State Space Representations

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Spring 2008

Notation

s_t: state variables

y_t: observed variables

Y_t: $\{y_j\}_{j=1}^{t}$

Note: variables are expressed in levels (detrended when appropriate)

State Space Representations

State-transition equation:

$$s_t = \gamma(s_{t-1}, Y_{t-1}, \upsilon_t)$$

Associated density:

$$f(s_t | s_{t-1}, Y_{t-1})$$

Measurement equation:

$$y_t = \delta(s_t, Y_{t-1}, u_t)$$

Associated density:

$$f(y_t | s_t, Y_{t-1})$$

Initialization:

$$f(s_0)$$

Likelihood Evaluation and Filtering

- **Filtering objective:** construct $f\left(s_t \middle| Y_t\right)$, which can then be used to approximate $E_t\left(h(s_t) \middle| Y_t\right)$.

- **Likelihood evaluation** obtains as a by-product of the filtering process.

Likelihood Evaluation and Filtering, cont.

- From Bayes' theorem, $f(s_t|Y_t)$ is given by

$$f(s_t|Y_t) = \frac{f(y_t, s_t|Y_{t-1})}{f(y_t|Y_{t-1})} = \frac{f(y_t|s_t, Y_{t-1})\, f(s_t|Y_{t-1})}{f(y_t|Y_{t-1})},$$

- where $f(s_t|Y_{t-1})$ is given by

$$f(s_t|Y_{t-1}) = \int f(s_t|s_{t-1}, Y_{t-1})\, f(s_{t-1}|Y_{t-1})\, ds_{t-1},$$

- and $f(y_t|Y_{t-1})$ is given by

$$f(y_t|Y_{t-1}) = \int f(y_t|s_t, Y_{t-1})\, f(s_t|Y_{t-1})\, ds_t.$$

Schematic of the Filtering Process

Taking $f\left(s_{t-1}|Y_{t-1}\right)$ as given, initialized with $f\left(s_0|Y_0\right) \equiv f(s_0)$, filtering and likelihood evaluation proceed recursively:

- Prediction: $f\left(s_{t-1}|Y_{t-1}\right)$ combines with $f\left(s_t|s_{t-1}, Y_{t-1}\right)$ to yield

$$f(s_t|Y_{t-1}) = \int f\left(s_t|s_{t-1}, Y_{t-1}\right) f\left(s_{t-1}|Y_{t-1}\right) ds_{t-1} \longrightarrow (4)$$

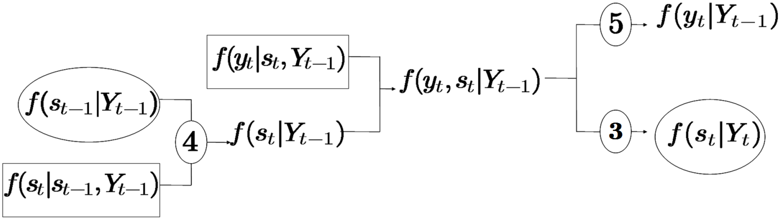
- Forecasting: $f(s_t|Y_{t-1})$ combines with $f\left(y_t|s_t, Y_{t-1}\right)$ to yield

$$f\left(y_t|Y_{t-1}\right) = \int f\left(y_t|s_t, Y_{t-1}\right) f\left(s_t|Y_{t-1}\right) ds_t. \longrightarrow (5)$$

- Updating: Bayes' Rule yields

$$f(s_t|Y_t) = \frac{f\left(y_t|s_t, Y_{t-1}\right) f\left(s_t|Y_{t-1}\right)}{f\left(y_t|Y_{t-1}\right)} \longrightarrow (3)$$

Schematic, cont.



One-Tree Model

Recall that with p_t representing $\frac{p_t}{e^{gt}}$, etc., the model is

$$
\begin{aligned}
p_t &= \beta e^{(1-\gamma)g} E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} (d_{t+1} + p_{t+1}) \right] && (1) \\
c_t &= d_t + q_t && (2) \\
d_t &= \overline{d} e^{\omega_{dt}}, \quad \omega_{dt} = \rho_d \omega_{dt-1} + \varepsilon_{dt}, && (3) \\
q_t &= \overline{q} e^{\omega_{qt}}, \quad \omega_{qt} = \rho_q \omega_{qt-1} + \varepsilon_{qt}. && (4)
\end{aligned}
$$

State: $s_t = [d_t \quad q_t]'$

Shocks: $\upsilon_t = [\varepsilon_{dt} \quad \varepsilon_{qt}]'$

Controls: $c_t = [c_t \quad p_t]'$

One-Tree Model, cont.

