Chapter 1

Basic Notions

1. Vector Spaces

Exercise 1.1. Let $\mathbf{x} = (1,2,3)^{\mathrm{T}}$, $\mathbf{y} = (y_1, y_2, y_3)^{\mathrm{T}}$ and $\mathbf{z} = (4,2,1)^{\mathrm{T}}$. Compute $2\mathbf{x}$, $3\mathbf{y}$, $\mathbf{x} + 2\mathbf{y} - 3\mathbf{z}$.

Proof. Little calculation reveals that

$$2\mathbf{x} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \quad 3\mathbf{y} = \begin{pmatrix} 3y_1 \\ 3y_2 \\ 3y_3 \end{pmatrix}, \quad \mathbf{x} + 2\mathbf{y} - 3\mathbf{z} = \begin{pmatrix} 2y_1 - 11 \\ 2y_2 - 4 \\ 2y_3 \end{pmatrix}$$

□

Exercise 1.2. Which of the following sets (with natural addition and multiplication by a scalar) are vector spaces? Justify your answers.

1. The set of all continuous functions on the interval $[0,1]$;
2. The set of all non-negative functions on the interval $[0,1]$;
3. The set of all polynomials of degree *exactly* n;
4. The set of all symmetric $n \times n$ matrices, i.e. the set of matrices $A = \{a_{j,k}\}_{j,k=1}^{n}$ such that $A^{\mathrm{T}} = A$.

Proof.

1. Let $\mathcal{C}[0,1]$ be the set of all continuous functions on $[0,1]$. For any $f, g \in \mathcal{C}[0,1]$ and $\alpha \in \mathbb{R}$, we define

   $$(f+g)(x) := f(x) + g(x) \quad \text{and} \quad (\alpha f)(x) := \alpha \cdot f(x)$$

   for each $x \in [0,1]$. Therefore, $(\mathcal{C}[0,1], +, \cdot)$ is closed under addition and scalar multiplication. It's immediate to see that all the eight properties of vector space are all satisfied, that is

- $f + g = g + f$
- $f + (g + h) = (f + g) + h$
- $f + 0 = f$
- $f + (-f) = 0$
- $1f = f$
- $\alpha(\beta f) = (\alpha\beta)f$
- $(\alpha + \beta)f = \alpha f + \beta f$
- $\alpha(f + g) = \alpha f + \beta g$

Note that the function $0 \in \mathcal{C}[0, 1]$ such that $0(x) = 0$ for each $x \in [0, 1]$.

2. Let $\mathcal{B}$ is the set of all non-negative functions on $[0, 1]$. Then $(\mathcal{B}, +\cdot)$ is not a vector space because it's not closed under scalar multiplication, i.e. if $f \in \mathcal{B}$, hence $f > 0$ yet

$$-f = (-1) \cdot f < 0.$$

(Even if we restrict the scalar be to positive real numbers, this set still won't be a vector space, because it fails to have an inverse.)

3. Let $\mathcal{P}$ be the set of all polynomials of degree exactly n, then $(\mathcal{P}, +, \cdot)$ is *not* a vector space, because the addtive indentity is the polynomial 0. However, $0 \notin \mathcal{P}$.

4. Let $\mathrm{sym}(n)$ be the set of all symmetric $n \times n$ matrices. The addition and scalar multiplication are defined as an *entrywise* operations. Hence, $\mathrm{sym}(n)$ is closed under $(+)$ and $(\cdot)$. The additive indentity is the matrix

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{sym}(n).$$

We could easily see that all the eight properties of vector spaces are all satisfied.

□

Exercise 1.3. True or false:

1. Every vector space contains a zero vector; (**True.**)
2. A vector space can have more than one zero vector; (**False.** The zero vector is unique.)
3. An $m \times n$ matrix has m rows and n columns; (**True.**)
4. If f and g are polynomials of degree n, then $f + g$ is also a polynomial of degree n. (**False.** consider t^n and $t - t^n$.)
5. If f and g are polynomials of degree atmost n, the $f + g$ is also a polynomial of degree atmost n. (**True.**)

Exercise 1.4. Prove that a zero vector $\mathbf{0}$ of a vector space V is unique.

Proof. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are the zero vectors of V. From the *Axioms of Vector Space*, we obtain that

$$\begin{aligned}
\mathbf{a} &= \mathbf{a} + \mathbf{b} && (\mathbf{b} \text{ is the zero vector}) \\
&= \mathbf{b} + \mathbf{a} && (\text{commutitativity}) \\
&= \mathbf{b} && (\mathbf{a} \text{ is the zero vector})
\end{aligned}$$

Hence, a zero vector of a vector space is unique, and we usually denote it by $\mathbf{0}$. □

Exercise 1.5. What is the zero matrix of the space $M_{2\times 3}$?

