Chapter 1

Basic Notions

1. Vector Spaces

Exercise 1.1. Let $\mathbf{x} = (1,2,3)^{\mathrm{T}}$, $\mathbf{y} = (y_1, y_2, y_3)^{\mathrm{T}}$ and $\mathbf{z} = (4,2,1)^{\mathrm{T}}$. Compute $2\mathbf{x}$, $3\mathbf{y}$, $\mathbf{x} + 2\mathbf{y} - 3\mathbf{z}$.

Proof. Little calculation reveals that

$$2\mathbf{x} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \quad 3\mathbf{y} = \begin{pmatrix} 3y_1 \\ 3y_2 \\ 3y_3 \end{pmatrix}, \quad \mathbf{x} + 2\mathbf{y} - 3\mathbf{z} = \begin{pmatrix} 2y_1 - 11 \\ 2y_2 - 4 \\ 2y_3 \end{pmatrix}$$

□

Exercise 1.2. Which of the following sets (with natural addition and multiplication by a scalar) are vector spaces? Justify your answers.

a). The set of all continuous functions on the interval $[0,1]$;

b). The set of all non-negative functions on the interval $[0,1]$;

c). The set of all polynomials of degree *exactly* n;

d). The set of all symmetric $n \times n$ matrices, i.e. the set of matrices $A = \{a_{j,k}\}_{j,k=1}^{n}$ such that $A^{\mathrm{T}} = A$.

Proof.

a). Let $C[0,1]$ be the set of all continuous functions on $[0,1]$. For any $f, g \in C[0,1]$ and $\alpha \in \mathbb{R}$, we define

$$(f+g)(x) := f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) := \alpha \cdot f(x)$$

for each $x \in [0,1]$. Therefore, $(C[0,1], +, \cdot)$ is closed under addition and scalar multiplication. It's immediate to see that all the eight properties of vector space are all satisfied, that is

- $f+g=g+f$
- $f+(g+h)=(f+g)+h$
- $f+0=f$
- $f+(-f)=0$
- $1f=f$
- $\alpha(\beta f)=(\alpha\beta)f$
- $(\alpha+\beta)f=\alpha f+\beta f$
- $\alpha(f+g)=\alpha f+\beta g$

Note that the function $0 \in \mathcal{C}[0,1]$ such that $0(x)=0$ for each $x \in [0,1]$.

b). Let $\mathcal{B}$ is the set of all non-negative functions on $[0,1]$. Then $(\mathcal{B},+\cdot)$ is not a vector space because it's not closed under scalar multiplication, i.e. if $f \in \mathcal{B}$, hence $f>0$ yet

$$-f=(-1)\cdot f<0.$$

(Even if we restrict the scalar be to positive real numbers, this set still won't be a vector space, because it fails to have an inverse.)

c). Let $\mathcal{P}$ be the set of all polynomials of degree exactly n, then $(\mathcal{P},+,\cdot)$ is *not* a vector space, because the addtive indentity is the polynomial 0. However, $0 \notin \mathcal{P}$.

d). Let $\mathrm{sym}(n)$ be the set of all symmetric $n \times n$ matrices. The addition and scalar multiplication are defined as an *entrywise* operations. Hence, $\mathrm{sym}(n)$ is closed under $(+)$ and $(\cdot)$. The additive indentity is the matrix

$$\mathbf{0}=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{sym}(n).$$

We could easily see that all the eight properties of vector spaces are all satisfied.

□

Exercise 1.3. True or false:

a). Every vector space contains a zero vector; (**True.**)

b). A vector space can have more than one zero vector; (**False.** The zero vector is unique.)

c). An $m \times n$ matrix has m rows and n columns; (**True.**)

d). If f and g are polynomials of degree n, then $f+g$ is also a polynomial of degree n. (**False.** consider t^n and $t-t^n$.)

e). If f and g are polynomials of degree atmost n, the $f+g$ is also a polynomial of degree atmost n. (**True.**)

Exercise 1.4. Prove that a zero vector $\mathbf{0}$ of a vector space V is unique.

Proof. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are the zero vectors of V. From the *Axioms of Vector Space*, we obtain that

$$\begin{aligned}
\mathbf{a} &= \mathbf{a} + \mathbf{b} && (\mathbf{b} \text{ is the zero vector}) \\
&= \mathbf{b} + \mathbf{a} && (\text{commutitativity}) \\
&= \mathbf{b} && (\mathbf{a} \text{ is the zero vector})
\end{aligned}$$

Hence, a zero vector of a vector space is unique, and we usually denote it by $\mathbf{0}$. ☐

Exercise 1.5. What is the zero matrix of the space $M_{2\times3}$?

Answer. In the space $M_{2\times3}$, the zero matrix is

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

☐

Exercise 1.6. Prove that the additive inverse of a vector space is unique.

Proof. Let $\mathbf{a}$ be an arbitrary vector. Assume the $\mathbf{a}$ has two inverses, namely $\mathbf{x}$ and $\mathbf{y}$. Hence

$$\begin{aligned}
\mathbf{x} &= \mathbf{x} + \mathbf{0} \\
&= \mathbf{x} + (\mathbf{a} + \mathbf{y}) && (\mathbf{y} \text{ is an inverse}) \\
&= (\mathbf{x} + \mathbf{a}) + \mathbf{y} && (\text{associativity}) \\
&= \mathbf{0} + \mathbf{y} && (\mathbf{x} \text{ is an inverse}) \\
&= \mathbf{y}.
\end{aligned}$$

Therefore, the inverse of any vector $\mathbf{a} \in V$ is unique, and is usually denoted by $-\mathbf{a}$. ☐

Exercise 1.7. Prove that $0\mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v} \in V$.

Proof. Let $\mathbf{v} \in V$ and $\mathbf{b}$ is an inverse of $0\mathbf{v}$. Therefore,

$$\begin{aligned}
0 &= 0\mathbf{v} + b \\
&= (0+0)\mathbf{v} + b \\
&= (0\mathbf{v} + 0\mathbf{v}) + b && (\text{distributivity}) \\
&= 0\mathbf{v} + (0\mathbf{v} + b) && (\text{associativity}) \\
&= 0\mathbf{v} + \mathbf{0} && (\mathbf{b} \text{ is an inverse of } 0\mathbf{v}) \\
&= 0\mathbf{v}
\end{aligned}$$

for any $\mathbf{v} \in V$. ☐

Exercise 1.8. Prove that for any vector $\mathbf{v}$ its additive inverse $-\mathbf{v}$ is given by $(-1)\mathbf{v}$.

Proof. As proved in the above exercise for any $\mathbf{v} \in V$,

$$\mathbf{0} = 0\mathbf{v} = (1-1)\mathbf{v} = \mathbf{v} + (-1)\mathbf{v}$$

where the last equallity derives from the distributive property. Because $-\mathbf{v}$ is the inverse of $\mathbf{v}$, then

$$\begin{aligned} -\mathbf{v} &= -\mathbf{v} + \mathbf{0} \\ &= -\mathbf{v} + \left[\mathbf{v} + (-1)\mathbf{v}\right] \\ &= (\underbrace{-\mathbf{v} + \mathbf{v}}_{\mathbf{0}}) + (-1)\mathbf{v} \\ &= (-1)\mathbf{v} \end{aligned}$$

as desired. □

2. Linear Combination, bases

Exercise 2.1. Find the basis in the space of 3×2 matrices $M_{3\times 2}$.

