SIMPLE NEW KEYNESIAN MODEL

This note derives a simple New Keynesian (NK) model that produces a modern version of the classical Keynesian model with an IS curve, a Phillips curve, and a monetary policy rule. The model is a standard RBC model—in its simplest form, without capital—augmented with price frictions. The households' side of the model is identical to that in standard models. The difference is in the production side of the economy.

To model price stickiness we need to depart from the assumption of perfect competition. Indeed, price stickiness requires firms to set prices while perfect competition imposes price taking behavior. New Keynesian models assume that at least one sector is composed of monopolistic competitive firms with some pricing power. Furthermore, NK models assume that, for some reason, changing nominal prices is costly, or that some firms are simply unable to reset their prices every single period.

The model consists of a representatitive household, a final goods producing firm, a continuum of intermediate good producers (this is the sector with monopolistic power and price stickiness), and a monetary authority. In the end we obtain a very simple system of three linear difference equations. But the process of deriving them requires some work.

Households

The household sector is standard. We consider a model with money in the utility function in order to motivate a demand for currency (some authors even ignore the demand for money). The utility function of the representative household is

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \eta \frac{l_t^{1+\nu}}{1+\nu} + \psi \frac{m_t^{1-\xi}}{1-\xi} \right)$$

where c_t is consumption, l_t is labor, $m_t = M_t/P_t$ denotes the real money demand, and β is the discount factor. The parameters σ, ν, η, ξ are all positive.

The budget constraint of the household in nominal terms is

$$P_t c_t + B_t + M_t = W_t l_t + D_t + R_{t-1} B_{t-1} + M_{t-1}$$

where P_t is the nominal price of goods, B_t are nominal bonds, M_t is the nominal demand for money, W_t is the nominal wage, R_{t-1} is the nominal gross interest rate on bonds between periods $t-1$ and t, and D_t are the profits from the firms owned by the household.

It is convenient to write the constraint in real terms. Dividing both sides of the previous equation by P_t gives

$$c_t + b_t + m_t = w_t l_t + d_t + \frac{R_{t-1}}{\pi_t} b_{t-1} + \frac{m_{t-1}}{\pi_t}.$$

where $w_t = W_t/P_t$ is the real wage rate, $b_t = B_t/P_t$ is the demand for bonds in real terms, d_t are the real dividends, and π_t denotes the **gross** inflation rate between periods $t-1$ and t (that is $\pi_t = P_t/P_{t-1}$).

The problem of the consumer is to maximize his utility subject to a sequence of budget constraint. Using $\beta^t \lambda_t$ for the Lagrange multiplier on the budget constraint, the Lagrangian for the household's problem can be written as

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \frac{c_t^{1-\sigma}}{1-\sigma} - \eta \frac{l_t^{1+\nu}}{1+\nu} + \psi \frac{m_t^{1-\xi}}{1-\xi} \\ -\lambda_t \left[c_t + b_t + m_t - w_t l_t - d_t - R_{t-1} \frac{b_{t-1}}{\pi_t} - \frac{m_{t-1}}{\pi_t} \right] \end{array} \right\}.$$

The first order conditions are

$$\frac{\partial L}{\partial c_t} = 0 \Leftrightarrow c_t^{-\sigma} = \lambda_t$$

$$\frac{\partial L}{\partial l_t} = 0 \Leftrightarrow \eta l_t^{\nu} = \lambda_t w_t$$

$$\frac{\partial L}{\partial b_t} = 0 \Leftrightarrow \lambda_t = \beta E_t \left[\lambda_{t+1} \frac{R_t}{\pi_{t+1}} \right]$$

$$\frac{\partial L}{\partial m_t} = 0 \Leftrightarrow \psi m_t^{-\xi} = \lambda_t - \beta E_t \left[\frac{\lambda_{t+1}}{\pi_{t+1}} \right]$$

However, because R_t is known at time t, the first order condition for b_t can be written as

$$\frac{\lambda_t}{R_t} = \beta E_t \left[\frac{\lambda_{t+1}}{\pi_{t+1}} \right]$$

Replacing this expression into the first order condition with respect to m_t gives

$$\psi m_t^{-\xi} = \lambda_t \frac{R_t - 1}{R_t}.$$

We can eliminate the multiplier λ_t and write the first order conditions as

$$\eta l_t^{\nu} = c_t^{-\sigma} w_t \tag{1}$$

$$\psi m_t^{-\xi} = c_t^{-\sigma} \frac{R_t - 1}{R_t} \tag{2}$$

$$c_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma} \frac{R_t}{\pi_{t+1}} \right] \tag{3}$$

Production sector

There are two producing sectors: a competitive sector that produces final goods and an intermediate goods sector with a large number (a continuum) of firms each of which produces a differentiated intermediate input.

Final good firms.—

There is a perfectly competitive representative firm that produces final goods using a continuum of intermediate goods indexed by numbers in the unit interval $[0,1]$. The production function is given by

$$y_t = \left(\int_0^1 y_t(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}}, \tag{4}$$

where $\theta > 1$ represents the elasticity of substitution between pairs of intermediate inputs. Note that this production function has constant returns to scale and, therefore, firms earn zero profits in equilibrium.

