Complex Analysis Notes

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1 Conformal Mapping

A conformal map is a holomorphic and a bijection. It turned out that its inverse is then automatically holorphic. To prove this, consider the following:

Lemma 1. *If $f : U \to V$ is holomorphic and injective, then $f'(z) \neq 0$.*

Proof. Assume by contradiction that there is $f'(z_0) = 0$ for some $z_0 \in U$. Then we can choose small R so that $f'(z) \neq 0$ for all $z \neq z_0$ and $z \in D_R(z_0)$, and f has power series

$$\begin{aligned} f(z) &= f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \\ &= f(z_0) + a(z - z_0)^k + G(z) \end{aligned}$$

converges in the disc $\overline{D_R}(z_0)$, where $a \neq 0, k \geq 2$ and $G(z)$ has zero at $z = z_0$ of oder $k + 1$. Next we choose w small such that $|G(z)| < |a(z - z_0)^k - w|$ on some smaller circle $|z - z_0| = r < R$.

- Write $G(z) = (z-z_0)^{k+1}H(z)$ where $H(z)$ is also holomorphic, then it has a maximum on the closure $\overline{D_R}(z_0)$, say $|H(z)| < M$ for all $z \in D_R(z_0)$. Then we have

  $$|G(z)| = |(z - z_0)^{k+1}H(z)| < r^{k+1}M$$

  on the disc $|z - z_0| \leq R$.

- Denote $F(z) := a(z - z_0)^k - w$, where we choose $r < R$ so that $r < \frac{|a|}{2M}$, and w so that $w = \frac{a}{2}r^k$, then for $|z - z_0| = r$

  $$\begin{aligned} |F(z)| &\geq ||a|r^k - |w|| \\ &= \frac{|a|}{2}r^k > Mr^{k+1} > |G(z)| \end{aligned}$$

Thus we've found $r < R$ such that $|G(z)| < |F(z)|$ on the circle $|z - z_0| = r$, moreover $F(z) = a(z - z_0)^k - \frac{a}{2} r^k$ has at least two zeros inside this circle, then by Rouché's theorem

$$f(z) - f(z_0) - w = F(z) + G(z)$$

has at least two zero inside this small circle. Let $\alpha \in D_r(z_0), \alpha \neq z_0$ be a zero of $f(z) - f(z_0) - w$. Because f is injective, then α has to be a double zero, then we can write $f(z) = (z - \alpha)^2 J(z)$, and differentiating both sides gives us $f'(w_0) = 0$ which contradicts the fact $f'(z) \neq 0$. $\square$

So if $f : U \to V$ is conformal, then let $g = f^{-1}$ denote the inverse of f. Suppose $w_0 := f(z_0) \in V$ and $w := f(z)$ is close to w_0, then if $w \neq w_0$

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\frac{w - w_0}{g(w) - g(w_0)}} = \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}}.$$

Since $f'(z_0) \neq 0$, we may let $z \to z_0$ and conclude that $g'(w_0) = 1/f'(g(w_0))$. Hence the invere f^{-1} is also holomorphic.

Theorem 2. *The upper half-plane is conformally equivalent to the unit circle by the map*

$$F(z) := \frac{i - z}{i + z}.$$

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2 Automorphisms of the disc

We say a conformal map from an open set Ω to itself an *automorphism* of Ω. In this section, we want to characterize all the automorphims of the unit disc, denoted by $\mathrm{Aut}(\mathbb{D})$. Some important maps in $\mathrm{Aut}(\mathbb{D})$ are:

- the map $\mathrm{rot}_\theta : z \mapsto e^{i\theta} z$.
- the map $\psi_\alpha : z \mapsto \frac{\alpha - z}{1 - \overline{\alpha} z}$, where $\alpha \in \mathbb{D}$. This map ψ_α interchanges 0 and α, namely $\psi_\alpha(0) = \alpha$ and $\psi_\alpha(\alpha) = 0$. Often it's called *Blaschke factors*.

The result in this chapter is that the maps rot_θ and ψ_α exhaust all of $\mathrm{Aut}(\mathbb{D})$. Before proving this result, let us first present

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Lemma 3 (Schwarz Lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$, and if some $z_0 \neq 0$ is such that $|f(z_0)| = |z_0|$ then f is a rotation.*

Proof. Expand power series of f centered at $z = 0$,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

where the series converges in all of $\mathbb{D}$. Since $f(0) = 0$ we get $a_0 = 0$, and therefore $f(z)/z$ has removable singularity at $z = 0$, hence it's holomorphic. Let $r < 1$. Then on the circle $|z| = r$ we have

$$\left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}.$$

By Maximum Modulus principle, we conclude that this expression is true for all $|z| \leq r$. Letting $r \to 1$, we get $|f(z)| \leq |z|$.

