3

Robust Digital Controller Design Methods

In this chapter the design of model based robust digital controllers is discussed. The design of digital PID controllers is first presented, emphasizing the general structure of digital controllers (three branched structure known as RST), the special features of the digital approach and the limitations of the digital PID. The following design methods are then presented: pole placement, tracking and regulation with independent objectives and tracking and regulation with internal model control. The presentation is done from the perspective of robust control. These methods permit the control of systems of any order with or without time delay. The last section of the chapter presents a general methodology for the design of robust digital controllers by means of sensitivity functions shaping.

3.1 Introduction

The use of a digital computer or microprocessor in control loops offers several advantages. These include:

- Considerable choice of strategies for controller design
- Possibility of using algorithms which are both more complex and more efficient than the PID
- Technique perfectly suited for the control of systems *with time dela;*
- Technique well suited for the control of systems characterized by linear dynamic models of high order (including systems with multiple low damped vibration modes)

Moreover, by combining the controller design methods with the system model identification techniques, a rigorous, high performance controller design procedure can be implemented. These aspects are covered in Chapters 7, 8 and 9.

The digital controller design methods that are presented in this chapter relate to single input-single output control in the presence of deterministic disturbances.

These methods are:

- Digital PID

- Pole placement
- Tracking and regulation with independent objectives
- Tracking and regulation with internal model control
- Pole placement with shaping of sensitivity functions

All the controllers, irrespective of their design method, will have the same RST three-branched structure (see Figure 2.27). Only the *memory* of the controller (number of coefficients) will vary depending on the complexity of the system.

Note that in the case of simple systems (at most second-order and small time delay), the various controllers designed by means of the pole placement, tracking and regulation with independent objectives and internal model control, correspond to digital PID controllers having differently tuned parameters.

The *tracking and regulation with independent objectives* and *tracking and regulation with internal model control* can be considered as particular cases of the *pole placement*. They result from a particular choice of the desired closed loop poles.

The robustness of the designed control system with respect to plant model uncertainties is a very important issue. The *pole placement with shaping of sensitivity functions* is a general methodology of digital control design that allows one to take into account simultaneously robustness and performances specifications for the closed loop.

The digital control design by *pole placement* (and the various particular cases) is a *predictive control* (the controller implicitly contains a predictor of the plant output). This will be illustrated in Sections 3.4.4 and 3.5.3 and in Appendix B.

The *design and tuning of digital controllers require the knowledge of the discrete-time model of the plant to be controlled (model based control).* It is not possible to implement effectively a high performance control loop without identifying the plant model. Fortunately the system identification methodology is a mature subject and toolboxes or dedicated software are available. System identification is discussed in Chapters 5, 6, 7 and 9.

If the continuous-time model of the plant to be controlled is available, the discrete-time model of the sampled-data system can be obtained by using appropriate discretization techniques. The functions *cont2disc.sci* (Scilab) and *cont2disc.m* (MATLAB®), available on the book website, can be used for this purpose.

3.2 Digital PID Controller

The basic version of the digital PID controller considered in this book results from the discretization of the continuous-time PID controller presented in Section 1.3.2. Another version, which provides some advantages, will also be presented.

The methodology for the digital PID controllers design to be presented is rigorously applicable only to[1]:

- Plants that can be modeled by a continuous-time system characterized by a transfer function of maximum degree equal to 2, with or without time delay
- Plants having a time delay which is less than one sampling period

Although continuous-time PID parameters can be recovered from the digital PID controller design, in some cases, the discrete-time methodologies have not been developed for the tuning of continuous-time (or pseudo-digital) PID controllers. Specific methodologies exist and should be used for the tuning of continuous time PID controllers (see Chapter 1, Section 1.3). Moreover, it should be pointed out that certain tunings of digital PID controller parameters offering excellent performances have no counterpart in terms of continuous-time PID controller parameters.

3.2.1 Structure of the Digital PID 1 Controller

Consider the transfer function of the continuous-time PID controller (Equation 1.3.2):

$$H_{PID}(s) = K\left[1 + \frac{1}{T_i s} + \frac{T_d s}{1 + \frac{T_d}{N} s}\right] \tag{3.2.1}$$

This controller is characterized by four tuning parameters:

- K – proportional gain
- T_i – integral action
- T_d – derivative action
- T_d/N – filtering of the derivative action

Several discretization methods may be used to derive the structure of the digital PID controller. The relations between the continuous-time and discrete-time parameters will depend on the method used, but the structure of the digital controller will remain the same.

Since in our case the design and implementation of the controller will be in the discrete-time, the discretization method is not essential. For this reason we will use the *backward difference approximation*. It follows that s (derivative) will be approximated by $(1 - q^{-1})/T_s$ and $1/s$ (integration) will be approximated by $T_s/(1 - q^{-1})$ (see Section 2.3, Equations 2.3.6 to 2.3.11).

---

[1] A method for the design of digital PID controllers for plants characterized by high order models is presented in Chapter 10, Section 10.5. It is based on the complexity reduction of a model based controller.

This produces

$$\frac{1}{T_i s} = \frac{T_s}{T_i} \cdot \frac{1}{1-q^{-1}} \tag{3.2.2}$$

$$T_d s = \frac{T_d}{T_s} \cdot (1-q^{-1}) \tag{3.2.3}$$

$$\frac{1}{1+\frac{T_d}{N}s} = \frac{1}{1+\frac{T_d}{NT_s}(1-q^{-1})} = \frac{\frac{NT_s}{T_d+NT_s}}{1-\frac{T_d}{T_d+NT_s}q^{-1}} \tag{3.2.4}$$

By introducing these expressions in Equation (3.2.1), the pulse transfer function (operator) of the digital PID 1 controller is obtained[2]:

$$H_{PID1}(q^{-1}) = \frac{R(q^{-1})}{S(q^{-1})} = K\left[1+\frac{T_s}{T_i}\cdot\frac{1}{1-q^{-1}}+\frac{\frac{NT_s}{T_d+NT_s}(1-q^{-1})}{1-\frac{T_d}{T_d+NT_s}q^{-1}}\right] \tag{3.2.5}$$

The expression in terms of the ratio of two polynomials is obtained by summing up the three terms. Polynomials $R(q^{-1})$ and $S(q^{-1})$ have the form

$$R(q^{-1}) = r_0 + r_1 q^{-1} + r_2 q^{-2} \tag{3.2.6}$$

$$S(q^{-1}) = (1 - q^{-1})(1 + s'_1 q^{-1}) = (1 + s_1 q^{-1} + s_2 q^{-2}) \tag{3.2.7}$$

where

$$s'_1 = -\frac{T_d}{T_d+NT_s} \quad ; \quad r_0 = K\left(1+\frac{T_s}{T_i}-N\frac{T_s}{T_d}s'_1\right)$$

$$r_1 = K\left[s_1\left(1+\frac{T_s}{T_i}+2N\frac{T_s}{T_d}\right)-1\right] \quad ; \quad r_2 = -Ks_1(1+N\frac{T_s}{T_d})$$

[2] As indicated in Section 2.5.3, for systems with constant coefficients the notation q^{-1} will be used both for the backward delay operator and the complex variable z^{-1}, with the exception of cases that require an interpretation in the frequency domain.

The digital PID 1 controller has four parameters (r_0, r_1, r_2, s'_1) as the continuous-time PID controller.

Note that the pulse transfer function (operator) of the digital PID 1 controller contains as a common factor of the denominator the term ($1-q^{-1}$), which assures the behavior of the numerical integration. The denominator also contains the term ($1+s'_1q^{-1}$), which is a digital filter that plays the role of the filter [$1 + (T_d/N)\, s$] in the continuous-time PID controller.

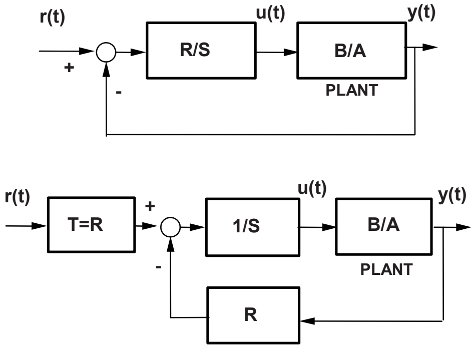

Figure 3.1. Equivalent block diagrams of a digital control loop using the digital PID 1 controller

The equivalent block diagram is given in the upper part of Figure 3.1. By taking $T(q^{-1}) = R(q^{-1})$, the digital PID 1 controller can take the standard three branched structure of the RST controller, as shown in the lower part of Figure 3.1.

The pulse transfer function (operator) of the closed loop relating the reference $r(t)$ and the output $y(t)$ is

$$H_{CL}(q^{-1}) = \frac{B(q^{-1})R(q^{-1})}{A(q^{-1})S(q^{-1}) + B(q^{-1})R(q^{-1})} = \frac{B(q^{-1})R(q^{-1})}{P(q^{-1})} \tag{3.2.8}$$

in which the polynomial $P(q^{-1})$ defines the desired closed loop poles (directly linked to the desired regulation performances).

The product $B(q^{-1})R(q^{-1})$ defines the closed loop zeros. The digital PID 1 controller, in general, does not simplify the plant zeros (unless $B(q^{-1})$ is chosen as a factor of $P(q^{-1})$) and thus can be used for the regulation of plants having a discrete-time model with unstable zeros (a situation occurring, for example, if there is a fractional time delay greater than half of a sampling period, see Section 2.3.7).

Furthermore, the digital PID 1 controller introduces additional zeros defined by $R(q^{-1})$ that will depend on $A(q^{-1})$, $B(q^{-1})$ and $P(q^{-1})$ and thus which cannot be specified a priori. In certain situations, these zeros may produce undesirable overshoots during the transient (see the examples given later on in Section 3.2.3).

3.2.2 Design of the Digital PID 1 Controller

The computation of the parameters involves several stages:

1. Determination of discrete time plant model
2. Specification of the performances
3. Computation of the controller parameters (the coefficients of the polynomial $R(q^{-1})$ and $S(q^{-1})$)
4. Verification of the achieved robustness margins and sensitivity functions

Discrete-Time Plant Model

This is the pulse transfer function $H(z^{-1})$ of the sampled model of a plant having the transfer function $H(s)$, and controlled through a zero order hold as is indicated in Figure 3.2.

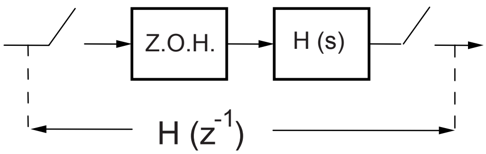

Figure 3.2. Pulse transfer function of a plant

Continuous-time transfer functions of the following form may be considered:

$$H(s) = \frac{Ge^{-s\tau}}{1+sT} \tag{3.2.9}$$

or

$$H(s) = \frac{\omega_0^2 e^{-s\tau}}{\omega_0^2 + 2\zeta\omega_0 s + s^2} \tag{3.2.10}$$

with the restriction

$$\tau < T_S \tag{3.2.11}$$

Note moreover that for the first-order system the sampling period T_s must be smaller than T ($T_S < T$ – see Equation 2.2.4). It thus means that the digital PID controller design can be correctly applied only to the first-order system with a time delay verifying the condition $\tau < T$.

For these two types of continuous-time models, discretization with hold results in a pulse transfer function (operator) of the form

$$H(q^{-1}) = \frac{b_1 q^{-1} + b_2 q^{-2}}{1 + a_1 q^{-1} + a_2 q^{-2}} = \frac{B(q^{-1})}{A(q^{-1})} \tag{3.2.12}$$

The transfer function $H(q^{-1})$ may be obtained:

- Directly by identification of the discrete-time plant model
- From knowledge of $H(s)$ and T_s using discretization routines (as for example *cont2disc.sci* (Scilab) and *cont2disc.m* (MATLAB®)) or transformation tables

Specification of the Performances

As a general rule, the desired closed loop system performances can be expressed in terms of the parameters of a pulse transfer function. This may be expressed by the condition

$$H_{CL}(q^{-1}) = \frac{B(q^{-1})R(q^{-1})}{A(q^{-1})S(q^{-1}) + B(q^{-1})R(q^{-1})} = \frac{B_M(q^{-1})}{P(q^{-1})} \tag{3.2.13}$$

However, $B_M(q^{-1})$ cannot be specified *a priori* since, in general, $B(q^{-1})$ is not simplified (unless $B(q^{-1})$ is stable, which corresponds to cases where the time delay is negligible); moreover, the controller itself introduces zeros by means of $R(q^{-1})$. The closed loop polynomial remains to be specified. The polynomial $P(q^{-1})$ is chosen of the form

$$P(q^{-1}) = 1 + p'_1 q^{-1} + p'_2 q^{-2} \tag{3.2.14}$$

A recommended method for defining p'_1 and p'_2 consists first in considering a second-order normalized continuous-time model (see Section 1.1.6), enabling a rise time (t_R) or a settling time (t_S) and a maximum overshoot (M) to be obtained in accordance with the specifications. This choice may be done using the diagrams given in Figures 1.10 and 1.11 or using the functions *omega_dmp.sci* (Scilab) and *omega_dmp.m* (MATLAB®)[3]. That allows determining the parameters ω_0 and ζ of the second-order system. The sampling period T_s and the natural frequency ω_0 should verify the condition

$$0.25 \le \omega_0 T_s \le 1.5 \quad ; \quad 0.7 \le \zeta \le 1 \tag{3.2.15}$$

The discretized model with hold can be computed either by means of the functions *cont2disc.sci* (Scilab) and *cont2disc.m* (MATLAB®), or by means of

[3] Available from the book website.

transformation tables (see Section 2.3.6). The denominator of the pulse transfer function thus obtained will represent the polynomial $P(q^{-1})$.

Computation of the Coefficients of the Digital Controller

From Equation 3.2.13 it results that the following polynomial equation must be solved:

$$P(q^{-1}) = A(q^{-1})\, S(q^{-1}) + B(q^{-1})\, R(q^{-1}) \tag{3.2.16}$$

in the unknown polynomials $S(q^{-1})$ and $R(q^{-1})$. In Equation 3.2.16 $P(q^{-1})$ is given by Equation 3.2.14 and $A(q^{-1})$ and $B(q^{-1})$ are given by Equation 3.2.12 . The structures of $R(q^{-1})$ and $S(q^{-1})$ are given respectively by Equations 3.2.6 and 3.2.7. Equation 3.2.16 is a Bezout polynomial equation.

The detailed version of Equation 3.2.16 is

$$\begin{aligned} P(q^{-1}) &= 1 + p_1' q^{-1} + p_2' q^{-2} = A(q^{-1}) S(q^{-1}) + B(q^{-1}) R(q^{-1}) \\ &= (1 + a_1 q^{-1} + a_2 q^{-2})(1 - q^{-1})(1 + s_1' q^{-1}) \\ &+ (b_1 q^{-1} + b_2 q^{-2})(r_0 + r_1 q^{-1} + r_2 q^{-2}) \\ &= A'(q^{-1}) S'(q^{-1}) + B(q^{-1}) R(q^{-1}) \end{aligned} \tag{3.2.17}$$

where

$$\begin{aligned} A'(q^{-1}) &= A(q^{-1})(1 - q^{-1}) = (1 + a_1' q^{-1} + a_2' q^{-2} + a_3' q^{-3}) \\ &= (1 + (a_1 - 1) q^{-1} + (a_2 - a_1) q^{-2} - a_2 q^{-3}) \end{aligned} \tag{3.2.18}$$

$$S'(q^{-1}) = 1 + s_1' q^{-1} \tag{3.2.19}$$

Solving a polynomial equation implies that coefficients related to the powers of q should be equal on both sides. One observes that the maximum order on the right hand side is q^{-4} (it is a system of four equations in four variables that must be solved). The higher order of the left side of the equation is q^{-2}, that corresponds to a zero value for the coefficients of q^{-3} and q^{-4} powers in the polynomial $P(q^{-1})$. In fact, it is possible to impose non-zero values for these coefficients.

In general one imposes

$$\begin{aligned} P(q^{-1}) &= (1 + p_1 q^{-1} + p_2 q^{-2} + p_3 q^{-3} + p_4 q^{-4}) = \\ &= (1 + p_1' q^{-1} + p_2' q^{-2})(1 + \alpha_1 q^{-1})(1 + \alpha_2 q^{-2}) \end{aligned} \tag{3.2.20}$$

where coefficients p_1' and p_2' result from the discretization of a second-order continuous-time system with ω_0 and ζ corresponding to the specified nominal performances, and with α_1, α_2 corresponding to aperiodic "auxiliary poles" $-\alpha_1$

and $-\alpha_2$ located on the real axis (inside the unit circle) and corresponding to a frequency higher than $\omega_0 / 2\pi$ (see Figure 3.3).

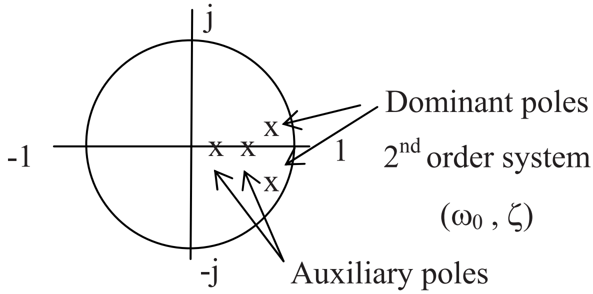

Figure 3.3. Dominant and auxiliary poles for the design of a digital PID controller

One observes that $-\alpha_1$ and $-\alpha_2$ are smaller than the real part of the dominant poles (then they are faster poles). The introduction of auxiliary poles allows improving the robustness of the controller. In practice, a typical choice is either $\alpha_1 = \alpha_2$, or $\alpha_2 = 0$ and the ranges for their values are

$$-0.05 \leq \alpha_1, \alpha_2 \leq -0.5 \tag{3.2.21}$$

The system of four equations to solve is

$$p_1 = b_1 r_0 + s_1' + a_1'$$

$$p_2 = b_2 r_0 + b_1 r_1 + s_1' a_1' + a_2'$$

$$p_3 = b_2 r_1 + b_1 r_2 + s_1' a_2' + a_3'$$

$$p_4 = b_2 r_2 + s_1' a_3'$$

The equivalent matrix form of this system of equations is

$$Mx = p \tag{3.2.22}$$

where

$$x^T = [1, s'_1, r_0, r_1, r_2] \tag{3.2.23}$$

$$p^T = [1, p_1, p_2, p_3, p_4] \tag{3.2.24}$$

and the matrix M is of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a'_1 & 1 & b_1 & 0 & 0 \\ a'_2 & a'_1 & b_2 & b_1 & 0 \\ a'_3 & a'_2 & 0 & b_2 & b_1 \\ 0 & a'_3 & 0 & 0 & b_2 \end{bmatrix}$$

The solution of Equation 3.2.22 is expressed by

$$x = M^{-1} p \tag{3.2.25}$$

where M^{-1} is the inverse matrix of M. In order to assure the existence of this inverse, it is necessary that the determinant of M is non-null. It can be shown that this condition is verified if, and only if, $A(q^{-1})$ and $B(q^{-1})$ are coprime polynomials (no simplification between zeros and poles).

Exercise
Let

$$B(q^{-1}) = b_1 q^{-1} + b_2 q^{-2} = b_1 q^{-1}(1 + \frac{b_2}{b_1} q^{-1})$$

$$A(q^{-1}) = (1 + \frac{b_2}{b_1} q^{-1})(1 + c q^{-1})$$

Show that in this case $\det M = 0$.

In order to solve the Bezout equation, one can use the function *bezoutd.sci* (Scilab) or the function *bezoutd.m* (MATLAB®) available on the book website[4].

The parameters of the continuous-time PID controller, which by discretization with a sampling period equal to T_s gives the digital PID controller with polynomials $R(q^{-1})$ and $S(q^{-1})$, are computed by means of the following relations:

$$K = \frac{r_0 s'_1 - r_1 - (2 + s'_1) r_2}{(1 + s'_1)^2} \tag{3.2.26}$$

---

[4] In Scilab and MATLAB® environments one must specify $A'(q^{-1})$ and $B(q^{-1})$ to obtain $S'(q^{-1})$ and $R(q^{-1})$. $S(q^{-1})$ is obtained by using Equation 3.2.7.

$$T_i = T_s \cdot \frac{K(1+s_1')}{r_0 + r_1 + r_2} \tag{3.2.27}$$

$$T_d = T_s \cdot \frac{s_1' r_0 - s_1' r_1 + r_2}{K(1+s_1')^3} \tag{3.2.28}$$

$$\frac{T_d}{N} = \frac{-s_1' T_s}{1+s_1'} \tag{3.2.29}$$

For the digital PID controller to be equivalent to a continuous-time PID controller, it is necessary that the coefficient s'_1 verify the condition

$$-1 < s'_1 \leq 0 \tag{3.2.30}$$

In the opposite case ($0 < s'_1 < 1$), the digital filter $1/(1+s'_1 q^{-1})$ is stable but without a continuous-time equivalent (as a first-order filter). The digital PID controller in this case may provide very good performances but an equivalent continuous-time PID controller cannot be obtained (see examples given in Section 3.2.3).

3.2.3 Digital PID 1 Controller: Examples

We consider the case of the regulation of a first-order plant with time delay having the following characteristics:

- Gain (G) = *1*; Time Constant (T) = *10* s; Pure time delay (τ) = *3 s*
- A sampling period T_s = *5 s* is chosen in order to verify the conditions: $\tau < T_s$ and $T_s < T$

The objective of the design is to obtain the best closed loop performances without overshoot. The damping factor of the model that specifies the performances is fixed at $\zeta = 0.8$, and the natural frequency ω_0 will be chosen in the range: $0.25 \leq \omega_0 T_s \leq 1.5$.

For $\omega_0 = 0.05$ ($\omega_0 T_s = 0.25$), the results obtained are summed up in Table 3.1.

The model output *y(t)* and the control signal *u(t)* are displayed in Figure 3.4. It can be observed that the closed loop step response (about *13Ts= 65 s*) is slower than the open loop step response (about $2.2T + \tau = 25$ *s*) since the control *u(t)* sent

Table 3.1. Digital PID 1 controller, $\omega_0 = 0.05$ rad/s

Plant:

- $B(q^{-1}) = 0.1813\, q^{-1} + 0.2122\, q^{-2}$
- $A(q^{-1}) = 1 - 0.6065\, q^{-1}$
- $T_S = 5s,\ G = 1,\ T = 10s,\ \tau = 3$

Performances → $T_S = 5s$, $\omega_0 = 0.05$ rad/s, $\zeta = 0.8$

***** CONTROL LAW *****

$$S(q^{-1}) \,.\, u(t) + R(q^{-1}) \,.\, y(t) = T(q^{-1}) \,.\, r(t)$$

Controller:

- $R(q^{-1}) = 0.0621 + 0.0681\, q^{-1}$
- $S(q^{-1}) = (1 - q^{-1}) \cdot (1 - 0.0238\, q^{-1})$
- $T(q^{-1}) = R(q^{-1})$

Gain margin: 7.712 Phase margin: 67.2 deg
Modulus margin: 0.751 (-2.49dB) Delay margin: 45.4 s
Cont. time PID: k = -0.073, Ti = -2.735, Td = - 0.122, Td/N = 0.122

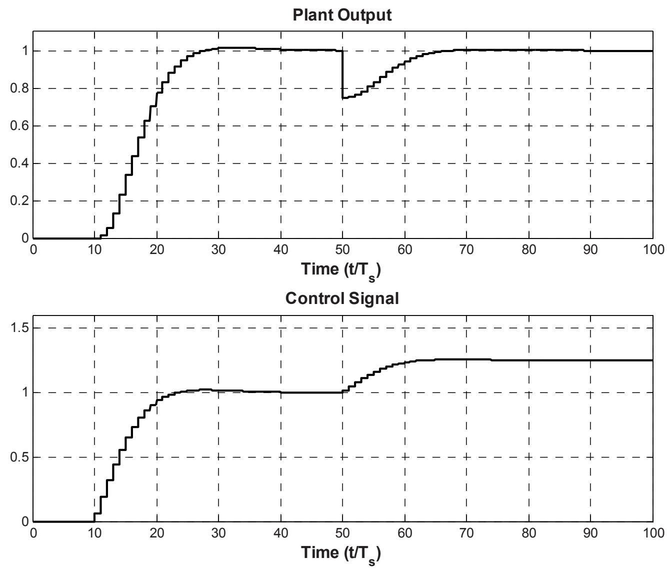

Figure 3.4. Performances of the digital PID 1 controller, $\omega_0 = 0.05$ rad/s for tracking and regulation

by the controller represents a filtered step. This situation is well known in practice. For time delays greater than 20 or 25% of the time constant, the continuous time PID controller slows down the closed loop system with respect to the open loop system. Can a digital PID controller therefore provide improved performances?

