1) a) $\Omega = \{ (d_1, d_2, d_3) : d_i \in \{1,2,3\} \}$

$\# \Omega = 27.$

b) $A = \{ (1,3,3) ; (2,3,3); (3,1,3) ; (3,2,3) ; (3,3,1); (3,3,2) \}$

c) $B = A \cup \{(3,3,3)\}$

d) $C = \{ (1,1,3); (1,2,3); (2,1,3); (2,2,3) ; (3,1,3) ; (3,2,3)$
$(1,3,1); (1,3,2); (2,3,1) ; (2,3,2) ; (3,3,1) ; (3,3,2) \}$

e) $A \cap C = \{ (3,1,3) ; (3,2,3); (3,3,1) ; (3,3,2) \}$

2) a) $p_1 = \frac{10! \, 2}{11!} = \frac{2}{11}$, $p_2 = \frac{9! . 2}{11!} = \frac{2}{110}$ $\quad p_1 > p_2$

b) $p_1 = \frac{27^3}{27^5}$, $p_2 = \frac{27^3}{27^6}$ $\quad p_1 > p_2.$

3) $P(M_1|G) = 0.12 \Rightarrow P(B|G) = P(M_1 \cap M_2 | G) = P(M_1|G) \cdot P(M_2|G) = 0.12$

Consigna

$P(C) = 0.0021$

$P(B | G \cap C^c) = 0.01$

a) $P(B|G) = P(M_1 \cap M_2 | G) = P(M_1|G) \cdot P(M_2|G) = 0.12^2$

$= 0.0144$

b) $P(B \cap G \cap C^c) \overset{\text{Regla de la multiplicación}}{=} P(B|G \cap C^c)\, P(G \cap C^c)$

$= P(B|G \cap C^c)\, P(C^c)\, P(G) = 0.01 \cdot (1-0.0021)\, P(G)$

independencia de C y G
(⇒ indep de C^c y G)

c) $P(B \cap G) = P(M_1 \cap M_2 \cap G) = P(M_1 \cap M_2 | G)\, P(G)$

$= P(M_1|G) \cdot P(M_2|G)\, P(G)$

$= [P(M_1|G)]^2\, P(G)$

$= 0.12^2\, P(G)$

d) $P(C^c | B \cap G) = \dfrac{0.01 \cdot (1-0.0021)\, \cancel{P(G)}}{0.12^2\, \cancel{P(G)}} \approx 0.69$

El argumento del fiscal no es válido.

4) Definiendo

$\Omega = \{\text{soldados de GOT}\}$

$O = \{\text{soldados que pierden un ojo}\}$

$J = \{\text{soldados que pierden una oreja}\}$

$P = \{\text{soldados que pierden una pierna}\}$

$$P(\underbrace{O^c \cap J^c \cap P^c}_{\substack{\text{soldados que} \\ \text{no pierden ni un ojo} \\ \text{ni una oreja ni una pierna}}}) = P((O \cup J \cup P)^c) = 1 - P(O \cup J \cup P) =$$

Por PIE

$$\overset{\downarrow}{=} 1 - P(O) - P(J) - P(P) + P(O \cap J) + P(O \cap P) + P(J \cap P)$$

$$- P(O \cap J \cap P)$$

$$= 1 - 0.4 - 0.4 - 0.4 + 0.2 + 0.2 + 0.2 - 0.1$$

$$= 0.3.$$

5) a) $X_1 + X_2$ resultado de la suma de dos dados 1 y 2

$$P(X_1 + X_2 = 5 \text{ u } 8) = P(X_1 + X_2 = 5) + P(X_1 + X_2 = 8)$$

$$= P\left(\substack{\text{salen} \\ (1,4),\ (2,3)(3,2)(4,1)}\right) + P\left(\substack{\text{salen } (2,6)\ (3,5) \\ (4,4)\ (5,3)\ (6,2)}\right)$$

$$= \frac{4}{36} + \frac{5}{36} = \frac{9}{36} = \frac{1}{4}.$$

b) $X \sim Ge(1/4)$

$P(X=x) = \frac{3}{4}^{x-1} \cdot \frac{1}{4}$ $\quad$ Sop $X = \{1, 2, 3, 4, \dots\}$

c) $E(X) = \frac{1}{1/4} = 4$ En promedio Alan tendrá que esperar 4 tiradas para ser liberado.

