Stochastic Signal Systems Homework 4

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1

$$
\begin{aligned}
E[X|X>0] &= \int_{-\infty}^{\infty} xf(x|X>0)dx && (1)\\
&= \int_{0}^{\infty} \frac{2x}{\sqrt{2\pi}\sigma} exp(-\frac{x^2}{2\sigma^2})dx && (2)\\
&= -\frac{2\sigma}{\sqrt{2\pi}} exp(-\frac{x^2}{2\sigma^2})|_0^{\infty} && (3)\\
&= \frac{2\sigma}{\sqrt{2\pi}} && (4)
\end{aligned}
$$

$$
\begin{aligned}
E[X^2|X>0] &= \int_{-\infty}^{\infty} x^2 f(x|X>0)dx && (5)\\
&= \int_{0}^{\infty} \frac{2x^2}{\sqrt{2\pi}\sigma} exp(-\frac{x^2}{2\sigma^2})dx && (6)\\
&= -\int_{0}^{\infty} \frac{2x\sigma}{\sqrt{2\pi}} d(exp(-\frac{x^2}{2\sigma^2})) && (7)\\
&= -\frac{2x\sigma}{\sqrt{2\pi}} exp(-\frac{x^2}{2\sigma^2})|_0^{\infty} + \int_{0}^{\infty} \frac{2\sigma}{\sqrt{2\pi}} exp(-\frac{x^2}{2\sigma^2})dx && (8)\\
&= \sigma^2 && (9)
\end{aligned}
$$

So, $VAR[X|X>0] = E[X^2|X>0] - E[X|X>0]^2 = \sigma^2 - \frac{2\sigma^2}{\pi} = \frac{\pi-2}{\pi}\sigma^2$.

2 For a Gaussian distribution, we know its characteristic function is $\Phi_X(\omega) = exp(j\omega m - \frac{1}{2}\sigma^2\omega^2)$. So,

$$
\begin{aligned}
E[g(x)] &= \int_{-\infty}^{\infty} f(x)\cos x dx && (10)\\
&= \frac{1}{2}\int_{-\infty}^{\infty} f(x)(e^{jx} + e^{-jx})dx && (11)\\
&= \frac{1}{2}(\Phi_X(1) + \Phi_X(-1)) && (12)\\
&= \frac{1}{2}(exp(jm - \frac{1}{2}\sigma^2) + \exp(-jm - \frac{1}{2}\sigma^2)) && (13)\\
&= \cos m \exp(-\frac{1}{2}\sigma^2) && (14)
\end{aligned}
$$

3

$$
\Phi_X(\omega) = \int_{-b}^{b} \frac{e^{j\omega x}}{2b}dx = \frac{e^{j\omega b} - e^{-j\omega b}}{2j\omega b} \qquad (15)
$$

$$
\begin{aligned}
E[X] &= -j\frac{d}{d\omega}\Phi_X(\omega)|_{\omega=0} && (16)\\
&= -j\frac{j\omega b e^{j\omega b} + j\omega b e^{-j\omega b} - e^{j\omega b} + e^{-j\omega b}}{2j\omega^2 b}|_{\omega=0} && (17)\\
&= -j\lim_{\omega\to 0}\frac{(j\omega b e^{j\omega b} + j\omega b e^{-j\omega b} - e^{j\omega b} + e^{-j\omega b})''}{(2j\omega^2 b)''} && (18)\\
&= 0 && (19)
\end{aligned}
$$

4

$$\begin{aligned}
E[g(X)] &= \int_{-\infty}^{\infty} f(x)g(x)dx && (20)\\
&= \int_0^{\infty} e^{-x}\frac{\sin x}{x}dx && (21)\\
&= \int_0^{\infty} e^{-x}\frac{1}{2}\int_{-1}^{1} e^{jux}dudx && (22)\\
&= \int_{-1}^{1}\int_0^{\infty} \frac{1}{2}e^{-x}e^{jux}dxdu && (23)\\
&= \int_{-1}^{1} \frac{1}{2(ju-1)}e^{jux-x}|_0^{\infty}du && (24)\\
&= -\int_{-1}^{1} \frac{1}{2(ju-1)}du && (25)\\
&= -\frac{1}{2j}(ln|2j-2| - ln|-2j-2|) && (26)\\
&= -\frac{1}{2j}(ln2\sqrt{2} + \frac{3}{4}j\pi - ln2\sqrt{2} - \frac{5}{4}j\pi) && (27)\\
&= \frac{\pi}{4} && (28)
\end{aligned}$$

