Formulas of the Quarter

1 Review 307

1. **First Degree Ordinary Differential Equation - ODE**: $\frac{dy}{dt} = f(t,y)$. **Existence and Uniqueness** of solution to Initial Condition Problem ICP requires $f, \partial f/\partial y$ continuous on an interval containing $t_0 \in I$
   - **Separable**: $y' = M(t)N(y) \Rightarrow \int \frac{1}{N} dy = \int M dt + c$.
   - **Exact**: $M + Ny' = 0$ where $M_y = N_t$. Then $y = \int M dt + \int \left(N - \frac{\partial}{\partial y} \int M dt\right) dy + c$.
2. **Linear ODE**: $\frac{dy}{dt} + p(t)y = g(t)$. **Existence and Uniqueness** of solution requires p, g continuous on $t_0 \in I$.
   - **Integrating Factor** for Linear: μ such that $\mu' = \mu p \Rightarrow y = \frac{1}{\mu}\left(\int_{t_0}^{t} \mu g dt + c\right)$.
3. **Second Degree ODE**: $\frac{d^2y}{dt^2} = f(t, y, y')$
4. **Second Order Linear ODE**: $y'' + p(t)y' + q(t)y = g(t)$. Homogeneous if $g(t) = 0$. **Existence and Uniqueness** requires p, q, g continuous on $t_0 \in I$. The general solution is the sum of the general solution of the homogeneous counterpart plus a particular solution $y_g = h_{hg} + y_p$.
   - **Constant Coefficients Homogeneous**: $ay'' + by' + c = 0$. Solve $am^2 + bm + c = 0$.
     - Real Solutions $r_1 \neq r_2$: $y = Ae^{r_1 t} + Be^{r_2 t}$.
     - Real Solutions equal $r_1 = r_2 = r$: $y = Ate^{rt} + Be^{rt}$.
     - Complex Conjugate Solutions $\alpha \pm \beta i$: $y = e^{\alpha t}[A\cos(\beta t) + B\sin(\beta t)]$.
   - Important Theorems:
     - **Superposition** for Homogeneous: Any linear combination of solutions is a solution.
     - **Wronskian**: $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$. **Abel's Theorem**: $W = ce^{-\int p\, dt}$ for some c.
     - **Fundamental Set of Solutions** or *basis of solutions*: $y_g = c_1 y_1 + c_2 y_2$. Either $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$. In particular $W(y_1, y_2)(t_0) = I_2$.
   - **Reduction of Order** for homogeneous: Knowing one solution y_1, assume $y_2 = vy_1$, then $y_1 v'' + (2y_1' + py_1)v' = 0$. Solve for v' and then integrate.
   - **Undetermined Coefficients** for Constant coefficients and $g(t)$ of specific kinds or types:
     - $g(t) = P_n(t) \Rightarrow y_p(t) = U_n(t)$. Except if 1, or 1 and t are already solutions of the homogeneous. Then multiply by t or t^2 respectively.
     - $g(t) = e^{\alpha t} P_n(t) \Rightarrow y_p = e^{\alpha t} U_n(t)$. Except if $e^{\alpha t}$, or $e^{\alpha t}$ and $te^{\alpha t}$ are already solutions of the homogeneous. Then multiply by t or t^2 respectively.
     - $g(t) = e^{\alpha t} P_n(t)\cos(\beta t)$ or $e^{\alpha t} P_n(t)\sin(t)$ or a linear combination $\Rightarrow y_p = e^{\alpha t} U_n(t)\cos(\beta t) + e^{\alpha t} V_n(t)\cos(\beta t)$. Except if $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ are already solutions of the homogeneous. Then multiply by t.

