Switching Regime Estimation

Series de TIempo

UTDT

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- The economy (the time series) often behaves very differently in periods such as booms and recessions. Markov Switching models are a useful way of characterizing this phenomena.

Uses of Markov Switching Models:

- Estimating models which are state dependent.
- Use the structure of the model to assess the probability that a state takes place, say a boom.
- Incorporate this feature in Rational Expectations models, pricing, derivatives, etc.

STRUCTURE OF THE LECTURE

- Definition and Properties.
- Estimation and testing
- Multivariate extensions and a R.E. example

Introduction

- Underlying assumption in all econometric models is that all observations have been drawn from the same distribution conditional on some constant parameter set
- The standard approach consists of trying to detect the existence of the regime changes and then imposing dummies.
    - Models with too many dummies

How should we model a change in the process?

- Suppose that the series under scrutiny has a break in its unconditional mean at time t_1. For data prior to t_1 we might use:

$$y_t - \mu_0 = \phi(y_{t-1} - \mu_0) + \varepsilon_t \quad \textit{for } t_1 < t$$

  and for data after t_1

$$y_t - \mu_1 = \phi(y_{t-1} - \mu_1) + \varepsilon_t \quad \textit{for } t \geq t_1$$

- Even if this captures the break, it is not a satisfactory model:
  - A complete time series model would include a description of the probability law governing the change from μ_0 to μ_1.

- We might consider the process to be influenced by an unobserved random variable x_t
  - x_t is the *state or regime*
- In the example above, we could regard x_t as:

$$x_t = \begin{cases} 0 \text{ if the process has mean } \mu_0 \\ 1 \text{ if the process has mean } \mu_1 \end{cases}$$

- Thus, we could write

$$y_t - \mu_{x_t} = \phi_1(y_{t-1} - \mu_{x_{t-1}}) + \varepsilon_t$$

where $\mu_{x_t} = (1 - x_t)\mu_0 + x_t\mu_1$

- Process for the unobserved variable: *Markov chain*

Properties of the Markov Process

Definition

Let x_t be a random variable that can take values 0 and 1. If the probability that x_t takes a particular value at time t, only depends on its value at $t-1$, this variable is governed by a *Markov process* of order 1.

$$P((x_t = i|x_{t-1} = j, x_{t-2} = k...) = P((x_t = i|x_{t-1} = j)$$

Thus the process is summarized by the probabilities:

$$P(x_t = 0|x_{t-1} = 0) = q \ , \ P(x_t = 1|x_{t-1} = 1) = p.$$

The transition Matrix

	0	1	(time t-1)
0	q	$(1-p)$	
1	$(1-q)$	p	
(time t)			

Autoregressive Representation of Markov Process

$$\begin{bmatrix} 1 - x_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} q & (1-p) \\ (1-q) & p \end{bmatrix} \begin{bmatrix} 1 - x_t \\ x_t \end{bmatrix} + \begin{bmatrix} \zeta_{1,t+1} \\ \zeta_{2,t+1} \end{bmatrix}$$

- The second row gives

$$x_{t+1} = (1-q) + (-1+p+q)x_t + \zeta_{2,t+1}$$

- This expression can be recognized as an AR(1) process with constant term $1-q$ and autoregressive coefficient $(-1+p+q)$

- Expected value of x_t :

$$E(x_{t+1}) = (1-q) + (-1+p+q)E(x_t)$$

or

$$E(x_t) = \frac{1-q}{2-p-q}$$

since $E(x_{t+1}) = E(x_t)$ for a stationary process.

- **Unconditional probabilities** of being in state 1 and 0.
- Notice that $E(x_t) = 0P(x_t = 0) + 1P(x_t = 1) = \dfrac{1-q}{2-p-q}$.
- Then,

$$P(x_t = 1) = \frac{1-q}{2-p-q},$$
$$P(x_t = 0) = \frac{1-p}{2-p-q}.$$

Conditional and Unconditional Probabilities of States 0 and 1

An alternative derivation

- It can be shown that the unconditional probabilities at time zero multiplied by the matrix of transition probabilities are equal to the unconditional probabilities at time one:

$$\begin{bmatrix} P(x_1 = 0) \\ P(x_1 = 1) \end{bmatrix} = \begin{bmatrix} q & (1-p) \\ (1-q) & p \end{bmatrix} \begin{bmatrix} P(x_0 = 0) \\ P(x_0 = 1) \end{bmatrix}$$

Equilibrium Probabilities

- If the process is stationary, there exist state probabilities $\{\pi_0, \pi_1\}$ that satisfy:

$$\Pi = \mathbf{P}\Pi$$

where $\mathbf{P}$ is the matrix of transition probabilities.

$$\begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix} = \begin{bmatrix} q & (1-p) \\ (1-q) & p \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix}$$

where $\pi_0 = P(x_{t-j} = 0)$ and $\pi_1 = P(x_{t-j} = 1)$ for all values of j.

