Projection Methods

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Motivation

To date, we have developed tools for

- converting model environments into non-linear first-order systems of the form

$$\Gamma\left(E_t z_{t+1}, z_t, v_t\right) = 0, \tag{1}$$

- and approximating the solution to these systems (log-)linearly, in the form

$$\begin{aligned} x_{t+1} &= Fx_t + Gv_t \\ &= Fx_t + e_t, \end{aligned} \tag{2}$$

where, e.g., $x_{it} = \ln \frac{z_{it}}{z_i}$.

Motivation, cont.

For many purposes, the approximation error associated with linear approximations is unacceptable. Specifically:

- non-linearities associated with policy functions are often of direct interest to the question at hand (e.g., in making judgements regarding the importance of precautionary motives in driving consumption/savings decisions: Hugget and Ospina, 1993 *JME*, "Aggregate Precautionary Savings: When is the Third Derivative Irrelevant?")
- desired measurements can be highly sensitive to model approximation errors (e.g., second-order model approximation errors accumulate into first-order approximation errors in associated likelihood functions: Fernandex-Villaverde and Rubio-Ramirez, 2005 *JAE*; 2007 REStud).

Motivation, cont.

In such cases, we must obtain more accurate (non-linear) model approximations. There are (at least) three alternative methods for doing-so:

- value/policy function iterations
- higher-order Taylor Series approximations (i.e., perturbation methods; Judd and Guu, 1997 *JEDC*; Schmitt-Grohe and Uribe, 2004 *JEDC*)
- projection methods

Notation

In seeking non-linear approximations, we typically work with variables expressed in levels (detrended when appropriate). Let s_t denote the vector of state variables contained in z_t, with law of motion determined by

$$s_t = f(s_{t-1}, v_t), \tag{3}$$

and let c_t denote the vector of control variables contained in z_t.

Notation, cont.

The solution to the model we seek is a policy function of the form

$$c_t = c(s_t). \tag{4}$$

The policy function is derived as the solution to

$$F(c(s)) = 0, \tag{5}$$

where $F()$ is an operator defined over function spaces.

Notation, cont.

For example, in the one-tree model

$$
\begin{aligned}
p_t &= \beta e^{(1-\gamma)g} E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} (d_{t+1} + p_{t+1}) \right] && (6)\\
c_t &= d_t + q_t && (7)\\
d_t &= \overline{d} e^{u_{dt}}, \quad u_{dt} = \rho_d u_{dt-1} + \varepsilon_{dt} && (8)\\
q_t &= \overline{q} e^{u_{qt}}, \quad u_{qt} = \rho_q u_{qt-1} + \varepsilon_{qt}, && (9)
\end{aligned}
$$

(8) and (9) jointly represent (3), (7) trivially represents the policy function for consumption, and the policy function for p_t emerges as the solution to the functional equation (6).

Overview

References: Ch. 10.2, Judd (1998), Ch. 6.

Projection methods involve the construction of approximated policy functions $\widehat{c}(s)$ such that the associated approximation

$$F(\widehat{c}(s)) = 0$$

is 'satisfactory'.
Specific methods differ along two dimensions:

- the functional form over s used to construct $\widehat{c}(s)$;
- the criterion used to judge whether the approximation $F(\widehat{c}(s)) = 0$ is satisfactory.

Overview, cont.

Justification for this approach stems from the Weierstrass Theorem:

> *Any continuous function $c()$ may be approximated uniformly well over the range $s \in [\underline{s}, \overline{s}]$ by a sequence of r polynomials $p_r(s)$. That is,*
>
> $$\lim_{r \to \infty} \max_{s \in [\underline{s}, \overline{s}]} |c(s) - p_r(s)| = 0.$$

Examples

For now, let s be one-dimensional

Example 1: Linear combination of monomials in s, $\{s^0, s^1, ..., s^r\}$:

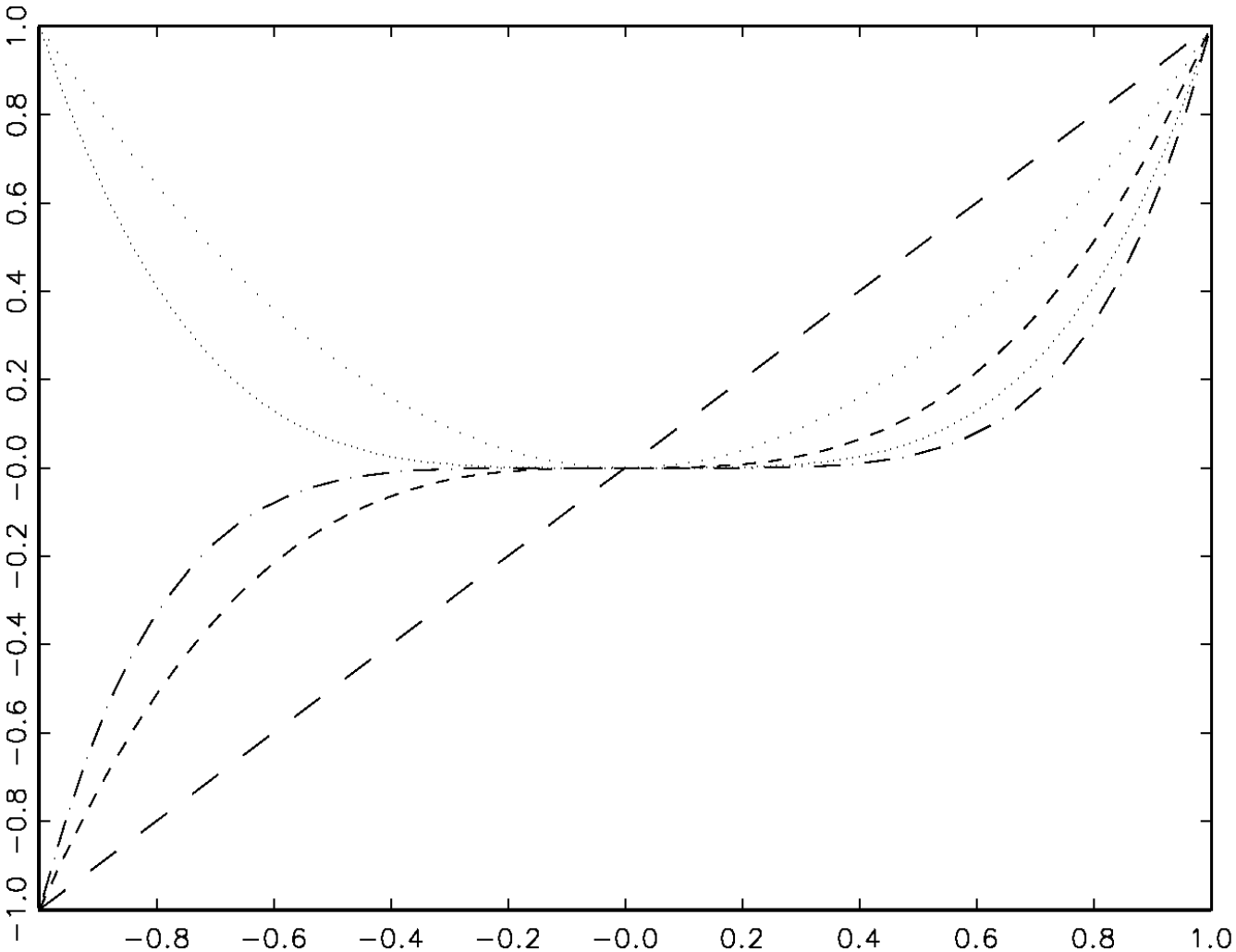
$$p_r(s) = \sum_{i=0}^{r} \chi_i s^i.$$

In using $p_r(s)$ to construct $\widehat{c}(s, \chi)$, the goal is to choose the parameters collected in the vector χ to provide an optimal characterization of $c(s)$.

This turns out to provide poor approximations in general, due to similarities in the behavior of the individual elements of $p_r(s)$.

Examples, cont.

Behavior of $\left\{s^0, s^1, ..., s^r\right\}$ over $\left[-1 \quad 1\right]$:



Examples, cont.

Example 2: Tent Function

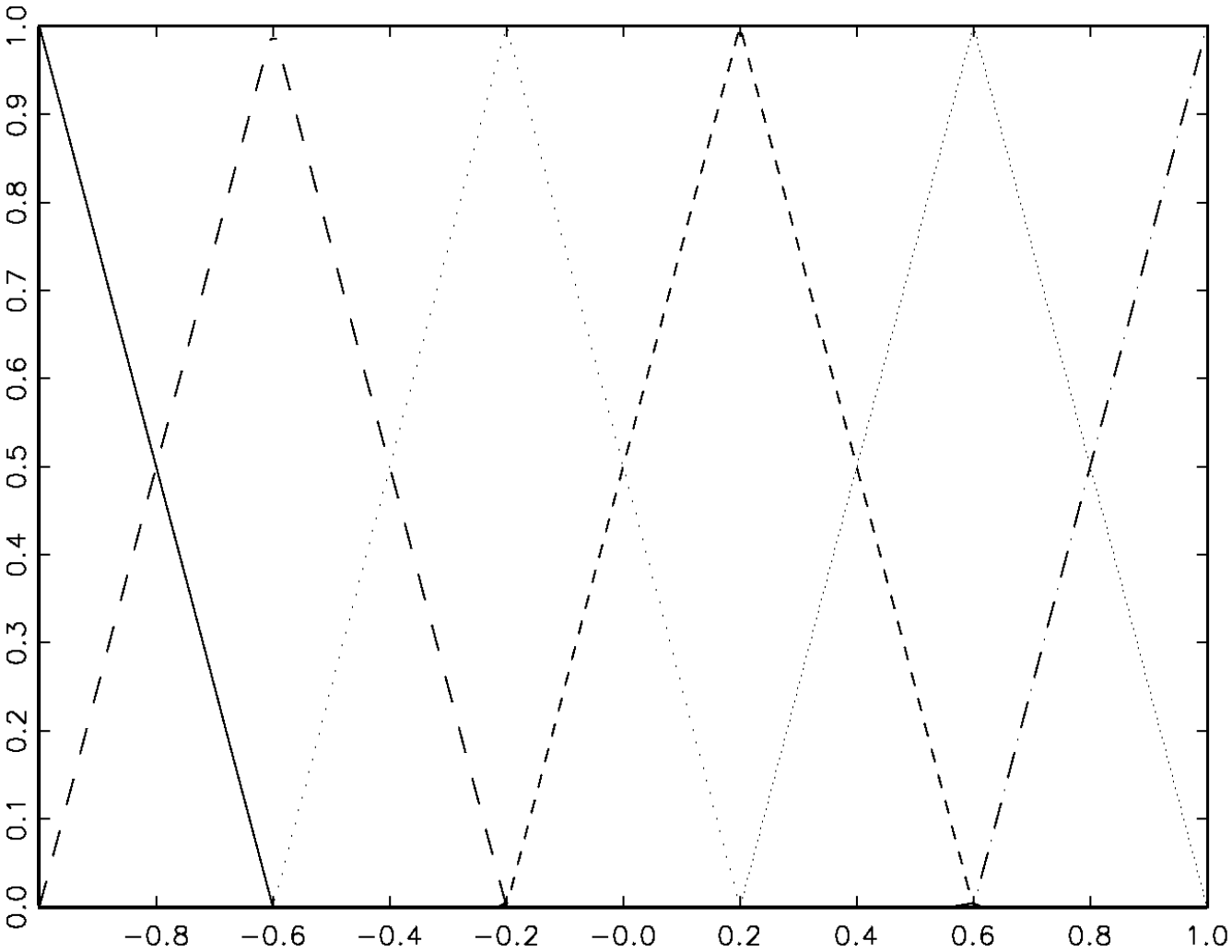
$$p_r(s) = \sum_{i=0}^{r} \chi_i p_r^i(s),$$

