Modeling of Complex Networks

Lecture 2: Introduction to Graph Theory

S8101003Q (Sem A, Fall 2023)

What is a **Graph** ?

- A ***graph*** is a diagrammatical representation of some physical structure

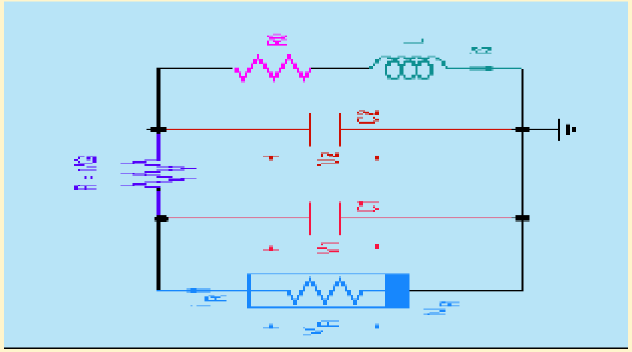
  such as:

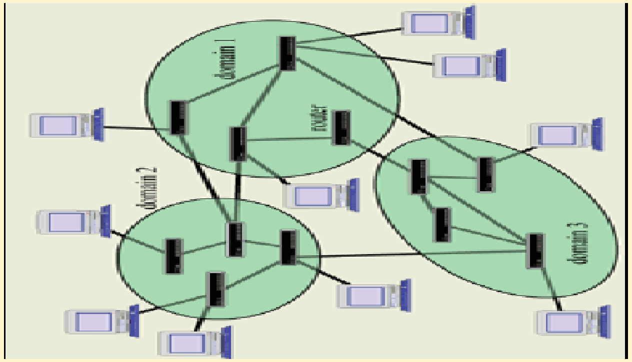
  a circuit

  a computer network

  a human relation network

  …and so on.



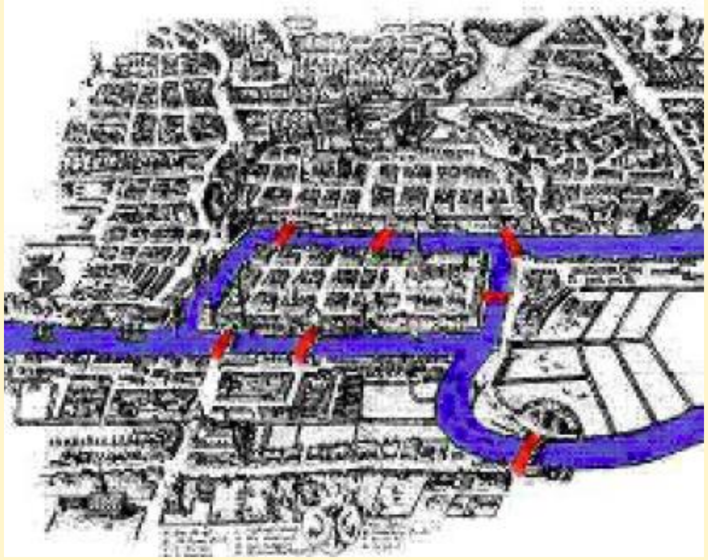


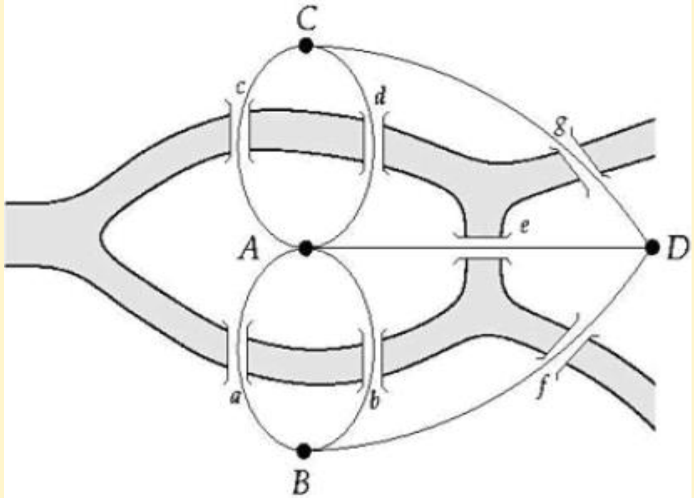
"Crowd Counting by Using Top-k Relations: A Mixed Ground-Truth CNN Framework." L Dong, H Zhang, K Yang, D Zhou, J Shi, J Ma, IEEE Transactions on Consumer Electronics 68 (3), 307-316, 2022.
https://ieeexplore.ieee.org/abstract/document/9828527

Example:
The seven-bridge problem of Königsburg

- Eüler (1707-1783) studied and solved this famous seven-bridge problem of Königsburg

  -- no solution!



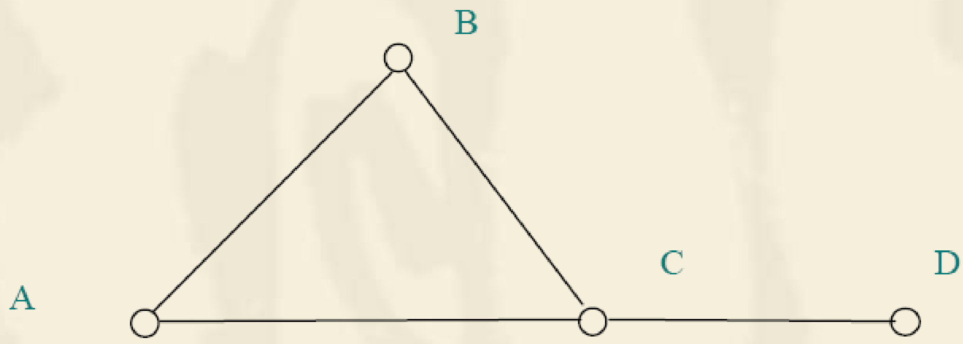


- Eüler was the pioneer of **Graph Theory**

Notation

- Let G be a non-empty graph with at least one **node** (or vertex).
- In a non-isolated case, G has at least one **edge** (or link); thus, it has at least two nodes.
- Let $N(G)$ and $M(E)$ denote the set of its nodes and the set of its edges, respectively.
- In general, $N(G)$ and $M(E)$ are finite sets.
- Such a non-trivial pair $(N(G), M(E))$ is referred to as **a simple graph** G
- A simple graph, also called a strict graph (Tutte 1998), is an unweighted, undirected graph containing no graph loops or multiple edges (Gibbons 1985; West 2000;). A simple graph may be either connected or disconnected.

Examples of Simple Graph



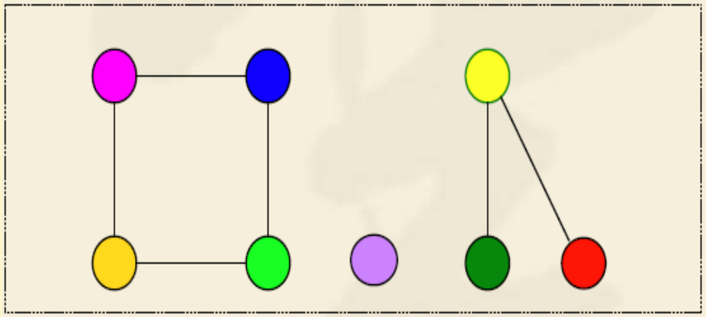
$$N(G) = \{A, B, C, D\}$$

$$M(E) = \{AB, AC, BC, CD\}$$

Subgraph: $ABC \quad AB \quad D$ etc. (needed not to be connected)

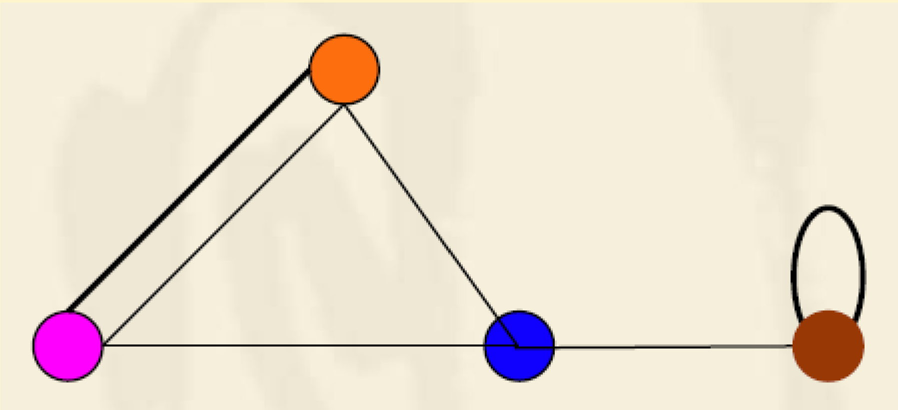
Loop (Circuit): ABC

Component: A self-connected subgraph, but un-connected with other parts of the same graph

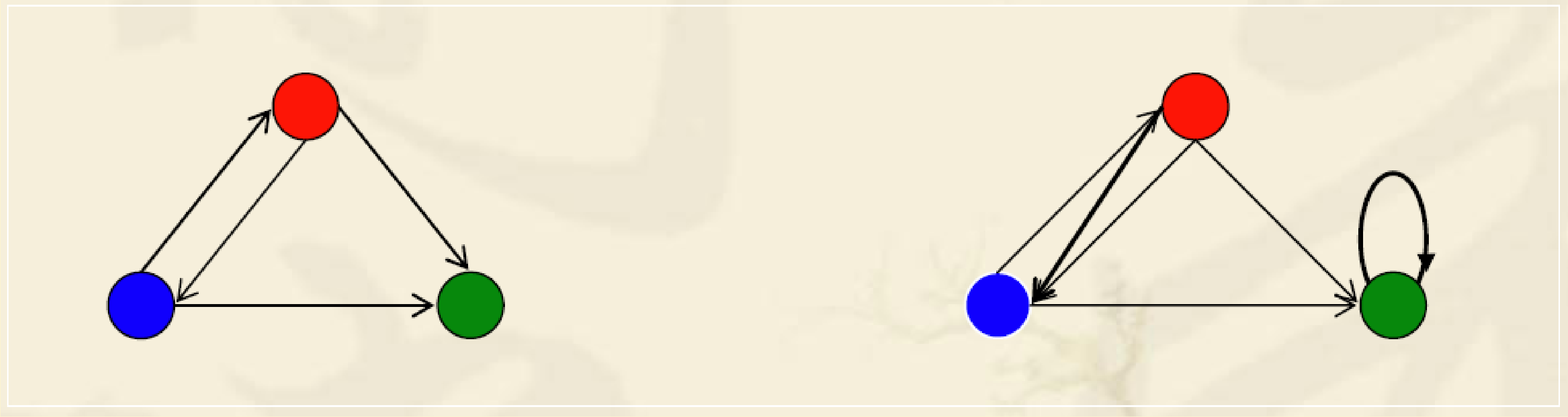


A simple graph with 3 components

Examples of General (non-simple) Graphs



graph with a self-loop



digraph – directed graph

general (non-simple) digraph

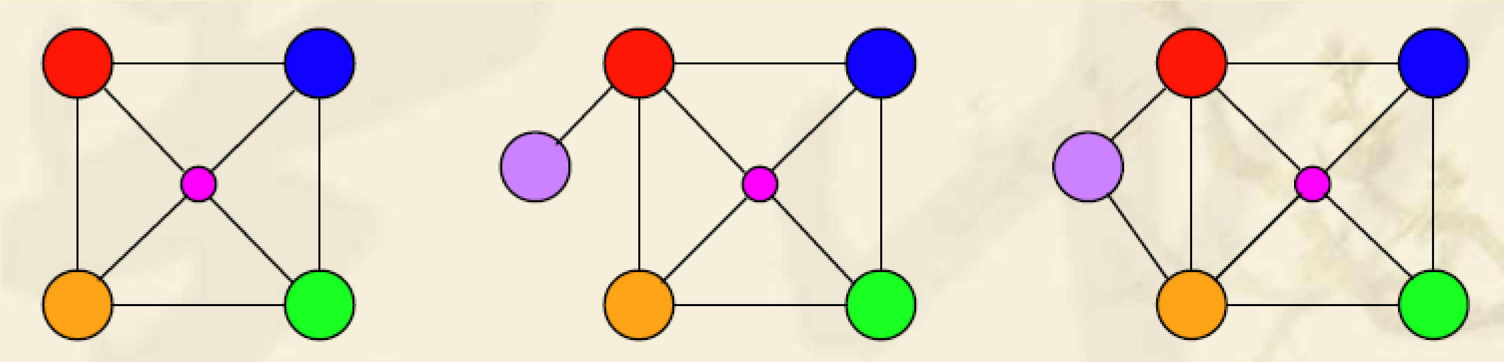
All such complex graphs are not studied in this course - basically

Some Basic Results

- **Theorem 2-1 (Handshaking Lemma)** *The total node degree of a graph is always an even number.*

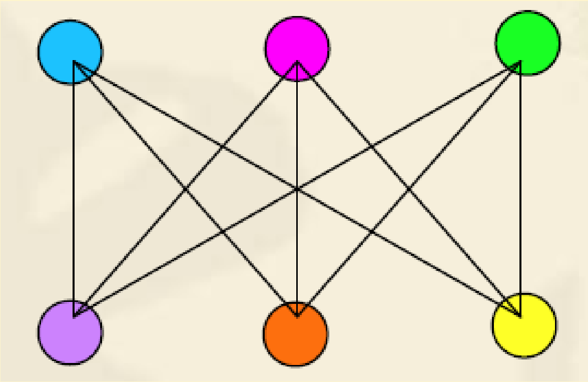
Proof. Since every edge joins two nodes, so the total node degree is twice of the number of edges.

- **Corollary 2-2** *In any graph, the number of nodes of odd degrees must be even.*



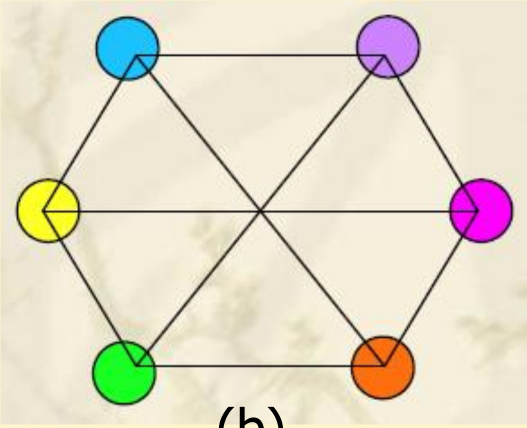
Isomorphism

- Two graphs G_1 and G_2 are said to be **isomorphic**, if there is a one-one correspondence between the nodes of G_1 and those of G_2, with the property that the number of edges joining any two nodes of G_1 is equal to the number of edges joining the two corresponding nodes of G_2.
- Example:



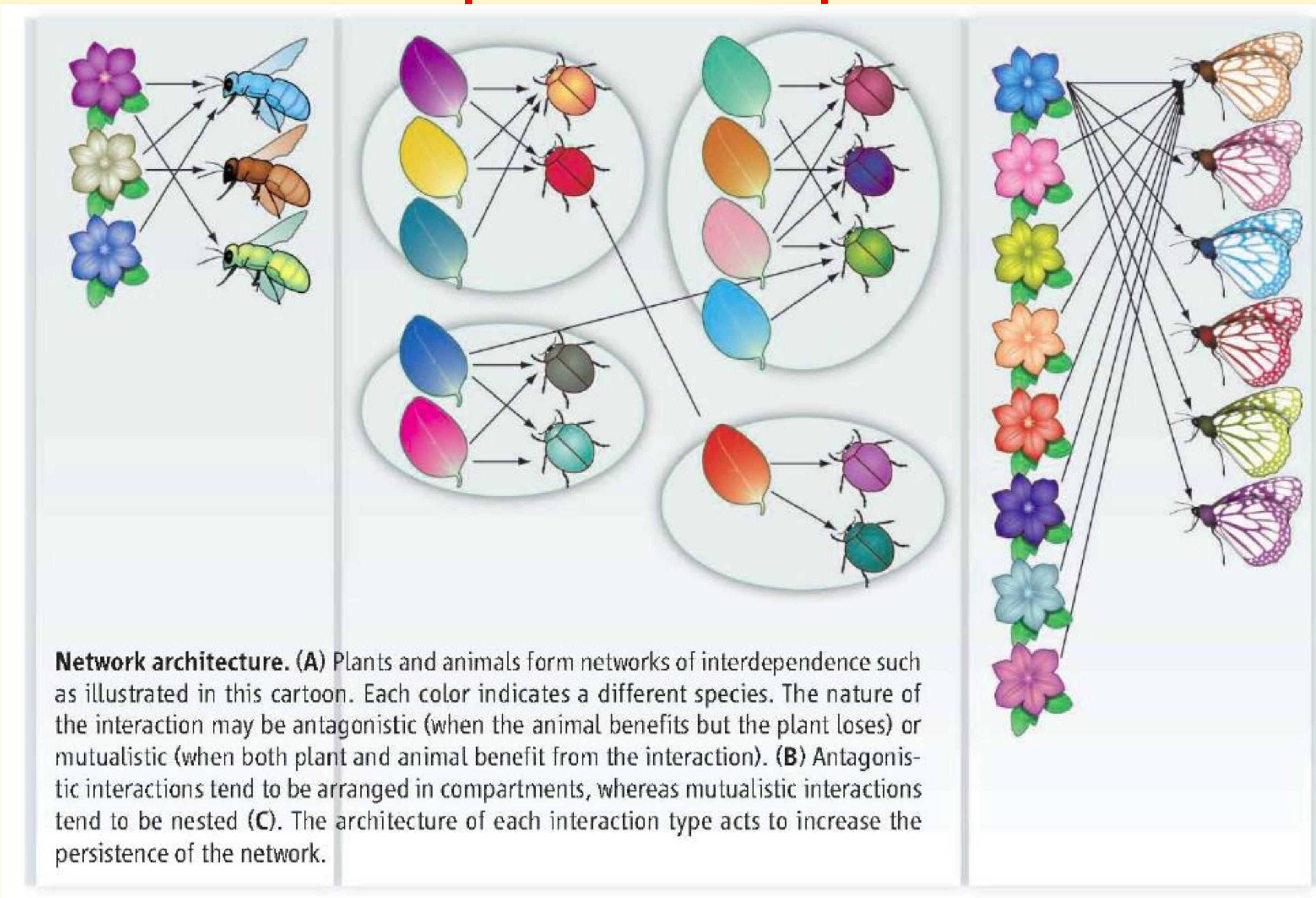
(a)

bipartite graph



(b)

Bipartite Graphs

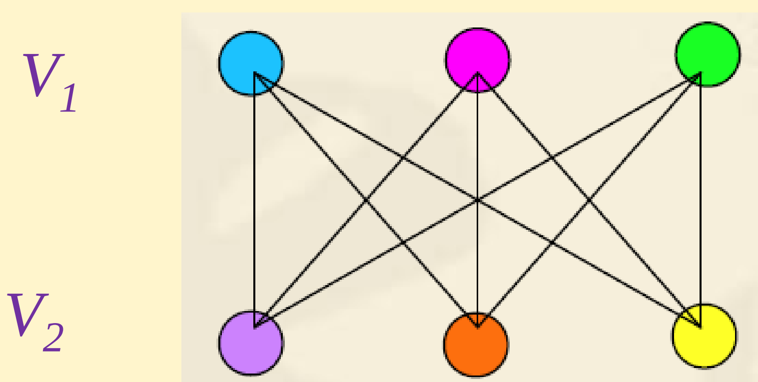


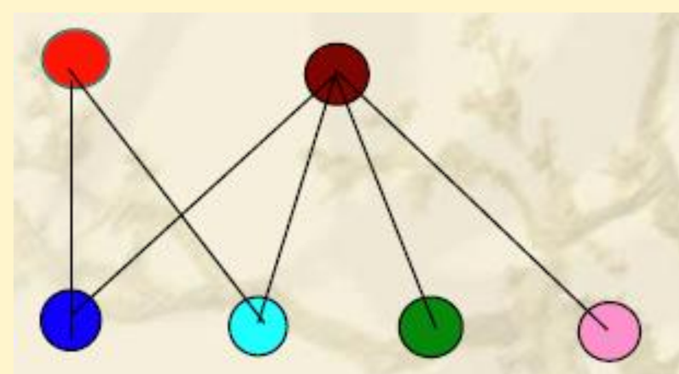
Network architecture. **(A)** Plants and animals form networks of interdependence such as illustrated in this cartoon. Each color indicates a different species. The nature of the interaction may be antagonistic (when the animal benefits but the plant loses) or mutualistic (when both plant and animal benefit from the interaction). **(B)** Antagonistic interactions tend to be arranged in compartments, whereas mutualistic interactions tend to be nested **(C)**. The architecture of each interaction type acts to increase the persistence of the network.

Circuits in Bipartite Graphs

- Theorem 2-6 *In any bipartite graph, every loop (circuit) has an even number of edges in the path.*

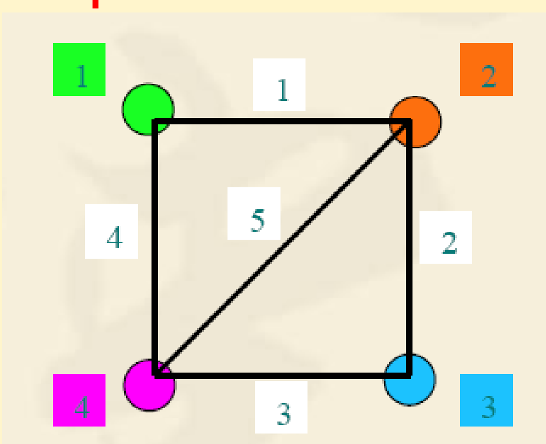
Proof. Let $v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_n \rightarrow v_1$ be a circuit in the bipartite graph $G = G(V_1,V_2)$. Assume, without loss of generality, that $v_1 \in V_1$. Then, since G is bipartite, one must have $v_2 \in V_2$, $v_3 \in V_1$ etc. Finally, one must have $v_n \in V_2$ because the network is bipartie, which forms a circuit, yielding an even number of paths.





Adjacency and Incidence Matrices

- For a graph G with nodes $N(G) = \{1,2,\ldots,n\}$, its adjacency matrix A is defined to be the n x n constant matrix those ij-th entry is 1 if node i connects node j; or 0 otherwise. [connectivity matrix]
- Further, let G have edges $M(E) = \{1,2,\ldots,m\}$. Then, its incidence matrix M is defined to be the n x m constant matrix whose ij-th entry is 1 if node i connects edge j ; or 0 otherwise.
- Example:



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

(always square and symmetrical)

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(maybe non-square)

Laplacian Matrix

Definition: **Laplacian matrix** (or, **admittance matrix** or Kirchhoff matrix),

denoted , $L = \lfloor L_{ij} \rfloor$ is defined as

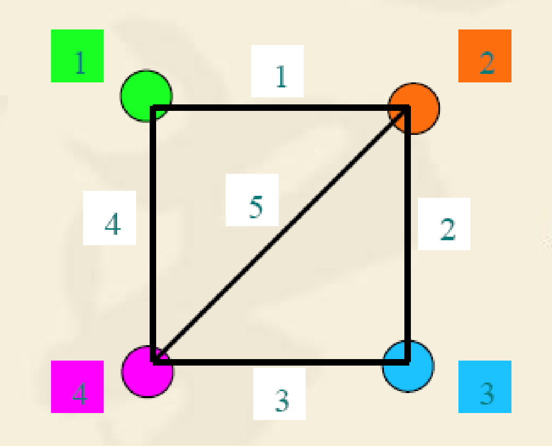
$$L_{ij} = \begin{cases} k(v_i) & if \quad\quad\quad\quad i = j \\ -1 & if \quad i \neq j, \quad v_i \quad adjacent \quad v_j \\ 0 & \quad\quad otherwise \end{cases}$$

where $k(v_i)$ is the degree of node v_i

It is always semi-positive definite; with eigenvalues $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_n$

Example:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$



$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - A$$

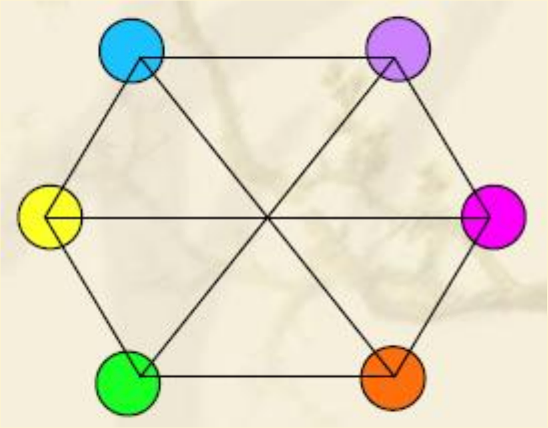
Regular Graphs

- A graph in which every node has the same degree is called a regular graph; if every node has degree r then the graph is called a regular graph of degree r
- **Theorem 2.3** *A regular graph of degree* r *with* N *nodes has* $r\,N\,/\,2$ *edges.*

  Proof. Since every node connects with r edges, there are $r\,N$ connecting edges. However, each edge has been doubly counted, so it should be divided by two.

