A Basic RBC model

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RBC model

- These slides shows, step by step, how to solve and approximate numerically a simple RBC model.
- A Planner chooses the allocation to maximize the household's welfare.
  - Second Welfare Theorem

Steps to solve a model

General procedure to solve a DSGE model:

1. Find equilibrium conditions of the model.
2. Find the steady state and calibrate the parameters of the model (often done simultaneously).
3. Log-linearize the equilibrium conditions around the steady state.
4. Write the linearized system of difference equations as

$$\mathbf{A}\mathbb{E}_t\left[\mathbf{z}_{t+1}\right] = \mathbf{B}\mathbf{z}_t,$$

   where $\mathbf{z}_t$ is a vector with all the variables ordered in a particular way (see below), and $\mathbf{A}$ and $\mathbf{B}$ are square matrices.
5. Call a routine that performs the QZ decomposition. A Matlab program `solab.m` does this.
6. Compute impulse responses, simulate the model, etc.

Planner's problem

$$\max_{\{c_t, l_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, l_t\right)$$

subject to

$$\begin{aligned} c_t + k_{t+1} &= A_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta) k_t \\ & k_0, A_0 \text{ given,} \end{aligned}$$

with

$$u\left(c_t, l_t\right) = \log c_t - \eta \frac{l_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}}.$$

Lagrangian

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t - \eta \frac{l_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} - \lambda_t \left[c_t + k_{t+1} - A_t k_t^{\alpha} l_t^{1-\alpha} - (1-\delta) k_t \right] \right\}$$

First order conditions:

$$\frac{1}{c_t} = \lambda_t,$$

$$\eta l_t^{\frac{1}{\nu}} = \lambda_t (1-\alpha) A_t k_t^{\alpha} l_t^{-\alpha},$$

$$\lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1} \left(\alpha A_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta \right) \right],$$

$$c_t + k_{t+1} = A_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta) k_t.$$

Transversality condition:

$$\lim_{T\to\infty} E_0 \left[\beta^T \lambda_T k_{T+1} \right] = 0.$$

Log of TFP follows an AR(1) process

$$\log A_{t+1} = \rho \log A_t + \varepsilon_{t+1}$$

where ε_{t+1} is i.i.d. normal with mean 0 and variance σ_ε^2.

Equilibrium conditions

- The state variables are k_t and A_t.
- The control variables are c_t, l_t, and λ_t.
- But we are also interested in output and investment.
  - Output is
$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha}$$
  - Investment is
$$x_t = k_{t+1} - (1-\delta) k_t$$
- Marginal product of capital and labor:

$$\begin{aligned} (1-\alpha) A_t k_t^{\alpha} l_t^{-\alpha} &= (1-\alpha)\frac{y_t}{l_t}, \\ \alpha A_t k_t^{\alpha-1} l_t^{1-\alpha} &= \alpha \frac{y_t}{k_t}. \end{aligned}$$

Equilibrium conditions

$$\frac{1}{c_t} = \lambda_t \tag{1}$$

$$\eta l_t^{\frac{1}{\nu}} = \lambda_t (1-\alpha) \frac{y_t}{l_t} \tag{2}$$

$$\lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1} \left(\alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta \right) \right] \tag{3}$$

$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha} \tag{4}$$

$$c_t + x_t = y_t \tag{5}$$

$$x_t = k_{t+1} - (1-\delta) k_t \tag{6}$$

$$E_t \log A_{t+1} = \rho \log A_t. \tag{7}$$

Steady state

In steady state, equations (1)-(7) become

$$\frac{1}{\bar{c}} = \bar{\lambda} \tag{8}$$

$$\eta \bar{l}^{\frac{1}{\nu}} = \bar{\lambda}\left(1-\alpha\right)\frac{\bar{y}}{\bar{l}} \tag{9}$$

