Examen Inferencia (Dic - 2023).

1.a $$L(\theta) = \prod_{i=1}^{n} (2\pi\theta)^{-1/2} e^{-x_i^2/2\theta} = (2\pi\theta)^{-n/2} e^{-\frac{\sum_{i=1}^{n} x_i^2}{2\theta}}$$

$$\ell(\theta) = -\frac{n}{2}\ln(2\pi\theta) - \sum_{i=1}^{n}\frac{x_i^2}{2\theta}$$

(i) $\ell'_\theta : -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2 \Longrightarrow \boxed{\hat{\theta}_{MV} = \frac{1}{n}\sum_{i=1}^{n} X_i^2}$

(ii) $\ell''_\theta : \frac{n}{2\theta^2} - \frac{1}{\theta^3}\sum_{i=1}^{n} x_i^2$ (Se puede dem. que $\ell''_\theta(\hat{\theta}_{MV}) < 0$)

1.b) • Insesgado: $E\{\hat{\theta}_{MV}\} = \frac{1}{n}\sum_{i=1}^{n} E(X_i^2) \overset{iid}{=} E(X_i^2) = \theta$

• Consistente: Por la LGN $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} E(X^2) = \theta$

• Alternativamente: $ECM(\hat{\theta}_{MV};\theta) = Var(\hat{\theta}_{MV}) = E[\hat{\theta}_{MV}^2] - [E(\hat{\theta}_{MV})]^2$

(Calculamos el ECM y comp. que $\xrightarrow{n\to\infty} 0$).

$$= \frac{1}{n^2} E\left(\sum_{i=1}^{n} X_i^4 + \sum_{i\neq j} X_i^2 X_j^2\right) - \theta^2$$

$$\overset{ind}{=} \frac{(n+2)}{n}\theta^2 - \theta^2 = \frac{2\theta^2}{n} \xrightarrow{n\to\infty} 0$$

1.c) $I_1(\theta) = -E\left\{\underbrace{\frac{\partial^2}{\partial\theta^2}\ln f(x,\theta)}_{\ell''_\theta \text{ con } n=1}\right\}$

$$I_1(\theta) = -E\left\{\frac{1}{2\theta^2} - \frac{X_1^2}{\theta^3}\right\} = -\frac{1}{2\theta^2} + \frac{\theta}{\theta^3}$$

$$= \boxed{I_1(\theta) = \frac{1}{2\theta^2}}$$

Por otro lado

$$\text{Var}(\hat{\theta}_{MV}) = \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) \overset{\text{iid}}{=} \frac{1}{n^2}\, n \,\text{Var}(X_i^2)$$

$$= \frac{1}{n}\left(E(X^4) - [E(X^2)]^2\right)$$

$$\boxed{\text{Var}(\hat{\theta}_{MV}) = \frac{1}{n}\left(3\theta^2 - \theta^2\right) = \frac{2\theta^2}{n}}$$

$\times$ Como $\text{Var}(\hat{\theta}_{MV}) = \underbrace{\frac{1}{n I_1(\theta)}}_{CR}$ y $\hat{\theta}_{MV}$ es insesgado

$\Downarrow$

$\hat{\theta}_{MV}$ es el UMVUE.

d) Por normalidad asintótica

$$\sqrt{n}\,(\hat{\theta}_{MV} - \theta) \xrightarrow{d} N(0; 2\theta^2)$$

Luego $IC^{a}_{1-\alpha}(\theta) = \left[\hat{\theta}_{MV} - z_{1-\alpha/2}\sqrt{\frac{2\hat{\theta}_{MV}^{2}}{n}} \,;\, \cdot \,\right]$

. Con $\alpha = 0{,}05 \Rightarrow z_{0,975} = 1{,}96$

. $\hat{\theta}_{MV} = \frac{500}{100} = 5.$

$$IC^{a}_{0.95}(\theta) = \left[5 - 1{,}96\sqrt{\frac{2\times(5)^2}{100}} \,;\, 5 + 1{,}96\sqrt{\frac{2\times(5)^2}{100}}\right]$$

$$= [3{,}61 \,;\, 6.39].$$

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2) a) $H_0: \mu = 1 \Rightarrow \bar{X}_n \sim N\left(1; \frac{4}{n}\right)$

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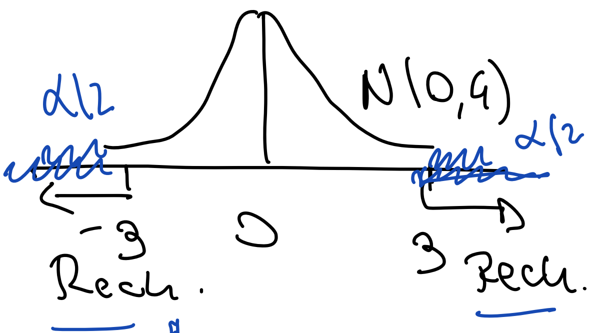

