Threshold Models

Series de Tiempo

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Introduction

- Economic variables may behave very differently in different states of the economy such as, for example, high/low inflation, high/low growth, or high/low stock prices (relative to dividends).
- A variety of nonlinear models have been proposed for describing the dynamics of economic time series subject to changes in regime [see, e.g., Tong (1983, 1990); Hamilton (1993); van Dijk et al. (2002); Dueker et al. (2007)].

TAR and SETAR Models

- Self-Exciting Threshold AutoRegressive (SETAR) models are an extension of autoregressive models which allow for a different behaviour of the series once the series enters a different regime.
- The switch from one regime to another depends on a past value of the series (hence the Self-Exciting portion of the name). crossing a threshold value. A possible parameterization of the SETAR model is

$$\begin{aligned} y_t &= y_{1t} && \text{if } y_{t-1} < k \\ &= y_{2t} && \text{if } y_{t-1} \geq k \end{aligned} \tag{1}$$

  where

$$y_{it} = \mu_i + \sum_{j=1}^{p} \alpha_j^{(i)} y_{t-j} + \sigma_i u_t, \quad i = 1, 2, \tag{2}$$

- To estimate the model a grid has to be created for k, and then choose the value of the grid with highest likelihood. Notice that the model can be estimated by OLS and evaluated using MLE.

- In general we can consider a model such as

$$
\begin{aligned}
y_t &= y_{1t} && \text{if } z_t < k \\
&= y_{2t} && \text{if } z_t \geq k
\end{aligned}
\tag{3}
$$

where z_t is a vector of exogenous or predetermined variables. When doing this we have to be careful with the exogeneity assumptions about z_t.

- To test for the existence of two states is not trivial because under the null that

$$
\begin{aligned}
\mu_1 &= \mu_2 \\
\alpha_j^{(1)} &= \alpha_j^{(2)} \text{ for } j = 1, ..., p.
\end{aligned}
$$

  the parameter k is not defined, or is what is called a nuisance parameter.

Smooth Transition Autoregressive Models

- One posible drawback of the SETAR models is that potential big changes on the conditional mean process are trigerred by small changes in the variable that drives the regimes.
- The STAR model may be thought of as a function of two (or more) autoregressive processes which are averaged, at any given point in time, according to some continuous function $G(\cdot)$ taking values in $[0,1]$.
- STAR models for the univariate time series $\{x_t\}$ may be formulated as

$$y_t = G(\mathbf{z}_{t-1})y_{1t} + [1 - G(\mathbf{z}_{t-1})]y_{2t}, \quad t = 1, 2, \ldots, \tag{4}$$

where $\mathbf{z}_{t-1}$ is a vector of exogenous and/or pre-determined variables and

$$y_{it} = \mu_i + \sum_{j=1}^{p} \alpha_j^{(i)} y_{t-j} + \sigma_i u_t, \quad i = 1, 2, \tag{5}$$

where p is a positive integer, $\{u_t\}$ are independent and identically distributed (i.i.d.) random variables such that u_t is independent of $(y_{t-p}, \ldots, y_0)$ $\mathbb{E}(u_t) = \mathbb{E}(u_t^2 - 1) = 0$, σ_1 and σ_2 are positive constants, and μ_i and $\alpha_j^{(i)}$ $(i = 1, 2;\; j = 1, \ldots, p)$ are real constants.

Logistic Smooth Transition Autoregressive Models

- STAR models like (4)–(5) have been used extensively in the analysis of economic and financial data. The main feature that differentiates alternative STAR specifications is the choice of the mixing (or transition) function $G(\cdot)$ and the transition variables $\mathbf{z}_{t-1}$ (see Teräsvirta, 1998; van Dijk et al., 2002). A popular choice for $G(\cdot)$ in (4) is the logistic specification

$$G(y_{t-1}) = \frac{\exp\{-\gamma(y_{t-1} - y^*)\}}{1 + \exp\{-\gamma(y_{t-1} - y^*)\}}, \qquad \gamma > 0, \tag{6}$$

  which gives rise to the pth-order LSTAR model, or LSTAR(p).
- The location parameter y^* in (6) may be interpreted as the threshold between the two regimes associated with the extreme values $\lim_{y_{t-1}\to\infty} G(y_{t-1}) = 0$ and $\lim_{y_{t-1}\to-\infty} G(y_{t-1}) = 1$, while the slope parameter γ determines the smoothness of the transitions between the two regimes.

Exponential Smooth Transition Autoregressive Models

- ESTAR models are intended to capture a different behavior of the time series when is close to the threshold from when it is far from the threshold.

$$G(y_{t-1}) = \exp\{-\gamma(y_{t-1} - y^*)^2\}, \qquad \gamma > 0, \tag{7}$$

  which gives rise to the pth-order ESTAR model, or ESTAR(p). Notice that the extreme values $\lim_{y_{t-1}\to\infty} G(y_{t-1}) = 0$ and $\lim_{y_{t-1}\to-\infty} G(y_{t-1}) = 0$, while $\lim_{y_{t-1}\to y^*} G(y_{t-1}) = 1$.
- This type of models are used for situations like target zones where the behaviour of the exchange rate in the middle of the band is different from that in the middle of the band.

Univariate Contemporaneous-Threshold Models

- The C-STAR model of Dueker et al. (2007) may be thought of as a function of two (or more) autoregressive processes which are averaged, at any given point in time, according to some continuous function $G(\cdot)$ taking values in $[0, 1]$.
- STAR models for the univariate time series $\{y_t\}$ may be formulated as

$$y_t = G(\mathbf{z}_{t-1})y_{1t} + [1 - G(\mathbf{z}_{t-1})]y_{2t}, \quad t = 1, 2, \ldots,$$

where $\mathbf{z}_{t-1}$ is a vector of exogenous and/or pre-determined variables and

$$y_{it} = \mu_i + \sum_{j=1}^{p} \alpha_j^{(i)} y_{t-j} + \sigma_i u_t, \quad i = 1, 2,$$

where p is a positive integer, $\{u_t\}$ are independent and identically distributed (i.i.d.) random variables such that u_t is independent of $(y_{t-p}, \ldots, y_0)$, $\mathbb{E}(u_t) = \mathbb{E}(u_t^2 - 1) = 0$, σ_1 and σ_2 are positive constants, and μ_i and $\alpha_j^{(i)}$ $(i = 1, 2;\ j = 1, \ldots, p)$ are real constants.

