Calculate the implied volatility

Calculate the implied volatility with Bisection method and Newton's method.
Use BS model and Binomial tree as option pricing model.

1 Algorithm

1.1 f(x) = c(sigma) - market price of the option

1.2 c() : BS Model, Binomial Tree

1.3 Inputs: S0, K, r, q, T, market price of the option, n, convergence criterion.
Outputs: Implied volatility

1.4 Build def of BS model and Binomial tree

1.5 Bisection method

1.6 Newton's method

1.6.1 Calculate VEGA as f' of BS model

1.6.2 Calculate slope with interval of 0.00000001 as f' of Binomial tree

- Bisection Method

  First find $[a_n, b_n]$ such that $f(a_n)f(b_n) < 0$. The iterative two steps to find $[a_{n+1}, b_{n+1}]$ are as follows.

  (i) Calcuate $x_n = a_n + \frac{b_n - a_n}{2}$

  (ii) If $f(a_n)f(x_n) < 0 \Rightarrow a_{n+1} = a_n, b_{n+1} = x_n$

  $\text{else} \Rightarrow a_{n+1} = x_n, b_{n+1} = b_n$

- Newton's Method

  $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

  Based on the first-order Taylor series: $f(x) = f(x_n) + f'(x_n)(x - x_n)$
  (Find the root of $f(x)$, i.e., solve $f(x) = 0$.)

  $$\Rightarrow -f(x_n) = f'(x_n)(x - x_n) \Rightarrow x = x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Quadratical convergence:

According to the second-order Taylor series, $f(x) = f(x_n) + f'(x_n)(x - x_n) + \frac{f''(\xi)}{2}(x - x_n)^2$,

where ξ is between x_n and x.

Solve $f(x) = 0$:

$\Rightarrow x - x_n + \frac{f(x_n)}{f'(x_n)} = -\frac{f''(\xi)}{2f'(x_n)}(x - x_n)^2$

$\Rightarrow x - x_{n+1} = -\frac{f''(\xi)}{2f'(x_n)}(x - x_n)^2$

(Suppose $|\frac{f''(\xi)}{2f'(x_n)}|$ is bounded by a finite number M, i.e., $|\frac{f''(\xi)}{2f'(x_n)}| \leq M < \infty$.)

$\Rightarrow |x - x_{n+1}| \leq M\,|x - x_n|^2$

(The error is smaller than the product of a finite number and the square of the error of the n-th iteration.)

- Vega $\upsilon \equiv \frac{\partial c}{\partial \sigma}$
  - ⊙ For calls and puts: $\upsilon = S_0 e^{-qT}\sqrt{T}\phi(d_1) = Ke^{-rT}\sqrt{T}\phi(d_2)$