W_n es un estimador de θ $(X \sim f(x;\theta))$

$$\tau(W_n;\theta) = \text{distancia entre } W_n \text{ y } \theta$$

$$\overset{ej}{=} (W_n - \theta)^2$$

$$R_\tau(W_n;\theta) = E\{\tau(W_n;\theta)\}$$

$$\overset{ej}{=} E\{(W_n - \theta)^2\}$$

$$= \underbrace{[E\{W_n\} - \theta]^2}_{\text{Sesgo}^2(W_n)} + \underbrace{E\{(W_n - E(W_n))^2\}}_{Var(W_n)}.$$

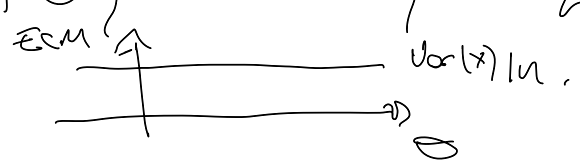
W_n insesgado p/ θ si

- $E\{W_n\} = \theta \quad \forall n \geq 1;\ \theta \in \Theta$

  $\text{Sesgo}(W_n) = 0 \quad \forall n \geq 1;\ \theta \in \Theta.$

Ej 1: $\theta = E(X)$ $\qquad \bar{X}_n \Rightarrow E\{\bar{X}_n\} = E(X) = \theta$

$$ECM(\bar{X}_n;\theta) = Var(\bar{X}_n) = \frac{Var(X)}{n}$$



Ej 2: $X \sim N(\mu; \sigma^2)$

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 = \frac{1}{n-1}\sum_{i=1}^{n} X_i^2 - \frac{n}{n-1}(\bar{X}_n)^2$$

$$ECM(S_n^2, \sigma^2) = Sesgo(S_n^2) + Var(S_n^2)$$

$$E\{S_n^2\} = \frac{1}{n-1}\sum_{i=1}^{n} E(X_i^2) - \frac{n}{n-1} E\left((\bar{X}_n)^2\right)$$

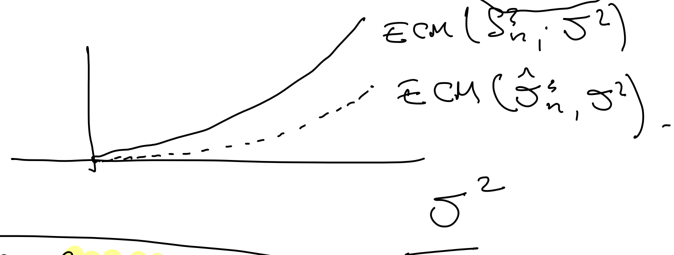
$$\boxed{\begin{array}{l} Var(Y) = E(Y^2) - [E(Y)]^2 \\ E(Y^2) = Var(Y) + [E(Y)]^2 \end{array}}$$

$$= \frac{n}{n-1}(\sigma^2 + \mu^2) - \frac{n}{n-1}\left[Var(\bar{X}_n) + (E(\bar{X}_n))^2\right]$$

$$= \frac{n}{n-1}(\sigma^2 + \mu^2) - \frac{n}{n-1}\left[\frac{\sigma^2}{n} + \mu^2\right]$$

$$E\{S_n^2\} = \sigma^2 \quad \text{(insesgada)}$$

$$ECM(S_n^2; \sigma^2) = Var(S_n^2) = \frac{2\sigma^4}{(n-1)}$$



$$\boxed{\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 = \frac{n-1}{n} S_n^2}$$

$$ECM(\hat{\sigma}_n^2; \sigma^2) = [E(\hat{\sigma}_n^2) - \sigma^2]^2 + Var(\hat{\sigma}_n^2)$$

$$E\{\hat{\sigma}_n^2\} = \frac{n-1}{n} E(S_n^2) = \frac{n-1}{n}\sigma^2 \neq \sigma^2$$

$$Sesgo(\hat{\sigma}_n^2) = \frac{n-1}{n}\sigma^2 - \sigma^2 = -\frac{1}{n}\sigma^2$$

