Theorem: Fixing a database $x \in X^*$, a quality function $u : X^* x\mathcal{R} \to \mathbb{R}$, approximation and probability parameters α, β.
denote: $OPT_u(x) = \max_{r\in\mathcal{R}}(u(x,r))$
if $|R_{OPT}| = 1$, Δu=1 and

$$OPT_u(x) \geq \frac{2}{\epsilon\alpha} \cdot ln\left(|R|\left(\frac{1}{\beta} - 2\right) + 1\right)$$

it holds that

$$Pr\left[u(\mathcal{M}(x,u,\mathcal{R})) \leq (1-\alpha) \cdot OPT_u(x)\right] \leq \beta$$

Proof:
denote:

- $R_{OPT} = \{r \in \mathcal{R} | u(x,r) = OPT_u(x)\}$
- $R_\alpha = \{r \in \mathcal{R} | u(x,r) \geq (1-\alpha) \cdot OPT_u(x)\}$
- $R'_\alpha = R_\alpha \backslash R_{OPT} = \{r \in \mathcal{R} | OPT_u(x) > u(x,r) \geq (1-\alpha) \cdot OPT_u(x)\}$
- $\bar{R_\alpha} = R \backslash R_\alpha = \{r \in \mathcal{R} | u(x,r) < (1-\alpha) \cdot OPT_u(x)\}$

we want to the parameters s.t.

$$Pr\left[u(\mathcal{M}(x,u,\mathcal{R})) \leq (1-\alpha) \cdot OPT_u\right] = Pr\left[\mathcal{M}(x,u,\mathcal{R}) \in \bar{R_\alpha}\right] \leq \beta$$

each element from $\bar{R_\alpha}$ has mass probability of at most $exp\left(\frac{\epsilon}{2}(1-\alpha) \cdot OPT_u(x)\right)$
so the entire set of "bad" elements has total probability mass of at most $\left|\bar{R_\alpha}\right| exp\left(\frac{\epsilon}{2}(1-\alpha) \cdot OPT_u(x)\right)$
on the other side elements that aren't in $\bar{R_\alpha}$ can be of two types:

1. $r \in R_{OPT}$ such elements has probability mass of $exp\left(\frac{\epsilon}{2} \cdot OPT_u(x)\right)$
2. $r \in R'_\alpha$ such elements has probability mass of at least $exp\left(\frac{\epsilon}{2}(1-\alpha) \cdot OPT_u(x)\right)$

so the entire set of "good" elements has total probability mass of at most

$$|R'_\alpha| \, exp\left(\frac{\epsilon}{2}(1-\alpha) \cdot OPT_u(x)\right) + |R_{OPT}| \, exp\left(\frac{\epsilon}{2} \cdot OPT_u(x)\right)$$

hence this is a lower bound on the normalization term and in total we get that

$$Pr\left[\mathcal{M}(x,u,\mathcal{R}) \in \bar{R_\alpha}\right] \leq \frac{\left|\bar{R_\alpha}\right| exp\left(\frac{\epsilon}{2}(1-\alpha) \cdot OPT_u(x)\right)}{|R'_\alpha| \, exp\left(\frac{\epsilon}{2}(1-\alpha) \cdot OPT_u(x)\right) + |R_{OPT}| \, exp\left(\frac{\epsilon}{2} \cdot OPT_u(x)\right)}$$

we defined the bound on the probability as β so it must hold that

$$\frac{\left|\bar{R_\alpha}\right| exp\left(\frac{\epsilon}{2}(1-\alpha) \cdot OPT_u(x)\right)}{|R'_\alpha| \, exp\left(\frac{\epsilon}{2}(1-\alpha) \cdot OPT_u(x)\right) + |R_{OPT}| \, exp\left(\frac{\epsilon}{2} \cdot OPT_u(x)\right)} \leq \beta \Rightarrow$$

$$\frac{\left|\bar{R_\alpha}\right|}{\beta} exp\left(\frac{\epsilon}{2}\left(1-\alpha\right)\cdot OPT_u\left(x\right)\right) \leq \left|R_\alpha\right| exp\left(\frac{\epsilon}{2}\left(1-\alpha\right)\cdot OPT_u\left(x\right)\right) + \left|R_{OPT}\right| exp\left(\frac{\epsilon}{2}\cdot OPT_u\left(x\right)\right)$$

$$\Rightarrow e^{\frac{\epsilon}{2}(1-\alpha)\cdot OPT_u(x)}\left(\frac{\left|\bar{R_\alpha}\right|}{\beta} - \left(|R| - \left|\bar{R_\alpha}\right| - 1\right)\right) \leq e^{\frac{\epsilon}{2}\cdot OPT_u(x)}$$

$$\Rightarrow \frac{\epsilon}{2}\left(1-\alpha\right)\cdot OPT_u\left(x\right) + ln\left(\frac{\left|\bar{R_\alpha}\right|}{\beta} - \left(|R| - \left|\bar{R_\alpha}\right| - 1\right)\right) \leq \frac{\epsilon}{2}\cdot OPT_u\left(x\right)$$

$$\Rightarrow \frac{2}{\epsilon\alpha}\cdot ln\left(\left|\bar{R_\alpha}\right|\left(\frac{1}{\beta} - 1\right) - |R| + 1\right) \leq OPT_u\left(x\right)$$

since by definition $\left|\bar{R_\alpha}\right| < |R|$ if

$$OPT_u\left(x\right) \geq \frac{2}{\epsilon\alpha}\cdot ln\left(|R|\left(\frac{1}{\beta} - 2\right) + 1\right)$$

we get the appropriate bound.

Remark 1: in the special case where $|R_{OPT}| = 1$ and

$$R_\alpha = R_{OPT} \Rightarrow \bar{R_\alpha} = R \backslash R_{OPT} \Rightarrow \left|\bar{R_\alpha}\right| = |R| - |R_{OPT}|$$

we get that if

$$OPT_u\left(x\right) \geq \frac{2}{\epsilon\alpha}\cdot ln\left(\left(|R| - 1\right)\left(\frac{1}{\beta} - 2\right)\right)$$

it holds that

$$Pr\left[u\left(\mathcal{M}\left(x,u,\mathcal{R}\right)\right) \neq OPT_u\left(x\right)\right] \leq \beta$$

Remark 2: *In Algorithmic Foundations of Differential Privacy*[1] the following additive bound had been proved:

$$Pr\left[u(\mathcal{M}_E(x,u,\mathcal{R})) \leq OPT_u(x) - \frac{2\Delta u}{\epsilon}\left(ln\left(\frac{|\mathcal{R}|}{|R_{OPT}|}\right) + t\right)\right] \leq e^{-t}$$

[1] C.Dwork , A.Roth