Take $\omega_0 = 0.1\ rad/s$ (twice faster desired closed loop response). The computation results are summed up in Table 3.2 and the evolutions of the plant output and of the control applied to the plant are represented in Figure 3.5.

By examining the results given in Table 3.2, first one can see that s'_1 is positive, and therefore an equivalent continuous-time PID controller does not exist.

An acceleration in the plant response is observed, but this is accompanied by the appearance of a slight overshoot (larger than the one corresponding to $\zeta = 0.8$).

The explanation of this overshoot is given by the coefficients of $T(q^{-1})=R(q^{-1})$. The polynomial $T(q^{-1})=R(q^{-1})$, which introduces zeros in the pulse transfer function, has now its second coefficient with a negative sign. As the zeros have influence on the first instants of the response, the contribution of $R(q^{-1})$ is a positive jump followed by a negative one, and the step on the reference filtered, by $R(q^{-1})$ is characterized by a peak related to the difference $r_0 - r_1$, that is not completely attenuated by the remaining components of the pulse transfer function.

In order to avoid an excessive overshoot with this structure of the PID, one needs (for $\zeta > 0.7$) to have all coefficients of $R(q^{-1})$ positive.

Taking $\omega_0 = 0.15\ rad/s$ (Table 3.3 and Figure 3.6), a significant acceleration of the step response is obtained, but the overshoot becomes larger. As in the previous case, there is no equivalent continuous-time PID controller. It is also observed that the difference $r_0 - r_1$ is greater, thus explaining the increased overshoot.

Table 3.2. Digital PID 1 controller, $\omega_0 = 0.1$ rad/s

Plant:

- $B(q^{-1}) = 0.1813\, q^{-1} + 0.2122\, q^{-2}$
- $A(q^{-1}) = 1 - 0.6065\, q^{-1}$
- $T_S = 5s,\ G = 1,\ T = 10s,\ \tau = 3$

Performances → $T_S = 5s\ ,\quad \omega_0 = 0.1\ rad/s,\quad \zeta = 0.8$

***** CONTROL LAW *****

$$S(q^{-1})\,.\,u(t) + R(q^{-1})\,.\,y(t) = T(q^{-1})\,.\,r(t)$$

Controller:

- $R(q^{-1}) = 0.8954 - 0.4671\, q^{-1}$
- $S(q^{-1}) = (1 - q^{-1})\,.\,(1 + 0.16343\, q^{-1})$
- $T(q^{-1}) = R(q^{-1})$

Gain margin: 6.046	Phase margin: 65.9 deg
Modulus margin: 0.759 (- 2, 39 dB)	Delay margin: 16.8 s
Cont. time PID: (no equivalent)	

Table 3.3. Digital PID 1 controller, $\omega_0 = 0.15$ rad/s

Plant: • $B(q^{-1}) = 0.1813\, q^{-1} + 0.2122\, q^{-2}$ • $A(q^{-1}) = 1 - 0.6065\, q^{-2}$ • $T_S = 5s,\ G = 1,\ T = 10s,\ \tau = 3$
Performances → $T_S = 5s$, $\omega_0 = 0.15$ rad/s, $\zeta = 0.8$
***** CONTROL LAW *****
$S(q^{-1}) \cdot u(t) + R(q^{-1}) \cdot y(t) = T(q^{-1}) \cdot r(t)$
Controller: • $R(q^{-1}) = 1.6874 - 0.8924\, q^{-1}$ • $S(q^{-1}) = (1 - q^{-1}) \cdot (1 + 0.3122\, q^{-1})$ • $T(q^{-1}) = R(q^{-1})$
Gain margin: 3.681   Phase margin: 58.4 deg
Modulus margin: 0.664 (- 3.56 dB)   Delay margin: 9.4 s
Cont. time PID: (no equivalent)

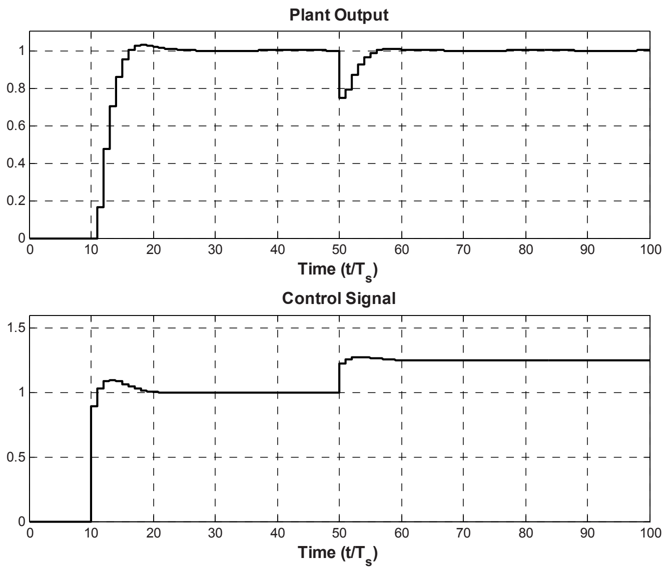

Figure 3.5. Performances of the digital PID 1 controller, $\omega_0 = 0.1$ rad/s for tracking and regulation

In order to eliminate the undesirable effect of the zeros introduced by the digital PID 1 controller, another structure must therefore be chosen for $T(q^{-1})$, which does not introduce additional zeros in the closed loop transfer function. That leads to the digital PID 2 controller.

3.2.4 Digital PID 2 Controller

This is a digital PID that does not introduce additional zeros.

The desired closed loop transfer function (between the reference and the output) will be of the form

$$H_{CL}(q^{-1}) = \frac{P(1)}{B(1)} \cdot \frac{B(q^{-1})}{P(q^{-1})} \tag{3.2.31}$$

in which $B(q^{-1})$ contains the plant zeros that will remain unchanged. $P(q^{-1})$ defines the desired closed loop poles and the term $P(1)/B(1)$ is introduced in order to ensure a unit gain between the reference and the output in steady state.

The controller will have the general structure

$$S(q^{-1})\, u(t) + R(q^{-1})\, y(t) = T(q^{-1})\, r(t) \tag{3.2.32}$$

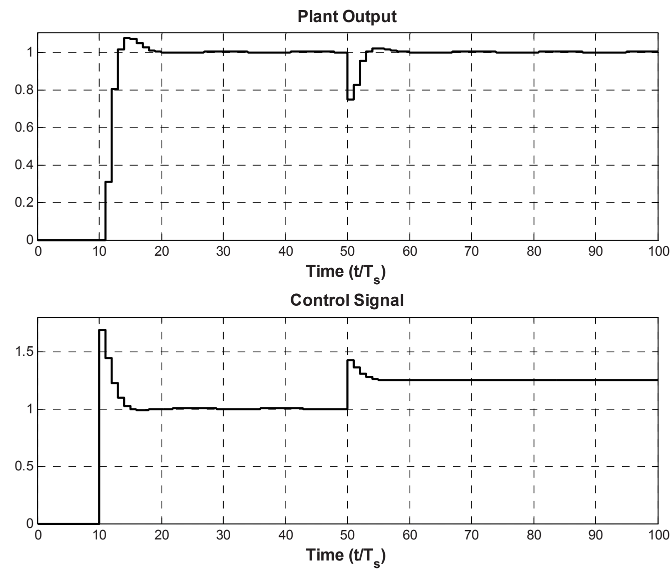

Figure 3.6. Performances of the digital PID 1 controller, $\omega_0 = 0.15$ rad/s for tracking and regulation

in which $S(q^{-1})$ and $R(q^{-1})$ are given by Equations 3.2.6 and 3.2.7 respectively.

The closed loop transfer function using the controller specified by Equation 3.2.32 will be

$$H_{CL}(q^{-1}) = \frac{T(q^{-1})B(q^{-1})}{A(q^{-1})S(q^{-1}) + B(q^{-1})R(q^{-1})} = \frac{[P(1)/B(1)]B(q^{-1})}{P(q^{-1})} \tag{3.2.33}$$

In the same way as the PID1, the coefficients of the polynomials $S(q^{-1})$ and $R(q^{-1})$ will be obtained by solving Equation 3.2.16 . It thus results from Equation 3.2.33 that

$$T(q^{-1}) = \frac{P(1)}{B(1)} = \frac{B(1)R(1)}{B(1)} = R(1) \tag{3.2.34}$$

since $S(1) = 0$ (which implies $P(1) = B(1)\,R(1)$). Then $T(q^{-1})$ will be a gain equal to the sum of the coefficients of $R(q^{-1})$.

To conclude: the digital PID 2 controller has the same polynomials $S(q^{-1})$ and $R(q^{-1})$ as the digital PID 1 controller; the only difference is that now $T(q^{-1}) = R(1)$ instead of $R(q^{-1})$, thereby preserving the unitary gain of the closed loop system in steady state without however introducing the effect of the zeros of $R(q^{-1})$.

The continuous-time PID (structure 2) that leads by discretization to the digital PID 2 controller is represented in Figure 3.7 (proportional and derivative actions only on the measure).

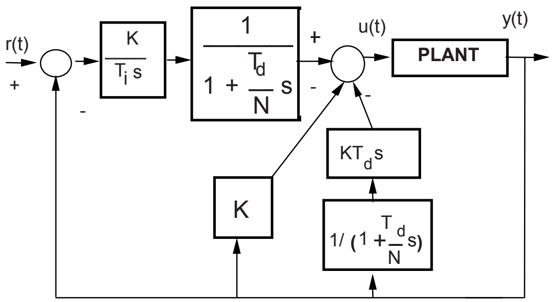

Figure 3.7. Continuous-time PID controller corresponding to digital PID 2 controller

The parameters of the continuous-time PID 2 controller, which leads by discretization with a sampling period equal to T_S to the digital PID 2 controller with polynomials $R(q^{-1})$ and $S(q^{-1})$, result from the following relations:

$$K = \frac{-(r_1 + 2r_2)}{1 + s_1'} \tag{3.2.35}$$

$$T_i = T_s \cdot \frac{-(r_1 + 2r_2)}{r_0 + r_1 + r_2} \tag{3.2.36}$$

$$T_d = T_s \cdot \frac{s_1' r_1 + (s_1' - 1) r_2}{(r_1 + 2r_2)(1 + s_1')} \tag{3.2.37}$$

$$\frac{T_d}{N} = \frac{-s_1' T_s}{1 + s_1'} \tag{3.2.38}$$

Just like the PID 1 digital controller, for the PID 2 digital controller to be equivalent to a continuous-time PID controller, the condition of Equation 3.2.30 on the coefficient s'_1 must be satisfied (that is $-1 < s'_1 \leq 0$).

Table 3.4 gives the computation results of the digital PID 2 controller for the same plant which was considered in the case of the digital PID 1 controller, and with a desired closed loop natural frequency $\omega_0 = 0.15\ rad/s$ (the results are to be compared with those given in Table 3.3; the values of the coefficients of $R(q^{-1})$ and $S(q^{-1})$ are the same).

Table 3.4. Digital PID controller (structure 2) $\omega_0 = 0.15$ rad/s

Plant:

- $B(q^{-1}) = 0.1813\ q^{-1} + 0.2122\ q^{-2}$
- $A(q^{-1}) = 1 - 0.6065\ q^{-1}$
- $T_S = 5s,\ G = 1,\ T = 10s,\ \tau = 3$

Performances → $T_S = 5s$, $\omega_0 = 0.15\ rad/s$, $\zeta = 0.8$

***** CONTROL LAW *****

$$S(q^{-1})\,u(t) + R(q^{-1})\,y(t) = T(q^{-1})\,r(t)$$

Controller:

- $R(q^{-1}) = 1.6874 - 0.8924\ q^{-1}$
- $S(q^{-1}) = (1 - q^{-1})\ (1 + 0.3122\ q^{-1})$
- $T(q^{-1}) = 0.795$

Gain margin: 3.681	Phase margin: 58.4 deg
Modulus margin: 0.664 (- 3.56 dB)	Delay margin: 9.4 s
Cont. time PID: (no equivalent)	

The performances obtained are illustrated in Figure 3.8, which must be compared with the curves of Figure 3.6. It can be observed that for the same values of the polynomials $R(q^{-1})$ and $S(q^{-1})$, the overshoot during the transient disappears when the digital PID 2 controller is used (for steps on the reference). Moreover, the response to a disturbance is the same for PID1 and PID2.

Note also that the values obtained for the required robustness margins (gain, phase, modulus, delay) are without doubt satisfactory.

3.2.5 Effect of Auxiliary Poles

Figures 3.9 and 3.10 show the frequency characteristics of $\left|S_{yp}\right|$ and $\left|S_{up}\right|$ for three PID controllers designed for the same plant model used in the previous sections with the following performances specifications:

- $\omega_0 = 0.25 \quad rad/s \quad ; \quad \zeta = 0.8$;
- $\omega_0 = 0.20 \quad rad/s \quad ; \quad \zeta = 0.8$;
- $\omega_0 = 0.25 \quad rad/s \quad ; \quad \zeta = 0.8$ and two auxiliary poles $-\alpha_1 = -\alpha_2 = 0.15$.

In Figure 3.9 the frequency region (low frequencies) where $\left|S_{yp}\right| < 0\,dB$ corresponds to the attenuation band for the disturbances. The frequency regions where $\left|S_{yp}\right| > 0\,dB$ correspond to an amplification of the disturbances. At the frequencies where $\left|S_{yp}\right| = 0dB$ the system behaves "in open loop" since the disturbances, at these frequencies, are neither amplified nor attenuated.

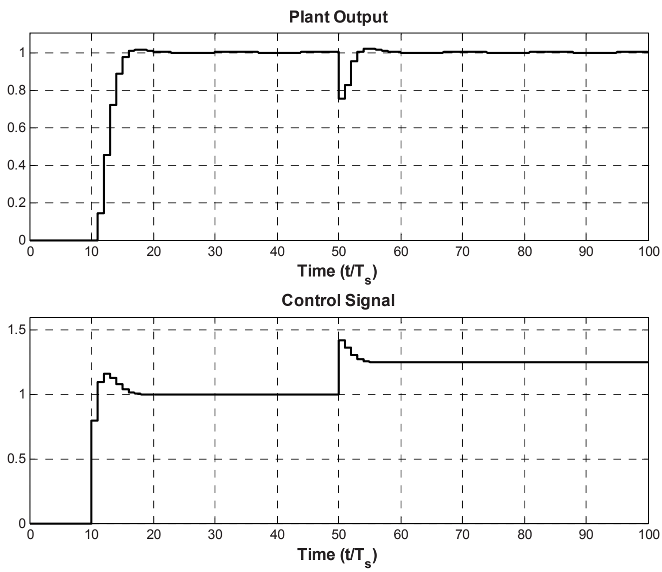

Figure 3.8. Performances of the digital PID 2 controller, $\omega_0 = 0.15$ rad/s for tracking and regulation

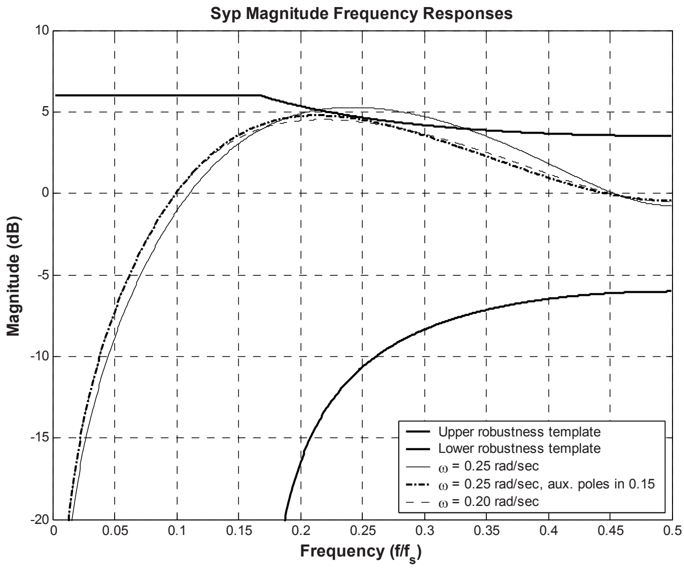

Figure 3.9. Frequency characteristics of the modulus of the output sensitivity function S_{yp} for different PID controllers

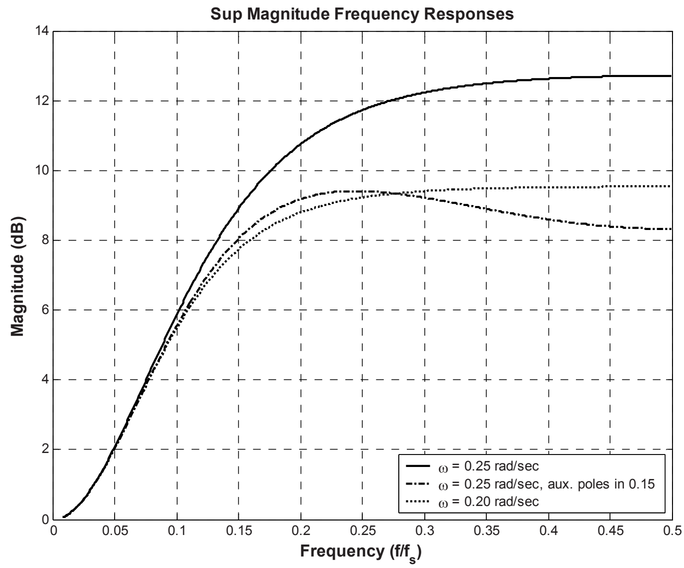

Figure 3.10. Frequency characteristics of the modulus of the input sensitivity function S_{up} for different PID controllers

The performance of a controller is enhanced by the augmentation of the attenuation band. But the augmentation of the attenuation band produces an increase of the amplification of the disturbances outside the attenuation band and an increase of the maximum of $\left|S_{yp}\right|$ which will reduce the modulus margin and the delay margin (it will be shown later in Section 3.6. that the surface of the attenuation band should be equal to the surface of the amplification band).

One observes that for a specified $\omega_0 = 0.25$ rad/s the attenuation band is larger than for $\omega_0 = 0.2$ rad/s but the $\left|S_{yp}\right|$ is beyond the acceptable value for the robustness (it crosses the template for $\Delta\tau = T_s$).

One also observes that the introduction of auxiliary poles is more or less equivalent to the reduction of the desired performances (the curve for $\omega_0 = 0.2$ rad/s is very close to the curve for $\omega_0 = 0.25$ rad/s and $\alpha_1 = \alpha_2 = -0.15$).

However, if one examines the frequency characteristics of the modulus of the input sensitivity function $\left|S_{up}\right|$, one observes that the introduction of the auxiliary poles for the case $\omega_0 = 0.25$ rad/s has a better effect than the reduction of the desired performance without introduction of auxiliary poles. The auxiliary poles allow reducing the value of $\left|S_{up}\right|$ in the high frequency region without affecting the regulation performances (see the frequency characteristics of $\left|S_{yp}\right|$). This means that a better robustness with respect to model uncertainties in high frequencies will be obtained together with a reduction of the actuator stress in high frequencies.

In order to assure a good robustness at high frequencies and to reduce the actuator stress, it will be desirable to reduce $\left|S_{up}\right|$ further in this frequency region, which implies a reduction of the controller gain at these frequencies. The introduction of auxiliary poles improves the situation but does not allow overcoming a fundamental limitation of PID controllers, namely the impossibility to obtain a very low gain at high frequencies. In order to achieve this it is necessary either to increase the order of $S(q^{-1})$ (which will allow the introduction of a second-order filter instead of a first-order filter), or "to open the loop", which implies the increase of the order of $R(q^{-1})$ in order to introduce zeros imposing a null gain of the controller at the frequency $0.5\, f_s$. The use of the pole placement, which do not impose any restrictions on the size of $R(q^{-1})$ and $S(q^{-1})$, will allow such a design.

3.2.6 Digital PID Controller: Conclusions

The discretization of the classical PID results in a digital controller with a canonical three-branched structure (RST) with $T(q^{-1}) = R(q^{-1})$ (PID1).

Based on the coefficient of $R(q^{-1})$ and $S(q^{-1})$, the coefficients of a continuous-time PID may be computed if the polynomial

$$S(q^{-1}) = 1 + s'_1\, q^{-1}$$

has $s'_1 \in]-1,0]$.

The model based digital PID controller considered can deal with first-or second-order systems with time delay, if this latter is less than T_s (sampling period).

For time delays $\tau \geq 0.25\ T$ (time constant of the system of the plant), the continuous-time PID leads to responses that are slower in closed loop than the ones in open loop!

For systems with time delay, the closed loop performances may be significantly improved by choosing coefficients of *R, S, T* that do not result in an equivalent continuous-time PID controller.

The overshoots that may appear in closed loop can be eliminated by replacing

$$T(q^{-1}) = R(q^{-1}) \text{ [PID 1]} \quad \text{with} \quad T(q^{-1}) = R(1) \text{ [PID 2]}$$

Digital PID controller design can be carried on the basis of the discrete-time plant model and of the desired closed loop performances.

Every PID design should be concluded by an analysis of the robustness margins and of the frequency response of the input sensitivity function at high frequencies.

3.3 Pole Placement

The computation of the digital PID controller parameters is a special case of the “pole placement” strategy.

The *pole placement* strategy allows the design of a RST digital controller both for stable and unstable systems:

- Without restriction on the degrees of the polynomials $A(q^{-1})$ and $B(q^{-1})$ of the discrete-time plant model (provided that they do not have common factors)
- Without restriction on the time delay
- Without restriction on the plant zeros (stable or unstable)

This method does not simplify the system zeros (this is why they can be unstable). The only restriction concerns the possible common factors of $A(q^{-1})$ and $B(q^{-1})$, which must be simplified before the computations are carried out.

3.3.1 Structure

The structure of the closed loop system is given in Figure 3.11.

The plant to be controlled is characterized by the pulse transfer function (irreducible)

$$H(q^{-1}) = \frac{q^{-d}B(q^{-1})}{A(q^{-1})} \tag{3.3.1}$$

in which d is the integer number of sampling periods contained in the time delay and

$$A(q^{-1}) = 1 + a_1 q^{-1} + \ldots + a_{n_A} q^{-n_A} \tag{3.3.2}$$

$$B(q^{-1}) = b_1 q^{-1} + b_2 q^{-2} + \ldots + b_{n_B} q^{-n_B} = q^{-1} B^*(q^{-1}) \tag{3.3.3}$$

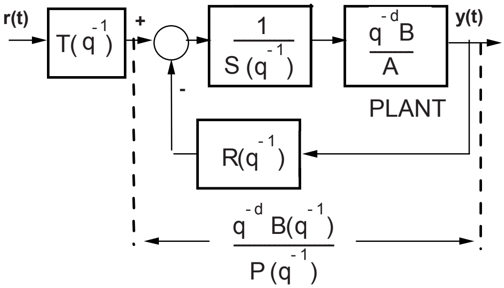

Figure 3.11. Pole placement with RST controller

The closed loop transfer function is given by

$$H_{CL}(q^{-1}) = \frac{q^{-d} T(q^{-1}) B(q^{-1})}{A(q^{-1}) S(q^{-1}) + q^{-d} B(q^{-1}) R(q^{-1})} = \frac{q^{-d} T(q^{-1}) B(q^{-1})}{P(q^{-1})} \tag{3.3.4}$$

in which

$$P(q^{-1}) = A(q^{-1})\, S(q^{-1}) + q^{-d}\, B(q^{-1})\, R(q^{-1}) = 1 + p_1 q^{-1} + p_2 q^{-2} + \ldots \tag{3.3.5}$$

defines the closed loop poles that play an essential role for the regulation behavior.