The firm takes all prices as given, in particular, the nominal input prices $P_t(j)$ as well as the output price P_t. The problem of the firm is to maximize profits period by period. The objective function in nominal terms is

$$\max_{y_t(j)} P_t y_t - \int_0^1 P_t(j) \, y_t(j) \, dj.$$

That is, the firm maximizes total revenues, $P_t y_t$, minus the total cost of the inputs that it uses, $\int_0^1 P_t(j) \, y_t(j) \, dj$. Introducing the production function (4) into the objective function gives

$$\max_{y_t(j)} P_t \left(\int_0^1 y_t(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}} - \int_0^1 P_t(j) \, y_t(j) \, dj.$$

The first order condition of this problem is to take the derivative with respect to $y_t(j)$ for all j and set it to zero. That is,

$$P_t \frac{\theta}{\theta-1} \left(\int_0^1 y_t(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}-1} \frac{\theta-1}{\theta} y_t(j)^{\frac{-1}{\theta}} - P_t(j) = 0 \text{ for all } j.$$

But note that

$$\left(\int_0^1 y_t(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}-1} = \left(\int_0^1 y_t(j)^{\frac{\theta-1}{\theta}} dj \right)^{\frac{1}{\theta-1}} = y_t^{\frac{1}{\theta}}.$$

Therefore, we can write the first order condition as

$$P_t y_t^{\frac{1}{\theta}} y_t(j)^{\frac{-1}{\theta}} = P_t(j).$$

From here we obtain the demands for the input $y_t(j)$ conditional on the level of output y_t

$$y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\theta} y_t. \tag{5}$$

This equation says that the demand for each intermediate good $j \in [0,1]$ depends negatively on the relative price of the input and positively on the level of production. Note that θ is also the demand elasticity of the intermediate good j.

The firm has constant returns to scale, so its profits must be zero. To see this formally, note that profits satisfy

$$\begin{aligned}
\text{Profits}_t &= P_t y_t - \int_0^1 P_t(j) y_t(j)\, dj \\
&= P_t y_t - \int_0^1 P_t(j) \left(\frac{P_t(j)}{P_t}\right)^{-\theta} y_t dj \\
&= \left[P_t - P_t^{\theta} \int_0^1 P_t(j)^{1-\theta}\, dj\right] y_t.
\end{aligned}$$

The term in square brackets does not depend on the firm's choices because the firm is a price taker. Suppose that the term in square brackets is negative. Then any level of positive output $y_t > 0$ will generate negative profits. Therefore, it will be optimal for the firm to choose $y_t = 0$. But this can't be an equilibrium because consumption will be zero and the marginal utility of consumption at zero is infinity. Therefore, the term in square brackets cannot be negative. Suppose now that the term in square brackets is positive. Then the firm could make infinite profits by producing an infinite amount of goods. This also can't be an equilibrium because this will violate feasibility (production is limited by the endowment of labor). Therefore, the only possibility is that the term in square brackets is zero and the firm makes zero profits. Equating the term in square brackets to zero gives

$$P_t = P_t^{\theta} \int_0^1 P_t(j)^{1-\theta}\, dj$$

or

$$P_t = \left[\int_0^1 P_t(j)^{1-\theta}\, dj\right]^{\frac{1}{1-\theta}} \tag{6}$$

This expression can be interpreted as a price index of the final good.

Intermediate good firms.—

There is a continuum of intermediate good firms indexed by $j \in [0,1]$ each of which is a price setter. Firm j produces an intermediate good using labor as the only input of production. The production function is given by

$$y_t(j) = A_t l_t^d(j) \tag{7}$$

where A_t is the level of technology, common across of intermediate good firms, and $l_t^d(j)$ is the amount of labor used by firm j. Labor markets are perfectly competitive.

Each firm is monopolistic in that it chooses its optimal price $P_t(j)$ internalizing that the demand they face is given by (5). To introduce price stickiness, we assume that firms face nominal rigidities in terms of a quadratic price adjustment cost

$$AC_t(j) = \frac{\omega}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi}\right)^2 y_t. \tag{8}$$

The parameter ω governs the price stickiness in the economy and $\bar{\pi}$ is the gross inflation rate in steady state. Note that the adjustment cost is measured in terms of the final consumption good. The assumption here is that if the firm chooses to increase or decrease the price it charges at a different rate from the long-run gross inflation rate $\bar{\pi}$, it incurrs in a quadratic adjustment cost.

The adjustment cost makes the firm's problem dynamic. The problem of the firm at time $t = 0$ is to maximize the present discounted value of future nominal dividends,

$$E_0 \sum_{t=0}^{\infty} Q_t D_t(j)$$

where Q_t is the nominal discount rate used by the firm to discount future nominal profits (more on this below) and

$$D_t(j) = P_t(j)\, y_t(j) - W_t l_t^d(j) - P_t \frac{\omega}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi}\right)^2 y_t$$

is the nominal dividend in period t.

It is convenient to write the firm's problem in real terms. The real present value of all future nominal dividends is given by

$$\max_{\{l_t(j), y_t(j), P_t(j)\}} E_0 \sum_{t=0}^{\infty} \frac{Q_t}{P_0} D_t(j) = \max_{\{l_t(j), y_t(j), P_t(j)\}} E_0 \sum_{t=0}^{\infty} Q_t \frac{P_t}{P_0} d_t(j)$$

where

$$d_t = \frac{D_t}{P_t} = \frac{P_t(j)}{P_t} y_t(j) - w_t l_t^d(j) - \frac{\omega}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi} \right)^2 y_t.$$

are the real dividends at time t and w_t is the real wage.

