

Chapter 1

Basic Notions

1. Vector Spaces

Exercise 1.1 Let $\mathbf{x} = (1, 2, 3)^T$, $\mathbf{y} = (y_1, y_2, y_3)^T$ and $\mathbf{z} = (4, 2, 1)^T$. Compute $2\mathbf{x}$, $3\mathbf{y}$, $\mathbf{x} + 2\mathbf{y} - 3\mathbf{z}$.

Proof. Little calculation reveals that

$$2\mathbf{x} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \quad 3\mathbf{y} = \begin{pmatrix} 3y_1 \\ 3y_2 \\ 3y_3 \end{pmatrix}, \quad \mathbf{x} + 2\mathbf{y} - 3\mathbf{z} = \begin{pmatrix} 2y_1 - 11 \\ 2y_2 - 4 \\ 2y_3 \end{pmatrix}$$

□

Exercise 1.2 Which of the following sets (with natural addition and multiplication by a scalar) are vector spaces? Justify your answers.

- a). The set of all continuous functions on the interval $[0, 1]$;
- b). The set of all non-negative functions on the interval $[0, 1]$;
- c). The set of all polynomials of degree *exactly* n ;
- d). The set of all symmetric $n \times n$ matrices, i.e. the set of matrices $A = \{a_{j,k}\}_{j,k=1}^n$ such that $A^T = A$.

Proof.

- a). Let $\mathcal{C}[0, 1]$ be the set of all continuous functions on $[0, 1]$. For any $f, g \in \mathcal{C}[0, 1]$ and $\alpha \in \mathbb{R}$, we define

$$(f + g)(x) := f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) := \alpha \cdot f(x)$$

for each $x \in [0, 1]$. Therefore, $(\mathcal{C}[0, 1], +, \cdot)$ is closed under addition and scalar multiplication. It's immediate to see that all the eight properties of vector space are all satisfied, that is

- | | |
|-------------------------------|--|
| ◦ $f + g = g + f$ | ◦ $1f = f$ |
| ◦ $f + (g + h) = (f + g) + h$ | ◦ $\alpha(\beta f) = (\alpha\beta)f$ |
| ◦ $f + 0 = f$ | ◦ $(\alpha + \beta)f = \alpha f + \beta f$ |
| ◦ $f + (-f) = 0$ | ◦ $\alpha(f + g) = \alpha f + \alpha g$ |

Note that the function $0 \in \mathcal{C}[0, 1]$ such that $0(x) = 0$ for each $x \in [0, 1]$.

- b). Let \mathcal{B} be the set of all non-negative functions on $[0, 1]$. Then $(\mathcal{B}, +, \cdot)$ is not a vector space because it's not closed under scalar multiplication, i.e. if $f \in \mathcal{B}$, hence $f > 0$ yet

$$-f = (-1) \cdot f < 0.$$

(Even if we restrict the scalar be to positive real numbers, this set still won't be a vector space, because it fails to have an inverse.)

- c). Let \mathcal{P} be the set of all polynomials of degree exactly n , then $(\mathcal{P}, +, \cdot)$ is *not* a vector space, because the additive identity is the polynomial 0. However, $0 \notin \mathcal{P}$.
- d). Let $\text{sym}(n)$ be the set of all symmetric $n \times n$ matrices. The addition and scalar multiplication are defined as an *entrywise* operations. Hence, $\text{sym}(n)$ is closed under $(+)$ and (\cdot) . The additive identity is the matrix

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \text{sym}(n).$$

We could easily see that all the eight properties of vector spaces are all satisfied.

□

Exercise 1.3 True or false:

- a). Every vector space contains a zero vector; (**True.**)
- b). A vector space can have more than one zero vector; (**False.** The zero vector is unique.)
- c). An $m \times n$ matrix has m rows and n columns; (**True.**)
- d). If f and g are polynomials of degree n , then $f + g$ is also a polynomial of degree n . (**False.** consider t^n and $t - t^n$.)
- e). If f and g are polynomials of degree atmost n , the $f + g$ is also a polynomial of degree atmost n . (**True.**)

Exercise 1.4 Prove that a zero vector $\mathbf{0}$ of a vector space V is unique.

Proof. Suppose that \mathbf{a} and \mathbf{b} are the zero vectors of V . From the *Axioms of Vector Space*, we obtain that

$$\begin{aligned} \mathbf{a} &= \mathbf{a} + \mathbf{b} && (\mathbf{b} \text{ is the zero vector}) \\ &= \mathbf{b} + \mathbf{a} && (\text{commutativity}) \\ &= \mathbf{b} && (\mathbf{a} \text{ is the zero vector}) \end{aligned}$$

Hence, a zero vector of a vector space is unique, and we usually denote it by $\mathbf{0}$. □

Exercise 1.5 What is the zero matrix of the space $M_{2 \times 3}$?

Answer. In the space $M_{2 \times 3}$, the zero matrix is

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

Exercise 1.6 Prove that the additive inverse of a vector space is unique.

Proof. Let \mathbf{a} be an arbitrary vector. Assume the \mathbf{a} has two inverses, namely \mathbf{x} and \mathbf{y} . Hence

$$\begin{aligned} \mathbf{x} &= \mathbf{x} + \mathbf{0} \\ &= \mathbf{x} + (\mathbf{a} + \mathbf{y}) && (\mathbf{y} \text{ is an inverse}) \\ &= (\mathbf{x} + \mathbf{a}) + \mathbf{y} && (\text{associativity}) \\ &= \mathbf{0} + \mathbf{y} && (\mathbf{x} \text{ is an inverse}) \\ &= \mathbf{y}. \end{aligned}$$

Therefore, the inverse of any vector $\mathbf{a} \in V$ is unique, and is usually denoted by $-\mathbf{a}$. □

Exercise 1.7 Prove that $0\mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v} \in V$.

Proof. Let $\mathbf{v} \in V$ and \mathbf{b} is an inverse of $0\mathbf{v}$. Therefore,

$$\begin{aligned} \mathbf{0} &= 0\mathbf{v} + \mathbf{b} \\ &= (0 + 0)\mathbf{v} + \mathbf{b} \\ &= (0\mathbf{v} + 0\mathbf{v}) + \mathbf{b} && (\text{distributivity}) \\ &= 0\mathbf{v} + (0\mathbf{v} + \mathbf{b}) && (\text{associativity}) \\ &= 0\mathbf{v} + \mathbf{0} && (\mathbf{b} \text{ is an inverse of } 0\mathbf{v}) \\ &= 0\mathbf{v} \end{aligned}$$

for any $\mathbf{v} \in V$. □

Exercise 1.8 Prove that for any vector \mathbf{v} its additive inverse $-\mathbf{v}$ is given by $(-1)\mathbf{v}$.

Proof. As proved in the above exercise for any $\mathbf{v} \in V$,

$$\mathbf{0} = 0\mathbf{v} = (1 - 1)\mathbf{v} = \mathbf{v} + (-1)\mathbf{v}$$

where the last equality derives from the distributive property. Because $-\mathbf{v}$ is the inverse of \mathbf{v} , then

$$\begin{aligned} -\mathbf{v} &= -\mathbf{v} + \mathbf{0} \\ &= -\mathbf{v} + [\mathbf{v} + (-1)\mathbf{v}] \\ &= (\underbrace{-\mathbf{v} + \mathbf{v}}_{\mathbf{0}}) + (-1)\mathbf{v} \\ &= (-1)\mathbf{v} \end{aligned}$$

as desired. □

2. Linear Combination, bases

Exercise 2.1 Find the basis in the space of 3×2 matrices $M_{3 \times 2}$.

Answer. Consider the vectors:

$$\begin{aligned} \mathbf{e}_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, & \mathbf{e}_2 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, & \mathbf{e}_3 &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ \mathbf{e}_4 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, & \mathbf{e}_5 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, & \mathbf{e}_6 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

and we're going to prove that the system of these vectors are a basis. Any matrix

$$\mathbf{v} = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \in M_{3 \times 2}$$

can be represented as the combination $\mathbf{v} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3 + d\mathbf{e}_4 + e\mathbf{e}_5 + f\mathbf{e}_6$ thus this system is generating. Next we're going to prove the uniqueness.

