# Structural Vector Autoregressions

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# Vector Autoregressions

- A vector autoregression (**VAR**) is a multivariate autoregressive process of the form

$$Y_t = D_1 Y_{t-1} + D_2 Y_{t-2} + ... + D_p Y_{t-p} + v_t; \quad v_t \sim (0, \Omega)$$

  where

  - $Y_t$ is an $n \times 1$ vector of observed variables,
  - $D_j$ are $n \times n$ matrices,
  - $\Omega$ is an $n \times n$ covariance matrix.

- Version with exogenous variables **VARX** (e.g. small open economy):

$$Y_t = D_1 Y_{t-1} + D_2 Y_{t-2} + \ldots + D_p Y_{t-p} + G_0 X_t + \ldots + G_q X_{t-q} + v_t; \quad v_t \sim (0, \Omega)$$

  where $X_t$ is a set of exogenous variables.

# Vector Autoregressions

- A (reduced form) VAR is a useful way to **summarize** the data:
  - Correlations and temporal dependence.
  - Simple to estimate. Simple to use.
  - All variables endogenous and depend on each other (no arbitrary exogeneity or exclusion restrictions).
  - Useful for forecasting.
- **Disadvantages:**
  - No economic interpretation of the shocks or the interrelation between the variables.
  - Not always possible to represent the solution of a structural model as a VAR.

# Structural VAR (SVAR)

- A **Structural VAR** imposes restrictions on the reduced form VAR to **identify** relevant economic shocks or policy changes
- Tool to answer relevant economic questions using **minimal** theoretical restrictions:
  - What is the effect of a policy intervention on macroeconomic variables of interest?
    - An increase in the policy interest rate, an increase in government spending, etc.
  - How does the economy respond to a particular shock.
  - As a tool to guide researchers in evaluating or constructing economic models.
    - E.g. do technology shocks lead to an increase or a fall in hours worked?

# Structural VAR (SVAR)

- Attaching a structural interpretation to a VAR requires imposing **identifying restrictions**
- Identification approaches:
  - **Short-run restrictions and policy rules:** restrictions on contemporaneous correlations (Sims; Christiano Eichenbaum, and Evans, etc.)
  - **Long-run restrictions** (Blanchard and Quah)
  - **Sign restrictions** (Canova, Uhlig)
  - Others
- Debates:
  - Are the identifying assumptions plausible?
  - Even if identifying assumptions are correct, can we recover the relevant effects using a finite amount of data?

# Where do VARs come from?

- Let $Y_t = (y_{1t}, y_{2t}, ..., y_{nt})'$ be a mean zero covariance stationary vector process.
- **Wold Representation Theorem:** we can (almost) always **represent** $Y_t$ as

$$Y_t = C_0 v_t + C_1 v_{t-1} + C_2 v_{t-2} + C_3 v_{t-3} + ... = C(L) v_t \tag{1}$$

  $C_j$ are $n \times n$ matrices, $C_0 = I$, $\sum_{j=0}^{\infty} C_j C_j' < \infty$, $v_t$ is white noise with covariance $\Omega$.

- $C(L)$ is a polynomial on the lag operator

$$C(L) = I + C_1 L + C_2 L^2 + C_3 L^3 + \dots$$

# Where do VARs come from?

- The MA coefficients $C_j$ are the impulse response function at $t+j$ to a unit shock at $t$:

$$\frac{\partial Y_{t+j}}{\partial v_t^h} = \mathbf{c}_j^h \tag{2}$$

  $v_t^h$ is the h-th element of the shock $v_t$ and $\boldsymbol{c}_j^h$ denotes column $h$ of the matrix $C_j$

- Problem with (2): the (Wold) shocks in $v_t$ are typically correlated and do not have a structural interpretation.

- The shocks $v_t$ are prediction errors (or "news") of a projection of $Y_t$ on its entire past history:

$$v_t \equiv Y_t - \text{Proj}\left[Y_t | Y_{t-1}, Y_{t-1}, Y_{t-2}, \ldots\right]$$

  where $\text{Proj}\left[Y_t | Y_{t-1}, Y_{t-1}, Y_{t-2} \ldots\right]$ is the projection of $Y_t$ on $\{Y_{t-1}, Y_{t-1}, Y_{t-2}, \ldots\}$.

# Where do VARs come from?

- But we are considering VARs, not MAs. The following result makes the connection (Rozanov, 1967):

  $\sum_{j=0}^{\infty} C_j C_j' < \infty$ implies that $C(L)$ is invertible (i.e all roots are greater than 1 in modulus).

- For this condition to hold, we need the coefficients $C_j$ to decay "rapidly" towards zero.

- Inverting the MA representation yields an infinite order stationary VAR

$$Y_t C\left(L\right)^{-1} = v_t$$

- VAR can be written as

$$\begin{aligned} Y_t =& D_1 Y_{t-1} + D_2 Y_{t-2} + D_3 Y_{t-3} + ... + v_t \\ =& D\left(L\right) Y_{t-1} + v_t \end{aligned} \tag{3}$$

- **Eqs (1) and (3) are equivalent representations of $Y_t$. Thus, under regularity conditions, any stationary process can be represented as a (typically infinite order) VAR.**

# A word of caution

- The Wold Representation Theorem says that we can represent a stationary process as an invertible infinite order moving average.

- But the theorem does not imply that the invertible MA process is the **true** data generating process.

- For instance, the true process could be stationary but non-linear or non-invertible.

- Suppose, for example, that the true process is generated by an equation of the form

$$Y_t = g(Y_{t-1}, \dots, Y_{t-p}) + \eta_t.$$

  The Wold theorem says that there exists a linear representation of that process, but the shocks we recover, $v_t$, will be different from the "true" shocks $\eta_t$. Moreover, it may be impossible to recover the true shocks using the procedures discussed in these lecture notes.

# Identification of structural shocks

- Identification is a feature of population moments (estimation is a different business).
- So, let's pretend we know the true parameters of the process.
- The $MA(\infty)$ and $VAR(\infty)$ are probabilistic representations of $Y_t$:
  - The shocks $v_t$ are projection errors and may be contemporaneously correlated.
  - The shocks $v_t$ do not have a structural interpretation.
- Identifying structural shocks requires extracting **economically relevant** shocks $\varepsilon_t$ from the **reduced form** shocks $v_t$ in equation (3).
- We often assume that the relevant structural shocks are:
  - Orthogonal
  - With economic meaning: technology shock, monetary policy shock, demand shock, etc.

# Identification of structural shocks

**Assumption 1:** The $n \times 1$ vector of reduced form shocks $v_t$ is a linear combination of an $n \times 1$ vector of structural shocks $\varepsilon_t$

$$v_t = S\varepsilon_t,$$

where $S$ is an $n \times n$ invertible matrix. We also may assume that structural shocks are orthogonal with covariance matrix $E(\varepsilon_t \varepsilon_t') = I$.

- Assumption 1 implies that, if we know $S$, we can recover the structural shocks from the reduced form shocks:

$$\varepsilon_t = S^{-1} v_t.$$

- $E(\varepsilon_t \varepsilon_t') = I$ is a normalization. If structural shocks don't have unit variances, we can make them have unit variances. Also, nicer interpretation in terms of shocks of one standard deviation.

- Example: if the first element of $Y_t$ is GDP growth and the structural shocks $\varepsilon_t$ include technology shocks and demand shocks, GDP growth will be affected by a **linear** combination of technology and demand shocks.

- What if there are more structural shocks than variables in the VAR?

# Identification of structural shocks

The structural model is obtained by replacing the reduced form shocks $v_t$ by the structural shocks $\varepsilon_t$. Two equivalent representations:

**Structural MA representation:** Using (11) into (1) gives

$$\begin{aligned} Y_t =& C_0 S\varepsilon_t + C_1 S\varepsilon_{t-1} + C_2 S\varepsilon_{t-2} + \ldots \\ =& A_0\varepsilon_t + A_1\varepsilon_{t-1} + A_2\varepsilon_{t-2} + \ldots \end{aligned}$$

where $A_j = C_j S$ and the shocks satisfy $E\varepsilon_t\varepsilon_t' = I$. In compact form:

$$Y_t = A(L)\,\varepsilon_t \tag{4}$$

**Structural VAR representation:** Premultiplying (3) by $S^{-1}$ yields

$$\begin{aligned} S^{-1}Y_t =& S^{-1}D_1 Y_{t-1} + S^{-1}D_2 Y_{t-2} + S^{-1}D_3 Y_{t-3} + ... + S^{-1}v_t \\ B_0 Y_t =& B_1 Y_{t-1} + B_2 Y_{t-2} + B_3 Y_{t-3} + ... + \varepsilon_t. \end{aligned}$$

where $B_0 = S^{-1}$ and $B_i = S^{-1}D_i$ for $i = 1, 2, \ldots$. In compact form:

$$B_0 Y_t = B(L)\,Y_{t-1} + \varepsilon_t \tag{5}$$

# Identification of structural shocks

- Identification is the process of "choosing" or "identifying" the $n^2$ elements of the matrix $S$ (or $B_0$)

- Does the reduced form model impose any constraint on the elements of $S$?

# Identification of structural shocks

- Identification is the process of "choosing" or "identifying" the $n^2$ elements of the matrix $S$ (or $B_0$)

- Does the reduced form model impose any constraint on the elements of $S$?

- **YES.** There is a relation between $S$ and the covariance matrix of $v_t$: from $v_t = S\varepsilon_t$ it follows

$$\begin{aligned} E(v_t v_t') =& E(S\varepsilon_t\varepsilon_t' S') \\ \Omega =& SS'. \end{aligned}$$

  - Since $\Omega$ has $n(n+1)/2$ independent elements, $\Omega = SS'$ imposes the same number of restrictions on $S$.

  - Given symmetry, $\Omega = SS'$ is a system of $n(n+1)/2$ nonlinear equations with $n^2$ unknowns.

# Identification of structural shocks

- Need to add $n(n-1)/2$ non-redundant equations to obtain full identification.
- The "order" condition is easily checked: count number of restrictions.
- The "rank" condition (non-redundant equations) may fail depending on the numerical values of the elements of $S$.
- Rubio-Ramirez, Waggoner, and Zha (2010) discuss rank conditions for global identification of SVAR models.
  - Will discuss this issue later.