Answer. Consider the vectors:

$$\mathbf{e_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{e_2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{e_3} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{e_4} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{e_5} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{e_6} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and we're going to prove that the system of thses vectors are a basis. Any matrix

$$\mathbf{v} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \in M_{3\times 2}$$

can be represented as the combination $\mathbf{v} = a\mathbf{e_1} + b\mathbf{e_2} + c\mathbf{e_3} + d\mathbf{e_4} + e\mathbf{e_5} + f\mathbf{e_6}$ thus this system is generating. Next we're going to prove the uniqueness.

Suppose that there are $\hat{a}, \hat{b}, \ldots, \hat{f}$ with

$$\mathbf{v} = \hat{a}\mathbf{e_1} + \hat{b}\mathbf{e_2} + \hat{c}\mathbf{e_3} + \hat{d}\mathbf{e_4} + \hat{e}\mathbf{e_5} + \hat{f}\mathbf{e_6}$$

$$\Longrightarrow \quad \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \\ \hat{e} & \hat{f} \end{bmatrix}$$

This implies that each corresponding entry is equals. Hence the representation is unique. Therefore this system is a basis.

□

Exercise 2.2. True or false:

a). Any set containing a zero vector is linearly dependent;

b). A basis must contain $\mathbf{0}$;

c). subsets of linearly dependent sets are linearly dependent;

d). subsets of linearly independent sets are linearly independent;

e). if $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v_2} + \cdots + \alpha_n\mathbf{v_n} = 0$ then all scalars α_k are zero.

Answer.

a). **True.** because $\mathbf{0}$ can be represented as a linear combination of the other vectors (simply put all the scalars to 0).

b). **No.** if so, they must be linearly dependent, which is not a base.

c). **No.** Take for example the system of linearly dependent $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ where $\mathbf{e_1} = (1,0)$, $\mathbf{e_2} = (0,1)$ and $\mathbf{e_3} = (1,1)$. The subset $\{\mathbf{e_1}, \mathbf{e_2}\}$ is a basis, which is clearly not linearly dependent.

d). **True.** Supppose that the system $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is a subset of the linearly independent system $\{\mathbf{v_1}, \ldots, \mathbf{v_p}, \ldots, \mathbf{v_n}\}$. Let α_k the real numbers such that $\alpha_1\mathbf{v_1} + \cdots + \alpha_p\mathbf{v_p} = \mathbf{0}$ hence

$$\alpha_1\mathbf{v_1} + \cdots + \alpha_p\mathbf{v_p} + 0\mathbf{v_{p+1}} + \cdots + 0\mathbf{v_n} = \mathbf{0}.$$

Because the system $\{\mathbf{v_1}, \ldots, \mathbf{v_p}, \ldots, \mathbf{v_n}\}$ is linearly independent, therefore all the scalars $\alpha_k = 0$. Thus, the system $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is also linearly independent.

e). **No.** Take, $\mathbf{e_1} = (2,2)$ and $\mathbf{e_2} = (1,1)$ for instance. We have $\mathbf{e_1} - 2\mathbf{e_2} = \mathbf{0}$ yet the scalars are non-zero.

□

Exercise 2.3. Recall, that a matrix is called *symmetric* if $A^{\mathrm{T}} = A$. Write down a basis in the space of *symmetric* 2×2 matrices (there are many possible answers). How many elements are there in the basis.

Answer. We are going to prove that the system $\{\mathbf{d_1}, \mathbf{d_2}, \mathbf{e_1}\}$ where

$$\mathbf{d_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{d_2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{e_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

is a basis. Observe that any symmetric matrix

$$\mathbf{v} = \begin{bmatrix} d_1 & e_1 \\ e_1 & d_2 \end{bmatrix}$$

can be represented as $\mathbf{v} = d_1\mathbf{d_1} + d_2\mathbf{d_2} + e_1\mathbf{e_1}$, hence it's generating. Note that the equation

$$d_1\mathbf{d_1} + d_2\mathbf{d_2} + e_1\mathbf{e_1} = \mathbf{0}$$
$$\begin{bmatrix} d_1 & e_1 \\ e_1 & d_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

holds only when all the scalars are all zero. Hence the system is linearly independent. Thus, it's a basis.

□

Exercise 2.4. Write down a basis for the space of

a). 3×3 symmetric matrices;

b). $n \times n$ symmetric matrices;

c). $n \times n$ antisymmetric matrices.

Answer.

a). we are going to prove that the system of vectors

$$\mathbf{d_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{d_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{d_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{e_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e_2} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{e_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

is the basis. First of, any symmetric matrix

$$\mathbf{v} = \begin{bmatrix} d_1 & e_1 & e_2 \\ e_1 & d_2 & e_3 \\ e_2 & e_3 & d_3 \end{bmatrix}$$

can be represented as

$$\mathbf{v} = d_1\mathbf{d_1} + d_2\mathbf{d_2} + d_3\mathbf{d_3} + e_1\mathbf{e_1} + e_2\mathbf{e_2} + e_3\mathbf{e_3}$$

yeilds that the system is generating. Similar to the previous problem, if the linear combination of these vectors equals $\mathbf{0}$, then all the scalars must equals zero. Thus it's linearly independent. Therefore it's a basis.

b). Working on it.

c). Working on it.

□

Exercise 2.5. Let a system of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be linearly independent but not generating. Show that it is possible to find a vector $\mathbf{v}_{r+1}$ such that the system $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}$ is linearly independent.

Proof. Because the system $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ is not generating, therefore there exists a vector $\mathbf{v}_{r+1}$ such that $\mathbf{v}_{r+1}$ cannot be represented as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. Let α_i be the scalars such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_r\mathbf{v}_r + \alpha_{r+1}\mathbf{v}_{r+1} = \mathbf{0} \tag{1.1}$$

Now we have to prove that all the scalars are all zero. If $\alpha_{r+1} \neq 0$ then

$$\mathbf{v}_{r+1} = -\sum_{i=1}^{r} \frac{\alpha_i}{\alpha_{r+1}} \cdot \mathbf{v}_i,$$

meaning $\mathbf{v}_{r+1}$ is the linear combination of the other vectors, a contradiction. Hence α_{r+1} must equals to zero. So the $r+1$ term in the equation (**??**) vanishes. And because the system $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ is linearly independent, all the scalars $\alpha_i = 0$ for all $i = 0, 1, \ldots, r$. Thus, the system

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}$$

is also *linearly independent*. □

Exercise 2.6. Is it possible that vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly dependent, but the vectors $\mathbf{w_1} = \mathbf{v_1} + \mathbf{v_2}$, $\mathbf{w_2} = \mathbf{v_2} + \mathbf{v_3}$ and $\mathbf{w_3} = \mathbf{v_3} + \mathbf{v_1}$ are linearly *independent*.