$$1 = \beta\left[\alpha\frac{\bar{y}}{\bar{k}} + 1 - \delta\right] \tag{10}$$

$$\bar{y} = \bar{A}\bar{k}^{\alpha}\bar{l}^{1-\alpha} \tag{11}$$

$$\bar{c} + \bar{x} = \bar{y} \tag{12}$$

$$\bar{x} = \delta\bar{k} \tag{13}$$

$$\bar{A} = 1. \tag{14}$$

Calibration and steady state

- Set parameter values to match certain features observed in the data.
- α is the capital share in output. NIPA accounts for the U.S. imply a value $\alpha \approx 1/3$.
- Set an average real interest rate of $\bar{R} = 1.01$ (1% per quarter). In the model, the (gross) steady state real interest rate is

$$\alpha \frac{\bar{y}}{\bar{k}} + 1 - \delta = \bar{R}. \tag{15}$$

  Given α, this is a restriction between $\bar{y}/\bar{k}$ and δ.
- Equation (10) implies that β must satisfy

$$\frac{1}{\beta} = \bar{R} \Rightarrow \beta = \frac{1}{1.01} \approx 0.99.$$

Calibration and steady state

Choose δ to match the average (long-run) investment rate $\bar{x}/\bar{y}$. Write (13) as

$$\frac{\bar{x}}{\bar{y}} = \delta \frac{\bar{k}}{\bar{y}}$$

But using (15) we can write

$$\frac{\bar{x}}{\bar{y}} = \delta \left(\frac{\alpha}{\bar{R} - (1 - \delta)} \right)$$

Solve for δ :

$$\delta = \frac{(\bar{R} - 1)(\bar{x}/\bar{y})}{\alpha - (\bar{x}/\bar{y})}.$$

Given a target value $\bar{x}/\bar{y} = 0.21$, and the calibrated values $\bar{R} = 1.01$ and $\alpha = 1/3$ we obtain

$$\delta = \frac{0.01 \times 0.21}{0.33 - 0.21} \approx 0.017.$$

Calibration and steady state

- Calibrate the model so that the steady state labor input is $\bar{l} = 1/3$, roughly the fraction of total weekly hours that workers spend working.
- Using (14), we can write (11) as

$$\bar{y} = \bar{k}^{\alpha}\bar{l}^{1-\alpha}.$$

  Dividing by $\bar{k}$

$$\frac{\bar{y}}{\bar{k}} = \left(\frac{\bar{l}}{\bar{k}}\right)^{1-\alpha}$$

- Using the calibration condition (15) we can write

$$\bar{k} = \bar{l}\left(\frac{\alpha}{\bar{R} - (1-\delta)}\right)^{\frac{1}{1-\alpha}} \tag{16}$$

Calibration and steady state

Given $\bar{l} = 1/3$ and the other calibrated parameters, this equation gives $\bar{k}$:

$$\bar{k} = \frac{1}{3} \left(\frac{1/3}{1.01 - (1 - 0.017)} \right)^{\frac{1}{1-1/3}} \approx 14.46.$$

The steady state level of output is thus

$$\bar{y} = \bar{k}^{\alpha} \bar{l}^{1-\alpha} \approx 1.17.$$

The steady state consumption $\bar{c}$ follows from feasibility (12)

$$\bar{c} = \bar{y} - \bar{x} = \bar{y} \left(1 - \frac{\bar{x}}{\bar{y}} \right) = 1.17\,(1 - 0.21) \approx 0.93.$$

Calibration and steady state

- It remains to calibrate η and ν.
- Write condition (9) as

$$\eta \bar{l}^{1+\frac{1}{\nu}} = (1-\alpha)\,\frac{\bar{y}}{\bar{c}}.$$

  In this equation we know $\bar{l}$, $\bar{c}$, $\bar{y}$, and α.
- We have one equation and two parameters: η and ν.
- Set the Frisch elasticity $\nu = 1$ (From micro studies. Some controversy here)
- Then we recover η.

Calibration and steady state

- Calibration of the parameters of the stochastic process ρ and σ_ε^2?
- Some possibilities:
  1. Run a first order autoregression on estimated Solow residuals to estimate ρ and σ_ε^2.
  2. Set ρ to some number and then choose σ_ε^2 to match the volatility of output in the data.
  3. Choose ρ and σ_ε^2 to match the volatility and persistence of output in the data (later in the course).