Note that $q_t \equiv Q_t \frac{P_t}{P_0}$ is the real discount rate that the firm uses to discount future real dividends. The discount rate measures the value that the firm attaches to a unit of real profits in period t—goods at time t—in terms of goods at time $t = 0$. Because the firm is owned by the household, the firm should discount future profits using the discount factor of its owners: the consumers. The discount factor of the consumers is the marginal rate of substitution between goods at time t in terms of goods at time 0. Therefore, the relevant real discount rate for the firm is

$$q_t = \beta^t \frac{c_t^{-\sigma}}{c_0^{-\sigma}}. \tag{9}$$

Using this insight, the firm's problem can be written as

$$\max_{P_t(j), y_t(j), l_t(j)} E_0 \sum_{t=0}^{\infty} q_t \left[\frac{P_t(j)}{P_t} y_t(j) - w_t l_t^d(j) - \frac{\omega}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi} \right)^2 y_t \right]$$

subject to the demand (5), the technology (7), and the adjustment cost (8).

We first use (7) and solve for labor as a function of output

$$l_t^d(j) = \frac{y_t(j)}{A_t}.$$

Therefore, real profits of firm j at time t are

$$d_t(j) = \frac{P_t(j)}{P_t} y_t(j) - \frac{w_t}{A_t} y_t(j) - \frac{\omega}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi} \right)^2 y_t$$

Note that the term w_t/A_t is the real marginal cost of production.

Next, we use the demand function (5) to replace $y_t(j)$ into the previous expression and write profits as

$$\begin{aligned} d_t(j) &= \frac{P_t(j)}{P_t} \left(\frac{P_t(j)}{P_t} \right)^{-\theta} y_t - \frac{w_t}{A_t} \left(\frac{P_t(j)}{P_t} \right)^{-\theta} y_t - \frac{\omega}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi} \right)^2 y_t \\ &= y_t \left[\frac{P_t(j)^{1-\theta}}{P_t^{1-\theta}} - \frac{w_t}{A_t} \frac{P_t(j)^{-\theta}}{P_t^{-\theta}} - \frac{\omega}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi} \right)^2 \right] \end{aligned}$$

It then follows that the present discounted value of future real profits is

$$\max_{P_t(j)} E_0 \sum_{t=0}^{\infty} q_t y_t \left[\frac{P_t(j)^{1-\theta}}{P_t^{1-\theta}} - \frac{w_t}{A_t} \frac{P_t(j)^{-\theta}}{P_t^{-\theta}} - \frac{\omega}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi} \right)^2 \right].$$

The maximization is taken with respecto to the nominal price $P_t(j)$.

The first order condition with respect to $P_t(j)$ is

$$\begin{aligned} 0 \;&=\; q_t y_t \left[(1-\theta) \frac{P_t(j)^{-\theta}}{P_t^{1-\theta}} + \theta \frac{w_t}{A_t} \frac{P_t(j)^{-\theta-1}}{P_t^{-\theta}} - \omega \left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi} \right) \frac{1}{P_{t-1}(j)} \right] \\ &\quad + E_t \left[q_{t+1} y_{t+1} \omega \left(\frac{P_{t+1}(j)}{P_t(j)} - \bar{\pi} \right) \frac{P_{t+1}(j)}{P_t(j)^2} \right]. \end{aligned} \tag{10}$$

Note that if $\omega = 0$—so that there are no adjustment costs—the above expression reduces to

$$0 = q_t y_t \left[(1-\theta) \frac{P_t(j)^{-\theta}}{P_t^{1-\theta}} + \theta \frac{w_t}{A_t} \frac{P_t(j)^{-\theta-1}}{P_t^{-\theta}} \right]$$

or

$$P_t(j) = \frac{\theta}{\theta - 1} \frac{P_t w_t}{A_t} = \frac{\theta}{\theta - 1} \frac{W_t}{A_t}$$

which says that the price is set as a constant mark-up over the nominal marginal cost. This is the usual result with a monopolistic producer facing a demand with a constant elasticity of θ.

Monetary Authority

We assume that monetary policy is described by a (modified) Taylor rule of the form

$$\log(R_t) = (1 - \rho_R) \log R_t^* + \rho_R \log R_{t-1} + v_t. \tag{11}$$

where $\log R_t$ is the net nominal interest rate (we always use $\log(\cdot)$ for the natural logarithm), $\log R_t^*$ is the nominal target rate, ρ_R is a smoothing parameter, and v_t is a monetary policy shock to be described below.

We assume that the target interest rate satisfies

$$\log R_t^* = \log \bar{R} + \phi_\pi \log \left(\frac{\pi_t}{\bar{\pi}} \right) + \phi_y \log \left(\frac{y_t}{\bar{y}} \right) \tag{12}$$

where $\bar{R}$, $\bar{\pi}$, and $\bar{y}$ are the steady state values of the gross nominal interest rate, inflation, and output respectively. In this specification, the target rate reacts to deviations of inflation and output from their steady state values $\bar{\pi}$ and $\bar{y}$. The higher is ϕ_π, the stronter is the response of the target rate to deviations of inflation from its long run value. The same interpretation holds for ϕ_y.

