Chapter 1

Sequence

1.1 Homework

Exercise 1.1.1 Let $(a)_n$ and (b_n) be real valued sequences such that $\lim a_n = a$ and $\lim b_n = b$ where $a, b \in \mathbb{R}$. Prove that

[1] $\lim(a_n + \lambda b_n) = a + \lambda b$, for all $\lambda \in \mathbb{R}$.

[2] $\lim a_n b_n = ab$.

[3] $\lim \frac{a_n}{b_n} = \frac{a}{b}$ whenever $b \neq 0$ and $b_n \neq 0$ for all n.

Proof. First we know that $\lim a_n = a$ and $\lim b_n = b$.

[1] Let $\lambda \in \mathbb{R}$. We want to make

$$\begin{aligned}|a_n + \lambda b_n - a - \lambda b| &= |a_n - a + \lambda(b_n - b)| \\ &\leq |a_n - a| + |\lambda| \cdot |b_n - b|\end{aligned}$$

as small as we want. Let $\epsilon > 0$. Because $\lim a_n = a$ and $\lim b_n = b$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - a| \leq \frac{\epsilon}{2}, \qquad \forall n \geq N_1$$

and

$$|b_n - b| \leq \frac{\epsilon}{2} \cdot \frac{1}{|\lambda| + 1}, \qquad \forall n \geq N_2$$

Note that these two inequalities both hold for all index n larger that $\max\{N_1, N_2\}$. Letting $N = \max\{N_1, N_2\}$, hence $\forall n \geq N$, we have

$$\begin{aligned}|a_n + \lambda b_n - a - \lambda b| &\leq |a_n - a| + |\lambda| \cdot |b_n - b| \\ &\leq \frac{\epsilon}{2} + \frac{|\lambda|}{|\lambda| + 1} \cdot \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

This shows that $\lim(a_n + \lambda b_n) = a + \lambda b$.

[2] Observe that

$$\begin{aligned}|a_nb_n - ab| &= |a_nb_n - a_nb + a_nb - ab| \\ &\leq |a_n||b_n - b| + |b||a_n - a|.\end{aligned}$$

As we'll soon prove (in later exercise) that a convergent sequence is bounded, hence there is $M > 0$ such that $|a_n| < M$ for all $n \in \mathbb{N}$. Now we can preceed the proof the same fashion as in the previous exercise, so I am going to omit the part $\max\{N_1, N_2\}$ for simplicity's sake.

Let $\epsilon > 0$. Because $\lim a_n = a$ and $\lim b_n = b$, there is a shared integer $N \in \mathbb{N}$ for which the inequalities

$$\begin{aligned}|a_n - a| &\leq \frac{1}{|b|+1} \cdot \frac{\epsilon}{2} \\ |b_n - b| &\leq \frac{\epsilon}{2M}\end{aligned}$$

both hold for all $n \geq N$. Therefore, for any $n \geq N$,

$$\begin{aligned}|a_nb_n - ab| &\leq |a_n||b_n - b| + |b||a_n - a| \\ &\leq M|b_n - b| + |b||a_n - a| \\ &\leq M \cdot \frac{\epsilon}{2M} + \frac{|b|}{|b|+1} \cdot \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

Thus $\boxed{\lim a_nb_n = ab.}$

[3] First, we claim that $\lim \frac{1}{b_n} = \frac{1}{b}$. Observe that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{1}{|b_n||b|} \cdot |b_n - b|$$

So we need to find the upper bound of $\frac{1}{|b_n|}$, i.e. the lower bound of $|b_n|$. Becase $\lim b_n = b \neq 0$, we can choose $\epsilon := \frac{|b|}{2}$ in the definition of the convergent b_n, hence there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, the inequality $|b_n - b| \leq \frac{|b|}{2}$ hold. From Triangle Inequality,

$$|b| \leq |b - b_n| + |b_n| \leq \frac{|b|}{2} + |b_n|$$

this shows that $|b_n| \geq \frac{|b|}{2}$. So we found that $\frac{1}{|b||b_n|} \leq \frac{2}{|b|^2}$ for each $n \geq N_1$.

Now let us fix an $\epsilon > 0$. Because $\lim b_n = b$, there is $N_2 \in \mathbb{N}$ such that

$$|b_n - b| \leq \frac{|b|^2}{2} \cdot \epsilon$$

for all $n \geq N_2$. Letting $N = \max\{N_1, N_2\}$, we obtain that for each $n \geq N$,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{1}{|b_n||b|} \cdot |b_n - b| \leq \frac{2}{|b|^2} \cdot \frac{|b|^2}{2} \cdot \epsilon = \epsilon$$

Now, we can finish the proof by writing

$$\lim \frac{a_n}{b_n} = \lim a_n \cdot \frac{1}{b_n} = a \cdot \frac{1}{b} = \frac{a}{b}.$$

$\square$

Exercise 1.1.2 Let (a_n) be a divergent sequence. Prove that $\lim \frac{1}{a_n} = 0$.

