Numerical approximation of DSGE Models

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Introduction

- Numerical approximation of DSGE (dynamic, stochastic, general equilibrium) models.
- Equilibrium conditions of the model is a system of non-linear stochastic difference equations.
- Solution of this system is the solution of the model.
- These notes consider linear approximations around the steady state.
- Workhorse example: real business cycle (RBC) model. But the method is general and can be applied to (most) models.

Basic RBC model

- Closed economy with a representative agent.
- No market failures: First and Second Welfare Theorems hold.
  - FWT: Competitive equilibrium is Pareto efficient.
  - SWT: Efficient allocation can be decentralized as a competitive equilibrium.
- Solve the planner's problem of maximizing utility subject to feasibility constraints.
- If we need prices, we get them from the appropriate conditions of the associated competitive equilibrium.

Basic RBC model

- Preferences over consumption c_t and leisure h_t

$$E_0 \sum_{t=0}^{\infty} \beta^t U\left(c_t, h_t\right).$$

- Technology:
  - Final goods
$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha}.$$
  - Productivity evolves as
$$\log\left(A_{t+1}\right) = \rho \log\left(A_t\right) + \varepsilon_{t+1},$$
    where $0 < \rho < 1$ and $\varepsilon_t \sim N\left(0, \sigma_{\varepsilon}^2\right)$
  - Capital accumulation:
$$k_{t+1} = (1-\delta) k_t + i_t,$$
    i_t is gross investment and $0 < \delta < 1$.
  - Timing convention: at time t, planner chooses k_{t+1}.

Basic RBC model

- Feasibility:
    - Final goods:
    $$c_t + i_t = y_t.$$
    - Labor allocation:
    $$l_t + h_t = 1.$$
- Initial stock of capital k_0 and technology A_0 are given.
- Replace i_t and h_t using the capital accumulation equation and labor feasibility.

Basic RBC model: planner's problem

- Planner's problem

$$\max_{\{c_t, l_t, k_{t+1}\}} E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, 1 - l_t)$$

subject to

$$\begin{aligned} c_t + k_{t+1} &= A_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta) k_t \\ \log A_{t+1} &= \rho \log A_t + \varepsilon_{t+1} \\ & k_0 \text{ and } A_0 \text{ given.} \end{aligned}$$

- We will solve this problem using the method of Lagrange multipliers.

Basic RBC model: planner's problem

- Lagrangian

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[U(c_t, 1 - l_t) - \lambda_t \left(c_t + k_{t+1} - A_t k_t^\alpha l_t^{1-\alpha} - (1 - \delta) k_t \right) \right].$$

  λ_t is a stochastic Lagrange multiplier "known" at time t.
- First order conditions with respect to c_t, l_t, k_{t+1}, and λ_t:

$$\begin{aligned} U_c(c_t, 1 - l_t) - \lambda_t &= 0, \\ -U_h(c_t, 1 - l_t) - \lambda_t (1 - \alpha) A_t k_t^\alpha l_t^{-\alpha} &= 0, \\ \beta E_t \left[\lambda_{t+1} \left(\alpha A_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta \right) \right] - \lambda_t &= 0, \\ c_t + k_{t+1} - A_t k_t^\alpha l_t^{1-\alpha} - (1 - \delta) k_t &= 0. \end{aligned}$$

  (U_c and U_h are the partial derivatives w.r.t. c and h)
- Plus a transversality condition:

$$\lim_{T \to \infty} E_0 \left[\beta^T \lambda_T k_{T+1} \right] = 0.$$

Basic RBC model: planner's problem

- Look for solutions in the form of time-invariant **policy functions** of the states.

- Categorize all variables either as state or control variables.

  - **State variables** (predetermined variables): set of variables that characterize the state of the economy at the beginning of time t. *RBC example:* k_t, A_t.

  - **Control variables** (jump variables, non-predetermined variables): the rest of the endogenous variables. *RBC example:* c_t, l_t, λ_t.

- Policy functions:

$$
\begin{aligned}
c_t &= \mathrm{c}\left(k_t, A_t\right), \\
l_t &= \mathrm{l}\left(k_t, A_t\right), \\
\lambda_t &= \lambda\left(k_t, A_t\right), \\
k_{t+1} &= \mathrm{k}\left(k_t, A_t\right).
\end{aligned}
$$

Basic RBC model: planner's problem

- In general, denote state variables by x_t and control variables by y_t.

- RBC example:

$$x_t = \begin{bmatrix} k_t \\ A_t \end{bmatrix}; \quad y_t = \begin{bmatrix} c_t \\ l_t \\ \lambda_t \end{bmatrix}$$

- Sometimes useful to separate state variables into exogenous and endogenous state variables (A_t and k_t respectively, in RBC example).

Basic RBC model: planner's problem

- Equilibrium conditions can be written as a system of expectational difference equations

$$E_t \left[f \left(x_{t+1}, y_{t+1}, x_t, y_t \right) \right] = \bar{0},$$

where $\bar{0}_{5\times 1}$ and $f : R^{2\times 2+2\times 3} \to R^5$ is given by

$$f\left(x_{t+1}, y_{t+1}, x_t, y_t\right) = \left[\begin{array}{c} U_c\left(c_t, 1-l_t\right) - \lambda_t \\ -U_h\left(c_t, 1-l_t\right) - \lambda_t\left(1-\alpha\right) A_t k_t^{\alpha} l_t^{-\alpha} \\ \beta\lambda_{t+1}\left(\alpha A_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta\right) - \lambda_t \\ c_t + k_{t+1} - A_t k_t^{\alpha} l_t^{1-\alpha} - \left(1-\delta\right) k_t \\ \log\left(A_{t+1}\right) - \rho\log\left(A_t\right) - \varepsilon_{t+1} \end{array} \right].$$

- More generally, the equilibrium conditions of a wide range of models can be written in this form.

Useful results from linear algebra

Let A be an arbitrary matrix. We use a_{ij} to denote element (i, j) of A.

If A is a square matrix of complex numbers, we denote by A^H the Hermitian transpose of A.

