Chapter 2

Systems of linear equations

1. Different faces of linear transformation

2. Solution of a linear system. Echelon forms

Exercise 2.1. Write the systems of equations below in matrix form.

Exercise 2.2. Find all solutions of the vector equation

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + x_3\mathbf{v_3} = \mathbf{0}$$

where $\mathbf{v_1} = (1,1,0)^T$, $\mathbf{x_2} = (0,1,1)^T$ and $\mathbf{v_3} = (1,0,1)^T$. What conclusion can you make about linear independence (dependence) of the system of vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.

Proof. The echelon form of the system is

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

so, clearly the solution to this equation is $x_1 = x_2 = x_3 = 0$.

Conclusion. If the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independence, then the above equation has unique solution, namely $x_1 = x_2 = x_3 = 0$.

If the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly dependence, then there exists α, β, γ (some of them are non-zero) such that

$$\alpha\mathbf{v_1} + \beta\mathbf{v_2} + \gamma\mathbf{v_3} = \mathbf{0}$$

Therefore, the solution to the above equation is

$$\begin{cases} x_1 = \alpha t \\ x_2 = \beta t \\ x_3 = \gamma t \end{cases}$$

for some $t \in \mathbb{R}$. □

3. Analyzing pivots

Exercise 3.6. Prove or disprove. If the columns of a square $(n \times n)$ matrix are linearly independence, so are the columns of A^2.

Proof. Since A is a square matrix, and columns vectors are linearly independence, so

$$\begin{aligned} & \text{has pivot in every column} \\ \Longrightarrow\ & \text{has pivot in evry row} \\ \Longrightarrow\ & A \text{ is invertable} \\ \Longrightarrow\ & A^2 \text{ also invertable} \\ \Longrightarrow\ & A^2 \text{ has pivot every column and row} \end{aligned}$$

Therefore, the column vectors of A^2 also independence. □

Exercise 3.7. Prove or disprove. If the columns of a square $(n \times n)$ matrix are linearly independence, so are the columns of A^3.

Exercise 3.8. Show that if the equation $A\mathbf{x} = \mathbf{0}$ has unique solution, then A is left invertable.

Proof. Because A has unique solution, then it has pivot in every column. Therfore, #col $\leq$ #row, so we let $m \times n$ be the size of A where $n \leq m$ Let R be the reduced echelon form of A, hence there exists $E = E_k \cdots E_2 E_1$ such that $R = EA$. Observe that, R would look like

$$R = \begin{pmatrix} \boxed{1} & 0 & \cdots & 0 \\ 0 & \boxed{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \boxed{1} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Using matrix muliplication gives us $R^{\mathrm{T}} R = I_n$. We obtain that

$$R^{\mathrm{T}} EA = R^{\mathrm{T}} T = I_n$$

Therfore, $\boxed{A \text{ is left invertable, and } R^{\mathrm{T}} E \text{ is its left inverse}}$.

□

4. Find A^{-1} by row reduction

5. Dimension

Exercise 5.1. True or false.

a). Every vector space that is generated by a finite set has a basis;

b). Every vector space has a (finite) basis;

c). A vector space cannot have more that one basis;

d). A vector space has a finite basis, then the number of vectors in every basis is the same;

e). The dimension of $\mathbb{P}_n$ is n;

f). The dimension of $M_{m\times n}$ is $m+n$;

g). If vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ generate (span) the vector space V, then every vector in V can be written as a linear combination of vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ in only one way;

h). Every subspace of a finite-dimensional space is finite-dimensional.

i). If V is a vector space having dimension n, then V has exactly one subspace of dimension 0, and exactly one subspace of dimension n.

