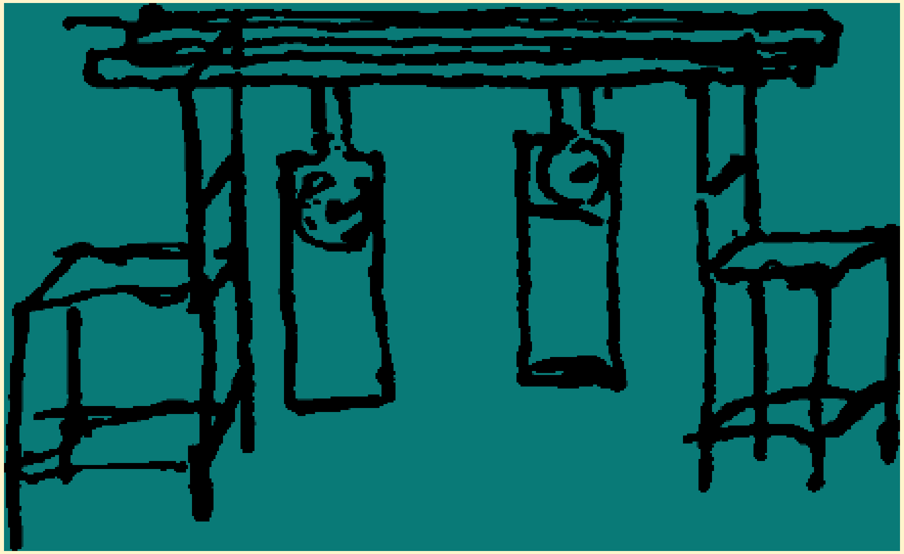
Modeling of Complex Networks

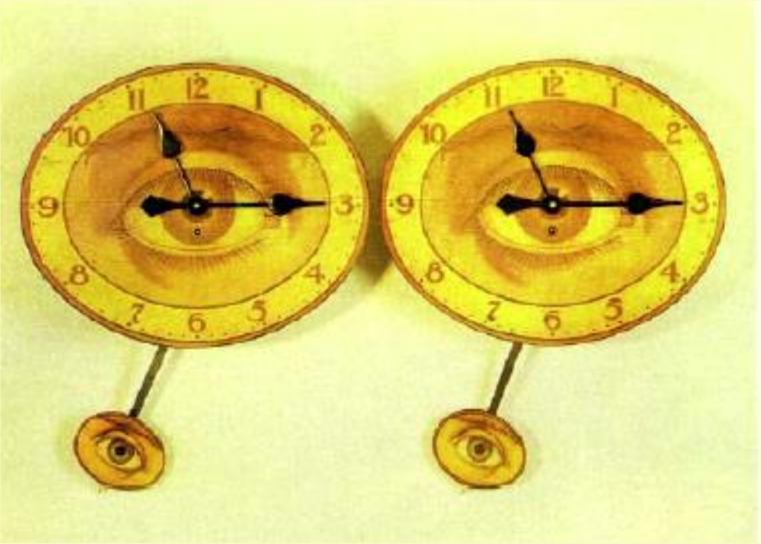
Lecture 7: Network Synchronization

S8101003Q-01(Sem A, Fall 2023)

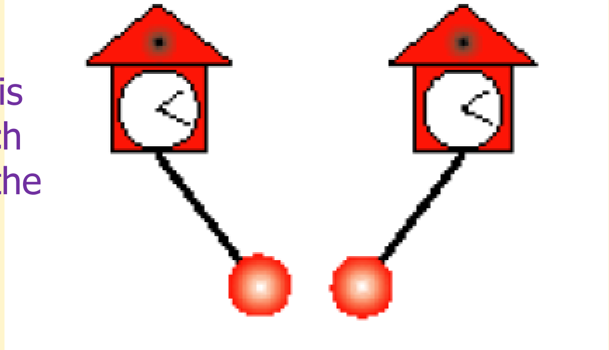
Synchronization

(Huygens, 1665)

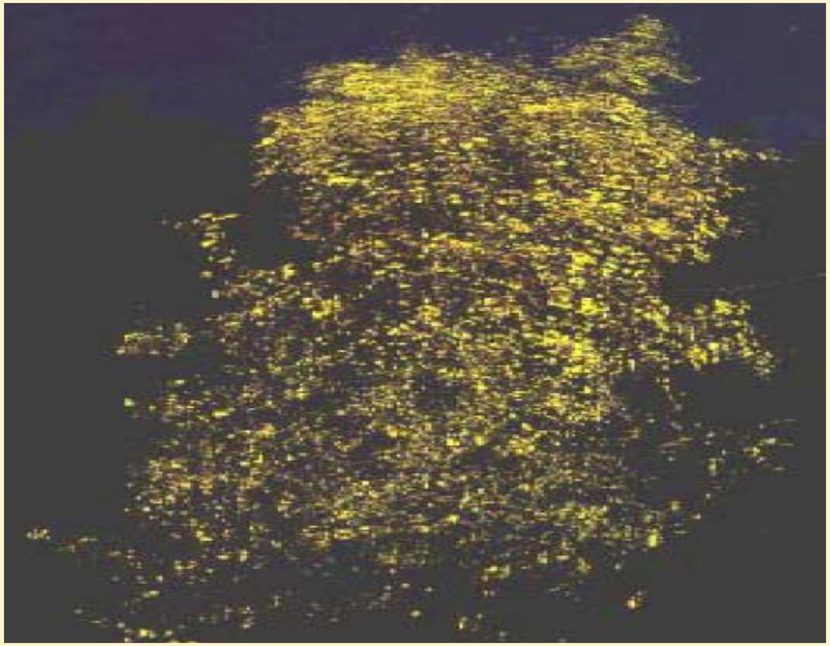




The term *synchronization* is bound with the name Christian Huygens, a Dutch mathematician. Huygens worked on the construction of accurate clocks, suitable for naval navigation, over a long period of his lifetime. His invention, the pendulum clock, was a breakthrough in timekeeping, which was patented in 1657. The most important work of Huygens relating to the concept of synchronization was his observation that two pendulums mounted on the same beam would come to swing in perfectly opposite directions, a phenomenon he referred to as odd sympathy.

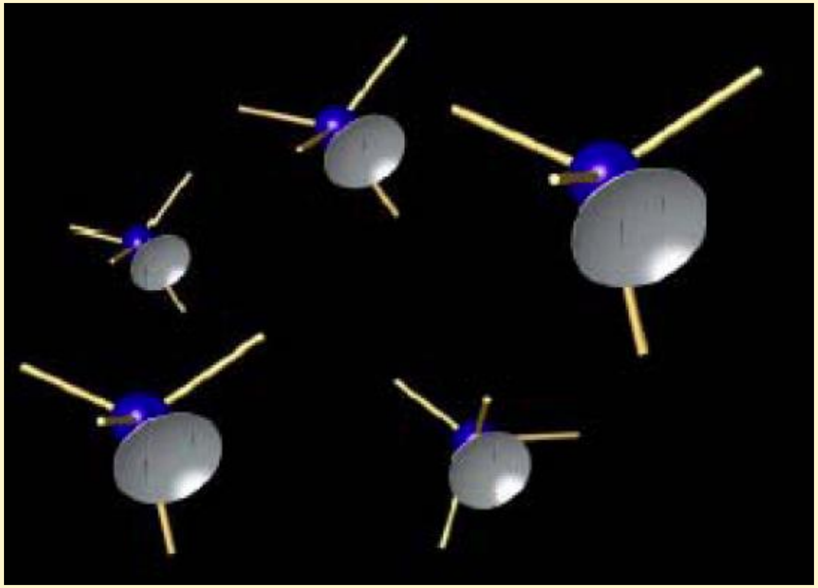


Fireflies Synchronization



Another interesting discovery was the observation of fireflies' synchronous flashing by another Dutch, a tourist named Kempfer. When he traveled through the River Naenam in Thailand in 1680, Kempfer found that those little insects could flash together fairly accurately in time.

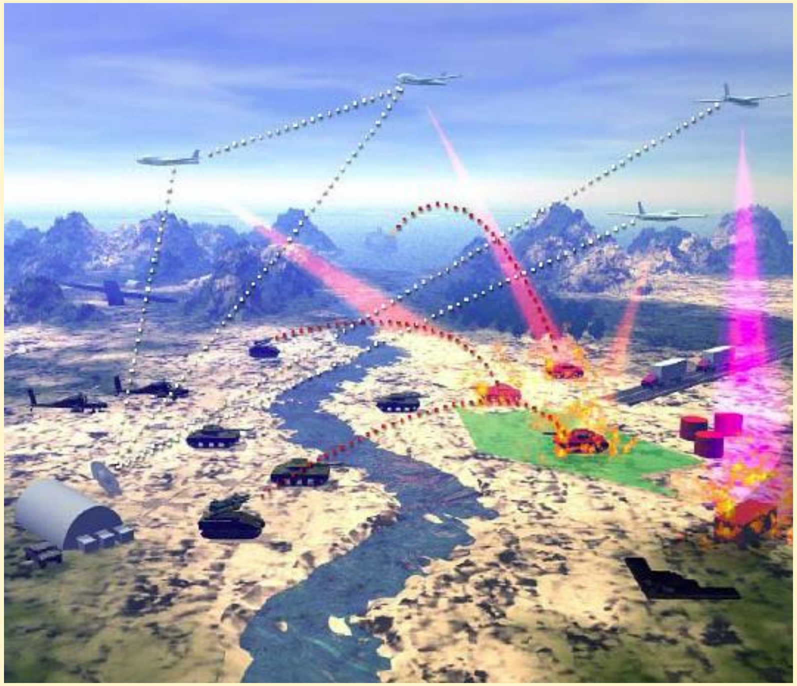
Attitude Alignment

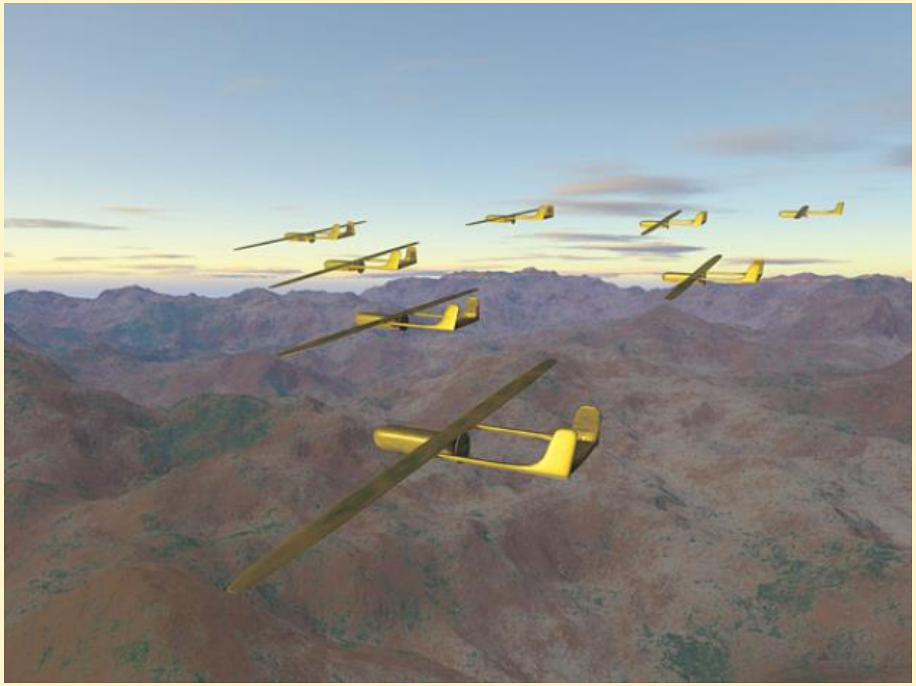


The attitude of each spacecraft is synchronized with its two adjacent neighbors via a bi-directional communication channel

Consensus

A position reached by a group as a whole





Battle space management scenario

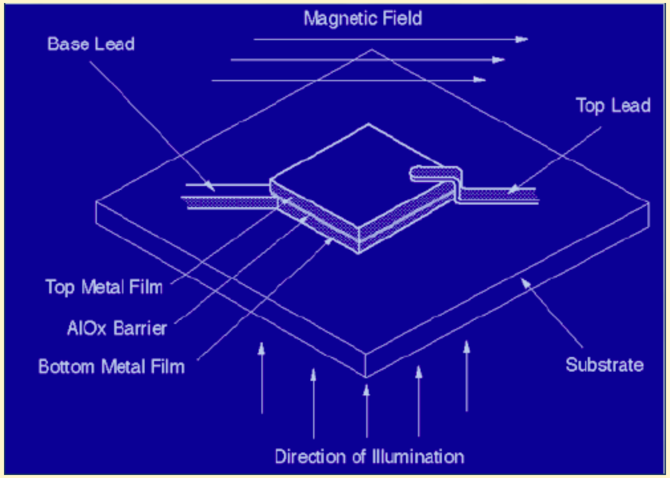
Synchronization-based Laser

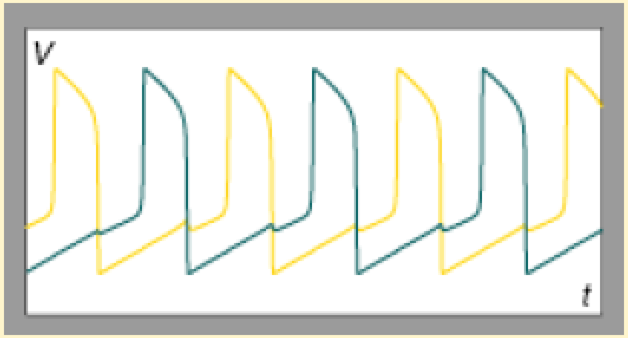
Astounding coherence of a laser beam comes from trillions of atoms pulsing in concert, all emitting photons of the same phase and frequency

