Clase 7 4/22

TP3

Ejercicio 18

$$A = \begin{pmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & 2 & c \end{pmatrix} \quad / \quad \lambda = 1 \text{ es autovalor}$$

y $(1,1,1) \in \{ v \in \mathbb{R}^3 : (I-A)v^t = 0 \}$ $\quad (1,1,1) \in Ker(A-I)$

a) $\lambda = 1$ es autovalor $\Rightarrow P_A(1) = 0$

$$A - I\lambda = \begin{pmatrix} 1-\lambda & 2 & a \\ 2 & 1-\lambda & b \\ 2 & 2 & c-\lambda \end{pmatrix} \rightarrow P_A(\lambda) = (1-\lambda)^2(c-\lambda) + 4b + 4a - [2a(1-\lambda) + 2b(1-\lambda) + 4(c-\lambda)]$$

$$P_A(1) = 0 \rightarrow 0 = 4b + 4a - 4(c-1) \rightarrow 0 = b + a - c + 1$$

$$\hookrightarrow c = a + b + 1$$

$$A - \underset{1}{\lambda} I = \begin{pmatrix} 0 & 2 & a \\ 2 & 0 & b \\ 2 & 2 & c-1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & a \\ 2 & 0 & b \\ 2 & 2 & a+b \end{pmatrix} \xrightarrow[F_3 - F_1 - F_2 \to F_3']{}$$

$$\left(\begin{array}{ccc|c} 0 & 2 & a & 0 \\ 2 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} 2y + az = 0 \\ 2x + bz = 0 \end{cases}$$

Para $(x,y,z) = (1,1,1)$ estas ecuaciones deben cumplirse, pues $(1,1,1)$ es autovector del autovalor $\lambda = 1$.

$$\begin{cases} 2 + a = 0 \\ 2 + b = 0 \end{cases} \longrightarrow a = -2,\ b = -2,\ c = -3$$

Reemplazamos a, b, c en $P_A(\lambda) \longrightarrow$ las raíces van a ser $\lambda_1 = 1$, $\lambda_2 = -1$ (doble)

autovector $(1,1,1) \longleftarrow \lambda_1 = 1$

$\lambda_2 = -1$: $\quad A - \lambda_2 Id = A + Id$

$$A=\begin{pmatrix}1&2&-2\\2&1&-2\\2&2&-3\end{pmatrix} \rightarrow A+Id=\begin{pmatrix}2&2&-2\\2&2&-2\\2&2&-2\end{pmatrix}\sim\begin{pmatrix}1&1&-1\\0&0&0\\0&0&0\end{pmatrix}$$

$$\Rightarrow (A+Id)v_2=0, \longrightarrow x+y-z=0 \rightarrow x+y=z$$

$$\Rightarrow Ker(A+Id)=\{(x,y,z)=(x,y,x+y),\ x,y\in\mathbb{R}\}$$
$$= x(1,0,1)+y(0,1,1)$$
$$=\langle \underbrace{(1,0,1),(0,1,1)}_{\text{autovectores}} \rangle$$

b) sí

$$D=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\ P=\begin{pmatrix}1&1&0\\1&0&1\\1&1&1\end{pmatrix}$$

c) A^{100}, A^{201}

$$A=PDP^{-1}\rightarrow A^2=PD\cancel{P^{-1}}\cancel{P}DP^{-1}=PD^2P^{-1}$$

$$\Rightarrow A^n=PD^nP^{-1},\quad D^{100}=\begin{pmatrix}1^{100}&0&0\\0&(-1)^{100}&0\\0&0&(-1)^{100}\end{pmatrix}=Id$$

$$D^{201}=\underbrace{D^{200}}_{Id}D=D$$

$$A^{100}=PD^{100}P^{-1}=PP^{-1}=Id$$
$$A^{201}=PD^{201}P^{-1}=PDP^{-1}=A$$

Ejercicio 19 $\quad A\in\mathbb{R}^{3\times 3}$, $\lambda=0,1,5$

a) Es diagonalizable: tres $\neq$ $\Rightarrow A=PDP^{-1}$

Si fuera invertible, $A^{-1}=PD^{-1}P^{-1}$ pero $D=\begin{pmatrix}0&0&0\\0&1&0\\0&0&5\end{pmatrix}$

