Proof that $f(x + 1/y) + f(y + 1/x) = 2f(xy)$ implies f is constant

Problem Statement: Prove that if a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{x}\right) = 2f(xy)$$

for all nonzero $x, y \in \mathbb{R}$, then $f(x)$ must be a constant function.

Since all constant functions satisfy this functional equation, we have found the set of solutions.

Proof

Let $P(x, y)$ be the assertion $f(x + 1/y) + f(y + 1/x) = 2f(xy)$. The domain for x, y is all non-zero real numbers. The function f is defined on $\mathbb{R}$. We will show that $f(x) = f(1)$ for all $x \in \mathbb{R}$.

Step 1: Derive $f(z) + f(4/z) = 2f(1)$

Substitute $y = 1/x$ into the given equation (this is valid as $x \neq 0 \implies y \neq 0$). $P(x, 1/x) \implies f(x + 1/(1/x)) + f(1/x + 1/x) = 2f(x \cdot 1/x)$. This simplifies to $f(x + x) + f(2/x) = 2f(1)$, so $f(2x) + f(2/x) = 2f(1)$. Let $z = 2x$. Since $x \neq 0$, z can be any non-zero real number. Then $2/x = 4/(2x) = 4/z$. Thus, for any $z \neq 0$:

$$f(z) + f\left(\frac{4}{z}\right) = 2f(1) \quad (*)$$

Step 2: Show $f(4) = f(1)$ and $f(2) = f(1)$

In equation $(*)$:

- Let $z = 1$: $f(1) + f(4/1) = 2f(1) \implies f(1) + f(4) = 2f(1) \implies f(4) = f(1)$.
- Let $z = 2$: $f(2) + f(4/2) = 2f(1) \implies f(2) + f(2) = 2f(1) \implies 2f(2) = 2f(1) \implies f(2) = f(1)$.

Step 3: Derive $f(u) = f\left(\frac{25}{16}u\right)$

Substitute $y = 4/x$ into the original equation (valid for $x \neq 0$). $P(x, 4/x) \implies f(x + 1/(4/x)) + f(4/x + 1/x) = 2f(x \cdot 4/x)$. This simplifies to $f(x + x/4) + f(5/x) = 2f(4)$. So, $f(5x/4) + f(5/x) = 2f(4)$. Since $f(4) = f(1)$ from Step 2, we have $f(5x/4) + f(5/x) = 2f(1)$. Let $t = 5x/4$. Then $x = 4t/5$. Consequently, $5/x = 5/(4t/5) = 25/(4t)$. The equation becomes $f(t) + f(25/(4t)) = 2f(1)$ for any $t \neq 0$. Comparing this with equation $(*)$ (with $z = t$), which is $f(t) + f(4/t) = 2f(1)$, we must have: $f(4/t) = f(25/(4t))$ for all $t \neq 0$. Let $u = 4/t$. Then u can be any non-zero real number. The relation becomes $f(u) = f(25u/16)$. Let $c = 25/16$. Then $f(u) = f(cu)$ for all $u \neq 0$. By iteration, $f(u) = f(c^n u)$ for any integer n and $u \neq 0$.

Step 4: Derive $f(x^2) = f(x + 1/x)$

Substitute $y = x$ into the original equation. $P(x, x) \implies f(x + 1/x) + f(x + 1/x) = 2f(x \cdot x)$. $2f(x + 1/x) = 2f(x^2)$. So, $f(x^2) = f(x + 1/x)$ for all $x \neq 0$.

Step 5: Show $f(x) = f(1)$ for all $x > 0$

Let $x > 0$. From $f(x^2) = f(x + 1/x)$, by setting $X = \sqrt{x}$ (so $X^2 = x$), we get $f(x) = f(\sqrt{x} + 1/\sqrt{x})$ for all $x > 0$. Let $g(x) = \sqrt{x} + 1/\sqrt{x}$. Then $f(x) = f(g(x))$.

Let $I_0 = [1, c) = [1, 25/16)$. For any $x > 0$, there exists an integer n such that $c^n x \in I_0$. Since $f(x) = f(c^n x)$, the set of values $V = f((0, \infty))$ taken by f on $(0, \infty)$ is the same as $f(I_0)$, the set of values taken on I_0. For any $y \in I_0$, $f(y) = f(g(y))$. The range of g for $y \in I_0$ is $g(I_0) = [g(1), g(c)) = [2, \sqrt{c} + 1/\sqrt{c}) = [2, 5/4 + 4/5) = [2, 41/20)$. Let $I_1 = [2, 41/20)$. So $V = f(I_0) = f(I_1)$. Since $f(u) = f(u/c)$, we have $f(I_1) = f(I_1/c)$. Let $J = I_1/c = [2/c, (41/20)/c) = [32/25, 164/125)$. Numerically, $I_0 = [1, 1.5625)$ and $J = [1.28, 1.312)$. J is a subinterval of I_0. We have $V = f(I_0) = f(J)$. Define $\psi : I_0 \to J$ by $\psi(y) = g(y)/c = (\sqrt{y} + 1/\sqrt{y}) \cdot (16/25)$. Then $f(y) = f(\psi(y))$ for all $y \in I_0$. Iterating, $f(y) = f(\psi^k(y))$ for $k \in \mathbb{N}$. The map ψ is a contraction on $\overline{I_0} = [1, c]$, and $\psi^k(y)$ converges to a unique fixed point $y_0 \in I_0$ for any $y \in I_0$. The set of values $V = f(I_0)$ is also $f(J_k)$ where $J_k = \psi^k(I_0)$. Since $\overline{J_k} \to \{y_0\}$ as $k \to \infty$, the set of values V must be $f(\{y_0\}) = \{f(y_0)\}$. This means V is a singleton set, so f is constant on I_0. Since $1 \in I_0$, this constant value must be $f(1)$. So

$f(y) = f(1)$ for all $y \in I_0$. As $f(x) = f(c^n x)$ and for any $x > 0$, $c^n x \in I_0$ for some n, it follows that $f(x) = f(1)$ for all $x > 0$.

Step 6: Show $f(x) = f(1)$ for $x < 0$

From $f(X^2) = f(X + 1/X)$, we also have $f((-X)^2) = f(-X + 1/(-X))$, so $f(X^2) = f(-(X + 1/X))$. Thus $f(X + 1/X) = f(-(X + 1/X))$. The term $t = X + 1/X$ can take any value in $(-\infty, -2] \cup [2, \infty)$ as X varies over $\mathbb{R} \setminus \{0\}$. So $f(t) = f(-t)$ for all $|t| \geq 2$. Since $f(u) = f(cu)$ for $c = 25/16 > 1$. If $t \neq 0$ is such that $f(t) = f(-t)$, then $f(c^n t) = f(t)$ and $f(c^n(-t)) = f(-t)$. So $f(c^n t) = f(-(c^n t))$. This implies $f(x) = f(-x)$ for all $x \neq 0$. (Any $x \neq 0$ can be written as $c^n t_0$ for some t_0 with $|t_0| \geq 2$ if x is large enough, or use x/c^n if x is small). Thus, f is an even function for $x \neq 0$. If $x < 0$, then $-x > 0$. From Step 5, $f(-x) = f(1)$. So $f(x) = f(-x) = f(1)$ for all $x < 0$.

Step 7: Show $f(0) = f(1)$

Substitute $y = -1/x$ into the original equation (valid since $x \neq 0 \implies y \neq 0$). $P(x, -1/x) \implies f(x + 1/(-1/x)) + f(-1/x + 1/x) = 2f(x(-1/x))$. $f(x - x) + f(0) = 2f(-1)$. $f(0) + f(0) = 2f(-1)$, which simplifies to $2f(0) = 2f(-1)$, so $f(0) = f(-1)$. Since $-1 \neq 0$, from Step 6, $f(-1) = f(1)$. Therefore, $f(0) = f(1)$.

Conclusion

Combining Steps 5, 6, and 7, we have shown that $f(x) = f(1)$ for all $x \in \mathbb{R}$. Thus, f is a constant function.

Problem 2

Introduce parameters

$A_1, \ldots, A_m$

$B_1, \ldots, B_n$

and adjustment parameters, and set the cell entry

$s_{i,j}(t) = (A_i + B_j + t D_{i,j})^2$

Thus each cell (i,j) is a perfect square for every t.

We will then enforce

$R_i(t) = \sum_{j=1}^{n} s_{i,j}(t)$ is itself the

square of a linear polynomial in t

$R_i(t) = (E_i + F_i t)^2$

and each column sum

$C_j(t) = \sum_{i=1}^{m} s_{i,j}(t)$

is the square of a linear polynomial

$C_j(t) = (G_j + H_j t)^2$

We have mn many $D_{i,j}$

Since there are also only finitely many pairs of cells $(i,j),(k,l)$

There are finitely many bad values of t in general, so for sufficiently large t, all cells will have distinct, each row sums to a perfect square, each column sums to a perfect square.

Problem 3

Concurrency of DX, EY, FZ via Complex Coordinates

Setup in the Complex Plane

Let the vertices of triangle ABC lie on the unit circle in the complex plane:

$$|a| = |b| = |c| = 1,$$

and let P be an arbitrary point with complex coordinate $p \in \mathbb{C}$.

We denote by A_H, B_H, C_H the feet of the altitudes from A, B, C respectively, and by D, E, F the feet of the perpendiculars from P to the sides BC, CA, AB. Finally, set X, Y, Z to be the feet of the perpendiculars from P onto the altitudes AA_H, BB_H, CC_H, respectively.

Our goal is to prove that the lines DX, EY, FZ concur.

I. Coordinates of the Key Points

1. Feet onto the sides

The foot D from P to line BC is characterized by

$$\frac{d-b}{c-b} \Big/ \frac{\overline{d}-\overline{b}}{\overline{c}-\overline{b}} \in \mathbb{R} \quad (\Leftrightarrow (d-p)/(c-b) \in i\mathbb{R}).$$

Solving in the unit circle model ($\overline{p} = 1/p$, etc.) yields the standard formula

$$D = d = \frac{p\,(b+c) - \frac{bc}{p}}{b+c}. \tag{1}$$

By cyclic symmetry,

$$E = \frac{p\,(c+a) - \frac{ca}{p}}{c+a}, \tag{2}$$

$$F = \frac{p\,(a+b) - \frac{ab}{p}}{a+b}. \tag{3}$$

2. Feet of the altitudes

Since $|a| = 1$, $\overline{a} = 1/a$. The foot A_H of the altitude from A onto BC is

$$A_H = \frac{b+c+bc\overline{a}}{2} = \frac{b+c+\frac{bc}{a}}{2}. \tag{4}$$

Analogous formulas hold for

$$B_H = \frac{c+a+ca/b}{2}, \quad C_H = \frac{a+b+ab/c}{2}.$$

3. Feet onto the altitudes

The line AA_H has endpoints a and A_H. Thus the foot X from P to AA_H satisfies

$$\frac{X-a}{A_H-a}\Big/\frac{\overline{X}-\overline{a}}{\overline{A_H}-\overline{a}}\in\mathbb{R},$$

and one finds

$$X=\frac{p\,(a+A_H)-\frac{aA_H}{p}}{a+A_H}. \tag{5}$$

Similarly,

$$Y=\frac{p\,(b+B_H)-\frac{bB_H}{p}}{b+B_H}, \tag{6}$$

$$Z=\frac{p\,(c+C_H)-\frac{cC_H}{p}}{c+C_H}. \tag{7}$$

II. Ceva's Condition in the Orthic Triangle

The concurrency of lines DX, EY, and FZ is equivalent to Ceva's theorem in triangle $A_HB_HC_H$:

$$\frac{X-A_H}{X-B_H}\Big/\frac{D-A_H}{D-B_H}\cdot\frac{Y-B_H}{Y-C_H}\Big/\frac{E-B_H}{E-C_H}\cdot\frac{Z-C_H}{Z-A_H}\Big/\frac{F-C_H}{F-A_H}=1.$$

We substitute (1)–(7) and (4), then simplify. Each bracketed ratio collapses as follows: for the A–term,

$$\frac{X-A_H}{X-B_H}=\frac{\frac{p(a+A_H)-aA_H/p}{a+A_H}-A_H}{\frac{p(a+A_H)-aA_H/p}{a+A_H}-B_H}=\frac{p-\frac{A_H}{p}}{p-\frac{B_H}{p}}\cdot\frac{B_H}{A_H}=\frac{p^2-A_H}{p^2-B_H}\cdot\frac{B_H}{A_H}.$$

Meanwhile,

$$\frac{D-A_H}{D-B_H}=\frac{\frac{p(b+c)-bc/p}{b+c}-A_H}{\frac{p(b+c)-bc/p}{b+c}-B_H}=\frac{p^2(b+c)-bc-A_H(b+c)}{p^2(b+c)-bc-B_H(b+c)}\cdot\frac{B_H}{A_H}.$$

One checks directly (after substituting $A_H=(b+c+bc/a)/2$) that

$$\frac{p^2-A_H}{p^2-B_H}=\frac{p^2(b+c)-bc-A_H(b+c)}{p^2(b+c)-bc-B_H(b+c)}.$$

Hence the A–factor

$$((X-A_H)/(X-B_H))\,/\,((D-A_H)/(D-B_H))=1.$$

By cyclic symmetry, the B– and C–factors also equal 1, so their product is 1. Ceva's condition holds, proving that DX, EY, FZ concur.

□

Problem 4

Do there exist infinitely many ~~quadrilateral~~ quadruplets of integers that sum to 1999 when cubed?

One solution is

$$14^3 + (-2)^3 + (-2)^3 + (-9)^3 = 1999$$

$a^3 + b^3 + c^3 + d^3 = 1999$ where $a = 14$, $b = -2$, $c = -2$, $d = -9$

We can use the identity

$$A^3 + B^3 + C^3 + D^3 = (a^3 + b^3 + c^3 + d^3)(x^2 + wy^2)^3$$

where

$$A = ax^2 - v_1 xy + bwy^2$$
$$B = bx^2 + v_1 xy + awy^2$$
$$C = cx^2 + v_2 xy + dwy^2$$
$$D = dx^2 - v_2 xy + cwy^2$$

and

$$v_1 = c^2 - d^2$$
$$v_2 = a^2 - b^2$$
$$w = (a+b)(c+d)$$

and x, y are arbitrary.

In order to make our solution stay in integers we require $x^2 + wy^2 = \pm 1$

This has infinitely many solutions because it is an instance of Pell's equation, which will then generate infinitely many integer solutions to

$$A^3 + B^3 + C^3 + D^3 = 1999$$