Symmetry from Control Flow

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Formal System Analysis, Eindhoven University of Technology

30-10-2025

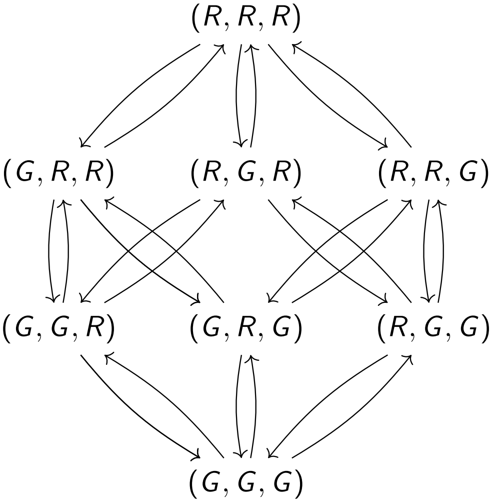
Introduction

$$
\begin{tikzcd}
 & (R, R) \arrow[dl, bend right] \arrow[dr, bend left] & \\
(R, G) \arrow[ur, bend right] \arrow[dr, bend right] & & (G, R) \arrow[ul, bend left] \arrow[dl, bend left] \\
 & (G, G) \arrow[ul, bend right] \arrow[ur, bend left] &
\end{tikzcd}
$$

Introduction

$$\begin{tikzcd}
(R, R) \arrow[d, shift right] \\
(G, R) \arrow[u, shift right] \arrow[d, shift right] \\
(G, G) \arrow[u, shift right]
\end{tikzcd}$$

Introduction



Introduction

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\begin{tikzcd}
 & (R,R,R) \arrow[dl, shift left] \arrow[d, shift left] \arrow[dr, shift left] & & & (R,R,R) \arrow[d, shift left] \\
(G,R,R) \arrow[ur, shift left] \arrow[d, shift left] \arrow[dr, shift left] & (R,G,R) \arrow[u, shift left] \arrow[dl, shift left] \arrow[dr, shift left] & (R,R,G) \arrow[ul, shift left] \arrow[dl, shift left] \arrow[d, shift left] & & (G,R,R) \arrow[u, shift left] \arrow[d, shift left] \\
(G,G,R) \arrow[u, shift left] \arrow[ur, shift left] \arrow[dr, shift left] & (G,R,G) \arrow[ul, shift left] \arrow[ur, shift left] \arrow[d, shift left] & (R,G,G) \arrow[ul, shift left] \arrow[u, shift left] \arrow[dl, shift left] & \longrightarrow & (G,G,R) \arrow[u, shift left] \arrow[d, shift left] \\
 & (G,G,G) \arrow[ul, shift left] \arrow[u, shift left] \arrow[ur, shift left] & & & (G,G,G) \arrow[u, shift left]
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$$

Introduction

Source: https://www.mechdes.nl/wp-content/uploads/2023/06/prinses-marijkesluis.jpg

Introduction

Source: Generated using https://copilot.cloud.microsoft.com

Introduction

This talk: identifying *some* symmetry

Introduction

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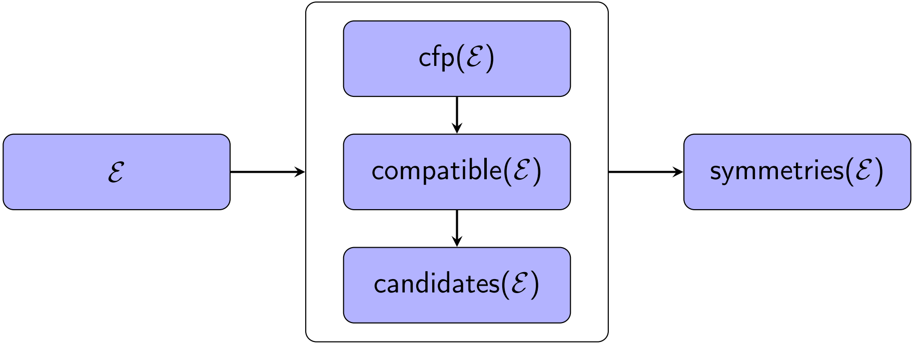
Up front: "Given a structured object X of any sort, a symmetry is a mapping of the object onto itself which preserves (certain parts of) the structure."

Source: https://en.wikipedia.org/wiki/Symmetry_in_mathematics

Overview


$\mathcal{E}$
cfp($\mathcal{E}$)
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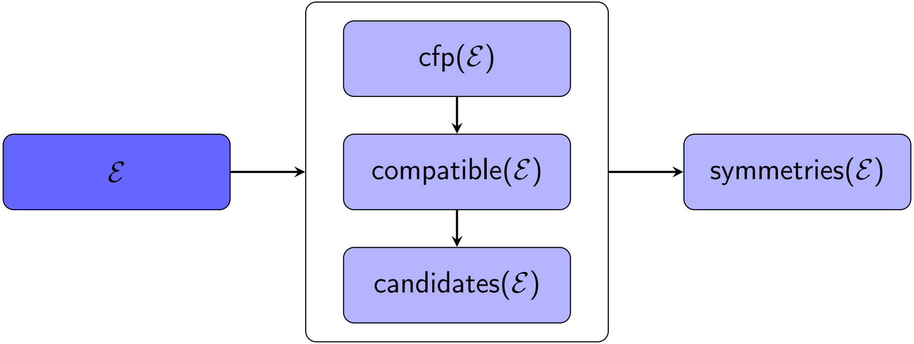
Overview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Parametrized Boolean Equation Systems

Definition

A Parametrized Boolean Equation System (PBES) is a sequence of fixpoint equations, inductively defined by the following grammar. Here $\sigma \in \{\nu, \mu\}$.

$$\mathcal{E} \quad := \quad \emptyset \mid (\sigma X(d : D) = \varphi)\mathcal{E}$$

$$\varphi \quad := \quad b \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid X(f) \mid \forall e \colon D. \varphi \mid \exists e \colon D. \varphi$$

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$$\begin{aligned}
\mu X(c_1 : B, d_1 : Z, c_2 : B, d_2 : Z) := \quad & (c_1 && \Rightarrow && Y(c_1, 1, c_2, d_2)) && (1)\\
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We assume initial configuration $X(\mathrm{true}, 0, \mathrm{true}, 0)$.

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$$X(\mathrm{true}, 0, \mathrm{true}, 0)$$

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$$X(\mathrm{true}, 0, \mathrm{true}, 0) \xrightarrow{\quad 1 \quad} Y(\mathrm{true}, 1, \mathrm{true}, 0)$$

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X(\text{true}, 0, \text{true}, 0) \arrow[r, "1"] & Y(\text{true}, 1, \text{true}, 0) \arrow[d, "3"] \\
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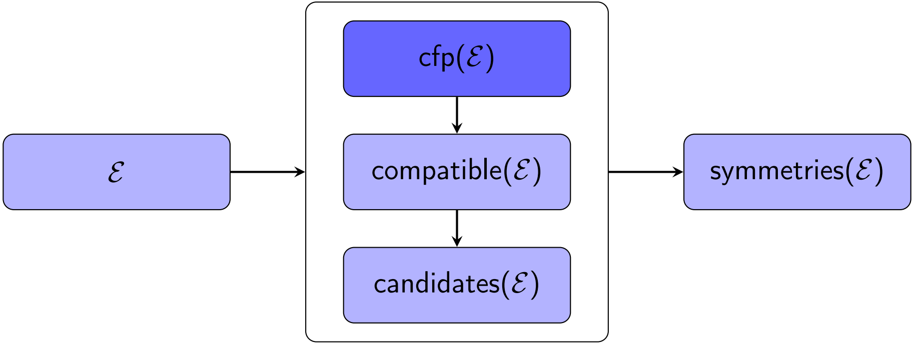
Overview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Control Flow Parameters

Definition

A parameter $c \in \mathrm{par}(\mathcal{E})$ is a *control flow parameter* (CFP) if we can deduce its unique value before and after every clause in $\mathcal{E}$.

Parameters that are not CFPs, are *data parameters* (DP). Notation: $\mathrm{cfp}(\mathcal{E})$, $\mathrm{dp}(\mathcal{E})$.

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$$\mathrm{cfp}(\mathcal{E}) = \{c_1, c_2\}$$

Control Flow Graphs

Definition

The control flow graph (CFG) of $c \in \mathrm{cfp}(\mathcal{E})$ is a directed graph $G_c = (V_c, E_c)$. The vertex set V_c is the set of values that c can attain throughout the system. The edge set E_c denotes which clause is responsible for a change in value.