- The Hermitian transpose is the generalization of the transpose of a real matrix: first transpose A and next take the complex conjugate of its elements: $a_{ij}^H = a_{ji}^*$.

Definition 1: A square matrix A of complex number is said to be unitary if $AA^H = A^H A = I$, where I is the identity matrix. (If A is real, we call it orthonormal).

Comment: the inverse of a unitary matrix A exists and equals A^H.

Result 1: A square matrix A is invertible if and only if all its eigenvalues are different from zero.

Useful results from linear algebra

Definition 2: A square matriz A is upper triangular if its entries below the main diagonal are zero.

Result 2: If A is an upper triangular matrix whose entries on the main diagonal are nonzero, then A is invertible.

Result 3: If A is upper triangular and invertible, then A^{-1} is also upper triangular. Moreover, the diagonal elements of A^{-1} are the reciprocal of the diagonal elements of A. That is, element (i, i) of A^{-1} is $1/a_{ii}$.

Result 4: If A and B are $n \times n$ upper triangular matrices, then AB is also upper triangular.

Useful results from linear algebra

Theorem 1: *(QZ Decomposition): Let A and B be $n \times n$ matrices. If there is a complex number z such that* $\det(B - Az) \neq 0$*, then there are matrices Q, Z, S, and T such that:*

1. *Q and Z are unitary, i.e. $Q^H Q = QQ^H = I$ and $Z^H Z = ZZ^H = I$,*
2. *T and S are upper triangular,*
3. *The matrices Q, Z, S, and T satisfy*

$$
\begin{aligned}
QAZ &= S \\
QBZ &= T,
\end{aligned}
$$

4. *There is no index i such that $s_{ii} = t_{ii} = 0$, and*
5. *The matrices Q, Z, S, and T can be chosen in such a way as to make the diagonal entries s_{ii} and t_{ii} appear in any desired order.*

Remark: the ratios t_{ii}/s_{ii} are called the *generalized eigenvalues* of the matrix pair (A, B). By convention, $s_{ii} = 0$ corresponds to an infinite generalized eigenvalue.

First order approximation to the solution of DSGE models

- $x_t \in R^n$: vector of state variables (predetermined variables).

- $y_t \in R^m$: vector of jump variables (control variables).

- Equilibrium conditions of a model can be expressed as

$$E_t \left[f \left(x_{t+1}, y_{t+1}, x_t, y_t \right) \right] = \bar{0}, \tag{1}$$

  - $f : R^{2n+2m} \to R^{n+m}$ contains all the equilibrium conditions.
  - f has image in R^{n+m} because there is one equation for each variable.

- Need two sets of conditions to find the solution to (1):

  1. x_0: initial conditions for the state variables at time $t = 0$.

  2. Transversality condition: state variables are bounded in an appropriate sense.

Partition the state vector x_t as

$$x_t = \left[\begin{array}{c} x_{1,t} \\ x_{2,t} \end{array} \right]$$

where

- $x_{1,t}$ $(n_1 \times 1)$ contains all endogenous state variables.

- $x_{2,t}$ $(n_2 \times 1)$ contains all exogenos state variables (shocks).

Exogenous state variables evolve according to

$$x_{2,t+1} = \Lambda x_{2,t} + \tilde{\eta}\varepsilon_{t+1},$$

where

- Λ is an $n_2 \times n_2$ stable matrix (all eigenvalues < 1 in absolute value)
- ε_{t+1} is an $n_2 \times 1$ vector of i.i.d. shocks, and
- $\tilde{\eta}$ is an $n_2 \times n_2$ matrix.

- We will solve the model using a first order perturbation around the non-stochastic steady state.
- Convenient to add an auxiliary parameter $\sigma \geq 0$ to control the "amount of uncertainty" in the model.
- Replace the exogenous state equation by

$$x_{2,t+1} = \Lambda x_{2,t} + \sigma \tilde{\eta} \varepsilon_{t+1}.$$

  - when $\sigma = 0$ the model becomes deterministic.
  - when $\sigma = 1$ we recover the original model.
- The perturbation approach involves approximating the model around the deterministic model $\sigma = 0$.

- Look for policy functions $g : R^n \times R^+ \to R^m$, $h : R^n \times R^+ \to R^n$ such that

$$y_t = g\left(x_t, \sigma\right), \tag{2}$$

$$x_{t+1} = h\left(x_t, \sigma\right) + \sigma \eta \varepsilon_{t+1}, \tag{3}$$

  where

$$\eta_{n \times n_2} = \begin{bmatrix} 0_{n_1 \times n_2} \\ \tilde{\eta}_{n_2 \times n_2} \end{bmatrix}.$$

  - Function $g(\cdot)$ determines the control variables y_t as a function of the state variables x_t.
  - Function $h(\cdot)$ determines the evolution of the state variables.
  - The block of zeros, $0_{n_1 \times n_2}$, appears because shocks only affect the exogenous state variables.

- Under regularity conditions, the policy functions are unique (more on this later).

Find the steady state

- Equilibrium conditions of the model

$$E_t \left[f \left(x_{t+1}, y_{t+1}, x_t, y_t \right) \right] = \bar{0}_{(m+n)\times 1}.$$

- We consider linear approximations around the steady state.

- **Steady state:** values of x_t and y_t that satisfy

$$x_{t+1} = x_t = \bar{x}$$

$$y_{t+1} = y_t = \bar{y}$$

  These values solve the system of equations

$$f \left(\bar{x}, \bar{y}, \bar{x}, \bar{y} \right) = \bar{0}.$$

  $n + m$ equations to find $n + m$ unknowns.

- Transversality Condition: Imposed by restricting attention to bounded solutions where the system converges to the steady state in the absence of shocks.

Find the approximate policy functions

- First order Taylor approximation of $g(x, \sigma)$ and $h(x, \sigma)$ around $(x, \sigma) = (\bar{x}, 0)$.
  - Approximate the solution around the steady state $(x_t = \bar{x})$ of the deterministic economy $(\sigma = 0)$.

