Examen Inferencia (Septiembre 23)

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a) $$X \sim f(x;\theta) = \underbrace{\frac{1}{6\theta^4}}_{c(\theta)} \underbrace{x^3}_{h(x)} \exp\Big\{-\underbrace{\frac{1}{\theta}}_{w(\theta)} \underbrace{x}_{t(x)}\Big\}$$

* El modelo pertenece a la flia. exponencial

$\Downarrow$

$$T = \sum_{i=1}^{n} t(X_i) = \sum_{i=1}^{n} X_i \text{ es suficiente p/ } \theta.$$

(minimal-suficiente y completo)

b) $$\mu_1 = g_1(\theta) \Rightarrow \underline{\bar{X}_n = 4\theta}$$

$$\boxed{\tilde{\theta}_n = \bar{X}_n / 4}$$

Como $\bar{X}_n \xrightarrow{P} E(X)$ por la LGN; entonces

$$\tilde{\theta}_n = \frac{\bar{X}_n}{4} \xrightarrow{P} \frac{E(X)}{4} = \frac{\cancel{4}\theta}{\cancel{4}} = \theta$$

$\Rightarrow$ $\tilde{\theta}_n$ es consistente p/ θ.

c) $Se = \sqrt{Var(\tilde{\theta}_n)}$; luego $Var(\tilde{\theta}_n) = Var\left(\frac{\bar{X}_n}{4}\right) = \frac{1}{16} Var(\bar{X}_n)$

prop. de la media muest. $= \frac{Var(X)}{16n}$

Como $Var(X) = E(X^2) - [E(X)]^2 = 20\theta^2 - 16\theta^2 = 4\theta^2 \Rightarrow \boxed{Var(\tilde{\theta}_n) = \frac{\theta^2}{4n}}$

$$\boxed{\text{Luego } se = \sqrt{\frac{\theta^2}{4n}} = \frac{\theta}{2\sqrt{n}}} \Rightarrow \hat{se} = \frac{\tilde{\theta}_n}{2\sqrt{n}} = \frac{\bar{X}_n}{8\sqrt{n}}$$

d) Notar que $L(\theta) = \prod_{i=1}^{n} \frac{X_i^3}{6\theta^4} e^{-\frac{X_i}{\theta}} = \frac{\left(\prod_{i=1}^{n} X_i\right)^3}{6^n \theta^{4n}} e^{-\frac{\sum_{i=1}^{n} X_i}{\theta}}$

$$\ell(\theta) = \ln L(\theta) = cte - 4n \ln(\theta) - \frac{\sum_{i=1}^{n} X_i}{\theta}.$$

CPO: $\ell'_\theta : -\frac{4n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} X_i = 0 \Rightarrow \boxed{\hat{\theta}_n = \frac{1}{4n} \sum_{i=1}^{n} X_i}$

CSO: $\ell''_\theta : \frac{4n}{\theta^2} - \frac{2}{\theta^3} \sum_{i=1}^{n} X_i$; $\ell''_\theta \big|_{\theta = \hat{\theta}_n} = -\frac{4n}{\hat{\theta}_n^2} < 0$ ✓

ya que $\sum_{i=1}^{n} X_i = 4n \hat{\theta}_n$.

e) • Notar que $Var(\hat{\theta}_n) = Var(\tilde{\theta}_n) = \frac{\theta^2}{4n}$

• Por otro lado: $E\{\hat{\theta}_n\} = \frac{1}{4} E(\bar{X}_n) = \frac{E(X)}{4} = \frac{4\theta}{4} = \theta$

Prop. de media muestral.

