Homework of Motion Planning for Mobile Robots

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Abstract

Optimal Boundary Value Problem (OBVP) using minimum principle, and Build an ego graph of the linear modeled robot.

Keywords: OBVP, Local lattice planner

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1. Optimal Boundary Value Problem

1.1. Necessary conditions of OBVP problem

The problem is:

$$\begin{aligned} &\max_{\boldsymbol{u}(t)\in\boldsymbol{\Omega}} \boldsymbol{J} = h(\boldsymbol{s}(T)) + \int_0^T g(\boldsymbol{s},\boldsymbol{u},t)dt \\ &\text{s.t.} \begin{cases} \dot{\boldsymbol{s}} = \boldsymbol{f}(\boldsymbol{s},\boldsymbol{u}) \\ \boldsymbol{s}(0) = \boldsymbol{s}_0 \\ \boldsymbol{\psi}\left[\boldsymbol{s}\left(T\right),T\right] = 0 \end{cases} \end{aligned} \tag{1}$$

Where terminal time T free or fixed, $\boldsymbol{\psi}(\cdot) \in \mathbf{R}^r, r \leqslant n$ is the constraint of the state at terminal time T, which is a r-vector. According to the minimum principle, there are non-zero constant vectors $\boldsymbol{\gamma} \in \mathbf{R}^r$ and $\boldsymbol{\lambda} \in \mathbf{R}^n$, such that the optimal solution meets the following necessary conditions[1]:

$$\dot{\boldsymbol{s}}(t) = \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{\lambda}}, \quad \dot{\boldsymbol{\lambda}}(t) = -\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{s}} \tag{2}$$

$$\boldsymbol{H}(\boldsymbol{s},\boldsymbol{u},\boldsymbol{\lambda}) = g(\boldsymbol{s},\boldsymbol{u},t) + \boldsymbol{\lambda}^T \boldsymbol{f}(\boldsymbol{s},\boldsymbol{u}) \tag{3}$$

$$\boldsymbol{s}\left(0\right) = \boldsymbol{s}_0, \quad \boldsymbol{\psi}\left[\boldsymbol{s}\left(T\right),T\right] = 0, \quad \boldsymbol{\lambda}\left(T\right) = \frac{\partial h}{\partial \boldsymbol{s}} + \frac{\partial \boldsymbol{\psi}^{\mathrm{T}}}{\partial \boldsymbol{s}} \gamma \bigg|_T \tag{4}$$

$$H\left(\boldsymbol{s}^*, \boldsymbol{u}^*, \boldsymbol{\lambda}, t\right) = \min_{\boldsymbol{u}\in\boldsymbol{\Omega}} H\left(\boldsymbol{s}^*, \boldsymbol{u}, \boldsymbol{\lambda}, t\right) \tag{5}$$

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1.2. The solution of the OBVP of quadrotors

Without loss of generality, we only consider the equation of motion of one axis. The quadrotors system equation:

$$\dot{\boldsymbol{s}} = \boldsymbol{f}(\boldsymbol{s}, \boldsymbol{u}) = (v, a, j), \quad \boldsymbol{s} = (p, v, a) \tag{6}$$

where $j = u$ is input vector, the initial state is $\boldsymbol{s}(0) = \boldsymbol{s}_0 = (p_0, v_0, a_0)$, the terminal constraints can be described as:

$$\psi\left[\boldsymbol{s}\left(T\right), T\right] = p(T) - C = 0 \tag{7}$$

where C is a constant value.

Objective:

$$J = \frac{1}{T}\int_0^T j(t)^2 dt \tag{8}$$

According to the necessary conditions of OBVP, we have

$$\dot{\lambda}(t) = \begin{bmatrix} 0 \\ -\lambda_1 \\ -\lambda_2 \end{bmatrix} \tag{9}$$

$$\boldsymbol{H} = \frac{1}{T}j^2 + \lambda_1 v + \lambda_2 a + \lambda_3 j \tag{10}$$

$$\begin{bmatrix} \lambda_1(T) \\ \lambda_2(T) \\ \lambda_3(T) \end{bmatrix} = \begin{bmatrix} \gamma \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

In this example, we can take $\lambda_1(t)$ as a constant while ignoring γ for now. Such that:

$$\lambda(t) = \frac{1}{T}\begin{bmatrix} -2\alpha \\ 2\alpha(t-T) \\ -\alpha(t-T)^2 \end{bmatrix} \tag{12}$$

$$\begin{aligned} j^*(t) = u^*(t) &= \arg\min_{j\in\Omega}\left[\frac{1}{T}j^2 + \lambda_1 v + \lambda_2 a + \lambda_3 j\right] \\ &= -\frac{\lambda_3 T}{2} = \frac{1}{2}\alpha(t-T)^2 \end{aligned} \tag{13}$$