- **Variation of Parameters**: $y_{gh} = c_1y_1 + c_2y_2$. Assume $y_p = u_1y_1 + u_2y_2$ with $u_1'y_1 + u_2'y_2 = 0$, and then
$$y_p = -y_1 \int \frac{y_2 g}{W} dt + y_2 \int \frac{y_1 g}{W} dt$$
- **Euler 2nd Order Equation**: $t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0$ for $t > 0$. Replace $t = e^x$ or $x = \ln(t)$ and the equation transforms $\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0$. Then the solution is $y(x) = y(\ln(t))$.
- **Laplace Transform**:
  - $L(f')(s) = sL(f)(s) - f(0)$.
  - Tail Bounded previous derivatives: $L(f^{(n)})(s) = s^n L(f)(s) - \sum_{i=1}^{n-1} s^i f^{(n-i)}(0)$.
  - $L(u_c(t)f(t-c)) = e^{-cs}L(f)(s)$. Moreover if $f(t) = L^{-1}(F)$, then $L^{-1}(e^{-cs}F(s))(t) = u_c(t)f(t - c)$.
  - $L(e^{ct}f(t)) = L(f)(s - c)$ for $s > a + c$. Moreover if $f(t) = L^{-1}(F)$, then $L^{-1}(F(s - c)) = e^{ct}f(t)$.
  - Table 6.2.1 on page 321.

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1. **Equivalence of Systems**:
   (1) Linear System of Equations: $\forall i : \sum_j a_{ij}x_j = b_j$.
   (2) Vector Equation: $\sum_j \vec{a}_j x_j = \vec{b}_j$.
   (3) Matrix System Equation $A\vec{x} = \vec{b}$.

2. **Gauss and Jordan Gauss Elimination**: Using Row Elementary Operations to reduce a system to either a stair-shaped System, or a stair-shaped system with leading coefficients 1 and pivot columns full of zeros expect by a one.

3. All Systems have either 0, 1, or infinitely many solutions. The general solution of a non-homogeneous system is equal to a particular solution plus the general solution of the homogeneous counterpart: $y_g = y_{gh} + y_p$.

4. **Minors**: $\det(A_{ij})$ where A_{ij} is the submatrix obtained by deleting the i-th row and j-th column. **Cofactor** $C_{ij}(A) = (-1)^{ij}\det(A_{ij})$ or signed minor. **The Cofactor Matrix**: the matrix whose ij-th entry is the ij-th cofactor $C_{ij}(A)$. The **Adjugate Matrix**: the transpose of the cofactor matrix.

5. **The Determinant**:
$$\det(A) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$
$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij}(A)$$

6. Inverse of a Matrix:
   (1) $[A|I]$, apply Jordan Gauss and obtain $[I|A^{-1}]$.
   (2) $Adj(A) \times A = \det(A)I_n$.

7. **Linear Independence**: $\sum_i x_i\vec{a}_i = \vec{0}$ has a unique solution $x_1 = x_2 = \ldots = 0$.

8. **Span of a list**: $Span(\vec{a}_1, \vec{a}_2, \ldots)$ is the set of all linear combinations (finite sums of constant multiples) of the vectors in the list.

9. **Eigenvalues, Eigenvectors, and Eigenspaces**:
(1) Roots of Characteristic Polynomial $\det(A - xI) = 0$ are eigenvalues.
(2) For each eigenvalue, $E_\lambda = Null(A - \lambda I)$.

$$1 \leq dim(E_\lambda) \leq mult(\lambda)$$

$dim(E_\lambda)$ is called the **geometric multiplicity**, and $mult(\lambda)$ is called the algebraic multiplicity.

10. **Big Theorem**: For a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ and the coefficient or associated matrix $A = [\vec{a}_j]_j$ where $\vec{a}_j \in \mathbb{R}^n$. That means $T(\vec{x}) = A\vec{x}$ where $A = (a_{ij})$ has by j-th column the vector $\vec{a}_j$. That means $T(\vec{0}_m) = \vec{0}_n$ and $T(a\vec{x} + b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$.