Plugging (12) into (11) gives the following Taylor rule

$$\log(R_t) = (1-\rho_R)\left[\log \bar{R} + \phi_\pi \log\left(\frac{\pi_t}{\bar{\pi}}\right) + \phi_y \log\left(\frac{y_t}{\bar{y}}\right)\right] + \rho_R \log R_{t-1} + v_t. \tag{13}$$

Shocks

We assume that technology and monetary policy shocks evolve according to first order autoregressive processes

$$\log A_{t+1} = \rho_A \log A_t + \varepsilon_{A,t+1}, \tag{14}$$

$$v_{t+1} = \rho_v v_t + \varepsilon_{v,t+1}. \tag{15}$$

where $\varepsilon_{A,t+1}$ and $\varepsilon_{v,t+1}$ are uncorrelated i.i.d. shocks with a standard deviation σ_A and σ_v. The innovations $\varepsilon_{A,t}$ and ε_{vt} are uncorrelated at all leads and lags.

Equilibrium

We focus on a **symmetric equilibrium** in which all intermediate good producers make identical choices. This implies that the subscript j in the corresponding expressions can be dropped. Moreover, expression (6) implies that the aggregate price level P_t is equal to the price chosen by the intermediate good producers, and condition (5) implies $y_t(j) = y_t$ for all j.

Feasibility in goods and labor markets are, respectively

$$y_t = c_t + AC_t \text{ and } l_t = l_t^d \tag{16}$$

Note that feasibility in goods markets include the adjustment cost since it is incurred in terms of final consumption goods.

We now perform some algebra on the price setting condition (10). Using **symmetry** implies that $P_t(j) = P_t$ for all j,

$$\begin{aligned} 0 \quad = \quad & q_t y_t \left[(1-\theta)\frac{P_t^{-\theta}}{P_t^{1-\theta}} + \theta \frac{w_t}{A_t}\frac{P_t^{-\theta-1}}{P_t^{-\theta}} - \omega\left(\frac{P_t}{P_{t-1}} - \bar{\pi}\right)\frac{1}{P_{t-1}}\right] \\ & + E_t\left[q_{t+1} y_{t+1} \omega \left(\frac{P_{t+1}}{P_t} - \bar{\pi}\right)\frac{P_{t+1}}{P_t^2}\right]. \end{aligned}$$

Canceling terms,

$$
\begin{aligned}
0 \quad = \quad & q_t y_t \left[(1-\theta) \frac{1}{P_t} + \theta \frac{w_t}{A_t} \frac{1}{P_t} - \omega \left(\pi_t - \bar{\pi} \right) \frac{1}{P_{t-1}} \right] \\
& + E_t \left[q_{t+1} y_{t+1} \omega \left(\pi_{t+1} - \bar{\pi} \right) \pi_{t+1} \frac{1}{P_t} \right].
\end{aligned}
$$

Multiplying the previous expression by P_t gives

$$
0 = q_t y_t \left[(1-\theta) + \theta \frac{w_t}{A_t} - \omega \left(\pi_t - \bar{\pi} \right) \pi_t \right] + E_t \left[q_{t+1} y_{t+1} \omega \left(\pi_{t+1} - \bar{\pi} \right) \pi_{t+1} \right].
$$

Using $q_t = \beta^t c_t^{-\sigma} / c_0^{-\sigma}$ gives

$$
0 = \beta^t \frac{c_t^{-\sigma}}{c_0^{-\sigma}} y_t \left[(1-\theta) + \theta \frac{w_t}{A_t} - \omega \left(\pi_t - \bar{\pi} \right) \pi_t \right] + E_t \left[\beta^{t+1} \frac{c_{t+1}^{-\sigma}}{c_0^{-\sigma}} y_{t+1} \omega \left(\pi_{t+1} - \bar{\pi} \right) \pi_{t+1} \right].
$$

Dividing both sides of the equation by $\beta^t \left(c_t^{-\sigma} / c_0^{-\sigma} \right) y_t$ and rearranging gives,

$$
\left(\pi_t - \bar{\pi} \right) \pi_t = \frac{\theta}{\omega} \left[\frac{w_t}{A_t} - \frac{\theta - 1}{\theta} \right] + E_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{y_{t+1}}{y_t} \left(\pi_{t+1} - \bar{\pi} \right) \pi_{t+1} \right] \tag{17}
$$

Therefore the equilibrium conditions of the model are summarized by the following system of nine equations

$$
\eta l_t^{\nu} = c_t^{-\sigma} w_t \tag{18}
$$

$$
\psi m_t^{-\xi} = c_t^{-\sigma} \frac{R_t - 1}{R_t} \tag{19}
$$

$$
c_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma} \frac{R_t}{\pi_{t+1}} \right] \tag{20}
$$

$$
y_t = A_t l_t \tag{21}
$$

$$
y_t = c_t + \frac{\omega}{2} \left(\pi_t - \bar{\pi} \right)^2 y_t \tag{22}
$$

$$
\left(\pi_t - \bar{\pi} \right) \pi_t = \frac{\theta}{\omega} \left[\frac{w_t}{A_t} - \frac{\theta - 1}{\theta} \right] + E_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{y_{t+1}}{y_t} \left(\pi_{t+1} - \bar{\pi} \right) \pi_{t+1} \right] \tag{23}
$$

$$
\log \left(R_t \right) = \left(1 - \rho_R \right) \left[\log \bar{R} + \phi_\pi \log \left(\frac{\pi_t}{\bar{\pi}} \right) + \phi_y \log \left(\frac{y_t}{\bar{y}} \right) \right] + \rho_R \log R_{t-1} + v_t \tag{24}
$$

$$
\log A_{t+1} = \rho_A \log A_t + \varepsilon_{A,t+1} \tag{25}
$$

$$
v_{t+1} = \rho_v v_t + \varepsilon_{v,t+1} \tag{26}
$$

We want to simplify this system. Use (18) and (21) to find the real marginal cost w_t/A_t as a function of output, consumption, and the level of technology

$$\frac{w_t}{A_t} \equiv mc_t = \frac{\eta l_t^{\nu} c_t^{\sigma}}{A_t} = \eta \frac{y_t^{\nu} c_t^{\sigma}}{A_t^{1+\nu}}. \tag{27}$$

