3 + 1 Finite Uniform Transform in GH

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1 3 + 1 Splitting

$$^{(4)}g_{\mu\nu} = g_{ab}X^a_\mu X^b_\nu - n_\mu n_\nu$$
$$^{(4)}g^{\mu\nu} = g^{ab}X^\mu_a X^\nu_b - n^\mu n^\nu$$

where

$$X^a_\mu = \delta^a_\mu + \beta^a \delta^0_\mu$$
$$X^\mu_a = \delta^\mu_a$$
$$n_\mu = -\alpha \delta^0_\mu$$
$$n^\mu = (\delta^\mu_0 - \beta^a \delta^\mu_a)/\alpha$$

2 Uniform Transform

Under this transformation, we have

$$t' = t'(t)$$
$$x^{a'} = x^{a'}(t, x^a)$$

and vice versa, we have

$$t = t(t')$$
$$x^a = x^a(t', x^{a'})$$

$$\begin{aligned}
g_{\mu'\nu'} &= g_{\mu\nu}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} \\
&= (g_{ab}X^a_\mu X^b_\nu - n_\mu n_\nu)\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}} \\
&= g_{ab}X^a_\mu \frac{\partial x^\mu}{\partial x^{\mu'}} X^b_\nu \frac{\partial x^\nu}{\partial x^{\nu'}} - n_\mu \frac{\partial x^\mu}{\partial x^{\mu'}} n_\nu \frac{\partial x^\nu}{\partial x^{\nu'}} \\
&= g_{ab}(\frac{\partial x^a}{\partial x^{\mu'}} + \beta^a \frac{\partial x^0}{\partial x^{\mu'}})(\frac{\partial x^b}{\partial x^{\nu'}} + \beta^b \frac{\partial x^0}{\partial x^{\nu'}}) - \alpha^2 \frac{\partial x^0}{\partial x^{\mu'}}\frac{\partial x^0}{\partial x^{\nu'}}
\end{aligned}$$

3 g_{ab}

According to the formula above,

$$\boxed{g_{a'b'} = g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}}$$

Therefore, g_{ab} transforms as a (0,2) tensor.

4 g^{ab}

Since

$$g^{ab}g_{ac} = \delta^b_c$$

is a (1,1) tensor, g^{ab} **transforms as a (2,0) tensor.**

$$\boxed{g^{a'b'} = g^{ab}\frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^{b'}}{\partial x^b}}$$

5 g

$$\boxed{g' = |g_{a'b'}| = |g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}| = |g_{ab}||\frac{\partial x^a}{\partial x^{a'}}||\frac{\partial x^b}{\partial x^{b'}}| = g|\frac{\partial x}{\partial x'}|^2}$$

6 β_a

$$\begin{aligned}
g_{a'0'} &= \beta_{a'} \\
&= g_{ab}\frac{\partial x^a}{\partial x^{a'}}(\frac{\partial x^b}{\partial t'} + \beta^b\frac{\partial t}{\partial t'}) \\
&= \beta_a\frac{\partial x^a}{\partial x^{a'}}\frac{\partial t}{\partial t'} + g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial t'}
\end{aligned}$$

$$\boxed{\beta_{a'} = \beta_a\frac{\partial x^a}{\partial x^{a'}}\frac{\partial t}{\partial t'} + g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial t'}}$$

7 β^a

$$\begin{aligned}
\beta^{a'} &= \beta_{b'}g^{a'b'} \\
&= (\beta_b\frac{\partial x^b}{\partial x^{b'}}\frac{\partial t}{\partial t'} + g_{cb}\frac{\partial x^b}{\partial x^{b'}}\frac{\partial x^c}{\partial t'})g^{ab}\frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^{b'}}{\partial x^b} \\
&= \beta^a\frac{\partial x^{a'}}{\partial x^a}\frac{\partial t}{\partial t'} + \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^a}{\partial t'}
\end{aligned}$$

$$\boxed{\beta^{a'} = \beta^a \frac{\partial x^{a'}}{\partial x^a} \frac{\partial t}{\partial t'} + \frac{\partial x^{a'}}{\partial x^a} \frac{\partial x^a}{\partial t'}}$$