By integrating the input, the optimal state expression is:

$$\boldsymbol{s}^*(t) = \begin{bmatrix} \frac{\alpha}{120}(t-T)^5 + \frac{1}{2}\left(a_0 + \frac{\alpha}{6}T^3\right)t^2 + \left(v_0 - \frac{\alpha}{24}T^4\right)t + \left(p_0 + \frac{\alpha}{120}T^5\right) \\ \frac{\alpha}{24}(t-T)^4 + \left(a_0 + \frac{\alpha}{6}T^3\right)t + \left(v_0 - \frac{\alpha}{24}T^4\right) \\ \frac{\alpha}{6}(t-T)^3 + \left(a_0 + \frac{\alpha}{6}T^3\right) \end{bmatrix} \tag{14}$$

Define the error position by end position $p(T) = p_f$:

$$\Delta p = p_f - p_0 - \frac{1}{2}a_0T^2 - v_0T \tag{15}$$

we have:

$$\alpha = \frac{20\Delta p}{T^5}, \quad J^* = \int_0^{\mathrm{T}} \frac{1}{\mathrm{T}} j^*(t)^2 dt = \frac{20\Delta P^2}{\mathrm{T}^6} \tag{16}$$

We use extreme conditions $\dfrac{\partial J}{\partial T} = 0$ to find the extreme points of J:

$$\frac{\partial J}{\partial T} = 20\frac{\frac{\partial \Delta p^2}{\partial T}T^6 - 6\Delta p^2 T^5}{T^{12}} = 0 \ \Rightarrow\ \frac{\partial \Delta p^2}{\partial T}T - 6\Delta p^2 = \Delta p\left(a_0 T^2 + 4v_0\ \mathrm{T} - 6p_f + 6p_0\right) = 0 \tag{17}$$

such that

$$T^* = \frac{-v_0 \pm \sqrt{v_0^2 + 2a_0\left(p_f - p_0\right)}}{a_0},\ or\quad T^* = \frac{-2v_0 \pm \sqrt{4v_0^2 + 6a_0\left(p_f - p_0\right)}}{a_0} \tag{18}$$

and finally, we select the real number T^* that makes J the smallest.

2. Local lattice planner

Use Mathematica to obtain the derivative of J:

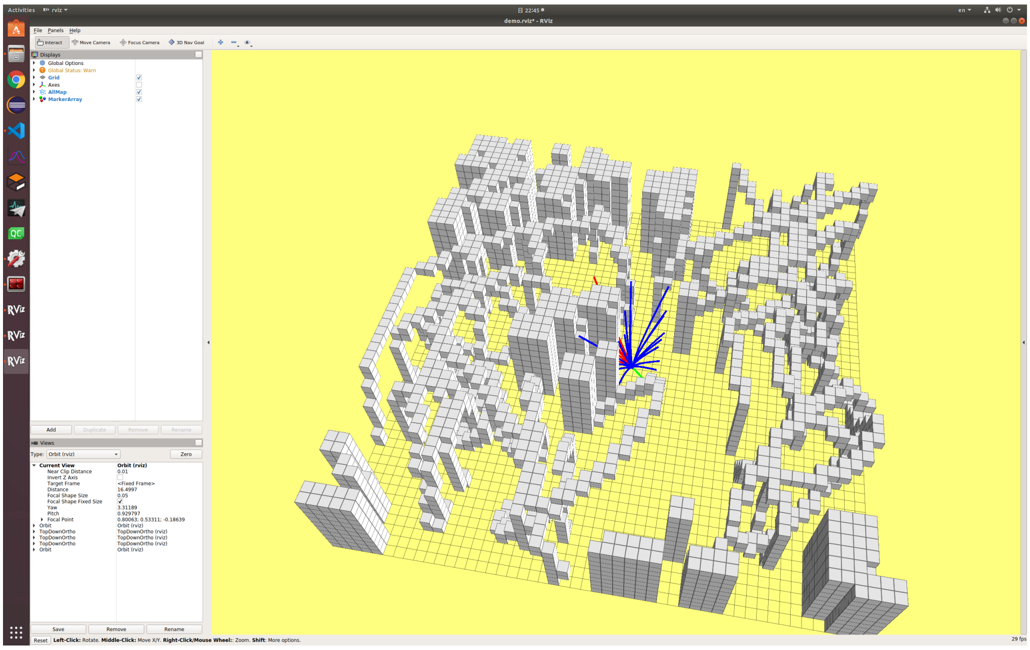
$$\begin{aligned}\frac{\partial J(T)}{\partial (T)} =& 1 - \frac{36}{T^4}\left(dx^2 + dy^2 + dz^2\right) + \frac{24}{T^3}\left(dxv_{x0} + dyv_{y0} + dzv_{z0}\right) \\ &- \frac{4}{T^2}\left(v_{x0}^2 + v_{y0}^2 + v_{z0}^2\right)\end{aligned} \tag{19}$$

where

$$\begin{aligned} dx &= p_{xf} - p_{x0} \\ dy &= p_{yf} - p_{y0} \\ dz &= p_{zf} - p_{z0} \end{aligned} \tag{20}$$

where p_{xf} is the end x-axis position, and so on. We use Eigen to solve the root of $\frac{\partial J(T)}{\partial (T)} = 0$.

The result is:



References

[1] A. E. Bryson, Y. C. Ho, G. M. Siouris, Applied Optimal Control: Optimization, Estimation, and Control, Applied Optimal Control: Optimization, Estimation, and Control, 1975.