3 + 1 Diffeomorphism in GH

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1 3+1 Splitting

$$
\begin{aligned}
{}^{(4)}g_{\mu\nu} &= g_{ab} X^a_\mu X^b_\nu - n_\mu n_\nu \\
{}^{(4)}g^{\mu\nu} &= g^{ab} X^\mu_a X^\nu_b - n^\mu n^\nu
\end{aligned}
$$

where

$$
\begin{aligned}
X^a_\mu &= \delta^a_\mu + \beta^a \delta^0_\mu \\
X^\mu_a &= \delta^\mu_a \\
n_\mu &= -\alpha \delta^0_\mu \\
n^\mu &= (\delta^\mu_0 - \beta^a \delta^\mu_a)/\alpha
\end{aligned}
$$

Time dependent spatial diffeomorphism vector

$$\xi^\mu = (0, \xi^a(t, x))$$

Time reparametrization vector

$$\xi^\mu = (\epsilon(t), 0)$$

2 Time Dependent Spatial Diffeomorphism

$$
\begin{aligned}
\delta g_{\mu\nu} &= \mathcal{L}_\xi g_{\mu\nu} \\
&= \xi^\sigma \partial_\sigma g_{\mu\nu} + g_{\mu\sigma} \partial_\nu \xi^\sigma + g_{\nu\sigma} \partial_\mu \xi^\sigma \\
&= \xi^c \partial_c g_{\mu\nu} + g_{\mu c} \partial_\nu \xi^c + g_{\nu c} \partial_\mu \xi^c
\end{aligned}
$$

2.1 g_{ab}

$$\boxed{\delta g_{ab} = \xi^c \partial_c g_{ab} + g_{ac} \partial_b \xi^c + g_{bc} \partial_a \xi^c = \mathcal{L}_\xi g_{ab}}$$

Therefore, g_{ab} transforms as a (0,2) tensor.

2.2 g^{ab}

Since

$$\begin{aligned}\delta(g_{bc}g^{ab}) &= \delta\delta^a_c = 0 \\ &= \xi^d\partial_d\delta^a_c + \delta^a_d\partial_c\xi^d - \delta^d_c\partial_d\xi^a \\ &= \partial_c\xi^a - \partial_c\xi^a \\ &= 0\end{aligned}$$

We claim that δ^a_c transforms as a (1,1) tensor, and **hence g^{ab} transforms as a (2,0) tensor.**

$$\boxed{\delta g^{ab} = \mathcal{L}_\xi g^{ab}}$$

2.3 g

g is the determinent of g_{ab} and we have that

$$\begin{aligned}\delta g &= gg^{ab}\delta g_{ab} = gg^{ab}(\xi^c\partial_c g_{ab} + g_{ac}\partial_b\xi^c + g_{bc}\partial_a\xi^c) \\ &= gg^{ab}\xi^c\partial_c g_{ab} + g\delta^b_c\partial_b\xi^c + g\delta^a_c\partial_a\xi^c \\ &= \xi^c\partial_c g + 2g\partial_c\xi^c\end{aligned}$$

Therefore, g transforms as a weight +2 scalar density.

$$\boxed{\delta g = \mathcal{L}_\xi g}$$

2.4 β^a

We have

$$g_{a0} = g_{ab}\beta^b$$

Hence

$$\begin{aligned}\delta g_{a0} &= \delta g_{a0} \\ \xi^c\partial_c g_{a0} + g_{ac}\partial_0\xi^c + g_{0c}\partial_a\xi^c &= g_{ab}\delta\beta^b + \beta^b\delta g_{ab} \\ \xi^c\beta^b\partial_c g_{ab} + \xi^c g_{ab}\partial_c\beta^b + g_{ac}\dot{\xi}^c + g_{cb}\beta^b\partial_a\xi^c &= g_{ab}\delta\beta^b + \beta^b\xi^c\partial_c g_{ab} + \beta^b g_{ac}\partial_b\xi^c + \beta^b g_{bc}\partial_a\xi^c \\ g_{ab}\delta\beta^b &= g_{ab}\xi^c\partial_c\beta^b - \beta^b g_{ac}\partial_b\xi^c + g_{ac}\dot{\xi}^c \\ \delta\beta^a &= \xi^b\partial_b\beta^a - \beta^b\partial_b\xi^a + \dot{\xi}^a\end{aligned}$$

$$\boxed{\delta\beta^a = \mathcal{L}_\xi\beta^a + \dot{\xi}^a}$$

2.5 $\Delta\beta^a$

If we define

$$\Delta\beta^a \equiv \beta^a - \bar{\beta}^a$$

we can see that

$$\boxed{\delta\Delta\beta^a = \mathcal{L}_\xi \beta^a + \dot{\xi}^a - \mathcal{L}_\xi \bar{\beta}^a - \dot{\xi}^a = \mathcal{L}_\xi \Delta\beta^a}$$

$\Delta\beta^a$ transforms as a (1,0) tensor.

