-\gamma)-1} = \Lambda_t \left(1 - \tau_{lt}\right) W_t \tag{2}$$

$$\Lambda_t = E_t \left[\beta \Lambda_{t+1} \left(R_{t+1} + 1 - \delta\right)\right] \tag{3}$$

Firms

Firms produce output Y_t using the technology

$$Y_t = K_t^{\theta} (Z_t L_t)^{1-\theta}$$

where $\theta \in (0, 1)$ and Z_t is a labor-augmenting productivity shock.

Firm's problem is static: rent labor and capital from households to maximize profits,

$$\max_{K_t, L_t} K_t^{\theta} (Z_t L_t)^{1-\theta} - W_t L_t - R_t K_t$$

First order conditions:

$$W_t = (1-\theta) K_t^{\theta} Z_t^{1-\theta} L_t^{-\theta} = (1-\theta) \frac{Y_t}{L_t} \tag{4}$$

$$R_t = \theta K_t^{\theta-1} (Z_t L_t)^{1-\theta} = \theta \frac{Y_t}{K_t} \tag{5}$$

Government

- The government sets taxes and transfers in such a way that its budget constraint is satisfied.

- The government does not consume goods and does not borrow or lend—WLOG given Ricardian equivalence.

- Therefore, the government follows a balanced budget

$$T_t = \tau_{lt} W_t L_t.$$

Feasibility and exogenous shocks

Resource constraint:

$$C_t + \underbrace{K_{t+1} - (1-\delta) K_t}_{=\text{investment}} = Y_t. \tag{6}$$

Exogenous shocks: Technology and demand shocks

$$\log Z_{t+1} = \mu_z + \log Z_t + \sigma_z \varepsilon^z_{t+1} \tag{7}$$

$$(\tau_{It+1} - \bar{\tau}_I) = \rho (\tau_{It} - \bar{\tau}_I) + \sigma_I \varepsilon^I_{t+1} \tag{8}$$

where $\varepsilon^z_t \sim N(0,1)$ and $\varepsilon^I_t \sim N(0,1)$ are i.i.d shocks.

- Productivity is $I(1)$ but the tax process is stationary, $|\rho| < 1$

Let

$$z_{t+1} \equiv \frac{Z_{t+1}}{Z_t} \tag{9}$$

Then, (7) implies

$$\log z_{t+1} = \mu_z + \sigma_z \varepsilon^z_{t+1}. \tag{10}$$

Summary of equilibrium conditions

$$
\begin{aligned}
C_t^{-\gamma}\left(1-L_t\right)^{\phi(1-\gamma)} &= \Lambda_t \\
\phi C_t^{1-\gamma}\left(1-L_t\right)^{\phi(1-\gamma)-1} &= \Lambda_t\left(1-\tau_{lt}\right) W_t \\
E_t\left[\beta \Lambda_{t+1}\left(R_{t+1}+1-\delta\right)\right] &= \Lambda_t \\
(1-\theta)\frac{Y_t}{L_t} &= W_t \\
\theta \frac{Y_t}{K_t} &= R_t \\
C_t+K_{t+1}-(1-\delta) K_t &= Y_t \\
Y_t &= K_t^{\theta}\left(Z_t L_t\right)^{1-\theta}
\end{aligned}
$$

Making the model stationary

- Technology shock is $I(1)$: (most) endogenous varibles will be $I(1)$.
- Model is stationary in terms of the following variables:

$$c_t \equiv \frac{C_t}{Z_t};\ y_t \equiv \frac{Y_t}{Z_t};\quad k_t \equiv \frac{K_t}{Z_{t-1}};\quad w_t = \frac{W_t}{Z_t};\quad \lambda_t = \Lambda_t Z_t^{\gamma}$$

1. Do not normalize labor L_t or the rental rate R_t, so we just define

$$l_t = L_t \text{ and } r_t = R_t.$$

2. Capital at time t is normalized by the level of technology at time $t-1$.
3. The transformation of the multiplier looks odd, but it is the transformation that works.

