A Big (but now is small) List of Problems/Solutions

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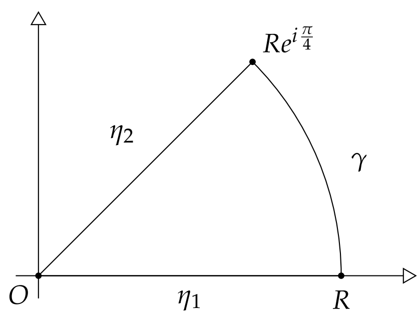
Chapter 1

Complex Analysis

Exercise 1.1 *(Stien, Ex 1, p. 64)* Prove that

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$

Solution Let $f(z) = e^{-z^2}$, and define C to be the positively oriented closed contour as below:



Observe the following:

- We parametrize γ by $\gamma(\theta) = Re^{i\theta}$ for $\theta \in [0, \frac{\pi}{4}]$. Estimate the integral on γ we get

$$\begin{aligned} \left| \int_\gamma f(z)dz \right| &\le \int_0^{\frac{\pi}{4}} \left| f(Re^{i\theta}) \right| \cdot \left| Rie^{i\theta} \right| d\theta \\ &= \int_0^{\frac{\pi}{4}} e^{-R^2 \cos 2\theta} \cdot R d\theta \end{aligned}$$

Next we want to bound $\cos 2\theta$ to some linear function, using Graphing software we can conclude that $\cos 2\theta \geq \frac{\pi}{4} - \theta$ for $\theta \in [0, \frac{\pi}{4}]$. Hence,

$$\begin{aligned}
\left| \int_{\gamma} f(z)dz \right| &\leq R \int_0^{\frac{\pi}{4}} e^{R^2(\theta - \frac{\pi}{4})} d\theta \\
&= R \cdot \frac{1}{R^2} \left[e^{R^2(\theta - \frac{\pi}{4})} \right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{R}(1 - e^{-R^2 \frac{\pi}{4}}) \to 0 \quad \text{as } R \to \infty
\end{aligned}$$

- Let η_3 be the reverse of η_2, i.e. we can parametrize η_3 as $\eta_3(t) = te^{i\frac{\pi}{4}}$ for $t \in [0, R]$, and thus

$$\begin{aligned}
\int_{\eta_2} f(z)dz &= -\int_{\eta_3} f(z)dz \\
&= -\int_0^R e^{-t^2 e^{i\frac{\pi}{2}}} \cdot e^{i\frac{\pi}{4}} dt \\
&= -e^{i\frac{\pi}{4}} \int_0^R e^{-ix^2} dx
\end{aligned}$$

Since f is holomophic there, by Cauchy's theorem we obtain

$$\oint_C f(z)dz = \int_{\eta_1} f(x)dx + \int_{\eta_2} f(z)dz + \int_{\gamma} f(z)dz = 0$$

Now, taking the limit as $R \to \infty$, we obtain

$$\begin{aligned}
& \int_0^\infty e^{-x^2} dx + 0 - e^{i\frac{\pi}{4}} \int_0^\infty e^{-ix^2} dx = 0 \\
\implies \quad & \int_0^\infty [\cos(x^2) - i\sin(x^2)] dx = e^{-i\frac{\pi}{4}} \cdot \frac{\sqrt{\pi}}{2}
\end{aligned}$$

Therefore,

$$\boxed{\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{\sqrt{2\pi}}{4}}$$

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Exercise 1.2 *(Stien, Ex 2, p. 64)* Show that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

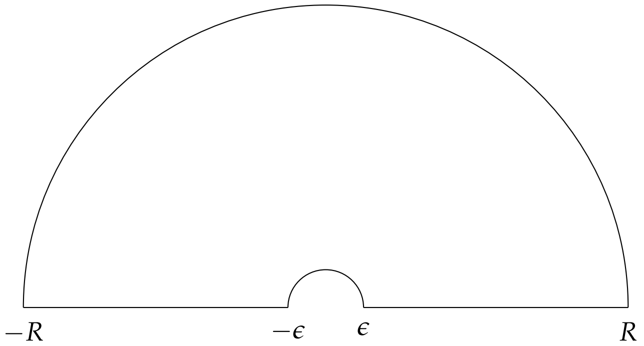
Solution From the hint in the book, observe that

$$\begin{aligned}
\frac{1}{2i} \int_{-\infty}^\infty \frac{e^{ix} - 1}{x} dx &= \frac{1}{2i} \int_{-\infty}^\infty \frac{\cos x - 1 + i\sin x}{x} dx \\
&= \frac{1}{2i} \int_{-\infty}^\infty \frac{\cos x - 1}{x} dx + \frac{1}{2} \int_{-\infty}^\infty \frac{\sin x}{x} dx
\end{aligned}$$