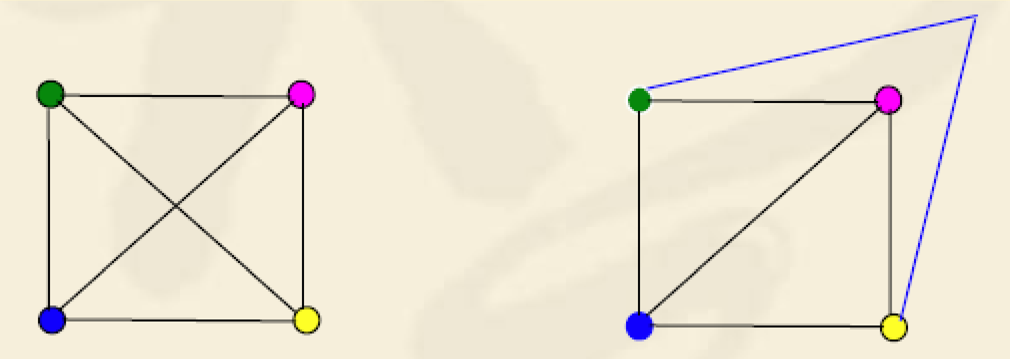
- Example:

  A regular graph of degree 3, with 6 nodes, which has $3 \times 6\,/\,2 = 9$ edges.



Graph Embedding

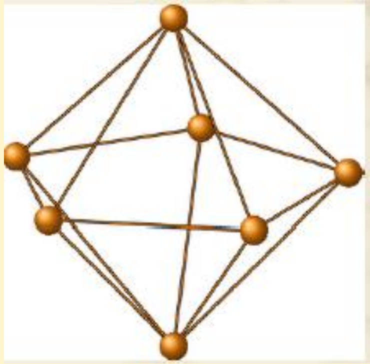
- Jordan curve on the plane: A continuous curve with no self-crossings.



(a) (b)

All the curves shown in Fig.(b) are Jordan curves on the plane, but the two in the middle of Fig.(a) are not.

- A graph G can be embedded in the Euclidean 3d space if it is isomorphic to a graph in the space with Jordan curve edges (i.e. there are no crossings in the resulting graph diagram in the Euclidean 3-space).

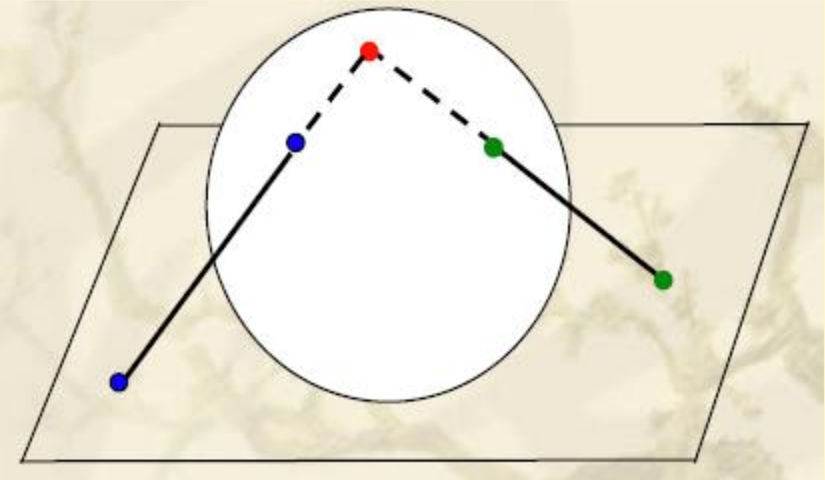


- **Theorem 2-4** Every simple graph can be embedded in the Euclidean 3d space. (the proof can be found in the textbook)

Planar Graphs

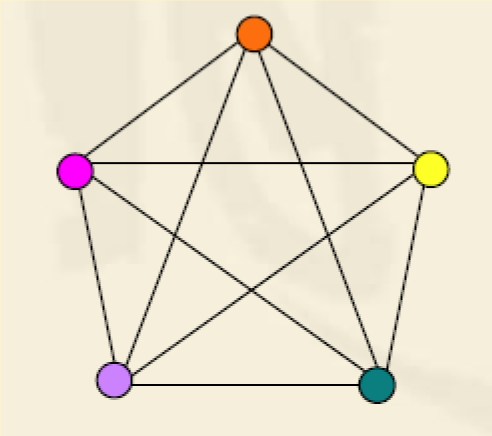
- Plane graph is the one that can be drawn on the plane without crossing edges
- Planar graph is the one that is isomorphic to a plane graph.
- Every planar graph can be embedded on a plane (within the Euclidean 3d space)

- **Theorem 2-5** *A graph is planar if and only if it can be embedded on the surface of a sphere.*
(the proof can be found in the textbook)

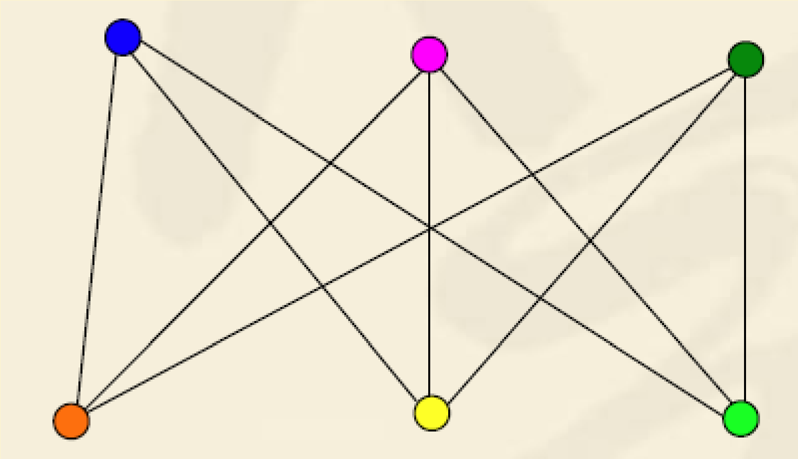


Non-Planar Graphs

- Two special yet important non-planar graphs:



Graph K_5

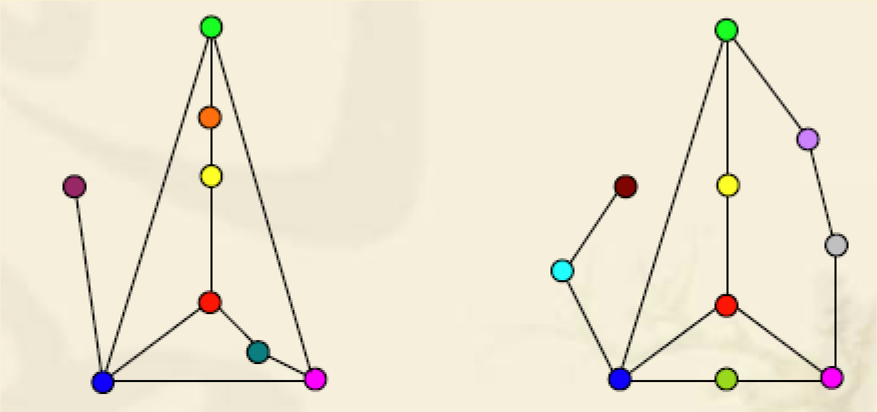


Graph $K_{3,3}$

K. Kuratowski (1896 -1980)

Homeomorphism

- Two graphs are said to be **homeomorphic**, if they can both be obtained from the same graph by inserting new nodes of degree two into edges (i.e., identical to within nodes of degree two).

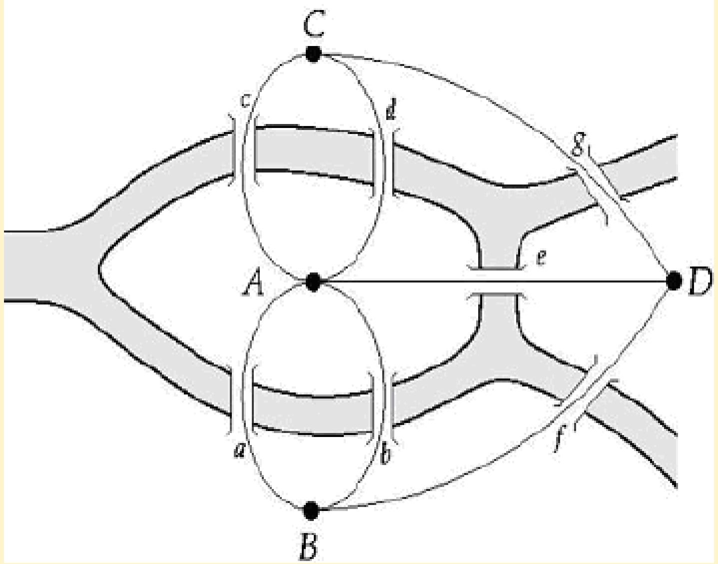


- **Theorem 2-22** (Kuratowski Theorem) A graph is planar if and only if it contains no subgraphs homeomorphic to K_5 or $K_{3,3}$

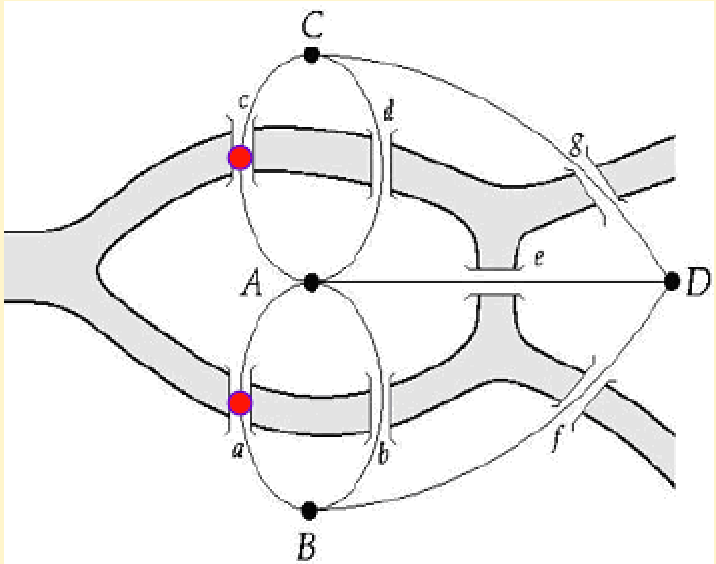
Proof (omitted)

An application Example

Recall: The seven-bridge problem of Königsburg



Homeomorphic

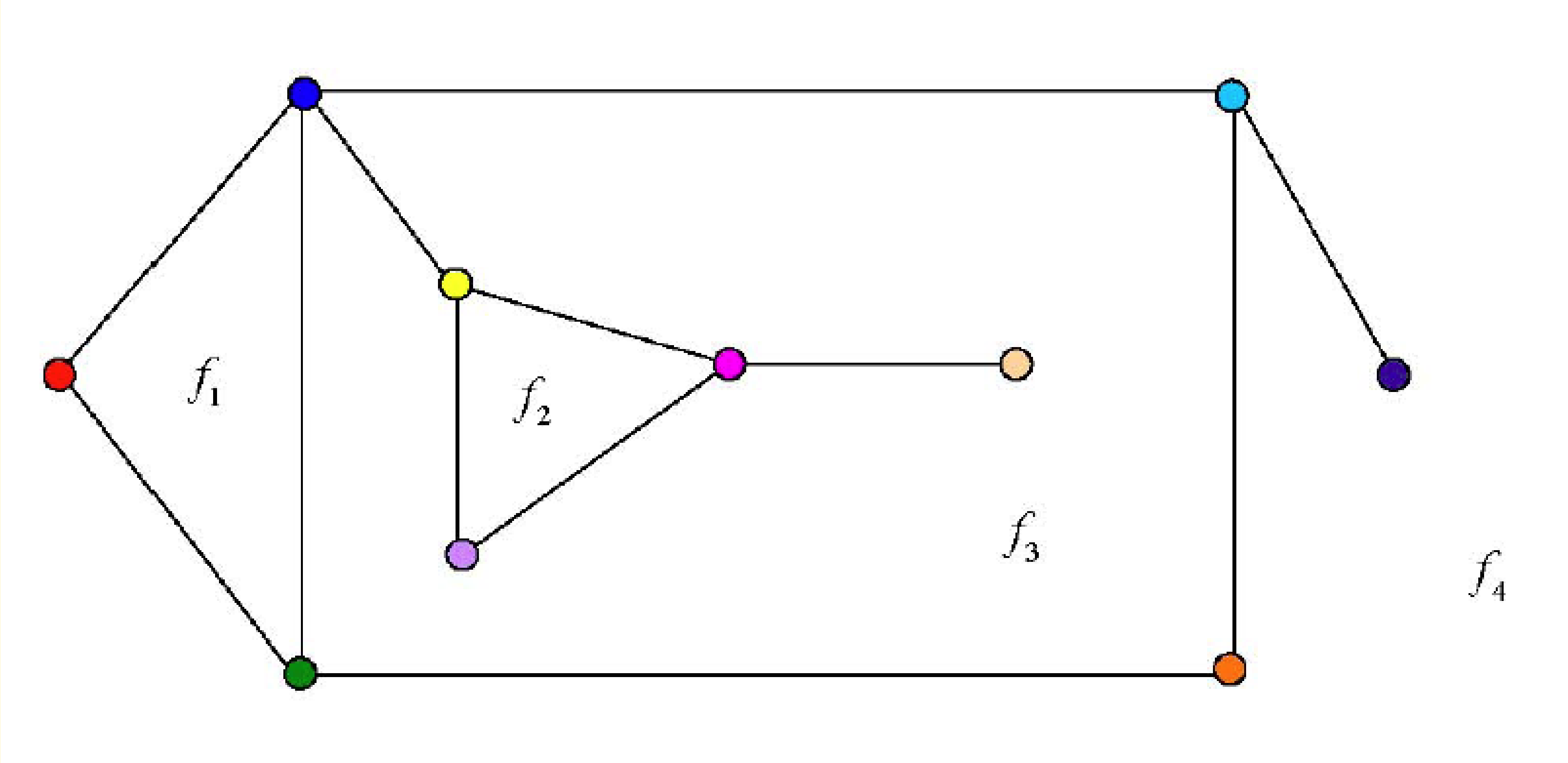


With multiple edges

A simple graph

Eüler Formula for Plane Graphs

When a graph is drawn without any crossing, any cycle that surrounds a region without any edges reaching from the cycle into the region forms a **face**