- State-transition equations:

$$\begin{aligned}
\ln d_t &= (1-\rho_d)\ln \overline{d} + \rho_d \ln d_{t-1} + \varepsilon_{dt}, \\
\ln q_t &= (1-\rho_q)\ln \overline{q} + \rho_q \ln q_{t-1} + \varepsilon_{qt},
\end{aligned}$$

$$v_t \equiv [\varepsilon_{dt} \quad \varepsilon_{qt}]' \sim iidN\left(0, \Sigma_v\right).$$

- Thus the state-transition density $f(s_t | s_{t-1}, Y_{t-1})$ is $N(0, \Sigma_v)$.
- Measurement equations:

$$\begin{aligned}
c_t &= d_t + q_t + u_{ct} \\
p_t &= p(d_t, q_t) + u_{pt} \\
d_t &= d_t,
\end{aligned}$$

$$u_t \equiv [u_{ct} \quad u_{pt}]' \sim iidN\left(0, \Sigma_u\right).$$

- Note that the measurement density $f(y_t | s_t, Y_{t-1})$ is partially degenerate.

RBC Model

$$\left(\frac{1-\varphi}{\varphi}\right)\frac{c_t}{l_t} = (1-\alpha)z_t\left(\frac{k_t}{n_t}\right)^{\alpha} \quad (5)$$

$$c_t^{\kappa} l_t^{\lambda} = \beta E_t\left\{*\right\} \quad (6)$$

$$y_t = z_t k_t^{\alpha} n_t^{1-\alpha} \quad (7)$$

$$y_t = c_t + i_t \quad (8)$$

$$\left(1+\frac{g}{1-\alpha}\right)k_{t+1} = i_t + (1-\delta)k_t \quad (9)$$

$$1 = n_t + l_t \quad (10)$$

$$\log z_t = (1-\rho)\log(z_0) + \rho\log z_{t-1} + \varepsilon_t \quad (11)$$

where $\kappa = \varphi(1-\phi) - 1$ and $\lambda = (1-\varphi)(1-\phi)$.

RBC Model

$$\{*\} = \left\{ \left(1 + \frac{g}{1-\alpha}\right)^{\kappa} c_{t+1}^{\kappa} l_{t+1}^{\lambda} \left[\alpha z_{t+1} \left(\frac{n_{t+1}}{k_{t+1}}\right)^{1-\alpha} + (1-\delta) \right] \right\}$$

RBC Model, cont.

A policy function $c(k,z)$ can be obtained by combining (5), (6) and (10) to eliminate (l,n) . Given $c(k,z)$, policy functions for (l,n,y,i) obtain from simple algebra. The state-transition equations are then

- $$\begin{aligned} \left(1+\frac{g}{1-\alpha}\right)k_{t+1} &= i\left(k_t,z_t\right)+(1-\delta)k_t \\ \log z_t &= (1-\rho)\log(z_0)+\rho\log z_{t-1}+\varepsilon_t . \end{aligned}$$
- Note that the transition density is partially degenerate.

RBC Model, cont.

Observation equations:

$$
\begin{aligned}
y_t &= z_t k_t^{\alpha} n(k_t, z_t)^{1-\alpha} + u_{yt} \\
c_t &= c(k_t, z_t) + u_{ct} \\
i_t &= i(k_t, z_t) + u_{it} \\
n_t &= n(k_t, z_t) + u_{nt},
\end{aligned}
$$

$u_t \sim N\left(0, \Sigma_u\right)$.

Generic Linear State Space Rep.

- State-transition equations:

$$\begin{aligned} x_t &= Fx_{t-1} + e_t, \\ e_t &= G\upsilon_t, \\ E(e_t e_t') &= GE\left(\upsilon_t \upsilon_t'\right) G' = Q. \end{aligned}$$

- Measurement equations:

$$X_t = H'x_t + u_t, \qquad Eu_t u_t' = \Sigma_u$$

- Note: x_t in general contains state variables s_t and control variables; X_t is directly analagous to y_t.

