Clase 4 (1/4)

Ejercicio 8

a) $\begin{cases} 2b\,x + y - z = 1 \\ x \quad - ay + z = 0 \\ 4x - by + az = 4 \end{cases} \Rightarrow \begin{pmatrix} 2b & 1 & -1 \\ 1 & -a & 1 \\ 4 & -b & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$

$(1,-2,3) \Rightarrow \begin{pmatrix} 2b & 1 & -1 \\ 1 & -a & 1 \\ 4 & -b & a \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 2b-5 \\ 2a+4 \\ 4+b2+3a \end{pmatrix} \Rightarrow \begin{cases} b=3 \\ a=-2 \\ 4 \stackrel{?}{=} 4+2b+3a = 4+6-6=4 \checkmark \end{cases}$

$\Rightarrow (a,b) = (-2,3)$

b) $\begin{cases} x + 2ay + z = 0 \\ ya - zb = 4 \\ x + yb + z(2a+b) = b \end{cases} \Rightarrow \begin{pmatrix} 1 & 2a & 1 \\ 0 & a & -b \\ 1 & b & 2a+b \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ b \end{pmatrix}$

$(1,-2,3)$ es solución:

$\begin{pmatrix} 0 \\ 4 \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2a & 1 \\ 0 & a & -b \\ 1 & b & 2a+b \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4-4a \\ -2a-3b \\ 1+6a+b \end{pmatrix}$

$\Rightarrow \left.\begin{cases} a=1 \\ b=-2 \\ \underset{-2}{\underset{\parallel}{b}} \stackrel{?}{=} 1+6a+b = 1+6-2 \quad \times \end{cases}\right\}$ Abs!

$\Rightarrow \nexists\,(a,b)$ / el sistema tiene la sol. que nos dice el enunciado

Ejercicio 10

e) $\begin{cases} x+ky+2z-w=k+2 \\ x+ky-2z \quad = 2 \\ \quad -4z+k^2w=-3k-2 \end{cases}$

$$\rightarrow \begin{pmatrix} 1 & k & 2 & -1 \\ 1 & k & -2 & 0 \\ 0 & 0 & -4 & k^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} k+2 \\ 2 \\ -3k-2 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & k & 2 & -1 & k+2 \\ 1 & k & -2 & 0 & 2 \\ 0 & 0 & -4 & k^2 & -3k-2 \end{array}\right) \xrightarrow[F_1-F_2\to F_1']{} \left(\begin{array}{cccc|c} 0 & 0 & 4 & -1 & k \\ 1 & k & -2 & 0 & 2 \\ 0 & 0 & -4 & k^2 & -3k-2 \end{array}\right)$$

$$\xrightarrow[F_3+F_1\to F_3']{} \left(\begin{array}{cccc|c} 0 & 0 & 4 & -1 & k \\ 1 & k & -2 & 0 & 2 \\ 0 & 0 & 0 & k^2-1 & -2k-2 \end{array}\right) \rightarrow \begin{pmatrix} 0 & 0 & 4 & -1 \\ 1 & k & -2 & 0 \\ 0 & 0 & 0 & k^2-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} k \\ 2 \\ -2k-2 \end{pmatrix}$$

$k^2-1=-2k-2 \rightarrow k^2+2k+1=0 \rightarrow k=\dfrac{-2\pm\sqrt{4-4}}{2}=-1$

Sólo puede ser compatible si $k=-1$

$\left.\begin{array}{l} 4z-w=-1 \\ x-y-2z=2 \end{array}\right\}$

$(k^2-1)\,w = -2k-2$

$\Downarrow$

$w=\dfrac{-2k-2}{k^2-1} \equiv w(k)$

$k=-1 \longrightarrow$ compatible indeterminado

$k=1 \longrightarrow$ incompatible

$k\neq -1$, $k\neq 1$ $\longrightarrow$ $\begin{cases} x+ky-2z=2 \longrightarrow \text{no hay problema} \\ 4z-w=k \\ 4z+\dfrac{+2k+2}{k^2-1}=k \longrightarrow z=\dfrac{1}{4}k-\dfrac{1}{4}\dfrac{2k+2}{k^2-1} \end{cases}$

