Notes on Differential Geometry

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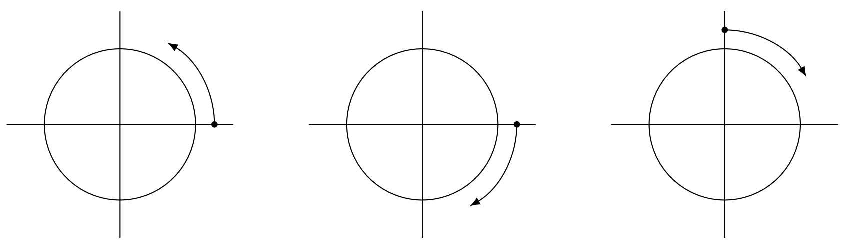
1 Curves

Exercise 1. Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2+y^2=1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0)=(0,1)$.

Solution. We start with the curve $r_1(t)=(\cos t, \sin t)$, where $t \in [0,2\pi)$.



$r_1(t)=(\cos t, \sin t)$ $\quad$ $r_2(t)=(\cos t, -\sin t)$ $\quad$ $r(t)=(\sin t, \cos t)$

By negating the sign of the second coordinate of r_1, we get the curve r_2 that starts at $(1,0)$ and runs clockwise. By replacing $t \mapsto t-\frac{\pi}{2}$, we get the curve r_3 that still runs clockwise but whose starting point is $(0,1)$. Therefore, $\alpha(t)=(\sin t, \cos t)$ is the desired parametrized curve. $\square$

Exercise 2. Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is a point of the trace of α closest to the origin and $\alpha'(t_0) \neq \mathbf{0}$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Proof. Because $|\alpha(t)| \neq 0$ for all t, the real-valued function $|\alpha|$ is differentiable whose derivative is

$$\begin{aligned}\frac{d}{dt}|\alpha(t)| &= \frac{d}{dt}\sqrt{\langle \alpha, \alpha \rangle} \\ &= \frac{\langle \alpha', \alpha \rangle + \langle \alpha, \alpha' \rangle}{2\sqrt{\langle \alpha, \alpha \rangle}} \\ &= \frac{1}{|\alpha|} \cdot \langle \alpha, \alpha' \rangle.\end{aligned}$$

Moreover $|\alpha|$ has minimun when $t = t_0$, thus the derivative of $|\alpha|$ is zero there. Hence

$$\langle \alpha(t_0), \alpha'(t_0) \rangle = 0,$$

thus the two vectors are orthogonal. □

Exercise 3. A parametrized curve $\alpha(t)$ has the property that its second derivative $\alpha''(t)$ is identically zero. What can be said about α?

Answer. My guess is that α is a line. □

Exercise 4. Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.

Proof. We have the following

$$\frac{d}{dt}\langle \alpha, v \rangle = \langle \alpha', v \rangle + \langle \alpha, v' \rangle = 0.$$

Thus the inner product $\langle \alpha(t), v \rangle = c$ is a constant for all t. Since $\alpha(0)$ is also orthogonal to v then $\langle \alpha(0), v \rangle = 0$. We must $c = 0$, which concludes the proof. □

Exercise 5. Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Answer. Use $\frac{d}{dt}|\alpha| = \frac{1}{\alpha}\langle \alpha, \alpha' \rangle$, I think. □

Exercise 6. Show that the tanget lines to the regular parametrized curve $\alpha(t) = (3t, 3t^2, 2t^3)$ make a constant angle with the line $y = 0, z = x$.

Proof. The tangent is $\alpha'(t) = (3, 6t, 6t^2)$. Because the angle of the tangent with the line $y = 0, z = x$ is the angle between α' and $n = (1, 0, 1)$, and if θ is this angle we then have

$$\cos\theta = \frac{\langle \alpha', n\rangle}{|\alpha'||n|} = \frac{3 + 6t^2}{\sqrt{9(1 + 4t^2 + 4t^4)} \cdot \sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Thus at any point, the angle is always constant. □

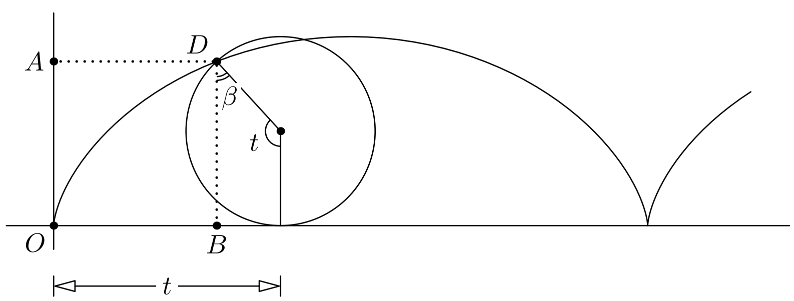
Exercise 7. A circular disk of radius 1 in the plane xy rolls without slipping along the x axis.

(a) Obtain a parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid, and determine its singular points.

(b) Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Solution.

(a) Because the cycloid the periodic in the period of 2π, it's then enough to only parametrize $\alpha : [0, 2\pi] \to \mathbb{R}^2$.



The angle $\beta = \pi - t$, using some trigs give us:

$$OA = 1 + \cos\beta = 1 - \cos t$$
$$OB = t - \sin\beta = t - \sin t$$

Therefore, we can parametrized the cycloid $\alpha : [0, 2\pi] \to \mathbb{R}^2$ by

$$\alpha(t) = (1 - \cos t, t - \sin t).$$

(b) The arc length correspond to a full rotation is

$$\begin{aligned}
L &= \int_0^{2\pi} |\alpha'(u)| du \\
&= \int_0^{2\pi} \sqrt{(1-\cos u)^2 + \sin^2 u} du \\
&= \int_0^{2\pi} \sqrt{1 - 2\cos u + \cos^2 u + \sin^2 u} du \\
&= \int_0^{2\pi} \sqrt{2} \cdot \sqrt{2\sin^2 \frac{u}{2}} du \\
&= \Big[-4\cos\frac{u}{2} \Big]_0^{2\pi} = 8
\end{aligned}$$

$\square$