Clase 8 (4/2a)

Ejercicio 3

a) $q = 4x^2 + 5y^2 + 8yx$

quiero $A / (x, y)\, A \begin{pmatrix} x \\ y \end{pmatrix} = q$; $A \in \mathbb{R}^{2\times 2}$

$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \longrightarrow (x\ y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x\ y)\begin{pmatrix} ax+by \\ bx+cy \end{pmatrix}$

$= ax^2 + bxy + bxy + cy^2$

$= ax^2 + 2b\, xy + cy^2 \Rightarrow$ $a=4$, $b=4$, $c=5$

$A = \begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix} \longrightarrow P_A(\lambda) = (4-\lambda)(5-\lambda) - 16$

$\longrightarrow \lambda_{1,2} = \dfrac{9 \pm \sqrt{65}}{2} > 0$

$\Rightarrow$ A es definida positiva

c) $q = 3x^2 - 2xy + 3xz + y^2 - 4yz + 3z^2$

$A = \underline{\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}} \Big/ \ (x\ y\ z)\, A \underline{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = q$

$(x\ y\ z)\begin{pmatrix} ax + by + cz \\ bx + dy + ez \\ cx + ey + fz \end{pmatrix} = ax^2 + byx + czx + bxy + dy^2 + ezy + cxz + eyz + fz^2$

$a = 3$ $\quad c = 3/2$ $\quad e = -2$

$b = -1$ $\quad d = 1$ $\quad f = 3$

$$A = \begin{pmatrix} 3 & -1 & 3/2 \\ -1 & 1 & -2 \\ 3/2 & -2 & 3 \end{pmatrix} \longrightarrow \lambda_1, \lambda_2 > 0, \ \lambda_3 < 0$$

$\rightarrow$ A es indeterminado

Ejercicio 4

a) $$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$\Delta_1 = -2 < 0$

$\Delta_2 = (-2)(-2) - 1 > 0$

$\Delta_3 = -8 + 1 + 1 - (-2 - 2 - 2)$

$= 0$

q es def neg. si $(-1)^i \Delta_i > 0$

pero si $\det(A) = 0$ y $(-1)^i \Delta_i > 0 \quad \forall\, i = 1, \dots, n-1$

$\Rightarrow$ q es semi-definido negativa

$\nwarrow$ este caso

c) $$C = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

$\Delta_1 = 0 \leftarrow$ es indeterminada

$\begin{cases} \text{sería def pos. si } \Delta_i > 0 \\ \text{y sería semi-def pos. si } \Delta_i > 0 \text{ con } \det(C) = 0 \end{cases}$

Ejercicio 5

a) $$A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

Quiero $\sqrt{A}$ / $\sqrt{A}\sqrt{A} = A$

$\lambda_1 + \lambda_2 = 10 \longrightarrow \lambda_1 = 1, \ \lambda_2 = 9$

$\lambda_1 \lambda_2 = 9$

$\lambda_1 = 1$

$$A - Id = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow x+y=0$$

$$\rightarrow x = -y \rightarrow E_1 = \langle (1,-1) \rangle$$

$\lambda_2 = 9$ $$A - 9Id = \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow x = y$$

$$\rightarrow E_9 = \langle (1,1) \rangle$$

$$A = \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}}_{D} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

$$\sqrt{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

$$\sqrt{A}\sqrt{A} = P\sqrt{D}\cancel{P^{-1}}\cancel{P}\sqrt{D}P^{-1} = P\sqrt{D}^2P^{-1} = PDP^{-1} = A$$

$$\sqrt{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Ejercicio 6

$$q = x^2 + \alpha y^2 + 2yz + \alpha z^2$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 1 & \alpha \end{pmatrix}$$

$\Delta_1 = 1$

$\Delta_2 = \alpha$

$\Delta_3 = \alpha^2 - 1$

Si $\alpha = -1, 1 \Rightarrow \Delta_3 = 0$

$\hookrightarrow \alpha = 1 \rightarrow \Delta_2 > 0 \rightarrow$ es semi-def positiva

$\hookrightarrow \alpha = -1 \rightarrow$ indeterminado

si $\alpha > 1 \rightarrow \Delta_1, \Delta_2, \Delta_3 > 0 \longrightarrow$ es definida positiva

Para el resto de los casos, indeterminada.