# Identification of structural models

- Example: in a bivariate VAR ($n = 2$) $\Omega$ imposes 3 restrictions on $S$:

$$\begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{bmatrix} = \begin{bmatrix} S_{11}^2 + S_{12}^2 & S_{11}S_{21} + S_{12}S_{22} \\ S_{21}S_{11} + S_{22}S_{12} & S_{21}^2 + S_{22}^2 \end{bmatrix}$$

$$\begin{aligned} \omega_{11} &= S_{11}^2 + S_{12}^2 \\ \omega_{12} &= S_{11}S_{21} + S_{12}S_{22} \\ \omega_{22} &= S_{21}^2 + S_{22}^2 \end{aligned}$$

- 3 independent left hand variables to determine 4 variables on the right hand side.
- We must impose 1 restriction (one additional equation) to fully identify the VAR.
- What kind of restrictions do we use to identify the free parameter in $S$?
  - **Use theory or some kind of institutional knowledge.**

# Partial identification of structural shocks

- Suppose that we are interested in estimating only the impact of a supply shock.
- We only need to identify the column of $S$ associated with the supply shock.
- Let the supply shock be the first element of $\varepsilon_t$ and consider the MA representation

  $$Y_{t+j} = C_0 S\varepsilon_{t+j} + C_1 S\varepsilon_{t+j-1} + ... + C_j S\varepsilon_t + \dots$$

  The impulse response function to the shock $\varepsilon_t^1$ is

  $$\frac{\partial Y_{t+j}}{\partial \varepsilon_t^1} = C_j \boldsymbol{s}^1$$

  where $\boldsymbol{s}^1$ is the first column of the matrix $S$.
- Need to impose restrictions in order to identify only the first column of $S$.

# Identification by short-run restrictions

**Procedure:**

1. Estimate reduced form parameters $D_1, D_2, \ldots, D_p$ and $\Omega$ of a VAR with $p$ lags using standard methods.
2. Need to identify the $n^2$ elements of $S$ (or $B_0$).
3. Condition $\Omega = SS'$ imposes $n(n+1)/2$ constraints on $S$. We need to add $n(n-1)/2$ additional restrictions to fully identify the model.
4. A traditional approach is to impose restrictions on selected elements of $S$ or $B_0$.

# Short-run restrictions: recursively identified models

- Popular way of identifying structural innovations $\varepsilon_t$ is to "orthogonalize" the reduced form shocks $v_t$.

- Orthogonalization means making the reduced-form errors uncorrelated by some algebraic procedure.

- Two mechanical ways of orthogonalizing shocks:
  1. Cholesky decomposition of the covariance matrix

     $$\Omega = SS'$$

     where $S$ is a lower triangular matrix with positive entries in the main diagonal.
  2. Spectral (eigenvalue-eigenvector) decomposition of the covariance matrix

     $$\Omega = PVP' = PV^{1/2}V^{1/2}P' = \tilde{P}\tilde{P}'.$$

     $P$ contains the eigenvectors of $\Omega$ and $V$ is a diagonal matrix with the eigenvalues of $\Omega$.

# Recursively Identified Models: Cholesky Decomposition

- Consider the Cholesky decomposition of the covariance matrix: $\Omega = SS'$.
  - In Matlab, `S=chol(Omega,'lower')`.
  - In Python, `S = numpy.linalg.cholesky(Omega)` or `S = scipy.linalg.cholesky(Omega)`
- Set $B_0 = S^{-1}$ or $B_0^{-1} = S$, where $\varepsilon_t = S^{-1} v_t$ are the structural shocks.
- Since $S$ is lower triangular, it has $n(n-1)/2$ zeros below the diagonal. This imposes $n(n-1)/2$ restrictions on the matrix $B_0^{-1}$ and, as a result, the order condition for identification is satisfied.
- Is it licit to perform a Cholesky decomposition?
  - Valid to the extent that the recursive structure embodied in $S$ can be justified on economic grounds.

# Recursively Identified Models: Cholesky Decomposition

- What are we doing when we perform a Cholesky decomposition?
  - $S$ lower triangular implies $B_0 = S^{-1}$ lower triangular.
  - Implies a causal chain in which some shocks affect some variables but not others.
- Example:

$$\begin{bmatrix} Y_t^1 \\ Y_t^2 \\ Y_t^3 \end{bmatrix} = D_1 Y_{t-1} + ... + D_p Y_{t-p} + \begin{bmatrix} S_{11} & 0 & 0 \\ S_{21} & S_{22} & 0 \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}.$$

- Cholesky decomposition implies
  - $Y_{1t}$ is only affected by the shock $\varepsilon_{1t}$.
  - $Y_{2t}$ is only affected by the shocks $\varepsilon_{1t}$ and $\varepsilon_{2t}$
  - $Y_{3t}$ is affected by the three shocks $\varepsilon_{1t}$, $\varepsilon_{2t}$, and $\varepsilon_{3t}$.

# Recursively Identified Models: Cholesky Decomposition

- Alternatively, letting $B_0 = S^{-1}$, we can write

$$\begin{bmatrix} B_{0,11} & 0 & 0 \\ B_{0,21} & B_{0,22} & 0 \\ B_{0,31} & B_{0,32} & B_{0,33} \end{bmatrix} \begin{bmatrix} Y_t^1 \\ Y_t^2 \\ Y_t^3 \end{bmatrix} = B_1 Y_{t-1} + \ldots + B_p Y_{t-p} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}$$

- Cholesky decomposition implies
    - $Y_{1t}$ does not depend contemporaneously on $Y_{2t}$ or $Y_{3t}$.
    - $Y_{2t}$ depends contemporaneously on $Y_{1t}$ but not on $Y_{3t}$
    - $Y_{3t}$ depends contemporaneously on, both, $Y_{1t}$ and $Y_{2t}$.

# Problems of the Cholesky Decomposition

- Multiplicity: there is a different $S$ for each possible ordering of the variables in the VAR.
  - We claim the ordering is recursive, but different orderings give different answers.
  - A VAR with $n = 4$ variables has $4! = 24$ possible permutations of the variables.
- The orderings may not have a credible economic interpretation.
- Most structural models do not generate a recursive ordering between the variables.
- To identify with Cholesky you must come up with a rationale for the particular ordering you choose.
  - Institutional knowledge.
  - Christiano, Eichembaum, and Evans (1999) is an example.

# Christiano, Eichembaum, and Evans (1999)

- Estimate the impact of monetary policy shocks.

- Suppose that the FED uses a policy rule of the form:

$$R_t = f(\Gamma_t) + e_t^R,$$

  $R_t$ is the nominal interest rate, $f(\Gamma_t)$ is a function of variables $\Gamma_t$ that the FED **observes** when setting monetary policy, and $e_t^R$ is a monetary policy shock.

- **Objetive:** measure the response of aggregate variables to an exogenous monetary policy shock $e_t^R$.

- **Problem:** Not enough assumptions **yet** to identify the monetary shock $e_t^R$.

- CEE assumptions:
  1. Policy shock $e_t^R$ is orthogonal to $\Gamma_t$.
  2. $\Gamma_t$ includes current prices and wages, aggregate quantities, lagged variables.

# Christiano, Eichembaum, and Evans (II)

- Economic content of the assumption:
  - FED observes current prices and output (all the variables in $\Gamma_t$) when choosing the current interest rate $R_t$.
  - *Therefore, prices and output (and all the variables in $\Gamma_t$) do not respond contemporaneously to the monetary shock $e_t^R$.*
- Under these assumptions, the shock $e_t^R$ can be estimated by OLS.
- Impulse responses can be obtained by regressing the variables of interest on current and lagged values of $e_t^R$.

# Christiano, Eichembaum, and Evans (III)

Identify the monetary policy shock using a SVAR under the previous assumptions:

- **Reduced form VAR:**

$$
\begin{aligned}
Y_t &= D_1 Y_{t-1} + ... + D_p Y_{t-p} + v_t \\
E(v_t v_t') &= \Omega.
\end{aligned}
$$

- **Structural VAR:**

$$
\begin{aligned}
B_0 Y_t &= B_1 Y_{t-1} + \ldots + B_p Y_{t-p} + \varepsilon_t \\
v_t &= S\varepsilon_t, \\
B_i &= B_0 D_i \\
SS' &= B_0^{-1} B_0^{-1\prime} = \Omega.
\end{aligned}
$$

# Christiano, Eichembaum, and Evans (IV)

Order the variables in $Y_t$ as follows:

$$Y_t = \begin{bmatrix} X_{1t} \\ R_t \\ X_{2t} \end{bmatrix}, \; B_0 = \begin{bmatrix} b_{11} & \mathbf{0} & \mathbf{0} \\ b_{21} & b_{22} & \mathbf{0} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \Rightarrow B_0 Y_t = \begin{bmatrix} b_{11} & \mathbf{0} & \mathbf{0} \\ b_{21} & b_{22} & \mathbf{0} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} X_{1t} \\ R_t \\ X_{2t} \end{bmatrix}$$

- $R_t$ is the interest rate equation (middle equation)
- $X_{1t}$: vector of $k_1$ variables whose current and lagged values appear in the policy rule.
- $X_{2t}$: vector of $k_2$ variables whose current values do not appear in the policy rule. These are variables that the FED does not observe when it sets $R_t$.
- Zero restrictions on $B_0$ implied by recursiveness assumption:
  - Zeros in middle row: FED does not observe $X_{2t}$ when setting monetary policy.
  - Zeros in first block ensure that monetary policy shocks do not affect $X_{1t}$.
    - First block of zeros: prevents direct effect via $R_t$.
    - Second block of zeros: prevents indirect effects via $X_{2t}$ (because $X_{2t}$ responds to $R_t$.)