Log-linearization

- We now approximate the policy functions around the steady state
- Rather than linearizing, most economists log-linearize their models:
  - log-linear equations often describes the data better
  - nicer interpretation as percentage deviations from steady state.
- For any x_t, define its log-deviation from the steady state as

$$\hat{x}_t = \log\left(x_t / \bar{x}\right).$$

x_t can then be written as

$$x_t = \bar{x}e^{\hat{x}_t}.$$

We linearize the system around $\hat{x}_t = 0$ for all varibles x_t.

Equation (1):

This equation is already log-linear. Write it as

$$1 = \lambda_t c_t$$

Taking logs

$$0 = \log \lambda_t + \log c_t$$

In steady state

$$0 = \log \bar{\lambda} + \log \bar{c}$$

Subtracting both equation and using $\hat{c}_t = \log(c_t/\bar{c})$ and $\hat{\lambda}_t = \log(\lambda_t/\bar{\lambda})$ gives

$$0 = \hat{c}_t + \hat{\lambda}_t. \tag{17}$$

Equation (2):

This equation is also log-linear. Write it as

$$\eta l_t^{1+\frac{1}{\nu}} = \lambda_t (1-\alpha) y_t$$

Taking logs

$$\log \eta + \left(1 + \frac{1}{\nu}\right) \log l_t = \log (1-\alpha) + \log \lambda_t + \log y_t$$

Subtracting the same equation evaluated at the steady state:

$$\left(1 + \frac{1}{\nu}\right) \hat{l}_t = \hat{\lambda}_t + \hat{y}_t$$

or

$$0 = (1 + 1/\nu)\, \hat{l}_t - \hat{\lambda}_t - \hat{y}_t. \tag{18}$$

Equation (3):

Disregard the expectation operator and write

$$\begin{aligned} 0 &= \beta\left[\lambda_{t+1}\left(\alpha\frac{y_{t+1}}{k_{t+1}}+1-\delta\right)\right]-\lambda_t \\ &= \beta\bar{\lambda}e^{\hat{\lambda}_{t+1}}\left(\alpha\left(\bar{y}/\bar{k}\right)e^{\hat{y}_{t+1}-\hat{k}_{t+1}}+1-\delta\right)-\bar{\lambda}e^{\hat{\lambda}_t}\end{aligned}$$

Linearize around $\left(\hat{\lambda}_{t+1},\hat{y}_{t+1},\hat{k}_{t+1},\hat{\lambda}_t\right)=(0,0,0,0)$

$$0\approx\beta\bar{\lambda}\left(\alpha\left(\bar{y}/\bar{k}\right)+1-\delta\right)\hat{\lambda}_{t+1}+\beta\bar{\lambda}\alpha\left(\bar{y}/\bar{k}\right)\left(\hat{y}_{t+1}-\hat{k}_{t+1}\right)-\bar{\lambda}\hat{\lambda}_t.$$

Dividing by $\bar{\lambda}$ and using than in steady state $\beta\left(\alpha\left(\bar{y}/\bar{k}\right)+1-\delta\right)=1$,

$$0\approx\hat{\lambda}_{t+1}+\beta\alpha\left(\bar{y}/\bar{k}\right)\left(\hat{y}_{t+1}-\hat{k}_{t+1}\right)-\hat{\lambda}_t.$$

Putting back the expectation operator gives

$$0\approx\mathbb{E}_t\left[\hat{\lambda}_{t+1}+\beta\alpha\left(\bar{y}/\bar{k}\right)\left(\hat{y}_{t+1}-\hat{k}_{t+1}\right)\right]-\hat{\lambda}_t. \tag{19}$$

Equation (4):

$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha}$$

Already log-linear:

$$\log y_t = \log A_t + \alpha \log k_t + (1-\alpha) \log l_t$$

Subtracting the same equation at the steady state,

$$\log \bar{y} = \log \bar{A} + \alpha \log \bar{k} + (1-\alpha) \log \bar{l}$$

gives

$$\hat{y}_t = \hat{A}_t + \alpha \hat{k}_t + (1-\alpha) \hat{l}_t \tag{20}$$

Equation (5):

$$\begin{aligned} 0 &= y_t - c_t - x_t \\ &= \bar{y} e^{\hat{y}_t} - \bar{c} e^{\hat{c}_t} - \bar{x} e^{\hat{x}_t} \end{aligned}$$