En el caso particular de la película Alan tiene que esperar 3 tiradas para ser liberado.

6) a) $f_X(x) = \int_0^1 3.6\, x^2 (y+y^2)\,dy = 3.6\, x^2 \left(\frac{y^2}{2} + \frac{y^3}{3}\right)\Big|_0^1$

$= 3.6\, x^2 \frac{5}{6} = 3x^2$ si $0 \leq x \leq 1$.

$E(X) = \int_0^1 3x^3\,dx = \frac{3}{4} x^4 \Big|_0^1 = \frac{3}{4}$.

b) $f_{Y|X=x}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)} = \dfrac{3.6\, x^2 (y+y^2)}{3x^2} = 1.2(y+y^2)$

c) $E(Y|X=x) = \int_0^1 y \cdot 1.2\,(y+y^2)\,dy = 1.2 \int_0^1 y^2+y^3\,dy$

$= 1.2 \left(\frac{y^3}{3} + \frac{y^4}{4}\right)\Big|_0^1 = 1.2 \cdot \frac{7}{12} = 0.7$.

d) $E\left(\frac{X}{Y}\right) = 3.6 \int_0^1 \int_0^1 \frac{x}{y}\, x^2 (y+y^2)\,dx\,dy = 3.6 \int_0^1 \int_0^1 x^3 (1+y)\,dx\,dy$

$= 3.6 \int_0^1 (1+y) \left.\frac{x^4}{4}\right|_0^1 dy = 3.6 \cdot \frac{1}{4} \int_0^1 1+y \, dy = 0.9 \left. y + \frac{y^2}{2} \right|_0^1$

$= 0.9 \cdot \frac{3}{2} = 1.35$

También se podría haber calculado $E\left(\frac{X}{Y}\right) = E(X) \cdot E\left(\frac{1}{Y}\right)$ porque X e Y son v.a. independientes. Sabemos que $E(X) = \frac{3}{4}$

faltaría calcular $E\left(\frac{1}{Y}\right) = \int_0^1 \frac{1}{y} f_Y(y) dy \underset{\downarrow}{=} \int_0^1 \frac{1}{y} 1.2 (y + y^2) dy$

Como X e Y son indep
$f_Y(y) = f_{Y|X=x}(y)$

$E\left(\frac{1}{Y}\right) = \int_0^1 1.2 (1+y) dy = 1.2 \left(y + \frac{y^2}{2}\right)\Big|_0^1$

$= 1.2 \cdot \frac{3}{2} = 1.8$

$\Rightarrow E(X/Y) = \frac{3}{4} \cdot 1.8 = 1.35.$

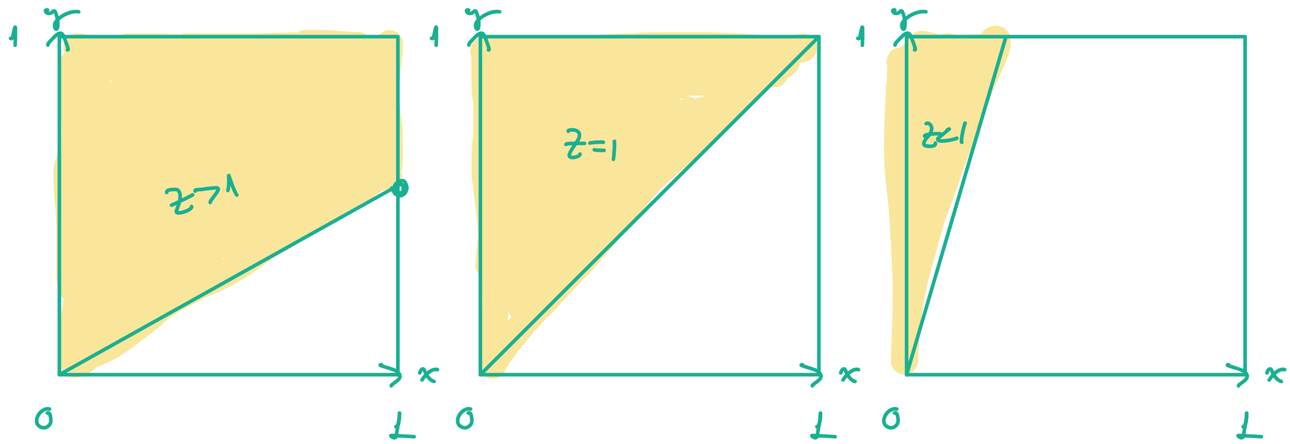
Forma más larga de calcular $E(X/Y)$

Sea $Z = \frac{X}{Y}$ $\qquad F_Z(z) = P\left(\frac{X}{Y} \leq z\right) = P(X \leq zY)$