$$\Rightarrow \nexists D^{-1}\Rightarrow \nexists A^{-1}$$

b) $B=(3A-4Id)^3$, $C=5A^T+4Id$

$$3A-4I=3PDP^{-1}-4PP^{-1}=P(3D-4Id)P^{-1}$$
$$B=(3A-4Id)^3=P(3D-4Id)^3P^{-1}$$

$$\shortparallel$$

$$\begin{pmatrix} 0&0&0\\0&3&0\\0&0&15\end{pmatrix} - \begin{pmatrix} 4&0&0\\0&4&0\\0&0&4\end{pmatrix} = \begin{pmatrix} -4&0&0\\0&-1&0\\0&0&11\end{pmatrix}$$

$$\begin{aligned} A^T &= (PDP^{-1})^T = \\ &= (P^{-1})^T D^T P^T \\ &= (P^T)^{-1} D^T P^T \text{; llamo } \tilde{P} = (P^T)^{-1} \\ &= \tilde{P} D \tilde{P}^{-1} \end{aligned}$$

$\lambda_B = -1, -64, 11^3$

$$\begin{aligned} C &= 5A^T + 4Id = \\ & 5\tilde{P}D\tilde{P}^{-1} + 4\tilde{P}\tilde{P}^{-1} \\ &= \tilde{P}(5D + 4Id)\tilde{P}^{-1} \\ &= \tilde{P}\left[\begin{pmatrix} 0&0&0\\0&15&0\\0&0&25\end{pmatrix} + \begin{pmatrix} 4&0&0\\0&4&0\\0&0&4\end{pmatrix}\right]\tilde{P}^{-1} \\ &= \tilde{P}\begin{pmatrix} 4&0&0\\0&19&0\\0&0&29\end{pmatrix}\tilde{P}^{-1} \leftarrow \text{autovalores} \end{aligned}$$

c) $H = A + I$ es inversible, y calcular λ de H^{-1}, $\det(H^{-1})$ y $tr(H^{-1})$

$$H = A + Id = PDP^{-1} + PP^{-1} = P(D + Id)P^{-1} = P\left[\begin{pmatrix} 0&0&0\\0&3&0\\0&0&5\end{pmatrix} + \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]P^{-1}$$

$$= P\begin{pmatrix} 1&0&0\\0&4&0\\0&0&6\end{pmatrix}P^{-1} \Rightarrow \exists H^{-1}$$

$$H^{-1} = P\begin{pmatrix} 1&0&0\\0&1/4&0\\0&0&1/6\end{pmatrix}P^{-1} \Rightarrow \text{autovalores}$$

Además, $tr(H) = 1 + \frac{1}{4} + \frac{1}{6}$

$\det(H^{-1}) = 1 \cdot \frac{1}{4} \cdot \frac{1}{6}$

d) $\alpha /\ \alpha A + 3Id$ no es inversible

$$\alpha A + 3Id = \alpha PDP^{-1} + 3PP^{-1} = P(\alpha D + 3Id)P^{-1}$$

$$= P\left[\alpha\begin{pmatrix} 0&0&0\\0&3&0\\0&0&5\end{pmatrix} + \begin{pmatrix} 3&0&0\\0&3&0\\0&0&3\end{pmatrix}\right]P^{-1}$$

$$= P \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3(1+\alpha) & 0 \\ 0 & 0 & 5\alpha+3 \end{bmatrix} P^{-1}$$

No es invertible si : $\left\{ \begin{array}{l} 3(1+\alpha)=0 \longrightarrow \alpha=-1 \\ \text{o} \\ 5\alpha+3=0 \longrightarrow \alpha=-3/5 \end{array} \right.$

