Chapter 7

Recurrence Relations and Generating functions

Summary

- **Linear homogeneous recurrence relations**
- **Generating functions**
- **Recurrences and generating functions**
- **A geometry example**
- **Exponential generating functions**
- **Assignments**

Linear Homogeneous Recurrence Relations

Linear Recurrence Relation

- **A sequence of numbers $h_0, h_1, \ldots, h_n, \ldots$ is said to satisfy a linear recurrence relation of order *k*, provided there exist quantities $a_1, a_2, \ldots, a_k$, with $a_k \neq 0$, and a quantity b_n (each of these quantities $a_1, a_2, \ldots, a_k, b_n$ may depend on n) such that $h_n = a_1h_{n-1}+a_2h_{n-2}+\ldots+a_kh_{n-k}+b_n, (n \geq k)$.**

Examples

- The sequence of derangement numbers $D_0, D_1, D_2, \ldots, D_n$ satisfy the two recurrence relations
- $D_n = (n-1)D_{n-1}+(n-1)D_{n-2}, \ (n \geq 2)$
- $D_n = nD_{n-1} + (-1)^n, \qquad (n \geq 1).$
- The first recurrence relation has order ??? and we have a_1 = ??? a_2 =??? and b_n =???
- The second ……..

Homogeneous

- **The linear recurrence relation $h_n = a_1 h_{n-1} + a_2 h_{n-2} + \ldots + a_k h_{n-k} + b_n$, $(n \geq k)$ is called homogeneous provided $b_n = 0$.**
- **The linear recurrence relation is said to have constant coefficients provided $a_1, a_2, \ldots, a_k$ are constants.**

Theorem 7.2.1

- Let *q* be a non-zero number. Then **hn = qn** is a solution of the linear homogeneous recurrence relation

  hn – *a*1hn-1 – *a*2hn-2 – … – *a*khn-k= 0, (*a*k ≠ 0, n ≥ k) (7.20)

  with constant coefficients iff q is a **root** of the polynomial equation

  xk – *a*1xk-1 – *a*2xk-2 – … – *ak* = 0. (7.21)

- If the polynomial equation has **k distinct** roots q1, q2, …, qk, then **hn = c1q1n + c2q2n + … + ckqkn (7.22)**

  is the general solution of (7.20) in the following sense: no matter what initial values for h0, h1, …, hk-1are given, there are constants c1, c2, …, ck so that (7.22) is the unique sequence which satisfies both the recurrence relation (7.20) and the initial conditions.

Comments

- The polynomial equation (7.21) is called the **characteristic equation** of the recurrence relation (7.20) and its k roots are the **characteristic roots**.
- If the characteristic roots are **distinct**, (7.22) is the **general solution** of (7.20).

Example

- **Solve the recurrence relation**

  hn = 2hn-1+hn-2 – 2hn-3, (n ≥ 3)

  subject to the initial values h0 = 1, h1 = 2 and h2 = 0.

Hints: the characteristic equation of the recurrence relation is x3 – 2x2 – x + 2 = 0 and its three roots are 1, -1 , 2. By Th.7.2.1, hn = c11n + c2(-1)n + c32n is the general solution.
How to continue??????

Example

- **Words of length n, using only the three letters a, b, c are to be transmitted over a communication channel subject to the condition that no word in which two a's appear consecutively is to be transmitted. Determine the number of words allowed by the communication channel.**

Hints

- Firstly, find the recurrence relation and then find its solution.
- Let hn denote the number of allowed words of length n. We have h0 = 1 and h1 = 3. Let n ≥ 2. If the first letter of the word is b or c, then the word can be completed in hn-1 ways. If the first letter is a , then second letter should be b or c. hence, hn = 2 hn-1 + 2hn-2, (n ≥ 2). Continue by yourself.

b					

a	b				

Exercises

- Consider a 1-by-n chessboard. Suppose we color each square with one of the two colors red and blue. Let hn denote the number of colorings in which no two squares that are colored red are adjacent. Find and verify a recurrence relation that hn satisfies. Then derive a formula for hn.
- Solve the recurrence relation

  hn = 4hn-1 – 4hn-2, (n ≥ 2) .