2.6 α

We have

$$g_{00} = -\alpha^2 + \beta^a \beta^b g_{ab}$$

Hence

$$\begin{aligned}
\delta g_{00} &= \delta g_{00} \\
\xi^c \partial_c g_{00} + 2 g_{0c} \partial_0 \xi^c &= -2\alpha\delta\alpha + \beta^a\beta^b \delta g_{ab} + 2 g_{ab}\beta^b \delta\beta^a \\
-2\xi^c \alpha \partial_c \alpha + \xi^c \beta^a \beta^b \partial_c g_{ab} + 2\xi^c g_{ab}\beta^b \partial_c \beta^a + 2 g_{ac}\beta^a \dot{\xi}^c &= -2\alpha\delta\alpha + \beta^a\beta^b \mathcal{L}_\xi g_{ab} + 2 g_{ab}\beta^b (\mathcal{L}_\xi \beta^a + \dot{\xi}^a)
\end{aligned}$$

Simplify this, we have

$$\boxed{\delta\alpha = \xi^c \partial_c \alpha = \mathcal{L}_\xi \alpha}$$

Therefore, α transforms as a scalar.

2.7 $\partial_c g_{ab}$

$$\begin{aligned}
\delta\partial_c g_{ab} &= \partial_c \delta g_{ab} \\
&= \partial_c (\xi^d \partial_d g_{ab} + g_{ad}\partial_b \xi^d + g_{bd}\partial_a \xi^d) \\
&= (\partial_c \xi^d)(\partial_d g_{ab}) + \xi^d \partial_c \partial_d g_{ab} + (\partial_c g_{ad})(\partial_b \xi^d) + g_{ad}\partial_c \partial_b \xi^d + (\partial_c g_{bd})(\partial_a \xi^d) + g_{bd}\partial_c \partial_a \xi^d \\
&= \mathcal{L}_\xi \partial_c g_{ab} + g_{ad}\partial_c \partial_b \xi^d + g_{bd}\partial_c \partial_a \xi^d
\end{aligned}$$

2.8 $\Gamma^a{}_{bc}$

$$
\begin{aligned}
\delta\Gamma^a{}_{bc} &= \delta[\frac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})] \\
&= \frac{1}{2}\delta g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) + \frac{1}{2}g^{ad}(\delta\partial_b g_{cd} + \delta\partial_c g_{bd} - \delta\partial_d g_{bc}) \\
&= \frac{1}{2}\mathcal{L}_\xi g^{cd}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) \\
&+ \frac{1}{2}g^{ad}(\mathcal{L}_\xi\partial_b g_{cd} + g_{ce}\partial_b\partial_d\xi^e + g_{de}\partial_b\partial_c\xi^e) \\
&+ \frac{1}{2}g^{ad}(\mathcal{L}_\xi\partial_c g_{bd} + g_{be}\partial_c\partial_d\xi^e + g_{de}\partial_c\partial_b\xi^e) \\
&- \frac{1}{2}g^{ad}(\mathcal{L}_\xi\partial_d g_{bc} + g_{be}\partial_d\partial_c\xi^e + g_{ce}\partial_d\partial_b\xi^e) \\
&= \frac{1}{2}\mathcal{L}_\xi g^{cd}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}) + \frac{1}{2}g^{ad}(\mathcal{L}_\xi\partial_b g_{cd} + \mathcal{L}_\xi\partial_c g_{bd} - \mathcal{L}_\xi\partial_d g_{bc}) + \partial_b\partial_c\xi^a \\
&= \mathcal{L}_\xi\Gamma^a{}_{bc} + \partial_b\partial_c\xi^a
\end{aligned}
$$

2.9 $\Delta\Gamma^a{}_{bc}$

If we define

$$\Delta\Gamma^a{}_{bc} \equiv \Gamma^a{}_{bc} - \bar{\Gamma}^a{}_{bc}$$

It is clear to see that

$$\boxed{\delta\Delta\Gamma^a{}_{bc} = \mathcal{L}_\xi\Gamma^a{}_{bc} + \partial_b\partial_c\xi^a - \mathcal{L}_\xi\bar{\Gamma}^a{}_{bc} - \partial_b\partial_c\xi^a = \mathcal{L}_\xi\Delta\Gamma^a{}_{bc}}$$

Therefore, $\Delta\Gamma^a{}_{bc}$ transforms as a (1,2) tensor.

