BASIC RBC MODEL

This note shows how to solve, step-by-step, a plain vanilla real business cycle (RBC) model using a log-linearization of the equilibrium conditions around the steady state. We will use Paul Klein's matlab function `solab.m`. This program uses the Generalized Schur decomposition as described in a previous note to solve the first order stochastic difference equation derived from the log-linearized model. The general procedure to solve a model consists of the following steps:

1. Find the equilibrium conditions of the model.

2. Calibrate and find the steady state of the model.

3. Log-linearize the equilibrium conditions of the model around the steady state.

4. Write the linearized system of difference equations in a format that can be used in the Matlab function `solab.m`. This requires writing the system of equations as

$$\mathbf{A} E_t \left[\mathbf{z}_{t+1} \right] = \mathbf{B} \mathbf{z}_t$$

   where $\mathbf{z}_t$ is a vector that contains all the variables in the model ordered in a particular way (see below), and $\mathbf{A}$ and $\mathbf{B}$ are square matrices conformable with the vector $\mathbf{z}_t$.

5. Call the `solab.m` routine to find the approximate policy functions. Then compute impulse responses, simulations, second moments, and so forth.

We consider a standard RBC model cast as the following planner's problem

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[\log c_t - \eta \frac{l_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} \right]$$

subject to

$$c_t + k_{t+1} = A_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta) k_t$$

given initial conditions A_0, k_0, and a law of motion for the technology process that we specify below. The parameter η is a constant affecting the disutility of working and ν is the Frisch elasticity of labor supply.

Let λ_t denote the multiplier on the constraint and write the Lagrangian

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left[\log c_t - \eta \frac{l_t^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} - \lambda_t \left[c_t + k_{t+1} - A_t k_t^{\alpha} l_t^{1-\alpha} - (1-\delta) k_t \right] \right]$$

The first order conditions with respect to c_t, l_t, and k_{t+1} are, respectively,

$$\frac{1}{c_t} = \lambda_t$$

$$\eta l_t^{\frac{1}{\nu}} = \lambda_t \left(1-\alpha\right) A_t k_t^{\alpha} l_t^{-\alpha}$$

$$\lambda_t = \beta E_t \left[\lambda_{t+1}\left(\alpha A_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + 1 - \delta\right)\right]$$

plus the feasibility constraint. The transversality condition of this problem is

$$\lim_{T\to\infty} E_0 \left[\beta^T \lambda_T k_{T+1}\right] = 0.$$

Shocks. The logarithm of TFP follows an AR(1) process

$$\log A_{t+1} = \rho \log A_t + \varepsilon_{t+1}$$

where ε_{t+1} is i.i.d. normal with mean 0 and variance σ_ε^2.

Equilibrium equations

As written, the control variables of the model are consumption c_t, labor l_t, and the multiplier λ_t. But we are also interested in output (y_t) and investment (x_t). We will augment the above system to include a few equations that will allow us to obtain the policy functions for output and investment in a direct fashion. In particular, recall that output is given by

$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha}$$

and investment is

$$x_t = k_{t+1} - \left(1-\delta\right) k_t$$

Then, we can write the marginal product of capital and labor as

$$\begin{aligned}\left(1-\alpha\right) A_t k_t^{\alpha} l_t^{-\alpha} &= \left(1-\alpha\right) y_t / l_t \text{ and}\\ \alpha A_t k_t^{\alpha-1} l_t^{1-\alpha} &= \alpha y_t / k_t.\end{aligned}$$

It then follows that we can write the equilibrium conditions as the following system of 7 equations

$$\frac{1}{c_t} = \lambda_t \tag{1}$$

$$\eta l_t^{\frac{1}{\nu}} = \lambda_t \left(1-\alpha\right) \frac{y_t}{l_t} \tag{2}$$

$$\lambda_t = \beta E_t \left[\lambda_{t+1} \left(\alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta \right) \right] \tag{3}$$