Comments

- **If the roots q1, q2, …, qk of the characteristic equation are not distinct, then hn = c1q1n+c2q2n+…+ckqkn is not a general solution of the recurrence relation.**

Theorem 7.2.2

Let q1, q2, …, qt be the distinct roots of the characteristic equation of the linear homogeneous recurrence relation (7.20) with constants coefficients. Then if qi is an si-fold root of the characteristic equation (7.21) , then the general solution of the recurrence relation is

hn = Hn(1) + Hn (2) + … + Hn(t), where

Hn(i) = c1qin + c2nqin + … + csinsi-1qin

= (c1 +c2n+…+csinsi-1)qin.

Example

- **Solve the recurrence relation**

  $hn = 4hn-1 – 4hn-2, (n \geq 2)$.

Hints: the characteristic equation is x2-4x+4 = 0. thus 2 is the twofold characteristic root. The general solution of the recurrence relation is

$hn = c12n + c2n2n$.

Exercise

- **Solve the recurrence relation**

  $h_n = -h_{n-1} + 3h_{n-2} + 5h_{n-3} + 2h_{n-4}$, $(n \geq 4)$.

Generating Functions

What generating functions do?

- **Count the number of possibilities for a problem by means of algebra**
- **Generating functions are Taylor series of infinitely differentiable functions**
- **If we can find the function and its Taylor series, then the coefficients of the Taylor series give the solution to the problem.**

Definition of generating functions

Let h0, h1, …, hn, …… be an infinite sequence of numbers. Its generating function is defined to be the infinite series

g(x) = h0 + h1x + h2x2 +…+ hnxn +……

The coefficient of xn in g(x) is the *n*th term hn of the sequence, and thus xn acts as a "place holder" for hn.

Examples

1. The generating function of the infinite sequence 1, 1, 1, …, 1, …, each of whose terms equals 1 is

$$g(x) = 1 + x + x2 + \ldots + xn + \ldots = 1/(1-x)$$

2. Let m be a positive integer. The generating function for the binomial coefficients C(m, 0), C(m, 1) C(m, 2),…., C(m, m) is

$$gm(x) = C(m, 0) + C(m, 1)x + C(m, 2)x2 + \ldots + C(m, m)xm$$

$$= (1+x)m \quad \text{(by the binomial theorem).}$$

Exercises

Let *a* be real number. By Newton's binomial theorem, what is the generating function for the infinite sequence of binomial coefficients C(*a*, 0), C(*a*, 1) , …, C(*a*, n),…?

Let k be an integer and let the sequence h0, h1, h2,…, hn, …be defined by letting hn equals the number of non-negative integral solution of e1+e2+…ek=n. What is the generating function for this sequence?

The generating function (using summation notation now) is

$$g(x) = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n.$$

From Chapter 5, we know that this generating function is

$$g(x) = \frac{1}{(1-x)^k}.$$

It is instructive to recall the derivation of this formula. We have

$$\begin{aligned}
\frac{1}{(1-x)^k} &= \frac{1}{1-x} \times \frac{1}{1-x} \times \cdots \times \frac{1}{1-x} \qquad (k \text{ factors}) \\
&= (1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots) \\
&= \left(\sum_{e_1=0}^{\infty} x^{e_1}\right)\left(\sum_{e_2=0}^{\infty} x^{e_2}\right)\cdots\left(\sum_{e_k=0}^{\infty} x^{e_k}\right). \qquad (7.11)
\end{aligned}$$

In the preceding notation, x^{e_1} is a typical term of the first factor, x^{e_2} is a typical term of the second factor, ... , x^{e_k} is a typical term of the kth factor. Multiplying these typical terms, we get

$$x^{e_1}x^{e_2}\cdots x^{e_k} = x^n, \text{ provided that}$$

$$e_1 + e_2 + \cdots + e_k = n. \qquad (7.12)$$

Thus, the coefficient of x^n in (7.11) equals the number of nonnegative integral solutions of (7.12), and this number we know to be