There are four faces: f_1, f_2, f_3 and f_4 (it is an infinite face)

Eüler Formula for Plane Graphs

- **Theorem 2-23** (Euler, 1750) Let n, e and f be the number of nodes, edges and faces of a connected plane graph, respectively. Then

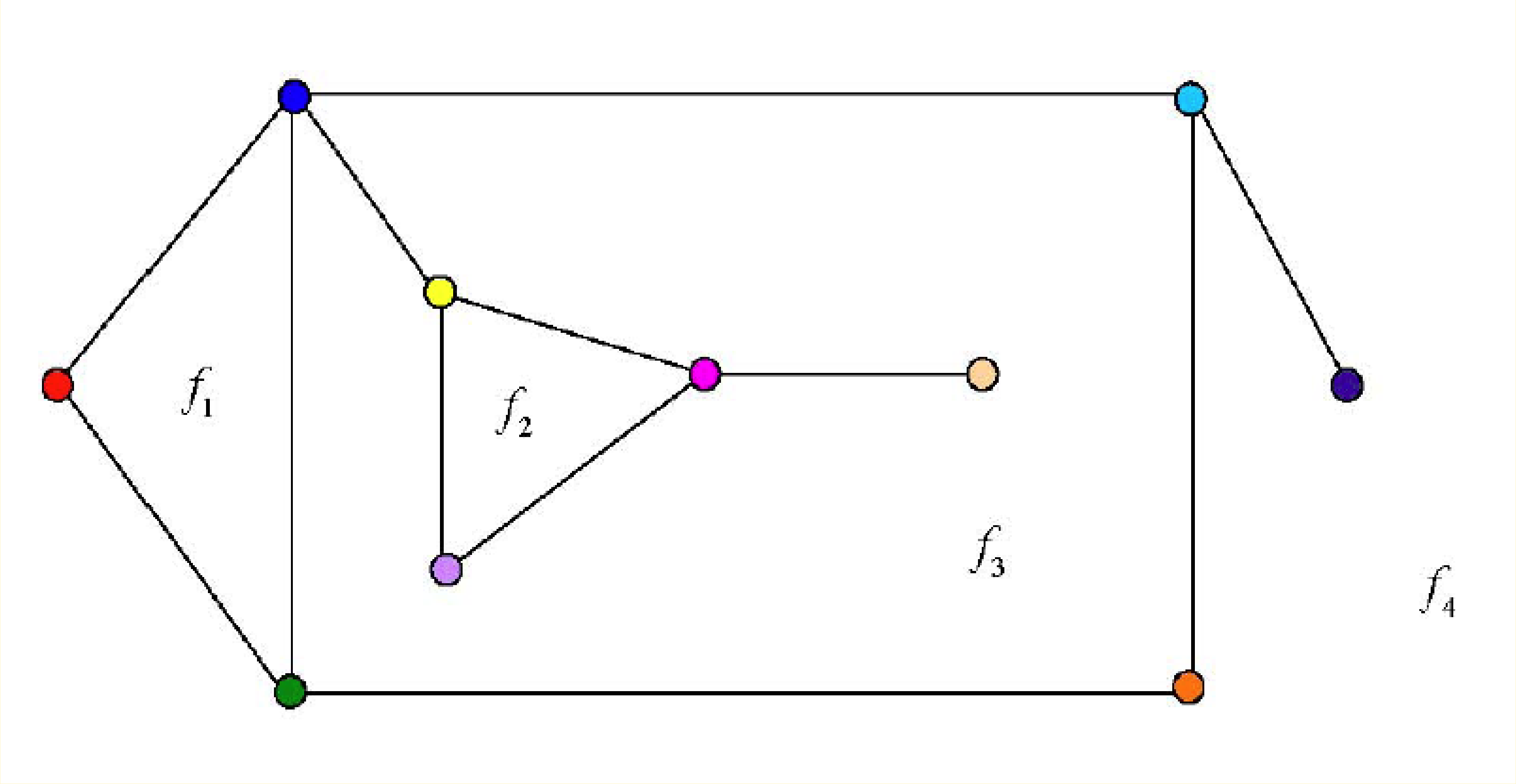
$$n - e + f = 2$$

Proof. Apply induction on the number of edges of the graph. If $e = 0$ then $n = 1$, since the graph is connected, and in this case $f = 1$, the infinite face. So the theorem is true. Suppose that the theorem is true for all graphs with at most $e - 1$ edges, and then consider a connected plane graph with e edges. If the graph is a tree, then $e = n - 1$ and $f = 1$, the infinite face, therefore the theorem is true. If the graph is not a tree, then remove an edge e from any circuit in the graph. This will result in a connected plane graph with n nodes, $e - 1$ edges and $f - 1$ faces, so that $n - (e - 1) + (f - 1) = 2$ by the induction hypothesis, which gives $n - e + f = 2$. This completes the induction. #

Eüler Formula for Plane Graphs

- **Theorem 2-23** (Euler, 1750) Let n, e and f be the number of nodes, edges and faces of a connected plane graph, respectively. Then

  $$n - e + f = 2$$



$n = 10, e = 12, f = 4 \rightarrow n - e + f = 2$

Some More Results

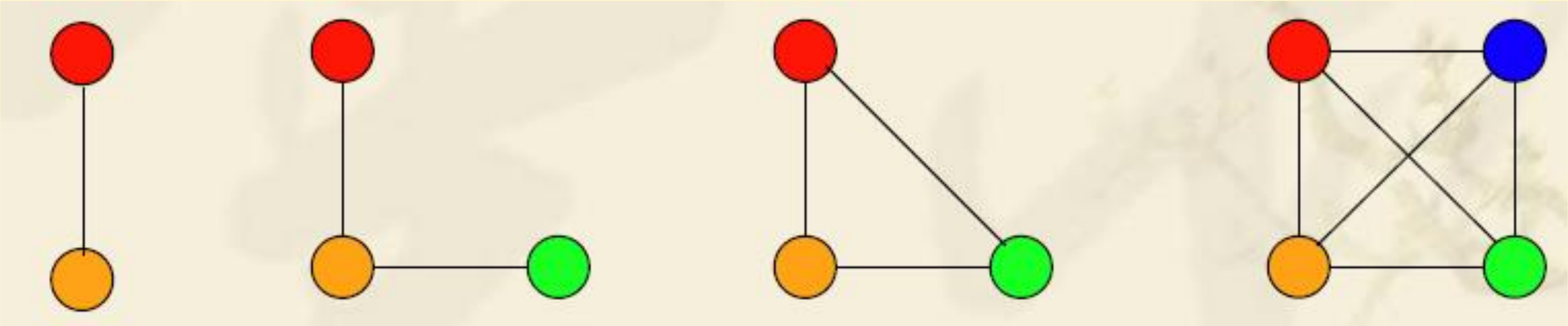
- **Theorem 2-7** If a simple graph G with N nodes has K components, then the number of edges, M, of G satisfies

  $$N - K \leq M \leq (1/2)(N - K)(N - K + 1)$$

  In particular, for a connected graph, it reduces to

  $$N - 1 \leq M \leq (1/2)N(N - 1)$$

- *Proof.* The general case is proved in the textbook, while the case of $K = 1$ is obvious: a connected graph with N nodes has at least $N - 1$ edges and at most $(1/2)N(N - 1)$ edges.



and More …

- **Corollary 2-12** If a simple graph with N nodes satisfies $M > (1/2)(N-1)(N-2)$ then it must be connected.

- *Proof.* If not connected, then $K \geq 2$ in $N - K \leq M \leq (1/2)(N-K)(N-K+1)$
  In case of $K = 2$: $N - 2 \leq M \leq (1/2)(N-1)(N-2)$
  But, now it is assumed $M > (1/2)(N-1)(N-2)$
  This is a contradiction.

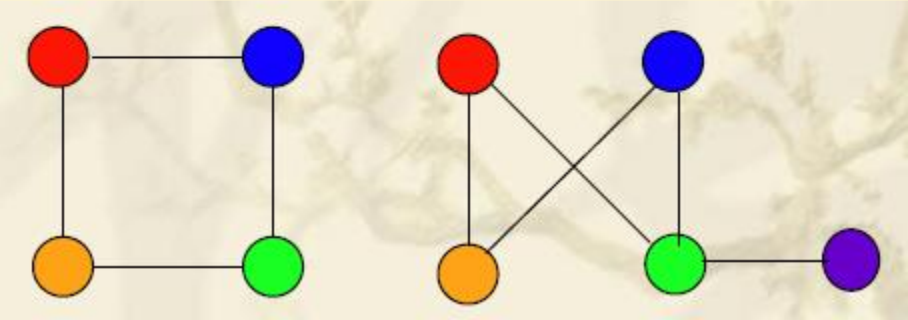
and More …

- **Lemma 2-13** If every node in a graph has degree $r \geq 2$ then this graph contains a loop (circuit).

- *Proof.*

  Consider a simple graph. Starting from any node v_0, construct a walk $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots$ such a way that v_1 is any adjacent node of v_0 and, for $i = 1,2,\ldots,$node v_{i+1} is any (except v_{i-1}) adjacent node of v_i. Since every node has degree $r \geq 2$, such a node v_{i+1} exists. Since the graph has finitely many nodes, the walk eventually connects to a node that has been chosen before. This walk yields a circuit in the graph.

The converse may not be true.



More Concepts

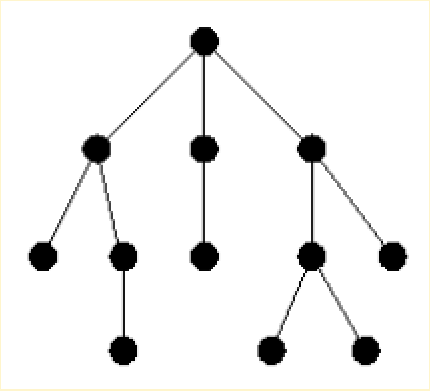
- Walk: A finite sequence of edges, one after another, in the form of $v_1 v_2$, $v_2 v_3$,…, $v_{n-1} v_n$ where $N(G) = \{v_1,v_2,\ldots,v_n\}$ are nodes.

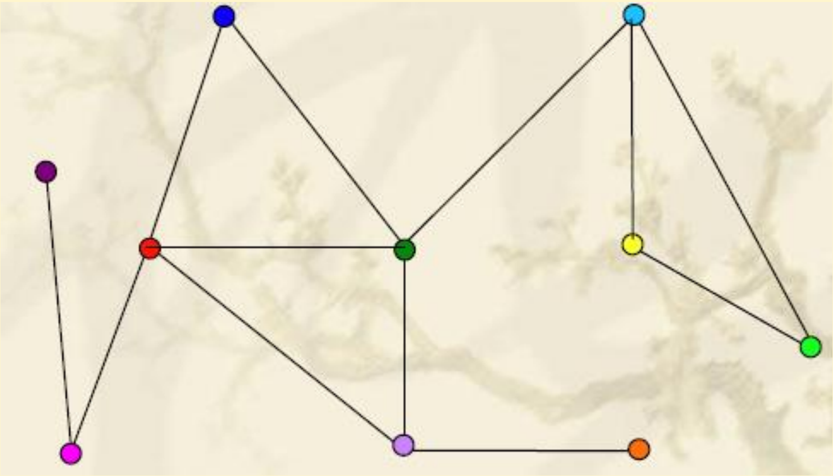
- A walk is denoted by $v_1 \rightarrow v_2 \rightarrow \ldots v_n$ and the number of edges in a walk is called its length.

- Trail: A walk in which all edges are distinct.

- Path: A trail in which all nodes are distinct, except perhaps $v_1 = v_n$ which is called a closed path, often called a circuit (or, sometimes, a loop or a cycle).

