The EIS Filter

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Spring 2008

Introduction

The **EIS Filter** (DeJong, Dharmarajan, Liesenfeld, Richard, 2007) extends the concept of efficient importance sampling (Richard & Zhang, 2007 *J. of Econometrics*) to applications involving state space representations.

The application of EIS in this context is non-trivial, due to the absence of an analytical expression for the targeted integrand.

Introduction, cont.

Recall the critical elements of the state space representation:

$$f\left(s_t \middle| Y_t\right) = \frac{f\left(y_t, s_t \middle| Y_{t-1}\right)}{f\left(y_t \middle| Y_{t-1}\right)} = \frac{f\left(y_t \middle| s_t, Y_{t-1}\right) f\left(s_t \middle| Y_{t-1}\right)}{f\left(y_t \middle| Y_{t-1}\right)}, \tag{1}$$

$$f\left(s_t \middle| Y_{t-1}\right) = \int f\left(s_t \middle| s_{t-1}, Y_{t-1}\right) f\left(s_{t-1} \middle| Y_{t-1}\right) ds_{t-1}, \tag{2}$$

$$f\left(y_t \middle| Y_{t-1}\right) = \int f\left(y_t \middle| s_t, Y_{t-1}\right) f\left(s_t \middle| Y_{t-1}\right) ds_t. \tag{3}$$

Issue:
We lack an analytical expression for $f\left(s_{t-1} \middle| Y_{t-1}\right)$ in (2), and for $f\left(s_t \middle| Y_{t-1}\right)$ in (3).

Introduction, cont.

We overcome the lack of an analytical expression for $f(s_{t-1}|Y_{t-1})$ using an EIS approximation constructed in the previous period.

We overcome the lack of an analytical expression for $f(s_t|Y_{t-1})$ by approximating

$$\int f(s_t|s_{t-1}, Y_{t-1}) f(s_{t-1}|Y_{t-1}) ds_{t-1}$$

via an additional EIS calculation ("inner EIS").

The EIS Filter

Objective:

$$f\left(y_t \middle| Y_{t-1}\right) = \int f\left(y_t \middle| s_t, Y_{t-1}\right) f\left(s_t \middle| Y_{t-1}\right) ds_t$$
$$= \int \varphi\left(s_t\right) ds_t$$

Approach to approximation: Introduce Importance Sampler

$$f\left(y_t \middle| Y_{t-1}\right) = \int \frac{\varphi\left(s_t\right)}{g\left(s_t \middle| a_t\right)} g\left(s_t \middle| a_t\right) ds_t,$$

with a_t determined from

$$\left(\widehat{a}_t^{l+1}, \widehat{c}_t^{l+1}\right) = \arg\min_{a_t, c_t} \sum_{i=1}^{R} \left[\ln \varphi_t(s_{t,l}^i) - c_t - \ln k(s_{t,l}^i; a_t)\right]^2.$$

The EIS Filter, cont.

Resulting approximation:

$$\widehat{f}_N(y_t|Y_{t-1}) = \frac{1}{N}\sum_{i=1}^{N}\frac{f\left(y_t|s_t^i, Y_{t-1}\right) f\left(s_t^i|Y_{t-1}\right)}{g(s_t^i;\widehat{a}_t)}.$$

The EIS Filter, cont.

Complication:

- $f\left(s_t \middle| Y_{t-1}\right)$ must be evaluated for each MC drawing s_t^i in constructing $\widehat{f}_N(y_t | Y_{t-1})$.
- But as noted, we lack an analytical expression for $f\left(s_t \middle| Y_{t-1}\right)$.

Resolution (Inner EIS): weighted-sum approximation of $f\left(s_t \middle| Y_{t-1}\right)$.

The EIS Filter, cont.

