Proof of unsatisfiability π for a set of clauses $\mathcal{C}$ = a DAG (V_π, E_π) where
(true without backjumping)

- V_π is a set of clauses s.t. $\forall c \in V_\pi$
  - $c \in \mathcal{C}$ and c is a root
  - c has exactly 2 predecessors c_1 and c_2 s.t. c is the resolvent of c_1 and c_2 i.e. $\dfrac{c_1 = v \vee A,\ c_2 = \neg v \vee B}{c = A \vee B}$ (v is called the pivot variable)
  - $\square$ is the unique leaf

Getting a proof of unsat from a run of DPLL:

$$x_1 \dots x_n \parallel F, C \Rightarrow \square \quad \text{if} \quad x_1, \dots, x_n \models \neg \overbrace{\underbrace{C}_{\text{conflict clause}}}^{= \neg x_1 \vee \dots \vee \neg x_n}$$

$\vdots$

$$x_1 \dots x_{i-1} \parallel F, C_i \vee x_i \Rightarrow x_1 \dots x_i \parallel F, \underbrace{C_i \vee x_i}_{\alpha(i)} \quad \text{if} \quad x_1 \dots x_{i-1} \models \neg C_i$$

$\vdots$

$$\neg x_1^d\, y_1 \dots y_n \parallel F, C' \Rightarrow x_1 \parallel F, C' \quad \text{if} \quad \neg x_1^d\, y_1 \dots y_n \models \neg \overbrace{\underbrace{C'}_{\text{conflict clause}}}^{= x_1 \vee \neg y_1 \vee \dots \vee \neg y_n}$$

$\vdots$

$$\neg x_1^d\, y_1 \dots y_{i-1} \parallel F, \Delta_i \vee y_i \Rightarrow \neg x_1^d\, y_1 \dots y_i \parallel F, \underbrace{\Delta_i \vee y_i}_{\beta(i)} \quad \text{if} \quad \neg x_1^d\, y_1 \dots y_{i-1} \models \neg \Delta_i$$

$$\dfrac{\dfrac{\overset{x_1}{\overset{\parallel}{\Delta_1 \vee y_1}},\ \overset{x_1 \vee \neg y_1}{\overset{\parallel}{\Delta_2 \vee y_2}}, \dots \quad C' = x_1 \vee \neg y_1 \vee \dots \vee \neg y_n}{x_1} \qquad \dfrac{\overset{\neg x_1}{\overset{\parallel}{C_1 \vee x_2}},\ \overset{\neg x_1 \vee \neg x_2}{\overset{\parallel}{C_2 \vee x_3}}, \dots \quad \overset{\neg x_1 \vee \dots \vee \neg x_n}{\overset{\parallel}{C}}}{\neg x_1}}{\square}$$

$$\dfrac{\dfrac{\overset{x_1}{\overset{\parallel}{\Delta_1 \vee y_1}} \quad C'}{x_1 \vee \neg y_2 \vee \dots \vee \neg y_n}\,y_1 \qquad \dfrac{\overset{x_1}{\overset{\parallel}{\Delta_1 \vee y_1}} \quad \overset{x_1 \vee \neg y_1}{\overset{\parallel}{\Delta_2 \vee y_2}}}{x_1 \vee y_2}\,y_1}{x_1 \vee \neg y_3 \vee \dots \vee \neg y_n}\,y_2$$

$\vdots$

$$\dfrac{\dfrac{\overset{\neg x_1}{\overset{\parallel}{C_1 \vee x_2}} \quad C}{\neg x_1 \vee \neg x_3 \vee \dots \vee \neg x_n}\,x_2 \qquad \dfrac{\overset{\neg x_1}{\overset{\parallel}{C_1 \vee x_2}} \quad \overset{\neg x_1 \vee \neg x_2}{\overset{\parallel}{C_2 \vee x_3}}}{\neg x_1 \vee x_3}\,x_2}{\neg x_1 \vee \neg x_4 \vee \dots \vee \neg x_n}\,x_3$$

$\vdots$

- if C is a clause

$C \downarrow B$ = keep only literals with symbols in B

$C \setminus B$ = remove literals with symbols in B

- proof in a theory: roots can be also theory lemmas (obtained from T-Backjump + T-Learn)
  - for every theory lemma C: I_C = interpolant for $(\neg C\setminus B, \neg C \downarrow B)$
  - for every clause: $I_C = \begin{cases} C\downarrow B & \text{if } C \in A \\ \text{true}, & \text{otherwise} \end{cases}$

say at the end

$$\frac{C_1 = p \vee \Delta_1 \qquad C_2 = \neg p \vee \Delta_2}{C = \Delta_1 \vee \Delta_2}$$

P6 of McMillan-TCS05

$$I_C = \begin{cases} I_{C_1} \vee I_{C_2} & \text{if } p \notin B \\ I_{C_1} \wedge I_{C_2} & \text{otherwise} \end{cases}$$

(split over a predicate of A ⟹ A proves a disjunction)

(split over a predicate in B ⟹ A must refute both cases of B)

$I_\square$ is the interpolant for A, B

<u>Invariants</u>

$A \models_T I_C \vee (C \setminus B)$

$B \wedge I_C \models C \downarrow B$ (Induction)

I_C has common symbols to A and B

from Ruemmer's slides: weakest/strongest interpolant, example of computing interpol. for theories

<u>Example</u>

$A = \{\neg a \vee b,\ a \vee c\}$

$B = \{\neg b, \neg c\}$

$a \in B\ (a \notin B)\ a$: $\dfrac{\overset{b\ \text{true}}{\neg a \vee b} \qquad \overset{c\ \text{true}}{a \vee c}}{b \vee c^{\ b\vee c\ \ \text{true}}}$

$b \in B\ (b \in B)\ b$: $\dfrac{b \vee c \qquad \neg b^{\ \text{true}\ \ \neg b}}{c^{\ (b\vee c)\wedge \text{true}\ \ \text{true}\wedge\neg b}}$

$c \in B\ (c \in B)\ c$: $\dfrac{c \qquad \neg c^{\ \text{true}\ \ \neg c}}{\square\ \ b \vee c\ \ \neg b \wedge \neg c}$

A proves a disjunction of facts (one for a, one for ¬a)

⇓

I is the disjunction

B proves a disjunction $b \vee c$ which must be refuted by A both

⇓

A must imply a conjunction

$A = \{\neg b, \neg c\}$

$B = \{\neg a \vee b,\ a \vee c\}$

$A = \{ \neg b,\ \neg a \vee b \vee c,\ a \}$ $\qquad$ $B = \{ \neg a \vee \neg c \}$

$I = a \wedge c$

$$\wedge\ a \in B \quad \dfrac{\overset{\neg a \vee c}{\neg a \vee b \vee c} \qquad \overset{a}{a}}{b \vee c \quad a \wedge (\neg a \vee c)}$$

$$\vee\ b \notin B \quad \dfrac{\neg b^{\text{false}} \qquad b \vee c}{c \quad a \wedge (\neg a \vee c) \vee \text{false}}$$

$$\wedge\ c \in B \quad \dfrac{c \qquad \neg a \vee \neg c^{\ \text{true}}}{\neg a \quad a \wedge (\neg a \vee c)}$$

$$\wedge\ a \in B \quad \dfrac{\neg a \qquad a^{\ a}}{\square \quad a \wedge (\neg a \vee c) \equiv a \vee c}$$

<u>Proof of</u> a) $A \models I_C \vee (C \setminus B)$

b) $B \wedge I_C \models C \downarrow B$

I_C has common symbols

<u>Case 1</u>

$$\dfrac{p \vee \Delta_1^{\ \phi} \qquad \neg p \vee \Delta_2^{\ \phi'}}{\Delta_1 \vee \Delta_2 \quad \phi \vee \phi'} \quad p \notin B$$

a) Ind hyp: $A \models \underbrace{\phi}_{\text{global}} \vee \overbrace{(p \vee \Delta_1 \setminus B)}^{\text{local to } A}$

$A \models \underbrace{\phi'}_{\text{global}} \vee \underbrace{(\neg p \vee \Delta_2 \setminus B)}_{\text{local to } A}$

$\Rightarrow A \models (\phi \wedge \phi') \vee \underbrace{\left((p \vee \Delta_1 \setminus B) \wedge (\neg p \vee \Delta_2 \setminus B) \right)}_{= (\Delta_1 \vee \Delta_2) \setminus B}$

b) Ind hyp: $B \wedge \phi \models \Delta_1 \downarrow B$ $\quad (p \notin B)$

$B \wedge \phi' \models \Delta_2 \downarrow B$

$\Downarrow$

Given B,

$\phi \vee \phi' \models \underbrace{(\Delta_1 \downarrow B) \vee (\Delta_2 \downarrow B)}_{= (\Delta_1 \vee \Delta_2) \downarrow B}$

<u>Case 2</u> Analogous

Theory interpolants: A, B

strongest interpolant: $\exists \vec{x} .\ A(\vec{x})$

weakest interpolant: $\forall \vec{y}\ \neg B(\vec{y})$

<u>Q-F interpolation</u>

$\rightarrow$ getting quantifier-free interpolants from quantifier-free formulas

if the theory admits quantifier-elimination this problem is solvable

converse??