Ejercicio 7

$$\begin{pmatrix} 2 & 2 & 4 \\ 2 & \alpha & 8 \\ 4 & 8 & 7 \end{pmatrix}$$

$\Delta_1 > 0$

$\Delta_2 = 2\alpha - 4 > 0 \Rightarrow \underline{\alpha > 2}$

$\Delta_3 = 14\alpha + \cancel{64 + 64} - [16\alpha + \cancel{2 \cdot 64} + 28]$

$= -2\alpha - 28 > 0 \Rightarrow -28 > 2\alpha \rightarrow -14 > \alpha$

$\underline{\hphantom{-14>\alpha}}$

$\alpha < -14$

$\Rightarrow \nexists \, \alpha$

Ejercicio 8

$$q = ax^2 + ay^2 + (a-1)z^2 + 2xy + 0 \cdot z \cdot x + 0 \cdot zy$$

$$A = \begin{pmatrix} a & 1 & 0 \\ 1 & a & 0 \\ 0 & 0 & a-1 \end{pmatrix}$$

$\Delta_1 = a$

$\Delta_2 = a^2 - 1 = (a-1)(a+1)$

$\Delta_3 = (a-1)(a^2-1) = (a-1)^2(a+1)$

$\Delta_3 = 0 \longrightarrow a = 1, -1$

$a > 0 \Rightarrow \Delta_1 > 0$

- $a > 1 \Rightarrow \Delta_2 > 0, \ \Delta_3 > 0 \longrightarrow$ definida > 0
- $a = 1 \Rightarrow \Delta_2 = 0$ indet.
- $a \in (0,1) \Rightarrow \Delta_2 < 0 \rightarrow$ indet

$(a = 0 \rightarrow \Delta_1 = 0 \rightarrow$ indet.$)$

$a < 0 \Rightarrow \Delta_1 < 0$

- $a < -1 \Rightarrow \Delta_2 > 0, \ \Delta_3 < 0 \longrightarrow$ definida < 0
- $a = -1 \Rightarrow \Delta_2 = 0 \rightarrow$ indet.
- $a \in (-1,0) \Rightarrow \Delta_2 < 0 \rightarrow$ indet.

def < 0 $\quad -1 \quad$ indeterminado $\quad 1 \quad$ def > 0

Ejercicio 9

$$q = x^2 + 4y^2 + 5z^2 + 2axy + 2xz + 4yz$$
$$\text{s.t. } x = y$$

$$\to q\Big|_{x=y} = x^2 + 4x^2 + 5z^2 + 2ax^2 + 2xz + 4xz$$
$$= (2a+5)x^2 + 6xz + 5z^2$$

$$A = \begin{pmatrix} 2a+5 & 3 \\ 3 & 5 \end{pmatrix}$$

$\Delta_1 = 2a+5$

$\Delta_2 = (2a+5)5 - 9$

$\Delta_2 = 10a + 25 - 9 = 10a + 16$

$\Delta_1 > 0 \iff a > -5/2$

$\hookrightarrow \Delta_2 \geq 0$: $10a + 25 - 9 \geq 0 \to 10a \geq -16 \to a \geq -\frac{16}{10}$

$\hookrightarrow a > -\frac{16}{10}$: definido > 0

$a = -\frac{16}{10}$: semi def > 0

$\left.\begin{array}{l} \Delta_1 < 0 \iff a < -5/2 \\ \hookrightarrow \Delta_2 \geq 0 : \; a \geq -\frac{16}{10} \end{array}\right\}$ absurdo

Para el resto de a, indet.

Ejercicio 10

a) $q = x^2 - 2xy + y^2$ s.t. $x + y = 0 \to x = -y$

$$q\Big|_{x=-y} = (-y)^2 - 2(-y)y + y^2$$
$$= 2y^2 + 2y^2 = 4y^2 \geq 0$$

> 0 para $y \neq 0$

b) $q = 2x^2 - 4xy + y^2$ s.t. $3x + 4y = 0 \rightarrow y = -\frac{3}{4}x$

$$q\Big|_{y=-\frac{3}{4}x} = 2x^2 - 4x\left(-\frac{3}{4}x\right) + \left(-\frac{3}{4}x\right)^2$$