Write $f\left(s_t \middle| Y_{t-1}\right)$ as

$$\begin{aligned} f\left(s_t \middle| Y_{t-1}\right) &= \int f\left(s_t \middle| s_{t-1}, Y_{t-1}\right) f\left(s_{t-1} \middle| Y_{t-1}\right) ds_{t-1} \\ &= \frac{\int f(s_t|s_{t-1}, Y_{t-1}) f(y_{t-1}|s_{t-1}, Y_{t-2}) f(s_{t-1}|Y_{t-2}) ds_{t-1}}{f\left(y_{t-1} \middle| Y_{t-2}\right)} \\ &= \frac{\int f(s_t|s_{t-1}, Y_{t-1}) f(y_{t-1}|s_{t-1}, Y_{t-2}) f(s_{t-1}|Y_{t-2}) ds_{t-1}}{\int f(y_{t-1}|s_{t-1}, Y_{t-2}) f(s_{t-1}|Y_{t-2}) ds_{t-1}} \end{aligned}$$

Note: We already have an optimized sampler for the denominator constructed in the previous period:

$$g(s_{t-1}; a_{t-1}).$$

Idea: Use this to approximate both the numerator and denominator.

The EIS Filter, cont.

Resulting approximation:

$$\begin{aligned}
\widehat{f}_N(s_t|Y_{t-1}) &= \frac{\sum_{i=1}^{S} f(s_t|s_{t-1}^{0,i}, Y_{t-1}) \cdot \omega(s_{t-1}^{0,i}; \widehat{a}_{t-1})}{\sum_{i=1}^{S} \omega(s_{t-1}^{0,i}; \widehat{a}_{t-1})}, \\
\omega(s_{t-1}; \widehat{a}_{t-1}) &= \frac{f(y_{t-1}|s_{t-1}, Y_{t-2}) f(s_{t-1}|Y_{t-2})}{g(s_{t-1}|\widehat{a}_{t-1})}.
\end{aligned}$$

The EIS Filter, cont.

Implementation algorithm:

- Inherit $\{s_{t-1}^{0,i}, \omega(s_{t-1}^{0,i}; \widehat{a}_{t-1})\}_{i=1}^{N}$ from previous period

The EIS Filter, cont.

Implementation algorithm:

- Inherit $\{s_{t-1}^{0,i}, \omega(s_{t-1}^{0,i}; \widehat{a}_{t-1})\}_{i=1}^{N}$ from previous period
- Obtain $\widehat{a}_t$ using

$$(\widehat{a}_t^{l+1}, \widehat{c}_t^{l+1}) = \arg\min_{a_t, c_t} \sum_{i=1}^{R} \left[\ln \varphi_t(s_{t,l}^i) - c_t - \ln k(s_{t,l}^i; a_t)\right]^2.$$

The EIS Filter, cont.

Implementation algorithm:

- Inherit $\{s_{t-1}^{0,i}, \omega(s_{t-1}^{0,i}; \widehat{a}_{t-1})\}_{i=1}^{N}$ from previous period
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$$(\widehat{a}_t^{l+1}, \widehat{c}_t^{l+1}) = \arg\min_{a_t, c_t} \sum_{i=1}^{R} \left[\ln \varphi_t(s_{t,l}^i) - c_t - \ln k(s_{t,l}^i; a_t)\right]^2 .$$

- Using $\widehat{a}_t$, generate $\{s_t^{0,i}\}_{i=1}^{N}$, and use to approximate

$$\widehat{f}_N(y_t | Y_{t-1}) = \frac{1}{N} \sum_{i=1}^{N} \frac{f\left(y_t | s_t^i, Y_{t-1}\right) f\left(s_t^i | Y_{t-1}\right)}{g(s_t^i; a_t)}.$$

The EIS Filter, cont.

Implementation algorithm:

- Inherit $\{s_{t-1}^{0,i}, \omega(s_{t-1}^{0,i}; \widehat{a}_{t-1})\}_{i=1}^{N}$ from previous period
- Obtain $\widehat{a}_t$ using

$$(\widehat{a}_t^{l+1}, \widehat{c}_t^{l+1}) = \arg\min_{a_t, c_t} \sum_{i=1}^{R} \left[\ln \varphi_t(s_{t,l}^i) - c_t - \ln k(s_{t,l}^i; a_t)\right]^2.$$

- Using $\widehat{a}_t$, generate $\{s_t^{0,i}\}_{i=1}^{N}$, and use to approximate

$$\widehat{f}_N(y_t | Y_{t-1}) = \frac{1}{N} \sum_{i=1}^{N} \frac{f\left(y_t | s_t^i, Y_{t-1}\right) f\left(s_t^i | Y_{t-1}\right)}{g(s_t^i; a_t)}.$$