Since $\frac{\cos x - 1}{x}$ is an odd function, and $\frac{\sin x}{x}$ is an even function, we conclude that

$$\boxed{\frac{1}{2i} \int_{-\infty}^\infty \frac{e^{ix} - 1}{x} dx = \int_0^\infty \frac{\sin x}{x} dx}$$

Now let $f(z) = \frac{e^{iz} - 1}{z}$, and the indented circle C as below



Let γ_ϵ and γ_R be the positively oriented semi circle centered at O with radii ϵ and R respectively.

- Using series expansion of e^z reveals that $\lim_{z\to 0} \dfrac{e^{iz}-1}{z} = i$, so if we set $f(0) = i$ we obtain that f is continuous around $z = 0$. Hence f is bounded there. Estimate the integral over γ_ϵ

$$\begin{aligned}\left|\int_{\gamma_\epsilon} f(z)dz\right| &\leq \sup_{z\in\gamma_\epsilon}|f(z)| \cdot \text{length}(\gamma_\epsilon)\\ &\leq M\cdot \pi\epsilon\end{aligned}$$

  hence this integral approaches to 0 as $\epsilon \to 0$.

- For γ_R,

$$\int_{\gamma_R} f(z)dz = \int_{\gamma_R} \frac{e^{iz}}{z}dz - \int_{\gamma_R} \frac{1}{z}dz$$

  but

$$\begin{aligned}\left|\int_{\gamma_R} \frac{e^{iz}}{z}dz\right| &\leq \int_0^\pi \left|\frac{e^{iRe^{i\theta}}}{Re^{i\theta}}\right| \cdot \left|Rie^{i\theta}\right|d\theta\\ &= \int_0^\pi e^{-R\sin\theta}d\theta = 2\int_0^{\frac{\pi}{2}} e^{-R\sin\theta}d\theta\end{aligned}$$

  From Jordan's inequality, $\sin\theta \geq \frac{2\theta}{\pi}$ for $\theta \in [0, \frac{\pi}{2}]$, then

$$\begin{aligned}\left|\int_{\gamma_R} \frac{e^{iz}}{z}dz\right| &\leq \int_0^{\frac{\pi}{2}} e^{-R\frac{2\theta}{\pi}}d\theta\\ &= -\frac{\pi}{2R}\left[e^{-R\frac{2\theta}{\pi}}\right]_0^{\frac{\pi}{2}}\\ &= -\frac{\pi}{2R}\cdot(e^{-R}-1) \to 0 \quad \text{as } R\to\infty\end{aligned}$$

Because f is holomorphic on and inside C, apply Cauchy's theorem and letting $R \to \infty$ and $\epsilon \to 0$, we obtain

$$\begin{aligned}
& \int_{-R}^{-\epsilon} f(x)dx + \int_{\epsilon}^{R} f(x)dx - \int_{\gamma_\epsilon} f(z)dz + \int_{\gamma_R} f(z)dz = 0 \\
\implies & \int_{e \le |x| \le R} f(x)dx = \int_{\gamma_\epsilon} f(z)dz - \int_{\gamma_R} \frac{e^{iz}}{z} + \int_{\gamma_R} \frac{1}{z}dz \\
\implies & \int_{-\infty}^{\infty} \frac{e^{ix}-1}{x}dx = 0 - 0 + i\pi \\
\implies & \int_{0}^{\infty} \frac{\sin x}{x}dx = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{ix}-1}{x}dx = \boxed{\frac{\pi}{2}}.
\end{aligned}$$

