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Generalized geodesy via geodesic time

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Abstract

The time necessary to cover a path on a grey-scale image is the sum of the grey-level values along the path. The geodesic time between two points in a grey-scale image is defined as the smallest amount of time allowing to link these points. The geodesic time allows the definition of generalized geodesic distances, erosions, dilations, and skeletons by influence zones. An application to minimal path extraction on grey-scale images is presented.

Keywords: Mathematical morphology; Distance; Geodesy; Minimal path

1. Introduction

In mathematical morphology (Serra, 1982), the term geodesy is used wherever an operator is controlled by some external constraints. For example, the geodesic distance between two points included in a set is the length of the shortest paths or geodesics (Serra, 1988, p. 48) linking these points and included in the set (Lantuéjoul and Beucher, 1980, 1981; Lantuéjoul and Maisonneuve, 1984). The binary geodesic dilation involves two images: a marker image and a control image or geodesic mask. The geodesic dilation of size n of a marker set within a geodesic mask is the set of points of the geodesic mask whose geodesic distance to the marker set is lower or equal to n. Geodesic dilations and erosions have been generalized by Beucher (1990, pp. 84–91) to grey-scale markers and geodesic masks and they are at the basis of powerful morphological reconstruction algorithms. An efficient implementation of these algorithms has been recently proposed by Vincent (1993). Finally, Beucher (1990, pp. 92–104) investigated geodesic operators over a refringence graph.

The scope of this letter is to propose a generalization of binary geodesic operators to grey-scale geodesic masks while preserving a binary marker set. The expansion and shrinking of this set are therefore controlled by the grey-scale geodesic mask. Our generalization is based on the concept of geodesic time, a notion closely related to the grey-weighted distance.

2. Geodesic time

Let us first consider a path $\mathscr{P}$ defined on the domain of definition of a grey-scale image or integrable function f. The time required to cover $\mathscr{P}$ is defined as the integral of the value of f along $\mathscr{P}$ and is denoted by $t_f(\mathscr{P})$:

$$t_f(\mathscr{P}) = \int_{\mathscr{P}} f(s)\,\mathrm{d}s\,.$$

This definition is illustrated in Fig. 1. Note that the designation "time" is justified by considering units equal to the inverse of that of a speed for the image f.

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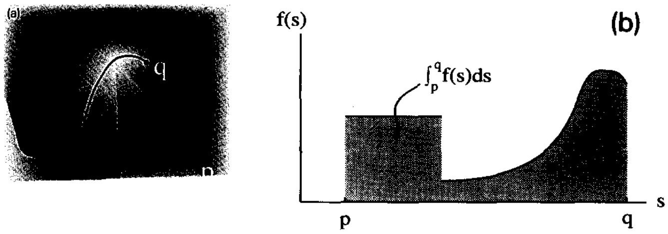

Fig. 1. The time required to cover the path $\mathcal{P}$ on the image f equals the integral of the value of f along $\mathcal{P}$. (a) Path $\mathcal{P}$ between p and q on an image f. (b) Values of f along $\mathcal{P}$.

the lower the grey-level, the faster the propagation is. A definition suited to discrete images is not presented.

We define a discrete grey-scale image f as an application of a subset $\mathcal{D}_f$ of the n-dimensional discrete space $\mathbb{Z}^n$ into the set $\bar{\mathbb{N}}_0$ of nonnegative integers union $+\infty$. The neighbourhood relationships between the points or pixels of a discrete image are defined by a graph (e.g., 4- or 8-connected graph for 2D square grids of points). A discrete path $\mathcal{P}$ of length l going from p to q in a graph $\mathcal{G}$ can be defined as a $(l+1)$-tuple $(p_0, p_1, ..., p_l)$ of vertices of $\mathcal{G}$ such that $p_0=p$, $p_l=q$, and (p_{i-1}, p_i) defines an arc of $\mathcal{G}$ for all $i\in[1, l]$.

The *time* $t_f(\mathcal{P})$ necessary to cover a discrete path $\mathcal{P}$ of length l defined on a discrete grey-scale image f equals the mean of the values of f taken two at a time along $\mathcal{P}$:

$$t_f(\mathcal{P}) = \sum_{i=1}^{l} \frac{f(p_{i-1})+f(p_i)}{2} = \frac{f(p_0)}{2} + \frac{f(p_l)}{2} + \sum_{i=1}^{l-1} f(p_i) \; . \tag{1}$$

$t_f(\mathcal{P})$ is therefore independent of the direction used to cover the path. An example is given in Fig. 2(a).