Proof. It's not possible, and we're going to prove this assertion via contradiction. Assume that there are such vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ satisfying the above conditions. Then there are numbers $x, y, z \in \mathbb{R}$ such that

$$|x| + |y| + |z| > 0 \quad \text{and} \quad x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{0}.$$

By letting

$$a = x + y - z, \quad b = y + z - x, \quad c = z + x - y$$

we obtain that

$$\begin{aligned} a\mathbf{w}_1 + b\mathbf{w}_2 + c\mathbf{w}_3 &= (x\mathbf{w}_1 + y\mathbf{w}_1 - z\mathbf{w}_1) + (y\mathbf{w}_2 + z\mathbf{w}_2 - x\mathbf{w}_2) \\ &\qquad + (x\mathbf{w}_3 + z\mathbf{w}_3 - y\mathbf{w}_3) \\ &= 2x\mathbf{v}_1 + 2y\mathbf{v}_2 + 2z\mathbf{v}_3 \\ &= \mathbf{0}. \end{aligned}$$

Since $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ are linearly independent, we must have $a = b = c = 0$. Hence

$$\begin{cases} x + y - z = 0 \\ y + z - x = 0 \\ z + x - y = 0 \end{cases}$$

adding all the 3 eqations, $x + y + z = 0$. Substituting back to the system of eqations above we get

$$x = y = z = 0$$

which contradicts to the fact that $|x| + |y| + |z| > 0$. □

Exercise 2.7. Any finite independent system is a subset of some basis.

Proof. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent. If this system is generating, then it's a base and we're done. If not, from exercise 2.5, there exists $\mathbf{v}_{n+1}$ such that

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{v}_{n+1}\}$$

is still linearly independent. Now if this new system is generating, then we're done. If not, we keep continue this process a finite steps, adding vectors $\mathbf{v}_{n+1}, \mathbf{v}_{n+2}, \ldots, \mathbf{v}_{n+r}$, and eventually the new system

$$\{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}_{n+1}, \ldots, \mathbf{v}_{n+r}\}$$

is now a basis. □

3. Linear Transformation

Homework 1. Prove that the transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ if and only if $T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$ for any scalars α, β and vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}$.

Proof. We need to prove this in two directions.

($\Rightarrow$) Suppose T is a linear transformation, then

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = T(\alpha\mathbf{x}) + T(\beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$$

as needed.

($\Leftarrow$) For this direction, we first assume that T has the property that $T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$ for all $\alpha, \beta, \mathbf{x}, \mathbf{y}$. We need to show that T has the property listed in the definition of the linear transformation. Observe that

- take $\alpha = \beta = 1$ then, $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
- take $\beta = 0$ then, $T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$

Hence T is a linear transformation, and the proof is completed.

□

Homework 2. Let $T : V \to W$ be a linear transformation. Prove that $T(\mathbf{0}) = \mathbf{0}$ and

$$TV = \{T\mathbf{v} \,:\, \mathbf{v} \in V\}$$

is a vector space.

Proof. Since T is linear, and as proved before $0 \cdot \mathbf{0} = \mathbf{0}$, it's easy to see that

$$T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0 \cdot T(\mathbf{0}) = \mathbf{0}.$$

To prove that TV is a vector space, we need to check that TV satisfies all the eight conditions listed in the definition of vector space.

We first need to prove that TV is closed. Because $TV \subset W$, hence TV is closed under scalar multiplication and vector addition. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$. Observe that

- $T\mathbf{x} + T\mathbf{y} = T\mathbf{y} + T\mathbf{x}$ (commutitativity of W)
- $(T\mathbf{x} + T\mathbf{y}) + T\mathbf{z} = T\mathbf{x} + (T\mathbf{y} + T\mathbf{z})$ (associativity of W)
- The vector $\mathbf{0} \in W$ is the indentity of TV because

$$T\mathbf{x} + \mathbf{0} = T\mathbf{x} + T\mathbf{0} = T(\mathbf{x} + \mathbf{0}) = T(\mathbf{x}), \quad \forall \mathbf{x} \in V$$

- The vector $T(-\mathbf{x})$ is the additive inverse of $T\mathbf{x}$ because

$$T\mathbf{x} + T(-\mathbf{x}) = T(\mathbf{x} - \mathbf{x}) = \mathbf{0}$$

- $1 \cdot T\mathbf{v} = T\mathbf{v}$ (multiplicative iden. in W)

Let α, β be scalars.

- multiplicative associativity

$$\begin{aligned}
(\alpha\beta)T\mathbf{x} &= T((\alpha\beta)\mathbf{x}) && \text{(linearity of } T\text{)}\\
&= T(\alpha(\beta\mathbf{x})) && \text{(mult. asso. of } V\text{)}\\
&= \alpha T(\beta\mathbf{x}) && \text{(linearity of } T\text{)}\\
&= \alpha \cdot \beta T\mathbf{x}
\end{aligned}$$

- scalar multiplication

$$\begin{aligned}
\alpha(T\mathbf{x} + T\mathbf{y}) &= \alpha T(\mathbf{x} + \mathbf{y}) && \text{(linearity of } T\text{)}\\
&= T(\alpha(\mathbf{x} + \mathbf{y}))\\
&= T(\alpha\mathbf{x} + \alpha\mathbf{y}) && \text{(scalar mult. in } V\text{)}\\
&= T(\alpha\mathbf{x}) + T(\alpha\mathbf{x}) && \text{(linearity of } T\text{)}\\
&= \alpha T\mathbf{x} + \alpha T\mathbf{y}
\end{aligned}$$

- scalar multiplication

$$\begin{aligned}
(\alpha + \beta)T\mathbf{x} &= T((\alpha + \beta)\mathbf{x}) && \text{(linearity of } T\text{)}\\
&= T(\alpha\mathbf{x} + \beta\mathbf{x}) && \text{(scalar mult. of } V\text{)}\\
&= T(\alpha\mathbf{x}) + T(\beta\mathbf{x})\\
&= \alpha T\mathbf{x} + \beta T\mathbf{x}
\end{aligned}$$

We see that TV has all eight properties to be a vector space, and the proof is completed.

$\square$

Homework 3. Let V, W be vector spaces. Prove that $\mathcal{L}(V, W)$, the set of all linear transformations $T : V \to W$, is also a vector space.

Proof. We first need to show that $\mathcal{L}(V, W)$ is closed. Let $T_1, T_2 \in \mathcal{L}(V, W)$ and a be a scalar. So we need to show the transformation $T_1 + T_2$ and aT_1 are both linear.