2.10 $C_\perp$, C_a

Since C_μ is a covector in spacetime, we have

$$\delta C_\mu = \mathcal{L}_\xi C_\mu = \xi^\sigma\partial_\sigma C_\mu + C_\sigma\partial_\mu\xi^\sigma = \xi^c\partial_c C_\mu +^{(4)} C_d\partial_\mu\xi^d$$

$$\delta C^\mu = \mathcal{L}_\xi C^\mu = \xi^\sigma\partial_\sigma C^\mu - C^\sigma\partial_\sigma\xi^\mu = \xi^c\partial_c C^\mu - C^\sigma\partial_\sigma\xi^\mu$$

We also have

$$C_a = C_\mu X^\mu_a = C_\mu\delta^\mu_a =^{(4)} C_a$$

Therefore,

$$\boxed{\delta C_a = \xi^c\partial_c C_a + C_d\partial_a\xi^d = \mathcal{L}_\xi C_a}$$

Hence we have, **C_a transforms as a (0,1) tensor.**

For $C_\perp$ we have

$$C_\perp = C^\mu n_\mu = -\alpha C^\mu\delta^0_\mu = -\alpha\ {}^{(4)}C^0$$

Hence,

$$\delta C_\perp = -\alpha\delta\,{}^{(4)}C^0 - {}^{(4)}C^0\delta\alpha = -\alpha\xi^c\partial_c\,{}^{(4)}C^0 - {}^{(4)}C^0\xi^c\partial_c\alpha = \xi^c\partial_c(-\alpha\,{}^{(4)}C^0) = \xi^c\partial_c C_\perp$$

$$\boxed{\delta C_\perp = \mathcal{L}_\xi C_\perp}$$

and $C_\perp$ **transforms as a scalar.**

2.11 π

We have

$$\pi = \frac{\sqrt{g}}{\alpha}C_\perp$$

Therefore, π **transforms as a weight +1 density.**

2.12 ρ_a

We have

$$\rho_a = \frac{\sqrt{g}}{\alpha}C_a$$

Therefore, ρ_a **transforms as a weight +1 (0,1) tensor density.**

2.13 $\dot{g}_{ab}$

$$\begin{aligned}
\delta\dot{g}_{ab} &= \frac{d}{dt}\delta g_{ab} \\
&= \frac{d}{dt}(\xi^c\partial_c g_{ab} + g_{ac}\partial_b\xi^c + g_{bc}\partial_a\xi^c) \\
&= \dot{\xi}^c\partial_c g_{ab} + \xi^c\partial_c\dot{g}_{ab} + \dot{g}_{ac}\partial_b\xi^c + g_{ac}\partial_b\dot{\xi}^c + \dot{g}_{bc}\partial_a\xi^c + g_{bc}\partial_a\dot{\xi}^c \\
&= \mathcal{L}_\xi\dot{g}_{ab} + \mathcal{L}_{\dot{\xi}}g_{ab}
\end{aligned}$$

2.14 $D_a\beta_b$

First consider $\delta\beta_b$,

$$\begin{aligned}
\delta\beta_b &= \delta g_{ab}\beta^a \\
&= (\delta g_{ab})\beta^a + g_{ab}(\delta\beta^a) \\
&= (\mathcal{L}_\xi g_{ab})\beta^a + g_{ab}(\mathcal{L}_\xi\beta^a + \dot{\xi}^a) \\
&= \mathcal{L}_\xi(g_{ab}\beta^a) + g_{ab}\dot{\xi}^a \\
&= \mathcal{L}_\xi\beta_b + g_{ab}\dot{\xi}^a
\end{aligned}$$