$\hat{\theta}_n$ es insesgado!

$$\Rightarrow ECM(\hat{\theta}_n, \theta) = \underbrace{[E\{\hat{\theta}_n\} - \theta]^2}_{=0} + \underbrace{Var(\hat{\theta}_n)}_{\frac{\theta^2}{4n}}$$

$ECM(\hat{\theta}_n, \theta) = \frac{\theta^2}{4n} \xrightarrow{n \to \infty} 0$ $\hat{\theta}_n$ es consistente e insesgado

f) $\hat{\theta}_n$ es insesgado y su varianza es $\frac{\theta^2}{4n}$

$$CR = \frac{1}{n\, i(\theta)} \qquad i(\theta) = -E\left\{\frac{\partial^2}{\partial\theta^2} \ln f(x_i;\theta)\right\}$$

$$= -E\left\{\frac{4}{\theta^2} - \frac{2x}{\theta^3}\right\}$$

$$= -\frac{4}{\theta^2} + \frac{2(4\theta)}{\theta^3} = \frac{4}{\theta^2}$$

$CR = \frac{\theta^2}{4n} \Rightarrow$ La varianza de el EMV alcanza la cota de CR.

$\Rightarrow \hat{\theta}_n$ es el UMVUE de θ.

* Si asumía que $Var(\hat{\theta}_n) = \frac{\theta^2}{n} > \frac{\theta^2}{4n} \Rightarrow$ el EMU no es el UMVUE.

g) i) Por invarianza $\hat{\psi}_n = \ln\left(\frac{\bar{X}_n}{4}\right) \xrightarrow{P} \ln(\theta) = \psi$

ii) Usando el método delta $\left(\underbrace{g(\theta)}_{\psi} = \ln(\theta)\right)$

$$\sqrt{n}\left(\underbrace{\ln\left(\frac{\bar{X}_n}{4}\right)}_{\hat{\psi}_n} - \underbrace{\ln(\theta)}_{\psi}\right) \xrightarrow{d} N\left(0, \frac{\theta^2}{4}\underbrace{\left(\frac{1}{\theta}\right)^2}_{(g'(\theta))^2}\right)$$

$$\boxed{\sqrt{n}(\hat{\psi}_n - \psi) \xrightarrow{d} N(0, \underbrace{1/4}_{\sigma^2_\psi})}$$

h) $\bar{X}_n = 4 \Rightarrow \hat{\theta}_n = 1 \Rightarrow \hat{\psi}_n = \ln(1) = 0$

$$IC^{(a)}_{0{,}95}(\psi) = \left[\hat{\psi}_n - 1{,}96\sqrt{\tfrac{1}{400}}\;;\; \hat{\psi}_n + 1{,}96\sqrt{\tfrac{1}{400}}\right]$$
$$= [0 - 0{,}098\;;\; 0 + 0{,}098]$$
$$= [-0{,}098\;;\; 0{,}098].$$

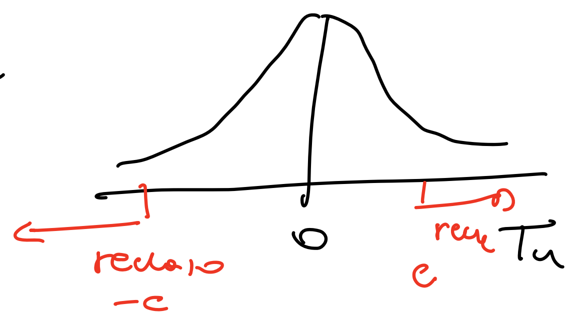
12) $X \sim N(\underset{E(X)}{\theta}, \underset{Var(X)}{\theta^2})$

$$\begin{cases} H_0: \theta = 1 \\ H_1: \theta \neq 1 \end{cases}$$

$$T_n = \frac{\sqrt{n}(\bar{X}_n - 1)}{1} \overset{H_0}{\sim} N(0,1)$$

$$\boxed{T_n = \sqrt{n}(\bar{X}_n - 1)}$$ Bajo H_0

$$\delta = \begin{cases} 1 & \text{si } |T_n| \geq c \\ 0 & \text{en otro caso} \end{cases}$$



a. Si $c = 1/2$

$$\alpha = P(\text{Rechazar } H_0 \mid \theta = 1)$$
$$= P(|T_n| \geq 1/2 \mid \theta = 1)$$
$$= 1 - P(-1/2 \leq T_n \leq 1/2) = \underline{0{,}603}$$

Sale de los cuantiles.