G_{c_1} :

$$\begin{tikzcd}
& X(\text{true}) \arrow[dl, "1"'] & \\
Y(\text{true}) \arrow[rr, "3"'] & & Y(\text{false}) \arrow[ul, "5"']
\end{tikzcd}$$

G_{c_2} :

$$\begin{tikzcd}
& X(\text{true}) \arrow[dl, "2"'] & \\
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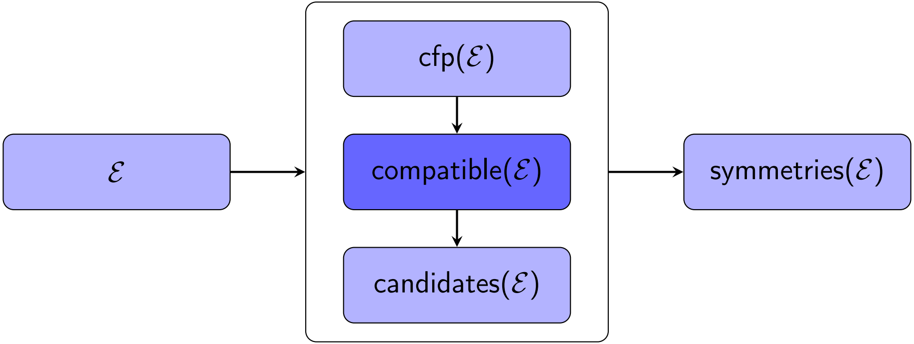
Overview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Data Parameters

Definition

Consider data parameter $d \in \mathrm{dp}(\mathcal{E})$, control flow parameter $c \in \mathrm{cfp}(\mathcal{E})$ and clause j in $\mathcal{E}$. Then d *plays a role for* c in j if d is used or changed by j.

$$
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\end{array}
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G_{c_1} :

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\begin{tikzcd}
 & X(\text{true}) \arrow[dl, "{1,\{d_1\}}"'] & \\
Y(\text{true}) \arrow[rr, "{3,\{d_1\}}"'] & & Y(\text{false}) \arrow[ul, "{5,\{d_1,d_2\}}"']
\end{tikzcd}
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G_{c_2} :

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\begin{tikzcd}
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Y(\text{true}) \arrow[rr, "{4,\{d_2\}}"'] & & Y(\text{false}) \arrow[ul, "{5,\{d_1,d_2\}}"']
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Compatibility

Definition

$$
\begin{aligned}
\mathrm{cmp}(c_1, c_2) \quad = \quad & \quad V_{c_1} = V_{c_2} \\
& \wedge \ \forall s, s' \in V_{c_1} \cup V_{c_2}.\#(c_1, s, s') = \#(c_2, s, s') \\
& \wedge \ \forall e \in E_{c_1} \cup E_{c_2}.|\mathrm{data}(c_1, e)| = |\mathrm{data}(c_2, e)|
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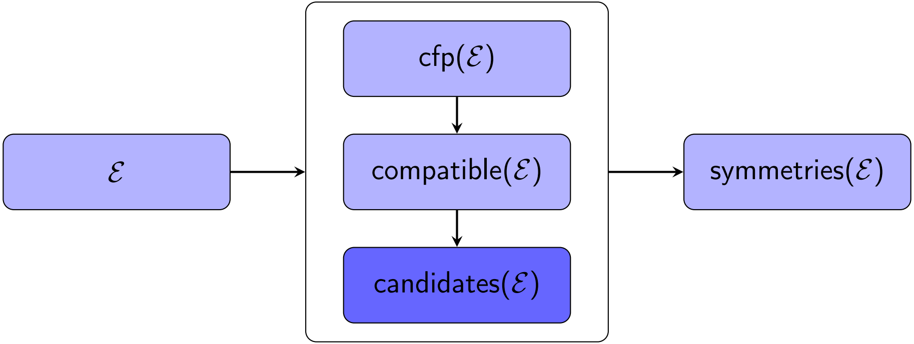
$$
\begin{aligned}
\#(c_1, Y(\text{false}), X(\text{true})) \quad &= \quad 1 \quad = \quad \#(c_2, Y(\text{false}), X(\text{true})) \\
|\mathrm{data}(c_1, (Y(\text{false}), 5, X(\text{true})))| \quad &= \quad 2 \quad = \quad |\mathrm{data}(c_2, (Y(\text{false}), 5, X(\text{true})))|
\end{aligned}
$$

$$
\mathrm{compatible}(\mathcal{E}) = \{c_1, c_2\}
$$

Overview



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Intermezzo - Permutations

Permutations

Let X be a set. A *permutation* is a bijection $\alpha \colon X \to X$. We denote the set of all permutations on X as $\mathrm{per}(X)$.

