Topological vector space

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In mathematics, a topological vector space (also called a linear topological space) is one of the basic structures investigated in functional analysis. As the name suggests the space blends a topological structure (a uniform structure to be precise) with the algebraic concept of a vector space.

The elements of topological vector spaces are typically functions or linear operators acting on topological vector spaces, and the topology is often defined so as to capture a particular notion of convergence of sequences of functions.

Hilbert spaces and Banach spaces are well-known examples.

Unless stated otherwise, the underlying field of a topological vector space is assumed to be either the complex numbers C or the real numbers R.

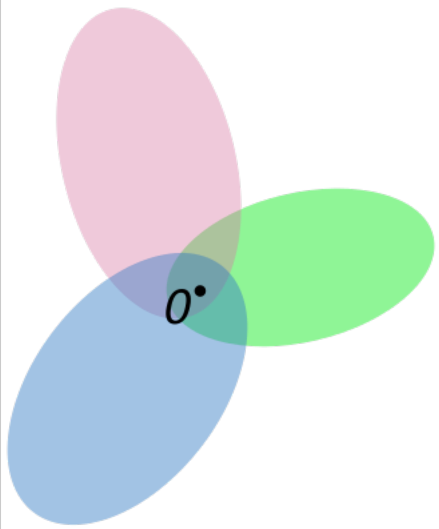
Contents

Definition

A topological vector space X is a vector space over a topological field K (most often the real or complex numbers with their standard topologies) that is endowed with a topology such that vector addition $X \times X \to X$ and scalar multiplication $K \times X \to X$ are continuous functions (where the domains of these functions are endowed with product topologies).

Some authors (e.g., Rudin) require the topology on X to be T_1; it then follows that the space is Hausdorff, and even Tychonoff. The topological and linear algebraic structures can be tied together even more closely with additional assumptions, the most common of which are listed below.

The category of topological vector spaces over a given topological field K is commonly denoted TVS_K or $TVect_K$. The objects are the topological vector spaces over K and the morphisms are the continuous K-linear maps from one object to another.



A family of neighborhoods of the origin with the above two properties determines uniquely a topological vector space. The system of neighborhoods of any other point in the vector space is obtained by translation.

Examples

Every normed vector space has a natural topological structure: the norm induces a metric and the metric induces a topology. This is a topological vector space because:

1. The vector addition $+ : V \times V \to V$ is jointly continuous with respect to this topology. This follows directly from the triangle inequality obeyed by the norm.
2. The scalar multiplication $\cdot : K \times V \to V$, where K is the underlying scalar field of V, is jointly continuous. This follows from the triangle inequality and homogeneity of the norm.

Therefore, all Banach spaces and Hilbert spaces, are examples of topological vector spaces.

There are topological vector spaces whose topology is not induced by a norm, but are still of interest in analysis. Examples of such spaces are spaces of holomorphic functions on an open domain, spaces of infinitely differentiable functions, the Schwartz spaces, and spaces of test functions and the spaces of distributions on them. These are all examples of Montel spaces. On the other hand, infinite-dimensional Montel spaces are never normable.

A topological field is a topological vector space over each of its subfields.

Product vector spaces

A cartesian product of a family of topological vector spaces, when endowed with the product topology, is a topological vector space. For instance, the set X of all functions f : $\mathbb{R} \to \mathbb{R}$: this set X can be identified with the product space $\mathbb{R}^{\mathbb{R}}$ and carries a natural product topology. With this topology, X becomes a topological vector space, called the *space of pointwise convergence*. The reason for this name is the following: if (f_n) is a sequence of elements in X, then f_n has limit f in X if and only if $f_n(x)$ has limit $f(x)$ for every real number x. This space is complete, but not normable: indeed, every neighborhood of 0 in the product topology contains lines, *i.e.*, sets $K f$ for $f \neq 0$.

Topological structure

A vector space is an abelian group with respect to the operation of addition, and in a topological vector space the inverse operation is always continuous (since it is the same as multiplication by −1). Hence, every topological vector space is an abelian topological group.

Let *X* be a topological vector space. Given a subspace $M \subset X$, the quotient space *X/M* with the usual quotient topology is a Hausdorff topological vector space if and only if *M* is closed.[1] This permits the following construction: given a topological vector space *X* (that is probably not Hausdorff), form the quotient space *X / M* where *M* is the closure of {0}. *X / M* is then a Hausdorff vector topological space that can be studied instead of *X*.

In particular, topological vector spaces are uniform spaces and one can thus talk about completeness, uniform convergence and uniform continuity. (This implies that every Hausdorff topological vector space is completely regular.[2]) The vector space operations of addition and scalar multiplication are actually uniformly continuous. Because of this, every topological vector space can be completed and is thus a dense linear subspace of a complete topological vector space.

The Birkhoff–Kakutani theorem gives that the following three conditions on a topological vector space *V* are equivalent:[3]

- The origin 0 is closed in *V*, and there is a countable basis of neighborhoods for 0 in *V*.
- *V* is metrizable (as a topological space).
- There is a translation-invariant metric on *V* that induces the given topology on *V*.

A metric linear space means a (real or complex) vector space together with a metric for which addition and scalar multiplication are continuous. By the Birkhoff–Kakutani theorem, it follows that there is an equivalent metric that is translation-invariant.

