

# Chapter 1

## Basic Notions

### 1. Vector Spaces

**Exercise 1.1.** Let  $\mathbf{x} = (1, 2, 3)^T$ ,  $\mathbf{y} = (y_1, y_2, y_3)^T$  and  $\mathbf{z} = (4, 2, 1)^T$ . Compute  $2\mathbf{x}$ ,  $3\mathbf{y}$ ,  $\mathbf{x} + 2\mathbf{y} - 3\mathbf{z}$ .

*Proof.* Little calculation reveals that

$$2\mathbf{x} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \quad 3\mathbf{y} = \begin{pmatrix} 3y_1 \\ 3y_2 \\ 3y_3 \end{pmatrix}, \quad \mathbf{x} + 2\mathbf{y} - 3\mathbf{z} = \begin{pmatrix} 2y_1 - 11 \\ 2y_2 - 4 \\ 2y_3 \end{pmatrix}$$

□

**Exercise 1.2.** Which of the following sets (with natural addition and multiplication by a scalar) are vector spaces? Justify your answers.

- The set of all continuous functions on the interval  $[0, 1]$ ;
- The set of all non-negative functions on the interval  $[0, 1]$ ;
- The set of all polynomials of degree *exactly*  $n$ ;
- The set of all symmetric  $n \times n$  matrices, i.e. the set of matrices  $A = \{a_{j,k}\}_{j,k=1}^n$  such that  $A^T = A$ .

*Proof.*

- Let  $\mathcal{C}[0, 1]$  be the set of all continuous functions on  $[0, 1]$ . For any  $f, g \in \mathcal{C}[0, 1]$  and  $\alpha \in \mathbb{R}$ , we define

$$(f + g)(x) := f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) := \alpha \cdot f(x)$$

for each  $x \in [0, 1]$ . Therefore,  $(\mathcal{C}[0, 1], +, \cdot)$  is closed under addition and scalar multiplication. It's immediate to see that all the eight properties of vector space are all satisfied, that is

- |                               |  |
|-------------------------------|--|
| ◦ $f + g = g + f$             | ◦ $1f = f$                                 |
| ◦ $f + (g + h) = (f + g) + h$ | ◦ $\alpha(\beta f) = (\alpha\beta)f$       |
| ◦ $f + 0 = f$                 | ◦ $(\alpha + \beta)f = \alpha f + \beta f$ |
| ◦ $f + (-f) = 0$              | ◦ $\alpha(f + g) = \alpha f + \alpha g$    |

Note that the function  $0 \in \mathcal{C}[0, 1]$  such that  $0(x) = 0$  for each  $x \in [0, 1]$ .

- Let  $\mathcal{B}$  is the set of all non-negative functions on  $[0, 1]$ . Then  $(\mathcal{B}, +, \cdot)$  is not a vector space because it's not closed under scalar multiplication, i.e. if  $f \in \mathcal{B}$ , hence  $f > 0$  yet

$$-f = (-1) \cdot f < 0.$$

(Even if we restrict the scalar be to positive real numbers, this set still won't be a vector space, because it fails to have an inverse.)

- c). Let  $\mathcal{P}$  be the set of all polynomials of degree exactly  $n$ , then  $(\mathcal{P}, +, \cdot)$  is *not* a vector space, because the additive identity is the polynomial 0. However,  $0 \notin \mathcal{P}$ .
- d). Let  $\text{sym}(n)$  be the set of all symmetric  $n \times n$  matrices. The addition and scalar multiplication are defined as an *entrywise* operations. Hence,  $\text{sym}(n)$  is closed under  $(+)$  and  $(\cdot)$ . The additive identity is the matrix

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \text{sym}(n).$$

We could easily see that all the eight properties of vector spaces are all satisfied.

□

**Exercise 1.3.** True or false:

- a). Every vector space contains a zero vector; (**True.**)
- b). A vector space can have more than one zero vector; (**False.** The zero vector is unique.)
- c). An  $m \times n$  matrix has  $m$  rows and  $n$  columns; (**True.**)
- d). If  $f$  and  $g$  are polynomials of degree  $n$ , then  $f + g$  is also a polynomial of degree  $n$ . (**False.** consider  $t^n$  and  $t - t^n$ .)
- e). If  $f$  and  $g$  are polynomials of degree atmost  $n$ , the  $f + g$  is also a polynomial of degree atmost  $n$ . (**True.**)

**Exercise 1.4.** Prove that a zero vector  $\mathbf{0}$  of a vector space  $V$  is unique.

*Proof.* Suppose that  $\mathbf{a}$  and  $\mathbf{b}$  are the zero vectors of  $V$ . From the *Axioms of Vector Space*, we obtain that

$$\begin{aligned} \mathbf{a} &= \mathbf{a} + \mathbf{b} && (\mathbf{b} \text{ is the zero vector}) \\ &= \mathbf{b} + \mathbf{a} && (\text{commutativity}) \\ &= \mathbf{b} && (\mathbf{a} \text{ is the zero vector}) \end{aligned}$$

Hence, a zero vector of a vector space is unique, and we usually denote it by  $\mathbf{0}$ .

□

**Exercise 1.5.** What is the zero matrix of the space  $M_{2 \times 3}$ ?

*Answer.* In the space  $M_{2 \times 3}$ , the zero matrix is

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

**Exercise 1.6.** Prove that the additive inverse of a vector space is unique.

*Proof.* Let  $\mathbf{a}$  be an arbitrary vector. Assume the  $\mathbf{a}$  has two inverses, namely  $\mathbf{x}$  and  $\mathbf{y}$ . Hence

$$\begin{aligned} \mathbf{x} &= \mathbf{x} + \mathbf{0} \\ &= \mathbf{x} + (\mathbf{a} + \mathbf{y}) && (\mathbf{y} \text{ is an inverse}) \\ &= (\mathbf{x} + \mathbf{a}) + \mathbf{y} && (\text{associativity}) \\ &= \mathbf{0} + \mathbf{y} && (\mathbf{x} \text{ is an inverse}) \\ &= \mathbf{y}. \end{aligned}$$

Therefore, the inverse of any vector  $\mathbf{a} \in V$  is unique, and is usually denoted by  $-\mathbf{a}$ .