$$p_r^i(s) = \left\{ \begin{array}{ll} \frac{s-s^{i-1}}{s^i-s^{i-1}}, & s \in \left[s^{i-1}, s^i\right] \\ \frac{s^{i+1}-s}{s^{i+1}-s^i}, & s \in \left[s^i, s^{i+1}\right] \\ 0, & \text{otherwise.} \end{array} \right\}$$

Approximation methods involving tent functions are known as finite element methods.

Examples, cont.

Behavior of tent function over $[-1 \quad 1]$:



Examples, cont.

Example 3: Chebyshev polynomial.
Background: The Chebyshev polynomial is an example of an orthogonal polynomial.
The definition of an orthogonal polynomial is based on the inner product between two functions f_1 and f_2, given the weighting function w :

$$\langle f_1, f_2 \rangle = \int_{\underline{s}}^{\overline{s}} f_1(s) f_2(s) w(s) ds.$$

The family of polynomials $\{\varphi_r\}$ is defined to be mutually orthogonal with respect to $w(s)$ if and only if $\left\langle \varphi_r, \varphi_q \right\rangle = 0$ for $r \neq q$.

Examples, cont.

The Chebyshev polynomial is defined over $s \in [-1, 1]$; its associated the weighting function is

$$w(s) = (1 - s^2)^{-1/2}.$$

It is given by

$$p_r(s) = \sum_{i=0}^{r} \chi_i T_i(s),$$

where

$$T_i(s) = \cos(i \cos^{-1}(s)).$$

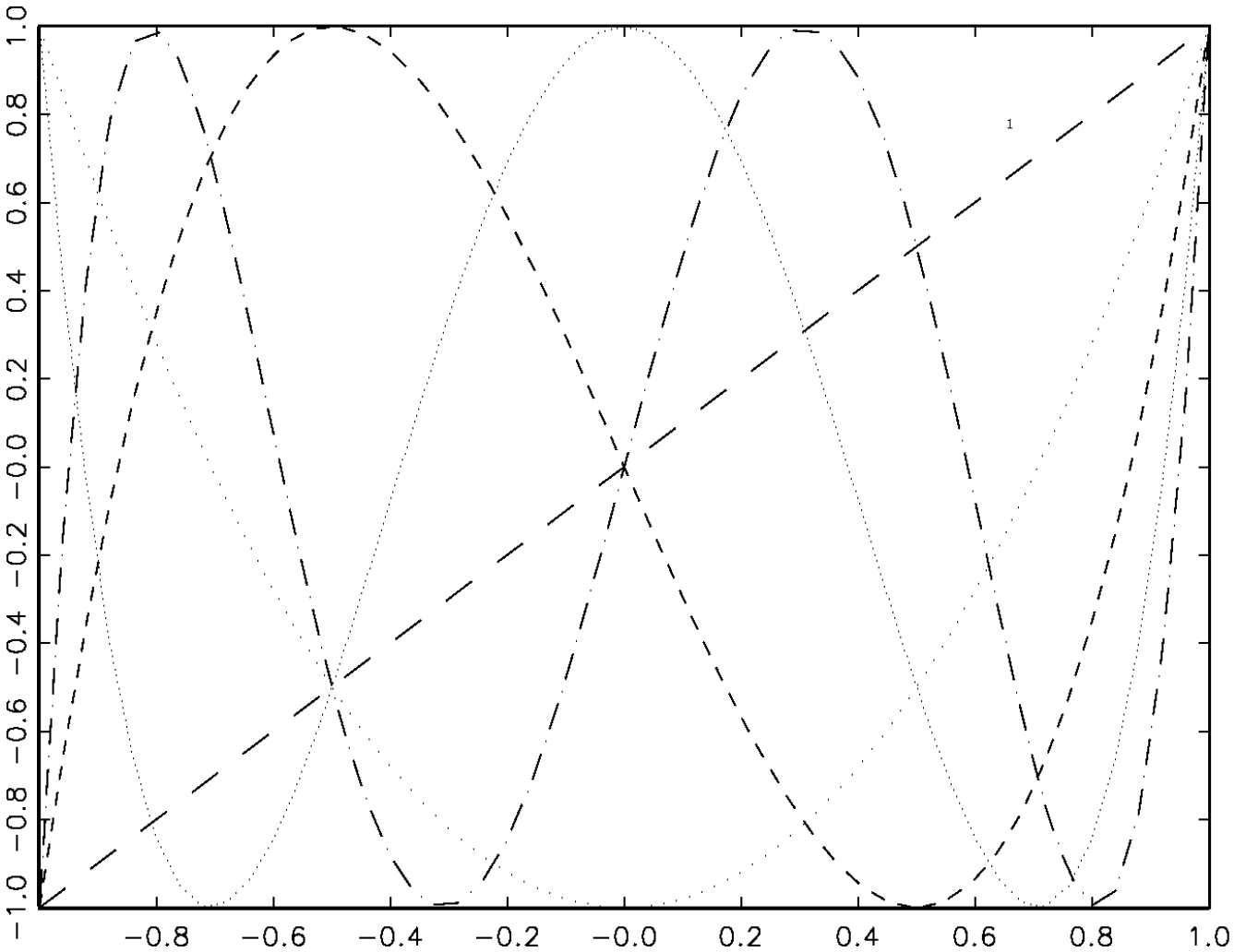
Examples, cont.

