Problems in Real Analysis

Sivmeng HUN

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Chapter 1

Real Numbers

Example 1.1. Let A be nonempty and bounded below. Prove that $\inf(A)$ exists.

Proof 1. Define $\mathcal{L}(A)$ to be the set of all lower bounds of A. Notice that $\mathcal{L}(A)$ is bounded above. Of course, let fix $a \in A$, therefore $b \leq a$ for all $b \in \mathcal{L}(A)$. This implies that a is an upper bound of $\mathcal{L}(A)$, and thus bounded above. By AoC, it has supremum. Let's denote it's supremum by m. Thus

$$m \leq a, \quad \forall a \in A.$$

If m_0 arbitrary lower bound of A, then $m_0 \in \mathcal{L}(A)$, then we must have $m_0 \leq \sup(\mathcal{L}(A)) = m$. This implies that $m = \sup(\mathcal{L}(A))$ is the greatest lower bound of A. Moreover, $\inf(A) = \sup(\mathcal{L}(A))$. $\square$

Proof 2. Let $B := \{-a \; : \; a \in A\}$. Let ℓ be arbitrary lower bound of A. Then $\ell \leq a \iff -\ell \geq -a$ for all $a \in A$. This implies that B is bounded above. By AoC, let $-s := \sup B$. We conclude that

$$\begin{cases} -s \geq -a, & \forall a \in A \\ \text{if } \forall a \in A, \; -s_0 \geq -a \implies -s_0 \geq s. \end{cases}$$

Muliply both eqations by -1, we obtain that s is the greatest lower bound of A. Moreover $\sup(-A) = -\inf(A)$. $\square$