ODE

OSCAR REULA

1. Preliminary material

1.1. **Linear equations with sources.** We shall look now at the solutions to:

$$y_t = \lambda y + a e^{\mu t} \qquad y(0) = y_0, \quad \lambda \neq \mu$$

The last term is called a forcing term. The corresponding inhomogeneous solution the forced solution.

$$y_p(t) := A e^{\mu t},$$

plugging it into the equation we get,

$$A\mu e^{\mu t} = \lambda A e^{\mu t} + a e^{\mu t} \quad A(\mu - \lambda) = a, \qquad A = \frac{a}{\mu - \lambda}$$

Thus,

$$y_p(t) := \frac{a}{\mu - \lambda} e^{\mu t},$$

When the forcing frequency is resonant ($\lambda = \mu$) then the solution can be obtained as a limit. In that case it is given by,

$$y_p(t) := A t e^{\lambda t}, \quad A = a.$$

It grows faster than the homogeneous solution.

$$y(t) = y_h(t) + y_p(t),$$

$$y_h(0) = y_o - y_p(0) = \begin{cases} y_o - \frac{a}{\mu - \lambda} & \mu \neq \lambda \\ y_o & \mu = \lambda \end{cases}$$

$$\begin{aligned}
y(t) &= (y_o - \frac{a}{\mu - \lambda}) e^{\lambda t} + \frac{a}{\mu - \lambda} e^{\mu t} && (1)\\
&= [(y_o - \frac{a}{\mu - \lambda}) + \frac{a}{\mu - \lambda}] e^{\lambda t} + \frac{a}{\mu - \lambda} [e^{\mu t} - e^{\lambda t}] && (2)\\
&= y_o e^{\lambda t} + \frac{a}{\mu - \lambda} e^{\lambda t} [e^{(\mu - \lambda) t} - 1] && (3)\\
&= y_o e^{\lambda t} + t a e^{\lambda t} \quad \text{in the limit} \quad \mu \to \lambda && (4)
\end{aligned}$$

Resonances would appear latter when studying hyperbolic systems.

In general, for a system of the form:

$$y_t = \lambda y + F(t)$$

The solution is given by,

$$y(t) = e^{\lambda t} y_o + \int_0^t e^{\lambda(t-\zeta)} F(\zeta)\, d\zeta$$

This expression can be further generalized noticing that if we define the map that takes an homogeneous solution at time ζ and move it to time t,

$$y(t) = S(t, \zeta) y(\zeta).$$

then

$$y_h(t) = e^{\lambda t} y_o = e^{\lambda(t-\zeta)} e^{\zeta t} y_o = e^{\lambda(t-\zeta)} y_h(\zeta) = S(t, \zeta) y_h(\zeta)$$

and

$$y(t) = S(t, 0) y_o + \int_0^t S(t, \zeta) F(\zeta)\, d\zeta$$

If we consider now the equation,

$$y_t = a(t) y + F(t)$$

and we define the homogeneous map as before, then, $y_h(t) = S(t, \zeta) y_h(\zeta)$, then the solution of the inhomogeneous equation is as before:

$$y(t) = S(t, 0) y_o + \int_0^t S(t, \zeta) F(\zeta)\, d\zeta.$$

To see this notice that

$$S(t, \zeta)_t = a(t) S(t, \zeta), \quad \text{and} \;\; S(\zeta, \zeta) = id$$

and so,

$$\begin{aligned}
y_t(t) &= a(t) S(t, 0) + S(t, t) F(t) + \int_0^t a(t) S(t, \zeta) F(\zeta)\, d\zeta && (5)\\
&= a(t) [S(t, 0) + \int_0^t S(t, \zeta) F(\zeta)\, d\zeta] + S(t, t) F(t) && (6)\\
&= a(t) y(t) + F(t) && (7)
\end{aligned}$$

But the homogeneous solution is also easy to find,

$$S(t, \zeta) = e^{\int_\zeta^t a(\tau)\, d\tau}$$

indeed,

$$S(t, \zeta)_t = a(t) e^{\int_\zeta^t a(\tau)\, d\tau} = a(t) S(t, \zeta)$$

2. One step methods: Euler's Explicit Method

We first discretize time, we let $t_n = t_o + dtn$ if the discretization is constant or $t_n = t_o \sum_{i=0}^n dt_i$ if the successive time steps are denoted by dt_i. Thus we have a discrete grid, $G \subset Z \to R$. Thus, if we have a function $y : I \to R$ we shall denote by y_n the restriction of that function to the grid G, $y_n = y(t_n)$.

Given an ordinary differential equation, (here we can consider y to be a vector in some region of R^n)

$$y_t = F(y) \qquad y(0) = y_o$$

Euler's explicit method approximate the time derivative by,

$$y_t(t_n) = \frac{y(t_{n+1}) - y(t_n)}{dt_n} = \frac{y_{n+1} - y_n}{dt_n},$$

therefore we have,

$$y_{n+1} = y_n + dt_n F(y_n),$$

which can be solved recursively starting with $y_0 = y_o$. If $F = F(y,t)$ then we extend the system with an extra variable t and use the same formulae.