    - $Rank(A) + Nullity(A) = dim(Ker(T)) + dim(Range(T)) = m$. Therefore, $Nullity(A) = dim(Ker(T)) \leq m$ and $Rank(A) = dim(Range(T)) \leq \min(m, n)$.
    - **Injective** or one-to-one: Requires $m \leq n$ and equivalent to any of the following: $E_0 = Null(A) = Ker(T) = \{\vec{0}\}$; $Nullity(A) = dim(Ker(T)) = 0$; $\{\vec{a}_i : i = 1, \ldots, m\}$ is LI; all systems $A\vec{x} = \vec{b}$ have at most one solution; 0 is not an eigenvalue. T has a surjective left inverse, or A has a left inverse. Computational Test: Jordan Gauss on A and the number of equations is equal to the number of variables.
    - **Surjective** or onto: Requires $n \leq m$ and equivalent to any of the following: $Col(A) = Range(T) = Span(\vec{a}_i)_i = \mathbb{R}^n$; $Rank(A) = dim(Range(T)) = n$; all systems $A\vec{x} = \vec{b}$ have at least one solution; T has an injective right inverse, or A has a right inverse. Computational Test: Jordan Gauss on A and the number of pivot columns is equal to n.
    - **Bijective**: Requires $m = n$ and equivalent to any of the following: Injective; Surjective; A is invertible or non-singular, $\det(A) \neq 0$, $\langle \vec{a}_i \rangle_i$ is a basis for $\mathbb{R}^n$.

11. **Affine Transformation**: $T : \mathbb{R}^m \to \mathbb{R}^n$ with $T(\vec{x}) = A\vec{x} + \vec{b}$. The image is not a subspace unless $\vec{b} = \vec{0}$.

12. The dot product $\vec{x} \cdot \vec{y} = \vec{x}^T \times \vec{y}$ (product as matrices) and $\vec{x} \perp \vec{y} \Leftrightarrow \vec{x} \cdot \vec{y} = 0$.

13. Gram-Schmidt: Given $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^m$. Then let $\vec{s}_1 = \vec{x}_1$ and $\forall k > 1$:

$$\vec{s}_k = \vec{x}_k - \sum_{i=1}^{k-1} Proj_{\vec{s}_i}(\vec{x}_k) = \vec{x}_k - \sum_{i=1}^{k-1} \frac{\vec{s}_i \cdot \vec{x}_k}{\|\vec{s}_i\|^2}\vec{s}_i$$

It returns a list of pairwise orthogonal vectors with $\forall 1 \leq d \leq m : Span(\vec{x}_1, \ldots, \vec{x}_d) = Span(\vec{s}_1, \ldots, \vec{s}_d)$.

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1. **General Solution to the Homogeneous** System $\vec{x}' = A\vec{x}$:

   - Two different real eigenvalues and corresponding eigenvectors:

     $$\vec{x}_{gh} = C_1\vec{v}_1 e^{\lambda_1 t} + C_2\vec{v}_2 e^{\lambda_2 t}$$

     Classification of $(0,0)$: if $\lambda_1, \lambda_2 < 0$: asymptotically stable node; if $\lambda_1, \lambda_2 > 0$: unstable node; $\lambda_1 > 0 > \lambda_2$; (unstable) saddle point. If $\lambda_1 = \lambda_2$ but $\vec{v}_1 \neq \vec{v}_2$, then Star.

   - Two complex conjugate eigenvectors $\lambda = \alpha \pm \beta i$ and $\vec{v} = \vec{a} + i\vec{b}$.

     $$\vec{x}_{gh} = C_1 e^{\alpha t}\left[\vec{a}\cos(\beta t) - \vec{b}\sin(\beta t)\right] + C_2 e^{\alpha t}\left[\vec{a}\sin(\beta t) + \vec{b}\cos(\beta t)\right]$$

     Classification of $(0,0)$: if $\alpha = 0$, then center; if $\alpha > 0$, then unstable spiral; if $\alpha < 0$, then asymptotically stable spiral.

   - Single Eigenvalue - Single eigenvector: Find generalized eigenvector $\vec{w}$ from the eigenvector $\vec{v}$ by solving $(A - \lambda I)\vec{w} = \vec{v}$.

     $$\vec{x}_{gh} = C_1\vec{v}e^{\lambda t} + C_2\left(\vec{v}te^{\lambda t} + \vec{w}e^{\lambda t}\right)$$

     Classification of $(0,0)$: if $\lambda > 0$, unstable improper node, if $\lambda < 0$, asymptotically stable improper node.

