Problem Set 3

Problems 1-7 correspond to "Linear algebra I: basic notation and dot products"

Problem 1

1/1 point (graded)

A data set consists of 200 points in $\mathbb{R}^{80}$. If we store these in a matrix, with one point per row, what is the dimension of the matrix?

● 200×80 ✓

○ 80×200

○ 200×1

○ 1×80

Submit

Problem 2

3/3 points (graded)

For $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$, compute

a) $A^T =$

○ $\begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$

○ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

● $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ ✓

○ $\begin{pmatrix} 6 & 3 \\ 5 & 2 \\ 4 & 1 \end{pmatrix}$

b) $A + B =$

○ $\begin{pmatrix} 2 & 2 & 4 \\ 4 & 6 & 6 \end{pmatrix}$

● $\begin{pmatrix} 0 & 2 & 4 \\ 5 & 4 & 6 \end{pmatrix}$ ✓

○ $\begin{pmatrix} 0 & 0 & 2 \\ 5 & 5 & 6 \end{pmatrix}$

○ $\begin{pmatrix} 6 & 3 & 1 \\ 2 & 4 & 7 \end{pmatrix}$

c) $A - B =$

○ $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

● $\begin{pmatrix} 2 & 2 & 2 \\ 3 & 6 & 6 \end{pmatrix}$ ✓

○ $\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 6 \end{pmatrix}$

○ $\begin{pmatrix} 2 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}$

Submit

Problem 3

2/2 points (graded)

Let $x = (1, 0, -1)$ and $y = (0, 1, -1)$.

a) What is $x \cdot y$?

1 ✓

1

b) What is the angle between these two vectors, in degrees (give a number in the range 0 to 180)?

60 ✓

60

Submit

Problem 4

2/2 points (graded)

For each pair of vectors below, say whether or not they are orthogonal.

a) $(1, 3, 0, 1)$ and $(-1, -3, 0, -1)$

not orthogonal ▼ ✓

b) $(1, 3, 0, 1)$ and $(1, 3, 0, -10)$

orthogonal ▼ ✓

Submit

Problem 5

1/1 point (graded)

Find the unit vector in the same direction as $\boldsymbol{x} = (1, 2, 3)$.

○ $(1, 2, 3)/6$

○ $(1, 2, 3)/14$

○ $(1, 2, 3)/\sqrt{7}$

● $(1, 2, 3)/\sqrt{14}$ ✓

Submit

Problem 6

1/1 point (graded)

Find all unit vectors in $\mathbb{R}^2$ that are orthogonal to $(1, 1)$.

☐ $(1, 1)/\sqrt{2}$ and $(-1, -1)/\sqrt{2}$

☐ $(1, -1)$ and $(-1, 1)$

☐ $(1, 1)/2$ and $(-1, -1)/2$

☑ $(1,-1)/\sqrt{2}$ and $(-1,1)/\sqrt{2}$

✓

Submit

Problem 7

1/1 point (graded)

How would you describe the set of all points $x \in \mathbb{R}^d$ with $x \cdot x = 25$? Select all that apply.

☑ All points of ℓ_2 length 5.

☐ The surface of a sphere that is centered at the origin, of radius 25.

☐ All points of ℓ_2 length 25.

☑ The surface of a sphere that is centered at the origin, of radius 5.

✓

Submit

Problems 8-17 correspond to "Linear algebra II: matrix products and linear functions"

Problem 8

1/1 point (graded)

Which of the following is a linear function of $x \in \mathbb{R}^3$? Select all that apply.

☐ $x_1^2 + 3x_2 + x_3$

☑ $2x_1 - 3x_2$

☐ $x_1 x_2 - x_3$

✓

Submit

Problem 9

1/1 point (graded)

True or false: the function $f(x) = 2x_1 - x_2 + 6x_3$ can be written as $w \cdot x$ for $x \in \mathbb{R}^3$, where $w = (2, -1, 6)$.

● True ✓

○ False

Submit

Problem 10

3/3 points (graded)

Consider the linear function that is expressed by the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix}$.

This function maps vectors in $\mathbb{R}^p$ to $\mathbb{R}^q$.

a) What is p?

3 ✓

3

b) What is q?

2 ✓

2

c) Which of the following vectors are mapped to zero?

