Clase 10 (5/13)

Ejercicio 5

a) $\begin{cases} y_{t+2} + 3y_{t+1} - \frac{7}{4} y_t = 9 \\ y_0 = 6 \\ y_1 = 3 \end{cases}$

Solución homogénea $\quad y_t = r^t$

$$y_{t+2} + 3y_{t+1} - \frac{7}{4} y_t = 0$$

$$\downarrow$$

$$r^{t+2} + 3r^{t+1} - \frac{7}{4} r^t = 0$$

$$r^t \left(\underbrace{r^2 + 3r - \frac{7}{4}}_{} \right) = 0$$

$$r_{1,2} = \frac{-3 \pm \sqrt{9 + \cancel{4} \cdot 7/\cancel{4}}}{2} = \frac{-3 \pm 4}{2}$$

$$r_1 = 1/2$$

$$r_2 = -7/2$$

$$\boxed{y_t^h = C_1 \left(\frac{1}{2}\right)^t + C_2 \left(-\frac{7}{2}\right)^t}$$

Solución particular $\quad y_t^p = k$

$$y_{t+2} + 3y_{t+1} - \frac{7}{4} y_t = 9$$

$$k + 3k - \frac{7}{4} k = 9$$

$$\frac{4 + 12 - 7}{4} k = 9$$

$$\frac{9}{4} k = 9 \longrightarrow k = \boxed{4 = y_t^p}$$

Solución general $y_t = 4 + C_1\left(\frac{1}{2}\right)^t + C_2\left(-\frac{7}{2}\right)^t$

$$\left.\begin{array}{l} y_0 = 6 \longrightarrow y_0 = 4 + C_1 + C_2 = 6 \\ y_1 = 3 \longrightarrow y_1 = 4 + \frac{C_1}{2} - \frac{7}{2}C_2 = 3 \xrightarrow[\times 2]{} 8 + C_1 - 7C_2 = 6 \end{array}\right\}$$

$$(4 + C_1 + C_2) - (8 + C_1 - 7C_2) = 6 - 6 = 0$$

$$4 + C_2 - 8 + 7C_2 = 0$$

$$-4 + 8C_2 = 0$$

$$8C_2 = 4$$

$$\boxed{C_2 = 1/2}$$

$$6 = 4 + C_1 + \overset{1/2}{C_2}$$

$$\boxed{\frac{3}{2} = C_1}$$

$$\Rightarrow \boxed{y_t = 4 + \frac{3}{2}\left(\frac{1}{2}\right)^t + \frac{1}{2}\left(-\frac{7}{2}\right)^t} \xrightarrow{t \to \infty} \text{no converge}$$

b) $\begin{cases} y_{t+2} - 2y_{t+1} + 2y_t = 1 \\ y_0 = 3 \\ y_1 = 4 \end{cases}$

Solución homogénea $y_t^h = r^t$

$$y_{t+2} - 2y_{t+1} + 2y_t = 0$$

$$r^{t+2} - 2r^{t+1} + 2r^t = 0$$

$$r^t(r^2 - 2r + 2) = 0$$

$$\searrow r_{1,2} = \frac{2 \pm \sqrt{4 - 4\cdot 2}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm \sqrt{4}\,i}{2}$$

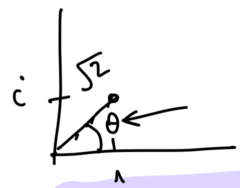
$$= \frac{2 \pm 2i}{2} = 1 \pm i$$

$$\rightarrow \boxed{\cos(\theta) = \frac{(-b/2)}{|r|}}, \quad \boxed{\sin(\theta) = \frac{\sqrt{4c - b^2}/2}{|r|}}$$

$$y_t^h = |r|^t \left[C_1 \cdot \cos(\theta t) + C_2 \sin(\theta t) \right]$$

$$\cos(\theta) = \frac{2/2}{|1 \pm i|} = \frac{1}{\sqrt{1^2 + (\pm 1)^2}} = \frac{1}{\sqrt{2}} \leftarrow \theta = \arccos\left(\frac{1}{\sqrt{2}}\right)$$

$$\theta = \frac{\pi}{4}$$



$$\rightarrow y_t^h = \left(\frac{1}{\sqrt{2}}\right)^t \left[C_1 \cos\left(\frac{\pi}{4} t\right) + C_2 \sin\left(\frac{\pi}{4} t\right) \right]$$