Electric Currents

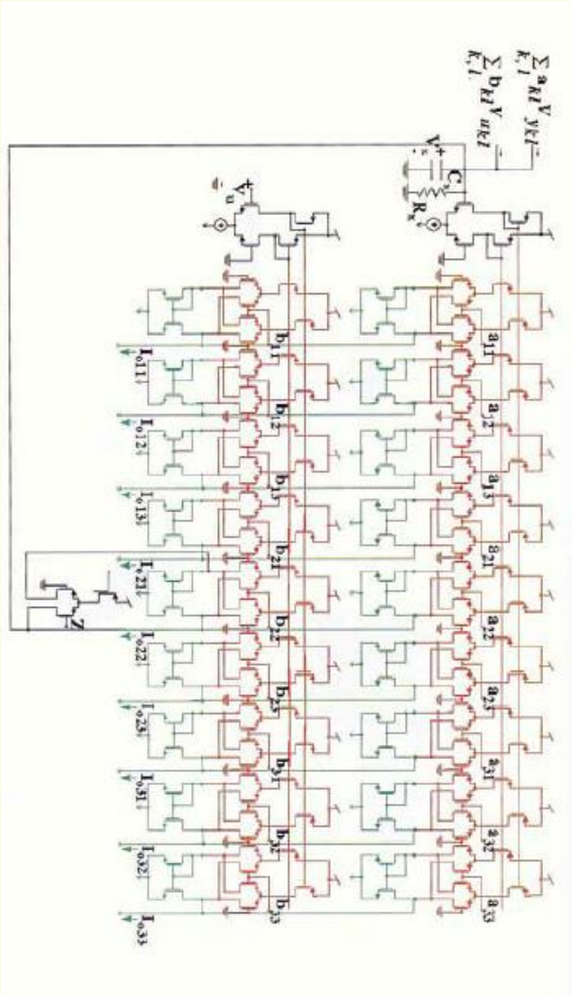
Through Josephson Junctions

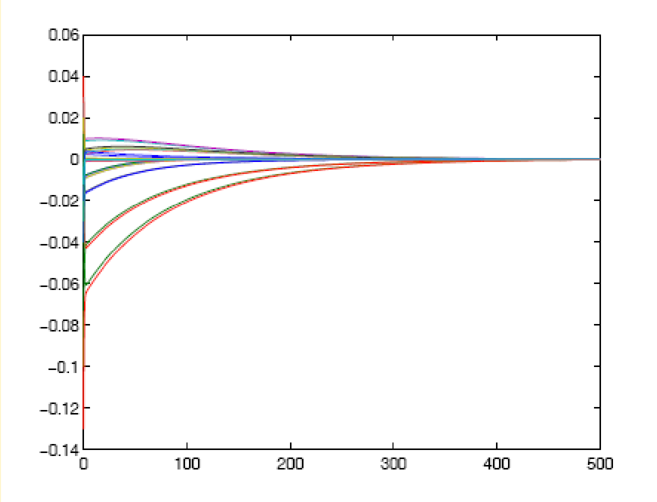
Oscillate as One





Phase Sync

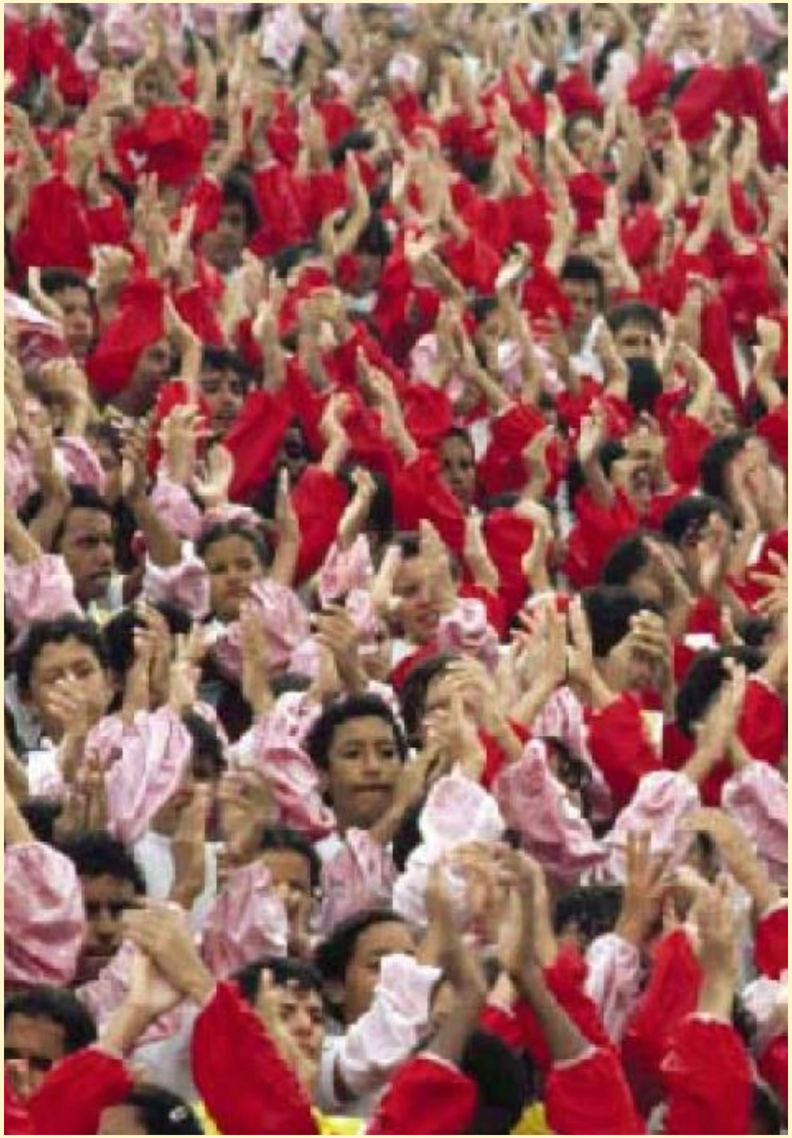




Simulation

Out of tumultuous applause:

Synchronized claps



Self-organization
in the concert-hall:
the dynamics of
rhythmic applause

--- *Nature (2001)*

Pedestrians Make London's Millennium Bridge Wobble

June 10, 2000

Yet, synchronization can be harmful. On June 10, year 2000, when the London Millennium Footbridge over the River Thames was first opened to the public, thousands of people walked on it for celebration but unexpectedly lateral vibration (resonant structural response) caused this 690-ton steal bridge to sway severely. This swaying motion earned it a nickname the Wobbly Bridge afterwards. Although the resonant vibrational modes have been well understood in bridge design after the famous event of collapse of the Tacoma Narrows Bridge in 1940, not much attention had been given to pedestrian-excited lateral motion, which was responsible for the big vibration of the millenium bridge. An extensive analysis was conducted thereafter, but it took more than one year and costed about 5-million pounds to finally fix the problems and then re-open the bridge on February 22, 2002.

source

Birds Flocking

Flocking: to congregate or travel in flock

Fish Swarming

- Swarming:
  to move or gather in group

Harmful Synchronization in Internet

- TCP window increase/decrease cycles
  Synchronization occurs when separate TCP connections share a common bottleneck router
- Synchronization to an external clock
  Two processes can become synchronized simply if they are both synchronized to the same external clock
- Client-server models
  Multiple clients can become synchronized as they wait for services from a busy (or recovering) server
- Periodic message routing
- And ……

Useful **Synchronization** in Engineering

Examples:

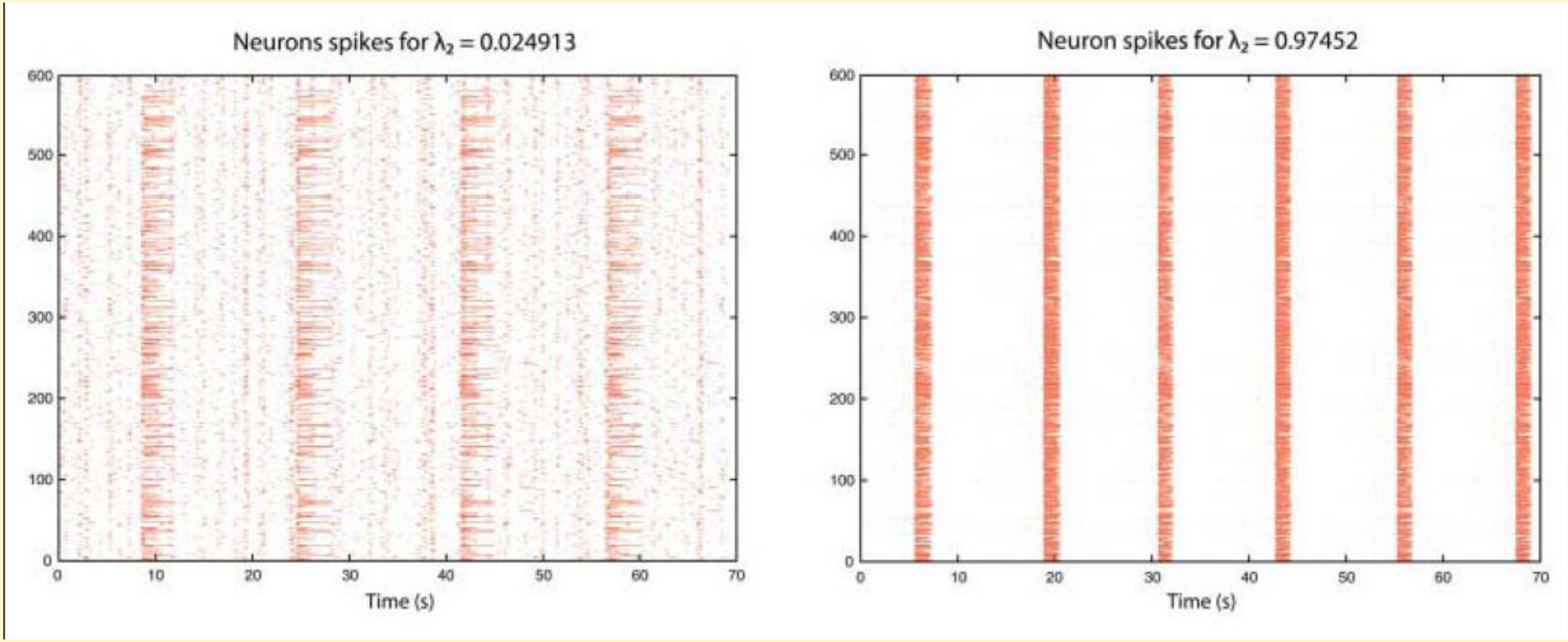
- Secure communications
- Generation of harmonic oscillations
- Language emergence and development:

  Synchronization in conversations -> common vocabulary
- Organization management:

  Agents' synchronization

  -> work efficiency
- Biological systems (brain, heart)
- ……

One Example



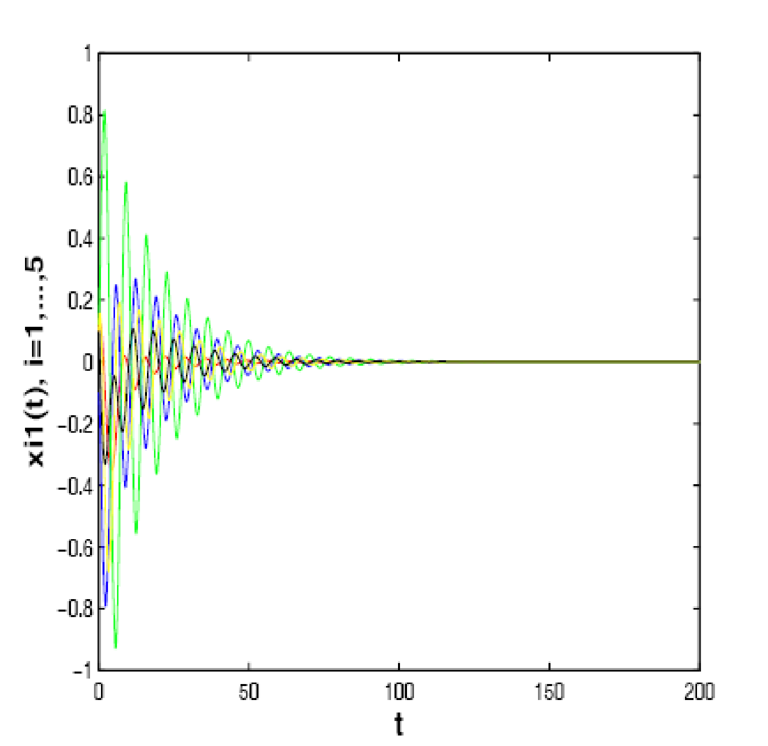
non-synchrous synchrous

In mammals, a small group of neurons in the brain stem, named pre-Bötzinger complex, is responsible for generating a regular rhythmic output to motor calls that initiate a breath.