Suppose that there are $\hat{a}, \hat{b}, \dots, \hat{f}$ with

$$\begin{aligned} \mathbf{v} &= \hat{a}\mathbf{e}_1 + \hat{b}\mathbf{e}_2 + \hat{c}\mathbf{e}_3 + \hat{d}\mathbf{e}_4 + \hat{e}\mathbf{e}_5 + \hat{f}\mathbf{e}_6 \\ \Rightarrow \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} &= \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \\ \hat{e} & \hat{f} \end{bmatrix} \end{aligned}$$

This implies that each corresponding entry is equal. Hence the representation is unique. Therefore this system is a basis. □

Exercise 2.2 True or false:

- a). Any set containing a zero vector is linearly dependent;
- b). A basis must contain $\mathbf{0}$;
- c). subsets of linearly dependent sets are linearly dependent;
- d). subsets of linearly independent sets are linearly independent;
- e). if $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0}$ then all scalars α_k are zero.

Answer.

- a). **True.** because $\mathbf{0}$ can be represented as a linear combination of the other vectors (simply put all the scalars to 0).
- b). **No.** if so, they must be linearly dependent, which is not a base.
- c). **No.** Take for example the system of linearly dependent $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$ and $\mathbf{e}_3 = (1, 1)$. The subset $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis, which is clearly not linearly dependent.
- d). **True.** Suppose that the system $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a subset of the linearly independent system $\{\mathbf{v}_1, \dots, \mathbf{v}_p, \dots, \mathbf{v}_n\}$. Let α_k the real numbers such that $\alpha_1 \mathbf{v}_1 + \cdots + \alpha_p \mathbf{v}_p = \mathbf{0}$ hence

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_p \mathbf{v}_p + 0\mathbf{v}_{p+1} + \cdots + 0\mathbf{v}_n = \mathbf{0}.$$

Because the system $\{\mathbf{v}_1, \dots, \mathbf{v}_p, \dots, \mathbf{v}_n\}$ is linearly independent, therefore all the scalars $\alpha_k = 0$. Thus, the system $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is also linearly independent.

- e). **No.** Take, $\mathbf{e}_1 = (2, 2)$ and $\mathbf{e}_2 = (1, 1)$ for instance. We have $\mathbf{e}_1 - 2\mathbf{e}_2 = \mathbf{0}$ yet the scalars are non-zero.

□

Exercise 2.3 Recall, that a matrix is called *symmetric* if $A^T = A$. Write down a basis in the space of *symmetric* 2×2 matrices (there are many possible answers). How many elements are there in the basis.

Answer. We are going to prove that the system $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{e}_1\}$ where

$$\mathbf{d}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{d}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

is a basis. Observe that any symmetric matrix

$$\mathbf{v} = \begin{bmatrix} d_1 & e_1 \\ e_1 & d_2 \end{bmatrix}$$

can be represented as $\mathbf{v} = d_1\mathbf{d}_1 + d_2\mathbf{d}_2 + e_1\mathbf{e}_1$, hence it's generating. Note that the equation

$$d_1\mathbf{d}_1 + d_2\mathbf{d}_2 + e_1\mathbf{e}_1 = \mathbf{0}$$

$$\begin{bmatrix} d_1 & e_1 \\ e_1 & d_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

holds only when all the scalars are all zero. Hence the system is linearly independent. Thus, it's a basis. □

Exercise 2.4 Write down a basis for the space of

- a). 3×3 symmetric matrices;
- b). $n \times n$ symmetric matrices;
- c). $n \times n$ antisymmetric matrices.

Answer.

- a). we are going to prove that the system of vectors

$$\mathbf{d}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{d}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{d}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

is the basis. First of, any symmetric matrix

$$\mathbf{v} = \begin{bmatrix} d_1 & e_1 & e_2 \\ e_1 & d_2 & e_3 \\ e_2 & e_3 & d_3 \end{bmatrix}$$

can be represented as

$$\mathbf{v} = d_1\mathbf{d}_1 + d_2\mathbf{d}_2 + d_3\mathbf{d}_3 + e_1\mathbf{e}_1 + e_2\mathbf{e}_2 + e_3\mathbf{e}_3$$

yields that the system is generating. Similar to the previous problem, if the linear combination of these vectors equals $\mathbf{0}$, then all the scalars must equals zero. Thus it's linearly independent. Therefore it's a basis.

- b). Working on it.
- c). Working on it.

□

Exercise 2.5 Let a system of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ be linearly independent but not generating. Show that it is possible to find a vector \mathbf{v}_{r+1} such that the system $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r, \mathbf{v}_{r+1}$ is linearly independent.

Proof. Because the system $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ is not generating, therefore there exists a vector \mathbf{v}_{r+1} such that \mathbf{v}_{r+1} cannot be represented as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$. Let α_i be the scalars such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_r \mathbf{v}_r + \alpha_{r+1} \mathbf{v}_{r+1} = \mathbf{0} \quad (1.1)$$

Now we have to prove that all the scalars are all zero. If $\alpha_{r+1} \neq 0$ then

$$\mathbf{v}_{r+1} = - \sum_{i=1}^r \frac{\alpha_i}{\alpha_{r+1}} \cdot \mathbf{v}_i,$$

meaning \mathbf{v}_{r+1} is the linear combination of the other vectors, a contradiction. Hence α_{r+1} must equals to zero. So the $r+1$ term in the equation (??) vanishes. And because the system $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ is linearly independent, all the scalars $\alpha_i = 0$ for all $i = 0, 1, \dots, r$. Thus, the system

$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r, \mathbf{v}_{r+1}$$

is also *linearly independent*. □

Exercise 2.6 Is it possible that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, but the vectors $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$ and $\mathbf{w}_3 = \mathbf{v}_3 + \mathbf{v}_1$ are linearly independent.

Proof. It's not possible, and we're going to prove this assertion via contradiction. Assume that there are such vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ satisfying the above conditions. Then there are numbers $x, y, z \in \mathbb{R}$ such that

$$|x| + |y| + |z| > 0 \quad \text{and} \quad x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{0}.$$

By letting

$$a = x + y - z, \quad b = y + z - x, \quad c = z + x - y$$

we obtain that

$$\begin{aligned} a\mathbf{w}_1 + b\mathbf{w}_2 + c\mathbf{w}_3 &= (x\mathbf{w}_1 + y\mathbf{w}_1 - z\mathbf{w}_1) + (y\mathbf{w}_2 + z\mathbf{w}_2 - x\mathbf{w}_2) \\ &\quad + (x\mathbf{w}_3 + z\mathbf{w}_3 - y\mathbf{w}_3) \\ &= 2x\mathbf{v}_1 + 2y\mathbf{v}_2 + 2z\mathbf{v}_3 \\ &= \mathbf{0}. \end{aligned}$$

Since $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ are linearly independent, we must have $a = b = c = 0$. Hence

$$\begin{cases} x + y - z = 0 \\ y + z - x = 0 \\ z + x - y = 0 \end{cases}$$

adding all the 3 equations, $x + y + z = 0$. Substituting back to the system of equations above we get

$$x = y = z = 0$$

which contradicts to the fact that $|x| + |y| + |z| > 0$. \square

Exercise 2.7 Any finite independent system is a subset of some basis.

Proof. Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly independent. If this system is generating, then it's a base and we're done. If not, from exercise 2.5, there exists \mathbf{v}_{n+1} such that

$$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n, \mathbf{v}_{n+1}\}$$

is still linearly independent. Now if this new system is generating, then we're done. If not, we keep continue this process a finite steps, adding vectors $\mathbf{v}_{n+1}, \mathbf{v}_{n+2}, \dots, \mathbf{v}_{n+r}$, and eventually the new system

$$\{\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}_{n+1}, \dots, \mathbf{v}_{n+r}\}$$

is now a basis. \square

3. Linear Transformation

Homework 1. Prove that the transformation $T : \mathbb{F}^n \rightarrow \mathbb{F}^m$ if and only if $T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$ for any scalars α, β and vectors $\mathbf{x}, \mathbf{y} \in \mathbb{F}$.

Proof. We need to prove this in two directions.

(\Rightarrow) Suppose T is a linear transformation, then

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = T(\alpha\mathbf{x}) + T(\beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$$

as needed.