• b

$\beta(\theta, n) = P_\theta(\text{No rechazar } H_0 \mid \theta \neq 1)$

$= P_\theta(|T_n| \leq 1/2 \mid \theta \neq 1)$

Dependerá de n

$= P_\theta\left(-1/2 \leq \sqrt{n}(\bar{X}_n - 1) \leq 1/2\right)$

$= P_\theta\left(1 - \frac{1}{2\sqrt{n}} \leq \bar{X}_n \leq 1 + \frac{1}{2\sqrt{n}}\right)$

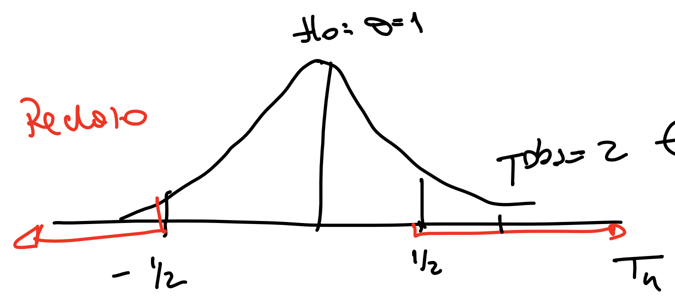
$= P\left(\frac{\sqrt{n}(1-\theta)}{\theta} - \frac{1}{2\theta} \leq Z \leq \frac{\sqrt{n}(1-\theta)}{\theta} + \frac{1}{2\theta}\right)$

$= \Phi\left(\frac{\sqrt{n}(1-\theta)}{\theta} + \frac{1}{2\theta}\right) - \Phi\left(\frac{\sqrt{n}(1-\theta)}{\theta} - \frac{1}{2\theta}\right)$ $\forall \theta > 0$ y $\theta \neq 1$.

• c

i) Con $\bar{X}_n = 1,5 \Rightarrow \boxed{T_{obs} = \sqrt{16}\,\frac{(1,5-1)}{1} = 2}$

• Región de Rechazo



$T_{obs} = 2 \in$ Rechazo $\Rightarrow$ Rechazo H_0

ii) $p\text{-val} = 2P(Z \geq 2)$ con $Z \sim N(0,1)$

$= 2 \times (0,023) = 0,045$

Por lo tanto con $\alpha = 0,01$ no alcanzamos a rechazar H_0 con los datos de la muestra ($p\text{-val} = 0,045 \geq 0,01$).

• d) Ordenamos los p-valores

Tam. del ful →

$\underbrace{P_{(1)} = 0,001}_{H^1}$, $\underbrace{P_{(2)} = 0,004}_{H^4}$, $\underbrace{P_{(3)} = 0,033}_{H^5}$; $\underbrace{P_{(4)} = 0,045}_{H^1}$; $\underbrace{P_{(5)} = 0.195}_{H^6}$; $\underbrace{P_{(6)} = 0.265}_{H^3}$

corresponden

- Con $\alpha = 0{,}05$ y $m = 6$ $\left(\frac{\beta}{m} = 1\right)$

$$P_{(1)} = 0{,}001 < \frac{0{,}05}{6} 1 = 0{,}008$$

$$\boxed{P_{(2)} = 0{,}004 < \frac{0{,}05}{6} \times 2 = 0{,}016} \rightarrow \text{Rechazamos los tst. 1 y 4}$$

$$P_{(3)} = 0{,}033 > \frac{0{,}05 \times 3}{6} = 0{,}025$$

$$P_{(4)} = 0{,}045 > \frac{0{,}05}{6} \times 4 = 0{,}033$$

$$P_{(5)} = 0{,}195 > \frac{0{,}05}{6} \times 5 = 0{,}041$$

$$P_{(6)} = 0{,}265 > \frac{0{,}05}{6} \times 6 = 0{,}05$$

③ . $X \sim f(x;\theta) = \text{Geom}(\theta)$ y nos interesa $E(X) = \frac{1-\theta}{\theta}$

