Introduction of Fast Fourier Transform [1]

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2022.7

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[1] J. W. Cooley and J. W. Tukey, "An algorithm for the machine calculation of complex Fourier series," Math. of Computation, vol. 19, pp. 297–301, 1965

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FFT Input and Output

真实的函数：

$$S = 2 + 3 * cos(2 * \pi * 50 * t - \pi * 30/180) + 1.5 * cos(2 * \pi * 75 * t + \pi * 90/180) \quad (1)$$

采样点数：设 1s 内采样点为 256 个，则采样频率为 256Hz，分辨 1Hz 频率

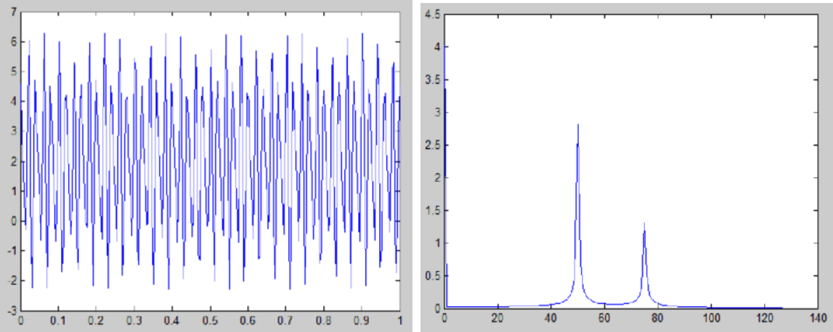


Figure: Origin and Spectrum

Output 计算公式

FFT 幅度：毕达哥拉斯公式

$$\text{Magnitude} = \sqrt{\text{Real}^{2} + \text{Imaginary}^{2}} \tag{2}$$

FFT 相位：反正切规则

$$\text{Phase} = \tan^{-1}\left(\frac{\text{Imaginary}}{\text{Real}}\right) \tag{3}$$

Output 计算结果

fft input: [256,], fft output: [256,], 采样定理：采样频率 $\leq 2*$ 信号频率

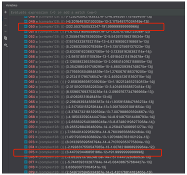


Figure: FFT Result

- 0Hz 分量复数表示：511.999
- 50Hz 分量复数表示：$332 - 191j$
- 75Hz 分量复数表示：$3.44e^{12} + 191j$
- 0Hz 模值表示：$511.999 * \frac{2}{N} = 4$
- 50Hz 模值表示：$384 * \frac{2}{N} = 3$
- 75Hz 模值表示：$192 * \frac{2}{N} = 1.5$

类似，相位计算得：0, -30, 90

因此，原函数

$S = 2 + 3 * cos(2 * \pi * 50 * t - \pi * 30/180) + 1.5 * cos(2 * \pi * 75 * t + \pi * 90/180)$

FFT Introduction

FFT Introduction

傅立叶变换 (FT)：连续时间 & 连续频率

$$X(f) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-i2\pi ft} \cdot dt \tag{4}$$

离散傅立叶变换 (DFT)：离散时间 & 离散频率

$$X(k) = \sum_{n=0}^{N-1} x(n) \times e^{-i2\pi \frac{k}{N} n}, k = 0, 1, ..., N-1 \tag{5}$$