Eigenvalues	Type of Critical Point	Stability
$r_1 > r_2 > 0$	Node	Unstable
$r_1 < r_2 < 0$	Node	Asymptotically Stable
$r_2 < 0 < r_1$	Saddle Point	Unstable
$r_1 = r_2 > 0$	Proper Node or Star: Two independent Eigenvectors	Unstable
	Improper or degenerate node: One Eigenvector	Unstable
$r_1 = r_2 < 0$	Proper Node or Star: Two independent Eigenvectors	Asymptotically stable
	Improper or degenerate node: One Eigenvector	Asymptotically stable
$r_1, r_2 = \lambda \pm \mu i$	Spiral Point	
$\lambda > 0$		Unstable
$\lambda < 0$		Asymptotically stable.
$r_1, r_2 = \pm\mu i$	Center	Stable.

Table 1: Stability And Classification of the Critical point at $\vec{x} = \vec{0}$.

2. **Fundamental Matrix** for Homogeneous: $\Phi = \left(\vec{x}^{(1)}, \vec{x}^{(2)}\right)$ where the columns are two independent solutions.

3. **General Solution to the Euler Homogeneous** system $t\vec{x}' = A\vec{x}$ and $t > 0$: Replace t by $\log(t)$ in the solution of $\vec{x}' = A\vec{x}$ to obtain the following:

   - Two different real eigenvalues and corresponding eigenvectors:

     $$\vec{x}_{gh} = C_1\vec{v}_1 t^{\lambda_1} + C_2\vec{v}_2 t^{\lambda_2}$$

- Two complex conjugate eigenvectors $\lambda = \alpha \pm \beta i$ and $\vec{v} = \vec{a} + i\vec{b}$.

$$\vec{x}_{gh} = C_1 t^{\alpha}\left[\vec{a}\cos(\beta\ln(t)) - \vec{b}\sin(\beta\ln(t))\right] + C_2 t^{\alpha}\left[\vec{a}\sin(\beta\ln(t)) + \vec{b}\cos(\beta\ln(t))\right]$$

- Single Eigenvalue - Single Eigenvector: Find the generalized eigenvector $\vec{w}$:

$$\vec{x}_{gh} = C_1\vec{v}t^{\lambda} + C_2\left(\vec{v}t^{\lambda}\ln(t) + \vec{w}t^{\lambda}\right)$$

4. **Diagonalization** or **Jordan Eigenform** for nonhomogeneous $\vec{x}' = A\vec{x} + \vec{g}(t)$:

   - Diagonalizable: eigenvalues λ_1, λ_2 and eigenvectors $\vec{v}_1, \vec{v}_2$. Then $T = (\vec{v}_1, \vec{v}_2), D = diag(\lambda_1, \lambda_2)$: Assume $\vec{x} = T\vec{y}$. Then $\vec{y}' = D\vec{y} + T^{-1}\vec{g}$. Both equations will be linear on a single variable. Recall that $y' + p(t)y = g(t)$ solves as $y = e^{-\int p(t)dt}\int e^{\int p(t)dt}g(t)dt$.
   - Nondiagonalizable: eigenvalue λ, eigenvector $\vec{v}$, generalized eigenvector: $\vec{w}$. Then $T = (\vec{v}, \vec{w})$ and $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$: Assume $\vec{x} = T\vec{y}$. Then $\vec{y}' = J\vec{y} + T^{-1}\vec{g}$. Solve first the second equation which only involves y_2, and then plug it in the first equation. Both equations will be then of the form $y' + p(t)y = g(t)$. Solve it as in the previous case.

   At the end, we can assume all constants of integration zero and obtain the particular solution $\vec{y}_p$, and $\vec{x}_p = T\vec{y}_p$.

5. **Variations of Parameters** for the non-homogeneous system: Assume Ψ is a fundamental matrix, where each column is an element of a fundamental set of solutions for the homogeneous corresponding system.

$$\vec{x}_p = \Psi(t)\int \Psi^{-1}(t)\vec{g}(t)dt$$

   Note: the indefinite integral happens on each entry of the product $\Psi^{-1}(t)\vec{g}(t)$.

6. **Exponential of a Matrix**: In order to find $\exp(At)$, one follows one of two processes:

   (a) **Using Differential Equations**: Use differential equations methods to find the general solution $\vec{x}_{gh} = C_1\vec{x}^{(1)} + C_2\vec{x}^{(2)}$ to $\vec{x}' = A\vec{x}$. Find the solutions $\vec{X}_1, \vec{X}_2$ that satisfy the initical conditions $\vec{X}_1(0) = \vec{e}_1; \vec{X}_2(0) = \vec{e}_2$ respectively. Then $\exp(At) = [\vec{X}_1, \vec{X}_2]$. Indeed, $\exp(At)$ satisfy the IVP $X' = AX$ with $X(0) = I$.