□

**Exercise 1.7.** Prove that  $0\mathbf{v} = \mathbf{0}$  for any vector  $\mathbf{v} \in V$ .

*Proof.* Let  $\mathbf{v} \in V$  and  $\mathbf{b}$  is an inverse of  $0\mathbf{v}$ . Therefore,

$$\begin{aligned}
 0 &= 0\mathbf{v} + \mathbf{b} \\
 &= (0 + 0)\mathbf{v} + \mathbf{b} \\
 &= (0\mathbf{v} + 0\mathbf{v}) + \mathbf{b} && \text{(distributivity)} \\
 &= 0\mathbf{v} + (0\mathbf{v} + \mathbf{b}) && \text{(associativity)} \\
 &= 0\mathbf{v} + \mathbf{0} && (\mathbf{b} \text{ is an inverse of } 0\mathbf{v}) \\
 &= 0\mathbf{v}
 \end{aligned}$$

for any  $\mathbf{v} \in V$ . □

**Exercise 1.8.** Prove that for any vector  $\mathbf{v}$  its additive inverse  $-\mathbf{v}$  is given by  $(-1)\mathbf{v}$ .

*Proof.* As proved in the above exercise for any  $\mathbf{v} \in V$ ,

$$\mathbf{0} = 0\mathbf{v} = (1 - 1)\mathbf{v} = \mathbf{v} + (-1)\mathbf{v}$$

where the last equality derives from the distributive property. Because  $-\mathbf{v}$  is the inverse of  $\mathbf{v}$ , then

$$\begin{aligned}
 -\mathbf{v} &= -\mathbf{v} + \mathbf{0} \\
 &= -\mathbf{v} + [\mathbf{v} + (-1)\mathbf{v}] \\
 &= \underbrace{(-\mathbf{v} + \mathbf{v})}_{\mathbf{0}} + (-1)\mathbf{v} \\
 &= (-1)\mathbf{v}
 \end{aligned}$$

as desired. □

## 2. Linear Combination, bases

**Exercise 2.1.** Find the basis in the space of  $3 \times 2$  matrices  $M_{3 \times 2}$ .

*Answer.* Consider the vectors:

$$\begin{aligned}
 \mathbf{e}_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, & \mathbf{e}_2 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, & \mathbf{e}_3 &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \\
 \mathbf{e}_4 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, & \mathbf{e}_5 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, & \mathbf{e}_6 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}
 \end{aligned}$$

and we're going to prove that the system of these vectors are a basis. Any matrix

$$\mathbf{v} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \in M_{3 \times 2}$$

can be represented as the combination  $\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 + d\mathbf{e}_4 + e\mathbf{e}_5 + f\mathbf{e}_6$  thus this system is generating. Next we're going to prove the uniqueness.

Suppose that there are  $\hat{a}, \hat{b}, \dots, \hat{f}$  with

$$\begin{aligned}
 \mathbf{v} &= \hat{a}\mathbf{e}_1 + \hat{b}\mathbf{e}_2 + \hat{c}\mathbf{e}_3 + \hat{d}\mathbf{e}_4 + \hat{e}\mathbf{e}_5 + \hat{f}\mathbf{e}_6 \\
 \implies \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} &= \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \\ \hat{e} & \hat{f} \end{bmatrix}
 \end{aligned}$$

This implies that each corresponding entry is equal. Hence the representation is unique. Therefore this system is a basis.  $\square$

**Exercise 2.2.** True or false:

- a). Any set containing a zero vector is linearly dependent;
- b). A basis must contain  $\mathbf{0}$ ;
- c). subsets of linearly dependent sets are linearly dependent;
- d). subsets of linearly independent sets are linearly independent;
- e). if  $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}$  then all scalars  $\alpha_k$  are zero.

*Answer.*

- a). **True.** because  $\mathbf{0}$  can be represented as a linear combination of the other vectors (simply put all the scalars to 0).
- b). **No.** if so, they must be linearly dependent, which is not a base.
- c). **No.** Take for example the system of linearly dependent  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  where  $\mathbf{e}_1 = (1, 0)$ ,  $\mathbf{e}_2 = (0, 1)$  and  $\mathbf{e}_3 = (1, 1)$ . The subset  $\{\mathbf{e}_1, \mathbf{e}_2\}$  is a basis, which is clearly not linearly dependent.
- d). **True.** Suppose that the system  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is a subset of the linearly independent system  $\{\mathbf{v}_1, \dots, \mathbf{v}_p, \dots, \mathbf{v}_n\}$ . Let  $\alpha_k$  the real numbers such that  $\alpha_1 \mathbf{v}_1 + \cdots + \alpha_p \mathbf{v}_p = \mathbf{0}$  hence

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_p \mathbf{v}_p + 0 \mathbf{v}_{p+1} + \cdots + 0 \mathbf{v}_n = \mathbf{0}.$$

Because the system  $\{\mathbf{v}_1, \dots, \mathbf{v}_p, \dots, \mathbf{v}_n\}$  is linearly independent, therefore all the scalars  $\alpha_k = 0$ . Thus, the system  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is also linearly independent.