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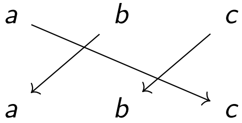
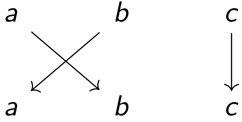
$\alpha = (acb)$



$\beta = (ab)(c)$



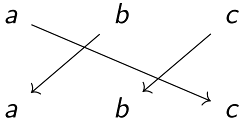
Intermezzo - Permutations

Permutations

Let X be a set. A *permutation* is a bijection $\alpha\colon X \to X$. We denote the set of all permutations on X as $\mathrm{per}(X)$.

Consider the set $X = \{a, b, c\}$, then $\mathrm{per}(X) = \{\mathrm{id}, (ab), (ac), (bc), (abc), (acb)\}$

$\alpha = (acb)$



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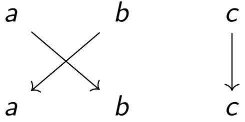

$\alpha \circ \beta =$

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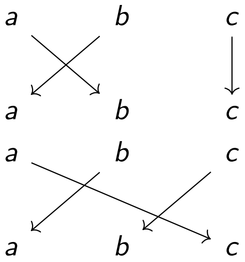

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$$\beta = (ab)(c) \qquad \begin{tikzcd} a \arrow[dr] & b \arrow[dl] & c \arrow[d] \\ a & b & c \end{tikzcd}$$

$$\alpha = (acb) \qquad \begin{tikzcd} a \arrow[drr] & b \arrow[dl] & c \arrow[dl] \\ a & b & c \end{tikzcd}$$

$$\alpha \circ \beta = (a)(bc) \qquad \begin{tikzcd} a \arrow[d] & b \arrow[dr] & c \arrow[dl] \\ a & b & c \end{tikzcd}$$

Candidate Permutations

Candidates

$$\text{candidates}(\mathcal{E}) = \{\alpha\beta \mid \alpha \in \text{per}(\text{compatible}(\mathcal{E})), \beta \in \text{per}(\text{dp}(\text{compatible}(\mathcal{E})))\}$$

Candidate Permutations

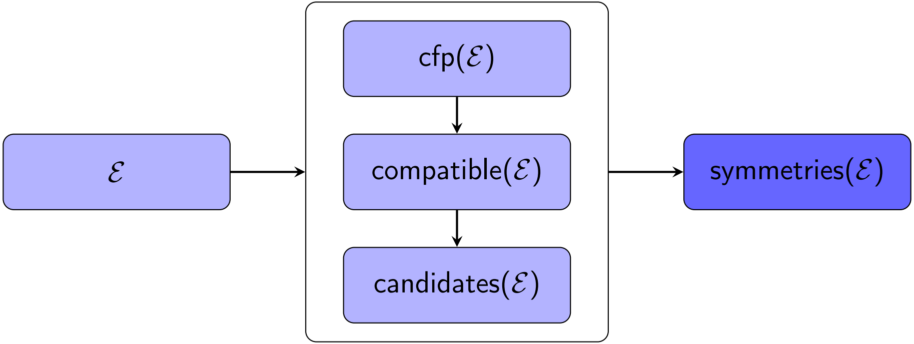
Candidates

$$\text{candidates}(\mathcal{E}) = \{\alpha\beta \mid \alpha \in \text{per}(\text{compatible}(\mathcal{E})), \beta \in \text{per}(\text{dp}(\text{compatible}(\mathcal{E})))\}$$