For a given value of s, $p_r(s)$ may be constructed as follows (Judd, 1998):

- Set $T_0(s) = 1$
- Set $T_1(s) = s$.
- Perform the recursion $T_{i+1}(s) = 2sT_i(s) - T_{i-1}(s)$, $i = 3, ..., r$.
- Collecting the $T_i(s)$ terms in the $r + 1 \times 1$ vector $T(s)$, calculate $p_r(s) = T(s)'\chi$.

Examples, cont.

Behavior of $T_i(s)$ over $[-1 \quad 1]$:



Examples, cont.

Note that while s must be constrained between $[-1, 1]$ in working with Chebyshev polynomials, a simple transformation may be used to map a state variable defined over a general range $[\underline{s}, \overline{s}]$ into a variable defined over the range $[-1, 1]$.
For example, for an element of s ranging $\pm\omega s^*$ units above and below the steady state value s^* the transformation

$$\widetilde{s} = \frac{s - s^*}{\omega s^*}$$

yields the desired range.

l-Dimensional State Space

Now consider $s = (s_1 \ s_2 \ \ldots \ s_\ell)'$.
Two leading approaches to approximation in this case involve the use of *tensor products* and *complete polynomials.*

Let $p_{r_j}(s_j)$ denote an r_j^{th}-order sequence of polynomials specified for the j^{th} element of s. Then the ℓ-dimensional tensor product of $p_{r_j}(s_j)$, $j = 1, \ldots, \ell$, is given by

$$P = \prod_{j=1}^{\ell} p_{r_j}(s_j).$$

I-Dimensional State Space, cont.

For example, with $\ell = 2$, $r_1 = 2$, $r_2 = 3$, and $p_{r_j}(s_j)$ representing Chebyshev polynomials, we have

$$P = (1 + s_1 + T_2(s_1))\,(1 + s_2 + T_2(s_2) + T_3(s_2))\,,$$

where $T_i(s_j)$ is the i^{th}-order term corresponding with the j^{th} element of s

I-Dimensional State Space, cont.

Using a given tensor product, the approximation $\widehat{c}(s, \chi)$ is constructed using

$$\widehat{c}(s, \chi) = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \dots \sum_{i_\ell=1}^{r_\ell} \chi_{i_1 i_2 \dots i_\ell} P_{i_1 i_2 \dots i_\ell}(s_1, s_2, \dots, s_\ell),$$

where

$$P_{i_1 i_2 \dots i_\ell}(s_1, s_2, \dots, s_\ell) = p_{i_1}(s_1) p_{i_2}(s_2) \dots p_{i_\ell}(s_\ell).$$

1-Dimensional State Space, cont.

In the example above, the elements $P_{i_1 i_2 \ldots i_\ell}(s_1, s_2, \ldots, s_\ell)$ can be constructed by forming the vectors

$$\begin{aligned} T^1 &= (1 \; s_1 \; T_2(s_1))', \\ T^2 &= (1 \; s_2 \; T_2(s_2) \; T_3(s_2))', \end{aligned}$$

and retrieving the elements of the 3×4 matrix $T = T^1 T^{2\prime}$. Moving to ℓ dimensions, construction may be achieve via recursion on

$$vec(T^i T^{i+1\prime}) T^{i+2}, \quad i = 1, \ldots, \ell - 2.$$

1-Dimensional State Space, cont.

An issue with the use of tensor products:
as ℓ increases, the number of elements $\chi_{i_1 i_2 \ldots i_\ell}$ that must be estimated to construct the approximation $\widehat{c}(s, \chi)$ increases exponentially. This is a manifestation of the curse of dimensionality.

One possible remedy: *complete polynomials*. For a polynomial of degree k, this refers to the collection of terms that appear in the k^{th}-order Taylor Series approximation of $c(s)$ about s^0.

l-Dimensional State Space, cont.

For $k = 1$, the complete set of polynomials for the ℓ-dimensional case is given by

$$P_1^\ell = \{1, s_1, s_2, ..., s_\ell\} .$$

For $k = 2$ the set expands to

$$P_2^\ell = P_1^\ell \cup \left\{ s_1^2, s_2^2, ..., s_\ell^2, s_1 s_2, s_1 s_3, ..., s_1 s_\ell, s_2 s_3, ..., s_{\ell-1} s_\ell \right\} ,$$

etc.
In the two-dimensional case, while the tensor product of third-order polynomials involves 16 terms, the complete set of third-degree polynomials involves only 10 terms.