☑ $(2, -1, 6)$

☑ $(-4, 2, -12)$

☐ $(1, 4, -1)$

☐ $(4, -2, 1)$

✓

Submit

Problem 11

3/3 points (graded)

Compute the product: $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 0 & 1 \\ 2 & 6 & 0 \end{pmatrix}$:

$$= \begin{pmatrix} 1 & a & 1 \\ 14 & b & c \end{pmatrix}$$

$a =$

-6 ✓

-6

$b =$

24 ✓

24

$c =$

5 ✓

5

Submit

Problem 12

4/4 points (graded)

For a certain pair of matrices A, B, the product AB has dimension 10×20. Suppose A has 30 columns.

a) $A \in \mathbb{R}^{m \times n}$

$m =$

10 ✓

10

$n =$

30 ✓

30

b) $B \in \mathbb{R}^{r \times s}$

$r =$

30 ✓

30

$s =$

20 ✓

20

Submit

Problem 13

3/3 points (graded)

We have n data points $x^{(1)}, \ldots, x^{(n)} \in \mathbb{R}^d$ and we store them in a matrix X, one point per row.

a) True or false: X has dimension $d \times n$.

○ True

● False ✓

b) True or false: $X^T X$ has dimension $d \times d$.

● True ✓

○ False

c) Which of the following is a matrix with (i, j) entry $x^{(i)} \cdot x^{(j)}$?

○ XX

○ $X^T X$

● XX^T ✓

○ $X^T X^T$

Submit

Problem 14

1/1 point (graded)

Vector x has length 10. What is $x^T x x^T x x^T x$?

1000000 ✓

1000000

Submit

Problem 15

5/5 points (graded)

Suppose $x = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$.

a) What is $x^T x$?

35 ✓

35

b) What is $x x^T$?

$$x x^T = \begin{pmatrix} 1 & a & b \\ 3 & 9 & c \\ 5 & 15 & d \end{pmatrix}$$

$a =$

3 ✓

3

$b =$

5 ✓

5

$c =$

15 ✓

15

$d =$

25 ✓

25

Submit

Problem 16

1/1 point (graded)

Vectors $x, y \in \mathbb{R}^d$ both have length 2. If $x^T y = 2$, what is the angle between x and y, in degrees (the answer is an integer in the range 0 to 180)?