$$G_{c_1}: \begin{tikzcd} & X(\text{true}) \arrow[dl, "1{,}\{d_1\}"'] & \\ Y(\text{true}) \arrow[rr, "3{,}\{d_1\}"'] & & Y(\text{false}) \arrow[ul, "5{,}\{d_1{,}d_2\}"'] \end{tikzcd} \qquad G_{c_2}: \begin{tikzcd} & X(\text{true}) \arrow[dl, "2{,}\{d_2\}"'] & \\ Y(\text{true}) \arrow[rr, "4{,}\{d_2\}"'] & & Y(\text{false}) \arrow[ul, "5{,}\{d_1{,}d_2\}"'] \end{tikzcd}$$

$$\begin{aligned} \text{per}(\text{compatible}(\mathcal{E})) &= \{\mathrm{id}, (c_1 c_2)\} \\ \text{per}(\text{dp}(\text{compatible}(\mathcal{E}))) &= \{\mathrm{id}, (d_1 d_2)\} \\ \text{candidates}(\mathcal{E}) &= \{\mathrm{id}, (c_1 c_2), (d_1 d_2), (c_1 c_2)(d_1 d_2)\} \end{aligned}$$

Overview



$\mathcal{E}$
cfp($\mathcal{E}$)
compatible($\mathcal{E}$)
candidates($\mathcal{E}$)
symmetries($\mathcal{E}$)

Symmetries

Definition

A permutation π is a symmetry for a PBES $\mathcal{E}$ iff for every righthand side φ it holds that

$$[\![\pi(\varphi)]\!]\eta\delta = [\![\varphi]\!]\eta\delta$$

$$\begin{aligned}
\mu X(c_1 : B, d_1 : Z, c_2 : B, d_2 : Z) :=\quad & (c_1 \Rightarrow Y(c_1, 1, c_2, d_2)) && (1)\\
\wedge\quad & (c_2 \Rightarrow Y(c_1, d_1, c_2, 1)) && (2)\\
\nu Y(c_1 : B, d_1 : Z, c_2 : B, d_2 : Z) :=\quad & (c_1 \wedge Y(\neg c_1, 2, c_2, d_2)) && (3)\\
\vee\quad & (c_2 \wedge Y(c_1, d_1, \neg c_2, 2)) && (4)\\
\vee\quad & ((\neg c_1 \wedge \neg c_2 \wedge (d_1 + d_2 \geq 4)) \wedge X(\neg c_1, d_1 - d_2, \neg c_2, d_2 - d_1)) && (5)
\end{aligned}$$

Symmetries

$$\pi = (c_1 c_2)(d_1 d_2)$$

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$$\begin{tikzcd} c_1 \Rightarrow Y(c_1, 1, c_2, d_2) \arrow[d, "\text{Swap positions}"] \\ c_1 \Rightarrow Y(c_2, d_2, c_1, 1) \end{tikzcd}$$

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$$\begin{tikzcd}
c_1 \Rightarrow Y(c_1, 1, c_2, d_2) \arrow[d, "\text{Swap positions}"] \\
c_1 \Rightarrow Y(c_2, d_2, c_1, 1) \arrow[d, "\text{Update parameters}"] \\
c_2 \Rightarrow Y(c_1, d_1, c_2, 1)
\end{tikzcd}$$

$$\pi(c_1 \Rightarrow Y(c_1, 1, c_2, d_2)) = c_2 \Rightarrow Y(c_1, d_1, c_2, 1)$$

Symmetries

Definition

A candidate permutation π is a symmetry for a PBES $\mathcal{E}$ iff for every righthand side φ it holds that

$$[\![\pi(\varphi)]\!]\eta\delta = [\![\varphi]\!]\eta\delta$$

$$\begin{aligned}
\mu X(c_1 : B, d_1 : Z, c_2 : B, d_2 : Z) :=\quad & (c_1 \;\Rightarrow\; Y(c_1, 1, c_2, d_2)) && (1)\\
\wedge\quad & (c_2 \;\Rightarrow\; Y(c_1, d_1, c_2, 1)) && (2)\\
\nu Y(c_1 : B, d_1 : Z, c_2 : B, d_2 : Z) :=\quad & (c_1 \;\wedge\; Y(\neg c_1, 2, c_2, d_2)) && (3)\\
\vee\quad & (c_2 \;\wedge\; Y(c_1, d_1, \neg c_2, 2)) && (4)\\
\vee\; ((\neg c_1 \wedge \neg c_2 \wedge (d_1 + d_2 \geq 4)) \quad & \wedge\; X(\neg c_1, d_1 - d_2, \neg c_2, d_2 - d_1)) && (5)
\end{aligned}$$