- Using $\pi_0 + \pi_1 = 1$:

$$\pi_0 = \frac{(1-p)}{(2-p-q)} \text{ and } \pi_1 = \frac{(1-q)}{(2-p-q)}$$

where π_0 and π_1 are the equilibrium unconditional probabilities.

- Given the following initial values:

$$\begin{bmatrix} p^0(0) \\ p^0(1) \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix}$$

  It can be shown (by multiplying n times by the transition probability matrix) that the unconditional probability vector at time n is:

$$\begin{bmatrix} p^n(0) \\ p^n(1) \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \pi_1 \end{bmatrix}$$

- Therefore, the distribution does not change with time and the stochastic process is always in equilibrium.

Forecasts for a Markov Chain

- A n- period ahead forecast for a Markov chain can be obtained simply by multiplying n times by the transition probability:

$$\begin{bmatrix} P(x_{t+n}=0) \\ P(x_{t+n}=1) \end{bmatrix} = \begin{bmatrix} q & (1-p) \\ (1-q) & p \end{bmatrix}^n \begin{bmatrix} P(x_t=0) \\ P(x_t=1) \end{bmatrix}$$

$\mathbf{P}^n$ is derived in the following way:

1) Find the eigenvalues of the transition probability Matrix.

$$\lambda_1 = 1, \quad \lambda_2 = -1 + p + q,$$

2) Find the associated eigenvectors.

$$\begin{bmatrix} \frac{(1-p)}{(2-p-q)} \\ \frac{(1-q)}{(2-p-q)} \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

3) Express **P** as $T\Lambda T^{-1}$, where

$$T = \begin{bmatrix} \frac{(1-p)}{(2-p-q)} & -1 \\ \frac{(1-q)}{(2-p-q)} & 1 \end{bmatrix}$$

is the matrix of eigenvectors and

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1+p+q \end{bmatrix}$$

a diagonal matrix of eigenvalues.

4) Use the result that

$$\mathbf{P}^n = T\Lambda^n T^{-1}$$

or

$$P^n = \begin{bmatrix} \frac{1-p}{(2-p-q)} & -1 \\ \frac{1-q}{2-p-q} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1+p+q \end{bmatrix}^n \begin{bmatrix} \frac{1-p}{(2-p-q)} & -1 \\ \frac{1-q}{2-p-q} & 1 \end{bmatrix}^{-1}$$

$$
\begin{array}{lcc}
 & 0 & 1 \quad (\,\mathrm{t}) \\
\begin{array}{l} 0 \\ 1 \end{array} & \multicolumn{2}{c}{\left[\begin{array}{c|c}
\frac{(1-p)}{(2-p-q)} + \frac{(1-q)(p+q-1)^n}{(2-p-q)} & \frac{(1-p)}{(2-p-q)} - \frac{(1-p)(p+q-1)^n}{(2-p-q)} \\
\frac{(1-q)}{(2-p-q)} - \frac{(1-q)(p+q-1)^n}{(2-p-q)} & \frac{(1-q)}{(2-p-q)} + \frac{(1-p)(p+q-1)^n}{(2-p-q)}
\end{array}\right]} \\
(\,\mathrm{t+n}) & &
\end{array}
$$

Note that by making $n = 1$ in the above matrix we end with the matrix of transition probabilities, **P**.

- In addition, we can derive the conditional expectations:

$$
\begin{aligned}
E(x_{t+n}|x_t &= 0) = \frac{(1-q)}{(2-p-q)} - \frac{(1-q)(p+q-1)^n}{(2-p-q)} \\
E(x_{t+n}|x_t &= 1) = \frac{(1-q)}{(2-p-q)} + \frac{(1-p)(p+q-1)^n}{(2-p-q)}
\end{aligned}
$$

- The expected value of x_t at time n conditional on s at time zero is:

$$E(x_{t+n}|x_t = x_t) = \frac{(1-q)}{(2-p-q)} + (x_t - \frac{(1-q)}{(2-p-q)})(p+q-1)^n$$

- This result is derived assuming that x_t is observed (conditional on x_t)

A brief description of Hamilton's non linear filter

- The procedure assumes that discrete states of the economy are not known: inferred from the data.
- States follow a discrete Markov process.

