Stochastic Signal Systems Homework 3

Mengfan Wang

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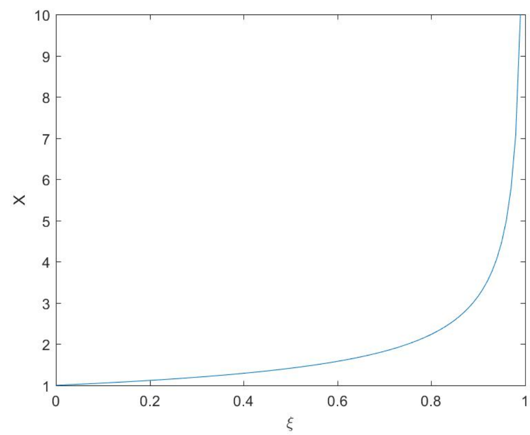

a

b

$$\begin{aligned} F_X(x) &= P[X \le x] && (1)\\ &= P[\frac{1}{\sqrt{1-\xi}} \le x] && (2)\\ &= P[1-\xi \ge \frac{1}{x^2}] && (3)\\ &= P[\xi \le 1 - \frac{1}{x^2}] && (4)\\ &= 1 - \frac{1}{x^2} && (5) \end{aligned}$$

So,

$$F_X(x) = \begin{cases} 1 - \frac{1}{x^2} & x \ge 1 \\ 0 & otherwise \end{cases} \quad (6)$$

c

$$P[X > 1] = 1 - F_X(1) = 1 - (1 - 1) = 1 \quad (7)$$

$$P[5 < X < 7] = F_X(7) - F_X(5^-) = 1/25 - 1/49 = 24/1225 \quad (8)$$

2 We have $X = \frac{Y-b}{a}$, then

$$\begin{aligned} f_Y(y) &= f_X(x)|\frac{dx}{dy}|_{x=\frac{y-b}{a}} && (9)\\ &= f_X(\frac{y-b}{a})/a && (10)\\ &= \frac{1}{\sqrt{2\pi}a} exp(-\frac{(y-b)^2}{2a^2}) && (11) \end{aligned}$$

So Y a Gaussian random variable with mean b and standard deviation a.

3 From the pdf of X we know that $P[X \leq 0||X > 0] = 0$. When $0 < X \leq 1$, we have $0 < Y = \sqrt{X} \leq 1$:

$$f_Y(y) = f_X(x)|\frac{dx}{dy}|_{x=y^2} \tag{12}$$
$$= \frac{1}{2y}2y \tag{13}$$
$$= 1 \tag{14}$$

So,

$$f_Y(y) = \begin{cases} 1 & 0 < y \leq 1 \\ 0 & otherwise \end{cases} \tag{15}$$

4 It's easy to know $Y \geq 0$:

$$f_Y(y) = f_X(x)|\frac{dx}{dy}|_{x=y} + f_X(x)|\frac{dx}{dy}|_{x=-\sqrt{y}} \tag{16}$$
$$= \frac{1}{\sqrt{2\pi}}exp(-\frac{y^2}{2}) + \frac{1}{2\sqrt{2\pi y}}exp(-\frac{y}{2}) \tag{17}$$

So,

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}}exp(-\frac{y^2}{2}) + \frac{1}{2\sqrt{2\pi y}}exp(-\frac{y}{2}) & y \geq 0 \\ 0 & y < 0 \end{cases} \tag{18}$$

5

$$p_Y(0) = P[Y = 0] = P[-1 \leq X \leq 1] = \Phi(\frac{1}{3}) - \Phi(-\frac{1}{3}) \tag{19}$$

$$p_Y(0.5) = p_Y(-0.5) = P[1 < X \leq 2] = \Phi(\frac{2}{3}) - \Phi(\frac{1}{3}) \tag{20}$$

$$p_Y(1) = p_Y(-1) = P[X > 2] = 1 - \Phi(\frac{2}{3}) \tag{21}$$

So,

$$p_Y(y) = \begin{cases} \Phi(\frac{1}{3}) - \Phi(-\frac{1}{3}) & y = 0 \\ \Phi(\frac{2}{3}) - \Phi(\frac{1}{3}) & y = \pm 0.5 \\ 1 - \Phi(\frac{2}{3}) & y = \pm 1 \end{cases} \tag{22}$$

6 $X = \ln Y$, so:

$$f_Y(y) = f_X(x)|\frac{dx}{dy}|_{x=\ln y} \tag{23}$$
$$= \frac{1}{\sqrt{2\pi}\sigma y}exp(-\frac{(\ln y - m)^2}{2\sigma^2}) \tag{24}$$

