Numerical Method for Ordinary Differential Equations with Initial Conditions and Its Application in Physics

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Abstract

In Newtonian mechanics, we usually need to solve a set of ordinary differential equations with some specific initial conditions. Most initial value problems do not have an analytical solution, especially for a many-body system with gravity. In this report we investigate the numerical methods for the initial value problem and their application in solving the solar system. Euler's forward and Velocity-Verlet methods are discussed and applied in the Earth-Sun, Earth-Sun-Jupiter and whole solar system. The comparison between these two methods shows that Verlet method is more stable and thus better to use. We also calculate the perihelion precession of Mercury by adding a relativistic correction to the gravitational force, and obtain almost the same result as the analytical solution.

1 Introduction

The initial value problem often appears in physics, especially in Newtonian mechanics. Applying Newton's second law on a n-body system with no external force, we obtain the equations of motion

$$m_i \frac{d^2\vec{r}_i}{dt^2} = \sum_{j=1,\ j\neq i}^{n} \vec{F}_{ij},\ \ i = 1, 2, \cdots n, \tag{1}$$

where m_i, $\vec{r}_i$ and $\vec{F}_{ij}$ are the mass of object i, coordinate of object i and force between object i and j, respectively. Here we have $\vec{F}_{ij} = -\vec{F}_{ji}$ $(i \neq j)$. Eq. 1 can be transformed into a set of first-order differential equations:

$$\begin{cases} \vec{v}_i = \frac{d\vec{r}_i}{dt} \\ \frac{d\vec{v}_i}{dt} = \frac{1}{m_i}\sum_{j=1,\ j\neq i}^{n} \vec{F}_{ij} \end{cases} \quad i = 1, 2, \cdots n, \tag{2}$$

where $\vec{v}_i$ is the velocity of object i. Eq. 2 should be solved with specified initial $\vec{r}_i$ and $\vec{v}_i$, namely

$$\begin{cases} \vec{r}_i(t_0) = \vec{r}_i^{(0)} \\ \vec{v}_i(t_0) = \vec{v}_i^{(0)} \end{cases} \quad i = 1, 2, \cdots n, \tag{3}$$

where t_0 is the initial time (usually $t_0 = 0$).

In the history of physics, the great success of Newtonian mechanics was partly from its (almost) perfect explanation of planets' motion in the solar system. Eq. 1 of a two-body system with gravity has an analytical solution, which gives a conic section orbit. However, a n-body $(n > 2)$ system with gravity cannot be solved analytically, making it necessary to explore the numerical methods for the initial value problem. In Sec. 2 we will give the equations for both two- and many-body systems with gravity, and discuss two numerical methods (Euler's forward and Velocity-Verlet methods) in Sec. 3. Sec. 4 will discuss and compare the results of applying these two methods on the Earth-Sun, Earth-Sun-Jupiter and whole solar system.

2 Examples of ordinary differential problems in physics

2.1 The Earth-Sun system

The Earth-Sun system is a two-body system governs by the attractive gravitational force between them

$$\vec{F}_G = \frac{GM_{\mathrm{S}}M_{\mathrm{E}}\hat{r}}{r^2}, \tag{4}$$

where G is the gravitational constant; M_{S} and M_{E} are mass of the Sun and the Earth; $\vec{r}$ is the displacement between them.

Given the fact the Sun is about 10^6 heavier than the Earth, we can safely keep the Sun as the center of mass (C.M.) in this problem. With a proper coordinate setup, the orbit of the Earth is co-planer in xy-plane. Using Newton's second law we get the following first-order ordinary differential equations (ODEs) for the Earth

$$\begin{cases} \frac{dx}{dt} = v_x \\ \frac{dx}{dt} = v_x \\ \frac{dv_x}{dt} = -\frac{GM_{\mathrm{S}}x}{r^3} \\ \frac{dv_y}{dt} = -\frac{GM_{\mathrm{S}}y}{r^3}. \end{cases} \tag{5}$$

By solving above equations, we can obtain the informations about the Earth's orbit we need.

2.2 Many-body problem

In the above Sec. 2.1, we simplify the calculation of the orbit of the Earth by taking into account only it's interaction with the Sun. This simplification is reasonable as the Sun is much heavier than other planets in the solar system.