- Pass $\{s_t^{0,i}, \omega(s_t^{0,i}; \widehat{a}_t)\}_{i=1}^{N}$ to the next period

Constructing Initial Samplers, Inner EIS

(Actually,begin with outer EIS) In establishing an initial sampler for

$$f\left(y_t \middle| Y_{t-1}\right) = \int f\left(y_t \middle| s_t, Y_{t-1}\right) f\left(s_t \middle| Y_{t-1}\right) ds_t,$$

disconnect between $f\left(y_t \middle| s_t, Y_{t-1}\right)$ and $f\left(s_t \middle| Y_{t-1}\right)$ can cause the EIS algorithm to "crash" (e.g., deliver non-positive-definite VCV matricies).

This very problem is the root of "sample impoverishment" associated with the Particle Filter.

Constructing Initial Samplers, cont.

A similar challenge arises given disconnect between $f\left(s_t | s_{t-1}, Y_{t-1}\right)$ and $f\left(s_{t-1} | Y_{t-1}\right)$ in establishing an initial sampler for

$$f\left(s_t \middle| Y_{t-1}\right) = \int f\left(s_t | s_{t-1}, Y_{t-1}\right) f\left(s_{t-1} \middle| Y_{t-1}\right) ds_{t-1}.$$

That is, for the inner EIS, the sampler established for $f\left(s_{t-1} \middle| Y_{t-1}\right)$ in period $t-1$ may not be effective in delivering an accurate approximation for $f\left(s_t \middle| Y_{t-1}\right)$.

Constructing Initial Samplers, cont.

Proposition: address these challenges by working with linear-Normal approximations to targeted densities.

(This is the subject of ongoing research; initial results appear highly promising in applications involving DSGE models.)

Constructing Initial Samplers, cont.

Return to the inner EIS problem

$$f\left(s_t | Y_{t-1}\right) = \int f\left(s_t | s_{t-1}, Y_{t-1}\right) f\left(s_{t-1} | Y_{t-1}\right) ds_{t-1}.$$

From period $t-1$, we have a normal approximation (constructed via EIS) for $f\left(s_{t-1} | Y_{t-1}\right)$:

$$f^*\left(s_{t-1} | Y_{t-1}\right) \sim N_k\left(\mu, \Omega\right).$$

Constructing Initial Samplers, cont.

In turn, $f\left(s_t | s_{t-1}, Y_{t-1}\right)$ is either linear-normal, or may be approximated as such (e.g., using the log-linear approximation of the model). Then it may be written as

$$f^*\left(s_t | s_{t-1}, Y_{t-1}\right) \sim N_k\left(\Phi s_{t-1}, \Psi\right).$$

Combining the distributions yields

$$\begin{aligned} f^*\begin{pmatrix} s_t \\ s_{t-1} \end{pmatrix} &\sim N_{2k}\left(\begin{bmatrix} \Phi_k \\ I_k \end{bmatrix}\mu, \begin{bmatrix} \Psi_{k\times k} & 0_{k\times k} \\ 0_{k\times k} & 0_{k\times k} \end{bmatrix} + \begin{bmatrix} \Phi \\ I \end{bmatrix}\Omega\begin{bmatrix} \Phi \\ I \end{bmatrix}'\right) \\ &= N_{2k}\left(m, \Sigma\right). \end{aligned}$$

Constructing Initial Samplers, cont.

Our goal is to implement $f^*\begin{pmatrix} s_t \\ s_{t-1} \end{pmatrix} \equiv f^*(s_t, s_{t-1})$ as a sampler for evaluating

$$f(s_t | Y_{t-1}) = \int f(s_t | s_{t-1}, Y_{t-1}) f(s_{t-1} | Y_{t-1}) ds_{t-1}.$$

Recall that we must do this conditionally for each value of s_t generated in the outer EIS integral. Therefore a final step is to factor $f^*(s_t, s_{t-1})$ as

$$f^*(s_t, s_{t-1}) = f^*(s_t) f^*(s_{t-1} | s_t).$$

Constructing Initial Samplers, cont.

