TP2

Clase 3 (3/25)

Ejercicio 1

Para probar que es un subespacio
1) $A \in S_i \Rightarrow kA \in S_i$
2) $A_1, A_2 \in S_i \Rightarrow A_1 + A_2 \in S_i$

a) $S_1 = \{ A \in \mathbb{R}^{n \times n} : A^t = A \}$

D/. • $A \in S_1$, $k \in \mathbb{R} \Rightarrow (kA)^t =$

$= k A^t \underset{A \in S_1}{=} kA \Rightarrow kA \in S_1$

• $A_1, A_2 \in S_1$, $(A_1 + A_2)^t = A_1^t + A_2^t \underset{A_1, A_2 \in S_1}{=} A_1 + A_2 \Rightarrow A_1 + A_2 \in S_1$ $\square$

c) $S_3 = \{ A \in \mathbb{R}^{n \times n} : A_{ij} = 0 \text{ si } i > j \}$

$\begin{pmatrix} \cdot & & \\ & \cdot & \\ & & \cdot \end{pmatrix}$

D/. • $A \in S_3 \Rightarrow A_{ij} = 0 \overset{i>j}{\Rightarrow} \forall k \in \mathbb{R}$, $kA_{ij} = 0 \; {}^{i>j} \Rightarrow kA \in S_3$

• $A, B \in S_3 \Rightarrow A_{ij} = 0, B_{ij} = 0, i > j \Rightarrow A_{ij} + B_{ij} = 0, i > j$

$\Rightarrow (A+B)_{ij} = 0, i > j \Rightarrow A + B \in S_3$ $\square$

f) $S_6 = \{ A \in \mathbb{R}^{n \times n} : tr(A) = 0 \}$

Propiedad: $A, B \in \mathbb{R}^{n \times n}$, $k \in \mathbb{R} \Rightarrow tr(kA + B) = k\, tr(A) + tr(B)$

D/. $tr(kA+B) = \sum_{i=1}^{n} (kA+B)_{ii} = \sum_{i=1}^{n} kA_{ii} + B_{ii} =$

$\left(\sum_{i=1}^{n} kA_{ii}\right) + \left(\sum_{i=1}^{n} B_{ii}\right) = k \sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii} = k\, tr(A) + tr(B)$ $\square$

S_6 es un subespacio:

D/. • $A \in S_6 \Rightarrow tr(A) = 0 \Rightarrow \forall k$, $k\, tr(A) = 0 \overset{\text{propiedad}}{\Rightarrow} tr(kA) = 0$

$\Rightarrow kA \in S_6$

• $A, B \in S_6 \Rightarrow tr(A) = 0, tr(B) = 0 \Rightarrow tr(A) + tr(B) = 0 \overset{\text{propiedad}}{\Rightarrow} tr(A+B) = 0$

$\Rightarrow A + B \in S_6$ $\square$

Ejercicio 2

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}$$

a) $$A+3B-3C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} - 3\begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 9 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix}$$ asociatividad + conmutatividad

$$= \begin{pmatrix} 7 & 2 \\ 12 & 4 \end{pmatrix}$$

Ejercicio 3

$$A \in \mathbb{R}^{4\times 5}, \quad B \in \mathbb{R}^{5\times 7}, \quad C \in \mathbb{R}^{7\times 5}$$

a) AB?

$$\underset{\mathbb{R}^{4\times 5}}{A} \cdot \underset{\mathbb{R}^{5\times 7}}{B} \Rightarrow \text{se puede } A\cdot B$$

$$A = 4\left\{\overbrace{\begin{pmatrix} \text{---} & v_1 & \text{---} \\ \text{---} & v_2 & \text{---} \\ \text{---} & v_3 & \text{---} \\ \text{---} & v_4 & \text{---} \end{pmatrix}}^{5}\right. \qquad B = \left.\overbrace{\begin{pmatrix} | & | & & | \\ b_1 & b_2 & \dots & b_7 \\ | & | & & | \end{pmatrix}}^{7}\right\}5$$

$$= 4\left\{\underbrace{\begin{pmatrix} \bullet & \bullet & \dots \\ \bullet & \bullet & \dots \\ & & \end{pmatrix}}_{7}\right. \Rightarrow A\cdot B \in \mathbb{R}^{4\times 7}$$

g) $B\cdot C\cdot B\cdot C$?

se puede ¡!

$$\underset{\mathbb{R}^{5\times 7}}{B} \cdot \underset{\mathbb{R}^{7\times 5}}{C} \cdot \underset{\mathbb{R}^{5\times 7}}{B} \cdot \underset{\mathbb{R}^{7\times 5}}{C} \in \mathbb{R}^{5\times 5}$$

h) $A \cdot A$?

$$\underset{\mathbb{R}^{4\times 5}}{\overset{}{A}} \cdot \underset{\mathbb{R}^{4\times 5}}{A} \quad \leftarrow \text{no puedo}$$

Ejercicio 4

$$A = \begin{pmatrix} 1 & -4 \\ -2 & 1 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -3 & 4 \\ -1 & 0 & 3 \end{pmatrix}$$