The geodesic time $t_f(p, q)$ separating two points p and q in a grey-scale image f is the smallest amount of time allowing to link p to q in f:

$$t_f(p, q) = \min\{t_f(\mathcal{P}) \mid \mathcal{P} \text{ links } p \text{ to } q\} \; . \tag{2}$$

The notion of grey-weighted distance transform briefly introduced by Rutovitz (1968, p. 128) is closely related to the concept of geodesic time. Grey-weighted distances have been used by Verbeek and Verwer (1990) for solving the Eikonal equation and Meyer (1992) for integrating images. Kimmel and Bruckstein (1993) proposed an implementation of the grey-weighted distance transform using an implicit representation of a curve evolution process (Osher and Sethian, 1988; Sapiro et al., 1993). The initial curve corresponds to the contours of the input set and it is embedded in a higher-dimensional function. The curve is then propagated as a level set of a bivariate function, the equations of motion being solved using sophisticated numerical techniques derived from hyperbolic conservation laws. The simplicity of our approach makes it more appealing for practical applications.

Contrary to the geodesic distance, the geodesic time does not define a geodesic metric for $\mathcal{D}_f$ since two different points may be separated by a null geodesic time. We refer to a geodesic metric for a function satisfying the three axioms of a metric but taking values in $\mathbb{R}_0^+ \cup +\infty$ rather than $\mathbb{R}_0^+$. If necessary, a geodesic metric can be obtained by constraining f to take values strictly greater than 0.

The geodesics of a grey-scale image are the paths linking points of the image in a minimum amount of time. An example of geodesic based on geodesic time calculations is shown in Fig. 2(b). Note that there may be more than one geodesic linking two points. An application involving the determination of these shortest paths is presented in Section 5.

The geodesic time between a point p and a set Y of points of an image f is the smallest amount of time allowing to link p to any point y of Y:

$$t_f(p, Y) = \min_{y\in Y} t_f(p, y) \; .$$

If p belongs to Y, the geodesic time from p to Y is zero. By associating to each point of an image f their geodesic time to a given set of points Y of $\mathcal{D}_f$, we define the geodesic time function $T_{f,Y}$. Examples are shown in Figs. 3(b) and 4(b) and 4(c).

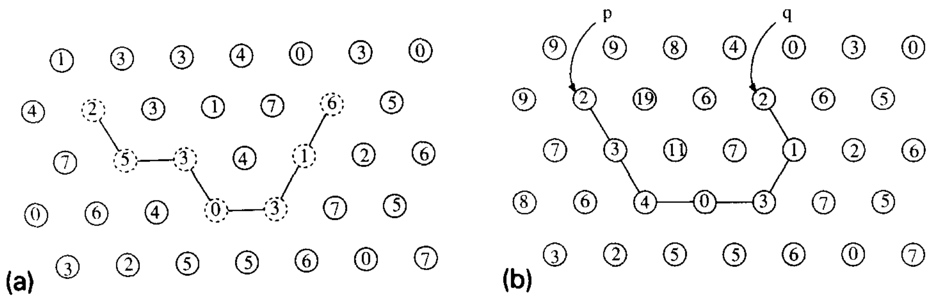

Fig. 2. (a) Discrete path $\mathscr{P}$ on a grey-scale image defined in the hexagonal grid: the time necessary to cover this path equals 16 (refer to Eq. (1)). (b) Geodesic or shortest path between two points p and q on a grey-scale image. It is a geodesic since there exists no other path linking p to q in a smaller amount of time.

3. Generalized geodesic distance

A new and interesting geodesic metric is defined by considering the actual length of the minimal paths defined on a grey-scale image. The generalized geodesic distance $d_f^*(p,q)$ between two points p and q of a grey-scale image f is the shortest length L of the path(s) linking p to q in a minimum amount of time if it is different from $+\infty$, and is $+\infty$ otherwise:

$$d_f^*(p,q) = \begin{cases} \inf\{L(\mathscr{P}) \mid \mathscr{P} \text{ links } p \text{ to } q \\ \quad \text{and } t_f(\mathscr{P}) = t_f(p,q)\} & \text{if } t_f(p,q) \neq +\infty, \\ +\infty & \text{otherwise}. \end{cases}$$

For instance, the generalized geodesic distance separating the points p and q of Fig. 2(b) equals 6. The generalized geodesic distance defines a geodesic metric for $\mathscr{D}_f$ since $d_f^*(p,q) \in \mathbb{R}_0^+ \cup \infty$ and

(i) $d_f^*(p,q) \geqslant 0$ and $d_f^*(p,q) = 0 \Leftrightarrow p = q$,

(ii) $d_f^*(p,q) = d_f^*(q,p)$,

(iii) $d_f^*(p,q) \leqslant d_f^*(p,r) + d_f^*(r,q)$.