Inserting this equation into the pricing condition (23) gives

$$(\pi_t - \bar{\pi})\,\pi_t = \frac{\theta}{\omega}\left[\eta \frac{y_t^{\nu} c_t^{\sigma}}{A_t^{1+\nu}} - \frac{\theta - 1}{\theta}\right] + E_t\left[\beta \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} \frac{y_{t+1}}{y_t}\,(\pi_{t+1} - \bar{\pi})\,\pi_{t+1}\right]$$

Therefore, we have reduced the system to the following seven equations

$$\psi m_t^{-\xi} = c_t^{-\sigma} \frac{R_t - 1}{R_t} \tag{28}$$

$$c_t^{-\sigma} = \beta E_t\left[c_{t+1}^{-\sigma} \frac{R_t}{\pi_{t+1}}\right] \tag{29}$$

$$y_t\left[1 - \frac{\omega}{2}\,(\pi_t - \bar{\pi})^2\right] = c_t \tag{30}$$

$$(\pi_t - \bar{\pi})\,\pi_t = \frac{\theta}{\omega}\left[\eta \frac{y_t^{\nu} c_t^{\sigma}}{A_t^{1+\nu}} - \frac{\theta - 1}{\theta}\right] + E_t\left[\beta \left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} \frac{y_{t+1}}{y_t}\,(\pi_{t+1} - \bar{\pi})\,\pi_{t+1}\right] \tag{31}$$

$$\log R_t = (1 - \rho_R)\left[\log \bar{R} + \phi_\pi \log\left(\frac{\pi_t}{\bar{\pi}}\right) + \phi_y \log\left(\frac{y_t}{\bar{y}}\right)\right] + \rho_R \log R_{t-1} + v_t \tag{32}$$

$$\log A_{t+1} = \rho_A \log A_t + \varepsilon_{A,t+1} \tag{33}$$

$$v_{t+1} = \rho_v v_t + \varepsilon_{v,t+1} \tag{34}$$

This system can be reduced further, but it is convenient to leave it as it is and reduce it after we perform the linearization of the equilibrium conditions. To linearize the model, we first need to find the steady state

Steady state

Evaluating the previous system of equations at the steady state gives

$$\psi \bar{m}^{-\xi} = \bar{c}^{-\sigma} \frac{\bar{R} - 1}{\bar{R}}$$

$$\bar{c}^{-\sigma} = \beta \bar{c}^{-\sigma} \frac{\bar{R}}{\bar{\pi}}$$

$$\bar{y} = \bar{c}$$

$$(\bar{\pi} - \bar{\pi})\,\bar{\pi} = \frac{\theta}{\omega}\left[\eta \frac{\bar{y}^{\nu}\bar{c}^{\sigma}}{\bar{A}^{1+\nu}} - \frac{\theta-1}{\theta}\right] + \beta\left(\frac{\bar{c}}{\bar{c}}\right)^{-\sigma}\frac{\bar{y}}{\bar{y}}\,(\bar{\pi} - \bar{\pi})\,\bar{\pi}$$

$$\log \bar{R} = (1-\rho_R)\left[\log \bar{R} + \phi_\pi \log\left(\frac{\bar{\pi}}{\bar{\pi}}\right) + \phi_y \log\left(\frac{\bar{y}}{\bar{y}}\right)\right] + \rho_R \log \bar{R}$$

$$\log \bar{A} = \rho_A \log \bar{A}$$

$$\bar{v} = \rho_v \bar{v}$$

This can be solved for the steady state quantities and prices

$$\bar{R} = \bar{\pi}/\beta \tag{35}$$

$$\bar{A} = 1 \tag{36}$$

$$\bar{v} = 0 \tag{37}$$

$$\bar{c} = \bar{y} = \left(\frac{1}{\eta}\frac{\theta-1}{\theta}\right)^{\frac{1}{\sigma+\nu}} \tag{38}$$

$$\bar{m} = \left[\psi \bar{y}^{\sigma}\frac{\bar{R}}{\bar{R}-1}\right]^{1/\xi} \tag{39}$$

Note that the steady state interest rate and money demand depend on the steady state inflation rate, which is chosen by the monetary authority. For example, if the monetary authority targets zero inflation, so that $\bar{\pi} = 1$, then $\bar{R} = 1/\beta$.

Log-linearization of the equilibrium conditions

We log-linearize the equilibrium conditions around the steady state. To that end we define, for any variable x_t, its log-deviation from the steady state as

$$\hat{x}_t = \log\left(x_t/\bar{x}\right)$$

which implies

$$x_t = \bar{x}e^{\hat{x}_t}.$$

We now rewrite the system of equations (28)–(33) in terms of the transformed variables. We do not transform the monetary policy shock v_t, which is already linear with mean zero.