- Let $\mathbf{x}, \mathbf{y}$ be arbitrary vectors in V and α, β be scalar. Denote $T := T_1 + T_2$. Observe that

$$\begin{aligned}
T(\alpha\mathbf{x} + \beta\mathbf{y}) =&(T_1 + T_2)(\alpha\mathbf{x} + \beta\mathbf{y})\\
&= T_1(\alpha\mathbf{x} + \beta\mathbf{y}) + T_2(\alpha\mathbf{x} + \beta\mathbf{y}) && \text{(by def. of } T_1 + T_2\text{)}\\
&= \alpha T_1\mathbf{x} + \beta T_1\mathbf{y} + \alpha T_2\mathbf{x} + \beta T_2\mathbf{y} && \text{(by lin. of } T_1 \text{ and } T_2\text{)}\\
&= (\alpha T_1\mathbf{x} + \alpha T_2\mathbf{x}) + (\beta T_1\mathbf{y} + \beta T_2\mathbf{y})\\
&= \alpha(T_1\mathbf{x} + T_2\mathbf{x}) + \beta(T_1\mathbf{y} + T_2\mathbf{y}) && \text{(by scalar mult. in } W\text{)}\\
&= \alpha(T_1 + T_2)\mathbf{x} + \beta(T_1 + T_2)\mathbf{y}\\
&= \alpha T\mathbf{x} + \beta T\mathbf{y}
\end{aligned}$$

  This shows that $T_1 + T_2$ is also a linear transformation, hence $\mathcal{L}(V, W)$ is closed under addition.

- Similarly, we let $\mathbf{x}, \mathbf{y} \in V$. For simplicity, we again denote $T := aT_1$. Hence for any scalars α, β

$$\begin{aligned} T(\alpha\mathbf{x} + \beta\mathbf{y}) &= (aT_1)(\alpha\mathbf{x} + \beta\mathbf{y}) \\ &= a \cdot T_1(\alpha\mathbf{x} + \beta\mathbf{y}) && \text{(by def. of } aT_1\text{)} \\ &= a \cdot (\alpha T_1\mathbf{x} + \beta T_1\mathbf{y}) && \text{(by lin. of } T_1\text{)} \\ &= \alpha a T_1\mathbf{x} + \beta a T_1\mathbf{y} \\ &= \alpha(aT_1)\mathbf{x} + \beta(aT_1)\mathbf{y} \\ &= \alpha T\mathbf{x} + \beta T\mathbf{y} \end{aligned}$$

  This suggests that aT_1 is also linear, hence $\mathcal{L}(V, W)$ is closed under scalar multiplication. Ultimately, we've proved that $\mathcal{L}(V, W)$ is closed as needed.

We are now ready to prove that $\mathcal{L}(V, W)$ is a vector space. Let $T_1, T_2, T_3 \in \mathcal{L}(V, W)$ we have

- $T_1 + T_2 = T_2 + T_1$, because for any $\mathbf{x} \in V$

$$(T_1 + T_2)\mathbf{x} = T_1\mathbf{x} + T_2\mathbf{x} = T_2\mathbf{x} + T_1\mathbf{x} = (T_2 + T_1)\mathbf{x}.$$

- $T_1 + (T_2 + T_3) = (T_1 + T_2) + T_3$, because for any $\mathbf{x} \in V$

$$\begin{aligned} (T_1 + (T_2 + T_3))\mathbf{x} &= T_1\mathbf{x} + (T_2 + T_3)\mathbf{x} \\ &= T_1\mathbf{x} + (T_2\mathbf{x} + T_3\mathbf{x}) \\ &= (T_1\mathbf{x} + T_2\mathbf{x}) + T_3\mathbf{x} && \text{(by asso. of } W\text{)} \\ &= (T_1 + T_2)\mathbf{x} + T_3\mathbf{x} \\ &= ((T_1 + T_2) + T_3)\mathbf{x} \end{aligned}$$

- Consider the transformation $0 : V \to W$ such that $0(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$. We're going to prove that this 0 is the indentity of $\mathcal{L}(V, W)$. But first, we need to know if 0 is linear or not. For any $\mathbf{v_1}, \mathbf{v_2} \in V$, we have

$$0(\alpha\mathbf{v_1} + \beta\mathbf{v_2}) = \mathbf{0} \quad \text{and} \quad \alpha 0\mathbf{v_1} + \beta 0\mathbf{v_2} = \alpha\mathbf{0} + \beta\mathbf{0} = \mathbf{0}.$$

  Hence $0(\alpha\mathbf{v_1} + \beta\mathbf{v_2}) = \alpha 0\mathbf{v_1} + \beta 0\mathbf{v_2}$, thus the transformation 0 is linear, i.e. $0 \in \mathcal{L}(V, W)$.

  Observe that for any $\mathbf{x} \in V$

$$(T_1 + 0)\mathbf{x} = T_1\mathbf{x} + 0\mathbf{x} = T_1\mathbf{x}$$

  This implies that $T_1 + 0 = T_1$ for any $T_1 \in \mathcal{L}(V, W)$. We conclude that 0 is the indentity of $\mathcal{L}(V, W)$.

- The transformation $-T_1 := (-1)T_1$ is the additive inverse of T_1 because for any $\mathbf{x} \in V$

$$T_1\mathbf{x} + (-T_1\mathbf{x}) = T_1\mathbf{x} + T_1(-\mathbf{x}) = T_1(\mathbf{x} - \mathbf{x}) = \mathbf{0} = 0(\mathbf{x}).$$

- $1 \cdot T_1 = T_1$ because $(1 \cdot T_1)\mathbf{x} = 1 \cdot T_1\mathbf{x} = T_1\mathbf{x}$ for any $\mathbf{x} \in V$.

- $(\alpha\beta)T_1 = \alpha(\beta T_1)$, because

$$[(\alpha\beta)T_1]\mathbf{x} = (\alpha\beta)T_1\mathbf{x} = T_1(\alpha\beta\mathbf{x}) = \alpha T_1(\beta\mathbf{x}) = \alpha(\beta T_1)\mathbf{x}$$

- $\alpha(T_1 + T_2) = \alpha T_1 + \alpha T_2$ because

$$[\alpha(T_1 + T_2)](\mathbf{x}) = \alpha T_1\mathbf{x} + \alpha T_2\mathbf{x} = (\alpha T_1 + \alpha T_2)(\mathbf{x})$$

□

Exercise 3.1. Multiply

a). $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 54 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+6+6 \\ 4+15+12 \end{pmatrix} = \begin{pmatrix} 13 \\ 31 \end{pmatrix}$

b). $\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+6 \\ 0+3 \\ 2+0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix}$

c). $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1+4+0+0 \\ 0+2+6+0 \\ 0+0+3+8 \\ 0+0+0+4 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 11 \\ 4 \end{pmatrix}$

d). $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$
can't be multiplied because the number of columns of the first matrix doesn't equal to the number of rows of the second matrix.

Exercise 3.2. Let a linear transformation in $\mathbb{R}^2$ be the reflection in the line $x_1 = x_2$. Find its matrix.