(\Leftarrow) For this direction, we first assume that T has the property that $T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y})$ for all $\alpha, \beta, \mathbf{x}, \mathbf{y}$. We need to show that T has the property listed in the definition of the linear transformation. Observe that

- take $\alpha = \beta = 1$ then, $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$
- take $\beta = 0$ then, $T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$

Hence T is a linear transformation, and the proof is completed. \square

Homework 2. Let $T : V \rightarrow W$ be a linear transformation. Prove that $T(\mathbf{0}) = \mathbf{0}$ and

$$TV = \{T\mathbf{v} : \mathbf{v} \in V\}$$

is a vector space.

Proof. Since T is linear, and as proved before $0 \cdot \mathbf{0} = \mathbf{0}$, it's easy to see that

$$T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0 \cdot T(\mathbf{0}) = \mathbf{0}.$$

To prove that TV is a vector space, we need to check that TV satisfies all the eight conditions listed in the definition of vector space.

We first need to prove that TV is closed. Because $TV \subset W$, hence TV is closed under scalar multiplication and vector addition. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$. Observe that

- $T\mathbf{x} + T\mathbf{y} = T\mathbf{y} + T\mathbf{x}$ (commutativity of W)
- $(T\mathbf{x} + T\mathbf{y}) + T\mathbf{z} = T\mathbf{x} + (T\mathbf{y} + T\mathbf{z})$ (associativity of W)
- The vector $\mathbf{0} \in W$ is the identity of TV because

$$T\mathbf{x} + \mathbf{0} = T\mathbf{x} + T\mathbf{0} = T(\mathbf{x} + \mathbf{0}) = T(\mathbf{x}), \quad \forall \mathbf{x} \in V$$

- The vector $T(-\mathbf{x})$ is the additive inverse of $T\mathbf{x}$ because

$$T\mathbf{x} + T(-\mathbf{x}) = T(\mathbf{x} - \mathbf{x}) = \mathbf{0}$$

- $1 \cdot T\mathbf{v} = T\mathbf{v}$ (multiplicative iden. in W)

Let α, β be scalars.

- multiplicative associativity

$$\begin{aligned} (\alpha\beta)T\mathbf{x} &= T((\alpha\beta)\mathbf{x}) && \text{(linearity of } T) \\ &= T(\alpha(\beta\mathbf{x})) && \text{(mult. asso. of } V) \\ &= \alpha T(\beta\mathbf{x}) && \text{(linearity of } T) \\ &= \alpha \cdot \beta T\mathbf{x} \end{aligned}$$

- scalar multiplication

$$\begin{aligned} \alpha(T\mathbf{x} + T\mathbf{y}) &= \alpha T(\mathbf{x} + \mathbf{y}) && \text{(linearity of } T) \\ &= T(\alpha(\mathbf{x} + \mathbf{y})) \\ &= T(\alpha\mathbf{x} + \alpha\mathbf{y}) && \text{(scalar mult. in } V) \\ &= T(\alpha\mathbf{x}) + T(\alpha\mathbf{y}) && \text{(linearity of } T) \\ &= \alpha T\mathbf{x} + \alpha T\mathbf{y} \end{aligned}$$

- scalar multiplication

$$\begin{aligned} (\alpha + \beta)T\mathbf{x} &= T((\alpha + \beta)\mathbf{x}) && \text{(linearity of } T) \\ &= T(\alpha\mathbf{x} + \beta\mathbf{x}) && \text{(scalar mult. of } V) \\ &= T(\alpha\mathbf{x}) + T(\beta\mathbf{x}) \\ &= \alpha T\mathbf{x} + \beta T\mathbf{x} \end{aligned}$$

We see that TV has all eight properties to be a vector space, and the proof is completed. □

Homework 3. Let V, W be vector spaces. Prove that $\mathcal{L}(V, W)$, the set of all linear transformations $T : V \rightarrow W$, is also a vector space.

Proof. We first need to show that $\mathcal{L}(V, W)$ is closed. Let $T_1, T_2 \in \mathcal{L}(V, W)$ and a be a scalar. So we need to show the transformation $T_1 + T_2$ and aT_1 are both linear.

- Let \mathbf{x}, \mathbf{y} be arbitrary vectors in V and α, β be scalar. Denote $T := T_1 + T_2$. Observe that

$$\begin{aligned}
 T(\alpha\mathbf{x} + \beta\mathbf{y}) &= (T_1 + T_2)(\alpha\mathbf{x} + \beta\mathbf{y}) \\
 &= T_1(\alpha\mathbf{x} + \beta\mathbf{y}) + T_2(\alpha\mathbf{x} + \beta\mathbf{y}) && \text{(by def. of } T_1 + T_2\text{)} \\
 &= \alpha T_1\mathbf{x} + \beta T_1\mathbf{y} + \alpha T_2\mathbf{x} + \beta T_2\mathbf{y} && \text{(by lin. of } T_1 \text{ and } T_2\text{)} \\
 &= (\alpha T_1\mathbf{x} + \alpha T_2\mathbf{x}) + (\beta T_1\mathbf{y} + \beta T_2\mathbf{y}) \\
 &= \alpha(T_1\mathbf{x} + T_2\mathbf{x}) + \beta(T_1\mathbf{y} + T_2\mathbf{y}) && \text{(by scalar mult. in } W\text{)} \\
 &= \alpha(T_1 + T_2)\mathbf{x} + \beta(T_1 + T_2)\mathbf{y} \\
 &= \alpha T\mathbf{x} + \beta T\mathbf{y}
 \end{aligned}$$

This shows that $T_1 + T_2$ is also a linear transformation, hence $\mathcal{L}(V, W)$ is closed under addition.

- Similarly, we let $\mathbf{x}, \mathbf{y} \in V$. For simplicity, we again denote $T := aT_1$. Hence for any scalars α, β

$$\begin{aligned}
 T(\alpha\mathbf{x} + \beta\mathbf{y}) &= (aT_1)(\alpha\mathbf{x} + \beta\mathbf{y}) \\
 &= a \cdot T_1(\alpha\mathbf{x} + \beta\mathbf{y}) && \text{(by def. of } aT_1\text{)} \\
 &= a \cdot (\alpha T_1\mathbf{x} + \beta T_1\mathbf{y}) && \text{(by lin. of } T_1\text{)} \\
 &= \alpha a T_1\mathbf{x} + \beta a T_1\mathbf{y} \\
 &= \alpha(aT_1)\mathbf{x} + \beta(aT_1)\mathbf{y} \\
 &= \alpha T\mathbf{x} + \beta T\mathbf{y}
 \end{aligned}$$

This suggests that aT_1 is also linear, hence $\mathcal{L}(V, W)$ is closed under scalar multiplication. Ultimately, we've proved that $\mathcal{L}(V, W)$ is closed as needed.

We are now ready to prove that $\mathcal{L}(V, W)$ is a vector space. Let $T_1, T_2, T_3 \in \mathcal{L}(V, W)$ we have

- $T_1 + T_2 = T_2 + T_1$, because for any $\mathbf{x} \in V$

$$(T_1 + T_2)\mathbf{x} = T_1\mathbf{x} + T_2\mathbf{x} = T_2\mathbf{x} + T_1\mathbf{x} = (T_2 + T_1)\mathbf{x}.$$

- $T_1 + (T_2 + T_3) = (T_1 + T_2) + T_3$, because for any $\mathbf{x} \in V$

$$\begin{aligned}
 (T_1 + (T_2 + T_3))\mathbf{x} &= T_1\mathbf{x} + (T_2 + T_3)\mathbf{x} \\
 &= T_1\mathbf{x} + (T_2\mathbf{x} + T_3\mathbf{x}) \\
 &= (T_1\mathbf{x} + T_2\mathbf{x}) + T_3\mathbf{x} && \text{(by asso. of } W\text{)} \\
 &= (T_1 + T_2)\mathbf{x} + T_3\mathbf{x} \\
 &= ((T_1 + T_2) + T_3)\mathbf{x}
 \end{aligned}$$

- Consider the transformation $0 : V \rightarrow W$ such that $0(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$. We're going to prove that this 0 is the identity of $\mathcal{L}(V, W)$. But first, we need to know if 0 is linear or not. For any $\mathbf{v}_1, \mathbf{v}_2 \in V$, we have

$$0(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \mathbf{0} \quad \text{and} \quad \alpha 0\mathbf{v}_1 + \beta 0\mathbf{v}_2 = \alpha\mathbf{0} + \beta\mathbf{0} = \mathbf{0}.$$

Hence $0(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \alpha 0\mathbf{v}_1 + \beta 0\mathbf{v}_2$, thus the transformation 0 is linear, i.e. $0 \in \mathcal{L}(V, W)$.