$A^t, B^t, (A\cdot B)^t, B^t \cdot A^t$

$$A^t = \begin{pmatrix} 1 & -2 & 2 \\ -4 & 1 & 0 \end{pmatrix}, \quad B^t = \begin{pmatrix} 2 & -1 \\ -3 & 0 \\ 4 & 3 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 1 & -4 \\ -2 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & -3 & 4 \\ -1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & -3 & -8 \\ -5 & 6 & -5 \\ 4 & -6 & 8 \end{pmatrix}$$

$$(A\cdot B)^t = \begin{pmatrix} 6 & -5 & 4 \\ -3 & 6 & -6 \\ -8 & -5 & 8 \end{pmatrix}$$

$$=$$

$$(A\cdot B)^t = B^t \cdot A^t$$

Ejercicio 5

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

a) $$A \begin{pmatrix} k & 0 \\ 0 & j \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & j \end{pmatrix} = \begin{pmatrix} ak & bj \\ ck & dj \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & j \end{pmatrix} = \begin{pmatrix} ak & bj \\ ck & dj \end{pmatrix}$$

$$\begin{pmatrix} k & 0 \\ 0 & j \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ak & bk \\ cj & dj \end{pmatrix}$$

d) $A\begin{pmatrix}1&0\\k&1\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&0\\k&1\end{pmatrix}=\begin{pmatrix}a+bk&b\\c+dk&d\end{pmatrix}$

$\begin{pmatrix}1&0\\k&1\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}a&b\\c+ak&d+bk\end{pmatrix}$

Ejercicio 6

a) "$A,B\in\mathbb{R}^{n\times n}\Rightarrow AB=BA$". Falso!

Contraejemplo: $n=2$, $A=\begin{pmatrix}1&1\\0&0\end{pmatrix}$, $B=\begin{pmatrix}1&0\\1&0\end{pmatrix}$

$\Rightarrow AB=\begin{pmatrix}2&0\\0&0\end{pmatrix}$, $BA=\begin{pmatrix}1&1\\1&1\end{pmatrix}$ $\square$

b) "$A\in\mathbb{R}^{n\times m}$, $B\in\mathbb{R}^{m\times p}$ $\wedge$ $AB=0\Rightarrow \underset{A=0}{\overset{B=0}{\vee}}$". Falso!

Contraejemplo: $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $B=\begin{pmatrix}0&0\\0&1\end{pmatrix}$

$AB=0$, $A,B\neq 0$

$A=\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}$, $B=\begin{pmatrix}0&0\\0&1\\0&0\end{pmatrix}\Rightarrow AB=0$

$a,b\in\mathbb{R}$ /
$ab=0$
$\Rightarrow a=0\vee b=0$

No funciona con matrices

c) "$AB=0\Rightarrow BA=0$". Falso

Contraejemplo: $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $B=\begin{pmatrix}0&0\\1&1\end{pmatrix}$

$\Rightarrow AB=0$, $BA=\begin{pmatrix}0&0\\1&0\end{pmatrix}$

d) "$A^k=0\Rightarrow A=0$". Falso

nilpotentes

Contraejemplo: $A=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, pero $A^2=0$

e) "$A^2=A\Rightarrow A=Id\vee A=0$". Falso

Contraejemplo: $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}\Rightarrow A^2=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ $\square$

f) "$tr(AA^t)=0\Rightarrow A=0$". Verdadero

Dl. $A=\begin{pmatrix}A_{11}&A_{12}&\cdots&A_{1n}\\A_{21}&A_{22}&\cdots&\\\vdots&&\ddots&\\A_{n1}&&&A_{nn}\end{pmatrix}\Rightarrow A^t=\begin{pmatrix}A_{11}&A_{21}&\cdots&A_{n1}\\A_{12}&A_{22}&&\\\vdots&&\ddots&\\A_{1n}&&&A_{nn}\end{pmatrix}$

$$\Rightarrow AA^t = \begin{pmatrix} A_{11}^2 + A_{12}^2 + \dots + A_{1n}^2 & \cdots & \\ \vdots & A_{21}^2 + A_{22}^2 + \dots + A_{2n}^2 & \cdots \\ \vdots & \vdots & \ddots \\ & & A_{n1}^2 + A_{n2}^2 + \dots + A_{nn}^2 \end{pmatrix}$$

$$\Rightarrow tr(AA^t) = \sum_{i=1}^{n} \left(A_{i1}^2 + A_{i2}^2 + A_{i3}^2 + \dots + A_{in}^2 \right)$$

$$= \sum_{i,j} A_{ij}^2 = 0 \Rightarrow A_{ij} = 0 \quad \forall i,j \Rightarrow A = 0 \quad \square$$