The behavior with respect to a disturbance is given by the output sensitivity function

$$S_{yp}(q^{-1}) = \frac{A(q^{-1}) S(q^{-1})}{A(q^{-1}) S(q^{-1}) + q^{-d} B(q^{-1}) R(q^{-1})} = \frac{A(q^{-1}) S(q^{-1})}{P(q^{-1})} \tag{3.3.6}$$

It can thus be seen that $P(q^{-1})$ corresponds to the denominator of the output sensitivity function and, thereby, it defines for a large extent the regulation behavior.

3.3.2 Choice of the Closed Loop Poles ($P(q^{-1})$)

We have seen for the case of a digital PID controller that one can specify a polynomial $P(q^{-1})$ defining the closed loop poles on the basis of a second-order continuous-time system with the desired natural frequency and damping (see Section 3.2.2). One can also directly specify the polynomial $P(q^{-1})$ from the desired performances. To illustrate the last statement, consider the following example.

Let

$$P(q^{-1}) = 1 + p_1 q^{-1} \quad \text{with} \quad p_1 = -0.5$$

When there is no reference, the free output response is defined by

$$y(t+1) = -p_1 y(t) = 0.5 y(t)$$

One thus obtains a relative decrease of 50% for the output amplitude at each sampling instant (see Figure 3.12).

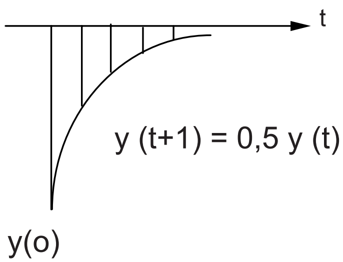

Figure 3.12. Responses for $P(q^{-1}) = 1-0.5\, q^{-1}$

Choosing p_1 between -0.2 and -0.8, it is clear that the disturbance rejection speed can be controlled.

Nevertheless, generally speaking, $P(q^{-1})$ is chosen in the form of a second-order polynomial by discretization of a second-order continuous time system, specifying ω_0, ζ and assuring that the condition

$$0.25 \leq \omega_0 T_s \leq 1.5 \quad ; \qquad 0.7 \leq \zeta \leq 1$$

is satisfied (see Section 3.2.2)[5].

The polynomial chosen from the desired closed loop performances will define the dominant closed loop poles and it will be named $P_D(q^{-1})$.

---

[5] The functions *fd2pol.sci*, *omega_dmp.sci* (Scilab) and *fd2pol.m*, *omega_dmp.m* (MATLAB®) can be used.

If it is desired to introduce a filtering action in certain frequency regions (or to reduce the effect of the noise on the measure, or to smooth the variations of the control signal, or to improve the robustness), the poles of the corresponding filter, defined by a polynomial $P_F(q^{-1})$, should also be the poles of the closed loop. As a consequence, the polynomial $P(q^{-1})$ defining the desired closed loop poles will be the product of the polynomials $P_D(q^{-1})$ and $P_F(q^{-1})$ specifying dominant and auxiliary closed loop poles, respectively.

$$P(q^{-1}) = P_D(q^{-1}) \,.\, P_F(q^{-1}) \tag{3.3.7}$$

As a general rule, the poles named "auxiliary poles" are faster than the "dominant poles". That is expressed, for the case of discrete-time models, by the property that the roots of $P_F(q^{-1})$ should have a real part smaller than the real part of the roots of $P_D(q^{-1})$.

3.3.3 Regulation (Computation of $R(q^{-1})$ and $S(q^{-1})$)

Once $P(q^{-1})$ is specified, in order to compute $R(q^{-1})$ and $S(q^{-1})$ according to Equation 3.3.4 , the following equation, known as "Bezout identity" (equation), must be solved:

$$A(q^{-1})\, S(q^{-1}) + q^{-d}\, B(q^{-1})\, R(q^{-1}) = P(q^{-1}) \tag{3.3.8}$$

Defining

$$n_A = deg\, A(q^{-1}) \quad ; \quad n_B = deg\, B(q^{-1}) \tag{3.3.9}$$

this polynomial equation has a unique solution with minimal degree (when ($A(q^{-1})$ and $B(q^{-1})$ do not have common factors) for

$$\begin{aligned} &n_P = deg\, P(q^{-1}) \le n_A + n_B + d - 1\, ; \\ &n_S = deg\, S(q^{-1}) = n_B + d - 1 \quad ; \; n_R = deg\, R(q^{-1}) = n_A - 1 \end{aligned} \tag{3.3.10}$$

in which

$$S(q^{-1}) = 1 + s_1 q^{-1} + \ldots + s_{n_S} q^{-n_S} = 1 + q^{-1} S^*(q^{-1}) \tag{3.3.11}$$

$$R(q^{-1}) = r_0 + r_1 q^{-1} + \ldots + r_{n_R} q^{-n_R} \tag{3.3.12}$$

In order to solve effectively Equation 3.3.8 , this latter is often put in the matrix form

$$M x = p \tag{3.3.13}$$

in which

$$x^T = [1, s_1, \ldots, s_{n_S}, r_0, \ldots, r_{n_R}] \tag{3.3.14}$$

$$p^T = [1, p_1, \ldots, p_i, \ldots, p_{n_P}, 0, \ldots, 0] \tag{3.3.15}$$

and the matrix M has the following form

$$\begin{array}{c}
\begin{array}{cc} \overbrace{\hphantom{1 \quad 0 \quad \cdots \quad 0 \quad}}^{n_B + d} & \overbrace{\hphantom{0 \quad \cdots \quad \cdots \quad 0 \quad}}^{n_A} \end{array} \\
\underbrace{\left[\begin{array}{cccccccc}
1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
a_1 & 1 & & . & b'_1 & & & \\
a_2 & & & 0 & b'_2 & & & b'_1 \\
 & & & 1 & . & & & b'_2 \\
 & & & a_1 & . & & & . \\
a_{n_A} & & & a_2 & b'_{n_B} & & & . \\
0 & & & . & 0 & . & . & . \\
0 & \cdots & 0 & a_{n_A} & 0 & 0 & 0 & b'_{n_B}
\end{array}\right]}_{n_A + n_B + d}
\end{array}
\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\} n_A + n_B + d$$

where:

$$b'_i = 0 \quad for \quad i = 0, 1 \ldots d \qquad ; \qquad b'_i = b_{i-d} \quad for \quad i \geq d+1$$

The vector x, which contains the coefficients of the polynomials $R(q^{-1})$ and $S(q^{-1})$, is obtained, after the inversion of the matrix M, by the formula

$$x = M^{-1} p \tag{3.3.16}$$

where M^{-1} is the matrix inverse of M. This inverse exists if the determinant of the matrix M is different from zero. One can prove that this is verified if, and only if, $A(q^{-1})$ and $B(q^{-1})$ are coprime polynomials (no simplifications between zeros and poles).

Several methods are used to solve Equation 3.3.8. These methods give better numerical performances with respect to a simple matrix inversion.

For different reasons the polynomials $R(q^{-1})$ and $S(q^{-1})$ may contain, in general, fixed parts specified before the resolution of Equation 3.3.8 . For example, if zero steady state error is required for a step on the reference, or for a step disturbance, the presence of an integrator in the open loop transfer function is required. This corresponds to the introduction of a term $(1-q^{-1})$ in the polynomial $S(q^{-1})$ (see Section 2.4).

In order to take into account these pre-specified fixed parts, the polynomials $R(q^{-1})$ and $S(q^{-1})$ are factorized in the form

$$R(q^{-1}) = R'(q^{-1})\, H_R\,(q^{-1}) \qquad (3.3.17)$$
$$S(q^{-1}) = S'\,(q^{-1})\, H_S\,(q^{-1}) \qquad (3.3.18)$$

where $H_R\,(q^{-1})$ and $H_S\,(q^{-1})$ are pre-specified polynomials and

$$R'(q^{-1}) = r_0' + r_1' q^{-1} + ... + r_{n_{R'}}' q^{-n_{R'}} \qquad (3.3.19)$$

$$S'(q^{-1}) = 1 + s_1' q^{-1} + ... + s_{n_{S'}}' q^{-n_{S'}} \qquad (3.3.20)$$

For this parameterization of polynomials $R(q^{-1})$ and $S(q^{-1})$, the closed loop transfer function will be

$$H_{CL}(q^{-1}) = \frac{q^{-d}T(q^{-1})B(q^{-1})}{A(q^{-1})S'(q^{-1})H_S(q^{-1}) + q^{-d}B(q^{-1})R'(q^{-1})H_R(q^{-1})}$$
$$= \frac{q^{-d}T(q^{-1})B(q^{-1})}{P(q^{-1})} \qquad (3.3.21)$$

and, instead of Equation 3.3.8 , one needs to solve the equation

$$A(q^{-1})\, H_S(q^{-1})\, S'(q^{-1}) + q^{-d}\, B(q^{-1})\, H_R(q^{-1})\, R'(q^{-1}) = P(q^{-1}) \qquad (3.3.22)$$

In order to solve Equation 3.3.22, one needs to solve Equation 3.3.8 after replacing: $A(q^{-1})$ by $A'(q^{-1}) = A(q^{-1})H_S(q^{-1})$ and $B(q^{-1})$ and $B'(q^{-1}) = q^{-d}B(q^{-1})$ $H_R(q^{-1})$ with the restriction that polynomials $[A(q^{-1})\, H_S(q^{-1})]$ and $[B(q^{-1})\, H_R(q^{-1})]$ are coprime.

The condition of Equation 3.3.10 on the orders of the polynomials that allow one to get a unique solution of minimal order, become in this case

$$n_P = deg\, P(q^{-1}) \leq n_A + n_{H_S} + n_B + n_{H_R} + d - 1\,;$$
$$n_{S'} = deg\, S'(q^{-1}) = n_B + n_{H_R} + d - 1\,;\; n_{R'} = deg\, R'(q^{-1}) = n_A + n_{H_S} - 1 \qquad (3.3.23)$$

For the controller implementation, $S(q^{-1})$ will be given by $S'(q^{-1})H_S(q^{-1})$ and $R(q^{-1})$ by $R'(q^{-1})\, H_R(q^{-1})$.

Equation 3.3.22 or 3.3.8 can be solved by means of the functions *bezoutd.sci* (Scilab) and *bezoutd.m* (MATLAB®) available on the book website.

Use of Fixed Parts of the Controller (H_R and H_S): Examples

Steady-State Error

As illustrated in the previous sections, $S(q^{-1})$ must contain a term $(1-q^{-1})$ in order to have a zero steady state error for a step input or disturbance ($S_{yp}(q^{-1})$ must be zero in steady state, *i.e.* for $q = 1$). Thus

$$S_{yp}(q^{-1}) = \frac{A(q^{-1})H_S(q^{-1})S'(q^{-1})}{P(q^{-1})} \tag{3.3.24}$$

and then one needs to choose:

$$H_S(q^{-1}) = 1 - q^{-1} \tag{3.3.25}$$

Rejection of a Sinusoidal Disturbance

If a perfect rejection of a sinusoidal disturbance is required at a specified frequency, it isessential that $S_{yp}(q^{-1})$ is zero at this frequency, which is equivalent to the requirement that $H_S(q^{-1})$ has a pair of *undamped* complex zeros at this frequency.

In this case:

$$H_S(q^{-1}) = 1 + \alpha q^{-1} + q^{-2} \tag{3.3.26}$$

with $\alpha = -2 \cos \omega T_s = -2 \cos (2 \pi f/f_s)$.

If one requires only a given attenuation, $H_S(q^{-1})$ must introduce a pair of *damped* complex zeros with a damping factor depending on the desired attenuation.

Signal Blocking

In some applications the measured signal contains signal components at particular frequencies for which the controller should not react (these signals may serve for process technological operation). This implies that at these frequencies one should have $S_{yp}(q^{-1})=1$. It results from Equation 3.3.24 and 3.3.6 that at these frequencies it is essential that $H_R(q^{-1})$ be null. As a consequence, the input sensitivity function $S_{up}(q^{-1})$ must be null (no effect of the disturbance upon the control signal).

The expression of $S_{up}(q^{-1})$ is given by

$$S_{up}(q^{-1}) = -\frac{A(q^{-1})H_R(q^{-1})R'(q^{-1})}{P(q^{-1})} \tag{3.3.27}$$

and one must choose the fixed part $R(q^{-1})$ named $H_R(q^{-1})$ such that it has a null gain at this frequency.

The fixed part of $R(q^{-1})$ will be of the form

$$H_R(q^{-1}) = 1 + \beta q^{-1} + q^{-2} \tag{3.3.28}$$

where $\beta = -2 \cos \omega_0 T_s = -2 \cos (2 \pi f / f_s)$.

This introduces a pair of undamped complex zeros at the frequency f or, more generally, in the form of a second order polynomial corresponding to a damped complex zeros pair, if a desired attenuation is accepted.

In many applications it is required that the controller does not react to signals close to $0.5 f_s$ (where the gain of the cascade actuator-plant is generally low). In this case one imposes:

$$H_R(q^{-1}) = (1 + \beta q^{-1})^n \qquad n=1,2 \tag{3.3.29}$$

with $0 < \beta \leq 1$.

Note that $(1 + \beta q^{-1})^2$ corresponds to a damped second-order system with a resonant frequency equal to πf_s (see Section 2.3.2):

$$\omega_0 \sqrt{1-\zeta^2} = \pi f_s$$

and the corresponding damping factor depends upon β through the relation

$$\beta = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}$$

For $\beta = 1$ it follows that $\zeta = 0$ and the closed loop system works in open loop at $0.5 f_s$ (even for $n=1$).

Robustness

In order to guarantee both robustness margins and specified closed loop performances, fixed parts $H_R(q^{-1})$ and $H_S(q^{-1})$ (as well as auxiliary poles) should be introduced for shaping the frequency characteristics of the sensitivity functions $S_{yp}(q^{-1})$ and $S_{up}(q^{-1})$ in specified frequency regions. This will be discussed in Section 3.6.

3.3.4 Tracking (Computation of T(q-1))

Ideally, when the reference changes, it is desired that the system output $y(t)$ follows a desired trajectory $y^*(t)$. This trajectory may be stored or generated each time the reference is changed by using a reference model (as indicated in Figure 3.13).

The transfer function of the reference model is

$$H_m(q^{-1}) = \frac{q^{-1}B_m(q^{-1})}{A_m(q^{-1})} \tag{3.3.30}$$

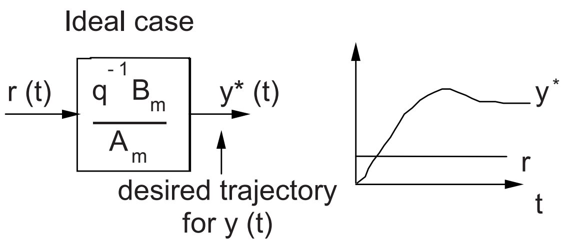

Figure 3.13. Desired trajectory y*(t) generation

It is generally determined from desired performances (rising time, overshoot, settling time). For example, a second-order normalized continuous-time model (parameters ω_0, ζ) can be defined by means of the curves given in Figure 1.10 starting from the desired performances. Once the continuous-time transfer function and the sampling period T_s, are known, the pulse transfer function of the reference model is obtained by discretization.

The pulse transfer function of the reference model will be of the form

$$H_m(q^{-1}) = \frac{q^{-1}B_m(q^{-1})}{A_m(q^{-1})} = \frac{q^{-1}(b_{m0} + b_{m1}q^{-1})}{1 + a_{m1} + a_{m2}q^{-2}} \tag{3.3.31}$$

This is the transfer function that the controller must achieve between the reference r and the output y, eventually multiplied by q^{-d} in the case of a time delay d in the plant model (the delay cannot be compensated). In the case of the *pole placement* this cannot be obtained because the plant zeros are maintained (polynomial $B(q^{-1})$).

The objective is then to approach the delayed model reference trajectory

$$y^*(t) = \frac{q^{-(d+1)}B_m(q^{-1})}{A_m(q^{-1})} r(t) \tag{3.3.32}$$

For this, firstly $y^*(t+d+1)$ is generated from $r(t)$:

$$y^*(t+d+1) = \frac{B_m(q^{-1})}{A_m(q^{-1})} r(t) \tag{3.3.33}$$

and one chooses $T(q^{-1})$ to impose:

- Unit static gain between y^* and y
- Compensation of the regulation dynamics defined by $P(q^{-1})$ (because the regulation dynamics is in general different from the tracking dynamics $A_m(q^{-1})$)

This leads to the choice

$$T(q^{-1}) = G\,P(q^{-1}) \tag{3.3.34}$$

where

$$G = \begin{cases} 1/B(1) & if \quad B(1) \neq 0 \\ 1 & if \quad B(1) = 0 \end{cases} \tag{3.3.35}$$

The control law equation becomes

$$S(q^{-1})\,u(t) + R(q^{-1})\,y(t) = T(q^{-1})\,y^*(t+d+1) \tag{3.3.36}$$

The full diagram for the pole placement is given in Figure 3.14.

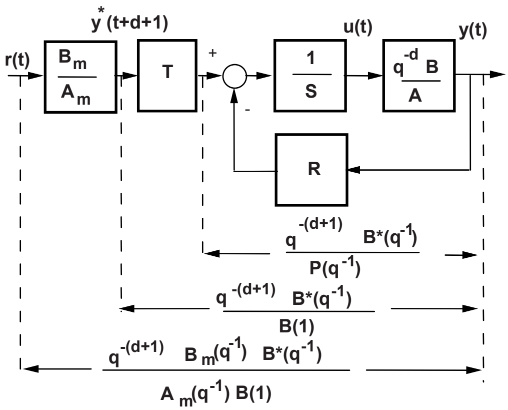

Figure 3.14. Pole placement - tracking and regulation

The transfer function (operator) between the reference and the output will be

$$H_{CL}(q^{-1}) = \frac{q^{-d+1} B_m(q^{-1})}{A_m(q^{-1})} \cdot \frac{B^*(q^{-1})}{B(1)} \tag{3.3.37}$$

In some cases one can consider a simplification of the polynomial T by taking into account just the dominant poles (as auxiliary poles are often at high frequencies with a small influence on the time response). In this case

$$H_{CL}(q^{-1}) = \frac{q^{-d+1}B_m(q^{-1})}{A_m(q^{-1})} \cdot \frac{B^*(q^{-1})}{B(1)} \cdot \frac{P_F(1)}{P_F(q^{-1})} \tag{3.3.38}$$

and

$$T(q^{-1}) = GP_D(q^{-1}) \tag{3.3.39}$$

where

$$G = \begin{cases} \dfrac{P_F(1)}{B(1)} & if \quad B(1) \neq 0 \\ 1 & if \quad B(1) = 0 \end{cases}$$

If the regulation dynamics is the same as the tracking dynamics, there is no need for a reference model of the form of Equation 3.3.33 and the polynomial $T(q^{-1})$ is replaced by a gain[6]:

$$T(q^{-1}) = G = \begin{cases} \dfrac{P\,(1)}{B(1)} & if \quad B(1) \neq 0 \\ 1 & if \quad B(1) = 0 \end{cases}$$

which guarantees a unit static gain between the reference trajectory and the output (if $B(1)$ is not null).

The controller equations for the pole placement under different forms are summed up in Table 3.5 (the recursive equations needed for the implementation are boxed).

[6] If $S(q^{-1})$ contains an integrator ($S(1)=0$) it follows from Equation 3.3.21 that $B(1)R(1) = P(1)$ and $P(1)/B(1) = R(1)$, respectively.

3.3.5 Pole Placement: Examples

Table 3.6 gives the results for a pole placement design. The considered example is the control of a plant model characterized by a second-order discrete-time model with two real poles (at 0.6 and 0.7) and an unstable zero.

The tracking dynamics (polynomials $A_m(q^{-1})$ and $B_m(q^{-1})$) has been obtained by discretization of a second-order normalized continuous-time model, with $\omega_0 = 0.5$ *rad/s* and $\zeta = 0.9$, ($T_s = 1s$). The regulation dynamics (polynomials $P(q^{-1})$) has been obtained by discretization of a second-order normalized continuous-time model with $\omega_0 = 0.4$ *rad/s* and $\zeta = 0.9$. The controller includes an integrator. Satisfactory robustness margins are obtained. The performances are shown in Figure 3.15.

Table 3.5. Pole placement - control law equations

$$u(t) = \frac{T(q^{-1})y^*(t+d+1) - R(q^{-1})y(t)}{S(q^{-1})}$$

$$S(q^{-1})u(t) + R(q^{-1})y(t) = GP(q^{-1})y^*(t+d+1) = T(q^{-1})y^*(t+d+1)$$

$$S(q^{-1}) = 1 + q^{-1}S^*(q^{-1})$$

$$\boxed{u(t) = P(q^{-1})Gy^*(t+d+1) - S^*(q^{-1})u(t-1) - R(q^{-1})y(t)}$$

$$y^*(t+d+1) = \frac{B_m(q^{-1})}{A_m(q^{-1})}r(t)$$

$$A_m(q^{-1}) = 1 + q^{-1}A_m^*(q^{-1})$$

$$\boxed{y^*(t+d+1) = -A_m^*(q^{-1})y(t+d) + B_m(q^{-1})r(t)}$$

$$B_m(q^{-1}) = b_{m0} + b_{m1}q^{-1} + \ldots$$

$$A_m(q^{-1}) = 1 + a_{m1}q^{-1} + a_{m2}q^{-2} + \ldots$$

Exercise:
What are the specifications for the “pole placement” technique that will lead to a PID 2 controller ?

Table 3.6. Pole placement results

Plant:

- $d=0$
- $B(q^{-1}) = 0.1\, q^{-1} + 0.2\, q^{-2}$
- $A(q^{-1}) = 1 - 1.3\, q^{-1} + 0.42\, q^{-2}$

Tracking dynamics:

- $B_m\,(q^{-1}) = 0.0927 + q^{-1} + 0.0687\, q^{-2}$
- $A_m\,(q^{-1}) = 1 - 1.2451q^{-1} + 0.4066\, q^{-2}$
- $T_S = 1s$, $\omega_0 = 0.5$ rad/s, $\zeta = 0.9$

Regulation dynamics → $P\,(q^{-1}) = 1 - 1.3741\, q^{-1} + 0.4867\, q^{-2}$
$T_S = 1s$, $\omega_0 = 0.4$ rad/s, $\zeta = 0.9$

Pre-specifications: Integrator

***** CONTROL LAW *****

$$S\,(q^{-1})\,u(t) + R\,(q^{-1})\,y(t) = T\,(q^{-1})\,y^*(t+d+1)$$

$$y^*(t+d+1) = [B_m(q^{-1})/A_m(q^{-1})]\; r(t)$$

Controller:

- $R(q^{-1}) = 3 - 3.94\, q^{-1} + 1.3141\, q^{-2}$
- $S(q^{-1}) = 1 - 0.3742\, q^{-1} - 0.6258\, q^{-2}$
- $T(q^{-1}) = 3.333 - 4.5806\, q^{-1} + 1.6225\, q^{-2}$

Gain margin: 2.703　　Phase margin: 65.4 deg
Modulus margin: 0.618 (- 4.19 dB)　　Delay margin: 2.1 s

3.4 Tracking and Regulation with Independent Objectives

This controller design method makes it possible to obtain the desired tracking behavior (changing of reference) independently of the desired regulation behavior (rejection of a disturbance). For example, the performance specifications illustrated in Figure 3.16 correspond to a situation for which the desired regulation response time (for a step disturbance) is significantly smaller than the desired tracking response time (for a step reference). Note that the reverse situation may be encountered. This method is a generalization of the so-called “model reference control”.