5

$$\begin{aligned}
f_X(x) &= \int_0^{\infty} xe^{-x(1+y)}dy && (29)\\
&= -e^{-x(1+y)}|_0^{\infty} && (30)\\
&= e^{-x} && (31)
\end{aligned}$$

So,

$$F_X(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x \leq 0 \end{cases} \tag{32}$$

$$\begin{aligned}
f_Y(y) &= \int_0^{\infty} xe^{-x(1+y)}dx && (33)\\
&= \int_0^{\infty} -\frac{x}{1+y}de^{-x(1+y)} && (34)\\
&= -\frac{x}{1+y}e^{-x(1+y)}|_0^{\infty} + \int_0^{\infty}\frac{1}{1+y}e^{-x(1+y)}dx && (35)\\
&= -\frac{1}{(1+y)^2}e^{-x(1+y)}|_0^{\infty} && (36)\\
&= \frac{1}{(1+y)^2} && (37)
\end{aligned}$$

So,

$$F_Y(y) = \begin{cases} \frac{1}{(1+y)^2} & y > 0 \\ 0 & y \leq 0 \end{cases} \tag{38}$$

6 According to the conditions, $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2}exp(-\frac{x^2+y^2}{2\sigma^2})$. Suppose $x = r\cos\theta$, $y = r\sin\theta$:

$$\begin{aligned}
P[X^2+Y^2 < q^2] &= \frac{1}{2\pi\sigma^2}\int_0^{2\pi}\int_0^{q} exp(-\frac{r^2}{2\sigma^2})rdrd\theta && (39)\\
&= \frac{1}{2\pi\sigma^2}\int_0^{2\pi}\sigma^2(1-exp(-\frac{q^2}{2\sigma^2}))d\theta && (40)\\
&= 1 - exp(-\frac{q^2}{2\sigma^2}) && (41)
\end{aligned}$$

7

a

$$F_{X,Y}(\pi/2,\pi/2) = c\int_0^{\pi/2}\int_0^{\pi/2} sin(x+y)dxdy \tag{42}$$
$$= c\int_0^{\pi/2} cosy - cos(\pi/2+y)dx \tag{43}$$
$$= 2c = 1 \tag{44}$$

So $c = 1/2$.

b When $x < 0$ or $y < 0$, $F_{X,Y}(x,y) = 0$; When $x > \pi/2, y > \pi/2$, $F_{X,Y}(x,y) = 1$. When $0 \le x \le \pi/2$ and $0 \le y \le \pi/2$,

$$F_{X,Y}(x,y) = \frac{1}{2}\int_0^x\int_0^y sin(x'+y')dy'dx' \tag{45}$$
$$= \frac{1}{2}\int_0^x cosx' - cos(x'+y)dx' \tag{46}$$
$$= \frac{1}{2}(sinx + siny - sin(x+y)) \tag{47}$$

When $0 \le x \le \pi/2$ but $y > \pi/2$:

$$F_{X,Y}(x,y) = \frac{1}{2}\int_0^{\pi/2}\int_0^x sin(x'+y')dx'dy' \tag{48}$$
$$= \frac{1}{2}\int_0^{\pi/2} cosy' - cos(x+y')dy' \tag{49}$$
$$= \frac{1}{2}(1 + sinx - sin(x+\pi/2)) \tag{50}$$

Similarly, when $0 \le y \le \pi/2$ but $x > \pi/2$, we have $F_{X,Y}(x,y) = \frac{1}{2}(1 + siny - sin(y+\pi/2))$ So,