Writing the equilibrium in terms of transformed variables

First equation:

$$C_t^{-\gamma}\left(1-L_t\right)^{\phi(1-\gamma)}=\Lambda_t$$

or

$$C_t^{-\gamma}\left(\frac{Z_t}{Z_t}\right)^{-\gamma}\left(1-L_t\right)^{\phi(1-\gamma)}=\Lambda_t$$

or

$$c_t^{-\gamma}\left(1-l_t\right)^{\phi(1-\gamma)}=Z_t^{\gamma}\Lambda_t$$

or

$$\boxed{c_t^{-\gamma}\left(1-l_t\right)^{\phi(1-\gamma)}=\lambda_t.}$$

Second equation:

$$\phi C_t^{1-\gamma} \left(1 - L_t\right)^{\phi(1-\gamma)-1} = \Lambda_t \left(1 - \tau_{lt}\right) W_t$$

or

$$\phi C_t^{1-\gamma} \frac{Z_t^{1-\gamma}}{Z_t^{1-\gamma}} \left(1 - L_t\right)^{\phi(1-\gamma)-1} = \Lambda_t Z_t \left(1 - \tau_{lt}\right) \frac{W_t}{Z_t}$$

or

$$\phi \left(\frac{C_t}{Z_t}\right)^{1-\gamma} \left(1 - L_t\right)^{\phi(1-\gamma)-1} = \Lambda_t Z_t^{\gamma} \left(1 - \tau_{lt}\right) \frac{W_t}{Z_t}$$

So that

$$\boxed{\phi c_t^{1-\gamma} \left(1 - l_t\right)^{\phi(1-\gamma)-1} = \lambda_t \left(1 - \tau_{lt}\right) w_t .}$$

Third equation

$$\Lambda_t = E_t \left[\beta \Lambda_{t+1} \left(R_{t+1} + 1 - \delta\right)\right]$$

or

$$Z_t^{\gamma} \Lambda_t = E_t \left[\beta \left(\frac{Z_t}{Z_{t+1}}\right)^{\gamma} Z_{t+1}^{\gamma} \Lambda_{t+1} \left(r_{t+1} + 1 - \delta\right)\right]$$

or

$$\lambda_t = E_t \left[\beta \left(\frac{Z_{t+1}}{Z_t}\right)^{-\gamma} \lambda_{t+1} \left(r_{t+1} + 1 - \delta\right)\right]$$

But using (9),

$$\frac{Z_{t+1}}{Z_t} = z_{t+1} \Rightarrow \left(\frac{Z_{t+1}}{Z_t}\right)^{-\gamma} = z_{t+1}^{-\gamma}$$

Hence,

$$\boxed{\lambda_t = E_t \left[\beta z_{t+1}^{-\gamma} \lambda_{t+1} \left(r_{t+1} + 1 - \delta\right)\right].}$$

...and so on...

Equilibrium in terms of transformed variables

The equilibrium conditions in terms of the transformed variables are

$$c_t^{-\gamma} (1 - l_t)^{\phi(1-\gamma)} = \lambda_t \tag{11}$$

$$\phi c_t^{1-\gamma} (1 - l_t)^{\phi(1-\gamma)-1} = \lambda_t (1 - \tau_{lt}) w_t \tag{12}$$

$$\lambda_t = E_t \left[\beta z_{t+1}^{-\gamma} \lambda_{t+1} (r_{t+1} + 1 - \delta) \right] \tag{13}$$

$$w_t = (1 - \theta) \frac{y_t}{l_t} \tag{14}$$

$$r_t = \theta \frac{z_t y_t}{k_t} \tag{15}$$

$$c_t + k_{t+1} - (1 - \delta) k_t z_t^{-1} = y_t \tag{16}$$

$$y_t = k_t^{\theta} z_t^{-\theta} l_t^{1-\theta}. \tag{17}$$

Steady state of the transformed system

$$\bar{c}^{-\gamma}\left(1-\bar{l}\right)^{\phi(1-\gamma)} = \bar{\lambda}$$

$$\phi\bar{c}^{1-\gamma}\left(1-\bar{l}\right)^{\phi(1-\gamma)-1} = \bar{\lambda}\left(1-\bar{\tau}_l\right)\bar{w}$$

$$\bar{\lambda} = \beta\bar{z}^{-\gamma}\bar{\lambda}\left(\bar{r}+1-\delta\right)$$

$$\bar{w} = \left(1-\theta\right)\frac{\bar{y}}{\bar{l}}$$

$$\bar{r} = \theta\frac{\bar{z}\bar{y}}{\bar{k}}$$

$$\bar{c}+\bar{k}-\left(1-\delta\right)\bar{k}\bar{z}^{-1} = \bar{y}$$

$$\bar{y} = \bar{k}^{\theta}\bar{z}^{-\theta}\bar{l}^{1-\theta}.$$

$$\bar{z} = e^{\mu_z}$$

Solve the steady state as usual. But note the next result:

- *The labor tax τ_l does not affects key steady state ratios. In particular, labor productivity $\bar{y}/\bar{l}$ does not depend on $\bar{\tau}_l$:*

$$\frac{\bar{y}}{\bar{l}} = \left(\frac{\theta}{\bar{z}^{\gamma}/\beta - (1-\delta)} \right)^{\frac{\theta}{\theta-1}}.$$

- Recalling that (i) $y_t = Y_t/Z_t$, (ii) $l_t = L_t$, and (iii) that labor is constant in a balanced growth path, we observe that labor productivity in the long-run satisfies

$$\frac{Y_t^{lr}/Z_t^{lr}}{L^{lr}} = \left(\frac{\theta}{\bar{z}^{\gamma}/\beta - (1-\delta)} \right)^{\frac{\theta}{\theta-1}}$$

  so that

$$\frac{Y_t^{lr}}{L^{lr}} = Z_t^{lr} \left(\frac{\theta}{\bar{z}^{\gamma}/\beta - (1-\delta)} \right)^{\frac{\theta}{\theta-1}}$$

- *Labor productivity in the long-run depends only on the level of technology Z_t^{lr} and not on the demand shock τ_{lt}.*
  - **Identification restriction for doing SVAR!**

Log-linearized equilibrium conditions

$$0 = \gamma \tilde{c}_t + \frac{\phi\,(1-\gamma)\,\bar{l}}{1-\bar{l}}\tilde{l}_t + \tilde{\lambda}_t$$

$$0 = (1-\gamma)\,\tilde{c}_t + [1-\phi\,(1-\gamma)]\left(\frac{\bar{l}}{1-\bar{l}}\right)\tilde{l}_t - \tilde{\lambda}_t - \tilde{w}_t + \frac{\tau_{lt}-\bar{\tau}_l}{1-\bar{\tau}_l}$$

$$0 = \tilde{y}_t - \tilde{l}_t - \tilde{w}_t$$

$$0 = \tilde{y}_t - (\tilde{k}_t - \tilde{z}_t) - \tilde{r}_t$$

$$0 = \theta(\tilde{k}_t - \tilde{z}_t) + (1-\theta)\,\tilde{l}_t - \tilde{y}_t$$

$$E_t\left[\tilde{\lambda}_{t+1} - \gamma \tilde{z}_{t+1} + \beta \bar{z}^{-\gamma}\bar{r}\tilde{r}_{t+1}\right] = \tilde{\lambda}_t$$

$$\bar{k}E_t[\tilde{k}_{t+1}] = (1-\delta)\,\bar{k}\bar{z}^{-1}(\tilde{k}_t - \tilde{z}_t) + \bar{y}\tilde{y}_t - \bar{c}\tilde{c}_t$$

where, as usual, for any variable x_t we define

$$\tilde{x}_t = \log\left(x_t/\bar{x}\right).$$

Log-linearization of equilibrium conditions

1. Note that
$$E_t\left[\tilde{z}_{t+1}\right] = 0$$
Therefore, we can write the linearized Euler equation as
$$E_t[\tilde{\lambda}_{t+1}] + \beta \bar{z}^{-\gamma} \bar{r} E_t[\tilde{r}_{t+1}] = \tilde{\lambda}_t.$$
2. Note that $\tilde{k}_t$ and $\tilde{z}_t$ always appear as $\tilde{k}_t - \tilde{z}_t$. This suggest that the relevant state variable is the difference
$$\hat{k}_t \equiv \tilde{k}_t - \tilde{z}_t$$
3. We need to fix the term in the last equation. Using $E_t \tilde{z}_{t+1} = 0$,
$$\begin{aligned} \bar{k} E_t[\tilde{k}_{t+1}] &= \bar{k} E_t[\tilde{k}_{t+1} - \tilde{z}_{t+1}] + \bar{k} E_t[\tilde{z}_{t+1}] \\ &= \bar{k} E_t[\tilde{k}_{t+1} - \tilde{z}_{t+1}] \\ &= \bar{k} E_t \hat{k}_{t+1}. \end{aligned}$$
4. Therefore, we can rewrite the system of linearized equations in terms of the variable $\hat{k}_t$ instead of $\tilde{k}_t$ and $\tilde{z}_t$ separately.