n 表示离散时间，k 表示离散频率

1. Vandermonde Matrix

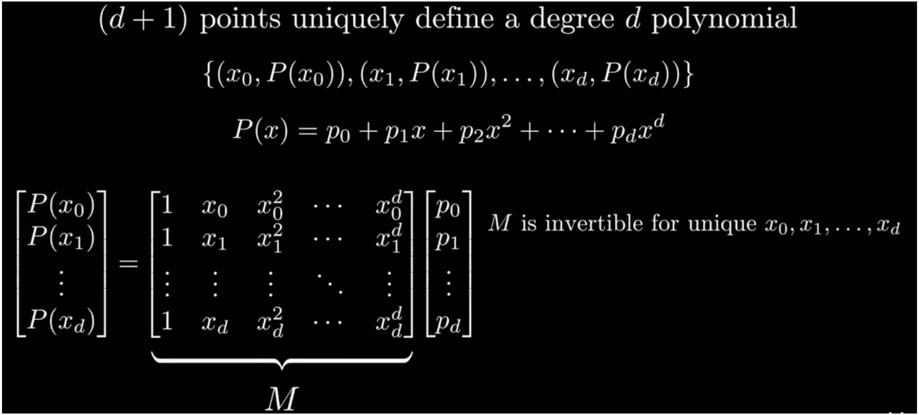


Figure: Vandermonde Matrix[2]

[2] https://www.youtube.com/watch?v=h7apO7q16V0feature=emb

1. Vandermonde Matrix

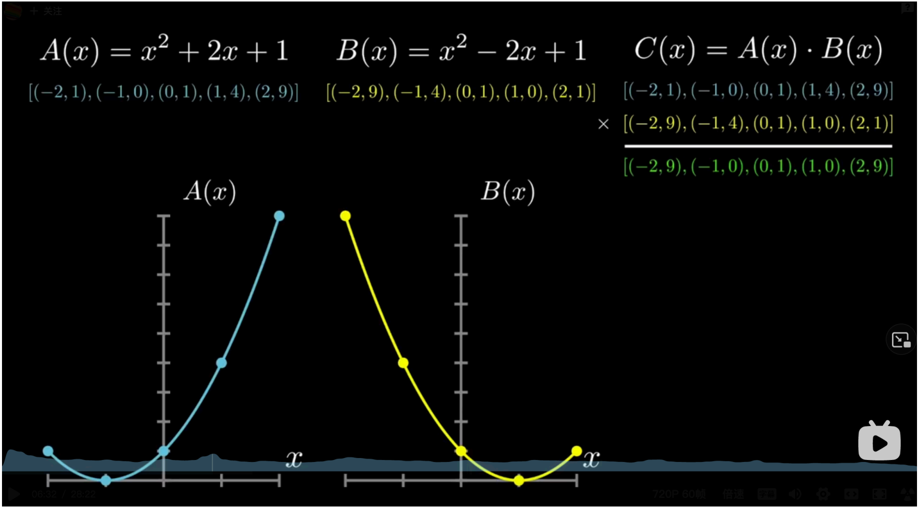


Figure: 点值表示法[3]

2. 对称结构

$$P(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_{n-1} x^{n-1}$$

Evaluate at n points $\pm x_1, \pm x_2, \ldots, \pm x_{n/2}$

$$P(x) = P_e(x^2) + x P_o(x^2)$$

$$\left.\boxed{\begin{aligned} P(x_i) &= P_e(x_i^2) + x_i P_o(x_i^2) \\ P(-x_i) &= P_e(x_i^2) - x_i P_o(x_i^2) \end{aligned}}\right\} \text{Lot of overlap!}$$

$P_e(x^2)$ and $P_o(x^2)$ have degree $n/2 - 1$!

$$\underbrace{\text{Evaluate } P_e(x^2) \text{ and } P_o(x^2) \text{ each at } x_1^2, x_2^2, \ldots, x_{n/2}^2 \ (n/2 \text{ points})}_{\text{Same process on simpler problem}}$$

Figure: 对称结构[4]