$\longrightarrow x=2-ky+2z(k)\equiv f(k)-ky$

$$S_{k\neq 1} = \Big\{ (x,y,z,w) \;/\; \begin{array}{l} k \neq -1,1 \to (x,y,z,w) = (f(k)-ky, y, z(k), w(k)) \\ \quad = (f(k),0,z(k),w(k)) + y(0,1,0,0) \\ \quad \text{para } y \in \mathbb{R} \\ k=-1 \to (x,y,z,w) = (y-\frac{w}{2}, y, -1+\frac{w}{4}, w) \\ \quad = y(1,1,0,0) + w(-\frac{1}{2},0,\frac{1}{4},1) \\ \quad + (0,0,-1,0) \end{array} \Big\}$$

$$S_{k=1} = \emptyset$$

f) $\begin{cases} 2x+3y-z=3 \\ x-y+3z=1 \\ 3x+7y-5z=k^2 \end{cases} \to \begin{pmatrix} 2 & 3 & -1 \\ 1 & -1 & 3 \\ 3 & 7 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ k^2 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 3 \\ 1 & -1 & 3 & 1 \\ 3 & 7 & -5 & k^2 \end{array}\right) \xrightarrow[F_1-2F_2 \to F_1']{F_3-(F_1+F_2)\to F_3'} \left(\begin{array}{ccc|c} 0 & 5 & -7 & 1 \\ 1 & -1 & 3 & 1 \\ 0 & 5 & -7 & k^2-4 \end{array}\right)$$

$$\xrightarrow[F_3-F_1 \to F_3']{} \left(\begin{array}{ccc|c} 0 & 5 & -7 & 1 \\ 1 & -1 & 3 & 1 \\ 0 & 0 & 0 & k^2-5 \end{array}\right)$$

Si $k^2=5$ ($\leftrightarrow k=\pm\sqrt{5}$) el sistema es compatible indeterminado

Si $k^2 \neq 5 \to$ incompatible

Ejercicio 11

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ b \end{pmatrix}$$

Si $a=1 \to \underline{b=-1}$ para que sea compatible

Pero de la segunda ecuación, $(1,1,1)(x,y,z)=1$

$\Rightarrow$ el sistema es incompatible

$\det(A) = (a-1)^2(-1) \leftarrow$ sólo es cero si $a=1$

Si $a \neq 1 \longrightarrow \det(A) \neq 0 \longrightarrow$ compatible determinado

Ejercicio 12

d) "$\underbrace{a_1x_1 + \ldots + a_nx_n = b}$ / $\underbrace{\exists i \text{ con } a_i \neq 0}$ $\Rightarrow$ hay sol."

D/. $\sum_{j=1}^{n} a_jx_j \underset{\uparrow}{=} a_ix_i + \sum_{j\neq i}^{n} a_jx_j \overset{?}{=} b$

$$\Rightarrow x_i = \frac{b - \sum_{j\neq i}^{n} a_jx_j}{a_i}$$

Puedo tomar $x_j = 0 \ \forall j \neq i$, $x_i = b/a_i \longrightarrow$ es sol. $\square$

e) Falso. $\begin{cases} x=1 \\ x=2 \end{cases}$ $\square$

Ejercicio 13

$$A = \begin{pmatrix} 3 & 0 & 2 \\ -3 & -3 & 3 \\ \alpha & 0 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix}$$

Hallar $\alpha \in \mathbb{R}$ / $Ax = 6x + b$ tiene + de 1 sol.

$Ax = 6\,Id\,x + b$

$Ax - 6\,Id\,x = b$

$$\Rightarrow (A - 6Id)x = b$$

$$\det(A-6Id) = \det\begin{pmatrix} -3 & 0 & 2 \\ -3 & -9 & 3 \\ \alpha & 0 & -2 \end{pmatrix} = \det\begin{pmatrix} 0 & 9 & -1 \\ -3 & -9 & 3 \\ \alpha & 0 & -2 \end{pmatrix}$$

$$= 0 + 27\alpha + 0 - [9\alpha + 54]$$

$$= 18\alpha - 54 = 0 \iff \alpha = 3$$

$$A - 6Id = \begin{pmatrix} -3 & 0 & 2 \\ -3 & -9 & 3 \\ 3 & 0 & -2 \end{pmatrix} \longrightarrow \left(\begin{array}{ccc|c} -3 & 0 & 2 & 4 \\ -3 & -9 & 3 & 0 \\ 3 & 0 & -2 & -4 \end{array}\right)$$