- The feature that differentiates alternative STAR models is the choice of the mixing function $G(\cdot)$ and transition variables $\mathbf{z}_{t-1}$ [cf. Teräsvirta (1998); van Dijk et al. (2002)].
- The C-STAR model of order p is obtained by defining the mixing function $G(\cdot)$ in (4) as

$$G(\mathbf{z}_{t-1}) = \frac{\mathbb{P}(y_{1t} < y^*|\mathbf{z}_{t-1};\boldsymbol{\vartheta}_1)}{\mathbb{P}(y_{1t} < y^*|\mathbf{z}_{t-1};\boldsymbol{\vartheta}_1) + \mathbb{P}(y_{2t} \geqslant y^*|\mathbf{z}_{t-1};\boldsymbol{\vartheta}_2)}$$

where $\boldsymbol{\vartheta}_i = (\mu_i, \alpha_1^{(i)}, \ldots, \alpha_p^{(i)}, \sigma_i^2)'$ is the vector of parameters associated with regime i, and $\mathbf{z}_{t-1} = (y_{t-1}, ..., y_{t-p})'$.

- Letting $\boldsymbol{\alpha}_i = (\alpha_1^{(i)}, \ldots, \alpha_p^{(i)})'$ $(i = 1, 2)$, the (conditionally) Gaussian two-regime

$$G(\mathbf{z}_{t-1}) = \frac{\Phi(\dfrac{y^* - \mu_1 - \boldsymbol{\alpha}_1'\mathbf{z}_{t-1}}{\sigma_1})}{\Phi(\dfrac{y^* - \mu_1 - \boldsymbol{\alpha}_1'\mathbf{z}_{t-1}}{\sigma_1}) + 1 - \Phi(\dfrac{y^* - \mu_2 - \boldsymbol{\alpha}_2'\mathbf{z}_{t-1}}{\sigma_2})}, \qquad (8)$$

where $\Phi(\cdot)$ denotes the standard normal distribution function and y^* is a threshold parameter.

- Hence, (4) may be rewritten as

$$y_t = \frac{\mathbb{P}(y_{1t} < y^* | \mathbf{z}_{t-1}; \boldsymbol{\vartheta}_1) y_{1t} + \mathbb{P}(y_{2t} \geq y^* | \mathbf{z}_{t-1}; \boldsymbol{\vartheta}_2) y_{2t}}{\mathbb{P}(y_{1t} < y^* | \mathbf{z}_{t-1}; \boldsymbol{\vartheta}_1) + \mathbb{P}(y_{2t} \geq y^* | \mathbf{z}_{t-1}; \boldsymbol{\vartheta}_2)}.$$

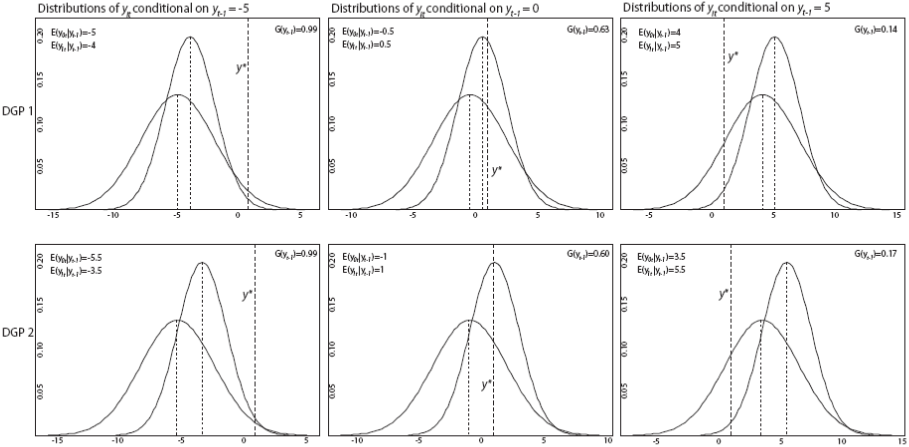
- As with conventional STAR models, a C-STAR model may be thought of as a regime-switching model that allows for two regimes associated with the two latent variables y_{1t} and y_{2t}.
- Alternatively, a C-STAR model may be thought of as allowing for a continuum of regimes, each of which is associated with a different value of $G(\mathbf{z}_{t-1})$.

- One of the main purposes of the C-STAR model is to address two somewhat arbitrary features of conventional STAR models:
- First, STAR models specify a delay such that the mixing function for period t consists of a function of y_{t-j} for some $j \geq 1$.
- Second, STAR models specify which of and in what way the model parameters enter the mixing function.
- C-STAR models address these twin issues in an intuitive way: they use a forecasting function such that the mixing function depends on the ex ante regime-dependent probabilities that y_t will exceed the threshold value(s). Furthermore, the mixing function makes use of all of the model parameters in a coherent way.

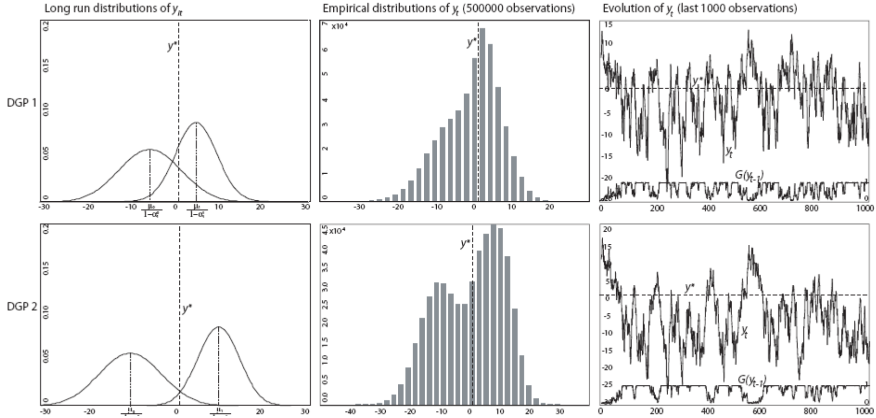
Table 2. DGPs

	μ_0	α_1^0	σ_0	μ_1	α_1^1	σ_1	y^*
DGP 1	-0.5	0.9	3	0.5	0.9	2	1
DGP 2	-1	0.9	3	1	0.9	2	1

