Stochastic Signal Systems Homework 5

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1 $z \in (0, \infty)$. When $0 < z \leq 1$:

$$F_Z(z) = \int_0^a \int_0^{zy} \frac{1}{a^2} dxdy \tag{1}$$
$$= \int_0^a \frac{zy}{a^2} dxdy \tag{2}$$
$$= \frac{z}{2} \tag{3}$$

When $z > 1$:

$$F_Z(z) = \int_0^{z/a} \int_0^{zy} \frac{1}{a^2} dxdy + \int_{a/z}^a \int_0^a \frac{1}{a^2} dxdy \tag{4}$$
$$= 1 - \frac{1}{2z} \tag{5}$$

So,

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} 0 & z \leq 0 \\ \frac{1}{2} & 0 < z \leq 1 \\ \frac{1}{2z^2} & z > 1 \end{cases} \tag{6}$$

2

a Because X and Y are independent, we have $f_Z(z|y) = f_X(x|y)|\frac{\partial x}{\partial z}| = f_X(z + y)$.

$$f_Z(z) = \int_{-\infty}^{\infty} f_Z(z|y) f_Y(y) dy \tag{7}$$
$$= \int_{-\infty}^{\infty} e^{-(z+2y)} U(z + y) U(y) dy \tag{8}$$
$$\tag{9}$$

When $z \geq 0$:

$$f_Z(z) = \int_0^{\infty} e^{-(z+2y)} dy \tag{10}$$
$$= \frac{1}{2} e^{-z} \tag{11}$$

When $z < 0$:

$$f_Z(z) = \int_{-z}^{\infty} e^{-(z+2y)} dy \tag{12}$$
$$= \frac{1}{2} e^{z} \tag{13}$$

So,

$$f_Z(z) = \begin{cases} \frac{1}{2} e^{z} & z < 0 \\ \frac{1}{2} e^{-z} & z \geq 0 \end{cases} \tag{14}$$

b $f_Z(z|y) = f_X(x|y)|\frac{\partial x}{\partial z}| = f_X(z/y)/y$.

$$f_Z(z) = \int_{-\infty}^{\infty} f_Z(z|y) f_Y(y) dy \tag{15}$$
$$= \int_{-\infty}^{\infty} \frac{1}{y} e^{-(z/y+y)} U(z/y) U(y) dy \tag{16}$$
$$\tag{17}$$

When $z < 0$, $U(z/y) = 0$ for $y > 0$. When $z \geq 0$:

$$f_Z(z) = \int_0^{\infty} \frac{1}{y} e^{-(z/y+y)} dy \tag{18}$$
$$= 2K_0(2\sqrt{z}), \tag{19}$$

while K_0 is modified Bessel functions of the second kind. This step is calculated by MATLAB. So,

$$f_Z(z) = \begin{cases} 0 & z < 0 \\ 2K_0(2\sqrt{z}) & z \geq 0 \end{cases} \tag{20}$$

c

$$F_Z(z) = P[X \leq z / Y \leq z] \tag{21}$$
$$= \int_0^z \int_0^{\infty} e^{-(x+y)} dx dy + \int_0^{\infty} \int_0^z e^{-(x+y)} dx dy - \int_0^z \int_0^z e^{-(x+y)} dx dy \tag{22}$$
$$= 1 - e^{-z} + 1 - e^{-z} - (1 - 2e^{-z} + e^{-2z}) \tag{23}$$
$$= 1 - e^{-2z} \tag{24}$$

So,

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} 0 & z < 0 \\ 2e^{-2z} & z \geq 0 \end{cases} \tag{25}$$

3 By calculating the shadow area under $X + Y = z$, we have:

$$F_Z(z) = \begin{cases} O & z < -1 \\ \frac{1}{2} - \frac{1}{2}z^2 & -1 \leq z < 0 \\ \frac{1}{2} + \frac{1}{2}z^2 & 0 \leq z < 1 \\ 1 & z > 1 \end{cases} \tag{26}$$

So,

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} -z & -1 \leq z < 0 \\ z & 0 \leq z \leq 1 \\ 0 & otherwise \end{cases} \tag{27}$$

4

a $f_Z(z|y) = f_X(x|y)|\frac{\partial x}{\partial z}| = |\frac{y}{(1-z)^2}| f_X(\frac{yz}{1-z})$.

$$f_Z(z) = \int_{-\infty}^{\infty} f_Z(z|y) f_Y(y) dy \tag{28}$$
$$= \int_{-\infty}^{\infty} \frac{|y|}{(1-z)^2} f_X(\frac{yz}{1-z}) f_Y(y) dy \tag{29}$$
$$= \int_{-\infty}^{\infty} \frac{|y|}{(1-z)^2} f_{X,Y}(\frac{yz}{1-z}, y) dy \tag{30}$$

b

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} \frac{|y|}{(1-z)^2} f_{X,Y}(\frac{yz}{1-z}, y)dy && (31)\\
&= \int_{0}^{\infty} \frac{y}{(1-z)^2} e^{-\frac{y}{1-z}} U(\frac{yz}{1-z})dy && (32)
\end{aligned}
$$

