BASIC RBC MODEL

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log c_t - \eta \frac{l_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} \right]$$

subject to

$$c_t + k_{t+1} = A_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta) k_t$$

given initial conditions A_0, k_0, and a law of motion for the technology process that we specify below.

Let λ_t denote the multiplier on the constraint and write the Lagrangian

$$L = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log c_t - \eta \frac{l_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} \right] - \lambda_t \left[c_t + k_{t+1} - A_t k_t^{\alpha} l_t^{1-\alpha} - (1-\delta) k_t \right]$$

The first order conditions with respect to c_t, l_t, and k_{t+1} are, respectively,

$$\frac{1}{c_t} = \lambda_t$$

$$\eta l_t^{\frac{1}{\nu}} = \lambda_t (1-\alpha) A_t k_t^{\alpha} l_t^{-\alpha}$$

$$\lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1} \left(\alpha A_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta \right) \right].$$

plus the feasibility constraint. The transversality condition of this problem is

$$\lim_{T\to\infty} E_0 \left[\beta^T \lambda_T k_{T+1} \right] = 0.$$

Shocks. The logarithm of TFP follows an AR(1) process

$$\log A_{t+1} = \rho \log A_t + \varepsilon_{t+1}$$

where ε_{t+1} is i.i.d. normal with mean 0 and variance σ_{ε}^2.

Equilibrium equations

We can write the equilibrium conditions as the following system of 7 equations

$$\frac{1}{c_t} = \lambda_t \tag{1}$$

$$\eta l_t^{\frac{1}{\nu}} = \lambda_t (1-\alpha) \frac{y_t}{l_t} \tag{2}$$

$$\lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1} \left(\alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta \right) \right] \tag{3}$$

$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha} \tag{4}$$

$$c_t + x_t = y_t \tag{5}$$

$$x_t = k_{t+1} - (1 - \delta) k_t \tag{6}$$

$$\log A_{t+1} = \rho \log A_t + \varepsilon_{t+1} \tag{7}$$

Steady state

The second step in the procedure consists of finding the non-stochastic steady steate of the economy and calibrating the model. In steady state the system (1)-(7) becomes

$$\frac{1}{\bar{c}} = \bar{\lambda} \tag{8}$$

$$\eta \bar{l}^{\frac{1}{\nu}} = \bar{\lambda} (1 - \alpha) \bar{y} / \bar{l} \tag{9}$$

$$1 = \beta \left(\alpha \bar{y} / \bar{k} + 1 - \delta \right) \tag{10}$$

$$\bar{y} = \bar{A} \bar{k}^{\alpha} \bar{l}^{1-\alpha} \tag{11}$$

$$\bar{c} + \bar{x} = \bar{y} \tag{12}$$

$$\bar{x} = \delta \bar{k} \tag{13}$$

$$\bar{A} = 1. \tag{14}$$

The system can be reduced to (get rid of $\bar{\lambda}$, $\bar{x}$)

$$\eta \bar{l}^{\frac{1}{\nu}} = \frac{1}{\bar{c}} (1 - \alpha) \frac{\bar{y}}{\bar{l}}$$

$$1 = \beta \left(\alpha \frac{\bar{y}}{\bar{k}} + 1 - \delta \right)$$

$$\frac{\bar{y}}{\bar{k}} = \left(\frac{\bar{l}}{\bar{k}} \right)^{1-\alpha}$$

$$\frac{\bar{c}}{\bar{k}} + \delta = \frac{\bar{y}}{\bar{k}}$$

From the second equation we find $\bar{y}/\bar{k}$:

$$\frac{\bar{y}}{\bar{k}} = \frac{1/\beta - (1 - \delta)}{\alpha}$$

and the third then gives $\bar{l}/\bar{k}$:

$$\frac{\bar{l}}{\bar{k}} = \left(\frac{\bar{y}}{\bar{k}}\right)^{\frac{1}{1-\alpha}} \Rightarrow \frac{\bar{l}}{\bar{k}} = \left(\frac{1/\beta - (1-\delta)}{\alpha}\right)^{\frac{1}{1-\alpha}}$$

The last equation then gives $\bar{c}/\bar{k}$

$$\frac{\bar{c}}{\bar{k}} = \frac{\bar{y}}{\bar{k}} - \delta \Rightarrow \frac{\bar{c}}{\bar{k}} = \frac{1/\beta - (1-\delta)}{\alpha} - \delta$$

So we are left with the last equation. We write it as

$$\eta \bar{l}^{\frac{1}{\nu}+1} = \frac{\bar{k}}{\bar{c}} (1-\alpha) \frac{\bar{y}}{\bar{k}}$$

Since we know $\bar{c}/\bar{k}$ and $\bar{y}/\bar{k}$, we know the right side. From here we solve for $\bar{l}$:

$$\bar{l} = \left(\frac{1-\alpha}{\eta} \frac{\bar{y}/\bar{k}}{\bar{c}/\bar{k}}\right)^{\frac{1}{1+1/\nu}} \tag{15}$$

Once we have $\bar{l}$, we recover $\bar{k}$ from $\bar{l}/\bar{k}$, and then we have the entire steady state

