ECE 5605: Stochastic Signals and Systems – Fall 2018

Homework #2

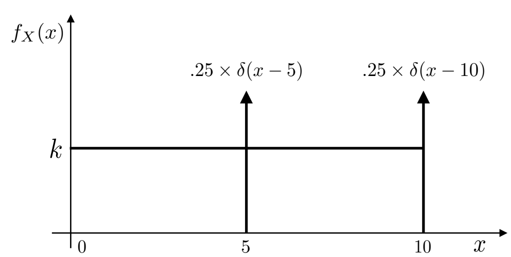
Due: September 13, 2018 (1:45pm)

Reading: Chapters 3 and 4 of the textbook.
Total points: 100

1. (5 pts) Suppose that a point is selected uniformly at random from inside the unit circle. Let Y be the distance of the point from the center of the unit circle.
   (a) (1 pt) Find the sample space S_Y of Y.
   (b) (1 pt) Find the equivalent event in the original sample space S for the event $\{Y \leq y\}$.
   (c) (3 pts) Find $P[Y \leq y]$.

2. (10 pts) Consider a random variable with cdf:
$$F_X(x) = \begin{cases} 1 & x > 0 \\ (1/3) + (2/3)(x+1)^2 & -1 \leq x \leq 0 \\ 0 & x < -1 \end{cases}$$
   Find the probability of the events: $A = \{X > 1/3\}, B = \{|X| \geq 1\}, C = \{|X - 1/3| < 1\}, D = \{X < 0\}$.

3. (10 pts) The pdf of a mixed random variable is shown below.



   (a) (4 pts) What is the value of the constant k?
   (b) (4 pts) Compute $P[X \leq 5]$ and $P[5 \leq X < 10]$.
   (c) (2 pts) Plot the cumulative distribution function (cdf).

   Note: $\delta(\cdot)$ is the delta function. Please refer to Section 4.2.1 of the textbook for more details.

4. (5 pts) Let X be is a continuous random variable with strictly increasing CDF $F_X(x)$. Show that $Y = F_X^{-1}(U)$ has the same distribution as X, where U is uniformly distributed between zero and one.
   Note: Using this method you can simulate non-uniform random variables if you know how to simulate uniform random variable.

5. (15 pts) Let X be the exponential random variable with pdf
$$f_X(x) = \lambda e^{-\lambda x}, x \geq 0 \text{ and } \lambda > 0.$$
   (a) (5 pts) Find and plot $F_X(x|X > t)$. How does $F_X(x|X > t)$ differ from $F_X(x)$?

(b) (5 pts) Find and plot $f_X(x|X > t)$.

(c) (5 pts) Show that $P[X > t+x|X > t] = P[X > x]$. Explain why this is termed *memoryless property.*

6. (10 pts) In a certain random experiment the random variable X has a pdf of the form

$$f_X(x) = \begin{cases} \frac{1}{2}\cos(x) & -\pi/2 < x < \pi/2 \\ 0 & |x| \geq \pi/2 \end{cases}$$

Let A be the event that $0 \leq X \leq \pi/6$.

(a) (4 pts) Find the cdf of X.

(b) (2 pts) What is the probability of event A?

(c) (2 pts) What is the conditional cdf of X (conditioned on A)?

(d) (2 pts) What is the conditional pdf of X (conditioned on A)?

7. (5 pts) Show that $P[A] = P[A|X \leq x]F_X(x) + P[A|X > x](1 - F_X(x))$.

8. (10 pts) The random variable X is Gaussian with mean 10 and unit standard deviation. Find

$$f_X(x|(X-10)^2 < 4).$$

9. (5 pts) The integer-valued random variable N has the probability mass function

$$P[N = k] = \begin{cases} \frac{c}{k^2} & k \geq 1 \\ 0 & k < 1 \end{cases}$$

(a) (2 pts) Find the normalization constant c. Note: $\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$.

(b) (3 pts) Conditioned on $N \leq 4$, find the *conditional* probability mass function of N.

10. (10 pts) A random variable X has cdf:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - \frac{1}{4}e^{-2x} & x \geq 0 \end{cases}$$

(a) (5 pts) Find $F_X(x|\{X > 0\})$.

(b) (5 pts) Find $F_X(x|\{X = 0\})$.

11. (15 pts) In this problem, we prove the continuous versions of Bayes's and total probability theorems.

(a) (2 pts) Assume that events $A_1, A_2 \ldots A_n$ form a partition of sample space S, i.e., $A_j \cap A_k = \varnothing$ for all $j \neq k$ and $\cup_{k=1}^{n} A_k = S$. Using total probability theorem, show that

$$F_X(x) = \sum_{k=1}^{n} F_X(x|A_k)P[A_k] \qquad f_X(x) = \sum_{k=1}^{n} f_X(x|A_k)P[A_k] \tag{1}$$

(b) (3 pts) Using Bayes' theorem, show that

$$P[A|x_1 < X \leq x_2] = \frac{F_X(x_2|A) - F_X(x_1|A)}{F_X(x_2) - F_X(x_1)}P[A]. \tag{2}$$

(c) (10 pts) As discussed in the class, the right way of handling $P[A|X = x]$ is in terms of the following limit (because $P[X = x]$ can in general be 0):

$$P[A|X = x] = \lim_{\Delta x \to 0} P[A|x < X \leq x + \Delta x]. \tag{3}$$

Using (2) and (3), show that

$$f_X(x|A) = \frac{P[A|X = x]}{P[A]} f_X(x). \tag{4}$$

Note that this is the continuous version of Bayes' theorem. Using (4), show that

$$P[A] = \int_{-\infty}^{\infty} P[A|X = x] f_X(x) \mathrm{d}x. \tag{5}$$

This is the continuous version of the total probability theorem.