[4] https://www.youtube.com/watch?v=h7apO7q16V0feature=emb_logo

2. 对称结构

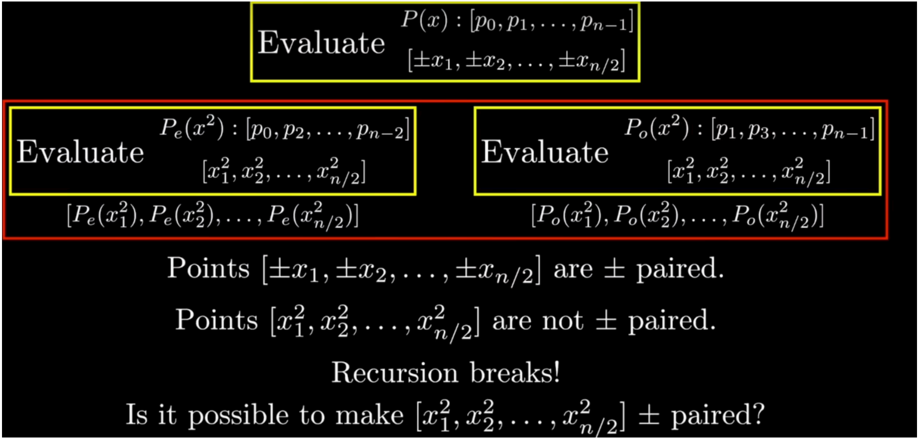


Figure: 对称结构[5]

[5] https://www.youtube.com/watch?v=h7apO7q16V0feature=emb_logo

3. 复数引入

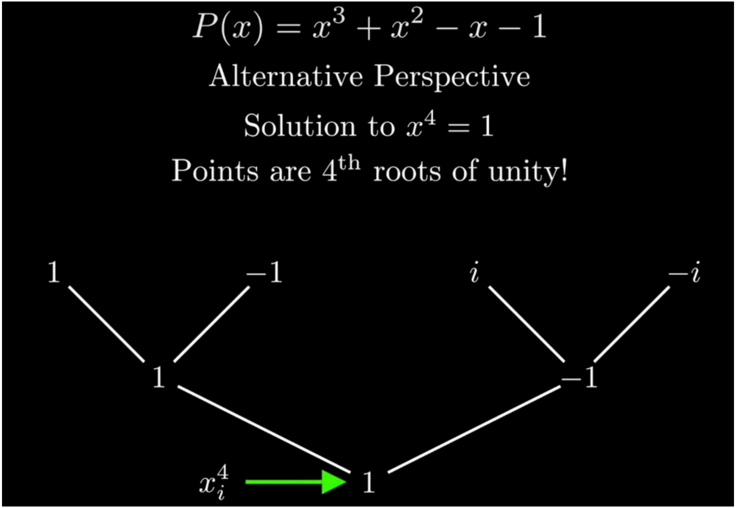


Figure: 复数引入[6]

3. 复数引入

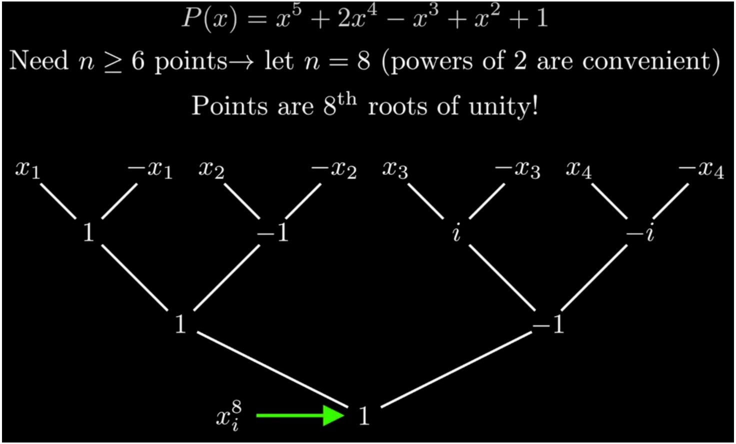


Figure: 复数引入[7]