However, if we want to go a step further to get a more precise description of the Earth's orbit. We have to include distortions from other seven planets as well as the Pluto. We should also abandon our previous static Sun approximation but using the real center of mass of the solar system. Until now, we have a new set of ODEs for the Earth

$$\begin{cases} \frac{dx}{dt} = v_x \\ \frac{dy}{dt} = v_y \\ \frac{dv_x}{dt} = \sum_{i=1}^{n} -\frac{GM_{\mathrm{i}}x_i}{r_i^3} \\ \frac{dv_y}{dt} = \sum_{i=1}^{n} -\frac{GM_{\mathrm{i}}y_i}{r_i^3}, \end{cases} \tag{6}$$

where i runs over all other celestial bodies except the Earth itself. Besides the Earth, we have similar sets of ODEs for every celestial body.

By solving these coupled ODEs, we can obtain a full description for the solar system.

3 Numerical methods

3.1 Euler's forward method

Suppose a first-order ordinary differential equation

$$\frac{dx}{dt} = f(x,t);\ t \in [t_0, t_1] \tag{7}$$

Input: $x_0 = 1$, $y_0 = 0$, $v_x^0 = 0$, $v_y^0 = 2\pi$, M_{E}, M_{S}, G
Output: $\vec{x}$=(x_0,x_1,...,x_N), $\vec{y}$, $\vec{v}_x$, $\vec{v}_y$, $\vec{E}_k$, $\vec{E}_p$, $\vec{E}$, $\vec{L}_z$

$r = \mathrm{sqrt}(x_0^2 {+} y_0^2)$;
// i is deferent time points $t_i = t_0 + ih$;
for $i = 1; i <= N; i++$ **do**
  $x_i = x_{i-1} + v_x^{i-1}h$; $y_i = y_{i-1} + v_y^{i-1}h$;
  $v_x^i = v_x^{i-1} - \frac{4\pi^2 x_{i-1}}{r_{i-1}^3}h$; $v_y^i = v_y^{i-1} - \frac{4\pi^2 y_{i-1}}{r_{i-1}^3}h$;
  $r = \mathrm{sqrt}(x_i^2 {+} y_i^2)$;
  // Kinetic energy E_k, Potential energy E_k, Total energy E, Angular momentum in $\hat{z}$ L_z;
  $E_k^i = 0.5 M_{\mathrm{E}}((v_x^i)^2 {+} (v_y^i)^2)$; $E_p^i = -\frac{G M_{\mathrm{E}} M_{\mathrm{S}}}{r}$;
  $E^i = E_p^i + E_k^i$; $L_z^i = M_{\mathrm{E}}(x_i v_y^i - y_i v_x^i)$;
end
return $\vec{x}$, $\vec{y}$, $\vec{v}_x$, $\vec{v}_y$, $\vec{E}_k$, $\vec{E}_p$, $\vec{E}$, $\vec{L}_z$;

Algorithm 1: The Euler's forward method for the Earth-Sun system. It initials from a circular orbit.

with a initial value x_0 at time t_0.

In order to solve this problem, we first discretize the region $[t_0, t_1]$ into N subintervals with a step h, so we get the relation

$$h = \frac{t - t_0}{N}. \tag{8}$$

Then, We have discretized $x_i = x(t_i = t_0 + ih)$ where i is a integer between 0 and N.

Using Taylor expansion, we get

$$x_{i+1} = x_i + \frac{dx_i}{dt}h + \frac{d^2 x_i}{dt^2}h^2 + O(h^3), \tag{9}$$

where i goes from 0 to $N - 1$. The Euler's forward method truncates at the second term of the above equation. Thus, Eq. 9 becomes

$$\begin{aligned} x_{i+1} &= x_i + \frac{dx_i}{dt}h + O(h^2) \\ &= x_i + f(x_i, t_i)h + O(h^2). \end{aligned} \tag{10}$$

We can see it's a one-step method with a local error $O(h^2)$.

Getting back to the Earth-Sun system, we can formulate Eq. 5 to

$$\begin{cases} x_{i+1} = x_i + v_x^i h \\ y_{i+1} = y_i + v_y^i h \\ v_x^{i+1} = v_x^i - \frac{4\pi^2 x_i}{r_i^3} h \\ v_y^{i+1} = v_y^i - \frac{4\pi^2 y_i}{r_i^3} h. \end{cases} \tag{11}$$

in unit of AU for length, year (yr) for time and M_{S} for mass. We will keep using this unit system in our report and calculations. Starting from some initial conditions, we can simply solve out time evolution of the Earth iteratively. Our implementation of this method with a circular orbit initial conditions is shown in Algorithm 1.

