Hamiltonian in GH

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1 Action

In 4D spacetime,

$$S = \int \sqrt{-^{(4)}g}(^{(4)}R - \frac{1}{2}C^\mu C_\mu)d^4x$$

Splitting in 3+1

$$\sqrt{-^{(4)}g} = \alpha\sqrt{g}$$

$$^{(4)}R = R + K^{ab}K_{ab} - K^2 + boundary\ \ term$$

$$\begin{aligned} C^\mu C_\mu &= C_\mu\ {}^{(4)}g^{\mu\nu}C_\nu \\ &= C_\mu C_\nu(g^{ab}X^\mu_a X^\nu_b - n^\mu n^\nu) \\ &= C^a C_a - C_\perp^2 \end{aligned}$$

Therefore, the action in 3+1 splitting is

$$S[g_{ab}, \alpha, \beta^a] = \int \alpha\sqrt{g}(R + K^{ab}K_{ab} - K^2 - \frac{1}{2}C^a C_a + \frac{1}{2}C_\perp^2)d^4x = \int \mathscr{L} d^4x$$

where

$$K_{ab} \equiv -\frac{1}{2\alpha}(\partial_\perp g_{ab})$$

$$\partial_\perp \equiv \partial_t - \mathcal{L}_\beta$$

$$C_\perp \equiv H_\perp + K + \frac{1}{\alpha^2}\partial_\perp\alpha$$

$$C_a \equiv H_a + \Delta\Gamma_{abc}g^{bc} - \frac{1}{\alpha}\partial_a\alpha - \frac{1}{\alpha^2}g_{ab}(\partial_t\beta^b - \beta^c\bar{D}_c\beta^b)$$

2 Momenta

To find the conjugate momenta, we have

$$P^{ab} = \frac{\partial \mathscr{L}}{\partial \dot{g}_{ab}}$$

$$\pi = \frac{\partial \mathscr{L}}{\partial \dot{\alpha}}$$

$$\rho_a = \frac{\partial \mathscr{L}}{\partial \dot{\beta}^a}$$

For the partial derivatives, we have

$$\frac{\partial K_{ab}}{\partial \dot{g}_{ab}} = -\frac{1}{2\alpha}$$

$$\frac{\partial K_{ab}}{\partial \dot{\alpha}} = \frac{\partial K_{ab}}{\partial \dot{\beta}^a} = 0$$

$$\frac{\partial C_\perp}{\partial \dot{g}_{ab}} = -\frac{1}{2\alpha} g^{ab}$$

$$\frac{\partial C_\perp}{\partial \dot{\alpha}} = \frac{1}{\alpha^2}$$

$$\frac{\partial C_\perp}{\partial \dot{\beta}^a} = 0$$

$$\frac{\partial C_a}{\partial \dot{g}_{ab}} = \frac{\partial C_a}{\partial \dot{\alpha}} = 0$$

$$\frac{\partial C_a}{\partial \dot{\beta}^b} = -\frac{1}{\alpha^2} g_{ab}$$

Therefore, we have

$$\boxed{P^{ab} = \sqrt{g}(K g^{ab} - K^{ab} - \frac{C_\perp}{2} g^{ab})}$$

$$\boxed{\pi = \frac{\sqrt{g}}{\alpha} C_\perp}$$

$$\boxed{\rho_a = \frac{\sqrt{g}}{\alpha} C_a}$$

As you can see here, P^{ab} is a weight 1 tensor density, π is a weight 1 scalar density and ρ_a is a weight 1 covector density.