Ejercicio 22

A diagonalizable , $\underbrace{P_A(t) = (t-1)^2 (t-3)^2}_{1 \text{ (doble)},\ 3 \text{ (doble)}}$

a) $rg(A-3Id)$

$$A = CDC^{-1} = C \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 3 & \\ 0 & & & 3 \end{pmatrix} C^{-1}$$

$$A-3Id = CDC^{-1} - 3CC^{-1}$$

$$= C \left[\begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 3 & \\ 0 & & & 3 \end{pmatrix} - \begin{pmatrix} 3 & & & 0 \\ & 3 & & \\ & & 3 & \\ 0 & & & 3 \end{pmatrix} \right] C^{-1}$$

$$= C \begin{bmatrix} -2 & & & 0 \\ & -2 & & \\ & & 0 & \\ 0 & & & 0 \end{bmatrix} C^{-1} \sim \begin{pmatrix} -2 & & & 0 \\ & -2 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix}$$

$\Rightarrow$ como $rg(D-3Id)=2 \Rightarrow rg(A-3Id)=2$

b) $B = A^2 - 4A + 5Id$

$$= \underbrace{(CDC^{-1})^2}_{CD^2C^{-1}} - 4(CDC^{-1}) + 5CC^{-1}$$

$$= CD^2C^{-1} - 4CDC^{-1} + 5CC^{-1}$$

$$= C\,(D^2 - 4D + 5Id)\,C^{-1}$$

$$= C\left(\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 9 & \\ & & & 9 \end{pmatrix} + \begin{pmatrix} -4 & & & \\ & -4 & & \\ & & -12 & \\ & & & -12 \end{pmatrix} + \begin{pmatrix} 5 & & & \\ & 5 & & \\ & & 5 & \\ & & & 5 \end{pmatrix} \right) C^{-1}$$

$$= C \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{pmatrix} C^{-1} = C\, 2\,Id\, C^{-1}$$

$$= 2\, C \cancel{Id} C^{-1} = 2\,Id = B$$

Ejercicio 23

$A \,/\, \dim(\ker(A)) = 1$, $\quad rg(A + 2Id) = 2$

$A \in \mathbb{R}^{3\times 3}$ $\quad A_{11} + A_{22} + A_{33} = 0$

a) $\dim(\ker(A)) = 1 > 0 \rightarrow \lambda_1 = 0$

$tr(A_{11} + A_{22} + A_{33}) = 0 = \lambda_1 + \lambda_2 + \lambda_3 \rightarrow \lambda_2 = -\lambda_3$

Si λ_2 y $\lambda_3 = 0 \rightarrow rg(A + 2Id) = 3 \rightarrow$ Absurdo

de $rg(A + 2Id) = 2 < 3 \Rightarrow$ x teorema de la dimensión,

$\ker(A + 2Id) = 1 > 0$

$\Rightarrow \lambda_2 = -2 \Rightarrow \lambda_3 = 2$

b) A es diagonalizable ✓

A no es inversible $(\lambda_1 = 0)$

Ejercicio 25 $\quad A \in \mathbb{R}^{4\times 4} \,/\, \ker(A + Id) \neq \{0\}$

$rg(A - 2Id) \leqslant 2$, $\quad P_A(1) = -4$

A es diagonalizable ?

Calcular $A^3 - 4A^2 + A + 6Id$

$\ker(A + Id) \neq \{0\} \Rightarrow \exists\, v \,/\, (A + Id)v = 0 \rightarrow Av = -v$

$\Rightarrow -1$ es autovalor

$rg(A-2Id) \leq 2 \longrightarrow$ x teo de la dimensión,

$\dim(\underline{Ker(A-2Id)}) \geq 2 \Rightarrow$ 2 es autovalor

y es doble, porque $\dim(E_2) \geq 2$

pero $\dim(E_\lambda) \leq$ multiplicidad del autovalor

$\Rightarrow$ al menos es doble

$$P_A(\lambda) = (-1-\lambda)(2-\lambda)^2(\lambda_3-\lambda)$$

$$P_A(1) = -4 = (-2)(1)^2(\lambda_3-1)$$

$$2 = \lambda_3 - 1 \rightarrow \lambda_3 = 3$$

Conseguimos 4 autovalores y nos alcanzan los autovalores para construir $P \Rightarrow A$ diagonalizable