$$\xrightarrow[F_1+F_3 \to F_3']{} \left(\begin{array}{ccc|c} -3 & 0 & 2 & 4 \\ -3 & -9 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{matrix} \leftarrow x(z) \\ \leftarrow y(z) \end{matrix}$$

es compatible indeterminado para $\alpha = 3$

Ejercicio 16

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 3 & 4 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & -4 \\ 3 & 7 \end{pmatrix}$$

$$\begin{pmatrix} -2a+3b+2c & a+4b+5c \\ -2d+3e+2f & d+4e+5f \\ -2g+3h+2i & g+4h+5i \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & -4 \\ 3 & 7 \end{pmatrix}$$

$$\xrightarrow[C_1+2C_2\to C_1']{} \begin{pmatrix} 11b+12c & a+4b+5c \\ 11e+12f & d+4e+5f \\ 11h+12i & g+4h+5i \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ -6 & -4 \\ 17 & 7 \end{pmatrix}$$

$$\Rightarrow\ b=\frac{(7-12c)}{11},\quad a+4b+5c=3 \to a+\frac{4\cdot 7}{11}b-\frac{12}{11}c+5c=3$$

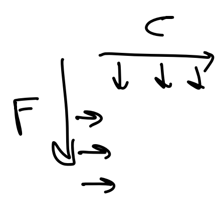
$$\Rightarrow\ a=\frac{(5-7c)}{11}$$

$$\Rightarrow\ e=\frac{(-6-12f)}{11},\quad d+\frac{4}{11}(-6-12f)+5f=-4$$

$$\Rightarrow\ d=\frac{(-20-7f)}{11}$$

$$\Rightarrow\ h=\frac{17-12i}{11},\quad g+4\frac{(17-12i)}{11}+5i=7$$

$$\Rightarrow\ g=\frac{9-7i}{11}$$



$$S=\left\{X\in\mathbb{R}^{3\times3} : X=\frac{1}{11}\begin{pmatrix} 5-7c & 7-12c & c \\ -20-7f & -6-12f & f \\ 9-7i & 17-12i & i \end{pmatrix}\right.$$

$$\left.=\frac{1}{11}\left[\begin{pmatrix} 5 & 7 & 0 \\ -20 & -6 & 0 \\ 9 & 17 & 0 \end{pmatrix}+\vec{x}\begin{pmatrix} -7 & -12 & 1 \end{pmatrix}\right],\ \text{para } \vec{x}\in\mathbb{R}^{3\times1}\right\}$$

Ejercicio 19

$$A = \left(\begin{array}{cc|cc} 7b & -a & 0 & 0 \\ 7d & -c & 0 & 0 \\ \hline 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & \alpha+2 \end{array}\right)$$

quiero $\alpha / \det(\alpha) = 1$, si $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 5$.

$$\det(A) = \det\begin{pmatrix} 7b & -a \\ 7d & -c \end{pmatrix} 3(\alpha+2)$$

$$= \underline{3}(\alpha+2)\underline{7} \det\begin{pmatrix} b & -a \\ d & -c \end{pmatrix} = 21(\alpha+2) \det\begin{pmatrix} b & a \\ d & c \end{pmatrix}(-1)$$

$$= 21(\alpha+2) \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= 21(\alpha+2)5 = 105(\alpha+2) = 1$$

$$\Rightarrow \alpha = \frac{1}{105} - 2$$

Ejercicio 21

$$A = \begin{pmatrix} a & b & c \\ 5 & 0 & 10 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 5a & -50b & 5c \\ 1 & 0 & 2 \\ 1 & -10 & 1 \end{pmatrix}$$

$\det(A) = 2$

$$\det(B) = 5 \det\begin{pmatrix} a & -10b & c \\ 1 & 0 & 2 \\ 1 & -10 & 1 \end{pmatrix} = -50 \det\begin{pmatrix} a & b & c \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= -10 \underbrace{\det\begin{pmatrix} a & b & c \\ 5 & 0 & 10 \\ 1 & 1 & 1 \end{pmatrix}}_{=A} = -20 \quad \square$$

$$\det(3A^TB) = 3^3 \det(A^T)\det(B) = 27 \det(A)\det(B)$$

$$= -1080$$

Ejercicio 22

a) " $A, B \in GL(n) \Rightarrow A+B \in GL(n)$". Falso

Contraejemplo : $A = id$, $B = -id \Rightarrow A+B = 0$ $\square$

b) " $A, B \in \mathbb{R}^{4\times4} \wedge det(A) = 2 \wedge det(B^{-1}) = 4$
$\Rightarrow det(2AB) = 8$ ". Verdadero ☺

