Coding Exercises

Forum for Pushing the Boundary

December 2023

Problem 1

Consider the function $f(x) = \cos(x^3 + \sin(x))$ defined on $\mathbb{R}$.

1. Graph the function f on the interval $[-5, 5]$.
2. Graph the function $f(f(x))$ on $[-5, 5]$.
3. Find the integral $I = \int_0^1 \cos(x^3 + \sin(x))dx$.

Problem 2

Giving a periodic function $f : [0, 2\pi] \to \mathbb{C}$, Fourier theory tell us that we can decompose it as sum of trigonometric function, ie

$$f(x) \sim S_N(f, x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx},$$

where the Fourier coefficients are defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx}dx$$

Now let's investigate this numerically.

1. Take $f(x) = e^{x(x-2\pi)}$. Find the Fourier coefficient $\hat{f}(n)$ for $n \in [-50 : 50]$.
2. Reconstruct $S_N(f, x)$ for $N = 2, 10, 30, 50$ and plot their graph along side with f.
3. Compute the error $\sup_{x\in[0,2\pi]} |f(x) - S_N(f, x)|$ for N above.

Problem 3

Consider a function $f_r(x) = rx(1-x)$ where $r > 1$ and $x \in [0, 1]$.

1. Now fix $r_0 = \sqrt{2}$ and $x_0 = 0.3$. Find the limit of the sequence (a_n) defined by $a_1 = f_{r_0}(x_0), a_{n+1} = f_{r_0}(a_n)$.
2. Now let $r_0 = \sqrt{2}$ fixed and $x \in [0, 1]$ varried. Let $\ell_{r_0}(x) = \lim_{n\to\infty} a_n(x)$ if the limit exist, where $(a_n(x))$ is defined by $a_1(x) = f_{r_0}(x)$ and $a_{n+1}(x) = f_{r_0}(a_n(x))$. Graph ℓ_{r_0}
3. Now let $x_0 = \sqrt{0.5}$ fixed and $r \in [1, 3.2]$ is varied. Study the limit behavior of the sequence $(p_n(r))$ defined by $p_1(r) = f_r(x_0)$ and $p_{n+1}(r) = f_r(p_n(r))$

Problem 4

Consider the polynomial $f(z) = z^2 + c$ where $z \in \mathbb{C}$. We say a complex number c is in the Mandelbrot set M if the sequence $z_1 = 0, z_{n+1} = f(z_n)$ is convergent.

1. Find the set $M \subset \mathbb{C}$.
2. Plot M.

Problem 5

1. Write a code for function Irrepoly(n, p) that produce all irreducible polynomial of degree n in the ring $\mathbb{Z}_p[X]$.
2. Consider the ring of Quadratic integer $\mathbb{Z}[i\sqrt{3}] = \{a + i\sqrt{3}b : a, b \in \mathbb{Z}\}$. Write a code for the function $\pi_{i\sqrt{3}}(n)$ = number of primes inside the disk $D(0, n)$. Can you guess a version of prime number theorem in this ring? **Note:** $\mathbb{Z}[i\sqrt{3}]$ is a Euclidean domain, so it make sense to talk about prime.