Ejercicio 7

a) $\begin{cases} 3x + 5y = 2 \\ -2x + 4y = 6 \end{cases} \Rightarrow \begin{array}{cc} x & y \\ \downarrow & \downarrow \end{array} \begin{pmatrix} 3 & 5 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$

$$\Rightarrow \left(\begin{array}{cc|c} 3 & 5 & 2 \\ -2 & 4 & 6 \end{array}\right) \xrightarrow[\frac{F_2}{2} \to F_2']{} \left(\begin{array}{cc|c} 3 & 5 & 2 \\ -1 & 2 & 3 \end{array}\right) \xrightarrow[F_1 + 2F_2 \to F_1']{} \left(\begin{array}{cc|c} 1 & 9 & 8 \\ -1 & 2 & 3 \end{array}\right)$$

$$\xrightarrow[F_1 + F_2 \to F_2']{} \left(\begin{array}{cc|c} 1 & 9 & 8 \\ 0 & 11 & 11 \end{array}\right) \xrightarrow[F_2/11 \to F_2']{} \left(\begin{array}{cc|c} 1 & 9 & 8 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow[F_1 - 9F_2 \to F_1']{} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

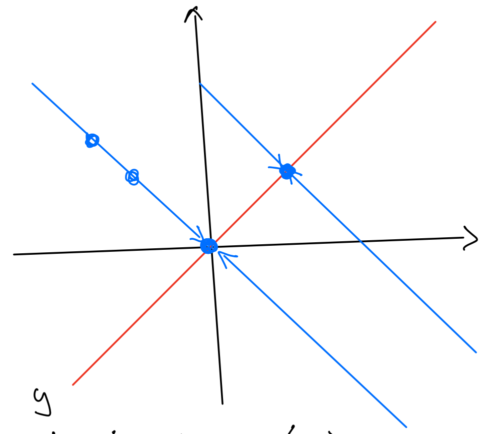
$\to$ el sistema tiene solución compatible determinado $\to \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

El sistema homogéneo asociado: $\begin{pmatrix} 3 & 5 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \left(\begin{array}{cc|c} 3 & 5 & 0 \\ -2 & 4 & 0 \end{array}\right)$

$\to$ hago los mismos pasos que antes: $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \Rightarrow$ solución: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Intuición:

$$\left.\begin{aligned} &v_1 \longrightarrow 0 \\ &v_2 \longrightarrow 0 \\ &\underline{v_3 \longrightarrow v_3'} \\ &v_1+v_3 \longrightarrow 0+v_3=v_3' \\ &\underline{v_2+v_3 \longrightarrow 0+v_3=v_3'} \end{aligned}\right\}$$



b) $\left\{\begin{aligned} x-2y&=0 \\ 2x+y&=0 \\ x+3y&=1 \end{aligned}\right. \Rightarrow \begin{array}{cc} x & y \end{array}$ $\begin{pmatrix} 1 & -2 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{array}\right) \xrightarrow{2F_1-F_2\to F_1'} \left(\begin{array}{cc|c} 0 & -5 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{array}\right) \xrightarrow[\substack{F_3+3F_1/5\to F_3' \\ F_2+F_1/5\to F_2'}]{F_1/5\to F_1'}$$

$$\left(\begin{array}{cc|c} 0 & -1 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{array}\right) \xrightarrow[F_3-F_2/2\to F_3']{} \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 2 & 0 & 0 \\ \underline{0} & \underline{0} & \underline{1} \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} 0 & -1 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow 0=1. \text{ Abs.}$$

El sistema es incompatible.

El " homogéneo asociado: $\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 0 \end{array}\right) \xrightarrow{\text{mismos pasos}} \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$\Rightarrow \begin{pmatrix} 0 & -1 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \to \begin{pmatrix} -y \\ 2x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow y=x=0 \Rightarrow$ solución es $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

g) $\begin{cases} x + 2y - z + 2t = 0 \\ x + y + z + 2t = 1 \\ -x + y - 5z - 2t = -3 \end{cases}$

$$\begin{array}{cccc} x & y & z & t \end{array}$$

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 1 & 1 & 1 & 2 \\ -1 & 1 & -5 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 0 \\ 1 & 1 & 1 & 2 & 1 \\ -1 & 1 & -5 & -2 & -3 \end{array}\right) \xrightarrow[\frac{F_3+F_1}{3} \to F_3']{F_1-F_2 \to F_2'} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 1 & -2 & 0 & -1 \end{array}\right)$$

$$\xrightarrow[F_2-F_3 \to F_3']{} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x+2y-z+2t=0 \\ y-2z=-1 \end{cases}$$

$$\to y = 2z-1 \to x+2(2z-1)-z+2t=0 \to x = 2-3z-2t$$

$$\Rightarrow S = \{(x,y,z,t) = (2-3z-2t,\ 2z-1,\ z,\ t)\}$$

sistema homogéneo: $\left(\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x+2y-z+2t=0 \\ y-2z=0 \end{cases}$

$$\Rightarrow y = 2z \longrightarrow x = -3z-2t$$

$$\Rightarrow S_h = \{(x,y,z,t) = (-3z-2t,\ 2z,\ z,\ t)\}$$

$$= z(-3,2,1,0) + t(-2,0,0,1)$$

$$\Rightarrow S_h = \langle (-3,2,1,0),\ (-2,0,0,1) \rangle$$