Time Reparametrization Invariance in GH

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1 3+1 Splitting

$$
\begin{aligned}
{}^{(4)}g_{\mu\nu} &= g_{ab}X^a_\mu X^b_\nu - n_\mu n_\nu \\
{}^{(4)}g^{\mu\nu} &= g^{ab}X^\mu_a X^\nu_b - n^\mu n^\nu
\end{aligned}
$$

where

$$
\begin{aligned}
X^a_\mu &= \delta^a_\mu + \beta^a \delta^0_\mu \\
X^\mu_a &= \delta^\mu_a \\
n_\mu &= -\alpha \delta^0_\mu \\
n^\mu &= (\delta^\mu_0 - \beta^a \delta^\mu_a)/\alpha
\end{aligned}
$$

Time reparametrization vector

$$
\xi^\mu = (\epsilon(t), 0)
$$

$$
\begin{aligned}
\delta g_{\mu\nu} &= \mathcal{L}_\xi g_{\mu\nu} \\
&= \xi^\sigma \partial_\sigma g_{\mu\nu} + g_{\mu\sigma}\partial_\nu \xi^\sigma + g_{\nu\sigma}\partial_\mu \xi^\sigma \\
&= \epsilon \dot{g}_{\mu\nu} + g_{\mu 0}\delta^0_\nu \dot{\epsilon} + g_{\nu 0}\delta^0_\mu \dot{\epsilon}
\end{aligned}
$$

$$
\begin{aligned}
\delta g^{\mu\nu} &= \mathcal{L}_\xi g^{\mu\nu} \\
&= \xi^\sigma \partial_\sigma g^{\mu\nu} - g^{\mu\sigma}\partial_\sigma \xi^\nu + g^{\nu\sigma}\partial_\sigma \xi^\mu \\
&= \epsilon \dot{g}^{\mu\nu} - g^{\mu 0}\delta^\nu_0 \dot{\epsilon} - g^{\nu 0}\delta^\mu_0 \dot{\epsilon}
\end{aligned}
$$

2 g_{ab}

g_{ab} transforms as a scalar

$$
\delta g_{ab} = \epsilon \dot{g}_{ab}
$$

3 β^a

$$g_{a0} = g_{ab}\beta^b$$

Hence, we have

$$\begin{aligned}
\delta g_{a0} &= \delta g_{a0} \\
\delta(g_{ab}\beta^b) &= \epsilon(g_{ab}\beta^b)^{\cdot} + g_{a0}\dot{\epsilon} \\
(\delta g_{ab})\beta^b + g_{ab}(\delta\beta^b) &= \epsilon(g_{ab}\beta^b)^{\cdot} + g_{ab}\beta^b\dot{\epsilon} \\
\epsilon\dot{g}_{ab}\beta^b + g_{ab}(\delta\beta^b) &= \epsilon\dot{g}_{ab}\beta^b + \epsilon g_{ab}\dot{\beta}^b + g_{ab}\beta^b\dot{\epsilon} \\
\delta\beta^b &= \epsilon\dot{\beta}^b + \dot{\epsilon}\beta^b
\end{aligned}$$

$$\boxed{\delta\beta^b = (\epsilon\beta^b)^{\cdot}}$$

β^a transforms as a weight +1 density

4 α

$$g_{00} = g_{ab}\beta^a\beta^b - \alpha^2$$

Hence, we have

$$\begin{aligned}
\delta g_{00} &= \delta g_{00} \\
\delta(g_{ab}\beta^a\beta^b - \alpha^2) &= \epsilon(g_{ab}\beta^a\beta^b - \alpha^2)^{\cdot} + 2(g_{ab}\beta^a\beta^b - \alpha^2)\dot{\epsilon} \\
\delta g_{ab}\beta^a\beta^b + g_{ab}\delta\beta^a\beta^b + g_{ab}\beta^a\delta\beta^b - 2\alpha\delta\alpha &= \epsilon\dot{g}_{ab}\beta^a\beta^b + \epsilon g_{ab}\dot{\beta}^a\beta^b + \epsilon g_{ab}\beta^a\dot{\beta}^b - 2\epsilon\alpha\dot{\alpha} + 2(g_{ab}\beta^a\beta^b - \alpha^2)\dot{\epsilon} \\
\epsilon\dot{g}_{ab}\beta^a\beta^b + g_{ab}(\epsilon\beta^a)^{\cdot}\beta^b + g_{ab}\beta^a(\epsilon\beta^b)^{\cdot} - 2\alpha\delta\alpha &= \epsilon\dot{g}_{ab}\beta^a\beta^b + g_{ab}(\beta^a\epsilon)^{\cdot}\beta^b + g_{ab}\beta^a(\epsilon\beta^b)^{\cdot} - 2\epsilon\alpha\dot{\alpha} - 2\alpha^2\dot{\epsilon} \\
\delta\alpha &= \epsilon\dot{\alpha} + \alpha\dot{\epsilon}
\end{aligned}$$