$$\binom{n+k-1}{n}.$$ 0

Review

- **Let *a* be a real number . Then for all *x* and *y* with 0 ≤ |x| <|y|,**

$$(x+y)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^{a-k} y^k$$

- **where**

$$\binom{a}{k} = \frac{a(a-1)(a-2)\boxed{?}(a-k+1)}{k!}$$

For |y|<1

$$(1+y)^{-k} = \sum_{n=0}^{\infty} (-1)^n \binom{n+k-1}{k-1} y^n$$

Set $y = -x$

$$(1-x)^{-k} = \sum_{n=0}^{\infty} (-1)^n \binom{n+k-1}{k-1} (-x)^n$$

$$= \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} ((-1)(-x))^n$$

$$= \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

More examples

For what sequence is (1+x+x2+x3+x4+x5)(1+x+x2)(1+x+x2+x3+x4) the generating function?

Let xe1(0≤e1≤5), xe2, (0≤e2≤2), and xe3 (0≤e3≤4) denote the typical terms in the first, second and third factors, respectively. Multiplying we obtain xe1xe2xe3 = xn, provided e1 +e2+e3 = n. Thus the coefficient of xn in the product is the number of hn of integral solutions of e1 +e2+e3 = n in which 0≤e1≤5, 0≤e2≤2 and 0≤e3≤4. (note that hn = 0 for n>11)

More examples (cont'd)

Determine the generating function for the number of n-combinations of apples, bananas, oranges, and pears where in each n-combination the number of apples is even, the number of bananas is odd, the number of oranges is between 0 and 4 and there is at least one pear.

Hints: the problem is equivalent to finding the number hn of non-negative integral solutions of

$$e1 + e2 + e3 + e4 = n.$$

where e1 is even (e1 counts the number of apples), e2 is odd, 0 ≤e3≤4, and e4 ≥1. We create one factor for each type of fruit where the exponents are the allowable number's in the n-combinations for that type of fruit:

g(x) = (1 + x2 + x4 +…)(x + x3 + x5 +…)(1 + x + x2 + x3 + x4) (x + x2 + x3 + x4 +…)

$$= \frac{1}{1-x^2}\,\frac{x}{1-x^2}\,\frac{1-x^5}{1-x}\,\frac{x}{1-x}$$

$$= \frac{x^2(1-x^5)}{(1-x^2)^2(1-x)^2}$$

Exercises

Determine the number hn of bags of fruit that can be made out of apples, bananas, oranges, and pears where in each bag the number of apples is even, the number of bananas is a multiple of 5, the number of oranges is at most 4 and the number of pears is 0 or 1.

Hints: This is to calculate the coefficient of xn for the generating functions of this problem.

$e_1 + e_2 + e_3 + e_4 = n \quad e_1 \cdot$

$(1+x^2+x^4+\cdots)(1+x^5+x^{10}+\cdots)(1+x+x^2+x^3+x^4)(1+x)$

$$= \frac{1}{1-x^2}\cdot\frac{1}{1-x^5}\cdot\frac{1-x^5}{1-x}\cdot(1+x)$$

$$= \frac{1+x}{(1-x^2)(1-x)}$$

Exercises (cont'd)

$$= \frac{1}{(1-x)^2} = (1-x)^{-2} = 1+2x+3x^2+\cdots+nx^{n-1}+\cdots$$

$h(n) = n+1$

Determine the generating function for the number hn of solutions of the equation e1 + e2 + … + ek = n in non-negative odd integers e1, e2, …, ek .

Hints: ∏k (x+x3+x5+x7+……).

Exercises (cont'd)

Let hn denote the number of non-negative integral solutions of the equation

3e1 + 4e2 + 2e3 + 5e4 = n. Find the generating function g(x) for h0, h1, h2, …, hn,…….

Hints: change the variable by let f1 = 3e1, f2 = 4e2, f3 = 2e3 and f4 =5e4. Then hn also equals the number of non-negative integral solutions of f1 + f2 + f3 + f4 = n where f1 is a multiple of 3, f2 is a multiple of 4, f3 is even and f4 is a multiple of 5.

CONTINUE BY YOURSELF.

$(1+x^3+x^6+\cdots)(1+x^4+x^8+\cdots)(1+x^2+x^4+\cdots)(1+x^5+x^{10}+\cdots)$

$= \frac{1}{1-x^3} \cdot \frac{1}{1-x^4} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^5}$

Recurrences and generating functions

What will be done?