Solution. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be this transformation. The basis of the domain is $\{\mathbf{e_1}, \mathbf{e_2}\}$ where $\mathbf{e_1} = (1,0)^{\mathrm{T}}$ and $\mathbf{e_2} = (0,1)^{\mathrm{T}}$. Because T reflect the line $x_1 = x_2$ then

$$T\mathbf{e_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad T\mathbf{e_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore, the matrix of this transformation is $[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

□

Exercise 3.3. For each linear transformation below, find its matrix

a). $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x,y)^{\mathrm{T}} = (x+2y, 2x-5y, 7y)^{\mathrm{T}}$

b). $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(x_1,x_2,x_3,x_4)^{\mathrm{T}} = (x_1+x_2+x_3+x_4, x_2-x_4, x_1+3x_2+6x_4)^{\mathrm{T}}$

c). $T : \mathbb{P}_n \to \mathbb{P}_n$ st $Tf(t) = f'(t)$ (find the matrix with respect to the standard basis $1, t, t^2, \ldots, t^n$)

d). $T : \mathbb{P}_n \to \mathbb{P}_n$ st $Tf(t) = 2f(t) + 3f'(t) - 4f''(t)$.

Proof. Find the matrix.

a). The standard basis in $\mathbb{R}^2$ is $\{\mathbf{e_1}, \mathbf{e_2}\}$ where $\mathbf{e_1} = (1,0)^{\mathrm{T}}$ and $\mathbf{e_2} = (0,1)^{\mathrm{T}}$. We have

$$T\mathbf{e_1} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad T\mathbf{e_2} = \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}$$

Hence $\begin{pmatrix} 1 & 2 \\ 2 & -5 \\ 0 & 7 \end{pmatrix}$ is its matrix.

b). Let $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}\}$ be the standard basis in $\mathbb{R}^4$. Hence

$$T\mathbf{e_1} = T(1,0,0,0)^{\mathrm{T}} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad T\mathbf{e_2} = T(0,1,0,0)^{\mathrm{T}} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$
$$T\mathbf{e_3} = T(0,0,1,0)^{\mathrm{T}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad T\mathbf{e_4} = T(0,0,0,1)^{\mathrm{T}} = \begin{pmatrix} 1 \\ -1 \\ 6 \end{pmatrix}$$

Therefore, $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 3 & 0 & 6 \end{pmatrix}$ is its matrix.

c). Let $E = \{t^n, t^{n-1}, \ldots, t, 1\}$ be the standard basis and $f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \in \mathbb{P}_n$. We write

$$f(t) = (a_n, a_{n-1}, \ldots, a_1, a_0)^{\mathrm{T}}$$

is base E. Since

$$T(t^n) = nt^{n-1}, \quad T(t^{n-1}) = (n-1)t^{n-2} \quad , \ldots, \quad T(t) = 1, \quad T(1) = 0$$

Therefore its matrix is

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ n & 0 & 0 & \cdots & 0 & 0 \\ 0 & n-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & n-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

d). $Tf(t) = 2f(t) + 3f'(t) - 4f''(t)$
Again, the standard basis is $\{t^n, t^{n-1}, \ldots, t, 1\}$. For each $i \in [0, n]$ we have

$$T(t^i) = 2t^i + 3it^{i-1} - 4i(i-1)t^{i-2}$$

Hence the matrix is achieved by stacking $[T(t^n), \ldots, T(t^i), \ldots, T(t), T(1)]$, therefore the matrix is

$$[T] = \begin{bmatrix} 2 & 0 & \cdots & 0 & 0 \\ 3n & 2 & \cdots & 0 & 0 \\ -4n(n-1) & 3(n-1) & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 2 & 0 \\ 0 & 0 & \cdots & 3 & 2 \end{bmatrix}$$

□

Exercise 3.4. Find 3×3 matrices representing the transformations of $\mathbb{R}^3$ which

a). project every vector onto x-y plane;

b). reflect every vector through x-y plane;

c). rotate the x-y plane through $30°$, leaving the z-axis alone.

Proof. In space $\mathbb{R}^3$, we shall use its standard basis $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ where $\mathbf{e_1} = (1,0,0)^{\mathrm{T}}$, $\mathbf{e_2} = (0,1,0)^{\mathrm{T}}$ and $\mathbf{e_3} = (0,0,1)^{\mathrm{T}}$.

a). Let T be this transformation. This means $T(x,y,z)^{\mathrm{T}} = (x,y,0)^{\mathrm{T}}$. We get

$$T\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad T\mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad T\mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

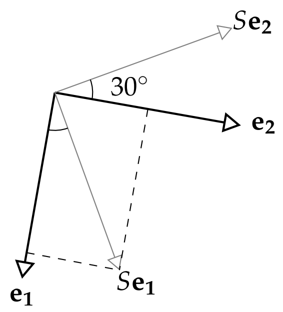
Therefore is matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

b). Let R be this transformation. Since R project every vector through x-y plane, hence $R(x,y,z)^{\mathrm{T}} = (x,y,-z)^{\mathrm{T}}$. We get

$$R\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad R\mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad R\mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Thus the matrix of R is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

c). Let S be this transformation. S moves the vectors $\mathbf{e_1}, \mathbf{e_2}$ to the point x', y' respectively.



Since $\cos 30° = \frac{\sqrt{3}}{2}$ and $\sin 30° = \frac{1}{2}$, we conclude that

$$S\mathbf{e_1} = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \quad S\mathbf{e_2} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}, \quad S\mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore the matrix is $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

□

Exercise 3.5. Let A be a linear transformation. If $\mathbf{z}$ is the center of the staight interval $[\mathbf{x}, \mathbf{y}]$, show that $A\mathbf{z}$ is the center of the interval $[A\mathbf{x}, A\mathbf{y}]$.

Proof. $\mathbf{z}$ is the center of $[\mathbf{x}, \mathbf{y}]$ iff $\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$. Therefore,

$$A\mathbf{z} = A\left(\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}\right) = \frac{1}{2}A\mathbf{x} + \frac{1}{2}A\mathbf{y}$$

Thus, $A\mathbf{z}$ is the center of the interval $[A\mathbf{x}, A\mathbf{y}]$. □

Exercise 3.6. The set $\mathbb{C}$ of complex numbers can be canonically identified with the space $\mathbb{R}^2$ by treating each $z = x + iy \in \mathbb{C}$ as a column $(x, y)^T \in \mathbb{R}^2$.

a). Treating $\mathbb{C}$ as a complex vector space, show that the multiplication by $\alpha = a + ib \in \mathbb{C}$ is a linear transformation in $\mathbb{C}$. What is its matrix.

b). Treating $\mathbb{C}$ as the real vector space $\mathbb{R}^2$ show that the multiplication by $\alpha = a + ib \in \mathbb{C}$ is a linear transformation there.

c). Define $T(x + iy) = 2x - y + i(x - 3y)$. Show that this tran is not a linear transformation in the complex vector space $\mathbb{C}$, but if we treat $\mathbb{C}$ as the real vector space $\mathbb{R}^2$ then it is a linear transformation there, then find its matrix.