A brief description of Hamilton's non linear filter

- The observed variable, y_t , is assumed to follow an $AR(m)$:

$$y_t - \mu_{x_t} = \phi_1(y_{t-1} - \mu_{x_{t-1}}) + .. + \phi_m(y_{t-m} - \mu_{x_{t-m}}) + \sigma_{x_t}\varepsilon_t$$

  ε_t is distributed N(0,1) and μ_{x_t} is parameterized as $\alpha_0 + \alpha_1 x_t$ and σ_{x_t} as $w_0 + w_1 x_t$

- The error ε_t is assumed to be independent of all $x_{t-j} \geq 0$

Step 1 Calculate the joint density of the m past states and the current state conditional on the information included in y_{t-1} and all past values of y, where y is the variable that is observed:

$$p(x_t, x_{t-1}, , ..., x_{t-m} | y_{t-1}, .., y_0) = p(x_t | x_{t-1}) p(x_{t-1}, , ..., x_{t-m} | y_{t-1}, ..., y_0)$$

Step 1

As in all the subsequent steps the second term on the right-hand-side is obtained from the preceding step of the filter. In this case, $p(x_{t-1}, x_{t-2}, ..., x_{t-m} | y_{t-1}, y_{t-2}, ..., y_0)$ is known from the input to the filter, which in turn represents the result of the iteration at date $t-1$ (from step 5).

Step 1

To begin with the iteration, it is necessary to assign some initial values to the parameters, and to impose some initial conditions on the Markov process. The unconditional distribution $p(x_{m-1}, x_{m-2}, ...x_0)$ has been chosen for the first observation:

$$p(x_{m-1}, x_{m-2}, ...x_0) = p(x_{m-1}|x_{m-2})...p(x_1|x_0)p(x_0)$$

where $p(x_0)$ are the equilibrium unconditional probabilities as defined above.

Step 2

Calculate the joint conditional distribution of y_t and $(x_t, x_{t-1}, ..., x_{t-m})$.

$$
\begin{aligned}
& p(y_t, x_t, x_{t-1}, ..., x_{t-m} | y_{t-1}, y_{t-2}, ..., y_0) \\
= \quad & p(y_t | x_t, x_{t-1}, ..., x_{t-m}, y_{t-1}, y_{t-2}, ..., y_0) \times \\
& p(x_t, x_{t-1}, x_{t-2}, ..., x_{t-m} | y_{t-1}, y_{t-2}, ..., y_0)
\end{aligned}
$$

We assume that

$$
\begin{aligned}
& p(y_t | x_t, x_{t-1}, .., x_{t-m}, y_{t-1}, y_{t-2}, .., y_0) \\
= \quad & \frac{1}{\sqrt{2\pi}(\omega_0 + \omega_1 x_t)} \exp[-\frac{1}{2[\omega_0 + \omega_1 x_t]^2}((y_t - \alpha_1 x_t - \alpha_0) \\
& -\phi_1(y_{t-1} - \alpha_1 x_{t-1} - \alpha_0) - .. - \phi_m(y_{t-m} - \alpha_1 x_{t-m} - \alpha_0))^2]
\end{aligned}
$$

Step 3

Marginalize the previous joint density with respect to the states giving the conditional density, from which the (conditional) likelihood function is calculated.

$$
\begin{aligned}
& p(y_t | y_{t-1}, y_{t-2}, ., y_0) \\
= \quad & \sum_{x_t=0}^{1} \sum_{x_{t-1}=0}^{1} .. \sum_{x_{t-m}=0}^{1} p(y_t, x_t, x_{t-1}, ., x_{t-m} | y_{t-1}, y_{t-2}, ., y_0)
\end{aligned}
$$

Step 4

Combine the results from steps 2 and 3 to calculate the joint density of the state conditional on the observed current and past realizations of y

$$
\begin{aligned}
& p(x_t, x_{t-1}, ., x_{t-m} | y_t, y_{t-1}, y_{t-2}, ., y_0) \\
= \quad & \frac{p(y_t, x_t, x_{t-1}, ., x_{t-m} | y_{t-1}, y_{t-2}, ., y_0)}{p(y_t | y_{t-1}, y_{t-2}, ., y_0)}
\end{aligned}
$$

Step 5

The desired output is then obtained from

$$p(x_t, ., x_{t-m+1} | y_t, y_{t-1} .., y_0) = \sum_{x_{t-m}=0}^{1} p(x_t, ., x_{t-m} | y_t, ., y_0)$$

The output of step 5 is used as an input to the filter in the next iteration.

Hamilton's Filter

Step 5

- Note that to iterate, estimates of the parameters are required.
- Maximum likelihood estimates can be obtained numerically from Step 3 as a by-product of the filter

$$\ln p(y_t, y_{t-1}, y_{t-2}, ., y_m | y_{m-1}, ., y_0) = \sum_{t=m}^{T} \ln p(y_t | y_{t-1}, ., y_0).$$

which can be maximized numerically with respect to the unknown parameters $(\alpha_1, \alpha_0, p, q, \omega_0, \omega_1, \phi_1, \phi_2 ... \phi_m)$.

Comments

- Notice that p and q, the parameters of the transition matrix, are also estimated by maximum likelihood.
- Hamilton's filter requires the numerical optimization of a very complicated non-linear function.