8 $\Delta\beta^a$

From the section above, we have that

$$\boxed{\Delta\beta^{a'} = \beta^{a'} - \bar{\beta}^{a'} = \Delta\beta^a \frac{\partial x^{a'}}{\partial x^a} \frac{\partial t}{\partial t'}}$$

9 α

$$\begin{aligned}
g_{0'0'} &= g_{0'0'} \\
-\alpha'^2 + g_{a'b'}\beta^{a'}\beta^{b'} &= g_{ab}(\frac{\partial x^a}{\partial t'} + \beta^a \frac{\partial t}{\partial t'})(\frac{\partial x^b}{\partial t'} + \beta^b \frac{\partial t}{\partial t'}) - \alpha^2 \frac{\partial t}{\partial t'} \frac{\partial t}{\partial t'} \\
\alpha' &= \alpha \frac{\partial t}{\partial t'}
\end{aligned}$$

$$\boxed{\alpha' = \alpha \frac{\partial t}{\partial t'}}$$

10 $\partial_c g_{ab}$

$$\begin{aligned}
\partial_{c'} g_{a'b'} &= \partial_{c'}(g_{ab} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}}) \\
&= \partial_c g_{ab} \frac{\partial x^c}{\partial x^{c'}} \frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}} + g_{ab} \partial_{c'}(\frac{\partial x^a}{\partial x^{a'}} \frac{\partial x^b}{\partial x^{b'}})
\end{aligned}$$

11 $\Gamma^c{}_{ab}$

$$
\begin{aligned}
\Gamma^{c'}{}_{a'b'} &= \frac{1}{2}g^{c'd'}(\partial_{a'}g_{d'b'} + \partial_{b'}g_{d'a'} - \partial_{d'}g_{a'b'}) \\
&= \frac{1}{2}g^{cd}\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^{d'}}{\partial x^d}(\partial_a g_{db}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^d}{\partial x^{d'}}\frac{\partial x^b}{\partial x^{b'}} + g_{db}\partial_{a'}(\frac{\partial x^d}{\partial x^{d'}}\frac{\partial x^b}{\partial x^{b'}}) \\
&+ \partial_b g_{da}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^d}{\partial x^{d'}}\frac{\partial x^b}{\partial x^{b'}} + g_{da}\partial_{b'}(\frac{\partial x^d}{\partial x^{d'}}\frac{\partial x^a}{\partial x^{a'}}) \\
&- \partial_d g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^d}{\partial x^{d'}}\frac{\partial x^b}{\partial x^{b'}} - g_{ab}\partial_{d'}(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}})) \\
&= \Gamma^c{}_{ab}\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} + \frac{1}{2}g^{cd}\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^{d'}}{\partial x^d}(g_{db}\partial_{a'}(\frac{\partial x^d}{\partial x^{d'}}\frac{\partial x^b}{\partial x^{b'}}) + g_{da}\partial_{b'}(\frac{\partial x^d}{\partial x^{d'}}\frac{\partial x^a}{\partial x^{a'}}) - g_{ab}\partial_{d'}(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}})) \\
&= \Gamma^c{}_{ab}\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} + \frac{1}{2}g^{cd}\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^{d'}}{\partial x^d}(g_{db}\frac{\partial x^d}{\partial x^{d'}}\frac{\partial^2 x^b}{\partial x^{b'}\partial x^{a'}} + g_{da}\frac{\partial x^d}{\partial x^{d'}}\frac{\partial^2 x^a}{\partial x^{a'}\partial x^{b'}} \\
&+ g_{db}\frac{\partial x^b}{\partial x^{b'}}\frac{\partial^2 x^d}{\partial x^{d'}\partial x^{a'}} + g_{da}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial^2 x^d}{\partial x^{d'}\partial x^{b'}} - g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial^2 x^b}{\partial x^{b'}\partial x^{d'}} - g_{ab}\frac{\partial x^b}{\partial x^{b'}}\frac{\partial^2 x^a}{\partial x^{a'}\partial x^{d'}}) \\
&= \Gamma^c{}_{ab}\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} + \frac{\partial x^{c'}}{\partial x^d}\frac{\partial^2 x^d}{\partial x^{a'}\partial x^{b'}}
\end{aligned}
$$