We can see that the Euler's forward method is easy to be realized. However, it has vital defects that it violates the energy conservation and time reversibility. The total energy increases with time in the Euler's forward method. That's why we need the Velocity-Verlet method to describe physical systems.

3.2 Velocity-Verlet method

The Velocity-Verlet method is widely used in molecular dynamics calculation as it overcomes above defects. It conserves energy within small round-off errors[1].

Staring from the Taylor expansions under same discretization for both displacement and velocity

$$\begin{aligned} x_{i+1} &= x_i + x_i^{(1)} h + x_i^{(2)} h^2 + O(h^3), \\ v_x^{i+1} &= v_x^i + v_x^{i(1)} h + v_x^{i(2)} h^2 + O(h^3), \end{aligned} \tag{12}$$

with initial values x_0 and v_x^0 at time t_0. We truncate at the third term and evaluate $v_x^{i(2)} h \approx v_x^{i+1(1)} - v_x^{i(1)}$, from which and Eq. 12 we can see that the Velocity-Verlet method is a two-step method with a local error $O(h^3)$

In the Earth-Sun system, with this method, we can formulate Eq. 5 to

$$\begin{cases} x_{i+1} = x_i + v_x^i h - \frac{4\pi^2 x_i}{r_i^3} \frac{h^2}{2} \\ y_{i+1} = y_i + v_y^i h - \frac{4\pi^2 y_i}{r_i^3} \frac{h^2}{2} \\ v_x^{i+1} = v_x^i - (\frac{4\pi^2 x_i}{r_i^3} + \frac{4\pi^2 x_{i+1}}{r_{i+1}^3}) \frac{h}{2} \\ v_y^{i+1} = v_y^i - (\frac{4\pi^2 y_i}{r_i^3} + \frac{4\pi^2 y_{i+1}}{r_{i+1}^3}) \frac{h}{2}. \end{cases} \tag{13}$$

We show our realization of the Velocity-Verlet method in Algorithm 2. Compared with the Euler's forward method, we can see the processes in this method are more complicated with more calculations involved.

3.3 Object-oriented code development

For flexibility and extensivity of our codes, the idea of object-oriented programming is employed in the development. A class called "planet" stores all the properties of a planet, such as coordinate, velocity, acceleration, total force on it, angular momentum and energy. This class also contains functions to initialize before each time step, to calculate gravitational force between two objects, to add up these forces, to do one-step Euler's forward or Velocity-Verlet calculation and to update all the properties after each step. The user can fix one or more objects at a given position, or he/she can also call a friend function to move the center of mass to origin. All the parameters needed for planets in the solar system are given from command line and input file.

4 Results and discussion

4.1 Comparison between two methods

To test the stability of Euler's forward (Euler) and the Velocity-Verlet (VV) method, we initialize the Earth-Sun system with a circular orbit as stated in Algorithms 1&2.

We vary the step size h starting from 0.02 yr to 0.001 yr. The Earth's orbits calculated by these two methods for 10 years are show in Fig. 1. Globally speaking, we see orbits given by the Euler method expand in time. On the other hand, VV methods' orbits keep circular with some tiny fluctuations hardly seen in Fig. 1a. Such observations justify the our statements in Sec. 3 that the VV method conserves energy but the Euler method increases energy in the process of calculation.

Input: $x_0 = 1$, $y_0 = 0$, $v_x^0 = 0$, $v_y^0 = 2\pi$, M_{E}, M_{S}, G

Output: $\vec{x}$=(x_0,x_1,...,x_N), $\vec{y}$, $\vec{v}_x$, $\vec{v}_y$, $\vec{E}_k$, $\vec{E}_p$, $\vec{E}$, $\vec{L}_z$

$r = \mathrm{sqrt}(x_0^2 + y_0^2)$;

$a_x^0 = \frac{4\pi^2 x_{i-1}}{r_{i-1}^3}$; $a_y^0 = \frac{4\pi^2 y_{i-1}}{r_{i-1}^3}$;