60 ✓

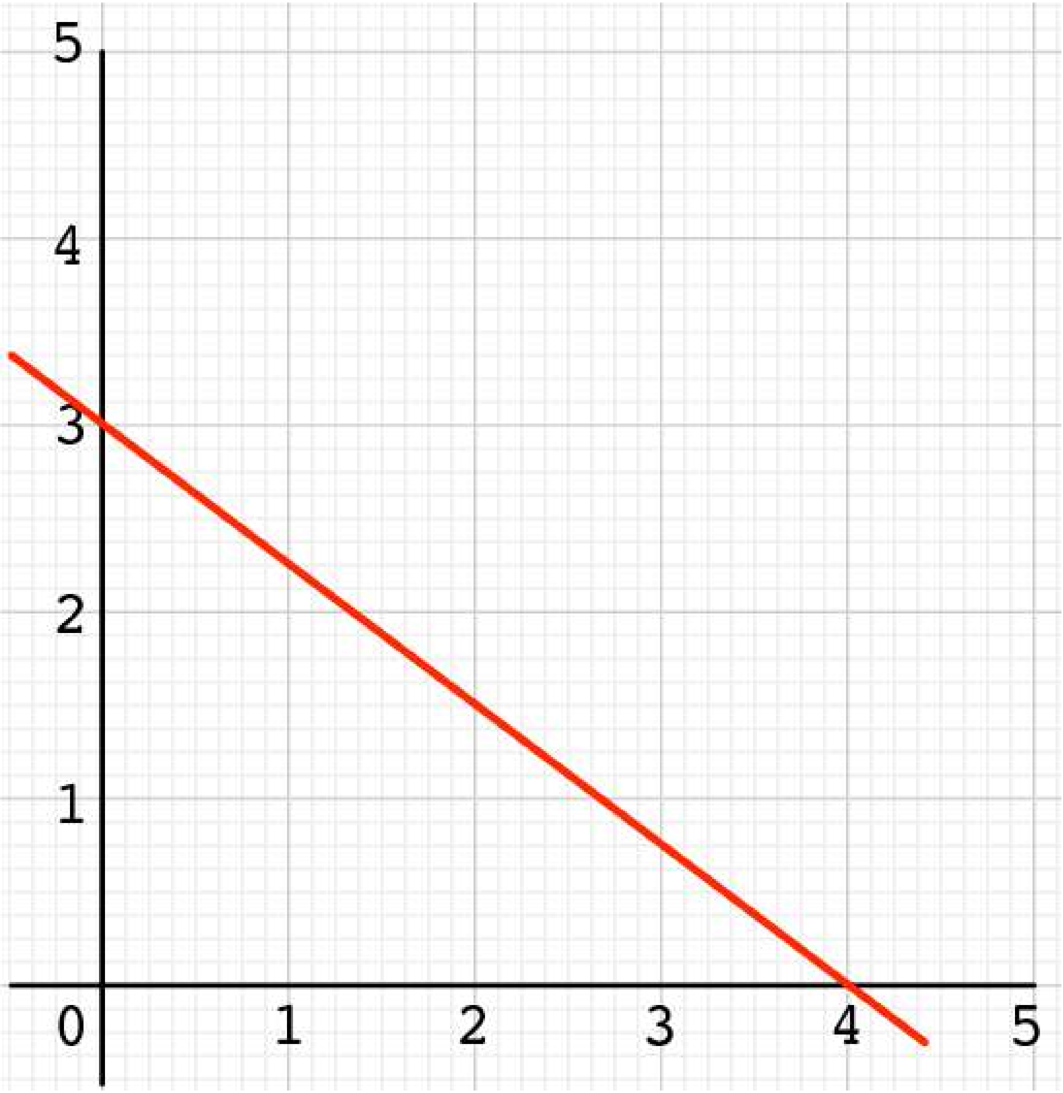
60

Submit

Problem 17

2/2 points (graded)

The line shown below can be expressed in the form $w \cdot x = 12$ for $x \in \mathbb{R}^2$. What is w?



$w = (w_1, w_2)$

$w_1 =$

3 ✓

3

$w_2 =$

4 ✓

4

Submit

Problems 18-24 correspond to "Linear algebra III: square matrices as quadratic functions"

Problem 18

4/4 points (graded)

The quadratic function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x) = 3x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_3^2$$

can be written in the form $x^T Mx$ for some **symmetric** matrix M. What are the missing entries in M?

$$M = \begin{pmatrix} a & 1 & b \\ 1 & c & 0 \\ -2 & d & 6 \end{pmatrix}$$

$a =$

3 ✓

3

$b =$

-2 ✓

-2

$c =$

0 ✓

0

$d =$

0 ✓

0

Submit

Problem 19

7/7 points (graded)

Answer the following questions about the quadratic function $f : \mathbb{R}^3 \to \mathbb{R}$ associated with the matrices A.

a) True or false: for $A = \textbf{diag}(6, 2, -1)$, $f(x_1, x_2, x_3) = 6x_1^2 + 2x_2^2 - x_3^2$.

● True ✓

○ False

b) $A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & -1 & 4 \\ 2 & -2 & 1 \end{pmatrix}$

Find the coefficients for

$f(x_1, x_2, x_3) = ax_1^2 + bx_1x_2 + cx_1x_3 + dx_2^2 + ex_2x_3 + fx_3^2$

$a =$

1 ✓

1

$b =$

4 ✓

4

$c =$

6 ✓

6

$d =$

-1 ✓

-1

$e =$

2 ✓

2

$f =$

1 ✓

1

Submit

Problem 20

0/1 point (graded)

Which of the following matrices is necessarily symmetric? Select all that apply.

☐ AA^T for arbitrary matrix A.

☐ A^TA for arbitrary matrix A.

☐ $A + A^T$ for arbitrary square matrix A.

☑ $A - A^T$ for arbitrary square matrix A.

✗

Submit

Problem 21

2/2 points (graded)

Let $A = \text{diag}(1, 2, 3, 4, 5, 6, 7, 8)$.

a) What is $|A|$?

40320 ✓

40320

b) True or false: $A^{-1} = \text{diag}(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8})$

● True ✓

○ False

Submit

Problem 22

2/2 points (graded)

Vectors $u_1, \ldots, u_d \in \mathbb{R}^d$ all have unit length and are orthogonal to each other. Let U be the $d \times d$ matrix whose rows are the u_i.

a) What is UU^T?

○ U

○ U^T

○ U^{-1}

● I_d ✓

b) What is U^{-1}?

○ U

● U^T ✓

○ U^{-1}

○ I_d

Submit

Problem 23

1/1 point (graded)

Matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & z \end{pmatrix}$ is singular. What is z?