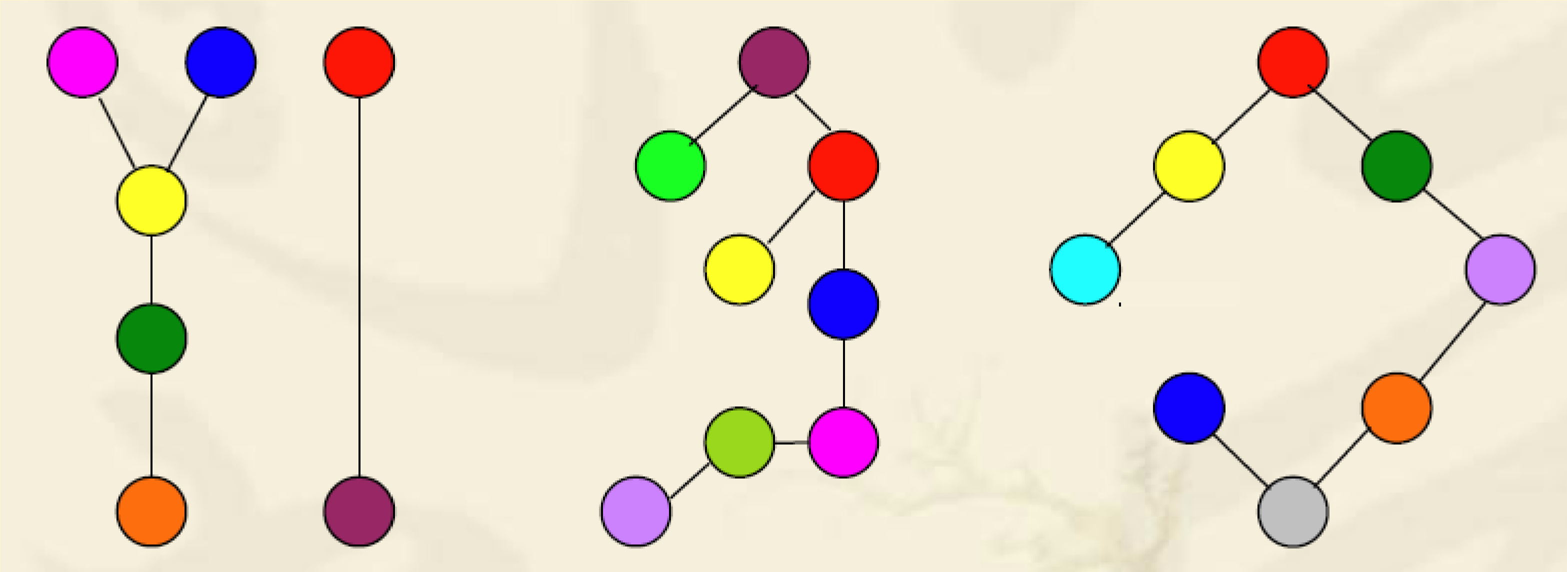
- Tree: A graph with no circuits.





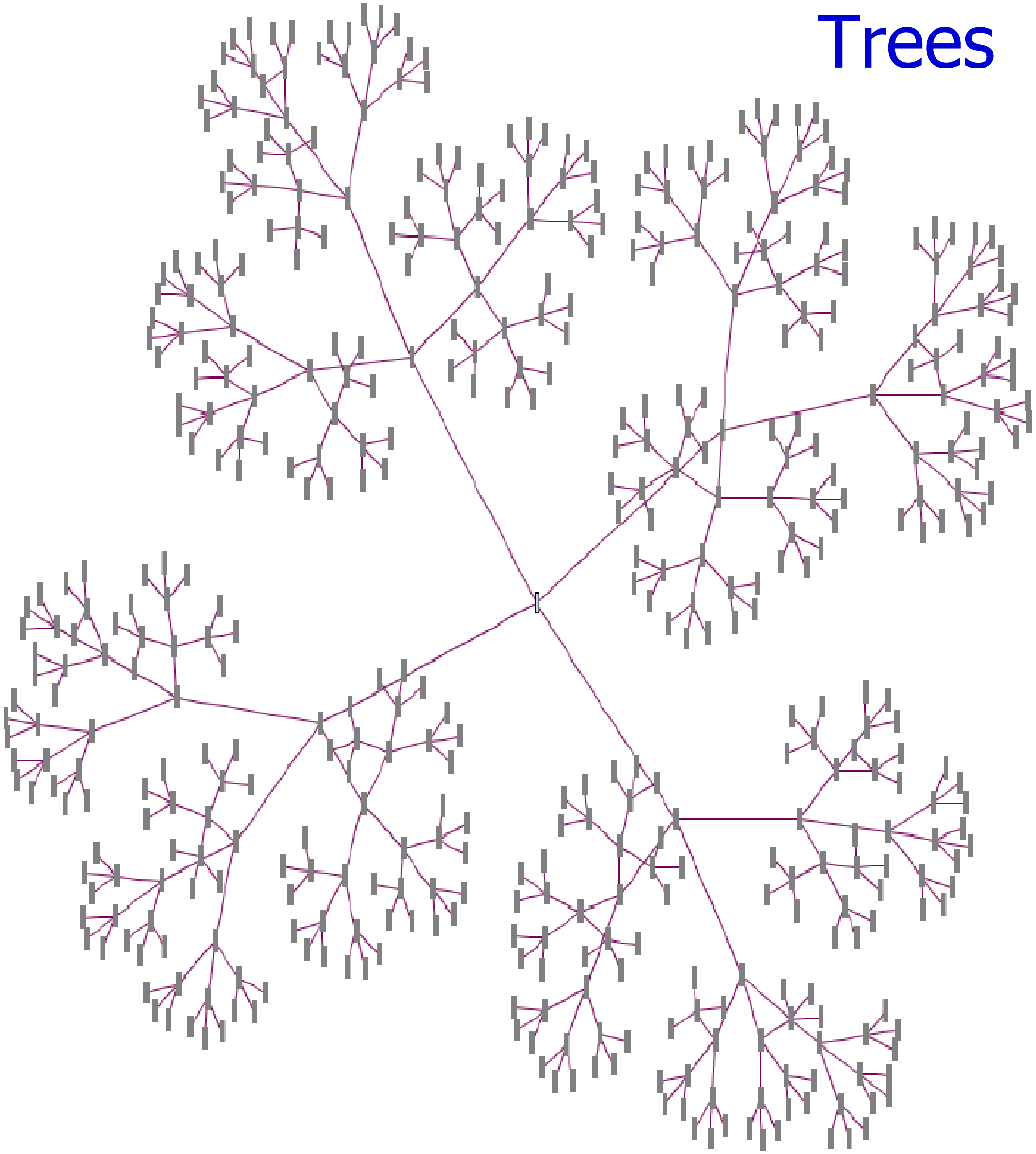
Trees

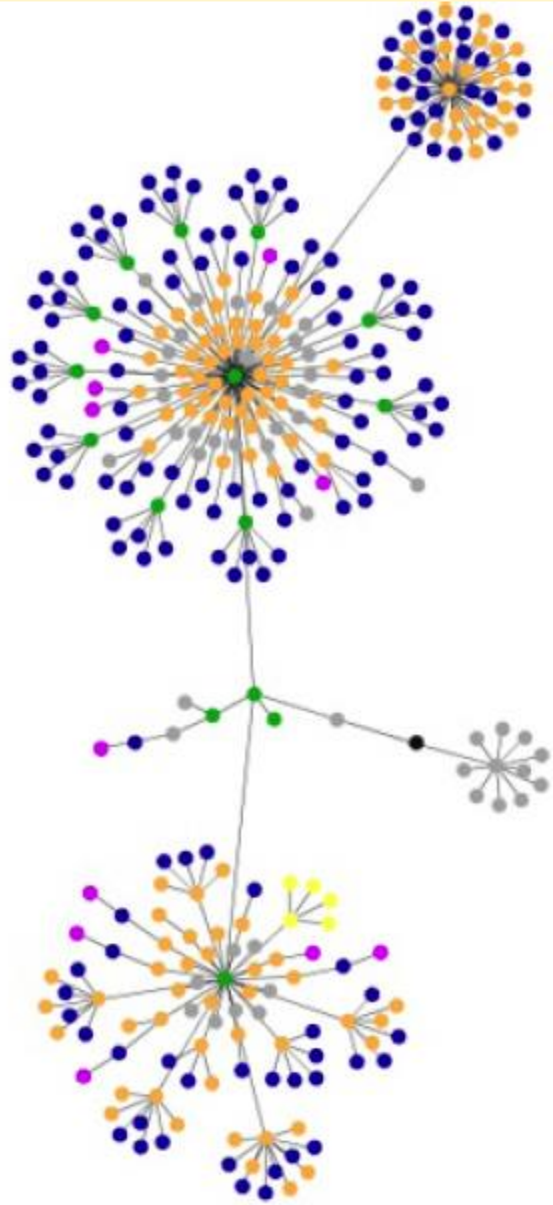
- Tree: A connected graph without circuits.
- Forest: A family of unconnected trees.



- A tree with N nodes has N-1 edges
- Sum of node degrees in a tree
  = 2 x (number of edges) $= 2\ (N - 1)$

Trees





Some Basic Results

- Theorem 2-18: *Let T be a graph with N nodes. Then, the following statements are equivalent:*

1) T is a tree;

2) T has N -1 edges but contains no circuits;

3) T has N -1 edges and is connected;

4) T is connected, but the removal of any edge will disconnect the graph;

5) every pair of nodes of T are connected by exactly one path;

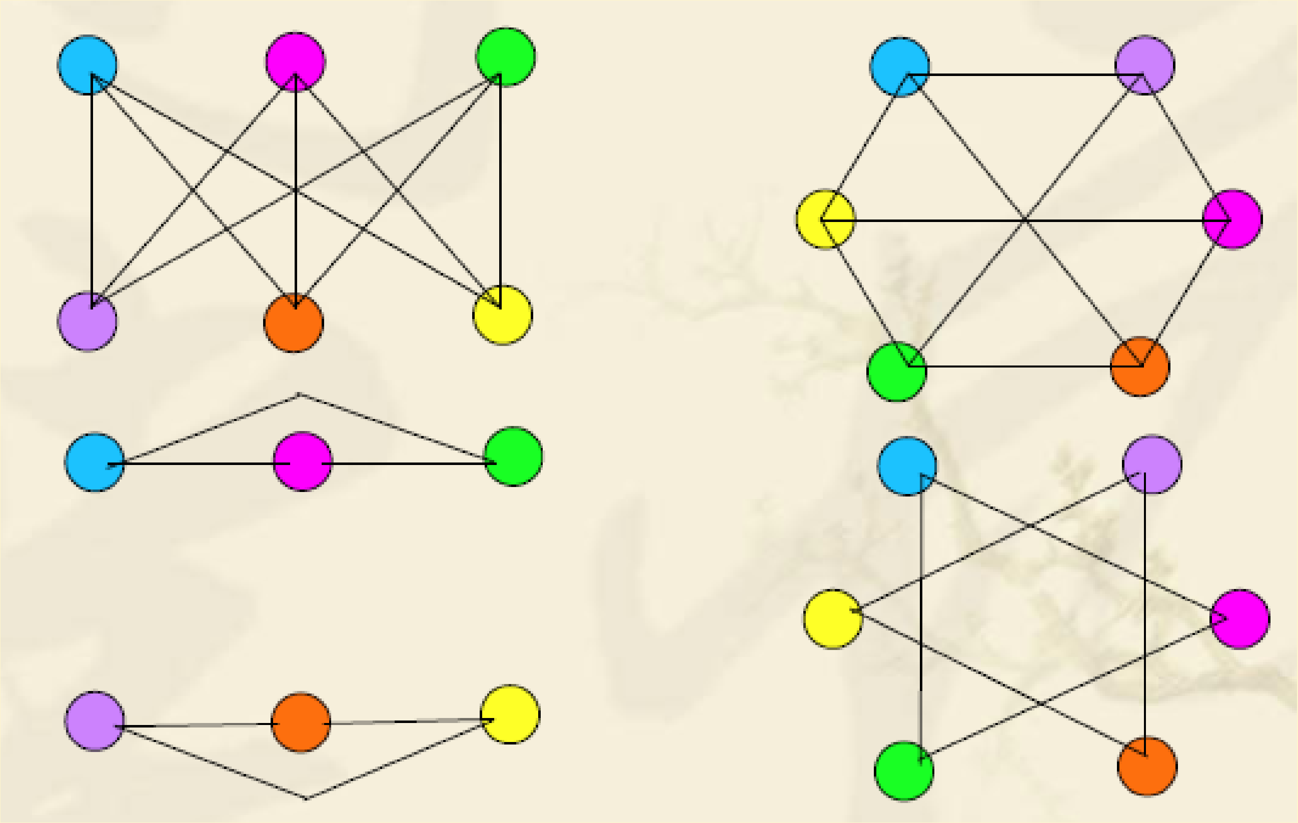
6) T contains no circuits, but the addition of any new edge creates exactly one circuit.