D/ $det(2AB) = 2^4\, det(A)\, det(B) =$

$= 2^4 \dfrac{det(A)}{det(B^{-1})} = 2^4 \dfrac{2}{2^2} = 2^3 = 8$ $\square$

c) " $A \in \mathbb{R}^{5\times3} / \{x \in \mathbb{R}^3 : Ax = 0\} = \langle (1,3,4), (0,0,4) \rangle$
$\Rightarrow rg(A) = 1$ ". Verdadero

D/. Por el teorema de la dimensión

$(Ker(A) = \{x \in V : Ax = 0\})$

$dim(Ker(A)) + \underbrace{dim(Im(A))}_{rg(A)} = dim(dom(A))$

$2 + rg(A) = 3$

$\Rightarrow rg(A) = 1$ $\square$

d) " $A \in GL(2) \Rightarrow A5 + A^2 \in GL(2)$". Falso

$det(A)\, \underbrace{det(A + 5Id)}$
$det(A) \neq 0$ ✓

$A = -5Id \Rightarrow A5 + A^2 \notin GL(2)$ $\square$

e) "$A \in GL(3) \Rightarrow rg(A^3-2A) = rg(A^2-2I)$". Verdadero

D/. x teorema de la dimensión:

$$\begin{cases} \dim(\operatorname{Ker}(A^3-2A)) + rg(A^3-2A) = \dim(\operatorname{dom}(A^3-2A)) \\ \dim(\operatorname{Ker}(A^2-2I)) + rg(A^2-2I) = \dim(\operatorname{dom}(A^2-2I)) \end{cases}$$

$\dim(\operatorname{dom}(A^3-2A)) = \dim(\operatorname{dom}(A^2-2I)) = 3$

Para probar el enunciado,
basta con ver $\dim(\operatorname{Ker}(A^3-2A)) = \dim(\operatorname{Ker}(A^2-2I))$

$\geq$: $x \in \operatorname{Ker}(A^2-2I) \Rightarrow (A^2-2I)(x) = 0 \Rightarrow$

$A \underbrace{(A^2-2I)(x)}_{=0} = 0 \Rightarrow (A^3-2A)(x) = 0$

$\Rightarrow x \in \operatorname{Ker}(A^3-2A) \Rightarrow \operatorname{Ker}(A^2-2I) \subset \operatorname{Ker}(A^3-2A)$

$\Rightarrow \underline{\dim(\operatorname{Ker}(A^2-2I)) \leq \dim(\operatorname{Ker}(A^3-2A))}$

$\leq$: $x \in \operatorname{Ker}(A^3-2A) \Rightarrow (A^3-2A)(x) = 0 \Rightarrow$

$A \underbrace{(A^2-2I)(x)} = 0 \xrightarrow[A \in GL(n)]{} (A^2-2I)(x) = 0$

$\Rightarrow x \in \operatorname{Ker}(A^2-2I) \Rightarrow \operatorname{Ker}(A^3-2A) \subset \operatorname{Ker}(A^2-2I)$

$\Rightarrow \underline{\dim(\operatorname{Ker}(A^3-2A)) \leq \dim(\operatorname{Ker}(A^2-2I))}$

$\Rightarrow \dim(\operatorname{Ker}(A^3-2A)) = \dim(\operatorname{Ker}(A^2-2I))$

$\Rightarrow rg(A^3-2A) = rg(A^2-2I)$ $\square$

Ejercicio 14

f)
$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}}_{\equiv A}\left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right) \xrightarrow{F_3 - F_1 \to F_3'} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{matrix}\right)$$

$$\xrightarrow{F_2 - 2F_3 \to F_2'} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\left|\begin{matrix} 1 & 0 & 0 \\ +2 & 1 & -2 \\ -1 & 0 & 1 \end{matrix}\right)$$

$$\xrightarrow{F_1 - 2F_2 - 3F_3 \to F_1'} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\left|\underbrace{\begin{matrix} 0 & -2 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{matrix}}_{\equiv A^{-1}}\right)$$

$$AA^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 0 & -2 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

Duda

$$\det\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 2$$

$$\det\begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} = 2$$