Then consider $\delta(D_a\beta_b)$

$$\begin{aligned}\delta(D_a\beta_b) &= \delta(\partial_a\beta_b - \Gamma^c{}_{ab}\beta_c) \\ &= \partial_a\delta\beta_b - (\delta\Gamma^c{}_{ab})\beta_c - \Gamma^c{}_{ab}(\delta\beta_c) \\ &= \partial_a(\xi^c\partial_c\beta_b + \beta_c\partial_b\xi^c + g_{cb}\dot{\xi}^c) - (\mathcal{L}_\xi\Gamma^c{}_{ab} + \partial_a\partial_b\xi^c)\beta_c - \Gamma^c{}_{ab}(\mathcal{L}_\xi\beta_c + g_{cd}\dot{\xi}^d) \\ &= \xi^c\partial_c\partial_a\beta_b + (\partial_c\beta_b)(\partial_a\xi^c) + (\partial_a\beta_c)(\partial_b\xi^c) - \beta_c\mathcal{L}_\xi\Gamma^c{}_{ab} - \Gamma^c{}_{ab}\mathcal{L}_\xi\beta_c + \partial_a(g_{cb}\dot{\xi}^c) - \Gamma^c{}_{ab}(g_{cd}\dot{\xi}^d) \\ &= \mathcal{L}_\xi(\partial_a\beta_b) + \mathcal{L}_\xi(\Gamma^c{}_{ab}\beta_c) - D_a(g_{cb}\dot{\xi}^c) \\ &= \mathcal{L}_\xi(D_a\beta_b) + g_{cb}D_a\dot{\xi}^c\end{aligned}$$

2.15 K_{ab}

$$\begin{aligned}K_{ab} &= -\frac{1}{2\alpha}(\dot{g}_{ab} - \mathcal{L}_\beta g_{ab}) \\ &= -\frac{1}{2\alpha}(\dot{g}_{ab} - D_a\beta_b - D_b\beta_a)\end{aligned}$$

Therefore,

$$\begin{aligned}\delta K_{ab} &= -\delta(\frac{1}{2\alpha})(\dot{g}_{ab} - D_a\beta_b - D_b\beta_a) - \frac{1}{2\alpha}(\delta\dot{g}_{ab} - \delta(D_a\beta_b) - \delta(D_b\beta_a)) \\ &= -\mathcal{L}_\xi(\frac{1}{2\alpha})(\dot{g}_{ab} - D_a\beta_b - D_b\beta_a) - \frac{1}{2\alpha}(\mathcal{L}_\xi\dot{g}_{ab} + \mathcal{L}_{\dot{\xi}}g_{ab} - \mathcal{L}_\xi(D_a\beta_b) - g_{ac}D_b\dot{\xi}^c - \mathcal{L}_\xi(D_b\beta_a) - g_{bc}D_a\dot{\xi}^c) \\ &= -\mathcal{L}_\xi(\frac{1}{2\alpha})(\dot{g}_{ab} - D_a\beta_b - D_b\beta_a) - \frac{1}{2\alpha}\mathcal{L}_\xi(\dot{g}_{ab} - D_a\beta_b - D_b\beta_a)\end{aligned}$$

$$\boxed{\delta K_{ab} = \mathcal{L}_\xi K_{ab}}$$

Therefore, K_{ab} **transforms as a (0,2) tensor.**

2.16 P_{ab}

According to the formula

$$P^{ab} = \sqrt{g}(Kg^{ab} - K^{ab} - \frac{C_\perp}{2}g^{ab})$$

We can say that P^{ab} **transforms as a weight +1 (2,0) tensor density.**

2.17 $\dot{\alpha}$

$$\delta\dot{\alpha} = \frac{d}{dt}\delta\alpha = \frac{d}{dt}(\xi^c\partial_c\alpha) = \dot{\xi}^c\partial_c\alpha + \xi^c\partial_c\dot{\alpha} = \mathcal{L}_\xi\dot{\alpha} + \dot{\xi}^a\partial_a\alpha$$

2.18 $\partial_a\alpha$

$$\delta(\partial_a\alpha) = \partial_a\delta\alpha = \partial_a(\xi^c\partial_c\alpha) = \xi^c\partial_c\partial_a\alpha + (\partial_c\alpha)(\partial_a\xi^c) = \mathcal{L}_\xi(\partial_a\alpha)$$

Therefore, $\partial_a\alpha$ **transforms as a (0,1) tensor.**

2.19 $\partial_\perp \alpha$

$$\partial_\perp \alpha \equiv \dot{\alpha} - \mathcal{L}_\beta \alpha = \dot{\alpha} - \beta^a \partial_a \alpha$$

