A Small-Scale New Keynesian Model

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New Keynesian model

Modern version of a Keynesian model with an IS curve, a Phillips curve, and a monetary policy rule.

Model:

- Households.
- Perfectly competitive final good producers.
- Continuum of intermediate good producers with pricing power.
- Monetary authority.
- Price stickiness:
  - Changing prices is costly (Rotemberg).
  - Firms that can change prices are randomly selected (Calvo).
  - Both models imply identical log-linear equilibrium conditions. But Rotemberg model is easier to derive.

Final equations are very simple. But deriving them requires some work.

Households

- Preferences over consumption c_t, labor l_t, and real money balances $m_t = M_t / P_t$:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \eta \frac{l_t^{1+\nu}}{1+\nu} + \psi \frac{m_t^{1-\xi}}{1-\xi} \right)$$

- Budget constraint in nominal terms

$$P_t c_t + B_t + M_t = W_t l_t + D_t + R_{t-1} B_{t-1} + M_{t-1}$$

P_t : nominal prices, B_t: nominal bonds, M_t : money, W_t: nominal wage, R_{t-1} gross nominal interest rate between $t-1$ and t; D_t: profits.

- Budget constraint in real terms (divide through by P_t)

$$c_t + b_t + m_t = w_t l_t + d_t + \frac{R_{t-1}}{\pi_t} b_{t-1} + \frac{m_{t-1}}{\pi_t}$$

where $\pi_t = P_t / P_{t-1}$ is the **gross** inflation rate.

Lagrangian of the household's problem

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \frac{c_t^{1-\sigma}}{1-\sigma} - \eta \frac{l_t^{1+\nu}}{1+\nu} + \psi \frac{m_t^{1-\xi}}{1-\xi} \\ -\lambda_t \left[c_t + b_t + m_t - w_t l_t - d_t - \frac{R_{t-1}}{\pi_t} b_{t-1} - \frac{m_{t-1}}{\pi_t} \right] \end{array} \right\}$$

First order conditions w.r.t. c_t, l_t, b_t, m_t:

$$c_t^{-\sigma} = \lambda_t$$

$$\eta l_t^{\nu} = \lambda_t w_t$$

$$\lambda_t = \beta E_t \left[\lambda_{t+1} \frac{R_t}{\pi_{t+1}} \right]$$

$$\psi m_t^{-\xi} = \lambda_t - \beta E_t \frac{\lambda_{t+1}}{\pi_{t+1}}$$

using the Euler equation into the last condition implies

$$\psi m_t^{-\xi} = \lambda_t \left(\frac{R_t - 1}{R_t} \right)$$

Production sector

Two sectors:

- A competitive sector that produces final goods
- An intermediate goods sector with a continuum of firms each producing a differentiated intermediate input.
  - Firms are monopolistically competitive and choose prices
  - Price stickiness in intermediate good sector
  - Changing nominal prices is costly: quadratic adjustment cost.

Final goods sector

- A representative firm produces final goods using a continuum of intermediate goods indexed by $j \in [0, 1]$ with the technology

$$y_t = \left(\int_0^1 y_t(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}}$$

- The firm takes prices as given: the nominal input prices $P_t(j)$ and the output price P_t.

- The problem of the firm is to maximize profits:

$$\max_{y_t(j)} P_t y_t - \int_0^1 P_t(j) y_t(j) dj.$$

Final goods sector

- Using the production function the problem is

$$\max_{y_t(j)} P_t \left(\int_0^1 y_t(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}} - \int_0^1 P_t(j) y_t(j) dj$$

- First order condition w.r.t $y_t(j)$

$$\frac{\theta}{\theta-1} P_t \left(\int_0^1 y_t(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}-1} \frac{\theta-1}{\theta} y_t(j)^{\frac{-1}{\theta}} - P_t(j) = 0$$

which implies the conditional demands

$$y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\theta} y_t. \tag{1}$$

Final goods sector

- Constant returns to scale imply that profits are zero

$$\begin{aligned}\text{Profits}_t &= P_t y_t - \int_0^1 P_t(j) y_t(j) dj \\ &= \left[P_t - P_t^{\theta} \int_0^1 P_t(j)^{1-\theta} dj\right] y_t \\ &= 0.\end{aligned}$$

- From here we obtain the relevant price index

$$P_t = \left[\int_0^1 P_t(j)^{1-\theta} dj\right]^{\frac{1}{1-\theta}}. \tag{2}$$

Intermediate goods producers

- There is a continuum of firms producing an intermediate good indexed by $j \in [0, 1]$, each of which is a price setter.

- Firm j produces the intermediate good using the technology

$$y_t(j) = A_t l_t^d(j). \tag{3}$$

- Each firm is monopolistic in that it chooses its optimal price $P_t(j)$ internalizing that the demand they face is given by (1).

- **Price stickiness:** firms face quadratic price adjustment cost

$$AC_t(j) = \frac{\omega}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi}\right)^2 y_t. \tag{4}$$

Intermediate goods producers

- Adjustment cost makes the firm's problem dynamic.

- Maximize the present discounted value of future nominal dividends,

$$E_0 \sum_{t=0}^{\infty} Q_t D_t (j)$$

$$D_t (j) = P_t (j)\, y_t (j) - W_t l_t^d (j) - P_t \frac{\omega}{2} \left(\frac{P_t (j)}{P_{t-1} (j)} - \bar{\pi} \right)^2 y_t$$

- In real terms,

$$\max E_0 \sum_{t=0}^{\infty} q_t \left[\frac{P_t (j)}{P_t} y_t (j) - w_t l_t^d (j) - \frac{\omega}{2} \left(\frac{P_t (j)}{P_{t-1} (j)} - \bar{\pi} \right)^2 y_t \right]$$

where $q_t = Q_t \frac{P_t}{P_0}$ is the real discount rate.

Intermediate goods producers

- Use $y_t(j) = A_t l_t^d(j)$ to solve for labor demand as a function of output

$$l_t^d(j) = \frac{y_t(j)}{A_t}.$$

- Real profits at time t are thus

$$d_t(j) = \frac{P_t(j)}{P_t} y_t(j) - \frac{w_t}{A_t} y_t(j) - \frac{\omega}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi}\right)^2 y_t$$

- Use the demand (1) to replace $y_t(j)$ as a function of prices and write profits as

$$d_t(j) = \left[\frac{P_t(j)^{1-\theta}}{P_t^{1-\theta}} - \frac{w_t}{A_t}\frac{P_t(j)^{-\theta}}{P_t^{-\theta}} - \frac{\omega}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi}\right)^2\right] y_t$$

Intermediate goods producers

- Now let's simplify this a little more. Let's define the price of intermediate good j relative to the aggregate price index as

$$\widetilde{p}_t(j) = \frac{P_t(j)}{P_t}.$$

  We can think of the firm as either choosing $\widetilde{p}_t(j)$ or $P_t(j)$.

- Then, real profits can be written as

$$d_t(j) = \left[\widetilde{p}_t(j)^{1-\theta} - \frac{w_t}{A_t}\widetilde{p}_t(j)^{-\theta} - \frac{\omega}{2}\left(\frac{\widetilde{p}_t(j)}{\widetilde{p}_{t-1}(j)}\pi_t - \bar{\pi}\right)^2\right] y_t$$

Intermediate goods producers

- Firm's problem becomes

$$\max_{\widetilde{p}_t(j)} E_0 \sum_{t=0}^{\infty} q_t y_t \left[\widetilde{p}_t(j)^{1-\theta} - \frac{w_t}{A_t} \widetilde{p}_t(j)^{-\theta} - \frac{\omega}{2} \left(\frac{\widetilde{p}_t(j)}{\widetilde{p}_{t-1}(j)} \pi_t - \bar{\pi} \right)^2 \right]$$

- The foc is

$$q_t y_t \left[(1-\theta) \widetilde{p}_t(j)^{-\theta} + \theta \frac{w_t}{A_t} \widetilde{p}_t(j)^{-\theta-1} - \omega \left(\frac{\widetilde{p}_t(j)}{\widetilde{p}_{t-1}(j)} \pi_t - \bar{\pi} \right) \frac{\pi_t}{\widetilde{p}_{t-1}(j)} \right]$$
$$+ E_t \left[q_{t+1} y_{t+1} \omega \left(\frac{\widetilde{p}_{t+1}(j)}{\widetilde{p}_t(j)} \pi_{t+1} - \bar{\pi} \right) \frac{\widetilde{p}_{t+1}(j) \pi_{t+1}}{\widetilde{p}_t(j)^2} \right] = 0.$$

Intermediate goods producers

- When $\omega = 0$, there is no adjustment cost in prices.

- The first order condition collapses to

$$\widetilde{p}_t(j) = \frac{\theta}{\theta - 1} \frac{w_t}{A_t}.$$

  or

$$P_t(j) = \frac{\theta}{\theta - 1} \frac{W_t}{A_t}.$$

  where we used $\widetilde{p}_t(j) = P_t(j)/P_t$.

- Price is set as a constant mark-up over the nominal marginal cost.

- This is the usual result with a monopolistic producer that faces a demand with a constant elasticity θ.

Monetary Authority

- Monetary policy is described by a Taylor rule of the form

$$\log R_t = (1-\rho_R)\log R_t^* + \rho_R \log R_{t-1} + v_t. \tag{5}$$

  - $\log R_t$ is the net nominal interest rate (we always use $\log(\cdot)$ for the natural logarithm),
  - $\log R_t^*$ is the target nominal interest rate,
  - ρ_R is a smoothing parameter
  - v_t is monetary policy shock.

- Target interest rate satisfies

$$\log R_t^* = \log \bar{R} + \phi_\pi \log\left(\frac{\pi_t}{\bar{\pi}}\right) + \phi_y \log\left(\frac{y_t}{\bar{y}}\right). \tag{6}$$

Monetary Authority

- Using (6) into (5) the Taylor rule becomes

$$\log\left(\frac{R_t}{\bar{R}}\right) = (1-\rho_R)\left[\phi_\pi \log\left(\frac{\pi_t}{\bar{\pi}}\right) + \phi_y \log\left(\frac{y_t}{\bar{y}}\right)\right] + \rho_R \log\left(\frac{R_{t-1}}{\bar{R}}\right) + v_t.$$

Shocks

- Technology:
$$\log A_{t+1} = \rho_A \log A_t + \varepsilon_{A,t+1}, \tag{7}$$
- Monetary policy:
$$v_{t+1} = \rho_v v_t + \varepsilon_{v,t+1}. \tag{8}$$
- Innovations $\varepsilon_{A,t+1}$ and $\varepsilon_{v,t+1}$ are uncorrelated i.i.d. shocks with a standard deviation σ_A and σ_v.

Equilibrium in goods and labor markets

- Focus on a **symmetric equilibrium**: all intermediate good producers make same choices.

- The subscript j can be dropped.

- The aggregate price level P_t is equal to the price chosen by the intermediate good producers so that $\widetilde{p}_t(j) = 1$. Also, condition (1) implies $y_t(j) = y_t$ for all j.

Feasibility in goods and labor markets are

$$y_t = c_t + AC_t \tag{9}$$

$$l_t = l_t^d$$

Feasibility in goods markets include the adjustment cost since it is incurred in terms of final consumption goods.

Now some algebra

- Using symmetry, $\widetilde{p}_t(j) = 1$ for all j, pricing condition becomes

$$0 = q_t y_t \left[(1-\theta) + \theta \frac{w_t}{A_t} - \omega \left(\pi_t - \bar{\pi} \right) \pi_t \right] + E_t \left[q_{t+1} y_{t+1} \omega \left(\pi_{t+1} - \bar{\pi} \right) \pi_{t+1} \right].$$

- The relevant discount rate q_t for the firms is the marginal valuation of goods at time 0 of its owners, the households. Thus,

$$q_t = \beta^t \frac{c_t^{-\sigma}}{c_0^{-\sigma}} \Rightarrow \frac{q_{t+1}}{q_t} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma}.$$

- Therefore, the pricing equation becomes

$$\left(\pi_t - \bar{\pi} \right) \pi_t = \frac{\theta}{\omega} \left[\frac{w_t}{A_t} - \frac{\theta - 1}{\theta} \right] + E_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{y_{t+1}}{y_t} \left(\pi_{t+1} - \bar{\pi} \right) \pi_{t+1} \right]$$

Summary of equilibrium conditions (I)

$$\eta l_t^{\nu} = c_t^{-\sigma} w_t \tag{10}$$

$$\psi m_t^{-\xi} = c_t^{-\sigma} \frac{R_t - 1}{R_t} \tag{11}$$

$$c_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma} \frac{R_t}{\pi_{t+1}} \right] \tag{12}$$

$$y_t = A_t l_t \tag{13}$$

$$y_t = c_t + \frac{\omega}{2} \left(\pi_t - \bar{\pi} \right)^2 y_t \tag{14}$$

$$\left(\pi_t - \bar{\pi} \right) \pi_t = \frac{\theta}{\omega} \left[\frac{w_t}{A_t} - \frac{\theta - 1}{\theta} \right] + E_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{y_{t+1}}{y_t} \left(\pi_{t+1} - \bar{\pi} \right) \pi_{t+1} \right] \tag{15}$$