Answer. In the space $M_{2\times 3}$, the zero matrix is

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

□

Exercise 1.6. Prove that the additive inverse of a vector space is unique.

Proof. Let $\mathbf{a}$ be an arbitrary vector. Assume the $\mathbf{a}$ has two inverses, namely $\mathbf{x}$ and $\mathbf{y}$. Hence

$$\begin{aligned}
\mathbf{x} &= \mathbf{x} + \mathbf{0} \\
&= \mathbf{x} + (\mathbf{a} + \mathbf{y}) && (\mathbf{y} \text{ is an inverse}) \\
&= (\mathbf{x} + \mathbf{a}) + \mathbf{y} && (\text{associativity}) \\
&= \mathbf{0} + \mathbf{y} && (\mathbf{x} \text{ is an inverse}) \\
&= \mathbf{y}.
\end{aligned}$$

Therefore, the inverse of any vector $\mathbf{a} \in V$ is unique, and is usually denoted by $-\mathbf{a}$. □

Exercise 1.7. Prove that $0\mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v} \in V$.

Proof. Let $\mathbf{v} \in V$ and $\mathbf{b}$ is an inverse of $0\mathbf{v}$. Therefore,

$$\begin{aligned}
0 &= 0\mathbf{v} + b \\
&= (0+0)\mathbf{v} + b \\
&= (0\mathbf{v} + 0\mathbf{v}) + b && (\text{distributivity}) \\
&= 0\mathbf{v} + (0\mathbf{v} + b) && (\text{associativity}) \\
&= 0\mathbf{v} + \mathbf{0} && (\mathbf{b} \text{ is an inverse of } 0\mathbf{v}) \\
&= 0\mathbf{v}
\end{aligned}$$

for any $\mathbf{v} \in V$. □

Exercise 1.8. Prove that for any vector $\mathbf{v}$ its additive inverse $-\mathbf{v}$ is given by $(-1)\mathbf{v}$.

Proof. As proved in the above exercise for any $\mathbf{v} \in V$,

$$\mathbf{0} = 0\mathbf{v} = (1-1)\mathbf{v} = \mathbf{v} + (-1)\mathbf{v}$$

where the last equallity derives from the distributive property. Because $-\mathbf{v}$ is the inverse of $\mathbf{v}$, then

$$\begin{aligned} -\mathbf{v} &= -\mathbf{v} + \mathbf{0} \\ &= -\mathbf{v} + \left[\mathbf{v} + (-1)\mathbf{v}\right] \\ &= \underbrace{(-\mathbf{v} + \mathbf{v})}_{\mathbf{0}} + (-1)\mathbf{v} \\ &= (-1)\mathbf{v} \end{aligned}$$

as desired. □

2. Linear Combination, bases

Exercise 2.1. Find the basis in the space of 3×2 matrices $M_{3\times 2}$.

Answer. The basis of $M_{3\times 2}$ has six vetors,

$$\mathbf{e_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{e_2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{e_3} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathbf{e_4} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{e_5} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{e_6} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

□

Exercise 2.2. True or false:

a) Any set containing a zero vector is linearly dependent;

b) A basis must contain $\mathbf{0}$;

c) subsets of linearly dependent sets are linearly dependent;

d) subsets of linearly independent sets are linearly independent;

e) if $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v_2} + \cdots + \alpha_n\mathbf{v_n} = 0$ then all scalars α_k are zero.

Answer.

a) **True.** because $\mathbf{0}$ can be represented as a linear combination of the other vectors (simply put all the scalars to 0).

b) **No.** if so, they must be linearly dependent, which is not a base.

c) **No.** Take for example the system of linearly dependent $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ where $\mathbf{e_1} = (1,0)$, $\mathbf{e_2} = (0,1)$ and $\mathbf{e_3} = (1,1)$. The subset $\{\mathbf{e_1}, \mathbf{e_2}\}$ is a basis, which is clearly not linearly dependent.

d) **True.** Supppose that the system $\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$ is a subset of the linearly independent system $\{\mathbf{v_1}, \dots, \mathbf{v_p}, \dots, \mathbf{v_n}\}$. Let α_k the real numbers such that $\alpha_1 \mathbf{v_1} + \cdots + \alpha_p \mathbf{v_p} = \mathbf{0}$ hence

$$\alpha_1 \mathbf{v_1} + \cdots + \alpha_p \mathbf{v_p} + 0\mathbf{v_{p+1}} + \cdots + 0\mathbf{v_n} = \mathbf{0}.$$

Because the system $\{\mathbf{v_1}, \dots, \mathbf{v_p}, \dots, \mathbf{v_n}\}$ is linearly independent, therefore all the scalars $\alpha_k = 0$. Thus, the system $\{\mathbf{v_1}, \dots, \mathbf{v_p}\}$ is also linearly independent.

e) **No.** It's true only when $\{\mathbf{v_1}, \dots, \mathbf{v_n}\}$ are linearly independent.