Specification Tests

- Specification tests based on the properties of the standardized residuals, $\widehat{\varepsilon}_t = \frac{y_t - E\widehat{(y_t|I_{t-1})}}{\widehat{\sigma}_t}$.
- First calculate the   of y_t given information at time $t-1$:

$$\begin{aligned} E(y_t|I_{t-1}) &= \alpha_0 + \alpha_1 E(x_t|I_{t-1}) + \phi_1(y_{t-1} - \alpha_1 E(x_{t-1}|I_{t-1}) - \alpha_0) \\ &\quad +... + \phi_m(y_{t-m} - \alpha_1 E(x_{t-m}|I_{t-1}) - \alpha_0), \end{aligned}$$

  where

$$E(x_t|I_{t-1}) = \frac{(1-q)}{(2-p-q)} + (P(x_{t-1}|I_{t-1}) - \frac{(1-q)}{(2-p-q)})(p+q-1),$$
$$E(x_{t-m}|I_{t-1}) = P(x_{t-m} = 1|I_{t-1}), m \geq 1.$$

- $P(x_{t-m}|I_{t-1})$, for $m > 1$, are called "*smoothing probabilities*"
  - They can be calculated from the "filtering probabilities".

Computing the standard deviation

Step 1 First make use of the autoregressive representation of the Markov process

$$x_t = (1-q) + (-1+p+q)x_{t-1} + \zeta_{2,t}$$

For this process the error, conditional on $x_{t-1} = 1$, can be characterized as

$$\zeta_{2,t} = \begin{array}{l} (1-p) \text{ with probability } p \\ -p \text{ with probability } 1-p \end{array}$$

and conditional on $x_{t-1} = 0$

$$\zeta_{2,t} = \begin{array}{l} -(1-q) \text{ with probability } q \\ q \text{ with probability } 1-q \end{array}$$

Computing the standard deviation

Step 2 Calculate the variance of the error term, $\zeta_{2,t}$, conditional on the state at $t-1$.

$$\begin{aligned} E(\zeta_{2,t}^2|x_{t-1} &= 1) = (1-p)^2 p + p^2(1-p) = p(1-p) \\ E(\zeta_{2,t}^2|x_{t-1} &= 0) = (1-q)^2 q + q^2(1-q) = q(1-q) \end{aligned}$$

Computing the standard deviation

Step 3 Calculate the conditional variance (conditional on $I_{t-1} = \{y_{t-1}, ...y_0\}$)

We start by calculating the state dependent variance $\sigma^2_{x_t}$ as a function of the Markov switching parameters.

Computing the standard deviation

Step 3 Conditional on $x_{t-1} = 1$, the switching variance can be written as:

$$\sigma^2_{x_t} = E(\sigma^2_{x_t} | x_{t-1} = 1) + V(\mu_{x_t} | x_{t-1} = 1)$$

where

$$E(\sigma^2_{x_t} | x_{t-1} = 1) = (E(\sigma_{x_t} || x_{t-1} = 1))^2 + Var((\sigma_{x_t} | x_{t-1} = 1))$$

(since σ_{x_t} is a random variable) and

$$Var((\sigma_{x_t} | x_{t-1} = 1)) = E(\sigma^2_{x_t} | x_{t-1} = 1) - (E(\sigma_{x_t} | x_{t-1} = 1))^2$$

Computing the standard deviation

Step 3 Then using that

$$(E(\sigma_{x_t}|x_{t-1}=1))^2=(w_0+w_1E(x_t|x_{t-1}=1)^2=(w_0+w_1p)^2$$

$$\begin{aligned} Var((\sigma_{x_t}|x_{t-1} &= 1))=Var(w_0+w_1x_t|x_{t-1}=1) \\ &= w_1^2p(1-p). \end{aligned}$$

$$\begin{aligned} V(\mu_{x_t}|x_{t-1} &= 1)=Var(\alpha_0+\alpha_1x_t|x_{t-1}=1) \\ &= \alpha_1^2p(1-p) \end{aligned}$$

Computing the standard deviation

Step 3 Collecting all these terms we can see that

$$\sigma^2_{x_t} = (w_0 + w_1 p)^2 + w_1^2 p(1-p) + \alpha_1^2 p(1-p)$$

We can obtain a similar formulae for the variance conditional on $x_{t-1} = 0$.

$$\sigma^2_{x_t} = E(\sigma^2_{x_t} | x_{t-1} = 0) + V(\mu_{x_t} | x_{t-1} = 0)$$

and doing the same transformations for state 0 we obtain

$$\sigma^2_{x_t} = (w_0 + w_1(1-q))^2 + w_1^2 q(1-q) + \alpha_1^2 q(1-q).$$

Computing the standard deviation

Step 3 Clearly the state is not observed at time $t-1$ but we can use the filtered probabilities to make an inference of the unobserved state.