$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha} \tag{4}$$

$$c_t + x_t = y_t \tag{5}$$

$$x_t = k_{t+1} - (1 - \delta) k_t \tag{6}$$

$$\log A_{t+1} = \rho \log A_t + \varepsilon_{t+1} \tag{7}$$

Steady state and calibration

The second step in the procedure consists of finding the non-stochastic steady steate of the economy and calibrating the model. In steady state the system (1)-(7) becomes

$$\frac{1}{\bar{c}} = \bar{\lambda} \tag{8}$$

$$\eta \bar{l}^{\frac{1}{\nu}} = \bar{\lambda} (1 - \alpha) \bar{y} / \bar{l} \tag{9}$$

$$1 = \beta \left(\alpha \bar{y} / \bar{k} + 1 - \delta \right) \tag{10}$$

$$\bar{y} = \bar{A} \bar{k}^{\alpha} \bar{l}^{1-\alpha} \tag{11}$$

$$\bar{c} + \bar{x} = \bar{y} \tag{12}$$

$$\bar{x} = \delta \bar{k} \tag{13}$$

$$\bar{A} = 1. \tag{14}$$

The steady state quantities satisfy the previous system of equations. One possibility is to solve the system as a function of the parameters of the model. It is easier, however, to perform a calibration that simultaneously sets the parameters of the model and delivers the steady state quantities. The following is one possibility:

Calibration: We set some numbers to match certain features of the data. For example, the parameter α is the capital share in output. NIPA accounts for the U.S. imply a capital share of about $\alpha = 1/3$. Second, we calibrate the a long-run (gross) real interest rate of $\bar{R} = 0.01$ (1% per quarter). In the model, the gross real interest rate in steady state satisfies

$$\alpha \frac{\bar{y}}{\bar{k}} + 1 - \delta = \bar{R}. \tag{15}$$

Given the value for α chosen before, this equation imposes a relation between the steady state output to capital ratio $\left(\bar{y}/\bar{k}\right)$ and the depreciation rate δ. Equation (10) then implies that β must satisfy

$$\frac{1}{\beta} = \bar{R} \rightarrow \beta = \frac{1}{1.01} \approx 0.99.$$

Next, we want to calibrate the model to match some average (long-run) investment rate $\bar{x}/\bar{y}$. Equation (13) can be written as

$$\frac{\bar{x}}{\bar{y}} = \delta \frac{\bar{k}}{\bar{y}}$$

But using the calibration relation (15) we can write the last equation as

$$\frac{\bar{x}}{\bar{y}} = \delta \left(\frac{\alpha}{\bar{R} - (1-\delta)} \right)$$

We can use this equation to solve for the value of δ :

$$\delta = \frac{\left(\bar{R} - 1\right)(\bar{x}/\bar{y})}{\alpha - (\bar{x}/\bar{y})}.$$

Given a target value for the investment rate $\bar{x}/\bar{y} = 0.21$ (this is roughly the value for the US for a broad concept of investment that includes durable consumption goods), and the values $\bar{R} = 1.01$ and $\alpha = 1/3$ we obtain

$$\delta = \frac{0.01 \times 0.21}{0.33 - 0.21} \approx 0.017.$$

This implies an annualized depreciation rate of about 6.6%:

$$(1 - 0.017)^4 = 1 - \delta^{annual} \Rightarrow \delta^{annual} = 1 - (1 - 0.017)^4 \cong 0.0663.$$

(In any case, in the model we used the quarterly depreciation rate.)

Next, we calibrate the model to deliver an average labor input value of $\bar{l} = 1/3$. This is roughly the fraction of total weekly hours that workers spend working. Using (14), we can write (11) as

$$\bar{y} = \bar{k}^{\alpha} \bar{l}^{1-\alpha}.$$

Dividing by $\bar{k}$

$$\frac{\bar{y}}{\bar{k}} = \left(\frac{\bar{l}}{\bar{k}} \right)^{1-\alpha}$$

and using the calibration condition (15) we can write

$$\left(\frac{\bar{R}-(1-\delta)}{\alpha}\right)=\left(\frac{\bar{l}}{\bar{k}}\right)^{1-\alpha}$$

or

$$\bar{k}=\bar{l}\left(\frac{\alpha}{\bar{R}-(1-\delta)}\right)^{\frac{1}{1-\alpha}} \tag{16}$$

Given the target value of $\bar{l}=1/3$ and the other parameters already calibrated, this equation delivers the steady state level of capital. For the parameter values described above, we find

$$\bar{k}=\frac{1}{3}\left(\frac{1/3}{1.01-(1-0.017)}\right)^{\frac{1}{1-1/3}}\approx 14.46.$$

The steady state level of output is thus

$$\bar{y}=\bar{A}\bar{k}^{\alpha}\bar{l}^{1-\alpha}\approx 1.17.$$

The steady state level of consumption follows from the feasibility condition (12)

$$\bar{c}=\bar{y}-\bar{x}=\bar{y}\left(1-\frac{\bar{x}}{\bar{y}}\right)=1.17\,(1-0.21)\approx 0.93.$$

Given $\bar{c}$, the steady state condition (8) determines the steady state level of the multiplier $\bar{\lambda}$.