Exploring Synchronization

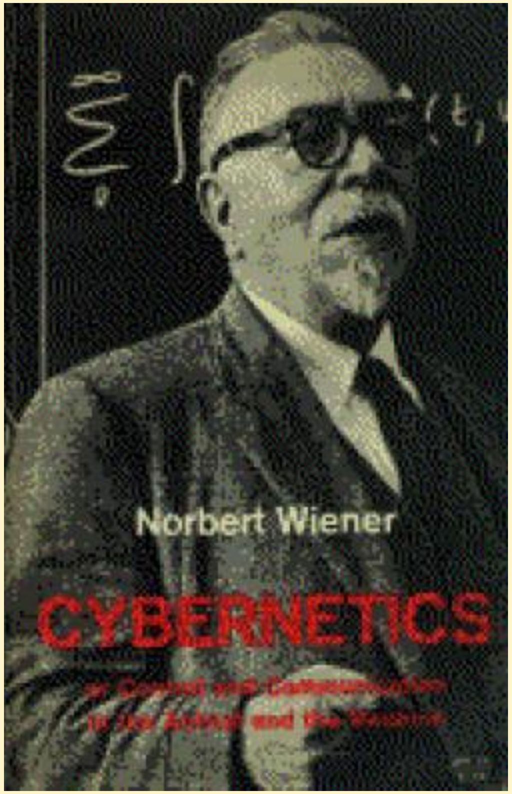
First, **how to describe the individual dynamics of each isolated node?**

Then, **how to describe the synchronous dynamics of the coupled network?**



Norbert Wiener

(1894-1964)



- Collective synchronization was first studied mathematically by Wiener, who recognized its ubiquity in the natural world, and speculated that it was involved in the generation of alpha rhythms in the brain.
- Unfortunately, Wiener's mathematical approach based on Fourier integrals has turned out to be a dead end.

- N. Wiener, Nonlinear Problems in Random Theory, MIT Press, 1958
- N. Wiener, Cybernetics, MIT Press, 1st ed., 1948, 2nd ed., 1961.

Network Synchronization

- Network Synchronization versus Individual Dynamics
  - Typical networks: coupled map lattices, cellular neural networks
  - Typical dynamics: Turing patterns, autowaves, spiral waves, spatiotemporal chaos

- Network topology and the dynamics of individual nodes determine the network dynamical behaviors
-> Synchronization

A General Dynamical Network Model

Linearly and diffusively coupled:

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij} H(x_j) \qquad x_i = \begin{bmatrix} x_i^1 \\ x_i^2 \\ \vdots \\ x_i^n \end{bmatrix} \in R^n \quad i = 1, 2, \ldots, N$$

$f(.)$ – Lipschitz (exist constant K, satisfying ‖f(x1)-f(x2)‖<=K‖x1-x2‖)

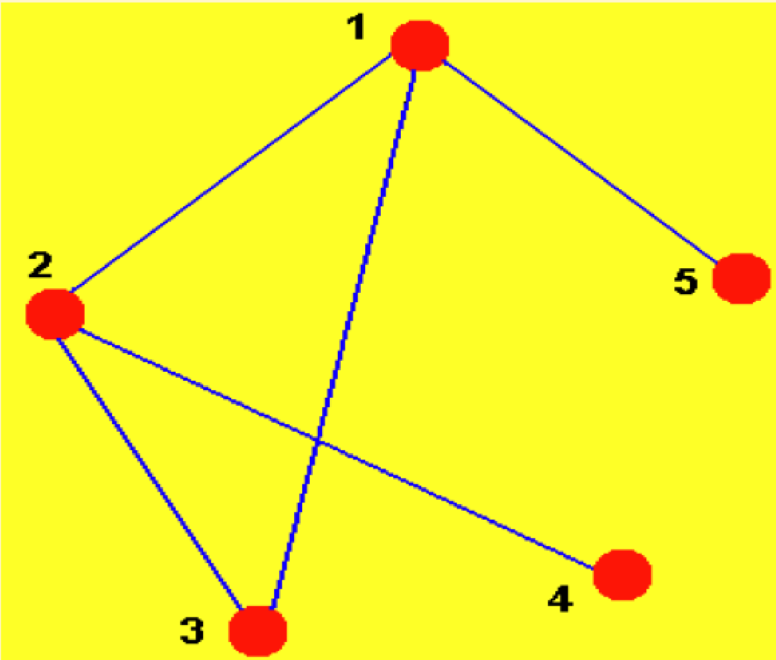
coupling strength $c > 0$

coupling matrices $A = [a_{ij}]_{N\times N} \quad H(x_j) = \begin{bmatrix} H_1(x_j) \\ H_2(x_j) \\ \vdots \\ H_n(x_j) \end{bmatrix}$ e.g. $H = \Gamma = \begin{bmatrix} r_{11} & & & 0 \\ & r_{22} & & \\ & & \ddots & \\ 0 & & & r_{nn} \end{bmatrix}$

If there is a connection between node i and node j ($j \neq i$), then $a_{ij}=a_{ji}=1$;otherwise, $a_{ij}=a_{ji}=0$ and $a_{ii}=-k_i$ (diffusive)

Laplacian matrix $L=-A$

Simple Examples

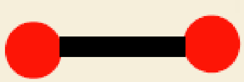


$$A=\begin{bmatrix} -3 & 1 & 1 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

symmetrical, diffusive, irreducible, zero row-sum

eigenvalues of A: $\lambda_5 \leq \ldots \leq \lambda_2 < \lambda_1 = 0$

eigenvalues of L: $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_5$



$$A=\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

eigenvalues of A: $\lambda_1 = 0 \quad \lambda_2 = -2$

eigenvalues of L: $\lambda_1 = 0 \quad \lambda_2 = 2$

Network Synchronization: Definition

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij} H(x_j) \qquad x_i \in R^n \qquad i = 1, 2, \ldots, N \qquad (1)$$

Complete state synchronization:

$$\lim_{t\to\infty} \| x_i(t) - x_j(t) \|_2 = 0, \qquad i, j = 1, 2, \ldots, N$$

State synchrony:

$$x_1(t) \to x_2(t) \to \ldots \to x_N(t) \to s(t)$$

$$\dot{s}(t) = f(s(t))$$

(e.g., equilibrium point, periodic orbit, chaotic orbit)

$x_1 = x_2 = \ldots = x_N$ -- Synchronization manifold

$s(t) \in R^n$ -- Synchronized state

Synchronization Criteria

Linearizing equation (1) at the synchronized state *s(t)* and then letting ξ_i be the variation of the state vector of node *i* lead to

$$\dot{\xi}_i = \partial f(s)\xi_i + \sum_{j=1}^{N} c a_{ij} \partial H(s)\xi_j, \quad i = 1, \dots, N$$

where ∂f(s) and ∂H(s) are the Jacobi matrices of f(s) and H(s) evaluated at s, respectively. Setting $\xi = \left[\xi_1, \xi_2, \dots, \xi_N\right]$ will transform the above to

$$\dot{\xi} = \partial f(s)\xi + c\partial H(s)\xi A^T$$

Furthermore, diagonalize $A^T = S\Lambda S^{-1}$ with a diagonal matrix

$$\Lambda = diag(\lambda_1, \dots, \lambda_N),$$

where $\{\lambda_k\}_{k=1}^{N}$ are eigenvalues of matrix $A = [a_{ij}]$, with $\lambda_1 = 0$. Then, by denoting a new vector $\eta = \left[\eta_1, \eta_2, \dots, \eta_N\right] = \xi S$, one has

$$\dot{\eta} = \partial f(s)\eta + c\partial H(s)\eta\Lambda$$

which is equivalent to

$$\dot{\eta}_k = [\partial f(s) + c\lambda_k \partial H(s)]\eta_k, \quad k = 2, \dots, N \qquad (2)$$

Synchronization Criteria

Since the function *f(.)* is Lipschitz, it is possible to make the term $c\lambda_k\, \partial H(s)$ dominant over the Jacobian $[\partial f(s)]$. To this end, a criterion for the synchronization manifold to be locally stable is that all the transversal Lyapunov exponents of the variational equation (2) are strictly negative. Clearly, these Lyapunov exponents depend on the node dynamics *f(.)*, the network coupling strength *c* and the coupling matrics *A* and *H*.

Note that in equation (2), each term has the same form and only η_k and λ_k depend on *k*. For the case of $k=1$, the variational equation (not shown in (2)) corresponds to the synchronization manifold associated with $\lambda_1 = 0$; for all other $k=2,\ldots,N$, they correspond to the transversal eigenvectors that span some subspaces, referred to as transversal modes.

Note also that when matrix *A* is asymmetric (in directed networks, for instance), its eigenvalues can have complex values, so it is natural to replace $c\lambda_k$ by a complex parameter $(\alpha+i\beta)$, which is a function of where the eignvalues are generally complex, in each equation and then define the following so-called master stability equation for all $k=2,\ldots,N$:

$$\dot{y} = [\partial f(s) + (\alpha + i\beta)\partial H(s)]y$$

Synchronization Criteria

Since this may be a time-varying system, particularly if $s(t)$ is a time function, its eigenvalues may not be useful for determining the stability. Thus, the maximum Lyapunov exponent L_{max} of the system will be used instead, which is a function of α and β, and is called the master stability function. Similarly, these Lyapunov exponents depend on the node dynamics $f(.)$, the network coupling strength c and the coupling matrices A and H.

For a given and fixed coupling strength c, one can find a point $c\,\lambda_k$ on the parameter plane (α, β), at which the sign of L_{max} indicates the stability of the corresponding transversal mode: a negative sign means stability and a positive sign means instability of the transversal mode. Thus, it is necessary to require all transversal modes of λ_k be stable, $k=2,\ldots,N$, so that the synchronization manifold of the original network *(1)* is stable, implying the network achieves synchronization. In other words, the negative master stability function is a convenient necessary condition for network synchronization.

Synchronization Criteria

Eigenvalues of $L = -A$: $0=\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$

Transversal subspace ($\perp$ synchronization manifold):

$$\Theta = \left\{ x = \left[x_1^T, x_2^T, \ldots, x_N^T \right]^T \in R^{nN} : \quad \sum_{j=1}^{N} \xi_j x_j = 0 \right\}$$

Here, $\left[\xi_1, \xi_2, \ldots, \xi_N \right]^T$ is the left eigenvector of $\lambda_1 = 0$

For undirected and unweighted networks, the master stability equation can be simplified to

$$\dot{y} = [\partial f(s) + \alpha \partial H(s)] y$$

and the corresponding maximum Lyapunov exponent L_{max} is a function of the real parameter α, a function of $c\,\lambda_k$, $k=2,\ldots,N$.

The range of that guarantees L_{max} be negative is called *a synchronized region*, denoted by S.

Synchronization Criteria

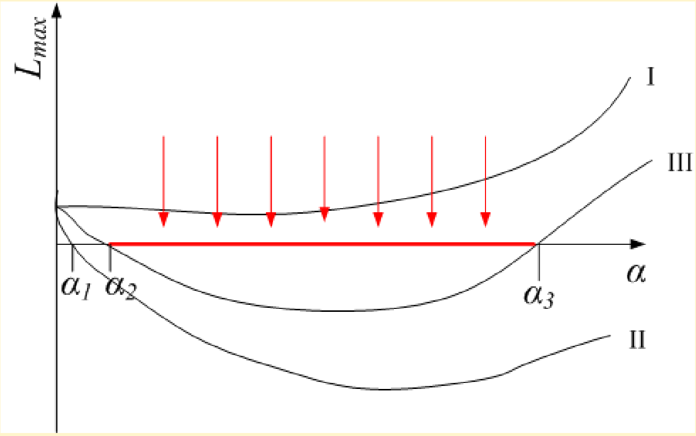
- Clearly, this region is determined not only by the real parameter α, which depends on the outer coupling strength c, but also by the node dynamics $f(.)$ and the inner coupling function $H(.)$.
- But, the synchronized region can be restricted onto a one-dimensional set with respect only to α.
- Thus, if the nonzero eigenvalues satisfy

$$c\lambda_k \in S, \quad k = 2, \ldots, N$$

Then the synchronization manifold will be stable, i.e., the network will synchronize.

synchronizing if $c\lambda_2 > \alpha_1$ or if $\frac{\alpha_2}{\lambda_2} < c < \frac{\alpha_3}{\lambda_N}$ or $c > \frac{\alpha_2}{\alpha_3}\frac{\lambda_N}{\lambda_2} > 0$

Case I:
No sync



Case II
Sync region:
$S_1 = (\alpha_1, \infty)$

Case III
Sync region:
$S_2 = (\alpha_2, \alpha_3)$

Synchronization Criteria

Case II: Sync region $S_1 = (\alpha_1, \infty)$

- Condition: $c\lambda_2 > \alpha_1$
- The synchronizability is determined by the smallest nonzero eigenvalue λ_2 of its Laplacian matrix *L*.
- The larger the λ_2, the smaller the *c* is needed, so the better or stronger the synchronizability of the network.
- Here, the constant α_1 is not explicitly given.

