Chapter 1

Introduction

1.1 Complex Number

Theorem 1. *The polynomial with complex-valued coefficient*

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

has at root in $\mathbb{C}$*.*

Proof. We break into two parts; firstly we show that $|p(z)|$ has a minimum, that is there exists a $z_0 \in \mathbb{C}$ such that $|p(z_0)| \leq |p(z)|$ for any $z \in \mathbb{C}$. Next, we show that z_0 is indeed the root of p by arguing that the case when $|p(z_0)| \neq 0$ is not possible, this forces $|p(z_0)| = 0$ and hence $p(z_0) = 0$.

Claim 1. There is an $R > 0$ such that $|p(z)| \geq |a_0|$ for any $z \in \mathbb{C} \setminus D_R(0)$.

Proof: First, let's denote $A = \max\{|a_{n-1}|, \ldots, |a_0|\}$, and we choose

$$R := \max\{1,\ A(n+1)\}.$$

Thus for any $|z| > R$, we have

$$\begin{aligned}
|p(z)| &= |z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| \\
&\geq |z^n| - |a_{n-1}z^{n-1} + \cdots + a_1 z + a_0| && \text{(Triangle inequality)} \\
&\geq |z|^n - (|a_{n-1}||z^{n-1}| + \cdots + |a_1|z + |a_0|) \\
&\geq |z|^n - M(|z|^{n-1} + \cdots + |z| + 1) \\
&\geq R^n - M(R^{n-1} + \cdots + R + 1) && \text{(becuse } |z| > R\text{)} \\
&\geq R^n - M(\underbrace{R^{n-1} + \cdots + R^{n-1} + R^{n-1}}_{n}) && \text{(because } R > 1\text{)} \\
&= R^{n-1} \cdot (R - Mn) \\
&\geq 1 \cdot (M(n+1) - Mn) = M > |a_0|
\end{aligned}$$

Thus the claim is proved. □

This claim helps us to say that $|p|$ has a minimum. To see why, observe that $D_R(0)$ is compact, and because $|p|$ is continues, then $\{|p(z)| \ : \ z \in D_R(0)\}$ is also compact. Thus it contains its minimum, say z_0, that is $|p(z_0)| \leq |p(z')|$ for any $z' \in D_R(0)$. But $0 \in D_R(0)$, hence $|p(z_0)| \leq |p(0)| = |a_0|$. Now for abitrary $z \notin D_R(0)$, that is $|z| > R$, we have $|p(z)| \geq |a_0|$ by the above claim. Hence $|p(z_0)| \leq |a_0| \leq |p(z)|$.

This shows that $|p|$ has a global minimum value at $z_0 \in \mathbb{C}$, i.e. $|p(z_0)| \leq |p(z)|$ for any $z \in \mathbb{C}$.

Claim 2. For that minimum value z_0, $|p(z_0)| = 0$.

Proof: Assume on the contrary that $|p(z_0)| \neq 0$. Let $z = z_0 + u$ for some $u \in \mathbb{C}$, then

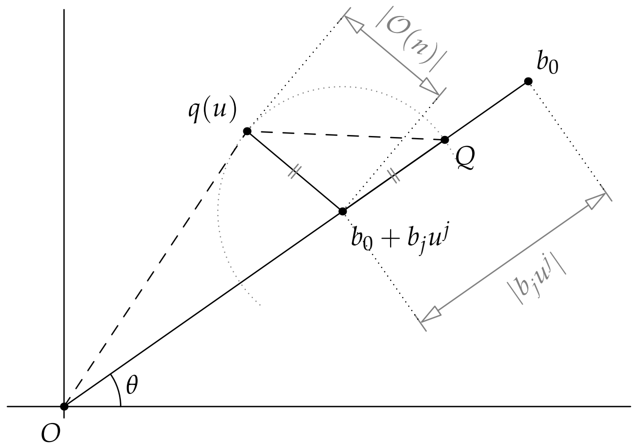
$$\begin{aligned} q(u) &:= p(z_0 + u) \\ &= (z_0 + u)^n + a_{n-1}(z_0 + u)^{n-1} + \cdots + a_1(z_0 + u) + a_0 \\ &= u^n + b_{n-1}u^{n-1} + \cdots + b_1 u + b_0 \end{aligned}$$

