# Real Business Cycles in a Small Open Economy (Mendoza, *AER*, 1991)

Consider the following model of a small open economy. The differences between this model and the simple model with capital accumulation and adjustment costs analyzed in class are: (1) Productivity innovations are stochastic; (2) Output is produced using capital and labor services, not just capital. (3) Physical capital depreciates; and (4) The model analyzed here displays stationarity via Uzawa-type preferences.

## Preferences

$$E_0 \sum_{t=0}^{\infty} \theta_t \, U(c_t, h_t),$$

$$\theta_0 = 1,$$

$$\theta_{t+1} = \beta(c_t, h_t)\theta_t \qquad t > 0,$$

$\beta_c < 0$, $\beta_h > 0$. This preference specification allows the model to be stationary, in the sense that the non-stochastic steady state is independent of the initial conditions (namely, the initial level of financial wealth, physical capital, and technology).

## Technology

The evolution of financial wealth, $b_t$, is given by

$$b_{t+1} = (1+r)b_t + tb_t,$$

where $r$ denotes the world interest rate, and $tb_t$ denotes the trade balance. In turn, the trade balance is given by

$$tb_t = y_t - c_t - i_t - \Phi(k_{t+1} - k_t),$$

where $y_t$ denotes domestic output, $c_t$ denotes consumption, $i_t$ denotes gross investment, and $\Phi(\cdot)$ denotes an adjustment cost function satisfying $\Phi(0) = \Phi'(0) = 0$. Output is produced by means of a linearly homogeneous production function that takes services of physical capital and labor as inputs. Formally,

$$y_t = A_t F(k_t, h_t),$$

where $h_t$ denotes labor services and $\theta_t$ is a stochastic productivity shock. The stock of capital evolves according to

$$i_t = k_{t+1} - (1-\delta)k_t,$$

where $\delta \in (0,1)$ denotes the rate of depreciation of physical capital.

## 0.1 The Lagrangean

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \theta_t \left\{ U(c_t, h_t) + \lambda_t \left[(1+r)b_t + A_t F(k_t, h_t) + (1-\delta)k_t - \Phi(k_{t+1} - k_t) - k_{t+1} - c_t - b_{t+1}\right] + \eta_t \left[\frac{\theta_{t+1}}{\theta_t} - \beta(c_t, h_t)\right] \right\}$$

## First-order conditions

$$\lambda_t = \beta(c_t, h_t)(1+r)E_t\lambda_{t+1}$$

$$\lambda_t = U_c(c_t, h_t) - \eta_t\beta_c(c_t, h_t)$$

$$\eta_t = -E_tU(c_{t+1}, h_{t+1}) + E_t\eta_{t+1}\beta(c_{t+1}, h_{t+1})$$

$$-U_h(c_t, h_t) + \eta_t\beta_h(c_t, h_t) = \lambda_t A_t F_h(k_t, h_t)$$

$$\lambda_t[1 + \Phi'(k_{t+1} - k_t)] = \beta(c_t, h_t)E_t\lambda_{t+1}\left[A_{t+1}F_k(k_{t+1}, h_{t+1}) + 1 - \delta + \Phi'(k_{t+2} - k_{t+1})\right]$$

Note:

$$\eta_t = -E_t\sum_{j=1}^{\infty}\left(\frac{\theta_{t+j}}{\theta_{t+1}}\right)U(c_{t+j}, h_{t+j})$$

## Functional forms

$$U(c,h) = \frac{\left[c - \omega^{-1}h^{\omega}\right]^{1-\gamma} - 1}{1-\gamma}$$

$$\beta(c,h) = \left[1 + c - \omega^{-1}h^{\omega}\right]^{-\psi}$$

$$F(k,h) = k^{\alpha}h^{1-\alpha}$$

$$\Phi(x) = \frac{\phi}{2}x^2; \quad \phi > 0$$

## Non-stochastic steady state

$$F_k(k,h) - \delta = r \implies \frac{h}{k} = \left(\frac{r+\delta}{\alpha}\right)^{\frac{1}{1-\alpha}}$$

