Thorem in algorithm RecConcave[1] for privacy parameters ϵ, δ ,promise r database X, domain R, sensetivity-1 utility function u and recursion bound $N = 2$ then: the step 6 will returns a "good interval" in probability at least $1 - \beta$ if $r > \frac{16}{\epsilon\alpha} ln\left(\frac{log_2(T)}{\beta}\right)$

Proof notice that when $N = 2$ the recurssion call actually calls the $Exponential-Mechanism$ and retrives its answer

we want to make sure that the $Exponential - Mechanism$ outputs with high probability a good interval

meaning that

$$Pr\left[q\left(S, j\right) < \frac{3\alpha}{\epsilon} r\right] < \beta$$

by The Algorithmic Foundations of Differential Privacy[2] Theorem 3.11

$$Pr\left[q(M_E(x, q, R)) \leq OPT_q(x) - \frac{2\Delta q}{\epsilon}\left(ln\left(\frac{|R|}{|R_{OPT}|}\right) + t\right)\right] \leq e^{-t}$$

and if we switch $e^{-t} = \beta$ and take the specific parameters used in our case$\Delta u = 1,\ |R| = log_2\left(T\right)$ and $|R_{OPT}| = 1$

we get

$$Pr\left[q(M_E(x, q, R)) \leq OPT_q(x) - \frac{2}{\epsilon} ln\left(\frac{log_2\left(T\right)}{\beta}\right)\right] \leq \beta$$

we also know that $\frac{\alpha}{2} r \leq OPT_q(x) \leq r$

if we combine all the above we get that we want

$$\frac{3\alpha}{8} r < \frac{\alpha}{2} r - \frac{2}{\epsilon} ln\left(\frac{log_2\left(T\right)}{\beta}\right) \Rightarrow r > \frac{16}{\epsilon\alpha} ln\left(\frac{log_2\left(T\right)}{\beta}\right)$$

Remark we saw that for $A_{dist}$to run we must have

$$r > \frac{8ln\left(\frac{1}{\beta\delta}\right)}{3\epsilon\alpha}$$

notice that this bound will be less than the one above iff $\delta > \frac{\beta^5}{\log_2^6(T)}$

Proof

$$\frac{8ln\left(\frac{1}{\beta\delta}\right)}{3\epsilon\alpha} < \frac{16}{\epsilon\alpha} ln\left(\frac{log_2\left(T\right)}{\beta}\right) \Rightarrow ln\left(\frac{1}{\beta\delta}\right) < 6ln\left(\frac{log_2\left(T\right)}{\beta}\right) \Rightarrow \frac{1}{\beta\delta} < \left(\frac{log_2\left(T\right)}{\beta}\right)^6$$

$$\Rightarrow \delta > \frac{\beta^5}{\log_2^6\left(T\right)}$$

[1]A. Beimel, K. Nissim, and U. Stemmer. Private learning and sanitization- Pure vs. Approximate Differential Privacy

[2]C.Dwork , A.roth