# Detalles del algoritmo RWZ (algorithm 2 en las slides)

This note fills in some of the details of the efficient algorithm in RWZ. We use the notation that we have in the slides

Let $n$ be the number of variables.

- Short run restrictions are zero restrictions on $S$. Define

$$L_0 \equiv S$$

- Long-run restrictions are zeros imposed on $IR_\infty$. Define

$$L_\infty = (1 - \mathbf{D})^{-1} S$$

  where $\mathbf{D} = \sum_{j=1}^{p} D_j$.

- We can embed short and long run restrictions in

$$f(S, D) = \begin{bmatrix} L_0 \\ L_\infty \end{bmatrix}$$

  Let $F = f(S, D)$. Note that $F$ is a $2n \times n$ matrix.

- The identification constraints can be written as

$$Q_i F e_i = \mathbf{0}_{n\times 1}.$$

Let's dig into the algorithm.

1. Assume that the model is exactly identified
2. Let $F$ be the matrix with short and long-run impact matrices. Let $Q_1$, $Q_2, ..., Q_n$ represent the identifying restrictions and assume that we have ordered the shocks so that $q_i = n - i$ for $i = 1, 2, ..., n$ is the rank of $Q_i$.

3. Perform a Cholesky decomposition of the covariance matrix of reduced form residuals,

$$V = chol\,(\Omega)$$

where $VV' = \Omega$.

4. Construct initial short and long run response matrices

$$\begin{aligned} L_0^* &= V \\ L_\infty^* &= (I - \mathbf{D})^{-1} V \end{aligned}$$

And let

$$F^* = \begin{bmatrix} L_0^* \\ L_\infty^* \end{bmatrix}$$

Of course, $F^*$ need not satisfy the identifying assumptions.

5. By Theorem 1, there is a rotation matrix $P$ such that $L_0 = L^* P$ and $L_\infty = L_\infty^* P$. Equivalently,

$$F = F^* P$$

satisfy the identifying assumptions.

6. RWZ provide an algorithm to construct $P$.

Set $j = 1$. Then, according to the algorithm,

$$\tilde{Q}_1 = Q_1 F^*.$$

Note that $Q_1$ has rank $n - 1$, so that $Q_1 F^*$ has rank$< n$, $rank\left(\tilde{Q}_1\right) < n$. This implies that there exist an infinite number of non-trivial solutions to the system of equations $\tilde{Q}_1 \mathbf{x} = \mathbf{0}$. Then, there exist a unit-length vector $\mathbf{p}_1 \neq 0$ such that

$$\tilde{Q}_1 \mathbf{p}_1 = \mathbf{0}.$$

Note that

$$Q_1 F^* \mathbf{p}_1 = \mathbf{0}. \tag{1}$$

Set $j = 2$ and let

$$\underbrace{\tilde{Q}_2}_{n+1\times n} = \begin{bmatrix} Q_2 F^* \\ \mathbf{p}_1' \end{bmatrix}$$

Note that $\tilde{Q}_2$ is of size $n+1 \times n$. Moreover, since $rank\,(Q_2) = n-2$, $rank\,(Q_2 F^*) \leq n-2$. Therefore, $rank(\tilde{Q}_2) < n$. Hence, there exist a unit-length vector $\mathbf{p}_2$ such that

$$\tilde{Q}_2 \mathbf{p}_2 = \mathbf{0}_{n+1\times 1}$$

This implies two things:

$$\begin{aligned} Q_2 F^* \mathbf{p}_2 &= \mathbf{0}_{n\times 1} \\ \mathbf{p}_1' \mathbf{p}_2 &= 0. \end{aligned} \tag{2}$$

Set $j = 3$ and let

$$\underbrace{\tilde{Q}_3}_{n+2\times n} = \begin{bmatrix} Q_3 F^* \\ \mathbf{p}_1' \\ \mathbf{p}_2' \end{bmatrix}.$$

Now, since $rank\,(Q_3) = n-3$, then $rank\,(Q_3 F^*) \leq n-3$ and, hence, $rank(\tilde{Q}_3) < n$. Therefore, there exist a unit-length vector $\mathbf{p}_3$ such that

$$\tilde{Q}_3 \mathbf{p}_3 = \mathbf{0}_{n+2\times 1}.$$

This implies:

$$\begin{aligned} Q_3 F^* \mathbf{p}_3 &= \mathbf{0}_{n\times 1} \\ \mathbf{p}_1' \mathbf{p}_3 &= 0 \\ \mathbf{p}_2' \mathbf{p}_3 &= 0. \end{aligned} \tag{3}$$

Continue doing this $n$ times. In this way, we are constructing a matrix with columns $\mathbf{p}_1$, $\mathbf{p}_2$, ....,

$\mathbf{p}_n$ such that

$$
\begin{aligned}
Q_j F^* \mathbf{p}_j &= \mathbf{0}_{n\times 1} \\
\mathbf{p}_j' \mathbf{p}_j &= 1 \\
\mathbf{p}_j' \mathbf{p}_k &= 0
\end{aligned}
\tag{4}
$$

Let

$$P = \left[ \begin{array}{ccccc} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & ... & \mathbf{p}_n \end{array} \right]$$

so that

$$PP' = I.$$

Now let

$$F = F^* P.$$

And let's check that the identifying assumptions are satisfied for the matrix $F$. We must have

$$Q_j F e_j = 0_{n\times 1}$$

for $j = 1, 2, ..., n$. But

$$
\begin{aligned}
Q_j F e_j &= Q_j F^* P e_j \\
&= Q_j F^* \left[ \begin{array}{ccccc} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 & ... & \mathbf{p}_n \end{array} \right] \times \left[ \begin{array}{c} 0 \\ \vdots \\ 1 \text{ (row } j) \\ \vdots \\ 0 \end{array} \right] \\
&= Q_j F^* \mathbf{p}_j \\
&= \mathbf{0}
\end{aligned}
$$

where the last equality follows from equation (4). Since this holds for all $j$, we have

$$Q_j F e_j = \mathbf{0}$$

for $j = 1, 2, ..., n$ once we set $F = F^* P$ with $P$ constructed as above. This gives the desired rotation matrix $P$ and we set $S = VP$.