Case III: Sync region $S_2 = (\alpha_2, \alpha_3)$

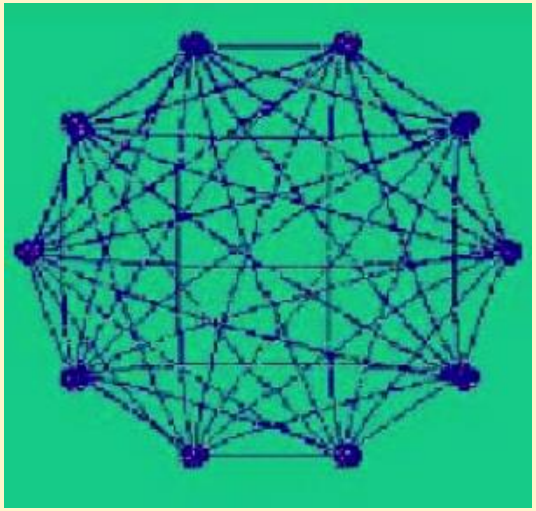
- Condition:

$$\frac{\alpha_2}{\lambda_2} < c < \frac{\alpha_3}{\lambda_N} \quad \text{or} \quad c > \frac{\alpha_2}{\alpha_3}\frac{\lambda_N}{\lambda_2} > 0$$

- The inequalites can be simply written as $0 < \dfrac{\lambda_N}{\lambda_2} < c\alpha$
- In this case, $\dfrac{\lambda_N}{c\lambda_2} \in S$
- The synchronizability is characterized by the ratio λ_N / λ_2 of the largest and smallest nonzero eigenvalues of the Laplacian matrix *L*.
- The smaller the ratio λ_N / λ_2, the smaller the *c* is needed, so the better the synchronizability of the network.

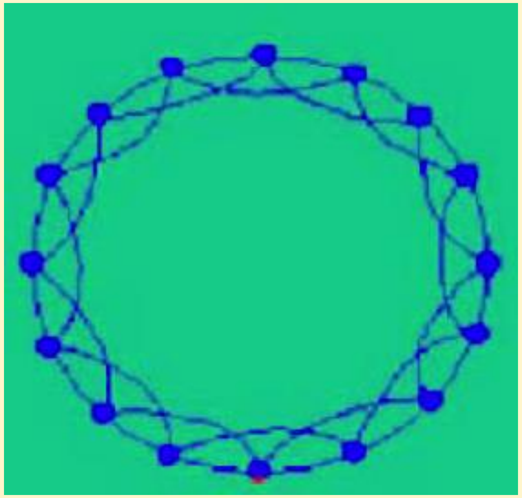
Synchronization in Regularly Coupled Networks

Globally coupled network: $\lambda_2 = \ldots = \lambda_N = N$



Usually, no matter how small the coupling strength c is, a global coupled network will synchronize if its size is sufficiently large

Locally coupled network: $\lambda_2 = 4\sum_{j=1}^{K/2} \sin^2(j\pi / N)$

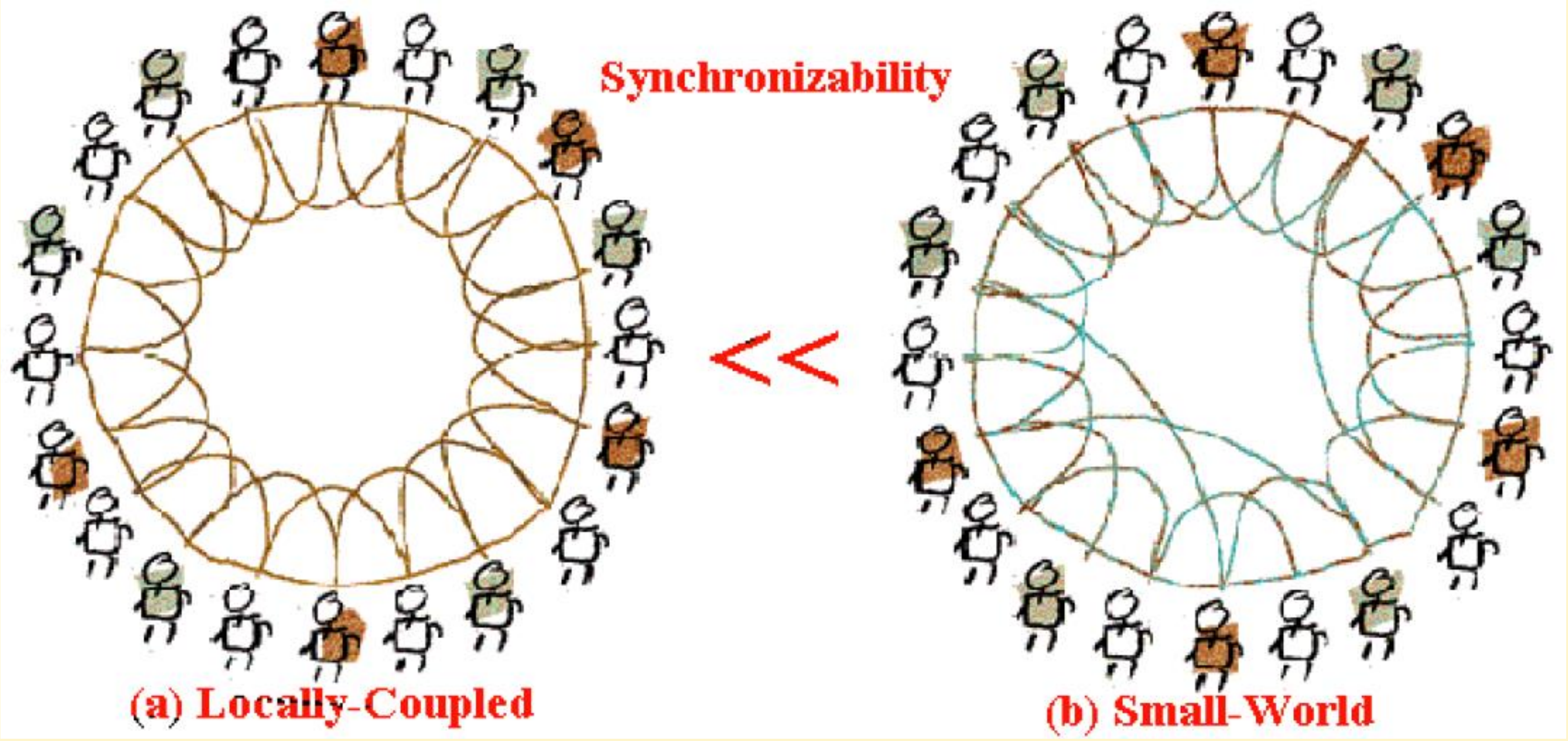


Usually, no matter how large the coupling strength c is, a locally coupled network will not synchronize if its size is too large

Wang and Chen (2002)

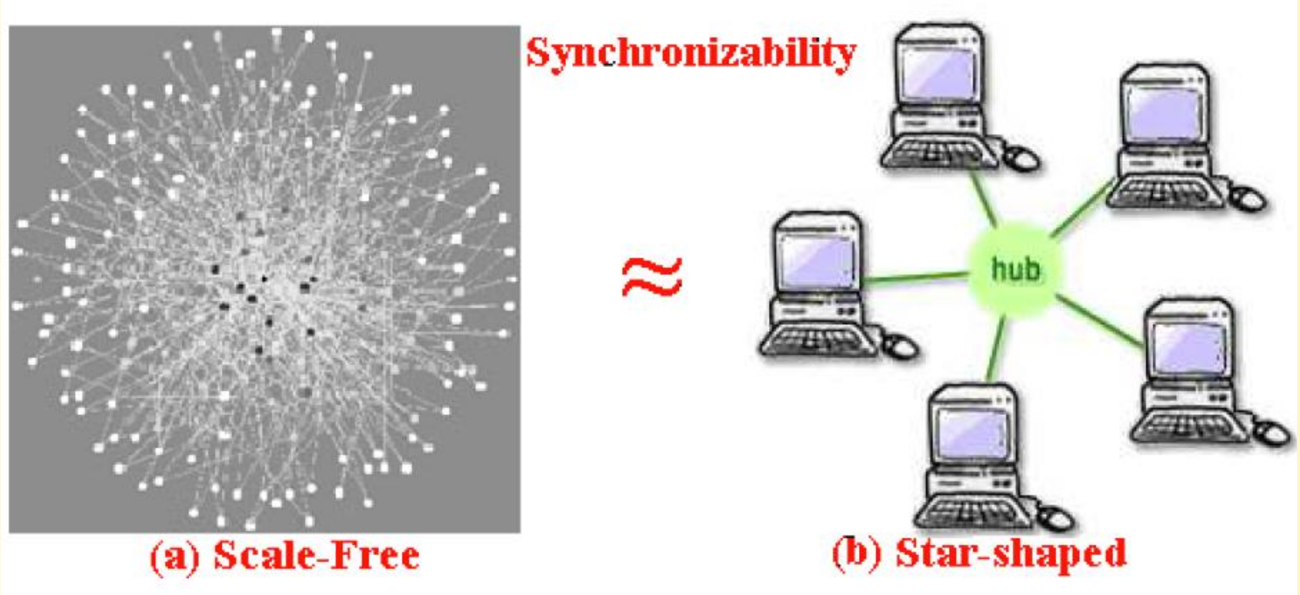
Synchronization in Small-World Networks

Synchronizability can be greatly enhanced by adding a tiny fraction of long-range connections, revealing an advantage of small-world network for synchronization



Wang and Chen (2002)

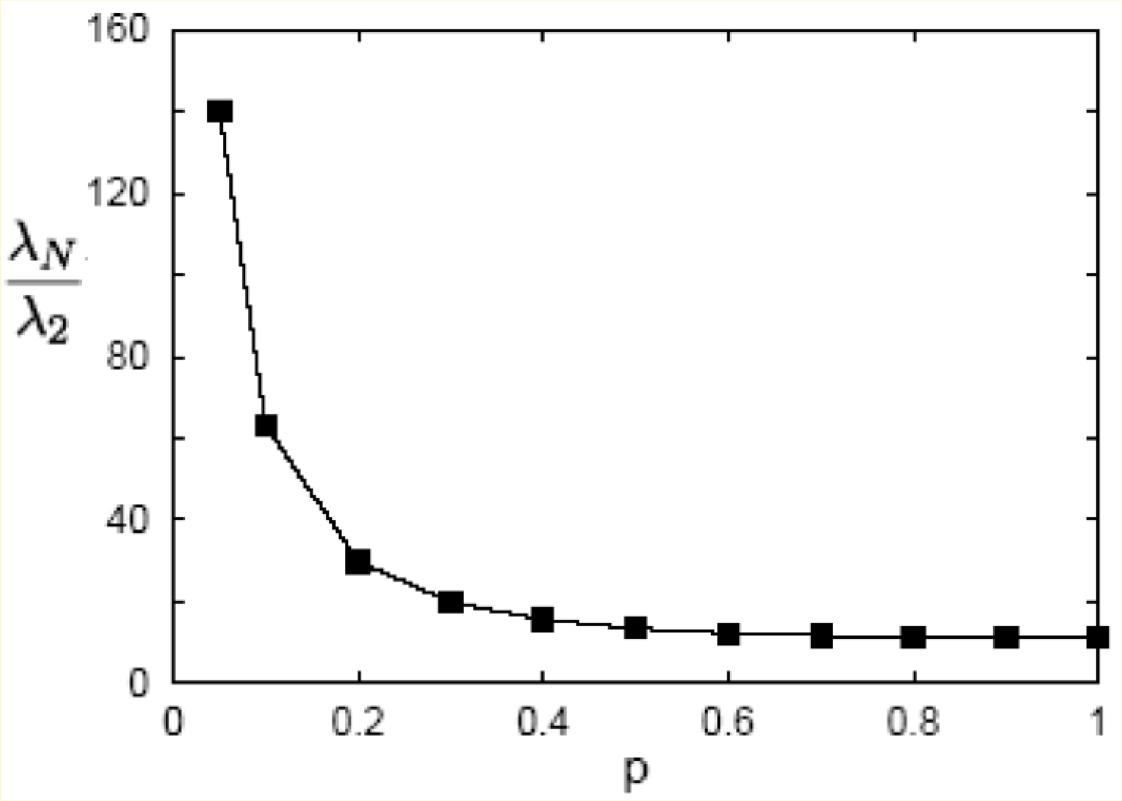
Synchronization in Scale-Free Networks



- The synchronizability of a SF network is about the same as that of a star-shaped network, mainly determined by the (one-to-one) relations between the central node and each of the neighboring nodes
- This is due to the extremely inhomogeneous connectivity distribution of an SF network: a few hubs in an SF network play a similar role as a single center in a star-shaped network