The Kalman Filter.

Likelihood evaluation and filtering is achieved in the linear-normal case via the Kalman filter. Given linearity/normality, targeted densities are fully characterized by means and covariance matricies.

Notation:

$$\begin{aligned} x_{t|t-j} &= E[x_t| \{X_1, ..., X_{t-j}\}], \\ P_{t|t-j} &= E\left[(x_t - x_{t|t-j})(x_t - x_{t|t-j})'\right], \\ j &= 0, 1. \end{aligned}$$

The Kalman Filter, cont.

Background I: Linear Projections

$$
\begin{aligned}
P\left(\underbrace{x}_{nx1}\middle|\underbrace{X}_{mx1}\right) &= \underbrace{a}_{nxm}{}'\underbrace{X}_{mx1} \\
&= E\left[x|X\right] \text{ given lin/norm,}
\end{aligned}
$$

where

$$a = \arg\min E\left[\left(x - a'X\right)^2\right]$$

The Kalman Filter, cont.

FONC for a (Normal Equations/Orthogonality Conditions):

$$E\left[\left(x - a'X\right)X'\right] = 0$$

$$\begin{aligned}
&\rightarrow \quad E\left(xX'\right) = a'E\left(XX'\right) \\
&\rightarrow \quad a' = \left(ExX'\right)\left(E\left(XX'\right)\right)^{-1} \\
&\rightarrow \quad \widehat{x} = \left(ExX'\right)\left(E\left(XX'\right)\right)^{-1}x
\end{aligned}$$

The Kalman Filter, cont.

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Background II: Updating

$$P\left(x|\left\{X_t, X_{t-1}, ...\right\}\right) = \underbrace{P\left(x|\left\{X_{t-1}, ...\right\}\right)}_{\text{old forecast}} +$$

$$P\left(\underbrace{x - P\left(x|\left\{X_{t-1}, ...\right\}\right)}_{\text{forecast error}} | \underbrace{X_t - P(X_t|\left\{X_{t-1}, ...\right\})}_{\text{new information}}\right)$$

The Kalman Filter, cont.

Kalman Filter I: Initialization $(f(s_0))$
Unconditional mean:

$$
\begin{aligned}
Ex_t &= FEx_{t-1} = FEx_t \\
&\rightarrow (I - F) Ex_t = 0 \\
&\rightarrow Ex_t \equiv x_{1|0} = 0
\end{aligned}
$$

The Kalman Filter, cont.

Unconditional VCV:

$$
\begin{aligned}
E\left(x_{1|0}x_{1|0}'\right) &\equiv P_{1|0} \\
&= E\left[\left(Fx_{t-1}+e_t\right)\left(Fx_{t-1}+e_t\right)'\right] \\
&= F\left(E\left(x_{t-1}x_{t-1}'\right)\right)F' + E\left(e_t e_t'\right) \\
&= FP_{1|0}F' + Q
\end{aligned}
$$

Thus

$$vec(P_{1|0}) = (I - F \otimes F')^{-1} vec(Q)$$

The Kalman Filter, cont.

Kalman Filter II: Forecasting $(f(y_t|Y_{t-1}))$
Given $(x_{t|t-1}, P_{t|t-1})$ (initially $(x_{1|0}, P_{1|0})$):

- $$X_{t|t-1} = H' x_{t|t-1}$$
- MSE:

$$\begin{aligned}
\Omega_{t|t-1} &= E\left[\left(H'\left(x_t - x_{t|t-1}\right) + u_t\right)\left(H'\left(x_t - x_{t|t-1}\right) + u_t\right)'\right] \\
&= E\left[H'\left(x_t - x_{t|t-1}\right)\left(x_t - x_{t|t-1}\right)' H + u_t u_t'\right] \\
&= H' P_{t|t-1} H + \Sigma_u
\end{aligned}$$

The Kalman Filter, cont.