Judging Approximation Quality

Issue: for a given $\widehat{c}(s,\chi) = p_r(s)$, what criterion should be used to select χ, and to judge the quality of the approximation this selection provides?
Specifically, in seeking to achieve

$$F(\widehat{c}(s_i,\chi)) = 0,$$

the approximation we seek is defined by the parameter vector χ that minimizes

$$\langle F(\widehat{c}(s,\chi)), f(s)\rangle = \int_{\underline{s}}^{\overline{s}} F(\widehat{c}(s,\chi)) f(s) w(s) ds.$$

Alternative specifications for $f(s)$ and $w(s)$ differentiate projection methods along this second dimension.

Judging Approximation Quality, cont.

Three leading specifications;

- Weighted least squares (WLS)
- Galerkin method
- Collocation Method

WLS: $f(s) = F(\widehat{c}(s, \chi))$, yielding the single-valued objective function

$$\langle F(\widehat{c}(s, \chi)), f(s) \rangle = \sum_{i=1}^{N} F(\widehat{c}(s_i, \chi))^2 w(s_i).$$

In this case, the optimal χ can be obtained via a numerical optimization procedure (e.g., Gauss's `optmum`).

Judging Approximation Quality, cont.

Galerkin: Finite element method (e.g., $\widehat{c}(s,\chi)$ is constructed via a tent function), with $w(s) = 1$, and $f(s)$ is the sequence of basis functions $p_r^i(s)$, $i = 1, \ldots r$ used to construct $\widehat{c}(s,\chi)$.

Here, the objective function is a system of r equations, which are to be solved by choice of the r-dimensional vector of coefficients χ:

$$\left\langle F(\widehat{c}(s,\chi)), p_r^i(s) \right\rangle = \int_{\underline{s}}^{\overline{s}} F(\widehat{c}(s,\chi)) p_r^i(s) ds = 0, \quad i = 1, \ldots r.$$

The integrals may be approximated using a sum over a range of distinct values chosen for s. Derivative-based methods are available for solving non-linear systems of this form (e.g., Gauss's `nlsys`).

Judging Approximation Quality, cont.

Collocation: Set $w(s) = 1$, and specify $f(s)$ as the indicator function $\delta(s - s_i)$, $i = 1, \ldots r$, with

$$\delta(s - s_i) = \begin{Bmatrix} 1, & s = s_i \\ 0, & s \neq s_i \end{Bmatrix}.$$

Under $\delta(s - s_i)$, the functional equation $F(\cdot) = 0$ is restricted to hold exactly at r fixed points.

Judging Approximation Quality, cont.

When using Chebyshev polynomials for constructing $\widehat{c}(s,\chi)$, the *Chebyshev Interpolation Theorem* provides a means of optimizing the r choices of s used to construct $\delta(s-s_i)$.

Letting $\{\widetilde{s}_i\}_{i=1}^{r}$ denote the roots of the r^{th}-order component $T_r(s)$, if $F(\widehat{c}(\widetilde{s}_i,\chi))=0$ for $i=1,...,r$ and $F(\widehat{c}(s,\chi))$ is continuous, $F(\widehat{c}(s,\chi))$ will be be close to 0 over the entire range $[-1,1]$.

The r roots of $T_r(s)$ are given by

$$\widehat{s}_j=\cos\left(\frac{(2j-1)}{r}\frac{\pi}{2}\right),\quad j=1,2,...,r.$$

Calculating Expectations

Issue: The objective function

$$F(\widehat{c}(s)) = 0$$

often contains an expectations operator. E.g.,

$$p_t = \beta e^{(1-\gamma)g} E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} (d_{t+1} + p_{t+1}) \right].$$

Lacking knowledge of the properties of p_t, this expectations calculation cannot be performed analytically.

Calculating Expectations, cont.

Four general approches to calculating expectations:

- Work with Markov processes for SPs
- Work with Markov chain approximations of continuous stochastic processes (Tauchen, 1986 *Econ. Letters*)
- Use quadrature methods to approximate expectations
- Monte Carlo simulation

Markov Processes

Definition: A SP is said to have the *Markov property* if for all $k \geq 1$ and for all t,

$$\Pr\left(x_{t+1}|x_t, x_{t-1}, ..., x_{t-k}\right) = \Pr\left(x_{t+1}|x_t\right).$$

Definition: A *Markov chain* is defined by:

- A vector x with r unique values x_i, $i = 1, ..., r$.
- A transition matrix P, with (i, j) *th* element

$$P_{ij} = \Pr\left(x_{t+1} = e_j|x_t = e_i\right).$$

- An initialization vector π_0, with *ith* element

$$\pi_{0i} = \Pr\left(x_0 = e_i\right).$$

Calculating Expectations, cont.

For a one-dimensional state space, given $x_t = x_i$, the conditional expectation

$$E_t f\left(x_{t+1}\right)$$

is given by the weighted average

$$\sum_{j=1}^{r} f\left(x_j\right) P_{ij}.$$

Given $x_t = \left[x_i^1 | x_{ii}^2 | ... | x_{ii...i}^n\right]$, the $n-$dimensional case generalizes to

$$\sum_j \sum_k ... \sum_z f\left(x_j^1 | x_k^2 | ... | x_z^n\right) P_{ij}^1 P_{iik}^2 ... P_{ii...iz}^n$$

Quadrature Methods

Quadrature methods comprise a wide class of numerical tools available for approximating specific examples of integrals of the form

$$\int_a^b f(x)dx.$$

Gaussian quadrature methods are an important subclass. For univariate cases, they take the form

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i),$$

where the nodes x_i and weights w_i are chosen so that if $f(\cdot)$ were a polynomial of degree $2n-1$, then the approximation will be exact given the use of n nodes and weights.

Quadrature Methods, cont.