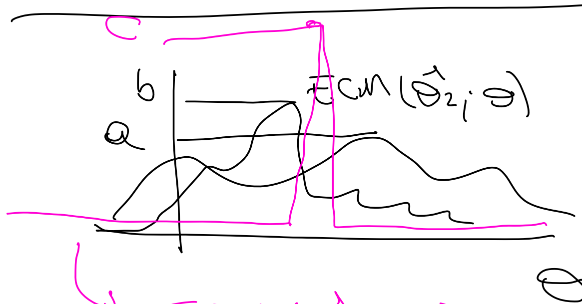
$$\operatorname{Var}(\hat{\sigma}^2_n) = \operatorname{Var}\left(\frac{n-1}{n} S^2_n\right) = \frac{(n-1)^2}{n^2} \frac{2\sigma^4}{(n-1)}$$

$$ECM(\hat{\sigma}^2_n; \sigma^2) = \frac{(2n-1)}{n^2} \sigma^2$$

$$\frac{ECM(\hat{\sigma}^2_n, \sigma^2)}{ECM(S^2_n; \sigma^2)} = 1 - \underbrace{\frac{3n-1}{2n^2}}_{\geq 0 \ \forall n \geq 1} < 1$$

$$ECM(\hat{\sigma}^2_n; \sigma^2) < ECM(S^2_n; \sigma^2) \quad \forall n \geq 1$$

$$Q = \max_{\theta} ECM(\hat{\theta}_1; \theta)$$



$ECM(\hat{\theta}_1, \theta)$

$ECM(\hat{\theta}_3; \theta)$

$$X \sim Pois(\lambda) \Rightarrow E(X) = V(X) = \lambda$$

Hay 2 est. insesgados de λ

① $\bar{X}_n \Rightarrow E\{\bar{X}_n\} = E(X) = \lambda$

② $S^2_n \Rightarrow E(S^2_n) = Var(X) = \lambda$

$$Var(\bar{X}_n) = \frac{\lambda}{n} \qquad Var(S_n^2) = ?$$

$$W_{\alpha,n} = \alpha \bar{X}_n + (1-\alpha) S_n^2 \qquad \alpha \in (0,1)$$

$$W_n \in C_0 \Rightarrow Var(W_n) \geq \text{Cota}$$

CR

$$I_n(\lambda) = n\, i(\lambda) = n\, \underbrace{E\left\{\left[\frac{\partial}{\partial\lambda} \ln f(x;\lambda)\right]^2\right\}}_{i(\lambda)}$$

$$\ln f(x;\lambda) = \ln \frac{e^{-\lambda}\lambda^x}{x!} = -\lambda + x\ln(\lambda) - \ln(x!)$$

$$\frac{\partial}{\partial\lambda} \ln f(x;\lambda) = -1 + \frac{x}{\lambda} \Rightarrow \frac{\partial^2}{\partial\lambda^2} \ln f(x;\lambda) = -\frac{x}{\lambda^2}$$

$$i(\lambda) = E\left\{\left[-1 + \frac{x}{\lambda}\right]^2\right\}$$

$$= \frac{1}{\lambda^2} \underbrace{E\{[x-\lambda]^2\}}_{Var(x)} = \frac{\lambda}{\lambda^2} \Rightarrow i(\lambda) = \frac{1}{\lambda}$$

$$I_n(\lambda) = n/\lambda$$

Si W_n es un est. insesg. de λ

$$\Rightarrow V(W_n) \geq I_n^{-1}(\lambda) = \lambda/n$$

$$Var(\overline{X}_n) = \frac{Var(X)}{n} = \frac{\lambda}{n} \longrightarrow \text{Alcanza la cota}$$

$$\Downarrow$$

$$\underline{\text{UMVUE}}.$$

$$i(\lambda) = -E\left\{\frac{\partial^2}{\partial \lambda^2} \ln f(x;\lambda)\right\} = -E\left\{-\frac{X}{\lambda^2}\right\} = \frac{E(X)}{\lambda^2} = \underline{\frac{1}{\lambda}}$$

$$I_n(\lambda) = n/\lambda.$$

$$X \sim Exp(\theta) \qquad \theta = E(X) \qquad \overline{X}_n$$

$$E\{\overline{X}_n\} = E(X) = \theta \quad y \quad \boxed{Var(\overline{X}_n) = \frac{Var(X)}{n} = \frac{\theta^2}{n}}$$

$$I_n(\theta) = \underline{-n\, E\left\{\frac{\partial^2}{\partial \theta^2} \ln f(x;\theta)\right\}}$$

$$\ln f(x;\theta) = \ln \frac{1}{\theta} e^{-x/\theta} = -\ln(\theta) - \frac{x}{\theta}$$

$$\frac{\partial^2}{\partial \theta^2} \ln f(x;\theta) = +\frac{1}{\theta^2} - \frac{2X}{\theta^3}$$

$$\underline{I_n(\theta)} = -n\, E\left\{\frac{1}{\theta^2} - \frac{2X}{\theta^3}\right\} = \frac{-n}{\theta^2} + \frac{2n}{\theta^2}$$

$$= \frac{n}{\theta^2}$$

$$\text{Si } W_n \in \underline{C_\theta} \Rightarrow \boxed{Var(W_n) \geq \theta^2/n}$$

$\Rightarrow \bar{X}_n$ es el UMVUE de θ.

