# Elementary Real Analysis

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32 Novembre 2024

## 1 What is the set $\mathbb{R}$?

In this lecture, we assume there are sets

$$\begin{aligned} \mathbb{N} &= \{1, 2, 3, \dots\} \\ \mathbb{Z} &= \{0, \pm 1, \pm 2, \dots\} \\ \mathbb{Q} &= \{\tfrac{p}{q} \colon p \in \mathbb{Z},\ q \in \mathbb{N}\}. \end{aligned}$$

Moreover, we assume there is another $\mathbb{R}$ that is bigger than $\mathbb{Q}$ and it satisfies something called "completeness property". This property says that

> Completeness Property: Every subset of $\mathbb{R}$ that is bounded above has a least upper bound.

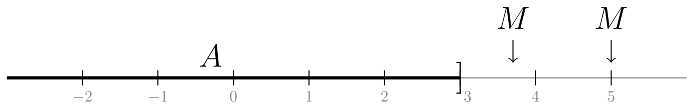
Believe it or not, this short statement could explains all the irrational numbers, numbers like $\sqrt{2}, \sqrt[3]{5}, \pi, \ln 2$, etc. So, it is very important that we spend some time and try to understand what this statements is actually saying. This statement contains the following new notions:

- set that is bounded above;
- upper bound;
- least upper bound.

We will spend quite some time in the next section introducing these new definitions. Please also note that, this is *not* a rigorous introduction to real analysis. The author will quite frequently waving hands and assume some basic properties to be true. In this case, we assume that we could do operations like $+, -, \times, \div$ and comparisions like $>, \geq, <, \leq$ on the set $\mathbb{R}$. That being said, we develop our studies based on the fact that the set $\mathbb{R}$ satisfies completeness property and from there we will prove that $\sqrt{2}$ exists, how to make sense of $5^{\sqrt{2}}$, how to define special numbers like $e, \pi$ et cetera. So, let us begin.

## 1.1 Bounded sets

Look at the set $A = \{x \in \mathbb{R} \colon x \leq 3\} = (-\infty, 3]$. Now let $M = 5$. We observe that if we draw the set in the real line, we see that every number in $A$ is less than $M$. We call the set $A$ "bounded above" and the number $M = 5$ is called an "upper bound" of the set $A$. Note that there are a lot of uppper bounds of $A$, for example $5.2, 4, 3.7$ or even 3 are upper bounds of $A$.




So whenever we talk about set that is bounded above, you are encouraged to picture it just like the one above. Below, we give the formal definition

**Definition 1** (Bounded above)
Let $A \subset \mathbb{R}$ and $M \in \mathbb{R}$.

- We say that $M$ is an *upper bound* of $A$ if $\forall a \in A, a \leq M$.
- We say that the set $A$ is *bounded above* if $A$ has an upper bound; in symbol we mean $\exists M \in \mathbb{R},\ \forall a \in A,\ a \leq M$.

Similarly, there is a notion of bounded below and the notion of lower bound. Imagine the set $A$ and a number $m$ as below.


**Definition 2** (Bounded below)
Let $A \subset \mathbb{R}$ and $m \in \mathbb{R}$.

- We say that $m$ is a *lower bound* of $A$ if $\forall a \in A, a \geq m$.
- We say that the set $A$ is *bounded below* if $A$ has a lower bound; in symbol we mean $\exists m \in \mathbb{R},\ \forall a \in A,\ a \geq m$.

Don't memorise the definition! Try to imagine them as pictures in your head, and formulate which one is smaller, which one is bigger. To better understand the definition, we need to pracise with some examples.

**Example 1.** The set $A = (-\infty, 3]$ is bounded above. To prove this, we need to choose a number $M$ that is an upper bound of $A$. There are a lot of choices of $M$, so any one of them is fine. Here we decided to choose choose $M = 5$ and we see that $\forall a \in A$ we have $a \leq 3 < M$. Therefore $A$ is bounded above.

**Example 2.** Let $A = (-1, 3]$. Is $M = 2.8$ an upper bound of $A$? Looking at the picture, it seems that $M$ is *not* an upper bound of $A$.


Since we are not a kid anymore, we need to prove this using the definition. But, what does it mean for $M = 2.8$ not an upper bound? Think about it with the analogy this way:

- less than or equals to = obey
- 2.8 is upper bound if every numbers in $A$ has to obey 2.8.
- So if *there is* some numbers is $A$ that doesn't obey 2.8, then we can conclude that 2.8 is no longer an uppper bound, i.e.

  $$\text{2.8 not uppper bound} \iff \exists a \in A,\ a > 2.8$$

  In general, $\boxed{M \text{ is } \textit{not} \text{ upper bound of } A \text{ if } \exists a \in A, a > M.}$ This is called negating the definition. It is a good pracise for you to negate a definition whenever you encounter one.

Now, back to our example when $A = (-1, 3]$ and $M = 2.8$. We can see now that 2.8 is not an upper bound since there is an element $a = 2.99 \in A$ such that $a > M$. Note that we can choose a different $a$ as long as it is in $A$ and bigger than $M$. For example we can instead choose $a = 2.81$ or 2.9 or 2.992024 or even 3, et cetera.

**Example 3.** Let $A = (3, +\infty) = \{x \in \mathbb{R} \colon x > 3\}$. Is this set bounded above? Drawing the pictures, you will see that elements of $A$ go forever to the right, which means that $A$ is not bounded above. Again, we are no longer a kiddo. We need to prove this assertion using a definition. Now, what does it mean to be unbounded above? We need to negate the definition!

- $A$ is bounded above means: $A$ has an upper bound.

- So $A$ is unbounded above means: $A$ doesn't have an upper bound. To be more precise, this means that any number $M \in \mathbb{R}$, this number is never ever an upper bound of $A$. We get the following.

$$\begin{aligned}
A \text{ unbounded above} &\iff A \text{ has no upper bound} \\
&\iff \forall M \in \mathbb{R},\ M \text{ is not an upper bound of } A \\
&\iff \boxed{\forall M \in \mathbb{R},\ \exists a \in A,\ a > M.}
\end{aligned}$$

Let's imagine the boxed statement as a game between you and me. The game goes as follow: I give you a number $M$, you must respond by choosing a number $a \in A$ that is bigger than $M$. Ready?

- If I give you $M = 10$, what is your response? Well, you have a lot of response! You can choose $a = 11$ or $12$ or $10.1291$. These numbers are all in $A$ and are bigger than $M$.
- If I give you $M = 100$, you can response by simply choosing $a = 101$.
- In the end, you need to have a formula, this formula could tell you what to respond to my challenge $M$.
- If I give you $M$ in general, what is your response? Some of you might think of the response $a = M + 1$. This is fine if the value $M$ is larger than 3.
- If I give you $M = 1$, then the response $a = M + 1$ is no longer valid, because now $a = 2 > M$ but $a \notin A$. But this is easily fixed. If I give you any $M > 3$, you could choose the response $a = M + 1$. But if I give you $M \leq 3$, you can response by choosing $a = 4 \in A$.

Now we prove that the set $A = (3, +\infty)$ is unbounded above. Let $M \in \mathbb{R}$ be an arbitrary number, now

- if $M \geq 3$, we choose $a = M + 1$. We see that $a > M$, and also $a \in A$;
- if $M < 3$, we choose $a = 4$. We see that $a > M$ and $a \in A$.

In both cases we get, $\forall M \in \mathbb{R}, \exists a \in A, a > M$. Therefore, $A$ is unbounded above.