Symmetries

Definition

A candidate permutation π is a symmetry for a PBES $\mathcal{E}$ iff for every righthand side φ it holds that

$$[\![\pi(\varphi)]\!]\eta\delta = [\![\varphi]\!]\eta\delta$$

$$\begin{aligned}
\mu X(c_1 : B, d_1 : Z, c_2 : B, d_2 : Z) :=\ & (c_1 \Rightarrow Y(c_1, 1, c_2, d_2)) && (1)\\
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\nu Y(c_1 : B, d_1 : Z, c_2 : B, d_2 : Z) :=\ & (c_1 \wedge Y(\neg c_1, 2, c_2, d_2)) && (3)\\
\vee\ & (c_2 \wedge Y(c_1, d_1, \neg c_2, 2)) && (4)\\
\vee\ & ((\neg c_1 \wedge \neg c_2 \wedge (d_1 + d_2 \geq 4)) \wedge X(\neg c_1, d_1 - d_2, \neg c_2, d_2 - d_1)) && (5)
\end{aligned}$$

The candidate permutation $\pi = (c_1 c_2)(d_1 d_2)$ is a symmetry for $\mathcal{E}$.

$$\text{symmetries}(\mathcal{E}) = \{\mathrm{id}, (c_1 c_2)(d_1 d_2)\}$$

Solving with Symmetries

Theorem

A permutation π is a symmetry for a PBES $\mathcal{E}$ iff $[\![\pi(\mathcal{E})]\!]\eta\delta = [\![\mathcal{E}]\!]\eta\delta$

Theorem

Let $\mathcal{E}$ be a PBES and $\pi \in \mathsf{per}([n])$ be a symmetry for $\mathcal{E}$. Then for all predicate variables X and values $v \in \mathbb{D}^n$ it holds that

$$v \in [\![\mathcal{E}]\!]\eta\delta(X) \iff \pi(v) \in [\![\mathcal{E}]\!]\eta\delta(X)$$

So we can conclude whether $\pi(v)$ is a solution for $\mathcal{E}$ by considering v.

Solving with Symmetries

$$\pi = (c_1 c_2)(d_1 d_2)$$

$$\begin{tikzcd}
Y(t,1,t,0) \arrow[d, "3"'] \arrow[dr, "4"] & & X(t,0,t,0) \arrow[ll, "1"'] \arrow[rr, "2"] & & Y(t,0,t,1) \arrow[dl, "3"'] \arrow[d, "4"] \\
Y(f,2,t,0) \arrow[drr, bend right, "4"'] & Y(t,1,f,2) \arrow[dr, "3"'] & & Y(f,2,t,1) \arrow[dl, "4"] & Y(t,0,f,2) \arrow[dll, bend left, "3"] \\
& & Y(f,2,f,2) \arrow[uu, "5"] & &
\end{tikzcd}$$

Solving with Symmetries

$$\pi = (c_1 c_2)(d_1 d_2)$$

$$\begin{tikzcd}
Y(t,1,t,0) \arrow[d, "3"'] \arrow[dr, "4"] & & X(t,0,t,0) \arrow[ll, "1"'] \arrow[rr, "2"] & & Y(t,0,t,1) \arrow[dl, "3"'] \arrow[d, "4"] \\
Y(f,2,t,0) \arrow[drr, bend right, "4"'] & Y(t,1,f,2) \arrow[dr, "3"'] & & Y(f,2,t,1) \arrow[dl, "4"] & Y(t,0,f,2) \arrow[dll, bend left, "3"] \\
& & Y(f,2,f,2) \arrow[uu, "5"] & &
\end{tikzcd}$$

Concluding

Summary

- Control flow as a basis for finding symmetry
- Generate candidate permutations
- Check which candidates are symmetries
- Solving for reduced state space

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Summary

- Control flow as a basis for finding symmetry
- Generate candidate permutations
- Check which candidates are symmetries
- Solving for reduced state space

Further directions

- What if there is no control flow parameters that we can distinguish from syntax?
- Modify the control flow in a system such that more symmetry can be identified
- Techniques to identify semantic symmetries

Thanks!

Extra slide