$$\boxed{\delta\alpha = (\epsilon\alpha)^{\cdot}}$$

α transforms as a weight +1 density

5 g^{ab}

$$\begin{aligned}
\delta(g^{ab} - \beta^a\beta^b/\alpha^2) &= \delta(g^{ab} - \beta^a\beta^b/\alpha^2) \\
\delta g^{ab} - \delta(\beta^a\beta^b/\alpha^2) &= \epsilon(g^{ab} - \beta^a\beta^b/\alpha^2)^{\cdot}
\end{aligned}$$

Due to the transform property of β^a and α, $\beta^a\beta^b/\alpha^2$ should transform as a scalar. Therefore, we have

$$\delta g^{ab} - \epsilon(\beta^a\beta^b/\alpha^2)^{\cdot} = \epsilon\dot{g}^{ab} - \epsilon(\beta^a\beta^b/\alpha^2)^{\cdot}$$

$$\boxed{\delta g^{ab} = \epsilon \dot{g}^{ab}}$$

g^{ab} transforms as a scalar

? short cut
$g^{ab} g_{bc} = \delta^a_c$ should transform as a scalar, and g_{bc} transforms as a scalar, so g^{ab} transforms as a scalar.

6 $\Gamma^a{}_{bc}$

$\Gamma^a{}_{bc}$ is a combination of g^{ab} and $\partial_c g_{ab}$, so **it should also transform as a scalar**.

7 $\dot{g}_{ab}$,$\dot{g}^{ab}$

$$\boxed{\delta \dot{g}_{ab} = (\delta g_{ab})^{\cdot} = (\epsilon \dot{g}_{ab})^{\cdot}}$$

$$\boxed{\delta \dot{g}^{ab} = (\delta g^{ab})^{\cdot} = (\epsilon \dot{g}^{ab})^{\cdot}}$$

Both $\dot{g}_{ab}$ and $\dot{g}^{ab}$ transform as weight +1 densities.

8 $\sqrt{g}$

$$\boxed{\delta g = g g^{ab} \delta g_{ab} = g g^{ab} \epsilon \dot{g}_{ab} = \epsilon \dot{g}}$$

Therefore, both g and $\sqrt{g}$ transform as a scalar.

9 R

Since R is all about g_{ab}, g^{ab} and $\Gamma^a{}_{bc}$, **R should transform as a scalar**.

10 $D_a \beta_b$

$$\begin{aligned} \delta(D_a \beta_b) &= \delta(\partial_a \beta_b - \Gamma^c{}_{ab} \beta_c) \\ &= \partial_a(\delta \beta_b) - \delta \Gamma^c{}_{ab} \beta_c - \Gamma^c{}_{ab} \delta \beta_c \end{aligned}$$

Since $\beta_a = \beta^b g_{ab}$ and g_{ab} transforms as a scalar, β^b transforms as a weight +1 density, β_a transforms as a weight +1 density. Hence, we have

$$\begin{aligned}\delta(D_a\beta_b) &= \partial_a(\epsilon\beta_b)^{\cdot} - \epsilon\dot{\Gamma}^c{}_{ab}\beta_c - \Gamma^c_{ab}(\epsilon\beta_c)^{\cdot} \\ &= (\partial_a(\epsilon\beta_b) - \epsilon\Gamma^c{}_{ab}\beta_c)^{\cdot} \\ &= (\epsilon(\partial_a\beta_b - \Gamma^c_{ab}\beta_c))^{\cdot} \\ &= (\epsilon D_a\beta_b)^{\cdot}\end{aligned}$$

$$\boxed{\delta(D_a\beta_b) = (\epsilon D_a\beta_b)^{\cdot}}$$

$D_a\beta_b$ **transforms as a weight +1 density**.

11 K_{ab}

$$K_{ab} = -\frac{1}{2\alpha}(\dot{g}_{ab} - \mathcal{L}_\beta g_{ab}) = -\frac{1}{2\alpha}(\dot{g}_{ab} - D_a\beta_b - D_b\beta_a)$$

Therefore, K_{ab} **transforms as a scalar**.