Conditional Distributions



Long run, Empirical Distributions and Simulated Time Series



Stability Properties of the Skeleton of the C-STAR

- As Chan and Tong (1985) pointed out, we can analyze the properties of a nonlinear time series by considering the deterministic part of the model alone.
- This part is usually called the skeleton of the model and is defined as $y_t = F(y_{t-1}, \Theta)$, where

$$F(y_{t-1}, \Theta) = G(y_{t-1})(\mu_0 + \alpha_1^0 y_{t-1}) + (1 - G(y_{t-1}))(\mu_1 + \alpha_1^1 y_{t-1}), \quad (9)$$

  and $\Theta = \{\Theta_0, \Theta_1, y^*\}$.
- Then a fixed point of the skeleton of the model is any value, y_L, that satisfies

$$y_L = F(y_L, \Theta) = G(y_L)(\mu_0 + \alpha_1^0 y_L) + (1 - G(y_L))(\mu_1 + \alpha_1^1 y_L). \quad (10)$$

- Since the C-STAR(1) is a nonlinear model, there may be one, several or no equilibrium values that satisfy equation (10).
- Then, assessing which of the equilibria of the nonlinear first-order difference equation are stable is crucial for learning about the stability properties of the C-STAR model.
- We use each of the DGPs presented in Table 2 to assess:
- *i*) the number of equilibria and
- *ii*) the stability of the equilibria.

- We find the number of equilibria for the different DGPs presented in Table 1, using a grid of starting values to solve equation (10) numerically.
- For each equilibrium, we analyze whether it is locally stable by considering the following expansion around the fix point

$$\begin{aligned} y_t - y_L &= F(y_{t-1}, \Theta) - F(y_L, \Theta) \\ &\simeq \frac{\partial F(y_{t-1}, \Theta)}{\partial y_{t-1}}(y_{t-1} - y_L). \end{aligned} \tag{11}$$

- Whenever $\left|\frac{\partial F(y_{t-1},\Theta)}{\partial y_{t-1}}\right| < 1$, the equilibrium is locally stable and $F(y_{t-1}, \Theta)$ is a contraction in the neighborhood of $y = y_L$.

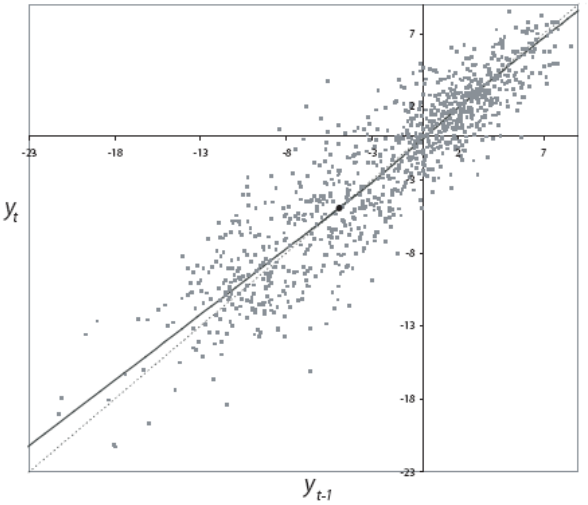
- Where

$$\frac{\partial F(y_L,\Theta)}{\partial y_{t-1}} = \alpha_1^1 + (\alpha_1^0 - \alpha_1^1)G(y_L) + \left[(\mu_0 - \mu_1) + (\alpha_1^0 - \alpha_1^1)y_L\right]\frac{\partial G(y_L)}{\partial y_{t-1}} \tag{12}$$

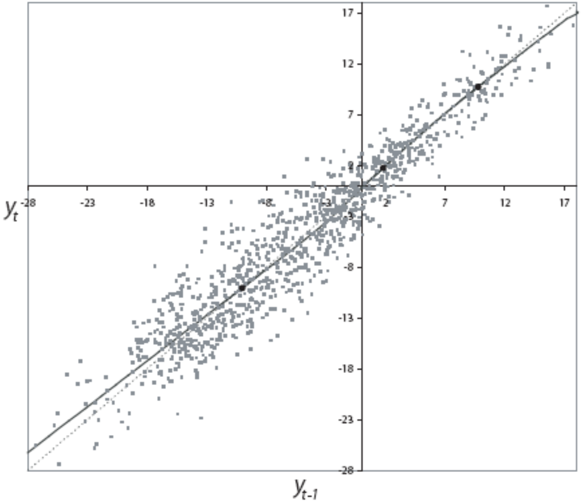
and

$$\begin{aligned}\frac{\partial G(y_L)}{\partial y_{t-1}} &= \frac{-\left(\frac{\alpha^0}{\sigma_0}\phi(w_0^L)\Phi(w_1^L) + \frac{\alpha^1}{\sigma_1}\phi(w_1^L)\Phi(w_0^L)\right)}{\left(\Phi(w_0^L) + [1-\Phi(w_1^L)]\right)^2}, \text{ where } \phi = \Phi', \\ w_0^L &= (y^* - \mu_0 - \alpha_1^0 y_L)/\sigma_0 \text{ and } w_1^L = (y^* - \mu_1 - \alpha_1^1 y_L)/\sigma_1.\end{aligned}$$

DGP 1



DGP 2



State Dependent Contemporaneous-Threshold Models

- Here we investigate the possibility that the separation of regimes implied by nonlinear regime-switching autoregressive models is better characterized in relative terms rather than being dictated by the (constant) level of some variables.
- We consider the threshold as function of variables which potentially affect the evolution of the time series under consideration.
- For example, interest rates may be considered high or low not in absolute terms but relative to relevant macroeconomic variables that describe the state of the economy. A rate of interest which is considered to be high when the economy is in a state of low inflation and high output growth may be considered too low when the economy in a state of high inflation and low output growth.
- We propose to model such behavior using variants of smooth transition autoregressive (STAR) models which allow the interest rate threshold to evolve over time as a function of inflation and output growth.

State Dependent Contemporaneous-Threshold Models

- We generalize the standard C-STAR specifications to allow for state dependency of the threshold.
- We define y^*_{t-1} as a time-varying threshold. This is specified as a linear combination of the elements of a k-dimensional vector $\mathbf{x}_{t-1} = (x_{1,t-1}, \ldots, x_{k,t-1})'$ of observable exogenous and/or predetermined variables, that is,

$$y^*_{t-1} = y^* + \boldsymbol{\delta}'\mathbf{x}_{t-1}, \tag{13}$$

where y^* is an unknown threshold intercept and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)'$ is a vector of unknown parameters.