$= \int_0^1 P(X \leq zy \mid Y = y) \; f_Y(y) dy$

$= \int_0^1 P(X \leq zy) \; f_Y(y) dy$ $\qquad$ X e Y indep

$= \int_0^1 \int_0^{\min\{z \cdot y, 1\}} f_X(x) dx f_Y(y) dy = \int_0^1 \int_0^{\min\{zy, 1\}} f_{XY}(x,y) dx d$



$x = zy \qquad \frac{x}{y} \leq z$

$y = \frac{1}{z}x$

Si $z \geq 1 \Rightarrow F_Z(z) = 1 - \int_0^1 \int_0^{1/z\,x} f_{XY}(x,y)\,dy\,dx$

$\left(\begin{array}{l} 0<x<1 \\ 0<y<1/z \end{array} \right)$

$$= 1 - \int_0^1 3.6\,x^2 \left(\frac{y^2}{2} + \frac{y^3}{3} \right) \Big|_0^{1/z\,x} dx$$

$$= 1 - \int_0^1 3.6 \left(\frac{x^4}{2z^2} + \frac{x^5}{3z^3} \right) dx = 1 - 3.6 \left(\frac{x^5}{10z^2} + \frac{x^6}{18z^3} \right) \Big|_0^1$$

$$= 1 - \frac{0.36}{z^2} - \frac{0.2}{z^3}.$$

Si $z < 1 \Rightarrow F_Z(z) = \int_0^1 \int_0^{zy} f_{XY}(x,y)\,dx\,dy$

$0 < x < z$

$0 < y < 1$

$$= \int_0^1 3.6 \frac{x^3}{3} (y + y^2) \Big|_0^{zy} dy$$

$x < yz$

$$= \int_0^1 1.2\,z^3\,y^3 (y + y^2)\,dy$$

$$= 1.2\,z^3 \left(\frac{y^5}{5} + \frac{y^6}{6} \right) \Big|_0^1 = \frac{11}{25} z^3$$

$$F_Z(z) = \begin{cases} \frac{11}{25} z^3 & \text{si } z \in (0,1) \\ 1 - \frac{0.36}{z^2} - \frac{0.2}{z^3} \end{cases} \Rightarrow f_Z(z) = \begin{cases} \frac{33}{25} z^2 & \text{si } z \in (0,1) \\ \frac{0.72}{z^3} + \frac{0.6}{z^4} & \text{si } z \geq 1 \end{cases}$$

$$\Rightarrow E(Z) = \int_0^{+\infty} f_Z(z)\,dz = \int_0^1 \frac{33}{25} z^3\,dz + \int_1^{+\infty} \frac{0.72}{z^2} + \frac{0.6}{z^3}\,dz$$

$$= \frac{33}{100} z^4 \Big|_0^1 - \left(0.72 z^{-1} + 0.3 z^{-2}\right)\Big|_1^{+\infty}$$

$$= 0.33 - (0 - 0.72 - 0.3) = 0.33 + 0.72 + 0.3$$

$$= 1.35$$

7) a) $E(X) \underset{\text{LEI}}{=} E(E(X|N)) = E(E(D_1 + \dots + D_N | N)) =$

$$= E(E(D_1|N) + E(D_2|N) + \dots + E(D_N|N))$$

$$= E(\underbrace{\tfrac{1}{\lambda} + \dots + \tfrac{1}{\lambda}}_{N \text{ veces}}) = E\left(\tfrac{1}{\lambda} \cdot N\right)$$

$$= \frac{1}{\lambda} E(N) = \frac{1}{\lambda} \cdot \frac{1}{p} = \frac{1}{\lambda p}.$$

Con $\lambda = 2$ y $p = 0.1$ $\quad \boxed{E(X) = \frac{1}{0.2} = 5}$

en promedio hay que esperar 5 minutos hasta que llegue el primer mail de spam.

b) $Var(X) = E(Var(X|N)) + Var(E(X|N))$

$D_i \sim iid \leftarrow = E(Var(D_1|N) + \dots + Var(D_N|N)) + Var(E(D_1|N) + \dots + E(D_N|N))$