$$A^3 - 4A^2 + A + 6Id = PD^3P^{-1} - 4PD^2P^{-1} + PDP^{-1} + 6PP^{-1}$$

$$= P\left(D^3 - 4D^2 + D + 6Id\right)P^{-1} \; ; \quad D = \begin{pmatrix} -1 & & & \\ & 2 & & \\ & & 2 & \\ & & & 3 \end{pmatrix}$$

$$= P\left[\begin{pmatrix} -1 & & & \\ & 8 & & \\ & & 8 & \\ & & & 27 \end{pmatrix} - 4\begin{pmatrix} 1 & & & \\ & 4 & & \\ & & 4 & \\ & & & 9 \end{pmatrix} + \begin{pmatrix} -1 & & & \\ & 2 & & \\ & & 2 & \\ & & & 3 \end{pmatrix} + \begin{pmatrix} 6 & & & \\ & 6 & & \\ & & 6 & \\ & & & 6 \end{pmatrix}\right]P^{-1}$$

$$= P\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}P^{-1} = 0_{4\times4}$$

Ejercicio 26

u_t = volumen de c/ fieno x año

$$u_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} 5\cdot 10^8$$

$$\mathcal{A} = \begin{pmatrix} \overset{A\to A}{0.3} & \overset{B\to A}{0.6} \\ \underset{A\to B}{0.7} & \underset{B\to B}{0.4} \end{pmatrix} \; ; \quad u_{t+1} = \mathcal{A} u_t$$

Nos piden $u_5 = \mathcal{A} u_4 = \mathcal{A}\mathcal{A} u_3 = \dots = \mathcal{A}^5 u_0$

$\longrightarrow$ los autovalores son 1 y $-0{,}3$

y los autovectores son $\begin{pmatrix} 6 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\mathcal{A} = \begin{pmatrix} 6 & 1 \\ 7 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -0{,}3 \end{pmatrix} \begin{pmatrix} 6 & 1 \\ 7 & -1 \end{pmatrix}^{-1}$$

$$\mathcal{A}^5 = \begin{pmatrix} 6 & 1 \\ 7 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -0{,}3^5 \end{pmatrix} \begin{pmatrix} 6 & 1 \\ 7 & -1 \end{pmatrix}^{-1}$$

$$\mathcal{A}^5 \simeq \begin{pmatrix} 0{,}46023 & 0{,}4266 \\ 0{,}5397 & 0{,}53734 \end{pmatrix}$$

$$u_5 = \mathcal{A}^5 u_0 = \begin{pmatrix} 0{,}461202 \\ 0{,}538798 \end{pmatrix} 5\cdot 10^8$$

Ejercicio 27

$$\begin{array}{c} \begin{array}{cc} A\to A & \quad B\to A \end{array} \\ \mathcal{A} = \begin{pmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{pmatrix} \\ A\to B \end{array} \quad ; \quad u_0 = 50\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

en octubre del 18. Cómo es u_t en agosto del 19 $(t=10)$

$$u_t = \mathcal{A}^t u_0 \longrightarrow u_{10} = \mathcal{A}^{10} u_0$$

$$\mathcal{A} = \frac{1}{4}\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$$

$$\mathcal{A} - \lambda Id = (1/2-\lambda)(1/4-\lambda) - 3/4 \; 1/2$$

$$0 = (1-2\lambda)(1-4\lambda) - 3$$

$$= 1 - 6\lambda + 8\lambda^2 - 3$$

$$= -2 - 6\lambda + 8\lambda^2 \to \lambda = \frac{6 \pm \sqrt{36+16}}{16}$$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{42}}{16}$$