System in terms of new state variables

$$0 = \gamma \tilde{c}_t + \frac{\phi (1-\gamma) \bar{l}}{1-\bar{l}} \tilde{l}_t + \tilde{\lambda}_t$$

$$0 = (1-\gamma) \tilde{c}_t + [1 - \phi (1-\gamma)] \left(\frac{\bar{l}}{1-\bar{l}} \right) \tilde{l}_t - \tilde{\lambda}_t - \tilde{w}_t + \frac{\tau_{lt} - \bar{\tau}_l}{1-\bar{\tau}_l}$$

$$0 = \tilde{y}_t - \tilde{l}_t - \tilde{w}_t$$

$$0 = \tilde{y}_t - \hat{k}_t - \tilde{r}_t$$

$$0 = \theta \hat{k}_t + (1-\theta) \tilde{l}_t - \tilde{y}_t$$

$$E_t[\tilde{\lambda}_{t+1}] + \beta \bar{z}^{-\gamma} \bar{r} E_t[\tilde{r}_{t+1}] = \tilde{\lambda}_t$$

$$\bar{k} E_t[\hat{k}_{t+1}] = (1-\delta) \bar{k} \bar{z}^{-1} \hat{k}_t + \bar{y} \tilde{y}_t - \bar{c} \tilde{c}_t$$

$$E_t \left[\tau_{lt+1} - \bar{\tau}_l \right] = \rho \left(\tau_{lt} - \bar{\tau}_l \right)$$

Solve the linearized model

Let

$$X_t = \left[\hat{k}_t, \tau_{lt} - \bar{\tau}_l, \tilde{y}_t, \tilde{c}_t, \tilde{l}_t, \tilde{r}_t, \tilde{w}_t, \tilde{\lambda}_t\right]'$$

denote the vector of relevant variables, where the first two are the state variables. Then, we can write the log-linearized system in the following form

$$\mathbf{A} E_t \left[X_{t+1}\right] = \mathbf{B} X_t$$

where $\mathbf{A}$ and $\mathbf{B}$ are 8×8 matrices given by

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \bar{z}^{-\gamma} \bar{r} & 0 & 1 \\ \bar{k} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

Solve the linearized model

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & \gamma & \phi\left(1-\gamma\right)\frac{\bar{l}}{1-\bar{l}} & 0 & 0 & 1 \\ 0 & \frac{1}{1-\bar{\tau}_l} & 0 & 1-\gamma & \frac{1-\phi(1-\gamma)}{1-\bar{l}}\bar{l} & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ \theta & 0 & -1 & 0 & 1-\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \left(1-\delta\right)\frac{\bar{k}}{\bar{z}} & 0 & \bar{y} & -\bar{c} & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- The state variables of this system are

$$x_t = \begin{bmatrix} \hat{k}_t \\ \tau_{lt} - \bar{\tau}_l \end{bmatrix}$$

- The control variables are

$$u_t = \begin{bmatrix} \tilde{y}_t & \tilde{c}_t & \tilde{l}_t & \tilde{r}_t & \tilde{w}_t & \tilde{\lambda}_t \end{bmatrix}'.$$

- The QZ decomposition delivers policy functions of the form

$$E_t\left[x_{t+1}\right] = Px_t \tag{18}$$

$$u_t = Fx_t \tag{19}$$

where P is 2×2 and F is 6×2.

- The evolution of the state variables with the shock is then given by

$$x_{t+1} = Px_t + \begin{bmatrix} 0 \\ \sigma_l \end{bmatrix} \varepsilon^l_{t+1} \tag{20}$$

where $\varepsilon_{t+1} \sim N\left(0, \sigma_l^2\right)$.