Network Topology versus Synchronizability

Connection Probability:
Synchronizability of small-world networks will increase as the small-world feature increases



Larger p
→
Smaller λ_N / λ_2
(Criterion III)
→
Better synchrony

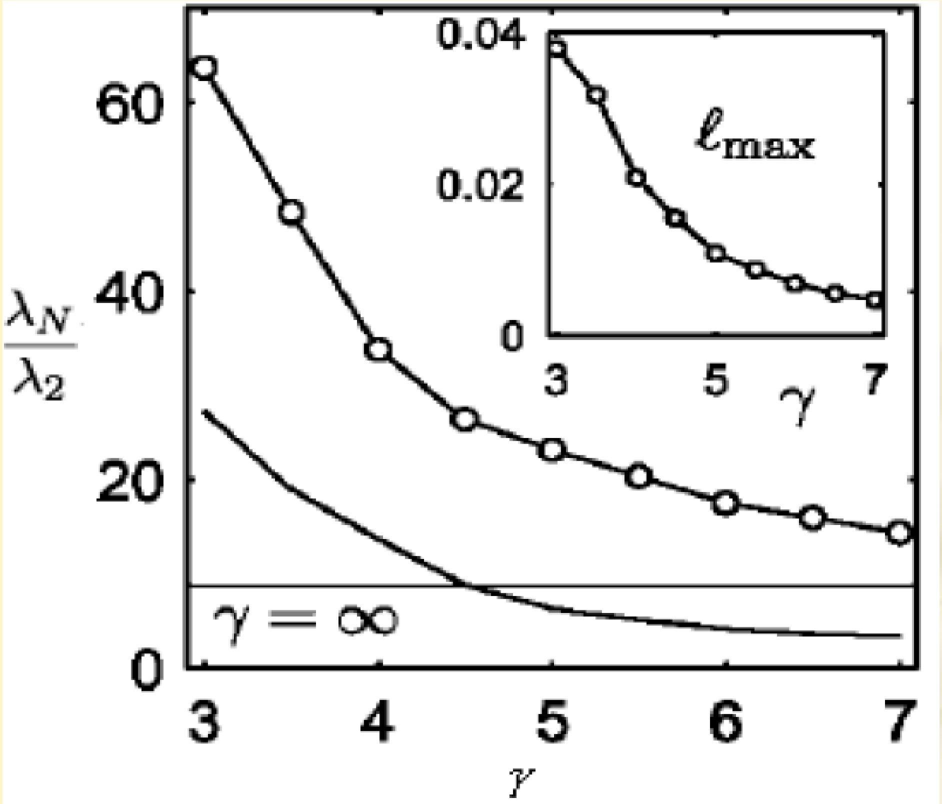
$p = 1$ → fully connected

Hong, et al. (2004)

Network Topology versus Synchronization

Power-Law Exponent:

Synchronizability of scale-free networks will increase as the power-law exponent increases



Larger γ

→

Smaller λ_N / λ_2 (Criterion III)

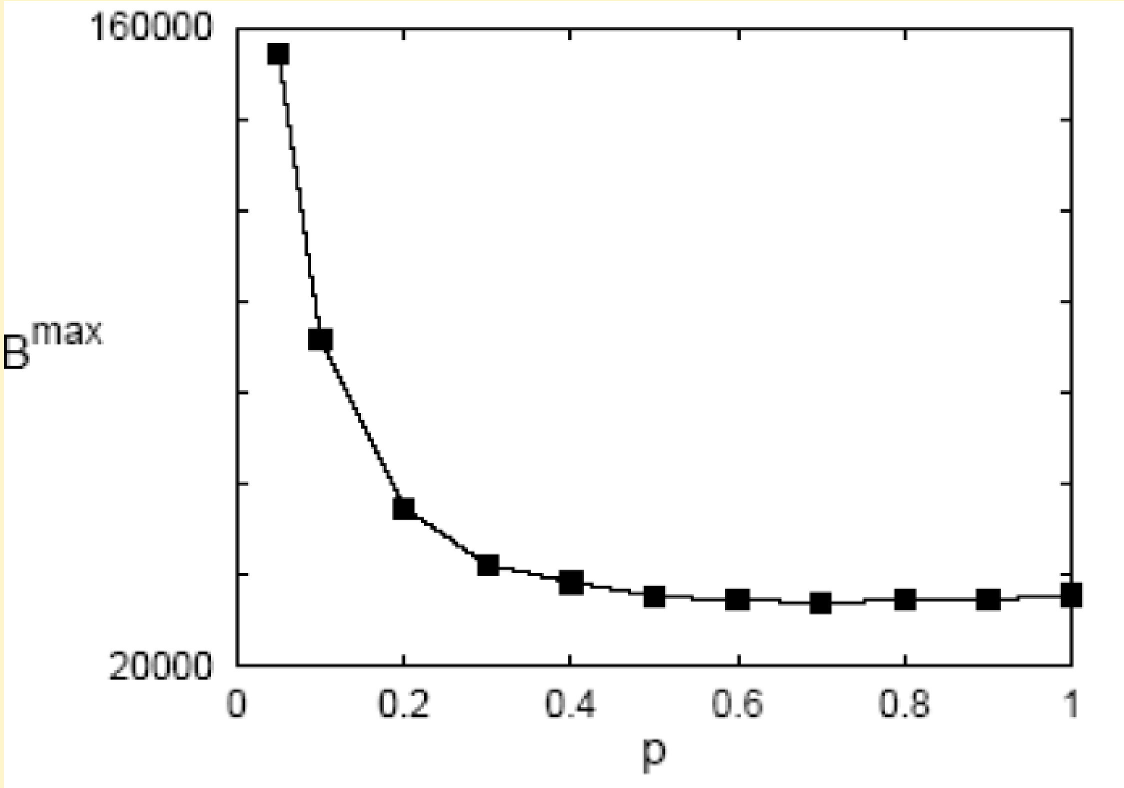
→

Better synchrony

Nishikawa, et al. (2003)

Network Topology versus Synchronization

Betweenness Centrality:
Synchronizability of small-world and scale-free networks will increase as the node betweenness decreases

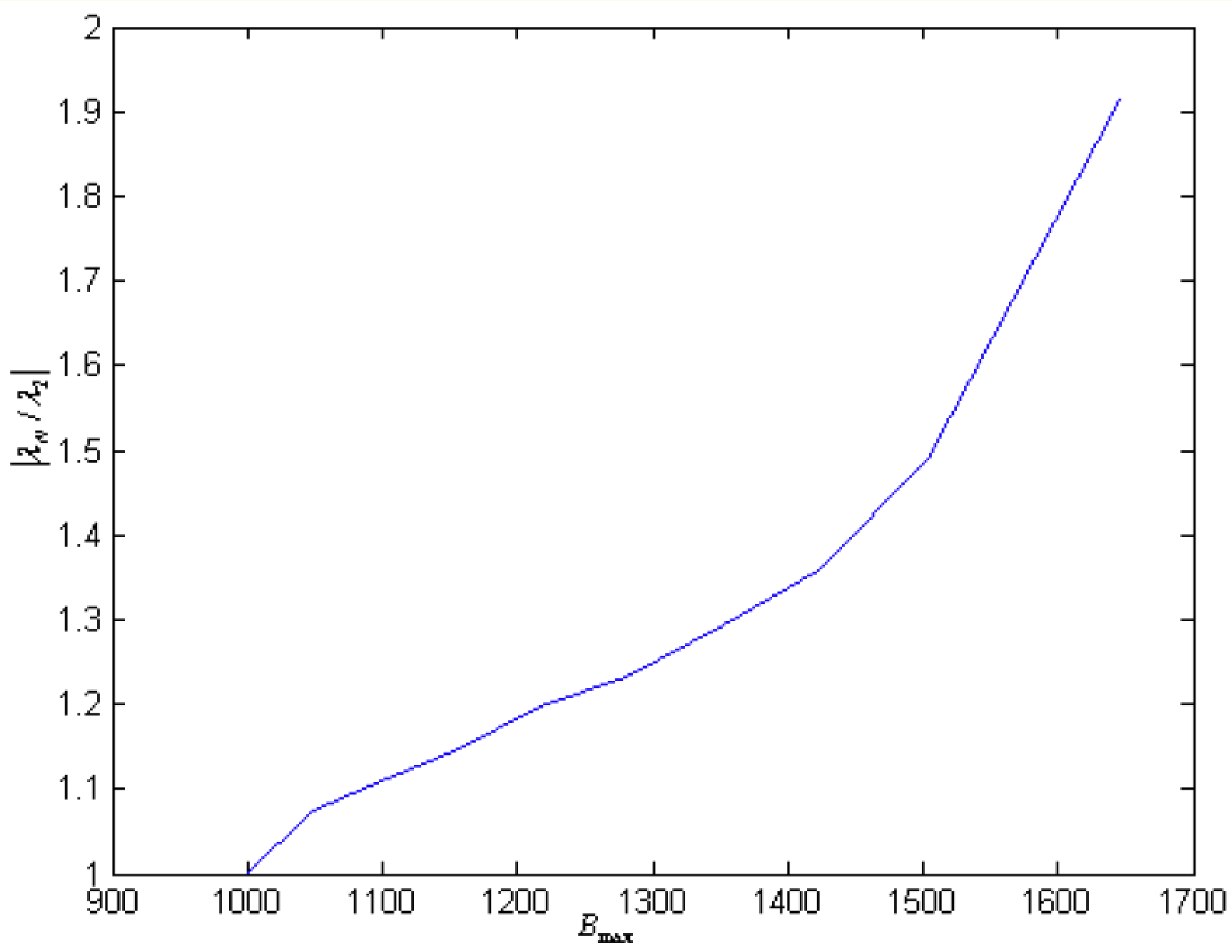


Larger p
→
Smaller
Node betweenness
→
Better synchrony

$p = 1$ → fully connected

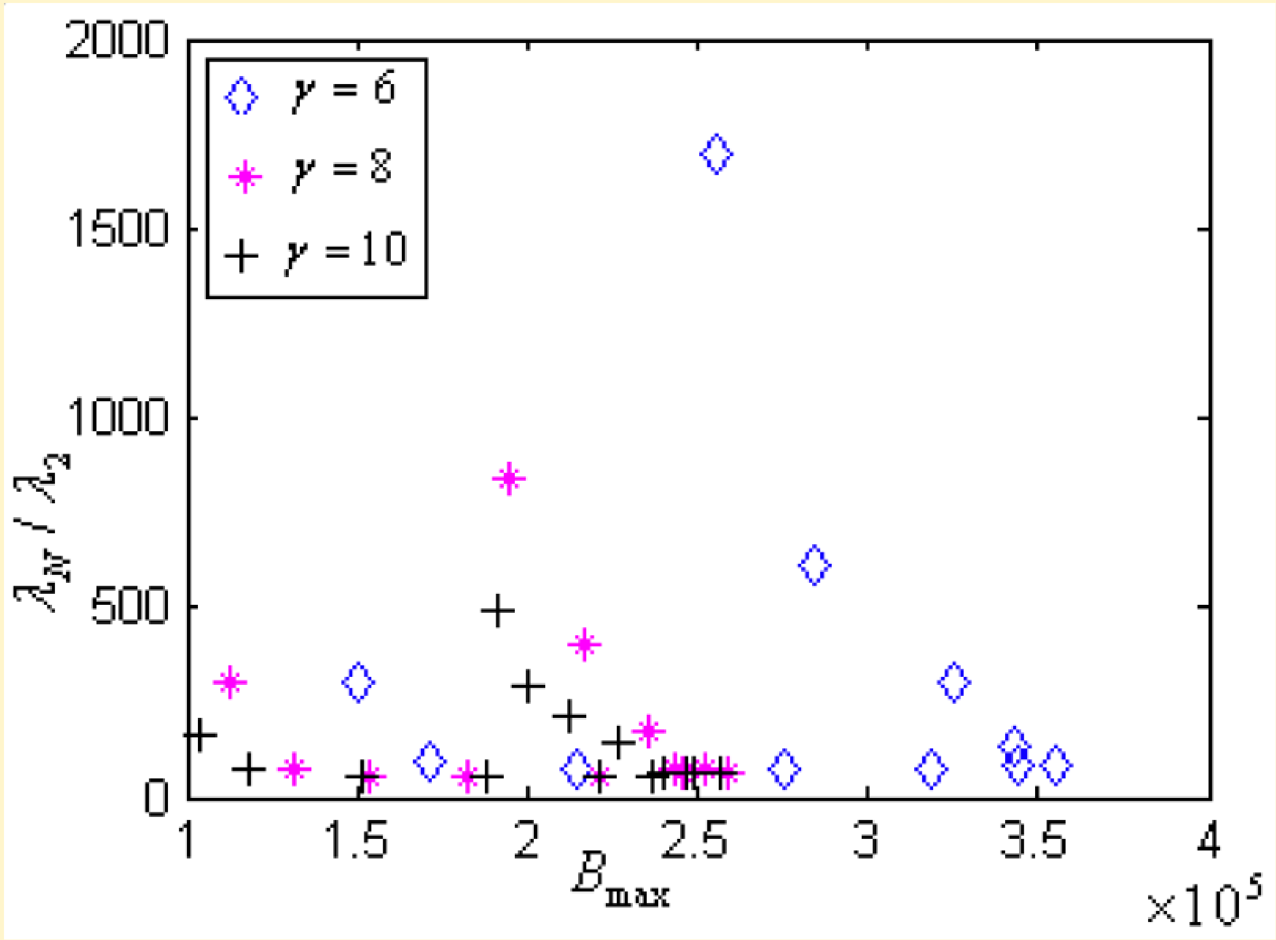
Newman (2001)