Partitioning:

$$m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

with m_i $k \times 1$ and V_{ij} $k \times k$, the distributions we seek are

$$\begin{aligned} f^*(s_t) &\sim N_k(m_1, \Sigma_{11}), \\ f^*(s_{t-1}|s_t) &\sim N_k(m_{2\cdot 1} + \Delta_{21} s_t, \Sigma_{22\cdot 1}), \end{aligned}$$

where

$$\begin{aligned} \Delta_{21} &= \Sigma_{21}\Sigma_{11}^{-1}, \\ m_{2\cdot 1} &= m_2 - \Delta_{21} m_1, \\ \Sigma_{22\cdot 1} &= \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}. \end{aligned}$$

(e.g., see Bauwens, Lubrano, Richard, 1999, OUP).

Constructing Initial Samplers, cont.

Having partitioned,

$$
\begin{aligned}
f\left(s_t \middle| Y_{t-1}\right) &= \int f\left(s_t \middle| s_{t-1}, Y_{t-1}\right) f\left(s_{t-1} \middle| Y_{t-1}\right) ds_{t-1} \\
&= \int \frac{f\left(s_t \middle| s_{t-1}, Y_{t-1}\right) f\left(s_{t-1} \middle| Y_{t-1}\right)}{f^*\left(s_t, s_{t-1}\right)} f^*\left(s_t, s_{t-1}\right) ds_{t-1} \\
&= \int \frac{f\left(s_t \middle| s_{t-1}, Y_{t-1}\right) f\left(s_{t-1} \middle| Y_{t-1}\right)}{f^*\left(s_{t-1} \middle| s_t\right) f^*\left(s_t\right)} f^*\left(s_{t-1} \middle| s_t\right) f^*\left(s_t\right) ds_{t-1} \\
&= f^*\left(s_t\right) \int \frac{f\left(s_t \middle| s_{t-1}, Y_{t-1}\right) f\left(s_{t-1} \middle| Y_{t-1}\right)}{f^*\left(s_{t-1} \middle| s_t\right) f^*\left(s_t\right)} f^*\left(s_{t-1} \middle| s_t\right) ds_{t-1}
\end{aligned}
$$

Constructing Initial Samplers, cont.

Implementation: Obtain drawings $\left\{s_{t-1}^{i}\right\}$ from $f^{*}\left(s_{t-1}|s_{t}\right)$, approximate $f\left(s_{t}|Y_{t-1}\right)$ as

$$\begin{aligned} f\left(s_{t}|Y_{t-1}\right) &\approx f^{*}\left(s_{t}\right)\frac{1}{R}\sum_{i=1}^{R}\frac{f\left(s_{t}|s_{t-1}^{i},Y_{t-1}\right)f\left(s_{t-1}^{i}|Y_{t-1}\right)}{f^{*}\left(s_{t-1}^{i}|s_{t}\right)f^{*}\left(s_{t}\right)} \\ &\equiv f^{*}\left(s_{t}\right)\frac{1}{R}\sum_{i=1}^{R}w\left(s_{t-1}^{i}\right). \end{aligned}$$

If $f^{*}\left(s_{t,}s_{t-1}\right)$ is a good approximation for $f\left(s_{t}|s_{t-1},Y_{t-1}\right)f\left(s_{t-1}|Y_{t-1}\right)$, then the $w\left(s_{t-1}^{i}\right)'s$ will be roughly 1's, and

$$f\left(s_{t}|Y_{t-1}\right)\approx f^{*}\left(s_{t}\right).$$

Dirac Distributions

It is often the case that $f\left(s_t \middle| s_{t-1}, Y_{t-1}\right)$ is degenerate along certain dimensions.

For example, in the RBC model, the transition equation

$$\left(1+\frac{g}{1-\alpha}\right) k_t = i\left(k_{t-1}, z_{t-1}\right) + \left(1-\delta\right) k_{t-1}$$

implies degeneracy in the k_t dimension.

Dirac Distributions, cont.

In this case, implementation of inner EIS requires an additional step:

the sampler $f^*\left(s_{t,}s_{t-1}\right)$ must be altered so that conditional on s_t, draws of s_{t-1} are obtained that satisfy equality restrictions implied by the degeneracy of $f\left(s_t|s_{t-1}, Y_{t-1}\right)$.

Dirac Distributions, cont.

