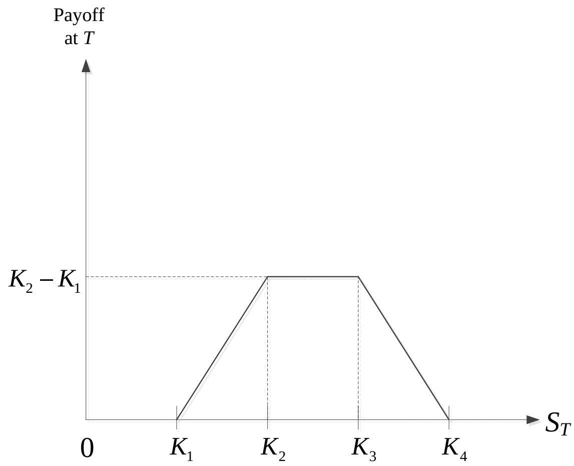
Homework 1

Derive the closed-form formula for an option with the following payoff function:



- Basic requirement (80 points):

  (i) Derive the closed-form formula using the martingale pricing method by hands.
  (ii) Based on the formula you derive, implement a program to price this option.

  (Inputs: S_0, r, q, σ, T, K_1, K_2, K_3, K_4. Output: Option value.)

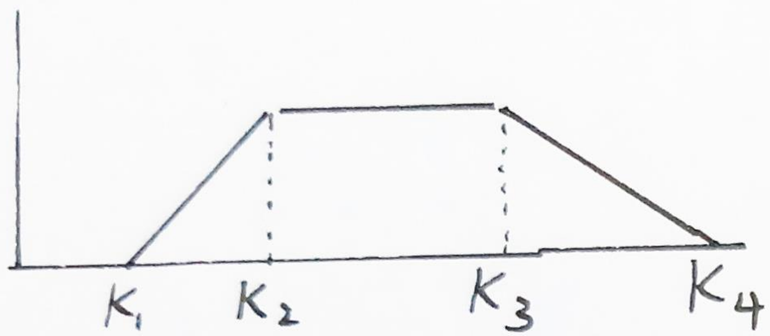
- Bonus (10 points):

  Employ the Monte Carlo simulation to price this option.

  Based on $\ln S_T \sim ND(\ln S_0 + (r - q - \sigma^2/2)T, \sigma^2 T)$, draw 10,000 random samples for S_T to compute an option price. Repeat the above step 20 times to obtain the 95% confidence interval for the option value:

  [mean of 20 repetitions $-$ 2$\times$(s.d. of 20 repetitions), mean of 20 repetitions + 2$\times$(s.d. of 20 repetitions)].

1. 將 payoff 分三段，分別以 Girsanov theorem 推導



第一段： $K_1 \leq S_T < K_2$

第二段： $K_2 \leq S_T < K_3$

第三段： $K_3 \leq S_T < K_4$

第一段： $1_A \begin{cases} 1 & \text{if } K_2 > S_T \geq K_1 \\ 0 & \text{o/w} \end{cases}$ ， $c(S_0, 0) = e^{-rT} \underbrace{E^Q[S_T \times 1_A]}_{(1)} - Ke^{-rT} \underbrace{E^Q[1_A]}_{(2)}$

$$(2) = E^Q[1_A] = Pr^Q(K_2 > S_T \geq K_1) = Pr^Q(\ln K_2 > \ln S_T \geq \ln K_1)$$

$$= Pr^Q\left(\frac{\ln K_2 - \ln S_0 - (r - q - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} > \frac{\Delta Z^Q(T)}{\sqrt{T}} \geq \frac{\ln K_1 - \ln S_0 - (r - q - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$$= Pr^Q\left(\frac{\ln(\frac{S_0}{K_2}) + (r - q - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} < -\frac{\Delta Z^Q(T)}{\sqrt{T}} \leq \frac{\ln(\frac{S_0}{K_1}) + (r - q - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$$= Pr^Q(d_2(K_2) < ND(0,1) \leq d_2(K_1)) = N(d_2(K_1)) - N(d_2(K_2))$$

$\ldots(d_2(k_1))) = N(d_2(k_1)) - N(d_2(k_2)))$

(1) $= S_0 e^{(r-q)T} \cdot E^R[1_A] = S_0 e^{(r-q)T} \cdot P_r^R(k_2 > S_T \geq k_1)$

