Homework

Deadline: 17-April-2022

1. Let $(a_n)_n$ and $(b_n)_n$ be real-sequences such that $a_n \xrightarrow[n\to\infty]{} a$ and $b_n \xrightarrow[n\to\infty]{} b$ where $a, b \in \mathbb{R}$. Prove that,
   i. $(a_n + \lambda b_n)_n$ converges to $a + \lambda b$, for all $\lambda \in \mathbb{R}$.
   ii. $(a_n b_n)_n$ converges to ab.
   iii. $\left(\dfrac{a_n}{b_n}\right)_n$ converges to $\dfrac{a}{b}$ if $b \neq 0$ and $b_n \neq 0, \forall n$.
2. Let $(a_n)_n$ be a divergent real-sequence. Prove that $\lim\limits_{n\to\infty} \dfrac{1}{a_n} = 0$.
3. If a real-sequence is convergent, prove that its limit is unique.
4. Prove that a convergent real-sequence is bounded. Is the converse true?
5. Prove that convergence of $(a_n)_n$ implies convergence of $(|a_n|)_n$. Is the converse true?
6. Prove the density of $\mathbb{Q}$ in $\mathbb{R}$; that is to prove that for every $a, b \in \mathbb{R}, a < b$, there is $q \in \mathbb{Q}$ such that $a < q < b$.
7. By definition, prove that
   i. $\lim\limits_{n\to\infty} \dfrac{2n+1}{n-2} = 2$
   ii. $\lim\limits_{n\to\infty} \dfrac{\sqrt{n}-1}{\sqrt{n}+1} = 1$
   iii. $\lim\limits_{n\to\infty} \dfrac{\sqrt[3]{n^3-3}}{2n} = \dfrac{1}{2}$.
   iv. $\lim\limits_{n\to\infty} \dfrac{2\sqrt{n}+3}{n-3} = 0$.
   v. $\lim\limits_{n\to\infty} (n - \sqrt{n}) = \infty$.

* Note that any homework received after the deadline is not considered.