Summary of equilibrium conditions (II)

$$\log\left(\frac{R_t}{\bar{R}}\right) = (1-\rho_R)\left[\phi_\pi \log\left(\frac{\pi_t}{\bar{\pi}}\right) + \phi_y \log\left(\frac{y_t}{\bar{y}}\right)\right] + \rho_R \log\frac{R_{t-1}}{\bar{R}} + v_t \tag{16}$$

$$\log A_{t+1} = \rho_A \log A_t + \varepsilon_{A,t+1} \tag{17}$$

$$v_{t+1} = \rho_v v_t + \varepsilon_{v,t+1} \tag{18}$$

Simplifying the system

- Ignore the money demand equation (11) because once we know the policy functions for c_t and R_t, we can recover the policy function for m_t. That is, m_t does not affect any other equation.

- Introducing (13) into (10) gives

$$\eta \frac{y_t^{\nu} c_t^{\sigma}}{A_t^{\nu}} = w_t$$

- And use this equation to get rid of w_t in equation (15)

System reduced to 6 equations

$$c_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma} \frac{R_t}{\pi_{t+1}} \right] \tag{19}$$

$$y_t \left[1 - \frac{\omega}{2} (\pi_t - \bar{\pi})^2 \right] = c_t \tag{20}$$

$$(\pi_t - \bar{\pi}) \pi_t = \frac{\theta}{\omega} \left[\eta \frac{y_t^{\nu} c_t^{\sigma}}{A_t^{1+\nu}} - \frac{\theta - 1}{\theta} \right] + E_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{y_{t+1}}{y_t} (\pi_{t+1} - \bar{\pi}) \pi_{t+1} \right] \tag{21}$$

$$\log \left(\frac{R_t}{\bar{R}} \right) = (1 - \rho_R) \left[\phi_\pi \log \left(\frac{\pi_t}{\bar{\pi}} \right) + \phi_y \log \left(\frac{y_t}{\bar{y}} \right) \right] + \rho_R \log \frac{R_{t-1}}{\bar{R}} + v_t \tag{22}$$

$$E_t [\log A_{t+1}] = \rho_A \log A_t \tag{23}$$

$$E_t [v_{t+1}] = \rho_v v_t \tag{24}$$

Steady state

Evaluating the system of equations at the steady state gives

$$\bar{R} = \bar{\pi}/\beta \tag{25}$$

$$\bar{c} = \bar{y} = \left[\frac{1}{\eta}\left(\frac{\theta - 1}{\theta}\right)\right]^{\frac{1}{\sigma+\nu}} \tag{26}$$

$$\bar{A} = 1 \tag{27}$$

$$\bar{v} = 0 \tag{28}$$

- The steady state depends on the steady state inflation rate, $\bar{\pi}$, which is chosen by the monetary authority.
- If the monetary authority targets zero inflation, so that $\bar{\pi} = 1$, then $\bar{R} = 1/\beta$.

Log-linearization of the equilibrium conditions

- We log-linearize the equilibrium conditions around the steady state.
- Define, for any variable x_t, its log-deviation from the steady state by

$$\hat{x}_t = \log\left(x_t/\bar{x}\right)$$

so that

$$x_t = \bar{x}e^{\hat{x}_t}.$$

- Rewrite the system of equations in terms of the variables $\hat{x}_t$.
- Do not transform the monetary policy shock v_t, which is already linear with mean zero.

Euler Equation (19)

$$0 = \beta E_t \left[c_{t+1}^{-\sigma} \frac{R_t}{\pi_{t+1}} \right] - c_t^{-\sigma}$$

$$0 = \beta E_t \left[\frac{\bar{c}^{-\sigma} \bar{R}}{\bar{\pi}} e^{-\sigma \hat{c}_{t+1} + \hat{R}_t - \hat{\pi}_{t+1}} \right] - \bar{c}^{-\sigma} e^{-\sigma \hat{c}_t}$$