12 $D_a\beta_b$

$$
\begin{aligned}
D_{a'}\beta_{b'} &= \partial_{a'}\beta_{b'} - \Gamma^{c'}{}_{a'b'}\beta_{c'} \\
&= \partial_{a'}(\beta_b\frac{\partial x^b}{\partial x^{b'}}\frac{\partial t}{\partial t'} + g_{cb}\frac{\partial x^b}{\partial x^{b'}}\frac{\partial x^c}{\partial t'}) - (\Gamma^c{}_{ab}\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} + \frac{\partial x^{c'}}{\partial x^d}\frac{\partial^2 x^d}{\partial x^{a'}\partial x^{b'}})(\beta_c\frac{\partial x^c}{\partial x^{c'}}\frac{\partial t}{\partial t'} + g_{ce}\frac{\partial x^c}{\partial x^{c'}}\frac{\partial x^e}{\partial t'}) \\
&= D_a\beta_b\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}\frac{\partial t}{\partial t'} + D_{a'}(g_{cb}\frac{\partial x^b}{\partial x^{b'}}\frac{\partial x^c}{\partial t'}) \\
&= D_a\beta_b\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}\frac{\partial t}{\partial t'} + g_{c'b'}D_{a'}(\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^c}{\partial t'})
\end{aligned}
$$

13 $\dot{g}_{ab}$

$$
\begin{aligned}
\frac{\partial}{\partial t'}g_{a'b'} &= \frac{\partial}{\partial t'}(g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}) \\
&= (\frac{\partial}{\partial t'}g_{ab})\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} + g_{ab}\frac{\partial}{\partial t'}(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}) \\
&= \partial_t g_{ab}\frac{\partial t}{\partial t'}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} + \frac{\partial x^c}{\partial t'}\partial_c g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} + g_{ab}\frac{\partial}{\partial t'}(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}})
\end{aligned}
$$

14 K_{ab}

$$
K_{ab} = -\frac{1}{2\alpha}(\dot{g}_{ab} - 2D_{(a}\beta_{b)}) = -\frac{1}{2\alpha}(\dot{g}_{ab} - D_a\beta_b - D_b\beta_a)
$$