Kalman Filter III: Updating $(f(s_t|Y_t))$

- Using the updating equation from Background II:

$$x_{t|t} = x_{t|t-1} + E\left[\left(x_t - x_{t|t-1}\right) \mid \left(X_t - X_{t|t-1}\right)\right]$$

- Using the Normal Equations from Background I:

$$E\left[\left(x_t - x_{t|t-1}\right) \mid \left(X_t - X_{t|t-1}\right)\right] = E\left(x_t - x_{t|t-1}\right)\left(X_t - X_{t|t-1}\right)' \times$$

$$\left(E\left(X_t - X_{t|t-1}\right)\left(X_t - X_{t|t-1}\right)'\right)^{-1} E\left(X_t - X_{t|t-1}\right)$$

The Kalman Filter, cont.

Kalman Filter III: Updating $(f(s_t|Y_t))$

- Thus

$$
\begin{aligned}
x_{t|t} &= x_{t|t-1} + P_{t|t-1} H \Omega_{t|t-1}^{-1} \times \\
&\quad \left(X_t - H' x_{t|t-1} \right)
\end{aligned}
$$

- MSE:

$$
\begin{aligned}
P_{t|t} &= E\left[\left(x_t - x_{t|t} \right) \left(x_t - x_{t|t} \right)' \right] \\
&= E\left[\left(x_t - x_{t|t-1} \right) \left(x_t - x_{t|t-1} \right)' \right] - \\
&\quad E\left(x_t - x_{t|t-1} \right) \left(X_t - X_{t|t-1} \right)' \times \\
&\quad \left(E\left(X_t - X_{t|t-1} \right) \left(X_t - X_{t|t-1} \right)' \right)^{-1} \times \\
&\quad E\left(X_t - X_{t|t-1} \right) \left(x_t - x_{t|t-1} \right)' \\
&= P_{t|t-1} - P_{t|t-1} H \Omega_{t|t-1}^{-1} H' P_{t|t-1}
\end{aligned}
$$

The Kalman Filter, cont.

Kalman Filter IV: Prediction $\left(f\left(s_{t+1} \middle| Y_t\right)\right)$

- Plugging $x_{t|t}$ into the state equation:

$$\begin{aligned} x_{t+1|t} &= F x_{t|t} \\ &= F x_{t|t-1} + F P_{t|t-1} H \Omega_{t|t-1}^{-1} \times \\ &\quad \left(X_t - H' x_{t|t-1}\right) \end{aligned}$$

- MSE:

$$\begin{aligned} P_{t+1|t} &= E\left[(x_{t+1} - x_{t+1|t})(x_{t+1} - x_{t+1|t})'\right] \\ &= E\left[\left(F x_t + e_{t+1} - F x_{t|t}\right)\left(F x_t + e_{t+1} - F x_{t|t}\right)'\right] \\ &= F E\left[\left(x_t - x_{t|t}\right)\left(x_t - x_{t|t}\right)'\right] F' + E\left(e_{t+1} e_{t+1}'\right) \\ &= F P_{t|t} F' + Q \end{aligned}$$

The Kalman Filter, cont.

Summary (means and covariances)

- Initialization:
  $x_{1|0} = 0, \; vec(P_{1|0}) = (I - F \otimes F')^{-1} vec(Q)$
- Forecasting: $X_{t|t-1} = H' x_{t|t-1}, \; H' P_{t|t-1} H + \Sigma_u$
- Updating:

$$\begin{aligned} x_{t|t} &= x_{t|t-1} + P_{t|t-1} H \Omega_{t|t-1}^{-1} \times \\ & \left(X_t - H' x_{t|t-1} \right), \end{aligned}$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1} H \Omega_{t|t-1}^{-1} H' P_{t|t-1}$$

The Kalman Filter, cont.

- Prediction:

$$\begin{aligned} x_{t+1|t} &= F x_{t|t} \\ &= F x_{t|t-1} + F P_{t|t-1} H \Omega_{t|t-1}^{-1} \times \\ &\quad \left(X_t - H' x_{t|t-1} \right), \\ P_{t+1|t} &= F P_{t|t} F' + Q \end{aligned}$$

The Kalman Filter, cont.

Code:

- `kalman.prc` $(\Sigma_u = 0)$
- `kalmanm.prc` $(\Sigma_u \neq 0)$

The Kalman Filter, cont.

Exercise:

Consider the AR(p) representation for a generic variable y_t:

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \ldots + \rho_p y_{t-p} + \varepsilon_t.$$

- Map this into the form of a state-space representation.
- Generate artificial data using the model as a DGP.
- Show that OLS estimates and ML estimates of $\rho(L)$ coincide.