Log-linearization of the model

Log-linearized model

$$\begin{aligned}
0 &= \hat{c}_t + \hat{\lambda}_t && (16)\\
0 &= (1+\frac{1}{\nu})\hat{l}_t - \hat{\lambda}_t - \hat{y}_t && (17)\\
0 &= \hat{y}_t - \hat{A}_t - \alpha \hat{k}_t - (1-\alpha)\,\hat{l}_t && (18)\\
0 &= \bar{y}\hat{y}_t - \bar{c}\hat{c}_t - \bar{x}\hat{x}_t && (19)\\
\mathbb{E}_t[\hat{k}_{t+1}] &= (1-\delta)\,\hat{k}_t + \delta \hat{x}_t && (20)\\
\mathbb{E}_t\left[\hat{\lambda}_{t+1} + \beta\alpha\left(\bar{y}/\bar{k}\right)\left(\hat{y}_{t+1} - \hat{k}_{t+1}\right)\right] &= \hat{\lambda}_t && (21)\\
\mathbb{E}_t\left[\hat{A}_{t+1}\right] &= \rho \hat{A}_t. && (22)
\end{aligned}$$

Note that I wrote $\mathbb{E}_t[\hat{k}_{t+1}]$ even though $\hat{k}_{t+1}$ is chosen (and therefore already known) at time t. This is just notation that will allows us to write the model as the following first order vector expectational difference equation

$$\mathbf{A}\mathbb{E}_t\left[\mathbf{z}_{t+1}\right] = \mathbf{B}\mathbf{z}_t \tag{23}$$

where the vector $\mathbf{z}_t$ contains all the variables in the economy and $\mathbf{A}$ and $\mathbf{B}$ are square matrices.

We solve numerically this model using the Matlab program `solab.m`. We order the variables $\mathbf{z}_t$ as follows:

$$\mathbf{z}_t = \begin{bmatrix} \text{endogenous states variables} \\ \text{exogenous states variables} \\ \text{jump variables} \end{bmatrix}$$

In the RBC model described above, the only endogenous state variable is the stock of capital $\hat{k}_t$ and the only exogenous state variable is the level of technology $\hat{A}_t$. Therefore, the variable $\mathbf{z}_t$ is given by

$$\mathbf{z}_t = \left[\hat{k}_t,\ \hat{A}_t,\ \ \hat{y}_t,\ \hat{c}_t,\ \hat{l}_t,\ \hat{x}_t,\ \hat{\lambda}_t\right]'. \tag{24}$$

Please note that the order within each group of variables does not matter (e.g. we could put c_t before y_t in the vector $\mathbf{z}_t$).

In addition, we must tell the program how many of the variables in $\mathbf{z}_t$ are state variables. In our case, it is 2: $\hat{k}_t$ and $\hat{A}_t$. Note that, in this case, $\mathbf{A}$ and $\mathbf{B}$ are 7×7 matrices. If we let $\boldsymbol{\kappa}_t \equiv [\hat{k}_t, \hat{A}_t]'$ denote the vector of state variables and $\mathbf{u}_t = [\hat{y}_t,\ \hat{c}_t,\ \hat{l}_t,\ \hat{x}_t,\ \hat{\lambda}_t]$, the vector of jump variables, the solver delivers the equilibrium of the "certainty equivalent" model in the form

$$\begin{aligned} \mathbf{u}_t &= \mathbf{F}\boldsymbol{\kappa}_t \\ \boldsymbol{\kappa}_{t+1} &= \mathbf{P}\boldsymbol{\kappa}_t \end{aligned}$$

The "stochastic" solution of the model is obtained by replacing the second equation above with

$$\boldsymbol{\kappa}_{t+1} = \mathbf{P}\boldsymbol{\kappa}_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}$$

which simply recovers the stochastic shock $\hat{A}_{t+1} = \rho\hat{A}_t + \varepsilon_{t+1}$.

For the ordering (24), the matrices $\mathbf{A}$ and $\mathbf{B}$ of the system (23) are given by

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\beta\alpha\left(\bar{y}/\bar{k}\right) & 0 & \beta\alpha\left(\bar{y}/\bar{k}\right) & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \text{equation (16)} \\ \text{equation (17)} \\ \text{equation (18)} \\ \text{equation (19)} \\ \text{equation (20)} \\ \text{equation (21)} \\ \text{equation (22)} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & (1+\frac{1}{\nu}) & 0 & -1 \\ -\alpha & -1 & 1 & 0 & -(1-\alpha) & 0 & 0 \\ 0 & 0 & \bar{y} & -\bar{c} & 0 & -\bar{x} & 0 \\ 1-\delta & 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \text{equation (16)} \\ \text{equation (17)} \\ \text{equation (18)} \\ \text{equation (19)} \\ \text{equation (20)} \\ \text{equation (21)} \\ \text{equation (22)} \end{bmatrix}$$

Using the calibrated parameter values, the model delivers the following solution:

$$\mathbf{F} = \begin{bmatrix} 0.22 & 1.33 \\ 0.57 & 0.34 \\ -0.17 & 0.50 \\ -1.10 & 5.07 \\ -0.57 & -0.34 \end{bmatrix}; \quad \mathbf{P} = \begin{bmatrix} 0.96 & 0.09 \\ 0 & 0.95 \end{bmatrix}$$

Which, in other words, implies the following policy functions:

$$\begin{aligned} \hat{y}_t &= 0.22\hat{k}_t + 1.33\hat{A}_t \\ \hat{c}_t &= 0.57\hat{k}_t + 0.34\hat{A}_t \\ \hat{l}_t &= -0.17\hat{k}_t + 0.50\hat{A}_t \\ \hat{x}_t &= -1.10\hat{k}_t + 5.07\hat{A}_t \\ \hat{k}_{t+1} &= 0.96\hat{k}_t + 0.09\hat{A}_t \\ \hat{A}_{t+1} &= 0.95\hat{A}_t + \varepsilon_{t+1}. \end{aligned}$$

Once we have this solution, we can compute impulse responses, variance decompositions, simulations, compute spectral densities, and so forth.