It remains to calibrate the parameters ν and η. To that end, write condition (9) as

$$\eta\bar{l}^{1+\frac{1}{\nu}}=(1-\alpha)\,\frac{\bar{y}}{\bar{c}}.$$

In this equation we know $\bar{l}$, $\bar{c}$, and $\bar{y}$. We thus have one equation for the two parameters η and ν. The Frisch elasticity ν is typically calibrated based on microeconomic studies of labor supply elasticity. There is controversy regarding the value for ν. We will set $\nu=1$. Given this, we can use the previous equation to determine the constant η.

How do we calibrate the parameters of the stochastic process ρ and σ_{ε}^{2}? This may be done running a first order autoregression on estimated Solow residuals. Another possibility is to set ρ to some number and then choose σ_{ε}^{2} to match the volatility of output in the data. For this latter approach we must solve the model many times for different values of σ_{ε}^{2} and choose the one that makes the volatility of simulated output to match that observed in the data.

Log-linearization of the model

We will solve the model approximating the policy functions around the steady state. Most economists choose to log-linearize rather than to linearize their models. This gives log-linear equations that often seem to better describe data. Furthermore, log-linear policy functions have a nice economic interpretation as the percentage deviation of the particular variable being considered from its steady state value. Therefore, define for any (positive) variable x_t, its log-deviation from the steady state value

$$\hat{x}_t = \log\left(x_t/\bar{x}\right).$$

This implies that the level of the variable can be written in terms of the its log-deviation as

$$x_t = \bar{x}e^{\hat{x}_t}.$$

We will linearize the equilibrium conditions around $\hat{x}_t = 0$ for all varibles x_t.

Equation (1):

Write equation (1) in terms of the log-deviation from the steady state as

$$0 = \frac{1}{\bar{c}}e^{-\hat{c}_t} - \bar{\lambda}e^{\hat{\lambda}_t}$$

Performing a first order Taylor expansion around $(\hat{c}_t, \hat{\lambda}_t) = (0, 0)$ gives

$$0 \approx \frac{1}{\bar{c}} - \bar{\lambda} - \frac{1}{\bar{c}}\hat{c}_t - \bar{\lambda}\hat{\lambda}_t.$$

Using that in steady state $\frac{1}{\bar{c}} = \bar{\lambda}$ we obtain

$$0 \approx \hat{c}_t + \hat{\lambda}_t. \tag{17}$$

Note that the constant of the Taylor expansion dissapears because it is zero in steady state. This happens in all the equations and, therefore, from now on we ignore that constant term of the Taylor expansion.

Equation (2):

Write equation (2) in terms of the log-deviations from the steady state

$$0 = \eta\bar{l}^{\frac{1}{\nu}}e^{\frac{1}{\nu}\hat{l}_t} - \bar{\lambda}\left(1-\alpha\right)\frac{\bar{y}}{\bar{l}}e^{\hat{\lambda}_t+\hat{y}_t-\hat{l}_t}$$

Linearizing around $(\hat{l}_t, \hat{\lambda}_t, \hat{y}_t) = (0,0,0)$ gives

$$0 \approx \eta \bar{l}^{\frac{1}{\nu}} \frac{1}{\nu} \hat{l}_t - \bar{\lambda} (1-\alpha) \frac{\bar{y}}{\bar{l}} [\hat{\lambda}_t + \hat{y}_t - \hat{l}_t]$$

But in steady state $\eta \bar{l}^{\frac{1}{\nu}} = \bar{\lambda} (1-\alpha) \left(\bar{y}/\bar{l}\right)$, therefore

$$0 \approx \frac{1}{\nu} \hat{l}_t - \hat{\lambda}_t - \hat{y}_t + \hat{l}_t$$

or

$$0 \approx (1 + \frac{1}{\nu}) \hat{l}_t - \hat{\lambda}_t - \hat{y}_t. \tag{18}$$

Equation (3): Disregard for the moment the expectation operator—we will put it back later—and write the equation as

$$0 = \beta \bar{\lambda} e^{\hat{\lambda}_{t+1}} \left(\alpha \left(\bar{y}/\bar{k}\right) e^{\hat{y}_{t+1} - \hat{k}_{t+1}} + 1 - \delta \right) - \bar{\lambda} e^{\hat{\lambda}_t}.$$

