Elementary Real Analysis

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1 Completeness Property

1.1 Relation between $\mathbb{N}$ and $\mathbb{R}$

Here we assume that there exists $\mathbb{N}$ a subset of $\mathbb{R}$ with the following properties:

(i). $1 \in \mathbb{N}$, and is the smallest element in $\mathbb{N}$;

(ii). If $n \in \mathbb{N}$, then $n+1 \in \mathbb{N}$;

(iii). If $n, m \in \mathbb{N}$ such that $n \neq m$, then $|n-m| \geq 1$.

Theorem 1 (Archimedean Property): *The set $\mathbb{N}$ is not bounded above. In other words, for any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.*

Proof. Assume by contradiction that the above statement is wrong, that is the set $\mathbb{N}$ is bounded above. Thus $\alpha = \sup \mathbb{N}$ exists. By definition of supremum, there is $n \in \mathbb{N}$ so that

$$n > \alpha - 1 \implies \alpha < n+1.$$

However, this is a contradiction since $n+1 \in \mathbb{N}$ and α is supposed to be an upper bound of $\mathbb{N}$. $\square$

Theorem 2 (Well-ordering property): *Any non-empty subset $S \subset \mathbb{N}$ has a minimal element; in other words $\min S \in S$.*

Proof. Let $S \subset \mathbb{N}$ and $S \neq \varnothing$. Since $\mathbb{N}$ bounded below, then so does S. We conclude from completeness property that $\alpha = \inf S \in \mathbb{R}$ exists. It suffices to prove that $\alpha \in S$. Argue by contradiction and suppose that $\alpha \notin S$. In particular, α is an a natural number. From definition of infimum, there is an element $s \in S$ so that $\alpha \leq s < \alpha + 1$. Moreover we cannot have $\alpha = s$, since we assumed $\alpha \notin S$. Thus we have found $s \in S$ such that

$$\alpha < s < \alpha + 1.$$

Notice that s is a number that is greater than $\alpha = \inf S$. Using definition of infimum again, we conclude that there is $s' \in S$ with $\alpha \leq s' < s$. Using the same argument as above, we must have $\alpha < s'$. Combining all inequalities, we obtain

$$\alpha < s' < s < \alpha + 1.$$

Hence $0 < s - s' < 1$. This is a contradiction to the (iii) property of $\mathbb{N}$. $\square$

1.2 Relation between $\mathbb{Q}$ and $\mathbb{R}$

Theorem 3 (Density of $\mathbb{Q}$ in $\mathbb{R}$): *For any two distinct real numbers $x, y \in \mathbb{R}$ with $x < y$, there is a rational number $r \in \mathbb{Q}$ satisfying $x < r < y$.*

We say that a set $A \subset \mathbb{R}$ is dense provided that for any $x < y$, there exists $a \in A$ with $x < a < y$. From the above theorem, the set $\mathbb{Q}$ has this exact property. In other words, this means that no matter how you choose x and y, there is always an element in $\mathbb{Q}$ sits between them. Visually, the elements of $\mathbb{Q}$ are densely put into $\mathbb{R}$, that is why we say $\mathbb{Q}$ is dense in $\mathbb{R}$.

The machinery of the proof is to use Archimedean property. To shorten the proof a bit, we are going to use the fact that, for any two numbers that are strictly of distance 1 apart, there is an integer sits between them. This fact is left as an exercise, as you can see in TD n°1.

Proof. Let $x < y$ be real numbers. From Archimedean property, there is an natural number $n \in \mathbb{N}$ such that $n > \frac{1}{y-x}$. Equivalently,

$$ny - nx > 1.$$

Since ny and nx are strictly of distance 1 part, there is an integer $m \in \mathbb{Z}$ such that $nx < m < ny$. Thus

$$x < \frac{m}{n} < y.$$

Therefore, we have found a rational number $\frac{m}{n}$ that is between x and y. This concludes the proof. $\square$