Complementary Graph

For a given graph G, its *complementary graph* G^c is the graph containing all the nodes of G and all the edges that are not in G (is a complementary graph unique for G?)

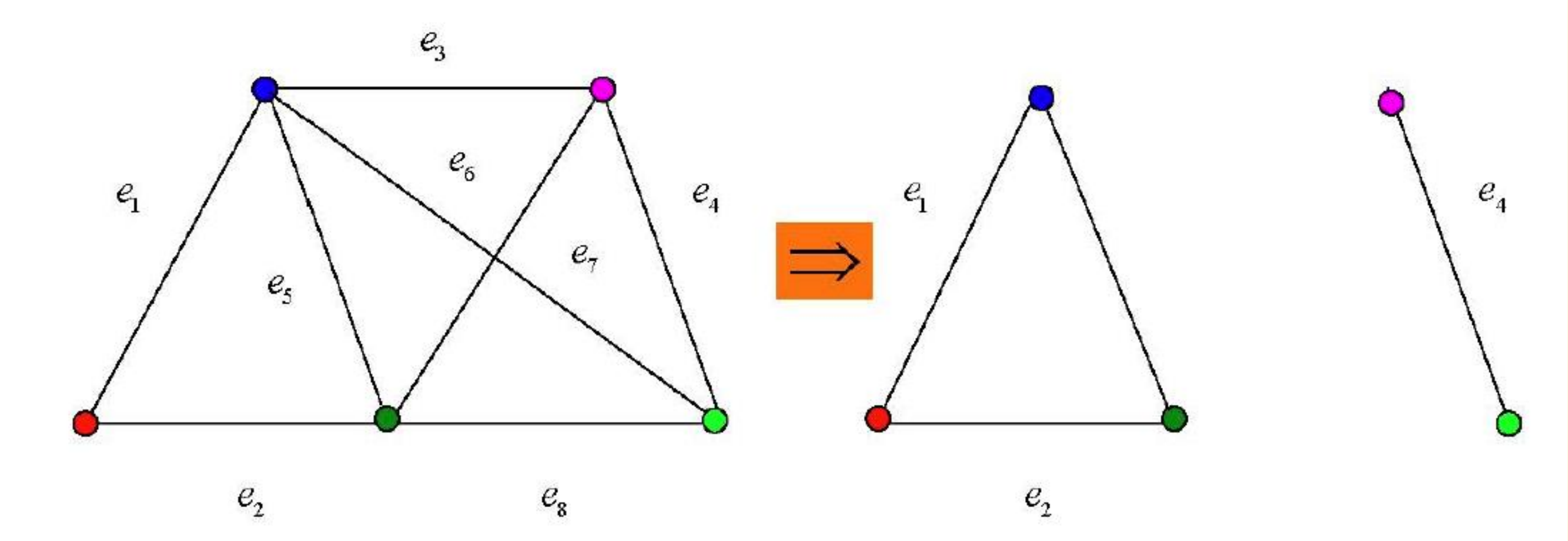
G

G^c



Graph Connectivity

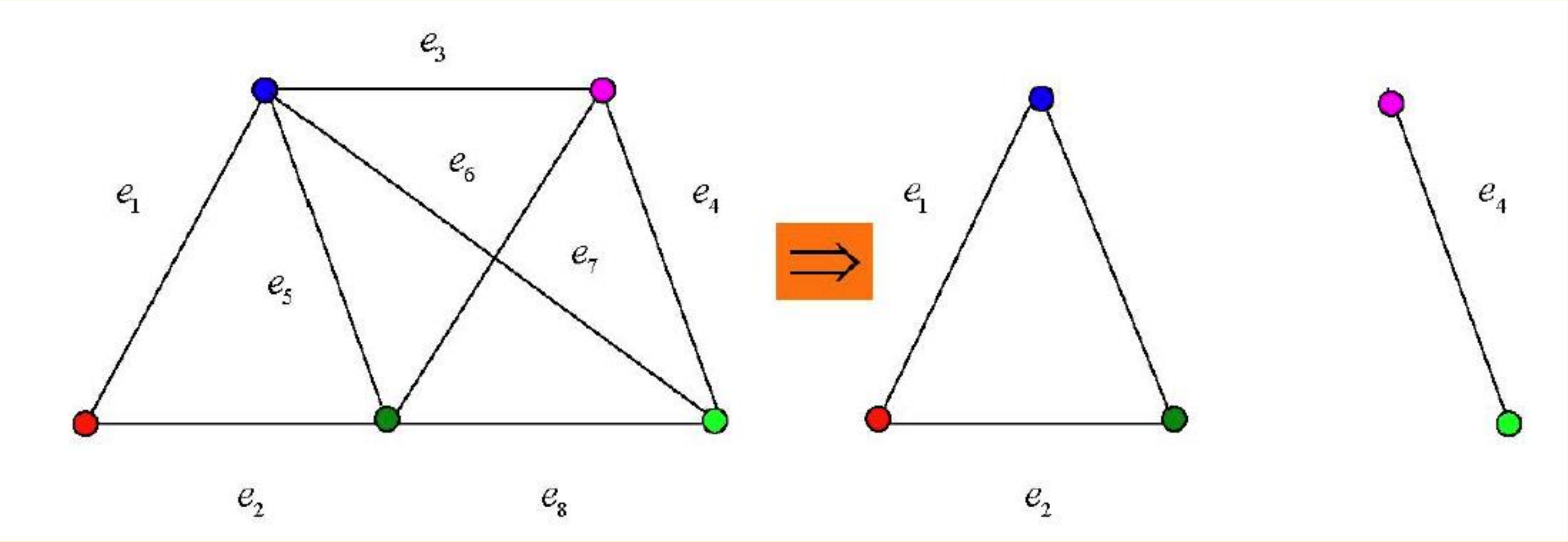
- **Q:** How many edges or nodes must be removed in order to disconnect an originally connected graph?
- Note: If a node is removed, then all edges joining it will also be removed; but the converse is not so.



One example (among several possible cases)

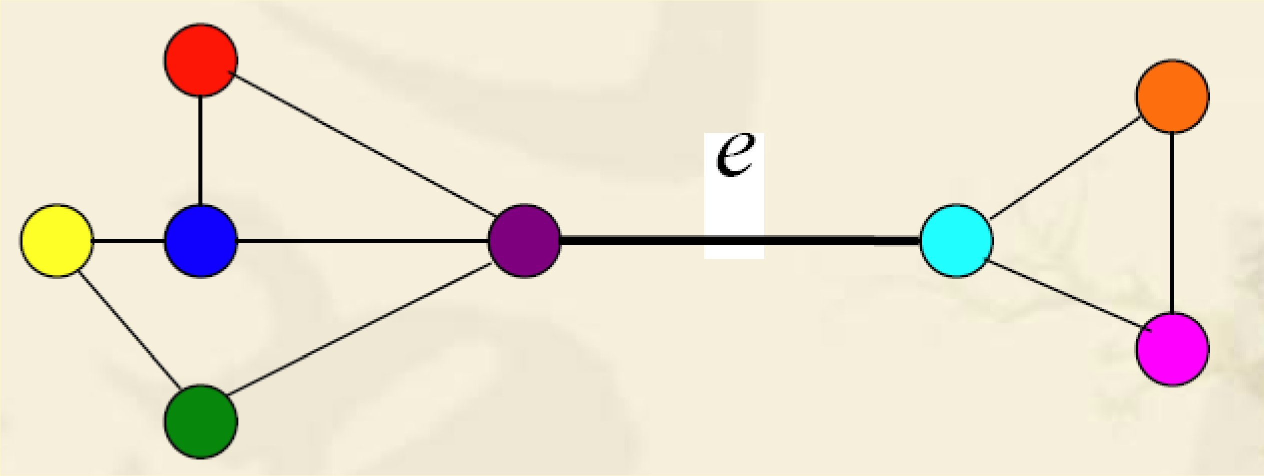
Disconnecting Sets and Cut-Sets

- **Disconnecting set**: A set of edges, $E_0(G)$, after it is being removed, the graph G will become unconnected.
- **Cut-Set**: The smallest disconnecting set, i.e., no proper subset of which is a disconnecting set.
- Example: $E_0^{1}(G)= \{e_1, e_2\}$ $E_0^{2}(G) =\{e_1, e_2, e_5\}$ $E_0^{3}(G)= \{e_3, e_6, e_7, e_8\}$ are disconnecting sets, in which both $E_0^{1}(G)$ and $E_0^{3}(G)$ are cut-sets.



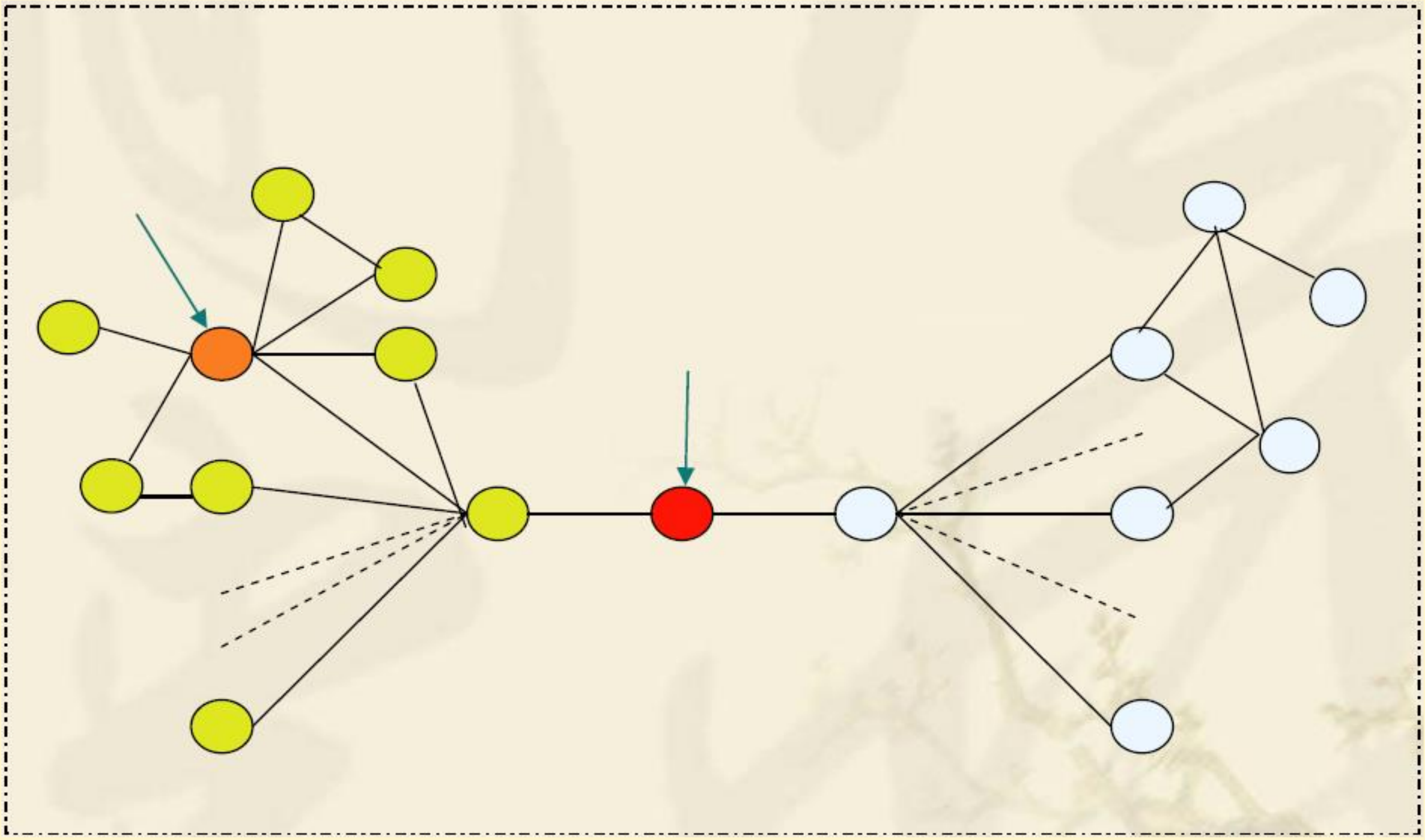
Bridges

- **Bridge**: A cut-set with only one edge
- **Example**: cut-set $\{e\}$ below is a bridge



