Chapter 3

Determinant

3. Constructing the determinant

Exercise 3.4 A square $(n \times n)$ matrix is called skew-symmetric (or anti-symmetric) if $A^{\mathrm{T}} = -A$. Prove that if A is skew-symmetric and n is odd, then $\det A = 0$. Is it true for even n?

Proof. When n is odd, we have

$$\det A = \det A^{\mathrm{T}} = \det(-A) = (-1)^n \det A = -\det A$$

This implies that $\det A = 0$ whenever n is odd. For even n, the determinant $\det A$ doesn't neccessarily zero, for instance

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is a skew-symmetric matrix, yet $\det B = 1$. $\square$

Exercise 3.5 A square matrix is called *nilpoten* if $A^k = 0$ for some $k \in \mathbb{N}$. Show that if A is nilpoten, then $\det A = 0$.

Proof. We use the property of determinant,

$$\det 0 = \det A^k = (\det A)^k$$

This shows that $\det A = 0$. $\square$

Exercise 3.6 Prove that if the matrices A and B are similar, then $\det A = \det B$.

Proof. Since $A \sim B$, hence there's invertable Q such that $A = QBQ^{-1}$. Therefore

$$\begin{aligned} \det A = \det QBQ^{-1} &= (\det Q)(\det B)(\det Q^{-1}) \\ &= \det(Q)(\det Q^{-1})(\det B) = \det(QQ^{-1})(\det B) \\ &= \det B. \end{aligned}$$

□

Exercise 3.7 A real square matrix Q is called *othogonal* if $Q^{\mathrm{T}}Q = I$. Prove that if Q is a n orthogonal matrix then $\det Q = \pm 1$.

Proof. We have

$$1 = \det Q^{\mathrm{T}}Q = (\det Q^{\mathrm{T}})(\det Q) = (\det Q)^2$$

this implies that $\det Q = \pm 1$. □

4. Formal definition

Exercise 4.2 Let P be a permutation matrix.

- Can you describe the correspoding linear transformation?
- Show that P is invertable. Can you describe P^{-1}.
- Show that for some $N > 0$, $P^N = I$.

Proof. Suppose that P is an $n \times n$ matrix.

- The linear transformation looks like it's swapping the axis in $\mathbb{R}^n$.
- If we interchange columns of P, we'll get the indentity matrix hence $\det P = \pm 1$. The direct computation shows that $P^{-1} = P^{\mathrm{T}}$.
- Because there are finitely many permutations, the sequence $\{P^n\}$ will eventually have repetitions. Hence we're sure there is i, j with $i > j$ such that $P^i = P^j$. Since P is invertable, we can multiply both sides by P^{-j}. Therefore
$$P^{i-j} = I,$$
now choose $N := i - j$ and this completes the proof.

□

Exercise 4.3 Why is there an even number of permutations of $(1, 2, \ldots, 9)$ and why are exactly half of them odd permutations?

Proof. There are 9! permutations of $(1, 2, \ldots, 9)$, which is an even number. To prove that there are exactly half of them odd (and other half even) we consider a 9×9 matrix A with all the entries are 1. The determinant is $\det A = 0$. However,

$$\det A = \sum_{\sigma \in \mathrm{perm}(9)} a_{\sigma(1),1} \cdots a_{\sigma(n),n} \operatorname{sign}(\sigma)$$

Since all the $a_{jk} = 1$, we get

$$\sum_{\sigma} \operatorname{sign} \sigma = \det A = 0$$

Because there are even number of permutations, and $\operatorname{sign} \sigma = \pm 1$ we must have half of them has sign $= 1$ and the other half has sign $= -1$. □

5. Cofactor

6. Minor and rank

Exercise 6.2 Let A be an $n \times n$ matrix. How are $\det(3A)$, $\det(-A)$ and $\det(A^2)$ related to $\det A$?

Proof. It follows immediately that

$$\begin{aligned}\det(-A) &= (-1)^n \det A,\\ \det(3A) &= 3^n \det A,\\ \det(A^2) &= (\det A)^2.\end{aligned}$$

□

Exercise 6.3 If the entries of both A and A^{-1} are integers, is it possible that $\det A = 3$?

Proof. It's impossible. If it was such a case, we'll get $\det A^{-1} = 1/\det A = 1/3$. Since all the entries of A^{-1} are integers, so

$$\det A^{-1} = \sum_{\sigma} a_{\sigma(1),1} \cdots a_{\sigma(n),n} \operatorname{sign} \sigma \in \mathbb{Z}$$

which is a contradiction that $\det A^{-1} = 1/3$. □

Exercise 6.4 Let $\mathbf{v_1}, \mathbf{v_2}$ be vectors in $\mathbb{R}^2$ and let A be the 2×2 matrix with columns $\mathbf{v_1}, \mathbf{v_2}$. Prove that $|\det A|$ is the area of parallelogram with two sides given by vectors $\mathbf{v_1}$ and $\mathbf{v_2}$.

Proof. Let α be angle between $\mathbf{v_1}$ and the x-axis, and let R be the matrix of rotation by $-\alpha$ angle. Denote

$$\tilde{\mathbf{v}}_\mathbf{1} := R\mathbf{v_1} = (a, 0) \quad \text{and} \quad \tilde{\mathbf{v}}_\mathbf{2} := R\mathbf{v_2} = (b, c).$$

for some reals a, b, c. Note that after the transformation, the area stays the same, i.e. $\operatorname{area}(\mathbf{v_1}, \mathbf{v_2}) = \operatorname{area}(\tilde{\mathbf{v}}_\mathbf{1}, \tilde{\mathbf{v}}_\mathbf{2})$. It's easy to see that the area of the new parallelogram is

$$|ac| = |\det(\tilde{\mathbf{v}}_\mathbf{1}, \tilde{\mathbf{v}}_\mathbf{2})|$$

(it has base $|a|$, and height $|c|$). But

$$\det(\tilde{\mathbf{v}}_\mathbf{1}, \tilde{\mathbf{v}}_\mathbf{2}) = \det(R\mathbf{v_1}, R\mathbf{v_2}) = \det(RA) = \det A$$

because $\det R = 1$ (rotation matrix). This implies that the area of the parallelogram is $|\det A|$. □

Exercise 6.5 Let $\mathbf{v_1}, \mathbf{v_2}$ be vectors in $\mathbb{R}^2$. Show that $\det(\mathbf{v_1}, \mathbf{v_2}) > 0$ if and only if there's a rotation T_α such that $T_\alpha \mathbf{v_1}$ parallel to $\mathbf{e_1}$ and $T_\alpha \mathbf{v_2}$ is in the upper half-plane.

Proof.

($\Rightarrow$) For this direction, the proof is almost identical to the previous exercise. We claim that $T_\alpha = R$ (R defined in the previous exercise). We now wanna show that $\tilde{\mathbf{v}}_2 = (b, c)$ is in the upper-half plane, meaning $c > 0$. But this immediately true from the fact that $a > 0$, $\det(\mathbf{v_1}, \mathbf{v_2}) > 0$ and

$$\det(\mathbf{v_1}, \mathbf{v_2}) = \det(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2) = ac > 0$$

($\Leftarrow$)

$\square$