Proof.

a). Let T be this transformation. For any $\mathbf{x} \in \mathbb{C}$, we have $T\mathbf{x} = \alpha\mathbf{x} \in \mathbb{C}$. Thus $T : \mathbb{C} \to \mathbb{C}$, and we'll prove that T is a linear transformation. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}$ be two vectors, and $z \in \mathbb{C}$ be a scalar (complex). Observe that

- $T(\mathbf{x}+\mathbf{y}) = \alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y} = T\mathbf{x} + T\mathbf{y}$ (distributivity of complex numbers)
- $T(z\mathbf{x}) = \alpha(z\mathbf{x}) = z(\alpha\mathbf{x}) = zT\mathbf{x}$

This shows that this transformation T is a linear one. To find its matrix, we only need to know the basis of $\mathbb{C}$. Since any vector $\mathbf{x} \in \mathbb{C}$ we be written as

$$\mathbf{x} = 1 \cdot \underbrace{\mathbf{x}}_{\text{scalar}}$$

and because this representation is unique, we obtain that $\{1\} \subset \mathbb{C}$ is a basis of $\mathbb{C}$. Thus the matrix is

$$[T] = [T(1)] = [\alpha \cdot 1] = [\alpha].$$

b). Because we treat $\mathbb{C}$ as $\mathbb{R}^2$, then any complex number $\mathbf{x} = x + iy$ can be represented as $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Let T be this transformation. Thus T would look like

$$\begin{aligned} T\begin{pmatrix} x \\ y \end{pmatrix} &= T(\mathbf{x}) = \alpha\mathbf{x} \\ &= (a+ib)(x+iy) \\ &= (ax-by) + i(ay+bx) \\ &= \begin{pmatrix} ax-by \\ ay+bx \end{pmatrix} \in \mathbb{R}^2 \end{aligned}$$

Thus $T : \mathbb{R}^2 \to \mathbb{R}^2$. We need to show that T is in fact linear. Let $\mathbf{x_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{x_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and be two arbitrary vectors. We have

$$T\mathbf{x_1} + T\mathbf{x_2} = \begin{pmatrix} ax_1 - by_1 \\ ay_1 + bx_1 \end{pmatrix} + \begin{pmatrix} ax_2 - by_2 \\ ay_2 + bx_2 \end{pmatrix} = \begin{pmatrix} a(x_1+x_2) - b(y_1+y_2) \\ a(y_1+y_2) + b(x_1+x_2) \end{pmatrix} = T(\mathbf{x_1}+\mathbf{x_2}),$$

and for any scalar $r \in \mathbb{R}$,

$$rT\mathbf{x} = r\begin{pmatrix} ax-by \\ ay+bx \end{pmatrix} = \begin{pmatrix} rax-rby \\ ray+rbx \end{pmatrix} = T(r\mathbf{x}).$$

This shows that T is a linear transformation. To find the matrix, we first need to find a bisis in $\mathbb{R}^2$. Luckily, as we've proved earlier we could choose $\{\mathbf{e_1}, \mathbf{e_2}\}$ to be a basis where

$$\mathbf{e_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{e_2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Therefore

$$T\mathbf{e_1} = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad T\mathbf{e_2} = \begin{pmatrix} -b \\ a \end{pmatrix}$$

Thus the matrix of this transformation is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

c). Define $T(x+iy) = 2x - y + i(x-3y)$

- We'll prove that T is not linear in complex vector space. Observe that

$$T(i) = T(0+i) = -1-3i \quad \text{and} \quad T(1) = T(1+0i) = 2+i$$

cleary $T(i) \neq iT(1)$, this implies that T is not a linear transformation in $\mathbb{C}$.

- In $\mathbb{R}^2$ the transformation would look like

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x-y \\ x-3y \end{pmatrix}.$$

For any vectors $\mathbf{x_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{x_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, we have

$$T\mathbf{x_1} + T\mathbf{x_2} = \begin{pmatrix} 2x_1-y_1 \\ x_1-3y_1 \end{pmatrix} + \begin{pmatrix} 2x_2-y_2 \\ x_2-3y_2 \end{pmatrix} = \begin{pmatrix} 2(x_1+x_2)-(y_1+y_2) \\ (x_1+x_2)-3(y_1+y_2) \end{pmatrix} = T(\mathbf{x_1}+\mathbf{x_2})$$

and for any scalar $r \in \mathbb{R}$,

$$rT\mathbf{x} = r\begin{pmatrix} 2x-y \\ x-3y \end{pmatrix} = \begin{pmatrix} 2rx-ry \\ rx-3ry \end{pmatrix} = T(r\mathbf{x})$$

this shows that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear. Because $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a basis in $\mathbb{R}^2$ and

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix},$$

thus the matrix of this transformation is $\begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix}$.

□

Exercise 3.7. Show that any linear transformation in $\mathbb{C}$ (treated as a complex vector space) is a multiplication by $\alpha \in \mathbb{C}$.

Proof. Let $T : \mathbb{C} \to \mathbb{C}$ be this transformation. For any $\mathbf{x} \in \mathbb{C}$

$$T\mathbf{x} = T(\mathbf{x} \cdot \mathbf{1}) = \mathbf{x} \cdot \underbrace{T(\mathbf{1})}_{\text{scalar}}$$

and the proof is completed. □

4. Linear transformation as a Vector

Let set $\mathcal{L}(V, W)$ is a vector space with addition and scalar multiplication (as proved above).

5. Composition

Homework 4. Let A and B be matrices of size $m \times n$ and $n \times m$ respectively. Then

$$\operatorname{trace}(AB) = \operatorname{trace}(BA).$$

Proof. We'd like to prove this theorem *less* computationally. Let $X \in M_{n\times m}$. Consider the mapping $T,\ T_1 : M_{n\times m} \to \mathbb{F}$ defined by

$$T(X) = \operatorname{trace}(AX) \quad \text{and} \quad T_1(X) = \operatorname{trace}(XA).$$

To prove the theorem it is sufficient to show that T, T_1 are linear and they are the same. so by substituting $X = B$ gives the theorem.

Claim 1. The transformations T, T_1 defined above are linear.

Proof. For $X, Y \in M_{n\times m}$,

- From the properties of matrix, $A(X+Y) = AX + AY$. Because AX and BX are both square matrices with size $m \times m$, and since we add the matrices $AX + AY$ entrywise, it follows that

$$\begin{aligned} T(X+Y) &= \operatorname{trace}(A(X+Y)) = \operatorname{trace}(AX+AY) \\ &= \operatorname{trace}(AX) + \operatorname{trace}(AY) \\ &= T(X) + T(Y) \end{aligned}$$

- Similarly for any scalar $\alpha \in \mathbb{F}$,

$$T(\alpha X) = \operatorname{trace}(A \cdot \alpha X) = \operatorname{trace}(\alpha AX) = \alpha \operatorname{trace}(AX) = \alpha T(X)$$

This implies that T is a linear transformation. With simply proof, we conclude that T_1 is also a linear transformation. □

We choose $\mathbf{e_{11}}, \mathbf{e_{21}}, \dots, \mathbf{e_{nm}}$ to be the standard basis of $M_{n\times m}$, meaning the vector

$$\mathbf{e_{ij}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \cdots & & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

is a matrix whose entries are zero, except at the entry at row i and column j, which is 1. Then we only need to show that $T\mathbf{e_{ij}} = T_1\mathbf{e_{ij}}$ for all i, j. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1m} \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{im} \\ a_{nn} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nm} \end{pmatrix}$$

Hence

$$A\mathbf{e_{ij}} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a_{ij} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdot & \cdot \\ \cdot & a_{ij} & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}$$

and

$$\mathbf{e_{ij}}A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a_{ij} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & a_{ij} & \cdot \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

This implies that $T\mathbf{e_{ij}} = T_1\mathbf{e_{ij}}$ for all i, j, and hence $T = T_1$. $\square$

Exercise 5.1. Working on it.