$= E\left(\frac{1}{\lambda^2} N\right) + Var\left(\frac{1}{\lambda} N\right)$

$= \frac{1}{\lambda^2} E(N) + \frac{1}{\lambda^2} Var(N)$

$= \frac{1}{\lambda^2} \cdot \frac{1}{p} + \frac{1}{\lambda^2} \cdot \frac{(1-p)}{p^2}$

$= \frac{1}{\lambda^2 p^2}(p + 1 - p) = \frac{1}{\lambda^2 p^2}$

$Var(X) = 25 \Rightarrow \sqrt{Var(X)} = 5$ minutos.

c) $\boxed{M_X(t) =} E(e^{tX}) = E(e^{t(D_1 + \dots + D_N)}) = E(e^{tD_1} \cdot e^{tD_2} \cdots e^{tD_N})$

$= E(M_{D_1}(t) \cdots M_{D_N}(t)) = E([M_{D_1}(t)]^N)$

$= \sum_{n=1}^{\infty} \overbrace{p \cdot (1-p)^{n-1}}^{P(N=n)} \cdot [M_{D_1}(t)]^n$

sop $N = \{1, 2, 3, \dots\}$

$= \frac{p}{1-p} \sum_{n=1}^{\infty} (1-p)^n \frac{\lambda^n}{(\lambda - t)^n} = \frac{p}{1-p} \sum_{n=1}^{\infty} \left[\frac{\lambda(1-p)}{\lambda - t}\right]^n =$

notemos que

$a = \frac{\lambda(1-p)}{\lambda - t}$ cumple que $\underbrace{0 < \frac{\lambda(1-p)}{\lambda - t} < 1}$

si $t < \lambda p$.

$= \frac{p}{1-p} \cdot \frac{\frac{\lambda(1-p)}{\lambda - t}}{1 - \frac{\lambda(1-p)}{\lambda - t}}$

$= \frac{p}{1-p} \cdot \frac{\frac{\lambda(1-p)}{\cancel{\lambda - t}}}{\frac{\lambda - t - \lambda + \lambda p}{\cancel{\lambda - t}}} = \frac{p}{\cancel{1-p}} \frac{\lambda \cancel{(1-p)}}{\lambda p - t} = \frac{\lambda p}{\lambda p - t}$.

O sea $X \sim Exp(\lambda p)$.

8) $Var(X \cdot Y) = E(X^2 Y^2) - [E(XY)]^2$

$$= E(X^2)\,E(Y^2) - [E(X)\,E(Y)]^2$$

X^2 e Y^2 son indep

X e Y son indep

$$Var(X \cdot Y) = \underbrace{\left(E(X^2) - [E(X)]^2\right)}_{Var(X)} E(Y^2) + [E(X)]^2 E(Y^2) - [E(X)\,E(Y)]^2$$

$$= Var(X)\,E(Y^2) + [E(X)]^2\left[E(Y^2) - [E(Y)]^2\right]$$

$$= Var(X)\,E(Y^2) + [E(X)]^2\, Var(Y)$$

$$= Var(X)\left[E(Y^2) - [E(Y)]^2\right] + Var(X)\,[E(Y)]^2 + [E(X)]^2\, Var(Y)$$

$$= Var(X)\,Var(Y) + Var(X)\,[E(Y)]^2 + [E(X)]^2\, Var(Y)$$

otra forma:

$$Var(XY) = Var\left(E(XY \mid X)\right) + E\left(Var(XY \mid X)\right)$$

$$= Var\left(X\,E(Y \mid X)\right) + E\left(X^2\, Var(Y \mid X)\right)$$

$$= Var\left(X \cdot E(Y)\right) + E\left(X^2\, Var(Y)\right)$$

X e Y indep

$= E(Y^2)\, Var(X) + Var(Y)\, E(X^2)$

$= E(Y^2)\, Var(X) + Var(Y)\left[Var(X) + E(X)^2\right]$

9) a) $V_n + W_n \xrightarrow{P} 1 + (-2) = -1$ (se pueden sumar los resultados)

b) $\ln(V_n) \xrightarrow{P} \ln(1) = 0$ (TFC)

c) $W_n \cdot X_n \xrightarrow{D} -2\, N(3,4) = N(-6,16)$ por Slutsky

$V_n + W_n \cdot X_n \xrightarrow{D} 1 + N(-6,16) = N(-5,16)$ por Slutsky