- **Use generating functions to solve linear homogeneous recurrence relations with constant coefficients.**
- **Newton's binomial theorem will be applied.**

Review: Newton's binomial theorem

If n is a positive integer and r is a non-zero real number, then

$$(1 - rx)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-rx)^k$$

$$= \sum_{k=0}^{\infty} (-1)^k \binom{-n}{k} r^k x^k$$

$$= \sum_{k=0}^{\infty} \binom{n+k-1}{k} r^k x^k , \qquad (|x| < |r|^{-1})$$

Examples

Determine the generating function for the sequence of squares 0, 1, 4, …, n2,…..

Solution: by the above Newton's binomial theorem with n =2 and r =1,

$(1-x)-2 = 1+2x+3x2+\ldots+nxn-1+\ldots.$

Hence $x/(1-x)2=x+2x2 + 3x3+\ldots+nxn +\ldots..$

Differentiating, we obtain

$(1+x)/(1-x)3=1+22x+32x2+\ldots+n2xn-1+\ldots..$

Multiplying by x, we obtain the desired generating function $x(1+x)/(1-x)3$.

Examples (cont'd)

- **Solve the recurrence relation**

  $h_n = 5h_{n-1} - 6 h_{n-2}$, ($n \geq 2$) subject to the initial values $h_0 = 1$ and $h_1 = -2$.

$x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0$

$x_1 = 2$

$x_2 = 3$

Hints: let $g(x) = h_0 + h_1x + h_2x^2 + \ldots + h_nx^n + \ldots$. be the generating function for $h_0, h_1, h_2, \ldots, h_n \ldots$ we then have the following equations

$h_n = 5h_{n-1} - 6h_{n-2}, (n \geq 2)$

$g(x) = h_0+h_1x+h_2x^2+\ldots+h_nx^n+\ldots.$

$-5xg(x) = -5h_0x -5h_1x^2 - 5h_2x^3 -\ldots- 5h_{n-1}x^n -\ldots.$

$6x^2 g(x) = 6h_0x^2 +6h_1x^3 +6h_2x^4 +\ldots+6h_{n-2}x^n +\ldots.$

Adding these three equations, we obtain

$(1-5x+6x^2)g(x) = h_0+(h_1-5h_0)x+(h_2-5h_1+6h_0)x^2+\ldots+(h_n-5h_{n-1}+6h_{n-2})x^n+\ldots.$

$= h_0+(h_1-5h_0)x = 1-7x$ (by assumptions)

Hence, $g(x) = (1-7x)/(1-5x+6x^2)$

$= 5/(1-2x) - 4/(1-3x)$

By Newton’s binomial theorem

(1-2x)-1 = 1+2x+22x2+…+2nxn…..

(1-3x)-1 = 1+3x+32x2+…+3nxn…..

Therefore,

g(x) = 1 + (-2)x + (-15)x2 +…+ (5×2n – 4×3n)xn+…

and we obtain

hn = 5×2n – 4×3n   (n = 0, 1, 2, …).

$h(n) = h_0 + h_1 x + h_2 x^2 + \cdots + h_n x^n + \cdots$

$-3x^2 h(n) = -3h_0 x^2 - 3h_1 x^3 - 3h_2 x^4 + \cdots - 3h_n x^{n+2} + \cdots$

$2x^3 h(n) = 2h_0 x^3 + 2h_1 x^4 + \cdots + 2h_n x^{n+3}$

Exercise

$(1-3x^2+2x^3)h(n) = h_0 + h_1 x + (h_2 - 3h_0)x^2 + (h_3 - 3h_1 + 2h_0)x^3 + \cdots + (h_n - 3h_{n-2} + 2h_{n-3})x^n$

$= h_0 + h_1 x + (h_2 - 3h_0)x^2 = 1 - 3x^2$

$h(n) = \dfrac{1-3x^2}{(1-3x^2+2x^3)}$

- **Solve the recurrence relation hn = 3hn-2 - 2hn-3 (n≥3) subject to the initial values h0 = 1, h1 = 0 and h2 = 0**
- **Solve the recurrence relation hn + hn-1 – 16 hn-2+20hn-3 = 0 (n≥3) subject to the initial values h0 = 0, h1 = 1 and h2 = -1. (refer to the book)**