- Ignore expectation operator for the moment.
- Taylor expansion around $\left(\hat{R}, \hat{c}_{t+1}, \hat{\pi}_{t+1}, \hat{c}_t\right) = (0, 0, 0, 0)$:

$$0 \approx \beta \bar{c}^{-\sigma} \frac{\bar{R}}{\bar{\pi}} \left[\hat{R}_t - \hat{\pi}_{t+1} - \sigma \hat{c}_{t+1} \right] + \bar{c}^{-\sigma} \sigma \hat{c}_t$$

- Rearranging and reinserting the expectation operator

$$E_t \hat{c}_{t+1} - \hat{c}_t \approx \frac{1}{\sigma} \left[\hat{R}_t - E_t \hat{\pi}_{t+1} \right]. \tag{29}$$

Feasibility condition (20)

$$0 = y_t \left[1 - \frac{\omega}{2} \left(\pi_t - \bar{\pi}\right)^2\right] - c_t$$

$$0 = \bar{y} e^{\hat{y}_t} \left[1 - \frac{\omega}{2} \left(\bar{\pi} e^{\hat{\pi}_t} - \bar{\pi}\right)^2\right] - \bar{c} e^{\hat{c}_t}$$

- Linearizing around $(\hat{y}_t, \hat{\pi}_t, \hat{c}_t) = (0, 0, 0)$ gives

$$0 \approx \bar{y}\hat{y}_t - \bar{y}\omega \left(\bar{\pi} - \bar{\pi}\right) \bar{\pi}\hat{\pi}_t - \bar{c}\hat{c}_t.$$

- Using $\bar{y} = \bar{c}$ gives

$$\hat{c}_t \approx \hat{y}_t. \tag{30}$$

Pricing Equation (21)

$$0 = \frac{\theta}{\omega}\left[\eta \frac{y_t^{\nu} c_t^{\sigma}}{A_t^{1+\nu}} - \frac{\theta - 1}{\theta}\right] - (\pi_t - \bar{\pi})\,\pi_t + E_t\left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\frac{y_{t+1}}{y_t}(\pi_{t+1} - \bar{\pi})\,\pi_{t+1}\right]$$

$$\begin{aligned} 0 = {} & \frac{\theta}{\omega}\left[\eta \frac{\bar{y}^{\nu}\bar{c}^{\sigma}}{\bar{A}^{1+\nu}} e^{\nu\hat{y}_t + \sigma\hat{c}_t - (1+\nu)\hat{A}_t} - \frac{\theta - 1}{\theta}\right] - \left(\bar{\pi}e^{\hat{\pi}_t} - \bar{\pi}\right)\bar{\pi}e^{\hat{\pi}_t} \\ & + E_t\left[\beta\bar{\pi}e^{-\sigma(\hat{c}_{t+1} - \hat{c}_t) + (\hat{y}_{t+1} - \hat{y}_t) + \hat{\pi}_{t+1}}\left(\bar{\pi}e^{\hat{\pi}_{t+1}} - \bar{\pi}\right)\right] \end{aligned}$$

Linearizing around $\left(\hat{\pi}_t, \hat{y}_t, \hat{c}_t, \hat{A}_t, \hat{\pi}_{t+1}, \hat{y}_{t+1}, \hat{c}_{t+1}\right) = \bar{0}$ gives

$$0 \approx \frac{\theta}{\omega}\eta\frac{\bar{y}^{\nu}\bar{c}^{\sigma}}{\bar{A}^{1+\nu}}\left[\nu\hat{y}_t + \sigma\hat{c}_t - (1+\nu)\hat{A}_t\right] - \bar{\pi}^2\hat{\pi}_t + \beta\bar{\pi}^2 E_t\hat{\pi}_{t+1}$$

Pricing Equation (21)

- But in steady state

$$\eta \frac{\bar{y}^{\nu}\bar{c}^{\sigma}}{\bar{A}^{1+\nu}} = \frac{\theta - 1}{\theta}$$

- Dividing by $\bar{\pi}^2$ and rearranging gives

$$\beta E_t\left[\hat{\pi}_{t+1}\right] \approx \hat{\pi}_t - \frac{\theta - 1}{\bar{\pi}^2\omega}\left[\nu\hat{y}_t + \sigma\hat{c}_t - (1+\nu)\hat{A}_t\right]$$

- But using $\hat{y}_t = \hat{c}_t$,

$$\hat{\pi}_t = \beta E_t\left[\hat{\pi}_{t+1}\right] + \kappa\left[(\nu + \sigma)\hat{y}_t - (1+\nu)\hat{A}_t\right], \tag{31}$$

where

$$\kappa \equiv \frac{\theta - 1}{\bar{\pi}^2\omega}.$$

Taylor rule and shocks

- The Taylor rule is log-linear:

$$\hat{R}_t = (1 - \rho_R)\left[\phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t\right] + \rho_R \hat{R}_{t-1} + v_t \tag{32}$$