# Christiano, Eichembaum, and Evans (V)

- There are many matrices $B_0$ with a pattern of zeros that satisfy $B_0^{-1} B_0^{-1\prime} = \Omega$.
    - For example: lower triangular $B_0$ with positive diagonal elements.
    - In this case, $B_0^{-1}$ is the lower triangular Cholesky decomposition of $\Omega$.
- However, CEE show:
    - All $B_0$ matrices that satisfy $B_0^{-1} B_0^{-1\prime} = \Omega$ and the zero restrictions imposed above imply the same value for the column of $B_0^{-1}$ corresponding to $e_t^R$.
    - Thus, one can work without loss of generality with the lower triangular Cholesky decomposition of $\Omega$.
    - If we change the ordering of $X_{1t}$ and $X_{2t}$ but always pick a lower triangular Cholesky decomposition of $\Omega$, the impulse responses all variables to a monetary shock $e_t^R$ are always the same.

# VAR with inflation, unemployment, and the FED fund rate

Stock and Watson, 2001

- Two different orderings:
  - Ordering 1 (Stock and Watson)

$$Y_t = \begin{bmatrix} \text{Inflation}_t \\ \text{Unemployment}_t \\ \text{Fed fund rate}_t \end{bmatrix}$$

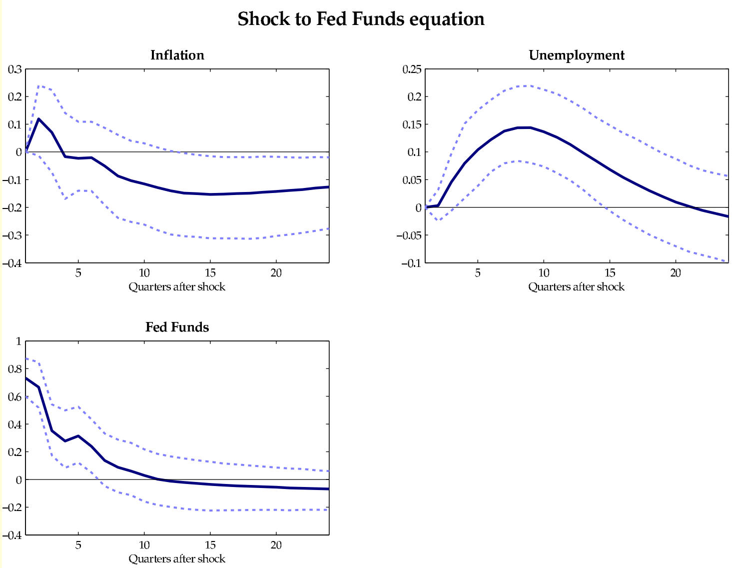
    Economic content: FED observes current inflation and unemployment when setting the interest rate.

  - Ordering 2 (us)

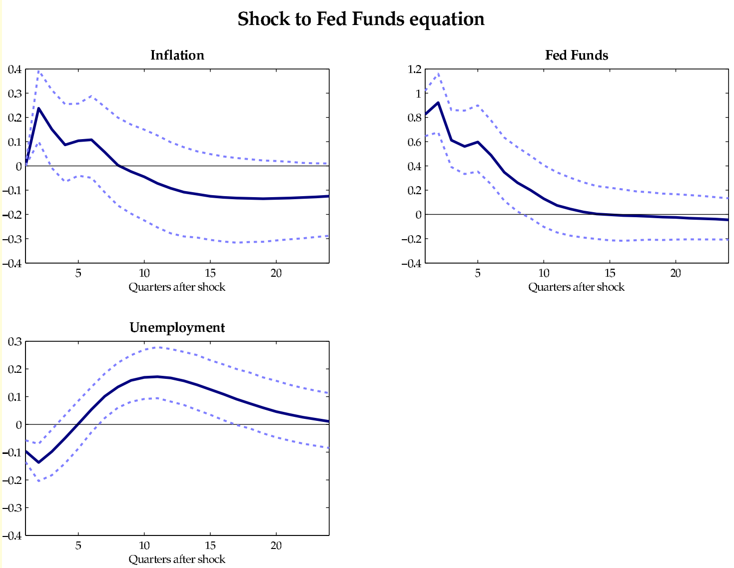
$$Y_t = \begin{bmatrix} \text{Inflation}_t \\ \text{Fed fund rate}_t \\ \text{Unemployement}_t \end{bmatrix}$$

    Economic content: FED does not observe (or does not respond to) current unemployment when setting the interest rate.

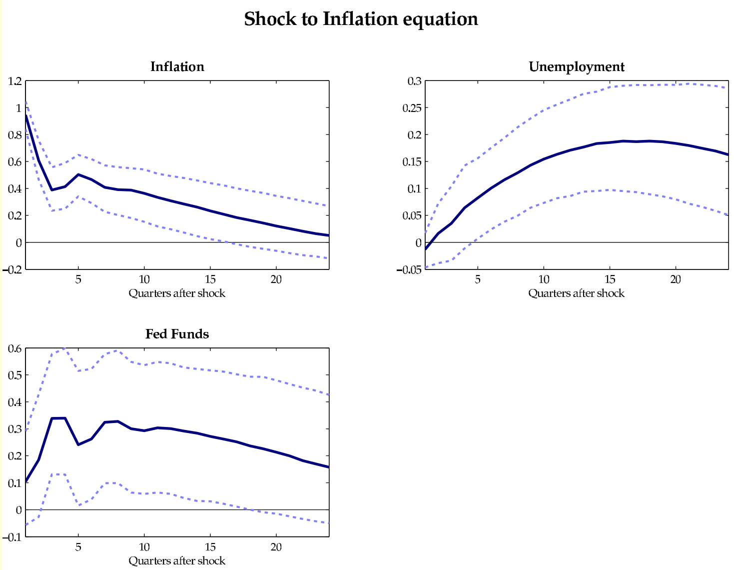
# Impact to a monetary policy shock: ordering 1



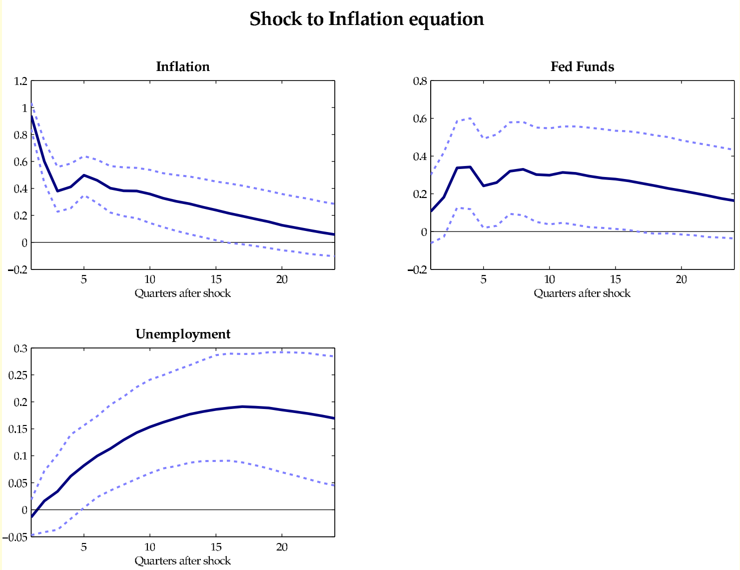
# Impact to a monetary policy shock: ordering 2



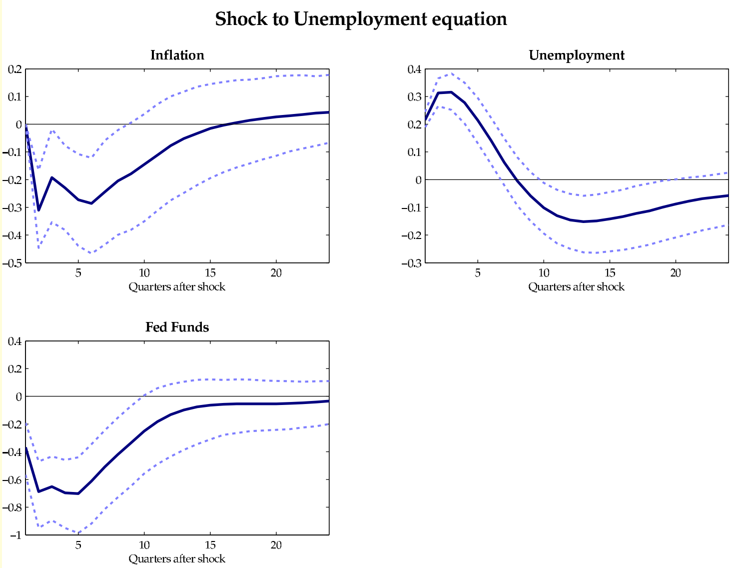
# Impact to an inflation shock: ordering 1



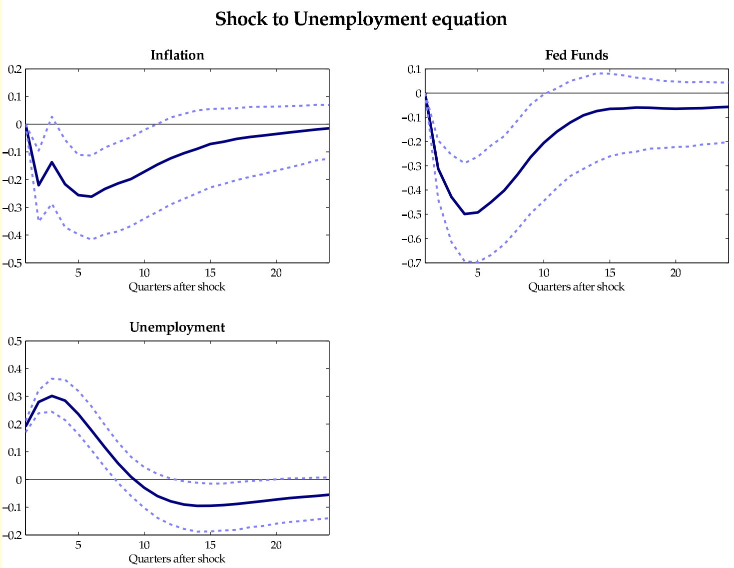
# Impact to an inflation shock: ordering 2



# Impact to an unemployment shock: ordering 1



# Impact to an unemployment shock: ordering 2



# Blanchard and Perotti (2002): non-recursively identified models

- Short-run restrictions on the relevant matrices are non-recursive.
- Blanchard and Perotti: **estimate the impact of a fiscal policy shock on output** (expenditure and tax multipliers).
- Quarterly model with $Y_t = (\log T_t, \log G_t, \log X_t)$ where $T_t$ denotes taxes, $G_t$ is government spending, and $X_t$ is GDP.
  - All variables $I(1)$
  - Data are either detrended or cointegration is assumed between $\log T_t$ and $\log G_t$ (cointegration relation is $\log T_t - \log G_t$)
- Reduced form VAR:

$$Y_t = D(L)Y_{t-1} + v_t$$

$v_t = (v_t^T, v_t^G, v_t^X)$ are the reduced form residuals and $\varepsilon_t = (\varepsilon_t^T, \varepsilon_t^G, \varepsilon_t^X)$ are the structural (orthogonal) shocks.