$$= 2x^2 + 3x^2 + \frac{9}{16}x^2 > 0 \quad \forall x \neq 0$$

$\rightarrow$ def > 0

Ejercicio 1d

$$q = a(x^2 + z^2) + 2y^2 + 4xy$$

$$q\Big|_{x=y} = a(x^2+z^2) + 2x^2 + 4x^2 = (6+a)x^2 + az^2$$

$$A = \begin{pmatrix} 6+a & 0 \\ 0 & a \end{pmatrix}$$

$\Delta_1 = 6+a$

$\Delta_2 = (6+a)a$

$\Delta_2 = 0 \Leftrightarrow a = 0, -6$

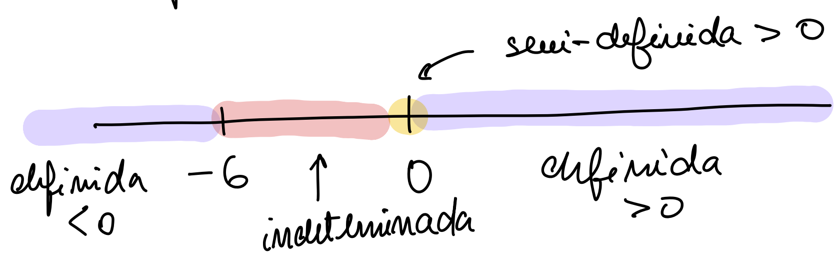
$a = -6 \rightarrow \Delta_1 = 0 \rightarrow$ indet.

$a = 0 \rightarrow \Delta_1 > 0 \rightarrow$ semi-def > 0

$\Delta_2 > 0 \rightarrow a < -6$ o $a > 0$

$a < -6 \rightarrow \Delta_1 < 0 \rightarrow$ def < 0

$a > 0 \rightarrow \Delta_1 > 0 \rightarrow$ def > 0



Consulta

$$q = \underline{x^2 + 2xy + y^2}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$(x \;\; y) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x \;\; y) \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

$$= ax^2 + bxy + cxy + dy^2$$

$$= \underline{ax^2 + (b+c)xy + dy^2}$$

$$b = c$$

Ejercicio 23. TP3 $\qquad A \in \mathbb{R}^{3\times 3} \leftarrow$

$\dim(\{x : Ax = 0\}) = 1 > 0 \longrightarrow$ un autovalor es cero

$\underline{rg(A+2I) = 2} < 3 \xrightarrow{\dim} (\ker(A+2I)) = 1 > 0$

$\longrightarrow$ puedo encontrar un vector $v \neq 0$ / $(A+2I)v = 0$

$\longrightarrow Av + 2Iv = 0$

$\longrightarrow Av = -2v$ uno de los autovalores es -2

$tr(A) = 0 \longrightarrow \underbrace{\lambda_1}_{0} + \underbrace{\lambda_2}_{-2} + \lambda_3 = 0 \Rightarrow \lambda_3 = 2$

Ejercicio 10

c) $q = 2x^2 + y^2 - 4xy + 2yz\,, \quad x - y + z = 0$

$\rightarrow z = y - x$

$$q\Big|_{z=y-x} = 2x^2 + y^2 - 4xy + 2y(y-x)$$

$$= 2x^2 + y^2 - 4xy + 4y^2 - 2yx$$

$$= 2x^2 - 6xy + 5y^2$$

$$A = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$$

$\Delta_1 = 2 > 0$

$\Delta_2 = 10 - 9 = 1 > 0$

$\rightarrow$ es definido > 0

Consulta

Ejercicio 25 TP3

$A \in \mathbb{R}^{4\times 4}, \quad \operatorname{Ker}(A+I) \neq \{0\}$

$\operatorname{rg}(A-2I) \leq 2, \quad p_A(1) = -4$

1) $\operatorname{Ker}(A+I) \supseteq \{0\}$ como es $\neq \{0\} \rightarrow \operatorname{Ker}(A+I) \supsetneq \{0\}$

$\Rightarrow \exists\, v \neq 0,\; v \in \operatorname{Ker}(A+I) \Rightarrow (A+I)v = 0$

$\Rightarrow \quad Av = -v = -1 \cdot v$ $\quad -1 = \lambda_1$

2) $\operatorname{rg}(A-2I) \leq 2 \rightarrow \dim(\operatorname{Ker}(A-2I)) \geq 2$

$\rightarrow \exists\; v_1 \perp v_2 \neq 0 \;/\; \begin{matrix}(A-2I)v_1 = 0 \\ (A-2I)v_2 = 0\end{matrix}$

$\rightarrow \quad \begin{matrix} Av_1 = -2v_1 \\ Av_2 = -2v_2 \end{matrix} \quad \rightarrow \quad \begin{matrix} \lambda_2 = -2 \\ \lambda_3 = -2 \end{matrix}$

3) $p_A(\lambda) = (-1-\lambda)(-2-\lambda)^2(\lambda_4-\lambda)$

$-p_A(+1) = +4 = 72\,(-3)^2(\lambda_4-1)$

$-\frac{2}{8} = \lambda_4 - 1 \longrightarrow -\frac{2}{8} + \frac{8}{8} = \lambda_4 = \frac{3}{4}$

Consulta: