3 + 1 Finite Transform in GH

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1 3 + 1 Splitting

$$^{(4)}g_{\mu\nu} = g_{ab}X^a_\mu X^b_\nu - n_\mu n_\nu$$
$$^{(4)}g^{\mu\nu} = g^{ab}X_a^\mu X_b^\nu - n^\mu n^\nu$$

where

$$X^a_\mu = \delta^a_\mu + \beta^a\delta^0_\mu$$
$$X_a^\mu = \delta_a^\mu$$
$$n_\mu = -\alpha\delta^0_\mu$$
$$n^\mu = (\delta_0^\mu - \beta^a\delta_a^\mu)/\alpha$$

2 Time Dependent Spatial Finite Transform

Under this time dependent spatial finite transformation, we have

$$t' = t$$
$$x^{a'} = x^{a'}(t, x^a)$$

and vice versa, we have

$$t = t'$$
$$x^a = x^a(t', x^{a'})$$

$$\begin{aligned}
g_{\mu'\nu'} &= g_{\mu\nu}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} \\
&= (g_{ab}X^a_\mu X^b_\nu - n_\mu n_\nu)\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} \\
&= g_{ab}X^a_\mu\frac{\partial x^\mu}{\partial x^{\mu'}}X^b_\nu\frac{\partial x^\nu}{\partial x^{\nu'}} - n_\mu\frac{\partial x^\mu}{\partial x^{\mu'}}n_\nu\frac{\partial x^\nu}{\partial x^{\nu'}} \\
&= g_{ab}(\frac{\partial x^a}{\partial x^{\mu'}} + \beta^a\frac{\partial x^0}{\partial x^{\mu'}})(\frac{\partial x^b}{\partial x^{\nu'}} + \beta^b\frac{\partial x^0}{\partial x^{\nu'}}) - \alpha^2\frac{\partial x^0}{\partial x^{\mu'}}\frac{\partial x^0}{\partial x^{\nu'}}
\end{aligned}$$

Useful identity

$$\delta^a_b = \frac{\partial x^a}{\partial x^b} = \frac{\partial x^a}{\partial x^{\mu'}}\frac{\partial x^{\mu'}}{\partial x^b} = \frac{\partial x^a}{\partial t'}\frac{\partial t'}{\partial x^b} + \frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^{a'}}{\partial x^b}$$

since $t' = t$, we have

$$\frac{\partial t'}{\partial x^b} = 0$$

and hence,

$$\boxed{\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^{a'}}{\partial x^b} = \delta^a_b}$$

and vice versa,

$$\boxed{\frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^a}{\partial x^{b'}} = \delta^{a'}_{b'}}$$

2.1 g_{ab}

According to the formula above,

$$\boxed{g_{a'b'} = g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}}$$

Therefore, g_{ab} transforms as a (0,2) tensor.

2.2 g^{ab}

Since

$$g^{ab}g_{ac} = \delta^b_c$$

is a (1,1) tensor, g^{ab} **transforms as a (2,0) tensor.**

2.3 g

$$g' = |g_{a'b'}| = |g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}| = |g_{ab}||\frac{\partial x^a}{\partial x^{a'}}||\frac{\partial x^b}{\partial x^{b'}}| = g|\frac{\partial x}{\partial x'}|^2$$

Hence, g transforms as a weight $+2$ density.

2.4 β_a

$$\begin{aligned} g_{a'0'} &= \beta_{a'} \\ &= g_{ab}\frac{\partial x^a}{\partial x^{a'}}(\frac{\partial x^b}{\partial t'} + \beta^b\frac{\partial t}{\partial t'}) \\ &= \beta_a\frac{\partial x^a}{\partial x^{a'}} + g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial t'} \end{aligned}$$

2.5 β^a

$$\begin{aligned}\beta^{a'} &= \beta_{b'} g^{a'b'} \\ &= (\beta_b \frac{\partial x^b}{\partial x^{b'}} + g_{cb}\frac{\partial x^b}{\partial x^{b'}}\frac{\partial x^c}{\partial t'})g^{ab}\frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^{b'}}{\partial x^b} \\ &= \beta^a \frac{\partial x^{a'}}{\partial x^a} + \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^a}{\partial t'}\end{aligned}$$

2.6 $\Delta\beta^a$

From the section above, we have that

$$\boxed{\Delta\beta^{a'} = \beta^{a'} - \bar{\beta}^{a'} = \Delta\beta^a \frac{\partial x^{a'}}{\partial x^a}}$$

Therefore, $\Delta\beta^a$ transforms as a (1,0) tensor.

2.7 α

$$\begin{aligned}g_{0'0'} &= g_{0'0'} \\ -\alpha'^2 + g_{a'b'}\beta^{a'}\beta^{b'} &= g_{ab}(\frac{\partial x^a}{\partial x^{\mu'}} + \beta^a\frac{\partial x^0}{\partial x^{\mu'}})(\frac{\partial x^b}{\partial x^{\nu'}} + \beta^b\frac{\partial x^0}{\partial x^{\nu'}}) - \alpha^2\frac{\partial x^0}{\partial x^{\mu'}}\frac{\partial x^0}{\partial x^{\nu'}} \\ \alpha' &= \alpha\end{aligned}$$

Therefore, α transforms as a scalar.

2.8 $\partial_c g_{ab}$

$$\begin{aligned}\partial_{c'} g_{a'b'} &= \partial_{c'}(g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}) \\ &= \partial_c g_{ab}\frac{\partial x^c}{\partial x^{c'}}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} + g_{ab}\partial_{c'}(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}})\end{aligned}$$

2.9 $\Gamma^c{}_{ab}$

$$
\begin{aligned}
\Gamma^{c'}{}_{a'b'} &= \frac{1}{2} g^{c'd'} (\partial_{a'} g_{d'b'} + \partial_{b'} g_{d'a'} - \partial_{d'} g_{a'b'}) \\
&= \frac{1}{2} g^{cd} \frac{\partial x^{c'}}{\partial x^c} \frac{\partial x^{d'}}{\partial x^d} (\partial_a g_{db} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^d}{\partial x^{d'}} \frac{\partial x^b}{\partial x^{b'}} + g_{db} \partial_{a'} (\frac{\partial x^d}{\partial x^{d'}} \frac{\partial x^b}{\partial x^{b'}}) \\
&+ \partial_b g_{da} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^d}{\partial x^{d'}} \frac{\partial x^b}{\partial x^{b'}} + g_{da} \partial_{b'} (\frac{\partial x^d}{\partial x^{d'}} \frac{\partial x^a}{\partial x^{a'}}) \\
&- \partial_d g_{ab} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^d}{\partial x^{d'}} \frac{\partial x^b}{\partial x^{b'}} - g_{ab} \partial_{d'} (\frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}})) \\
&= \Gamma^c{}_{ab} \frac{\partial x^{c'}}{\partial x^c} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} + \frac{1}{2} g^{cd} \frac{\partial x^{c'}}{\partial x^c} \frac{\partial x^{d'}}{\partial x^d} (g_{db} \partial_{a'} (\frac{\partial x^d}{\partial x^{d'}} \frac{\partial x^b}{\partial x^{b'}}) + g_{da} \partial_{b'} (\frac{\partial x^d}{\partial x^{d'}} \frac{\partial x^a}{\partial x^{a'}}) - g_{ab} \partial_{d'} (\frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}})) \\
&= \Gamma^c{}_{ab} \frac{\partial x^{c'}}{\partial x^c} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} + \frac{1}{2} g^{cd} \frac{\partial x^{c'}}{\partial x^c} \frac{\partial x^{d'}}{\partial x^d} (g_{db} \frac{\partial x^d}{\partial x^{d'}} \frac{\partial^2 x^b}{\partial x^{b'} \partial x^{a'}} + g_{da} \frac{\partial x^d}{\partial x^{d'}} \frac{\partial^2 x^a}{\partial x^{a'} \partial x^{b'}} \\
&+ g_{db} \frac{\partial x^b}{\partial x^{b'}} \frac{\partial^2 x^d}{\partial x^{d'} \partial x^{a'}} + g_{da} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial^2 x^d}{\partial x^{d'} \partial x^{b'}} - g_{ab} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial^2 x^b}{\partial x^{b'} \partial x^{d'}} - g_{ab} \frac{\partial x^b}{\partial x^{b'}} \frac{\partial^2 x^a}{\partial x^{a'} \partial x^{d'}}) \\
&= \Gamma^c{}_{ab} \frac{\partial x^{c'}}{\partial x^c} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} + \frac{\partial x^{c'}}{\partial x^d} \frac{\partial^2 x^d}{\partial x^{a'} \partial x^{b'}}
\end{aligned}
$$

2.10 $D_a \beta_b$

$$
\begin{aligned}
D_{a'} \beta_{b'} &= \partial_{a'} \beta_{b'} - \Gamma^{c'}{}_{a'b'} \beta_{c'} \\
&= \partial_{a'} (\beta_b \frac{\partial x^b}{\partial x^{b'}} + g_{cb} \frac{\partial x^b}{\partial x^{b'}} \frac{\partial x^c}{\partial t'}) - (\Gamma^c{}_{ab} \frac{\partial x^{c'}}{\partial x^c} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} + \frac{\partial x^{c'}}{\partial x^d} \frac{\partial^2 x^d}{\partial x^{a'} \partial x^{b'}}) (\beta_c \frac{\partial x^c}{\partial x^{c'}} + g_{ce} \frac{\partial x^c}{\partial x^{c'}} \frac{\partial x^e}{\partial t'}) \\
&= D_a \beta_b \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} + D_{a'} (g_{cb} \frac{\partial x^b}{\partial x^{b'}} \frac{\partial x^c}{\partial t'}) \\
&= D_a \beta_b \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} + g_{c'b'} D_{a'} (\frac{\partial x^{c'}}{\partial x^c} \frac{\partial x^c}{\partial t'})
\end{aligned}
$$

2.11 $\dot{g}_{ab}$

$$
\begin{aligned}
\frac{\partial}{\partial t'} g_{a'b'} &= \frac{\partial}{\partial t'} (g_{ab} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}}) \\
&= (\frac{\partial}{\partial t'} g_{ab}) \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} + g_{ab} \frac{\partial}{\partial t'} (\frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}}) \\
&= (\frac{\partial}{\partial t} g_{ab}) \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} + \frac{\partial x^c}{\partial t'} \partial_c g_{ab} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} + g_{ab} \frac{\partial}{\partial t'} (\frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}})
\end{aligned}
$$

2.12 K_{ab}

$$
K_{ab} = -\frac{1}{2\alpha} (\dot{g}_{ab} - 2 D_{(a} \beta_{b)}) = -\frac{1}{2\alpha} (\dot{g}_{ab} - D_a \beta_b - D_b \beta_a)
$$

Therefore, we have

$$
\begin{aligned}
K_{a'b'} &= -\frac{1}{2\alpha'}(\frac{\partial}{\partial t'}g_{a'b'} - D_{a'}\beta_{b'} - D_{b'}\beta_{a'}) \\
&= -\frac{1}{2\alpha}[\dot{g}_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} - D_a\beta_b\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} - D_b\beta_a\frac{\partial x^b}{\partial x^{b'}}\frac{\partial x^a}{\partial x^{a'}} \\
&+ g_{ab}\frac{\partial}{\partial t'}(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}) + \frac{\partial x^c}{\partial t'}\partial_c g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} - g_{c'b'}D_{a'}(\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^c}{\partial t'}) - g_{c'a'}D_{b'}(\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^c}{\partial t'})] \\
&= K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} \\
&- \frac{1}{2\alpha}[g_{ab}\frac{\partial}{\partial t'}(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}) - g_{c'b'}\partial_{a'}(\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^c}{\partial t'}) - g_{c'a'}\partial_{b'}(\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^c}{\partial t'}) \\
&- g_{c'b'}\Gamma^{c'}{}_{a'd'}\frac{\partial x^{d'}}{\partial x^c}\frac{\partial x^c}{\partial t'} - g_{c'a'}\Gamma^{c'}{}_{b'd'}\frac{\partial x^{d'}}{\partial x^c}\frac{\partial x^c}{\partial t'} + \frac{\partial x^c}{\partial t'}\partial_c g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}] \\
&= K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} \\
&+ \frac{1}{2\alpha}(g_{c'b'}\frac{\partial x^c}{\partial t'}\partial_{a'}\frac{\partial x^{c'}}{\partial x^c} + g_{c'a'}\frac{\partial x^c}{\partial t'}\partial_{b'}\frac{\partial x^{c'}}{\partial x^c} + \partial_{d'}g_{a'b'}\frac{\partial x^{d'}}{\partial x^c}\frac{\partial x^c}{\partial t'} - \frac{\partial x^c}{\partial t'}\partial_c g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}) \\
&= K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} \\
&+ \frac{1}{2\alpha}(g_{c'b'}\frac{\partial x^c}{\partial t'}\partial_{a'}\frac{\partial x^{c'}}{\partial x^c} + g_{c'a'}\frac{\partial x^c}{\partial t'}\partial_{b'}\frac{\partial x^{c'}}{\partial x^c} + \partial_c g_{a'b'}\frac{\partial x^c}{\partial t'} - \frac{\partial x^c}{\partial t'}\partial_c g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}) \\
&= K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} \\
&+ \frac{1}{2\alpha}[g_{c'b'}\partial_{a'}\frac{\partial x^{c'}}{\partial x^c} + g_{c'a'}\partial_{b'}\frac{\partial x^{c'}}{\partial x^c} + g_{ab}\partial_c(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}})]\frac{\partial x^c}{\partial t'} \\
&= K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} \\
&+ \frac{1}{2\alpha}[g_{db}\frac{\partial x^b}{\partial x^{b'}}\frac{\partial x^d}{\partial x^{c'}}\partial_{a'}\frac{\partial x^{c'}}{\partial x^c} + g_{da}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^d}{\partial x^{c'}}\partial_{b'}\frac{\partial x^{c'}}{\partial x^c} + g_{ab}\partial_c(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}})]\frac{\partial x^c}{\partial t'} \\
&= K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} \\
&+ \frac{1}{2\alpha}[-g_{db}\frac{\partial x^b}{\partial x^{b'}}\frac{\partial x^{c'}}{\partial x^c}\frac{\partial^2 x^d}{\partial x^{c'}\partial x^{a'}} - g_{da}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^{c'}}{\partial x^c}\frac{\partial^2 x^d}{\partial x^{c'}\partial x^{b'}} + g_{ab}\frac{\partial x^{c'}}{\partial x^c}\partial_{c'}(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}})]\frac{\partial x^c}{\partial t'} \\
&= K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}
\end{aligned}
$$

$$\boxed{K_{a'b'} = K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}}$$

Therefore, K_{ab} transforms as a (0,2) tensor.

2.13 $C_\perp$, C_a

Since

$$C_{\mu'} = C_\mu \frac{\partial x^\mu}{\partial x^{\mu'}}$$

we have

$$\boxed{C_{a'} = C_\mu \frac{\partial x^\mu}{\partial x^{a'}} = C_0 \frac{\partial t}{\partial x^{a'}} + C_a \frac{\partial x^a}{\partial x^{a'}} = C_a \frac{\partial x^a}{\partial x^{a'}}}$$

$$\boxed{C_{\perp'} = C^{\mu'} n_{\mu'} = C^\mu \frac{\partial x^{\mu'}}{\partial x^\mu} n_\mu \frac{\partial x^\mu}{\partial x^{\mu'}} = C^\mu n_\mu = C_\perp}$$

Therefore, C_a transforms as a (0,1) tensor and $C_\perp$ transforms as a scalar.

2.14 P^{ab}

According to the formula

$$P^{ab} = \sqrt{g}(K g^{ab} - K^{ab} - \frac{C_\perp}{2} g^{ab})$$

We can say that P^{ab} **transforms as a weight +1 (2,0) tensor density.**

2.15 π

We have

$$\pi = \frac{\sqrt{g}}{\alpha} C_\perp$$

Therefore, π **transforms as a weight +1 density.**

2.16 ρ_a

We have

$$\rho_a = \frac{\sqrt{g}}{\alpha} C_a$$

Therefore, ρ_a **transforms as a weight +1 (0,1) tensor density.**

2.17 $\dot{\alpha}$

$$\begin{aligned} \partial_{t'} \alpha' &= \partial_{t'} \alpha \\ &= \partial_t \alpha + \frac{\partial x^a}{\partial t'} \partial_a \alpha \end{aligned}$$

2.18 $\partial_a\alpha$

$$\boxed{\partial_{a'}\alpha' = \frac{\partial x^a}{\partial x^{a'}}\partial_a\alpha' = \frac{\partial x^a}{\partial x^{a'}}\partial_a\alpha}$$

Therefore, $\partial_a\alpha$ transforms as a (0,1) tensor.

2.19 $\partial_\perp\alpha$

$$\partial_\perp\alpha = \partial_t\alpha - \beta^a\partial_a\alpha$$

Hence

$$\begin{aligned}
\partial_{\perp'}\alpha' &= \partial_{t'}\alpha' - \beta^{a'}\partial_{a'}\alpha' \\
&= \partial_{t'}\alpha - \beta^{a'}\partial_{a'}\alpha \\
&= \partial_t\alpha + \frac{\partial x^a}{\partial t'}\partial_a\alpha - (\beta^a\frac{\partial x^{a'}}{\partial x^a} + \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^a}{\partial t'})\frac{\partial x^a}{\partial x^{a'}}\partial_a\alpha \\
&= \partial_t\alpha - \beta^a\partial_a\alpha + \frac{\partial x^a}{\partial t'}\partial_c\alpha - \frac{\partial x^a}{\partial t'}\partial_c\alpha \\
&= \partial_\perp\alpha
\end{aligned}$$

Therefore, $\partial_\perp\alpha$ transforms as a scalar.

2.20 $\Delta\dot{\beta}^a$ and extension

$$\begin{aligned}
\frac{\partial}{\partial t'}\Delta\beta^{a'} &= \frac{\partial}{\partial t'}(\Delta\beta^a\frac{\partial x^{a'}}{\partial x^a}) \\
&= (\frac{\partial}{\partial t'}\Delta\beta^a)\frac{\partial x^{a'}}{\partial x^a} + \Delta\beta^a\frac{\partial}{\partial t'}\frac{\partial x^{a'}}{\partial x^a} \\
&= (\frac{\partial}{\partial t}\Delta\beta^a)\frac{\partial x^{a'}}{\partial x^a} + \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^c}{\partial t'}\partial_c\Delta\beta^a + \Delta\beta^{c'}\frac{\partial x^a}{\partial x^{c'}}\frac{\partial}{\partial t'}\frac{\partial x^{a'}}{\partial x^a} \\
&= (\frac{\partial}{\partial t}\Delta\beta^a)\frac{\partial x^{a'}}{\partial x^a} + \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^c}{\partial t'}\partial_c\Delta\beta^a - \Delta\beta^{c'}\frac{\partial x^{a'}}{\partial x^a}\frac{\partial^2 x^a}{\partial t'\partial x^{c'}}
\end{aligned}$$

To construct an invariance form of $\Delta\dot{\beta}^a$, consider a term as

$$\beta^c\bar{D}_c\beta^a$$

It transforms as

$$\begin{aligned}
\beta^{c'}\bar{D}_{c'}\beta^{a'} &= (\beta^c\frac{\partial x^{c'}}{\partial x^c} + \frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^c}{\partial t'})(\bar{D}_c\beta^a\frac{\partial x^c}{\partial x^{c'}}\frac{\partial x^{a'}}{\partial x^a} + \bar{D}_{c'}(\frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^a}{\partial t'})) \\
&= \beta^c\bar{D}_c\beta^a\frac{\partial x^{a'}}{\partial x^a} + \frac{\partial x^c}{\partial t'}\frac{\partial x^{a'}}{\partial x^a}\bar{D}_c\beta^a + \beta^c\frac{\partial x^{c'}}{\partial x^c}\bar{D}_{c'}(\frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^a}{\partial t'}) + \frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^c}{\partial t'}\bar{D}_{c'}(\frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^a}{\partial t'})
\end{aligned}$$

One can notice that there are some extra terms can be cancelled if we arrange the terms correctly.

We claim the invariant form of $\Delta\dot{\beta}^a$ is $\dot{B}^a$, where $\dot{B}^a$

$$\dot{B}^a = \Delta\dot{\beta}^a + c_1\beta^c\bar{D}_c\beta^a + c_2\beta^c\bar{D}_c\bar{\beta}^a + c_3\bar{\beta}^c\bar{D}_c\beta^a + c_4\bar{\beta}^c\bar{D}_c\bar{\beta}^a$$

Hence we have

$$\begin{aligned}
\partial_{t'}B^{a'} &= (\partial_t B^a)\frac{\partial x^{a'}}{\partial x^a} + \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^c}{\partial t'}\partial_c\Delta\beta^a - \Delta\beta^{c'}\frac{\partial x^{a'}}{\partial x^a}\frac{\partial^2 x^a}{\partial t'\partial x^{c'}} \\
&+ (c_1+c_3)\frac{\partial x^c}{\partial t'}\frac{\partial x^{a'}}{\partial x^a}\bar{D}_c\beta^a + (c_2+c_4)\frac{\partial x^c}{\partial t'}\frac{\partial x^{a'}}{\partial x^a}\bar{D}_c\bar{\beta}^a \\
&+ (c_1+c_2)\beta^c\frac{\partial x^{c'}}{\partial x^c}\bar{D}_{c'}(\frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^a}{\partial t'}) + (c_3+c_4))\bar{\beta}^c\frac{\partial x^{c'}}{\partial x^c}\bar{D}_{c'}(\frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^a}{\partial t'}) \\
&+ (c_1+c_2+c_3+c_4)\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^c}{\partial t'}\bar{D}_{c'}(\frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^a}{\partial t'})
\end{aligned}$$

Then we require

$$\begin{aligned}
c_1+c_3 &= -1 \\
c_2+c_4 &= 1 \\
c_1+c_2 &= 1 \\
c_3+c_4 &= -1 \\
c_1+c_2+c_3+c_4 &= 0
\end{aligned}$$

Then we have

$$
\begin{aligned}
\partial_{t'} B^{a'} &= \partial_t B^a \frac{\partial x^{a'}}{\partial x^a} + \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^c}{\partial t'}\partial_c \Delta\beta^a - \Delta\beta^{c'}\frac{\partial x^{a'}}{\partial x^a}\frac{\partial^2 x^a}{\partial t' \partial x^{c'}} \\
&- \frac{\partial x^c}{\partial t'}\frac{\partial x^{a'}}{\partial x^a}\bar{D}_c \Delta\beta^a + \Delta\beta^{c'}\bar{D}_{c'}(\frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^a}{\partial t'}) \\
\\
&= \partial_t B^a \frac{\partial x^{a'}}{\partial x^a} + \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^c}{\partial t'}\partial_c \Delta\beta^a - \Delta\beta^{c'}\frac{\partial x^{a'}}{\partial x^a}\frac{\partial^2 x^a}{\partial t' \partial x^{c'}} \\
&- \frac{\partial x^c}{\partial t'}\frac{\partial x^{a'}}{\partial x^a}\partial_c \Delta\beta^a - \frac{\partial x^c}{\partial t'}\frac{\partial x^{a'}}{\partial x^a}\bar{\Gamma}^a{}_{cb}\Delta\beta^b \\
&+ \Delta\beta^{c'}\frac{\partial x^{a'}}{\partial x^a}\frac{\partial^2 x^a}{\partial x^{c'}\partial t'} + \Delta\beta^{c'}\frac{\partial x^a}{\partial t'}\frac{\partial}{\partial x^{c'}}\frac{\partial x^{a'}}{\partial x^a} + \Delta\beta^{c'}\frac{\partial x^{b'}}{\partial x^a}\frac{\partial x^a}{\partial t'}\bar{\Gamma}^{a'}{}_{c'b'} \\
\\
&= \partial_t B^a \frac{\partial x^{a'}}{\partial x^a} - \frac{\partial x^c}{\partial t'}\frac{\partial x^{a'}}{\partial x^a}\bar{\Gamma}^a{}_{cb}\Delta\beta^b + \Delta\beta^{c'}\frac{\partial x^a}{\partial t'}\frac{\partial}{\partial x^{c'}}\frac{\partial x^{a'}}{\partial x^a} + \Delta\beta^{c'}\frac{\partial x^{b'}}{\partial x^a}\frac{\partial x^a}{\partial t'}\bar{\Gamma}^{a'}{}_{c'b'} \\
\\
&= \partial_t B^a \frac{\partial x^{a'}}{\partial x^a} - \frac{\partial x^c}{\partial t'}\frac{\partial x^{a'}}{\partial x^a}\bar{\Gamma}^a{}_{cb}\Delta\beta^b + \Delta\beta^{c'}\frac{\partial x^a}{\partial t'}\frac{\partial}{\partial x^{c'}}\frac{\partial x^{a'}}{\partial x^a} \\
&+ \Delta\beta^{c'}\frac{\partial x^{b'}}{\partial x^a}\frac{\partial x^a}{\partial t'}(\bar{\Gamma}^d{}_{bc}\frac{\partial x^{a'}}{\partial x^d}\frac{\partial x^c}{\partial x^{c'}}\frac{\partial x^b}{\partial x^{b'}} + \frac{\partial x^{a'}}{\partial x^d}\frac{\partial^2 x^d}{\partial x^{b'}\partial x^{c'}}) \\
\\
&= \partial_t B^a \frac{\partial x^{a'}}{\partial x^a} - \frac{\partial x^c}{\partial t'}\frac{\partial x^{a'}}{\partial x^a}\bar{\Gamma}^a{}_{cb}\Delta\beta^b + \Delta\beta^{c'}\frac{\partial x^a}{\partial t'}\frac{\partial}{\partial x^{c'}}\frac{\partial x^{a'}}{\partial x^a} \\
&+ \Delta\beta^c\frac{\partial x^b}{\partial t'}\frac{\partial x^{a'}}{\partial x^d}\bar{\Gamma}^d{}_{bc} + \frac{\partial x^{a'}}{\partial x^d}\frac{\partial^2 x^d}{\partial x^{b'}\partial x^{c'}}\Delta\beta^{c'}\frac{\partial x^{b'}}{\partial x^a}\frac{\partial x^a}{\partial t'} \\
\\
&= \partial_t B^a \frac{\partial x^{a'}}{\partial x^a} + \Delta\beta^{c'}\frac{\partial x^a}{\partial t'}\frac{\partial}{\partial x^{c'}}\frac{\partial x^{a'}}{\partial x^a} - \Delta\beta^{c'}\frac{\partial x^a}{\partial t'}\frac{\partial x^{b'}}{\partial x^a}\frac{\partial x^d}{\partial x^{b'}}\frac{\partial}{\partial x^{c'}}\frac{\partial x^{a'}}{\partial x^d} \\
\\
&= \partial_t B^a \frac{\partial x^{a'}}{\partial x^a}
\end{aligned}
$$

Let

$$
\begin{aligned}
c_1 &= \sigma \\
c_2 &= 1 - \sigma \\
c_3 &= -(1 + \sigma) \\
c_4 &= \sigma
\end{aligned}
$$

and we have

$$
\dot{B}^a = \Delta\dot{\beta}^a + \sigma\beta^c\bar{D}_c\beta^a + (1-\sigma)\beta^c\bar{D}_c\bar{\beta}^a - (1+\sigma)\bar{\beta}^c\bar{D}_c\beta^a + \sigma\bar{\beta}^c\bar{D}_c\bar{\beta}^a
$$

and

$$\boxed{\partial_{t'} B^{a'} = \partial_t B^a \frac{\partial x^{a'}}{\partial x^a}}$$

$\dot{B}^a$ transforms as a (1,0) tensor.

3 Finite Time Reparameterization

$$\begin{aligned} t' &= t'(t) \\ x^{a'} &= x^a \end{aligned}$$

and vice versa

$$\begin{aligned} t &= t(t') \\ x^a &= x^{a'} \end{aligned}$$

$$\begin{aligned} g_{\mu'\nu'} &= g_{\mu\nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \\ &= (g_{ab} X^a_\mu X^b_\nu - n_\mu n_\nu) \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \\ &= g_{ab} X^a_\mu \frac{\partial x^\mu}{\partial x^{\mu'}} X^b_\nu \frac{\partial x^\nu}{\partial x^{\nu'}} - n_\mu \frac{\partial x^\mu}{\partial x^{\mu'}} n_\nu \frac{\partial x^\nu}{\partial x^{\nu'}} \\ &= g_{ab} (\frac{\partial x^a}{\partial x^{\mu'}} + \beta^a \frac{\partial x^0}{\partial x^{\mu'}})(\frac{\partial x^b}{\partial x^{\nu'}} + \beta^b \frac{\partial x^0}{\partial x^{\nu'}}) - \alpha^2 \frac{\partial x^0}{\partial x^{\mu'}} \frac{\partial x^0}{\partial x^{\nu'}} \end{aligned}$$

3.1 g_{ab}

According to the formula above, we have

$$\boxed{g_{a'b'} = g_{ab}}$$

Therefore, g_{ab} transforms as a scalar.

3.2 g^{ab}

Since

$$g^{ab} g_{bc} = \delta^a_c$$

and

$$\delta^{a'}_{c'} = \delta^a_c \frac{\partial x^{a'}}{\partial x^a} \frac{\partial x^c}{\partial x^{c'}} = \delta^a_c$$

δ^a_c transforms as a scalar, and **hence g^{ab} transforms as a scalar.**

3.3 g

$$\boxed{g' = |g_{a'b'}| = |g_{ab}| = g}$$

Therefore, g transforms as a scalar.

3.4 β_a

$$\boxed{\beta_{a'} = g_{a'0'} = g_{ab}\beta^b \frac{\partial t}{\partial t'} = \beta_a \frac{\partial t}{\partial t'}}$$

Therefore, β_a transforms as a weight $+1$ density.

3.5 β^a

Since

$$\beta^a = \beta_b g^{ab}$$

β^a transforms as a weight $+1$ density.

3.6 α

$$g_{0'0'} = -\alpha'^2 + g_{a'b'}\beta^{a'}\beta^{b'}$$
$$g_{ab}\beta^a\beta^b \frac{\partial t}{\partial t'}\frac{\partial t}{\partial t'} - \alpha^2 \frac{\partial t}{\partial t'}\frac{\partial t}{\partial t'} = -\alpha'^2 + g_{ab}\beta^a\beta^b \frac{\partial t}{\partial t'}\frac{\partial t}{\partial t'}$$
$$\alpha' = \alpha \frac{\partial t}{\partial t'}$$

Therefore, α transforms as a weight $+1$ density.

3.7 $\Gamma^a{}_{bc}$

Since $\Gamma^a{}_{bc}$ is a combination of g_{ab} and $\partial_c g_{ab}$ and g_{ab} transforms as a scalar, **$\Gamma^a{}_{bc}$ transforms as a scalar.**

3.8 C_a, $C_\perp$

Since we have

$$C_{\mu'} = C_\mu \frac{\partial x^\mu}{\partial x^{\mu'}}$$

$$\boxed{C_{a'} = C_\mu \frac{\partial x^\mu}{\partial x^{a'}} = C_0 \frac{\partial t}{\partial x^{a'}} + C_a \frac{\partial x^a}{\partial x^{a'}} = C_a}$$

$$
\begin{aligned}
C_{\perp'} &= C^{\mu'} n_{\mu'} \\
&= -C^{\mu} \frac{\partial x^{\mu'}}{\partial x^{\mu}} \alpha \frac{\partial t}{\partial t'} \delta^{t'}_{\mu'} \\
&= -C^{0} \frac{\partial t'}{\partial t} \alpha \frac{\partial t}{\partial t'} \\
&= -C^{0} \alpha \\
&= -C^{\mu} \alpha \delta^{0}_{\mu} \\
&= C^{\mu} n_{\mu} \\
&= C_{\perp}
\end{aligned}
$$

Therefore, both $C_{\perp}$ and C_a transform as scalars.

3.9 $\dot{g}_{ab}$

$$
\partial_{t'} g_{a'b'} = \partial_{t'} g_{ab} = \frac{\partial x^{\mu}}{\partial t'} \partial_{\mu} g_{ab} = \frac{\partial t}{\partial t'} \partial_t g_{ab}
$$

Therefore, $\partial_t g_{ab}$ transforms as a weight +1 density.

3.10 $D_a \beta_b$

$$
\begin{aligned}
D_{a'} \beta_{b'} &= \partial_{a'} \beta_{b'} - \Gamma^{c'}_{a'b'} \beta_{c'} \\
&= \partial_{a'} (\beta_b \frac{\partial t}{\partial t'}) - \Gamma^{c}_{ab} \beta_c \frac{\partial t}{\partial t'} \\
&= (\partial_{a'} \beta_b) \frac{\partial t}{\partial t'} - \Gamma^{c}_{ab} \beta_c \frac{\partial t}{\partial t'} \\
&= \frac{\partial x^{a}}{\partial x^{a'}} \partial_a \beta_b \frac{\partial t}{\partial t'} + \frac{\partial t}{\partial x^{a'}} \partial_t \beta_b \frac{\partial t}{\partial t'} - \Gamma^{c}_{ab} \beta_c \frac{\partial t}{\partial t'} \\
&= \partial_a \beta_b \frac{\partial t}{\partial t'} - \Gamma^{c}_{ab} \beta_c \frac{\partial t}{\partial t'} \\
&= D_a \beta_b \frac{\partial t}{\partial t'}
\end{aligned}
$$

Therefore, $D_a \beta_b$ transforms as a weight +1 density.

3.11 K_{ab}

Since

$$
K_{ab} = -\frac{1}{2\alpha} (\partial_t g_{ab} - D_a \beta_b - D_b \beta_a)
$$

Therefore, K_{ab} transforms as a scalar.

3.12 P^{ab}

According to the formula of P^{ab}

$$P^{ab} = \sqrt{g}(Kg^{ab} - K^{ab} - \frac{C_\perp}{2} g^{ab})$$

P^{ab} **should transform as a scalar**.

3.13 π

Since we have

$$\pi = \frac{\sqrt{g}}{\alpha} C_\perp$$

π **transforms as a weight -1 density**.

3.14 ρ_a

Since we have

$$\rho_a = \frac{\sqrt{g}}{\alpha} C_a$$

ρ_a **transforms as a weight -1 density**.

3.15 $\dot{\alpha}$

$$\begin{aligned}
\frac{\partial}{\partial t'}\alpha' &= \frac{\partial}{\partial t'}(\alpha \frac{\partial t}{\partial t'}) \\
&= (\frac{\partial}{\partial t'}\alpha)\frac{\partial t}{\partial t'} + \alpha \frac{\partial^2 t}{\partial t'^2} \\
&= (\frac{\partial}{\partial t}\alpha)\frac{\partial t}{\partial t'}^2 + \alpha \frac{\partial^2 t}{\partial t'^2}
\end{aligned}$$

3.16 $\mathring{\alpha}$

The covariant derivative of α in time domain is defined as

$$\mathring{\alpha} = \dot{\alpha} - \frac{\alpha}{\bar{\alpha}}\dot{\bar{\alpha}}$$

Hence we have

$$\begin{aligned}
\partial_{t'}\alpha' - \frac{\alpha'}{\bar{\alpha}'}\partial_{t'}\bar{\alpha}' &= (\frac{\partial}{\partial t}\alpha)\frac{\partial t}{\partial t'}^2 + \alpha\frac{\partial^2 t}{\partial t'^2} - \frac{\alpha}{\bar{\alpha}}[(\frac{\partial}{\partial t}\bar{\alpha})\frac{\partial t}{\partial t'}^2 + \bar{\alpha}\frac{\partial^2 t}{\partial t'^2}] \\
&= (\partial_t \alpha - \frac{\alpha}{\bar{\alpha}}\partial_t \alpha)\frac{\partial t}{\partial t'}^2
\end{aligned}$$

$$\mathring{\alpha}' = \mathring{\alpha}\frac{\partial t}{\partial t'}^2$$

Therefore, $\mathring{\alpha}$ **transforms as a weight +2 density.**

3.17 $\mathring{\beta}^a$

Define

$$\mathring{\beta}^a = \dot{\beta}^a - \frac{\beta^a}{\bar{\alpha}}\dot{\bar{\alpha}}$$

Following the same logic as previous section, we claim that $\mathring{\beta}^a$ **transforms as a weight +2 density.**

4 Conclusion

According to the discussion above, we need to require $\dot{\alpha}$ appears in the following form

$$\boxed{(\dot{\alpha} - \beta^c \partial_c \alpha) - \frac{\alpha}{\bar{\alpha}}(\dot{\bar{\alpha}} - \bar{\beta}^c \partial_c \bar{\alpha})}$$

and $\dot{\beta}^a$ should appear in the following form

$$\boxed{\Delta\dot{\beta}^a - \frac{\Delta\beta^a}{\bar{\alpha}}(\dot{\bar{\alpha}} - \bar{\beta}^c \partial_c \bar{\alpha}) + \sigma\beta^c \bar{D}_c \beta^a + (1-\sigma)\beta^c \bar{D}_c \bar{\beta}^a - (1+\sigma)\bar{\beta}^c \bar{D}_c \beta^a + \sigma\bar{\beta}^c \bar{D}_c \bar{\beta}^a}$$