- e). **No.** Take,  $\mathbf{e}_1 = (2, 2)$  and  $\mathbf{e}_2 = (1, 1)$  for instance. We have  $\mathbf{e}_1 - 2\mathbf{e}_2 = \mathbf{0}$  yet the scalars are non-zero.  $\square$

**Exercise 2.3.** Recall, that a matrix is called *symmetric* if  $A^T = A$ . Write down a basis in the space of *symmetric*  $2 \times 2$  matrices (there are many possible answers). How many elements are there in the basis.

*Answer.* We are going to prove that the system  $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_1\}$  where

$$\mathbf{d}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{d}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

is a basis. Observe that any symmetric matrix

$$\mathbf{v} = \begin{bmatrix} d_1 & e_1 \\ e_1 & d_2 \end{bmatrix}$$

can be represented as  $\mathbf{v} = d_1 \mathbf{d}_1 + d_2 \mathbf{d}_2 + e_1 \mathbf{e}_1$ , hence it's generating. Note that the equation

$$\begin{aligned} d_1 \mathbf{d}_1 + d_2 \mathbf{d}_2 + e_1 \mathbf{e}_1 &= \mathbf{0} \\ \begin{bmatrix} d_1 & e_1 \\ e_1 & d_2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

holds only when all the scalars are all zero. Hence the system is linearly independent. Thus, it's a basis.  $\square$

**Exercise 2.4.** Write down a basis for the space of

- a).  $3 \times 3$  symmetric matrices;
- b).  $n \times n$  symmetric matrices;
- c).  $n \times n$  antisymmetric matrices.

*Answer.*

- a). we are going to prove that the system of vectors

$$\mathbf{d}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{d}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{d}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

is the basis. First of, any symmetric matrix

$$\mathbf{v} = \begin{bmatrix} d_1 & e_1 & e_2 \\ e_1 & d_2 & e_3 \\ e_2 & e_3 & d_3 \end{bmatrix}$$

can be represented as

$$\mathbf{v} = d_1 \mathbf{d}_1 + d_2 \mathbf{d}_2 + d_3 \mathbf{d}_3 + e_1 \mathbf{e}_1 + e_2 \mathbf{e}_2 + e_3 \mathbf{e}_3$$

yeilds that the system is generating. Similar to the previous problem, if the linear combination of these vectors equals  $\mathbf{0}$ , then all the scalars must equals zero. Thus it's linearly independent. Therefore it's a basis.

- b). Working on it.
- c). Working on it.

□

**Exercise 2.5.** Let a system of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$  be linearly independent but not generating. Show that it is possible to find a vector  $\mathbf{v}_{r+1}$  such that the system  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r, \mathbf{v}_{r+1}$  is linearly independent.

*Proof.* Because the system  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$  is not generating, therefore there exists a vector  $\mathbf{v}_{r+1}$  such that  $\mathbf{v}_{r+1}$  cannot be represented as a linear combination of  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ . Let  $\alpha_i$  be the scalars such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_r \mathbf{v}_r + \alpha_{r+1} \mathbf{v}_{r+1} = \mathbf{0} \quad (1.1)$$

Now we have to prove that all the scalars are all zero. If  $\alpha_{r+1} \neq 0$  then

$$\mathbf{v}_{r+1} = - \sum_{i=1}^r \frac{\alpha_i}{\alpha_{r+1}} \cdot \mathbf{v}_i,$$

meaning  $\mathbf{v}_{r+1}$  is the linear combination of the other vectors, a contradiction. Hence  $\alpha_{r+1}$  must equals to zero. So the  $r + 1$  term in the equation (1.1) vanishes. And because the system  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$  is linearly independent, all the scalars  $\alpha_i = 0$  for all  $i = 0, 1, \dots, r$ . Thus, the system

$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r, \mathbf{v}_{r+1}$$

is also *linearly independent*.

□

**Exercise 2.6.** Is it possible that vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly dependent, but the vectors  $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$ ,  $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$  and  $\mathbf{w}_3 = \mathbf{v}_3 + \mathbf{v}_1$  are linearly *independent*.

*Proof.* It's not possible, and we're going to prove this assertion via contradiction. Assume that there are such vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  satisfying the above conditions. Then there are numbers  $x, y, z \in \mathbb{R}$  such that

$$|x| + |y| + |z| > 0 \quad \text{and} \quad x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{0}.$$

By letting

$$a = x + y - z, \quad b = y + z - x, \quad c = z + x - y$$

we obtain that

$$\begin{aligned} a\mathbf{w}_1 + b\mathbf{w}_2 + c\mathbf{w}_3 &= (x\mathbf{w}_1 + y\mathbf{w}_1 - z\mathbf{w}_1) + (y\mathbf{w}_2 + z\mathbf{w}_2 - x\mathbf{w}_2) \\ &\quad + (x\mathbf{w}_3 + z\mathbf{w}_3 - y\mathbf{w}_3) \\ &= 2x\mathbf{v}_1 + 2y\mathbf{v}_2 + 2z\mathbf{v}_3 \\ &= \mathbf{0}. \end{aligned}$$

Since  $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$  are linearly independent, we must have  $a = b = c = 0$ . Hence

$$\begin{cases} x + y - z = 0 \\ y + z - x = 0 \\ z + x - y = 0 \end{cases}$$

adding all the 3 equations,  $x + y + z = 0$ . Substituting back to the system of equations above we get

$$x = y = z = 0$$

which contradicts to the fact that  $|x| + |y| + |z| > 0$ . □

**Exercise 2.7.** Any finite independent system is a subset of some basis.

*Proof.* Let  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  is linearly independent. If this system is generating, then it's a base and we're done. If not, from exercise 2.5, there exists  $\mathbf{v}_{n+1}$  such that

$$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \mathbf{v}_{n+1}\}$$

is still linearly independent. Now if this new system is generating, then we're done. If not, we keep continue this process a finite steps, adding vectors  $\mathbf{v}_{n+1}, \mathbf{v}_{n+2}, \dots, \mathbf{v}_{n+r}$ , and eventually the new system

$$\{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}_{n+1}, \dots, \mathbf{v}_{n+r}\}$$

is now a basis. □

### 3. Linear Transformation

**Homework 1.** Prove that the transformation  $T : \mathbb{F}^n \rightarrow \mathbb{F}^m$  if and only if  $T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$  for any scalars  $\alpha, \beta$  and vectors  $\mathbf{x}, \mathbf{y} \in \mathbb{F}$ .