Mapping the model to a state-space representation for ML

- The model has two shocks, a productivity shock and a labor tax (demand) shock
$$\log Z_t \text{ and } \tau^l_t$$
- To avoid stochastic singularity, we can perform MLE with two observable variables.
  - Alternative is to add measurement errors or other shocks. Canova et.al. (2014) claim that this is not a terribly good idea.
  - There are many pair of variables to choose. We will start with **output** and **labor**
- The data has trends, so to perform MLE we need to apply some trend-removal transformation.
- We also need a mapping from the observable variables into the variables in the model.
- We will need to create a new state-space representation of the model to be able to estimate it.

Mapping the model to a state-space representation for ML

- **First observable variable**: growth of GDP per capita (output)
- Model: delivers predictions for $\tilde{y}_t = \log\left(y_t / \bar{y}\right)$ where

$$y_t = Y_t / Z_t .$$

- Decision rule for this variable is given by

$$\tilde{y}_t = \psi_y^k \hat{k}_t + \psi_y^\tau \left(\tau_{It} - \bar{\tau}_I\right) .$$

Here,

$$\psi_y^k = F_{1,1} \text{ and } \psi_y^\tau = F_{1,2}$$

where $F_{i,j}$ denotes row i and column j of the matrix F derived from the QZ decomposition.

- Using $\hat{k}_t = \tilde{k}_t - \tilde{z}_t$, we have

$$\log y_t - \log \bar{y} = \psi_y^k(\tilde{k}_t - \tilde{z}_t) + \psi_y^\tau\left(\tau_{lt} - \bar{\tau}_l\right)$$

- Moreover, $\log y_t = \log\left(Y_t / Z_t\right)$ implies

$$\log Y_t - \log Z_t - \log \bar{y} = \psi_y^k(\tilde{k}_t - \tilde{z}_t) + \psi_y^\tau\left(\tau_{lt} - \bar{\tau}_l\right)$$

- Subtracting the same expression at time $t-1$ gives

$$\begin{gathered}\Delta \log Y_t - (\log Z_t - \log Z_{t-1}) = \psi_y^k(\tilde{k}_t - \tilde{z}_t) + \psi_y^\tau\left(\tau_{lt} - \bar{\tau}_l\right) \\ -\psi_y^k(\tilde{k}_{t-1} - \tilde{z}_{t-1}) - \psi_y^\tau\left(\tau_{lt-1} - \bar{\tau}_{l-1}\right).\end{gathered}$$

- But $\log Z_t - \log Z_{t-1} = \mu_z + \tilde{z}_t$. Then

$$\begin{aligned}\Delta \log Y_t - \mu_z - \tilde{z}_t &= \psi_y^k(\tilde{k}_t - \tilde{z}_t) + \psi_y^\tau\left(\tau_{lt} - \bar{\tau}_l\right) \\ &\quad -\psi_y^k(\tilde{k}_{t-1} - \tilde{z}_{t-1}) - \psi_y^\tau\left(\tau_{lt-1} - \bar{\tau}_{l-1}\right)\end{aligned}$$

- Rearranging we have

$$\begin{aligned}\Delta \log Y_t - \mu_z &= \psi_y^k \tilde{k}_t + \left(1 - \psi_y^k\right) \tilde{z}_t + \psi_y^\tau \left(\tau_{lt} - \bar{\tau}_l\right) \quad (21) \\ &\quad - \psi_y^k \tilde{k}_{t-1} + \psi_y^k \tilde{z}_{t-1} - \psi_y^\tau \left(\tau_{lt-1} - \bar{\tau}_{l-1}\right).\end{aligned}$$

- Given a guess for the parameter μ_z, the left hand side is observed ($\Delta \log Y_t$ is per-capita output growth).

- This equation relates an observable to a linear combination of current and lagged state variables.

- We can accomodate this system using an extended state space model where the state vector is

$$s_t \equiv \left[\tilde{k}_t, \tilde{z}_t, \tau_{lt} - \bar{\tau}_l, \tilde{k}_{t-1}, \tilde{z}_{t-1}, \tau_{lt-1} - \bar{\tau}_l\right]'.$$

- **Second observable variable**: labor input l_t.
- The decision rule for labor is

$$\log \tilde{l}_t = \psi_l^k (\tilde{k}_t - \tilde{z}_t) + \psi_l^\tau (\tau_{lt} - \bar{\tau}_l)$$

or

$$\log l_t - \log \bar{l} = \psi_l^k \tilde{k}_t - \psi_l^k \tilde{z}_t + \psi_l^\tau (\tau_{lt} - \bar{\tau}_l)$$