$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 0/y < 0 \\ \frac{1}{2}(sinx + siny - sin(x+y)) & 0 \le x \le \pi/2 \& 0 \le y \le \pi/2 \\ \frac{1}{2}(1 + sinx - sin(x+\pi/2)) & 0 \le x \le \pi/2 \& y > \pi/2 \\ \frac{1}{2}(1 + siny - sin(y+\pi/2)) & x > \pi/2 \& 0 \le y \le \pi/2 \\ 1 & x > \pi/2 \& y > \pi/2 \end{cases} \tag{51}$$

c When $0 \le x \le \pi/2$:

$$f_X(x) = \frac{1}{2}\int_0^{\pi/2} sin(x+y')dy \tag{52}$$
$$= \frac{1}{2}(cosx - cos(x+\pi/2)) \tag{53}$$

So,

$$f_X(x) = \begin{cases} \frac{1}{2}(cosx - cos(x+\pi/2)) & 0 \le x \le \pi/2 \\ 0 & otherwise \end{cases} \tag{54}$$

Similarly,

$$f_Y(y) = \begin{cases} \frac{1}{2}(cosy - cos(y+\pi/2)) & 0 \le y \le \pi/2 \\ 0 & otherwise \end{cases} \tag{55}$$

8

a

$$
\begin{aligned}
f_X(x) &= \int_0^1 k(x+y)dy && (56)\\
&= k(xy + \frac{1}{2}y^2)|_0^1 && (57)\\
&= k(x + \frac{1}{2}) && (58)
\end{aligned}
$$

Similarly, $f_Y(y) = k(y + \frac{1}{2})$. So, $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$, they are not independent.

b

$$
\begin{aligned}
f_Y(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} && (59)\\
&= \frac{k(x+y)}{k(x+\frac{1}{2})} && (60)\\
&= \frac{x+y}{x+\frac{1}{2}} && (61)
\end{aligned}
$$

9 Firstly, we have

$$
f_X(x) = \int_x^\infty e^{-y}dy = e^{-x} \tag{62}
$$

and

$$
f_Y(y) = \int_0^y e^{-y}dx = ye^{-y} \tag{63}
$$

So,

$$
\begin{aligned}
E[X|y] &= \int_0^y x\frac{f_{X,Y}(x,y)}{f_Y(y)}dx && (64)\\
&= \int_0^y \frac{x}{y}dx && (65)\\
&= \frac{y}{2} && (66)
\end{aligned}
$$

$$
\begin{aligned}
E[Y|x] &= \int_x^\infty y\frac{f_{X,Y}(x,y)}{f_X(x)}dy && (67)\\
&= \int_x^\infty ye^{x-y}dy && (68)\\
&= \int_x^\infty -yde^{x-y} && (69)\\
&= -ye^{x-y}|_x^\infty + \int_x^\infty e^{x-y}dy && (70)\\
&= x+1 && (71)
\end{aligned}
$$

10

$$
\begin{aligned}
f_Y(y) &= \int_0^1 6(1-x-y)dx && (72)\\
&= 3(1-y)^2 && (73)
\end{aligned}
$$

$$
\begin{aligned}
E[X|y] &= \int_0^{1-y} x\frac{f_{X,Y}(x,y)}{f_Y(y)}dx && (74)\\
&= \int_0^{1-y} x\frac{2(1-x-y)}{(1-y)^2}dx && (75)\\
&= \frac{1}{3}(1-y) && (76)
\end{aligned}
$$

$$E[X^2|y] = \int_0^{1-y} x^2 \frac{f_{X,Y}(x,y)}{f_Y(y)} dx \tag{77}$$

$$= \int_0^{1-y} x^2 \frac{2(1-x-y)}{(1-y)^2} dx \tag{78}$$

$$= \frac{1}{6}(1-y)^2 \tag{79}$$

11

$$P[X < Y] = \int_0^1 \int_0^y e^{-x} dx dy \tag{80}$$

$$= \int_0^1 1 - e^{-y} dy \tag{81}$$

$$= e^{-1} \tag{82}$$

So, $P[X \geq Y] = 1 - P[X < Y] = 1 - e^{-1}$.