- First order Taylor approximation:

$$g(x, \sigma) \approx g(\bar{x}, 0) + g_x(\bar{x}, 0)(x - \bar{x}) + g_\sigma(\bar{x}, 0)\sigma \tag{4}$$

$$h(x, \sigma) \approx h(\bar{x}, 0) + h_x(\bar{x}, 0)(x - \bar{x}) + h_\sigma(\bar{x}, 0)\sigma \tag{5}$$

  - $g_x(\bar{x}, 0)$ $(m \times n)$ Jacobian of partial derivatives $\partial g_i / \partial x_j$.
  - $g_\sigma(\bar{x}, 0)$ $(m \times 1)$ vector with component $\partial g_i / \partial \sigma$.
  - $h_x(\bar{x}, 0)$ $(n \times n)$ Jacobian of partial derivatives $\partial h_i / \partial x_j$.
  - $h_\sigma(\bar{x}, 0)$ $(n \times 1)$ vector with entries $\partial h_i / \partial \sigma$.
  - In all cases, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

- Derivatives are evaluated at the steady state $(\bar{x}, 0)$.

- By definition of the steady state

$$g(\bar{x}, 0) = \bar{y}$$
$$h(\bar{x}, 0) = \bar{x}.$$

- Thus, the approximate policy functions are

$$y_t \approx \bar{y} + g_x(\bar{x}, 0)(x_t - \bar{x}) + g_\sigma(\bar{x}, 0)\sigma \tag{6}$$
$$x_{t+1} \approx \bar{x} + h_x(\bar{x}, 0)(x_t - \bar{x}) + h_\sigma(\bar{x}, 0)\sigma + \sigma\eta\varepsilon_{t+1} \tag{7}$$

- We need to find the matrices

$$g_x(\bar{x}, 0)\,;\ g_\sigma(\bar{x}, 0)\,;\ h_x(\bar{x}, 0)\,;\ \text{and } h_\sigma(\bar{x}, 0)\,.$$

- Substitute the true (unknown) policy functions $g(x_t, \sigma)$ and $h(x_t, \sigma)$ into the equilibrium conditions (1).

- Define an "error" function $F : R^n \times R^+ \to R^{n+m}$ as

$$
\begin{aligned}
F(x_t, \sigma) &\equiv E_t\left[f\left(x_{t+1}, y_{t+1}, x_t, y_t\right)\right] \\
&= E_t\left[f\left(h(x_t, \sigma) + \sigma\eta\varepsilon_{t+1}, g(h(x_t, \sigma) + \sigma\eta\varepsilon_{t+1}, \sigma), x_t, g(x_t, \sigma)\right)\right] \\
&= \bar{0}.
\end{aligned}
\tag{8}
$$

- **Evaluated at the solution**, $F(x, \sigma)$ equals zero for all (x, σ).

- Thus, all the derivatives of $F(x, \sigma)$ must be zero as well

$$
F_\sigma(x, \sigma) = \bar{0}_{(n+m)\times 1} \tag{9}
$$

$$
F_x(x, \sigma) = \bar{0}_{(n+m)\times n}. \tag{10}
$$

- We use (8) to find the derivatives of the Taylor approximation.

- Differentiating (8) with respect to σ and evaluating at $(\bar{x}, 0)$ gives

$$\begin{aligned}
F_\sigma(\bar{x}, 0) &= E_t\left[f_{x'}\left(h_\sigma + \eta\varepsilon_{t+1}\right) + f_{y'}\left(g_x\left(h_\sigma + \eta\varepsilon_{t+1}\right) + g_\sigma\right) + f_y g_\sigma\right] \\
&= f_{x'}\left(h_\sigma + \eta E_t\left[\varepsilon_{t+1}\right]\right) + f_{y'}\left(g_x\left(h_\sigma + \eta E_t\left[\varepsilon_{t+1}\right]\right) + g_\sigma\right) + f_y g_\sigma \\
&= f_{x'} h_\sigma + f_{y'}\left(g_x h_\sigma + g_\sigma\right) + f_y g_\sigma \\
&= \left[f_{x'} + f_{y'} g_x\right] h_\sigma + \left[f_{y'} + f_y\right] g_\sigma,
\end{aligned}$$

- $f_{x'}$, $f_{y'}$, and f_y are evaluated at the steady state and, thus, are known matrices. E.g.

$$f_{x'} = f_{x'}\left(\bar{x}, \bar{y}, \bar{x}, \bar{y}\right), \text{ etc.}$$

Find $g_\sigma(\bar{x},0)$ and $h_\sigma(\bar{x},0)$

- $F_\sigma(\bar{x},0)=\bar{0}$ implies

$$\left[f_{x'}+f_{y'}g_x\right]h_\sigma+\left[f_{y'}+f_y\right]g_\sigma=\bar{0} \text{ or}$$

$$\left[\begin{array}{cc} f_{x'}+f_{y'}g_x & f_{y'}+f_y \end{array}\right]\left[\begin{array}{c} h_\sigma \\ g_\sigma \end{array}\right]=\bar{0}.$$

- This is a homogeneous linear system of equations.

- $h_\sigma=0$ and $g_\sigma=0$ is *one possible solution*.

- If there is a solution $\tilde{h}_\sigma\neq 0$ or $\tilde{g}_\sigma\neq 0$, then $\alpha\tilde{h}_\sigma$ and $\alpha\tilde{g}_\sigma$ is also a solution for any α.

- Since we are looking for a *unique* pair of policy functions, it then must be the case that $h_\sigma=0$ and $g_\sigma=0$.

- **Certainty equivalence principle**: in a linear approximation to the policy functions, the amount of uncertainty in the model —summarized by σ— is irrelevant.

Find $g_x(\bar{x}, 0)$ and $h_x(\bar{x}, 0)$

- Differentiate (8) with respect to x and evaluate at $(\bar{x}, 0)$ to obtain

$$\begin{aligned}
\bar{0}_{(n+m)\times n} &= F_x(\bar{x}, 0) \\
&= f_{x'} h_x + f_{y'} g_x h_x + f_x + f_y g_x \\
&= \left(f_{x'} + f_{y'} g_x\right) h_x + f_x + f_y g_x \\
&= \begin{bmatrix} f_{x'} & f_{y'} \end{bmatrix} \begin{bmatrix} I \\ g_x \end{bmatrix} h_x + \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} I \\ g_x \end{bmatrix}.
\end{aligned}$$

  Rearranging gives

$$\begin{bmatrix} f_{x'} & f_{y'} \end{bmatrix} \begin{bmatrix} I \\ g_x \end{bmatrix} h_x = -\begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} I \\ g_x \end{bmatrix}. \tag{11}$$

- This is a system of $(n+m) \times n$ quadratic equations in $(n+m) \times n$ unknowns given by the elements of g_x and h_x.