State Dependent Contemporaneous-Threshold Models

- The SDC-STAR(p) model
- To generalize the C-STAR(p) specification to allow for a state-dependent threshold, the mixing weights in (8) are replaced with

$$G(\mathbf{z}_{t-1}) = \frac{F_1(\frac{y^*_{t-1} - \mu_1 - \boldsymbol{\alpha}_1' \mathbf{y}_{t-1}}{\sigma_1})}{F_1(\frac{y^*_{t-1} - \mu_1 - \boldsymbol{\alpha}_1' \mathbf{y}_{t-1}}{\sigma_1}) + 1 - F_2(\frac{y^*_{t-1} - \mu_2 - \boldsymbol{\alpha}_2' \mathbf{y}_{t-1}}{\sigma_2})}, \tag{14}$$

  where y^*_{t-1} is a time-varying threshold depending on the observable exogenous and/or predetermined variables $\mathbf{x}_{t-1}$, and $\mathbf{z}_{t-1} = (\mathbf{y}'_{t-1}, \mathbf{x}'_{t-1})'$. As before, the threshold y^*_{t-1} is specified to be the linear combination of the elements of $\mathbf{x}_{t-1}$ given in (13).

State Dependent Contemporaneous-Threshold Models

- Notice that

$$G(\mathbf{z}_{t-1}) = \frac{\mathbb{P}(y_{1t} < y^*_{t-1}|\mathbf{z}_{t-1})}{\mathbb{P}(y_{1t} < y^*_{t-1}|\mathbf{z}_{t-1}) + \mathbb{P}(y_{2t} \geq y^*_{t-1}|\mathbf{z}_{t-1})}$$

  and

$$1 - G(\mathbf{z}_{t-1}) = \frac{\mathbb{P}(y_{2t} \geq y^*_{t-1}|\mathbf{z}_{t-1})}{\mathbb{P}(y_{1t} < y^*_{t-1}|\mathbf{z}_{t-1}) + \mathbb{P}(y_{2t} \geq y^*_{t-1}|\mathbf{z}_{t-1})}.$$

- Hence, under the assumptions in (13)–(14), the SDC-STAR(p) model may be rewritten as

$$y_t = \frac{\mathbb{P}(y_{1t} < y^*_{t-1}|\mathbf{z}_{t-1})y_{1t} + \mathbb{P}(y_{2t} \geq y^*_{t-1}|\mathbf{z}_{t-1})y_{2t}}{\mathbb{P}(y_{1t} < y^*_{t-1}|\mathbf{z}_{t-1}) + \mathbb{P}(y_{2t} \geq y^*_{t-1}|\mathbf{z}_{t-1})}. \qquad (15)$$

  As in the case of the C-STAR, the mixing weights involve the probability that the contemporaneous value of y_{1t} (y_{2t}) is smaller (greater) than some threshold level y^*_{t-1}.
- The SDC-STAR(p) model reduces to a C-STAR(p) under the restriction $\delta = \mathbf{0}$.

State Dependent Contemporaneous-Threshold Models

- A SDC-STARX(p) Model.
- This model includes exogenous and/or predetermined variables $\mathbf{x}_{t-1}$ in the equation that describes the dynamics of the latent variables y_{1t} and y_{2t}. More specifically, let

$$y_{it} = \mu_i + \boldsymbol{\alpha}^{(i)\prime}\mathbf{y}_{t-1} + \boldsymbol{\delta}^{(i)\prime}\mathbf{x}_{t-1} + \sigma_i u_{it}, \qquad i = 1, 2, \tag{16}$$

where $\boldsymbol{\delta}^{(i)} = (\delta_1^{(i)}, \ldots, \delta_k^{(i)})'$ are unknown parameters, and take $G(\cdot)$ to have the form

$$G(\mathbf{z}_{t-1}) = \frac{F_1(\dfrac{y^*_{t-1} - \mu_1 - \boldsymbol{\beta}_1'\mathbf{z}_{t-1}}{\sigma_1})}{F_1(\dfrac{y^*_{t-1} - \mu_1 - \boldsymbol{\beta}_1'\mathbf{z}_{t-1}}{\sigma_1}) + 1 - F_2(\dfrac{y^*_{t-1} - \mu_2 - \boldsymbol{\beta}_2'\mathbf{z}_{t-1}}{\sigma_2})}, \tag{17}$$

with $\boldsymbol{\beta}_i = (\boldsymbol{\alpha}_i', \delta_i')'$ $(i = 1, 2)$. Equations (4), (13), (16) and (17) define a pth-order state-dependent C-STAR model with exogenous variables.

State Dependent Contemporaneous-Threshold Models

- The SDC-STARX(p) model nests several specifications.
- If $\delta^{(1)} = \delta^{(2)}$, $\delta = \mathbf{0}$, we obtain a restricted SDC-STARX(p) model, or *RSDC-STARX*(p), the mixing weights of which are the same as those of a SDC-STAR(p) model with threshold $y^*_{t-1} = y^* - \delta^{(1)\prime}\mathbf{x}_{t-1}$; the two specifications are not, however, equivalent because they imply different conditional distributions for the latent variables y_{it}.
- If $\delta^{(1)} = \delta^{(2)} = \mathbf{0}$, the SDC-STARX($p$) model becomes a SDC-STAR(p).
- If $\delta = \mathbf{0}$, we obtain a pth-order C-STAR model with exogenous variables, or C-STARX(p), and constant threshold y^*.
- Finally, if $\delta^{(1)} = \delta^{(2)} = \delta = \mathbf{0}$, the SDC-STARX($p$) reduces to a C-STAR($p$) with constant threshold y^*.
- The validity of these restrictions on $\delta^{(1)}$, $\delta^{(2)}$ and δ can be tested using, for example, conventional likelihood ratio (LR) tests.