Observe that for any $\mathbf{x} \in V$

$$(T_1 + 0)\mathbf{x} = T_1\mathbf{x} + 0\mathbf{x} = T_1\mathbf{x}$$

This implies that $T_1 + 0 = T_1$ for any $T_1 \in \mathcal{L}(V, W)$. We conclude that 0 is the identity of $\mathcal{L}(V, W)$.

- The transformation $-T_1 := (-1)T_1$ is the additive inverse of T_1 because for any $\mathbf{x} \in V$

$$T_1\mathbf{x} + (-T_1)\mathbf{x} = T_1\mathbf{x} + T_1(-\mathbf{x}) = T_1(\mathbf{x} - \mathbf{x}) = \mathbf{0} = 0(\mathbf{x}).$$

- $1 \cdot T_1 = T_1$ because $(1 \cdot T_1)\mathbf{x} = 1 \cdot T_1\mathbf{x} = T_1\mathbf{x}$ for any $\mathbf{x} \in V$.

- $(\alpha\beta)T_1 = \alpha(\beta T_1)$, because

$$[(\alpha\beta)T_1]\mathbf{x} = (\alpha\beta)T_1\mathbf{x} = T_1(\alpha\beta\mathbf{x}) = \alpha T_1(\beta\mathbf{x}) = \alpha(\beta T_1)\mathbf{x}$$

- $\alpha(T_1 + T_2) = \alpha T_1 + \alpha T_2$ because

$$[\alpha(T_1 + T_2)](\mathbf{x}) = \alpha T_1\mathbf{x} + \alpha T_2\mathbf{x} = (\alpha T_1 + \alpha T_2)(\mathbf{x})$$

□

Exercise 3.1 Multiply

a). $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 54 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+6+6 \\ 4+15+12 \end{pmatrix} = \begin{pmatrix} 13 \\ 31 \end{pmatrix}$

b). $\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+6 \\ 0+3 \\ 2+0 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix}$

c). $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1+4+0+0 \\ 0+2+6+0 \\ 0+0+3+8 \\ 0+0+0+4 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 11 \\ 4 \end{pmatrix}$

d). $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

can't be multiplied because the number of columns of the first matrix doesn't equal to the number of rows of the second matrix.

Exercise 3.2 Let a linear transformation in \mathbb{R}^2 be the reflection in the line $x_1 = x_2$. Find its matrix.

Solution. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be this transformation. The basis of the domain is $\{\mathbf{e}_1, \mathbf{e}_2\}$ where $\mathbf{e}_1 = (1, 0)^T$ and $\mathbf{e}_2 = (0, 1)^T$. Because T reflect the line $x_1 = x_2$ then

$$T\mathbf{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad T\mathbf{e}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore, the matrix of this transformation is $[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

□

Exercise 3.3 For each linear transformation below, find its matrix

- a). $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by $T(x, y)^T = (x + 2y, 2x - 5y, 7y)^T$
- b). $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ defined by $T(x_1, x_2, x_3, x_4)^T = (x_1 + x_2 + x_3 + x_4, x_2 - x_4, x_1 + 3x_2 + 6x_4)^T$
- c). $T : \mathbb{P}_n \rightarrow \mathbb{P}_n$ st $Tf(t) = f'(t)$ (find the matrix with respect to the standard basis $1, t, t^2, \dots, t^n$)
- d). $T : \mathbb{P}_n \rightarrow \mathbb{P}_n$ st $Tf(t) = 2f(t) + 3f'(t) - 4f''(t)$.

Proof. Find the matrix.

- a). The standard basis in \mathbb{R}^2 is $\{\mathbf{e}_1, \mathbf{e}_2\}$ where $\mathbf{e}_1 = (1, 0)^T$ and $\mathbf{e}_2 = (0, 1)^T$. We have

$$T\mathbf{e}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \text{and} \quad T\mathbf{e}_2 = \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}$$

Hence $\begin{pmatrix} 1 & 2 \\ 2 & -5 \\ 0 & 7 \end{pmatrix}$ is its matrix.

- b). Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard basis in \mathbb{R}^4 . Hence

$$\begin{aligned} T\mathbf{e}_1 &= T(1, 0, 0, 0)^T = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, & T\mathbf{e}_2 &= T(0, 1, 0, 0)^T = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \\ T\mathbf{e}_3 &= T(0, 0, 1, 0)^T = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & T\mathbf{e}_4 &= T(0, 0, 0, 1)^T = \begin{pmatrix} 1 \\ -1 \\ 6 \end{pmatrix} \end{aligned}$$

Therefore, $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 3 & 0 & 6 \end{pmatrix}$ is its matrix.

c). Let $E = \{t^n, t^{n-1}, \dots, t, 1\}$ be the standard basis and $f(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0 \in \mathbb{P}_n$. We write

$$f(t) = (a_n, a_{n-1}, \dots, a_1, a_0)^T$$

is base E . Since

$$T(t^n) = nt^{n-1}, \quad T(t^{n-1}) = (n-1)t^{n-2}, \dots, \quad T(t) = 1, \quad T(1) = 0$$

Therefore its matrix is

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ n & 0 & 0 & \dots & 0 & 0 \\ 0 & n-1 & 0 & \dots & 0 & 0 \\ 0 & 0 & n-2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

- d). $Tf(t) = 2f(t) + 3f'(t) - 4f''(t)$
 Again, the standard basis is $\{t^n, t^{n-1}, \dots, t, 1\}$. For each $i \in [0, n]$ we have

$$T(t^i) = 2t^i + 3it^{i-1} - 4i(i-1)t^{i-2}$$

Hence the matrix is achieved by stacking $[T(t^n), \dots, T(t^i), \dots, T(t), T(1)]$, therefore the matrix is

$$[T] = \begin{bmatrix} 2 & 0 & \cdots & 0 & 0 \\ 3n & 2 & \cdots & 0 & 0 \\ -4n(n-1) & 3(n-1) & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & 2 & 0 \\ 0 & 0 & \cdots & 3 & 2 \end{bmatrix}$$

□

Exercise 3.4 Find 3×3 matrices representing the transformations of \mathbb{R}^3 which

- a). project every vector onto x - y plane;
- b). reflect every vector through x - y plane;
- c). rotate the x - y plane through 30° , leaving the z -axis alone.

Proof. In space \mathbb{R}^3 , we shall use its standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ where $\mathbf{e}_1 = (1, 0, 0)^T$, $\mathbf{e}_2 = (0, 1, 0)^T$ and $\mathbf{e}_3 = (0, 0, 1)^T$.

- a). Let T be this transformation. This means $T(x, y, z)^T = (x, y, 0)^T$. We get

$$T\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad T\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad T\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

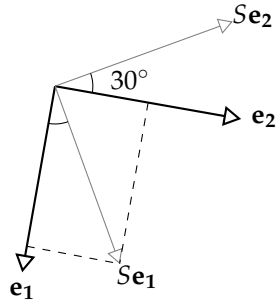
Therefore is matrix is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

- b). Let R be this transformation. Since R project every vector through x - y plane, hence $R(x, y, z)^T = (x, y, -z)^T$. We get

$$R\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad R\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad R\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Thus the matrix of R is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

- c). Let S be this transformation. S moves the vectors $\mathbf{e}_1, \mathbf{e}_2$ to the point x', y' respectively.



Since $\cos 30^\circ = \frac{\sqrt{3}}{2}$ and $\sin 30^\circ = \frac{1}{2}$, we conclude that

$$Se_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \quad Se_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{pmatrix}, \quad Se_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore the matrix is $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

□

Exercise 3.5 Let A be a linear transformation. If z is the center of the straight interval $[x, y]$, show that Az is the center of the interval $[Ax, Ay]$.

