Notes on Real Analysis

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Chapter 1

Real Numbers

1.1 Properties of Real Numbers

For now, let me just assume that the set $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are already exist. Just note that they're an *"Order Field"*. The thing is, there must be some number system that is quite *larger* than $\mathbb{Q}$, because for example, there is no $q \in \mathbb{Q}$ such that $q^2 = 2$.

To extend from $\mathbb{Q}$, we're going to make up a new number system (kinda cheat a little bit, dun you think?) denoted by $\mathbb{R}$, which has the addition operation $(+)$ and multiplication $(\cdot)$ such that for all $a, b \in \mathbb{R}$

$$a + b \in \mathbb{R} \quad \text{and} \quad ab := a \cdot b \in \mathbb{R}.$$

Since we extended from $\mathbb{Q}$, this new set $\mathbb{R}$ is going to inherit all the properties from $\mathbb{Q}$ listed below

[A1] for any $a, b \in \mathbb{R}$, then $a + b = b + a$ and $ab = ba$.

[A2] for any $a, b, c \in \mathbb{R}$, then $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$

[A3] there is a number (*indentity*) $\theta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ such that $a + \theta = a\gamma = a$ for all $a \in \mathbb{R}$.

[A4] for any $a \in \mathbb{R}$,

- there exist an *additive inverse* α such that $a + \alpha = \theta$.
- if $a \neq \theta$, there exist a *multiplicative inverse* β such that $a\beta = \gamma$.

[A5] for any $a, b, c \in \mathbb{R}$, then $a(b + c) = ab + ac$.

Well because $\mathbb{Q} \subset \mathbb{R}$ (subspace), the indentity θ and γ are the same of those in $\mathbb{Q}$, which we already know they're simply the numbers $0, 1$. Note also that, we must assume that $0 \neq 1$.

Theorem 1. *For any $a, b, c \in \mathbb{R}$, the following holds*

- *the* additive *and the* multiplicative *indentity are unique.* (*later denoted them by* 0 *and* 1 *resp.*)
- *the addition and multiplicative inverses are unique.* (*later denoted them by* $-a$ *and* a^{-1} *resp.*)
- *if* $a + c = b + c \iff a = b$.
- $a \cdot 0 = 0, \ -a = (-1)a$ *and* $-(-a) = a$.
- *if* $ab = 0$ *then* $a = 0$ *or* $b = 0$.

The above theorem is not that hard to prove. However, I promise to come back to this point to provide a full proof about it.

What now? The set $\mathbb{R}$ here behaves the same as $\mathbb{Q}$. How is it possible that $\mathbb{R}$ is bigger that $\mathbb{Q}$? Let's find the condition that $\mathbb{Q}$ is lack of.

A long the way, we're introduced *bounded sets.*

Definition 1.1 (Maximum and Minimum)**.** Let $S \subset \mathbb{R}$.

- if there exists $M \in S$ such that $M \geq s$ for all $s \in S$, then M is said to be the maximum of S and is denoted by $\max S$.
- if there exists $m \in S$ such that $m \leq s$ for all $s \in S$, then m is said to be the minimum of S and is denoted by $\min S$.

Definition 1.2 (Bounded Sets)**.** The set $S \subseteq \mathbb{R}$ is called

- *bounded above* if $\exists M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$.
- *bounded below* if $\exists m \in \mathbb{R}$ such that $m \leq s$ for all $s \in S$.
- *bounded* if it is both bounded below and above, that is $\exists M > 0$ such that $|s| < M$ for all $s \in S$.

Definition 1.3 (Upper and Lower bounds)**.** Let $S \subseteq \mathbb{R}$ be a bounded set. We denote

- the set of all upper bounds of S by $\mathcal{U}(S) = \{M \in \mathbb{R} \ : \ M \geq s \text{ for all } s \in S\}$
- the set of all lower bounds of S by $\mathcal{L}(S) = \{m \in \mathbb{R} \ : \ m \leq s \text{ for all } s \in S\}$

Definition 1.4 (Infimum and Supremum)**.** Let S be a bounded set. If

- the set $\mathcal{U}(S)$ has a minimum α, then the set S is said to have a *supremum*, that is $\sup S := \alpha$.
- the set $\mathcal{L}(S)$ has a maximum β, then the set S is said to have an *infimum*, that is $\inf S := \beta$.

Axiom 1.5 (Axiom of Completeness)**.** Every non-empty subset of $\mathbb{R}$ that is bounded above has a supremum.

Example 1. The function $f(x) = 2x + 1$ converges over $\mathbb{R}$. Note that $\forall x \in \mathbb{R}$

$$|f(x) - f(x_0)| = 2|x - x_0|$$

therefore we can choose $\delta = \frac{\epsilon}{2}$ which does not depend on the choise of x_0.

Example 2. The function $f(x) = x^2$ converges over $\mathbb{R}$. Note that $\forall x_0 \in \mathbb{R}$

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0| \cdot |x + x_0|$$

Suppose that $|x - x_0| < 1$ and hence

$$|x + x_0| < |x - x_0| + |2x_0| < 1 + |2x_0|$$

So we can choose $\delta = \min\left\{1, \frac{\epsilon}{1+2|x_0|}\right\}$ which depends on the choise of x_0.

Definition 1.6 (Absolutely Convergent)**.** The function $f : \mathcal{D} \to \mathbb{R}$ is said to be *Absolutely Convergent* iff for each $\epsilon > 0$ there exists $\delta > 0$ such that $\forall x, y \in \mathcal{D}$

$$|x - x_0| < \delta \implies |f(x) - f(y)| < \epsilon.$$