$\dfrac{}{(1-x)^2(1+2x)}$

$\dfrac{A}{(1-x)^2} + \dfrac{B}{1+2x} + \dfrac{C}{1-x}$

$$h_n = 3h_{n-2} - 2h_{n-3}$$

$$f(x) = \frac{1-3x^2}{1-3x^2+2x^3}$$

$$= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+2x}$$

$$= \frac{A(1-x)(1+2x) + B(1+2x) + C(1-x)^2}{(1-x)^2(1+2x)}$$

$$= \frac{(A+B+C) + (A+2B-2C)x + (-2A+C)x^2}{(1-x)^2(1+2x)}$$

$$\begin{cases} A+B+C = 1 \\ A+2B-2C = 0 \\ -2A+C = -3 \end{cases}$$

$$A = 14/9,\ B = -2/3,\ C = 1/9$$

$A(1+2x) + B(1-x)^2$
$+ (1-x)(1+2x)C$
$= 1-3x^2$

$A + 2Ax + B + Bx^2 - 2Bx + C + Cx$
$-2Cx^2$
$= 1-3x^2$

$B - 2C = -3$
$2A - 2B + C = 0$
$A + B + C = 1$

$B = 2C - 3$

$2A - 2(2C-3) + C = 0$
$A + 2C - 3 + C = 1$

$\cancel{2A - 4C + 6 + C = 0}$
$2A - 3C + 6 = 0$
$A + 3C = 4$

0

$\frac{14}{3}$

$3A = -2$

$A = -\frac{2}{3}$ $C = \frac{14}{9}$

$B = \frac{1}{9}$

A geometry example

A set K of points in the plane or in space is said to be *convex*, provided that for any two points p and q in K, all of the points on the line segment joining p and q are in K. Triangular regions, circular regions, and rectangular regions in the plane are all convex sets of points. On the other hand, the region on the left in Figure 7.1 is not convex since, for the two points p and q shown, the line segment joining p and q goes outside the region.