*Proof.* We need to prove this in two directions.

( $\Rightarrow$ ) Suppose  $T$  is a linear transformation, then

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = T(\alpha\mathbf{x}) + T(\beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$$

as needed.

( $\Leftarrow$ ) For this direction, we first assume that  $T$  has the property that  $T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$  for all  $\alpha, \beta, \mathbf{x}, \mathbf{y}$ . We need to show that  $T$  has the property listed in the definition of the linear transformation. Observe that

- take  $\alpha = \beta = 1$  then,  $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
- take  $\beta = 0$  then,  $T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$

Hence  $T$  is a linear transformation, and the proof is completed. □

**Homework 2.** Let  $T : V \rightarrow W$  be a linear transformation. Prove that  $T(\mathbf{0}) = \mathbf{0}$  and

$$TV = \{T\mathbf{v} : \mathbf{v} \in V\}$$

is a vector space.

*Proof.* Since  $T$  is linear, and as proved before  $0 \cdot \mathbf{0} = \mathbf{0}$ , it's easy to see that

$$T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0 \cdot T(\mathbf{0}) = \mathbf{0}.$$

To prove that  $TV$  is a vector space, we need to check that  $TV$  satisfies all the eight conditions listed in the definition of vector space.

We first need to prove that  $TV$  is closed. Because  $TV \subset W$ , hence  $TV$  is closed under scalar multiplication and vector addition. Let  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ . Observe that

- $T\mathbf{x} + T\mathbf{y} = T\mathbf{y} + T\mathbf{x}$  (commutativity of  $W$ )
- $(T\mathbf{x} + T\mathbf{y}) + T\mathbf{z} = T\mathbf{x} + (T\mathbf{y} + T\mathbf{z})$  (associativity of  $W$ )
- The vector  $\mathbf{0} \in W$  is the identity of  $TV$  because

$$T\mathbf{x} + \mathbf{0} = T\mathbf{x} + T\mathbf{0} = T(\mathbf{x} + \mathbf{0}) = T(\mathbf{x}), \quad \forall \mathbf{x} \in V$$

- The vector  $T(-\mathbf{x})$  is the additive inverse of  $T\mathbf{x}$  because

$$T\mathbf{x} + T(-\mathbf{x}) = T(\mathbf{x} - \mathbf{x}) = \mathbf{0}$$

- $1 \cdot T\mathbf{v} = T\mathbf{v}$  (multiplicative iden. in  $W$ )

Let  $\alpha, \beta$  be scalars.

- multiplicative associativity

$$\begin{aligned} (\alpha\beta)T\mathbf{x} &= T((\alpha\beta)\mathbf{x}) && \text{(linearity of } T) \\ &= T(\alpha(\beta\mathbf{x})) && \text{(mult. asso. of } V) \\ &= \alpha T(\beta\mathbf{x}) && \text{(linearity of } T) \\ &= \alpha \cdot \beta T\mathbf{x} \end{aligned}$$

- scalar multiplication

$$\begin{aligned} \alpha(T\mathbf{x} + T\mathbf{y}) &= \alpha T(\mathbf{x} + \mathbf{y}) && \text{(linearity of } T) \\ &= T(\alpha(\mathbf{x} + \mathbf{y})) \\ &= T(\alpha\mathbf{x} + \alpha\mathbf{y}) && \text{(scalar mult. in } V) \\ &= T(\alpha\mathbf{x}) + T(\alpha\mathbf{y}) && \text{(linearity of } T) \\ &= \alpha T\mathbf{x} + \alpha T\mathbf{y} \end{aligned}$$

- scalar multiplication

$$\begin{aligned} (\alpha + \beta)T\mathbf{x} &= T((\alpha + \beta)\mathbf{x}) && \text{(linearity of } T) \\ &= T(\alpha\mathbf{x} + \beta\mathbf{x}) && \text{(scalar mult. of } V) \\ &= T(\alpha\mathbf{x}) + T(\beta\mathbf{x}) \\ &= \alpha T\mathbf{x} + \beta T\mathbf{x} \end{aligned}$$

We see that  $TV$  has all eight properties to be a vector space, and the proof is completed.  $\square$

**Homework 3.** Let  $V, W$  be vector spaces. Prove that  $\mathcal{L}(V, W)$ , the set of all linear transformations  $T : V \rightarrow W$ , is also a vector space.

*Proof.* We first need to show that  $\mathcal{L}(V, W)$  is closed. Let  $T_1, T_2 \in \mathcal{L}(V, W)$  and  $a$  be a scalar. So we need to show the transformation  $T_1 + T_2$  and  $aT_1$  are both linear.