- The evolution of productivity is log-linear:

$$E_t\left[\hat{A}_{t+1}\right] = \rho_A \hat{A}_t.$$

- The monetary policy shock is linear:

$$E_t\left[v_{t+1}\right] = \rho_v v_t.$$

Summary log-linear conditions

$$E_t \hat{c}_{t+1} - \hat{c}_t = \frac{1}{\sigma}[\hat{R}_t - E_t \hat{\pi}_{t+1}]$$

$$0 = \hat{c}_t - \hat{y}_t$$

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \kappa \left[(\nu + \sigma) \hat{y}_t - (1 + \nu) \hat{A}_t \right]$$

$$\hat{R}_t = (1 - \rho_R) [\phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t] + \rho_R \hat{R}_{t-1} + v_t$$

$$E_t \left[\hat{A}_{t+1} \right] = \rho_A \hat{A}_t.$$

$$E_t [v_{t+1}] = \rho_v v_t$$

We can simplify by eliminating consumption from the system

Basic New-Keynesian model

- **Dynamic IS curve:**

$$\boxed{E_t\hat{y}_{t+1} = \hat{y}_t + \frac{1}{\sigma}[\hat{R}_t - E_t\hat{\pi}_{t+1}]}$$

- **New-Keynesian Phillips curve:**

$$\boxed{\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \kappa \underbrace{\left[(\nu+\sigma)\hat{y}_t - (1+\nu)\hat{A}_t\right]}_{=\widehat{mc}_t}}$$

- **Taylor rule** (replaces the LM curve):

$$\boxed{\hat{R}_t = (1-\rho_R)\left[\phi_\pi\hat{\pi}_t + \phi_y\hat{y}_t\right] + \rho_R\hat{R}_{t-1} + v_t}$$

- **Shocks:**

$$E_t\left[\hat{A}_{t+1}\right] = \rho_A\hat{A}_t$$

$$E_t\left[v_{t+1}\right] = \rho_v v_t$$

Solving the model numerically

- Write the system in the form

$$\mathbf{A} E_t \left[\mathbf{z}_{t+1} \right] = \mathbf{B} \mathbf{z}_t$$

- Need to identify the state and control variables.
  - The state, or predetermined, variables are those variables that agents cannot affect at time t and that help predict the future evolution of the economy.

- State variables in this model:

$$\mathbf{x}_t = \left[\hat{R}_{t-1}, \hat{A}_t, v_t \right]'.$$

- Control or non-predetermined variables:

$$\mathbf{y}_t = \left[\hat{y}_t, \hat{\pi}_t \right]'.$$

Solving the model numerically

- Our vector of variables is

$$\mathbf{z}_{t+1} = \begin{bmatrix} \hat{R}_t \\ \hat{A}_{t+1} \\ v_{t+1} \\ \hat{y}_{t+1} \\ \hat{\pi}_{t+1} \end{bmatrix} \quad \mathbf{z}_t = \begin{bmatrix} \hat{R}_{t-1} \\ \hat{A}_t \\ v_t \\ \hat{y}_t \\ \hat{\pi}_t \end{bmatrix}$$

- Therefore, in the system

$$\mathbf{A} E_t \left[\mathbf{z}_{t+1} \right] = \mathbf{B} \mathbf{z}_t$$

the matrices **A** and **B** are of dimension 5×5.

Write the system of equations as

$$E_t\hat{y}_{t+1} + \frac{1}{\sigma}\left(E_t\hat{\pi}_{t+1} - E_t\hat{R}_t\right) = \hat{y}_t$$

$$\beta E_t\hat{\pi}_{t+1} = \hat{\pi}_t - \kappa\left[(\nu+\sigma)\hat{y}_t - (1+\nu)\hat{A}_t\right]$$

$$E_t\hat{R}_t = (1-\rho_R)\left[\phi_\pi\hat{\pi}_t + \phi_y\hat{y}_t\right] + \rho_R\hat{R}_{t-1} + v_t$$

$$E_t\hat{A}_{t+1} = \rho_A\hat{A}_t$$

$$E_t v_{t+1} = \rho_R v_t.$$

Matrices A and B are

$$\mathbf{A} E_t \left[\mathbf{z}_{t+1} \right] = \mathbf{B} \mathbf{z}_t$$

$$\mathbf{z}_t = \left[\hat{R}_{t-1}, \hat{A}_t, v_t, \hat{y}_t, \hat{\pi}_t \right]'$$