   (b) **Using Diagonalization Matrix Theory**: Find the diagonal decomposition of A, say $A = PDP^{-1}$. Then $\exp(At) = P\exp(Dt)P^{-1}$. Recall:

$$D = diag(\lambda_1, \ldots, \lambda_n) \Rightarrow \exp(Dt) = diag(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$$

   Jordan Canonical form for 2×2 Matrix:

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \Rightarrow \exp(Jt) = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$

   Uses: If $\exp(At) = [\vec{a}_1, \vec{a}_2]$, then the general solution to the homogenous system $\vec{x}' = A\vec{x}$ is $\vec{x}_{gh} = C_1\vec{a}_1 + C_2\vec{a}_2$. Moreover, the solution to the IVP $\vec{x}' = A\vec{x}$ with $\vec{x}(0) = \vec{x}_0$ is given by $\exp(At)\vec{x}_0$.

Uses: If $\exp(At) = [\vec{a}_1, \vec{a}_2]$, then the general solution to the homogenous system $\vec{x}' = A\vec{x}$ is $\vec{x}_{gh} = C_1\vec{a}_1 + C_2\vec{a}_2$. Moreover, the solution to the IVP $\vec{x}' = A\vec{x}$ with $\vec{x}(0) = \vec{x}_0$ is given by $\exp(At)\vec{x}_0$.

7. **Fourier Series Expansion**: The set $\{\sin(n\pi x/L), \cos(n\pi x/L) : n \in \mathbb{Z}^+\}$ is a basis for the set of periodic functions f of period $2L$ where f, f' are piecewise continuous (ignoring the points of discontinuity, where the series converges to the average of the left and right limit of the function).
$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos(n\pi x/L) + \sum_{n\geq 1} b_n \sin(n\pi x/L)$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)dx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)dx$$
Moreover, if f is even, then $\forall n : b_n = 0$; and if f is odd, then $\forall n : a_n = 0$. Also Parseval's identity:
$$\frac{1}{L}\int_{-L}^{L} f(x)^2 dx = \frac{a_0^2}{2} + \sum_{n\geq 1}(a_n^2 + b_n^2)$$

8. **Eigen-pairs**: $y'' + \lambda y = 0$.
   - $y(0) = 0$ and $y(L) = 0$: $(\lambda_n, y_n)_{n>0} = (n^2\pi^2/L^2, \sin(n\pi x/L))$.
   - $y(0) = 0$ and $y'(L) = 0$: $(\lambda_n, y_n)_{n:odd>0} = (n^2\pi^2/4L^2, \sin(n\pi x/2L))$.
   - $y'(0) = 0$ and $y(L) = 0$: $(\lambda_n, y_n)_{n:odd>0} = (n^2\pi^2/4L^2, \cos(n\pi x/2L))$.
   - $y'(0) = 0$ and $y'(L) = 0$: $(\lambda_n, y_n)_{n\geq 0} = (n^2\pi^2/L^2, \cos(n\pi x/L))$.

9. **Extensions**:
   - f symmetric about $x = L$ on the interval $(0, 2L)$, i.e.: $f(2L - x) = f(x)$; odd, and of period $4L$ makes f have a Fourier expansion involving only $\sin(\pi nx/2L)$ where n is odd with coefficients $\frac{2}{L}\int_0^L f(x)\sin(\pi nx/2L)dx$
   - f anti-symmetric about $x = L$ on the interval $(0, 2L)$, i.e.: $f(2L - x) = -f(x)$; even, and of period $4L$ makes f have a Fourier expansion involving only $\cos(\pi nx/2L)$ where n is odd with coefficients $\frac{2}{L}\int_0^L f(x)\cos(\pi nx/2L)dx$

3.1 Applications of Separation of Variables

3.1.1 Heat Equation

The equation is $\alpha^2 u_{xx} = u_t$ where the initial temperature is known $u(x, 0)$ for $x \in [0, L]$.