// i is deferent time points $t_i = t_0 + ih$;

for $i = 1; i <= N; i++$ **do**

    $x_i = x_{i-1} + v_x^{i-1}h - \frac{a_x^0 h^2}{2}$; $y_i = y_{i-1} + v_y^{i-1}h - \frac{a_y^0 h^2}{2}$;

    $r = \mathrm{sqrt}(x_i^2 + y_i^2)$;

    $a_x^1 = \frac{4\pi^2 x_i}{r_i^3}$; $a_y^1 = \frac{4\pi^2 y_i}{r_i^3}$;

    $v_x^i = v_x^{i-1} - \frac{(a_x^0 + a_x^1)h}{2}h$; $v_y^i = v_y^{i-1} - \frac{(a_y^0 + a_y^1)h}{2}h$;

    $a_x^0 = a_x^1$; $a_y^0 = a_x^1$;

    // Kinetic energy E_k, Potential energy E_k, Total energy E, Angular momentum in $\hat{z}$ L_z;

    $E_k^i = 0.5M_{\mathrm{E}}((v_x^i)^2 + (v_y^i)^2)$;

    $E_p^i = -\frac{GM_{\mathrm{E}}}{r}$; $E^i = E_p^i + E_k^i$;

    $L_z^i = M_{\mathrm{E}}(x_i v_y^i - y_i v_x^i)$;

end

return $\vec{x}$, $\vec{y}$, $\vec{v}_x$, $\vec{v}_y$, $\vec{E}_k$, $\vec{E}_p$, $\vec{E}$, $\vec{L}_z$;

Algorithm 2: The Velocity-Verlet method for the Earth-Sun system. It initials from a circular orbit.

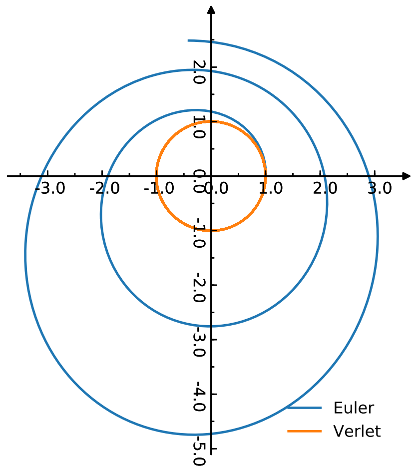
Table 1: The Euler forward method: step size; kinetic energy; potential energy; total energy; angular momentum in z direction, distance from the Sum and execution time from left to the right. Energies are in the unit of $M_{\mathrm{S}}\cdot\mathrm{AU}^2/\mathrm{yr}^2$. Angular momentum is in the unit of $M_{\mathrm{S}}\cdot\mathrm{AU}^2/\mathrm{yr}$.

h (yr)	E_{kin}	E_{pot}	E_{tot}	L_z	r (AU)	execution time (ms)
2.000E-02	3.300E-05	-4.700E-05	-1.400E-05	3.500E-05	2.520E+00	2.500E-02
1.000E-02	2.900E-05	-4.900E-05	-2.000E-05	3.200E-05	2.432E+00	5.000E-02
2.000E-03	3.200E-05	-6.500E-05	-3.300E-05	2.500E-05	1.824E+00	9.400E-02
1.000E-03	4.000E-05	-7.900E-05	-4.000E-05	2.300E-05	1.493E+00	1.680E-01

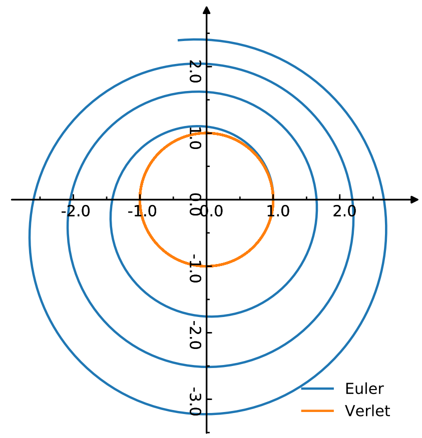
For a large step size in Fig. 1a, the Euler method is very unstable. Its orbit deviates from circle both in distance and shape apparently. As the step size becomes smaller, we can see that the Euler method works better; as the orbits expand slower and slower from Fig. 1a to Fig. 1d. The trends we observed agree with the statement that the error in the Euler method goes down with decreasing h. It also worth noticing that we can hardly see differences between orbits yielded by the VV method, which indicates its stability.