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Exercise 1.3 *(Stein, Ex 7, p. 65)* Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w \in \mathbb{D}} |f(z) - f(w)|$ of the image f f satisfies

$$2|f'(0)| \le d.$$

Solution Let $0 < r < 1$. From Cauchy formula

$$f'(0) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z^2}dz.$$

Let $g(z) = -f(-z)$, we then have

$$\frac{1}{2\pi i}\int_{C_r} \frac{-f(-z)}{z^2}dz = \frac{1}{2\pi i}\int_{C_r} \frac{g(z)}{z^2}dz = g'(0) = f'(0)$$

Adding these two quantities,

$$\begin{aligned}
2f'(0) &= \frac{1}{2\pi i}\int_{C_r} \frac{f(z)-f(-z)}{z^2}dz \\
\implies \quad 2|f'(0)| &\le \frac{1}{2\pi}\sup_{z \in \mathbb{D}}\left|\frac{f(z)-f(-z)}{z^2}\right| \cdot 2\pi r \\
&\le \frac{1}{r}\sup_{z \in \mathbb{D}}|f(z)-f(-z)| \le \frac{d}{r}
\end{aligned}$$

Thus $2|f'(0)|$ is a lower bound of $\{\frac{d}{r} : 0 < r < 1\}$, therefore

$$2|f'(0)| \le \inf_{0<r<1} \frac{d}{r} = d$$

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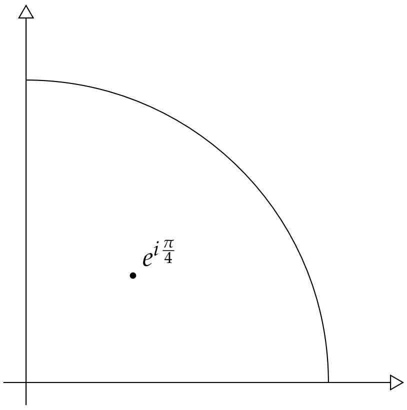
Exercise 1.4 *(Stien, Ex 2, p. 103)* Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

Solution Consider the integral of the function $f(z) = \frac{1}{1+z^4}$ over the contour C as below:



- The point $z_0 = e^{i\frac{\pi}{4}}$ is a simple pole of f and its residue is

$$\mathrm{res}_{z_0} = \lim_{z \to z_0} \frac{z - z_0}{1 + z^4} = \lim_{z \to z_0} \frac{1}{4z^3} = \frac{1}{4e^{i\frac{3\pi}{4}}}$$

- Let γ_1 be the line from O to R, and we let

$$I_R = \int_0^R \frac{1}{1+x^4} dx$$

- Let γ_2 be the above arc, observe that

$$\left| \int_{\gamma_2} f(z) dz \right| \leq \frac{\pi}{2} R \cdot \sup_{z \in \gamma_2} \left| \frac{1}{1+z^4} \right| \leq \frac{\pi}{2} R \cdot \frac{1}{R^4 - 1} \to 0$$

  as $R \to \infty$.

- Let γ_3 be the line from O to iR and we can parametrize it as $t \mapsto it$ when $t \in [0, R]$. Thus the integral

$$\int_{\gamma_3} f(z) dz = \int_0^R \frac{1}{1+(it)^4} \cdot i dt = i I_R$$

Applying Cauchy theorem

$$\oint_C f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_1} f(z) dz - \int_{\gamma_1} f(z) dz = 2\pi i \cdot \mathrm{res}_{z_0}$$

Letting $R \to \infty$, we obtain that

$$\begin{aligned} (1-i) I &= 2\pi i \cdot \frac{e^{-i\frac{3\pi}{4}}}{4} \\ \Longrightarrow \quad I &= \frac{\pi i}{2\sqrt{2}} e^{-i\frac{3\pi}{4}} \cdot e^{i\frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}}. \end{aligned}$$

Because $\frac{1}{1+x^4}$ is an even function we get

$$\int_{\{} -\infty\}^{\infty} f(x)dx = 2\int_0^\infty \frac{1}{1+x^4} = \frac{\pi}{2}.$$