- Define the following matrices and variables

$$
\begin{aligned}
A &= \left[\begin{array}{cc} f_{x'}\left(\bar{x},\bar{y},\bar{x},\bar{y}\right) & f_{y'}\left(\bar{x},\bar{y},\bar{x},\bar{y}\right) \end{array}\right] \\
B &= -\left[\begin{array}{cc} f_{x}\left(\bar{x},\bar{y},\bar{x},\bar{y}\right) & f_{y}\left(\bar{x},\bar{y},\bar{x},\bar{y}\right) \end{array}\right] \\
\hat{x}_{t+j} &= E_t\left[x_{t+j}\right] - \bar{x} \\
\hat{y}_{t+j} &= E_t\left[y_{t+j}\right] - \bar{y}
\end{aligned}
$$

  A and B are $(n+m)\times(n+m)$ square matrices.

- When $j=0$, $\hat{x}_t = x_t - \bar{x}$ and $\hat{y}_t = y_t - \bar{y}$.

- When $j=1$, $\hat{x}_{t+1} = E_t\left[x_{t+1} - \bar{x}\right]$ and $\hat{y}_{t+1} = E_t\left[y_{t+1} - \bar{y}\right]$.

- Rewrite (11) as

$$A \begin{bmatrix} I \\ g_x \end{bmatrix} h_x = B \begin{bmatrix} I \\ g_x \end{bmatrix}$$

- Post-multiply both sides by $\hat{x}_t$ to obtain

$$A \begin{bmatrix} I \\ g_x \end{bmatrix} h_x \hat{x}_t = B \begin{bmatrix} I \\ g_x \end{bmatrix} \hat{x}_t$$

or

$$A \begin{bmatrix} h_x \hat{x}_t \\ g_x h_x \hat{x}_t \end{bmatrix} = B \begin{bmatrix} \hat{x}_t \\ g_x \hat{x}_t \end{bmatrix}. \tag{12}$$

We now prove that

$$\begin{aligned} \hat{x}_{t+1} &\approx h_x \hat{x}_t, \\ \hat{y}_t &\approx g_x \hat{x}_t, \\ \hat{y}_{t+1} &\approx g_x h_x \hat{x}_t. \end{aligned}$$

Consider first

$$h_x \hat{x}_t = h_x(\bar{x}, 0)(x_t - \bar{x}).$$

- The Taylor approximation (7) and $h_\sigma = 0$ implies

$$x_{t+1} - \bar{x} \approx h_x \hat{x}_t + \eta \varepsilon_{t+1}. \tag{13}$$

- Taking expectations at time t gives

$$E_t[x_{t+1}] - \bar{x} \equiv \hat{x}_{t+1} \approx h_x \hat{x}_t.$$

- Likewise, using (6) and $g_\sigma = 0$ implies

$$y_t - \bar{y} \equiv \hat{y}_t \approx g_x \hat{x}_t$$

  and

$$\hat{y}_{t+1} \approx g_x \hat{x}_{t+1} \approx g_x h_x \hat{x}_t.$$

- Recall (12)

$$A\begin{bmatrix} h_x\hat{x}_t \\ g_x h_x \hat{x}_t \end{bmatrix} = B\begin{bmatrix} \hat{x}_t \\ g_x\hat{x}_t \end{bmatrix}$$

- Using the previous results gives

$$A\begin{bmatrix} \hat{x}_{t+1} \\ \hat{y}_{t+1} \end{bmatrix} = B\begin{bmatrix} \hat{x}_t \\ \hat{y}_t \end{bmatrix}. \tag{14}$$

- This is a standard representation of the equilibrium of a linearized rational expectations model.

- Equivalently, this condition can be written as

$$AE_t\begin{bmatrix} x_{t+1}-\bar{x} \\ y_{t+1}-\bar{y} \end{bmatrix} = B\begin{bmatrix} x_t-\bar{x} \\ y_t-\bar{y} \end{bmatrix}. \tag{15}$$

- We will use the *QZ decomposition* to solve this linear system of difference equations.
- Define the vector
$$\hat{w}_t = \begin{bmatrix} \hat{x}_t \\ \hat{y}_t \end{bmatrix}$$
and write the linear system as
$$A\hat{w}_{t+1} = B\hat{w}_t. \tag{16}$$
- We look for bounded solutions that satisfy
$$\lim_{j \to \infty} \hat{w}_{t+j} = 0.$$
That is, the system is expected to converge to the steady state.

- We want to solve

$$A\hat{w}_{t+1} = B\hat{w}_t$$

- By **Theorem 1**, there are unitary matrices Q and Z, and upper triangular matrices S and T such that

$$\begin{aligned} QAZ &= S \\ QBZ &= T. \end{aligned}$$

  - Order the pairs (s_{ii}, t_{ii}) as follows: those satisfying $|s_{ii}| > |t_{ii}|$ appear in the first block of diagonal elements of S and T.
  - We call these pairs of elements the *stable generalized eigenvalues*.

- We also impose:

  Assumption 1: There is no index i such that $|s_{ii}| = |t_{ii}|$.

- Assumption 1 implies that the system (16) does not have unit roots.

- Premultiply $A\hat{w}_{t+1} = B\hat{w}_t$ by Q and use that Z is unitary ($ZZ^H = I$) to obtain

$$\begin{aligned} QA\hat{w}_{t+1} &= QB\hat{w}_t \\ QAZZ^H\hat{w}_{t+1} &= QBZZ^H\hat{w}_t \\ SZ^H\hat{w}_{t+1} &= TZ^H\hat{w}_t, \end{aligned}$$

where the last line uses the QZ decomposition.