Stability

- The stability properties of SDC-STAR and C-STARX models follow as special cases of the SDC-STARX.
- Consider a SDC-STARX(1) model with a threshold which depends on a single exogenous variable x_{t-1} $(k = 1)$. The skeleton of such a model is defined as

$$Y_t = S(Y_{t-1}, X_{t-1}), \tag{18}$$

where

$$\begin{aligned} S(Y_{t-1}, X_{t-1}) &= G(Y_{t-1}, X_{t-1})\{\mu_1 + \alpha_1^{(1)} Y_{t-1} + \delta_1^{(1)} X_{t-1}\} \\ &\quad + \{1 - G(Y_{t-1}, X_{t-1})\}\{\mu_2 + \alpha_1^{(2)} Y_{t-1} + \delta_1^{(2)} X_{t-1}\}, \end{aligned} \tag{19}$$

and $G(Y_{t-1}, X_{t-1})$ is given by (17) with $\mathbf{z}_{t-1} = (Y_{t-1}, X_{t-1})'$ and $y^*_{t-1} = y^* + \delta x_{t-1}$.

Stability

- Assuming $\{x_t\}$ is stationary, a fixed point of the skeleton is any value Y_e which satisfies the equation

$$Y_e = S(Y_e, X_e), \tag{20}$$

  where $X_e = \mathbb{E}(x_t)$. The value Y_e is said to be an equilibrium point of the SDC-STARX(1) model and, since the model is nonlinear, there may be one, several or no equilibrium points satisfying (20).

- An examination of the local stability of each of the equilibrium points can be carried out by considering a first-order Taylor expansion about the fixed point,

$$Y_t - Y_e \approx \lambda(Y_e, X_e)(Y_{t-1} - Y_e), \tag{21}$$

  where

$$\lambda(Y_e, X_e) = \left. \frac{\partial S(Y_{t-1}, X_e)}{\partial Y_{t-1}} \right|_{Y_{t-1}=Y_e}. \tag{22}$$

  If $|\lambda(Y_e, X_e)| < 1$, then the equilibrium is locally stable and Y_t is a contraction in the neighbourhood of (Y_e, X_e).

- It is straightforward to verify that

$$\frac{\partial S(Y_{t-1}, X_e)}{\partial Y_{t-1}} = \alpha_1^{(2)} + \{\alpha_1^{(1)} - \alpha_1^{(2)}\} G(Y_{t-1}, X_e)$$
$$+ \left(\mu_1 - \mu_2 + \{\alpha_1^{(1)} - \alpha_1^{(2)}\} Y_{t-1} + \{\delta_1^{(1)} - \delta_1^{(2)}\} X_e\right) \frac{\partial G(Y_{t-1}, X_e)}{\partial Y_{t-1}},$$

where

$$\frac{\partial G(Y_{t-1}, X_e)}{\partial Y_{t-1}} = -\frac{\sigma_1^{-1}\alpha_1^{(1)} f_1(w_1)[1 - F_2(w_2)] + \sigma_2^{-1}\alpha_1^{(2)} f_2(w_2) F_1(w_1)}{\{F_1(w_1) + 1 - F_2(w_2)\}^2}, \tag{23}$$

$f_i(\cdot)$ is the density of $F_i(\cdot)$ $(i = 1, 2)$, and

$$w_i = \sigma_i^{-1}\left(y^* - \mu_i - \alpha_1^{(i)} Y_{t-1} + \{\delta - \delta_1^{(i)}\} X_e\right), \qquad i = 1, 2.$$

- A numerical illustration
- Consider a SDC-STARX(1) model where u_{it} $(i = 1, 2)$ have the Student-t distribution with ν_i degrees of freedom. Assume that $\{x_t\}$ is the autoregressive model

$$x_t = 0.2 + 0.9x_{t-1} + \varepsilon_t,$$

where $\{\varepsilon_t\}$ are i.i.d. random variables, independent of $\{u_{1t}\}$ and $\{u_{2t}\}$, having the standardized Student-t distribution with 5 degrees of freedom.
- The parameters of the model take the following values:

$$\mu_1 = -0.5, \quad \mu_2 = 0.5, \quad \alpha_1^{(1)} = 0.9, \quad \alpha_1^{(2)} = 0.9, 0.99,$$
$$\sigma_1 = 3, \quad \sigma_2 = 2, \nu_1 = 3, \quad \nu_2 = 4, \quad y^* = 0,$$
$$\delta = 0, 0.3, \quad \delta_1^{(1)} = 0, 0.1, \quad \delta_1^{(2)} = 0, -0.1.$$

For each parameter configuration, we use a grid of starting values to solve equation (20) numerically and find the equilibrium points; the local stability of each equilibrium point is then examined by considering the expansion in (21)–(23).