Hence,

$$\begin{aligned}
\delta \partial_\perp \alpha &= \delta \dot{\alpha} - (\delta \beta^a) \partial_a \alpha - \beta^a (\delta \partial_a \alpha) \\
&= \mathcal{L}_\xi \dot{\alpha} + \dot{\xi}^a \partial_a \alpha - (\mathcal{L}_\xi \beta^a + \dot{\xi}^a) \partial_a \alpha - \beta^a \mathcal{L}_\xi (\partial_a \alpha) \\
&= \mathcal{L}_\xi \dot{\alpha} - \mathcal{L}_\xi (\beta^a \partial_a \alpha)
\end{aligned}$$

$$\boxed{\delta(\partial_\perp \alpha) = \mathcal{L}_\xi (\partial_\perp \alpha)}$$

$\partial_\perp \alpha$ transforms as a scalar.

2.20 $\Delta \dot{\beta}^a$ and extension

$$\delta \Delta \dot{\beta}^a = \frac{d}{dt} \delta \Delta \beta^a = \frac{d}{dt} (\xi^c \partial_c \Delta \beta^a - \Delta \beta^c \partial_c \xi^a) = \mathcal{L}_\xi \Delta \dot{\beta}^a + \dot{\xi}^c \partial_c \Delta \beta^a - \Delta \beta^c \partial_c \dot{\xi}^a$$

In order to construct a formula containing $\dot{\beta}^a$ which transforms as a (1,0) tensor, try this:

$$\dot{B}^a = \Delta \dot{\beta}^a + c_1 \beta^c \bar{D}_c \beta^a + c_2 \beta^c \bar{D}_c \bar{\beta}^a + c_3 \bar{\beta}^c \bar{D}_c \beta^a + c_4 \bar{\beta}^c \bar{D}_c \bar{\beta}^a$$

where c_1, c_2, c_3 and c_4 are just constants.

We have

$$\begin{aligned}
\delta \dot{B}^a = {} & \mathcal{L}_\xi \dot{B}^a + \dot{\xi}^c \partial_c \Delta \beta^a - \Delta \beta^c \partial_c \dot{\xi}^a \\
& + (c_1 + c_3) \dot{\xi}^c \bar{D}_c \beta^a + (c_2 + c_4) \dot{\xi}^c \bar{D}_c \bar{\beta}^a + (c_1 + c_2) \beta^c \bar{D}_c \dot{\xi}^a + (c_3 + c_4) \bar{\beta}^c \bar{D}_c \dot{\xi}^a
\end{aligned}$$

We should have

$$\begin{aligned}
c_1 + c_3 &= -1 \\
c_2 + c_4 &= 1 \\
c_1 + c_2 &= 1 \\
c_3 + c_4 &= -1
\end{aligned}$$

and one can check that under this choice, all the Christoffel symbol terms will cancel each other.

Let $c_1 = \sigma$ and we have

$$\begin{aligned}
c_1 &= \sigma \\
c_2 &= 1 - \sigma \\
c_3 &= -(1 + \sigma) \\
c_4 &= \sigma
\end{aligned}$$

Therefore, we have

$$\dot{B}^a = \Delta\dot{\beta}^a + \sigma\beta^c\bar{D}_c\beta^a + (1-\sigma)\beta^c\bar{D}_c\bar{\beta}^a - (1+\sigma)\bar{\beta}^c\bar{D}_c\beta^a + \sigma\bar{\beta}^c\bar{D}_c\bar{\beta}^a$$

and

$$\boxed{\delta\dot{B}^a = \mathcal{L}_\xi\dot{B}^a}$$

$\dot{B}^a$ transforms as a (1,0) tensor.

3 Time Reparametrization Invariance

$$\begin{aligned}
\delta g_{\mu\nu} &= \mathcal{L}_\xi g_{\mu\nu} \\
&= \xi^\sigma\partial_\sigma g_{\mu\nu} + g_{\mu\sigma}\partial_\nu\xi^\sigma + g_{\nu\sigma}\partial_\mu\xi^\sigma \\
&= \epsilon\dot{g}_{\mu\nu} + g_{\mu0}\delta^0_\nu\dot{\epsilon} + g_{\nu0}\delta^0_\mu\dot{\epsilon}
\end{aligned}$$

$$\begin{aligned}
\delta g^{\mu\nu} &= \mathcal{L}_\xi g^{\mu\nu} \\
&= \xi^\sigma\partial_\sigma g^{\mu\nu} - g^{\mu\sigma}\partial_\sigma\xi^\nu + g^{\nu\sigma}\partial_\sigma\xi^\mu \\
&= \epsilon\dot{g}^{\mu\nu} - g^{\mu0}\delta^\nu_0\dot{\epsilon} - g^{\nu0}\delta^\mu_0\dot{\epsilon}
\end{aligned}$$

