Theorem for privacy parameters ϵ, δ ,database X, domain R and sensetivity-1 utility function u :

if the domain R is such that $|R| \leq \sqrt{\frac{\beta}{\delta}}$ and the gap between $OPT_u(x)$ and the second highest score is less than $\frac{log\left(\frac{1}{\delta\beta}\right)}{\epsilon}$ then the probability that the exponential mechanism will fail to output $OPT_u(x)$ is less than β

proof we want to use the exponential mechanism in a way that in every case that A-dist outputs a value the exponential mechanism will output that value with at least the same probability

ergo from the theorem about exponential mechanism

$$Pr\left[u(M_E(x,u,R)) \leq OPT_u(x) - \frac{2\Delta u}{\epsilon}\left(ln\left(\frac{|R|}{|R_{OPT}|}\right) + t\right)\right] \leq e^{-t}$$

when f is $\frac{log\left(\frac{1}{\delta\beta}\right)}{\epsilon}$-stable, in our case meaning that this is the minimum real-gap between $OPT_u(x)$ to the second highest score, then A-dist fails to return a value in probability β

so we want to calculate the probability that if that is the case the exponential mechanism will output something other then $OPT_u(x)$

we can assume that $|R_{OPT}| = 1$ and that $\Delta u = 1$ and set $e^{-t} = \beta$

so we want to get

$$Pr\left[u(M_E(x,u,R)) \leq OPT_u(x) - \frac{log\left(\frac{1}{\delta\beta}\right)}{\epsilon}\right] \leq Pr\left[u(M_E(x,u,R)) \leq OPT_u(x) - \frac{2}{\epsilon}\left(ln\left(|R|\right) + t\right)\right] \leq$$

or in other words

$$-log\left(\frac{1}{\delta\beta}\right) \leq -2\left(ln\left(|R|\right) + t\right) \Rightarrow 2t \leq log\left(\frac{1}{\delta\beta}\right) - 2ln\left(|R|\right) = ln\left(\frac{1}{\delta\beta|R|^2}\right)$$

$$\Rightarrow e^{2t} \leq \frac{1}{\delta\beta|R|^2} \Rightarrow |R|^2 \leq \frac{1}{\delta\beta e^{2t}} = \frac{e^{-t}e^{-t}}{\delta\beta} = \frac{\beta}{\delta} \Rightarrow |R| \leq \sqrt{\frac{\beta}{\delta}} \qquad \blacksquare$$