Linearizing around $(\hat{y}_t, \hat{c}_t, \hat{x}_t) = (0, 0, 0)$ gives

$$0 \approx \bar{y}\hat{y}_t - \bar{c}\hat{c}_t - \bar{x}\hat{x}_t \tag{21}$$

Equation (6):

$$
\begin{aligned}
0 &= k_{t+1} - (1-\delta)\, k_t - x_t \\
&= \bar{k} e^{\hat{k}_{t+1}} - (1-\delta)\, \bar{k} e^{\hat{k}_t} - \bar{x} e^{\hat{x}_t}
\end{aligned}
$$

Linearizing this equation gives

$$0 \approx \bar{k}\hat{k}_{t+1} - (1-\delta)\, \bar{k}\hat{k}_t - \bar{x}\hat{x}_t$$

But in steady state $\bar{x} = \delta\bar{k}$ which implies

$$0 \approx \hat{k}_{t+1} - (1-\delta)\, \hat{k}_t - \delta\hat{x}_t. \tag{22}$$

Equation (7):

TFP equation is already log-linear

$$0 = \log A_{t+1} - \rho \log A_t - \varepsilon_{t+1}.$$

Subtracting the same equation at the steady state,

$$0 = \log \bar{A} - \rho \log \bar{A}$$

gives

$$0 = \hat{A}_{t+1} - \rho \hat{A}_t - \varepsilon_{t+1}$$

Taking the conditional expectation as of time t then gives

$$0 = \mathbb{E}_t \hat{A}_{t+1} - \rho \hat{A}_t \tag{23}$$

Summary of log-linear system of equations:

$$
\begin{aligned}
0 &= \hat{c}_t + \hat{\lambda}_t \\
0 &= (1 + \frac{1}{\nu})\hat{l}_t - \hat{\lambda}_t - \hat{y}_t \\
0 &= \hat{y}_t - \hat{A}_t - \alpha \hat{k}_t - (1-\alpha)\,\hat{l}_t \\
0 &= \bar{y}\hat{y}_t - \bar{c}\hat{c}_t - \bar{x}\hat{x}_t \\
0 &= \mathbb{E}_t[\hat{k}_{t+1}] - (1-\delta)\,\hat{k}_t - \delta\hat{x}_t \\
0 &= \mathbb{E}_t\left[\hat{\lambda}_{t+1} + \beta\alpha\,(\bar{y}/\bar{k})\left(\hat{y}_{t+1} - \hat{k}_{t+1}\right)\right] - \hat{\lambda}_t \\
0 &= \mathbb{E}_t\left[\hat{A}_{t+1}\right] - \rho\hat{A}_t .
\end{aligned}
$$

Note that I wrote $\mathbb{E}_t[\hat{k}_{t+1}]$ even though $\hat{k}_{t+1}$ is chosen (and therefore already known) at time t. This is just notation, but it will be useful below.

Numerical solution of the model

We want to write the model as the following first order vector expectational difference equation

$$\mathbf{A}\mathbb{E}_t\left[\mathbf{z}_{t+1}\right] = \mathbf{B}\mathbf{z}_t \tag{24}$$

$\mathbf{z}_t$ contains all the variables in the economy and **A** and **B** are square matrices.
We will solve this model using the program `solab.m` written by Paul Klein. To that end, let's order the variables $\mathbf{z}_t$ as follows:

$$\mathbf{z}_t = \left[\begin{array}{c} \text{endogenous states variables} \\ \text{exogenous states variables} \\ \text{jump variables} \end{array}\right]$$

and **A** and **B** are square matrices.
The only endogenous state variable is $\hat{k}_t$ and the only exogenous state variable is $\hat{A}_t$. Therefore, $\mathbf{z}_t$ is given by

$$\mathbf{z}_t = \left[\hat{k}_t,\ \hat{A}_t,\ \ \hat{y}_t,\ \hat{c}_t,\ \hat{l}_t,\ \hat{x}_t,\ \hat{\lambda}_t\right]'. \tag{25}$$