$W_n = \sqrt{n}\,(\bar{X}_n - 1) \overset{H_0}{\sim} N(0,4)$

"bajo H_0"

b) $\underline{\text{Rech}}: \{\bar{X}_n : |\sqrt{n}\,(\bar{X}_n - 1)| \geq 3\}$

$$\alpha = \max_{\theta \in H_0} P(\text{Rechazar } H_0)$$

$$= P_{\mu=1}\left(|\sqrt{n}\,(\bar{X}_n - 1)| \geq 3\right)$$

$$= P\left(\left| \frac{\sqrt{n}(\bar{x}_n - 1)}{2} \right| \geq \frac{3}{2} \right) = P(|Z| \geq 1{,}5)$$

$$= 2P(Z \leq -1{,}5) \approx \boxed{0{,}133}$$

c) Con $n = 16$ $\bar{x}_n = 2$ $\Rightarrow \left|\sqrt{16}(2-1)\right| = \underbrace{|4|}_{\geq 3} \in \text{Rech.}$

(i) Con un nivel de sig. de $\alpha = 0{,}133$ rech. H_0, ya que $w_{obs} = |4| \in$ Reg. rech.

(ii) p-val $= 2P\left(\sqrt{16}(\bar{x}_n - 1) \geq 4\right)$

$$= 2P(Z \geq 2) = \boxed{2 \times 0{,}023} = 0{,}046$$

(iii) No porque el $\underbrace{\text{p-val}}_{0{,}046} \gg 0{,}01$.

d) $\beta(\mu) = P(\text{No rechazar } H_0 \mid H_0 \text{ falsa})$

$$\beta(\mu = 1{,}5) = P_{\mu=1{,}5}\left(\left|\sqrt{16}(\bar{x}_n - 1)\right| \leq 3 \right)$$

$$= P_{\mu=1{,}5}\left(-\frac{3}{4} \leq \bar{x}_n - 1 \leq \frac{3}{4} \right)$$

$$= P_{\mu=1,5}\left(1-\frac{1}{4} \leq \bar{X}_n \leq 1+\frac{3}{4}\right)$$

$$= P\left(\frac{1-1,5}{2/\sqrt{16}} - \frac{3\sqrt{16}}{8} \leq \frac{\bar{X}_n - 1,5}{2/\sqrt{16}} \leq \frac{1-1,5}{2/\sqrt{16}} + \frac{3\sqrt{16}}{8}\right)$$

$$= P\left(-2,5 \leq Z \leq +0,5\right)$$

$$= 0,698 - 0,006 =$$

$$= \boxed{0,692}$$

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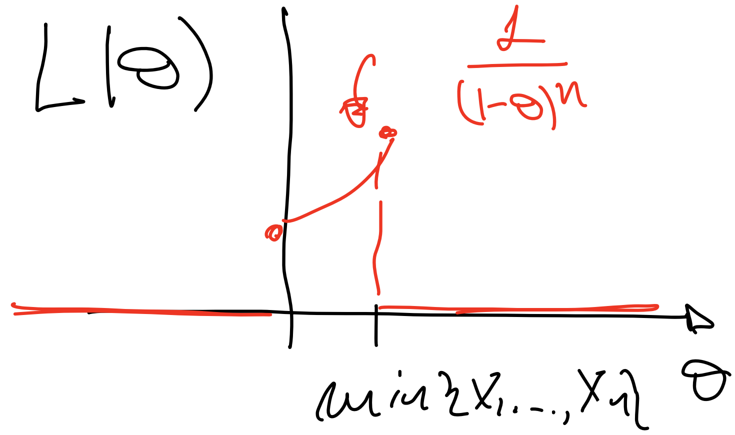
3) $X \sim \text{Unif}(\theta, 1)$ con $0 < \theta < 1$

a) $$L(\theta) = \underbrace{\frac{1}{(1-\theta)^n}}_{\substack{\text{Creciente} \\ \text{en } \theta \in (0,1)}} \prod_{i=1}^{n} \mathbb{1}_{\{[\theta,1]\}}(x_i)$$

$$\mathbb{1}_{\{[\theta,1]\}}(x_i) = \begin{cases} 1 & \text{si } \theta \leq x_i \leq 1 \\ 0 & \text{en otro caso} \end{cases} \Rightarrow \prod_{i=1}^{n} \mathbb{1}_{[\cdot]}(x_i) = \begin{cases} 1 & \text{si } \theta \leq x_1,\ \theta \leq x_2,\ \ldots,\ \theta \leq x_n \\ 0 & \text{en otro caso} \end{cases}$$