Then the conditional variance (on information on time $t-1$) is

$$\sigma_t^2 = \begin{array}{c} ((w_0 + w_1 p)^2 + w_1^2 p(1-p) \\ +\alpha_1^2 p(1-p))P(x_{t-1} = 1|I_{t-1})+ \\ ((w_0 + w_1(1-q))^2 + w_1^2 q(1-q)+ \\ \alpha_1^2 q(1-q))(1 - P(x_{t-1} = 1|I_{t-1})) \end{array} .$$

Computing the standard deviation

Step 3 Then the standardized residuals are simply, $v_t = \varepsilon_t / \sigma_t$ and we may conduct standard specification tests for these residuals.

Number of tests and specification tests

- Crucial: rightly identify the number of states or regimes.
- Hamilton proposes to use simple specification tests as a mean of assessing whether the estimated equation contains the right number of states.

A Bivariate model with regime switching

- We consider a VAR process in two variables, with 1 lag, with the feature that the means of each equation and the variance-covariance matrix are allowed to switch endogenously between two possible states.
- The two equations that define the VAR are influenced by the same state variable.
- The state is not observed and has to be inferred from a filter.

$$S_t' = \Phi S_{t-1}' + \psi D_{t-1}' + (\omega_0 + \omega_1 x_t)\nu_t$$

$$D_t' = \varphi S_{t-1}' + \Omega D_{t-1}' + (\tau_0 + \tau_1 x_t)\varepsilon_t$$

A Bivariate model with regime switching

- The centered variables are defined by the two following equations:

$$S'_t = S_t - \alpha_0 - \alpha_1 x_t$$

$$D'_t = D_t - \beta_0 - \beta_1 x_t$$

- A prime (') is used to denote centred variables in the remainder of the presentation
- x_t denotes the unobserved state of the system and takes values 0 and 1.

A Bivariate model with regime switching

- x_t is governed by a Markov process

$$
\begin{aligned}
P(x_t &= 0|x_{t-1} = 0) = q \\
P(x_t &= 1|x_{t-1} = 1) = p
\end{aligned}
$$

- The errors ν_t, ε_t are assumed to be independent of all x_{t-j}. $j \geq 0$.

A Bivariate model with regime switching

Substituting the centered variables into the VAR and rearranging terms, we obtain the following expression for S_t and D_t

$$\begin{aligned} S_t &= \alpha_0(1-\Phi) - \beta_0\psi + \Phi S_{t-1} + \psi D_{t-1} \\ &\quad + \alpha_1(x_t - \Phi x_{t-1}) - \beta_1\psi x_{t-1} + (\omega_0 + \omega_1 x_t)\nu_t \end{aligned}$$

$$\begin{aligned} D_t &= -\alpha_0\varphi + \beta_0(1-\Omega) + \varphi S_{t-1} + \Omega D_{t-1} \\ &\quad - \alpha_1\varphi x_{t-1} + \beta_1(x_t - \Omega x_{t-1}) + (\tau_0 + \tau_1 x_t)\varepsilon_t \end{aligned}$$

Testing The Term Structure of Interest Rates

- The process that drives the spread and the short-term interest rate difference is the VAR of equation described above in which D_t denotes the first difference of the three month rate, $R_{1t} - R_{1t-1}$ and S_t denotes the yield spread $R_{2t} - R_{1t}$.
- The expectations hypothesis of the term structure of the interest rates can be written as
$$S_t = (1/2)E_t D_{t+1} + \theta + u_t$$
- The restrictions imposed by the expectations model are presented below. Both an unrestricted and a restricted VAR can be estimated, and the restrictions tested using a likelihood ratio test.

Derivation of the restrictions in the regime-shifting VAR.

$$\begin{bmatrix} S'_t \\ D'_t \end{bmatrix} = \begin{bmatrix} \Phi & \psi \\ \varphi & \Omega \end{bmatrix} \begin{bmatrix} S'_{t-1} \\ D'_{t-1} \end{bmatrix} + \begin{bmatrix} (\omega_0 + \omega_1 x_t)\nu_t \\ (\tau_0 + \tau_1 x_t)\varepsilon_t \end{bmatrix}$$

To find the restrictions we condition on information available at $t-1$ on both sides of the term structure equation.

$$E(S_t | I^*_{t-1}) = (1/2)E(D_{t+1} | I^*_{t-1}) + \theta$$

were $I^*_{t-1} = \{S_{t-1}, S_{t-2}, ., D_{t-1}, D_{t-2}, ., x_{t-1}, ., \}$.

