Functional (mathematics)

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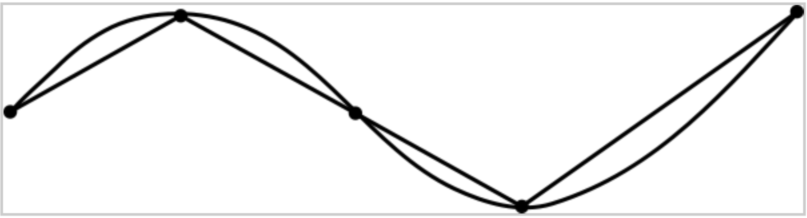
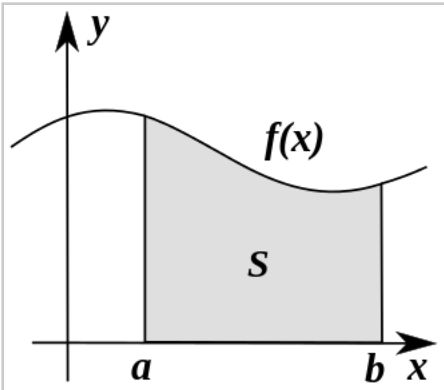
In mathematics, and particularly in functional analysis and the calculus of variations, a functional is a function from a vector space into its underlying field of scalars. Commonly the vector space is a space of functions; thus the functional takes a function for its input argument, then it is sometimes considered a *function of a function* (a higher-order function). Its use originates in the calculus of variations, where one searches for a function that minimizes a given functional. A particularly important application in physics is searching for a state of a system that minimizes the energy functional.



The arc length functional has as its domain the vector space of rectifiable curves (a subspace of $C([0,1],\mathbb{R}^3)$), and outputs a real scalar. This is an example of a non-linear functional.



The Riemann integral is a linear functional on the vector space of Riemann-integrable functions from $\mathbb{R}$ to $\mathbb{R}$.

Contents

Functional details

Duality

The mapping

$$x_0 \mapsto f(x_0)$$

is a function, where x_0 is an argument of a function f. At the same time, the mapping of a function to the value of the function at a point

$$f \mapsto f(x_0)$$

is a *functional*, here x_0 is a parameter.

Provided that f is a linear function from a linear vector space to the underlying scalar field, the above linear maps are dual to each other, and in functional analysis both are called linear functionals.

Definite integral

Integrals such as

$$f \mapsto I[f] = \int_\Omega H(f(x), f'(x), \ldots)\ \mu(\mathrm{d}x)$$

form a special class of functionals. They map a function f into a real number, provided that H is real-valued. Examples include

- the area underneath the graph of a positive function f

$$f \mapsto \int_{x_0}^{x_1} f(x)\ \mathrm{d}x$$

- L^p norm of functions

$$f \mapsto \left(\int |f|^p\ \mathrm{d}x\right)^{1/p}$$

- the arclength of a curve in 2-dimensional Euclidean space

$$f \mapsto \int_{x_0}^{x_1} \sqrt{1 + |f'(x)|^2}\ \mathrm{d}x$$

Vector scalar product

Given any vector $\vec{x}$ in a vector space X, the scalar product with another vector $\vec{y}$, denoted $\vec{x} \cdot \vec{y}$ or $\langle \vec{x}, \vec{y} \rangle$, is a scalar. The set of vectors $\vec{x}$ such that $\vec{x} \cdot \vec{y}$ is zero is a vector subspace of X, called the *null space* or kernel of X.

Locality

If a functional's value can be computed for small segments of the input curve and then summed to find the total value, the functional is called local. Otherwise it is called non-local. For example:

$$F(y) = \int_{x_0}^{x_1} y(x)\ \mathrm{d}x$$

is local while

$$F(y) = \frac{\int_{x_0}^{x_1} y(x)\ \mathrm{d}x}{\int_{x_0}^{x_1} (1 + [y(x)]^2)\ \mathrm{d}x}$$

is non-local. This occurs commonly when integrals occur separately in the numerator and denominator of an equation such as in calculations of center of mass.

Functional equation

The traditional usage also applies when one talks about a functional equation, meaning an equation between functionals: an equation $F = G$ between functionals can be read as an 'equation to solve', with solutions being themselves functions. In such equations there may be several sets of variable unknowns, like when it is said that an *additive* function f is one *satisfying the functional equation*

$$f(x + y) = f(x) + f(y).$$

Functional derivative and functional integration

Functional derivatives are used in Lagrangian mechanics. They are derivatives of functionals: i.e. they carry information on how a functional changes when the input function changes by a small amount.

Richard Feynman used functional integrals as the central idea in his sum over the histories formulation of quantum mechanics. This usage implies an integral taken over some function space.

See also

- Linear functional
- Optimization (mathematics)
- Tensor

References

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- Lang, Serge (2002), "III. Modules, §6. The dual space and dual module", *Algebra*, Graduate Texts in Mathematics, 211 (Revised third ed.), New York: Springer-Verlag, pp. 142–146, ISBN 978-0-387-95385-4, Zbl 0984.00001, MR 1878556

Categories: Types of functions