Because $z \in (0,1)$, $U(\frac{yz}{1-z}) = 1$:

$$
\begin{aligned}
f_Z(z) &= \int_{0}^{\infty} \frac{y}{(1-z)^2} e^{-\frac{y}{1-z}} dy && (33)\\
&= -\frac{y}{1-z} e^{\frac{-y}{1-z}} |_0^{\infty} + \int_0^{\infty} \frac{1}{1-z} e^{\frac{-y}{1-z}} dy && (34)\\
&= 1 && (35)
\end{aligned}
$$

5

$$
\begin{aligned}
F_P(p) &= \int_0^{\sqrt{p}} \int_0^1 12ri(1-i)drdi + \int_{\sqrt{p}}^1 \int_0^{p/i^2} 12ri(1-i)drdi && (36)\\
&= \int_0^{\sqrt{p}} 6i(1-i)di + \int_{\sqrt{p}}^1 6p^2i^{-3} - 6p^2i^{-2}di && (37)\\
&= 3p^2 + 6p - 8p^{\frac{3}{2}} && (38)
\end{aligned}
$$

So,

$$
f_P(p) = \frac{d}{dp}F_P(p) = \begin{cases} 6p + 6 - 12p^{\frac{1}{2}} & 0 \le p \le 1 \\ 0 & otherwise \end{cases} \quad (39)
$$

6

a

$$
\begin{aligned}
F_Z(z) &= \int_0^z \int_0^z \lambda_1\lambda_2 e^{-(\lambda_1 x + \lambda_2 y)}dxdy && (40)\\
&= \int_0^z -\lambda_2 e^{-(\lambda_1 z + \lambda_2 y)} + \lambda_2 e^{-\lambda_2 y}dy && (41)\\
&= e^{-(\lambda_1+\lambda_2)z} - e^{-\lambda_1 z} - e^{-\lambda_2 z} + 1 && (42)
\end{aligned}
$$

So,

$$
f_Z(z) = \frac{d}{dz}F_Z(z) = \lambda_1 e^{-\lambda_1 z} + \lambda_2 e^{-\lambda_2 z} - (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)z} \quad (43)
$$

The three terms are all exponentially distribution, so:

$$
E[Z] = \int_0^{\infty} zf_Z(z)dz = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2} \quad (44)
$$

b

$$
\begin{aligned}
F_W(w) &= P[X \le w / Y \le w] && (45)\\
&= \int_0^w \int_0^{\infty} \lambda_1\lambda_2 e^{-(\lambda_1 x + \lambda_2 y)}dxdy + \int_0^{\infty} \lambda_1\lambda_2 e^{-(\lambda_1 x + \lambda_2 y)}dxdy - \int_0^z \int_0^z \lambda_1\lambda_2 e^{-(\lambda_1 x + \lambda_2 y)}dxdy && (46)\\
&= 1 - e^{-\lambda_1 z} + 1 - e^{-\lambda_2 z} - (e^{-(\lambda_1+\lambda_2)z} - e^{-\lambda_1 z} - e^{-\lambda_2 z} + 1) && (47)\\
&= 1 - e^{-(\lambda_1+\lambda_2)z} && (48)
\end{aligned}
$$

So,

$$
f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} 0 & z < 0 \\ (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)z} & z \ge 0 \end{cases} \quad (49)
$$

7

a Suppose X is the distance between the location of the ambulance and a terminal, Y is the distance between the location of the accident and the same terminal. Z is the distance between the ambulance and the accident. The we have:

$$F_Z(z) = P[|X - Y| < z] = \frac{L - (L - z)^2}{L^2} = \frac{2zL - z^2}{L^2} \tag{50}$$

So,

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} \frac{2L-2z}{L^2} & 0 \le z \le L \\ 0 & otherwise \end{cases} \tag{51}$$

b

$$\begin{aligned} E[Z] &= \int_0^L z f_Z(z) dz && (52) \\ &= \int_0^L \frac{2L - 2z}{L^2} dz && (53) \\ &= \frac{1}{3} L && (54) \end{aligned}$$

The average time is $E[Z]/v = \frac{L}{3v}$.

8

$$\begin{aligned} P[X \le zY] &= \int_0^\infty f_Y(y) \int_0^{zy} f_X(x) dx dy && (55) \\ &= \int_0^\infty \frac{y}{\sigma_Y^2} e^{-\frac{y^2}{2\sigma_Y^2}} (1 - e^{-\frac{z^2 y^2}{2\sigma_X^2}}) dy && (56) \\ &= \frac{4\sigma_X^2 \sigma_Y^2}{2\sigma_Y^2 (2z^2 \sigma_Y^2 + 2\sigma_X^2)} e^{-\frac{z^2}{2\sigma_X^2} + \frac{1}{2\sigma_Y^2}} - e^{\frac{-y^2}{2\sigma_Y^2}} |_0^\infty && (57) \\ &= 1 - \frac{\sigma_x^2}{z^2 \sigma_Y^2 + \sigma_X^2} && (58) \end{aligned}$$