3.1 g_{ab}

g_{ab} transforms as a scalar

$$\delta g_{ab} = \epsilon\dot{g}_{ab}$$

3.2 β^a

$$g_{a0} = g_{ab}\beta^b$$

Hence, we have

$$\begin{aligned}
\delta g_{a0} &= \delta g_{a0} \\
\delta(g_{ab}\beta^b) &= \epsilon(g_{ab}\beta^b)^\cdot + g_{a0}\dot{\epsilon} \\
(\delta g_{ab})\beta^b + g_{ab}(\delta\beta^b) &= \epsilon(g_{ab}\beta^b)^\cdot + g_{ab}\beta^b\dot{\epsilon} \\
\epsilon\dot{g}_{ab}\beta^b + g_{ab}(\delta\beta^b) &= \epsilon\dot{g}_{ab}\beta^b + \epsilon g_{ab}\dot{\beta}^b + g_{ab}\beta^b\dot{\epsilon} \\
\delta\beta^b &= \epsilon\dot{\beta}^b + \dot{\epsilon}\beta^b
\end{aligned}$$

$$\boxed{\delta\beta^b = (\epsilon\beta^b)^\cdot}$$

β^a transforms as a weight +1 density

3.3 α

$$g_{00} = g_{ab}\beta^a\beta^b - \alpha^2$$

Hence, we have

$$\begin{aligned}
\delta g_{00} &= \delta g_{00} \\
\delta(g_{ab}\beta^a\beta^b - \alpha^2) &= \epsilon(g_{ab}\beta^a\beta^b - \alpha^2)^\cdot + 2(g_{ab}\beta^a\beta^b - \alpha^2)\dot{\epsilon} \\
\delta g_{ab}\beta^a\beta^b + g_{ab}\delta\beta^a\beta^b + g_{ab}\beta^a\delta\beta^b - 2\alpha\delta\alpha &= \epsilon\dot{g}_{ab}\beta^a\beta^b + \epsilon g_{ab}\dot{\beta}^a\beta^b + \epsilon g_{ab}\beta^a\dot{\beta}^b - 2\epsilon\alpha\dot{\alpha} + 2(g_{ab}\beta^a\beta^b - \alpha^2)\dot{\epsilon} \\
\epsilon\dot{g}_{ab}\beta^a\beta^b + g_{ab}(\epsilon\beta^a)^\cdot\beta^b + g_{ab}\beta^a(\epsilon\beta^b)^\cdot - 2\alpha\delta\alpha &= \epsilon\dot{g}_{ab}\beta^a\beta^b + g_{ab}(\beta^a\epsilon)^\cdot\beta^b + g_{ab}\beta^a(\epsilon\beta^b)^\cdot - 2\epsilon\alpha\dot{\alpha} - 2\alpha^2\dot{\epsilon} \\
\delta\alpha &= \epsilon\dot{\alpha} + \alpha\dot{\epsilon}
\end{aligned}$$

$$\boxed{\delta\alpha = (\epsilon\alpha)^\cdot}$$

α transforms as a weight +1 density

3.4 g^{ab}

$$\begin{aligned}
\delta(g^{ab} - \beta^a\beta^b/\alpha^2) &= \delta(g^{ab} - \beta^a\beta^b/\alpha^2) \\
\delta g^{ab} - \delta(\beta^a\beta^b/\alpha^2) &= \epsilon(g^{ab} - \beta^a\beta^b/\alpha^2)^\cdot
\end{aligned}$$

Due to the transform property of β^a and α, $\beta^a\beta^b/\alpha^2$ should transform as a scalar. Therefore, we have

$$\delta g^{ab} - \epsilon(\beta^a\beta^b/\alpha^2)^\cdot = \epsilon\dot{g}^{ab} - \epsilon(\beta^a\beta^b/\alpha^2)^\cdot$$

$$\boxed{\delta g^{ab} = \epsilon\dot{g}^{ab}}$$

g^{ab} transforms as a scalar

? short cut
$g^{ab}g_{bc} = \delta^a_c$ should transform as a scalar, and g_{bc} transforms as a scalar, so g^{ab} transforms as a scalar.