- Define a new variable

$$z_t \equiv Z^H\hat{w}_t. \tag{17}$$

and rewrite the system as:

$$Sz_{t+1} = Tz_t.$$

- This is progress because now the relevant matrices are triangular.

- Using that S and T are upper triangular, write the system in block form as

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} z^s_{t+1} \\ z^u_{t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} z^s_t \\ z^u_t \end{bmatrix}. \tag{18}$$

- The partition is such that S_{11} and T_{11} contain the pairs of elements (s_{ii}, t_{ii}) such that $|s_{ii}| > |t_{ii}|$.

- z_t is partitioned accordingly.

- **Assumption 1** implies that the diagonal elements of S_{22} and T_{22} satisfy $|s_{ii}| < |t_{ii}|$.

Solving the lower (unstable) block of the system

- Consider the lower block of the system

$$S_{22} z_{t+1}^{u} = T_{22} z_{t}^{u},$$

- The QZ decomposition implies that there is no i such that $s_{ii} = t_{ii} = 0$.
- Thus, $|s_{ii}| < |t_{ii}| \Rightarrow$ diagonal elements of T_{22} are non-zero.
- T_{22} upper triangular $\Rightarrow$ T_{22} is invertible (Result 2).
- Pre-multiply both sides of the previous expression by T_{22}^{-1} to obtain

$$T_{22}^{-1} S_{22} z_{t+1}^{u} = z_{t}^{u}.$$

- Then, (by Results 3 and 4) $T_{22}^{-1} S_{22}$ is upper triangular with diagonal elements s_{ii} / t_{ii}.

Solving the lower (unstable) block of the system

- Therefore, $T_{22}^{-1} S_{22}$ *has all its eigenvalues smaller than one in absolute value*.

- Then, unless $z_t^u = 0$ for all t, at least one element of z_t^u has to explode to infinity in absolute value.

- In other words, the only stable solution of the lower block of the system (18) is $z_t^u = 0$ for all t.

Solving the upper (stable) block of the system

- Using $z_t^u = 0$ for all t, the first block of (18) implies

$$S_{11} z_{t+1}^s = T_{11} z_t^s.$$

- $|s_{ii}| > |t_{ii}| \Rightarrow$ the diagonal elements of S_{11} are different from zero.

- Then, Result 2 implies that S_{11} is invertible.

- We thus have

$$z_{t+1}^s = S_{11}^{-1} T_{11} z_t^s. \tag{19}$$

- By Results 3 and 4, $S_{11}^{-1} T_{11}$ is upper triangular with diagonal elements $|t_{ii}/s_{ii}| < 1$ for all i. Hence, $S_{11}^{-1} T_{11}$ is a stable matrix.

- Therefore, (19) converges to zero as $t \to \infty$ for any value of z_0^s.

- That is, equation (19) is the solution to the first block of the system (18).

Finding the solution in terms of the original variables

- But we are interested in the solution in terms of the variables $\hat{w}_t$.

- Recall the definition
$$z_t = Z^H \hat{w}_t$$

- Since the inverse of Z^H is Z, then
$$\hat{w}_t \equiv \begin{bmatrix} \hat{w}_t^s \\ \hat{w}_t^u \end{bmatrix} = Z \begin{bmatrix} z_t^s \\ z_t^u \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} z_t^s \\ z_t^u \end{bmatrix} \tag{20}$$

- That is, $\hat{w}_t$ is a linear combination of z_t^s and z_t^u.

- But $z_t^u = 0$ and $z_t^s \to 0$ as $t \to \infty$ for any z_t^s implies that $\hat{w}_t$ also converges to zero as $t \to \infty$.

- This proves that the solution for $\hat{w}_t$ is also stable.

Finding the solution in terms of the original variables

- Consider the first block of (20), where we use $z_t^u = 0$ for all t,

$$\hat{w}_t^s = Z_{11} z_t^s.$$

Assumption 2: Z_{11} is invertible.

- With this assumption,

$$z_t^s = Z_{11}^{-1} \hat{w}_t^s.$$

- Combining this expression with

$$z_{t+1}^s = S_{11}^{-1} T_{11} z_t^s$$

gives

$$Z_{11}^{-1} \hat{w}_{t+1}^s = S_{11}^{-1} T_{11} Z_{11}^{-1} \hat{w}_t^s.$$

Solution in terms of the original variables: stable block

- Pre-multiplying both sides of this expression by Z_{11} gives

$$\hat{w}^s_{t+1} = H\hat{w}^s_t. \tag{21}$$

  where

$$H = Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}.$$

- Under Assumption 2, the eigenvalues of H are the same as the eigenvalues of $S_{11}^{-1}T_{11}$
- The eigenvalues of H satisfy $|t_{ii}/s_{ii}| < 1$ so (21) is a stable difference equation.
- This is the solution to the first block of equations $\hat{w}^s_t$.

Solution in terms of the original variables: unstable block

- We now find the solution for $\hat{w}_t^u$.
- The second block of (20) together with $z_t^s = Z_{11}^{-1}\hat{w}_t^s$ implies

$$\hat{w}_t^u = Z_{21} z_t^s = Z_{21} Z_{11}^{-1} \hat{w}_t^s, \tag{22}$$

- This gives the solution of $\hat{w}_t^u$ as a function of $\hat{w}_t^s$.
- Given an initial condition for w_t^s we have found the solution for the entire vector $\hat{w}_t$.