Therefore, we have

$$
\begin{aligned}
K_{a'b'} &= -\frac{1}{2\alpha'}(\frac{\partial}{\partial t'}g_{a'b'} - D_{a'}\beta_{b'} - D_{b'}\beta_{a'}) \\
&= -\frac{1}{2\alpha'}[\dot{g}_{ab}\frac{\partial t}{\partial t'}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} - D_a\beta_b\frac{\partial t}{\partial t'}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} - D_b\beta_a\frac{\partial t}{\partial t'}\frac{\partial x^b}{\partial x^{b'}}\frac{\partial x^a}{\partial x^{a'}} \\
&+ g_{ab}\frac{\partial}{\partial t'}(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}) + \frac{\partial x^c}{\partial t'}\partial_c g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} - g_{c'b'}D_{a'}(\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^c}{\partial t'}) - g_{c'a'}D_{b'}(\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^c}{\partial t'})] \\
&= K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} \\
&- \frac{1}{2\alpha'}[g_{ab}\frac{\partial}{\partial t'}(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}) - g_{c'b'}\partial_{a'}(\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^c}{\partial t'}) - g_{c'a'}\partial_{b'}(\frac{\partial x^{c'}}{\partial x^c}\frac{\partial x^c}{\partial t'}) \\
&- g_{c'b'}\Gamma^{c'}{}_{a'd'}\frac{\partial x^{d'}}{\partial x^c}\frac{\partial x^c}{\partial t'} - g_{c'a'}\Gamma^{c'}{}_{b'd'}\frac{\partial x^{d'}}{\partial x^c}\frac{\partial x^c}{\partial t'} + \frac{\partial x^c}{\partial t'}\partial_c g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}] \\
&= K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} \\
&+ \frac{1}{2\alpha'}(g_{c'b'}\frac{\partial x^c}{\partial t'}\partial_{a'}\frac{\partial x^{c'}}{\partial x^c} + g_{c'a'}\frac{\partial x^c}{\partial t'}\partial_{b'}\frac{\partial x^{c'}}{\partial x^c} + \partial_{d'}g_{a'b'}\frac{\partial x^{d'}}{\partial x^c}\frac{\partial x^c}{\partial t'} - \frac{\partial x^c}{\partial t'}\partial_c g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}) \\
&= K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} \\
&+ \frac{1}{2\alpha'}(g_{c'b'}\frac{\partial x^c}{\partial t'}\partial_{a'}\frac{\partial x^{c'}}{\partial x^c} + g_{c'a'}\frac{\partial x^c}{\partial t'}\partial_{b'}\frac{\partial x^{c'}}{\partial x^c} + \partial_c g_{a'b'}\frac{\partial x^c}{\partial t'} - \frac{\partial x^c}{\partial t'}\partial_c g_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}) \\
&= K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} \\
&+ \frac{1}{2\alpha'}[g_{c'b'}\partial_{a'}\frac{\partial x^{c'}}{\partial x^c} + g_{c'a'}\partial_{b'}\frac{\partial x^{c'}}{\partial x^c} + g_{ab}\partial_c(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}})]\frac{\partial x^c}{\partial t'} \\
&= K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} \\
&+ \frac{1}{2\alpha'}[g_{db}\frac{\partial x^b}{\partial x^{b'}}\frac{\partial x^d}{\partial x^{c'}}\partial_{a'}\frac{\partial x^{c'}}{\partial x^c} + g_{da}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^d}{\partial x^{c'}}\partial_{b'}\frac{\partial x^{c'}}{\partial x^c} + g_{ab}\partial_c(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}})]\frac{\partial x^c}{\partial t'} \\
&= K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}} \\
&+ \frac{1}{2\alpha'}[-g_{db}\frac{\partial x^b}{\partial x^{b'}}\frac{\partial x^{c'}}{\partial x^c}\frac{\partial^2 x^d}{\partial x^{c'}\partial x^{a'}} - g_{da}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^{c'}}{\partial x^c}\frac{\partial^2 x^d}{\partial x^{c'}\partial x^{b'}} + g_{ab}\frac{\partial x^{c'}}{\partial x^c}\partial_{c'}(\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}})]\frac{\partial x^c}{\partial t'} \\
&= K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}
\end{aligned}
$$

$$\boxed{K_{a'b'} = K_{ab}\frac{\partial x^a}{\partial x^{a'}}\frac{\partial x^b}{\partial x^{b'}}}$$

Therefore, K_{ab} transforms as a (0,2) tensor.

15 $C_\perp$, C_a

Since

$$C_{\mu'} = C_\mu \frac{\partial x^\mu}{\partial x^{\mu'}}$$

we have

$$\boxed{C_{a'} = C_\mu \frac{\partial x^\mu}{\partial x^{a'}} = C_0 \frac{\partial t}{\partial x^{a'}} + C_a \frac{\partial x^a}{\partial x^{a'}} = C_a \frac{\partial x^a}{\partial x^{a'}}}$$

$$\boxed{C_{\perp'} = C^{\mu'} n_{\mu'} = C^\mu \frac{\partial x^{\mu'}}{\partial x^\mu} n_\mu \frac{\partial x^\mu}{\partial x^{\mu'}} = C^\mu n_\mu = C_\perp}$$

Therefore, C_a transforms as a (0,1) tensor and $C_\perp$ transforms as a scalar.