Gauss-Hermite nodes and weights are tailored specifically for cases in which integrals are of the form

$$\int_{-\infty}^{\infty} f(x) e^{-x^2} dx,$$

which arises naturally in working with normal random variables.

Quadrature Methods, cont.

Example. Consider a simplification of the one-tree model in which q is eliminated, and utility is linear:

$$\begin{aligned} p_t(d_t) &= \beta e^{(1-\gamma)g} E_t\left[d_{t+1} + p_{t+1}(d_{t+1})\right], \\ \ln d_{t+1} &= \rho \ln d_t + \varepsilon_{t+1}. \end{aligned}$$

Then

$$\begin{aligned} p_t(d_t) &= \beta e^{(1-\gamma)g} E_t\left[e^{\rho \ln d_t + \varepsilon_{t+1}} + p_{t+1}(d_{t+1})\right] \\ &= \beta e^{(1-\gamma)g} \frac{1}{\sigma_\varepsilon \sqrt{2\pi}} \int_{-\infty}^{\infty} f(\varepsilon_{t+1}|d_t) e^{-\frac{\varepsilon_{t+1}^2}{2\sigma_\varepsilon^2}} d\varepsilon_{t+1}, \\ f(\varepsilon_{t+1}|d_t) &= e^{\rho \ln d_t + \varepsilon_{t+1}} + p_{t+1}(d_{t+1}). \end{aligned}$$

Quadrature Methods, cont.

Defining

$$\begin{aligned} \varepsilon &= \sqrt{2}\sigma_\varepsilon x \\ &\equiv g(x), \end{aligned}$$

and applying the change-of-variables forumula

$$\int_a^b f(\varepsilon)d\varepsilon = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(x))g'(x')dx,$$

we obtain

$$\begin{aligned} p(d_t) &= \beta e^{(1-\gamma)g} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(\sqrt{2}\sigma_\varepsilon x | d_t) e^{-x^2} dx \\ &\approx \beta e^{(1-\gamma)g} \frac{1}{\sqrt{\pi}} \sum_{i=1}^{n} w_i f\left(\sigma_\varepsilon \sqrt{2} x_i\right). \end{aligned}$$

Monte Carlo Integration

MC Integration entails the approximation of integrals via simulation. Given the ability to obtain drawings of s_{t+1}^i from the known conditional pdf $p\left(s_{t+1}|\Omega_t\right)$, conditional expectations of the form

$$E_t f\left(s_{t+1}\right) = \int f\left(s_{t+1}\right) p\left(s_{t+1}|\Omega_t\right) ds_{t+1}$$

may be approximated as

$$E_t f\left(s_{t+1}\right) \approx \frac{1}{N}\sum_{i=1}^{N} f\left(s_{t+1}^i\right).$$

Monte Carlo Integration, cont.

In our example, we must simulate drawings $\begin{bmatrix} d_{t+1} & q_{t+1} \end{bmatrix}'$ given $\begin{bmatrix} d_t & q_t \end{bmatrix}'$.

This amounts to obtaining drawings $\begin{bmatrix} \varepsilon_{dt+1} & \varepsilon_{qt+1} \end{bmatrix}'$ from a $N\left(0, \quad \Sigma\right)$ distribution, and computing

$$\begin{aligned} d_{t+1} &= \exp\left(\left(1-\rho_d\right)\overline{d} + \rho_d \ln d_t + \varepsilon_{dt+1}\right), \\ q_{t+1} &= \exp\left(\left(1-\rho_q\right)\overline{q} + \rho_q \ln q_t + \varepsilon_{qt+1}\right). \end{aligned}$$

Monte Carlo Integration, cont.

In GAUSS, drawings $\begin{bmatrix} \varepsilon_{dt+1} & \varepsilon_{qt+1} \end{bmatrix}'$ may be obtained as follows:

- `sqrtsig=chol(sig)';`
- `epsdraw = sqrtsig*rndn(2,1);`

Starting Values

Issue: Achievement of fit can be sensitive to starting values χ_0. Recommended solution: select χ_0 using linear approximation.

From the approximation

$$x_{t+1} = Fx_t + Gv_t,$$

the elements of F are elasticities.

Starting Values, cont.

For example, from our example, the two non-zero coefficients in the price equation are (σ_d, σ_q).

In terms of an approximation in levels, these coefficients appear as

$$\begin{aligned} p &\approx p^* + \frac{p^*}{d^*}\sigma_d(d - d^*) + \frac{p^*}{q^*}\sigma_q(q - q^*) \\ &\quad + \frac{1}{2}\left(\frac{p^*}{d^*}\sigma_d\right)\left(\frac{p^*}{q^*}\sigma_q\right)(d - d^*)(q - q^*). \end{aligned}$$

Starting Values, cont.

Given the use of a tensor product representation, the corresponding approximation of $\widehat{p}(d, q, \chi)$ we seek is of the form

$$\begin{aligned}\widehat{p}(d, q, \chi) &\approx \chi_{11} + \chi_{12}\left(\frac{d - d^*}{\omega_d}\right) + \chi_{21}\left(\frac{q - q^*}{\omega_q}\right) \\ &\quad + \chi_{22}\left(\frac{d - d^*}{\omega_d}\right)\left(\frac{q - q^*}{\omega_q}\right) + \ldots.\end{aligned}$$

Matching terms yields the suggested starting values

$$\begin{aligned}\chi_{11} &= p^*, \quad \chi_{12} = \sigma_d \omega_d \frac{p^*}{d^*}, \quad \chi_{21} = \sigma_q \omega_q \frac{p^*}{q^*}, \\ \chi_{22} &= \frac{1}{2}\left(\sigma_d \omega_d \frac{p^*}{d^*}\right)\left(\sigma \omega_q \frac{p^*}{q^*}\right).\end{aligned}$$

Establishing Ranges for Approximation

State variables are stochastic processes, so approximation ranges are best constructed on the basis of corresponding pdfs (e.g., centered at steady state values, and ranging $\pm x$ standard deviations around these values).
Information regarding pdfs is available once again from the log-linear approximation

$$x_t = Fx_{t-1} + e_t,$$

$$x_t = [\widehat{x}_1, \widehat{x}_2, ...]' \qquad \widehat{x}_i = \ln \frac{x_i}{\overline{x}_i}.$$

Approximation Ranges, cont.