Proof. z is the center of $[x, y]$ iff $z = \frac{1}{2}x + \frac{1}{2}y$. Therefore,

$$Az = A\left(\frac{1}{2}x + \frac{1}{2}y\right) = \frac{1}{2}Ax + \frac{1}{2}Ay$$

Thus, Az is the center of the interval $[Ax, Ay]$.

□

Exercise 3.6 The set \mathbb{C} of complex numbers can be canonically identified with the space \mathbb{R}^2 by treating each $z = x + iy \in \mathbb{C}$ as a column $(x, y)^T \in \mathbb{R}^2$.

- Treating \mathbb{C} as a complex vector space, show that the multiplication by $\alpha = a + ib \in \mathbb{C}$ is a linear transformation in \mathbb{C} . What is its matrix.
- Treating \mathbb{C} as the real vector space \mathbb{R}^2 show that the multiplication by $\alpha = a + ib \in \mathbb{C}$ is a linear transformation there.
- Define $T(x + iy) = 2x - y + i(x - 3y)$. Show that this tran is not a linear transformation in the complex vector space \mathbb{C} , but if we treat \mathbb{C} as the real vector space \mathbb{R}^2 then it is a linear transformation there, then find its matrix.

Proof.

- a). Let T be this transformation. For any $\mathbf{x} \in \mathbb{C}$, we have $T\mathbf{x} = \alpha\mathbf{x} \in \mathbb{C}$. Thus $T : \mathbb{C} \rightarrow \mathbb{C}$, and we'll prove that T is a linear transformation. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}$ be two vectors, and $z \in \mathbb{C}$ be a scalar (complex). Observe that

- $T(\mathbf{x} + \mathbf{y}) = \alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y} = T\mathbf{x} + T\mathbf{y}$ (distributivity of complex numbers)
- $T(z\mathbf{x}) = \alpha(z\mathbf{x}) = z(\alpha\mathbf{x}) = zT\mathbf{x}$

This shows that this transformation T is a linear one. To find its matrix, we only need to know the basis of \mathbb{C} . Since any vector $\mathbf{x} \in \mathbb{C}$ we be written as

$$\mathbf{x} = 1 \cdot \underbrace{\mathbf{x}}_{\text{scalar}}$$

and because this representation is unique, we obtain that $\{1\} \subset \mathbb{C}$ is a basis of \mathbb{C} . Thus the matrix is

$$[T] = [T(1)] = [\alpha \cdot 1] = [\alpha].$$

- b). Because we treat \mathbb{C} as \mathbb{R}^2 , then any complex number $\mathbf{x} = x + iy$ can be represented as $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Let T be this transformation. Thus T would look like

$$\begin{aligned} T \begin{pmatrix} x \\ y \end{pmatrix} &= T(\mathbf{x}) = \alpha\mathbf{x} \\ &= (a + ib)(x + iy) \\ &= (ax - by) + i(ay + bx) \\ &= \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} \in \mathbb{R}^2 \end{aligned}$$

Thus $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. We need to show that T is in fact linear. Let $\mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ and be two arbitrary vectors. We have

$$T\mathbf{x}_1 + T\mathbf{x}_2 = \begin{pmatrix} ax_1 - by_1 \\ ay_1 + bx_1 \end{pmatrix} + \begin{pmatrix} ax_2 - by_2 \\ ay_2 + bx_2 \end{pmatrix} = \begin{pmatrix} a(x_1 + x_2) - b(y_1 + y_2) \\ a(y_1 + y_2) + b(x_1 + x_2) \end{pmatrix} = T(\mathbf{x}_1 + \mathbf{x}_2),$$

and for any scalar $r \in \mathbb{R}$,

$$rT\mathbf{x} = r \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} rax - rby \\ ray + rbx \end{pmatrix} = T(r\mathbf{x}).$$

This shows that T is a linear transformation. To find the matrix, we first need to find a basis in \mathbb{R}^2 . Luckily, as we've proved earlier we could choose $\{\mathbf{e}_1, \mathbf{e}_2\}$ to be a basis where

$$\mathbf{e}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{e}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Therefore

$$T\mathbf{e}_1 = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad T\mathbf{e}_2 = \begin{pmatrix} -b \\ a \end{pmatrix}$$

Thus the matrix of this transformation is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

c). Define $T(x + iy) = 2x - y + i(x - 3y)$

- We'll prove that T is not linear in complex vector space. Observe that

$$T(i) = T(0 + i) = -1 - 3i \quad \text{and} \quad T(1) = T(1 + 0i) = 2 + i$$

clearly $T(i) \neq iT(1)$, this implies that T is not a linear transformation in \mathbb{C} .

- In \mathbb{R}^2 the transformation would look like

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ x - 3y \end{pmatrix}.$$

For any vectors $\mathbf{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, we have

$$T\mathbf{x}_1 + T\mathbf{x}_2 = \begin{pmatrix} 2x_1 - y_1 \\ x_1 - 3y_1 \end{pmatrix} + \begin{pmatrix} 2x_2 - y_2 \\ x_2 - 3y_2 \end{pmatrix} = \begin{pmatrix} 2(x_1 + x_2) - (y_1 + y_2) \\ (x_1 + x_2) - 3(y_1 + y_2) \end{pmatrix} = T(\mathbf{x}_1 + \mathbf{x}_2)$$

and for any scalar $r \in \mathbb{R}$,

$$rT\mathbf{x} = r \begin{pmatrix} 2x - y \\ x - 3y \end{pmatrix} = \begin{pmatrix} 2rx - ry \\ rx - 3ry \end{pmatrix} = T(r\mathbf{x})$$

this shows that $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is linear. Because $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a basis in \mathbb{R}^2 and

$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix},$$

thus the matrix of this transformation is $\begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix}$.

□

Exercise 3.7 Show that any linear transformation in \mathbb{C} (treated as a complex vector space) is a multiplication by $\alpha \in \mathbb{C}$.

Proof. Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be this transformation. For any $\mathbf{x} \in \mathbb{C}$

$$T\mathbf{x} = T(\mathbf{x} \cdot \mathbf{1}) = \mathbf{x} \cdot \underbrace{T(\mathbf{1})}_{\text{scalar}}$$

and the proof is completed. □

4. Linear transformation as a Vector

Let set $\mathcal{L}(V, W)$ is a vector space with addition and scalar multiplication (as proved above).

5. Composition

Homework 4. Let A and B be matrices of size $m \times n$ and $n \times m$ respectively. Then

$$\text{trace}(AB) = \text{trace}(BA).$$

Proof. We'd like to prove this theorem *less* computationally. Let $X \in M_{n \times m}$. Consider the mapping $T, T_1 : M_{n \times m} \rightarrow \mathbb{F}$ defined by

$$T(X) = \text{trace}(AX) \quad \text{and} \quad T_1(X) = \text{trace}(XA).$$

To prove the theorem it is sufficient to show that T, T_1 are linear and they are the same. so by substituting $X = B$ gives the theorem.

Claim 1. The transformations T, T_1 defined above are linear.

Proof. For $X, Y \in M_{n \times m}$,

- From the properties of matrix, $A(X + Y) = AX + AY$. Because AX and BX are both square matrices with size $m \times m$, and since we add the matrices $AX + AY$ entrywise, it follows that

$$\begin{aligned} T(X + Y) &= \text{trace}(A(X + Y)) = \text{trace}(AX + AY) \\ &= \text{trace}(AX) + \text{trace}(AY) \\ &= T(X) + T(Y) \end{aligned}$$

- Similarly for any scalar $\alpha \in \mathbb{F}$,

$$T(\alpha X) = \text{trace}(A \cdot \alpha X) = \text{trace}(\alpha AX) = \alpha \text{trace}(AX) = \alpha T(X)$$

This implies that T is a linear transformation. With simply proof, we conclude that T_1 is also a linear transformation. \square

We choose $\mathbf{e}_{11}, \mathbf{e}_{21}, \dots, \mathbf{e}_{nm}$ to be the standard basis of $M_{n \times m}$, meaning the vector

$$\mathbf{e}_{ij} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

is a matrix whose entries are zero, except at the entry at row i and column j , which is 1. Then we only need to show that $T\mathbf{e}_{ij} = T_1\mathbf{e}_{ij}$ for all i, j . Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1m} \\ \vdots & & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{im} \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nm} \end{pmatrix}$$

Hence

$$A\mathbf{e}_{ij} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a_{ij} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdot & \cdot \\ \cdot & a_{ij} & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}$$

and

$$\mathbf{e}_{ij}A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & a_{ij} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot \\ \cdot & \cdot & a_{ij} & \cdot \\ \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

This implies that $T\mathbf{e}_{ij} = T_1\mathbf{e}_{ij}$ for all i, j , and hence $T = T_1$. \square

Exercise 5.1 Working on it.