# Blanchard and Perotti (II)

- Reduced form shocks are a linear combination of three components:
    1. **Automatic response** of taxes and government spending to changes in output.
    2. **Systematic discretionary response** of $T$ and $G$ to news in macro variables. This is a policymaker's choice. For example, increase $G_t$ during recessions.
    3. **Exogenous shocks** to taxes and government spending. These are the "structural" fiscal shocks that we want to identify.

- Blanchard and Perotti model the reduced form residuals as

$$
\begin{aligned}
v_t^T &= a_1 v_t^X + a_2 \varepsilon_t^G + \varepsilon_t^T \\
v_t^G &= b_1 v_t^X + b_2 \varepsilon_t^T + \varepsilon_t^G \\
v_t^X &= c_1 v_t^T + c_2 v_t^G + \varepsilon_t^X
\end{aligned}
$$

# Blanchard and Perotti (III)

- The coefficients $a_1$ and $b_1$ capture simultaneously two different effects of economic activity on taxes and spending:

  1. The automatic effects of economic activity on taxes and spending under existing fiscal policy rules
  2. Any discretionary adjustment made to fiscal policy in response to unexpected events within the quarter.

- Blanchard and Perotti rely on institutional knowledge about tax, transfer, and spending programs to estimate $a_1$ and $b_1$.

- **Main identifying assumption:**
  - The use of quarterly data eliminates the second channel: it takes more than a quarter for discretionary fiscal policy to respond to unexpected shocks.

- Thus, $a_1$ and $b_1$ only reflect the automatic response of taxes and spending to output shocks.

# Blanchard and Perotti (IV)

- With this assumption, $a_1$ and $b_1$ can be computed using the (average) elasticities of government purchases and taxes to output.
  1. Blanchard and Perotti could not find evidence about any automatic feedback from economic activity to government spending, so they set $b_1 = 0$.
  2. To find $a_1$, they estimate the elasticity of taxes to output using external information (see the paper for details). They find $a_1 = 2.08$.

- Given $a_1$ and $b_1$, BP construct cyclically adjusted reduced form tax and spending residuals

$$\eta_t^T = v_t^T - a_1 v_t^X = a_2 \varepsilon_t^G + \varepsilon_t^T$$
$$\eta_t^G = v_t^G - b_1 v_t^X = b_2 \varepsilon_t^T + \varepsilon_t^G.$$

- Although $\eta_t^T$ and $\eta_t^G$ may be correlated with each other, they are uncorrelated with $\varepsilon_t^X$.

- Thus, we can use $\eta_t^T$ and $\eta_t^G$ as instruments to estimate $c_1$ and $c_2$ in a regression of $v_t^X$ on $v_t^T$ and $v_t^G$.

# Blanchard and Perotti (V)

- Two additional parameters to estimate: $a_2$ and $b_2$.
- Blanchard and Perotti find no convincing arguments to identify $a_2$ and $b_2$ from the correlation between $\eta_t^T$ and $\eta_t^G$:
  - If the government increases taxes and spending at the same time, are taxes responding to spending (i.e. $a_2 \neq 0$ and $b_2 = 0$) or is spending responding to the higher taxes (i.e. $a_2 = 0$ and $b_2 \neq 0$?
- They consider both possibilities
  1. Set $a_2 = 0$ and use $\eta_t^T$ as an instrument for $\varepsilon_t^T$; run a regression of $\eta_t^G$ on $\eta_t^T$ to estimate $b_2$.
  2. Set $b_2 = 0$ and use $\eta_t^G$ as an instrument for $\varepsilon_t^G$; run a regression of $\eta_t^T$ on $\eta_t^G$ to estimate $a_2$.
- Once you do that, we have estimates of $a_1$, $a_2$, $b_1$, $b_2$, $c_1$ and $c_2$.

# Blanchard and Perotti (VI)

- The relation between structural and reduced form shocks satisfies

$$\begin{bmatrix} 1 & 0 & -a_1 \\ 0 & 1 & -b_1 \\ -c_1 & -c_2 & 1 \end{bmatrix} \begin{bmatrix} v_t^T \\ v_t^G \\ v_t^X \end{bmatrix} = \begin{bmatrix} 1 & a_2 & 0 \\ b_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_t^T \\ \varepsilon_t^G \\ \varepsilon_t^X \end{bmatrix}$$

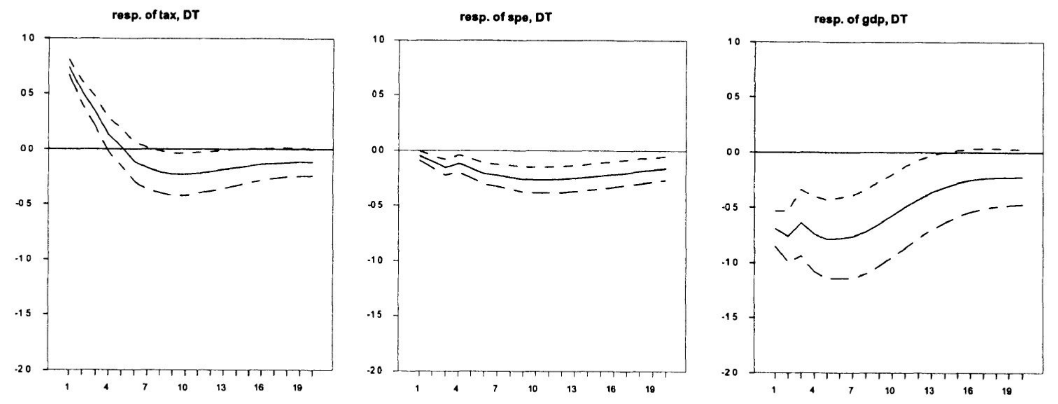
- In terms of our previous notation,

$$S = \begin{bmatrix} 1 & 0 & -a_1 \\ 0 & 1 & -b_1 \\ -c_1 & -c_2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & a_2 & 0 \\ b_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

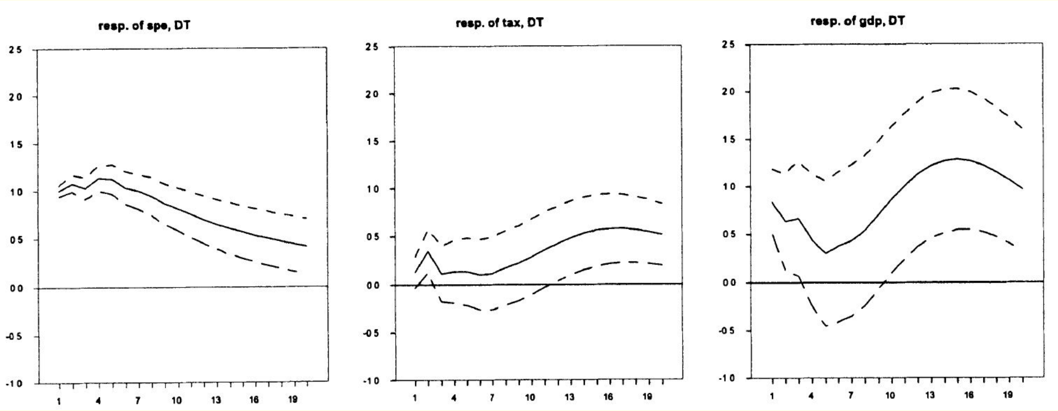
or

$$B_0 = \begin{bmatrix} 1 & a_2 & 0 \\ b_2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & -a_1 \\ 0 & 1 & -b_1 \\ -c_1 & -c_2 & 1 \end{bmatrix}.$$

# Blanchard and Perotti (VII): Response to a tax shock



# Blanchard and Perotti (VIII): Response to an spending



# Identification by long-run restrictions (Blanchard and Quah, 1989)

- Many models have strong implications about the long-run.
- Example: VAR with output and unemployment and two shocks:
  - Supply shock
  - Demand shock
- **Identifying assumption:** Demand shocks have no effects on the long-run level of output. In other words, demand shocks are transitory.
- Economic idea: long-run supply curve is vertical and depends only on technological factors.
- Identification based on long-run restrictions are rooted on such kind of theoretical considerations.

# Identification by long-run restrictions (II)

**Example:** VAR with log output growth ($\Delta y_t$) and unemployment ($u_t$)

$$Y_t = D(L)Y_{t-1} + v_t;$$

$$Ev_t v_t' = \Omega$$

$$v_t = S\varepsilon_t$$

$$Y_t = \begin{bmatrix} \Delta y_t \\ u_t \end{bmatrix};\ S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix};\ \varepsilon_t = \begin{bmatrix} \varepsilon_t^s \\ \varepsilon_t^d \end{bmatrix}.$$

We label $\varepsilon_t^s$ a “supply shock”.

- Estimate $D_1, D_2, \ldots, D_p$, and $\Omega$ by OLS.
- Identification problem: $SS' = \Omega$ gives 3 equations but there are 4 parameters in $S$.
- Identifying assumption: $\varepsilon_{2t}$ **has no long-run impact on the <u>level</u> of** $y_t$.
- Long-run restriction plus a “sign” convention gives the additional restriction to identify $S$.