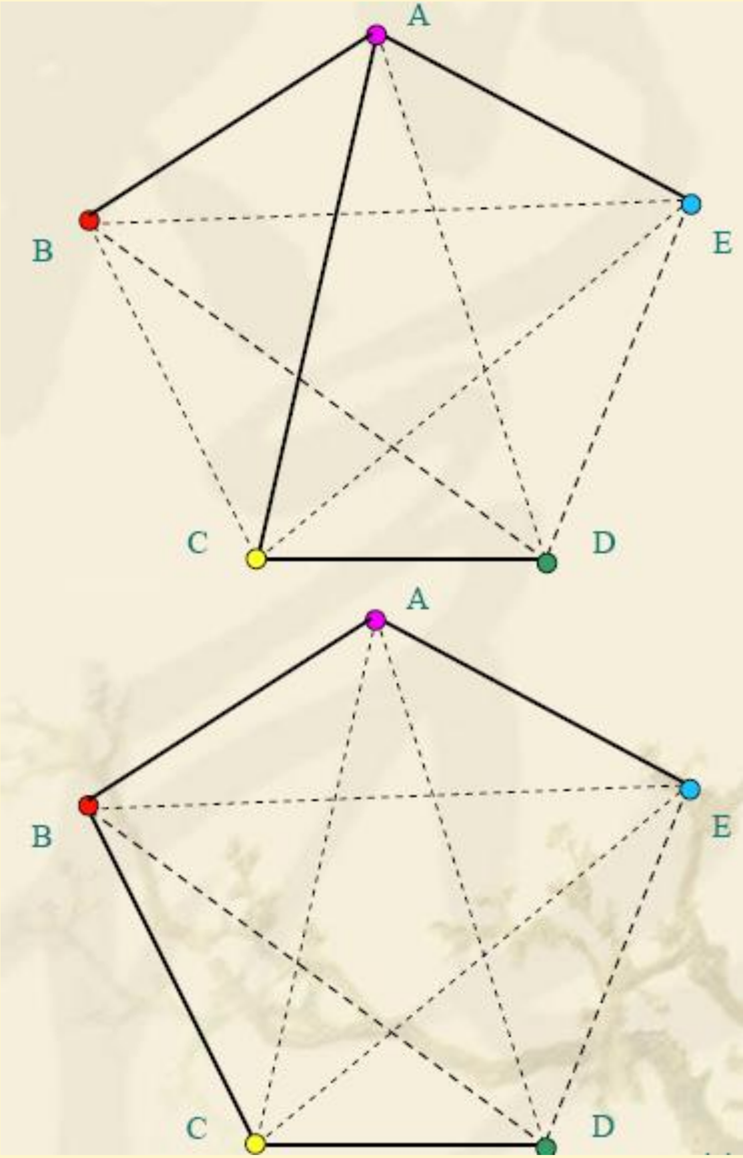
Importance of Bridges

In a network, a node of low degree may be more important than a node of high degree, for example, on a bridge:



More About Trees

- Starting from a given graph G, if it has a circuit, then remove one edge from the circuit. (Clearly, the resulting graph remains to be connected.)
- Repeat this procedure until no circuits are left out.
- The final resulting graph is a tree. This tree is called a **spanning tree** of graph G.
- Spanning tree usually is not unique

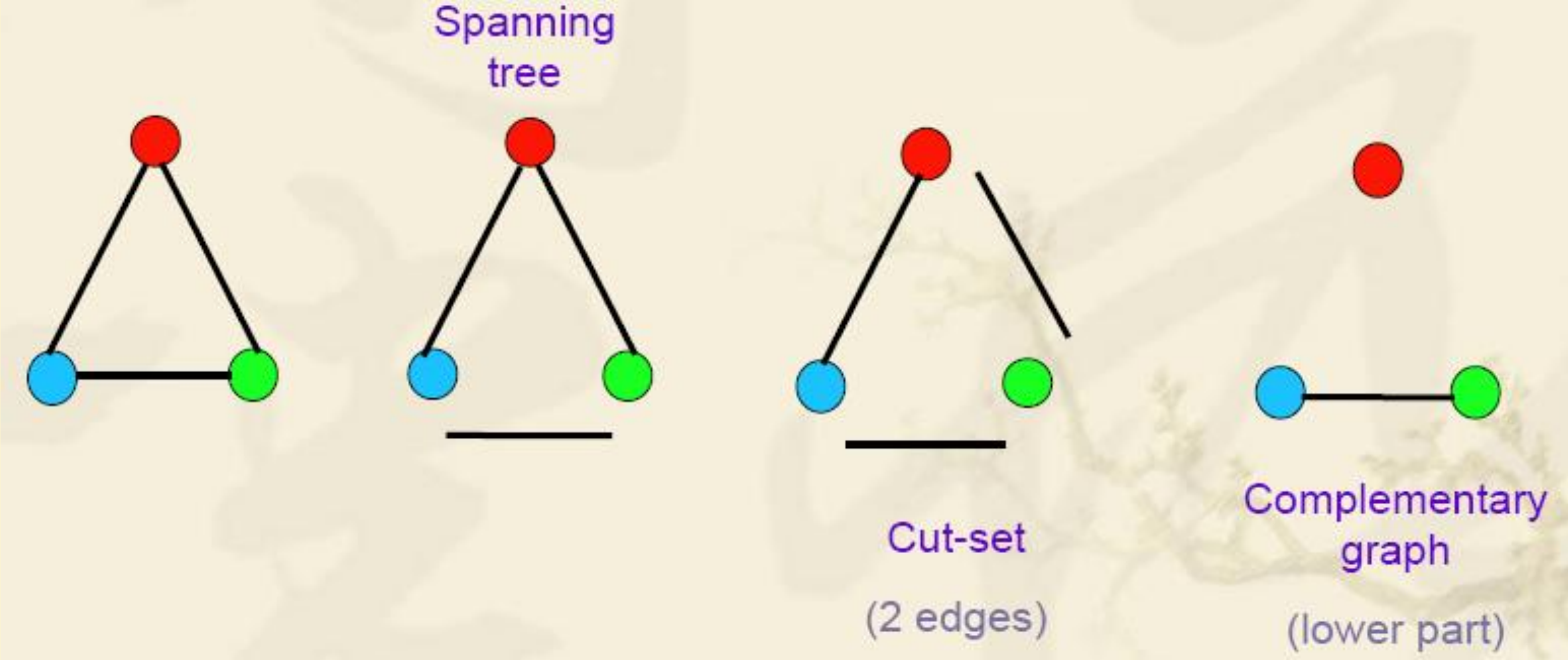


Some Results

- **Theorem 2-20** *Let T be any spanning tree of graph G. Then* →

  1) every cut-set of G has an edge in common with T ;

  2) every circuit of G has an edge in common with the complementary graph of T .

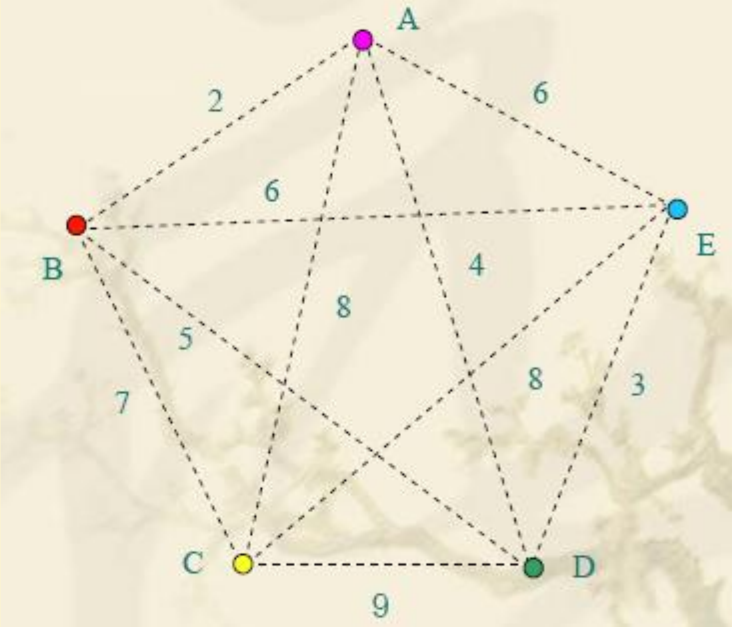


Minimum Connector Problem

- **Minimum connector problem**: Suppose that one wants to build a highway network connecting N given cities, in such a way that a car can travel from any city to any other city, but the total mileage of the highways is minimum.

• Clearly, the graph formed by taking the N cities as nodes and the connecting highways as edges must be a tree, because any more highways will be extra.

•The problem is to find an efficient algorithm to decide which tree connecting these cities reaches the minimum total mileage, given that the distance between any pair of cities is known.

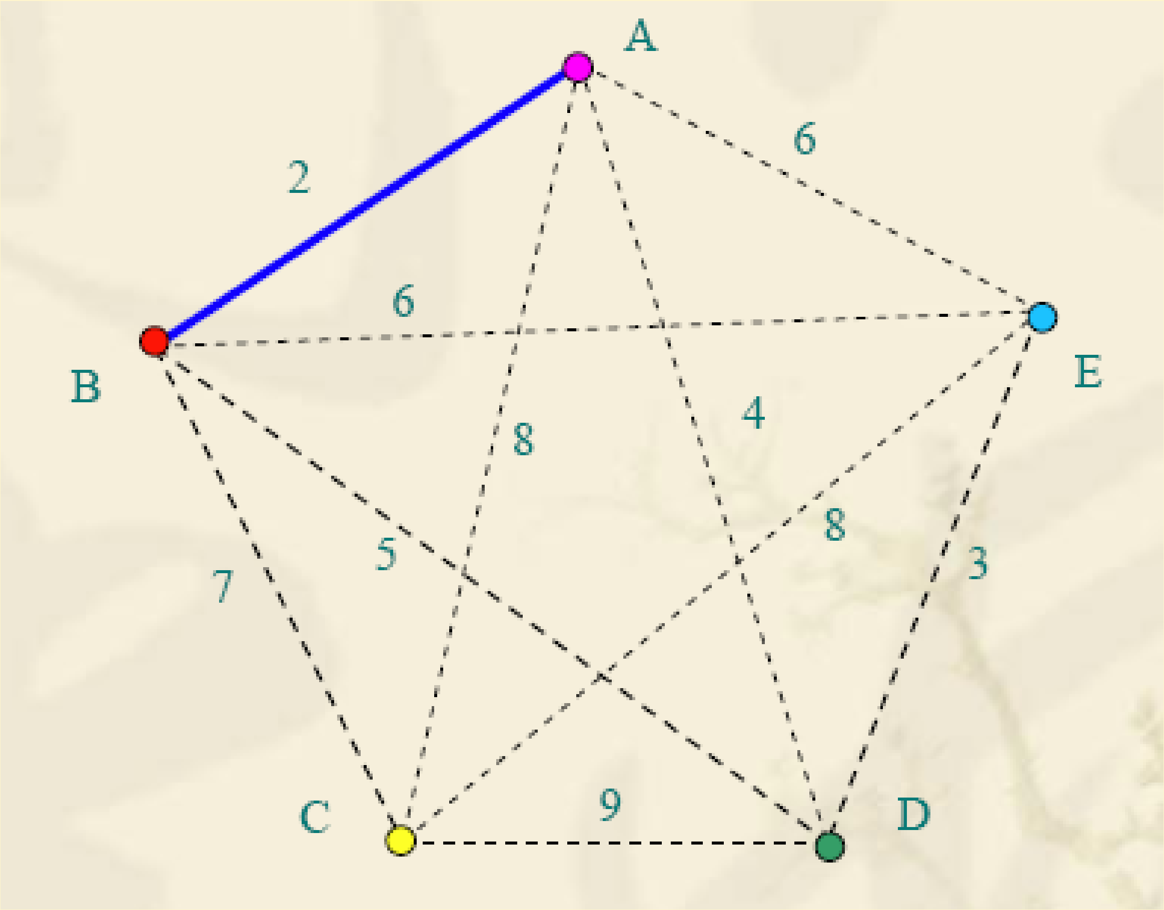


Greedy Algorithm

- **Theorem 2-21** (Kruskal Greedy Algorithm) Let G be a connected graph with N nodes. Then, the following constructive scheme yields a solution to the minimum connector problem:
- Let e_1 be an edge of G with the smallest weight;
- Choose $e_2, \ldots, e_{N-1}$ one by one, by choosing an edge e_i (not previously chosen) with a smallest weight, subject to the condition that it forms no circuit with all the previous edges $\{e_1, \ldots, e_{i-1}\}$;
- repeat this procedure until no more edge can be chosen
- the resulting graph is a spanning tree, i.e., the subgraph of G with edges $\{e_1, \ldots, e_{N-1}\}$

