TD n°1 – Set of real numbers

Exercice 1. We say the set $A \subset \mathbb{R}$ is *bounded* provided A is both bounded above and bounded below. Prove that A is bounded $\iff \exists M > 0,\ \forall a \in A,\ |a| \leq M$.

Exercice 2. Determine whether the following sets are bounded below, or bounded above. (Note: You are allowed to use $\lfloor \cdot \rfloor$ in this exercise.)

(1). $A = \{\frac{n}{2} + 1 \colon n \in \mathbb{N}\}$

(2). $B = \{\frac{n}{2} + 1 \colon n \in \mathbb{Z}\}$

(3). $C = \{\frac{1}{n} \colon n \in \mathbb{N}\}$

(4). $D = \{(-1)^n n \colon n \in \mathbb{N}\}$

(5). $E = \{x \in \mathbb{R} \colon x^2 \leq 3\}$ (Indication: For the last exercise even though it is tempting to use the $\sqrt{\cdot}$ in order to solve for x, you are advised *not* to use it. This is because up until now we haven't showed that the *number* $\sqrt{3}$ exist or not.)

Exercice 3 (Finding supremum and infimum)**.** (Note: You are allowed to use $\lfloor \cdot \rfloor$ in this exercise.)

(1). Let $A = (-\infty, 5]$. Prove that $\sup A = 5$.

(2). Let $B = (2, \infty)$. Prove that $\inf B = 2$.

(3). Let $C = (1, 3]$. Prove that $\sup C = 3$ and $\inf C = 1$.

(4). Let $D = \{\frac{1}{n} \colon n \in \mathbb{N}\}$. Prove that $\sup D = 1$ and $\inf D = 0$.

Exercice 4. Let $x \in \mathbb{R}$ be a real number. Prove that there exists an integer $n \in \mathbb{N}$ such that $nx > 1$.

Exercice 5. Let $x, y \in \mathbb{R}$ be real numbers satisfying $x < y$. Prove that there exists an integer $n \in \mathbb{N}$ such that $x + \frac{1}{n} < y$.

Exercice 6 (Existence of roots)**.**

(1). Let $A = \{x \in \mathbb{R} \colon x^2 < 3\}$. Prove that A is bounded above and $(\sup A)^2 = 3$.

(2). Let $n \in \mathbb{N}$ and $a \in \mathbb{R}$ with $a > 0$. We denote the set $A = \{x \in \mathbb{R} \colon x^n < a\}$. Prove that A is bounded above and $(\sup A)^n = a$.

Exercice 7 (Infimum)**.** The goal of this exercise is to prove that any subset of $\mathbb{R}$ that is bounded below has an infimum. Let $A \subset \mathbb{R}$ be a subset that is bounded below. We denote

$$B = \{-a \ : a \in A\}.$$

(1). Prove that B is bounded above. Let $\beta = \sup B$.

(2). Prove that $\inf A = -\beta$. (Thus proving that infimum of any set that is bounded below exists.)

Exercice 8 (Existence of integer part function)**.** Let $x \in \mathbb{R}$. Prove that there exists a unique integer $N \in \mathbb{Z}$ satisfying $N \leq x < N + 1$. We denote this integer by $N = \lfloor x \rfloor$.

Exercice 9. Let $x, y \in \mathbb{R}$ with $y - x > 1$. Prove that there exists an integer $m \in \mathbb{Z}$ such that $x < m < y$.

Exercice 10. Prove that the set $A = \{\frac{p}{2^n} : p \in \mathbb{Z},\ n \in \mathbb{N}\}$ is dense in $\mathbb{R}$.

Exercice 11. Prove that the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.