Now if there is such a $z_0 \in \mathbb{D}$ for which $|f(z_0)| = |z_0|$, that means $f(z)/z$ attains maximum in the interior of $\mathbb{D}$, thus it must be constant, i.e.

$$f(z) = cz, \quad \text{for some} c \in \mathbb{C}$$

Replacing $z = z_0$ then take absolute value we obtain $|c| = 1$, thus we can write $c = e^{i\theta}$. Therefore, $f(z) = e^{i\theta} z$ is a rotation. □

The corollary of the Schwarz lemma is of out great interest, it states:

Lemma 4. *Any automorphism of $\mathbb{D}$ that fixes the origin is a rotation.*

Proof. Let $f \in \mathrm{Aut}(\mathbb{D})$ that fixes the origin. Because both f and f^{-1} are holomorphic and because $f(0) = 0$, $f^{-1}(0) = 0$, applying Schwarz lemma gives us

$$|f(z)| \leq |z|,$$

and for any $w = f(z)$,

$$|f^{-1}(w)| \leq |w| \implies |z| \leq |f(z)|,$$

we conclude that $|f(z)| = |z|$. Thus f is a rotation. □

Now we are ready to prove the result promised above.

Theorem 5. *If $f \in \mathrm{Aut}(\mathbb{D})$, then there exist $\alpha \in \mathbb{D}$ and $\theta \in \mathbb{R}$ such that*

$$f(z) = (\mathrm{rot}_\theta \circ \psi_\alpha)(z) = e^{i\theta} \cdot \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Proof. Since $f \in \mathrm{Aut}(\mathbb{D})$, then there is a unique $\alpha \in \mathbb{D}$ so that $f(\alpha) = 0$. Then the map $g := f \circ \psi_\alpha$ is in $\mathrm{Aut}(\mathbb{D})$ and $g(0) = 0$. By Schwarz lemma, any automorphism that fixes the origin is a rotation, then we obtain that $g(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. Replace z by $\psi_\alpha(z)$, we deduce that

$$e^{i\theta}\psi_\alpha(z) = g(\psi_\alpha(z)) = f(\psi_\alpha(\psi_\alpha(z))) = f(z).$$

Thus we conclude that $f = \mathrm{rot}_\theta \circ \psi_\alpha$ for some $\alpha \in \mathbb{D}$ and $\theta \in \mathbb{R}$. □

3 Automorphisms of the upper half-plane

Previously we've seen that the upper half-plane $\mathbb{H}$ could be mapped conformally to the unit circle $\mathbb{D}$ by the map $F : z \mapsto \frac{i-z}{i+z}$. It turned out that automorphisms of $\mathbb{H}$ could be identified by automorphisms of $\mathbb{D}$. And moreover $\mathrm{Aut}(\mathbb{D}) \simeq \mathrm{Aut}(\mathbb{H})$.

To see why this is true, consider the map $\Gamma : \mathrm{Aut}(\mathbb{D}) \to \mathrm{Aut}(\mathbb{H})$ by

$$\Gamma(\varphi) = F^{-1}\varphi F$$

conjugation by F. It can be shown that Γ is indeed an isomorphism between these two groups. Next we want to characterize $\mathrm{Aut}(\mathbb{H})$ like we did for $\mathrm{Aut}(\mathbb{D})$ earlier. But first we define

$$\mathrm{SL}_2(\mathbb{R}) := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ a, b, c, d \in \mathbb{R} \text{ and } \det M = 1 \right\}$$

called the *special linear group.* Given a matrix $M \in \mathrm{SL}_2(\mathbb{R})$ we define the mapping f_M by

$$f_M(z) = \frac{az + b}{cz + d}.$$

Theorem 6. *Every automorphism of $\mathbb{H}$ takes the form f_M for some $M \in SL_2(\mathbb{R})$. Conversely, every map of this form is an automorphism of $\mathbb{H}$.*

Proof.

($\Leftarrow$) "*every map of this form is an automorphism of $\mathbb{H}$*"

For $M, N \in \mathrm{SL}_2(\mathbb{R})$, tedious calculations showed that

- ▷ $\mathrm{Im}(f_M(z)) > 0$ i.e. f_M maps the upper half-plane to itself.
- ▷ $f_M \circ f_N = f_{MN}$
- ▷ $(f_N)^{-1} = f_{N^{-1}}$

As a consequence, f_M is an automorphism of $\mathbb{H}$.

($\Rightarrow$) "*every automorphim of $\mathbb{H}$ takes the form f_M*"

First, let $f \in \mathrm{Aut}(\mathbb{H})$. Then there is a unique $\beta \in \mathbb{H}$ such that $f(\beta) = i$. We then claim the following facts, which we won't give any proof for the moment.

Claim 1. There is a matrix $N \in \mathrm{SL}_2(\mathbb{R})$ such that $f_N(i) = \beta$.

Claim 2. Every rotation in $\mathbb{D}$ corresponds to $F f_M F^{-1}$ for some $M \in \mathrm{SL}_2(\mathbb{R})$.

With these claims at hand, we can proceed the proof as follows: Let $g = f \circ f_N$, then $g(i) = i$, and therefore FgF^{-1} is an automorphism of the disc that fixes the origin. By Schwarz lemma, it must be a rotation. Thus by the second claim, we conclude that $FgF^{-1} = Ff_MF^{-1}$ for some $M \in \mathrm{SL}_2(\mathbb{R})$. Hence $g = f_M$, i.e.

$$f = f_M(f_N)^{-1} = f_M f_{N^{-1}} = f_{MN^{-1}}$$

as desired. $\square$