The unconditional VCV matrix of x solves

$$\begin{aligned}\Sigma_x &= Exx' \\ &= EF(xx')F' + Eee' \\ &= F\Sigma_x F' + Q,\end{aligned}$$

and is thus obtained from

$$vec(\Sigma_x) = (I - F \otimes F')^{-1} + Q.$$

Square roots of the diagonal elements of Σ_x yield $\sigma(\widehat{x}_i)$.

Approximation Ranges, cont.

Then $\sigma(x_i)$ is obtained from $\sigma(\widehat{x}_i)$ using

$$\Delta\widehat{x}_i = \Delta \ln \frac{x_i}{\overline{x}_i} \approx \frac{\Delta x_i}{x_i},$$

which implies

$$\sigma(x_i) \approx \overline{x}_i \sigma(\widehat{x}_i).$$

Judging Quality of Fit

Given a proposal for χ, fit can be judged simply by plotting

$$F(\widehat{c}(s, \chi))$$

over a range chosen for s, and judging proximity to zero.

Application

Returning to the functional equation

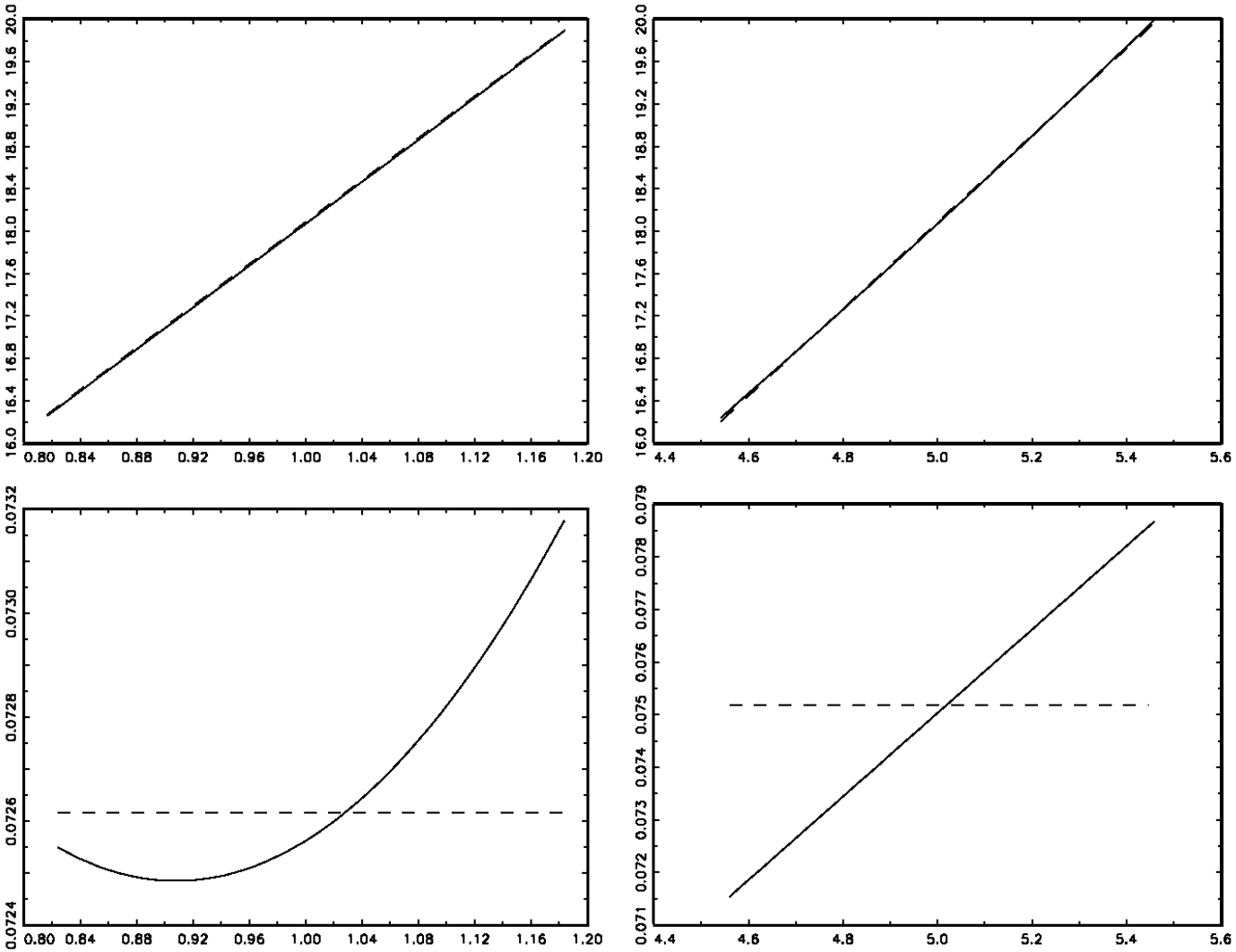
$$p_t = \beta e^{(1-\gamma)g} E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} (d_{t+1} + p_{t+1}) \right],$$

approximate using a tensor product of Chebyshev polynomials over $\begin{bmatrix} d_t & q_t \end{bmatrix}'$ (3rd and 4th order, respectively). Approximate E_t via MC simulation, with N=10,000. Use starting values χ_0 constructed from F as described above.