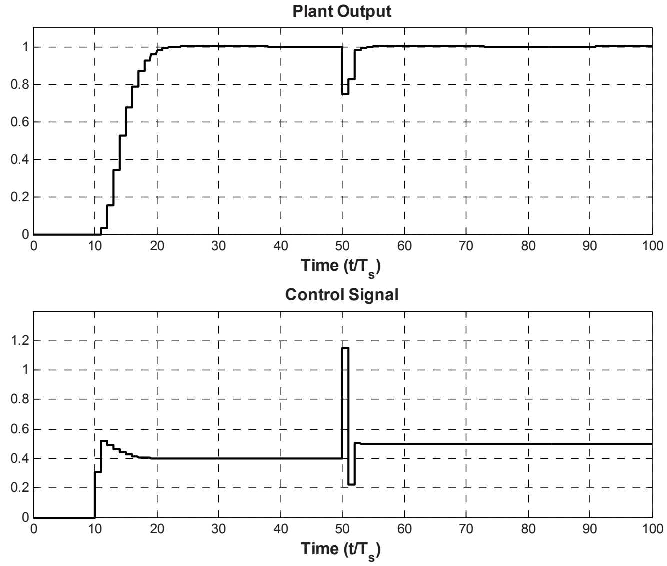

Figure 3.15. Pole placement performances

Unlike the "pole placement" method (Section 3.3), *this method leads to the simplification of the zeros of the discrete-time plant model*. This enables the tracking and regulation performances to be achieved without approximation.

This control strategy permits a RST digital controller to be designed for both stable and unstable systems:

- Without restriction on the degrees of the polynomials $A(q^{-1})$ and $B(q^{-1})$ characterizing the pulse transfer function of the plant
- Without restriction on the time delay d of the system

As a result of the simplification of the zeros, however, this strategy can only be applied to discrete time models with stable zeros.

This method cannot be applied to systems with fractional delay greater than $0.5T_S$. In the case of a fractional delay greater than $0.5T_S$, however, one can consider approximating (in the identification phase) the model with a fractional delay by a model with an augmented integer delay.

Note that unstable zeros can be the consequence of a too fast sampling of continuous time systems characterized by a difference greater than 2 between the numerator and denominator degree of the continuous time transfer function (even if the continuous time zeros are stable) (Åström and Wittenmark 1997).

This method can be considered as a particular case of the "pole placement" method. This equivalence can be obtained by imposing that the closed loop poles contain the zeros of the discrete time plant model (defined by the polynomial

$B^*(q^{-1})$). This is the reason for which the zeros of the plant model must be stable (see Section 3.4.2)[7].

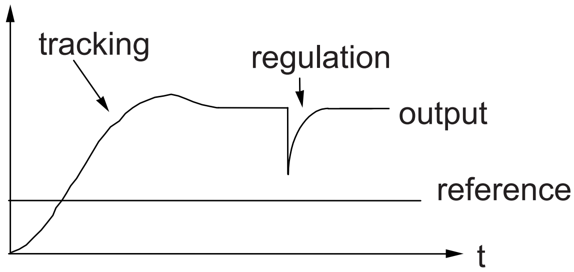

Figure 3.16. Tracking and regulation performances

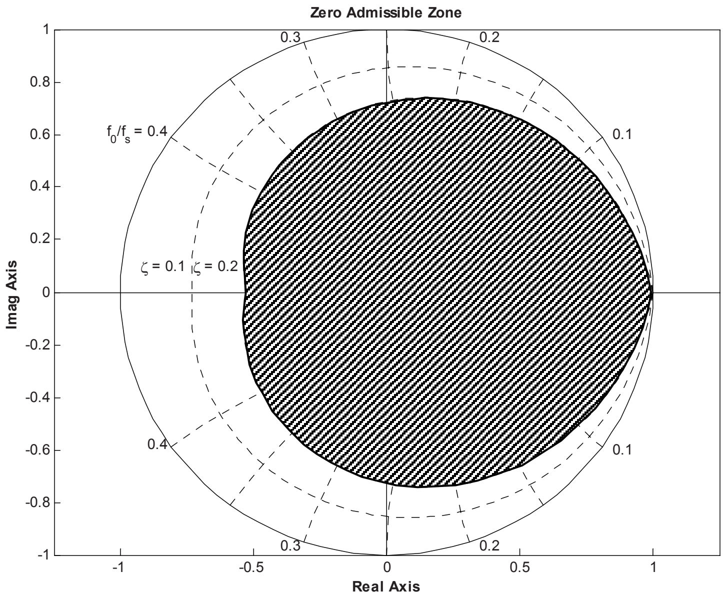

Figure 3.17. Admissible domain (hatching zone) for the zeros of the discrete-time plant model (tracking and regulation with independent objectives)

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[7] An approximation of the tracking and regulation with independent objectives, for the case of unstable zeros, can be obtained by pole placement, where the unstable zeros, which should be specified as closed loop poles, are approximated by stable zeros. This technique is presented in the stochastic context in Section 4.3.

It is convenient then to verify that, before the application of this method, the zeros of $B^*(q^{-1})$ are stable, and, moreover, that complex zeros have a sufficiently high damping factor ($\zeta > 0.2$). In other words, the zeros should lie inside a region defined by the cardioids related to the constant damping factor $\zeta = 0.2$ (see Figure 2.21). This admissibility domain is shown in Figure 3.17.

3.4.1 Structure

The structure of the closed loop system is presented in Figure 3.18.

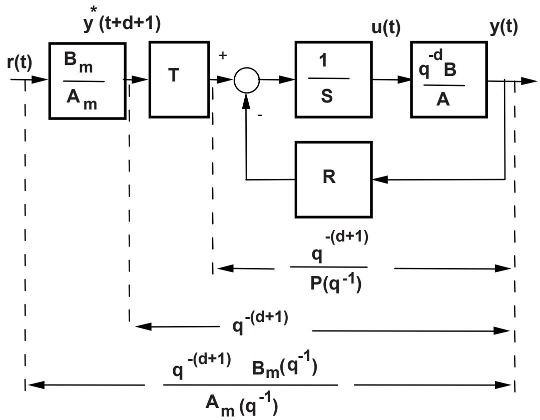

Figure 3.18. Tracking and regulation with independent objectives

The closed loop poles are defined by the polynomial $P(q^{-1})$ that almost completely specifies the desired behavior for the regulation.

As a general rule, $P(q^{-1})$ is the product of the two polynomials

$$P(q^{-1}) = P_D(q^{-1}) \,.\, P_F(q^{-1})$$

where $P_D(q^{-1})$ is determined as a function of the desired performances and $P_F(q^{-1})$ represents the auxiliary poles (for more details see Section 3.3.2).

The desired transfer function between the reference $r(t)$ and the plant output $y(t)$, which defines the tracking dynamics, is

$$q^{-(d+1)} B_m(q^{-1}) / A_m(q^{-1})$$

The output of the tracking model $B_m(q^{-1}) / A_m(q^{-1})$ specifies the desired trajectory y^* with $d+1$ steps ahead.

The plant to be controlled is characterized by the pulse transfer function given in Equation 3.3.1, where the polynomials $A(q^{-1})$ and $B(q^{-1})$ are specified by

Equations 3.3.2 and 3.3.3. Note that in this case polynomials $A(q^{-1})$ and $B(q^{-1})$ can have common factors.

The computation of $R(q^{-1})$, $S(q^{-1})$, $T(q^{-1})$ is done in two stages. First, by means of polynomials $R(q^{-1})$ and $S(q^{-1})$, the closed loop poles will be placed at the desired values specified by the polynomial $P(q^{-1})$, and the zeros of the discrete-time plant model will be simplified. Second, the pre-filter $T(q^{-1})$ is computed in order to obtain the tracking of the reference trajectory y^*, delayed by d steps. Note that for $P(q^{-1}) = 1$ this method corresponds to the "model reference control".

3.4.2 Regulation (Computation of R(q^{-1}) and S(q^{-1}))

Without the pre-filter $T(q^{-1})$, the closed loop transfer function is:

$$H_{CL}(q^{-1}) = \frac{q^{-d+1}B^*(q^{-1})}{A(q^{-1})S(q^{-1}) + q^{-d+1}B^*(q^{-1})R(q^{-1})} = \frac{q^{-d+1}}{P(q^{-1})}$$
$$= \frac{q^{-d+1}B^*(q^{-1})}{B^*(q^{-1})P(q^{-1})} \tag{3.4.1}$$

The closed loop poles should be those defined by $P(q^{-1})$ and the system zeros should be cancelled (in order to obtain a perfect tracking at a later stage).

From Equation 3.4.1, it results that the closed loop poles must contain in addition the zeros of the plant model.

Once $P(q^{-1})$ is specified, it results from Equation 3.4.1 that, in order to compute $R(q^{-1})$ and $S(q^{-1})$, the following equation must be solved:

$$A(q^{-1})\, S(q^{-1}) + q^{-(d+1)}\, B^*(q^{-1})\, R(q^{-1}) = B^*(q^{-1})\, P(q^{-1}) \tag{3.4.2}$$

Equation 3.4.2 corresponds to the pole placement with a particular choice of the desired closed loop poles (that contain n_B-1 additional poles corresponding to the plant zeros).

However, in order to solve Equation 3.4.2, one observes that $S(q^{-1})$ should have $B^*(q^{-1})$ as a common factor:

$$S(q^{-1}) = s_0 + s_1 q^{-1} + \ldots + s_{n_S} q^{-n_S} = B^*(q^{-1})S'(q^{-1}) \tag{3.4.3}$$

Introducing the expression of $S(q^{-1})$ given by Equation 3.4.3 in Equation 3.4.2, and after simplification by $B^*(q^{-1})$, one obtains

$$A(q^{-1})\, S'(q^{-1}) + q^{-(d+1)}\, R(q^{-1}) = P(q^{-1}) \tag{3.4.4}$$

This polynomial equation has a unique solution for

$$n_P = deg\, P(q^{-1}) = n_A + d \; ; \quad deg\, S'(q^{-1}) = d \; ; \quad deg\, R(q^{-1}) = n_A - 1 \quad (3.4.5)$$

in which

$$R(q^{-1}) = r_0 + r_1 q^{-1} + \ldots + r_{n_A - 1} q^{-n_A + 1} \quad (3.4.6)$$

and

$$S'(q^{-1}) = 1 + s'_1 q^{-1} + \ldots + s'_d q^{-d} \quad (3.4.7)$$

Equation 3.4.4 corresponds, under the conditions of Equation 3.4.5, to the matrix equation

$$M x = p \quad (3.4.8)$$

where M is a lower triangular matrix of dimension (n_A+d+1) x (n_A+d+1)

$$\left.\left[\;\overbrace{\begin{array}{cccccccc}
1 & 0 & & & & & & 0 \\
a_1 & 1 & & & & & & . \\
a_2 & a_1 & & 0 & & & & . \\
: & : & & 1 & & & & . \\
a_d & a_{d-1} & \ldots & a_1 & 1 & & & . \\
a_{d+1} & a_d & & & a_1 & 1 & & . \\
a_{d+2} & a_{d+1} & & & a_2 & 0 & & . \\
 & & & & . & & & . \\
 & & & & . & & & 0 \\
0 & 0 & \ldots & 0 & a_{n_A} & 0 & 0 & 1
\end{array}}^{n_A + d + 1}\;\right]\right\} n_A + d + 1$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{d+1} \qquad \underbrace{\qquad\qquad}_{n_A}$$

$$x^T = [1, s'_1, \ldots, s'_d, r_0, r_1, \ldots, r_{n-1}] \quad (3.4.9)$$

$$p^T = [1, p_1, p_2, \ldots, p_{n_A}, p_{n_A+1}, \ldots, p_{n_A+d}] \quad (3.4.10)$$

Some coefficients p_i can be zero. Since M is a lower triangular matrix, Equation 3.4.8 (and respectively Equation 3.4.4) always has a solution.

In order to solve Equation 3.4.4 one can use *predisol.sci* (Scilab) and *predisol.m* (MATLAB®) functions available on the book website.

One observes that, because of the nature of the design methodology, $S(q^{-1})$ already contains a fixed part (Equation 3.4.3) specified before solving Equation 3.4.4.

Thus, it is convenient to consider, as in the pole placement case, a parameterization of $S(q^{-1})$ and $R(q^{-1})$ in the form

$$R(q^{-1}) = H_R\,(q^{-1})\,R'(q^{-1}) \tag{3.4.11}$$

$$S(q^{-1}) = H_S\,(q^{-1})\,S'(q^{-1}) \tag{3.4.12}$$

In which $H_R(q^{-1})$ and $H_S(q^{-1})$ represent the pre-specified parts of $R(q^{-1})$ and $S(q^{-1})$.

In the case of tracking and regulation with independent objectives, $H_S(q^{-1})$ will have the form

$$H_S(q^{-1}) = B^*(q^{-1})\,.\,H'_S(q^{-1}) \tag{3.4.13}$$

and Equation 3.4.4, in the general case, becomes

$$A(q^{-1})\,H'_S(q^{-1})\,S'(q^{-1}) + q^{-(d+1)}\,H_R(q^{-1})\,R'(q^{-1}) = P(q^{-1}) \tag{3.4.14}$$

For solving 3.4.14 one can also use *predisol.sci* (Scilab) and *predisol.m* (MATLAB®) functions. Note also that pole placement equations can be used for computing $S'(q^{-1})$ and $R'(q^{-1})$ by introducing $B^*(q^{-1})$ in the expression of $P(q^{-1})$.

Steady-State Error

In order to have a zero steady state error for a step input or a step disturbance, the open loop cascade must contain a digital integrator, *i.e.* the polynomial $S(q^{-1})$ must contain a term $(1\text{-}q^{-1})$:

$$S(q^{-1}) = B^*(q^{-1})\,(1 - q^{-1})\,S'(q^{-1}) = B^*(q^{-1})\,H'_S(q^{-1})\,S'(q^{-1}) \tag{3.4.15}$$

Introducing this expression in Equation 3.4.2 and after simplification by $B^*(q^{-1})$ one gets

$$A(q^{-1})\,(1\text{-}q^{-1})\,S'(q^{-1}) + q^{-(d+1)}\,R(q^{-1}) = P(q^{-1}) \tag{3.4.16}$$

which must be solved in order to obtain the corresponding coefficients of $S'(q^{-1})$ and $R(q^{-1})$ when an integrator is used.

3.4.3 Tracking (Computation of $T(q^{-1})$)

The pre-filter $T(q^{-1})$ is computed in order to achieve (in accordance with Figure 3.17), between reference $r(t)$ and $y(t)$, a transfer function:

$$H_{CL}(q^{-1}) = \frac{q^{-(d+1)}B_m(q^{-1})}{A_m(q^{-1})} = \frac{B_m(q^{-1})T(q^{-1})q^{-(d+1)}}{A_m(q^{-1})P(q^{-1})} \tag{3.4.17}$$

From Equation 3.4.17

$$T(q^{-1}) = P(q^{-1}) \tag{3.4.18}$$

The input of $T(q^{-1})$ is the $(d+1)$ steps ahead prediction of the desired trajectory $y^*(t+d+1)$, obtained by filtering $r(t)$ with the tracking model $B_m(q^{-1}) / A_m(q^{-1})$.

$$y^*(t+d+1) = \frac{B_m(q^{-1})}{A_m(q^{-1})} r(t) \tag{3.4.19}$$

and the controller equation will be given by

$$S(q^{-1})\, u(t) + R(q^{-1})\, y(t) = P(q^{-1})\, y^*(t+d+1) \tag{3.4.20}$$

Equation 3.4.20 may also take the form

$$u(t) = \frac{P(q^{-1})y^*(t+d+1) - R(q^{-1})y(t)}{S(q^{-1})} \tag{3.4.21}$$

Since $S(q^{-1})$ is of the form

$$\begin{aligned} S(q^{-1}) &= s_0 + s_1 q^{-1} + \ldots + s_{n_S} q^{-n_S} = s_0 + q^{-1} S^*(q^{-1}) \\ &= B^*(q^{-1}) S'(q^{-1}) \end{aligned} \tag{3.4.22}$$

and considering the expressions of $B^*(q^{-1})$ and $S'(q^{-1})$ (given by Equation 3.4.7):

$$s_0 = b_1 \tag{3.4.23}$$

Therefore Equation 3.4.20 may also take the form[8]

[8] One can also normalize the parameters of the controller by dividing all parameters by $s_0 = b_1$. This avoids a multiplication operation in real time implementation.

$$u(t) = \frac{1}{b_1}\left[P(q^{-1})y^*(t+d+1) - S^*(q^{-1})u(t-1) - R(q^{-1})y(t)\right] \tag{3.4.24}$$

As for the pole placement case, if the desired dynamics is the same both for tracking and regulation, the reference model is no longer necessary and the polynomial $T(q^{-1})$ is replaced by a simple gain to guarantee unit static gain between the reference trajectory and the output:

$$T(q^{-1}) = G = P(1)$$

If $S(q^{-1})$ contains an integrator, then the polynomial $T(q^{-1})$ becomes

$$T(q^{-1}) = G = R(1)$$

3.4.4 Tracking and Regulation with Independent Objectives: Examples

Table 3.7 gives the results of the design for tracking and regulation with independent objectives.

The example considered here concerns the control of a plant characterized by a second-order discrete-time model with two poles at $z = 0.6$ and at $z = 0.7$, and with a stable zero. The desired tracking dynamics (polynomials $A_m\,(q^{-1})$ and $B_m\,(q^{-1})$) has been obtained by discretization of a second-order normalized continuous-time model with $\omega_0 = 0.5\ rad/s$ and $\zeta = 0.9$ ($T_s = 1s$). The desired regulation dynamics (polynomial $P(q^{-1})$) has been obtained by discretization of a second-order normalized continuous-time model with $\omega_0 = 0.4\ rad/s$ and $\zeta = 0.9$. The controller contains an integrator. The computed values of $R(q^{-1})$ and $S(q^{-1})$ are given in the lower part of Table 3.3, and the simulation results, for a reference tracking and a load disturbance (step) on the output, are presented in Figure 3.19. Note that the position of the zero is close to the boundaries of the admissibility domain ($z_0 = -0.5$ corresponds to $\zeta = 0.2$), and this explains the damped oscillations observed on the control signal when a step disturbance occurs.

Table 3.8 gives the computed values of the controller parameters for a plant having the same dynamics of the previous one but with a time delay $d = 3$ (instead of $d = 0$). All the desired performances have been maintained. Polynomial $S(q^{-1})$ has a higher order. Closed loop system responses are presented in Figure 3.20. One observes that the output response is the same as in the case of $d = 0$ (Figure 3.20) except that they are shifted by d steps. Moreover, by comparing Figures 3.19 and 3.20, one observes that the control signal is the same. In other words, even if the same output value is recovered with a delay of three sampling periods, the controller computes a signal that is the same as in the case of $d = 0$. In fact, *for the case with time delay, the controller contains a three step ahead predictor* (for details see Appendix B).

Table 3.7. Tracking and regulation with independent objectives (d=0)

Plant:

- $d = 0$
- $B(q^{-1}) = 0.2\, q^{-1} + 0.1\, q^{-2}$
- $A(q^{-1}) = 1 - 1.3\, q^{-1} + 0.42\, q^{-2}$

Tracking dynamics:

- $B_m(q^{-1}) = 0.0927 q^{-1} + 0.0687\, q^{-2}$
- $A_m(q^{-1}) = 1 - 1.2451 q^{-1} + 0.4066\, q^{-2}$
- $T_S = 1s$, $\omega_0 = 0.5$ rad/s, $\zeta = 0.9$

Regulation dynamics → $P(q^{-1}) = 1 - 1.3741\, q^{-1} + 0.4867\, q^{-2}$
$T_S = 1s$, $\omega_0 = 0.4$ rad/s, $\zeta = 0.9$

Pre-specifications: Integrator

***** CONTROL LAW *****

$$S(q^{-1})\, u(t) + R(q^{-1})\, y(t) = T(q^{-1})\, y^*(t+d+1)$$

$$y^*(t+d+1) = [B_m(q^{-1})/A_m(q^{-1})]\, .\, r(t)$$

Controller:

- $R(q^{-1}) = 0.9258 - 1.2332\, q^{-1} + 0.42\, q^{-2}$
- $S(q^{-1}) = 0.2 - 0.1\, q^{-1} - 0.1\, q^{-2}$
- $T(q^{-1}) = P(q^{-1})$

Gain margin: 2.109 Phase margin: 65.3 deg
Modulus margin: 0.526 (- 5.58 dB) Delay margin: 1.2 s

One observes that the delay margin is smaller than one sampling period. In this case, one should specify high frequency auxiliary poles in $P(q^{-1})$ that lightly affect the low frequency behavior but improve the robustness at high frequencies (in particular the delay margin).

The maximum number of closed loop poles (the maximum degree of polynomial $P(q^{-1})$) is 5. Having already specified a pair of dominant poles, one can impose, for example, $P_F(q^{-1}) = (1 - 0.1\, q^{-1})^3$. In this case, one obtains a gain margin of 2.157, a modulus margin of 0.534 (-5.45dB), a phase margin of 58.5 degrees and a delay margin of 1.19s (greater than one sampling period).

The closed loop system responses are presented in Figure 3.21. Tracking performances are the same as for the case without auxiliary pole, and regulation performances are almost unchanged. One also remarks that the introduction of these poles reduces the stress on the actuator in the transient for the disturbance rejection.

Table 3.8. Tracking and regulation with independent objectives (d=3)

Plant:

- $d = 3$
- $B(q^{-1}) = 0.2\, q^{-1} + 0.1\, q^{-2}$
- $A(q^{-1}) = 1 - 1.3\, q^{-1} + 0.42\, q^{-2}$

Tracking dynamics:

- $B_m\,(q^{-1}) = 0.0927 + 0.0687\, q^{-1}$
- $Am\,(q^{-1}) = 1 - 1.2451 q^{-1} + 0.4066\, q^{-2}$
- $T_s = 1s\;,\;\omega_0 = 0.5\ rad/s,\;\; \zeta = 0.9$

Regulation dynamics → $P\,(q^{-1}) = 1 - 1.3741\, q^{-1} + 0.4867\, q^{-2}$
$T_S = 1s\;,\;\omega_0 = 0.4\ rad/s,\;\; \zeta = 0.9$

Pre-specifications: Integrator

***** CONTROL LAW *****

$$S\,(q^{-1})\,u(t) + R\,(q^{-1})\,y(t) = T\,(q^{-1})\,y^*(t+d+1)$$

$$y^*(t+d+1) = [B_m\,(q^{-1})/A_m\,(q^{-1})\,]\,.ref(t)$$

Controller:

- $R(q^{-1}) = 0.8914 - 1.1521\, q^{-1} + 0.3732\, q^{-2}$
- $S(q^{-1}) = 0.2 + 0.0852\, q^{-1} - 0.0134\, q^{-2} - 0.0045\, q^{-3} - 0.1785\, q^{-4} - 0.0888\, q^{-5}$
- $T(q^{-1}) = P(q^{-1})$

Gain margin : 2.078 Phase margin: 58 deg
Modulus margin: 0.518 (- 5.71 dB) Delay margin: 0.7 s

As a general rule, when the plant has a time delay, one needs to specify not only the dominant poles of $P(q^{-1})$, but also auxiliary poles at values other than 0 in order to improve the robustness with respect to the possible variation of the delay (this remark is also valid for the other methods presented in this chapter).