Let s_t partition into $s_t = (p_t, q_t)$, so that the transition equations consist of two parts:

- a proper transition density $f(p_t | s_{t-1}, Y_{t-1})$ for p_t
- an identity for $q_t | p_t, s_{t-1}$:

$$q_t \equiv \phi(p_t, p_{t-1}, q_{t-1}) = \phi(p_t, s_{t-1}).$$

Dirac Distributions, cont.

Assuming that $\phi\left(p_t, s_{t-1}\right)$ is differentiable and strictly monotone in q_{t-1}, with inverse

$$q_{t-1} = \psi\left(p_t, q_t, p_{t-1}\right) = \psi\left(s_t, p_{t-1}\right),$$

the predictive density becomes

$$\begin{aligned} f\left(s_t \middle| Y_{t-1}\right) &= \int J\left(s_t, p_{t-1}\right) f\left(p_t \middle| s_{t-1}, Y_{t-1}\right) \cdot \\ & f\left(p_{t-1}, q_{t-1} \middle| Y_{t-1}\right) \Big|_{q_{t-1}=\psi\left(s_t, p_{t-1}\right)} dp_{t-1}, \end{aligned}$$

where

$$J\left(s_t, p_{t-1}\right) = \frac{\partial}{\partial q_t} \psi\left(s_t, p_{t-1}\right).$$

Dirac Distributions, cont.

Consider the RBC model. We continue to have a normal approximation (constructed via EIS) for $f\left(s_{t-1} | Y_{t-1}\right)$:

$$f^{*}\left(s_{t-1} | Y_{t-1}\right) \sim N_{2}\left(\mu, \Omega\right).$$

In addition, we have a proper transition density for $\ln z_t$, and a degenerate transition for k_t :

$$f\left(z_{t} | s_{t-1}\right) \sim N_{1}\left(\left[\begin{array}{cc} 0 & \rho \end{array}\right] s_{t-1}, \sigma_{\varepsilon}^{2}\right).$$

Dirac Distributions, cont.

Combining these densities yields

$$
\begin{aligned}
f^{*}\begin{pmatrix} z_t \\ s_{t-1} \end{pmatrix} &\sim N_3\left(\begin{bmatrix} 0 & \rho \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mu, V \right), \\
V &= \sigma_{\varepsilon}^{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \rho \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \Omega \begin{bmatrix} 0 & \rho \\ 1 & 0 \\ 0 & 1 \end{bmatrix}'
\end{aligned}
$$

(Compare with $f^{*}(s_t, s_{t-1})$ above.)

Dirac Distributions, cont.

Having obtained $f^* \begin{pmatrix} z_t \\ s_{t-1} \end{pmatrix} \equiv f^* \left(z_t, s_{t-1} \right)$, we next seek a transformation into $f^* \left(s_t, z_{t-1} \right)$.

Recall why: we must compute the inner EIS for each candidate s_t to be evaluated in the outer EIS step.

Dirac Distributions, cont.

To accomplish the transformation, we exploit the linear approximation (in order to retain a linear-normal sampler) of the degenerate state-transition equation

$$k_t = f_{kk} k_{t-1} + f_{kz} z_{t-1},$$

which yields the mapping

$$\begin{aligned} \begin{bmatrix} s_t \\ z_{t-1} \end{bmatrix} &= A \begin{bmatrix} z_t \\ s_{t-1} \end{bmatrix}, \\ A &= \begin{bmatrix} 0 & f_{kk} & f_{kz} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ |A| &= |f_{kk}| \end{aligned}$$

(Note: (z, k) are understood here to represent logged deviations from ss.)

Dirac Distributions, cont.

Using this mapping, $f^*(z_t, s_{t-1})$ transforms into

$$
\begin{aligned}
f^*(s_t, z_{t-1}) &\sim N_{2k}(m, \Sigma), \\
m &= B\mu, \\
B &= A \begin{bmatrix} 0 & \rho \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \\
\Sigma &= AVA'
\end{aligned}
$$

Note that since

$$|\Sigma|^{-1} = |f_{kk}||V|^{-1},$$

the multiplicative term $|f_{kk}|$ will enter into $f^*(s_t, z_{t-1})$ (applying the usual change-of-variables formula).

Dirac Distributions, cont.