Stability

Generated data, Histogram and Skeleton for the SDC−STARX

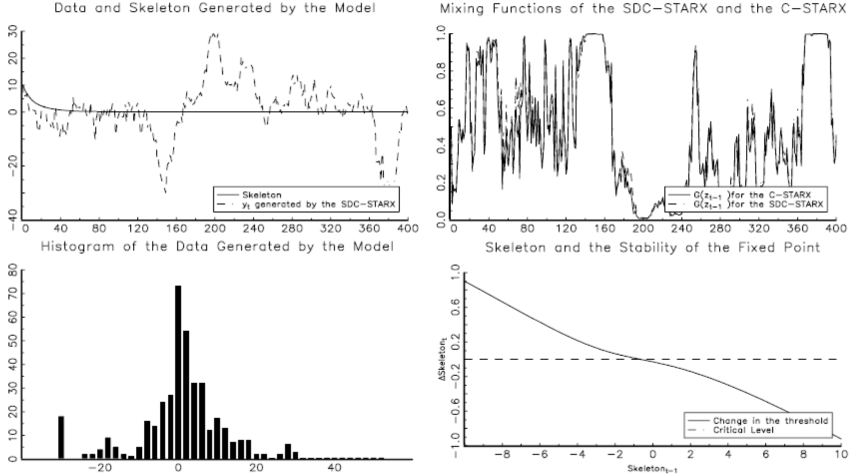


Figure 1

Stability

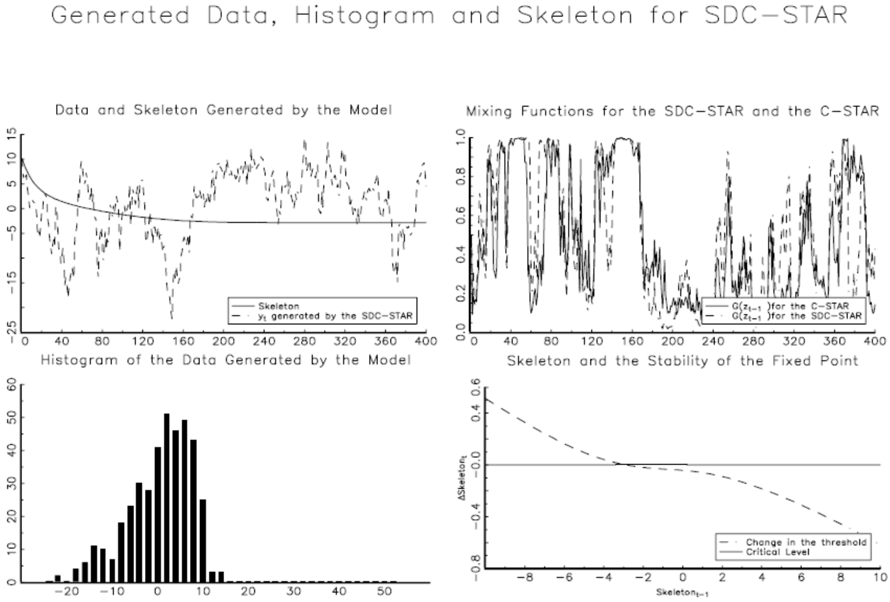


Figure 2

Estimation for the SDC-STAR model

- Given u_{1t} and u_{2t}, the parameters can be estimated by the ML method.
- Letting $\boldsymbol{\theta}$ denote the vector of all the unknown parameters, the conditional log-likelihood function associated with a sample $(y_1, \ldots, y_T)$ from the is

$$\mathcal{L}(\boldsymbol{\theta}) = \sum_{t=p+1}^{T} \ln \ell_t(\boldsymbol{\theta}),$$

where

$$\begin{aligned} \ell_t(\boldsymbol{\theta}) &= \frac{G(\mathbf{z}_{t-1})}{\sigma_1} f_1\left(\frac{y_t - \mu_1 - \boldsymbol{\alpha}_1' \mathbf{y}_{t-1}}{\sigma_1}\right) \\ &\quad + \frac{1 - G(\mathbf{z}_{t-1})}{\sigma_2} f_2\left(\frac{y_t - \mu_2 - \boldsymbol{\alpha}_2' \mathbf{y}_{t-1}}{\sigma_2}\right), \end{aligned} \tag{24}$$

with $G(\mathbf{z}_{t-1})$ given by (14) and $f_i(\cdot)$ denoting, as before, the probability density function of u_{it} $(i = 1, 2)$.

Estimation for the SDC-STARX

- Under this model, the contribution of the t-th observation to the conditional likelihood is

$$\begin{aligned}\ell_t(\boldsymbol{\theta}) &= \frac{G(\mathbf{z}_{t-1})}{\sigma_1} f_1\left(\frac{y_t - \mu_1 - \boldsymbol{\beta}_1' \mathbf{z}_{t-1}}{\sigma_1}\right) \\ &\quad + \frac{1 - G(\mathbf{z}_{t-1})}{\sigma_2} f_2\left(\frac{y_t - \mu_2 - \boldsymbol{\beta}_2' \mathbf{z}_{t-1}}{\sigma_2}\right),\end{aligned}$$

  with $G(\mathbf{z}_{t-1})$ given by (17).

- In the remainder of the paper, we use the standardized Student-t distribution with $\nu_i > 2$ degrees of freedom, i.e.,

$$f_i(z) = \frac{\Gamma\left(\{\nu_i + 1\}/2\right)}{\Gamma(\nu_i/2)\sqrt{(\nu_i - 2)\pi}} \left(1 + \frac{z^2}{\nu_i - 2}\right)^{-(\nu_i+1)/2}, \qquad -\infty < z < \infty,$$

  where $\Gamma(\cdot)$ is the gamma function. This specification has greater flexibility to accommodate possible outliers and other heavy-tailed characteristics in the data.

An Empirical Application

- We investigate whether the dynamics of U.S. short-term interests can be adequately described using C-STAR model with constant threshold or whether they are better represented by the SDC-STAR and SDC-STARX models with a threshold which is determined by inflation and output growth.
- Our data set consists of quarterly observations from 1947:2 to 2008:4 on the secondary market rate on three-month U.S. Treasury bills (y_t), the growth rate of real gross domestic product (g_t), and the consumer price inflation rate (π_t).

Estimation Results

- We first investigate the presence of nonlinearities in the dynamics of the interest rate. Specifically, using the modified Hansen LR-based method, we test a linear AR(4) model for the interest rate against C-STAR(4) alternative.
- The results of the tests show that the standardized LR statistic has, for all choices of the bandwidth parameter M, a P-value smaller than 0.05 (0.10) for the C-STAR(4) model. In view of the fact that the test is conservative by construction, these results provide significant evidence in favour of the nonlinear models.
- The state-dependent threshold y_{t-1}^* is specified as in (13) with $\mathbf{x}_{t-1} = (g_{t-1}, \pi_{t-1})'$.

Table 4: C-STAR, SDC-STAR and C-STARX Models

	C-STAR(4)	SDC-STAR(4)	RSDC-STARX(4)	C-STARX(4)	SDC-STARX(4)
μ_1	0.0491 (0.0503)	0.1440 (0.0479)	−0.0465 (0.0603)	0.0063 (0.0558)	−0.0607 (0.0513)
μ_2	−0.1046 (0.3341)	−0.0381 (0.2771)	−0.2268. (0.3734)	−0.5744 (0.2242)	−0.8349 (0.2893)
$\alpha_1^{(1)}$	1.3072 (0.0812)	1.2230 (0.0634)	1.3080 (0.1000)	1.3003 (0.0718)	1.2706 (0.0644)
$\alpha_2^{(1)}$	−0.3042 (0.1313)	−0.2114 (0.0857)	−0.3358 (0.1615)	−0.2928 (0.1163)	−0.2508 (0.0792)
$\alpha_3^{(1)}$	0.1142 (0.1035)	0.1102 (0.0846)	0.1152 (0.1108)	0.0980 (0.0956)	0.0537 (0.0783)
$\alpha_4^{(1)}$	−0.1023 (0.0635)	−0.0859 (0.0565)	−0.0708 (0.0650)	−0.0889 (0.0590)	−0.0397 (0.0531)
$\alpha_1^{(2)}$	1.0726 (0.1248)	1.2128 (0.1288)	1.0613. (0.1309)	0.9639 (0.09483)	0.9949 (0.1055)
$\alpha_2^{(2)}$	−0.0771 (0.1885)	−0.0989 (0.1719)	−0.0657 (0.1932)	−0.0773 (0.1458)	−0.0319 (0.1501)
$\alpha_3^{(2)}$	0.2700 (0.1604)	0.2859 (0.1445)	0.2969 (0.1693)	0.4162 (0.1320)	0.4729 (0.1376)
$\alpha_4^{(2)}$	−0.2693 (0.1053)	−0.4070. (0.0818)	−0.2940 (0.1062)	−0.3273 (0.0838)	−0.4510 (0.0728)