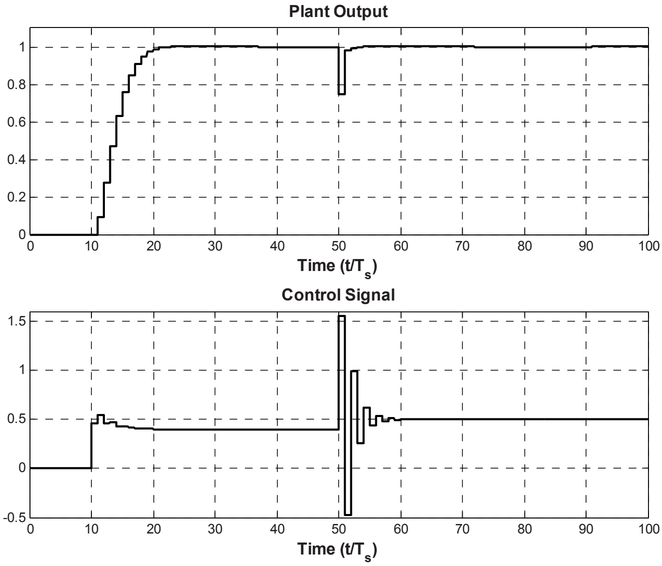

Figure 3.19. Performances of tracking and regulation with independent objectives (d = 0)

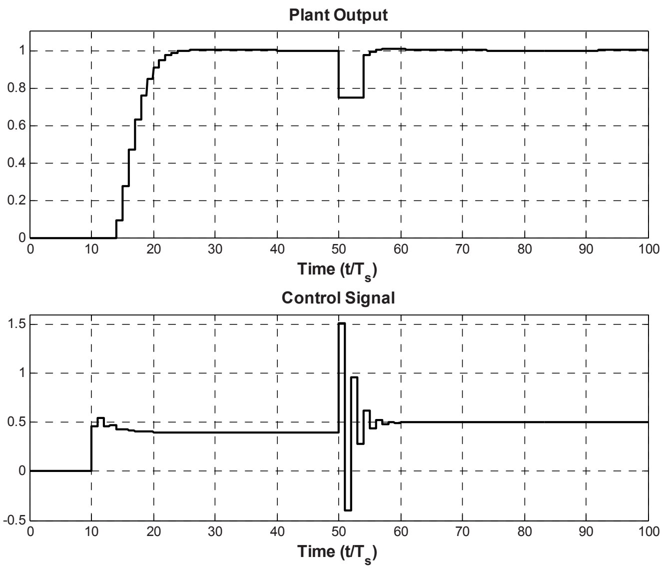

Figure 3.20. Performances of tracking and regulation with independent objectives in presence of a time delay (d = 3)

3.5 Internal Model Control (Tracking and Regulation)

The internal model control (IMC) strategy (not to be confused with the *internal model principle*) is a particular case of the pole placement technique. In the internal mode control the poles of the plant model are chosen as the desired dominant poles of the closed loop. As a consequence, the closed loop system will not be faster than the open loop system. Since there will not be an acceleration of the closed loop time response with respect to the open loop time response, one can expect a better closed loop robustness with respect to plant model uncertainties. As the closed loop poles include the poles of the plant model, *this technique can be applied only to stable and well damped discrete time plant models.* This technique is often used for the control of stable plants with a large time delay with respect to the open loop rise time (the rise time does not include the time delay!).
The control structure is the same as for the pole placement (Figure 3.14).

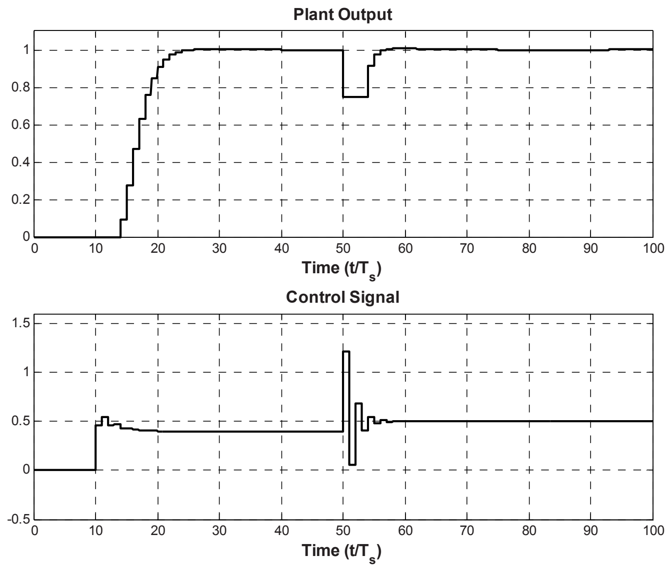

Figure 3.21. Performances of tracking and regulation with independent objectives with auxiliary poles in presence of a pure time delay (d = 3)

3.5.1 Regulation

In the case of the internal model control Equation 3.3.22 of the pole placement becomes

$$A(q^{-1})S(q^{-1}) + q^{-d}B(q^{-1})R(q^{-1}) = A(q^{-1})P_F(q^{-1}) = P(q^{-1}) \qquad (3.5.1)$$

Often in practice one selects

$$P_F(q^{-1}) = \left(1 + \alpha q^{-1}\right)^{n_{P_F}} \tag{3.5.2}$$

or

$$P_F(q^{-1}) = \left(1 + \alpha_1 q^{-1}\right)\left(1 + \alpha q^{-1}\right)^{n_{P_F}-1} \tag{3.5.3}$$

By examining the structure of Equation 3.5.1 one observes that $R(q^{-1})$ should have as a factor $A(q^{-1})$:

$$R(q^{-1}) = A(q^{-1})R'(q^{-1}) \tag{3.5.4}$$

and Equation 3.5.1, after elimination of the common factor $A(q^{-1})$, becomes

$$S(q^{-1}) + q^{-d}B(q^{-1})R'(q^{-1}) = P_F(q^{-1}) \tag{3.5.5}$$

which presents similarities with Equation 3.4.4 for tracking and regulation with independent objectives (by replacing in Equation 3.4.4 S' by R', A by $q^{-d}B$, $q^{-(d+1)}R$ by S and $P(q^{-1})$ by $P_F(q^{-1})$). Equation 3.5.5 has a unique minimal order solution for

$$n_{P_F} = \deg P_F(q^{-1}) \leq n_B + d$$

$$n_S = \deg S(q^{-1}) = n_B + d \qquad ; \qquad n_{R'} = \deg R'(q^{-1}) = 0$$

The solution of Equation 3.5.5 becomes simpler if one considers the typical case where an integrator is introduced in the controller, *i.e.*:

$$S(q^{-1}) = \left(1 - q^{-1}\right)S'(q^{-1}) \tag{3.5.6}$$

In this case, for $q=1$ Equation 3.5.5 becomes

$$B(1)R'(1) = P_F(1) \tag{3.5.7}$$

since $S(1)=0$ and one obtains

$$R'(q^{-1}) = R'(1) = \frac{P_F(1)}{B(1)} \tag{3.5.8}$$

and

$$R(q^{-1}) = A(q^{-1})\frac{P_F(1)}{B(1)} \tag{3.5.9}$$

$$S(q^{-1}) = (1-q^{-1})S'(q^{-1}) = P_F(q^{-1}) - q^{-d}B(q^{-1})\frac{P_F(1)}{B(1)} \tag{3.5.10}$$

From Equation 3.5.9 it follows that in this design the poles of the plant model (defined by $A(q^{-1})$) will be simplified by the zeros introduced by the controller. This implies that *the poles of the plant model should be stable* (see Chapter 2, Section 2.5.2).

One interesting aspect of this design is the possibility to characterize the set of all the controllers as a function of $P_F(q^{-1})$ without having to solve a polynomial equation.

If, in addition to the integrator, a fixed part $H_R(q^{-1})$ is imposed in $R(q^{-1})$, $S(q^{-1})$ has the structure given in Equation 3.5.6 and $R(q^{-1})$ will have the structure

$$R(q^{-1}) = A(q^{-1})H_R(q^{-1})R'(q^{-1}) \tag{3.5.11}$$

Equation 3.5.5 becomes

$$S(q^{-1}) + q^{-d}B(q^{-1})H_R(q^{-1})R'(q^{-1}) = P_F(q^{-1}) \tag{3.5.12}$$

with the condition

$$P_F(1) = B(1)H_R(1)R'(1) \tag{3.5.13}$$

from which one obtains

$$R'(q^{-1}) = R'(1) = \frac{P_F(1)}{B(1)H_R(1)} \tag{3.5.14}$$

$$R(q^{-1}) = A(q^{-1})H_R(q^{-1})\frac{P_F(1)}{B(1)H_R(1)} \tag{3.5.15}$$

$$S(q^{-1}) = P_F(q^{-1}) - q^{-d}B(q^{-1})H_R(q^{-1})R'(q^{-1}) \tag{3.5.16}$$

3.5.2 Tracking

In the case of internal model control the polynomial $T(q^{-1})$ used for tracking is given by (like in the pole placement)

$$T(q^{-1}) = A(q^{-1})P_F(q^{-1})/B(1) \tag{3.5.17}$$

But if one chooses the same dynamics for tracking and regulation, the tracking reference model can be eliminated and $T(q^{-1})$ will be given by

$$T(q^{-1}) = T(1) = \frac{A(1)P_F(1)}{B(1)} \tag{3.5.18}$$

which guarantees a unit steady state gain between the desired tracking trajectory and the controlled output.

3.5.3 An Interpretation of the Internal Model Control

Let us consider first the case $H_R(q^{-1})=1$. The control law equation (Table 3.5) is written using the expressions of *R, S, T* given by Equations 3.5.9, 3.5.10 and 3.5.17:

$$\begin{gathered} S(q^{-1})u(t) = \left[P_F(q^{-1}) - \frac{P_F(1)}{B(1)}q^{-d}B(q^{-1})\right]u(t) = \\ \left[\frac{1}{B(1)}A(q^{-1})P_F(q^{-1})y^*(t+d+1) - \frac{P_F(1)}{B(1)}A(q^{-1})y(t)\right] \end{gathered} \tag{3.5.19}$$

which can be re-written as

$$\begin{gathered} P_F(q^{-1})u(t) = \frac{1}{B(1)}A(q^{-1})P_F(q^{-1})y^*(t+d+1) - \\ \frac{P_F(1)}{B(1)}\left[A(q^{-1})y(t) - q^{-d}B(q^{-1})u(t)\right] \end{gathered} \tag{3.5.20}$$

Taking into account that by hypothesis $A(q^{-1})$ is asymptotically stable one obtains

$$\begin{gathered} S_0(q^{-1})u(t) = P_F(q^{-1})u(t) = \frac{1}{B(1)}A(q^{-1})P_F(q^{-1})y^*(t+d+1) - \\ \frac{P_F(1)}{B(1)}A(q^{-1})\left[y(t) - \frac{q^{-d}B(q^{-1})}{A(q^{-1})}u(t)\right] = \\ T_0(q^{-1})y^*(t+d+1) - R_0(q^{-1})\left[y(t) - \frac{q^{-d}B(q^{-1})}{A(q^{-1})}u(t)\right] \end{gathered} \tag{3.5.21}$$

where

$$R_0(q^{-1}) = \frac{P_F(1)}{B(1)} A(q^{-1})$$

$$S_0(q^{-1}) = P_F(q^{-1})$$

$$T_0(q^{-1}) = \frac{1}{B(1)} P(q^{-1}) = \frac{1}{B(1)} A(q^{-1}) P_F(q^{-1})$$

This leads to the equivalent representation of the closed loop behavior shown in Figure 3.22.

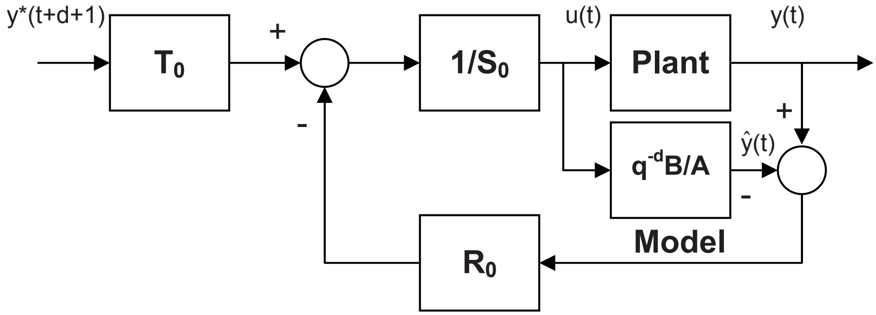

Figure 3.22. Equivalent diagram of the internal model control

One can see that an equivalent implementation of the RST controller can be considered. This implementation explicitly features the model of the plant (prediction model) as a component of the control scheme. It is the error signal between the plant output $y(t)$ and the predictor output $\hat{y}(t)$, which is fed-back. Note also that the computation of R_0 and S_0 does not require the solution of a polynomial equation.

For the case of $H_R(q^{-1}) \neq 1$, using Equations 3.5.14, 3.5.15, 3.5.16 and 3.5.17 one obtains a similar result. Only the expression of $R_0(q^{-1})$ changes (it will contain in addition the factor $H_R(q^{-1})/H_R(1)$).

3.5.4 The Sensitivity Functions

In the case of internal model control the sensitivity functions have a particular structure, as a consequence of the choice for the dominant poles of the closed loop (they are equal to the plant model poles):

$$S_{yp}(z^{-1}) = \frac{S(z^{-1})}{P_F(z^{-1})} = 1 - \frac{z^{-d} B(z^{-1}) H_R(z^{-1}) P_F(1)}{B(1) H_R(1) P_F(z^{-1})} \tag{3.5.22}$$

$$S_{yb}(z^{-1}) = -\frac{z^{-d}B(z^{-1})R(z^{-1})}{A(z^{-1})P_F(z^{-1})} = -\frac{z^{-d}B(z^{-1})H_R(z^{-1})P_F(1)}{B(1)H_R(1)P_F(z^{-1})} \tag{3.5.23}$$

$$S_{up}(z^{-1}) = -\frac{R(z^{-1})}{P_F(z^{-1})} = -\frac{A(z^{-1})H_R(z^{-1})P_F(1)}{B(1)H_R(1)P_F(z^{-1})} \tag{3.5.24}$$

$$S_{yv}(z^{-1}) = -\frac{z^{-d}B(z^{-1})S(z^{-1})}{A(z^{-1})P_F(z^{-1})} = -\frac{z^{-d}B(z^{-1})}{A(z^{-1})}S_{yp} \tag{3.5.25}$$

Equation 3.5.25 clearly indicates that the plant model should be asymptotically stable in open loop. One can also see the direct influence of the transfer function $H_R(q^{-1})/P_F(z^{-1})$ (*i.e.* the choice of H_R and P_F) upon the sensitivity functions.

3.5.5 Partial Internal Model Control (Tracking and Regulation)

In many applications the plant model is characterized by low frequency dominant poles (located within or close to the desired attenuation band) and secondary poles located outside the desired attenuation band.

If the dominant poles are too slow, or if they have a too low damping (often the case for mechanical systems), the internal model control cannot be applied as such.

In this case one uses a mixture of pole placement and internal model control. The pole placement will be used to assign the desired dominant poles of the closed loop but it will not move the secondary poles of the plant model (which will become poles of the closed loop).

Suppose $A(q^{-1})$ has the form

$$A(q^{-1}) = A_1(q^{-1})A_2(q^{-1}) \tag{3.5.26}$$

where $A_1(q^{-1})$ defines the dominant poles of the plant model. In the case of partial internal model control, the equation defining the closed loop poles will be

$$\begin{aligned} &A_1(q^{-1})A_2(q^{-1})S(q^{-1}) + q^{-d}B(q^{-1})R(q^{-1}) \\ &= P_D(q^{-1})A_2(q^{-1})P_F(q^{-1}) \end{aligned} \tag{3.5.27}$$

Examining the structure of Equation 3.5.27 it is seen that $R(q^{-1})$ will be of the form

$$R(q^{-1}) = A_2(q^{-1})R'(q^{-1}) \tag{3.5.28}$$

and Equation 3.5.27 (after elimination of the common factor $A_2(q^{-1})$) becomes

$$A_1(q^{-1})S(q^{-1}) + q^{-d}B(q^{-1})R'(q^{-1}) = P_D(q^{-1})P_F(q^{-1}) \tag{3.5.29}$$

This technique can also be interpreted as a simplification of the plant model secondary poles (defined by $A_2(q^{-1})$) by the zeros of the controller.

3.5.6 Internal Model Control for Plant Models with Stable Zeros

If $B(q^{-1})$ has all its zeros inside the unit circle (and these zeros are sufficiently damped) one can use the tracking and regulation with independent objectives design but using as desired poles the dominant poles of the plant model.

One considers that $S(q^{-1})$ has the structure given by Equation 3.4.15 (presence of an integrator) and one supposes $H_R(q^{-1})=1$ (in order to simplify the presentation).

Equation 3.4.16 becomes

$$A(q^{-1})\,(1-q^{-1})\,S'(q^{-1}) + q^{-(d+1)}\,R(q^{-1}) = A(q^{-1})\,P_F(q^{-1}) \tag{3.5.30}$$

This implies that $R(q^{-1})$ will have the form

$$R(q^{-1})=A(q^{-1})R'(q^{-1}) \tag{3.5.31}$$

and Equation 3.5.30 becomes

$$(1-q^{-1})\,S'(q^{-1}) + q^{-(d+1)}\,R'(q^{-1}) = P_F(q^{-1}) \tag{3.5.32}$$

For $q=1$one has

$$R'(1) = R'(q^{-1}) = P_F(1) \tag{3.5.33}$$

It results that

$$(1-q^{-1})\,S'(q^{-1}) = P_F(q^{-1}) - q^{-(d+1)}\,P_F(1) \tag{3.5.34}$$

from which one gets

$$S(q^{-1}) = B^*(q^{-1})\,[P_F(q^{-1}) - q^{-(d+1)}\,P_F(1)] \tag{3.5.35}$$

$$R(q^{-1}) = A(q^{-1})\,P_F(1) \tag{3.5.36}$$

3.5.7 Example: Control of Systems with Time Delay

Internal model control is often used for the control of systems with large time delay compared to the rise time of the system without time delay. If $d > t_R$ (where t_R is expressed in sampling periods), the reduction of the rise time in closed loop does not produce a significant improvement of the total response time, since this is mainly determined by d. Therefore, one is able to use internal model control. However, even when one takes $P_D(q^{-1})=A(q^{-1})$, the introduction of the auxiliary poles and/or of the fixed part $H_R(q^{-1})$ is necessary in order to assure the robustness of the closed loop with respect to the time delay variations.

Consider the case $P_F(q^{-1})=1$, $H_R(q^{-1})=1$ and $B(q^{-1})= b_1 q^{-1}$ $(b_1>0)$. From Equations 3.5.22 and 3.5.23 one gets

$$S_{yp}(z^{-1}) = 1 - z^{-d-1} = (1 - z^{-1})(1 + z^{-1} + z^{-2} + \ldots + z^{-d}) \tag{3.5.37}$$

$$S_{yb}(z^{-1}) = z^{-d-1} \tag{3.5.38}$$

$S_{yp}(z^{-1})$ has a zero at $z = 1$ $(\omega = 0)$ and d zeros on the unit circle if d is even, or $d-1$ zeros on the unit circle plus a zero at $z = -1$ $(\omega = \pi)$ if d is odd. From Equation 3.5.37, since $| z^{-d-1} |$ is always equal to 1, it results that

$$\left| S_{yp}(e^{-j\omega}) \right|_{\max} \leq 2 \tag{3.5.39}$$

and then the modulus margin $\Delta M = 0.5$ is always achieved.

However

$$\left| S_{yb}(e^{-j\omega}) \right| \equiv 1 \qquad \text{for } 0 \leq \omega \leq \pi \tag{3.5.40}$$

and therefore the frequency characteristic of $| S_{yb}|$ will cross the template for $\Delta\tau = T_s$ (see Figure 2.39: $| S_{yb}|$ should be smaller than 1 starting from $f = 0.17 f_s$).

The condition for satisfying the delay margin in Figure 2.39 is in this case

$$\left| z^{-d-1} \right| \leq \frac{1}{\left| z^{-1} - 1 \right|} \tag{3.5.41}$$

and clearly this will not be satisfied at all frequencies.

We will examine next the beneficial effect of the introduction of the auxiliary poles $P_F(q^{-1})$ and of a fixed part $H_R(q^{-1})$ in the controller.

Auxiliary Poles $(P_F(q^{-1}) \neq 1,\ H_R(q^{-1}) = 1)$

Let us select:

$$P_F(q^{-1}) = (1 + \alpha\, q^{-1}) \qquad -1 < \alpha < 0 \tag{3.5.42}$$

and search for the value of α assuring the delay margin $\Delta\tau = T_s$. In this case S_{yb} is given by Equation 3.5.23 and the condition for the delay margin $\Delta\tau = T_s$ takes the form

$$\left|S_{yb}(z^{-1})\right| = \left|\frac{z^{-d-1}P_F(1)}{P_F(z^{-1})}\right| < \frac{1}{\left|1 - z^{-1}\right|} \qquad z = e^{j\omega} \quad 0 \le \omega \le \pi \tag{3.5.43}$$

which for $P_F(q^{-1})$ given by Equation 3.5.42 becomes

$$\left|\frac{1+\alpha}{1+\alpha\, z^{-1}}\right| < \frac{1}{\left|1 - z^{-1}\right|} \qquad z = e^{j\omega} \quad 0 \le \omega \le \pi \tag{3.5.44}$$

Replacing z by $z = e^{j\omega}$ one observes that the worst situation occurs at $\omega = \pi$ (i.e. $f=0.5f_S$) where 3.5.44 becomes:

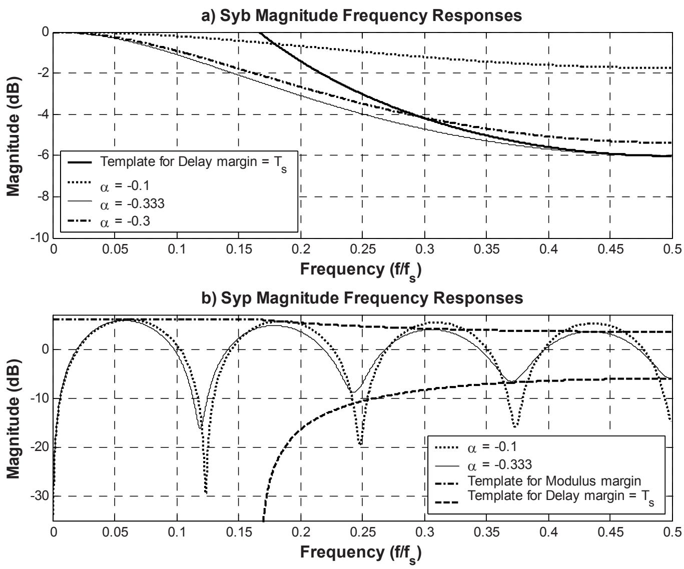

Figure 3.23a,b. Auxiliary poles effect on the delay margin: **a** noise- input sensitivity function (S_{yb}); **b** output sensitivity function (S_{yp})

$$\left|\frac{1+\alpha}{1-\alpha}\right| < 0.5 \quad \Rightarrow \quad \alpha \le -0.333 \tag{3.5.45}$$

Figure 3.23a,b shows the frequency characteristics of $|S_{yb}|$ and $|S_{yp}|$ (controller with integrator) for a plant model characterized by $d=7$, $B(q^{-1})=q^{-1}$ ($b_1=1$), $A(q^{-1})=1+a_1q^{-1}$ with $a_1=-0.2$ and for three values of α: $\alpha=-0.1$; $\alpha=-0.3$; $\alpha=-0.333$. For $\alpha=-0.1$ and $\alpha=-0.3$ one gets the delay margin $\Delta\tau = 0.52T_s$ and $\Delta\tau = 0.91T_s$, respectively, and the corresponding curves of S_{yb} and S_{yp} intersect the robustness template for $\Delta\tau = T_s$. For $\alpha=-0.333$ the effective delay margin is $\Delta\tau = T_s$ and it can be seen that the corresponding sensitivity functions do not cross the robustness template for the delay margin $\Delta\tau = T_s$.