Betweenness versus Synchronizability



homogeneous networks – there is a clear correlation between betweenness and synchronizability

Betweenness versus Synchronizability



heterogeneous networks –
there is no clear correlation between betweenness and synchronizability

Enhancing the Network Synchronizability

Two examples:

- Perturbing the network structure

  To eliminate the maximal betweenness

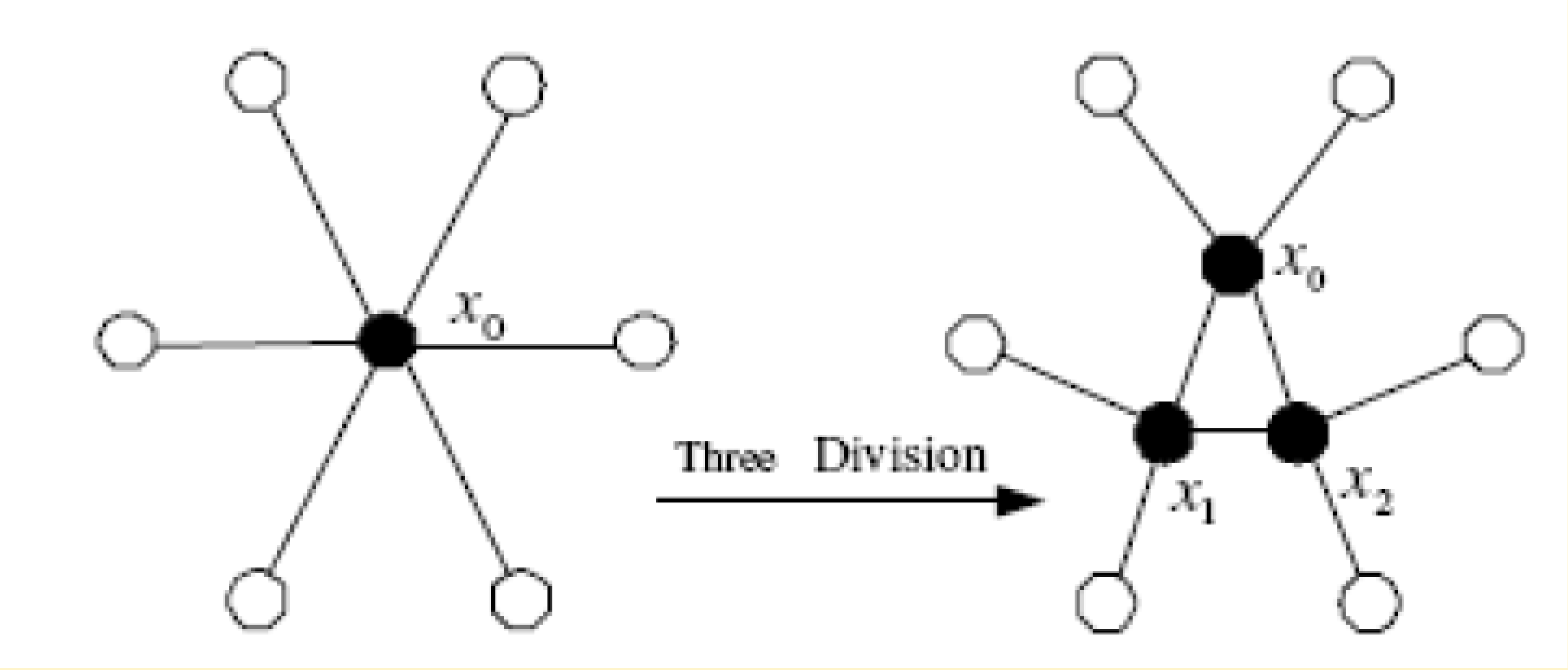
- Modifying the coupling structure

  To reduce the impact of the heterogeneity of degree and betweenness distributions

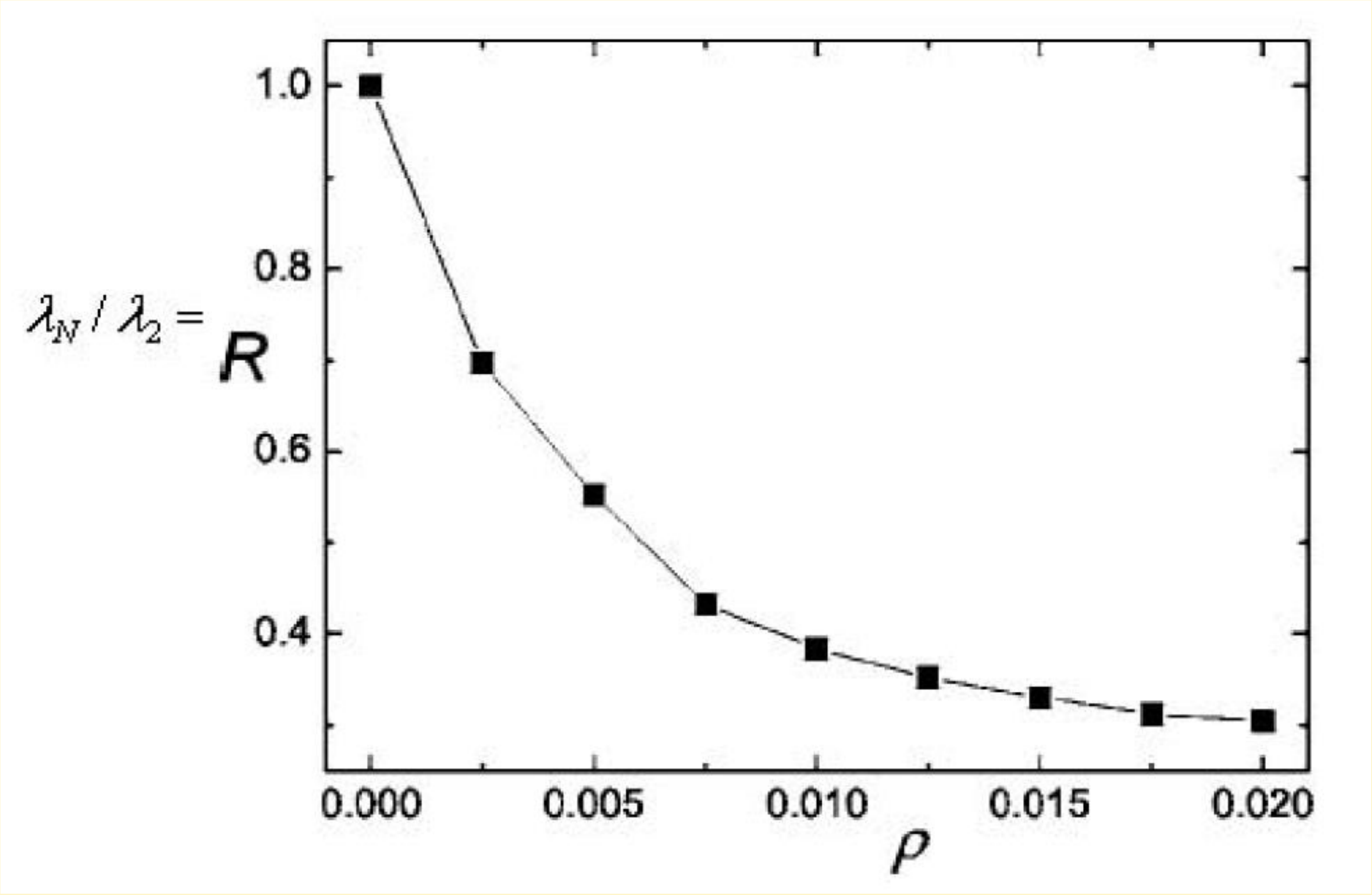
Perturbing the Network Topology

(Zhao, et al. Phys. Rev. E 2005, 72: 057102)

The node with the largest betweenness is replaced by several connected nodes, so that the shortest paths that passed through the original node will now only pass one or two new nodes, which will reduce the maximal betweenness dramatically.



Simulation Results



Fraction of divided nodes

Decoupling Method
(scale-free networks)

(CY Yin, WX Wang, G Chen, BH Wang, Phys. Rev. E 2006, 74: 047102)

Not only the nodes with large betweenness, but also the edges with large loads can cause data-traffic congestion.

Hence, if such heavily-loaded edges are decoupled, the data-traffic will be redistributed so become more efficient.

Algorithm:

1. Define and calculate the significance of all edges (i,j) by $k_i k_j$
2. Rank the edges from large to small
3. At each time step, cut an edge with the highest rank (which will decouple two nodes of higher degrees)

Simulation Results

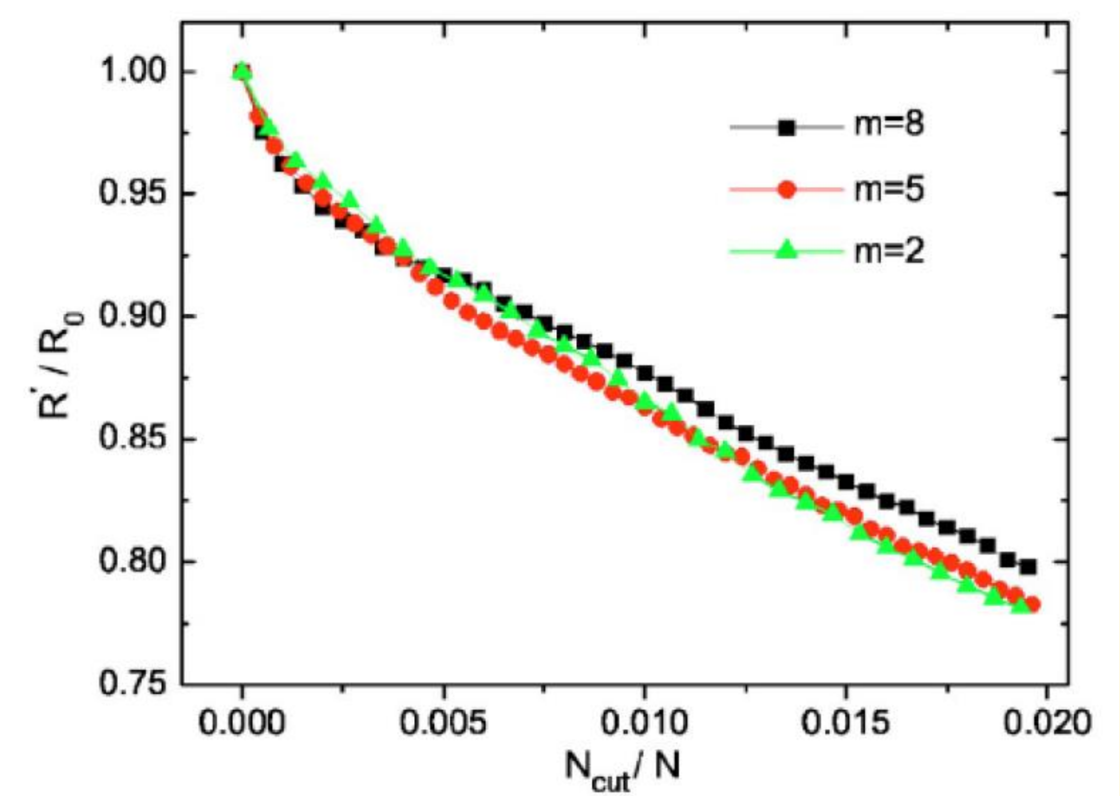


FIG. 1. (Color online) The change of synchronizability as a function of the proportion of cut edges N_{cut}/N for different values of m. Network size $n=2000$.