Summary

- System to solve

$$A\hat{w}_{t+1} = B\hat{w}_t$$

- Given w_t^s, the non-explosive solution is found by setting

$$\hat{w}_t^u = G\hat{w}_t^s$$
$$\hat{w}_{t+1}^s = H\hat{w}_t^s,$$

where

$$G = Z_{21}Z_{11}^{-1},$$
$$H = Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}.$$

Existence, local uniqueness, and multiplicity

- We were able to find a solution to the linear stochastic difference equation

$$A\hat{w}_{t+1} = B\hat{w}_t$$

- However, we didn't find yet the coefficient matrices $h_x(\bar{x}, 0)$ and $g_x(\bar{x}, 0)$ of the linear approximation

$$\begin{aligned} y_t - \bar{y} &\approx g_x(\bar{x}, 0)(x_t - \bar{x}) \\ x_{t+1} - \bar{x} &\approx h_x(\bar{x}, 0)(x_t - \bar{x}) + \eta\varepsilon_{t+1}. \end{aligned}$$

Blanchard-Kahn condition

Blanchard and Kahn condition: *the number of stable generalized eigenvalues of the matrix pair (A, B) (that is, the number of elements i such that $|s_{ii}| > |t_{ii}|$ is exactly equal to the number of state variables n.*

- If the Blanchard and Kahn condition is satisfied, the equilibrium of the DSGE model exists and is locally unique.
- In this case Z_{11} is of size $n \times n$ and $\hat{w}_t^s$ is of size $n \times 1$, the same dimension of $\hat{x}_t$.
- But given

$$\hat{w}_t = \begin{bmatrix} \hat{x}_t \\ \hat{y}_t \end{bmatrix} = \begin{bmatrix} \hat{w}_t^s \\ \hat{w}_t^u \end{bmatrix}$$

  then

$$\begin{aligned} \hat{w}_{t+1}^s &= \hat{x}_{t+1} = E_t\left[x_{t+1}\right] - \bar{x} \\ \hat{w}_t^s &= \hat{x}_t = x_t - \bar{x}. \end{aligned}$$

Local uniqueness

- Then, the solution

$$\hat{w}_{t+1}^{s} = H\hat{w}_{t}^{s}.$$

  implies

$$E_t\left[x_{t+1}\right] - \bar{x} = H\left(x_t - \bar{x}\right),$$

- Dropping the expectation operator leads to

$$x_{t+1} = \bar{x} + H\left(x_t - \bar{x}\right) + \eta\varepsilon_{t+1}. \tag{23}$$

- Therefore the matrix $h_x\left(\bar{x}, 0\right)$ that we were looking for is

$$h_x\left(\bar{x}, 0\right) = H = Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}.$$

Local uniqueness

- Using $\hat{w}_t^u = y_t - \bar{y}$ (because $\hat{w}_t^s = \hat{x}_t$), equation

$$\hat{w}_t^u = G\hat{w}_t^s$$

  implies

$$y_t - \bar{y} = G\left(x_t - \bar{x}\right), \tag{24}$$

- Therefore,

$$g_x\left(\bar{x}, 0\right) = Z_{21}Z_{11}^{-1}.$$

- Since y_t is uniquely determined from x_t, the policy function for the control variables also exists and is locally unique.

No local existence of the equilibrium

- Suppose that the number of stable generalized eigenvalues of (A, B) ($|s_{ii}| > |t_{ii}|$) is **smaller** than the number of state variables n.
  - Z_{11} is of size $(n-q) \times (n-q)$ for $0 < q < n$.
- In this case, $\hat{w}_t^s$ has less elements than the state vector $\hat{x}_t$.
- The vectors $\hat{w}_t^s$ and $\hat{w}_t^u$ take the form

$$\hat{w}_t^s = \hat{x}_t^a \text{ and } \hat{w}_t^u = \begin{bmatrix} \hat{x}_t^b \\ \hat{y}_t \end{bmatrix},$$

where

$$\hat{x}_t = \begin{bmatrix} \hat{x}_t^a \\ \hat{x}_t^b \end{bmatrix}.$$

No local existence of the equilibrium

- The solution of the difference equation $A\hat{w}_{t+1} = B\hat{w}_t$,

$$\hat{w}^s_{t+1} = H\hat{w}^s_t$$
$$\hat{w}^u_t = G\hat{w}^s_t$$

  implies

$$\hat{x}^a_{t+1} = H\hat{x}^a_t$$
$$\begin{bmatrix} \hat{x}^b_t \\ \hat{y}_t \end{bmatrix} = G\hat{x}^a_t.$$

- $\hat{x}^b_t$ is determined by $\hat{x}^a_t$. But this is impossible because $\hat{x}^b_t$ is a predetermined variable independent of $\hat{x}^a_t$.

- Therefore, the equilibrium does not exist in this case.

Local indeterminacy of the equilibrium

- Suppose that the number of generalized eigenvalues of (A, B) with absolute value less than one ($|s_{ii}| > |t_{ii}|$) is **greater** than the number of state variables, n.

- There are $n + q > n$ generalized eigenvalues with absolute value less than one. Then Z_{11} is size $(n + q) \times (n + q)$.

- $\hat{w}_t^s$ has more elements than $\hat{x}_t$ and $\hat{w}_t^u$ has less elements than $\hat{y}_t$,

$$\hat{w}_t^s = \begin{bmatrix} \hat{x}_t \\ \hat{y}_t^a \end{bmatrix}; \ \hat{w}_t^u = \hat{y}_t^b; \ \ \hat{y}_t = \begin{bmatrix} \hat{y}_t^a \\ \hat{y}_t^b \end{bmatrix}$$

  where $\hat{y}_t^a$ is a vector with the first q elements of the vector $\hat{y}_t$, and $\hat{y}_t^b$ is a vector with the remaning $m - q$ elements of $\hat{y}_t$.

Local indeterminacy of the equilibrium

According to

$$\hat{w}^s_{t+1} = H\hat{w}^s_t$$
$$\hat{w}^u_t = G\hat{w}^s_t$$

the solution in terms of the original variables is

$$\begin{bmatrix} \hat{x}_{t+1} \\ \hat{y}^a_{t+1} \end{bmatrix} = H \begin{bmatrix} \hat{x}_t \\ \hat{y}^a_t \end{bmatrix} \tag{25}$$

$$\hat{y}^b_t = G \begin{bmatrix} \hat{x}_t \\ \hat{y}^a_t \end{bmatrix}. \tag{26}$$

- But $\hat{y}^a_t$ is not predetermined at time t.
- Thus, we can choose an arbitrary $\hat{y}^a_0$ and solve the system in the variables $\hat{x}_t$ and $\hat{y}_t$.
- This means that the equilibrium is indeterminate (infinite solutions).