[7] https://www.youtube.com/watch?v=h7apO7q16V0feature=emb$_logo$

4. 如何选取复数采样点

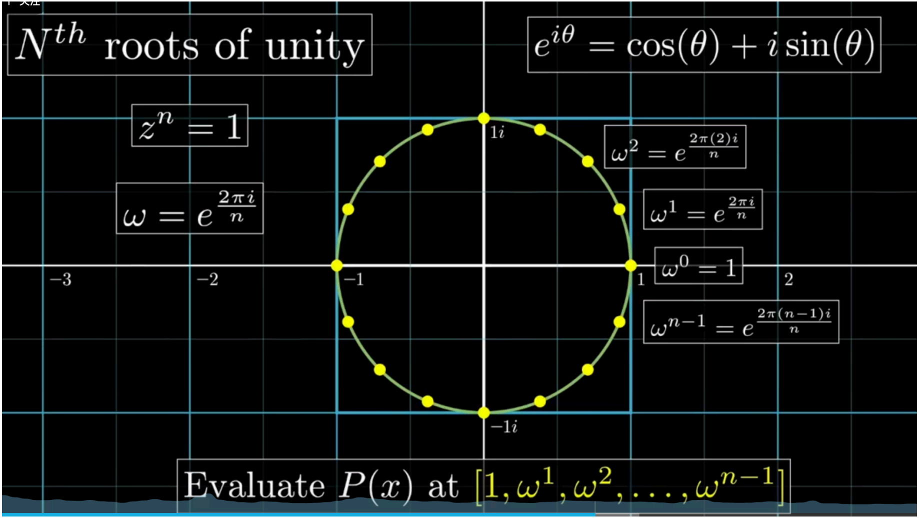
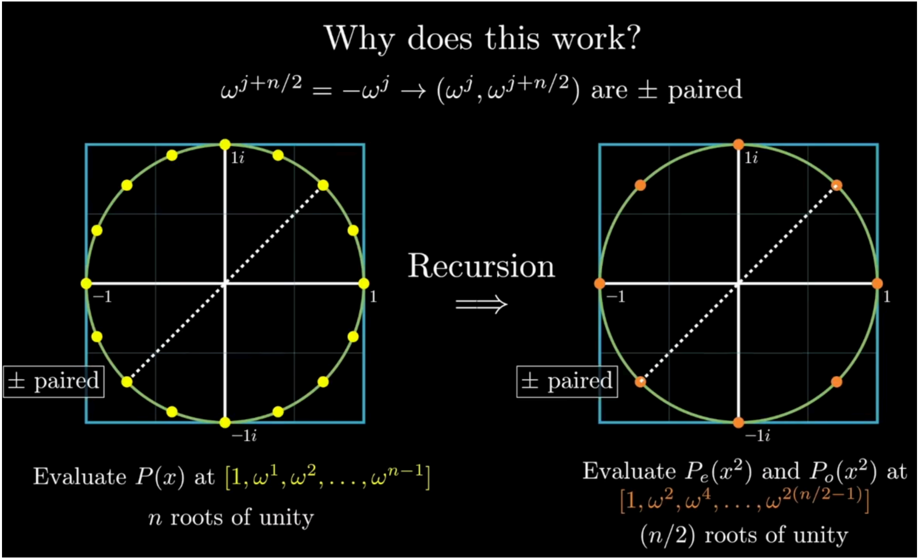


Figure: 复数引入[8]

4. 如何选取复数采样点


Why does this work?
$\omega^{j+n/2} = -\omega^j \rightarrow (\omega^j, \omega^{j+n/2})$ are $\pm$ paired
1i
−1
1
−1i
± paired
Recursion
$\Longrightarrow$
1i
−1
1
−1i
± paired
Evaluate $P(x)$ at $[1, \omega^1, \omega^2, \ldots, \omega^{n-1}]$
n roots of unity
Evaluate $P_e(x^2)$ and $P_o(x^2)$ at
$[1, \omega^2, \omega^4, \ldots, \omega^{2(n/2-1)}]$
$(n/2)$ roots of unity

FFT Pseudocode

```
def FFT(P):
    # P - [p_0, p_1, ..., p_{n-1}] coeff representation
    n = len(P) # n is a power of 2
    if n == 1:
        return P
    ω = e^(2πi/n)
    P_e, P_o = [p_0, p_2, ..., p_{n-2}], [p_1, p_3, ..., p_{n-1}]
    y_e, y_o = FFT(P_e), FFT(P_o)
    y = [0] * n
    for j in range(n/2):
        y[j] = y_e[j] + ω^j y_o[j]
        y[j + n/2] = y_e[j] - ω^j y_o[j]
    return y
```

Figure: FFT Pseudocode[10]