7 When $0 < X \leq \pi/2$, $X = \arcsin(Y)$; when $\pi/2 < X < \pi$, $X = \pi - \arcsin(Y)$. So,

$$f_Y(y) = f_X(x)|\frac{dx}{dy}|_{x=\arcsin(y)} + f_X(x)|\frac{dx}{dy}|_{x=\pi-\arcsin(y)} \tag{25}$$
$$= \frac{2}{\pi\sqrt{1-y^2}} \tag{26}$$

So,

$$f_Y(y) = \begin{cases} \frac{2}{\pi\sqrt{1-y^2}} & 0 < y \leq 1 \\ 0 & otherwise \end{cases} \tag{27}$$

8 $P[X > b] = P[X < b] = 1 - \Phi(1)$. When $-b \le x \le b$, $f_Y(y) = f_X(x)$. So,

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}b} exp(-\frac{y^2}{2b^2}) + (1 - \Phi(1))(\delta(y-b) + \delta(y+b)) & -b \le y \le b \\ 0 & otherwise \end{cases} \tag{28}$$

9 When $0 \le x < 1$, $x = \sqrt{y}$; When $-1 < x < 0$,$x = -\sqrt{y}$.

$$f_Y(y) = f_X(x)|\frac{dx}{dy}|_{x=\sqrt{y}} + f_X(x)|\frac{dx}{dy}|_{x=-\sqrt{y}} \tag{29}$$

$$= \frac{1}{4\sqrt{y}} + \frac{1}{2}\delta(y - \frac{1}{16}) \tag{30}$$

So,

$$f_Y(y) = \begin{cases} \frac{1}{4\sqrt{y}} + \frac{1}{2}\delta(y - \frac{1}{16}) & 0 \le x < 1 \\ 0 & otherwise, \end{cases} \tag{31}$$

And,

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{\sqrt{y}}{2} & 0 \le y < \frac{1}{16} \\ \frac{\sqrt{y}}{2} + \frac{1}{2} & \frac{1}{16} \le y \le 1 \\ 1 & y > 1 \end{cases} \tag{32}$$

10

$$E[X] = \sum_{k=0}^{\infty} k\frac{a^k}{k!}e^{-a} \tag{33}$$

$$= e^{-a}a\sum_{k=0}^{\infty}\frac{a^{k-1}}{(k-1)!} \tag{34}$$

$$= a \tag{35}$$

$$VAR[X] = E[X^2] - E[X]^2 \tag{36}$$

$$= \sum_{k=0}^{\infty} k^2\frac{a^k}{k!}e^{-a} - a^2 \tag{37}$$

$$= e^{-a}\sum_{k=0}^{\infty}\frac{(k-1+1)a^k}{(k-1)!} - a^2 \tag{38}$$

$$= e^{-a}(\sum_{k=0}^{\infty}\frac{a^k}{(k-1)!} + \sum_{k=0}^{\infty}\frac{a^k}{(k-2)!}) - a^2 \tag{39}$$

$$= e^{-a}(ae^a + a^2e^a) - a^2 \tag{40}$$

$$= a \tag{41}$$

11

$$E[X^n] = \int_0^1 x^n dx = \frac{x^{n+1}}{n+1}|_1 - \frac{x^{n+1}}{n+1}|_0 = \frac{1}{n+1} \tag{42}$$

$$E[X^n] = \int_a^b x^n \frac{1}{b-a}dx = \frac{1}{b-a}(\frac{x^{n+1}}{n+1}|_b - \frac{x^{n+1}}{n+1}|_a) = \frac{1}{b-a}\frac{b^{n+1} - a^{n+1}}{n+1} \tag{43}$$

12

$$
\begin{aligned}
E[X] &= \int_0^\infty x\lambda e^{-\lambda x} dx && (44) \\
&= \frac{1}{\lambda}\int_0^\infty x\lambda^2 e^{-\lambda x} dx && (45) \\
&= \frac{1}{\lambda}[(-e^{-\lambda x} - \lambda x e^{-\lambda x})|_\infty - (-e^{-\lambda x} - \lambda x e^{-\lambda x})|_0] && (46) \\
&= \frac{1}{\lambda} && (47)
\end{aligned}
$$

13

a

$$E[X] = \frac{1}{8}\sum_{k=-3}^{4} k = \frac{1}{2} \tag{48}$$

$$VAR[X] = E[X^2] - E[X]^2 = \frac{1}{8}\sum_{k=-3}^{4} k^2 - \frac{1}{4} = \frac{21}{4} \tag{49}$$

b

$$E[Y] = -2E[X^2] + 3 = -8 \tag{50}$$

$$VAR[Y] = E[Y^2] - E[Y]^2 = \frac{1}{8}\sum_{k=-3}^{4}(-2k^2+3) - E[Y]^2 = 105 \tag{51}$$