Deeper sense of local indeterminacy: sunspots

- Since there is nothing that ties $\hat{y}_t^a$ to previous decisions, we can drop the expectation operator and write the above system as

$$\begin{bmatrix} x_{t+1} \\ y_{t+1}^a \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \bar{y}^a \end{bmatrix} + H \begin{bmatrix} x_t - \bar{x} \\ y_t^a - \bar{y}^a \end{bmatrix} + \begin{bmatrix} \eta & 0 \\ \nu_\varepsilon & \nu_\mu \end{bmatrix} \begin{bmatrix} \varepsilon_{t+1} \\ \mu_{t+1} \end{bmatrix}$$

$$y_t^b - \bar{y}^b = G \begin{bmatrix} x_t - \bar{x} \\ y_t^a - \bar{y}^a \end{bmatrix},$$

  where μ_{t+1} is an arbitrary mean zero shock of size $q \times 1$ and variance covariance matrix equal to the identity matrix. The matrices ν_ε and ν_μ are arbitrary.

- This solves the difference equation: take conditional expectations and we return to the system (25) and (26).

Unconditional Second Moments

- Variables as deviations from their steady state

$$\tilde{x}_t = x_t - \bar{x}$$
$$\tilde{y}_t = y_t - \bar{y}$$

- Solution of the (linearized) model

$$\tilde{x}_{t+1} = h_x \tilde{x}_t + \eta \varepsilon_{t+1} \tag{27}$$

$$\tilde{y}_t = g_x \tilde{x}_t \tag{28}$$

- We show how to compute second moments and the spectral density.

Covariance matrix of states

- We want to compute

$$\Sigma_x = E\left[\tilde{x}_t \tilde{x}_t'\right]$$

- From (27),

$$\begin{aligned}
\tilde{x}_{t+1}\tilde{x}_{t+1}' &= \left(h_x \tilde{x}_t + \eta \varepsilon_{t+1}\right)\left(h_x \tilde{x}_t + \eta \varepsilon_{t+1}\right)' \\
&= h_x \tilde{x}_t \tilde{x}_t' h_x' + \eta \varepsilon_{t+1} \tilde{x}_t' h_x' + h_x \tilde{x}_t \varepsilon_{t+1}' \eta' + \eta \varepsilon_{t+1} \varepsilon_{t+1}' \eta'
\end{aligned}$$

- Taking expectations, using stationarity, and defining $\Sigma_\varepsilon = \eta\eta'$,

$$\begin{aligned}
E\left[\tilde{x}_{t+1}\tilde{x}_{t+1}'\right] &= h_x E\left[\tilde{x}_t \tilde{x}_t'\right] h_x' + \eta E\left[\varepsilon_{t+1}\varepsilon_{t+1}'\right]\eta' \\
\Sigma_x &= h_x \Sigma_x h_x' + \Sigma_\varepsilon .
\end{aligned}$$

- Need to solve for Σ_x.

Covariance matrix of states

Method 1:

- Uses property of the vec operator.

- Let ABC be conformable matrices, then

$$\text{vec}\,(ABC) = \left(C' \otimes A\right)\text{vec}(B)$$

- Taking the vec operator to the covariance matrix equation,

$$\begin{aligned}
\Sigma_x &= h_x \Sigma_x h_x' + \Sigma_\varepsilon. \\
\text{vec}\,(\Sigma_x) &= \text{vec}\left(h_x \Sigma_x h_x'\right) + \text{vec}\,(\Sigma_\varepsilon) \\
&= (h_x \otimes h_x)\,\text{vec}\,(\Sigma_x) + \text{vec}\,(\Sigma_\varepsilon)\,.
\end{aligned}$$

- Solving gives

$$\text{vec}\,(\Sigma_x) = (I - h_x \otimes h_x)^{-1}\,\text{vec}\,(\Sigma_\varepsilon)\,.$$

Covariance matrix of states

Method 2:

- By iteration.

- Start with a guess Σ_x^0 (e.g. $\Sigma_x^0 = I$) and set $j = 0$

- Iterate over $j = 1, 2, \ldots$ until convergence

$$\Sigma_x^j = h_x \Sigma_x^{j-1} h_x' + \Sigma_\varepsilon.$$

- Stop when $\Sigma_x^j \approx \Sigma_x^{j-1}$.

Autocovariances of the states

- We want to compute

$$\Sigma_x(\tau) = E\left[\tilde{x}_t \tilde{x}'_{t-\tau}\right].$$

- Let $\mu_t = \eta\varepsilon_t$. From the state equation,

$$\begin{aligned}
\tilde{x}_t &= h_x \tilde{x}_{t-1} + \mu_t \\
&= h_x^2 \tilde{x}_{t-2} + h_x \mu_{t-1} + \mu_t \\
&= h_x^3 \tilde{x}_{t-3} + h_x^2 \mu_{t-2} + h_x \mu_{t-1} + \mu_t \\
&= \ldots \\
&= h_x^\tau \tilde{x}_{t-\tau} + h_x^{\tau-1} \mu_{t-(\tau-1)} + \ldots + h_x^2 \mu_{t-1} + h_x \mu_{t-1} + \mu_t \\
&= h_x^\tau \tilde{x}_{t-\tau} + \sum_{j=0}^{\tau-1} h_x^j \mu_{t-j}.
\end{aligned}$$

Autocovariances of the states

- Therefore

$$
\begin{aligned}
\Sigma_x(\tau) &= E\left[\tilde{x}_t \tilde{x}'_{t-\tau}\right] \\
&= E\left[\left(h_x^\tau \tilde{x}_{t-\tau} + \sum_{j=0}^{\tau-1} h_x^j \mu_{t-j}\right) \tilde{x}'_{t-\tau}\right] \\
&= h_x^\tau E\left[\tilde{x}_{t-\tau}\tilde{x}'_{t-\tau}\right] + \sum_{j=0}^{\tau-1} h_x^j E\left[\mu_{t-j}\tilde{x}'_{t-\tau}\right] \\
&= h_x^\tau \Sigma_x(0)
\end{aligned}
$$

where we used $h_x^j E\left[\mu_{t-j}\tilde{x}'_{t-\tau}\right] = h_x^j E\left[\eta\varepsilon_{t-j}\tilde{x}'_{t-\tau}\right] = 0$ for all t and j.