3 Hamiltonian

For the Hamiltonian, we have

$$H = \int (P^{ab}\dot{g}_{ab} + \pi\dot{\alpha} + \rho_a\dot{\beta}^a - \mathscr{L})d^3x$$

We have to write $\mathcal{H}$ in terms of $g_{ab}, P^{ab}, \alpha, \pi, \beta^a, \rho_a$, so we need to do some inversions as following

$$\boxed{C_\perp = \frac{\alpha}{\sqrt{g}}\pi}$$

$$\boxed{C_a = \frac{\alpha}{\sqrt{g}}\rho_a}$$

Contract P^{ab}, we have

$$P = \sqrt{g}(2K - \frac{3}{2}C_\perp) = \sqrt{g}(2K - \frac{3\alpha}{2\sqrt{g}}\pi)$$

Therefore, we have

$$\boxed{K = \frac{P}{2\sqrt{g}} + \frac{3\alpha}{4\sqrt{g}}\pi}$$

$$\boxed{K^{ab} = -\frac{P^{ab}}{\sqrt{g}} + \frac{Pg^{ab}}{2\sqrt{g}} + \frac{\alpha\pi g^{ab}}{4\sqrt{g}}}$$

Then use the definition of K^{ab} to rewrite $\dot{g}_{ab}$

$$\boxed{\dot{g}_{ab} = \mathcal{L}_\beta g_{ab} + \frac{2\alpha}{\sqrt{g}}P_{ab} - \frac{\alpha P}{\sqrt{g}}g_{ab} - \frac{\alpha^2\pi}{2\sqrt{g}}g_{ab}}$$

Use the definition of $C_\perp$ to rewrite $\dot{\alpha}$, we have

$$\boxed{\dot{\alpha} = \mathcal{L}_\beta\alpha + \frac{\alpha^3\pi}{4\sqrt{g}} - H_\perp\alpha^2 - \frac{\alpha^2 P}{2\sqrt{g}}}$$

Use the definition of C_a to rewrite $\dot{\beta}_a$, we have

$$\boxed{\dot{\beta}^a = \alpha^2 H^a + \alpha^2\Delta\Gamma^a{}_{bc}g^{bc} - \alpha g^{ab}\partial_b\alpha + \beta^b\bar{D}_b\beta^a - \frac{\alpha^3}{\sqrt{g}}\rho^a}$$

Plug in, we have the formula of Hamiltonian,

$$
\begin{aligned}
H &= \int \mathcal{H} d^3x \\
&\equiv \int (P^{ab}\dot{g}_{ab} + \pi\dot{\alpha} + \rho_a \dot{\beta}^a - \mathscr{L}) d^3x \\
&= \int [\frac{\alpha}{\sqrt{g}}(P^{ab}P_{ab} - \frac{1}{2}P^2 - \frac{1}{2}\alpha\pi P + \frac{1}{8}\alpha^2\pi^2 - \frac{1}{2}\alpha^2\rho^a\rho_a) \\
&- \alpha^2\pi H_\perp + \pi\mathcal{L}_\beta\alpha + \alpha^2 H^a\rho_a + \alpha^2\rho_a\Delta\Gamma^a{}_{bc}g^{bc} - \alpha\rho^a\partial_a\alpha + \rho_a\beta^b\bar{D}_b\beta^a - \alpha\sqrt{g}R - 2\beta^a D_b P^b_a] d^3x
\end{aligned}
$$

4 Hamilton's Equations

So we have the following Hamilton's Equations

$$
\begin{aligned}
\dot{P}^{ab} &= -\frac{\delta\mathcal{H}}{\delta g_{ab}} \\
&= \mathcal{L}_\beta P^{ab} + \frac{\alpha}{\sqrt{g}}(\frac{1}{2}P^{cd}P_{cd}g^{ab} - \frac{1}{4}P^2 g^{ab} - 2P^{ac}P^{bd}g_{cd} + PP^{ab} \\
&- \frac{1}{4}\alpha\pi P g^{ab} + \frac{1}{16}\alpha^2\pi^2 g^{ab} - \frac{1}{4}\alpha^2\rho^c\rho_c g^{ab} + \frac{1}{2}\alpha\pi P^{ab} + \frac{1}{2}\alpha^2\rho^a\rho^b) \\
&+ \alpha^2\rho_e\Delta\Gamma^e{}_{cd}g^{ac}g^{bd} - \frac{1}{2}D_c(\rho^c\alpha^2)g^{ab} + D^{(a}(\rho^{b)}\alpha^2) - \frac{1}{2}\rho^{(a}D^{b)}\alpha^2 \\
&- \alpha\sqrt{g}G^{ab} + \sqrt{g}D^aD^b\alpha - \sqrt{g}g^{ab}D_cD^c\alpha
\end{aligned}
$$