Numerical solution of the model

- We must tell the program how many of the variables in $\mathbf{z}_t$ are state variables. In our case, two: $\hat{k}_t$ and $\hat{A}_t$.
- **A** and **B** are 7×7 matrices.
- Let $\mathbf{x}_t \equiv [\hat{k}_t, \hat{A}_t]'$ be the state variables and $\mathbf{y}_t = [\hat{y}_t,\ \hat{c}_t,\ \hat{l}_t,\ \hat{x}_t,\ \hat{\lambda}_t]$, the jump variables.
- The solver delivers the equilibrium of the *certainty equivalent* model in the form

$$\begin{aligned} \mathbf{y}_t &= \mathbf{F}\mathbf{x}_t \\ \mathbf{x}_{t+1} &= \mathbf{P}\mathbf{x}_t \end{aligned}$$

- The *stochastic* solution is obtained by replacing the second equation above with

$$\mathbf{x}_{t+1} = \mathbf{P}\mathbf{x}_t + \left[\begin{array}{c} 0 \\ 1 \end{array}\right] \varepsilon_{t+1}$$

Rewrite the system of equations as follows

$$
\begin{aligned}
0 &= \hat{c}_t + \hat{\lambda}_t \\
0 &= (1 + \frac{1}{\nu})\hat{l}_t - \hat{\lambda}_t - \hat{y}_t \\
0 &= \hat{y}_t - \hat{A}_t - \alpha \hat{k}_t - (1-\alpha)\,\hat{l}_t \\
0 &= \bar{y}\hat{y}_t - \bar{c}\hat{c}_t - \bar{x}\hat{x}_t \\
\mathbb{E}_t[\hat{k}_{t+1}] &= (1-\delta)\,\hat{k}_t + \delta\hat{x}_t \\
\mathbb{E}_t\left[\hat{\lambda}_{t+1} + \beta\alpha\,(\bar{y}/\bar{k})\left(\hat{y}_{t+1} - \hat{k}_{t+1}\right)\right] &= \hat{\lambda}_t \\
\mathbb{E}_t\left[\hat{A}_{t+1}\right] &= \rho\hat{A}_t.
\end{aligned}
$$

Finding the required matrices

The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\beta\alpha\left(\bar{y}/\bar{k}\right) & 0 & \beta\alpha\left(\bar{y}/\bar{k}\right) & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & \left(1+\frac{1}{\nu}\right) & 0 & -1 \\ -\alpha & -1 & 1 & 0 & -\left(1-\alpha\right) & 0 & 0 \\ 0 & 0 & \bar{y} & -\bar{c} & 0 & -\bar{x} & 0 \\ 1-\delta & 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Approximate policy functions

Using the calibrated parameter values, the model delivers

$$\mathbf{F} = \begin{bmatrix} 0.22 & 1.33 \\ 0.57 & 0.34 \\ -0.17 & 0.50 \\ -1.10 & 5.07 \\ -0.57 & -0.34 \end{bmatrix}; \quad \mathbf{P} = \begin{bmatrix} 0.96 & 0.09 \\ 0 & 0.95 \end{bmatrix}$$

Equivalently, the policy functions are

$$\begin{aligned}
\hat{y}_t &= 0.22\hat{k}_t + 1.33\hat{A}_t \\
\hat{c}_t &= 0.57\hat{k}_t + 0.34\hat{A}_t \\
\hat{l}_t &= -0.17\hat{k}_t + 0.50\hat{A}_t \\
\hat{x}_t &= -1.10\hat{k}_t + 5.07\hat{A}_t \\
\hat{k}_{t+1} &= 0.96\hat{k}_t + 0.09\hat{A}_t \\
\hat{A}_{t+1} &= 0.95\hat{A}_t + \varepsilon_{t+1}.
\end{aligned}$$

Once we have the solution, we can compute impulse responses, simulations, second moments, spectral densities, etc.