$$h^{\omega-1} = F_h(k,h) \implies h = \left[(1-\alpha)\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}\right]^{\frac{1}{\omega-1}}$$

$$\beta(c,h)(1+r) = 1 \implies c = (1+r)^{1/\psi} + \frac{h^{\omega}}{\omega} - 1$$

$$i = \delta k \implies i$$

$$tb = F(k,h) - c - i \implies tb$$

$$b = -\frac{tb}{r} \implies b$$

## Calibration

$$\gamma = 1.001 \text{ or } 2$$

$$\omega = 1.455$$

$$\psi = .11 \quad \text{(match } rb/y\text{)}$$

$$\alpha = .32$$

$$\phi \in [0.023, 0.028] \quad \text{(match } \sigma_i\text{)}$$

$$r = 0.04$$

$$\delta = 0.1$$

Let $\hat{A}_t = \ln(A_t/A)$. Assume that

$$\hat{A}_t = \rho \hat{A}_{t-1} + \epsilon_t$$

Set $\rho$ and $\sigma_\epsilon$ so as to match the standard deviation and first-order serial correlation of detrended output. Mendoza analyzes a number of parameter specifications. The one we will focus on features $\gamma = 2$, $\sigma_e = 1.29$ percent, $\rho = 0.42$, and $\phi = 0.028$ (Table 6).

## Log-linearization

Let $\widehat{x}_t \equiv \log(x_t/\bar{x})$ denote the log-deviation of $x_t$ from its steady-state value. Then, taking a first-order log-linearization of the model we get:

$$\widehat{\lambda}_t = \epsilon_{\beta c}\widehat{c}_t + \epsilon_{\beta h}\widehat{h}_t + E_t\widehat{\lambda}_{t+1}$$

$$\widehat{\lambda}_t = \frac{(1-\beta)\epsilon_c}{(1-\beta)\epsilon_c - \beta\epsilon_{\beta c}}[\epsilon_{cc}\widehat{c}_t + \epsilon_{ch}\widehat{h}_t] - \frac{\beta\epsilon_{\beta c}}{(1-\beta)\epsilon_c - \beta\epsilon_{\beta c}}[\widehat{\eta}_t + \epsilon_{\beta cc}\widehat{c}_t + \epsilon_{\beta ch}\widehat{h}_t]$$

$$\widehat{\eta}_t = (1-\beta)[\epsilon_c E_t\widehat{c}_{t+1} + \epsilon_h E_t\widehat{h}_{t+1}] + \beta[E_t\widehat{\eta}_{t+1} + \epsilon_{\beta c}\widehat{c}_t + \epsilon_{\beta h}\widehat{h}_t]$$

$$\frac{(1-\beta)\epsilon_h}{(1-\beta)\epsilon_h + \beta\epsilon_{\beta h}}[\epsilon_{hc}\widehat{c}_t + \epsilon_{hh}\widehat{h}_t] + \frac{\beta\epsilon_{\beta h}}{(1-\beta)\epsilon_h + \beta\epsilon_{\beta h}}[\widehat{\eta}_t + \epsilon_{\beta hc}\widehat{c}_t + \epsilon_{\beta hh}\widehat{h}_t] = \widehat{\lambda}_t + \widehat{A}_t + \alpha\widehat{k}_t - \alpha\widehat{h}_t$$

$$\begin{aligned}\widehat{\lambda}_t + \phi k\widehat{k}_{t+1} - \phi k\widehat{k}_t \quad = \quad & \epsilon_{\beta c}\widehat{c}_t + \epsilon_{\beta h}\widehat{h}_t + E_t\widehat{\lambda}_{t+1} + \beta(\beta^{-1} + \delta - 1)[E_t\widehat{A}_{t+1} \\ & +(1-\alpha)E_t\widehat{h}_{t+1} - (1-\alpha)\widehat{k}_{t+1} + \beta\phi k E_t\widehat{k}_{t+2} - \beta\phi k\widehat{k}_{t+1}\end{aligned}$$