Linearizing about $\left(\hat{\lambda}_{t+1}, \hat{y}_{t+1}, \hat{k}_{t+1}, \hat{\lambda}_t\right) = (0,0,0,0)$ we obtain

$$0 \approx \beta \bar{\lambda} \left(\alpha \left(\bar{y}/\bar{k}\right) + 1 - \delta \right) \hat{\lambda}_{t+1} + \beta \bar{\lambda} \alpha \left(\bar{y}/\bar{k}\right) \left(\hat{y}_{t+1} - \hat{k}_{t+1} \right) - \bar{\lambda} \hat{\lambda}_t.$$

Dividing by $\bar{\lambda}$ and using that in steady state $\beta \left(\alpha \left(\bar{y}/\bar{k}\right) + 1 - \delta \right) = 1$ we have

$$0 \approx \hat{\lambda}_{t+1} + \beta \alpha \left(\bar{y}/\bar{k}\right) (\hat{y}_{t+1} - \hat{k}_{t+1}) - \hat{\lambda}_t.$$

Putting back the expectation operator we have

$$0 \approx E_t \left[\hat{\lambda}_{t+1} + \beta \alpha \left(\bar{y}/\bar{k}\right) (\hat{y}_{t+1} - \hat{k}_{t+1}) \right] - \hat{\lambda}_t \tag{19}$$

Equation (4). This equation is already log-linear. Taking the logarithm of the equation gives

$$\log y_t = \log A_t + \alpha \log k_t + (1-\alpha) \log l_t$$

Subtracting the same equation at the steady state and rearranging gives

$$0 = \hat{y}_t - \hat{A}_t - \alpha \hat{k}_t - (1-\alpha) \hat{l}_t \tag{20}$$

Equation (5):

$$0 = \bar{y}e^{\hat{y}_t} - \bar{c}e^{\hat{c}_t} - \bar{x}e^{\hat{x}_t}$$

Linearizing about $(\hat{y}_t, \hat{c}_t, \hat{x}_t) = (0, 0, 0)$ gives

$$0 \approx \bar{y}\hat{y}_t - \bar{c}\hat{c}_t - \bar{x}\hat{x}_t \tag{21}$$

Equation (6)

$$0 = \bar{k}e^{\hat{k}_{t+1}} - (1-\delta)\,\bar{k}e^{\hat{k}_t} - \bar{x}e^{\hat{x}_t}$$

Linearizing this equation gives

$$0 \approx \bar{k}\hat{k}_{t+1} - (1-\delta)\,\bar{k}\hat{k}_t - \bar{x}\hat{x}_t$$

But in steady state $\bar{x} = \delta\bar{k}$ which implies

$$0 \approx \hat{k}_{t+1} - (1-\delta)\,\hat{k}_t - \delta\hat{x}_t. \tag{22}$$

Equation (7)

Finally, the TFP equation is already loglinear and given by

$$0 = \log A_{t+1} - \rho \log A_t - \varepsilon_{t+1}.$$

Subtrating the same equation at the steady state and taking the conditional expectation as of time t on both side of the equation then gives

$$0 = E_t[\hat{A}_{t+1}] - \rho\hat{A}_t \tag{23}$$

Summarizing, the linearized rational expectations model can be written as follows (changing somewhat the order of the equations)

$$
\begin{aligned}
0 &= \hat{c}_t + \hat{\lambda}_t && (24)\\
0 &= (1+\frac{1}{\nu})\hat{l}_t - \hat{\lambda}_t - \hat{y}_t && (25)\\
0 &= \hat{y}_t - \hat{A}_t - \alpha\hat{k}_t - (1-\alpha)\,\hat{l}_t && (26)\\
0 &= \bar{y}\hat{y}_t - \bar{c}\hat{c}_t - \bar{x}\hat{x}_t && (27)\\
E_t[\hat{k}_{t+1}] &= (1-\delta)\,\hat{k}_t + \delta\hat{x}_t && (28)\\
E_t\left[\hat{\lambda}_{t+1} + \beta\alpha\left(\bar{y}/\bar{k}\right)\left(\hat{y}_{t+1} - \hat{k}_{t+1}\right)\right] &= \lambda_t && (29)\\
E_t[\hat{A}_{t+1}] &= \rho\hat{A}_t && (30)
\end{aligned}
$$

Note that I wrode $E_t\left[\hat{k}_{t+1}\right]$ even though $\hat{k}_{t+1}$ is chosen (and therefore aldeary known) at time t. This is just notation that will allow us to write the model as the following first order vector expectational difference equation

$$\mathbf{A}E_t\left[\mathbf{z}_{t+1}\right] = \mathbf{B}\mathbf{z}_t \tag{31}$$

where the vector $\mathbf{z}_t$ contains all the variables in the economy and $\mathbf{A}$ and $\mathbf{B}$ are square matrices.