6 ✓

6

Submit

Problem 24

0/1 point (graded)

The *trace* of a $d \times d$ matrix A is defined to be $\mathrm{tr}(A) = \sum_{i=1}^{d} A_{ii}$. Which of the following statements is true, for arbitrary $d \times d$ matrices A, B? Select all that apply.

☑ $\mathrm{tr}(A) = \mathrm{tr}(A^T)$.

☐ $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$.

☐ $\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B)$.

☑ $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

✗

Submit

Problems 25-27 correspond to "The multivariate Gaussian"

Problem 25

1/1 point (graded)

A spherical Gaussian has mean $\mu = (1, 0, 0)$. At which of the following points will the density be the same as at $(1, 1, 0)$? Select all that apply.

☑ $(0, 0, 0)$

☐ $(1, 1, 1)$

☑ $(2, 0, 0)$

☑ $(1, 0, 1)$

✓

Submit

Problem 26

1/1 point (graded)

How many real-valued parameters are needed to specify a diagonal Gaussian in $\mathbb{R}^d$?

○ d

● $2d$ ✓

○ $\frac{1}{2}d^2$

○ d^2

Submit

Problem 27

1/1 point (graded)

A set of random variables $X_1, \ldots, X_d$ has the following properties:

$E(X_i) = 1$ for all i

$E(X_i X_j)$ is 2 whenever $i = j$, $3/2$ whenever $|i - j| = 1$, and 1 otherwise.

What multivariate Gaussian $N(\mu, \Sigma)$ would you fit to the d-dimensional distribution of $X = (X_1, \ldots, X_d)$? Just give μ and Σ in the specific case $d = 4$.

○ $\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{pmatrix}$

○ $\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \end{pmatrix}$

● $$\mu = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}$$ ✓

○ $$\mu = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{pmatrix}$$

Submit

Problems 28-29 correspond to "Gaussian generative models"

Problem 28

3/3 points (graded)

Suppose we solve a classification problem with k classes by using a Gaussian generative model in which the jth class is specified by parameters π_j, μ_j, Σ_j. In each of the following situations, say whether the decision boundary is **`linear`**, **`spherical`**, or **`other quadratic`**.

a) We compute the empirical covariance matrices of each of the k classes, and then set $\Sigma_1 = \Sigma_2 = \cdots = \Sigma_k$ to the **average** of these matrices.

linear ▼ ✓

b) The covariance matrices Σ_j are all **diagonal**, but no two of them are the same.

other quadratic ▼ ✓

c) There are two classes (that is, $k = 2$) and the covariance matrices Σ_1 and Σ_2 are multiples of the identity matrix.

spherical ▼ ✓

Submit

✓ Correct (3/3 points)

Problem 29

2/2 points (graded)

Consider a binary classification problem in which we fit a Gaussian to each class and find that they are both centered at the origin but different covariances: $\mu_1 = \mu_2 = 0$ and $\Sigma_1 \neq \Sigma_2$. Derive the precise form of the **decision boundary**, that is, the points x for which the two classes are equally likely. You will find that it is

$$x^T(\Sigma_2^{-1} - \Sigma_1^{-1})x = a \ln \frac{|\Sigma_1|}{|\Sigma_2|} + b \ln \frac{\pi_1}{\pi_2}.$$

What are a and b?

$a =$

1 ✓

1

$b =$

-2 ✓

-2

Submit

✓ Correct (2/2 points)

Problem 30 corresponds to "More generative modeling"

Problem 30

4/5 points (graded)

For each of the situations below, say which of the following distributions would be the best model for the data: **`Gaussian`**, **`gamma`**, **`beta`**, **`Poisson`**, or **`categorical`**.

a) You collect the number of airplane landings at Los Angeles International Airport during each one hour interval over the course of a week (thus, a total of 268 data points).

beta ▼ ✗

b) For your favorite sports team, you compute the fraction of games they won each year, during the period 1980-2015 (thus, a total of 36 data points).

beta ▼ ✓

c) Your local pet store has mammals, reptiles, birds, amphibians, and fish. You measure the fraction of each (thus, a total of five numbers).

categorical ▼ ✓

d) You collect the pollution levels (positive real numbers reflecting concentrations of particulate matter) recorded in your city over the past year (thus, a total of 365 numbers).

gamma ▼ ✓

e) Like (d), but instead you use the log of these values.

Gaussian ▼ ✓

Submit

✱ Partially correct (4/5 points)