$$\frac{s_{tb}}{r}\widehat{b}_{t+1} = \frac{s_{tb}}{r}(1+r)\widehat{b}_t - \widehat{A}_t - \alpha\widehat{k}_t - (1-\alpha)\widehat{h}_t + s_c\widehat{c}_t + \frac{s_i}{\delta}\widehat{k}_{t+1} - \frac{(1-\delta)s_i}{\delta}\widehat{k}_t$$

$$\hat{A}_t = \rho\hat{A}_{t-1} + \epsilon_t,$$

where $\epsilon_{\beta c} \equiv c\beta_c/\beta$, $\epsilon_{\beta h} \equiv h\beta_h/\beta$, $\epsilon_{\beta cc} \equiv c\beta_{cc}/\beta_c$, $\epsilon_{\beta ch} \equiv h\beta_{ch}/\beta_c$, $\epsilon_c \equiv cU_c/U$, $\epsilon_{cc} = cU_{cc}/U_c$, $\epsilon_{ch} = hU_{ch}/U_c$, $s_{tb} \equiv tb/y$, $s_c \equiv c/y$, $s_i = i/y$. In the log-linearization we are using the particular forms assumed for the production function and the capital adjustment cost function.

The 7 equations that form the above linearized equilibrium model contain 3 state variables, $\widehat{k}_t$, $\widehat{b}_t$, and $\widehat{A}_t$. State variables are variables whose values in each period $t \geq 0$ are either predetermined (determined before $t$) as is the case with $\widehat{k}_t$ and $\widehat{b}_t$, or are determined in $t$ but in an exogenous fashion, as is the case of $\widehat{A}_t$. Of these state variables, two are endogenous, $\widehat{k}_t$ and $\widehat{b}_t$, and the other is exogenous, $\widehat{A}_t$. In addition, the model posseses 4 co-state variables (variables that are non-predetermined in preiod $t$), $\hat{c}_t$, $\hat{h}_t$, $\hat{\lambda}_t$, and $\hat{\eta}_t$, all of which are endogenous. All the coefficients of the linear system are known functions of the deep structural parameters of the model to which we assigned values in the calibration section. Thus, the value of all coefficients of the linear model are known to us. We can write the linear system in matrix form as follows:

$$AE_t x_{t+1} = Bx_t$$

We look for solutions to this linear rational expectations model that converge to the steady state, so we impose the boudary condition

$$\lim_{j\to\infty} |E_t x_{t+j}| = 0$$

where $A$ and $B$ are $7 \times 7$ matrices whose elements are functions of the all of the parameters that enter in the coefficients of our linear equilibrium system. The system has has three (given) initial conditions, $\widehat{k}_0$, $\widehat{b}_0$ and $\widehat{A}_0$. We will discuss in detail techniques available to solve linear systems like the one above in the following few lectures. We will also study how to compute second moments and impulse response functions associated with the solution of the model.

Figure 1 displays some unconditional second moments of interest implied by our model.

Table 1: Business-cycle properties: Data and Model

| Variable | Canadian Data | | | Model | | |
|---|---|---|---|---|---|---|
| | $\sigma_{x_t}$ | $\rho_{x_t,x_{t-1}}$ | $\rho_{x_t,GDP_t}$ | $\sigma_{x_t}$ | $\rho_{x_t,x_{t-1}}$ | $\rho_{x_t,GDP_t}$ |
| $y$ | 2.8 | 0.61 | 1 | 3.1 | 0.61 | 1 |
| $c$ | 2.5 | 0.7 | 0.59 | 2.3 | 0.7 | 0.94 |
| $i$ | 9.8 | 0.31 | 0.64 | 9.1 | 0.07 | 0.66 |
| $h$ | 2 | 0.54 | 0.8 | 2.1 | 0.61 | 1 |
| $\frac{tb}{y}$ | 1.9 | 0.66 | -0.13 | 1.5 | 0.33 | -0.012 |
| $\frac{ca}{y}$ | | | | 1.5 | 0.3 | 0.026 |

Note. The first three columns were taken from Mendoza (AER, 1991). $\sigma_{x_t}$ is measured in percent.

Figure 1. displays the impulse responses of a number of variables of interest to a technology shock equal to 1 in period 0.

Figure 1: Small Open Economy Model: Impulse Responses to a Positive Technology Shock