# Identification by long-run restrictions (III)

- How do we compute the "long-run" effect of a shock on a given variable, say $z_t$?

- Suppose that the $n \times 1$ vector $Y_t$ can be partitioned as

$$Y_t = \begin{bmatrix} \Delta z_t \\ x_t \end{bmatrix}$$

  where $\Delta z_t$ is the first element written as a difference (typically log-difference) and $x_t$ is an $(n-1) \times 1$ vector with the other variables (which could also be in differences).

- Write the VAR in its moving average representation

$$Y_t = (I - D(L))^{-1} v_t = C(L) v_t.$$

- The impulse response to a (non-structural ) shock $v_t$ at horizon $j$ is

$$E_t Y_{t+j} - E_{t-1} Y_{t+j} = C_j v_t.$$

# Identification by long-run restrictions (IV)

- The impact on the first element of $Y_t$, $\Delta z_t$, at horizon $j$ can be obtained as

$$E_t \Delta z_{t+j} - E_{t-1} \Delta z_{t+j} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} C_j v_t$$

$\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$ is a row vector with a 1 in the first element and zeros elsewhere.

- But we want the impact on the **level of $z_t$** (not on the difference $\Delta z_t$). This is given by

$$E_t z_{t+j} - E_{t-1} z_{t+j} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \left( \sum_{h=0}^{j} C_h \right) v_t.$$

- The **long-run** impact impact of the shocks $v_t$ on the level of $z_t$ is obtained by taking the limit

$$\begin{aligned} \lim_{j \to \infty} \left[ E_t z_{t+j} - E_{t-1} z_{t+j} \right] &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \left( \sum_{h=0}^{\infty} C_h \right) v_t. \\ &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} C(1) v_t. \end{aligned}$$

# Identification by long-run restrictions (V)

- In terms of the structural shocks,

$$\lim_{j\to\infty}\left[E_t z_{t+j} - E_{t-1} z_{t+j}\right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} C(1) S \varepsilon_t.$$
$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} (I - D(1))^{-1} S \varepsilon_t,$$

  where the second line uses the relation between the VAR and MA representations,

$$C(1) = (I - D(1))^{-1} \equiv \bar{C}.$$

- Long-run effect is the sum of the impulse responses.

**Identifying assumption (version 1):**

- Only one structural shock (say $\varepsilon_{1t}$) has a long-run impact on the **level** of the variable $z_t$ (the first variable in the system).

- Long-run identifying assumption imposes the constraint

$$\left[\bar{C}S\right]_{11} \neq 0 \text{ and } \left[\bar{C}S\right]_{12} = \left[\bar{C}S\right]_{13} = \ldots = \left[\bar{C}S\right]_{1n} = 0,$$

  where $\left[\bar{C}S\right]_{ij}$ denotes element $(i, j)$ of $\bar{C}S$.

# Identification by long-run restrictions (VI)

Blanchard and Quah

$$\begin{bmatrix} \Delta y_t \\ u_t \end{bmatrix} = \begin{bmatrix} C_{11}(L) & C_{12}(L) \\ C_{21}(L) & C_{22}(L) \end{bmatrix} S \begin{bmatrix} \varepsilon_t^s \\ \varepsilon_t^d \end{bmatrix}$$

where $C(L) = (I - D(L))^{-1}$, $\varepsilon_t^s$ is a structural supply shock and $\varepsilon_t^d$ is a structural demand shock. The elements $C_{ij}(L)$ are polynominals in the lag-operator.

- **Identifying assumption:** only supply shocks have a long-run effect on unemployment.
- This imposes the constraint that element $(1, 2)$ of the matrix $C(1)S = \bar{C}S$ is zero. But

  $$\begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} \\ \bar{C}_{21} & \bar{C}_{22} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} \times & 0 \\ \times & \times \end{bmatrix}.$$

  The long-run restriction is

  $$\bar{C}_{11} S_{12} + \bar{C}_{12} S_{22} = 0.$$

- Sign convention: positive supply shock increases GDP $\bar{C}_{11} S_{11} + \bar{C}_{12} S_{21} > 0$.

# Identification by long-run restrictions (VII): Finding $S$ the hard way

- Long-run restriction provides one constraint:

$$\bar{C}_{11}S_{12} + \bar{C}_{12}S_{22} = 0.$$

- Covariance matrix of reduced form residuals provide three additional constraints : $SS' = \Omega$

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{bmatrix} = \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{12} & \omega_{22} \end{bmatrix}$$

.

- Given estimates of $\bar{C}_{11}$, $\bar{C}_{12}$, $\omega_{11}$, $\omega_{12}$, and $\omega_{22}$, we need to solve the following system of 4 equations in the 4 unknowns $S_{11}$, $S_{12}$, $S_{21}$, and $S_{22}$:

$$\begin{aligned} \bar{C}_{11}S_{12} + \bar{C}_{12}S_{22} =& 0 \\ S_{11}^2 + S_{12}^2 =& \omega_{11} \\ S_{11}S_{21} + S_{12}S_{22} =& \omega_{12} \\ S_{21}^2 + S_{22}^2 =& \omega_{22}. \end{aligned}$$

# Identification by long-run restrictions (VII): Finding $S$ the smart way

Let

$$F \equiv (I - D(1))^{-1} S = \bar{C} S$$

Note that

$$\begin{aligned} FF' &= \bar{C} S S' \bar{C}' \\ &= \bar{C} \Omega \bar{C}' \\ &\equiv W. \end{aligned}$$

Importantly, while $F$ is still unknown, the $2 \times 2$ (symmetric) matrix $W$ can be estimated from data (because $\bar{C}$ and $\Omega$ can be estimated).

- **Long-run restriction:** only technology shocks affect output:

$$F = \begin{bmatrix} f_{11} & 0 \\ f_{21} & f_{22} \end{bmatrix}$$

so

$$FF' = \begin{bmatrix} f_{11} & 0 \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} f_{11} & f_{21} \\ 0 & f_{22} \end{bmatrix} = \begin{bmatrix} f_{11}^2 & f_{11} f_{21} \\ f_{21} f_{11} & f_{21}^2 + f_{22}^2 \end{bmatrix} = \begin{bmatrix} W_{11} & W_{21} \\ W_{21} & W_{22} \end{bmatrix}.$$

# Identification by long-run restrictions (VIII): Finding $S$ the smart way

- **Sign restriction:** technology shocks drive an increase in output: $f_{11} > 0$.

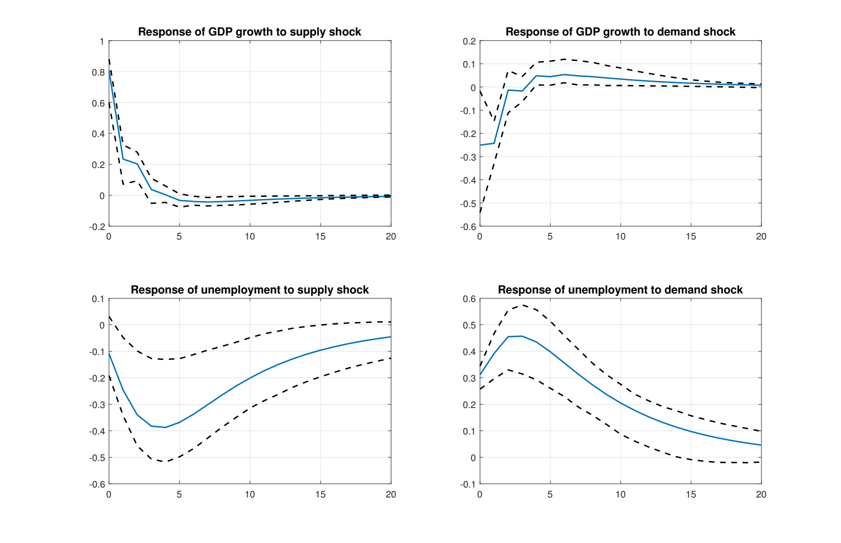
- Then, we can compute the elements of $F$ by solving

$$\begin{aligned} f_{11} &= \sqrt{W_{11}} \\ f_{21} &= W_{21} / f_{11} \\ f_{22} &= \sqrt{W_{22} - f_{21}^2}. \end{aligned}$$

- Once we have $F$, we compute $S$ using

$$S = \bar{C}^{-1} F.$$

# Blanchard and Quah example: effect of supply and demand shocks



# Limitations of long-run restrictions

- Identification by long-run restrictions requires an accurate estimate of the impulse response at the infinite horizon.
    - Difficult to estimate long-run behavior of a time series from finite (and persistent) data.
- Numerical estimates of VAR models identified by long-run restrictions are identified up to a sign convention.
    - This matters only if we are interested in estimating the sign of the response
- Conclusions often depend a lot on whether the second variable is entered in levels or as a difference.

# Identification by sign restrictions

- Structural shocks are (set) identified by restricting the sign of the responses of selected variables to structural shocks.
- Identification by sign restrictions requires each identified shock to be associated with a **unique sign pattern of the impulse responses**.
- Advantage of sign restrictions: we have robust theoretical predictions about the sign of certain shocks.
- Examples:
  - Monetary policy (Uhlig, 2005): contractionary monetary policy shock does not lead to (i) a decrease in the federal funds rate; (ii) an increase in prices; (iii) an increase in non-borrowed reserves for a number of periods.
  - Fiscal shocks (Canova and Pappa, 2007; Mountford and Uhlig, 2009; Pappa, 2009.
  - Technology shocks (Dedola and Neri, 2007).
  - Shocks in open economies (Canova and De Nicolo, 2002; Scholl and Uhlig, 2008), in oil markets (Baumeister and Peersman, 2010; Kilian and Murphy, 2011), etc.

# Identification by sign restrictions

## Steps to identify a SVAR using sign restrictions:

1. Determine a set of sign restrictions that the impulse responses of a VAR has to satisfy. This comes from theory.
2. Choose a matrix $S$ so that reduced form shocks are $v_t = S\varepsilon_t$ (more on this below).
3. Compute impulse responses using the matrix $S$ and check whether sign restrictions are satisfied. If they are, save $S$, otherwise discard it.
4. Repeat steps 2 and 3 a large number of times. Save each $S$ that satisfies the sign restrictions (and the corresponding impulse responses).
5. The set of all $S$ and impulse responses that satisfy the sign restrictions represent the set of admissible structural VARs.
6. Report some summary statistic or some particular model from the set of admissible structural VARs.

