Chapter 1

Basic Notions

1.1 Vector Spaces

Exercise 1.1. Let $\mathbf{x} = (1,2,3)^{\mathrm{T}}$, $\mathbf{y} = (y_1, y_2, y_3)^{\mathrm{T}}$ and $\mathbf{z} = (4,2,1)^{\mathrm{T}}$. Compute $2\mathbf{x}$, $3\mathbf{y}$, $\mathbf{x} + 2\mathbf{y} - 3\mathbf{z}$.

Proof. Little calculation reveals that

$$2\mathbf{x} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \quad 3\mathbf{y} = \begin{pmatrix} 3y_1 \\ 3y_2 \\ 3y_3 \end{pmatrix}, \quad \mathbf{x} + 2\mathbf{y} - 3\mathbf{z} = \begin{pmatrix} 2y_1 - 11 \\ 2y_2 - 4 \\ 2y_3 \end{pmatrix}$$

□

Exercise 1.2. Which of the following sets (with natural addition and multiplication by a scalar) are vector spaces? Justify your answers.

1. The set of all continuous functions on the interval $[0,1]$;
2. The set of all non-negative functions on the interval $[0,1]$;
3. The set of all polynomials of degree *exactly* n;
4. The set of all symmetric $n \times n$ matrices, i.e. the set of matrices $A = \{a_{j,k}\}_{j,k=1}^{n}$ such that $A^{\mathrm{T}} = A$.

Proof.

1. Let $\mathcal{C}[0,1]$ be the set of all continuous functions on $[0,1]$. For any $f, g \in \mathcal{C}[0,1]$ and $\alpha \in \mathbb{R}$, we define

   $$(f+g)(x) := f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) := \alpha \cdot f(x)$$

   for each $x \in [0,1]$. Therefore, $(\mathcal{C}[0,1], +, \cdot)$ is closed under addition and scalar multiplication. It's immediate to see that all the eight properties of vector space are all satisfied, that is

- $f+g=g+f$
- $f+(g+h)=(f+g)+h$
- $f+0=f$
- $f+(-f)=0$
- $1f=f$
- $\alpha(\beta f)=(\alpha\beta)f$
- $(\alpha+\beta)f=\alpha f+\beta f$
- $\alpha(f+g)=\alpha f+\beta g$

Note that the function $0 \in \mathcal{C}[0,1]$ such that $0(x)=0$ for each $x \in [0,1]$.

2. Let $\mathcal{B}$ is the set of all non-negative functions on $[0,1]$. Then $(\mathcal{B},+\cdot)$ is not a vector space because it's not closed under scalar multiplication, i.e. if $f \in \mathcal{B}$, hence $f > 0$ yet
$$-f = (-1)\cdot f < 0.$$
(Even if we restrict the scalar be to positive real numbers, this set still won't be a vector space, because it fails to have an inverse.)

3. Let $\mathcal{P}$ be the set of all polynomials of degree exactly n, then $(\mathcal{P},+,\cdot)$ is *not* a vector space, because the addtive indentity is the polynomial 0. However, $0 \notin \mathcal{P}$.

4. Let $\mathrm{sym}(n)$ be the set of all symmetric $n \times n$ matrices. The addition and scalar multiplication are defined as an *entrywise* operations. Hence, $\mathrm{sym}(n)$ is closed under $(+)$ and $(\cdot)$. The additive indentity is the matrix
$$\mathbf{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{sym}(n).$$
We could easily see that all the eight properties of vector spaces are all satisfied.

□

Exercise 1.3. True or false:

1. Every vector space contains a zero vector; (**True.**)
2. A vector space can have more than one zero vector; (**False.** The zero vector is unique.)
3. An $m \times n$ matrix has m rows and n columns; (**True.**)
4. If f and g are polynomials of degree n, then $f+g$ is also a polynomial of degree n. (**False.** consider t^n and $t-t^n$.)
5. If f and g are polynomials of degree atmost n, the $f+g$ is also a polynomial of degree atmost n. (**True.**)

Exercise 1.4. Prove that a zero vector $\mathbf{0}$ of a vector space V is unique.

Proof. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are the zero vectors of V. From the *Axioms of Vector Space*, we obtain that

$$\begin{aligned}
\mathbf{a} &= \mathbf{a} + \mathbf{b} && (\mathbf{b} \text{ is the zero vector}) \\
&= \mathbf{b} + \mathbf{a} && (\text{commutitativity}) \\
&= \mathbf{b} && (\mathbf{a} \text{ is the zero vector})
\end{aligned}$$

Hence, a zero vector of a vector space is unique, and we usually denote it by $\mathbf{0}$. □

Exercise 1.5. What is the zero matrix of the space $M_{2\times 3}$?

Answer. In the space $M_{2\times 3}$, the zero matrix is

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

Exercise 1.6. Prove that the additive inverse of a vector space is unique.

Proof. Let $\mathbf{a}$ be an arbitrary vector. Assume the $\mathbf{a}$ has two inverses, namely $\mathbf{x}$ and $\mathbf{y}$. Hence

$$\begin{aligned}
\mathbf{x} &= \mathbf{x} + \mathbf{0} \\
&= \mathbf{x} + (\mathbf{a} + \mathbf{y}) && (\mathbf{y} \text{ is an inverse}) \\
&= (\mathbf{x} + \mathbf{a}) + \mathbf{y} && (\text{associativity}) \\
&= \mathbf{0} + \mathbf{y} && (\mathbf{x} \text{ is an inverse}) \\
&= \mathbf{y}.
\end{aligned}$$

Therefore, the inverse of any vector $\mathbf{a} \in V$ is unique, and is usually denoted by $-\mathbf{a}$. □

Exercise 1.7. Prove that $0\mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v} \in V$.

Proof. Let $\mathbf{v} \in V$ and $\mathbf{b}$ is an inverse of $0\mathbf{v}$. Therefore,

$$\begin{aligned}
0 &= 0\mathbf{v} + b \\
&= (0 + 0)\mathbf{v} + b \\
&= (0\mathbf{v} + 0\mathbf{v}) + b && (\text{distributivity}) \\
&= 0\mathbf{v} + (0\mathbf{v} + b) && (\text{associativity}) \\
&= 0\mathbf{v} + \mathbf{0} && (\mathbf{b} \text{ is an inverse of } 0\mathbf{v}) \\
&= 0\mathbf{v}
\end{aligned}$$

for any $\mathbf{v} \in V$. □

Exercise 1.8. Prove that for any vector $\mathbf{v}$ its additive inverse $-\mathbf{v}$ is given by $(-1)\mathbf{v}$.

Proof. As proved in the above exercise for any $\mathbf{v} \in V$,

$$\mathbf{0} = 0\mathbf{v} = (1-1)\mathbf{v} = \mathbf{v} + (-1)\mathbf{v}$$

where the last equallity derives from the distributive property. Because $-\mathbf{v}$ is the inverse of $\mathbf{v}$, then

$$\begin{aligned} -\mathbf{v} &= -\mathbf{v} + \mathbf{0} \\ &= -\mathbf{v} + \left[\mathbf{v} + (-1)\mathbf{v}\right] \\ &= (\underbrace{-\mathbf{v} + \mathbf{v}}_{\mathbf{0}}) + (-1)\mathbf{v} \\ &= (-1)\mathbf{v} \end{aligned}$$

as desired. $\square$