Exercise 5.2 Let T_γ be the rotation matrix by γ in \mathbb{R}^2 . Check by matrix multiplication that $T_\gamma T_{-\gamma} = T_{-\gamma} T_\gamma = I$.

Proof. Working on it. \square

Exercise 5.3 Multiply two rotation matrices T_α and T_β . Deduce formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ from here.

Proof. Working on it. \square

Exercise 5.4 Find the matrix of the orthogonal projection in \mathbb{R}^2 on to the line $x_1 = -2x_2$.

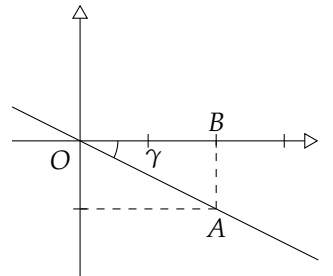
Proof. Let T be this transformation. Let R_γ and P_x be the transformations of rotation by γ and projection to x -axis, respectively. Therefore $T = R_\gamma P_x R_{-\gamma}$. Note that

$$R_\gamma = \begin{pmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{pmatrix}, \quad R_{-\gamma} = \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix}, \quad \text{and} \quad P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus,

$$\cos \gamma = \frac{\overline{OB}}{\overline{OA}} = \frac{2}{\sqrt{5}} \quad \text{and} \quad \sin \gamma = \frac{\overline{AB}}{\overline{OA}} = \frac{-1}{\sqrt{5}}.$$

We get



$$\begin{aligned}
 T &= \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \\
 &= \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \\
 &= \frac{1}{5} \begin{pmatrix} 4 & -2 \\ -2 & -2 \end{pmatrix}
 \end{aligned}$$

□

Exercise 5.5 Find linear transformations $A, B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $AB = \mathbf{0}$ but $BA \neq \mathbf{0}$.

Solution. Consider the following

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

however

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \neq \mathbf{0}$$

Therefore, these two matrices are the ones we wish to find.

□

Exercise 5.6 Prove that $\text{trace}(AB) = \text{trace}(BA)$.

Proof. See on page ??.

□

Exercise 5.7 Construct a non-zero matrix A such that $A^2 = \mathbf{0}$

Proof. Let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Thus perform the multiplication, we get

$$\begin{cases} a^2 + bc = 0 \\ ac + cd = 0 \\ ab + bd = 0 \\ bc + d^2 = 0 \end{cases}$$

for simplicity, we'll choose $a = 1$. Hence $bc = -1$ and

$$\begin{cases} c(d+1) = 0 \\ b(d+1) = 1 \\ d^2 = 1 \end{cases}$$

this suggests that $d = -1$, and $bc = -1$. Here, we'll choose $b = 1$ and $c = -1$. Therefore, the matrix

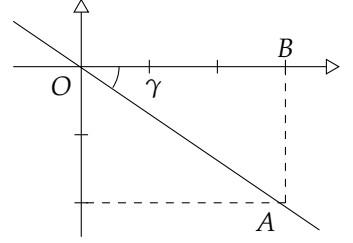
$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

□

Exercise 5.8 Find the matrix of the reflection through the line $y = -2x/3$.

Proof. Let T be this transformation and γ be the angle between the x -axis and the line $y = -2x/3$. Hence $T = R_\gamma T_0 R_\gamma$. We then have $\cos \gamma = OB/OA = 3/\sqrt{13}$ and $\sin \gamma = -AB/OA = -2/\sqrt{13}$. Thus

$$\begin{aligned} T &= \begin{pmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix} \\ &= \frac{1}{13} \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} \\ &= \frac{1}{13} \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -2 & -3 \end{pmatrix} \\ &= \frac{1}{15} \begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix}. \end{aligned}$$



□

6. Isomorphism

Exercise 6.1 Prove that if $A : V \rightarrow W$ is an isomorphism and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is a basis in V , then $A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n$ is a basis in W .

Proof. Since $A : V \rightarrow W$ is an isomorphism, hence it's invertible i.e. there is a linear transformation $A^{-1} : W \rightarrow V$ such that $AA^{-1} = A^{-1}A = I$. Thus for any $\mathbf{w} \in W$, there is a $\mathbf{v} \in V$ such that $A^{-1}\mathbf{w} = \mathbf{v}$. Recall that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis in V , then there are unique scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n.$$

This implies that

$$\begin{aligned} A^{-1}\mathbf{w} &= \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n \\ AA^{-1}\mathbf{w} &= A(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n) \\ \mathbf{w} &= \alpha_1 A\mathbf{v}_1 + \alpha_2 A\mathbf{v}_2 + \dots + \alpha_n A\mathbf{v}_n \end{aligned}$$

Because $\alpha_1, \alpha_2, \dots, \alpha_n$ are unique, we conclude that any $\mathbf{w} \in W$ can be represented as a unique linear combination of $A\mathbf{v}_1, A\mathbf{v}_2, \dots, A\mathbf{v}_n$. Thus the proof is completed. □

Exercise 6.2 Find all right inverses of the 1×2 matrix (row) $A = (1, 1)$. Conclude from here that the row A is not left invertible.

Exercise 6.3 Find all the left inverses of the column $(1, 2, 3)^T$.

Proof. Let $A = (1, 2, 3)^T$. Because A is a 3×1 matrix, then its inverse, say B is a 1×3 matrix. Let $B = (x \ y \ z)$. Hence

$$AB = (x \ y \ z) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (1)$$

This implies that $x + 2y + 3z = 1$ or $x = 1 - 2y - 3z$. Thus all the left inverses of A is in the form

$$B = (1 - 2y - 3z \ y \ z)$$

where y, z are arbitrary real numbers. \square

Exercise 6.4 Is the column $(1, 2, 3)^T$ right invertable?

Solution. The column $(1, 2, 3)^T$ is not right invertable, because as proved in previous exercise the column $(1, 2, 3)^T$ has more than one left inverses. \square

Exercise 6.5 Find two matrices A and B that AB is invertable, but A and B are not.

Solution. Consider: $A = (2 \ 1 \ -1)$ and $B = (1 \ 2 \ 3)^T$ Note that

$$AB = (2 \ 1 \ -1) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (2 + 2 - 3) = (1)$$

However, as proved in previous exercise, we know that the matrix A is not invertable. And we wish to prove that B is not invertable either. To achieved this, we have to find two matrices that are right invertable to B . Observe that

$$(2 \ 1 \ -1) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = (1) \quad \text{and} \quad (2 \ 1 \ -1) \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix} = (1)$$

This suggests that B is not invertable. Therefore, we've found matrices A and B such that AB is invertable, yet A and B are not. \square

Exercise 6.6 Suppose the product AB is invertable. Show that A is right invertable, and B is left invertable.

Proof. Because AB is invertable, then matrix $(AB)^{-1}$ is defined. Observe that

$$A \cdot B(AB)^{-1} = AB \cdot (AB)^{-1} = I$$

and

$$(AB)^{-1}A \cdot B = (AB)^{-1} \cdot AB = I$$

This shows that A is right invertable, and B is left invertable, as expected. \square

Exercise 6.7 Let A and AB be invertible. Prove that B is also invertible.

Proof. We claim that $(AB)^{-1}A$ is the inverse of B . To prove this, observe that

$$(AB)^{-1}A \cdot B = (AB)^{-1} \cdot AB = I$$

and

$$B \cdot (AB)^{-1}A = A^{-1}AB \cdot (AB)^{-1}A = A^{-1}IA = I$$

This shows that $(AB)^{-1}A$ is both the left and the right inverses of B . Thus, B is invertible. \square

Exercise 6.8 Let A be $n \times n$ matrix. Prove that if $A^2 = \mathbf{0}$ then A is not invertible.