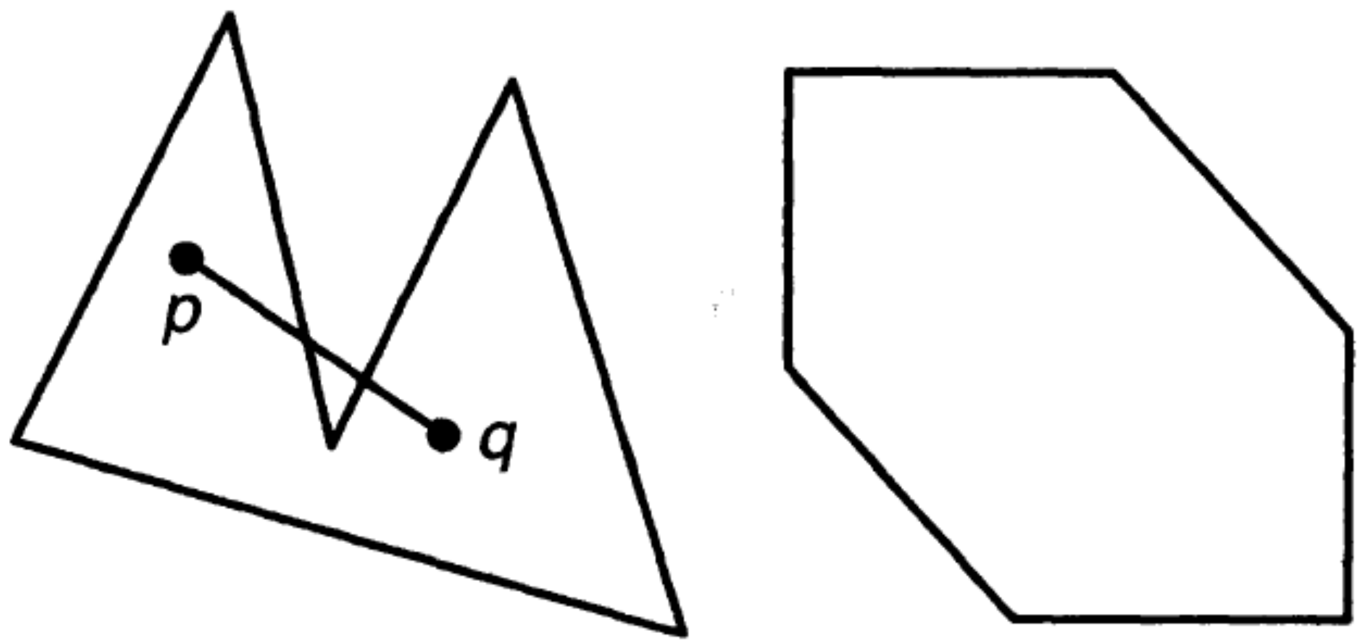


Figure 7.1

Ways to dividing a convex polygonal region

Let hn denote the number of ways of dividing a convex polygonal region with n+1 sides into triangular regions by inserting diagonals which do not intersect in the interior.

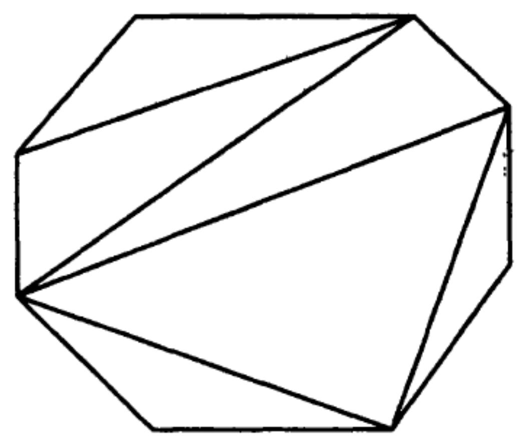
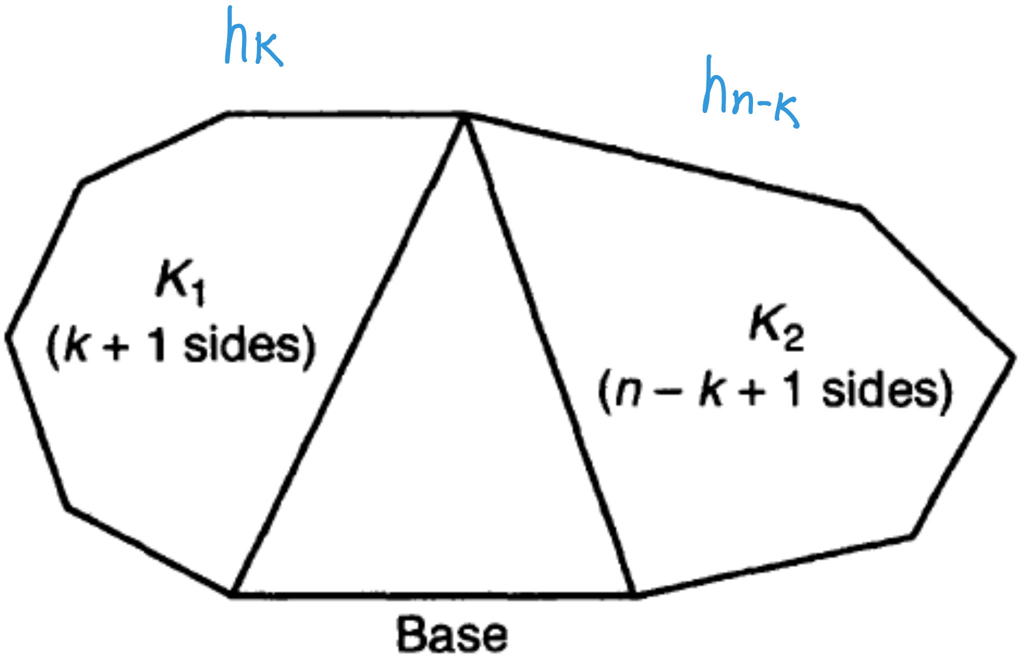