Having established $f^*(s_t, z_{t-1})$, we partition as in the non-degenerate case. Specifically, letting

$$m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \sigma_\varepsilon^2 \end{pmatrix},$$

with m_1 2×1, m_2 1×1, etc., we have

$$f^*(s_t, z_{t-1}) = f^*(s_t) f^*(z_{t-1} | s_t),$$

with

Dirac Distributions, cont.

$$\begin{aligned} f^*(s_t) &\sim N_k(m_1, \Sigma_{11}), \\ f^*(s_{t-1} | s_t) &\sim N_k(m_{2\cdot 1} + \Delta_{21} s_t, \Sigma_{22\cdot 1}), \end{aligned}$$

where

$$\begin{aligned} \Delta_{21} &= \Sigma_{21}\Sigma_{11}^{-1}, \\ m_{2\cdot 1} &= m_2 - \Delta_{21} m_1, \\ \Sigma_{22\cdot 1} &= \sigma_\varepsilon^2 - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}. \end{aligned}$$

Dirac Distributions, cont.

Having partitioned,

$$
\begin{aligned}
f\left(s_t \middle| Y_{t-1}\right) &= \int J\left(s_t, z_{t-1}\right) f\left(z_t \middle| s_{t-1}, Y_{t-1}\right) \cdot \\
&\quad f\left(z_{t-1}, k_{t-1} \middle| Y_{t-1}\right) \big|_{k_{t-1}=\psi(s_t, z_{t-1})} dz_{t-1} \\
&\approx \int J\left(s_t, z_{t-1}\right) \left|f_{kk}\right| f^*\left(s_t\right) f^*\left(z_{t-1} \middle| s_t\right) dz_{t-1} \\
&= f^*\left(s_t\right) \int J\left(s_t, z_{t-1}\right) \left|f_{kk}\right| f^*\left(z_{t-1} \middle| s_t\right) dz_{t-1}
\end{aligned}
$$

Dirac Distributions, cont.

Implementation: Obtain drawings $\left\{z_{t-1}^{i}\right\}$ from $f^{*}\left(z_{t-1}\middle|s_{t}\right)$, approximate $f\left(s_{t}\middle|Y_{t-1}\right)$ as

$$f\left(s_{t}\middle|Y_{t-1}\right) \approx f^{*}\left(s_{t}\right)\left|f_{kk}\right|\frac{1}{R}\sum_{i=1}^{R}J\left(s_{t}, z_{t-1}\right)$$

Constructing Effective Initial Samplers, Outer EIS

Recall the objective for outer EIS:

$$f\left(y_t \middle| Y_{t-1}\right) = \int f\left(y_t \middle| s_t, Y_{t-1}\right) f\left(s_t \middle| Y_{t-1}\right) ds_t.$$

Above, we constructed

$$f^*\left(s_t\right) \sim N_k\left(m_1, \Sigma_{11}\right)$$

which serves to approximate $f\left(s_t \middle| Y_{t-1}\right)$ (since we expect

$$\left|f_{kk}\right| \frac{1}{R} \sum_{i=1}^{R} J\left(s_t, z_{t-1}\right)$$

to very nearly be one).

Constructing Effective Initial Samplers, Outer EIS, cont.

Next, we seek a linear-Normal approximation $f^*(y_t | s_t, Y_{t-1}) \sim N(\mu(s_t), V)$. We obtain from our log-linear model approximation

$$\begin{aligned} y_{it} &= \bar{y}_{it} \exp\left(f_{ik} \widetilde{k}_{t-1} + f_{iz} \widetilde{z}_{t-1} \right) \\ &\approx \bar{y}_{it} \left(1 + f_{ik} \widetilde{k}_{t-1} + f_{iz} \widetilde{z}_{t-1} \right), \\ \widetilde{k}_{it-1} &= \ln\left(\frac{k_{t-1}}{\bar{k}} \right). \end{aligned}$$

Note that in terms of expected values, we may express $\widetilde{k}_t$ and $\widetilde{z}_t$ as

$$\widetilde{k}_t = \frac{\widetilde{k}_{t-1}}{\rho_k}, \quad \widetilde{z}_t = \frac{\widetilde{z}_{t-1}}{\rho_z}.$$

Constructing Effective Initial Samplers, Outer EIS, cont.