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{\sigma} & 0 & 0 & 1 & \frac{1}{\sigma} \\ 0 & 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & \kappa(1+\nu) & 0 & -\kappa(\nu+\sigma) & 1 \\ \rho_R & 0 & 1 & (1-\rho_R)\phi_y & (1-\rho_R)\phi_\pi \\ 0 & \rho_A & 0 & 0 & 0 \\ 0 & 0 & \rho_v & 0 & 0 \end{bmatrix}$$

Policy functions

- The Matlab program `New_Keynesian.m` solves the model using the QZ decomposition:

```
[F,P]=solab_ch(A,B,3)
```

  3 refers to the number of state variables

- The Jupyter notebook `New_Keynesian.ipynb` also solves the model.

- Policy functions are stored in F and P. In particular,

$$\begin{bmatrix} \hat{R}_t \\ \hat{A}_{t+1} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ 0 & \rho_A & 0 \\ 0 & 0 & \rho_v \end{bmatrix} \begin{bmatrix} \hat{R}_{t-1} \\ \hat{A}_t \\ v_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{A,t+1} \\ \varepsilon_{v,t+1} \end{bmatrix}$$

$$\begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \end{bmatrix} \begin{bmatrix} \hat{R}_{t-1} \\ \hat{A}_t \\ v_t \end{bmatrix}$$

Policy functions

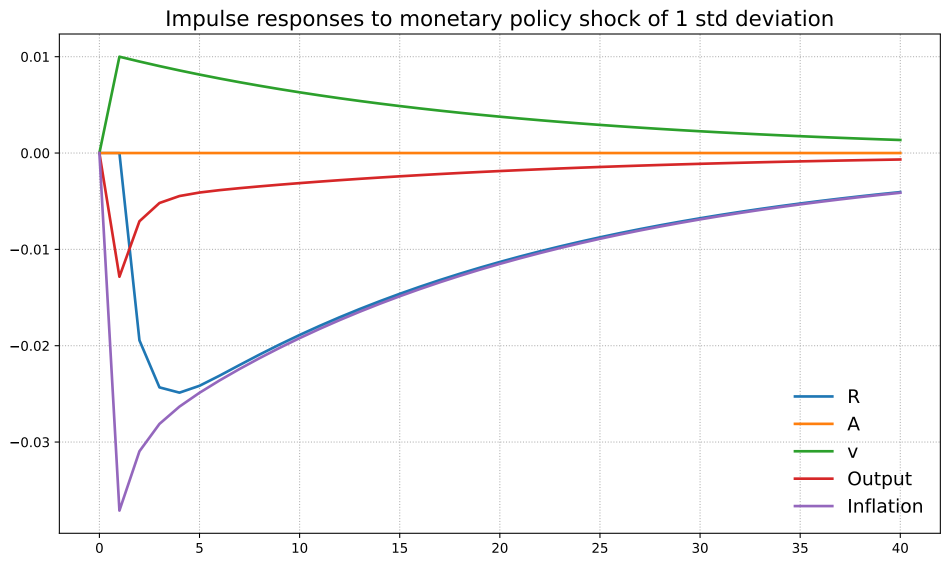
- For some parametrization (see the program), the model delivers

$$\begin{bmatrix} \hat{y}_t \\ \hat{\pi}_t \end{bmatrix} = \begin{bmatrix} -0.26 & 0.50 & -1.28 \\ -0.22 & -0.44 & -3.71 \end{bmatrix} \begin{bmatrix} \hat{R}_{t-1} \\ \hat{A}_t \\ v_t \end{bmatrix}$$

$$\begin{bmatrix} \hat{R}_t \\ \hat{A}_{t+1} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} 0.30 & -0.27 & -1.94 \\ 0 & 0.95 & 0 \\ 0 & 0 & 0.95 \end{bmatrix} \begin{bmatrix} \hat{R}_{t-1} \\ \hat{A}_t \\ v_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{A,t+1} \\ \varepsilon_{v,t+1} \end{bmatrix}$$

- Once we have the solution, we can compute impulse responses, simulations, etc.

Impulse response to monetary policy shock



Impulse response to productivity shock