More strongly: a topological vector space is said to be *normable* if its topology can be induced by a norm. A topological vector space is normable if and only if it is Hausdorff and has a convex bounded neighborhood of 0.[4]

A linear operator between two topological vector spaces which is continuous at one point is continuous on the whole domain. Moreover, a linear operator *f* is continuous if *f(V)* is bounded for some neighborhood *V* of 0.

A hyperplane on a topological vector space *X* is either dense or closed. A linear functional *f* on a topological vector space *X* has either dense or closed kernel. Moreover, *f* is continuous if and only if its kernel is closed.

Every Hausdorff finite-dimensional topological vector space is isomorphic to K^n for some topological field K. In particular, a Hausdorff topological vector space is finite-dimensional if and only if it is locally compact.

Local notions

A subset E of a topological vector space X is said to be

- *balanced* if $tE \subset E$ for every scalar $|t| \leq 1$
- *bounded* if for every neighborhood V of 0, then $E \subset tV$ when t is sufficiently large.[5]

The definition of boundedness can be weakened a bit; E is bounded if and only if every countable subset of it is bounded. Also, E is bounded if and only if for every balanced neighborhood V of 0, there exists t such that $E \subset tV$. Moreover, when X is locally convex, the boundedness can be characterized by seminorms: the subset E is bounded iff every continuous semi-norm p is bounded on E.

Every topological vector space has a local base of absorbing and balanced sets.

A sequence $\{x_n\}$ is said to be Cauchy if for every neighborhood V of 0, the difference $x_m - x_n$ belongs to V when m and n are sufficiently large. Every Cauchy sequence is bounded, although Cauchy nets or Cauchy filters may not be bounded. A topological vector space where every Cauchy sequence converges is sequentially complete but may not be complete (in the sense Cauchy filters converge). Every compact set is bounded.

Types

Depending on the application additional constraints are usually enforced on the topological structure of the space. In fact, several principal results in functional analysis fail to hold in general for topological vector spaces: the closed graph theorem, the open mapping theorem, and the fact that the dual space of the space separates points in the space.

Below are some common topological vector spaces, roughly ordered by their *niceness*.

- F-spaces are complete topological vector spaces with a translation-invariant metric. These include L^p spaces for all $p > 0$.
- Locally convex topological vector spaces: here each point has a local base consisting of convex sets. By a technique known as Minkowski functionals it can be shown that a space is locally convex if and only if its topology can be defined by a family of semi-norms. Local convexity is the minimum requirement for "geometrical" arguments like the Hahn–Banach theorem. The L^p spaces are locally convex (in fact, Banach spaces) for all $p \geq 1$, but not for $0 < p < 1$.
- Barrelled spaces: locally convex spaces where the Banach–Steinhaus theorem holds.
- Bornological space: a locally convex space where the continuous linear operators to any locally convex space are exactly the bounded linear operators.
- Stereotype space: a locally convex space satisfying a variant of reflexivity condition, where the dual space is endowed with the topology of uniform convergence on totally bounded sets.
- Montel space: a barrelled space where every closed and bounded set is compact
- Fréchet spaces: these are complete locally convex spaces where the topology comes from a translation-invariant metric, or equivalently: from a countable family of semi-norms. Many interesting spaces of functions fall into this class. A locally convex

F-space is a Fréchet space.

- LF-spaces are limits of Fréchet spaces. ILH spaces are inverse limits of Hilbert spaces.
- Nuclear spaces: these are locally convex spaces with the property that every bounded map from the nuclear space to an arbitrary Banach space is a nuclear operator.
- Normed spaces and semi-normed spaces: locally convex spaces where the topology can be described by a single norm or semi-norm. In normed spaces a linear operator is continuous if and only if it is bounded.
- Banach spaces: Complete normed vector spaces. Most of functional analysis is formulated for Banach spaces.
- Reflexive Banach spaces: Banach spaces naturally isomorphic to their double dual (see below), which ensures that some geometrical arguments can be carried out. An important example which is *not* reflexive is L^1, whose dual is L^∞ but is strictly contained in the dual of L^∞.
- Hilbert spaces: these have an inner product; even though these spaces may be infinite-dimensional, most geometrical reasoning familiar from finite dimensions can be carried out in them. These include L^2 spaces.
- Euclidean spaces: R^n or C^n with the topology induced by the standard inner product. As pointed out in the preceding section, for a given finite n, there is only one n-dimensional topological vector space, up to isomorphism. It follows from this that any finite-dimensional subspace of a TVS is closed. A characterization of finite dimensionality is that a Hausdorff TVS is locally compact if and only if it is finite-dimensional (therefore isomorphic to some Euclidean space).

Dual space

Every topological vector space has a continuous dual space—the set V^* of all continuous linear functionals, i.e. continuous linear maps from the space into the base field K. A topology on the dual can be defined to be the coarsest topology such that the dual pairing each point evaluation $V^* \to K$ is continuous. This turns the dual into a locally convex topological vector space. This topology is called the weak-* topology. This may not be the only natural topology on the dual space; for instance, the dual of a normed space has a natural norm defined on it. However, it is very important in applications because of its compactness properties (see Banach–Alaoglu theorem). Caution: Whenever V is a not-normable locally convex space, then the pairing map $V^* \times V \to K$ is never continuous, no matter which vector space topology one chooses on V^*.

Notes

1. In particular, X is Hausdorff if and only if the set {0} is closed (i.e., X is a T_1 space).
2. H. Schaefer, 16

3. Kőthe (1983), section 15.11.
4. http://eom.springer.de/T/t093180.htm
5. Rudin

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