Proof. First fix an $\epsilon > 0$. Because (a_n) diverges, then there exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n| > \frac{1}{\epsilon}$. This implies that

$$\left| \frac{1}{a_n} \right| < \epsilon.$$

Hence $\lim a_n = 0$ as expected. $\square$

Exercise 1.1.3 If a sequence converges, prove that its limit is unique.

Proof. Let (a_n) be a convergent sequence, and suppose that $a, a_0 \in \mathbb{R}$ be its limit. To prove that its limit is unique, we want to show that $a = a_0$. Assume by the contrary that $a \neq a_0$, by the definition of convergent sequence, we can choose $\epsilon := \frac{|a-a_0|}{2} \neq 0$ so there's an $N \in \mathbb{N}$ such that both inequalities

$$|a_n - a| < \frac{|a - a_0|}{2} \quad \text{and} \quad |a_n - a_0| < \frac{|a - a_0|}{2}$$

hold for all $n \geq N$. Using Triangle Inequality, we get

$$\begin{aligned} |a - a_0| &= |a - a_n + a_n - a_0| \\ &\leq |a_n - a| + |a_n - a_0| \\ &< \frac{|a - a_0|}{2} + \frac{|a - a_0|}{2} \\ &= |a - a_0| \end{aligned}$$

a contradiction. Therefore, we must have that $a = a_0$. $\square$

Exercise 1.1.4 Prove that a convergent sequence is bounded. Is the converse true?

Proof. Let (a_n) converges to $a \in \mathbb{R}$. From definition, we can choose $\epsilon = 1$, then there's $N \in \mathbb{N}$ for which $|a_n - a| < \epsilon = 1$ holds for all $n \geq N$. Therefore, for all $n \geq N$

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a|.$$

Now, we let

$$M = \max\{|a_0|, |a_1|, \ldots, |a_{N-1}|, 1 + |a|\}$$

this simply shows that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Hence (a_n) is bounded. The converse statement is not generally true. Take for example the sequence (a_n) : $1, -1, 1, -1, \ldots$ is a bounded sequence yet it failed to have a limit. $\square$

> **Exercise 1.1.5** Prove that the convergence of (a_n) implies the convergence of $|a_n|$. Is the converse true?

Proof. Suppose that (a_n) converges to $a \in \mathbb{R}$. We claim that $\lim|a_n| = |a|$. Let $\epsilon > 0$ be an arbitrary positive real number. Because $\lim a_n = a$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \epsilon$. Using triangle inequality, we get

$$||a_n| - |a|| \leq |a_n - a| < \epsilon$$

for all $n \geq N$. This proves that $\lim|a_n| = |a|$.

The converse is not generally true. In fact, we can find a counter example to disprove the claim. Consider the sequence $a_n = (-1)^n$. It's easy to see that $|a_n| = 1$ converges, yet a_n diverges. $\square$

> **Exercise 1.1.6** Prove the density of $\mathbb{Q}$ in $\mathbb{R}$, i.e. that is to prove that for every $a, b \in \mathbb{R}$ with $a < b$, there is $q \in \mathbb{Q}$ such that $a < q < b$.

Proof. Without loss of generality, we can assume that $0 < a < b$. From Archimedean property, we can find $n \in \mathbb{N}$ such that $n > \frac{1}{b-a}$ or

$$\frac{1}{n} < b - a.$$

Let us define $K = \{k \in \mathbb{N} : \frac{k}{n} \leq a\}$ and let $m := \max K$. Therefore, $\frac{m}{n} \leq a$, and since m is the maximal element, it's next term must not in the set K, in other words $m + 1 \notin K$. Hence

$$a < \frac{m+1}{n} = \frac{m}{n} + \frac{1}{n} < a + (b - a) = b$$

We had just found $m, n \in \mathbb{N}$ such that $a < \frac{m+1}{n} < b$. This proves that there's a rational between any two real numbers. $\square$

> **Exercise 1.1.7** Prove the following via the definition of convergence:
>
> [1] $\lim \frac{2n+1}{n-2} = 2$
>
> [2] $\lim \frac{\sqrt{n-1}}{\sqrt{n}+1} = 1$
>
> [3] $\lim \frac{\sqrt[3]{n^3-3}}{2n} = \frac{1}{2}$
>
> [4] $\lim \frac{2\sqrt{n}+3}{n-3} = 0$
>
> [5] $\lim(n - \sqrt{n}) = \infty$.