$\underline{\text{si}}$ $\theta \leq \underline{\min\{x_1, \ldots, x_n\}}$

Entonces:



$$\boxed{\hat{\theta}_{MO} = \min\{X_1, \dots, X_n\}}$$

b) Para obtener $\tilde{\theta}_n$: $E(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$\boxed{\frac{1+\theta}{2} = \bar{X}_n}$$

$$\Downarrow$$

$$\boxed{\tilde{\theta}_n = 2\bar{X}_n - 1}$$

Por otro lado $ECM(\tilde{\theta}_n, \theta) = bias^2(\tilde{\theta}_n) + Var(\tilde{\theta}_n)$

$$E\{\tilde{\theta}_n\} = 2E\{\bar{X}_n\} - 1 \overset{prop}{=} 2E(X) - 1$$

$$= 2\left(\frac{1+\theta}{2}\right) - 1 = \theta \quad (\text{Insesg.})$$

$$Var(\tilde{\theta}_n) = Var(2\bar{X}_n - 1)$$

$$= 4\,Var(\bar{X}_n) \overset{\text{prop. media muestral}}{=} 4\,\frac{Var(X)}{n} = \frac{4(1-\theta)^2}{12n}$$

$$ECM(\tilde{\theta}_n; \theta) = \frac{(1-\theta)^2}{3n} \xrightarrow{n \to \infty} 0$$

$\tilde{\theta}_n$ es consistente

c) Como $\hat{\theta}_{MV} = 0{,}1$ por invarianza

$$\Rightarrow \hat{\mu}_{M} = \frac{1+0{,}1}{2}$$

$$\Rightarrow \hat{\sigma}^2_{MV} = \frac{(1-0{,}1)^2}{12}$$

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4) a) $L(\theta) = \prod_{i=1}^{n} (1-\theta)^{x_i}\theta = (1-\theta)^{\sum_{i=1}^{n} x_i}\theta^n$

$$\pi(\theta \mid x; D) \propto \underbrace{\left((1-\theta)^{\sum_{i=1}^{n} x_i}\theta^n\right)}_{\text{Verosim.}} \times \underbrace{\left(\theta^{\alpha-1}(1-\theta)^{\beta-1}\right)}_{\text{Prior}}$$

$$\propto (1-\theta)^{\alpha+\sum_{i=1}^{n} x_i - 1}\,\theta^{n+\beta-1}$$

$$\pi(\theta \mid x, D) = \text{Beta}\left(\alpha + \sum_{i=1}^{n} x_i \,;\, n+\beta\right)$$

b) $$E(\theta \mid D) = \frac{\alpha + \sum_{i=1}^{n} x_i}{\alpha + \sum_{i=1}^{n} x_i + n + \beta}$$

$V(\theta \mid D) =$ idem.

c) Como la prior y la posterior son beta, el modelo es conjugado

d) $\begin{cases} H_0 = \mu \leq 1 \\ H_1 : \mu > 1 \end{cases}$

Como $E(X) = \frac{1-\theta}{\theta} \Rightarrow H_0 : \frac{1-\theta}{\theta} \leq 1$

$$\begin{cases} H_0 : \theta \geq 1/2 \\ H_1 : \theta < 1/2 \end{cases}$$

Para resolver construimos:

Posterior

$$FB(H_1, H_0) = \frac{P(\overbrace{\theta < 1/2}^{H_1} \mid \text{Datos})}{P(\theta \geq 1/2 \mid \text{Datos})} \times \frac{P(\theta \geq 1/2)}{P(\underbrace{\theta < 1/2}_{H_1})}$$

prior

$$P(\theta < 1/2 \mid \text{Datos}) = \int_0^{1/2} \underbrace{\pi(\theta \mid M_0; \text{Datos})}_{\text{posterior}}\, d\theta = 0{,}95$$

$$P(\theta < 1/2) = \int_0^{1/2} \underbrace{\pi(\theta \mid M_0)}_{\text{prior}}\, d\theta = 0{,}25$$

$$\Rightarrow FB = \frac{0{,}95}{0{,}05} \times \frac{0{,}75}{0{,}25} = 57 > 20$$

- Evidencia "fuerte" en favor de H_1

↳ Rechazo H_0 y afirmo que resulta razonable que $\underbrace{E(X) < 1}_{H_1}$.