16 P^{ab}

According to the formula

$$P^{ab} = \sqrt{g}(K g^{ab} - K^{ab} - \frac{C_\perp}{2} g^{ab})$$

$$\boxed{P^{a'b'} = P^{ab} \frac{\partial x^{a'}}{\partial x^a} \frac{\partial x^{b'}}{\partial x^b} |\frac{\partial x}{\partial x'}|}$$

17 π

We have

$$\pi = \frac{\sqrt{g}}{\alpha} C_\perp$$

$$\boxed{\pi' = \pi |\frac{\partial x}{\partial x'}| \frac{\partial t'}{\partial t}}$$

18 ρ_a

We have

$$\rho_a = \frac{\sqrt{g}}{\alpha} C_a$$

$$\boxed{\rho_{a'} = \rho_a \frac{\partial x^a}{\partial x^{a'}} |\frac{\partial x}{\partial x'}| \frac{\partial t'}{\partial t}}$$

19 $\partial_a \alpha$

$$\partial_{a'}\alpha' = \frac{\partial x^a}{\partial x^{a'}}\partial_a(\alpha\frac{\partial t}{\partial t'}) = \frac{\partial t}{\partial t'}\frac{\partial x^a}{\partial x^{a'}}\partial_a\alpha$$

20 $\dot{\alpha}$

$$\begin{aligned}
\partial_{t'}\alpha' &= \partial_{t'}(\alpha\frac{\partial t}{\partial t'}) \\
&= (\partial_{t'}\alpha)\frac{\partial t}{\partial t'} + \alpha\frac{\partial^2 t}{\partial t'^2} \\
&= (\frac{\partial t}{\partial t'})^2\partial_t\alpha + \frac{\partial x^a}{\partial t'}\frac{\partial t}{\partial t'}\partial_a\alpha + \alpha\frac{\partial^2 t}{\partial t'^2}
\end{aligned}$$

21 $\partial_\perp \alpha$

$$\partial_\perp\alpha = \dot{\alpha} - \beta^a\partial_a\alpha$$

Therefore, we have

$$\begin{aligned}
\partial_{\perp'}\alpha' &= \partial_{t'}\alpha' - \beta^{a'}\partial_{a'}\alpha' \\
&= (\frac{\partial t}{\partial t'})^2\partial_t\alpha + \frac{\partial x^a}{\partial t'}\frac{\partial t}{\partial t'}\partial_a\alpha + \alpha\frac{\partial^2 t}{\partial t'^2} - (\beta^a\frac{\partial x^{a'}}{\partial x^a}\frac{\partial t}{\partial t'} + \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^a}{\partial t'})\frac{\partial t}{\partial t'}\frac{\partial x^a}{\partial x^{a'}}\partial_a\alpha \\
&= (\frac{\partial t}{\partial t'})^2\partial_t\alpha - (\frac{\partial t}{\partial t'})\beta^a\partial_a\alpha + \alpha\frac{\partial^2 t}{\partial t'^2} \\
&= (\frac{\partial t}{\partial t'})^2\partial_\perp\alpha + \alpha\frac{\partial^2 t}{\partial t'^2}
\end{aligned}$$

22 $\mathring{\alpha}$

Construct the invariant form of $\dot{\alpha}$ as

$$\mathring{\alpha} = (\dot{\alpha} - \beta^a\partial_a\alpha) - \frac{\alpha}{\bar{\alpha}}(\dot{\bar{\alpha}} - \bar{\beta}^a\partial_a\bar{\alpha})$$