We solve numerically this model using the Matlab program `solab.m`. We order the variables $\mathbf{z}_t$ as follows:

$$\mathbf{z}_t = \begin{bmatrix} \text{endogenous states variables} \\ \text{exogenous states variables} \\ \text{jump variables} \end{bmatrix}$$

In the RBC model described above, the only endogenous state variable is the stock of capital $\hat{k}_t$ and the only exogenous state variable is the level of technology $\hat{A}_t$. Therefore, the variable $\mathbf{z}_t$ is given by

$$\mathbf{z}_t = \left[\hat{k}_t,\ \hat{A}_t,\ \ \hat{y}_t,\ \hat{c}_t,\ \hat{l}_t,\ \hat{x}_t,\ \hat{\lambda}_t\right]'. \tag{32}$$

Please note that the order within each group of variables does not matter (e.g. we could put c_t before y_t in the vector $\mathbf{z}_t$).

In addition, we must tell the program how many of the variables in $\mathbf{z}_t$ are state variables. This is to check the Blanchard-Kahn conditions to determine if the model has a unique solution or not. In our case, it is 2: $\hat{k}_t$ and $\hat{A}_t$. Note that, in this case, $\mathbf{A}$ and $\mathbf{B}$ are 7×7 matrices. If we let $\boldsymbol{\kappa}_t \equiv [\hat{k}_t, \hat{A}_t]'$ denote the vector of state variables and $\mathbf{u}_t = [\hat{y}_t,\ \hat{c}_t,\ \hat{l}_t,\ \hat{x}_t,\ \hat{\lambda}_t]$, the vector of jump variables, the solver delivers the equilibrium of the "certainty equivalent" model in the form

$$\begin{aligned} \mathbf{u}_t &= \mathbf{F}\boldsymbol{\kappa}_t \\ \boldsymbol{\kappa}_{t+1} &= \mathbf{P}\boldsymbol{\kappa}_t \end{aligned}$$

The "stochastic" solution of the model is obtained by replacing the second equation above with

$$\boldsymbol{\kappa}_{t+1} = \mathbf{P}\boldsymbol{\kappa}_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}$$

which simply recovers the stochastic shock $\hat{A}_{t+1} = \rho \hat{A}_t + \varepsilon_{t+1}$.

For the ordering (32), the matrices $\mathbf{A}$ and $\mathbf{B}$ of the system (31) are given by

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\beta\alpha\left(\bar{y}/\bar{k}\right) & 0 & \beta\alpha\left(\bar{y}/\bar{k}\right) & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \text{equation (24)} \\ \text{equation (25)} \\ \text{equation (26)} \\ \text{equation (27)} \\ \text{equation (28)} \\ \text{equation (29)} \\ \text{equation (30)} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & \left(1+\frac{1}{\nu}\right) & 0 & -1 \\ -\alpha & -1 & 1 & 0 & -\left(1-\alpha\right) & 0 & 0 \\ 0 & 0 & \bar{y} & -\bar{c} & 0 & -\bar{x} & 0 \\ 1-\delta & 0 & 0 & 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \rho & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \text{equation (24)} \\ \text{equation (25)} \\ \text{equation (26)} \\ \text{equation (27)} \\ \text{equation (28)} \\ \text{equation (29)} \\ \text{equation (30)} \end{bmatrix}$$

Using the calibrated parameter values, the model delivers the following solution:

$$\mathbf{F} = \begin{bmatrix} 0.22 & 1.33 \\ 0.57 & 0.34 \\ -0.17 & 0.50 \\ -1.10 & 5.07 \\ -0.57 & -0.34 \end{bmatrix}; \quad \mathbf{P} = \begin{bmatrix} 0.96 & 0.09 \\ 0 & 0.95 \end{bmatrix}$$

Which, in other words, implies the following policy functions:

$$
\begin{aligned}
\hat{y}_t &= 0.22\hat{k}_t + 1.33\hat{A}_t \\
\hat{c}_t &= 0.57\hat{k}_t + 0.34\hat{A}_t \\
\hat{l}_t &= -0.17\hat{k}_t + 0.50\hat{A}_t \\
\hat{x}_t &= -1.10\hat{k}_t + 5.07\hat{A}_t \\
\hat{k}_{t+1} &= 0.96\hat{k}_t + 0.09\hat{A}_t \\
\hat{A}_{t+1} &= 0.95\hat{A}_t + \varepsilon_{t+1}.
\end{aligned}
$$

Once we have this solution, we can compute impulse responses, variance decompositions, simulations, compute spectral densities, and so forth.