$\square$

Exercise 2.3. Recall, that a matrix is called *symmetric* if $A^{\mathrm{T}} = A$. Write down a basis in the space of *symmetric* 2×2 matrices (there are many possible answers). How many elements are there in the basis.

Answer. A basis of of this space is $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ where

$$\mathbf{e_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{e_2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{e_3} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

There are 3 vectors in this basis. $\square$

Exercise 2.4. Write down a basis for the space of

a). 3×3 symmetric matrices;

b). $n \times n$ symmetric matrices;

c). $n \times n$ antisymmetric matrices.

Answer.

a). the basis of 3×3 matrices contains 6 vectors, namely

$$\mathbf{e_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{e_4} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e_5} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{e_6} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

b). For the $n \times n$ symmetric matrices, we are going to prove that the basis of this space contains $\frac{n(n+1)}{2}$ vectors. It's easy to see that this is true for $n = 2$ and $n = 3$. Let's assume this is true for some integer n. Therefore, the numbers of vectors in a basis of an $(n+1) \times (n+1)$ symmetric matrices is

□

Exercise 2.5. Let a system of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be linearly independent but not generating. Show that it is possible to find a vector $\mathbf{v}_{r+1}$ such that the system $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}$ is linearly independent.

Proof. Because the system $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ is not generating, therefore there exists a vector $\mathbf{v}_{r+1}$ such that $\mathbf{v}_{r+1}$ cannot be represented as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$. Let α_i be the scalars such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_r \mathbf{v}_r + \alpha_{r+1} \mathbf{v}_{r+1} = \mathbf{0} \tag{1.1}$$

Now we have to prove that all the scalars are all zero. If $\alpha_{r+1} \neq 0$ then

$$\mathbf{v}_{r+1} = -\sum_{i=1}^{r} \frac{\alpha_i}{\alpha_{r+1}} \cdot \mathbf{v}_i,$$

meaning $\mathbf{v}_{r+1}$ is the linear combination of the other vectors, a contradiction. Hence α_{r+1} must equals to zero. So the $r+1$ term in the equation (1.1) vanishes. And because the system $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ is linearly independent, all the scalars $\alpha_i = 0$ for all $i = 0, 1, \ldots, r$. Thus, the system

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}$$

is also *linearly independent*. □

Exercise 2.6. Is it possible that vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly dependent, but the vectors $\mathbf{w_1} = \mathbf{v_1} + \mathbf{v_2}$, $\mathbf{w_2} = \mathbf{v_2} + \mathbf{v_3}$ and $\mathbf{w_3} = \mathbf{v_3} + \mathbf{v_1}$ are linearly *independent*.

Proof. It's not possible, and we're going to prove this assertion via contradiction. Assume that there are such vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ satisfying the above conditions. Then there are numbers $x, y, z \in \mathbb{R}$ such that

$$|x| + |y| + |z| > 0 \quad \text{and} \quad x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3 = \mathbf{0}.$$

By letting

$$a = x + y - z, \quad b = y + z - x, \quad c = z + x - y$$

we obtain that

$$\begin{aligned} a\mathbf{w}_1 + b\mathbf{w}_2 + c\mathbf{w}_3 &= (x\mathbf{w}_1 + y\mathbf{w}_1 - z\mathbf{w}_1) + (y\mathbf{w}_2 + z\mathbf{w}_2 - x\mathbf{w}_2) \\ &\qquad + (x\mathbf{w}_3 + z\mathbf{w}_3 - y\mathbf{w}_3) \\ &= 2x\mathbf{v}_1 + 2y\mathbf{v}_2 + 2z\mathbf{v}_3 \\ &= \mathbf{0}. \end{aligned}$$

Since $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ are linearly independent, we must have $a = b = c = 0$. Hence

$$\begin{cases} x + y - z = 0 \\ y + z - x = 0 \\ z + x - y = 0 \end{cases}$$

adding all the 3 eqations, $x + y + z = 0$. Substituting back to the system of eqations above we get

$$x = y = z = 0$$

which contradicts to the fact that $|x| + |y| + |z| > 0$. $\square$