For scale-free networks, smaller eigen-ratio R'/R0 implies better synchronizability, where *m* is the number of new edges in the scale-free networks generation

Graph-Theoretic Approach

- Relationships between structural parameters and the synchronizability of complex networks
- Relationships between graph theory and the synchronizability of complex networks

- Disconnected synchronized regions
- Enhancing the network synchronizability

Structure versus Synchronizability

- Network Model:

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij} H(x_j) \qquad x_i \in R^n \qquad i = 1, 2, \ldots, N \qquad (1)$$

- Laplacian matrix *L*: symmetric, diffusive and irreducible, with eigenvalues

$$0 = \lambda_1 < \lambda_2 \le \ldots \le \lambda_N \qquad (2)$$

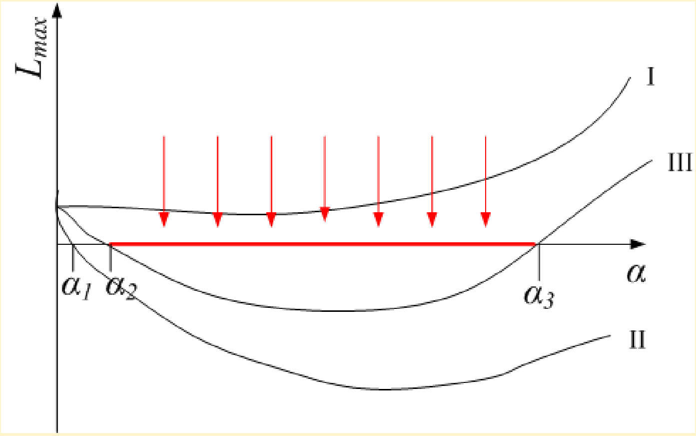
- Synchronization of network (1) :

$$x_1(t) \to x_2(t) \to \ldots \to x_N(t), \quad \text{as } t \to \infty \qquad (3)$$

- Synchronized region S: the region S of negative real α over which the network synchronizes.

- Remark 1: The synchronized region S can be an unbounded region, a bounded region, an empty set, or a union of several disconnected regions.

- Stability condition: The synchronous solution of network (1) is locally asymptotically stable: $c\lambda_k \in S,\ k = 2, 3, \ldots, N$



synchronizing if $c\lambda_2 \in S_1$

or if $\dfrac{\lambda_N}{\lambda_2} \in S_2$

- Remark 1: Key factors influencing the synchronizability:

  (i) inner-coupling matrix H

  (ii) eigenvalues of outer-coupling matrix A

- Synchronizability in terms of (ii):

  1. bounded region (M. Banahona and L. M. Pecora, 2002)

  $$\lambda_N / \lambda_2, \ 0 = \lambda_1 < \lambda_2 \le \ldots \le \lambda_N$$

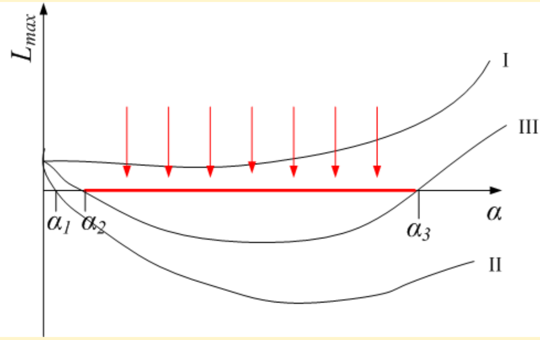
  2. unbounded region (X. F. Wang and G. Chen, 2002)

  $$c\lambda_2, \ 0 = \lambda_1 < \lambda_2 \le \ldots \le \lambda_N$$

  3. union of several disconnected regions

  (A. Stefanski, P. Perlikowski, and T. Kapitaniak, 2007)

  (Z. S. Duan, C. Liu, G. Chen, and L. Huang, 2007)



Recall:

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij} H(x_j), \ i = 1, 2, \ldots, N \qquad (1)$$

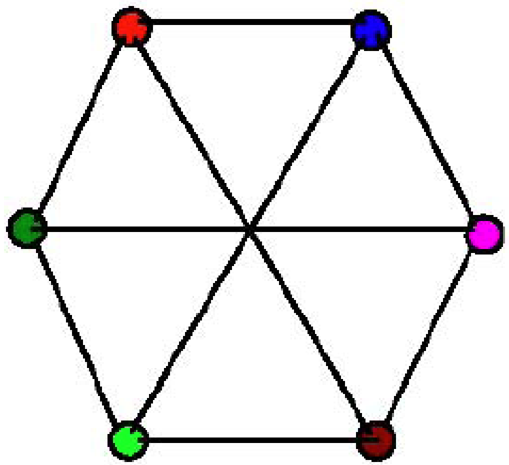
- Structural parameters that may affect the synchronizability:

  average distance, degree distribution, clustering coefficient, node betweenness, ...

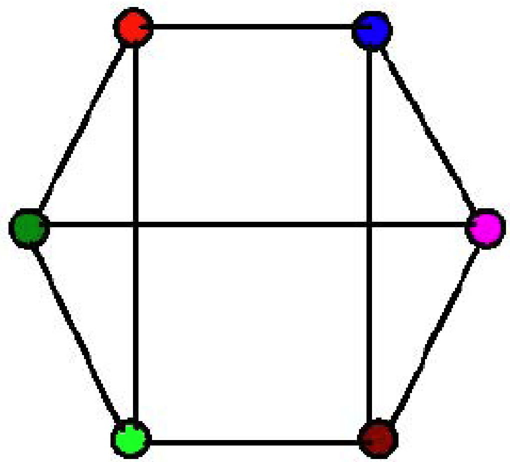
- Question:

  How does synchronizability depend on various structural parameters?

Two simple graphs:



Graph G_1



Graph G_2

- They have the same structural characteristics: Graphs G_1 and G_2 have
  - the same degree sequence: all node degrees: 3
  - the same average distance: $7/5$
  - the same node betweenness centrality: 2

- But they have different synchronizabilities:

  eigenvalues of G_1 are $0,3,3,3,3$ and 6

  eigenvalues of G_2 are $0,2,3,3,5$ and 5

$$\lambda_2(G_1) = 3 > \lambda_2(G_2) = 2$$

  Let $\gamma = \lambda_2 / \lambda_N$. Then

$$\gamma(G_1) = 0.5 > \gamma(G_2) = 0.4$$

About edge-adding:

Lemma 1: For any given connected undirected graph G, all its nonzero eigenvalues increase monotonically with the number of added edges, i.e., by adding any edge e, one has $\lambda_i(G+e) \geq \lambda_i(G)$.

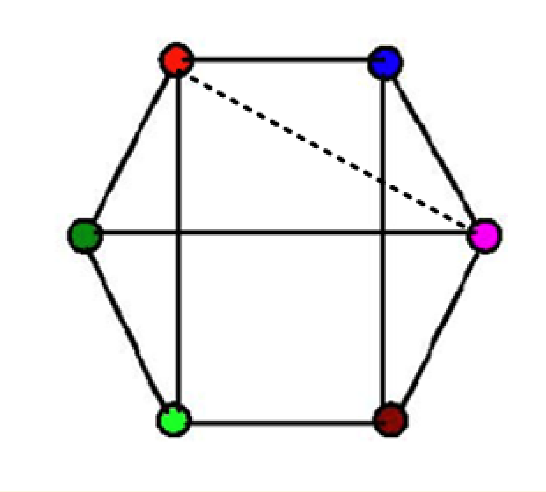
Z. Duan, G. Chen, and L. Huang (2007)

Lemma 1 → If the synchronized region is unbounded, then adding edges never decreases the synchronizability.

Main reason: $\lambda_i(G+e) \geq \lambda_i(G).$

However, for bounded synchronized regions, this is not necessarily true because the eigen-ratio can either increase or decrease

Example 1: Adding an edge between red node and pink node in graph G_2, $e\{red,pink\}$ leads to a new graph $G_2 + e\{red,pink\}$, whose eigenvalues are $0, 2.2679, 3, 4, 5$ and 5.7321, with $\gamma(G2 + e\{red,pink\})=0.3956$, smaller than $\gamma(G2)=0.4$, the synchronizability decreases



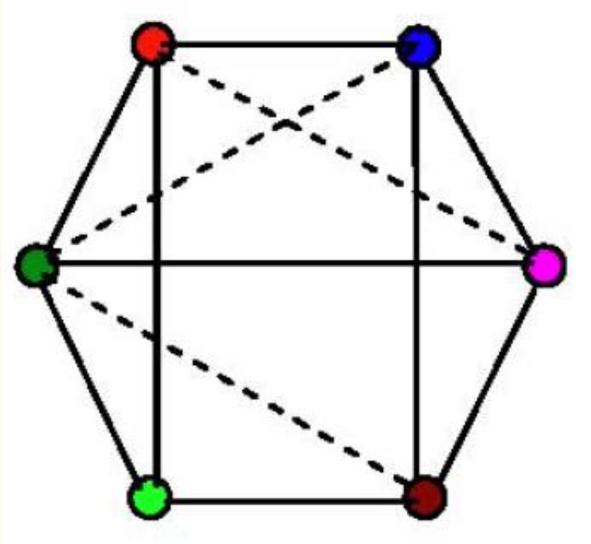
◆ Example 1: what about betweeness?

Consider the new graph

$$G = G_1 + e\{red,pink\} + e\{blue, green\} + e\{green, brown\}$$

Maximum node betweenness centrality of G is $11/6$ smaller than that of graph G_1 , 2 , but they have the same synchronizability.

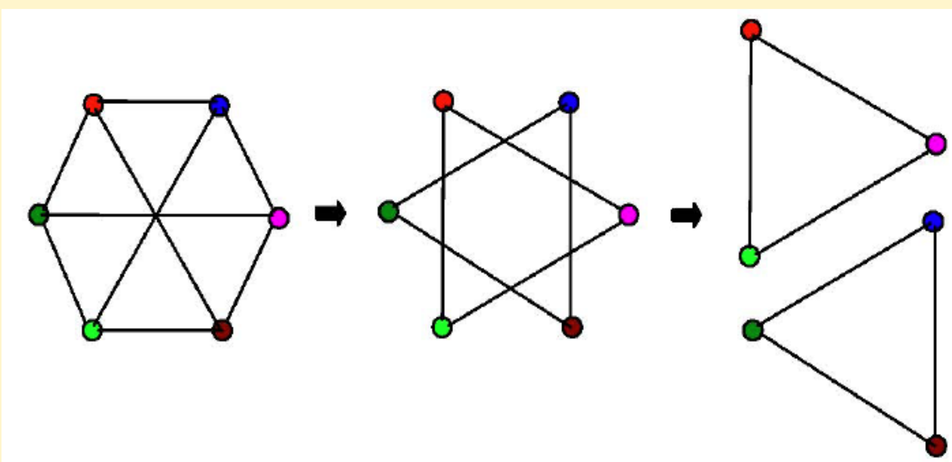
◆ Remark 3: The two simple graphs in Examples 1 and 2 show the complexity in the relationship between the synchronizability and the network structural parameters.

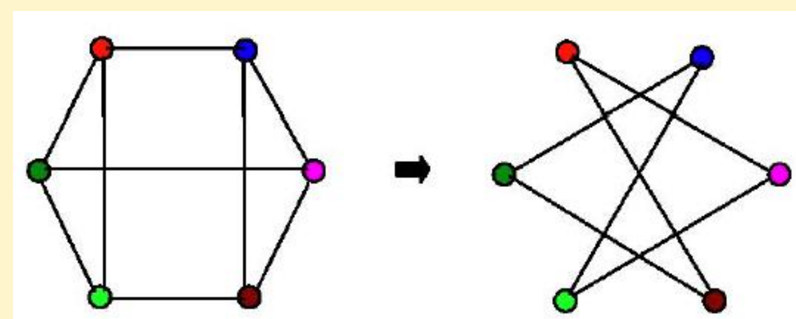


→ The synchronizability may have other (yet unknown) relationships to the common network structural parameters.

Complementary Graphs:
particularly when they are disconnected

- For a given graph G, the complement of G is the graph containing all the nodes of G, and the edges that are <u>not</u> in G, denoted by G^c
- The complements of G_1 and G_2:

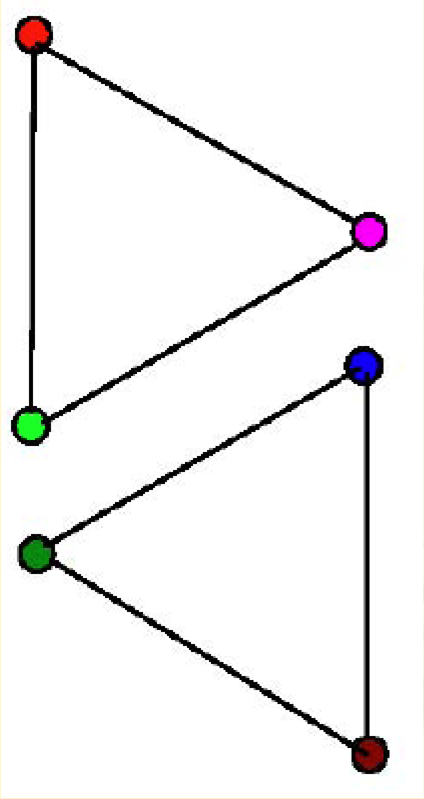




G_1 $G_1^{\,c}$ G_2 $G_2^{\,c}$

◆ Lemma 2: For any given graph G:
(i)the largest eigenvalue of G, $\lambda_N(G)$, satisfies $\lambda_N(G) \leq N$
(ii) $\lambda_N(G) = N$ if and only if G^c is disconnected. Moreover, if G^c has (exactly) q connected components, then the multiplicity of $\lambda_N(G) = N$ as an eigenvalue of G is q -1
(iii) $\lambda_N(G^c) = N - \lambda_{N-i+2}(G)$, $2 \leq i \leq N$

- For an example G_1, q=2, then an eigenvalue of G is 1*N



◆ G_1^c is disconnected. The largest eigenvalue of G_1 is 6, which remains the same for the graph with any more edges being added. Hence, Lemma 1→the synchronizability of all the networks built on graph G_1 never decrease with edge-adding.