- Let  $\mathbf{x}, \mathbf{y}$  be arbitrary vectors in  $V$  and  $\alpha, \beta$  be scalar. Denote  $T := T_1 + T_2$ . Observe that

$$\begin{aligned}
 T(\alpha\mathbf{x} + \beta\mathbf{y}) &= (T_1 + T_2)(\alpha\mathbf{x} + \beta\mathbf{y}) \\
 &= T_1(\alpha\mathbf{x} + \beta\mathbf{y}) + T_2(\alpha\mathbf{x} + \beta\mathbf{y}) && \text{(by def. of } T_1 + T_2\text{)} \\
 &= \alpha T_1\mathbf{x} + \beta T_1\mathbf{y} + \alpha T_2\mathbf{x} + \beta T_2\mathbf{y} && \text{(by lin. of } T_1 \text{ and } T_2\text{)} \\
 &= (\alpha T_1\mathbf{x} + \alpha T_2\mathbf{x}) + (\beta T_1\mathbf{y} + \beta T_2\mathbf{y}) \\
 &= \alpha(T_1\mathbf{x} + T_2\mathbf{x}) + \beta(T_1\mathbf{y} + T_2\mathbf{y}) && \text{(by scalar mult. in } W\text{)} \\
 &= \alpha(T_1 + T_2)\mathbf{x} + \beta(T_1 + T_2)\mathbf{y} \\
 &= \alpha T\mathbf{x} + \beta T\mathbf{y}
 \end{aligned}$$

This shows that  $T_1 + T_2$  is also a linear transformation, hence  $\mathcal{L}(V, W)$  is closed under addition.

- Similarly, we let  $\mathbf{x}, \mathbf{y} \in V$ . For simplicity, we again denote  $T := aT_1$ . Hence for any scalars  $\alpha, \beta$

$$\begin{aligned}
 T(\alpha\mathbf{x} + \beta\mathbf{y}) &= (aT_1)(\alpha\mathbf{x} + \beta\mathbf{y}) \\
 &= a \cdot T_1(\alpha\mathbf{x} + \beta\mathbf{y}) && \text{(by def. of } aT_1\text{)} \\
 &= a \cdot (\alpha T_1\mathbf{x} + \beta T_1\mathbf{y}) && \text{(by lin. of } T_1\text{)} \\
 &= \alpha a T_1\mathbf{x} + \beta a T_1\mathbf{y} \\
 &= \alpha(aT_1)\mathbf{x} + \beta(aT_1)\mathbf{y} \\
 &= \alpha T\mathbf{x} + \beta T\mathbf{y}
 \end{aligned}$$

This suggests that  $aT_1$  is also linear, hence  $\mathcal{L}(V, W)$  is closed under scalar multiplication. Ultimately, we've proved that  $\mathcal{L}(V, W)$  is closed as needed.

We are now ready to prove that  $\mathcal{L}(V, W)$  is a vector space. Let  $T_1, T_2, T_3 \in \mathcal{L}(V, W)$  we have

- $T_1 + T_2 = T_2 + T_1$ , because for any  $\mathbf{x} \in V$

$$(T_1 + T_2)\mathbf{x} = T_1\mathbf{x} + T_2\mathbf{x} = T_2\mathbf{x} + T_1\mathbf{x} = (T_2 + T_1)\mathbf{x}.$$

- $T_1 + (T_2 + T_3) = (T_1 + T_2) + T_3$ , because for any  $\mathbf{x} \in V$

$$\begin{aligned}
 (T_1 + (T_2 + T_3))\mathbf{x} &= T_1\mathbf{x} + (T_2 + T_3)\mathbf{x} \\
 &= T_1\mathbf{x} + (T_2\mathbf{x} + T_3\mathbf{x}) \\
 &= (T_1\mathbf{x} + T_2\mathbf{x}) + T_3\mathbf{x} && \text{(by asso. of } W\text{)} \\
 &= (T_1 + T_2)\mathbf{x} + T_3\mathbf{x} \\
 &= ((T_1 + T_2) + T_3)\mathbf{x}
 \end{aligned}$$

- Consider the transformation  $0 : V \rightarrow W$  such that  $0(\mathbf{v}) = \mathbf{0}$  for all  $\mathbf{v} \in V$ . We're going to prove that this  $0$  is the identity of  $\mathcal{L}(V, W)$ . But first, we need to know if  $0$  is linear or not. For any  $\mathbf{v}_1, \mathbf{v}_2 \in V$ , we have

$$0(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \mathbf{0} \quad \text{and} \quad \alpha 0\mathbf{v}_1 + \beta 0\mathbf{v}_2 = \alpha\mathbf{0} + \beta\mathbf{0} = \mathbf{0}.$$

Hence  $0(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \alpha 0\mathbf{v}_1 + \beta 0\mathbf{v}_2$ , thus the transformation  $0$  is linear, i.e.  $0 \in \mathcal{L}(V, W)$ .



Observe that for any  $\mathbf{x} \in V$

$$(T_1 + 0)\mathbf{x} = T_1\mathbf{x} + 0\mathbf{x} = T_1\mathbf{x}$$

This implies that  $T_1 + 0 = T_1$  for any  $T_1 \in \mathcal{L}(V, W)$ . We conclude that  $0$  is the identity of  $\mathcal{L}(V, W)$ .