1. **Endpoints at constant temperatures** BC: $u(0, t) = T_1, u(L, t) = T_2$ (Non-homogeneous BC - Stabilizing assumption)
$$u(x,t) = \frac{T_2 - T_1}{L}x + T_1 + \sum_{n\geq 1} c_n \exp\left(-\frac{\alpha^2 n^2\pi^2}{L^2}t\right)\sin\left(\frac{n\pi x}{L}\right);$$
Extend $u(x, 0) - \left(\frac{T_2 - T_1}{L}x + T_1\right)$ to be odd and of period $2L$.
$$c_n = \frac{2}{L}\int_0^L \left[u(x,0) - \left(\frac{T_2 - T_1}{L}x + T_1\right)\right]\sin\left(\frac{n\pi x}{L}\right)dx$$

2. **Insulated at both Endpoints** BC: $u_x(0,t) = 0, u_x(L,t) = 0$ (Separation of Variables) Extend $u(x,0)$ to be even of period $2L$, then

$$u(x,t) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \exp\left(-\frac{\alpha^2 n^2 \pi^2}{L^2} t\right) \cos\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(n\pi x/L)dx$$

3. **One endpoint fixed and the other is insulated**:

   (a) BC: $u(0,t) = T$ and $u_x(L,t) = 0$:

$$u(x,t) = T + \sum_{n:odd>0} c_n \sin\left(\frac{n\pi x}{2L}\right) e^{-\frac{n^2\pi^2\alpha^2}{4L^2}t}$$

$$c_n = \frac{2}{L}\int_0^L (f(x) - T)\sin\left(\frac{n\pi x}{2L}\right)dx$$

   Extend $u(x,0) - T$ symmetric about $x = L$ on $(0, 2L)$, odd, and of period $4L$.

   (b) BC: $u_x(0,t) = 0$ and $u_x(L,t) = T$:

$$u(x,t) = T + \sum_{n:odd>0} c_n \cos\left(\frac{n\pi x}{2L}\right) e^{-\frac{n^2\pi^2\alpha^2}{4L^2}t}$$

$$c_n = \frac{2}{L}\int_0^L (f(x) - T)\cos\left(\frac{n\pi x}{2L}\right)dx$$

   Extend $u(x,0) - T$ anti-symmetric about $x = L$ on $(0, 2L)$, even, and of period $4L$.

3.1.2 1-dimensional Wave Equation

$a^2 u_{xx} = u_{tt}$ where a is the velocity of propagation of the wave. BC: $u(0,t) = u(L,t) = 0$; $u(x,0) = f(x)$; $u_t(x,0) = g(x)$. The general solution is of the form:

$$u(x,t) = \sum_n \sin\left(\frac{n\pi x}{L}\right)\left[c_n \cos\left(\frac{n\pi at}{L}\right) + d_n \sin\left(\frac{n\pi at}{L}\right)\right]$$

Extend both f and g to be odd and of period $2L$.

$$c_n = b_n(f) = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx; d_n = \frac{L}{n\pi a}b_n(g) = \frac{2}{n\pi a}\int_0^L g(x)\sin(n\pi x/L)dx$$

3.1.3 Laplace's Equation

2-dimensional Dirichlet Problem for a Rectangle: $u_{xx} + u_{yy} = 0$ for $(x,y) \in (0,A)\times(0,B)$. The boundary conditions are:

- Left $u(0,y) = f_1(y)$ for $0 \leq y \leq B$.

- Right $u(A, y) = f_2(y)$ for $0 \le y \le B$.
- Bottom $u(x, 0) = g_1(x)$ for $0 \le x \le A$.
- Top $u(x, B) = g_2(x)$ for $0 \le x \le A$.