Recall: G_1 *has eigenvalues: 0, 3, 3, 3, 3, 6*

Remark 4: The multiplicity of the largest eigenvalue of a graph G is related to the number of connected components of its complement G^c. In order to reduce the number of edges needed to enhance the synchronizability, the multiplicity of the largest eigenvalue of G^c (i.e., the multiplicity of the least nonzero eigenvalue of G) should be large.

Therefore, better understanding and careful manipulation of complementary graphs are useful for enhancing the network synchronizability; and, at least for dense networks, the complementary graphs are easier to analyze than the original graphs, e.g., G_1^c is simpler than G_1.

Disconnected Synchronized Regions of complex dynamical networks

- Typical types of synchronized regions:
  - Type I: unbounded $S_1 = (\alpha_1, \infty)$
  - Type II: bounded $S_2 = (\alpha_1, \alpha_2)$
  - Type III: empty
  - Type IV: union of some of the above

$(\alpha_1, \alpha_2) \cup (\alpha_3, \infty) \quad (\alpha_1, \alpha_2) \cup (\alpha_3, \alpha_4) \cup (\alpha_5, \alpha_6)$ and so on

A. Stefanski, P. Perlikowski, and T. Kapitaniak, (2007)
Z. Duan, G. Chen, and L. Huang, (2007)

- When the synchronous state is an equilibrium, the synchronized region problem reduces to a stability problem of the matrix pencil $F+\alpha H$ with respect to parameter α. (F is the Jacobian and H is the inner-linking matrix).
- Theorem: For any natural number n, there are two matrices F and H of order $(2n\text{-}1)$ such that $F+\alpha H$ has n disconnected stable regions with respect to parameter α.

Z. Duan, G. Chen, and L. Huang, (2007)

Examples of networks with disconnected synchronized regions

- Example 3: Consider network

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij} H(x_j),\ \ i = 1, 2, \ldots, N \qquad (1)$$

with third-order smooth Chua's circuits, namely,

$$\begin{aligned} \dot{x}_{i1} &= -k\alpha x_{i1} + k\alpha x_{i2} - k\alpha(a x_{i1}^3 + b x_{i1}) \\ \dot{x}_{i2} &= k x_{i1} - k x_{i2} + k x_{i3} \\ \dot{x}_{i3} &= -k\beta x_{i2} - k\gamma x_{i3} \end{aligned} \qquad (4)$$

Linearizing (4) at the zero equilibrium yields

$$dx_i / dt = Fx_i, \quad F = \begin{pmatrix} -k\alpha - k\alpha b & k\alpha & 0 \\ k & -k & k \\ 0 & -k\beta & -k\gamma \end{pmatrix}$$

Take $k = 1, \alpha = -0.1, \beta = -1, \gamma = 1, a = 1, b = -25$, and

$$H = \begin{pmatrix} 0.8348 & 9.6619 & 2.6519 \\ 0.1002 & 0.0694 & 0.1005 \\ -0.3254 & -8.5837 & -0.9042 \end{pmatrix}$$

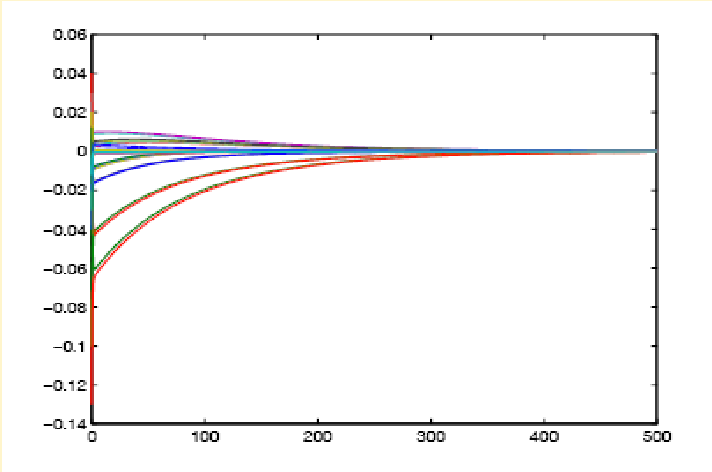
Then $F+\alpha H$ has a disconnected stable region:

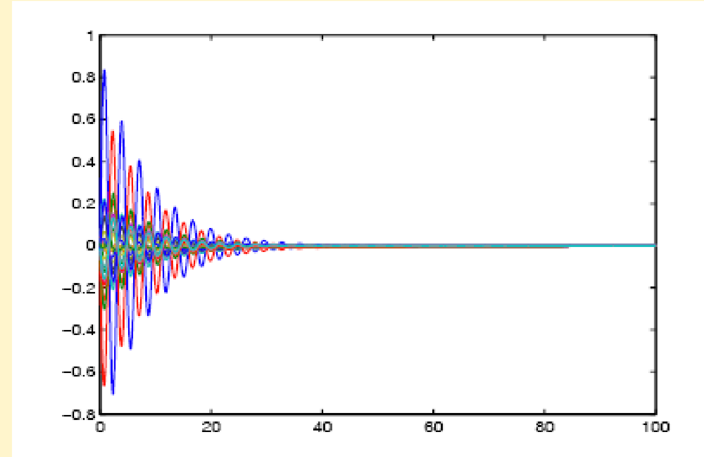
$$S = [-0.0099, 0] \cup [-2.225, -1)$$

- Suppose that the number of nodes of network (1) is N=10 and the outer coupled matrix A is a globally coupled matrix. Then network (1) with the above data achieves local synchronization when the coupling strength c satisfies

$$c \in [0, 0.00099] \quad \text{or} \quad c \in (0.1, 0.2225]$$

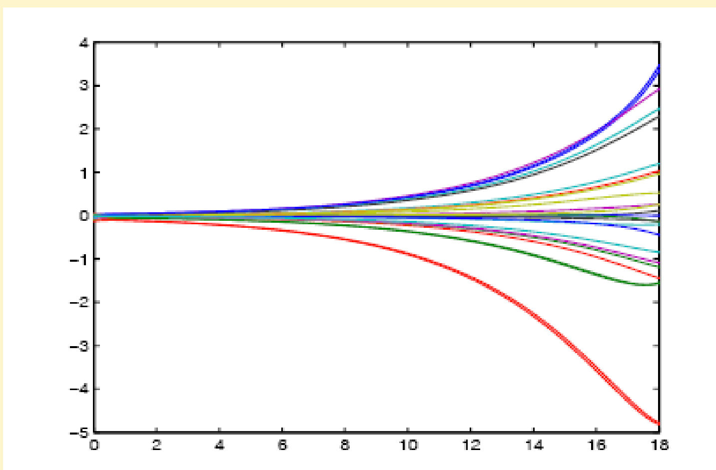
- Figures below visualize the synchronization and non-synchronization behaviors of this network.

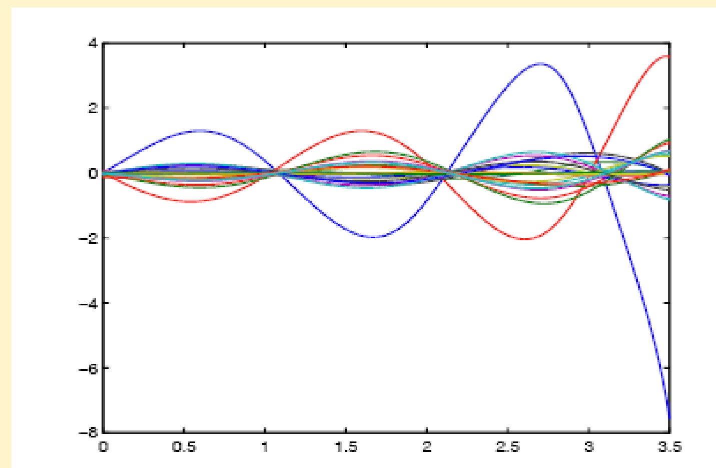




$(a)\ c = 0.0005 \in [0, 0.00099].$ $(b)\ c = 0.2 \in (0.1, 0.2225].$

Fig. 1 Synchronization of network (1) with different coupling strengths.





$(c)\ c = 0.02 \in (0.001, 0.1).$ $(d)\ c = 0.3 \in (0.2225, +\infty).$

Fig. 2 Non-synchronization of network (1) with different coupling strengths.

- Example 4: An interesting example of the coexistence of unbounded and bounded synchronization regions in the form of $S=(-\infty,-\alpha_1)\cup(-\alpha_2,-\alpha_3)$

Consider a linearly and diffusively coupled dynamical network with the state equations

$$\dot{x}_i = f(x_i)+c\sum_{j=1}^{N} a_{ij}H(x_j),\ \ i=1,2,\dots,N \qquad (5)$$

where

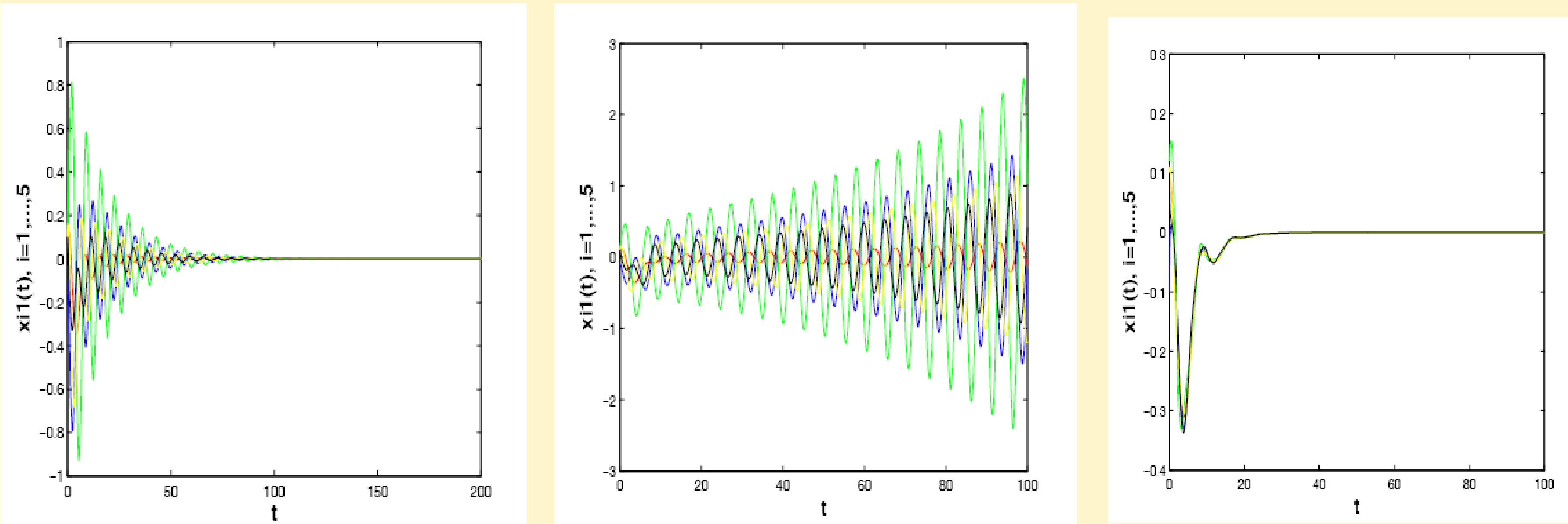
$$F=\begin{pmatrix} \frac{1}{4} & 0 & 1 \\ 0 & \frac{7}{16} & 1 \\ 0 & -3-\frac{7}{16}\cdot\frac{23}{16} & -\frac{23}{16} \end{pmatrix},\quad H=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- The synchronization region of network (5) with is

$$S = \left(-\infty, \frac{1}{4}(-3-\sqrt{2})\right) \cup \left(\frac{1}{4}\left((-3+\sqrt{2}\right), 0\right]$$

- Suppose that the number of nodes of network (5) is $N = 5$ and the outer coupled matrix A is a globally coupled matrix. Then, network (5) achieves local synchronization when the coupling strength c satisfies $c \in [0, 0.07929)$ or $c \in (0.22071, +\infty)$.

- Figures below visualize the synchronization and non-synchronization behaviors of this network.



$(a)\ c = 0.04 \in [0, 0.07929).$ $(b)\ c = 0.14 \in (0.07929, 0.22071).$ $(c)\ c = 1 \in (0.22071, +\infty).$

Fig. 3 Synchronization and non-synchronization of the first state variables of the five nodes in network (5) in the global coupling configuration: $x_{i1}(t), i = 1,2,\ldots,5.$