**Question:** what model to report? How do we compute confidence bands of IRs? Literature leaning towards Bayesian interpretation.

# Identification by sign restrictions: interpretation issues

- Fundamental problem in interpreting SVARs identified by sign restrictions:
    - There is not a unique point estimate of the structural impulse response function.
    - Follows from the observation that (usually) many structural VARs satisfy the sign restrictions.
- Sign restrictions identify sets of models. Without further assumptions, it is not clear which of these models is more likely or how to select one of many.
- Fry and Pagan (2011) is a critique of the literature along these lines.

# Solutions to the interpretation problem

- Early approach (Faust, 1998): focus on the admissible model that is most favorable to the hypothesis of interest (useful to "reject" models)

- Report vector of medians of the admissible IRFs or some measure of the dispersion across the admissible IRFs.

  - Shortcoming: vector of median responses does not correspond to the response of any admissible model. Thus, median impulse responses lack a structural interpretation

- **Bayesian approach.** Impose a prior on the structural parameters and obtain posteriors using the sign restrictions. Then we can report the most likely model as the posterior mode of the joint distribution of admissible models and/or "credible intervals". See for example Inoue and Kilian (2011), Baumeister and Hamilton (2015), Arias, Rubio-Ramirez and Waggoner (2016).

- Disregard point estimates and report confidence sets. See Granziera, Moon, and Schorfheide (2018) along classical inference.

# A device to generate observationally equivalent SVARs

- How do we “draw” different $S$ matrices to implement the algorithm?

**Definition:** Consider a SVAR $B_0 Y_t = B(L) Y_{t-1} + \varepsilon_t$ where $\varepsilon_t \sim IID(0, I)$. Two sets of structural parameters $\{B_i\}_{i=0}^{\infty}$ and $\{\tilde{B}_i\}_{i=0}^{\infty}$ are observationally equivalent if they generate the same reduced form VAR:

$$Y_t = \sum_{i=1}^{\infty} D_i Y_{t-i} + v_t; \text{ where } E(v_t v_t') = \Omega.$$

We then have

**Theorem 1:** (Rubio-Ramirez, Waggoner, and Zha): Two structural parameters $\{B_i\}_{i=0}^{\infty}$ and $\{\tilde{B}_i\}_{i=0}^{\infty}$ are observationally equivalent if and only if there exists an orthogonal matrix $P$ ($P'P = I$) such that $\tilde{B}_i = PB_i$ for $i = 0, 1, 2, ..., \infty$. In particular, using $S = B_0^{-1}$ and $\tilde{S} = \tilde{B}_0^{-1}$, the identification matrices satisfy $S = \tilde{S}P$

# Identification by sign restrictions

- Theorem 1 shows that any structural model can be obtained from an arbitrary identified SVAR by an appropriate rotation. This suggests the following algorithm:

**Algorithm 1:**

1. Obtain an arbitrary orthogonalization of the covariance matrix of the reduced form VAR. For example, use a Cholesky decomposition $\Omega = \tilde{S}\tilde{S}'$. The matrix $\tilde{S}$ is lower triangular and does not have any economic interpretation.
2. Draw an $n \times n$ matrix $L$ of independent normal $N(0, 1)$ random variables. Obtain the $QR$ decomposition of $L$ such that

$$L = QR \text{ and } QQ' = I.$$

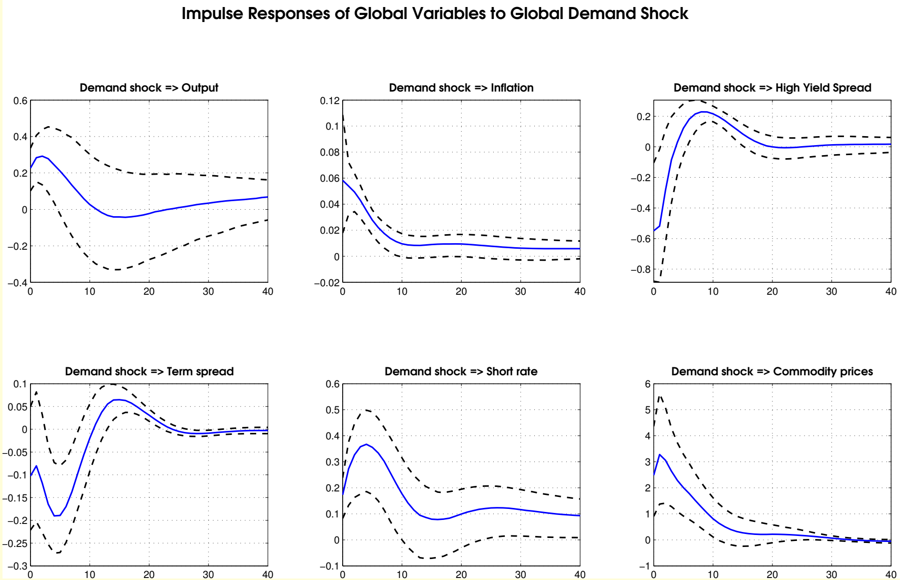
3. Let $P = Q'$. Compute impulse response functions using the orthogonalization $S = \tilde{S}P$. If all impulse responses satisfy the sign restrictions, keep $P$. Otherwise, discard $P$.
4. Repeat steps 2 and 3 a large number of times, saving each $P$ that satisfies the restrictions and the corresponding impulse responses.

# Example: monetary, demand, and supply shocks using sign restrictions

- We want to identify monetary, demand, and supply shocks using sign restrictions.
- US data: GDP, inflation, credit premium, term premium, commodity prices, short term interest rate.
- Sign restrictions:

| | Output | Inflation | Term premium | Credit premium | Comm. prices | Short rate |
|---|---|---|---|---|---|---|
| Supply shock | + | - | ? | ? | ? | ? |
| Demand shock | + | + | ? | ? | + | + |
| Monetary shock | + | + | ? | ? | + | - |

# Example: monetary, demand, and supply shocks using sign restrictions



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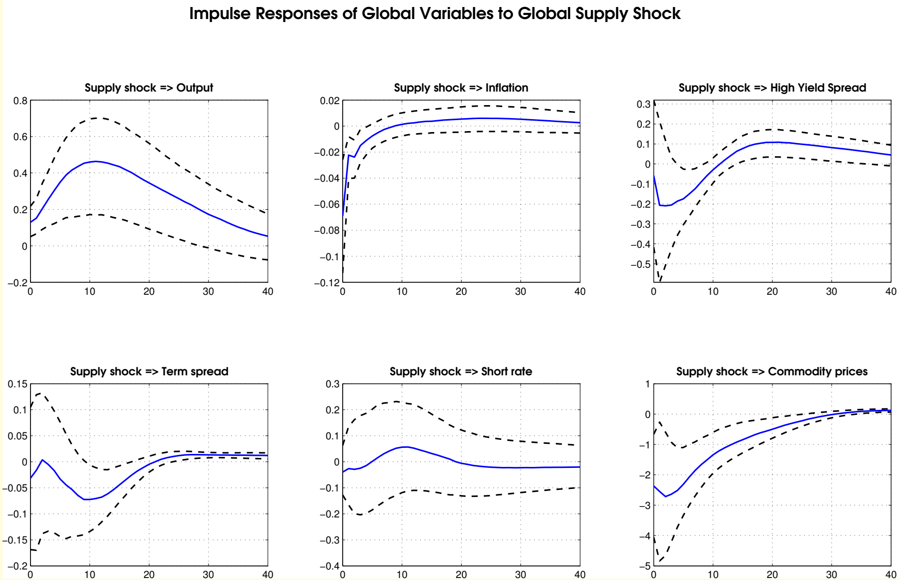
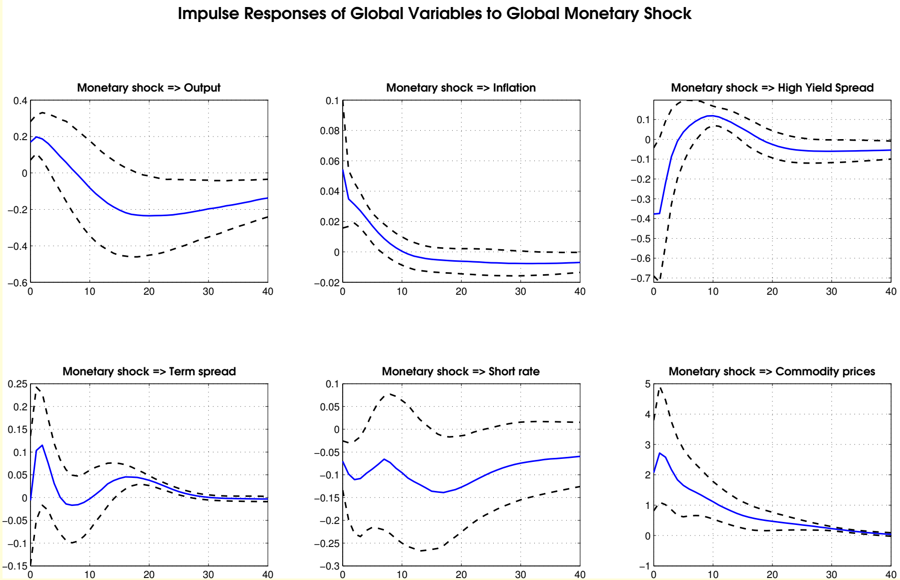


# Example: monetary, demand, and supply shocks using sign restrictions



# Global identification and efficient algorithms

- Rubio-Ramirez, Waggoner, and Zha (2010).

- We often simultaneously impose short-run and long-run restrictions. The traditional way of doing this is cumbersome.

- RWZ provide a simple check for global identification and a simple algorithm for imposing short and long-run restrictions simultaneously.