2.1. **Example.**

$$y_t = \lambda y + ae^{\mu t}$$

We start with the homogeneous equation. Euler's method gives, (for dt constant)

$$y^h_{n+1} = y^h_n + dt\lambda y^h_n = (1 + dt\lambda)y^h_n$$

which can be solved to be,

$$y^h_n = (1 + dt\lambda)^n (y_o - y^p_o)$$

where y^p_o is the initial value of the particular solution to be found next.

Notice that, using $\ln(1+\varepsilon) = \ln(1) + \frac{\varepsilon}{1} - \frac{\varepsilon^2}{2} + \cdots$, $1 + \varepsilon = e^{\varepsilon} e^{O(\varepsilon^2)}$,

$$(1 + dt\lambda)^n = (1 + dt\lambda)^{t_n/dt} = (e^{dt\lambda} e^{O(dt^2\lambda^2)})^{(t_n/dt)} = e^{\lambda t_n} e^{O(dt\lambda^2 t_n)}$$

Therefore,

$$y^h_n = e^{\lambda t_n} e^{dt\lambda^2 t_n}(y_o - y^p_o) = e^{\lambda t_n}[1 + O(dt\lambda^2 t_n)](y_o - y^p_o) = y^h(t_n) + e^{\lambda t_n} O(dt\lambda^2 t_n)(y_o - y^p_o),$$

or

$$max_{0 \le t_n \le T} |y^h_n - y^h(t_n)| \le C(\lambda T)dt$$

We now look at the particular solution: in Euler's approximation we get,

$$y^p_{n+1} = y^p_n + dt[\lambda y^p_n + ae^{\mu t_n}].$$

As in the analytic case we look for a solution of the form,

$$y^p_n = \tilde{A}e^{\mu t_n}.$$

Substitution in the recursion gives,

$$\tilde{A}e^{\mu(t_n+dt)} = (1 + dt\lambda)Ae^{\mu t_n} + dtae^{\mu t_n}, \qquad \tilde{A} = \frac{dta}{e^{\mu dt} - (1 + dt\lambda)}$$

Using that $e^{\mu dt} = 1 + \mu dt + \frac{\mu^2 dt^2}{2} + O(\mu^3 dt^3)$ we have,

$$\tilde{A} = \frac{a}{\mu - \lambda + \frac{\mu^2 dt}{2} + O(\mu^3 dt^2)}$$

and so,

$$y^p_n = \frac{ae^{\mu t_n}}{\mu - \lambda + \frac{\mu^2 dt}{2} + O(\mu^3 dt^2)} \tag{8}$$

$$= y^p(t_n) + ae^{\mu t_n}[\frac{1}{\mu - \lambda + \frac{\mu^2 dt}{2} + O(\mu^3 dt^2)} - \frac{1}{\mu - \lambda}] \tag{9}$$

$$= y^p(t_n) - \frac{dt}{2}\frac{\mu^2}{(\mu-\lambda)^2}ae^{\mu t_n} + O(\mu dt^2)\frac{\mu^2}{(\mu-\lambda)^2}ae^{\mu t_n} \tag{10}$$

Therefore, in a finite interval $[0, T]$ we have,

$$max_{0 \leq t_n \leq T}|y^p(t_n) - y^p_n| \leq C(\lambda, \mu, T)dt$$

Adding the two solutions with the correct boundary conditions we find that for finite time intervals the error is bounded linearly in dt and so, taking dt small enough we can approximate the solution as much as we want.

So we have the following lemma:

Lemma 2.1. *Let $y_t = \lambda y + ae^{\mu t}$, $\mu \neq \lambda$, $T > 0$, and y_o. Then for some constant C Euler's approximation satisfies*

$$max_{0 \leq t_n \leq T}|y(t_n) - y_n| \leq Cdt.$$

For the case $\Re(\lambda) < 0$, $\Re(\mu) < 0$ we can bound the solution for all times.

If we use a further order for the homogeneous approximation, namely,

$$(1 + dt\lambda)^n = (1 + dt\lambda)^{t_n/dt} \tag{11}$$

$$= (e^{dt\lambda}e^{-\frac{dt^2\lambda^2}{2}}e^{O(dt^3\lambda^3)})^{(t_n/dt)} \tag{12}$$

$$= e^{\lambda t_n}e^{-\frac{dt\lambda^2 t_n}{2}}e^{O(dt^2\lambda^3 t_n)}) \tag{13}$$

$$= e^{\lambda t_n}[1 - \frac{dt\lambda^2 t_n}{2} + O(dt^2\lambda^3)] \tag{14}$$

we can refine the error expression as follows:

$$y(t_n) - y_n = -\frac{dt}{2}[\lambda^2 t_n e^{\lambda t_n} + \frac{a\mu^2}{(\mu-\lambda)^2}e^{\mu t_n}] + O(dt^2) = dt\phi_1(t_n) + O(dt^2).$$

where the function ϕ_1 does not depends on dt.