$$
\begin{aligned}
\dot{\pi} &= -\frac{\delta\mathcal{H}}{\delta\alpha} \\
&= \mathcal{L}_\beta\pi + \frac{1}{\sqrt{g}}(-P^{ab}P_{ab} + \frac{P^2}{2} + \alpha\pi P - \frac{3}{8}\alpha^2\pi^2 + \frac{3}{2}\alpha^2\rho^a\rho_a) + 2\alpha\pi H_\perp \\
&- 2\alpha H^a\rho_a - 2\alpha\rho_a\Delta\Gamma^a{}_{bc}g^{bc} - \alpha(D_a\rho^a) + \sqrt{g}R
\end{aligned}
$$

$$
\begin{aligned}
\dot{\rho}_a &= -\frac{\delta\mathcal{H}}{\delta\beta^a} \\
&= -\pi D_a\alpha + 2(D_b P^{cb})g_{ac} + \bar{D}_b(\rho_a\beta^b) - \rho_b(\bar{D}_a\beta^b)
\end{aligned}
$$

5 Extended Hamiltonian

Introduce new variables

$$
\begin{aligned}
\Lambda &\equiv -\alpha^2 H_\perp \\
\Omega^a &\equiv \alpha^2 H^a
\end{aligned}
$$

Rewrite the Hamiltonian in the following form

$$H = \int [\alpha \mathscr{H} + \beta^a \mathscr{M}_a + \pi(\Lambda + \hat{\Lambda}) + \rho_a(\Omega^a + \hat{\Omega}^a)] d^3x$$

where

$$\mathscr{H} = \frac{1}{\sqrt{g}}(P^{ab}P_{ab} - \frac{P^2}{2}) - \sqrt{g}R$$

$$\mathscr{M}_a = -2D_b P_a^b$$

$$\hat{\Lambda} = \mathcal{L}_\beta \alpha - \frac{\alpha^2}{2\sqrt{g}}P + \frac{\alpha^3}{8\sqrt{g}}\pi$$

$$\hat{\Omega}^a = \alpha^2 \Delta\Gamma^a{}_{bc} g^{bc} - \alpha g^{ab}\partial_b \alpha + \beta^b \bar{D}_b \beta^a - \frac{\alpha^3}{2\sqrt{g}}\rho^a$$

Introduce the congugate momenta of Λ and Ω^a

$$\mathscr{P} = \frac{\partial \mathscr{L}}{\partial \dot{\Lambda}} = 0$$

$$\mathscr{P}_a = \frac{\partial \mathscr{L}}{\partial \dot{\Omega}^a} = 0$$

Now add new constraint terms into Hamiltonian

$$H = \int [\alpha \mathscr{H} + \beta^a \mathscr{M}_a + \pi(\Lambda + \hat{\Lambda}) + \rho_a(\Omega^a + \hat{\Omega}^a) + \mathscr{P}(\lambda + \hat{\lambda}) + \mathscr{P}_a(\omega^a + \hat{\omega}^a)] d^3x$$

Therefore, the new set of Hamilton's equations are

$$\begin{aligned}\dot{g}_{ab} &= \mathcal{L}_\beta g_{ab} + \frac{2\alpha}{\sqrt{g}}P_{ab} - \frac{\alpha P}{\sqrt{g}}g_{ab} - \frac{\alpha^2\pi}{2\sqrt{g}}g_{ab} \\ &+ \frac{\partial \hat{\lambda}}{\partial P^{ab}}\mathscr{P} + \frac{\partial \hat{\omega}^c}{\partial P^{ab}}\mathscr{P}_c\end{aligned}$$