If one uses auxiliary poles $P_F(z^{-1})$ of the form Equation 3.5.3:

$$P_F(q^{-1}) = (1+\alpha q^{-1})(1+\alpha' q^{-1})^{n_{P_F}-1}$$

one selects $-1<\alpha\le 0$ and $-0.25<\alpha'\le -0.05$ with $n_{P_F} \le n_B + d$. In this case one introduces $n_F - 1$ high frequency auxiliary poles. This choice leads to a further contraction of S_{yp} around *0 dB* in the high frequency region, and a reduced effect upon the reduction of the performance at low frequencies.

Figures 3.24a,b shows the frequency characteristics of the modulus of the sensitivity functions S_{yb} and S_{yp} for the same plant model, using either an auxiliary pole of the form of Equation 3.5.42 with $\alpha=-0.5$ or auxiliary poles of the form of Equation 3.5.3, with $\alpha=-0.3$, $\alpha'=-0.1$, $n_{P_F}-1=7$. The corresponding delay margins are $2.09T_S$ and $2.14T_S$, respectively. The second solution is much more efficient for attenuating the sensitivity functions at high frequencies (improving the robustness and reducing the stress on the actuator).

Introduction of $H_R(q^{-1})$ ($P_F(q^{-1}) = 1$, $H_R(q^{-1}) \neq 1$)
In this case, from Equations 3.5.14 and 3.5.15 one has:

$$R(q^{-1}) = A(q^{-1})H_R(q^{-1})R'(q^{-1}) \tag{3.5.46}$$

with

$$H_R(q^{-1}) = 1+\beta q^{-1} \tag{3.5.47}$$

which leads to

$$R'(q^{-1}) = \frac{1}{B(1)(1+\beta)} \tag{3.5.48}$$

and (using Equation 3.5.16):

$$S(q^{-1}) = 1 - \frac{q^{-d}B(q^{-1})(1+\beta q^{-1})}{B(1)(1+\beta)} \tag{3.5.49}$$

For $B(q^{-1}) = b_1 q^{-1}$ one gets

$$S_{yb}(z^{-1}) = -\frac{z^{-d-1}(1+\beta z^{-1})}{1+\beta} \tag{3.5.50}$$

and the condition for getting a delay margin $\Delta\tau = T_s$ becomes

$$\left|\frac{1+\beta\, z^{-1}}{1+\beta}\right| < \frac{1}{\left|1-z^{-1}\right|} \qquad z = e^{j\omega} \quad 0 \le \omega \le \pi \tag{3.5.51}$$

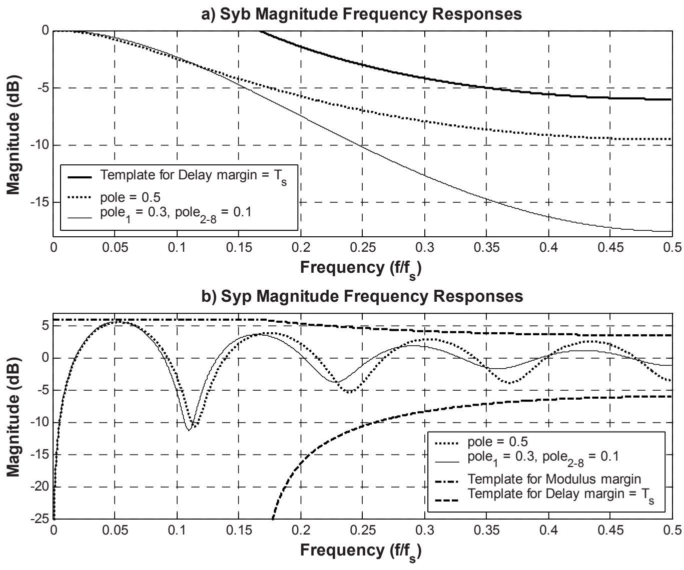

Figure 3.24a,b. Effect of high frequency auxiliary poles on the sensitivity functions of plant models with time delay: **a** noise- input sensitivity function (S_{yb}); **b** output sensitivity function (S_{yp})

The worst situation occurs at $\omega = \pi/2$ ($f=0.25f_s$) where the condition of Equation 3.5.51 becomes (replacing $<$ by $\leq$)

$$\frac{1+\beta^2}{(1+\beta)^2} \leq 0.5 \quad \Rightarrow \quad \beta = 1 \tag{3.5.52}$$

Therefore one should choose in this case

$$H_R(q^{-1}) = 1 + q^{-1} \tag{3.5.53}$$

in order to assure a delay margin of (almost) one sampling period.

It is interesting to note that Equation 3.5.53 corresponds to the opening of the closed loop at $0.5f_s$ (see Section 3.3). This solution leads to $S_{up} = 0$ at $f = 0.5f_s$ and therefore a significant increase of the robustness at high frequencies. Figure 3.25a,b presents comparatively the frequency characteristics of the modulus of S_{yb} and S_{yp} for the cases: 1) $H_R(q^{-1}) = 1 + q^{-1}$, $P_F(q^{-1}) = 1$ and 2) $H_R(q^{-1}) = 1$, $P_F(q^{-1}) = 1 - 0.333z^{-1}$.

Of course one can use simultaneously $P_F(q^{-1}) \neq 1$ and $H_R(q^{-1}) \neq 1$.

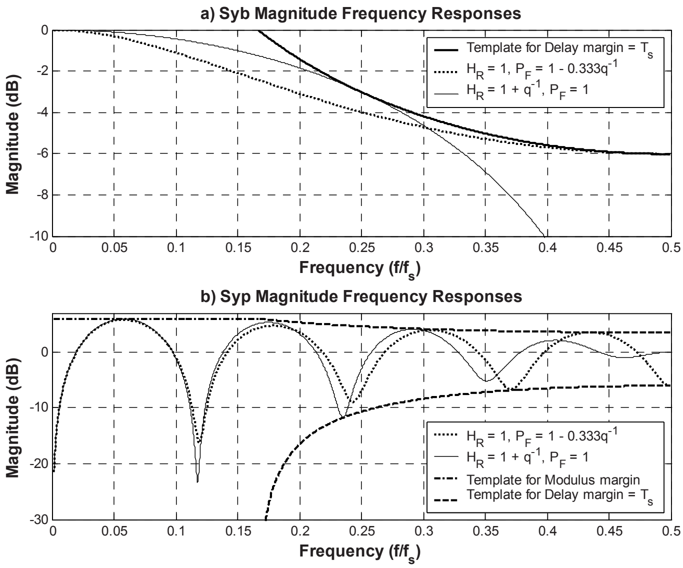

Figure 3.25a,b. Effect of the fixed part $H_R(q^{-1}) = 1 + q^{-1}$ on the sensitivity function of a plant model with time delay: **a** noise- input sensitivity function (S_{yb}); **b** output sensitivity function (S_{yp})

3.6 Pole Placement with Sensitivity Function Shaping

In many applications, in order to satisfy simultaneously the imposed performance in regulation and the robustness margins, one is obliged to shape the modulus of the output and input sensitivity functions in the frequency domain. The shaping of the sensitivity functions is done by the appropriate selection of the desired closed loop poles and the introduction of pre-specified filters in the controllers.

The modulus of the output sensitivity function $|S_{yp}(z^{-1})|$[9] is a significant indicator of both disturbance rejection properties and robustness properties of the closed loop (see Chapter 2, Section 2.6).

The modulus of the input sensitivity functions $|S_{up}(z^{-1})|$ is an indicator of the actuator stress in various frequency regions as well as an indicator of the tolerance to the additive uncertainties upon the plant model (see Chapter 2, Section 2.6).

Using a digital RST controller, the output sensitivity function has the expression

$$S_{yp}(z^{-1}) = \frac{A(z^{-1})S(z^{-1})}{A(z^{-1})S(z^{-1}) + z^{-d}B(z^{-1})R(z^{-1})} \tag{3.6.1}$$

and the input sensitivity function has the expression

$$S_{up}(z^{-1}) = -\frac{A(z^{-1})R(z^{-1})}{A(z^{-1})S(z^{-1}) + z^{-d}B(z^{-1})R(z^{-1})} \tag{3.6.2}$$

where

$$R(z^{-1}) = H_R(z^{-1})\, R'(z^{-1}) \tag{3.6.3}$$

$$S(z^{-1}) = H_S(z^{-1})\, S'(z^{-1}) \tag{3.6.4}$$

and

$$A(z^{-1})\, S(z^{-1}) + z^{-d}\, B(z^{-1})\, R(z^{-1}) = P_D(z^{-1})\,.\, P_F(z^{-1}) = P(z^{-1}) \tag{3.6.5}$$

In Equations 3.6.3 and 3.6.4, $H_R(z^{-1})$ and $H_S(z^{-1})$ correspond to pre-specified fixed filters incorporated in $R(z^{-1})$ and $S(z^{-1})$ respectively. $S(z^{-1})$ and $R(z^{-1})$ (more precisely $R'(z^{-1})$ and $S'(z^{-1})$) are the solutions of Equation 3.6.5 where the polynomial $P(z^{-1})$ defines the desired closed loop poles. The polynomial $P(z^{-1})$ is factorized in order to emphasize the dominant poles defined by $P_D(z^{-1})$ and the auxiliary poles defined by $P_F(z^{-1})$.

---

[9] In this section the notation "z" (complex variable) will be used instead of "q" when we will examine the properties of the various transfer functions in the frequency domain.

In what follows we will examine the properties of the sensitivity functions in the frequency domain $(q = z = e^{j\omega}, \omega = 2\pi f / f_s)$.

The various properties of the sensitivity functions will be illustrated by means of the following example ($T_s = 1s$).

Plant model:

$A(q^{-1}) = 1 - 0.7\, q^{-1}$, $B(q^{-1}) = 0.3\, q^{-1}$, $d = 2$.

Specified performance (polynomial $P(q^{-1})$):

Defined by the discretization of a continuous time second-order system with

- $\omega_0 = 0.4$ *or* 0.6 *or* 1 *rad/s*
- $\zeta = 0.9$ (constant)

3.6.1 Properties of the Output Sensitivity Function

Property 1
The modulus of the output sensitivity function at a certain frequency gives the amplification or the attenuation of the disturbance.

At the frequencies where $|S_{yp}(\omega)| = 1$ ($0\ dB$), there is neither amplification nor attenuation of the disturbance (operation like in open loop).

At the frequencies where $|S_{yp}(\omega)| < 1$ ($0\ dB$), the disturbance is attenuated.

At the frequencies where $|S_{yp}(\omega)| > 1$ ($0\ dB$), the disturbance is amplified.

Property 2
For asymptotically stable closed loop system, and stable open loop, the integral of the logarithm of the modulus of the output sensitivity function from 0 to 0.5 f_s satisfies[10]

$$\int_0^{0.5 f_s} \log \left| S_{yp}(e^{-j2\pi f/f_s}) \right| df = 0$$

In other terms, the sum of the areas between the modulus of the sensitivity function and the 0 dB axis, taken with their sign, is null. As a consequence, the attenuation of disturbances in a certain frequency region implies necessarily the amplification of disturbances in other frequency regions. Figure 3.26 illustrates this phenomenon.

[10] For a proof of this property see Sung and Hara (1988). In the case of unstable open loop systems but stable in closed loop, the value of this integral is positive.

The output sensitivity functions shown in Figure 3.26 correspond to the example mentioned earlier for various values of ω_0 *(0.4; 0.6; 1 rad/s)* but ζ = *constant (0.9)*.

It follows that increasing the value of attenuation in a frequency region, or widening the attenuation band, will lead to a higher amplification of disturbances outside the attenuation band. Figure 3.26 clearly emphasizes this phenomenon.

Property 3
The inverse of the maximum of the modulus of the output sensitivity function corresponds to the modulus margin ΔM:

$$\Delta M = (|S_{yp}(e^{-j\omega})|_{max})^{-1} \tag{3.6.6}$$

The modulus margin is defined as the minimal distance between the Nyquist plot of the open loop transfer function and the critical point [*-1, j0*]. The typical values for the modulus margin (see Section 2.6) are

$$\Delta M \geq 0.5 \ (-6\ dB) \ \ [min: 0.4 \ (-8\ dB)]$$

Recall that a modulus margin $\Delta M \geq 0.5$ implies a gain margin $\Delta G \geq 2$ and a phase margin $\Delta\phi > 29^o$. To assure a safe modulus margin, it is necessary that:

$$|S_{yp}(e^{-j\omega})|_{max} \leq 6\ dB \ (or\ exceptionally\ 8\ dB)$$

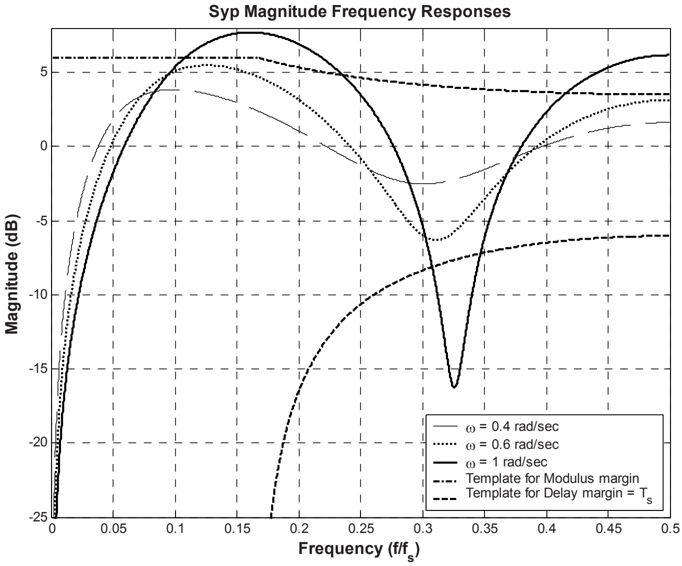

Figure 3.26. Modulus of the output sensitivity function for different attenuation bands of the disturbance

From property 2, it is seen that the increase of the attenuation in a certain frequency region, or the increase of the attenuation band, will in general lead to the increase of $|S_{yp}(e^{-j\omega})|_{max}$ and therefore a reduction of the modulus margin (and of the system robustness).

Property 4
Cancellation of the disturbance effect on the output ($|S_{yp}|= 0$) is obtained at the frequencies where

$$A(e^{-j\omega})S(e^{-j\omega}) = A(e^{-j\omega})H_S(e^{-j\omega})S'(e^{-j\omega}) = 0 \quad ; \quad \omega = 2\pi f / f_s \tag{3.6.7}$$

This results immediately from Equation 3.6.1. Equation 3.6.7 with $q= z= e^{j\omega}$ defines the zeros of the output sensitivity function in the frequency domain.

The fixed pre-specified part of $S(q^{-1})$, denoted $H_S(q^{-1})$, allows one to introduce the zeros at the desired frequencies.

For example

$$H_S(q^{-1}) = 1 - q^{-1}$$

introduces a zero at the zero frequency and allows a perfect rejection of constant disturbances

$$H_S(q^{-1}) = 1 + \alpha q^{-1} + q^{-2}$$

with $\alpha = -2cos\ (\omega T_S) = -2cos\ (2\pi f/f_S)$ introduces a pair of undamped complex zeros at the frequency f (more precisely the normalized frequency f/f_S), while

$$H_S (q^{-1}) = 1 + \alpha_1 q^{-1} + \alpha_2 q^{-2}$$

allows one to introduce complex zeros with non-null damping. The damping is selected as a function of the desired attenuation at a given frequency.

In Figure 3.27 the output sensitivity functions are shown for the cases $H_S(q^{-1}) = 1 - q^{-1}$ and $H_S(q^{-1}) = (1 - q^{-1})(1 + q^{-1})$. The closed loop poles are defined for both cases by ($\omega_0=0.6\ rad/s$ and $\zeta=0.9$). The second choice for H_S introduces, in addition to the integrator, a pair of undamped ($\zeta=0$) complex zeros at $0.25\ f_S$. One observes a very strong attenuation both at null frequency and at $0.25\ f_s$ ($<-100\ dB$).

Property 5
The modulus of the output sensitivity functions is equal to 1, that is

$$|S_{yp}(e^{-j\omega})| = 1 \ (0 \text{ dB})$$

at the frequencies where

$$B^*(e^{-j\omega})R(e^{-j\omega}) = B^*(e^{-j\omega})H_R(e^{-j\omega})R'(e^{-j\omega}) = 0 \ ; \ \omega = 2\pi f / f_s \qquad (3.6.8)$$

This results immediately from Equation 3.6.1 since under the condition of Equation 3.6.8 one gets $S_{yp}(j\omega) = 1$.

The specified fixed part of $R(q^{-1})$, denoted $H_R(q^{-1})$, allows one to obtain a null gain for $R(q^{-1})$ at certain frequencies, assuring at these frequencies $|S_{yp}(e^{-j\omega})| = 1$ (open loop type operation).

For example,

$$H_R(q^{-1}) = 1 + q^{-1}$$

introduces a zero at $0.5 f_s$ implying $|S_{yp}(e^{-j\pi})| = 1$.

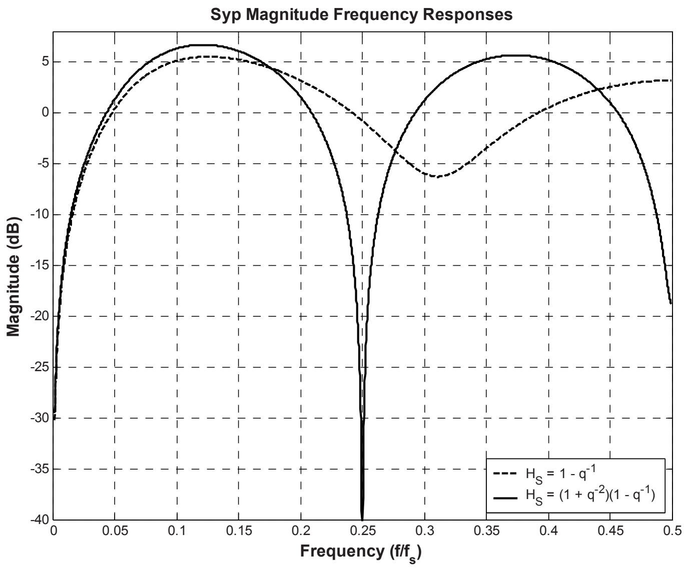

Figure 3.27. Output sensitivity function for the case $H_S(q^{-1}) = (1 - q^{-1})$ and $H_S(q^{-1}) = (1 - q^{-1})(1 + q^{-2})$

$$H_R(q^{-1}) = 1 + \beta q^{-1} + q^{-2}$$

with $\beta = -2cos\ (\omega T_s) = -2cos\ 2\pi f/f_s$ introduces a pair of undamped complex zeros at the normalized frequency f/f_s leading to $|S_{yp}(e^{-j\pi f/fs})| = 1$.

$$H_R(q^{-1}) = 1 + \beta_1 q^{-1} + \beta_2 q^{-2}$$

introduces a pair of complex zeros with a non null damping, allowing to influence the attenuation of the disturbance at a certain frequency.

Figure 3.28 illustrates the effect of the $H_R(q^{-1}) = 1 + q^{-2}$, which introduces a pair of complex zeros with null damping at $f = 0.25\ f_S$. One can see that, in the presence of $H_R(q^{-1})$, one has $|S_{yp}(e^{-j\omega})| = 1\ (0\ dB)$ at this frequency, while, without introducing $H_R(q^{-1})$ one has at $f = 0.25\, f_s$ a gain $|S_{yp}(e^{-j\omega})| = 3\ dB$.

Note also that $R(z^{-1})$ defines some of the zeros of the input sensitivity function $S_{up}(z^{-1})$ (given in Equation 3.6.2). Therefore at the frequencies where $R(z^{-1}) = 0$, this sensitivity function will be null.

Property 6

The introduction of asymptotically stable auxiliary poles $P_F(z^{-1})$ leads in general to the reduction of $|S_{yp}(z^{-1})|$ in the attenuation band of $1/P_F(z^{-1})$.

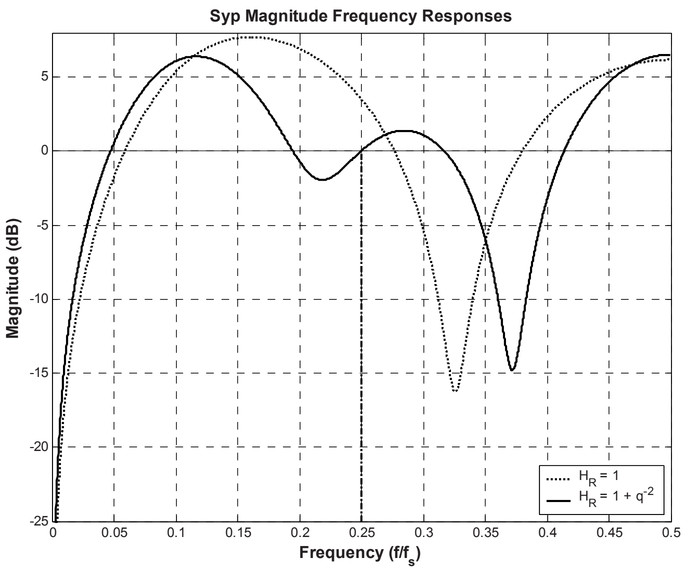

Figure 3.28. Output sensitivity function for the case $H_R(q^{-1}) = 1$ and $H(q^{-1}) = 1 + q^{-2}$

From the expressions of the sensitivity function in Equation 3.6.1 and of the closed loop poles, one can see that $1/(P_D(z^{-1})\ P_F(z^{-1}))$ will introduce a stronger attenuation in the frequency domain than $1/P_D(z^{-1})$, provided that the auxiliary poles defined by $P_F(z^{-1})$ are asymptotically stable aperiodic poles. However, since $S'(z^{-1})$ will depend upon these poles through Equation 3.6.4, this property cannot be guaranteed for all possible values of $P_F(z^{-1})$.

The auxiliary poles are in general selected as real poles located at high frequencies, and they take the form

$$P_F(q^{-1}) = (1 + p'q^{-1})^{n_{P_F}} \qquad -0.5 \le p' \le -0.05$$

where

$$n_{P_F} \le n_P - n_{P_D} \quad ; \quad n_P = (deg\ P)_{max} \quad ; \quad n_{P_D} = deg\ P_D$$

The effect of auxiliary poles is illustrated in Figure 3.29.

Remark:in many applications, the introduction of high frequency auxiliary poles is enough in order to assure the imposed robustness margins.