Equation (28):

$$
\begin{aligned}
0 &= \bar{c}^{-\sigma} e^{-\sigma \hat{c}_t} \frac{\bar{R} e^{\hat{R}_t} - 1}{\bar{R} e^{\hat{R}_t}} - \psi \bar{m}^{-\xi} e^{-\xi \hat{m}_t} \\
&= \frac{\bar{c}^{-\sigma}}{\bar{R}} e^{-\sigma \hat{c}_t} \left(\bar{R} e^{\hat{R}_t} - 1 \right) e^{-\hat{R}_t} - \psi \bar{m}^{-\xi} e^{-\xi \hat{m}_t}
\end{aligned}
$$

Linearizing this equation around $\left(\hat{c}_t, \hat{R}_t, \hat{m}_t\right) = (0, 0, 0)$ (we ignore the constant because it is zero in steady state) gives

$$
\begin{aligned}
0 &\approx \frac{\bar{c}^{-\sigma}}{\bar{R}} \left(-\sigma \hat{c}_t\right) \left(\bar{R} - 1\right) + \frac{\bar{c}^{-\sigma}}{\bar{R}} \left[\bar{R} \hat{R}_t - \left(\bar{R} - 1\right) \hat{R}_t \right] - \psi \bar{m}^{-\xi} \left(-\xi \hat{m}_t\right) \\
&\approx -\sigma \hat{c}_t \bar{c}^{-\sigma} \frac{\bar{R} - 1}{\bar{R}} + \bar{c}^{-\sigma} \frac{\bar{R} - 1}{\bar{R}} \frac{\hat{R}_t}{\bar{R} - 1} + \psi \bar{m}^{-\xi} \xi \hat{m}_t
\end{aligned}
$$

But in steady state

$$
\bar{c}^{-\sigma} \frac{\bar{R} - 1}{\bar{R}} = \psi \bar{m}^{-\xi}
$$

then we have

$$
0 \approx -\sigma \hat{c}_t + \frac{\hat{R}_t}{\bar{R} - 1} + \xi \hat{m}_t \tag{40}
$$

Equation (29):

$$
\begin{aligned}
0 &= \beta E_t \left[c_{t+1}^{-\sigma} \frac{R_t}{\pi_{t+1}} \right] - c_t^{-\sigma} \\
&= \beta E_t \left[\bar{c}^{-\sigma} e^{-\sigma \hat{c}_{t+1}} \frac{\bar{R} e^{\hat{R}_t}}{\bar{\pi} e^{\hat{\pi}_{t+1}}} \right] - \bar{c}^{-\sigma} e^{-\sigma \hat{c}_t} \\
&= \beta E_t \left[\frac{\bar{c}^{-\sigma} \bar{R}}{\bar{\pi}} e^{-\sigma \hat{c}_{t+1} + \hat{R}_t - \hat{\pi}_{t+1}} \right] - \bar{c}^{-\sigma} e^{-\sigma \hat{c}_t}
\end{aligned}
$$

Ignoring for the moment the expectation operator (we put it back after doing the algebra) and linearizing around $\left(\hat{R}, \hat{c}_{t+1}, \hat{\pi}_{t+1}, \hat{c}_t\right) = (0, 0, 0, 0)$ gives

$$
0 \approx \beta \frac{\bar{c}^{-\sigma} \bar{R}}{\bar{\pi}} \left[-\sigma \hat{c}_{t+1} + \hat{R}_t - \hat{\pi}_{t+1} \right] + \bar{c}^{-\sigma} \sigma \hat{c}_t
$$

$$
0 \approx \beta \bar{c}^{-\sigma} \frac{\bar{R}}{\bar{\pi}} \left[\hat{R}_t - \hat{\pi}_{t+1} - \sigma \hat{c}_{t+1} \right] + \bar{c}^{-\sigma} \sigma \hat{c}_t
$$

But in steady state $\beta\bar{c}^{-\sigma}\frac{\bar{R}}{\bar{\pi}} = \bar{c}^{-\sigma}$ so that

$$0 \approx \hat{R}_t - \hat{\pi}_{t+1} - \sigma\hat{c}_{t+1} + \sigma\hat{c}_t$$

$$0 \approx \hat{R}_t - \hat{\pi}_{t+1} - \sigma\hat{c}_{t+1} + \sigma\hat{c}_t.$$

Reinserting the expectation operator and rearranging gives

$$E_t\hat{c}_{t+1} \approx \hat{c}_t + \frac{1}{\sigma}\left[\hat{R}_t - E_t\hat{\pi}_{t+1}\right]. \tag{41}$$

which is the linearized Euler equation.

Equation (30):

$$\begin{aligned} 0 &= y_t\left[1 - \frac{\omega}{2}\left(\pi_t - \bar{\pi}\right)^2\right] - c_t \\ &= \bar{y}e^{\hat{y}_t}\left[1 - \frac{\omega}{2}\left(\bar{\pi}e^{\hat{\pi}_t} - \bar{\pi}\right)^2\right] - \bar{c}e^{\hat{c}_t} \end{aligned}$$

Linearizing around $(\hat{y}_t, \hat{\pi}_t, \hat{c}_t) = (0,0,0)$ gives

$$0 \approx \bar{y}\hat{y}_t - \bar{y}\omega\left(\bar{\pi} - \bar{\pi}\right)\hat{\pi}_t - \bar{c}\hat{c}_t$$