3.5 $\Gamma^a{}_{bc}$

$\Gamma^a{}_{bc}$ is a combination of g^{ab} and $\partial_c g_{ab}$, so **it should also transform as a scalar**.

3.6 $\dot{g}_{ab}$,$\dot{g}^{ab}$

$$\boxed{\delta\dot{g}_{ab} = (\delta g_{ab})^\cdot = (\epsilon\dot{g}_{ab})^\cdot}$$

$$\boxed{\delta\dot{g}^{ab} = (\delta g^{ab})^\cdot = (\epsilon\dot{g}^{ab})^\cdot}$$

Both $\dot{g}_{ab}$ and $\dot{g}^{ab}$ transform as weight +1 densities.

3.7 $\sqrt{g}$

$$\boxed{\delta g = g g^{ab} \delta g_{ab} = g g^{ab} \epsilon \dot{g}_{ab} = \epsilon \dot{g}}$$

Therefore, both g and $\sqrt{g}$ transform as a scalar.

3.8 R

Since R is all about g_{ab}, g^{ab} and $\Gamma^a{}_{bc}$, **R should transform as a scalar**.

3.9 $D_a\beta_b$

$$\begin{aligned}
\delta(D_a\beta_b) &= \delta(\partial_a\beta_b - \Gamma^c{}_{ab}\beta_c) \\
&= \partial_a(\delta\beta_b) - \delta\Gamma^c{}_{ab}\beta_c - \Gamma^c{}_{ab}\delta\beta_c
\end{aligned}$$

Since $\beta_a = \beta^b g_{ab}$ and g_{ab} transforms as a scalar, β^b transforms as a weight $+1$ density, β_a transforms as a weight $+1$ density. Hence, we have

$$\begin{aligned}
\delta(D_a\beta_b) &= \partial_a(\epsilon\beta_b)^{\cdot} - \epsilon\dot{\Gamma}^c{}_{ab}\beta_c - \Gamma^c_{ab}(\epsilon\beta_c)^{\cdot} \\
&= (\partial_a(\epsilon\beta_b) - \epsilon\Gamma^c{}_{ab}\beta_c)^{\cdot} \\
&= (\epsilon(\partial_a\beta_b - \Gamma^c_{ab}\beta_c))^{\cdot} \\
&= (\epsilon D_a\beta_b)^{\cdot}
\end{aligned}$$

$$\boxed{\delta(D_a\beta_b) = (\epsilon D_a\beta_b)^{\cdot}}$$

$D_a\beta_b$ transforms as a weight $+1$ density.

3.10 K_{ab}

$$K_{ab} = -\frac{1}{2\alpha}(\dot{g}_{ab} - \mathcal{L}_\beta g_{ab}) = -\frac{1}{2\alpha}(\dot{g}_{ab} - D_a\beta_b - D_b\beta_a)$$

Therefore, **K_{ab} transforms as a scalar**.

3.11 $C_\perp$, C_a

Since C_μ is a covector in spacetime, we have

$$\begin{aligned}
\delta C_\mu &= \mathcal{L}_\xi C_\mu \\
&= \xi^\sigma\partial_\sigma C_\mu + C_\sigma\partial_\mu\xi^\sigma \\
&= \epsilon\dot{C}_\mu + C_0\dot{\epsilon}\delta^0_\mu
\end{aligned}$$

Therefore, we can see that the spatial component of C_μ transform as a scalar, the time component of C_μ transforms as a weight $+1$ density.

Hence, we have

$$\boxed{\delta C_a = \delta(C_\mu X^\mu_a) = \delta^\mu_a\delta C_\mu = \epsilon\dot{C}_a}$$

C_a **transforms as a scalar**.

$$n^\mu = (\delta_0^\mu - \beta^a \delta_a^\mu)/\alpha$$

We can tell from the above equation that the time component of n^μ transforms as a weight -1 density and the spatial component of n^μ transform as scalars. Therefore, $C_\perp = C_\mu n^\mu$ **should transform as a scalar**.

3.12 P^{ab}

According to the formula of P^{ab}

$$P^{ab} = \sqrt{g}(Kg^{ab} - K^{ab} - \frac{C_\perp}{2} g^{ab})$$

P^{ab} **should transform as a scalar**.