- The transformation  $-T_1 := (-1)T_1$  is the additive inverse of  $T_1$  because for any  $\mathbf{x} \in V$

$$T_1\mathbf{x} + (-T_1)\mathbf{x} = T_1\mathbf{x} + T_1(-\mathbf{x}) = T_1(\mathbf{x} - \mathbf{x}) = \mathbf{0} = 0(\mathbf{x}).$$

- $1 \cdot T_1 = T_1$  because  $(1 \cdot T_1)\mathbf{x} = 1 \cdot T_1\mathbf{x} = T_1\mathbf{x}$  for any  $\mathbf{x} \in V$ .
- $(\alpha\beta)T_1 = \alpha(\beta T_1)$ , because

$$[(\alpha\beta)T_1]\mathbf{x} = (\alpha\beta)T_1\mathbf{x} = T_1(\alpha\beta\mathbf{x}) = \alpha T_1(\beta\mathbf{x}) = \alpha(\beta T_1)\mathbf{x}$$

- $\alpha(T_1 + T_2) = \alpha T_1 + \alpha T_2$  because

$$[\alpha(T_1 + T_2)](\mathbf{x}) = \alpha T_1\mathbf{x} + \alpha T_2\mathbf{x} = (\alpha T_1 + \alpha T_2)(\mathbf{x})$$

□

**Exercise 3.1.** Multiply

$$\text{a). } \begin{pmatrix} 1 & 2 & 3 \\ 4 & 54 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+6+6 \\ 4+15+12 \end{pmatrix} = \begin{pmatrix} 13 \\ 31 \end{pmatrix}$$

$$\text{b). } \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+6 \\ 0+3 \\ 2+0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix}$$

$$\text{c). } \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1+4+0+0 \\ 0+2+6+0 \\ 0+0+3+8 \\ 0+0+0+4 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 11 \\ 4 \end{pmatrix}$$

$$\text{d). } \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

can't be multiplied because the number of columns of the first matrix doesn't equal to the number of rows of the second matrix.

**Exercise 3.2.** Let a linear transformation in  $\mathbb{R}^2$  be the reflection in the line  $x_1 = x_2$ . Find its matrix.

*Solution.* Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be this transformation. The basis of the domain is  $\{\mathbf{e}_1, \mathbf{e}_2\}$  where  $\mathbf{e}_1 = (1, 0)^T$  and  $\mathbf{e}_2 = (0, 1)^T$ . Because  $T$  reflect the line  $x_1 = x_2$  then

$$T\mathbf{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad T\mathbf{e}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore, the matrix of this transformation is  $[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

□

**Exercise 3.3.** For each linear transformation below, find its matrix

$$\text{a). } T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \text{ defined by } T(x, y)^T = (x + 2y, 2x - 5y, 7y)^T$$

- b).  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  defined by  $T(x_1, x_2, x_3, x_4)^T = (x_1 + x_2 + x_3 + x_4, x_2 - x_4, x_1 + 3x_2 + 6x_4)^T$
- c).  $T : \mathbb{P}_n \rightarrow \mathbb{P}_n$  st  $Tf(t) = f'(t)$  (find the matrix with respect to the standard basis  $1, t, t^2, \dots, t^n$ )
- d).  $T : \mathbb{P}_n \rightarrow \mathbb{P}_n$  st  $Tf(t) = 2f(t) + 3f'(t) - 4f''(t)$ .

*Proof.* Find the matrix.

- a). The standard basis in  $\mathbb{R}^2$  is  $\{\mathbf{e}_1, \mathbf{e}_2\}$  where  $\mathbf{e}_1 = (1, 0)^T$  and  $\mathbf{e}_2 = (0, 1)^T$ . We have

$$T\mathbf{e}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad T\mathbf{e}_2 = \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}$$

Hence  $\begin{pmatrix} 1 & 2 \\ 2 & -5 \\ 0 & 7 \end{pmatrix}$  is its matrix.

- b). Let  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$  be the standard basis in  $\mathbb{R}^4$ . Hence

$$\begin{aligned} T\mathbf{e}_1 &= T(1, 0, 0, 0)^T = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, & T\mathbf{e}_2 &= T(0, 1, 0, 0)^T = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \\ T\mathbf{e}_3 &= T(0, 0, 1, 0)^T = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & T\mathbf{e}_4 &= T(0, 0, 0, 1)^T = \begin{pmatrix} 1 \\ -1 \\ 6 \end{pmatrix} \end{aligned}$$

Therefore,  $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 3 & 0 & 6 \end{pmatrix}$  is its matrix.

- c). Let  $E = \{t^n, t^{n-1}, \dots, t, 1\}$  be the standard basis and  $f(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0 \in \mathbb{P}_n$ . We write

$$f(t) = (a_n, a_{n-1}, \dots, a_1, a_0)^T$$

is base  $E$ . Since

$$T(t^n) = nt^{n-1}, \quad T(t^{n-1}) = (n-1)t^{n-2}, \dots, \quad T(t) = 1, \quad T(1) = 0$$

Therefore its matrix is

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ n & 0 & 0 & \cdots & 0 & 0 \\ 0 & n-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & n-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

d).  $Tf(t) = 2f(t) + 3f'(t) - 4f''(t)$

Again, the standard basis is  $\{t^n, t^{n-1}, \dots, t, 1\}$ . For each  $i \in [0, n]$  we have

$$T(t^i) = 2t^i + 3it^{i-1} - 4i(i-1)t^{i-2}$$

Hence the matrix is achieved by stacking  $[T(t^n), \dots, T(t^i), \dots, T(t), T(1)]$ , therefore the matrix is

$$[T] = \begin{bmatrix} 2 & 0 & \cdots & 0 & 0 \\ 3n & 2 & \cdots & 0 & 0 \\ -4n(n-1) & 3(n-1) & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 2 & 0 \\ 0 & 0 & \cdots & 3 & 2 \end{bmatrix}$$

□

**Exercise 3.4.** Find  $3 \times 3$  matrices representing the transformations of  $\mathbb{R}^3$  which

- project every vector onto  $x$ - $y$  plane;
- reflect every vector through  $x$ - $y$  plane;
- rotate the  $x$ - $y$  plane through  $30^\circ$ , leaving the  $z$ -axis alone.