- Consider the reduced form VAR

$$Y_t = D_1 Y_{t-1} + D_2 Y_{t-2} + \dots + D_p Y_{t-p} + v_t$$

  - $Y_t$ is an $n \times 1$ vector and $v_t$ is the vector of reduced form shocks that satisfies

$$v_t = S\varepsilon_t;\ E\left[\varepsilon_t \varepsilon_t'\right] = I;\ SS' = \Omega;$$

    $\Omega$ is the covariance matrix of the reduced form shocks, $\varepsilon_t$ is the vector of structural shocks, and the matrix $S$ relates reduced form with structural shocks.

# Global identification and efficient algorithms (II)

- Short-run restrictions are zero restrictions imposed on $S$. Define

$$L_0 \equiv S.$$

- Long-run restrictions are zeros (or other constraints) imposed on $IR_\infty =$. Define

$$L_\infty \equiv (I - \mathbf{D})^{-1} S$$

  where $\mathbf{D} \equiv \sum_{j=1}^{p} D_j$.

- We now define notation that captures both types of restrictions at the same time.

# Global identification and efficient algorithms (III)

- Identifying restrictions on short-run and long-run impulse responses can be imposed on the following $2n \times n$ matrix

$$
f(S, \mathbf{D}) = \begin{bmatrix} L_0 \\ L_\infty \end{bmatrix} = \begin{bmatrix} S \\ (I - \mathbf{D})^{-1} S \end{bmatrix} = \begin{array}{c} \\ \text{Var 1} \\ \text{Var 2} \\ \vdots \\ \text{Var n} \\ \text{Var 1} \\ \text{Var 2} \\ \vdots \\ \text{Var n} \end{array} \begin{array}{c} \begin{array}{cccc} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n \end{array} \\ \begin{bmatrix} \times & \times & \cdots & \times \\ \times & \times & \cdots & \times \\ \vdots & \vdots & & \vdots \\ \times & \times & \times & \times \\ \times & \times & \cdots & \times \\ \times & \times & \cdots & \times \\ \vdots & \vdots & & \vdots \\ \times & \times & \cdots & \times \end{bmatrix} \end{array}
$$

- Labels in rows are variables (Var i = $Y_i$); labels in columns represent structural shocks.
- Short run and long-run restrictions are zeros in the matrix $f(S, \mathbf{D})$.
- Rows 1 through $n$ are the short run restrictions. Rows $n+1$ through $2n$ are the long-run restrictions.

# Global identification and efficient algorithms (IV)

- Restrictions on the impact of the structural shock $\varepsilon_{it}$ are imposed using a $n \times 2n$ matrix $Q_i$ (which contains zeros and ones) such that

$$\underbrace{Q_i}_{n\times 2n}\underbrace{f(S,\mathbf{D})}_{2n\times n}\underbrace{e_i}_{n\times 1} = \underbrace{\mathbf{0}}_{n\times 1}$$

  where $e_i$ is a column vector with a 1 in the $i^{th}$ element and zeros elsewhere. (We will see an example in a few slides).

- Because there are $n$ shocks, global identification requires $n$ matrices $Q_i$.

# Global identification and efficient algorithms (V)

Example (Peersman and Smets, 2003): VAR with

$$\begin{bmatrix} \text{GDP growth} \\ \text{Inflation} \\ \text{Interest rate} \\ \text{Depr Exc. Rate} \end{bmatrix} = \begin{bmatrix} \Delta \log Y_t \\ \Delta \log P_t \\ R_t \\ \Delta \log Ex_t \end{bmatrix}; \begin{bmatrix} \text{Ex. rate shock} \\ \text{Mon. pol shock} \\ \text{Demand shock} \\ \text{Supply shock} \end{bmatrix} = \begin{bmatrix} \varepsilon_t^{Ex} \\ \varepsilon_t^{R} \\ \varepsilon_t^{D} \\ \varepsilon_t^{S} \end{bmatrix}$$

**Short run restrictions:**

1. Monetary policy shocks have no contemporaneous effect on output.
2. Exchange rate shocks have no contemporaneous effect on output.
3. Exchange rate shocks have no contemporaneous effect on the interest rate.

**Long run restrictions:**

1. Demand shocks have no long-run effect on output.
2. Monetary policy shocks have no long-run effect on output.
3. Exchange rate shocks have no long-run effect on output.

# Global identification and efficient algorithms (VI)

The matrix $f(S, \mathbf{D})$ satisfies

$$
f(S,\mathbf{D}) = \begin{bmatrix} L_0 \\ L_\infty \end{bmatrix} = \begin{array}{c} \\ \Delta \log Y \\ \Delta \log P \\ R \\ \Delta \log Ex \\ \Delta \log Y \\ \Delta \log P \\ R \\ \Delta \log Ex \end{array} \begin{array}{c} \begin{array}{cccc} \varepsilon^{Ex} & \varepsilon^{R} & \varepsilon^{D} & \varepsilon^{S} \end{array} \\ \begin{bmatrix} 0 & 0 & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & 0 & 0 & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \end{array}
$$

- Let's write these constraints as $Q_i f(S, \mathbf{D}) e_i = 0_{n\times 1}$.

# Global identification and efficient algorithms (VII): constraints on $\varepsilon^{Ex}$

$$Q_1 f(S,\mathbf{D}) e_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \\ f_{51} & f_{52} & f_{53} & f_{54} \\ f_{61} & f_{62} & f_{63} & f_{64} \\ f_{71} & f_{72} & f_{73} & f_{74} \\ f_{81} & f_{82} & f_{83} & f_{84} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{41} \\ f_{51} \\ f_{61} \\ f_{71} \\ f_{81} \end{bmatrix} = \begin{bmatrix} f_{11} \\ f_{31} \\ f_{51} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# Global identification and efficient algorithms (VIII): constraints on $\varepsilon^R$

$$Q_2 f(S, \mathbf{D}) e_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \\ f_{51} & f_{52} & f_{53} & f_{54} \\ f_{61} & f_{62} & f_{63} & f_{64} \\ f_{71} & f_{72} & f_{73} & f_{74} \\ f_{81} & f_{82} & f_{83} & f_{84} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} f_{12} \\ f_{22} \\ f_{32} \\ f_{42} \\ f_{52} \\ f_{62} \\ f_{72} \\ f_{82} \end{bmatrix} = \begin{bmatrix} f_{12} \\ f_{52} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# Global identification and efficient algorithms (IX): constraints on $\varepsilon^D$

$$Q_3 f(S, \mathbf{D}) e_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \\ f_{51} & f_{52} & f_{53} & f_{54} \\ f_{61} & f_{62} & f_{63} & f_{64} \\ f_{71} & f_{72} & f_{73} & f_{74} \\ f_{81} & f_{82} & f_{83} & f_{84} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} f_{13} \\ f_{23} \\ f_{33} \\ f_{43} \\ f_{53} \\ f_{63} \\ f_{73} \\ f_{83} \end{bmatrix} = \begin{bmatrix} f_{53} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# Global identification and efficient algorithms (X)

In this example the matrices $Q_i$ are

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \; Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \; Q_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Please note:

$$rank(Q_1) = 3 \geq rank(Q_2) = 2 \geq rank(Q_3) = 1 \geq rank(Q_4) = 0.$$

# Global identification and efficient algorithms (XI)

**Important:** Let $q_i = rank(Q_i)$. Because the ordering of the shocks is arbitrary, WLOG we can always rearrange the shocks so that the restriction matrices $Q_i$ satisfy

$$q_1 \geq q_2 \geq q_3 \geq q_4.$$

- So far this is just notation. The advantage of this is the following:

**Theorem 2:** Consider a SVAR with restrictions represented by the matrices $Q_i$, for $i = 1, 2, \ldots, n$. The SVAR is exactly identified **if and only if** the total number of restrictions is equal to $n(n-1)/2$ and the rank of the matrices $Q_i$ satisfy $q_i = n - i$ for $i = 1, 2, \ldots, n$.

- In the previous example, the SVAR is exactly identified:
  - Order condition is satisfied: there are $6 = n(n-1)/2 = 4 \times 3/2$ restrictions.
  - Rank condition is satisfied: $q_1 = 4 - 1 = 3$, $q_2 = 4 - 2 = 2$, $q_3 = 4 - 3 = 1$, and $q_4 = 4 - 4 = 0$.

# Efficient algorithm for exactly identified models (RWZ)

This algorithm is useful for imposing short- and long-run restrictions simultaneously

1. Assume that the model is exactly identified.
2. Let $f(S, \mathbf{D})$ be the matrix with the short-run and long-run impact matrices, let $Q_1, Q_2, ..., Q_n$ represent the identifying restrictions, and assume that we have ordered the shocks so that $q_i = n - i$ for $i = 1, 2, .., n$.
3. Perform a Cholesky decomposition of the covariance matrix of the reduced form residuals $V = chol(\Omega)$, where $VV' = \Omega$.
4. Construct initial short-run and long-run response matrices: $L_0^* = V$ and $L_\infty^* = (I - \mathbf{D})^{-1} V$, and let
$$F^* = \begin{bmatrix} L_0^* \\ L_\infty^* \end{bmatrix} = \begin{bmatrix} V \\ (I - \mathbf{D})^{-1} V \end{bmatrix}$$
5. By Theorem 1 there is a rotation matrix $P$ ($PP' = I_n$) such that $L_0 = L_0^* P$ and $L_\infty = L_\infty^* P$ satisfies the identifying restrictions (in point 2 above). Equivalently, find a matrix $F$ such that
$$F = F^* P.$$

This step is performed with the following algorithm:

# Efficient algorithm for exactly identified models (II)

## Algorithm 2 (RWZ):

1. Step 1: Set j=1.
2. Step 2: for j=1,2,...,n, form the matrix

$$\tilde{Q}_j = \begin{bmatrix} Q_j F^* \\ \boldsymbol{p}_1' \\ \vdots \\ \boldsymbol{p}_{j-1}' \end{bmatrix}$$

   when j=1, $\tilde{Q}_1 = Q_1 F^*$. Note that $\tilde{Q}_j$ has n columns (the number of rows increases with j).
3. Step 3: Since $rank(Q_j) = n - j$, then $rank(\tilde{Q}_j) < n$. Hence, there exists an $n \times 1$ unit-length vector $\boldsymbol{p}_j$ such that $\tilde{Q}_j \boldsymbol{p}_j = \mathbf{0}$. . You can find this vector using the LU decomposition.
4. Step 4: If j=n, stop; otherwise set j=j+1 and go to Step 2.
5. Step 5:The required identification matrix is $S = VP$, where

$$P = [\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n].$$

# Confidence bands for impulse responses (or other statistics)

- Asymptotic confidence bands have poor small sample performance.