$$\begin{aligned}\dot{P}^{ab} &= -\frac{\delta \mathcal{H}}{\delta g_{ab}} \\ &= \mathcal{L}_\beta P^{ab} + \frac{\alpha}{\sqrt{g}}(\frac{1}{2}P^{cd}P_{cd}g^{ab} - \frac{1}{4}P^2 g^{ab} - 2P^{ac}P^{bd}g_{cd} + PP^{ab} \\ &- \frac{1}{4}\alpha\pi P g^{ab} + \frac{1}{16}\alpha^2\pi^2 g^{ab} - \frac{1}{4}\alpha^2\rho^c\rho_c g^{ab} + \frac{1}{2}\alpha\pi P^{ab} + \frac{1}{2}\alpha^2\rho^a\rho^b) \\ &+ \alpha^2\rho_e\Delta\Gamma^e{}_{cd}g^{ac}g^{bd} - \frac{1}{2}D_c(\rho^c\alpha^2)g^{ab} + D^{(a}(\rho^{b)}\alpha^2) - \frac{1}{2}\rho^{(a}D^{b)}\alpha^2 \\ &- \alpha\sqrt{g}G^{ab} + \sqrt{g}D^aD^b\alpha - \sqrt{g}g^{ab}D_cD^c\alpha \\ &- \frac{\partial \hat{\lambda}}{\partial g_{ab}}\mathscr{P} - \frac{\partial \hat{\omega}^c}{\partial g_{ab}}\mathscr{P}_c \\ &+ \partial_d[\frac{\partial \hat{\lambda}}{\partial(\partial_d g_{ab})}\mathscr{P}] + \partial_d[\frac{\partial \hat{\omega}^c}{\partial(\partial_d g_{ab})}\mathscr{P}_c]\end{aligned}$$

$$
\begin{aligned}
\dot{\alpha} = {} & \mathcal{L}_\beta \alpha + \frac{\alpha^3 \pi}{4\sqrt{g}} - H_\perp \alpha^2 - \frac{\alpha^2 P}{2\sqrt{g}} \\
& + \frac{\partial \hat{\lambda}}{\partial \pi} \mathscr{P} + \frac{\partial \hat{\omega}^c}{\partial \pi} \mathscr{P}_c
\end{aligned}
$$

$$
\begin{aligned}
\dot{\pi} &= -\frac{\delta \mathcal{H}}{\delta \alpha} \\
&= \mathcal{L}_\beta \pi + \frac{1}{\sqrt{g}}(-P^{ab} P_{ab} + \frac{P^2}{2} + \alpha \pi P - \frac{3}{8} \alpha^2 \pi^2 + \frac{3}{2} \alpha^2 \rho^a \rho_a) + 2\alpha \pi H_\perp \\
&- 2\alpha H^a \rho_a - 2\alpha \rho_a \Delta \Gamma^a{}_{bc} g^{bc} - \alpha (D_a \rho^a) + \sqrt{g} R \\
&- \frac{\partial \hat{\lambda}}{\partial \alpha} \mathscr{P} - \frac{\partial \hat{\omega}^c}{\partial \alpha} \mathscr{P}_c \\
&+ \partial_d [\frac{\partial \hat{\lambda}}{\partial (\partial_d \alpha)} \mathscr{P}] + \partial_d [\frac{\partial \hat{\omega}^c}{\partial (\partial_d \alpha)} \mathscr{P}_c]
\end{aligned}
$$

$$
\begin{aligned}
\dot{\beta}^a = {} & \alpha^2 H^a + \alpha^2 \Delta \Gamma^a{}_{bc} g^{bc} - \alpha g^{ab} \partial_b \alpha + \beta^b \bar{D}_b \beta^a - \frac{\alpha^3}{\sqrt{g}} \rho^a \\
& + \frac{\partial \hat{\lambda}}{\partial \rho_a} \mathscr{P} + \frac{\partial \hat{\omega}^c}{\partial \rho_a} \mathscr{P}_c
\end{aligned}
$$