Then, we need to calculate expected values $E[D_{t+1}|I^*_{t-1}]$ and $E[S_t|I^*_{t-1}]$. These can be calculated in the following way:

$$E[D_{t+1}|I^*_{t-1}] = \beta_0 + \beta_1 E(x_{t+1}|I^*_{t-1}) + E(D'_{t+1}|I^*_{t-1}),$$

where

$$\begin{aligned} E[D'_{t+1}|I^*_{t-1}] &= [0,1]\,\Delta^2 Z'_{t-1}, \\ E(x_{t+1}|I^*_{t-1}) &= [\rho + (x_{t-1}-\rho)\lambda^2] \end{aligned}$$

where $\rho = \frac{1-q}{2-p-q}$ and $\lambda = (p+q-1)$.

$$E[S_t|I^*_{t-1}] = \alpha_0 + \alpha_1 E(x_t|I^*_{t-1}) + E(S'_t|I^*_{t-1})$$

where

$$\begin{aligned} E(S'_t|I^*_{t-1}) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \Delta Z'_{t-1} \\ E(x_t|I^*_{t-1}) &= [\rho + (x_{t-1}-\rho)\lambda] \end{aligned}$$

Finally, substituting the expected (difference of the) short-term interest rates and the expected spread, the term structure of interest rates relationship can be expressed as;

$$\begin{aligned} & \alpha_0 + \alpha_1[\rho + (x_{t-1} - \rho)\lambda] + [1, 0]\, \Delta Z'_{t-1} && ((A4)) \\ = \quad & \frac{1}{2}[\beta_0 + \beta_1[\rho + (x_{t-1} - \rho)\lambda^2] + [0, 1]\, \Delta^2 Z'_{t-1}] + \theta \end{aligned}$$

The restrictions written out in full are:

$$\begin{aligned} \alpha_1 &= \frac{1}{2}\beta_1\lambda \\ \Phi &= \frac{\varphi\Omega}{2-\varphi} \\ \psi &= \frac{\Omega^2}{2-\varphi} \end{aligned}$$

Contagion

- Define:

$$z_t = [x_t, y_t]'$$

$$z_t = \mu + \Phi_{s_t} u_t$$

where $\mu = [\mu_x, \mu_y]'$ and u_t is a Gaussian process

- $\{s_t\}$ is modelled as a time-homogeneous Markov chain on $\{1, 2, 3, 4\}$, independent of $\{u_t\}$
- Thus,

$$z_t | (s_t = s) \sim N(\mu, \Omega_{s_t})$$

Contagion

- The variance covariance matrix are:

$$\Omega = \left\{ \Omega_{s=1} = \begin{bmatrix} \sigma_{xh}^2 & \sigma_{xh,yh} \\ \sigma_{yh,xh} & \sigma_{yh}^2 \end{bmatrix},\ \Omega_{s=2} = \begin{bmatrix} \sigma_{xh}^2 & \sigma_{xh,yl} \\ \sigma_{yl,xh} & \sigma_{yl}^2 \end{bmatrix} \right.$$

$$\left. \Omega_{s=3} = \begin{bmatrix} \sigma_{xl}^2 & \sigma_{xl,yh} \\ \sigma_{yh,xl} & \sigma_{yh}^2 \end{bmatrix},\ \Omega_{s=4} = \begin{bmatrix} \sigma_{xl}^2 & \sigma_{xl,yl} \\ \sigma_{yl,xl} & \sigma_{yl}^2 \end{bmatrix} \right\}$$

Contagion

- Transition matrix: 4×4 matrix Π (with elements $\pi_{ij} = \Pr(s_t = i | s_{t-1} = j)$, $i, j = 1, 2, 3, 4$)
- No contagion amounts to:

$$\begin{pmatrix} \pi_{xh}\pi_{yh} & \pi_{xh}(1-\pi_{yl}) & (1-\pi_{xl})\pi_{yh} & (1-\pi_{xl})(1-\pi_{yl}) \\ \pi_{xh}(1-\pi_{yh}) & \pi_{xh}\pi_{yl} & (1-\pi_{xl})(1-\pi_{yh}) & (1-\pi_{xl})\pi_{yl} \\ (1-\pi_{xh})\pi_{yh} & (1-\pi_{xh})(1-\pi_{yl}) & \pi_{xl}\pi_{yh} & \pi_{xl}(1-\pi_{yl}) \\ (1-\pi_{xh})(1-\pi_{yh}) & (1-\pi_{xh})\pi_{yl} & \pi_{xl}(1-\pi_{yh}) & \pi_{xl}\pi_{yl} \end{pmatrix}.$$

Contagion

- Contagion will occur when one of the countries leads (or lags) the other one.
- This hypothesis can be verified testing (using LR tests distributed as $\chi^2(10)$) if we can reduce the transition matrices to $\Pi_1^{xly} =$

$$\begin{pmatrix} \pi_{xh} & \pi_{xh} & 0 & 0 \\ 0 & 0 & (1-\pi_{xl}) & (1-\pi_{xl}) \\ (1-\pi_{xh}) & (1-\pi_{xh}) & 0 & 0 \\ 0 & 0 & \pi_{xl} & \pi_{xl} \end{pmatrix}$$

where Π_1^{xly} indicates x leads y one period

Markov Switching Causality

- The analysis of Granger causality between x_1 and x_2 is based on the following Markov switching VAR model:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} \mu_{10} + \mu_{11} s_{1,t} \\ \mu_{20} + \mu_{21} s_{2,t} \end{bmatrix} + \begin{bmatrix} \phi_{10} + \phi_{11} s_{1,t} & \psi_1 s_{1,t} \\ \psi_2 s_{2,t} & \phi_{20} + \phi_{21} s_{2,t} \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix}$$