$\longrightarrow$ dos autovectores v_1, v_2

$$\Rightarrow \mathcal{A} = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}^{-1}$$

$$\mathcal{A}^{10} = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \begin{pmatrix} \lambda_1^{10} & 0 \\ 0 & \lambda_2^{10} \end{pmatrix} \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}^{-1}$$

$$\mathcal{A}^{10} u_0 = u_{10}$$

Ejercicio 24

$A \in \mathbb{R}^{3\times 3} \quad \exists A^{-1} \quad \wedge \ tr(A) = -2$

$rg\left(A^{-1} - \frac{1}{2} Id\right) < 3 \qquad p_A(1) = -8$

$tr(A) = \lambda_1 + \lambda_2 + \lambda_3 = -2$

$rg\left(A^{-1} - \frac{1}{2} Id\right) < 3 \Rightarrow \dim\left(Ker\left(A^{-1} - \frac{1}{2} Id\right)\right) > 0$

$\Rightarrow \frac{1}{2}$ es autovalor de A^{-1}

$\Rightarrow 2 = \lambda_1$ es autovalor de A

$2 + \lambda_2 + \lambda_3 = -2 \longrightarrow \lambda_2 + \lambda_3 = -4 \longrightarrow \lambda_2 = -4 - \lambda_3$

$p_A(\lambda) = (2-\lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)$

$p_A(1) = -8 = 1 \cdot (\lambda_2 - 1)(\lambda_3 - 1)$

$-8 = (-4 - \lambda_3 - 1)(\lambda_3 - 1)$

$-8 = (-5 - \lambda_3)(\lambda_3 - 1)$

$-8 = -5\lambda_3 - \lambda_3^2 + 5 + \lambda_3$

$\lambda_3^2 + 4\lambda_3 - 13 = 0$

$\lambda_3 = \dfrac{-4 \pm \sqrt{16 + 52}}{2}$

$$\begin{array}{r} 16 \\ +52 \\ \hline 68 \end{array}$$

$= \dfrac{-4 \pm 2\sqrt{17}}{2} = -2 \pm \sqrt{17} = \lambda_3$

$\Rightarrow \lambda_2 = -4 - (-2 \pm \sqrt{17}) \Rightarrow \lambda_2 = -2 \mp \sqrt{17}$

Ejercicio 16

$a \ /\ A = \begin{pmatrix} a & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ no es diagonalizable

$$A - \lambda Id = \begin{pmatrix} a-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{pmatrix}$$

$$P_A(\lambda) = (a-\lambda)(1-\lambda)^2 - (a-\lambda) = [(1-\lambda)^2 - 1](a-\lambda)$$

$$= [\cancel{1} - 2\lambda + \lambda^2 - \cancel{1}](a-\lambda)$$

$$= \lambda(\lambda-2)(a-\lambda) \leftarrow \quad \begin{matrix} 0 = \lambda_1 \\ 2 = \lambda_2 \\ a = \lambda_3 \end{matrix}$$

Si $a \neq 0, a \neq 2 \rightarrow$ diagonalizable

$a=0$: $\quad \lambda = 0$: $\quad A - \lambda Id = A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \left(\begin{matrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \middle| \vec{0} \right)$

$\Rightarrow (x, y, z)$: $y + z = 0 \rightarrow y = -z$

$\Rightarrow E_0 = \{ (x, y, z) = (x, -z, z),\ x, z \in \mathbb{R} \}$

$= x(1,0,0) + z(0,-1,1)$

$E_0 = \langle (1,0,0), (0,-1,1) \rangle$

$\rightarrow A$ es diagonalizable

$a=2$: $\quad \lambda = 2$:

$$A - \lambda Id = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \sim \left(\begin{matrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{matrix} \middle| 0 \right)$$

$\Rightarrow \begin{matrix} y + z = 0 \longrightarrow 2z = 0 \rightarrow z = 0 \rightarrow y = 0 \\ -y + z = 0 \rightarrow y = z \end{matrix}$

$$E_2 = \{(x,y,z) = (x,0,0),\quad x \in \mathbb{R}\}$$

$\leftarrow$ dimensión $= 1 <$ multiplicidad de $\lambda = 2$

Para $a=2$, A no es diagonalizable.