where $b_0 = z_0^n + a_{n-1}z_0^{n-1} + \cdots + a_1 z_0 + a_0 = p(z_0)$. Let $j > 0$ the the smallest index such that $b_j \neq 0$. Thus we can rewrite

$$\begin{aligned} q(u) &= b_0 + b_j u^j + (b_{j+1}u^{j+1} + \cdots + u^n) \\ &=: b_0 + b_j u^j + \mathcal{O}(u) \end{aligned}$$

Now let's suppose that $b_0 = r_0 e^{i\theta}$. Our goal is to find $u = re^{i\phi}$ such that $|q(u)| < |p(z_0)|$. If we could such such u, then we would derive to a contradiction. We want to choose r and ϕ such that

$$\begin{cases} b_j u^j \parallel b_0 |b_j u^j| < |b_0|; \\ |\mathcal{O}(u)| < |b_j u^j| \end{cases}$$



- How to choose r?

  Since we want $|b_j u^j| < |b_0|$, we can just choose $r < \left|\frac{b_0}{b_j}\right|^{\frac{1}{j}}$. Next we also want $|\mathcal{O}(u| < |b_j|r^j$. Observe that

$$|\mathcal{O}(u)| \leq |b_{j+1}|r^{j+1} + \cdots + |b_{n-1}|r^{n-1} + r^n$$

So, if we choose $r < 1$ and denote $B := \max\{|b_{j+1}|, \dots, |b_{n-1}|, 1\}$ then we would obtain that

$$
\begin{aligned}
|\mathcal{O}(u)| &< B(r^{j+1} + \cdots + r^{n-1} + r^n) \\
&< B(\underbrace{r^{j+1} + \cdots + r^{j+1} + r^{j+1}}_{n-j}) \qquad \text{(becase we choose } r < 1) \\
&< B(n-j) r \cdot r^j
\end{aligned}
$$

Finally, if we choose $r < \frac{|b_j|}{B(n-j)}$, then we would have $|\mathcal{O}(u)| < |b_j| u^j$. Thus the choice of r is

$$
r := \min\left\{ \left(\frac{b_0}{b_j}\right)^{\frac{1}{j}}, \frac{|b_j|}{B(n-j)}, 1 \right\}.
$$

- How do we choose ϕ?

  Because $b_j u^j \parallel b_0$, then

$$
\begin{aligned}
& \frac{b_0 + b_j u^j}{|b_0 + b_j u^j|} = \frac{b_0}{|b_0|} \\
\implies \quad & \frac{b_0 + b_j u^j}{|b_0| - |b_j u^j|} = \frac{b_0}{|b_0|} \\
\implies \quad & u^j = \frac{\frac{b_0}{|b_0|}(|b_0| - |b_j| r^j) - b_0}{b_j} \\
\implies \quad & \mathrm{e}^{\mathrm{i}\phi j} = \frac{\frac{b_0}{|b_0|}(|b_0| - |b_j| r^j) - b_0}{b_j r^j} := w \\
\implies \quad & \phi = \frac{1}{j} \arg w
\end{aligned}
$$

Thus we have found $u = r\mathrm{e}^{i\phi}$. From triangle inequality,

$$
\begin{aligned}
p(z) = |q(u)| &\leq |b_0 + b_j u^j| + |\mathcal{O}(u)| \\
&= |OQ| < |b_0| = |p(z_0)|
\end{aligned}
$$

Thus we have found z such that $p(z) < |p(z_0)|$. But this is a contradiction becase z_0 is supposed to be the minimum. Therefore, $|p(z_0)| = 0$. $\square$

So far so good, now we're going to finish this mess by using the last claim we proved. Because $|p(z_0)| = 0$, we obtain that $p(z_0) = 0$ which means that z_0 is a root of p. ;) $\square$