Proof. Assume by contradiction that A is invertible, meaning there's a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. Thus

$$A = A^2A^{-1} = \mathbf{0}A^{-1} = \mathbf{0}$$

Then $I = AA^{-1} = \mathbf{0}A^{-1} = \mathbf{0}$, a contradiction. Therefore, A is not invertible. \square

Exercise 6.9 Suppose $AB = \mathbf{0}$ for some non-zero matrix B . Can A be invertible?

Proof. We claim that A is not invertible. To prove this, we assume by contradiction that A is invertible. Hence A^{-1} exists, and

$$B = A^{-1}A \cdot B = A^{-1} \cdot AB = A^{-1}\mathbf{0} = \mathbf{0}$$

which is a contradiction that B is non-zero. \square

Exercise 6.10 Write matrices of the linear transformations T_1 and T_2 in \mathbb{F}^5 , defined as follows: T_1 interchanges the coordinates x_2 and x_4 of the vector \mathbf{x} , and T_2 just adds to the coordinates $x_2 \rightarrow a$ times the coordinate x_4 . and does not change other coordinates, i.e.

$$T_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_4 \\ x_3 \\ x_2 \\ x_5 \end{pmatrix} \quad \text{and} \quad T_2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + ax_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

where a is a fixed number. Show that T_1 and T_2 are invertible transformations, and write the matrices of the inverses.

Proof. The matrix of this \square

7. Subspaces

Exercise 7.1 Let X and Y be subspaces of a vector space V . Prove that $X \cap Y$ is a subspace of V .

Proof. Let \mathbf{a} and \mathbf{b} be arbitrary vectors of $X \cap Y$. Because X and Y are themselves subspaces of V , hence

$$\begin{cases} \mathbf{a} \in X \\ \mathbf{b} \in X \end{cases} \implies \begin{cases} \alpha \mathbf{a} \in X \\ \beta \mathbf{b} \in X \end{cases}$$

this implies that $\alpha \mathbf{a} + \beta \mathbf{b} \in X$ for any scalars α, β . Similarly, $\alpha \mathbf{a} + \beta \mathbf{b} \in Y$. Thus $\alpha \mathbf{a} + \beta \mathbf{b} \in X \cap Y$. Therefore, $X \cap Y$ is also a subspace of V . \square

Exercise 7.2 Let V be a vector space. For $X, Y \subset V$ the sum $X + Y$ is the collection of all vectors \mathbf{v} which can be represented as $\mathbf{v} = \mathbf{x} + \mathbf{y}$ where $\mathbf{x} \in X$ and $\mathbf{y} \in Y$. Show that if X and Y are subspaces of V , then $X + Y$ is also a subspace of V .

Proof. Let $\mathbf{v}_1, \mathbf{v}_2 \in X + Y$ then there are $\mathbf{x}_1, \mathbf{x}_2 \in X$ and $\mathbf{y}_1, \mathbf{y}_2 \in Y$ such that

$$\mathbf{v}_1 = \mathbf{x}_1 + \mathbf{y}_1 \quad \text{and} \quad \mathbf{v}_2 = \mathbf{x}_2 + \mathbf{y}_2$$

Because X, Y are subspaces of V , then

$$\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 = \underbrace{(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2)}_{\text{vector of } X} + \underbrace{(\alpha \mathbf{y}_1 + \beta \mathbf{y}_2)}_{\text{vector of } Y}$$

Hence $\alpha \mathbf{v}_1 + \beta \mathbf{v}_2$ is also a vector of $X + Y$. Thus $X + Y$ is a subspace of V . \square

Exercise 7.3 Let X be a subspace of a vector space V , and let $\mathbf{v} \in V$ and $\mathbf{v} \notin X$. Prove that if $\mathbf{x} \in X$ then $\mathbf{x} + \mathbf{v} \notin X$.

Proof. We'll prove this by contradiction by assuming that $\mathbf{x} + \mathbf{v} \in X$. Because X is a subspace of V and $\mathbf{x} \in X$, hence $-\mathbf{x} \in X$. Thus

$$(\mathbf{x} + \mathbf{v}) + (-\mathbf{x}) \in X$$

and we conclude that $\mathbf{v} \in X$, which is a contradiction. \square

Exercise 7.4 Let X and Y be subspaces of a vector space V . Using the previous exercise, show that $X \cup Y$ is a subspace iff $X \subset Y$ or $Y \subset X$.

Proof. We need to prove this in two directions.

- If $X \subset Y$ or $Y \subset X$: Without loss of generality, we may assume that $Y \subset X$. Therefore $X \cup Y = X$ is a subspace of V .

- If $X \not\subset Y$ and $Y \not\subset X$, we now wish to show that $X \cup Y$ is not a subspace. Observe that if $X \not\subset Y$ and $Y \not\subset X$ that means there are $\mathbf{x}_0 \in X$ and $\mathbf{y}_0 \in Y$ such that $\mathbf{x}_0 \notin Y$ and $\mathbf{y}_0 \notin X$. Hence $\mathbf{x}_0, \mathbf{y}_0 \in X \cup Y$ and follow from the previous exercise, we conclude that

$$\mathbf{x}_0 + \mathbf{y}_0 \notin X \quad \text{and} \quad \mathbf{x}_0 + \mathbf{y}_0 \notin Y$$

Therefore, $\mathbf{x}_0 + \mathbf{y}_0 \notin X \cup Y$. This suggests that $X \cup Y$ is not a vector space. Hence the proof is completed.

□

Chapter 2

Systems of linear equations

1. Different faces of linear transformation
2. Solution of a linear system. Echelon forms

Exercise 2.1 Write the systems of equations below in matrix form.

Exercise 2.2 Find all solutions of the vector equation

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_3 \mathbf{v}_3 = \mathbf{0}$$

where $\mathbf{v}_1 = (1, 1, 0)^T$, $\mathbf{v}_2 = (0, 1, 1)^T$ and $\mathbf{v}_3 = (1, 0, 1)^T$. What conclusion can you make about linear independence (dependence) of the system of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Proof. The echelon form of the system is

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

so, clearly the solution to this equation is $x_1 = x_2 = x_3 = 0$.

Conclusion. If the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independence, then the above equation has unique solution, namely $x_1 = x_2 = x_3 = 0$.

If the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependence, then there exists α, β, γ (some of them are non-zero) such that

$$\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 + \gamma \mathbf{v}_3 = \mathbf{0}$$

Therefore, the solution to the above equation is

$$\begin{cases} x_1 = \alpha t \\ x_2 = \beta t \\ x_3 = \gamma t \end{cases}$$

for some $t \in \mathbb{R}$.

□

3. Analyzing pivots

Exercise 3.6 Prove or disprove. If the columns of a square $(n \times n)$ matrix are linearly independence, so are the columns of A^2 .

Proof. Since A is a square matrix, and columns vectors are linearly independence, so

\Rightarrow has pivot in every column
 \Rightarrow has pivot in every row
 \Rightarrow A is invertible
 \Rightarrow A^2 also invertible
 \Rightarrow A^2 has pivot every column and row

Therefore, the column vectors of A^2 also independence. □

Exercise 3.7 Prove or disprove. If the columns of a square $(n \times n)$ matrix are linearly independence, so are the columns of A^3 .

Exercise 3.8 Show that if the equation $Ax = \mathbf{0}$ has unique solution, then A is left invertible.

Proof. Because A has unique solution, then it has pivot in every column. Therefore, $\#col \leq \#row$, so we let $m \times n$ be the size of A where $n \leq m$. Let R be the reduced echelon form of A , hence there exists $E = E_k \cdots E_2 E_1$ such that $R = EA$. Observe that, R would look like

$$R = \begin{pmatrix} \boxed{1} & 0 & \cdots & 0 \\ 0 & \boxed{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \boxed{1} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Using matrix multiplication gives us $R^T R = I_n$. We obtain that

$$R^T E A = R^T T = I_n$$

Therefore, A is left invertible, and $R^T E$ is its left inverse. □

4. Find A^{-1} by row reduction

5. Dimension

Exercise 5.1 True or false.