- $\theta \mid (\alpha,\beta) \sim \text{Beta}(\alpha,\beta)$

a) $\pi(\theta \mid \text{Datos}) \propto L(\theta)\, \pi(\theta)$

$$\propto \theta^{n} (1-\theta)^{\sum_{i=1}^{n} x_i} \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

$$\propto \theta^{n+\alpha-1} (1-\theta)^{\sum_{i=1}^{n} x_i + \beta - 1}$$

$$\Rightarrow \pi(\theta \mid \text{Datos}) \sim \text{Beta}(\underbrace{n+\alpha}_{\alpha'} ; \underbrace{\sum_{i=1}^{n} x_i + \beta}_{\beta'})$$

Luego $E(\theta \mid \text{Datos}) = \frac{\alpha'}{\alpha'+\beta'} = \frac{n+\alpha}{n+\alpha+\sum_{i=1}^{n} x_i + \beta} = \frac{1+\alpha/n}{1+\frac{(\alpha+\beta)}{n}+\bar{X}_n}$

$$M_o(\theta \mid \text{Datos}) = \frac{\alpha'-1}{(\alpha'+\beta'-2)} = \alpha' n \Big/ \left(1 + \frac{(\alpha+\beta-2)}{n} + \bar{X}_n\right)$$

$$V(\theta \mid \text{Datos}) = \frac{(n+\alpha)\left(\sum_{i=1}^{n} x_i + \beta\right)}{\left(n+\alpha+\sum_{i=1}^{n} x_i + \beta\right)^2 \left(n+\alpha+\sum_{i=1}^{n} x_i + \beta - 1\right)}$$

b) De la ley de los grandes números

$$\bar{X}_n \xrightarrow{P} E(X) = \frac{1-\theta}{\theta}$$

$$\boxed{E(\theta \mid \text{Datos}) \xrightarrow{P} \frac{1+0}{1+0+E(\theta)} = \frac{1}{1+\frac{1-\theta}{\theta}} = \theta}$$

De la misma manera se tiene que

$$Var(\theta \mid \text{Datos}) \xrightarrow{P} 0$$

* La distribución a posteriori se concentra sobre el verdadero θ cuando $n \to \infty$.

c) Si $\alpha = \beta = 1$ (prior no informativa)

$$Mo(\theta \mid \text{Datos}) = \frac{\alpha' - 1}{\alpha' + \beta' - 2} = \frac{n+1-1}{n+1+\sum_{i=1}^{n} x_i + 1 - 2} = \frac{n}{n+\sum_{i=1}^{n} x_i}$$

$$= \frac{1}{1+\bar{X}_n} = \hat{\theta}_{MV}$$

- Si $\underbrace{\bar{X}_n}_{\text{Estimador máximo}}$ estima $E(X) = \frac{1-\theta}{\theta} \Rightarrow \frac{1}{1+\bar{X}_n}$ estima θ

verosímil de $E(X)$

Estimador máximo verosímil de θ

$$\theta = g(E(X)) = \frac{1}{1+E(X)} \Rightarrow \hat{\theta}_{MV} = \underbrace{\frac{1}{1+\bar{X}_n}}_{g(\bar{X}_n)}$$

d) $\begin{cases} H_0: E(X) = \frac{1-\theta}{\theta} \geq 2 \implies H_0: \theta \leq 1/3 \Rightarrow \Theta_0 = [0, 1/3] \\ H_1: E(X) < 2 \implies H_1: \theta > 1/3 \Rightarrow \Theta_1 = (1/3; 1] \end{cases}$

$$P(\theta \in \Theta_1) = \int_{1/3}^{1} \pi(\theta, \mu_0) = 0{,}25 \leftarrow \text{Dato}$$

$$P(\theta \in \Theta_0) = \int_0^{1/3} \pi(\theta; \mu_0) = 1 - 0{,}25 = 0{,}75$$

$$P(\theta \in \Theta_1 | \text{Datos}) = \int_{1/3}^{1} \pi(\theta | \text{Datos}; \mu_0) d\theta = 0{,}95 \leftarrow \text{Dato}$$

$$P(\theta \in \Theta_0 | \text{Datos}) = 1 - 0{,}95 = 0{,}05$$

De esta forma: $FB(H_1; H_0) = \frac{0{,}95}{0{,}05} \cdot \frac{0{,}75}{0{,}25}$

$$\underline{FB = 57}$$

Evidencia fuerte en favor de $\underline{H_1}$.