DETERMINACY IN SIMPLE NEW KEYNESIAN MODEL

This note discusses the determinacy of the equilibrium of the New Keynesian model with an interest rate rule. Recall the equilibrium conditions of the new Keynesian model that we discussed in a previous note,

$$E_t\left[\hat{y}_{t+1}\right] = \hat{y}_t + \frac{1}{\sigma}\left[\hat{R}_t - E_t\left[\hat{\pi}_{t+1}\right]\right]$$

$$\hat{\pi}_t = \beta E_t\left[\hat{\pi}_{t+1}\right] + \kappa\left[\left(\nu+\sigma\right)\hat{y}_t - \left(1+\nu\right)\hat{A}_t\right]$$

$$\hat{R}_t = \left(1-\rho_R\right)\left[\phi_\pi\hat{\pi}_t + \phi_y\hat{y}_t\right] + \rho_R\hat{R}_{t-1} + \upsilon_t$$

$$\hat{A}_{t+1} = \rho_A\hat{A}_t + \varepsilon_{A,t+1}$$

$$\upsilon_{t+1} = \rho_\upsilon\upsilon_t + \varepsilon_{\upsilon,t+1}$$

where

$$\kappa = \frac{\theta-1}{\bar{\pi}^2\omega}$$

and ω is the parameter associated with the adjustment cost.

To discuss the issue of determinacy of the equilibrium, we will simplify the model further assuming that $\hat{A}_t = 0$, $\upsilon_t = 0$ and $\rho_R = 0$. So basically we consider a model with no smoothing in the interest rate rule. With these assumptions the model can be written as

$$E_t\left[\hat{y}_{t+1}\right] + \frac{1}{\sigma}E_t\left[\hat{\pi}_{t+1}\right] = \hat{y}_t + \frac{1}{\sigma}\hat{R}_t \tag{1}$$

$$\beta E_t\left[\hat{\pi}_{t+1}\right] = \hat{\pi}_t - \gamma\hat{y}_t \tag{2}$$

$$\hat{R}_t = \phi_\pi\hat{\pi}_t + \phi_y\hat{y}_t \tag{3}$$

where we define $\gamma \equiv \kappa\left(\nu+\sigma\right)$.

Replacing the Taylor rule into the first equation gives

$$E_t\left[\hat{y}_{t+1}\right] + \frac{1}{\sigma}E_t\left[\hat{\pi}_{t+1}\right] = \hat{y}_t + \frac{1}{\sigma}\left[\phi_\pi\hat{\pi}_t + \phi_y\hat{y}_t\right]$$

$$E_t\left[\hat{\pi}_{t+1}\right] = \frac{1}{\beta}\hat{\pi}_t - \frac{\gamma}{\beta}\hat{y}_t$$

We can now replace the second equation into the first to get rid of $E_t\left[\hat{\pi}_{t+1}\right]$,

$$E_t\left[\hat{y}_{t+1}\right] + \frac{1}{\sigma}\left[\frac{1}{\beta}\hat{\pi}_t - \frac{\gamma}{\beta}\hat{y}_t\right] = \hat{y}_t + \frac{1}{\sigma}\left[\phi_\pi\hat{\pi}_t + \phi_y\hat{y}_t\right]$$

or

$$E_t\left[\hat{y}_{t+1}\right]=\left(1+\frac{\phi_y}{\sigma}+\frac{\gamma}{\sigma\beta}\right)\hat{y}_t+\frac{1}{\sigma}\left(\phi_\pi-\frac{1}{\beta}\right)\hat{\pi}_t.$$

We reduced the model to the following system of two equations:

$$\begin{aligned}
E_t\left[\hat{y}_{t+1}\right] &= \left(1+\frac{1}{\sigma}\left(\phi_y+\frac{\gamma}{\beta}\right)\right)\hat{y}_t+\frac{1}{\sigma}\left(\phi_\pi-\frac{1}{\beta}\right)\hat{\pi}_t \\
E_t\left[\hat{\pi}_{t+1}\right] &= \frac{1}{\beta}\hat{\pi}_t-\frac{\gamma}{\beta}\hat{y}_t
\end{aligned}$$

This can be written in matricial form as

$$E_t\left[\begin{array}{c}\hat{y}_{t+1}\\ \hat{\pi}_{t+1}\end{array}\right]=\left[\begin{array}{cc}1+\frac{1}{\sigma}\left(\phi_y+\frac{\gamma}{\beta}\right) & \frac{1}{\sigma}\left(\phi_\pi-\frac{1}{\beta}\right)\\ -\frac{\gamma}{\beta} & \frac{1}{\beta}\end{array}\right]\left[\begin{array}{c}\hat{y}_t\\ \hat{\pi}_t\end{array}\right] \tag{4}$$

This system does not have any state variable. Therefore, the Blanchard and Kahn condition implies that, for the equilibrium to be unique, both eigenvalues of the above coefficients matrix must be larger than one in absolute value. If one of the eigenvalues is smaller than one, the model has multiple equilibria.