The solution is given by:

$$\begin{aligned} u(x,y) &= \textstyle\sum_{n\ge1} \left[a_n \sinh\left(\frac{n\pi x}{B}\right) + b_n \sinh\left(\frac{n\pi(x-A)}{B}\right)\right] \sin\left(\frac{n\pi y}{B}\right) \\ &+ \textstyle\sum_{n\ge1} \left[c_n \sinh\left(\frac{n\pi y}{A}\right) + d_n \sinh\left(\frac{n\pi(y-B)}{A}\right)\right] \sin\left(\frac{n\pi x}{A}\right) \end{aligned}$$

where f_1, f_2 are extended to be odd and $2B$ periodic, and g_1, g_2 are extended to be odd and $2A$ periodic. And

$$a_n \sinh\left(\frac{n\pi A}{B}\right) = \frac{2}{B}\int_0^B f_2(y) \sin\left(\frac{n\pi y}{B}\right) dy$$

$$-b_n \sinh\left(\frac{n\pi A}{B}\right) = \frac{2}{B}\int_0^B f_1(y) \sin\left(\frac{n\pi y}{B}\right) dy$$

$$c_n \sinh\left(\frac{n\pi B}{A}\right) = \frac{2}{A}\int_0^A g_2(x) \sin\left(\frac{n\pi x}{A}\right) dx$$

$$-d_n \sinh\left(\frac{n\pi B}{A}\right) = \frac{2}{A}\int_0^A g_1(x) \sin\left(\frac{n\pi x}{A}\right) dy$$

2-dimensional Dirichlet Problem for the Circle centered at $(0, 0)$ and of radius $r = a$: $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ with IC $u(a, \theta) = f(\theta)$.

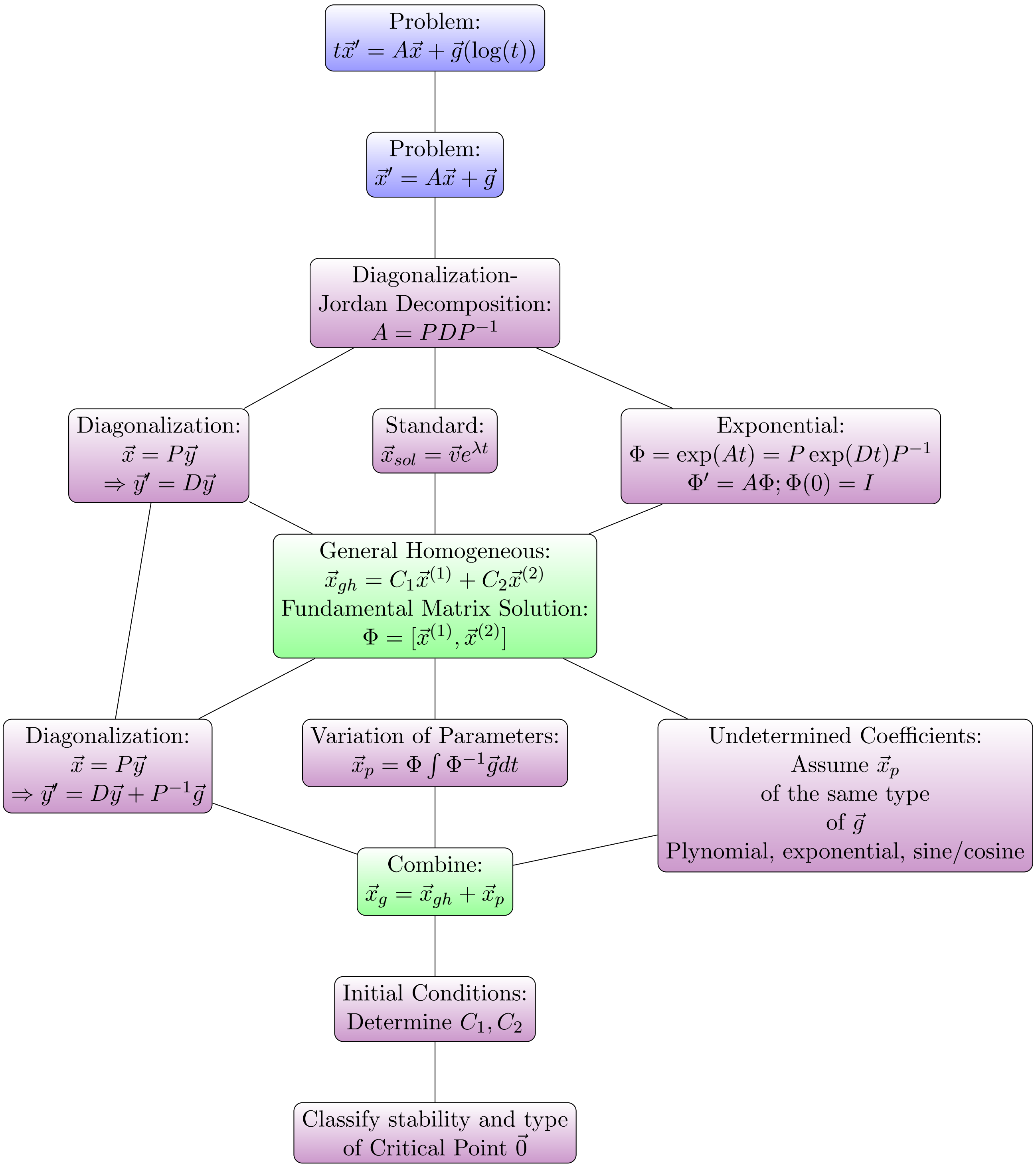
$$u(r,\theta) = \frac{c_0}{2} + \sum_{n\ge1} r^n (c_n \cos(n\theta) + k_n \sin(n\theta))$$