Proof. Prove the following limits via definition:

[1] Let us fix $\epsilon > 0$. Using Archimedean property, there is $N \in \mathbb{N}$ such that $N > \frac{5}{\epsilon} + 2$ or

$$\frac{5}{N-2} < \epsilon.$$

It's also clear that $N > 2$. Hence for any $n \geq N$,

$$\begin{aligned}
\left|\frac{2n+1}{n-2} - 2\right| &= \left|\frac{2n+1-2n+4}{n-2}\right| \\
&= \left|\frac{5}{n-2}\right| = \frac{5}{n-2} && \text{(because } n > 2\text{)} \\
&\leq \frac{5}{N-2} && \text{(because } n \geq N\text{)} \\
&< \epsilon
\end{aligned}$$

[2] First we claim that $\left|\frac{\sqrt{n-1}}{\sqrt{n}+1} - 1\right| < \frac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$ greater than 1. For any $n > 1$, we have

$$\begin{aligned}
& (n-1)^2 < n(n-1) \\
\implies \quad & n - 1 < \sqrt{n(n-1)} \\
\implies \quad & n - \sqrt{n(n-1)} + \sqrt{n} < 1 + \sqrt{n} \\
\implies \quad & \frac{\sqrt{n} - \sqrt{n-1} + 1}{1 + \sqrt{n}} < \frac{1}{\sqrt{n}} \\
\implies \quad & 1 - \frac{\sqrt{n-1}}{\sqrt{n}+1} < \frac{1}{\sqrt{n}}
\end{aligned}$$

Note also that $\frac{n-1}{\sqrt{n}-1} - 1 = \frac{\sqrt{n-1} - \sqrt{n} - 1}{\sqrt{n}+1} < 0$, then we must have

$$\left|\frac{\sqrt{n-1}}{\sqrt{n}+1} - 1\right| = 1 - \frac{\sqrt{n-1}}{\sqrt{n}+1} < \frac{1}{\sqrt{n}}$$

as claimed. Now fix an $\epsilon > 0$. By Archimedean property, there is $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon^2}$. Therefore for all $n \geq N$,

$$\left|\frac{\sqrt{n-1}}{\sqrt{n}+1} - 1\right| < \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \epsilon.$$

Thus, $\boxed{\lim \frac{\sqrt{n-1}}{\sqrt{n}+1} = 1}$.

[3] As before, we first claim that $\left|\frac{\sqrt[3]{n^3-3}}{n} - 1\right| < \frac{3}{n^3}$ for any $n > 2$.

Observe that if $n > 2$ then $n^3 > 3 \Rightarrow 1 - \frac{3}{n^3} > 0$. It's also easy to see that $\sqrt[3]{n^3 - 3} < n$, and hence

$$\begin{aligned}\left|\frac{\sqrt[3]{n^3-3}}{n} - 1\right| &= 1 - \frac{\sqrt[3]{n^3-3}}{n} \\ &= 1 - \sqrt[3]{1 - \frac{3}{n^3}} \\ &= \frac{1 - (1 - \frac{3}{n^3})}{1 + \sqrt[3]{a} + \sqrt[3]{a^2}}, \qquad \text{where } a = 1 - \frac{3}{n^3} > 0 \\ &< \frac{3}{n^3}\end{aligned}$$

for any $n > 2$ as expected. Now fix $\epsilon > 0$. By Archimedean property there's $N \in \mathbb{N}$ such that $N \geq \sqrt[3]{\frac{3}{2\epsilon}}$. Thus for any $n \geq \max\{2, N\}$, we have

$$\left|\frac{\sqrt[3]{n^3-3}}{2n} - \frac{1}{2}\right| < \frac{3}{2n^3} \leq \frac{3}{2N^3} < \epsilon.$$

Therefore, $\boxed{\lim \frac{\sqrt[3]{n^3-3}}{2n} = \frac{1}{2}.}$

[4] Note that for any $n \geq 10$, we have

$$\frac{2\sqrt{n}+3}{n-3} < \frac{2\sqrt{n}+6}{n-9} = \frac{2}{\sqrt{n}-3}$$

Fix an $\epsilon > 0$. From Archimedean property, there is $N \in \mathbb{N}$ such that

$$N \geq \left(3 + \frac{2}{\epsilon}\right)^2.$$

Thus for any $n \geq \max\{10, N\}$ we obtain that

$$\left|\frac{2\sqrt{n}+3}{n-3}\right| < \frac{2}{\sqrt{n}-3} < \frac{2}{\sqrt{N}-3} < \epsilon.$$

Hence, $\boxed{\lim \frac{2\sqrt{n}+3}{\sqrt{n}-3} = 0}$.

[5] Let $\epsilon > 0$, and let $a_n := n - \sqrt{n} = \sqrt{n}(\sqrt{n} - 1)$. It's clear that (a_n) is an increasing sequence. Also, we have

$$n - \sqrt{n} = \left(\sqrt{n} - \frac{1}{2}\right)^2 - \frac{1}{4},$$

then from Archimedean property, there is $N \in \mathbb{N}$ such that

$$N > \left(\frac{1}{2} + \sqrt{\epsilon + \frac{1}{4}}\right)^2.$$

Thus, for any $n \geq N$, we have

$$\begin{aligned} |a_n| &= n - \sqrt{n} \geq N - \sqrt{N} \\ &= \left(\sqrt{N} - \frac{1}{2}\right)^2 - \frac{1}{4} > \epsilon \end{aligned}$$

Therefore, $\boxed{\lim(n - \sqrt{n}) = \infty.}$

$\square$