σ_1	0.3078 (0.0924)	0.3998 (0.1215)	0.3100 (0.0874)	0.2303 (0.0481)	0.4458 (0.1269)
σ_2	1.2893 (0.2901)	1.2189 (0.2445)	1.2648 (0.2877)	1.5630 (0.4293)	0.8953 (0.1317)
y^*	5.4196 (0.3286)	4.7219 (1.2844)	5.5435 (0.3303)	5.3874 (0.2720)	5.0555 (0.9534)
$\delta_1^{(1)}$	—	—	0.0156 (0.0062)	0.0054 (0.0051)	0.0102 (0.0049)
$\delta_2^{(1)}$	—	—	−0.0026 (0.0087)	0.0033 (0.0082)	0.0027 (0.0089)
$\delta_1^{(2)}$	—	—	0.0156 (0.0062)	0.1025 (0.0182)	0.0988 (0.0226)
$\delta_2^{(2)}$	—	—	−0.0026 (0.0087)	−0.0122 (0.0242)	0.0174 (0.0311)
δ_1	—	0.9385 (0.1403)	—	—	0.4337 (0.0834)
δ_2	—	−0.9461 (0.2105)	—	—	−0.6798 (0.2732)
ν_1	3.4019 (1.4695)	3.0337 (1.1981)	3.4342 (1.8676)	5.7408 (5.5767)	2.4909 (0.3935)
ν_2	2.7620 (0.5728)	2.5497 (0.3183)	2.5194 (0.5290)	2.2577 (0.1821)	3.1796 (0.6434)
Q_{30}	27.0962 [0.4429]	39.8058 [0.1327]	30.2040 [0.5177]	33.1596 [0.3558]	41.2709 [0.1185]
Q_{30}^2	39.6623 [0.0753]	35.6250 [0.3810]	42.3384 [0.0986]	39.9753 [0.1279]	39.1482 [0.1320[
$\mathcal{L}_{\max}$	−191.184	−177.718	−186.350	−175.943	−165.534
AIC	412.369	389.437	[28] 406.701	389.885	373.069
BIC	464.765	413.355	466.083	456.253	446.423

Figures in parentheses (square brackets) are standard errors (P-values).

- The Ljung–Box statistic based on standardized residuals (Q_r) with $r = 30$ lags; the Ljung–Box statistic based on squared standardized residuals (Q_r^2); the maximized log-likelihood ($\mathcal{L}_{\max}$); the Akaike information criterion (AIC); the Bayesian information criterion (BIC).
- Since the asymptotic distribution of residual autocorrelations from nonlinear models such as those considered here is not generally the same as that obtained under linearity (see, e.g., Li, 1992; Hwang et al., 1994), the P-values of Q_r and Q_r^2 are computed from a bootstrap approximation to their null sampling distributions instead of the usual chi-squared asymptotic approximation.

- Table 4 reports the ML estimation results for C-STAR(4), SDC-STAR(4), RC-STARX(4), C-STARX(4) and SDC-STARX(4) models (under the assumption of Student-t conditional distributions).
- For the SDC-STAR and the SDC-STARX models, the coefficients on output growth (δ_1) and inflation (δ_2) in the equation which determines the state-dependent threshold are both significantly different from zero. The results suggest that, for both models, the interest rate threshold values are such that **regime 1 is favoured the lower output growth is and the higher inflation is, while regime 2 is favoured for high values of output growth and low values of inflation.**
- This is because high values of inflation (and low values of output growth) result in small values for the threshold, which makes it more likely that the interest rate would exceed the threshold value. Figure 6, shows that the SDC-STAR gives weights close to unity in several periods which can be mostly associated with economic recessions. We find that the C-STAR is a not a valid restriction of the SDC-STAR. The AIC and BIC also favour the SDC-STAR model over the C-STAR.