Solución particular

$$y_t^p = k$$

$$y_{t+2} - 2y_{t+1} + 2y_t = 1$$

$$k - 2k + 2k = 1$$

$$k = 1 = y_t^p$$

Solución general

$$y_t = 1 + \left(\frac{1}{\sqrt{2}}\right)^t \left[C_1 \cos\left(\frac{\pi}{4} t\right) + C_2 \sin\left(\frac{\pi}{4} t\right) \right]$$

$$y_0 = 3 \longrightarrow y_0 = 1 + C_1 = 3$$

$$y_1 = 4 \longrightarrow y_1 = 1 + \left(\frac{1}{\sqrt{2}}\right) \left[C_1 \frac{1}{\sqrt{2}} + C_2 \frac{1}{\sqrt{2}} \right]$$

$$= 1 + \frac{C_1}{2} + \frac{C_2}{2} = 4$$

$$\boxed{\sin\left(\frac{\pi}{4}\right) = \sqrt{1 - \cos\left(\frac{\pi}{4}\right)^2} = \sqrt{1 - 1/2} = \frac{1}{\sqrt{2}}}$$

$$\begin{cases} 1 + C_1 = 3 \longrightarrow C_1 = 2 \\ 1 + \underbrace{\frac{C_1}{2}}_{=1} + \frac{C_2}{2} = 4 \rightarrow 2 + \frac{C_2}{2} = 4 \rightarrow \frac{C_2}{2} = 2 \rightarrow C_2 = 4 \end{cases}$$

$$y_t = 1 + \underbrace{\left(\frac{1}{\sqrt{2}}\right)^t}_{\xrightarrow{t \to \infty} 0} \underbrace{\left[2 \cos\left(\frac{\pi}{4} t\right) + 4 \sin\left(\frac{\pi}{4} t\right) \right]}_{\text{acotado entre } -6,6} \xrightarrow{t \to \infty} 1$$

c) $\begin{cases} y_{t+2} - y_{t+1} + \frac{1}{4} y_t = 2 \\ y_0 = 4 \\ y_1 = 7 \end{cases}$

Solución homogénea $\quad y_t = r^t$

$$r^{t+2} - r^{t+1} + \frac{r^t}{4} = 0$$

$$r^t\left(r^2 - r + \frac{1}{4}\right) = 0$$

$$r_{1,2} = \frac{+1 \pm \sqrt{1 - 4/4}}{2} = \frac{1}{2}$$

$$y_t^h = c_1 \left(\frac{1}{2}\right)^t + c_2 t \left(\frac{1}{2}\right)^t$$

Solución particular $\quad y_t = k: \quad k - k + \frac{1}{4} k = 2 \rightarrow k = 8 = y_t^p$

Solución general: $\quad y_t = 8 + c_1 \left(\frac{1}{2}\right)^t + c_2 t \left(\frac{1}{2}\right)^t$

$y_0 = 4 \longrightarrow y_0 = 8 + c_1 = 4 \longrightarrow c_1 = 4$

$y_1 = 7 \longrightarrow y_1 = 8 + \underbrace{c_1}_{4} \left(\frac{1}{2}\right) + c_2 \left(\frac{1}{2}\right) = 7$

$10 + \frac{c_2}{2} = 7 \rightarrow c_2 = -6$

$$y_t = 8 + 4\left(\frac{1}{2}\right)^t - 6t\left(\frac{1}{2}\right)^t \xrightarrow{t \to \infty} 8$$

$$t\left(\frac{1}{2}\right)^t = \frac{t}{2^t}, \quad \lim_{t\to\infty} \frac{t}{2^t} \overset{LH}{=} \lim_{t\to\infty} \frac{1}{\log(2) 2^t} = 0$$

d) $\begin{cases} 2y_{t+2} + 2y_{t+1} + y_t = 2^{-t} \\ y_0 = 0 \\ y_1 = 0 \end{cases}$