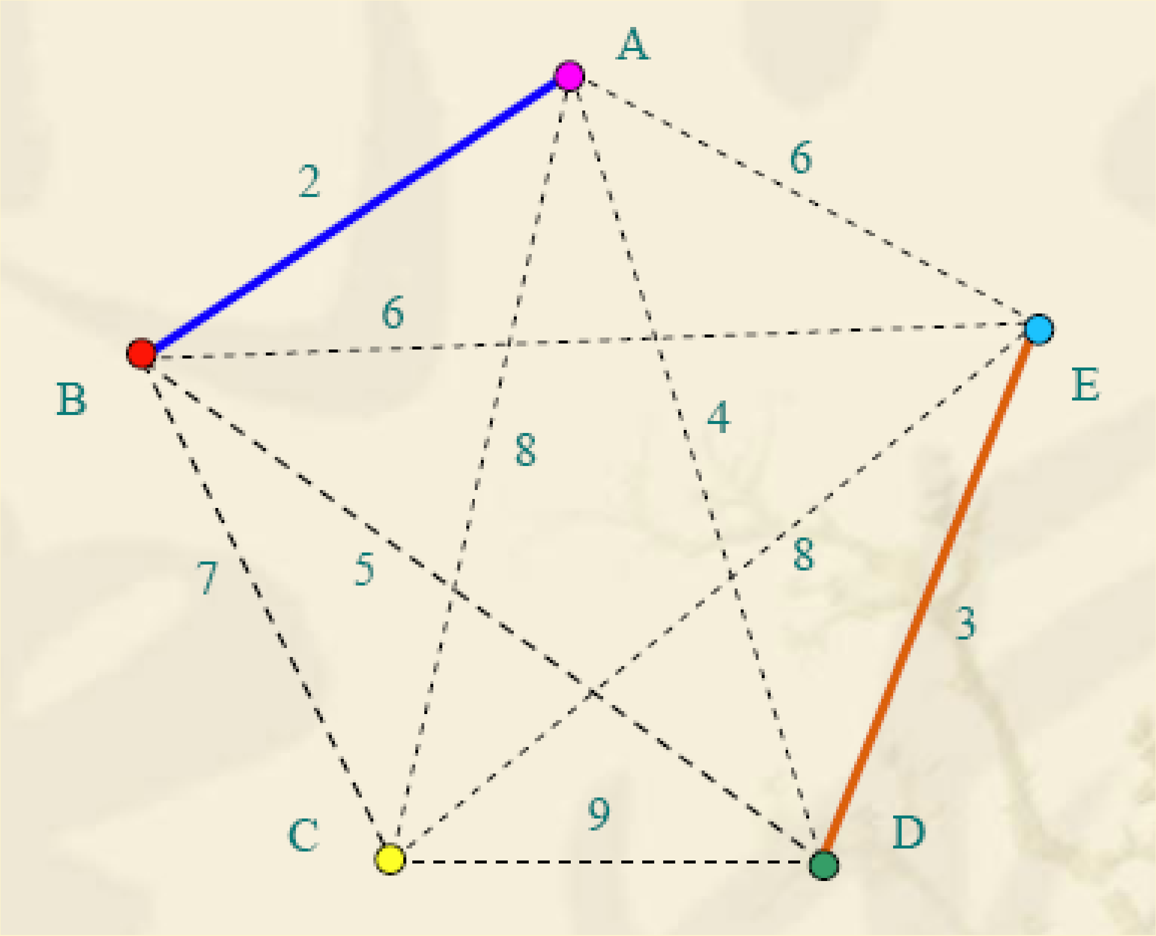
Example

Apply the greedy algorithm →



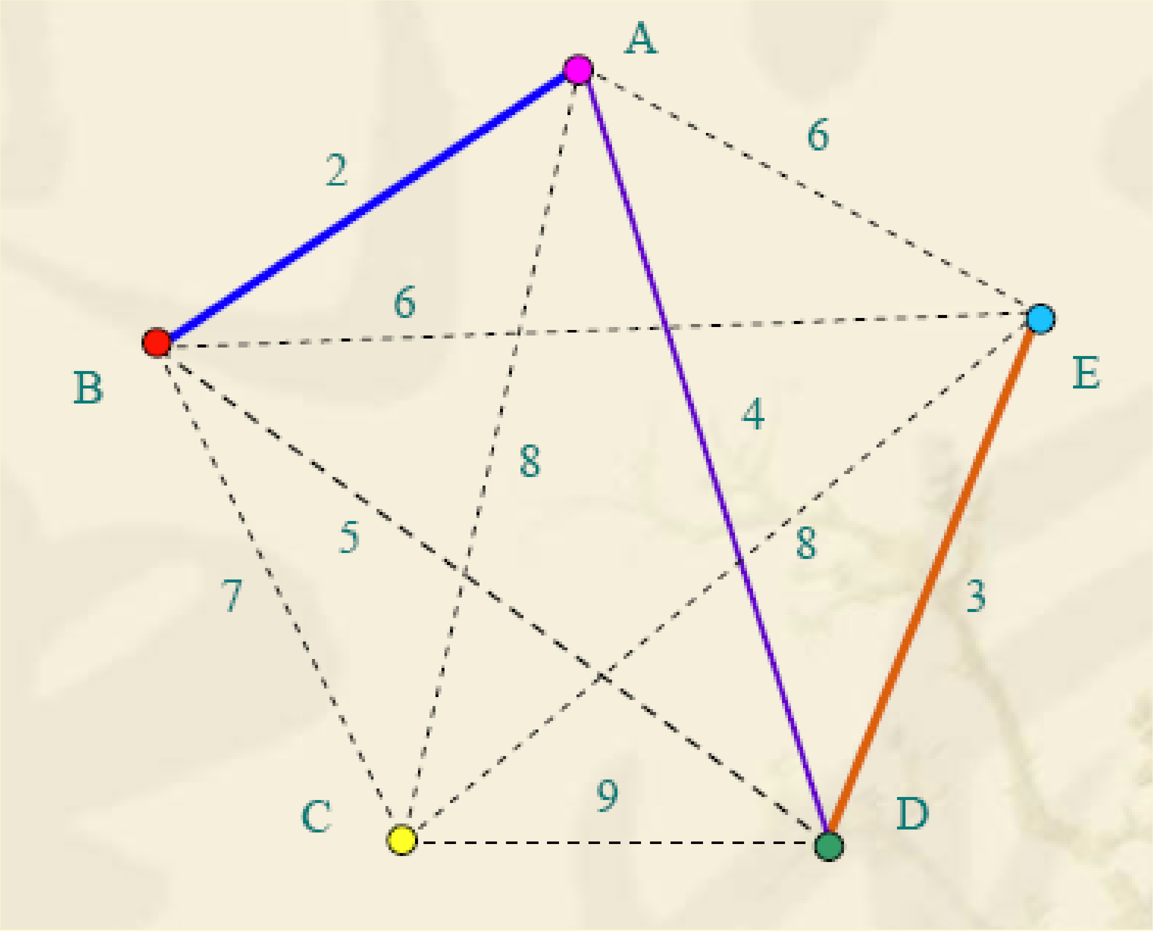
Example

Continue the greedy algorithm →



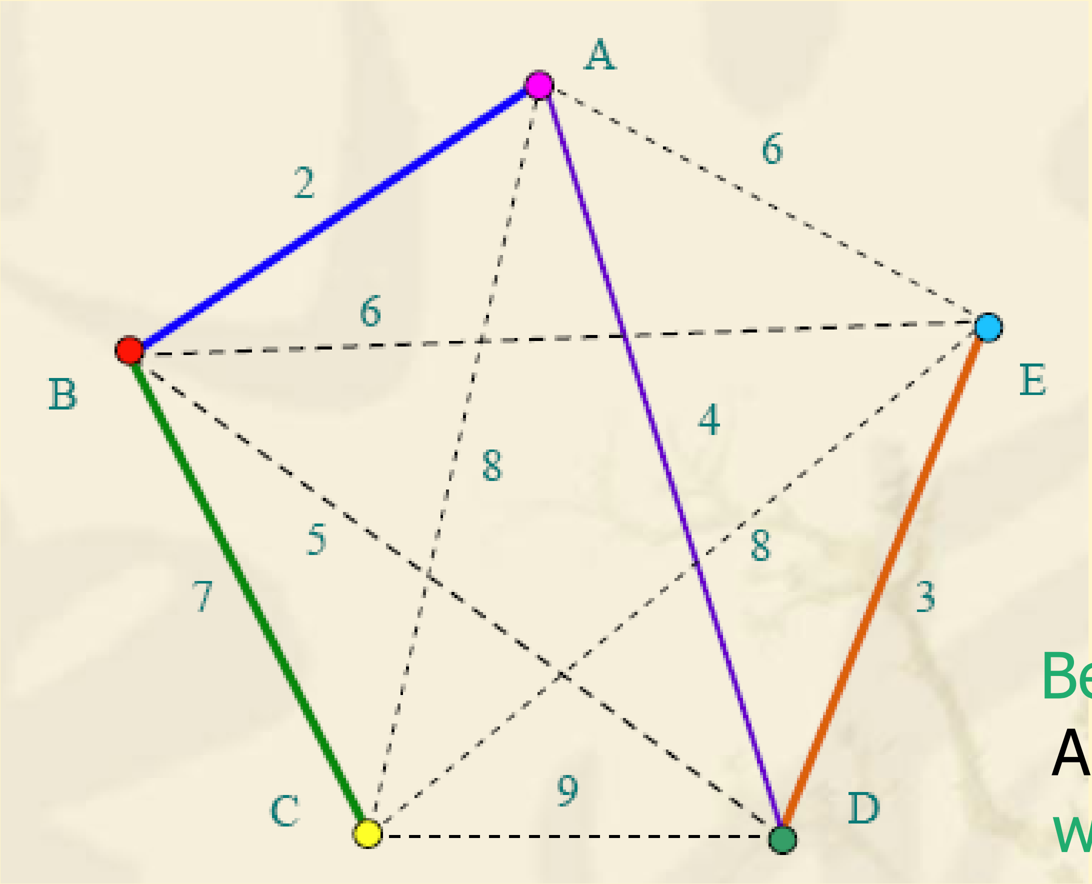
Example

Continue the greedy algorithm →



Example

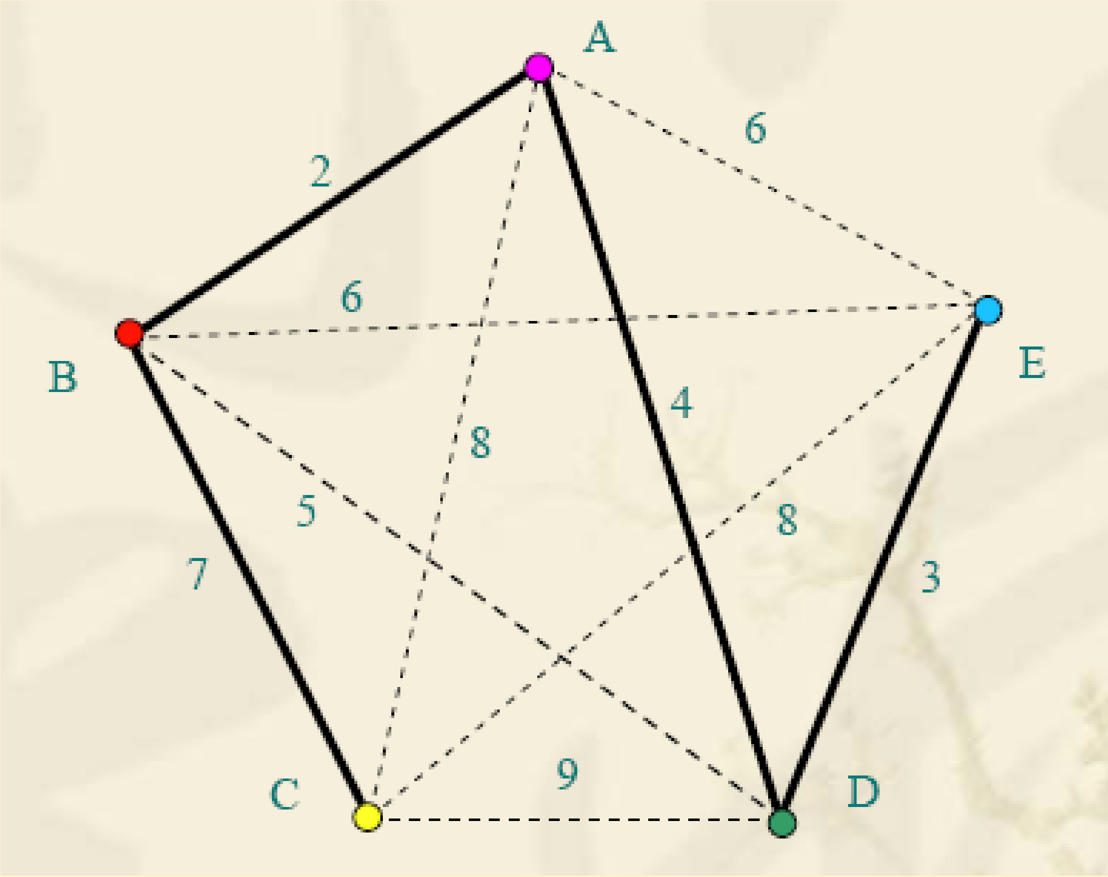
Continue the greedy algorithm →



Because BD =5 or AE = 6 or BE = 6 will form a circuit

Result of the Example

The greedy algorithm finally yields:



No more edge can be added