Data Alignment, Part II

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Motivation

Two Issues. #1:

- Empirical assessments of DSGEs require trend removal.
- Yet, levels and trends typically convey information regarding model parameterization, through implications regarding steady state relationships.
- How can we preserve this information, while accomplishing the detrending step?

Motivation, cont.

Issue #2: Upon detrending,

- Log-linear model representations are in terms of variables expressed as logged deviations from steady state;
- Non-linear representations are in terms of levels.
- How can we maintain direct comparability of likelihood functions across alternative representations?

Motivation, cont.

Consider two alternative state-space representations of the optimal growth model:

- Non-linear
- Log-Linear

Motivation, cont.

Log-linear:

$$
\begin{aligned}
x_t &= F x_{t-1} + e_t, \\
x_t &= \left[\widehat{y}, \widehat{c}, \widehat{i}, \widehat{k}, \widehat{z}\right]', \\
\widehat{a}_t &= \ln \frac{a_t}{\bar{a}_t}, \\
a_t &= \frac{A_t}{\left(1 + \frac{g}{1-\alpha}\right)^t}.
\end{aligned}
$$

Observation:

$$X_t = H' x_t + u_t, \qquad u_t \sim N(0, \Sigma_u).$$

Motivation, cont.

Non-linear:

$$
\begin{aligned}
\left(1+\frac{g}{1-\alpha}\right) k'(\widetilde{k}_t, \widetilde{z}_t) &= i(\widetilde{k}_t, \widetilde{z}_t) + (1-\delta) k_t \\
\log z_t &= (1-\rho)\log(z_0) + \rho \log z_{t-1} + \varepsilon_t \\
\widetilde{a} &= \frac{a_t - \bar{a}}{\omega_a} \in [-1, 1] .
\end{aligned}
$$

Observation (if direct comparability is to be maintained across model representations):

$$X_t = H' x_t + u_t , \qquad u_t \sim N(0, \Sigma_u).$$

Motivation, cont.

Steady state values:

$$
\begin{aligned}
\bar{k} &= \left(\frac{\alpha}{(\beta\zeta)^{-1} - (1-\delta)} \right)^{\frac{1}{1-\alpha}} \\
\bar{y} &= \bar{k}^{\alpha}, \\
\bar{i} &= \left(\delta + \frac{g}{1-\alpha} \right) \bar{k} \\
\bar{c} &= \bar{y} - \bar{i}
\end{aligned}
$$

Preserving Information

Let's take $[y, i]$ as observable. A convenient means of representing the model's characterization of these variables is in the form

$$a_t = \bar{a} e^{\left(\frac{g}{1-\alpha}\right)t} e^{\omega_{at}}, \qquad a = y, i,$$

where ω_{at} is a deterministic function of the stochastic component ω_t of the TFP process z_t, along with (k_0, z_0). The mapping from (ω_t, k_0, z_0) to ω_{at} is obtained implicitly from the policy function approximations described above.

Preserving Information, cont.

Recall that model variables are respresented in detrended form

$$\frac{a_t}{\left(1+\frac{g}{1-\alpha}\right)^t} \approx \frac{a_t}{e^{\left(\frac{g}{1-\alpha}\right)t}}.$$

Therefore, in working with the log-linear model approximation,

$$\begin{aligned} \widehat{a}_t &= \omega_{at} \\ &= \sigma_k \widehat{k}_{t-1} + \sigma_z \widehat{z}_{t-1} + e_{ct}; \end{aligned}$$

and in working with the non-linear approximation,

$$\begin{aligned} \frac{a_t}{\left(1+\frac{g}{1-\alpha}\right)^t} &= \overline{a} e^{\omega_{at}} \\ &= a\left(\widetilde{k}_t, \widetilde{z}_t\right). \end{aligned}$$

Preserving Information, cont.

Preserving information regarding g.

If the actual data truly followed a stable log-linear trend, detrending could be achieved as part of the model estimation process: for each candidate parameterization, use the relevant (α, g) to construct

$$\frac{a_t}{\left(1+\frac{g}{1-\alpha}\right)^t}.$$

However, the trend-break behavior we've noted makes this inappropriate.

Instead, we typically work with

$$\ln \frac{a_t}{HP_{at}},$$

which has no trend or intercept term.

Preserving Information, cont.

Remedy:

Impose $g = (1 - \alpha) \cdot \widehat{g_y}$, where $\widehat{g_y}$ is the sample estimate of the growth rate of y. (0.00475 for post-war quarterly data).

Preserving Information, cont.

Preserving information regarding steady states.

As noted,

$$\ln \frac{a_t}{HP_{at}}$$

has no trend or intercept term.

Preserving Information, cont.

Remedy:
Restore relative sample values to preserve information regarding steady states:

$$
\begin{aligned}
\ln y_t &= \ln \frac{Y_t}{HP_{yt}}, \\
\ln i_t &= \ln \frac{I_t}{HP_{it}} + \ln \left(\overline{\frac{i}{y}}_T \right),
\end{aligned}
$$

where $\left(\overline{\frac{i}{y}}_T \right)$ denotes the sample mean of $\frac{i_t}{y_t}$.

Model Alignment

Assuming interest in maintaining direct comparablility of likelihoods across models:

- For both representations, construct X_t by subtracting logged steady state ratios $\frac{\bar{a}}{\bar{y}}$, $a = y, i$ from logs of the observed data for each proposed model parameterization. The individual elements of X_t are thus

$$\begin{aligned}
\widehat{y_t} &= \ln \frac{Y_t}{HP_{yt}}, \\
\widehat{i_t} &= \ln \frac{I_t}{HP_{it}} + \ln \left(\overline{\frac{i}{y}}_T \right) - \ln \frac{\bar{i}}{\bar{y}},
\end{aligned}$$

- For the log-linear approximation, X_t is fed directly into the Kalman filter.

Model Alignment, cont.

- For the non-linear approximation, compare X_t with logged values of the ratio of corresponding model variables to their steady state values:

$$\ln \frac{a\left(\widetilde{k}_t, \widetilde{z}_t\right)}{\overline{a}}, \qquad a = y, i.$$

Differences between the elements of X_t and their model counterparts are independent and normally distributed, according to the measurement equation $X_t = H' x_t + u_t$.