$= S_0 e^{(r-q)T} P_r^R(\ln k_2 > S_T \geq \ln k_1)$

$$= S_0 e^{(r-q)T} P_r^R\left(\frac{\ln(\frac{S_0}{k_2}) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} < -\frac{\Delta Z^R(T)}{\sqrt{T}} \leq \frac{\ln(\frac{S_0}{k_1}) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$= S_0 e^{(r-q)T} \cdot (N(d_1(k_1)) - N(d_1(k_2)))$

第一段 value $= S_0 e^{-qT}(N(d_1(k_1)) - N(d_1(k_2)) - k_1 e^{-rT}(N(d_2(k_1)) - N(d_2(k_2))))$

第二段: $1_A \begin{cases} 1 & \text{if } k_3 > S_T \geq k_2 \\ 0 & \text{o/w} \end{cases}$, $c(S_0, 0) = \underbrace{E^Q[1_A]}_{(1)} \times (k_2 - k_1)e^{-rT}$

(1) $= E^Q[1_A] = P_r^Q(k_3 > S_T \geq k_2) = P_r^Q(\ln k_3 > \ln S_T \geq \ln k_2)$

$$= P_r^Q\left(\frac{\ln(\frac{S_0}{k_3}) + (r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} < -\frac{\sigma Z^Q(T)}{\sqrt{T}} \leq \frac{\ln(\frac{S_0}{k_2}) + (r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$= P_r^Q(d_2(k_3) < ND(0,1) \leq d_2(k_2)) = N(d_2(k_2)) - N(d_2(k_3))$

第二段 value $= (k_2 - k_1) \times [N(d_2(k_2)) - N(d_2(k_3))] e^{-rT}$

第三段：不知斜率，以 $\frac{k_2-k_1}{K_4-K_3}$ 當斜率

$$1A \begin{cases} 1 & K_4 > S_T \geq K_3 \\ 0 & o/w \end{cases}, \quad c(S_0,0) = E^Q\left[\frac{(K_4-S_T)(k_2-k_1)}{K_4-K_3} \times 1A \right] e^{-rT}$$

$$= \frac{k_2-k_1}{k_4-k_3} e^{-rT} \Big(K_4 \underbrace{E^Q[1A]}_{(1)} - \underbrace{E^Q[S_T \times 1A]}_{(2)} \Big)$$

$$(1) = E^Q[1A] = P_r^Q(K_4 > S_T \geq K_3) = P_r^Q(\ln K_4 > \ln S_T \geq \ln K_3)$$

$$= P_r^Q\left(\frac{\ln(\frac{S_0}{K_4}) + (r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} < -\frac{\Delta Z^Q(T)}{\sqrt{T}} \leq \frac{\ln(\frac{S_0}{K_3}) + (r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \right)$$

$$= P_r^Q(d_2(K_4) < ND(0,1) \leq d_2(K_3)) = N(d_2(K_3)) - N(d_2(K_4))$$

$$(2) = S_0 e^{(r-q)T} P_r^R(\ln K_4 > S_T \geq \ln K_3)$$

$$= S_0 e^{(r-q)T} \times P_r^R\left(\frac{\ln(\frac{S_0}{K_4}) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} < -\frac{\Delta Z^R(T)}{\sqrt{T}} \leq \frac{\ln(\frac{S_0}{K_3}) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \right)$$

$$= S_0 e^{(r-q)T} \cdot (N(d_1(K_3)) - N(d_1(K_4)))$$

第三段 value : $\frac{k_2-k_1}{k_4-k_3} \times \Big(k_4 e^{-rT}[N(d_2(k_3))-N(d_2(k_4))] - S_0 e^{-qT}(N(d_1(k_3)) - N(d_1(k_4))\Big)$

$$(i) = S_0 e^{-qT}(N(d_1(k_1))-N(d_1(k_2))) - k_1 e^{-rT}(N(d_2(k_1))-N(d_2(k_2))) +$$
$$(k_2-k_1) \times [N(d_2(K_2)-N(d_2(k_3)]\, e^{-rT} +$$
$$\frac{k_2-k_1}{k_4-k_3} \times \Big(k_4 e^{-rT}[N(d_2(k_3))-N(d_2(k_4))] - S_0 e^{-qT}(N(d_1(k_3))-N(d_1(k_4)))\Big)$$