Proof. a). **True.** That finite set which generated a vector space is the spanning set itself. Since it's finite, it contains a basis.

b). **False.** Take $\mathbb{R}[x]$ for example.

c). **False.** In $\mathbb{R}^2$, one can choose

$$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} \quad \text{or} \quad \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$$

as a basis.

d). **True.** As proved in the above theorems,

$$\begin{aligned}&\#\text{ independence vectors} \leq \dim V\\ &\#\text{ generating vectors} \geq \dim V\end{aligned}$$

hence the number of any basis in V must be exactly $\dim V$ vectors.

e). **False.** In $\mathbb{P}_n$, the standard basis is

$$1,\ t,\ t^2, \ldots, t^n$$

which has $n+1$ vectors. Hence $\boxed{\dim \mathbb{P}_n = n+1}$.

f). **False.** The standard basis in $M_{m\times n}$ is

$$\{\mathbf{e}_{11}, \mathbf{e}_{12}, \ldots, \mathbf{e}_{mn}\}$$

has $m \times n$ vectors. Hence $\boxed{\dim M_{m\times n} = mn}$.

g). **False.** span doesn't guarantee uniqueness.

h). **True.** Let W be a subspace of V. Because $\dim V$ finite, we can find

$$\mathcal{A} = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\} \subset V$$

that spans V. WLOG, we assume that none of vectors in $\mathcal{A}$ belongs to W (the unluckiest case.) We can choose $\mathbf{w_1} \in W$ such that $\mathbf{w_1} \neq \mathbf{0}$. Then

$$\mathbf{w_1} = \alpha_1 \mathbf{v_1} + \dots + \alpha_n \mathbf{v_n}$$

Because $\mathbf{w_1} \neq 0$, we're sure some of the α_i's are non-zero, say α_1. Then the new system

$$\mathcal{A}_1 = \{\mathbf{w_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$$

still spans the space V. Now, if $\mathcal{B}_1 := \{\mathbf{w_1}\}$ doesn't span W, we can repeat the above procedure and find $\mathbf{w_2}$. We can do this at most n times, because once we reach the nth step, we have the new system $\mathcal{A}_n \subset W$ that spans the whole space V.

Therfore, after some finite $k \leq n$ step, we have

$$\mathcal{B}_k = \{\mathbf{w_1}, \mathbf{w_2}, \dots, \mathbf{w_k}\} \subset W$$

spans W. Hence, W is finite dimensional.

i). **Not sure.**

□

Exercise 5.2. Prove that if V is a vector space having dimension n, then a system of vectors $\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}$ in V is linearly independent iff it spans V.

6. Change of basis

Exercise 6.3. Find the change of coordinates matrix that changes the coordinates in basis $\{1, 1+t\}$ in $\mathbb{P}_1$ to the coordinates in the basis $\{1-t, 2t\}$.

Proof. Let's denote $\mathcal{A} = \{1, 1+t\}$ and $\mathcal{B} = \{1-t, 2t\}$. Let $\mathcal{S}$ be the standard basis in $\mathbb{P}_1$. Therefore, the matrix that transforms from vector in basis $\mathcal{A}$ to basis $\mathcal{B}$ is $[\mathcal{BA}] = [\mathcal{BS}][\mathcal{SA}]$. We have

$$[\mathcal{SA}] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$[\mathcal{BS}] = [\mathcal{SB}]^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

Therefore,

$$[\mathcal{BA}] = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

□

Exercise 6.4. Let T be the ...

Proof. In standard basis, T looks like

$$[T] = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}.$$

Let $\mathcal{B} = \{(1,1)^{\mathrm{T}}, (1,2)^{\mathrm{T}}\}$. In basis $\mathcal{B}$, the transformation would look like

$$[T]_{\mathcal{B}\mathcal{B}} = [\mathcal{B}\mathcal{S}][T][\mathcal{S}\mathcal{B}]$$

And we have

$$[\mathcal{S}\mathcal{B}] = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

so

$$[\mathcal{B}\mathcal{S}] = [\mathcal{S}\mathcal{B}]^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

Therefore, the transformation $[T]_{\mathcal{B}\mathcal{B}}$ in basis $\mathcal{B}$ is

$$[T]_{\mathcal{B}\mathcal{B}} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

□

Exercise 6.5. Prove that if A and B are similar matrices, then trace $A =$ trace B.

Proof. Because A and B are similar, then there exists an invertable matrix Q such that $A = Q^{-1}BQ$. Observe that

$$A = Q^{-1} \cdot BQ \quad \text{and} \quad B = BQ \cdot Q^{-1}$$

This implies that trace $A =$ trace B. □