Exercise 5.2. Let T_γ be the rotation matrix by γ in $\mathbb{R}^2$. Check by matrix multiplication that $T_\gamma T_{-\gamma} = T_{-\gamma}T_\gamma = I$.

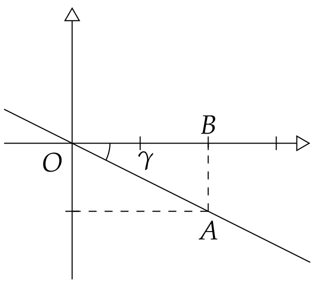
Proof. Working on it. $\square$

Exercise 5.3. Multiply two rotation matrices T_α and T_β. Deduce formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ from here.

Proof. Working on it. $\square$

Exercise 5.4. Find the matrix of the orthogonal projection in $\mathbb{R}^2$ on to the line $x_1 = -2x_2$.

Proof. Let T be this transformation. Let R_γ and P_x be the transformations of rotation by γ and projection to x-axis, respectively. Therefore $T = R_\gamma P_x R_{-\gamma}$. Note that



$$R_\gamma = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}, \quad R_{-\gamma} = \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix}, \quad \text{and} \quad P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus,

$$\cos\gamma = \frac{\overline{OB}}{\overline{OA}} = \frac{2}{\sqrt{5}} \quad \text{and} \quad \sin\gamma = \frac{\overline{AB}}{\overline{OA}} = \frac{-1}{\sqrt{5}}.$$

We get

$$\begin{aligned} T &= \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 4 & -2 \\ -2 & -2 \end{pmatrix} \end{aligned}$$

$\square$

Exercise 5.5. Find linear transformations $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ such that $AB = \mathbf{0}$ but $BA \neq \mathbf{0}$.

Solution. Consider the following

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

however

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \neq \mathbf{0}$$

Therefore, these two matrices are the ones we wish to find. ☐

Exercise 5.6. Prove that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$.

Proof. See on page **??**. ☐

Exercise 5.7. Construct a non-zero matrix A such that $A^2 = \mathbf{0}$

Proof. Let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Thus perform the multiplication, we get

$$\begin{cases} a^2 + bc = 0 \\ ac + cd = 0 \\ ab + bd = 0 \\ bc + d^2 = 0 \end{cases}$$

for simplicity, we'll choose $a = 1$. Hence $bc = -1$ and

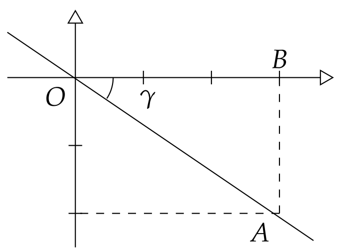
$$\begin{cases} c(d+1) = 0 \\ b(d+1) = 1 \\ d^2 = 1 \end{cases}$$

this suggests that $d = -1$, and $bc = -1$. Here, we'll choose $b = 1$ and $c = -1$. Therefore, the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

☐

Exercise 5.8. Find the matrix of the reflection through the line $y = -2x/3$.



Proof. Let T be this transformation and γ be the angle between the x-axis and the line $y = -2x/3$. Hence $T = R_\gamma T_0 R_\gamma$. We then have $\cos\gamma = OB/OA = 3/\sqrt{13}$ and $\sin\gamma = -AB/OA = -2/\sqrt{13}$. Thus

$$\begin{aligned} T &= \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} \\ &= \frac{1}{13} \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} \\ &= \frac{1}{13} \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -2 & -3 \end{pmatrix} \\ &= \frac{1}{15} \begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix}. \end{aligned}$$

☐

6. Isomophism

Exercise 6.1. Prove that if $A : V \to W$ is an isomorphism and $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ is a basis in V, then $A\mathbf{v_1}, A\mathbf{v_2}, \ldots, A\mathbf{v_n}$ is a basis in W.

Proof. Since $A : V \to W$ is an isomorphism, hence it's invertable i.e. there is a linear transformation $A^{-1} : W \to V$ such that $AA^{-1} = A^{-1}A = I$. Thus for any $\mathbf{w} \in W$, there is a $\mathbf{v} \in V$ such that $A^{-1}\mathbf{w} = \mathbf{v}$. Recall that $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a basis in V, then there are unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\mathbf{v} = \alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \cdots + \alpha_n\mathbf{v_n}.$$

This implies that

$$\begin{aligned} A^{-1}\mathbf{w} &= \alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \cdots + \alpha_n\mathbf{v_n} \\ AA^{-1}\mathbf{w} &= A(\alpha_1\mathbf{v_1} + \alpha_2\mathbf{v_2} + \cdots + \alpha_n\mathbf{v_n}) \\ \mathbf{w} &= \alpha_1 A\mathbf{v_1} + \alpha_2 A\mathbf{v_2} + \cdots + \alpha_n A\mathbf{v_n} \end{aligned}$$

Because $\alpha_1, \alpha_2, \cdots, \alpha_n$ are unique, we conclude that any $\mathbf{w} \in W$ can be represented as a unique linear combination of $A\mathbf{v_1}, A\mathbf{v_2}, \ldots, A\mathbf{v_2}$. Thus the proof is completed. □

Exercise 6.2. Find all right inverses of the 1×2 matrix (row) $A = (1, 1)$. Conclude from here thaat the row A is not left invertable.

Exercise 6.3. Find all the left inverses of the column $(1, 2, 3)^{\mathrm{T}}$.

Proof. Let $A = (1, 2, 3)^{\mathrm{T}}$. Because A is a 3×1 matrix, then its inverse, say B is a 1×3 matrix. Let $B = \begin{pmatrix} x & y & z \end{pmatrix}$. Hence

$$AB = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (1)$$

This implies that $x + 2y + 3z = 1$ or $x = 1 - 2y - 3z$. Thus all the left inverses of A is in the form

$$B = \begin{pmatrix} 1 - 2y - 3z & y & z \end{pmatrix}$$

where y, z are arbitrary real numbers. □

Exercise 6.4. Is the column $(1, 2, 3)^{\mathrm{T}}$ right invertable?

Solution. The column $(1, 2, 3)^{\mathrm{T}}$ is not right invertable, because as proved in previous exercise the column $(1, 2, 3)^{\mathrm{T}}$ has more than one left inverses. □

Exercise 6.5. Find two matrices A and B that AB is invertable, but A and B are not.