$$
\begin{aligned}
\dot{\rho}_a &= -\frac{\delta \mathcal{H}}{\delta \beta^a} \\
&= -\pi D_a \alpha + 2(D_b P^{cb}) g_{ac} + \bar{D}_b(\rho_a \beta^b) - \rho_b (\bar{D}_a \beta^b) \\
&- \frac{\partial \hat{\lambda}}{\partial \beta^a} \mathscr{P} - \frac{\partial \hat{\omega}^c}{\partial \beta^a} \mathscr{P}_c \\
&+ \partial_d [\frac{\partial \hat{\lambda}}{\partial (\partial_d \beta^a)} \mathscr{P}] + \partial_d [\frac{\partial \hat{\omega}^c}{\partial (\partial_d \beta^a)} \mathscr{P}_c]
\end{aligned}
$$

$$
\dot{\Lambda} = \lambda + \hat{\lambda} + \frac{\partial \hat{\lambda}}{\partial \mathscr{P}} \mathscr{P} + \frac{\partial \hat{\omega}^c}{\partial \mathscr{P}} \mathscr{P}_c
$$

$$
\begin{aligned}
\frac{d\mathscr{P}}{dt} = {} & -\pi - \frac{\partial \hat{\lambda}}{\partial \Lambda} \mathscr{P} - \frac{\partial \hat{\omega}^c}{\partial \Lambda} \mathscr{P}_c \\
& + \partial_d [\frac{\partial \hat{\lambda}}{\partial (\partial_d \Lambda)} \mathscr{P}] + \partial_d [\frac{\partial \hat{\omega}^c}{\partial (\partial_d \Lambda)} \mathscr{P}_c]
\end{aligned}
$$

$$
\dot{\Omega}^a = \omega^a + \hat{\omega}^a + \frac{\partial \hat{\lambda}}{\partial \mathscr{P}_a} \mathscr{P} + \frac{\partial \hat{\omega}^c}{\partial \mathscr{P}_a} \mathscr{P}_c
$$

$$\frac{d\mathscr{P}_a}{dt} = -\rho_a - \frac{\partial\hat{\lambda}}{\partial\Omega^a}\mathscr{P} - \frac{\partial\hat{\omega}^c}{\partial\Omega^a}\mathscr{P}_c + \partial_d[\frac{\partial\hat{\lambda}}{\partial(\partial_d\Omega^a)}\mathscr{P}] + \partial_d[\frac{\partial\hat{\omega}^c}{\partial(\partial_d\Omega^a)}\mathscr{P}_c]$$

$$\mathscr{P} = 0$$

$$\mathscr{P}_a = 0$$

6 Appendix

$$C_\perp = H_\perp + K + \frac{\partial_\perp\alpha}{\alpha^2} - \frac{\alpha}{\bar{\alpha}}\bar{K}_{ab}g^{ab} + \frac{\bar{K}_{ab}}{\alpha\bar{\alpha}}\Delta\beta^a\Delta\beta^b - \frac{1}{\alpha\bar{\alpha}}(\partial_t\bar{\alpha} + \bar{\beta}^a\partial_a\bar{\alpha} - 2\beta^a\partial_a\bar{\alpha})$$

$$C_i = H_i + \Delta\Gamma^a{}_{cd}g^{cd}g_{ai} - \frac{\partial_i\alpha}{\alpha} - \frac{g_{ai}}{\alpha^2}(\partial_t\beta^a - \beta^b\bar{D}_b\beta^a) + \frac{\bar{\alpha}}{\alpha^2}g_{ai}\bar{g}^{ab}\partial_b\bar{\alpha} + \frac{g_{ai}}{\alpha^2}(\partial_t\bar{\beta}^a + \bar{\beta}^b\bar{D}_b\bar{\beta}^a - 2\beta^b\bar{D}_b\bar{\beta}^a) + \frac{g_{ai}\Delta\beta^a}{\alpha^2\bar{\alpha}}(\partial_t\bar{\alpha} + \bar{\beta}^b\partial_b\bar{\alpha} - 2\beta^b\partial_b\bar{\alpha} + \alpha^2g^{cd}\bar{K}_{cd}) - \frac{g_{ai}\bar{K}_{bc}\Delta\beta^b}{\alpha^2\bar{\alpha}}(\Delta\beta^a\Delta\beta^c - 2\bar{\alpha}^2\bar{g}^{ca})$$