For detailed check and performance comparison, we list distances, energies, angular momentums together with execution times for these two methods in Table 1&2 with precision up to 10^{-6}. As conservation laws predicted by classical mechanics, physical quantities including kinetic, potential, total energies and angular momentum should be conserved in the Earth-Sun system. From these two tables, we can see conservations for the VV method and divergences for the Euler method. Thus, we can conclude that the VV method is stable and the Euler method is not.

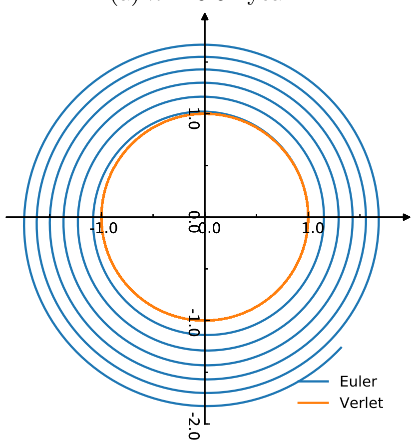
Compared the execution time of two methods, we find the Euler method is faster. Counting the FLOPS from these two algorithms, we find there is about $30N$ for the Euler method and $45N$ for the VV method. It explains speed advantage of the Euler method.



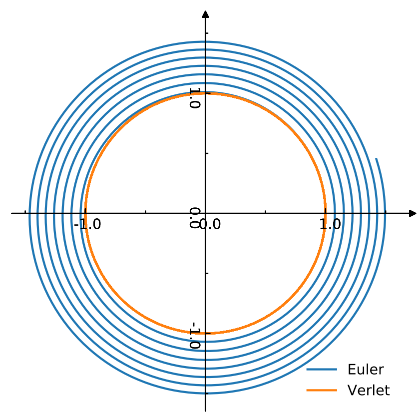
(a) $h = 0.02$ year



(b) $h = 0.01$ year



(c) $h = 0.002$ year



(d) $h = 0.001$ year

Figure 1: Comparison of different step size h of two methods for 10 years.

In order to obtain physically meaningful results, we will keep using the VV method in our following calculations.

4.2 Escape velocity from the Sun

In physics, the escape velocity from the Sun is defined as the minimum speed needed for an object to escape from Sun's gravitational influence. For an object starts from the Earth with a initial speed v_i; we assume it can goes to infinity where its potential energy is zero. At the same time, it has a minimum kinetic energy zero. Due to the energy conservation, we can generalize the equation for the escape velocity

$$\frac{mv_e^2}{2} - GM_\mathrm{S}m/r = 0 \Longrightarrow v_e = \sqrt{\frac{2GM_\mathrm{S}}{r}}. \tag{14}$$

We can see from Eq. 14 that the escape velocity v_e is independent of the mass of object. In our unit system $v_e = 2\sqrt{2}\pi$ which larger than the speed of circular v_c a factor of $\sqrt{2}$.

Table 2: The Velocity-Verlet method: step size; kinetic energy; potential energy; total energy; angular momentum in z direction, distance from the Sum and execution time from left to the right. Energies are in the unit of $M_{\rm S}{\cdot}{\rm AU}^2/{\rm yr}^2$. Angular momentum is in the unit of $M_{\rm S}{\cdot}{\rm AU}^2/{\rm yr}$.

h (yr)	E_{kin}	E_{pot}	E_{tot}	L_z	r (AU)	execution time (ms)
2.000E-02	5.900E-05	-1.180E-04	-1.180E-04	1.900E-05	1.000014	2.600E-02
1.000E-02	5.900E-05	-1.180E-04	-1.180E-04	1.900E-05	1.000E+00	5.600E-02
2.000E-03	5.900E-05	-1.180E-04	-1.180E-04	1.900E-05	1.000E+00	1.840E-01
1.000E-03	5.900E-05	-1.180E-04	-1.180E-04	1.900E-05	1.000E+00	3.350E-01

Table 3: The escape velocity factor $\alpha/\sqrt{2}$ in $v_e = 2\pi\alpha$ and total energy for different t_c. Energy is in the unit of $M_{\rm S}{\cdot}{\rm AU}^2/{\rm yr}^2$.