Solución homogénea

$$y_t = r^t$$

$$2y_{t+2} + 2y_{t+1} + y_t = 0$$

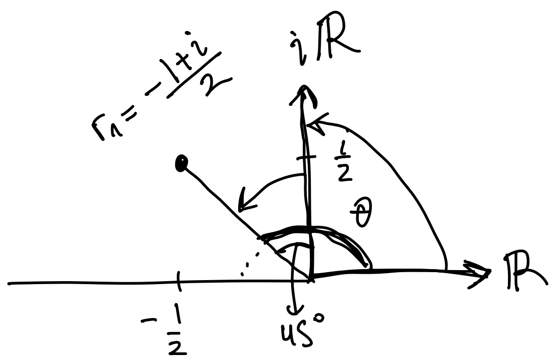
$$2r^{t+2} + 2r^{t+1} + r^t = 0$$

$$r^t(2r^2 + 2r + 1) = 0$$

$$\rightarrow \quad r_{1,2} = \frac{-2 \pm \sqrt{4-8}}{4} = \frac{-2 \pm \sqrt{-4}}{4} = \frac{-2 \pm \overset{\overset{2}{\shortparallel}}{\sqrt{4}}\, i}{4} = \frac{-1 \pm i}{2}$$

$$|r| = \sqrt{\left(-\tfrac{1}{2}\right)^2 + \left(\pm\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{1}{4} + \tfrac{1}{4}} = \frac{1}{\sqrt{2}}$$

$$\rightarrow \quad \theta = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3}{4}\pi = \theta$$



$$\boxed{y_t^h = \left(\frac{1}{\sqrt{2}}\right)^t \left[C_1 \cos\left(\tfrac{3}{4}\pi t\right) + C_2 \sin\left(\tfrac{3}{4}\pi t\right) \right]}$$

Solución particular

$$2y_{t+2} + 2y_{t+1} + y_t = \underline{2^{-t} \cdot 1}$$

Propongo $y_t^p = k\, 2^{-t} \rightarrow 2k\, 2^{-(t+2)} + 2k\, 2^{-(t+1)} + k\, 2^{-t} = 2^{-t}$

$$2^{-t}\left(2k\, \overset{\shortparallel}{2^{-2}} + 2k\, 2^{-1} + k\right) = 2^{-t}$$

$$\frac{5}{2}k = \frac{k}{2} + k + k = 1 \quad \hookrightarrow \quad k = 2/5 \qquad \leftarrow \quad \cancel{2^{-t}}\left(\frac{k}{2} + k + k\right) = \cancel{2^{-t}}$$

$$y_t^p = \frac{2}{5}\,2^{-t}$$

Solución general

$$y_t = \frac{2}{5}2^{-t} + \left(\frac{1}{\sqrt{2}}\right)^t \left[c_1 \cos\left(\frac{3}{4}\pi t\right) + c_2 \sin\left(\frac{3}{4}\pi t\right)\right]$$

$$y_0 = 0 \longrightarrow y_0 = \frac{2}{5} + c_1 = 0 \longrightarrow c_1 = -\frac{2}{5}$$

$$y_1 = 1 \Rightarrow y_1 = \frac{\cancel{2}}{5}\frac{1}{\cancel{2}} + \left(\frac{1}{\sqrt{2}}\right)\left[\left(-\frac{2}{5}\right)\left(-\frac{\sqrt{2}}{2}\right) + c_2 \frac{\sqrt{2}}{2}\right]$$

$$= \frac{1}{5} + \left(\frac{+2}{5}\right)\frac{1}{2} + c_2\frac{1}{2} = 0$$

$$= \frac{2}{5} + c_2\frac{1}{2} = 0$$

$$\rightarrow c_2 = -\frac{4}{5}$$

$$\sin\left(\frac{3}{4}\pi t\right) = \sqrt{1-\left(-\frac{\sqrt{2}}{2}\right)^2} = \sqrt{1-\frac{2}{4}} = \sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$y_t = \frac{2}{5}2^{-t} + \left(\frac{1}{\sqrt{2}}\right)^t \left[-\frac{2}{5}\cos\left(\frac{3}{4}\pi t\right) - \frac{4}{5}\sin\left(\frac{3}{4}\pi t\right)\right]$$

