Properties of Real Numbers

Jeng Inger, Hun Sivmeng

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Discussion

Suppose we want to solve $x^2 - 2 = 0$ in the field $\mathbb{Q}$. It turns out that this equation has no roots in $\mathbb{Q}$. So we need to extend this field to a new one — a *better* one, called the set of real numbers and is denoted by $\mathbb{R}$.

What are the properties of this new set $\mathbb{R}$, and how do we extend this exactly?

But before we proceed, we ask the audience to forget all the properties of $\mathbb{R}$ we learned in highschool because we're going to start from the bottom up.

First, because $\mathbb{R}$ is an extension of $\mathbb{Q}$, thus $\mathbb{R}$ must inherit all the properties of $\mathbb{Q}$.

Algebraic Properties

The set $\mathbb{R}$ has the following algebraic properties:

Algebraic Properties (Addition)

1. *(closed):* $a + b \in \mathbb{R}$
2. *(commutative):* $a + b = b + a$
3. *(associative):* $a + (b + c) = (a + b) + c$
4. *(additive identity):* $\exists 0 \in \mathbb{R}, \forall a \in \mathbb{R} : \; 0 + a = a$
5. *(additive inverse):* $\forall a \in \mathbb{R}, \; \exists a' \in \mathbb{R} : \; a + a' = 0.$

Algebraic Properties (Multiplication)

1. *(closed):* $ab \in \mathbb{R}$
2. *(commutative):* $ab = ba$
3. *(associative):* $a(bc) = (ab)c$
4. *(multiplicative identity):* $\exists 1 \in \mathbb{R}, \forall a \in \mathbb{R}: \ a \cdot 1 = a$
5. *(multiplicative inverse):* $\forall a \in \mathbb{R}, a \neq 0, \ \exists a^* \in \mathbb{R}: \ aa^* = 1$
6. *(distributive law):* $a(b + c) = ab + bc.$

Thus, $\mathbb{R}$ is a field with two operations, namely *addition* and *multiplication*.

From this algebraic properties, we can prove the basic arithmetic we learned in highschool with ease.

Lemma

- *additive identity/inverse is unique*
- *multiplicative identity/inverse is unique*
- $x + a = y + a \iff x = y$
- $a \cdot 0 = 0$
- $-a = (-1) \cdot a$
- $ab = 0 \iff a = 0$ *or* $b = 0$

Ordering Properties

There is a non-empty subset of $\mathbb{R}$, called the set of positive real numbers which is denoted by $\mathbb{R}^+$. This set has the properties:

Ordering Properties

- if $a \in \mathbb{R}$, then $a \in \mathbb{R}^+$ or $a = 0$ or $-a \in \mathbb{R}^+$;
- if $a, b \in \mathbb{R}^+$ then $a + b \in \mathbb{R}^+$;
- if $a, b \in \mathbb{R}^+$ then $ab \in \mathbb{R}^+$.

We then define relations $<$, $\leq$, $>$ and $\geq$ as follows:

- $a > b$ whenever $a - b \in \mathbb{R}^+$;
- $a \geq b$ whenever $a > b$ or $a = b$;
- $b < a$ whenever $a > b$;
- $b \leq a$ whenever $a \geq b$;

From this ordering properties, we can prove the following:

Lemma

Let $a, b, c \in \mathbb{R}$.

- *then* $a > b$ *or* $a = b$ *or* $a < b$.
- *if* $a > b$, *then* $a + c > b + c$.
- *if* $a > b$ *and* $b > c$, *then* $a > c$.
- *if* $a > b$ *and* $c > 0$, *then* $ac > bc$.
- *if* $a > b$ *and* $c < 0$, *then* $ac < bc$.

Completeness

Up until this point, the set $\mathbb{R}$ still behaves like $\mathbb{Q}$, namely $\mathbb{R}$ is an odered field just like $\mathbb{Q}$. How do we make $\mathbb{R}$ *better* than $\mathbb{Q}$?

Definition

Let $A \subset \mathbb{R}$. We call

- A is bounded above if $\exists M \in \mathbb{R}$ st $a \leq M$ for all $a \in A$.
- A is bounded below if $\exists m \in \mathbb{R}$ st $m \leq a$ for all $a \in A$.
- A is bounded if A is both bounded above and below, i.e. $\exists M > 0$ st $|a| \leq M$ for all $a \in A$.

We call M, m the upper bound and lower bound of A, respectively. The *Completeness* property of $\mathbb{R}$ states that

Axiom of Completeness – (AoC)

Let $A \subset \mathbb{R}$. If A is bounded above, then A has *least upper bound.*

This axiom is a mathematical way to say that $\mathbb{R}$ has no gaps. It also applies to sets that are bouned below.

Corollary

Let $B \subset \mathbb{R}$. If B is bouned below, then B has greatest lower bound.

We denote the

- least upper bound of A by $\sup A$
- greatest lower bound of A by $\inf A$

Lemma

There is $x \in \mathbb{R}$ such that $x^2 = 2$.

Proof.

Consider the set $S = \{x \in \mathbb{R} \ : \ x^2 \leq 2\}$. This set is clearly bouned above. Hence by AoC, S has supremum. Let $s = \sup S \in \mathbb{R}$. We claim that $s^2 = 2$. From ordering property of $\mathbb{R}$, one of the following must true

$$s^2 < 2, \quad \text{or} \quad s^2 > 2, \quad \text{or} \quad s^2 = 2.$$

Case if $s^2 < 2$: $\square$

Why study on $\mathbb{R}$

As we mentioned above, we can't do analysis on $\mathbb{Q}$ because some numbers simply don't exist in $\mathbb{Q}$.

1. allow us to do analysis
2. concept of limit is well behaved in $\mathbb{R}$
3. (need more ...)

Rational and real numbers

Theorem

For any real number x, there is a sequence (x_n) of rationals such that $x_n \neq x$ for all $n \in \mathbb{N}$ and

$$\lim x_n = x.$$

To prove this theorem, we use the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$. That is, for any two real numbers $x, y \in \mathbb{R}$ where $x < y$, there exists a rational $r \in \mathbb{Q}$ such that $x < r < y$.

Proof.

Let $x \in \mathbb{R}$. We wish to construct $(x_n) \subset \mathbb{Q}$ such that $x_n \to x$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, then for each $n \in \mathbb{N}$ we can find $x_n \in \mathbb{Q}$ such that $x < x_n < x + \frac{1}{n}$.

We claim that (x_n) does converge to x, and hence proved the theorem.

Let $\epsilon > 0$. Using Archimedean property, we can find $N \in \mathbb{N}$ for which $N > \frac{1}{\epsilon}$. Now for any $n \geq N$, and because $x_n \in (x, x + \frac{1}{n})$ we then obtain that

$$|x_n - x| < \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Thus $x_n \to x$ as required. $\square$