Therefore,

$$\mu\left(s_t\right) = r + Ps_t,$$

with

$$\begin{aligned} r_i &= \overline{y}_{it}, \\ P_{i,.} &= \begin{bmatrix} \dfrac{\overline{y}_i f_{ik}}{\rho_k} & \dfrac{\overline{y}_i f_{iz}}{\rho_z} \end{bmatrix}. \end{aligned}$$

Constructing Effective Initial Samplers, Outer EIS, cont.

So we have

$$\begin{aligned} f^*(s_t) &\sim N_k(m_1, \Sigma_{11}), \\ f^*(y_t | s_t, Y_{t-1}) &\sim N(\mu(s_t), V). \end{aligned}$$

Letting $Q = V^{-1}$, and $H = \Sigma_{11}^{-1}$, their product is Normal with mean

$$(H + P'QP)^{-1} \left[Hm_1 + P'Q(y_t - r) \right],$$

and variance

$$(H + P'QP)^{-1}.$$

Example: F-V/R-R (2005 JAE) RBC Model

Experiment: Generate artificial realizations of $\{y_t, i_t, n_t\}$, $T = 100$ from the RBC model.

Using both the Particle Filter and the EIS filter, recover likelihood estimates at actual parameter values.

Note: Std. dev.s of measurement errors for $\{y_t, i_t, n_t\}$: $[0.000158, 0.0011, 0.000866]$; in turn, $\sigma_\varepsilon = 0.007$.

Example, cont.

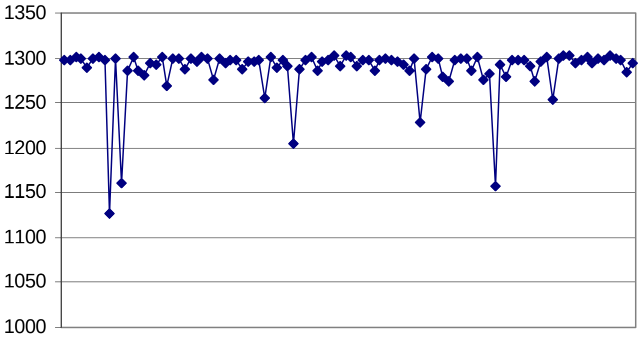
- Particle Filter: N=60,000, CPU time = 40.6 seconds per likelihood evaluation.
- EIS Filter: Outer EIS draws = 20, one EIS iteration; Inner EIS draws = 10 (no iterations necessary). CPU time = 0.22 seconds per evaluation

Mean, Std. Dev. of likelihood values over 1,000 MC replications:

Particle Filter:	1,285.51	33.48
EIS Filter:	1,299.81	0.00177

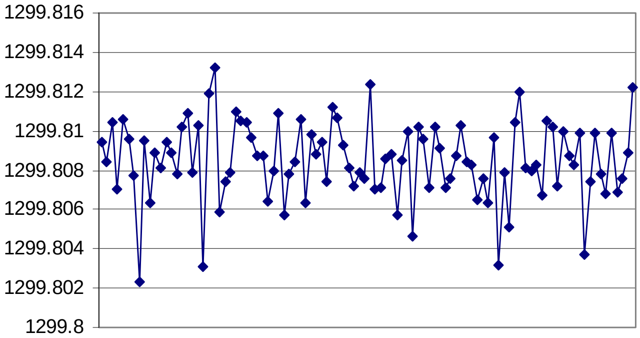
Example, cont.

Log-likelihoods, Particle Filter



Example, cont.

Log-likelihoods, Particle Filter

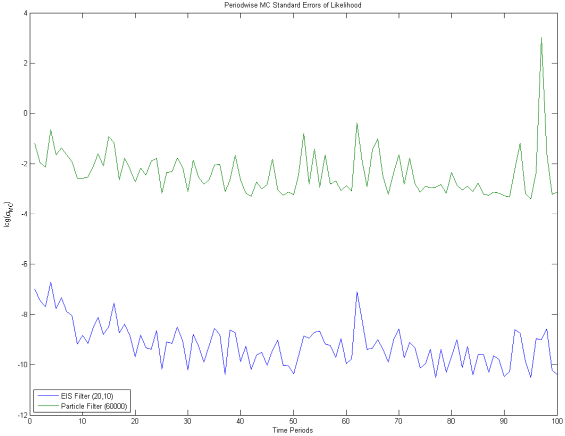


Example, cont.