$$+ \begin{bmatrix} \varphi_{10} + \varphi_{11} s_{1,t} \\ \varphi_{20} + \varphi_{21} s_{2,t} \end{bmatrix} z_{t-1} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}, \quad t = 1, \ldots, T.$$

  where $s_{1,t}$ $s_{2,t} \epsilon \{0, 1\}$ are unobserved random variables

- x_2 Granger causes x_1 when $s_{1,t} = 1$ and is Granger non-causal for x_1 when $s_{1,t} = 0$

Target Zone Credibility

- Dynamic of the nominal interest rate differential

$$r_t - \mu(s_t) = \sum_{j=1}^{m} \phi_j [r_{t-j} - \mu(s_{t-j})] + \sigma(s_t)\varepsilon_t, \qquad (t = 1, \ldots, T)$$

where $s_t \epsilon \{0, 1\}$ is the state variable with

$$\begin{aligned}
\Pr(s_t = 1 | s_{t-1} = 1) &= p, \\
\Pr(s_t = 0 | s_{t-1} = 1) &= 1 - p, \\
\Pr(s_t = 0 | s_{t-1} = 0) &= q, \\
\Pr(s_t = 1 | s_{t-1} = 0) &= 1 - q,
\end{aligned}$$

Target Zone Credibility

- $$\mu(s_t) = \alpha_0 + \alpha_1 s_t,$$

  and

  $$\sigma(s_t) = \sigma_0(1 - s_t) + \sigma_1 s_t.$$

  where $\alpha_1 - \alpha_0 > 0$

Target Zone Credibility

- To assess the links between target-zone credibility and macroeconomic variables:

$$\Pr(s_t = 1|s_{t-1} = 1, z_{t-1}) = p_t = \exp(c_1 + \beta_1 z_{t-1})/[1 + \exp(c_1 + \beta_1 z_{t-1})],$$
$$\Pr(s_t = 0|s_{t-1} = 0, z_{t-1}) = q_t = \exp(c_0 + \beta_0 z_{t-1})/[1 + \exp(c_0 + \beta_0 z_{t-1})],$$
$$\Pr(s_t = 0|s_{t-1} = 1, z_{t-1}) = 1 - \exp(c_1 + \beta_1 z_{t-1})/[1 + \exp(c_1 + \beta_1 z_{t-1})],$$
$$\Pr(s_t = 1|s_{t-1} = 0, z_{t-1}) = 1 - \exp(c_0 + \beta_0 z_{t-1})/[1 + \exp(c_0 + \beta_0 z_{t-1})],$$

where z_t is an economic variable that affects the state transition probabilities.

Target Zone Credibility

- Sample log-likelihood function

$$\log \mathfrak{L} = \sum_{t=m+1}^{T} \log f(r_t|\mathcal{F}_{t-1})$$

where $\mathcal{F}_i = (r_1, z_1, \ldots, r_i, z_i)$ $(i \geqslant 1)$ and $f(r_t|\mathcal{F}_{t-1})$ represents the conditional density of r_t given the set of information that is available at date $t-1$

Target Zone Credibility

- Inferences about the unobserved regimes $\{s_t\}$ may be made on the basis of the filter probabilities $\Pr(s_t|\mathcal{F}_t)$, obtained as:

$$\Pr(s_t|\mathcal{F}_t) = \sum_{s_{t-1}=0}^{1} \sum_{s_{t-2}=0}^{1} \cdots \sum_{s_{t-m}=0}^{1} \Pr(s_t, s_{t-1}, \ldots, s_{t-m}|\mathcal{F}_t).$$

Intrinsic Bubbles and Regime Switching

- $$P_t = e^{-r} E_t(D_t + P_{t+1}).$$
- Any rational bubble B_t in the stock price satisfies
$$B_t = e^{-r} E_t(B_{t+1}).$$
- The process that drives the log of dividends is assumed to be a random walk with drift μ :
$$d_{t+1} = \mu + d_t + \xi_{t+1},$$
where $\xi_{t+1} \backsim N(0, \sigma^2)$.
- The "intrinsic bubble":
$$B(D_t) = cD_t^{\lambda}.$$

- The parameter λ is the positive root of the quadratic equation

$$\frac{\sigma^2}{2}\lambda^2 + \mu\lambda - r = 0$$

- The present value (denoted by P_t^{pv}) is proportional to dividends:

$$P_t^{pv} = kD_t,$$

where $k = (e^r - e^{(\mu + \frac{1}{2}\sigma^2)})^{-1}$.