[10] https://www.youtube.com/watch?v=h7apO7q16V0feature=emb_logo

IFFT Vandermonde Matrix

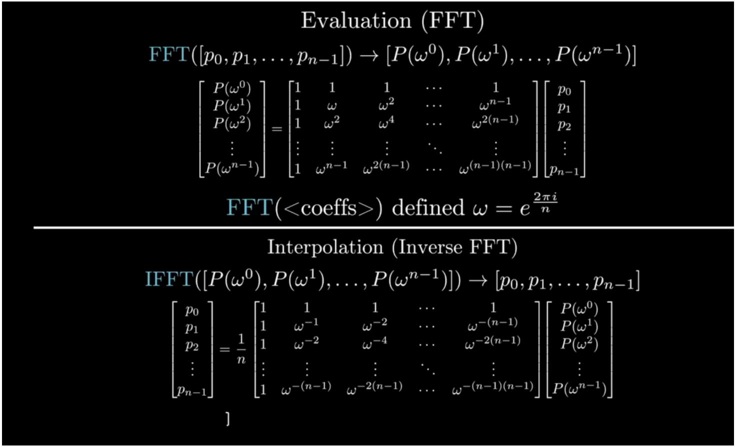


Figure: IFFT Vandermonde Matrix [11]

[11] https://www.youtube.com/watch?v=h7apO7q16V0feature=emb$_{logo}$

IFFT Pseudocode

+ 关注

$$\text{IFFT}(<\text{values}>) \Leftrightarrow \text{FFT}(<\text{values}>) \text{ with } \omega = \frac{1}{n} e^{\frac{-2\pi i}{n}}$$

```
def FFT(P):
    # P - [p_0, p_1, ..., p_{n-1}] coeff rep
    n = len(P) # n is a power of 2
    if n == 1:
        return P
    ω = e^{2πi/n}
    P_e, P_o = P[::2], P[1::2]
    y_e, y_o = FFT(P_e), FFT(P_o)
    y = [0] * n
    for j in range(n/2):
        y[j] = y_e[j] + ω^j y_o[j]
        y[j + n/2] = y_e[j] - ω^j y_o[j]
    return y
```

```
def IFFT(P):
    # P - [P(ω^0), P(ω^1), ..., P(ω^{n-1})] value rep
    n = len(P) # n is a power of 2
    if n == 1:
        return P
    ω = (1/n) * e^{-2πi/n}
    P_e, P_o = P[::2], P[1::2]
    y_e, y_o = IFFT(P_e), IFFT(P_o)
    y = [0] * n
    for j in range(n/2):
        y[j] = y_e[j] + ω^j y_o[j]
        y[j + n/2] = y_e[j] - ω^j y_o[j]
    return y
```

Figure: FFT Pseudocode[12]

[12] https://www.youtube.com/watch?v=h7apO7q16V0feature=emb

复数采样点的另一个优势

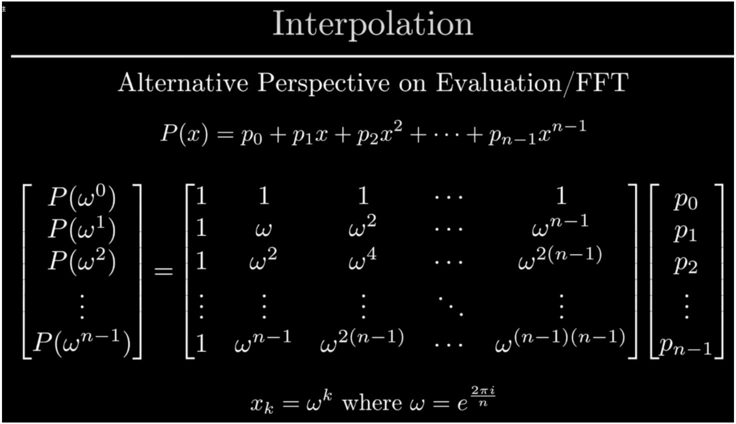


Figure: 另一个优势[13]

[13] https://www.youtube.com/watch?v=h7apO7q16V0feature=emb_logo

Thank you!