- We can trivially write this equation in terms of the extended state vector s_t as

$$\begin{aligned} \log l_t - \log \bar{l} &= \psi_l^k \tilde{k}_t - \psi_l^k \tilde{z}_t + \psi_l^\tau (\tau_{lt} - \bar{\tau}_l) + \\ & \quad 0 \times \tilde{k}_{t-1} + 0 \times \tilde{z}_{t-1} + 0 \times (\tau_{lt-1} - \bar{\tau}_l) . \end{aligned} \tag{22}$$

- Equations (21) and (22) constitute the observation equation of an extended state-space model
- It remains to derive the evolution of the extended state vector s_t.

- The equilibrium evolution of the stock of capital can be found from the first row of (18) as a function

$$E_t\left[\hat{k}_{t+1}\right]=\xi_k^k\hat{k}_t+\xi_k^\tau\left(\tau_{lt}-\bar{\tau}_l\right),$$

where

$$\xi_k^k=P_{1,1}\text{ and }\xi_k^\tau=P_{1,2}$$

- Using $\hat{k}_t=\tilde{k}_t-\tilde{z}_t$ implies

$$\begin{aligned}E_t\left[\tilde{k}_{t+1}-\tilde{z}_{t+1}\right] &= \xi_k^k\left[\tilde{k}_t-\tilde{z}_t\right]+\xi_k^\tau\left(\tau_{lt}-\bar{\tau}_l\right)\\ &= \xi_k^k\tilde{k}_t-\xi_k^k\tilde{z}_t+\xi_k^\tau\left(\tau_{lt}-\bar{\tau}_l\right).\end{aligned}$$

- However, using $E_t\left[\tilde{z}_{t+1}\right]=0$ and $E_t[\tilde{k}_{t+1}]=\tilde{k}_{t+1}$ we write the evolution of capital in terms of the enlarged state vector as

$$\begin{aligned}\tilde{k}_{t+1} &= \xi_k^k\tilde{k}_t-\xi_k^k\tilde{z}_t+\xi_k^\tau\left(\tau_{lt}-\bar{\tau}_l\right)\\ &\quad +0\times\tilde{k}_{t-1}+0\times\tilde{z}_{t-1}+0\times\left(\tau_{lt-1}-\bar{\tau}_l\right).\end{aligned}$$

- Note, also that the evolution of $\tilde{z}_{t+1}$ is simply given by

$$\tilde{z}_{t+1}=\sigma_z\varepsilon_{t+1}^z.$$

Enlarged state space system for MLE

State equation:

$$\begin{bmatrix} \tilde{k}_{t+1} \\ \tilde{z}_{t+1} \\ \tau_{lt+1} - \bar{\tau}_l \\ \tilde{k}_t \\ \tilde{z}_t \\ \tau_{lt} - \bar{\tau}_l \end{bmatrix} = \begin{bmatrix} \zeta_k^k & -\zeta_k^k & \zeta_k^\tau & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{k}_t \\ \tilde{z}_t \\ \tau_{lt} - \bar{\tau}_l \\ \tilde{k}_{t-1} \\ \tilde{z}_{t-1} \\ \tau_{lt-1} - \bar{\tau}_l \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ \sigma_z & 0 \\ 0 & \sigma_l \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{t+1}^z \\ \varepsilon_{t+1}^l \end{bmatrix}.$$

Enlarged state space system for MLE

Observation equation:

$$\begin{bmatrix} \Delta Y_t - \mu_z \\ \log\left(l_t / \bar{l}\right) \end{bmatrix} = \begin{bmatrix} \psi_y^k & 1-\psi_y^k & \psi_y^\tau & -\psi_y^k & \psi_y^k & -\psi_y^\tau \\ \psi_l^k & -\psi_l^k & \psi_l^\tau & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{k}_t \\ \tilde{z}_t \\ \tau_{lt} - \bar{\tau}_l \\ \tilde{k}_{t-1} \\ \tilde{z}_{t-1} \\ \tau_{lt-1} - \bar{\tau}_{l-1} \end{bmatrix}$$

- Then use the Kalman filter to evaluate (and maximize) the likelihood function of the enlarged state-space system.