Property 7
Simultaneous introduction of a fixed part H_{S_i} *and of a pair of auxiliary poles* P_{F_i} *in the form*

$$\frac{H_{S_i}(z^{-1})}{P_{F_i}(z^{-1})} = \frac{1+\beta_1 z^{-1} + \beta_2 z^{-2}}{1+\alpha_1 z^{-1} + \alpha_2 z^{-2}} \tag{3.6.9}$$

resulting from the discretization of the continuous-time filter

$$F(s) = \frac{s^2 + 2\zeta_{num}\omega_0 s + \omega_0^2}{s^2 + 2\zeta_{den}\omega_0 s + \omega_0^2} \tag{3.6.10}$$

using the bilinear transformation[11]

$$s = \frac{2}{T_s}\frac{1-z^{-1}}{1+z^{-1}} \tag{3.6.11}$$

---

[11] The bilinear transformation assures a better approximation of a continuous-time model by a discrete-time model in the frequency domain than the replacement of differentiation by a difference, *i.e.* $s = (1-z^{-1})\ T_S$ (see Equations 2.3.6 and 2.5.2).

introduces an attenuation (a "hole") at the normalized discretized frequency

$$\omega_{disc} = 2\arctan\left(\frac{\omega_0 T_s}{2}\right) \tag{3.6.12}$$

as a function of the ratio $\zeta_{num} / \zeta_{den} < 1$. *The attenuation at* ω_{disc} *is given by*

$$M_t = 20\log\left(\frac{\zeta_{num}}{\zeta_{den}}\right) \quad ; \qquad (\zeta_{num} < \zeta_{den}) \tag{3.6.13}$$

The effect upon the frequency characteristics of S_{yp} *at frequencies* $f << f_{disc}$ *and* $f >> f_{disc}$ *is negligible.*

Figure 3.30 illustrates the effect of the simultaneous introduction of a fixed part H_S and a pair of poles in P, corresponding to the discretization of a resonant filter of the form of Equation 3.6.10. One observes its weak effect on the frequency characteristics of S_{yp} , far from the resonance frequency of the filter.

This pole-zero filter is essential for an accurate shaping of the modulus of the sensitivity functions in the various frequency regions in order to satisfy the constraints. It allows one to reduce the interaction between the tuning in different regions.

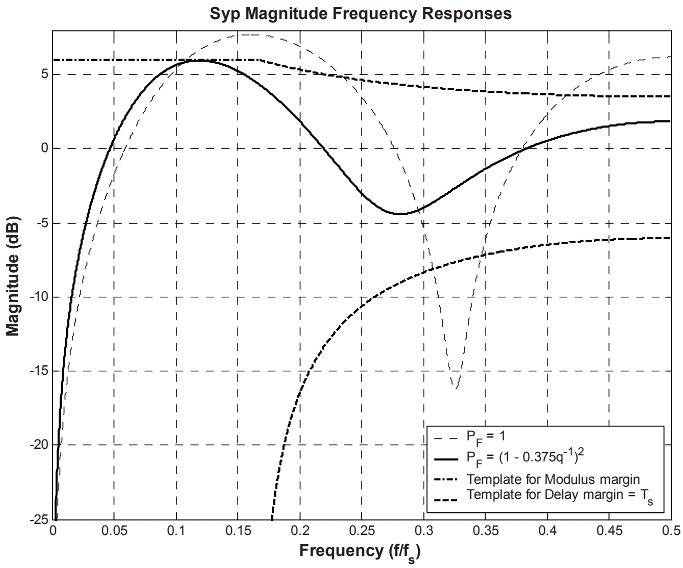

Figure 3.29. Effects of auxiliary poles on the output sensitivity function

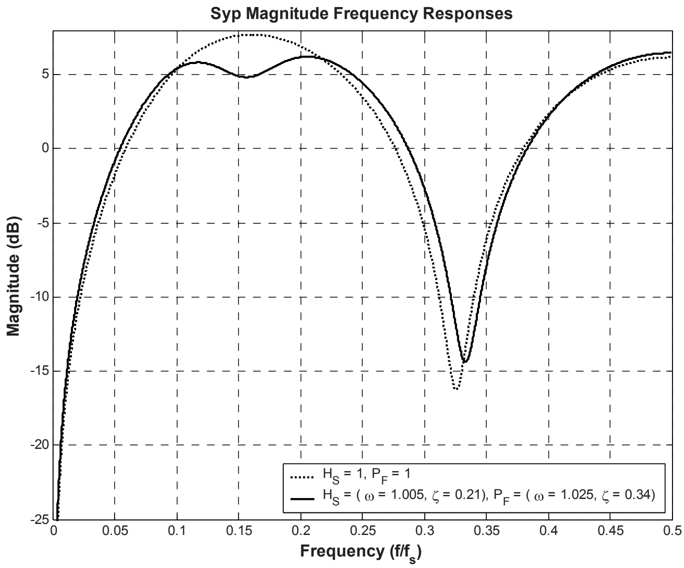

Figure 3.30. Effects of a resonant filter H_{S_i} / P_{F_i} on the output sensitivity functions.

Design of the Resonant Pole-Zero Filter H_{S_i} / P_{F_i}

The computation of the coefficients of H_{S_i} and P_{F_i} is done in the following way:

Specifications:

- Central normalized frequency f_{disc} ($\omega_{disc} = 2\pi f_{disc}$)
- Desired attenuation at frequency f_{disc}: M_t dB
- Minimum accepted damping for auxiliary poles

$P_{F_i} : (\zeta_{den})_{min}$ (≥ 0.3)

Step I: Design of the continuous-time filter

$$\omega_0 = \frac{2}{T_s} \tan\left(\frac{\omega_{disc}}{2}\right) \qquad 0 \leq \omega_{disc} \leq \pi \qquad \zeta_{num} = 10^{M_t/20} \zeta_{den}$$

Step II: Design of the discrete-time filter using the bilinear transformation of Equation 3.6.11.

Using Equation 3.6.11 one gets:

$$F(z^{-1}) = \frac{b_{z0} + b_{z1} z^{-1} + b_{z2} z^{-2}}{a_{z0} + a_{z1} z^{-1} + a_{z2} z^{-2}} = \gamma \frac{1 + \beta_1 z^{-1} + \beta_2 z^{-2}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}} \tag{3.6.14}$$

which will be effectively implemented as[12]

$$F(z^{-1}) = \frac{H_S(z^{-1})}{P_i(z^{-1})} = \frac{1+\beta_1 z^{-1} + \beta_2 z^{-2}}{1+\alpha_1 z^{-1} + \alpha_2 z^{-2}}$$

where the coefficients are given by

$$\begin{aligned}
b_{z0} &= \frac{4}{T_s^2} + 4\frac{\zeta_{num}\omega_0}{T_s} + \omega_0^2 \;\; ; \;\; b_{z1} = 2\omega_0^2 - \frac{8}{T_s^2} \\
b_{z2} &= \frac{4}{T_s^2} - 4\frac{\zeta_{num}\omega_0}{T_s} + \omega_0^2 \\
a_{z0} &= \frac{4}{T_s^2} + 4\frac{\zeta_{den}\omega_0}{T_s} + \omega_0^2 \;\; ; \;\; a_{z1} = 2\omega_0^2 - \frac{8}{T_s^2} \\
a_{z2} &= \frac{4}{T_s^2} - 4\frac{\zeta_{den}\omega_0}{T_s} + \omega_0^2
\end{aligned} \tag{3.6.15}$$

$$\begin{aligned}
\gamma &= \frac{b_{z0}}{a_{z0}} \\
\beta_1 &= \frac{b_{z1}}{b_{z0}} \;\; ; \;\; \beta_2 = \frac{b_{z2}}{b_{z0}} \\
\alpha_1 &= \frac{a_{z1}}{a_{z0}} \;\; ; \;\; \alpha_2 = \frac{a_{z2}}{a_{z0}}
\end{aligned} \tag{3.6.16}$$

The resulting filters H_{S_i} and P_{F_i} can be characterized by the undamped resonance frequency ω_0 and the damping ζ. Therefore, first we will compute the roots of numerator and denominator of $F(z^{-1})$. One gets

$$\begin{aligned}
z_{n1,2} &= \frac{-\beta_1 \pm j\sqrt{4\beta_2 - \beta_1^2}}{2} = A_n e^{j\varphi_n} \\
z_{d1,2} &= \frac{-\alpha_1 \pm j\sqrt{4\alpha_2 - \alpha_1^2}}{2} = A_d e^{j\varphi_d}
\end{aligned} \tag{3.6.17}$$

From Table 2.4 and expressions given in Section 2.3.8, one can establish the relation between the filter and the undamped resonance frequency and damping of an equivalent continuous-time filter (discretized with a ZOH). The roots of the second-order monic polynomial in z^{-1} have the expression

$$z_{1,2} = e^{-\zeta_{disc}\omega_{0disc}T_s} e^{\pm j\omega_{0disc}T_s\sqrt{1-\zeta_{disc}^2}} \tag{3.6.18}$$

[12] The factor γ has no effect on the final result (coefficients of *R* and *S*). It is possible, however, to implement the filter without normalizing the numerator coefficients.

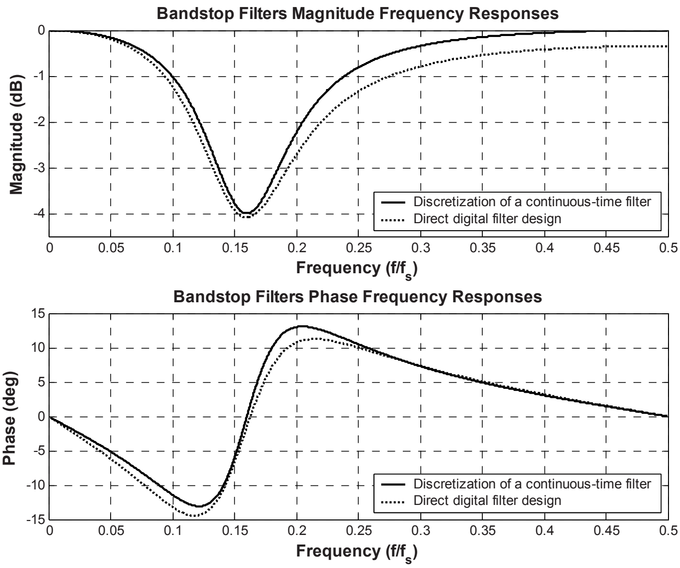

Figure 3.31. Frequency characteristics of the resonant filter H_S / P_F used in the example presented in Figure 3.30

One gets therefore for the numerator and denominator of $F(z^{-1})$

$$
\begin{aligned}
\omega_{0num} &= \sqrt{\frac{\varphi_n^2 + \ln^2 A_n}{T_s^2}} \quad ; \quad \zeta_{numd} = -\frac{\ln A_n}{\omega_{0num} T_s} \\
\omega_{0den} &= \sqrt{\frac{\varphi_d^2 + \ln^2 A_d}{T_s^2}} \quad ; \quad \zeta_{dend} = -\frac{\ln A_d}{\omega_{0den} T_s}
\end{aligned}
\tag{3.6.19}
$$

where the indexes "num" and "den" correspond to H_S and P_F, respectively. These filters can be computed using the functions *filter22.sci* (Scilab) *filter22.m* (MATLAB®) and also with *ppmaster* (MATLAB®)[13].

Remark: for frequencies below 0.17 f_S the design can be done with a very good precision directly in discrete-time. In this case, $\omega_0 = \omega_{0,den} = \omega_{0,num}$ and the damping of the discrete time filters H_{S_i} and P_{F_i} is computed as a function of the attenuation directly using Equation 3.6.13.

Figure 3.31 gives the frequency characteristics of a filter H_S / P_F obtained by the discretization of a continuous-time filter and used in Figure 3.30 (continuous line) as well as the characteristics of the discrete-time filter directly designed in discrete time (dashed line). The continuous-time filter is characterized by a natural

[13] To be download from the book website (*http://landau-bookic.lag.ensieg.inpg.fr*).

frequency $\omega_0 = 1$ rad/s $(f_0 = 0.159\, f_s)$, and dampings $\zeta_{num} = 0.25$ and $\zeta_{den} = 0.4$. The same specifications have been used for a direct design in discrete-time. One observes a small difference at high frequencies but this is not very significant. The differences will become obviously more important as ω_0 increases.

Remark: while H_S *is effectively implemented in the controller,* P_F *is only used indirectly.* P_F *will be introduced in Equation 3.6.5 and its effect will be reflected in the coefficients of R and S obtained as solutions of Equation 3.6.5.*

3.6.2 Properties of the Input Sensitivity Function

Property 1
Cancellation of the disturbance effect on the input (i.e. $S_{up}= 0$*) is obtained at frequencies where*

$$A(e^{-j\omega})H_R(e^{-j\omega})R'(e^{-j\omega}) = 0 \quad ; \quad \omega = 2\pi\, f / f_s \tag{3.6.20}$$

At these frequencies $S_{yp}= 1$ (see Property 5 of the output sensitivity function) and the system operates in open loop.

Figure 3.32 illustrates the effect of $H_R(q^{-1})$ on $| S_{up} |$ for

$$H_R(q^{-1}) = 1 + \beta q^{-1} \qquad 0 < \beta \le 1 \tag{3.6.21}$$

for $\beta=1$, one has $| S_{up} | = 0$ at $0.5 f_s$. Using $0< \beta <1$ one could reduce more or less the modulus of S_{up} around $0.5\ f_s$. Note that this structure for $H_R(q^{-1})$ is systematically used for reducing the modulus of the input sensitivity function at high frequencies.
One can also use

$$H_R(q^{-1}) = (1 + \beta q^{-1})^n \qquad 0 < \beta \le 1$$

(usually $n=1$ or 2).

Property 2
At frequencies where

$$A(e^{-j\omega})H_S(e^{-j\omega})S'(e^{-j\omega}) = 0 \quad ; \quad \omega = 2\pi\, f / f_s \tag{3.6.22}$$

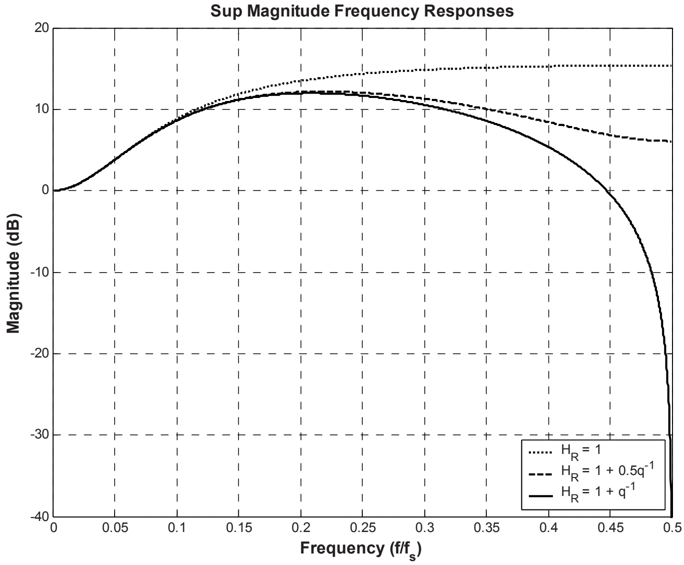

Figure 3.32. Effect of the filter $H_R(q^{-1}) = 1 + \beta q^{-1} \quad 0 < \beta \leq 1$ on the modulus of the input sensitivity function: a) $\beta = 0$; b) $\beta = 0.5$; c) $\beta = 1$

which correspond to frequencies where a perfect rejection of disturbances is achieved (S_{yp}= 0), one has

$$\left|S_{up}(e^{-j\omega})\right| = \left|\frac{A(e^{-j\omega})}{B(e^{-j\omega})}\right| \tag{3.6.23}$$

Equation 3.6.23 corresponds to the inverse of the gain of the system to be controlled. The implication of Equation 3.6.23 is that *cancellation (or in general an important attenuation) of disturbances on the output should be done only in frequency regions where the system gain is large enough*. If the gain of the controlled system is to low, $|S_{up}|$ will be large at these frequencies. Therefore, the robustness *vs* additive term uncertainties will be reduced and the stress on the actuator will become important. Equation 3.6.23 also implies that serious problems will occur if $B(z^{-1})$ has complex zeros close to the unit circle (stable or unstable zeros) at frequencies where an important attenuation of disturbances is required. It is mandatory to avoid attenuation of disturbances at these frequencies

Property 3
Simultaneous introduction of a fixed part H_{R_i} and of a pair of auxiliary poles P_{F_i} having the form

$$\frac{H_{R_i}(z^{-1})}{P_{F_i}(z^{-1})} = \frac{1 + \beta_1 z^{-1} + \beta_2 z^{-2}}{1 + \alpha_1 z^{-1} + \alpha_2 z^{-2}}$$

resulting from the discretization of the continuous time filter

$$F(s) = \frac{s^2 + 2\zeta_{num}\omega_0 s + \omega_0^2}{s^2 + 2\zeta_{den}\omega_0 s + \omega_0^2}$$

using the bilinear transformation

$$s = \frac{2}{T_s}\frac{1 - z^{-1}}{1 + z^{-1}}$$

introduces an attenuation (a " hole") at the normalized discretized frequency

$$\omega_{disc} = 2\arctan\left(\frac{\omega_0 T_s}{2}\right)$$

as a function of the ratio $\zeta_{num} / \zeta_{den} < 1$.
The attenuation at ω_{disc} is given by

$$M_t = 20\log\left(\frac{\zeta_{num}}{\zeta_{den}}\right) \quad ; \quad (\zeta_{num} < \zeta_{den})$$

The effect on the frequency characteristics of S_{yp} at frequencies $f << f_{disc}$ and $f >> f_{disc}$ is negligible.

The design of these filters is done with the method described in Section 3.6.1.

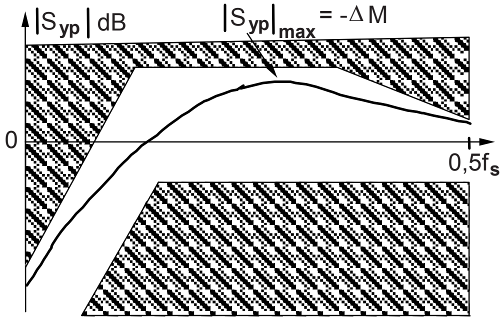

Figure 3.33. Desired template for the output sensitivity function (case of disturbances rejection at low frequencies)

3.6.3 Definition of the "Templates" for the Sensitivity Functions

The performance and robustness specifications lead to the definition of desired "templates" for the frequency characteristics of the sensitivity functions. Most often we are interested in attenuating the disturbances at low frequencies, while assuring certain values for the modulus and delay margins, in order to achieve both good robustness and low amplification of disturbances outside the attenuation band. Figure 3.33 shows a typical template for the modulus of the output sensitivity function.

This template is defined by accepted maximum values for the modulus of the output sensitivity function ("upper template"). One can also consider introducing a "lower template" since on one hand a too large attenuation in a certain frequency region will induce a too important increase of the modulus of the sensitivity function at other frequency regions (and $|S_{yp}|_{max}$ will increase) and, on the other hand, the delay margin requires a "lower template" to be considered.

However, more complex templates will result if, for example, the attenuation should occur in several frequency regions and if, in addition, it is required that the system operate in open loop at a specified frequency (neither amplification, nor attenuation of the disturbance). Such a type of template is shown in Figure 3.34.

The constraints on robustness and on the actuator also lead to the definition of an upper template for the modulus of the input sensitivity function S_{up}.

Recall first that the inverse of the modulus of S_{up} gives at each frequency the tolerated additive uncertainty for preserving the stability of the loop (see Chapter 2, Section 2.6). Therefore, *in the frequency regions where there are uncertainties upon the plant model used for the design, one should impose a very low acceptable value for the modulus of the input sensitivity function* S_{up}.

We recall that opening the loop ($S_{up}= 0$) may be required at specified frequencies either because the system should not react to a particular disturbance (see Section 3.6.6), or because one would not like to excite the plant at certain frequencies (for example the plant is characterized by high frequency vibration modes).

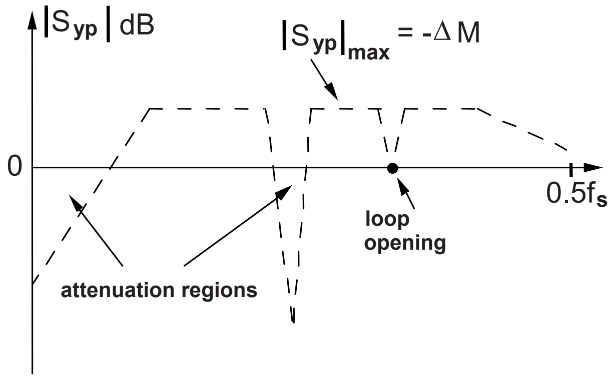

Figure 3.34. Desired templates for the modulus of the output sensitivity function (case of disturbances rejection at certain frequencies and open loop operation required at a specified frequency)

Moreover, for robustness reasons and insensitivity to noise at high frequencies, one systematically reduces the modulus of S_{up} close to $0.5\,f_s$ (often by opening the loop at $0.5\,f_s$).

The limitation of the actuator stress in certain frequency regions (often at high frequencies) induces a limitation of acceptable values of the modulus of S_{up} in these frequency regions.

An example of a template for $|S_{up}|$ is given in Figure 3.35.

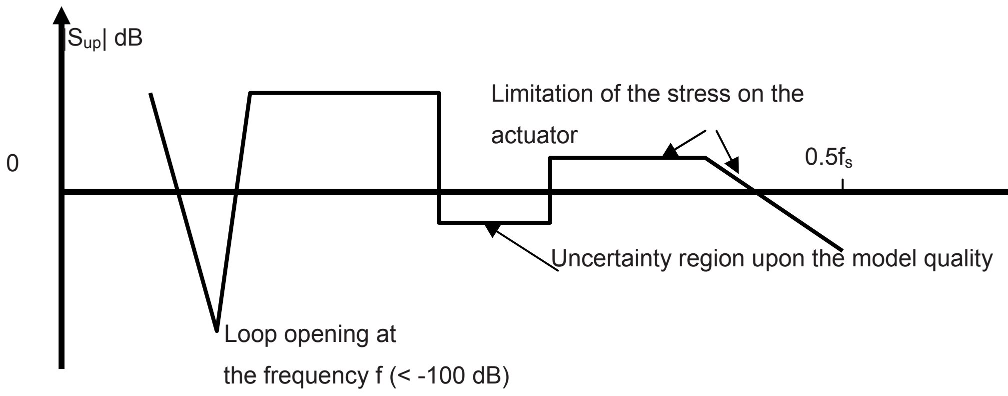

Figure 3.35. Example of a desired template for the modulus of the input sensitivity function

3.6.4 Shaping of the Sensitivity Functions

In order to achieve the shaping of the modulus of the sensitivity functions such that the constraints defined by the template are satisfied, one has the following possibilities:

1. Selection of the desired dominant and auxiliary poles of the closed loop
2. Selection of the fixed parts of the controller ($H_R(q^{-1})$ and $H_S(q^{-1})$)
3. Simultaneous selection of the auxiliary poles and of the fixed parts of the controller

Despite the coupling between the shaping of the output sensitivity function and the input sensitivity function, it is possible, in practice, to deal with these two problems almost independently.

In particular, it is useful to remember that at frequencies where $|S_{yp}|$ is close to 1 (0dB), $|S_{up}|$ is close to 0 (<-60dB).

Automatic methods using convex optimization techniques (Langer and Landau 1999; Langer and Constantinescu 1999; Adaptech 1998a) are available in order to solve the shaping problem[14]. However, in most of practical situations, it is relatively easy to calibrate the sensitivity functions using the tools previously indicated, and taking into account the properties of the sensitivity functions.

The iterative procedure that will be presented next is very efficient and helps to understand the operation of the controller. However, the use of a CACSD tool like *ppmaster*[15] (MATLAB®) (Prochazka and Landau 2001) will considerably accelerate the procedure.