3.13 π

Since we have

$$\pi = \frac{\sqrt{g}}{\alpha} C_\perp$$

π **transforms as a weight -1 density**.

3.14 ρ_a

Since we have

$$\rho_a = \frac{\sqrt{g}}{\alpha} C_a$$

ρ_a **transforms as a weight -1 density**.

3.15 $\mathring{\alpha}$

The covariant derivative of α in time domain is defined as

$$\mathring{\alpha} = \dot{\alpha} - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}} \alpha$$

So we have

$$
\begin{aligned}
\delta\mathring{\alpha} &= \delta\dot{\alpha} - \delta(\frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\alpha) \\
&= \delta\dot{\alpha} - \frac{\alpha}{\bar{\alpha}}\delta\dot{\bar{\alpha}} - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\delta\alpha + \frac{\dot{\bar{\alpha}}}{\bar{\alpha}^2}\alpha\delta\bar{\alpha} \\
&= (\delta\alpha)^{\cdot} - \frac{\alpha}{\bar{\alpha}}(\delta\bar{\alpha})^{\cdot} - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\delta\alpha + \frac{\dot{\bar{\alpha}}}{\bar{\alpha}^2}\alpha\delta\bar{\alpha} \\
&= (\epsilon\alpha)^{\cdot\cdot} - \frac{\alpha}{\bar{\alpha}}(\epsilon\bar{\alpha})^{\cdot\cdot} - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}(\epsilon\alpha)^{\cdot} + \frac{\dot{\bar{\alpha}}}{\bar{\alpha}^2}\alpha(\epsilon\bar{\alpha})^{\cdot} \\
&= \ddot{\epsilon}\alpha + 2\dot{\epsilon}\dot{\alpha} + \epsilon\ddot{\alpha} - \frac{\alpha}{\bar{\alpha}}\ddot{\epsilon}\bar{\alpha} - 2\frac{\alpha}{\bar{\alpha}}\dot{\epsilon}\dot{\bar{\alpha}} - \frac{\alpha}{\bar{\alpha}}\epsilon\ddot{\bar{\alpha}} - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\dot{\epsilon}\alpha - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\epsilon\dot{\alpha} + \frac{\dot{\bar{\alpha}}}{\bar{\alpha}^2}\alpha\dot{\epsilon}\bar{\alpha} + \frac{\dot{\bar{\alpha}}}{\bar{\alpha}^2}\alpha\epsilon\dot{\bar{\alpha}} \\
&= \epsilon(\ddot{\alpha} - \frac{\ddot{\bar{\alpha}}}{\bar{\alpha}}\alpha - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\dot{\alpha} + \frac{\dot{\bar{\alpha}}}{\bar{\alpha}^2}\alpha\dot{\bar{\alpha}}) + 2\dot{\epsilon}(\dot{\alpha} - \frac{\dot{\bar{\alpha}}}{\bar{\alpha}}\alpha) \\
&= \epsilon(\mathring{\alpha})^{\cdot} + 2\dot{\epsilon}\mathring{\alpha}
\end{aligned}
$$

Therefore we claim that $\mathring{\alpha}$ transforms as a weight +2 density

3.16 $\mathring{\beta}^a$

Define

$$\mathring{\beta}^a = \dot{\beta}^a - \frac{\beta^a}{\bar{\alpha}}\dot{\bar{\alpha}}$$

Following the same steps as previous section, we can show that **$\mathring{\beta}^a$ also transforms as a weight +2 density**.

4 Conclusion

According to the discussion above, we need to require $\dot{\alpha}$ appears in the following form

$$\boxed{(\dot{\alpha} - \beta^c\partial_c\alpha) - \frac{\alpha}{\bar{\alpha}}(\dot{\bar{\alpha}} - \bar{\beta}^c\partial_c\bar{\alpha})}$$

and $\dot{\beta}^a$ should appear in the following form

$$\boxed{\Delta\dot{\beta}^a - \frac{\Delta\beta^a}{\bar{\alpha}}(\dot{\bar{\alpha}} - \bar{\beta}^c\partial_c\bar{\alpha}) + \sigma\beta^c\bar{D}_c\beta^a + (1-\sigma)\beta^c\bar{D}_c\bar{\beta}^a - (1+\sigma)\bar{\beta}^c\bar{D}_c\beta^a + \sigma\bar{\beta}^c\bar{D}_c\bar{\beta}^a}$$