Figure 7.2



Polygonal region with $n + 1$ sides

Figure 7.3

Define $h_1 = 1$. Then h_n satisfies the recurrence relation

$$h_n = h_1 h_{n-1} + h_2 h_{n-2} + \ldots + h_{n-1} h_1, \quad (n \geq 2).$$

The solution of this recurrence relation is

$$h_n = \frac{1}{n-1} C(2n-2, n-1), \quad (n=1, 2, 3, \ldots).$$

Exponential generating functions

Review: Taylor's series for ex

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \boxed{?} + \frac{x^n}{n!} + \boxed{?}$$

Definition of exponential generating functions

- The exponential generating function for the sequence h0, h1, h2, …, hn, … is defined to be

$$g^{(e)}(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}$$

$$= h_0 + h_1 x + h_2 \frac{x^2}{2!} + \boxed{?} + h_n \frac{x^n}{n!} + \boxed{?}$$

Examples

- Let n be a positive integer. Determine the exponential generating function for the sequence of numbers P(n, 0), P(n, 1), P(n, 2), …, P(n, n), where P(n, k) denote the number of k-permutations of an n-element set, and thus has the value n!/(n-k)! For k = 0, 1, …, n. The exponential generating function is

$$g^{(e)}(x) = P(n,0) + P(n,1)x + P(n,2)\frac{x^2}{2!} + \boxed{?} P(n,n)\frac{x^n}{n!}$$

$$= 1 + nx + \frac{n!}{2!(n-2)!}x^2 + \boxed{?} + \frac{n!}{n!0!}x^n$$

$$= (1+x)^n$$

Thus (1+x)n is both the exponential generating function for the sequence P(n, 0), P(n, 1), P(n, 2), …, P(n, n) and, as we have seen in previous section, the ordinary generating function for the sequence C(n, 0), C(n, 1), C(n, 2), …, C(n, n).

Examples (cont'd)

- **The exponential generating function for the sequence 1, 1, 1, …, 1, …. is**

$$g^{(e)}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

- **More generally, if *a* is any real number, the exponential generating function for the sequence *a*0 = 1, *a, a2, …, an, …. is***

$$g^{(e)}(x) = \sum_{n=0}^{\infty} a^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(ax)^n}{n!} = e^{ax}.$$

A Theorem

Theorem 7.3.1 *Let S be the multiset $\{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$, where $n_1, n_2, \ldots, n_k$ are nonnegative integers. Let h_n be the number of n-permutations of S. Then the exponential generating function $g^{(e)}(x)$ for the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ is given by*

$$g^{(e)}(x) = f_{n_1}(x) f_{n_2}(x) \cdots f_{n_k}(x), \tag{7.18}$$

where, for $i = 1, 2, \ldots, k$,

$$f_{n_i}(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_i}}{n_i!}. \tag{7.19}$$

Proof. Let

$$g^{(e)}(x) = h_0 + h_1 x + h_2 \frac{x^2}{2!} + \cdots + h_n \frac{x^n}{n!} + \cdots$$

be the exponential generating function for $h_0, h_1, h_2, \ldots, h_n, \ldots$. Note that $h_n = 0$ for $n > n_1 + n_2 + \cdots + n_k$, so that $g^{(e)}(x)$ is a finite sum. From (7.19), we see that, when (7.18) is multiplied out, we get terms of the form

$$\frac{x^{m_1}}{m_1!} \frac{x^{m_2}}{m_2!} \cdots \frac{x^{m_k}}{m_k!} = \frac{x^{m_1+m_2+\cdots+m_k}}{m_1! m_2! \cdots m_k!}, \tag{7.20}$$

where

$$0 \le m_1 \le n_1,\ 0 \le m_2 \le n_2, \ldots, 0 \le m_k \le n_k.$$

Let $n = m_1 + m_2 + \cdots + m_k$. Then the expression in (7.20) can be written as

$$\frac{x^n}{m_1! m_2! \cdots m_k!}, = \frac{n!}{m_1! m_2! \cdots m_k!} \frac{x^n}{n!}.$$

