Positive-definite function

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In mathematics, the term positive-definite function may refer to a couple of different concepts.

Contents

In dynamical systems

A real-valued, continuously differentiable function *f* is positive definite on a neighborhood of the origin, *D*, if $f(0)=0$ and $f(x)>0$ for every non-zero $x\in D$.[1][2]

A function is negative definite if the inequality is reversed. A function is semidefinite if the strong inequality is replaced with a weak (≤ , ≥ 0)

A positive-definite function of a real variable *x* is a complex-valued function *f*:R → C such that for any real numbers x_1, ..., x_n the $n\times n$ matrix

$$A=(a_{i,j})_{i,j=1}^{n}\ ,\quad a_{i,j}=f(x_i-x_j)$$

is positive semi-definite (which requires *A* to be Hermitian; therefore *f*(-*x*) is the complex conjugate of *f*(*x*)).

In particular, it is necessary (but not sufficient) that

$$f(0)\geq 0\ ,\quad |f(x)|\leq f(0)$$

(these inequalities follow from the condition for *n*=1,2.)

Bochner's theorem

Positive-definiteness arises naturally in the theory of the Fourier transform; it is easy to see directly that to be positive-definite it is sufficient for *f* to be the Fourier transform of a function *g* on the real line with $g(y) \geq 0$.

The converse result is Bochner's theorem, stating that any continuous positive-definite function on the real line is the Fourier transform of a (positive) measure.[3]

Applications

In statistics, and especially Bayesian statistics, the theorem is usually applied to real functions. Typically, one takes *n* scalar measurements of some scalar value at points in R^d and one requires that points that are closely separated have measurements that are highly correlated. In practice, one must be careful to ensure that the resulting covariance matrix (an n-by-n matrix) is always positive definite. One strategy is to define a correlation matrix *A* which is then multiplied by a scalar to give a covariance matrix: this must be positive definite. Bochner's theorem states that if the correlation between two points is dependent only upon the distance between them (via function *f()*), then function *f()* must be positive definite to ensure the covariance matrix *A* is positive definite. See Kriging.

In this context, one does not usually use Fourier terminology and instead one states that *f(x)* is the characteristic function of a symmetric PDF.

Generalisation

One can define positive-definite functions on any locally compact abelian topological group; Bochner's theorem extends to this context. Positive-definite functions on groups occur naturally in the representation theory of groups on Hilbert spaces (i.e. the theory of unitary representations).

References

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- Wells, J. H.; Williams, L. R. *Embeddings and extensions in analysis*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 84. Springer-Verlag, New York-Heidelberg, 1975. vii+108 pp.

Notes

1. Verhulst, Ferdinand (1996). *Nonlinear Differential Equations and Dynamical Systems* (2nd ed.). Springer. ISBN 3-540-60934-2.
2. Hahn, Wolfgang (1967). *Stability of Motion*. Springer.
3. Bochner, Salomon (1959). *Lectures on Fourier integrals*. Princeton University Press.

External links

- Hazewinkel, Michiel, ed. (2001), "Positive-definite function", *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4

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