*Proof.* In space  $\mathbb{R}^3$ , we shall use its standard basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  where  $\mathbf{e}_1 = (1, 0, 0)^T$ ,  $\mathbf{e}_2 = (0, 1, 0)^T$  and  $\mathbf{e}_3 = (0, 0, 1)^T$ .

- a). Let  $T$  be this transformation. This means  $T(x, y, z)^T = (x, y, 0)^T$ . We get

$$T\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad T\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad T\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

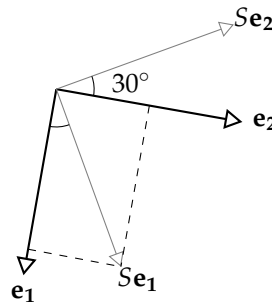
Therefore is matrix is  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ .

- b). Let  $R$  be this transformation. Since  $R$  project every vector through  $x$ - $y$  plane, hence  $R(x, y, z)^T = (x, y, -z)^T$ . We get

$$R\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad R\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad R\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Thus the matrix of  $R$  is  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

- c). Let  $S$  be this transformation.  $S$  moves the vectors  $\mathbf{e}_1, \mathbf{e}_2$  to the point  $x', y'$  respectively.



Since  $\cos 30^\circ = \frac{\sqrt{3}}{2}$  and  $\sin 30^\circ = \frac{1}{2}$ , we conclude that

$$S\mathbf{e}_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \quad S\mathbf{e}_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}, \quad S\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore the matrix is  $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ .

□

**Exercise 3.5.** Let  $A$  be a linear transformation. If  $\mathbf{z}$  is the center of the straight interval  $[\mathbf{x}, \mathbf{y}]$ , show that  $A\mathbf{z}$  is the center of the interval  $[A\mathbf{x}, A\mathbf{y}]$ .

*Proof.*  $\mathbf{z}$  is the center of  $[\mathbf{x}, \mathbf{y}]$  iff  $\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$ . Therefore,

$$A\mathbf{z} = A\left(\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}\right) = \frac{1}{2}A\mathbf{x} + \frac{1}{2}A\mathbf{y}$$

Thus,  $A\mathbf{z}$  is the center of the interval  $[A\mathbf{x}, A\mathbf{y}]$ .

□

**Exercise 3.6.** The set  $\mathbb{C}$  of complex numbers can be canonically identified with the space  $\mathbb{R}^2$  by treating each  $z = x + iy \in \mathbb{C}$  as a column  $(x, y)^T \in \mathbb{R}^2$ .

- Treating  $\mathbb{C}$  as a complex vector space, show that the multiplication by  $\alpha = a + ib \in \mathbb{C}$  is a linear transformation in  $\mathbb{C}$ . What is its matrix.
- Treating  $\mathbb{C}$  as the real vector space  $\mathbb{R}^2$  show that the multiplication by  $\alpha = a + ib \in \mathbb{C}$  is a linear transformation there.
- Define  $T(x + iy) = 2x - y + i(x - 3y)$ . Show that this tran is not a linear transformation in the complex vector space  $\mathbb{C}$ , but if we treat  $\mathbb{C}$  as the real vector space  $\mathbb{R}^2$  then it is a linear transformation there, then find its matrix.

*Proof.*

- Let  $T$  be this transformation. For any  $\mathbf{x} \in \mathbb{C}$ , we have  $T\mathbf{x} = \alpha\mathbf{x} \in \mathbb{C}$ . Thus  $T : \mathbb{C} \rightarrow \mathbb{C}$ , and we'll prove that  $T$  is a linear transformation. Let  $\mathbf{x}, \mathbf{y} \in \mathbb{C}$  be two vectors, and  $z \in \mathbb{C}$  be a scalar (complex). Observe that
  - $T(\mathbf{x} + \mathbf{y}) = \alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y} = T\mathbf{x} + T\mathbf{y}$  (distributivity of complex numbers)
  - $T(z\mathbf{x}) = \alpha(z\mathbf{x}) = z(\alpha\mathbf{x}) = zT\mathbf{x}$

This shows that this transformation  $T$  is a linear one. To find its matrix, we only need to know the basis of  $\mathbb{C}$ . Since any vector  $\mathbf{x} \in \mathbb{C}$  we be written as

$$\mathbf{x} = 1 \cdot \underbrace{\mathbf{x}}_{\text{scalar}}$$

and because this representation is unique, we obtain that  $\{1\} \subset \mathbb{C}$  is a basis of  $\mathbb{C}$ . Thus the matrix is

$$[T] = [T(1)] = [\alpha \cdot 1] = [\alpha].$$

- b). Because we treat  $\mathbb{C}$  as  $\mathbb{R}^2$ , then any complex number  $\mathbf{x} = x + iy$  can be represented as  $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ .  
Let  $T$  be this transformation. Thus  $T$  would look like

$$\begin{aligned} T \begin{pmatrix} x \\ y \end{pmatrix} &= T(\mathbf{x}) = a\mathbf{x} \\ &= (a + ib)(x + iy) \\ &= (ax - by) + i(ay + bx) \\ &= \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} \in \mathbb{R}^2 \end{aligned}$$

Thus  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ . We need to show that  $T$  is in fact linear. Let  $\mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$  and  $\mathbf{x}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$  and be two arbitrary vectors. We have

$$T\mathbf{x}_1 + T\mathbf{x}_2 = \begin{pmatrix} ax_1 - by_1 \\ ay_1 + bx_1 \end{pmatrix} + \begin{pmatrix} ax_2 - by_2 \\ ay_2 + bx_2 \end{pmatrix} = \begin{pmatrix} a(x_1 + x_2) - b(y_1 + y_2) \\ a(y_1 + y_2) + b(x_1 + x_2) \end{pmatrix} = T(\mathbf{x}_1 + \mathbf{x}_2),$$

and for any scalar  $r \in \mathbb{R}$ ,

$$rT\mathbf{x} = r \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} rax - rby \\ ray + rbx \end{pmatrix} = T(r\mathbf{x}).$$

This shows that  $T$  is a linear transformation. To find the matrix, we first need to find a basis in  $\mathbb{R}^2$ . Luckily, as we've proved earlier we could choose  $\{\mathbf{e}_1, \mathbf{e}_2\}$  to be a basis where

$$\mathbf{e}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{e}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Therefore

$$T\mathbf{e}_1 = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad T\mathbf{e}_2 = \begin{pmatrix} -b \\ a \end{pmatrix}$$

Thus the matrix of this transformation is  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ .

- c). Define  $T(x + iy) = 2x - y + i(x - 3y)$

- We'll prove that  $T$  is not linear in complex vector space. Observe that

$$T(i) = T(0 + i) = -1 - 3i \quad \text{and} \quad T(1) = T(1 + 0i) = 2 + i$$

clearly  $T(i) \neq iT(1)$ , this implies that  $T$  is not a linear transformation in  $\mathbb{C}$ .

- In  $\mathbb{R}^2$  the transformation would look like

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ x - 3y \end{pmatrix}.$$

For any vectors  $\mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$  and  $\mathbf{x}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ , we have

$$T\mathbf{x}_1 + T\mathbf{x}_2 = \begin{pmatrix} 2x_1 - y_1 \\ x_1 - 3y_1 \end{pmatrix} + \begin{pmatrix} 2x_2 - y_2 \\ x_2 - 3y_2 \end{pmatrix} = \begin{pmatrix} 2(x_1 + x_2) - (y_1 + y_2) \\ (x_1 + x_2) - 3(y_1 + y_2) \end{pmatrix} = T(\mathbf{x}_1 + \mathbf{x}_2)$$

and for any scalar  $r \in \mathbb{R}$ ,

$$rT\mathbf{x} = r \begin{pmatrix} 2x - y \\ x - 3y \end{pmatrix} = \begin{pmatrix} 2rx - ry \\ rx - 3ry \end{pmatrix} = T(r\mathbf{x})$$

this shows that  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is linear. Because  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  is a basis in  $\mathbb{R}^2$  and

$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix},$$

thus the matrix of this transformation is  $\begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix}$ .

□

**Exercise 3.7.** Show that any linear transformation in  $\mathbb{C}$  (treated as a complex vector space) is a multiplication by  $\alpha \in \mathbb{C}$ .

*Proof.* Let  $T : \mathbb{C} \rightarrow \mathbb{C}$  be this transformation. For any  $\mathbf{x} \in \mathbb{C}$

$$T\mathbf{x} = T(\mathbf{x} \cdot \mathbf{1}) = \mathbf{x} \cdot \underbrace{T(\mathbf{1})}_{\text{scalar}}$$

and the proof is completed. □

#### 4. Linear transformation as a Vector

Let set  $\mathcal{L}(V, W)$  is a vector space with addition and scalar multiplication (as proved above).

#### 5. Composition

**Homework 4.** Let  $A$  and  $B$  be matrices of size  $m \times n$  and  $n \times m$  respectively. Then

$$\text{trace}(AB) = \text{trace}(BA).$$

*Proof.* We'd like to prove this theorem *less* computationally. Let  $X \in M_{n \times m}$ . Consider the mapping  $T, T_1 : M_{n \times m} \rightarrow \mathbb{F}$  defined by

$$T(X) = \text{trace}(AX) \quad \text{and} \quad T_1(X) = \text{trace}(XA).$$

To prove the theorem it is sufficient to show that  $T, T_1$  are linear and they are the same. so by substituting  $X = B$  gives the theorem.

*Claim 1.* The transformations  $T, T_1$  defined above are linear.

*Proof.* For  $X, Y \in M_{n \times m}$ ,

- From the properties of matrix,  $A(X + Y) = AX + AY$ . Because  $AX$  and  $BX$  are both square matrices with size  $m \times m$ , and since we add the matrices  $AX + AY$  entrywise, it follows that

$$\begin{aligned} T(X + Y) &= \text{trace}(A(X + Y)) = \text{trace}(AX + AY) \\ &= \text{trace}(AX) + \text{trace}(AY) \\ &= T(X) + T(Y) \end{aligned}$$

- Similarly for any scalar  $\alpha \in \mathbb{F}$ ,

$$T(\alpha X) = \text{trace}(A \cdot \alpha X) = \text{trace}(\alpha AX) = \alpha \text{trace}(AX) = \alpha T(X)$$

This implies that  $T$  is a linear transformation. With simply proof, we conclude that  $T_1$  is also a linear transformation. □

We choose  $\mathbf{e}_{11}, \mathbf{e}_{21}, \dots, \mathbf{e}_{nm}$  to be the standard basis of  $M_{n \times m}$ , meaning the vector

$$\mathbf{e}_{ij} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

is a matrix whose entries are zero, except at the entry at row  $i$  and column  $j$ , which is 1. Then we only need to show that  $T\mathbf{e}_{ij} = T_1\mathbf{e}_{ij}$  for all  $i, j$ . Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1m} \\ \vdots & & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{im} \\ a_{nn} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nm} \end{pmatrix}$$

Hence

$$A\mathbf{e}_{ij} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a_{ij} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdot & \cdot \\ \cdot & a_{ij} & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}$$

and

$$\mathbf{e}_{ij}A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a_{ij} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & a_{ij} & \cdot \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

This implies that  $T\mathbf{e}_{ij} = T_1\mathbf{e}_{ij}$  for all  $i, j$ , and hence  $T = T_1$ . □