The geodesic distance as defined in the binary case (Lantuéjoul and Beucher, 1980) is a particular case of the generalized geodesic distance where f has only two values: 1 inside the geodesic mask and $+\infty$ outside.

The generalized geodesic distance separating a point p from a set Y is defined as follows:

$$d_f^*(p,Y) = \min_{y \in Y} d_f^*(p,y).$$

The generalized geodesic distance function $D_{f,Y}^*$ is defined by associating to each point of $\mathscr{D}_f$ their generalized geodesic distance to Y.

4. Generalized geodesic operators

Both the geodesic time and the generalized geodesic distance can be used to generalize the concepts of binary morphology to grey-scale geodesic masks. In this letter, we develop the generalization based on the geodesic time. Similar developments are obtained when using the generalized geodesic distance instead of the geodesic time (Soille, 1992).

Let X be a set defined on the domain of definition of a grey-scale image f. The geodesic dilation of size n of X with respect to f is denoted by $\delta_f^{(n)}(X)$ and is defined as the set of points of $\mathscr{D}_f$ such that their geodesic time to X is lower or equal to n:

$$\delta_f^{(n)}(X) = \{x \in \mathscr{D}_f \mid t_f(x,X) \leqslant n\}.$$

The geodesic erosion of size n of X in f is denoted by $\varepsilon_f^n(X)$ and is defined as the set of points of X such that their geodesic time to the complement of X, denoted by X^c, is strictly greater than n:

$$\varepsilon_f^{(n)}(X) = \{x \in X \mid t_f(x,X^c) > n\}.$$

Note that the proposed geodesic dilations and erosions are dual with respect to set complementation:

$$\varepsilon_f^{(n)}(X) = [\delta_f^{(n)}(X^c)]^c.$$

Finally, let X be a set composed of N disconnected components X_i. The geodesic influence zone of a connected component X_i of X is the set of points of the domain of definition of f whose geodesic time to X_i is

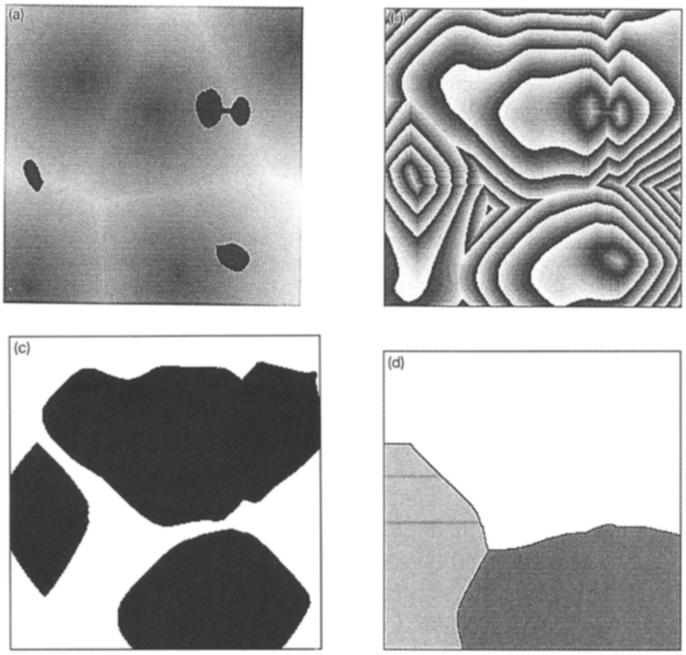

Fig. 3. Generalized geodesic operators for a binary marker defined over a grey-scale image. (a) Set X in a grey-scale geodesic mask f. (b) Time function $T_f(X)$. (c) Geodesic dilation $\delta_f^{(2795)}(X)$. (d) Geodesic influence zones $IZ_f(X)$.

smaller than those to other connected components of X:

$$IZ_f(X_i)=\{x\in\mathcal{D}_f \mid \forall j\in[1,N]\setminus\{i\},\ t_f(x,X_i)<t_f(x,X_j)\}.$$

The influence zones of X in f are denoted by $IZ_f(X)$ and defined as the union of the influence zones of all connected components of X. The boundaries of these influence zones correspond to the skeleton by influence zones or *SKIZ* of X in f. These definitions are illustrated in Fig. 3.

Efficient algorithms for computing generalized geodesic operators and using hierarchical (Meyer, 1990) or priority queues of pixels are detailed in (Soille, 1992).