State Dependent Threshold, Mixing Funtion for the SDC−STAR

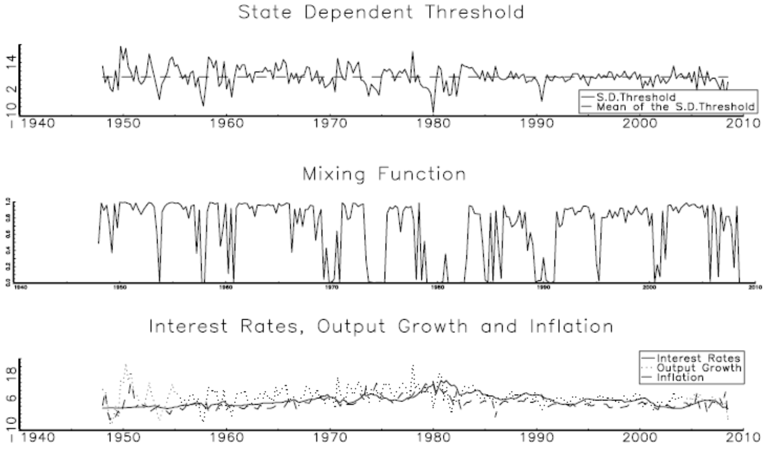


Figure 6

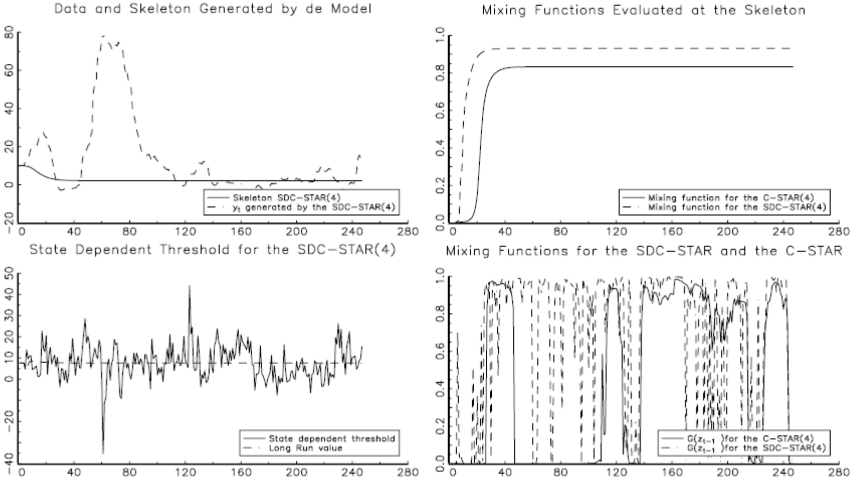

Data and Skeleton Generated by de Model
Skeleton SDC−STAR(4)
y_t generated by the SDC−STAR(4)
Mixing Functions Evaluated at the Skeleton
Mixing function for the C−STAR(4)
Mixing function for the SDC−STAR(4)
State Dependent Threshold for the SDC−STAR(4)
State dependent threshold
Long Run value
Mixing Functions for the SDC−STAR and the C−STAR
$G(z_{t-1})$ for the C−STAR(4)
$G(z_{t-1})$ for the SDC−STAR(4)

Figure 7

- The stability of the empirical SDC-STAR(4) is assessed by using numerical simulation. The skeleton of the model is found to have a single stable fixed point $Y_e = 2.2129$. The values of the skeleton are plotted in the top left panel of Figure 7, along with artificial data obtained by using the fitted model as the DGP.
- We note that the simulated data appear to replicate the qualitative features of the observed data – there is a fairly long period during which the series diverges from its long-run value, which takes the series above the threshold. The top right panel in Figure 7 shows the values of the mixing function $G(\cdot)$ evaluated under the skeleton of the SDC-STAR(4) and C-STAR(4) models, while the bottom left panel shows the evolution of the state-dependent threshold and its convergence to its long-run value 7.591. Finally, the bottom right panel shows the evolution of the mixing functions for the simulated data, which reveals substantial differences between the absolute and relative (to the exogenous variable) mixing functions.

- Within the class of SD-CSTARX models, there are several testable hypotheses of interest. We find that the C-STARX specification is not a valid reduction of the SD-CSTARX model since the LR statistic for $\delta_1 = \delta_2 = 0$ has a value of 20.818, leading to a rejection of the hypothesis of a constant threshold at the 1% significance level.
- The RSDC-STARX specification is not a valid simplification either since the LR statistic for $\delta_1^{(1)} - \delta_1^{(2)} = \delta_2^{(1)} - \delta_2^{(2)} = \delta = 0$ is 20.814. When the SD-CSTAR is tested against the SD-CSTARX, the LR statistic for $\delta_1^{(1)} = \delta_1^{(2)} = \delta_2^{(1)} = \delta_2^{(2)} = 0$ has a value of 24.368, implying that the model in which output growth and inflation are allowed to influence the regime-specific variables directly enjoys more support by the data.
- The SDC-STARX model is favoured by the AIC, while the SDC-STAR is favoured by the BIC. Overall, there is significant evidence in favour of specifications with a state-dependent threshold.
- The stability of the empirical SDC-STARX(4) is assessed in a similar way, with the corresponding plots shown in Figure 11. We find a single stable fixed point $Y_e = 5.50$ with a long-run value of 5.57 for the threshold.

State Dependent Threshold and Mixing Function for the SDC−STARX

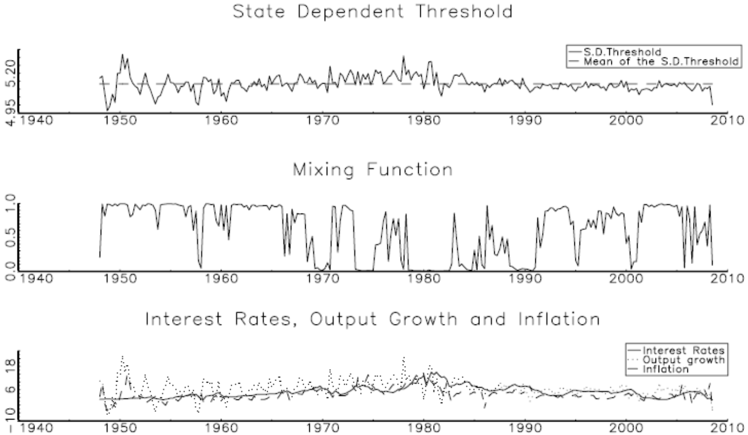


Figure 10

Generated Data and Skeleton for the Empirical DGP

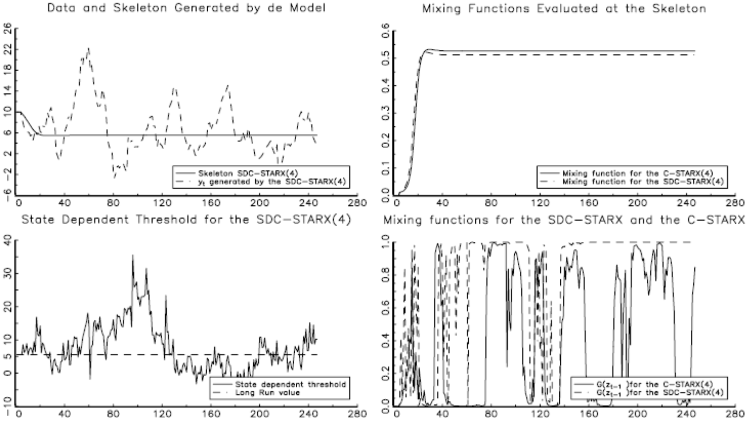


Figure 11

Summary

- In summary, the results presented in this section suggest that models with state-dependent thresholds, i.e., SDC-STAR and SDC-STARX models, are capable of characterizing the data successfully. In our application at least, what seems to matter is not so much the absolute value of the threshold but rather its level relative to other variables that dictate the evolution of the economy as a whole.
- We believe that, for economic variables such as those considered in this paper, a relative threshold with respect to other variables that dictate the evolution of the economy is of more importance for the dynamics of the variables under consideration than a fixed threshold. Furthermore, since constant-threshold specifications are special cases of state-dependent specifications, it seems reasonable to use state-dependent specifications as a first step in a modelling cycle.