Thus, the coefficient of $x^n/n!$ in (7.18) is

$$\sum \frac{n!}{m_1! m_2! \cdots m_k!}, \tag{7.21}$$

where the summation extends over all integers $m_1, m_2, \ldots, m_k$, with

$$0 \le m_1 \le n_1, 0 \le m_2 \le n_2, \ldots, 0 \le m_k \le n_k,$$

$$m_1 + m_2 + \cdots + m_k = n.$$

But from Section 3.4 we know that the quantity

$$\frac{n!}{m_1! m_2! \cdots m_k!} \text{ with } n = m_1 + m_2 + \cdots + m_k$$

in the sum (7.21) equals the number of n-permutations (or, simply, permutations) of the combination $\{m_1 \cdot e_1, m_2 \cdot e_2, \ldots, m_k \cdot e_k\}$ of S. Since the number of n-permutations of S equals the number of permutations taken over all such combinations with $m_1 + m_2 + \cdots + m_k = n$, the number h_n equals the number in (7.21). Since this is also the coefficient of $x^n/n!$ in (7.18), we conclude that

$$g^{(e)}(x) = f_{n_1}(x) f_{n_2}(x) \cdots f_{n_k}(x). \qquad \square$$

$(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots)($

Examples

Determine the number of ways to color the squares of a 1-by-n chessboard, using the colors red, white, and blue, if an even number of squares are to be colored red.

Hints: Let hn denote the number of such colorings where we define h0 =1. Then hn equals the number of n-permutations of a multiset of three colors, each with an infinite repetition number, in which red occurs an even number of times. Thus the exponential generating function for h0, h1, h2,…, hn ,… is the product of red, white and blue factors:

$$g^{(e)} = (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \boxed{?})(1 + \frac{x}{1!} + \frac{x^2}{2!} + \boxed{?})(1 + \frac{x}{1!} + \frac{x^2}{2!} + \boxed{?})$$

$$= \frac{1}{2}(e^x + e^{-x})e^x e^x = \frac{1}{2}(e^{3x} + e^x)$$

$$= \frac{1}{2}(\sum_{n=0}^{\infty} 3^n \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!})$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(3^n + 1)\frac{x^n}{n!}$$

Hence, hn = (3n+1)/2.

1, 3, 5, 7, 9

$(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots)(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots)(1+x+\frac{x^2}{2!}+\cdots)$

$(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots)(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots)$

Exercises

$(e^{2x}+e^{-2x}+2)\cdot e^{3x}$

$=\frac{1}{2}(e^{x}+e^{-x})\cdot\frac{1}{2}(e^{x}+e^{-x})\cdot e^{x}\cdot e^{x}\cdot e^{x}$

- **Determine the number hn of n digit numbers with each digit odd where the digits 1 and 3 occur an even number of times.**
- **Hints: let h0 =1. the number hn equals the number of n-permutations of the multiset S = {∞1, ∞3, ∞5, ∞7, ∞9}, in which 1 and 3 occur an even number of times. ……..**

$=\frac{1}{4}(e^{5x}+e^{x}+2e^{3x})$

$=\frac{1}{4}(\sum 5^n\frac{x^n}{n!}+\sum\frac{x^n}{n!}+2\sum 3^n\cdot\frac{x^n}{n!})$

$=\frac{1}{4}(\sum(5^n+1+2\cdot 3^n)\frac{x^n}{n!})$

$h(n)=\frac{5^n+1+2\cdot 3^n}{4}$

Exercises (cont'd)

Determine the number of ways to color the squares of a 1-by-n chessboard, using the colors red, white, and blue, if an even number of squares are to be colored red and there is at least one blue square.

$$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)$$

$$= \frac{1}{2}(e^x+e^{-x})\cdot(e^x-1)\cdot e^x$$

$$(e^{2x}-e^x)(e^x+e^{-x})$$

$$= \frac{1}{2}(e^{3x}-e^{2x}+e^x-1)$$

$$e^{3x}+e^x-e^{2x}-1$$

$$= \frac{1}{2}\left(\sum 3^n\frac{x^n}{n!}-\sum 2^n\frac{x^n}{n!}+\sum\frac{x^n}{n!}-1\right)$$

$=$

$$h(n) = \frac{3^n - 2^n + 1}{2}$$

Assignments

9

16

25

48(b)