Rearrange the terms so that it obeys the coordinate transform invariance, we have

$$C_\perp = H_\perp + K + \frac{1}{\alpha^2}(\partial_t\alpha - \beta^a\partial_a\alpha - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\alpha) - \frac{\alpha}{\bar{\alpha}}\bar{K}_{ab}g^{ab} + \frac{\bar{K}_{ab}}{\alpha\bar{\alpha}}\Delta\beta^a\Delta\beta^b + \frac{2}{\alpha\bar{\alpha}}\Delta\beta^a\partial_a\bar{\alpha} + \frac{1}{\alpha\bar{\alpha}}\bar{\beta}^a\partial_a\bar{\alpha}$$

$$C_i = H_i + \Delta\Gamma^a_{cd}g^{cd}g_{ai} - \frac{\partial_i\alpha}{\alpha} + \frac{\bar{\alpha}}{\alpha^2}g_{ai}\bar{g}^{ab}\partial_b\bar{\alpha} + \frac{g_{ai}\Delta\beta^a}{\alpha^2\bar{\alpha}}(\alpha^2g^{cd}\bar{K}_{cd} - 2\Delta\beta^b\partial_b\bar{\alpha} - \bar{\beta}^b\partial_b\bar{\alpha}) - \frac{g_{ai}\bar{K}_{bc}\Delta\beta^b}{\alpha^2\bar{\alpha}}(\Delta\beta^a\Delta\beta^c - 2\bar{\alpha}^2\bar{g}^{ac}) - \frac{g_{ai}}{\alpha^2}(\partial_t\beta^a - \beta^b\bar{D}_b\beta^a - \beta^a\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}) + \frac{g_{ai}}{\alpha^2}(\partial_t\bar{\beta}^a - \bar{\beta}^a\frac{\dot{\bar{\alpha}}}{\bar{\alpha}} + \bar{\beta}^b\bar{D}_b\bar{\beta}^a - 2\beta^b\bar{D}_b\bar{\beta}^a)$$

Rearrange the terms so that it obeys the coordinate transform invariance, we have

$$C_\perp = H_\perp + K + \frac{1}{\alpha^2}[\dot{\alpha} - \beta^a\partial_a\alpha - \frac{\alpha}{\bar{\alpha}}(\dot{\bar{\alpha}} - \bar{\beta}^a\partial_a\bar{\alpha})] - \frac{\alpha}{\bar{\alpha}}\bar{K}_{ab}g^{ab} + \frac{\bar{K}_{ab}}{\alpha\bar{\alpha}}\Delta\beta^a\Delta\beta^b + \frac{2}{\alpha\bar{\alpha}}\Delta\beta^a\partial_a\bar{\alpha}$$

$$C_i = H_i + \Delta\Gamma^a_{cd}g^{cd}g_{ai} - \frac{\partial_i\alpha}{\alpha} + \frac{\bar{\alpha}}{\alpha^2}g_{ai}\bar{g}^{ab}\partial_b\bar{\alpha} + \frac{g_{ai}\Delta\beta^a}{\alpha^2\bar{\alpha}}(\alpha^2g^{cd}\bar{K}_{cd} - 2\Delta\beta^b\partial_b\bar{\alpha}) - \frac{g_{ai}\bar{K}_{bc}\Delta\beta^b}{\alpha^2\bar{\alpha}}(\Delta\beta^a\Delta\beta^c - 2\bar{\alpha}^2\bar{g}^{ac}) - \frac{g_{ai}}{\alpha^2}[\dot{\beta}^a - \beta^b\bar{D}_b\beta^a - \frac{\beta^a}{\bar{\alpha}}(\dot{\bar{\alpha}} - \bar{\beta}^b\partial_b\bar{\alpha})] + \frac{g_{ai}}{\alpha^2}[\dot{\bar{\beta}}^a - \frac{\bar{\beta}^a}{\bar{\alpha}}(\dot{\bar{\alpha}} - \bar{\beta}^b\partial_b\bar{\alpha}) + \bar{\beta}^b\bar{D}_b\bar{\beta}^a - 2\beta^b\bar{D}_b\bar{\beta}^a]$$