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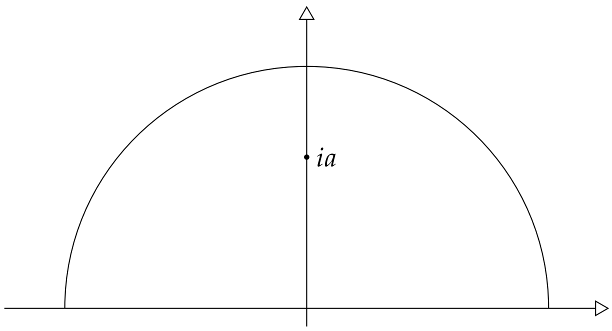
Exercise 1.5 *(Stien, Ex 3, p. 103)* Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx = \pi \frac{e^{-a}}{a}, \qquad \text{for } a > 0.$$

Solution We integrate the function $f(z) = \frac{e^{iz}}{z^2+a^2}$ along with the semi-circle shown below. In this contour, f has a simple pole at $z = ia$. The residue correspond to this pole is

$$\text{res}_{ia} = \lim_{z \to ia} \frac{e^{iz}}{z+ia} = \frac{e^{-a}}{2ia}$$



By Estimation theorem, the integral along the upper arc

$$\begin{aligned} \left| \int_{\text{arc}} f(z)dz \right| &\le \pi R \cdot \sup_{z \in \text{arc}} \left| \frac{e^{iz}}{z^2+a^2} \right| \\ &\le \pi R \cdot \sup_{\theta \in [0,\pi]} \frac{e^{-R\sin\theta}}{R^2-a^2} \end{aligned}$$

And as $R \to \infty$, the RHS approaches to zero. Thus the integral is also zero. Now applying Cauchy's residue theorem and let $R \to \infty$, we obtain that

$$\begin{aligned} & \int_{-R}^{R} f(x)dx + \int_{\text{arc}} f(z)dz = 2\pi i \cdot \text{res} \\ \implies & \int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x^2+a^2} dx = 2\pi i \cdot \frac{e^{-a}}{2ia} \\ \implies & \int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} dx = \pi \cdot \frac{e^{-a}}{a} \end{aligned}$$

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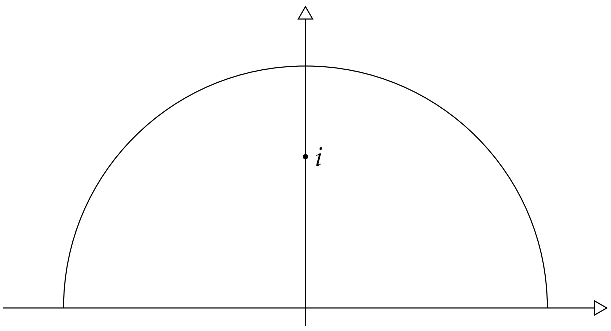
Exercise 1.6 *(Stien, Ex 6 p. 104)* Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}} dx = \frac{(2n-1)!!}{(2n)!!} \cdot \pi.$$

Solution We integrate the function $f(z) = \frac{1}{(1+z^2)^{n+1}}$ along with the semi-circle shown below. In this contour, f has a pole of oder $n+1$ at $z = i$. The residue correspond to this pole is

$$\begin{aligned}
\text{res}_i &= \lim_{z\to i} \frac{1}{n!} \frac{d^n}{dz^n} (z+i)^{-n-1} \\
&= \lim_{z\to i} \frac{1}{n!} (-n-1)(-n-2)\cdots(-n-n)(z+i)^{-2n-1} \\
&= \frac{1}{n!}(-1)^n (n+1)(n+2)\cdots(2n) \frac{1}{(2i)^{2n+1}} \\
&= \frac{1}{n!2^n} \cdot \frac{(n+1)(n+2)\cdots(2n)}{2^n} \cdot \frac{(-1)^n}{2i^{2n}i} \\
&= \frac{1}{(2n)!!} \cdot (2n-1)!! \cdot \frac{1}{2i}
\end{aligned}$$



Applying Cauchy Residue Theorem,

$$\int_{-R}^{R} f(x)dx + \int_{\text{arc}} f(z)dz = 2\pi i \cdot \text{res}$$

Letting $R \to \infty$ and arguing as above, we found that the integral along the arc is zero, hence

$$\begin{aligned}
&\implies \int_{-\infty}^{\infty} f(x)dx + 0 = 2\pi i \cdot \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2i} \\
&\implies \boxed{\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}} dx = \frac{(2n-1)!!}{(2n)!!} \cdot \pi}
\end{aligned}$$

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