Using $\bar{y} = \bar{c}$ we obtain

$$\hat{c}_t \approx \hat{y}_t. \tag{42}$$

Equation (31):

We can write this equation as

$$\begin{aligned} 0 = {} & \frac{\theta}{\omega}\left[\eta\frac{\bar{y}^{\nu}\bar{c}^{\sigma}}{\bar{A}^{1+\nu}}e^{\nu\hat{y}_t + \sigma\hat{c}_t - (1+\nu)\hat{A}_t} - \frac{\theta-1}{\theta}\right] - \left(\bar{\pi}e^{\hat{\pi}_t} - \bar{\pi}\right)\bar{\pi}e^{\hat{\pi}_t} \\ & + E_t\left[\beta\bar{\pi}e^{-\sigma(\hat{c}_{t+1} - \hat{c}_t) + (\hat{y}_{t+1} - \hat{y}_t) + \hat{\pi}_{t+1}}\left(\bar{\pi}e^{\hat{\pi}_{t+1}} - \bar{\pi}\right)\right] \end{aligned}$$

Ignoring for the moment the expectation operator and linearizing around $\left(\hat{\pi}_t, \hat{y}_t, \hat{c}_t, \hat{A}_t, \hat{\pi}_{t+1}, \hat{y}_{t+1}, \hat{c}_{t+1}\right) = (0,0,0,0,0,0,0)$ gives

$$0 \approx \frac{\theta}{\omega}\eta\frac{\bar{y}^{\nu}\bar{c}^{\sigma}}{\bar{A}^{1+\nu}}\left[\nu\hat{y}_t + \sigma\hat{c}_t - (1+\nu)\hat{A}_t\right] - \bar{\pi}^2\hat{\pi}_t + \beta\bar{\pi}^2\hat{\pi}_{t+1}$$

but in steady state

$$\eta\frac{\bar{y}^{\nu}\bar{c}^{\sigma}}{\bar{A}^{1+\nu}} = \frac{\theta-1}{\theta}$$

Therefore we have

$$0 \approx \frac{\theta - 1}{\omega} \left[\nu \hat{y}_t + \sigma \hat{c}_t - (1 + \nu) \hat{A}_t \right] - \bar{\pi}^2 \hat{\pi}_t + \beta \bar{\pi}^2 \hat{\pi}_{t+1}$$

Dividing by $\bar{\pi}^2$, inserting the expectation operator, and rearranging

$$\beta E_t [\hat{\pi}_{t+1}] \approx \hat{\pi}_t - \frac{\theta - 1}{\bar{\pi}^2 \omega} \left[\nu \hat{y}_t + \sigma \hat{c}_t - (1 + \nu) \hat{A}_t \right]$$

But using that $\hat{y}_t = \hat{c}_t$ (this is equation (42)), we can write this equation as

$$\hat{\pi}_t = \beta E_t [\hat{\pi}_{t+1}] + \kappa \left[(\nu + \sigma) \hat{y}_t - (1 + \nu) \hat{A}_t \right] \tag{43}$$

where

$$\kappa \equiv \frac{\theta - 1}{\bar{\pi}^2 \omega}.$$

Please note that the term $\left[(\nu + \sigma) \hat{y}_t - (1 + \nu) \hat{A}_t \right]$ is actually the log-deviation of the marginal cost of the firm from its steady state value

$$\widehat{mc}_t = (\nu + \sigma) \hat{y}_t - (1 + \nu) \hat{A}_t.$$

Using this observation, you can write (43) as

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \kappa \widehat{mc}_t \tag{44}$$

which is an expression that appears in many papers of the literature.

Equation (32):

This equation is already log-linear. Subtracting $\log(\bar{R})$ from both sides we have

$$\log\left(\frac{R_t}{\bar{R}}\right) = (1 - \rho_R) \left[\log\left(\frac{\bar{R}}{\bar{R}}\right) + \phi_\pi \log\left(\frac{\pi_t}{\bar{\pi}}\right) + \phi_y \log\left(\frac{y_t}{\bar{y}}\right) \right] + \rho_R \log\left(\frac{R_{t-1}}{\bar{R}}\right) + \upsilon_t$$

Using the definition of hatted variables,

$$\hat{R}_t = (1 - \rho_R) \left[\phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t \right] + \rho_R \hat{R}_{t-1} + \upsilon_t \tag{45}$$

Equation (33):

The equation for the evolution of productivity is already log-linear. We thus have,

$$\hat{A}_{t+1} = \rho_A \hat{A}_t + \varepsilon_{A,t+1} \tag{46}$$

We also keep the interest rate shock linear as in equation (34)

Summarizing, the linearized system of equations is

$$0 = -\sigma \hat{c}_t + \frac{\hat{R}_t}{\bar{R}-1} + \xi \hat{m}_t$$

$$E_t\left[\hat{c}_{t+1}\right] = \hat{c}_t + \frac{1}{\sigma}\left[\hat{R}_t - E_t\left[\hat{\pi}_{t+1}\right]\right]$$

$$0 = \hat{c}_t - \hat{y}_t$$

$$\hat{\pi}_t = \beta E_t\left[\hat{\pi}_{t+1}\right] + \kappa\left[\left(\nu+\sigma\right)\hat{y}_t - \left(1+\nu\right)\hat{A}_t\right]$$

$$\hat{R}_t = \left(1-\rho_R\right)\left[\phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t\right] + \rho_R \hat{R}_{t-1} + v_t$$