- a). Every vector space that is generated by a finite set has a basis;
- b). Every vector space has a (finite) basis;
- c). A vector space cannot have more than one basis;
- d). A vector space has a finite basis, then the number of vectors in every basis is the same;
- e). The dimension of \mathbb{P}_n is n ;
- f). The dimension of $M_{m \times n}$ is $m + n$;
- g). If vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ generate (span) the vector space V , then every vector in V can be written as a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in only one way;
- h). Every subspace of a finite-dimensional space is finite-dimensional.
- i). If V is a vector space having dimension n , then V has exactly one subspace of dimension 0, and exactly one subspace of dimension n .

Proof. a). **True.** That finite set which generated a vector space is the spanning set itself. Since it's finite, it contains a basis.

b). **False.** Take $\mathbb{R}[x]$ for example.

c). **False.** In \mathbb{R}^2 , one can choose

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \text{or} \quad \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

as a basis.

d). **True.** As proved in the above theorems,

$$\# \text{ independence vectors} \leq \dim V$$

$$\# \text{ generating vectors} \geq \dim V$$

hence the number of any basis in V must be exactly $\dim V$ vectors.

e). **False.** In \mathbb{P}_n , the standard basis is

$$1, t, t^2, \dots, t^n$$

which has $n + 1$ vectors. Hence $\boxed{\dim \mathbb{P}_n = n + 1}$.

f). **False.** The standard basis in $M_{m \times n}$ is

$$\{\mathbf{e}_{11}, \mathbf{e}_{12}, \dots, \mathbf{e}_{mn}\}$$

has $m \times n$ vectors. Hence $\boxed{\dim M_{m \times n} = mn}$.

g). **False.** span doesn't guarantee uniqueness.

h). **True.** Let W be a subspace of V . Because $\dim V$ finite, we can find

$$\mathcal{A} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} \subset V$$

that spans V . WLOG, we assume that none of vectors in \mathcal{A} belongs to W (the unluckiest case.) We can choose $\mathbf{w}_1 \in W$ such that $\mathbf{w}_1 \neq \mathbf{0}$. Then

$$\mathbf{w}_1 = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$$

Because $\mathbf{w}_1 \neq \mathbf{0}$, we're sure some of the α_i 's are non-zero, say α_1 . Then the new system

$$\mathcal{A}_1 = \{\mathbf{w}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$$

still spans the space V . Now, if $\mathcal{B}_1 := \{\mathbf{w}_1\}$ doesn't span W , we can repeat the above procedure and find \mathbf{w}_2 . We can do this at most n times, because once we reach the n th step, we have the new system $\mathcal{A}_n \subset W$ that spans the whole space V .

Therefore, after some finite $k \leq n$ step, we have

$$\mathcal{B}_k = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_k\} \subset W$$

spans W . Hence, W is finite dimensional.

i). **Not sure.**

□

Exercise 5.2 Prove that if V is a vector space having dimension n , then a system of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in V is linearly independent iff it spans V .

Proof.

(\implies) Suppose that v_1, \dots, v_n linearly independent in V . If it's not span, then we can complete this system to a basis, but then this new system will have the number of vectors more than n , but this is a contradiction because a basis (linearly independent) cannot have vectors more than n .

(\impliedby) Now suppose that v_1, \dots, v_n spans V . If it's not linearly independence, we can throw away some vectors, and become the basis. However, this new basis system has less than n vectors in it, which is a contradiction because a basis (span) must have more than n vectors in it.

□

Exercise 5.3 Prove that a linearly independent system of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in a vector space V is a basis iff $n = \dim V$.

Proof. We have $\mathbf{v}_1, \dots, \mathbf{v}_n$ linearly independent. If it's a basis then $\dim V = n$. Now suppose that $\dim V = n$. We need to show that the system is a basis. From the above exercise, this system has to span V . Thus a basis in V . □

Exercise 5.4 (An old problem revisited.) Is it possible that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, but the vectors $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$ and $\mathbf{w}_3 = \mathbf{v}_3 + \mathbf{v}_1$ are linearly independent.

Proof. Because $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ linearly independent, then

$$\dim \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} = 3.$$

Because $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$, it's easy to see that

$$\text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} \subseteq \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

But since the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ linearly dependent, then one vector, say \mathbf{v}_3 , is the linear combination of $\mathbf{v}_1, \mathbf{v}_2$. This suggests that $\dim \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \dim \text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \leq 2$, thus

$$\dim \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\} \leq \dim \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \leq 2$$

But this contradicts to the result we found above. Hence it's impossible that such $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ exists. \square

Exercise 5.5 Let vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be a basis in V . Show that $\mathbf{u} + \mathbf{v} + \mathbf{w}$, $\mathbf{v} + \mathbf{w}$, \mathbf{w} is also a basis in V .

Proof. It can be shown by using the fact from the exercise above i.e. show that the new system is either spans or LI (because it already has 3 vectors in it). But here I'm gonna use matrix and pivot stuff. Since V has three vectors as a basis, then $\dim V = 3$. Choose an isomorphism $A : V \rightarrow \mathbb{R}^3$ such that $A\mathbf{v} = (1, 0, 0)$, $A\mathbf{u} = (0, 1, 0)$ and $A\mathbf{w} = (0, 0, 1)$. Letting

$$\mathbf{a}_1 := A(\mathbf{u} + \mathbf{v} + \mathbf{w}) = (1, 1, 1)$$

$$\mathbf{a}_2 := A(\mathbf{v} + \mathbf{w}) = (0, 1, 1)$$

$$\mathbf{a}_3 := A(\mathbf{w}) = (0, 0, 1)$$

Observe that the Echelon form of the matrix formed by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has pivots in every row and every column, hence the system $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is basis in \mathbb{R}^3 . Using isomorphism theorem, we conclude that

$$\mathbf{u} + \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}, \mathbf{w}$$

also a basis in V . \square

6. Rank

Exercise 6.4 Prove that if $A : X \rightarrow Y$ and V is a subspace of X , then $\dim AV \leq \text{rank } A$. Then deduce that $\text{rank } AB \leq \text{rank } A$.

Proof. Observe that $AV \subseteq AX = \text{col } A$, hence

$$\dim AV \leq \dim \text{col } A = \text{rank } A.$$

To prove the next statement, we may assume that $B : Z \rightarrow X$. This implies that $\text{col } B = BZ \subseteq X$. Using the previous result we obtain that

$$\begin{aligned} & \dim A(\text{col } B) \leq \text{rank } A \\ \implies & \dim \text{col } AB \leq \text{rank } A \\ \implies & \text{rank } AB \leq \text{rank } A \end{aligned}$$

□

7. Change of basis

Exercise 7.3 Find the change of coordinates matrix that changes the coordinates in basis $\{1, 1+t\}$ in \mathbb{P}_1 to the coordinates in the basis $\{1-t, 2t\}$.

Proof. Let's denote $\mathcal{A} = \{1, 1+t\}$ and $\mathcal{B} = \{1-t, 2t\}$. Let S be the standard basis in \mathbb{P}_1 . Therefore, the matrix that transforms from vector in basis \mathcal{A} to basis \mathcal{B} is $[\mathcal{B}\mathcal{A}] = [\mathcal{B}S][S\mathcal{A}]$. We have

$$[S\mathcal{A}] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$[\mathcal{B}S] = [\mathcal{S}\mathcal{B}]^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

Therefore,

$$[\mathcal{B}\mathcal{A}] = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

□

Exercise 7.4 Let T be the ...

Proof. In standard basis, T looks like

$$[T] = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}.$$

Let $\mathcal{B} = \{(1, 1)^T, (1, 2)^T\}$. In basis \mathcal{B} , the transformation would look like

$$[T]_{\mathcal{B}\mathcal{B}} = [\mathcal{B}S][T][S\mathcal{B}]$$

And we have

$$[S\mathcal{B}] = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

so

$$[\mathcal{B}S] = [\mathcal{S}\mathcal{B}]^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Therefore, the transformation $[T]_{\mathcal{B}\mathcal{B}}$ in basis \mathcal{B} is

$$[T]_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

□

Exercise 7.5 Prove that if A and B are similar matrices, then $\text{trace } A = \text{trace } B$.

Proof. Because A and B are similar, then there exists an invertible matrix Q such that $A = Q^{-1}BQ$. Observe that

$$A = Q^{-1} \cdot BQ \quad \text{and} \quad B = BQ \cdot Q^{-1}$$

This implies that $\text{trace } A = \text{trace } B$.

□