t_c (yr)	$\alpha/\sqrt{2}$	E_{tot}
500	0.998968061	-5.9E-07
1000	0.998418985	-3.7E-07
2000	0.999002447	-2.4E-07
5000	0.999456311	-1.3E-07
10000	0.999655708	-0.8E-07
20000	0.999781297	-0.5E-07
50000	0.999879019	-0.3E-07

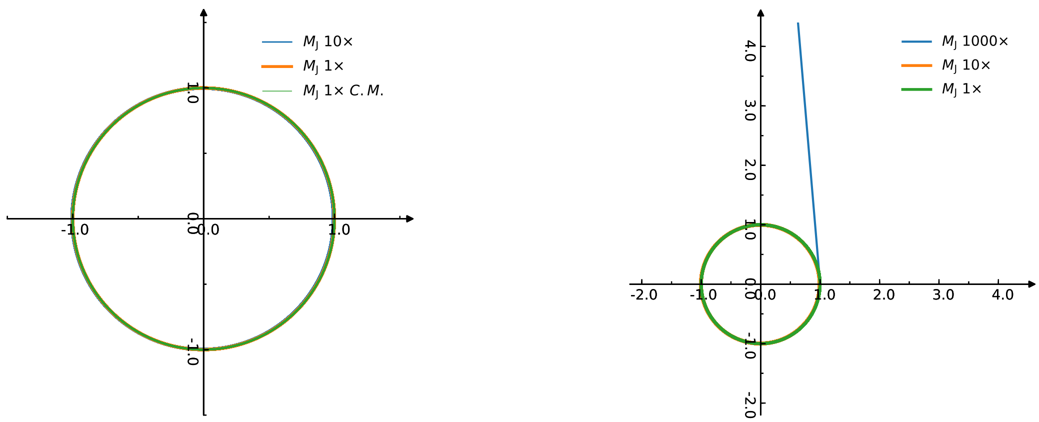
We would like to use a trial and error method to find the escape velocity. In our calculation, we fix the step size $h = 0.001$ and set up a criteria for escape. It is if an object doesn't start to turn back after time t_c, then it be regarded as a successful escape. We initialize our calculation the same as in Algorithm 2 except a different $v_y^0 = 2\pi\alpha$, where α is a constant larger than 1. With $1.3 < \sqrt{2} < 1.5$, we have lower and upper bounds for α for starting a binary search for v_e.

The results for different t_c are presented in Table 3. From the table, we find $\alpha/\sqrt{2}$ converges to one as increasing t_c. Moreover, the total energy gets closer to zero at the same time. In sum, We would expect v_e and E_tot converge their theoretical value $2\sqrt{2}\pi$ and zero eventually.

4.3 Extension to the whole solar system

After testing our VV solver for the two-body case, we extend our program to solve the Earth-Sun-Jupiter (three-body) system and then the whole solar system. This extension is done easily thanks to object-oriented programming. Fig. 2 shows the orbits of the Earth-Sun-Jupiter system with different Jupiter mass. The total time is 100 years. In Fig. 2a, when we fix the position of the Sun and set Jupiter mass as 1× and 10× the actual Jupiter mass, the orbit of the Earth is still a quite good circle, and we can hardly see any difference between 1× and 10× cases. When we do not fix the position of the Sun and fix the center of mass at origin, the orbit of the Earth hardly changes. So in the above three cases the influence of Jupiter on Earth can be treated perturbatively. But when we increase the mass of Jupiter to 1000× (Fig. 2b), the Earth flies away and the system seems to be unstable. In this case the Sun and Jupiter have almost the same mass, and thus the Sun should not be fixed and the whole system is an unstable "binary stars + one planet" system.

After confirming the stability of VV method in the three-body system, we simulate the whole solar system using the positions and velocities of the Sun and eight planets at A.D. 2018-Apr-02 00:00:00.0000 TDB [2] as the initial condition. Fig. 3 shows the orbits of the Sun and all the planets. The total time is 200 years, longer than the period of Neptune. Also, the center of mass is fixed at origin. In our



(a) (b)

Figure 2: Orbit of the Earth in the Earth-Sun-Jupiter system with different Jupiter mass. C.M. represents that the Sun is not fixed and the center of mass is at origin. Total time is 100 years. The unit of coordinate is AU.

simulation, motions in x, y and z directions are all calculated. But as the motion in z direction is not significant compared to that in $x - y$ plane, we only plot the projection of orbits in $x - y$ plane. From Fig. 3, we can see that the whole system is stable. Because of the large mass of the Sun, its orbit can hardly be seen in the figure. The orbit of Mercury is not a closed oval because of the impact from outter planets. The orbits of other planets are almost closed, and their relative eccentricities agree with the observation.