Shaping the Output Sensitivity Function.

The shaping objective can be summarized as follows: given a desired attenuation band, select the closed loop poles and the fixed parts $H_R(q^{-1})$ and $H_S(q^{-1})$ in order to "flatten" $|S_{yp}(e^{-j\omega})|$ outside the attenuation band, and to reduce $|S_{yp}(e^{-j\omega})|_{max}$ while satisfying Property 2.

Often the selection of auxiliary poles and of $H_R(q^{-1})$ and $H_S(q^{-1})$ is sufficient for matching the performance and robustness specifications. However, for a fine tuning, it may be also necessary to select simultaneously the fixed part $H_S(q^{-1})$ and the auxiliary poles. (see Property 7: second-order pole-zero filter).

The iterative procedure for the shaping of the output sensitivity function can be summarized as follows[16]:

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[14] To apply convex optimization techniques, the Youla-Kucera parametrization of the controller is used. See Appendix E.

[15] Available on the book website (*http://landau-bookic.lag.ensieg.inpg.fr*).

[16] The principles of the methodology for shaping the output sensitivity function can be extended to the shaping of other sensitivity functions.

Step I

- Selection of $P(q^{-1})$ and fixed parts $H_R(q^{-1})$ and $H_S(q^{-1})$ from the performance specifications. (Example: zero steady-state error will require the introduction of $H_S(q^{-1}) = 1 - q^{-1}$, opening the loop at a certain frequency will require $H_R(q^{-1}) = 1 + \alpha q^{-1} + q^{-2}$)
- Controller computation
- Analysis of the resulting output sensitivity function.

If the upper template corresponding to the desired modulus margin $\Delta M = 0.5$ and the delay margin $\Delta\tau = T_s$ are violated, one can distinguish three different solutions:

1. The maximum of the modulus of the sensitivity function is at high frequencies (the resulting controller is often unstable as a consequence of Property 2, since the area between the modulus of the sensitivity function and the $0\ dB$ axis is positive).
2. A local maximum of the modulus of the sensitivity function is located in a frequency region close to the attenuation band.
3. The modulus of the sensitivity function presents a maximum both at low and high frequencies.

Case 1 and 3

Step II
One introduces auxiliary poles at high frequencies:

$$P_F(q^{-1}) = (1 + p_1 q^{-1})^{n_F} \quad ; \quad - 0.05 \geq p_1 \geq - 0.5$$

with increasing values of $|p_1|$ starting from 0.05.

The number of auxiliary poles that can be introduced without increasing the size of the controller is given by

$$n_{P_F} \leq n_P - n_{P_D} \; ; \quad n_P = (\deg P)_{max} \; ; \; n_{P_D} = \deg P_D$$

The value of $|p_1|$ is increased in order to avoid a violation of the template of the sensitivity function at high frequencies (*i.e.* over $0.25 f_s$ or $0.30 f_s$).

Usually, the maximum of the modulus of the sensitivity function will move towards low frequencies. If this maximum is above the template we are in case 2 which will be discussed next.

Case 2

Step II – Use of second-order pole-zero resonant filters H_{S_i} / P_i.

One identifies first the frequency region where $|S_{yp}|$ is above the "upper template". One finds the value of the frequency at which the maximum of $|S_{yp}|$ occurs and the attenuation which should be introduced in order to be below the upper template. Then one computes the corresponding filter H_{S_i}/P_i on the basis of these specifications. The selection of the damping of the denominator depends to some extent upon the width of the frequency region where $|S_{yp}|$ is above the template.

Some Hints

- If the modulus of the sensitivity function presents a too strong minimum at high frequencies, or in the central frequency region, this can be raised by a few dB by adding a pair of complex auxiliary poles placed at the frequency corresponding to the minimum
- If the attenuation band increases, then one should impose slower dominant poles
- If the attenuation band is reduced as a consequence of the introduction of the auxiliary poles, then one can either raise the natural frequency of the dominant poles, or introduce a further zero-pole filter around the frequency corresponding to the desired attenuation band.

In order to match the constraints in the high frequency region corresponding to the imposed delay margin, one can either use auxiliary poles or impose the opening of the loop at $0.5\,f_S$ ($H_R(q^{-1}) = 1 + q^{-1}$). In practice, a simultaneous application of both solutions can be considered.

Step III
One iterates around the values of the dominant and auxiliary poles selected in step I and II in order to get the best result in terms of robustness and performance constraints matching.

Shaping the Input Sensitivity Function
The shaping of the input sensitivity function is usually performed after the shaping of the output sensitivity function. If the value of the modulus of the input sensitivity function is not enough low at high frequencies (frequencies close to $0.5\,f_S$), one introduces a fixed part $H_R(q^{-1})$ of the form

$$H_R(q^{-1}) = (1+\beta q^{-1})^n \qquad 0.5 < \beta < 1 \qquad n = 1,2$$

This will have, in general, a weak interaction with the shaping of the output sensitivity function.

In the frequency regions where there are uncertainties upon the plant model, $|S_{up}|$ should be low (upper bound defined by the template). If the resulting sensitivity function does not fulfill the specifications, one has to introduce a pole-zero resonant filter H_{R_i}/P_{F_i}. In order to match the template imposed on $|S_{up}|$ the resonance frequency of the filter is chosen close to that corresponding to the local maximum of $|S_{up}|$, and an appropriate damping is selected in order to provide the required attenuation.

Once the shaping of $|S_{up}|$ is achieved, one should go back to check if $|S_{yp}|$ still match the template (an appropriate CACSD like *ppmaster* allows to shape simultaneously both sensitivity functions).

3.6.5 Shaping of the Sensitivity Functions: Example 1

The considered plant model is characterized by

$$A(q^{-1}) = 1 - 0.7\, q^{-1} \quad ; \quad B(q^{-1}) = 0.3\, q^{-1} \quad ; \quad d = 2 \quad ; \quad T_s = 1s$$

An integrator is imposed in the controller. One considers the roots of the polynomial obtained from the discretization of a second-order continuous time system with $\omega_0 = 1\ rad/s$, and $\zeta = 0.9$ as the imposed dominant poles for the closed loop. The controller is designed by means of *pole placement.* The output sensitivity function corresponding to this design is shown in Figure 3.37 (curve A). It crosses the standard robustness template defined by $\Delta M = -6dB$ and $\Delta\tau = T_s$. Both obtained margins (modulus and delay) are lower than specifications (see Table 3.5). The upper frequency of the attenuation band is at 0.058 Hz.

The objective is to obtain the same attenuation band but with

$$\Delta M \geq -6dB \ \ (|S_{yp}|_{max} \leq 6dB) \ \ and \ \ \Delta\tau \geq T_s.$$

First we will add auxiliary real poles. From Equations 3.3.22 and 3.3.23 it results that one can assign a number of closed loop poles equal to:

$$n_P = deg\, P(q^{-1}) \leq n_A + n_{HS} + n_B + n_{HR} + d - 1$$

without increasing the size of the controller. Since in this example $n_A = 1$, $n_B = 1$, $n_{HS} = 1$, $d = 2$, one can assign four poles. Two poles have been already assigned as dominant poles and therefore we will add two auxiliary poles of the form

$$P_F(q^{-1}) = (1 - 0.4\, q^{-1})^2$$

The resulting sensitivity function is shown in Figure 3.36 (curve B). The constraints on the modulus margin and the delay margin are satisfied but the attenuation band has been reduced (0.045 Hz instead of 0.058 Hz).

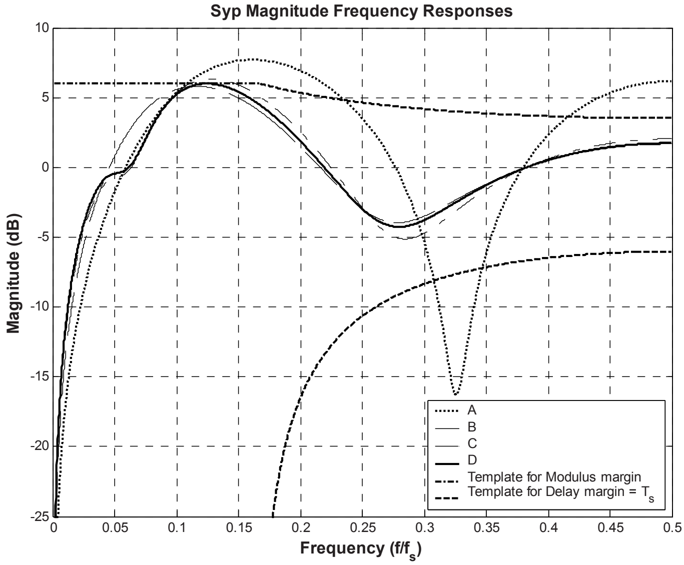

Figure 3.36. Output sensitivity functions corresponding to different RST controllers (see details in Table 3.5)

In order to increase the attenuation in this frequency region without influencing high frequency regions, one chooses a pole-zero filter H_S/P'_F centered at $\omega_0=0.4$ *rad/s* (*f=0.064 Hz*) with H_S resulting from the discretization of a second-order continuous-time system with $\omega_0 = 0.4$ *rad/s* and $\zeta = 0.3$, and P'_F resulting from the discretization of a second-order continuous-time system with $\omega_0 = 0.4$ *rad/s* and $\zeta = 0.5$ (we are below $0.17\, f_S$ and the direct design of the discrete filter is possible). Since $\zeta_{num} = 0.3$ and $\zeta_{den} = 0.5$, an attenuation will result at 0.064 Hz. The corresponding sensitivity function is shown in Figure 3.36 (curve C). One observes that the attenuation band matches the specifications but the maximum value of $|S_{yp}|$ is slightly higher than 6 dB. In order to match this constraint the value of the real auxiliary pole is increased from 0.4 to 0.44. The final sensitivity function is shown in Figure 3.36 (curve D).

3.6.6 Shaping of the Sensitivity Functions: Example 2[17]

The considered plant model is characterized by

$$A(q^{-1}) = 1 - q^{-1} \quad ; \quad B(q^{-1}) = 0.5\, q^{-1} \quad ; \quad d = 2 \quad ; \quad T_S = 1s$$

[17] This example and the corresponding specifications are related to the continuous casting of steel.

Table 3.9. Shaping of sensitivity functions- example 1

	$H_S(q^{-1})$	Closed loop poles		Modulus margin (dB)	Delay margin (Ts)	Attenuation band (Hz)
		Dominant	Auxiliary			
A	$1 - q^{-1}$	$\omega_0 = 1$ $\zeta = 0.9$	---	-7.71	0.4	0.058
B	$1 - q^{-1}$	$\omega_0 = 1$ $\zeta = 0.9$	$(1 - 0.4q^{-1})^2$	-5.81	3.07	0.045
C	$1 - q^{-1}$ $\omega_0=0.4$ $\zeta=0.3$	$\omega_0 = 1$ $\zeta = 0.9$	$(1-0.4q^{-1})^2$ $\omega_0=0.4$ $\zeta=0.5$	-6.33	5.01	0.063
D	idem	$\omega_0 = 1$ $\zeta = 0.9$	$(1-0.44q^{-1})^2$ $\omega_0=0.4$ $\zeta=0.5$	-5.99	5.34	0.060

The plant has an integrator behavior with delay. The plant is subject to two types of disturbances:

- Low frequency disturbances that should be attenuated
- A sinusoidal disturbance at 0.25 Hz that should not be compensated by the action of the controller

The block diagram related to the plant model is shown in Figure 3.37.

The robustness and performance specifications are as follows:

1. No attenuation of the sinusoidal disturbance at $f = 0.25\ Hz$ ($|S_{yp}| = 0\ dB$ at $0.25\ Hz$)
2. Attenuation band at low frequencies (frequency region corresponding to $|S_{yp}| < 0dB$): 0 to 0.03 Hz
3. Disturbances amplification at 0.07 Hz less than 3 dB ($|S_{yp}| < 3\ dB$ at $0.07\ Hz$)
4. Modulus margin ≥ -6 dB ($|S_{yp}|_{max} \leq 6\ dB$)
5. Delay margin ≥ 1s (T_s)
6. No integrator in the controller

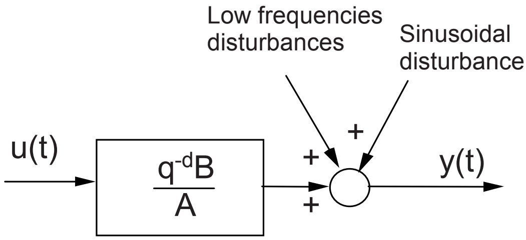

Figure 3.37. Block diagram for plant model and disturbances

The presence of an integrator has not been imposed because the plant model has already an integrator behavior.

One starts by designing the fixed parts of the controller, $H_R(q^{-1})$ and $H_S(q^{-1})$. In order to obtain $|S_{yp}| = 0dB$ at $0.25\ Hz = 0.25\ f_s$, one should introduce $H_R(q^{-1})$ with a pair of undamped complex zeros at this frequency. This is obtained by

$$H_R(q^{-1}) = 1 + \beta q^{-1} + q^{-2}$$

with

$$\beta = -2\cos\,(\omega T_s) = -2\cos\,(2\pi f/f_s)$$

From the condition

$$H_R(e^{-j\omega})\Big|_{\omega=\omega_s/4} = 0$$

it results that

$$\beta = -2\cos\,(2\pi .\ 0.25) = 0$$

and then

$$H_R(q^{-1}) = 1 + q^{-2}$$

Since there is no integrator imposed in the controller, as a first choice one chooses $H_S(q^{-1}) = 1$. One selects for the dominant poles a polynomial $P(q^{-1})$ resulting from the discretization of a second-order continuous-time system with $\omega_0 = 0.628\ rad/s$ ($f_0 = 0.1\ Hz$) and $\zeta = 0.9$.

The corresponding sensitivity function is shown in Figure 3.38 (curve A).

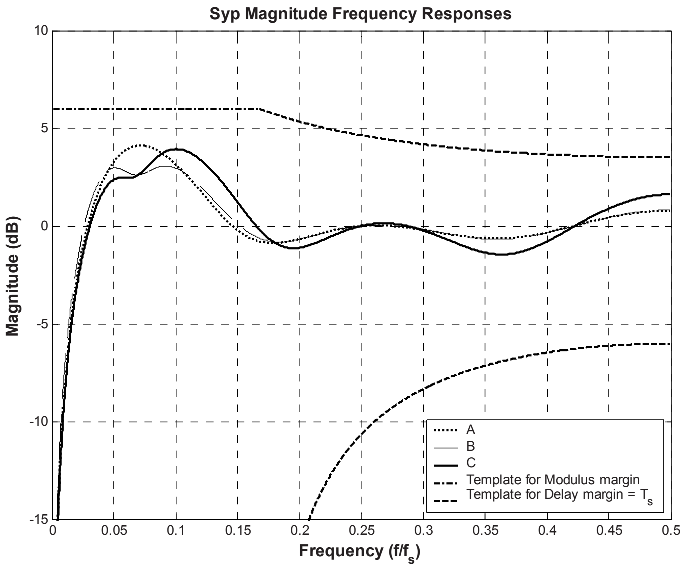

Figure 3.38. Output sensitivity functions corresponding to different RST controllers (see details in Table 3.6)

One observes that the specifications at 0.07 Hz are not satisfied. On the other hand one can remark that $|S_{yp}|$ is equal to 1 (0dB) for $f = 0.25\, f_s = 0.25\ Hz$, as required.

In order to reduce the modulus of the output sensitivity function at $f = 0.07\ Hz$, one considers the introduction of a narrow-band pole-zero discrete filter $H_S\,/P_F$ centered at $\omega_0 = 0.44\ rad/s$ ($f_0 = 0.07\ Hz$). Since the frequency f_0 is below $0.17\, f_s$, one can directly design the discrete-time filter. One chooses for H_S a damping $\zeta = 0.3$ and for P_F a damping $\zeta=0.4$, corresponding to a filter attenuation of about 2.4 dB around ω_0. Using this filter (that is H_S and P_F) one obtains the sensitivity function shown in Figure 3.38 (curve B). The attenuation introduced by this filter at 0.07 Hz is sufficient ($|S_{yp}|_{0.07Hz} = 2.6\ dB$) but the width of the attenuation band is smaller than the specified one. In order to correct that, it is sufficient to increase the natural frequency of the dominant poles. With $\omega_0 = 0.9\ rad/s$ (instead of 0.628) one obtains the sensitivity function shown in Figure 3.38 (curve C). All the specifications are fulfilled. The results are summarized in Table 3.10.

Note that the choice of $H_S\,/P_F$ is not critical. The specifications can also be satisfied with a discrete-time filter $H_S\,/P_F$ centered at $\omega_0 = 0.5\ rad/s$ with $\zeta_{num}=0.35$ and $\zeta_{den}=0.45$.

The same design steps will be followed for the case when an integrator is introduced in the controller.

Table 3.10. Shaping of the sensitivity functions - Example 2

	H_R	H_S	Closed loop poles		Attenuation band (Hz)	Modulus margin (dB)	Delay margin (Ts)	$\lvert S_{yp} \rvert$ 0.07 Hz (dB)
			Dominants	Auxiliaries				
A	$1+q^{-2}$	---	$\omega_0 = 0.628$ $\zeta = 0.9$		0.03	-4.12	6.52	4.11
B	$1+q^{-2}$	$\omega_0 = 0.44$ $\zeta = 0.3$	$\omega_0 = 0.628$ $\zeta = 0.9$	$\omega_0 = 0.44$ $\zeta = 0.4$	0.026	-3.06	7.61	2.6
C	$1+q^{-2}$	idem	$\omega_0 = 0.9$ $\zeta = 0.9$		0.03	-3.94	6.62	2.6

3.7 Concluding Remarks

In this chapter several digital control designs in a deterministic environment have been presented. All the digital controllers have a three-branched structure (RST), corresponding to a control law of the form

$$S(q^{-1})\, u(t) + R(q^{-1})\, y(t) = T(q^{-1})\, y^*(t+d+1)$$

where u is the control input, y is the output of the plant, y^* is the desired tracking trajectory and d is the time delay.

The design of the controller involves essentially two stages:

1. Computation of the polynomials $S(q^{-1})$ and $R(q^{-1})$ in order to match the desired regulation performances
2. Computation of the polynomial $T(q^{-1})$ in order to approach (or to match) the desired tracking performances

The digital controllers are perfectly suited for the control of plants characterized by high order models with time delay and/or resonant modes. The controller complexity (*i.e.* the degrees of the polynomials $R(q^{-1})$, $S(q^{-1})$, $T(q^{-1})$) depends upon the complexity of the polynomials of the plant model transfer function. The discrete-time plant model used for design can be either obtained directly by an identification technique, or by computation from the continuous-time model.

The *pole placement* control strategy is applied to plants having discrete-time models both with stable and unstable zeros. It makes possible to match the desired regulation performance and the desired tracking performance filtered by the plant zeros.

The *tracking and regulation with independent objectives* allows to match perfectly the desired tracking and regulation performances, but, in order to be applied, the pulse transfer function of the discrete-time plant model must have stable zeros.

The *internal model control* strategy is a particular case of the pole placement in which the plant model poles are chosen as closed loop system poles. This strategy can only be applied to enough damped stable systems. The closed loop dynamics is not faster than the open loop dynamics.

The digital PID controllers are directly designed by *pole placement* on the basis of discrete-time models of the plants to be controlled. This technique can be applied to plants characterized by discrete-time models of lower order ($n_{\max} \leq 2$). Two structures of digital PID controllers have been examined. The difference is only in the choice of the polynomial $T(q^{-1})$. It is advisable to use the choice $T(q^{-1}) = R(1)$ (corresponding to the digital PID 2), which offers better tracking performances. The digital controllers designed by the methods presented in this chapter implement *a predictive control* in the time domain. They implicitly contain a *predictor* of the plant to be controlled.

Independently of the design method used, it is required to verify the *robustness margins* of the closed loop (mainly the modulus margin and the delay margin). Taking into account simultaneously performance and robustness margins may require the shaping of the sensitivity functions (output and input sensitivity functions), by imposing pre-specified fixed parts in the controller and choosing appropriate closed loop system poles.

An iterative methodology for the shaping of the sensitivity functions has been presented in Section 3.6.4. Moreover, it is possible to obtain in an automatic way the controller, fulfilling the various constraints on the sensitivity functions, by means of a convex optimization procedure (see Langer and Landau 1999; Adaptech 1998b).

3.8 Notes and References

For different types of digital PID controllers and their computation by means of the pole placement see:

Åström K.J., Wittenmark B. (1997) Computer Controlled Systems Theory and Design, 3rd edition, Prentice Hall, N.J., U.S.A.

Åström K.J., Hägglund I. (1995) PID Controllers Theory, Design and Tuning, 2nd edition ISA, Research Triangle Park, N.C., U.S.A.

The pole placement and the solution of the Bezout identity is discussed in:

Goodwin G.C., Sin K.S. (1984) Adaptive Filtering Prediction and Control, Prentice-Hall, Englewood Cliffs, N.J.

Kailath T. (1980) Linear systems, Prentice Hall, Englewood Cliffs, N.J.

For the solution of the Bezout equation by means of recursive least squares see:

Lozano R., Landau I.D. (1982) Quasi-direct adaptive control for nonminimum phase systems, Transactions A.S.M.E., Journal of D.S.M.C., vol. 104, n°4, pp. 311-316, December.

For the numerical solutions of the Bezout equation see:

Press W.H., Vetterling W.T., Teukolsky S., Flanery B. (1992) Numerical recipes in C (The art of scientific computing),2nd edition, Cambridge University Press, Cambridge, Mass.

For the links between the pole placement and the optimal control with quadratic criterion see (Åström and Wittemark 1997).

Tracking and regulation with independent objectives is discussed in:

Landau I.D., Lozano R. (1981) Unification of Discrete-Time Explicit Model Reference Adaptive Control Designs, Automatica, vol. 12, pp. 593-611.

The interpretation and the design of RST controllers in the time domain are discussed in appendix B.

Robust control of systems with delay and links with the Smith predictor and the internal model control are discussed in:

Landau I.D. (1995) Robust digital control of systems with time delay (the Smith predictor revisited), Int. J. of Control, vol. 62, pp. 325-347.

For properties of the integral of the sensitivity functions in discrete-time see:

Sung H.K., Hara S. (1988) Properties of sensitivity and complementary sensitivity functions in single-input, single-output digital systems, Int. J. of Cont., vol. 48, n°6, pp. 2429-2439.

For examples about the iterative design methodology using pole placement with shaping of the sensitivity functions see:

Landau I.D., Langer J., Rey D., Barnier J. (1996) Robust control of a 360° flexible arm using the combined pole placement / sensitivity function shaping method, IEEE Trans. on Control Systems Tech., vol. 4, no. 4, pp. 369-383.

Landau I.D., Karimi A. (1998) Robust digital control using pole placement with sensitivity function shaping method, Int. J. of Robust and Nonlinear Control, vol. 8, pp. 191-210.

Prochazka H., LandauI.D. (2003) Pole placement with sensitivity function shaping using 2nd order digital notch filters, Automatica, Vol. 39, 6, pp. 1103-1107.

For an “automatic” solution based on convex optimization for designing robust digital controllers by means of the pole placement with shaping of the sensitivity functions see:

Langer J., Landau I.D. (1999) Combined pole placement / sensitivity function shaping method using convex optimization criteria, Automatica, vol. 35, no. 6, pp. 1111-1120.

Langer J., Constantinescu A. (1999) Pole placement design using convex optimization criteria for the flexible transmission benchmark, European Journal of Control, vol. 5, no. 2-4, pp. 193-207.

Adaptech (1998) Optreg – Software for automated design of robust digital controllers using convex optimization (for MATLAB®), Adaptech, St. Martin d’Hères, France.