Parameterization: $\beta = 0.96$, $\gamma = 2$, $g = 0.013$, $\rho's = 0.9$, $\sigma_d = 0.02$, $\sigma_q = 0.01$, $corr(d, q) = 0.4$.

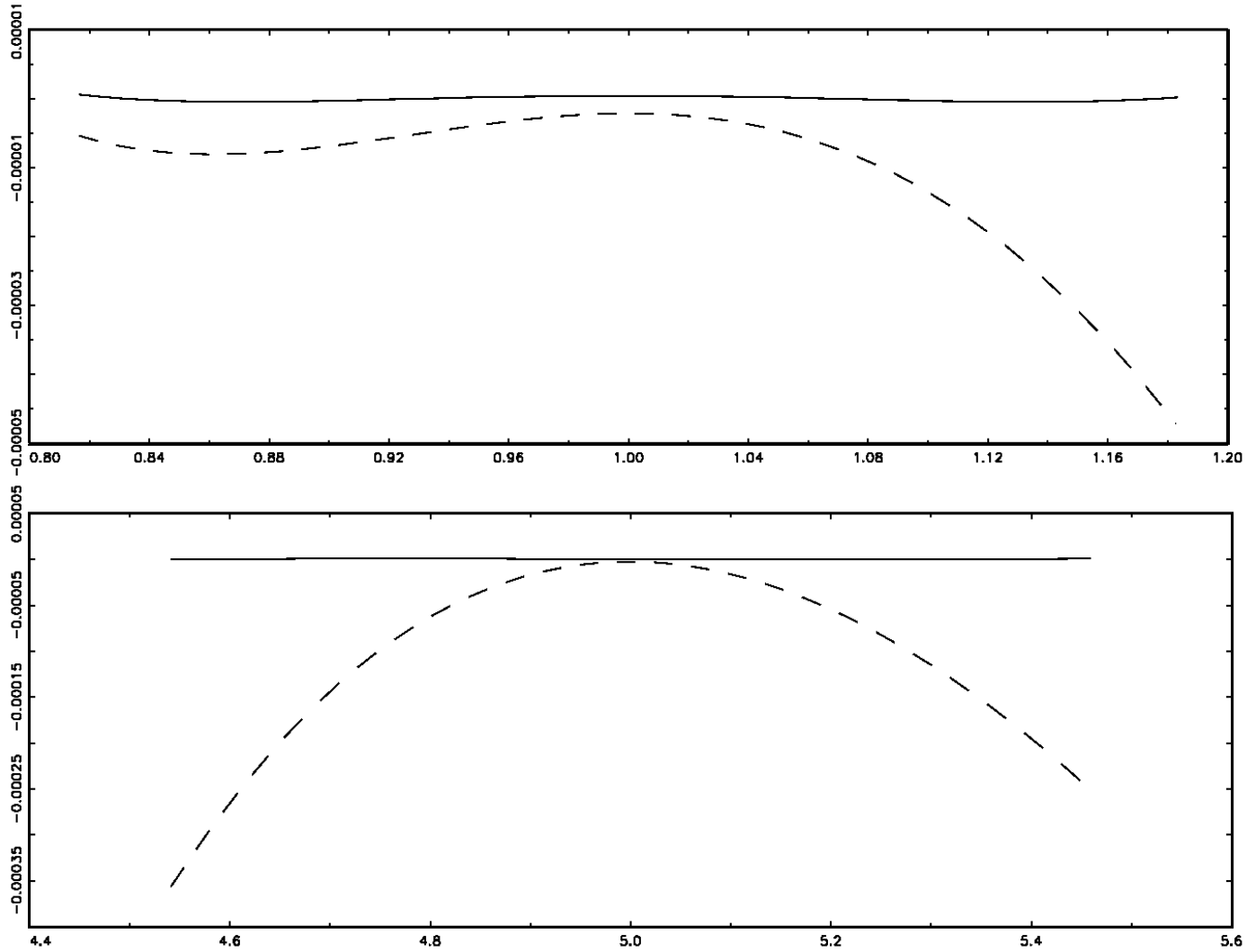
Application, cont.

Policy Functions and Slopes



Application, cont.

Fit



Application, cont.

Implications for Relative Standard Deviations, Nonlin Versus Lin

- Note: variables are measured in terms of logged deviations from steady state.
- For non-linear approximation, statistics obtained via simulation. Reference: Ch. 11.1, pp. 290-293.

σ_c/σ_p	0.537	0.544
σ_d/σ_p	1.101	1.074
σ_q/σ_p	0.527	0.537

Bottom line: differences are noticable in this context, but their size is hard to interpret. Stay tuned for likelihood comparisons....

Application, cont.

Exercise: Construct a non-linear approximation of the optimal growth model; compare fit and relative standard deviations with those obtained using a log-linear approximation.

Details:

- Use a 4th-order Chebyshev polynomial for capital, 3rd-order for TFP shock
- Approximate expectations using Gauss-Hermite approximation
- Experiment with starting values for χ
- Experiment with alternative model parameterizations.