One can check that

$$\begin{aligned}
\mathring{\alpha}' &= (\partial_{t'}\alpha' - \beta^{a'}\partial_{a'}\alpha') - \frac{\alpha'}{\bar{\alpha}'}(\partial_{t'}\bar{\alpha}' - \bar{\beta}^{a'}\partial_{a'}\bar{\alpha}') \\
&= (\frac{\partial t}{\partial t'})^2(\dot{\alpha} - \beta^a\partial_a\alpha) + \alpha\frac{\partial^2 t}{\partial t'^2} - \frac{\alpha}{\bar{\alpha}}(\frac{\partial t}{\partial t'})^2(\dot{\bar{\alpha}} - \bar{\beta}^a\partial_a\bar{\alpha}) - \alpha\frac{\partial^2 t}{\partial t'^2} \\
&= (\frac{\partial t}{\partial t'})^2\mathring{\alpha}
\end{aligned}$$

$$\boxed{\mathring{\alpha}' = \mathring{\alpha}(\frac{\partial t}{\partial t'})^2}$$

23 $\dot{\beta}^a$

$$\partial_{t'}\beta^{a'} = \partial_{t'}(\beta^a \frac{\partial x^{a'}}{\partial x^a}\frac{\partial t}{\partial t'} + \frac{\partial x^{a'}}{\partial x^a}\frac{\partial x^a}{\partial t'})$$

24 $\dot{B}^a$

One can show that

$$\dot{B}^a = \Delta\dot{\beta}^a + \sigma\beta^c\bar{D}_c\beta^a + (1-\sigma)\beta^c\bar{D}_c\bar{\beta}^a - (1+\sigma)\bar{\beta}^c\bar{D}_c\beta^a + \sigma\bar{\beta}^c\bar{D}_c\bar{\beta}^a$$

transforms as

$$\partial_{t'}B^{a'} = \partial_t B^a \frac{\partial x^{a'}}{\partial x^a}(\frac{\partial t}{\partial t'})^2 + \Delta\beta^a \frac{\partial x^{a'}}{\partial x^a}\frac{\partial^2 t}{\partial t'^2}$$

25 $\mathring{\beta}^a$

Construct the invariant form of $\dot{\beta}^a$ as

$$\mathring{\beta}^a = \dot{B}^a - \frac{\Delta\beta^a}{\bar{\alpha}}(\dot{\bar{\alpha}} - \bar{\beta}^a\partial_a\bar{\alpha})$$

One can show that

$$\begin{aligned}
\mathring{\beta}^{a'} &= \partial_{t'}B^{a'} - \frac{\Delta\beta^{a'}}{\bar{\alpha}'}(\partial_{t'}\bar{\alpha}' - \bar{\beta}^{a'}\partial_{a'}\bar{\alpha}') \\
&= \dot{B}^a \frac{\partial x^{a'}}{\partial x^a}(\frac{\partial t}{\partial t'})^2 + \Delta\beta^a\frac{\partial x^{a'}}{\partial x^a}\frac{\partial^2 t}{\partial t'^2} - \frac{\Delta\beta^a}{\bar{\alpha}}(\dot{\bar{\alpha}} - \bar{\beta}^a\partial_a\bar{\alpha})\frac{\partial x^{a'}}{\partial x^a}(\frac{\partial t}{\partial t'})^2 - \frac{\Delta\beta^a}{\bar{\alpha}}\frac{\partial x^{a'}}{\partial x^a}\bar{\alpha}\frac{\partial^2 t}{\partial t'^2} \\
&= \mathring{\beta}^a\frac{\partial x^{a'}}{\partial x^a}(\frac{\partial t}{\partial t'})^2
\end{aligned}$$

$$\boxed{\mathring{\beta}^{a'} = \mathring{\beta}^a\frac{\partial x^{a'}}{\partial x^a}(\frac{\partial t}{\partial t'})^2}$$