- Therefore,

$$\Sigma_x(\tau) = h_x^\tau \Sigma_x.$$

Second moments of the control variables

- Covariance

$$
\begin{aligned}
\Sigma_y &= E\left[\tilde{y}_t \tilde{y}_t'\right] \\
&= E\left[\left(g_x \tilde{x}_t\right)\left(g_x \tilde{x}_t\right)'\right] \\
&= g_x E\left[\tilde{x}_t \tilde{x}_t'\right] g_x' \\
&= g_x \Sigma_x g_x'.
\end{aligned}
$$

- Autocovariances

$$
\begin{aligned}
\Sigma_y\left(\tau\right) &= E\left[\tilde{y}_t \tilde{y}_{t-\tau}'\right] \\
&= E\left[\left(g_x \tilde{x}_t\right)\left(g_x \tilde{x}_{t-\tau}\right)'\right] \\
&= g_x E\left[\tilde{x}_t \tilde{x}_{t-\tau}'\right] g_x' \\
&= g_x \Sigma_x\left(\tau\right) g_x'.
\end{aligned}
$$

Second moments of the control variables

- Cross-covariance (contemporaneous)

$$\begin{aligned}\Sigma_{y,x} &= E\left[\tilde{y}_t \tilde{x}_t'\right] \\ &= E\left[\left(g_x \tilde{x}_t\right) \tilde{x}_t'\right] \\ &= g_x \Sigma_x .\end{aligned}$$

- Cross-covariance (lag τ)

$$\begin{aligned}\Sigma_{y,x}(\tau) &= E\left[\tilde{y}_t \tilde{x}_{t-\tau}'\right] \\ &= E\left[\left(g_x \tilde{x}_t\right) \tilde{x}_{t-\tau}'\right] \\ &= g_x E\left[\tilde{x}_t \tilde{x}_{t-\tau}'\right] \\ &= g_x \Sigma_x(\tau).\end{aligned}$$

Spectral densities

- Rewrite state equation $\tilde{x}_{t+1} = h_x \tilde{x}_t + \mu_{t+1}$ as

$$\tilde{x}_t = (1 - h_x L)^{-1} \eta \varepsilon_{t+1}$$

- This is a moving average process on μ_t. Using results from a previous lecture

$$\begin{aligned} S_{\tilde{x}}(\omega) &= \left(I - h_x e^{-i\omega}\right)^{-1} S_\mu(\omega) \left[\left(I - h_x e^{-i\omega}\right)^{-1}\right]^* \\ &= \left(I - h_x e^{-i\omega}\right)^{-1} \Sigma_\varepsilon \left[\left(I - h_x e^{-i\omega}\right)^*\right]^{-1} \\ &= \left(I - h_x e^{-i\omega}\right)^{-1} \Sigma_\varepsilon \left(I - h_x' e^{i\omega}\right)^{-1}. \end{aligned}$$

- Spectral density of the control variables: $\tilde{y}_t = g_x \tilde{x}_t \Rightarrow$

$$S_{\tilde{y}}(\omega) = g_x S_{\tilde{x}}(\omega) g_x'$$

$$S_{\tilde{y}}(\omega) = g_x \left(I - h_x e^{-i\omega}\right)^{-1} \Sigma_\varepsilon \left(I - h_x' e^{i\omega}\right)^{-1} g_x'$$

Impulse response function (IR)

Method 1: simulation

Solution of the model

$$\tilde{x}_{t+1} = h_x \tilde{x}_t + \eta \varepsilon_{t+1}$$
$$\tilde{y}_t = g_x \tilde{x}_t.$$

- Easiest way to compute an IR function is to simulate the model.
- Suppose we want to compute the IR to a 1 standard deviation shock to first element of $\varepsilon_t : \varepsilon_{1,t}$.
- Set $\varepsilon_{1,t} = 1$ for $t = 0$, $\varepsilon_{1,t} = 0$ for $t > 0$, and $\varepsilon_{j,t} = 0$ for all t and all $j \neq 1$.
- Then iterate on the previous equations.

Impulse response function (IR)

Method 2: analytical expression

- The response of z_t in period $t+j$ to an impulse in period t (i.e. an arbitrary shock to ε_t) is defined as
$$IR\left(z_{t+j}\right) = E_t\left[z_{t+j}\right] - E_{t-1}\left[z_{t+j}\right].$$
- The IR function tells us the new information that we acquire exactly at time t of the variable z_t in period $t+j$.
- Consider the state equation
$$\tilde{x}_{t+1} = h_x \tilde{x}_t + \eta \varepsilon_{t+1}.$$
- Iterating forward we have
$$E_t\left[\tilde{x}_{t+j}\right] = h_x^j \tilde{x}_t.$$

Impulse response function (IR)

Method 2: analytic expression

- Initial impulse at time $t = 0$ is

$$\tilde{x}_0 = h_x \tilde{x}_{-1} + \eta \varepsilon_0 = \eta \varepsilon_0.$$

(economy is in steady state at $t = -1 \Rightarrow \tilde{x}_{-1} = 0$).

- Using the law of iterated expectations and $E_{-1}\varepsilon_0 = 0$ we have

$$\begin{aligned} E_0 \left[\tilde{x}_t\right] &= h_x^t \tilde{x}_0 \\ E_{-1} \left[\tilde{x}_t\right] &= h_x^t E_{-1} \tilde{x}_0 = h_x^t \eta E_{-1} \varepsilon_0 = 0. \end{aligned}$$

- Therefore, IR response to x_t at time t to an impulse ε_0 at time 0 is

$$IR\left(\tilde{x}_t\right) = E_0\left[\tilde{x}_t\right] - E_{-1}\left[\tilde{x}_t\right] = h_x^t \tilde{x}_0.$$

- The impulse to the vector of control variables $\tilde{y}_t = g_x \tilde{x}_t$ is thus

$$IR\left(\tilde{y}_t\right) = g_x h_x^t \tilde{x}_0.$$