5. Application

Given two points p and q in an image f, we are looking for the geodesics linking these points. These geodesics may be called minimal paths since the sum of the grey-levels along these paths is minimum (see Section 2). A solution based on curve evolution theory is presented by Kimmel and Bruckstein (1993). Here, we propose to use simple geodesic time calculations.

By definition, the sum $t_f(x,p)+t_f(x,q)$ is constant and equals $t_f(p,q)$ for all points x belonging to the geodesics linking p to q. Hence, the points belonging to the paths can be extracted by thresholding the sum of the geodesic time functions from p and q at the value corresponding to $t_f(p,q)$ (i.e., the value of $T_f(p)$ at q of $T_f(q)$ at p). An example is shown in Fig. 4.

Kiryati and Székely (1993) proposed a method for estimating shortest paths on digitized three-dimensional surfaces. In principle, their approach can be applied to grey-scale images since graphs of two-dimensional grey-scale images can be viewed as three-dimensional surfaces. However, the resulting shortest paths depend on the scaling of intensity values of the original image (see for instance Figs. 4 and 6 in (Kiryati and Székely, 1993)). Indeed, length measurements on the graph of a function mix spatial and intensity units, and are therefore not dimensional (Soille et al., 1992). Geodesic time calculations are

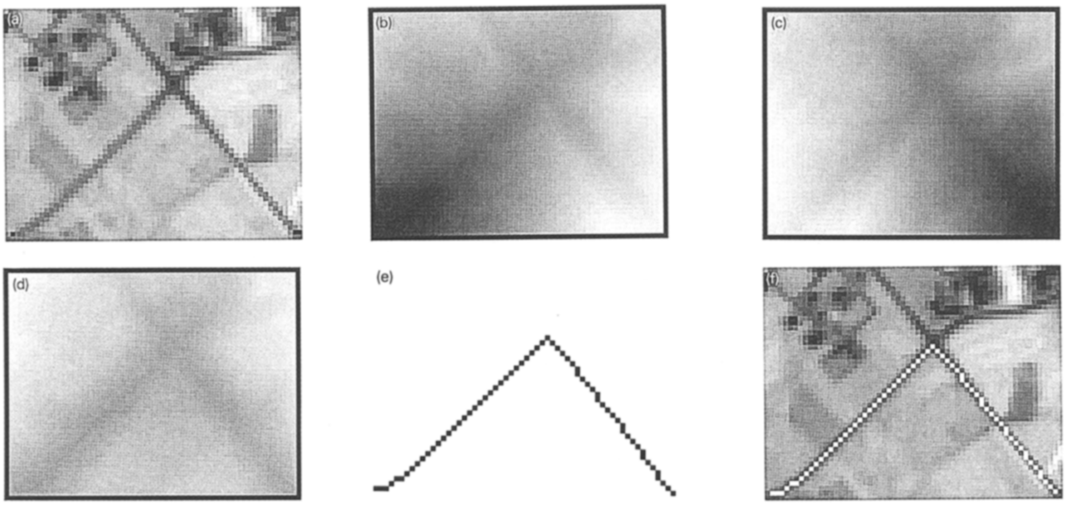

Fig. 4. Geodesic or minimal path detection. The minimal path (e) is obtained by thresholding the image (d), see text. (a) Grey-scale image or geodesic mask, f. (b) Geodesic time function from the lower left pixel q: $T_f(q)$. (c) Geodesic time function from the lower right pixel p: $T_f(p)$. (d) Sum of the time functions from p and q: $T_f(p)+T_f(q)$. (e) Geodesic or minimal path linking p to q in f. (f) Minimal path superimposed on the original image.

dimensional since they commute with scalings Λ of factor λ of the intensity values (see Eqs. (1) and (2)):

$$t_{\Lambda f}(p,q)=\lambda\, t_f(p,q)\ .$$

It follows that shortest paths calculated from time calculations are dimensional and even invariant to scalings of the intensity values of the original image.

6. Conclusion

Binary geodesic operators have been generalized for taking grey-scale geodesic masks into account. It is expected that the new operators introduced in this letter will be successfully applied to any image whose grey-scale values can be considered as a brake upon the expansion of some connected components of their domain of definition. Digital elevation models are a good example and interesting applications have already been demonstrated in (Soille, 1992). Applications of the generalized geodesic distance (see Section 4) to shape description and interpolation of contour data are presented in (Soille, 1994). The simulation of physical phenomena such as those studied by Jeulin et al. (1992) should also benefit of our developments.

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