4.4 The perihelion precession of Mercury

Historically, the discrepancy between the observed and calculated values of perihelion precession of Mercury were not able to be accounted by classical Newtonian mechanics. Until the introduction of general relativity, the problem was solved. It was also labeled as a great success of general relativity. The value results from relativistic correction is about $\delta\varphi \approx 43''$ per century[3]. In order to get this value, we introduce a correction to our gravitational force, so that the force becomes

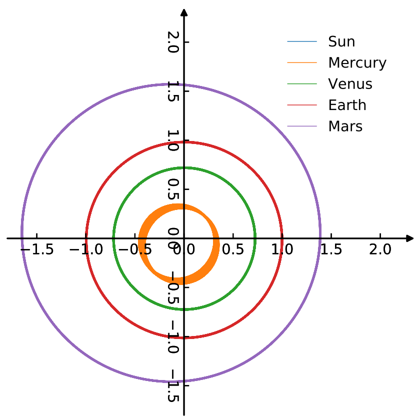
$$\vec{F}_G = \frac{GM_{\mathrm{S}}M_{\mathrm{M}}\hat{r}}{r^2}\left[1 + \frac{3l^2}{r^2c^2}\right], \tag{15}$$

where M_{M} is the mass of Mercury, l is $\vec{r} \times \vec{v}$ and c is the speed of light in vacuum.

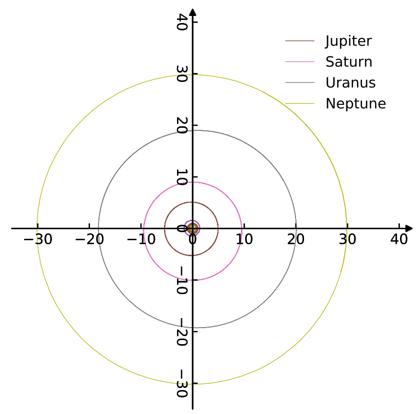
In our calculation, we apply the Mercury mass and new force to calculate the acceleration. The Sun-Mercury system initials from $x_0 = 0.3075$, $y_0 = 0$, $v_x^0 = 0$ and $v_y^0 = 12.44$. For this elliptical orbit, theoretical calculation gives a period $T_m \approx 0.24073$ yr. We also notice for a resolution higher to $1''$, we have to use a very small step size.

Before setting up the step size, we did a rough estimation. The Mercury moves about 420 circles in a century which is about 5.5E08 degrees. To distinguish $1''$ within a century movement, we need at least a step size $h = 100/5.5\mathrm{E}08 \approx 1.84\mathrm{E}$-07. Therefore, $h = 1.0\mathrm{E}$-07 would be a good choice.

We try different h in our calculations and the final results are given in Table 4. The outcomes justify our estimation that the error will less than $1''$ for $h = 1.0\mathrm{E}$-07. Eventually, our calculation gives $\delta\varphi = 42.9333''$ with h smaller to 5.0E-08 which is very close to the theoretical value $43''$.



(a) The Sun and inner planets.



(b) Outer planets.

Figure 3: Orbits of the Sun and all the planets. The center of mass is fixed at origin. Total time is 200 years. The unit of coordinates is AU.

Table 4: The perihelion precession angle $\delta\varphi$ of Mercury due to relativistic correction after one century.

h (yr)	$\delta\varphi$
5.00E-07	$53.2277''$
2.00E-07	$45.8665''$
1.00E-07	$42.8183''$
5.00E-08	$42.9333''$

5 Conclusions

In this work, we investigate two methods, Euler's forward and Velocity-Verlet methods, to solve a set of first-order ordinary differential equations with initial conditions. To test the performance of these algorithms, we use them to calculate the evolution of the Earth-Sun, Earth-Sun-Jupiter and whole solar system. Our comparison shows that VV method can better describe the characteristics of motion in these systems than Euler method. We conclude that VV method is more stable and thus better to use. We also use VV method to calculate the perihelion precession of Mercury by adding a relativistic correction to the gravitational force and obtain a $42.9''$ per century precession which is almost the same as the analytical solution.

Acknowledgments

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