$$\hat{A}_{t+1} = \rho_A \hat{A}_t + \varepsilon_{A,t+1}$$

$$v_{t+1} = \rho_v v_t + \varepsilon_{v,t+1}$$

This can be simplified. Use the third condition to get rid of all terms with $\hat{c}_t$:

$$0 = -\sigma \hat{y}_t + \frac{\hat{R}_t}{\bar{R}-1} + \xi \hat{m}_t \tag{47}$$

$$E_t\left[\hat{y}_{t+1}\right] = \hat{y}_t + \frac{1}{\sigma}\left[\hat{R}_t - E_t\left[\hat{\pi}_{t+1}\right]\right] \tag{48}$$

$$\hat{\pi}_t = \beta E_t\left[\hat{\pi}_{t+1}\right] + \kappa\left[\left(\nu+\sigma\right)\hat{y}_t - \left(1+\nu\right)\hat{A}_t\right] \tag{49}$$

$$\hat{R}_t = \left(1-\rho_R\right)\left[\phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t\right] + \rho_R \hat{R}_{t-1} + v_t \tag{50}$$

$$\hat{A}_{t+1} = \rho_A \hat{A}_t + \varepsilon_{A,t+1} \tag{51}$$

$$v_{t+1} = \rho_v v_t + \varepsilon_{v,t+1} \tag{52}$$

The set of equations (48), (49), and (50) is usually referred to as *The Three Equation New Keynesian Model*. Given the equilibrium values of $\hat{y}_t$ and $\hat{R}_t$, equation (47) can be used to recover the money demand $\hat{m}_t$, so we basically can ignore it for the purpose of finding the equilibrium. Therefore, from now on we focus on the equations (48), (49), (50), (51), and (52).

We want to put this system of equations in a form to be used in Paul Klein's Matlab function `solab.m`. We need to identify the state and the control variables. The state or predetermined variables are those variables that agents cannot affect at time t and that help predict the future evolution of the system. In the current model, at the beginning of period t neither R_{t-1}, A_t, nor v_t can be changed by anyone. Furthermore, those variables help predict the future evolution of the economy. Therefore, the state variables of this model are

$$\mathbf{x}_t = \left[\hat{R}_{t-1}, \hat{A}_t, v_t\right]'$$

The control variables are the rest of the variables in the economy—recall that we will recover the demand for money later on,

$$\mathbf{y}_t = \left[\hat{y}_t, \hat{\pi}_t\right]$$

so that

$$\mathbf{s}_{t+1} = \begin{bmatrix} \hat{R}_t \\ \hat{A}_{t+1} \\ v_{t+1} \\ \hat{y}_{t+1} \\ \hat{\pi}_{t+1} \end{bmatrix} \qquad \mathbf{s}_t = \begin{bmatrix} \hat{R}_{t-1} \\ \hat{A}_t \\ v_t \\ \hat{y}_t \\ \hat{\pi}_t \end{bmatrix}$$

We rewrite our system of equations in a form to be used in the `solab.m` routine

$$\begin{aligned}
E_t\left[\hat{y}_{t+1}\right] + \frac{1}{\sigma}E_t\left[\hat{\pi}_{t+1}\right] - \frac{1}{\sigma}E_t\left[\hat{R}_t\right] &= \hat{y}_t \\
\beta E_t\left[\hat{\pi}_{t+1}\right] &= \hat{\pi}_t - \kappa\left[\left(\nu+\sigma\right)\hat{y}_t - \left(1+\nu\right)\hat{A}_t\right] \\
E_t\left[\hat{R}_t\right] &= \left(1-\rho_R\right)\left[\phi_\pi\hat{\pi}_t + \phi_y\hat{y}_t\right] + \rho_R\hat{R}_{t-1} + v_t \\
E_t\left[\hat{A}_{t+1}\right] &= \rho_A\hat{A}_t \\
E_t\left[v_{t+1}\right] &= \rho_v v_t.
\end{aligned}$$

Note that we are adding the irrelevant expectation in the interest rate at time t ($R_t = E_t\left[R_t\right]$ is known at t,) because this is the notation that is used in the `solab.m` routine.

We want to put this system of equations in the form

$$\mathbf{A}E_t\left[\mathbf{s}_{t+1}\right] = \mathbf{B}\mathbf{s}_t$$

where $\mathbf{A}$ and $\mathbf{B}$ are 5×5 matrices. We have

$$\begin{bmatrix} -\frac{1}{\sigma} & 0 & 0 & 1 & \frac{1}{\sigma} \\ 0 & 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} E_t \begin{bmatrix} \hat{R}_t \\ \hat{A}_{t+1} \\ v_{t+1} \\ \hat{y}_{t+1} \\ \hat{\pi}_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & \kappa\left(1+\nu\right) & 0 & -\kappa\left(\nu+\sigma\right) & 1 \\ \rho_R & 0 & 1 & \left(1-\rho_R\right)\phi_y & \left(1-\rho_R\right)\phi_\pi \\ 0 & \rho_A & 0 & 0 & 0 \\ 0 & 0 & \rho_v & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{R}_{t-1} \\ \hat{A}_t \\ v_t \\ \hat{y}_t \\ \hat{\pi}_t \end{bmatrix}.$$

The Blanchard and Kahn condition requires that this system has exactly 3 stable generalized eigenvalues to obtain a unique rational expectations equilibrium. In another note we will discuss the issue of determinacy in a simpler version of this model.