$$P_t = kD_t + cD_t^{\lambda}.$$

Intrinsic Bubbles and Regime Switching

- Evolution of real dividends

$$d_{t+1} = d_t + \mu_0(1 - s_{t+1}) + \mu_1 s_{t+1} + (\sigma_0(1 - s_{t+1}) + \sigma_1 s_{t+1})\varepsilon_{t+1}$$

  where s_{t+1} follow an homogenous first order Markov Process and ε_{t+1} is an *iid* variable
- $p(s_t = 1|s_{t-1} = 1) = p$ and $p(s_t = 0|s_{t-1} = 0) = q$.
- The fundamental value of the stock is

$$P_t = \begin{cases} k_0 D_t \text{ if } s_t = 0 \\ k_1 D_t \text{ if } s_t = 1 \end{cases}$$

- Then k_0 and k_1 satisfy

$$
\begin{aligned}
k_0 &= e^{-r}(1 + qk_0 a_0 + (1-q)k_1 a_1). \\
&\quad \text{and} \\
k_1 &= e^{-r}(1 + pk_1 a_1 + (1-p)k_0 a_0)
\end{aligned}
$$

where $a_0 = e^{(\mu_0 + \frac{1}{2}\sigma_0^2)}$ and $a_1 = e^{(\mu_1 + \frac{1}{2}\sigma_1^2)}$.

Intrinsic Bubbles and Regime Switching

- Intrinsic Bubble:
$$B_t = c_i D_t^{\lambda} \text{ when } s_t = i$$
- It satisfies
$$B_t = e^{-r} E(B_{t+1}|\Omega_t)$$
- Putting everything together
$$P_{s_t} = P_{s_t}^{pv} + B_{s_t}(D_t)$$

where
$$P_{s_t}^{pv} = (k_0(1 - s_t) + k_1 s_t))D_t,$$
and
$$B_{s_t}(D_t) = (c_0(1 - s_t) + c_1 s_t))D_t^{\lambda}.$$

Instrumental Variables

- Standard CAPM

$$u'(C_t) = \beta E_t[(1 + r_{t+1})u'(C_{t+1})]$$

$$E_t\left[\left(\frac{F_t - S_{t+1}}{P_{t+1}}\right)\frac{u'(C_{t+1})}{u'(C_t)}\right] = 0$$

- When all the variables are jointly lognormally distributed, the equation may be rewritten as

$$f_t = E_t[s_{t+1}] + \tfrac{1}{2}\mathrm{Var}_t[s_{t+1}] + \mathrm{Cov}_t[R_{t+1}, s_{t+1}],$$

Instrumental Variables

- UFER Hypothesis

$$f_t = E_t[s_{t+1}]$$

and

$$s_{t+1} = E_t[s_{t+1}] + \eta_{t+1}$$

- These equations are often expressed as

$$\Delta s_{t+1} = \alpha + \beta(f_t - s_t) + e_{t+1}$$

Instrumental Variables

- Consumption

$$C_t = \mu_{x_t} + \sum_{j=1}^{h} \varphi_{j,x_t} C_{t-j} + \sigma_{x_t} \zeta_t,$$

where $\{\zeta_t\}$ is a white noise and $\{x_t\}$ are regime-indicator variables independent of $\{\zeta_t\}$

$$q = \Pr[x_t = 0 | x_{t-1} = 0], \qquad p = \Pr[x_t = 1 | x_{t-1} = 1]$$

Instrumental Variables

- Conditional on $x_t = 0$, the solution for the forward rate is

$$f_t = \begin{array}{c} E_t[s_{t+1}] + \frac{1}{2}\mathrm{Var}_t[s_{t+1}] + q\mathrm{Cov}_t[R^{(0)}_{t+1}, s_{t+1}] + \\ (1-q)\mathrm{Cov}_t[R^{(1)}_{t+1}, s_{t+1}] \end{array}$$

- Conditional on $x_t = 1$, the solution is

$$f_t = \begin{array}{c} E_t[s_{t+1}] + \frac{1}{2}\mathrm{Var}_t[s_{t+1}] + (1-p)\mathrm{Cov}_t[R^{(0)}_{t+1}, s_{t+1}] \\ + p\mathrm{Cov}_t[R^{(1)}_{t+1}, s_{t+1}] \end{array}$$

where $R^{(i)}_t = \ln(u'(C^{(i)}_{t+1})/P_{t+1})$ and
$C^{(i)}_{t+1} = \mu_i + \sum_{j=1}^{h} \varphi_{j,i} C_{t+1-j} + \sigma_i \zeta_{t+1}$ for $i \in \{0, 1\}$.

Instrumental Variables

- These equations yield:

$$f_t = E_t[s_{t+1}] + A_t^{(0)}(1 - x_t) + A_t^{(1)}x_t,$$

with $A_t^{(i)} = (1/2)\mathrm{Var}_t[s_{t+1}] + \mathrm{Cov}_t[R_{t+1}^{(i)}, s_{t+1}]$ for $i \in \{0, 1\}$