- Consider the var

$$D(L)Y_t = v_t \tag{6}$$

$$D(L) = I - D_1 L - D_2 L^2 - \dots - D_p L^p$$

  where $E(v_t v_t') = \Omega$; $t = -p+1, \dots, 0, 1, 2, \dots, T$.

- Denote the impulse response function by

$$\theta(D(L), \Omega; S)$$

  where we have imposed appropriate identifying restrictions through the matrix $S$: $\Omega = SS'$.

- Initial conditions $Y_0$, $Y_{-1}$, ..., $Y_{-p+1}$ are taken as given.

# Confidence bands for impulse responses (naive bootstrap)

**Algorithm 4:** (Runkle, 1987)

1. Estimate $\hat{D}(L)$ and $\hat{\Omega}$ by OLS and recover the fitted residuals $\hat{v}_t = \hat{D}(L)Y_t$ for $t = 1, 2, ..., T$.
2. Construct the point estimate of the impulse response $\hat{\theta}(\hat{D}(L), \hat{\Omega}; \hat{S})$.
3. Choose $M$ large. For $m = 1, 2, \ldots, M$:
   3.1 Draw a history $v^*_{t,m}$ of length $T$ by resampling OLS residuals $\hat{v}_t$ with replacement.
   3.2 Construct artificial data according to $\hat{D}(L)Y^*_{t,m} = v^*_{t,m}$.
   3.3 Estimate $D^*_m(L)$ using the artificial data $Y^*_{t,m}$ and construct the impulse response $\theta^*_m = \theta(D^*_m(L), \hat{\Omega}; S^*)$.
4. Report percentiles $\alpha$ and $1-\alpha$ (e.g. 16-84% or 2.5-97.5%) or other moments of the empirical distribution of impulse responses $\{\theta^*_m\}_{m=1}^M$

# Problems with the naive bootstrap

- Autoregressive models are biased in small samples. Therefore, two problems:
  1. Impulse responses (nonlinear function of VAR coefficients) are also biased.
  2. Naive bootstrap amounts to assuming that OLS estimates (and therefore the IRFs) are unbiased.

- It's like a squared bias problem:
  1. First bias when computing the initial IRF.
  2. Second bias. When bootstrapping, we pretend that OLS estimates $\hat{D}(L)$ is unbiased. But $\hat{D}(L)$ is biased, and fitting a VAR to a biased data generating process will give biased estimates of the biased estimate $\hat{D}(L)$.

- Confidence bands using the naive approach will be biased.

- Moreover, the confidence band may not include de IRF point estimate!

# Bootstrap con "trampa"

- Use the naive bootstrap to compute the empirical distribution of impulse responses $\{\theta_m^*\}_{m=1}^M$.

- Compute the standard deviation of the bootstrapped impulse responses:

$$\sigma_\theta = \sqrt{\frac{1}{M}\sum_{m=1}^{M}\left(\theta_m^* - \bar{\theta}^*\right)},$$

  where $\bar{\theta}^* = \frac{1}{M}\sum_{m=1}^M \theta_m^*$ is the average over the $M$ bootstrapped samples.

- Report

$$\hat{\theta} \pm 2 \times \sigma_\theta$$

- This procedure guarantees a symmetric confidence band around the point estimate.

# Confidence bands for impulse responses (bootstrap-after-bootstrap)

**Algorithm 5:** (Kilian, 1998)

1. Estimate $\hat{D}(L)$ and $\hat{\Omega}$ by OLS and recover the fitted residuals $\hat{v}_t$.
2. Construct bias correction for $\hat{D}(L)$:
   2.1 For $m = 1, 2, ..., M$, sample with replacement $v^*_{t,m}$ from $\hat{v}_t$ and construct artificial data $\hat{D}(L) Y^*_{t,m} = v^*_{t,m}$
   2.2 Estimate $\hat{D}^*_m(L)$ by OLS using the artificial data $Y^*_{t,m}$.
   2.3 Estimate the bias $\hat{\Psi} = \frac{1}{M} \sum_{m=1}^{M} \hat{D}^*_j(L) - \hat{D}(L)$.

# Confidence bands for impulse responses (bootstrap-after-bootstrap)

**Algorithm 5 (contd):**

3. Apply a bias correction to form $\tilde{D}(L)$ but preserve stationarity if $\hat{D}(L)Y_t$ is stationary. That is, compute roots of $\det(\hat{D}(L)) = 0$ and let $\lambda(\hat{D})$ be the largest root in absolute value.
   - 3.1 If $\lambda(\hat{D}) \leq 1$ (non-stationary VAR), set $\tilde{D}(L) = \hat{D}(L)$ (ignore bias due to superconsistency of estimates).
   - 3.2 If $\lambda(\hat{D}) > 1$ (stationary VAR), let $\delta \approx 0.05$ and construct the bias-corrected coefficient estimates $\tilde{D}(L) = \hat{D}(L) - \hat{\Psi}(1-\delta)^j$ where $j$ is the minimal non-negative integer number such that all roots of $\det(\tilde{D}(L)) = 0$ are outside the unit circle. This guarantees that $\tilde{D}(L)Y_t$ is also stationary.
4. Compute the bias-corrected impulse response function $\tilde{\theta} = \theta(\tilde{D}(L), \hat{\Omega}, \tilde{S})$. Here we use Kilian's suggestion of keeping the OLS estimate of $\hat{\Omega}$.

# Confidence bands for impulse responses (bootstrap-after-bootstrap)

**Algorithm 5 (contd):**

5. Bootstrap $\tilde{D}(L)$ and $\tilde{\theta} = \theta(\tilde{D}(L), \hat{\Omega}, \tilde{S})$:
   5.1 Compute the fitted residuals $\tilde{v}_t = \tilde{D}(L) Y_t$ for $t = 1, 2, ..., T$.
   5.2 For $m = 1, 2, ..., M$, sample with replacement $v^*_{t,m}$ from $\tilde{v}_t$ and construct artificial data $\tilde{D}(L) Y^*_{t,m} = v^*_{t,m}$.
   5.3 Estimate $\hat{\tilde{D}}^*_m(L)$ and $\hat{\Omega}^*_m$ by OLS.
   5.4 Bias correct $\hat{\tilde{D}}^*_m(L)$ to form $\tilde{\tilde{D}}^*_m(L)$. This leads to a nested bootstrap within a bootstrap. Use the initial bias correction $\hat{\Psi}$ if this is too costly.
   5.5 Compute the impulse response $\theta^*_m = \theta(\tilde{\tilde{D}}^*_m(L), \hat{\Omega}^*_m, \tilde{\tilde{S}})$.
6. Compute quantiles or other moments of the empirical distribution of impulse responses $\{\theta^*_m\}_{m=1}^{M}$ to form confidence bands.

# Relationship between economic models and VARs

- Fernandez-Villaverde, Rubio-Ramirez, Sargent, and Watson (2007).

- Approximate solution of DSGE models has the general form (note timing convention)

$$x_{t+1} = Ax_t + Bw_{t+1} \tag{7}$$
$$y_{t+1} = Cx_t + Dw_{t+1} \tag{8}$$

  $x_t$ is an $n \times 1$ vector of (possibly unobserved) state variables, $y_t$ is a $k \times 1$ vector of observed variables, $w_t$ is an $m \times 1$ vector of economic shocks and measurement errors. Shocks $w_t$ and $z_t$ are Gaussian with $E[w_t] = 0$, $E[w_t w_t'] = I$.

- The matrices $A$, $B$, $C$, and $D$ depend on the parameters of the DSGE model. Matrix $A$ is $n \times n$, $B$ is $n \times m$, $C$ is $k \times n$, and $D$ is $k \times m$.

- **Question:** *Under what conditions can we recover the economic shocks $w_t$ from the shocks of a SVAR?* That is, under what conditions

$$w_{t+1} = S\left(y_{t+1} - E(y_{t+1}|y_t, y_{t-1}, ....)\right)$$

  for some matrix $S$ with the appropriate identifying restrictions?

# A poor's man invertibility condition

- Suppose that the matrix $D$ in equation (8) is invertible. Then (8) implies

$$w_{t+1} = D^{-1}\left(y_{t+1} - Cx_t\right)$$

  Substituting this equation into (7) gives

$$x_t = Ax_{t-1} + Bw_t = Ax_{t-1} + B\left[D^{-1}\left(y_t - Cx_{t-1}\right)\right]$$

  or

$$x_t\left[I - \left(A - BD^{-1}C\right)L\right] = BD^{-1}y_t$$

- **Condition I:** The eigenvalues of $A - BD^{-1}C$ are strictly less than one in modulus.

# A poor's man invertibility condition (II)

- If Condition 1 is satisfied, invert the lag-polynomial and write $x_t$ as an infinite order moving average of the observables $y_t$

$$x_t = \left[I - BD^{-1}CL\right]^{-1} BD^{-1} y_t$$

$$x_t = \sum_{j=0}^{\infty} \left(A - BD^{-1}C\right)^j BD^{-1} y_{t-j}.$$

- Replacing the result into the observation equation (8) gives an infinite order VAR representation in terms of the observables $y_t$

$$y_t = C \sum_{j=0}^{\infty} \left(A - BD^{-1}C\right)^j BD^{-1} y_{t-1-j} + Dw_t$$

  where $Dw_t$ is a shock orthogonal to $y_{t-j}$ for all $j > 0$.

- Note that this is a very restricted VAR. Moreover, the matrices $A$, $B$, $C$, $D$ are non-linear functions of the deep parameters of the model.