Properties of Real Numbers

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Content

Introduction

Introduction

From previous group presentation we have constructed the set $\mathbb{R}$ from $\mathbb{Q}$. Now, we turn our attention to study and explore some properties of $\mathbb{R}$.

$\mathbb{R}$ is an Ordered Field

Algebraic Properties

We won't talk much the detail here because it was already presented. Basically $\mathbb{R}$ has two operations, namely that of addition $(+)$ and multiplication $(\times)$. This makes $(\mathbb{R}, +, \times)$ a *field.* In other words, $(\mathbb{R}, +)$ and $(\mathbb{R}^*, \times)$ are Abelian groups.

Ordering Properties

The ordering of $\mathbb{R}$ is the same as those in $\mathbb{Q}$. Moreover, $(\mathbb{R}, \leq)$ is an ordered field and also an extension from $(\mathbb{Q}, \leq)$.

(working on it)

Uncountabiliy of $\mathbb{R}$

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As shown in Group 3, the set $\mathbb{Q}$ is countable. How about the set $\mathbb{R}$?

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The idea is to produce a number in $(0,1)$ that is not already placed on the table shown below.

(continued)

n	$f(n)$
1	$0.\textcolor{blue}{1}12212411\ldots$
2	$0.0\textcolor{blue}{9}3240987\ldots$
3	$0.58\textcolor{blue}{7}103941\ldots$
4	$0.314\textcolor{blue}{4}14391\ldots$
5	$0.0981\textcolor{blue}{3}2081\ldots$

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Let $N = 0.a_1a_2a_3a_4\ldots$ where the n-th digit of N, namely a_n, is found by adding 1 to the n-th digit of $f(n)$ in modulo 10.

(continued)

n	$f(n)$
1	$0.112212411\ldots$
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And from this construction, we see that $N \neq f(n)$ for all $n \in \mathbb{N}$.

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Theorem

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- What is a *Cauchy sequence*?

So, first let's talk about sequence and convergence.

Sequence

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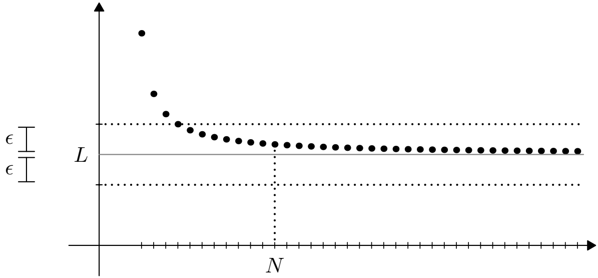

Absolute Value

We define the absolute value by

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0 \end{cases}$$

The distance from x to y is given by $|x - y|$. These are some properties of absolute value.

Properties

- $|x| \geq 0$
- $|x + y| \leq |x| + |y|$
- $|x - y| \geq ||x| - |y||$

Convergence

Okay, we have the language of distance. The phrase "a_n is close to L" can then be expressed by $|a_n - L| < \epsilon$ for any small ϵ.

Note that we only require it's true for larger and larger n. With all the ingredients at hand, we now present the

Definition (Convergence)

A sequence (a_n) is said to converge to $L \in \mathbb{R}$ if $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$.

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Proof.

We choose $\epsilon = 1$ in the definition of Cauchy sequence to obtain that there's an $N \in \mathbb{N}$ such that (choose $m = N$)

$$\begin{aligned} &|a_n - a_N| < 1, \quad (\forall n \geq N) \\ \implies &|a_n| < 1 + |a_N| \end{aligned}$$

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Then we denote $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |a_N|\}$,

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Then we denote $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |a_N|\}$, therefore, $|a_n| \leq M$ for all $n \in \mathbb{N}$. $\square$

Now we know that (a_n) is bounded. What can we say about bounded sequence? We borrow a theorem called *Bolzano Weierstrass* theorem.

Definition (Subsequence)

Let $n_1 < n_2 < n_3 < \cdots$ be increasing sequence of integers. We call (a_{n_k}) a subsequence of (a_n).

Theorem (Bolzano-Weierstrass)

Any bounded sequence has a convergent subsequence.