Period-by-period MC standard errors (log scale):

Standard

Errors



$_3$.*pdf*

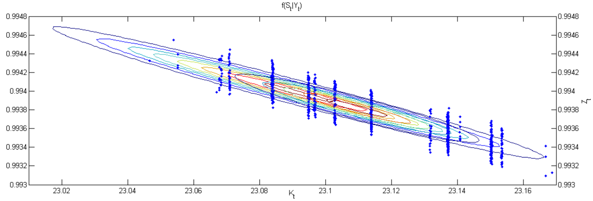
Example, cont.

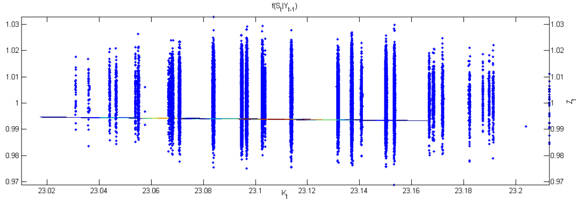
Diagnosis of the challenges facing the Particle Filter

Blue dots: EIS representations of $f\left(s_t \middle| Y_{t-1}\right)$, $f\left(s_t \middle| Y_t\right)$

Small ellipse: EIS representation of $f\left(y_t \middle| s_t, Y_{t-1}\right) f\left(s_t \middle| Y_{t-1}\right)$

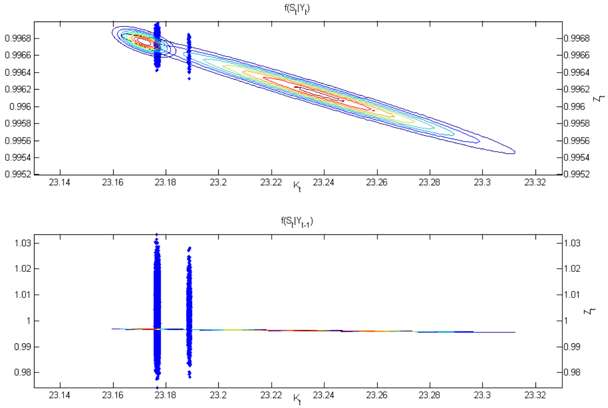
Large ellipse: $f\left(y_t \middle| s_t, Y_{t-1}\right)$, $t = 3$





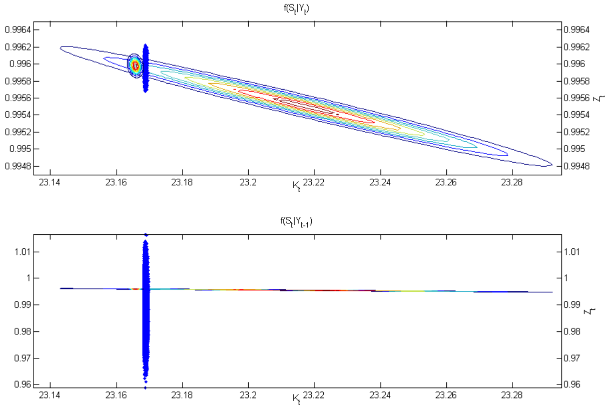
Example, cont.

$t = 18$



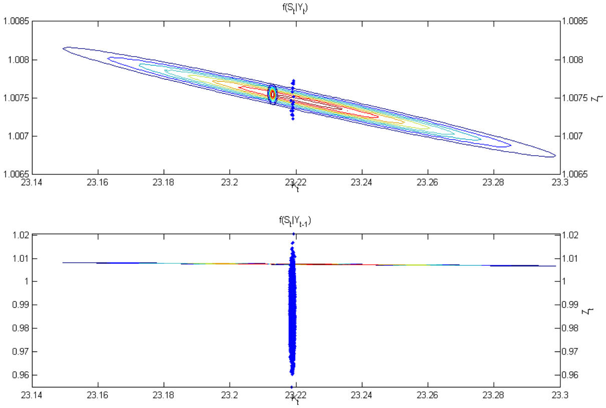
Example, cont.

$t = 53$



Example, cont.

$t = 66$



Example, cont.

Likelihood countours: α