$$\xrightarrow[t\to\infty]{} 0 \qquad \text{(converge)}$$

$$\lim_{t\to\infty} y_t = \lim_{t\to\infty}\left\{\frac{2}{5}2^{-t} + \left(\frac{1}{\sqrt{2}}\right)^t\left[-\frac{2}{5}\cos\left(\frac{3}{4}\pi t\right) - \frac{4}{5}\sin\left(\frac{3}{4}\pi t\right)\right]\right\}$$

$$= \underbrace{\frac{2}{5}\lim(2^{-t})}_{=0} + \underbrace{\lim\left\{\underbrace{\left(\frac{1}{\sqrt{2}}\right)^t}_{\to 0}\underbrace{\left[-\frac{2}{5}\cos\left(\frac{3}{4}\pi t\right) - \frac{4}{5}\sin\left(\frac{3}{4}\pi t\right)\right]}_{\text{acotado entre } -\frac{6}{5},\ \frac{6}{5}}\right\}}_{=0}$$

$$= 0$$

Ejercicio 6

h) $y_{t+2}+5y_{t+1}+2y_t = e^t+18+6t+8t^2$

Solución homogénea $y_t = r$:

$$r^{t+2}+5r^{t+1}+2r=0$$

$$r\left(r^2+5r+2\right)=0 \rightarrow r_{1,2} = \frac{-5\pm\sqrt{25-8}}{2}$$

$$\Rightarrow r_{1,2} = \frac{-5\pm\sqrt{17}}{2} \quad , \quad y_t^h = c_1\left(\frac{-5-\sqrt{17}}{2}\right)+c_2\left(\frac{-5+\sqrt{17}}{2}\right)$$

Solución particular Si del lado derecho tengo una constante, propongo una constante
tengo un polinomio, " un polinomio
" una exponencial, " una exponencial

$$(t+\alpha)^2 = t^2+2\alpha t+\alpha^2$$

$$y_t^p = ae^t+b+ct+dt^2$$

$y_{t+2}+5y_{t+1}+2y_t =$

$$= ae^{t+2}+b+c(t+2)+d(t+2)^2 + 5\left[ae^{t+1}+b+c(t+1)+d(t+1)^2\right]$$
$$+2\left[ae^t+b+ct+dt^2\right]$$

$$= ae^t(e^2+5e+2) + b+2c+tc+dt^2+4dt+4d$$
$$+5b+5ct+5c+5dt^2+10dt+5d$$
$$+2b+2ct+2dt^2$$

$$= ae^t(e^2+5e+2)+(8b+7c+9d)$$
$$+t(8c+14d)+t^2 8d$$

$$= e^t + 18 + 6t + 8t^2$$

de la ecuación
en diferencias

$$\begin{cases} t^2\,8d = 8t^2 \longrightarrow 8d = 8 \longrightarrow d = 1 \\ 6t = t(8c+14d) \longrightarrow 6 = 8c+14d \longrightarrow 6 = 8c+14 \longrightarrow c = -1 \\ 18 = 8b+7c+9d \longrightarrow \underset{(d=1,\, c=-1)}{18 = 8b-7+9} \longrightarrow 2 = b \\ e^t = ae^t(e^2+5e+2) \longrightarrow 1 = a(e^2+5e+2) \longrightarrow a = \frac{1}{e^2+5e+2} \end{cases}$$

$$y_t^p = \frac{e^t}{e^2+5e+2} + 2 - t + t^2$$

Solución general

$$y_t = C_1\left(\frac{-5-\sqrt{17}}{2}\right)^t + C_2\left(\frac{-5+\sqrt{17}}{2}\right)^t + \frac{e^t}{e^2+5e+2} + 2 - t + t^2$$