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From the above lemma (a_n) is bounded, then it has a convergent subsequence (from Bolzano-Weierstrass). Suppose that (a_{n_k}) be that convergent subsequence whose limit is $\lim a_{n_k} = a$.

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Let an $\epsilon > 0$, then there is $N \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\epsilon}{2}$ whenever $n, m \geq N$.

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Let an $\epsilon > 0$, then there is $N \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\epsilon}{2}$ whenever $n, m \geq N$. Because $(a_{n_k}) \to a$ and (n_k) is increasing, then there's $n_{k_0} > N$ such that $|a_{n_k} - a| < \frac{\epsilon}{2}$ whenever $k > k_0$.

(continued.)

We obtain that for any $n \geq N$,

$$\begin{aligned} |a_n - a| &\leq |a_n - a_{n_k}| + |a_{n_k} - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

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Thus $\lim a_n = a$. This completes the proof. $\square$

What Does Completeness Mean?

Completeness of $\mathbb{R}$ is quite nice for it tells us that any Cauchy sequence has to have a limit in $\mathbb{R}$. For instance, take the sequence

$$a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2}$$

Density of $\mathbb{Q}$ in $\mathbb{R}$

Density in language of sequence

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Recall the Archimedean property that for any given $x \in \mathbb{R}$, then there's an integer n such that $n \geq x$. In class, we've already exposed to the idea that $\mathbb{Q}$ is dense in $\mathbb{R}$. Today we're going to present the same thing but with different language.

Density in language of sequence

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Theorem

Given any $x \in \mathbb{R}$, then there's a sequence $(q_n) \subset \mathbb{Q}$ of rationals such that $\lim q_n = x$.

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$$x - \frac{1}{n} < \frac{k}{n} < x + \frac{1}{n}.$$

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Without any doubt, we immediately denote $q_n := \frac{k}{n}$, thus we have constructed (q_n) such that $|q_n - x| < \frac{1}{n}$. With this sequence (q_n) so constructed, we claim that its limit is exactly x.

(continued.)

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$$|q_n - x| < \frac{1}{n} \leq \frac{1}{N} \leq \epsilon.$$

This shows that $\lim q_n = x$ as advertised. $\square$

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Given any $x, y \in \mathbb{R}$ with $x < y$, then there exists a $q \in \mathbb{Q}$ such that $x < q < y$.

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We have proved these two theorems independently by invoking the Archimedean property. We now will prove that these two are in fact equivalent. Namely,

$$\text{Density 1} \iff \text{Density 2}$$

Proof. (Density 1 $\implies$ Density 2).

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Thus we have constructed (q_n) such that $|q_n - x| < \frac{1}{n}$ for each $n \in \mathbb{N}$. Using the same argument as the above proof, we could safely conclude that $\lim q_n = x$ as expected. $\square$

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Let $x, y \in \mathbb{R}$ with $x < y$. Then by Density 2, we're sure that there's $(q_n) \subset \mathbb{Q}$ such that $\lim q_n = \frac{x+y}{2}$.

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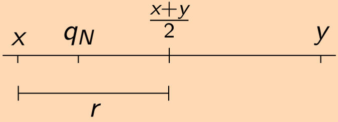

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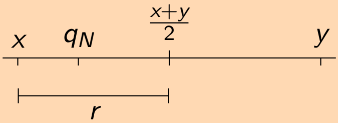

Then there exists $N \in \mathbb{N}$ such that

$$\left| q_N - \frac{x+y}{2} \right| < r$$

$$\implies -r + \frac{x+y}{2} < q_N < r + \frac{x+y}{2}$$

$$\implies x < q_N < y.$$

$\square$

Why Real Numbers?

Why study on $\mathbb{R}$

Why do we bother to extend from $\mathbb{Q}$ to $\mathbb{R}$ anyway? Well in real analysis, we would like to have a rigorous way to the study of calculus. Namely, we need a *good* definition of integral, differentiability, convergence, etc ... .

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This is to say that $\mathbb{R}$ is complete, or to put it in another way, every *Cauchy* sequence converges.

Referrences

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