12 $C_\perp$, C_a

Since C_μ is a covector in spacetime, we have

$$\begin{aligned}\delta C_\mu &= \mathcal{L}_\xi C_\mu \\ &= \xi^\sigma\partial_\sigma C_\mu + C_\sigma\partial_\mu\xi^\sigma \\ &= \epsilon\dot{C}_\mu + C_0\dot{\epsilon}\delta^0_\mu\end{aligned}$$

Therefore, we can see that the spatial component of C_μ transform as a scalar, the time component of C_μ transforms as a weight +1 density.

Hence, we have

$$\boxed{\delta C_a = \delta(C_\mu X^\mu_a) = \delta^\mu_a\delta C_\mu = \epsilon\dot{C}_a}$$

C_a **transforms as a scalar**.

$$n^\mu = (\delta^\mu_0 - \beta^a\delta^\mu_a)/\alpha$$

We can tell from the above equation that the time component of n^μ transforms as a weight -1 density and the spatial component of n^μ transform as scalars. Therefore, $C_\perp = C_\mu n^\mu$ **should transform as a scalar**.

13 P^{ab}

According to the formula of P^{ab}

$$P^{ab} = \sqrt{g}(Kg^{ab} - K^{ab} - \frac{C_\perp}{2}g^{ab})$$

P^{ab} **should transform as a scalar**.

14 π

Since we have

$$\pi = \frac{\sqrt{g}}{\alpha} C_{\perp}$$

π **transforms as a weight -1 density**.

15 ρ_a

Since we have

$$\rho_a = \frac{\sqrt{g}}{\alpha} C_a$$

ρ_a **transforms as a weight -1 density**.

16 $\mathring{\alpha}$

The covariant derivative of α in time domain is defined as

$$\mathring{\alpha} = \dot{\alpha} - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}} \alpha$$

So we have

$$\begin{aligned}
\delta\mathring{\alpha} &= \delta\dot{\alpha} - \delta(\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\alpha) \\
&= \delta\dot{\alpha} - \frac{\alpha}{\bar{\alpha}}\delta\dot{\bar{\alpha}} - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\delta\alpha + \frac{\dot{\bar{\alpha}}}{\bar{\alpha}^2}\alpha\delta\bar{\alpha} \\
&= (\delta\alpha)^{\cdot} - \frac{\alpha}{\bar{\alpha}}(\delta\bar{\alpha})^{\cdot} - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\delta\alpha + \frac{\dot{\bar{\alpha}}}{\bar{\alpha}^2}\alpha\delta\bar{\alpha} \\
&= (\epsilon\alpha)^{\cdot\cdot} - \frac{\alpha}{\bar{\alpha}}(\epsilon\bar{\alpha})^{\cdot\cdot} - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}(\epsilon\alpha)^{\cdot} + \frac{\dot{\bar{\alpha}}}{\bar{\alpha}^2}\alpha(\epsilon\bar{\alpha})^{\cdot} \\
&= \ddot{\epsilon}\alpha + 2\dot{\epsilon}\dot{\alpha} + \epsilon\ddot{\alpha} - \frac{\alpha}{\bar{\alpha}}\ddot{\epsilon}\bar{\alpha} - 2\frac{\alpha}{\bar{\alpha}}\dot{\epsilon}\dot{\bar{\alpha}} - \frac{\alpha}{\bar{\alpha}}\epsilon\ddot{\bar{\alpha}} - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\dot{\epsilon}\alpha - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\epsilon\dot{\alpha} + \frac{\dot{\bar{\alpha}}}{\bar{\alpha}^2}\alpha\dot{\epsilon}\bar{\alpha} + \frac{\dot{\bar{\alpha}}}{\bar{\alpha}^2}\alpha\epsilon\dot{\bar{\alpha}} \\
&= \epsilon(\ddot{\alpha} - \frac{\ddot{\bar{\alpha}}}{\bar{\alpha}}\alpha - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\dot{\alpha} + \frac{\dot{\bar{\alpha}}}{\bar{\alpha}^2}\alpha\dot{\bar{\alpha}}) + 2\dot{\epsilon}(\dot{\alpha} - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\alpha) \\
&= \epsilon(\mathring{\alpha})^{\cdot} + 2\dot{\epsilon}\mathring{\alpha}
\end{aligned}$$

Therefore we claim that $\mathring{\alpha}$ transforms as a weight +2 density

17 $\mathring{\beta}^a$

Following the same steps as previous section, we can show that $\mathring{\beta}^a$ **also transforms as a weight +2 density**.