Solution. Consider: $A = \begin{pmatrix} 2 & 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^{\mathrm{T}}$ Note that

$$AB = \begin{pmatrix} 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2+2-3 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}$$

However, as proved in previous exercise, we know that the matrix A is not invertable. And we wish to prove that B is not invertable either. To achieved this, we have to find two matrices that are right invertable to B. Observe that

$$\begin{pmatrix} 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}$$

This suggests that B is not invertable. Therefore, we've fond matrices A and B such that AB is invertable, yet A and B are not. □

Exercise 6.6. Suppose the product AB is invertable. Show that A is right invertable, and B is left invertable.

Proof. Because AB is invertable, then matrix $(AB)^{-1}$ is defined. Observe that

$$A \cdot B(AB)^{-1} = AB \cdot (AB)^{-1} = I$$

and

$$(AB)^{-1}A \cdot B = (AB)^{-1} \cdot AB = I$$

This shows that A is right invertable, and B is left invertable, as expected. □

Exercise 6.7. Let A and AB be invertable. Prove that B is also in invertable.

Proof. We claim that $(AB)^{-1}A$ is the inverse of B. To prove this, observe that

$$(AB)^{-1}A \cdot B = (AB)^{-1} \cdot AB = I$$

and

$$B \cdot (AB)^{-1}A = A^{-1}AB \cdot (AB)^{-1}A = A^{-1}IA = I$$

This shows that $(AB)^{-1}A$ is both the left and the right inverses of B. Thus, B is invertable. □

Exercise 6.8. Let A be $n \times n$ matrix. Prove that if $A^2 = \mathbf{0}$ then A is not invertable.

Proof. Assume by contradiction that A is invertable, meaning there's a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. Thus

$$A = A^2A^{-1} = \mathbf{0}A^{-1} = \mathbf{0}$$

Then $I = AA^{-1} = \mathbf{0}A^{-1} = \mathbf{0}$, a contradiction. Therefore, A is not invertable. □

Exercise 6.9. Suppose $AB = \mathbf{0}$ for some non-zero matrix B. Can A invertable?

Proof. We claim that A is not invertable. To prove this, we assume by contradiction that A is invertable. Hence A^{-1} exists, and

$$B = A^{-1}A \cdot B = A^{-1} \cdot AB = A^{-1}\mathbf{0} = \mathbf{0}$$

which is a contradiction that B is non-zero. $\square$

Exercise 6.10. Write matrices of the linear transformations T_1 and T_2 in $\mathbb{F}^5$, defined as follows: T_1 interchanges the coordinates X_2 and x_2 of the vector $\mathbf{x}$, and T_2 just adds to the coordinates $x_2 \to a$ times the coordinate x_4. and does not change other coordinates, i.e.

$$T_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_4 \\ x_3 \\ x_2 \\ x_5 \end{pmatrix} \quad \text{and} \quad T_2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + ax_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

where a is a fixed number. Show that T_1 and T_2 are invertable transformations, and write the matrices of the inverses.

Proof. The matrix of this $\square$

7. Subspaces

Exercise 7.1. Let X and Y be subspaces of a vector space V. Prove that $X \cap Y$ is a subspace of V.

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be arbitrary vectors of $X \cap Y$. Because X and Y are themselves subspaces of V, hence

$$\begin{cases} \mathbf{a} \in X \\ \mathbf{b} \in X \end{cases} \implies \begin{cases} \alpha\mathbf{a} \in X \\ \beta\mathbf{b} \in X \end{cases}$$

this implies that $\alpha\mathbf{a} + \beta\mathbf{b} \in X$ for any scalars α, β. Similarly, $\alpha\mathbf{a} + \beta\mathbf{b} \in Y$. Thus $\alpha\mathbf{a} + \beta\mathbf{b} \in X \cap Y$. Therefore, $X \cap Y$ is also a subspace of V. $\square$

Exercise 7.2. Let V be a vector space. For $X, Y \subset V$ the sum $X + Y$ is the collection of all vectors $\mathbf{v}$ which can be represented as $\mathbf{v} = \mathbf{x} + \mathbf{y}$ where $\mathbf{x} \in X$ and $\mathbf{y} \in Y$. Show that if X and Y are subspaces of V, then $X + Y$ is also a subspace of V.

Proof. Let $\mathbf{v_1}, \mathbf{v_2} \in X + Y$ then there are $\mathbf{x_1}, \mathbf{x_2} \in X$ and $\mathbf{y_1}, \mathbf{y_2} \in Y$ such that

$$\mathbf{v_1} = \mathbf{x_1} + \mathbf{y_1} \quad \text{and} \quad \mathbf{v_2} = \mathbf{x_2} + \mathbf{y_2}$$

Because X, Y are subspaces of V, then

$$\alpha\mathbf{v_1} + \beta\mathbf{v_2} = \underbrace{(\alpha\mathbf{x_1} + \beta\mathbf{x_2})}_{\text{vector of } X} + \underbrace{(\alpha\mathbf{y_1} + \beta\mathbf{y_2})}_{\text{vector of } Y}$$

Hence $\alpha\mathbf{v_1} + \beta\mathbf{v_2}$ is also a vector of $X + Y$. Thus $X + Y$ is a subspace of V. $\square$

Exercise 7.3. Let X be a subspace of a vector space V, and let $\mathbf{v} \in V$ and $\mathbf{v} \notin X$. Prove that if $\mathbf{x} \in X$ then $\mathbf{x} + \mathbf{v} \notin X$.

Proof. We'll prove this by contradiction by assuming that $\mathbf{x} + \mathbf{v} \in X$. Because X is a subspace of V and $\mathbf{x} \in X$, hence $-\mathbf{x} \in X$. Thus

$$(\mathbf{x} + \mathbf{v}) + (-\mathbf{x}) \in X$$

and we conclude that $\mathbf{v} \in X$, which is a contradiction. □

Exercise 7.4. Let X and Y be subspaces of a vector space V. Using the previous exercise, show that $X \cup Y$ is a subspace iff $X \subset Y$ or $Y \subset X$.

Proof. We need to prove this in two directions.

- If $X \subset Y$ or $Y \subset X$: Without loss of generality, we may assume that $Y \subset X$. Therefore $X \cup Y = X$ is a subspace of V.
- If $X \not\subset Y$ and $Y \not\subset Y$, we now wish to show that $X \cup Y$ is not a subspace. Observe that if $X \not\subset Y$ and $Y \not\subset X$ that means there are $\mathbf{x_0} \in X$ and $\mathbf{y_0} \in Y$ such that $\mathbf{x_0} \notin Y$ and $\mathbf{y_0} \notin X$. Hence $\mathbf{x_0}, \mathbf{y_0} \in X \cup Y$ and follow from the previous exercise, we conclude that

$$\mathbf{x_0} + \mathbf{y_0} \notin X \quad \text{and} \quad \mathbf{x_0} + \mathbf{y_0} \notin Y$$

  Therefore, $\mathbf{x_0} + \mathbf{y_0} \notin X \cup Y$. This suggests that $X \cup Y$ is not a vector space. Hence the proof is completed.

□