We now recall two useful facts about eigenvalues of a 2×2 matrix:

1. The product of the eigenvalues is equal to the determinant of the matrix
2. The sum of the eigenvalues is equal to the trace of the matrix.

The first result implies

$$\begin{aligned}
\lambda_1\lambda_2 &= \det\left[\begin{array}{cc}1+\frac{1}{\sigma}\left(\phi_y+\frac{\gamma}{\beta}\right) & \frac{1}{\sigma}\left(\phi_\pi-\frac{1}{\beta}\right)\\ -\frac{\gamma}{\beta} & \frac{1}{\beta}\end{array}\right] \\
&= \left[1+\frac{1}{\sigma}\left(\phi_y+\frac{\gamma}{\beta}\right)\right]\frac{1}{\beta}+\frac{\gamma}{\beta\sigma}\left(\phi_\pi-\frac{1}{\beta}\right) \\
&= \frac{1}{\beta}+\frac{\phi_y}{\sigma\beta}+\frac{\gamma}{\sigma\beta^2}+\frac{\gamma\phi_\pi}{\beta\sigma}-\frac{\gamma}{\beta^2\sigma} \\
&= \frac{1}{\beta}+\frac{\phi_y}{\sigma\beta}+\frac{\gamma\phi_\pi}{\beta\sigma}>0
\end{aligned} \tag{5}$$

using $\phi_\pi>0$ and $\phi_y>0$.

The second result implies

$$\begin{aligned}\lambda_1+\lambda_2 &= trace\begin{bmatrix} 1+\frac{1}{\sigma}\left(\phi_y+\frac{\gamma}{\beta}\right) & \frac{1}{\sigma}\left(\phi_\pi-\frac{1}{\beta}\right) \\ -\frac{\gamma}{\beta} & \frac{1}{\beta}\end{bmatrix} \\ &= 1+\frac{1}{\sigma}\left(\phi_y+\frac{\gamma}{\beta}\right)+\frac{1}{\beta}>0.\end{aligned} \tag{6}$$

From these results it follows that both eigenvalues are positive. We can also prove that they must be real as well. Since both eigenvalues are positive and real, for the equilibrium to be unique it must be the case that both eigenvalues are larger than 1. Therefore, the necessary condition for a unique equilibrium can be written as

$$(\lambda_1-1)(\lambda_2-1)>0.$$

Expanding this expression gives

$$\lambda_2\lambda_1-\lambda_2-\lambda_1+1>0$$

or

$$\lambda_1\lambda_2-(\lambda_1+\lambda_2)>-1.$$

Using (5) and (6) to replace $\lambda_1\lambda_2$ and $\lambda_1+\lambda_2$ gives

$$\frac{1}{\beta}+\frac{\phi_y}{\sigma\beta}+\frac{\gamma\phi_\pi}{\beta\sigma}-\left(1+\frac{1}{\sigma}\left(\phi_y+\frac{\gamma}{\beta}\right)+\frac{1}{\beta}\right)>-1$$

This implies

$$\frac{1}{\beta}+\frac{\phi_y}{\sigma\beta}+\frac{\gamma\phi_\pi}{\beta\sigma}-1-\frac{1}{\sigma}\left(\phi_y+\frac{\gamma}{\beta}\right)-\frac{1}{\beta}>-1$$

or

$$\frac{\phi_y}{\sigma\beta}+\frac{\gamma\phi_\pi}{\beta\sigma}-\frac{1}{\sigma}\left(\phi_y+\frac{\gamma}{\beta}\right)>0$$

Multiplying by σ and rearranging the inequality we obtain

$$\phi_y\left(\frac{1}{\beta}-1\right)+\frac{\gamma\phi_\pi}{\beta}>\frac{\gamma}{\beta}$$

Multiplying by β/γ then gives

$$\phi_y\left(\frac{1-\beta}{\gamma}\right)+\phi_\pi>1 \tag{7}$$

This is a necessary condition for uniqueness of the equilibrium of the New Keynesian model.

It is a constraints on the parameters of the Taylor rule. The Taylor principle $\phi_\pi > 1$ is a sufficient condition for the equibrium to be determinate. But you can also have $\phi_\pi < 1$ and ϕ_y sufficiently large and the above inequality still hold.