$X \sim Bern(\theta)$ $\quad \hat{\theta}_n = \bar{X}_n \Rightarrow E\{\bar{X}_n\} = E(X) = \theta$

$$\boxed{Var(\bar{X}_n) = \frac{\theta(1-\theta)}{n}}$$

C_θ.

$$l(\theta) = \ln(\theta)\sum_{i=1}^{n} X_i + \ln(1-\theta)\left(n - \sum_{i=1}^{n} X_i\right)$$

$$l'_\theta : \frac{\sum_{i=1}^{n} X_i}{\theta} - \frac{\left(n - \sum_{i=1}^{n} X_i\right)}{1-\theta} = 0 \Rightarrow \hat{\theta}_n = \bar{X}_n$$

$$l''_\theta : -\frac{\sum_{i=1}^{n} X_i}{\theta^2} - \frac{\left(n - \sum_{i=1}^{n} X_i\right)}{(1-\theta)^2} < 0 \quad \forall \theta \in (0,1)$$

$$I_n(\theta) = -E\{l''_\theta\}$$

$$= -E\left\{-\frac{\sum_{i=1}^{n} X_i}{\theta^2} - \frac{\left(n - \sum_{i=1}^{n} X_i\right)}{(1-\theta)^2}\right\}$$

$$= \frac{\sum_{i=1}^{n} E(X_i)}{\theta^2} + \frac{n - \sum_{i=1}^{n} E(X_i)}{(1-\theta)^2}$$

$$= \frac{n\cancel{\theta}}{\theta^2} + \frac{n\cancel{(1-\theta)}}{(1-\theta)^2}$$

$$I_n(\theta) = \frac{n}{\theta} + \frac{n}{(1-\theta)} = \frac{n}{\theta(1-\theta)}$$

Si $W_n \in C_\theta \Rightarrow Var(W_n) \geq \frac{\theta(1-\theta)}{n}$

$\bar{X}_n$ es insesgado y $var(\bar{X}_n) = \frac{\theta(1-\theta)}{n}$

$$\Downarrow$$

$\bar{X}_n$ es el UMVUE

Consistencia y $ECM(W_n;\theta) = Sesgo^2(W_n) + Var(W_n)$

$$W_n \xrightarrow{p} \theta \Rightarrow \lim_{n\to\infty} P(|W_n-\theta| \geq \varepsilon) = 0$$

Chebys: Si $g(x)$ es una función no-neg. y $\varepsilon > 0$

$$P(g(X) \geq \varepsilon) \leq \frac{E(g(X))}{\varepsilon}$$

$$P(|W_n-\theta| \geq \varepsilon) = P(\underbrace{|W_n-\theta|^2}_{g(W_n)} \geq \varepsilon^2) \leq \frac{\overbrace{E[(W_n-\theta)^2]}^{ECM(W_n;\theta)}}{\varepsilon^2}$$

$$\lim_{n\to\infty} P(|W_n-\theta|\geq \varepsilon) \leq \lim_{n\to\infty} \frac{\text{Sesgo}^2(W_n)+\text{Var}(W_n)}{\varepsilon^2}$$

$$ECM(W_n;\theta) \xrightarrow[n\to\infty]{} 0$$

Ej: $ECM(S_n^2;\sigma^2) = \dfrac{2\sigma^4}{(n-1)} \xrightarrow[n\to\infty]{} 0$

$X \sim N(\mu;\sigma^2)$

S_n^2 es consistente p/ σ^2

$S_n^2 \xrightarrow{P} \sigma^2$

$ECM(\hat{\sigma}_n^2;\sigma^2) = \dfrac{(2n-1)\sigma^4}{n^2} \xrightarrow[n\to\infty]{} 0$

Es consistente $\hat{\sigma}_n^2$ como est. de σ^2