where $a^n c_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta$ and $a^n k_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta$

3.2 Useful Integrals

$$\begin{aligned} \textstyle\int x^2 \cos(bx)\,dx &= \tfrac{1}{b}\,x^2 \sin(bx) + \tfrac{2}{b^2}\,x\cos(bx) - \tfrac{2}{b^3}\sin(bx) \\ \textstyle\int x\cos(bx)\,dx &= \tfrac{1}{b}\,x\sin(bx) + \tfrac{1}{b^2}\cos(bx) \\ \textstyle\int x^2\, sin(bx)\,dx &= -\tfrac{1}{b}\,x^2\cos(bx) + \tfrac{2}{b^2}\,x\sin(bx) + \tfrac{2}{b^3}\cos(bx) \\ \textstyle\int x\sin(bx)\,dx &= -\tfrac{1}{b}\,x\cos(bx) + \tfrac{1}{b^2}\sin(bx) \end{aligned}$$

4 Road Map for the Midterm



5 Worksheet 0: Inner Products and Orthogonality

- In $\mathbb{R}^n$, one defines $\vec{v} \cdot \vec{w} = \sum_i v_i w_i$.
- In $\mathbb{C}^n$, one defines $\vec{v} \cdot \vec{w} = \sum_i v_i \overline{w_i}$.

1. Assume $\vec{x} \in \mathbb{R}^n$. What is $\vec{x} \cdot \vec{x}$? Why is $\|\vec{x}\|^2 := \vec{x} \cdot \vec{x} \geq 0$? Is it symmetric, i.e. is it true that $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$?
2. Assume $\vec{x} = \vec{a} + \vec{b}i \in \mathbb{C}^n$. What is $\vec{x} \cdot \vec{x}$? Why is $\|\vec{x}\|^2 := \vec{x} \cdot \vec{x} \geq 0$? Is it symmetric, i.e. is it true that $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$?
3. Can you think of an inner product for continuous functions $f : [a, b] \to \mathbb{R}$ defined on a finite closed interval $[a, b]$ and with real values, i.e. in $\mathbb{R}$? Can you think of an inner product for continuous functions $f : [a, b] \to \mathbb{C}$ defined on a finite closed interval $[a, b]$ and with complex values, i.e. in $\mathbb{C}$?

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6 Worksheet 1: Review Ordinary Differential Equations

1. Find the general solution to the first order ordinary differential equations:

   - Separable: $\frac{dy}{dt} = e^{y+t}$.
   - Homogeneous: $\frac{dy}{dt} + yt = 0$
   - Non-homogeneous: $\frac{dy}{dt} + yt = t^2$. Suggestion: Multiply the equation by $e^{\int t dt}$ and identify the left hand side as $\frac{d}{dt}\left(ye^{\int t dt}\right)$.

     Solution: After multiplying by the integrating factor $\mu = e^{\int t dt} = e^{t^2/2}$, one obtains

     $$\frac{d}{dt}\left(ye^{t^2/2}\right) = t^2 e^{t^2/2}$$

     Integrating both parts, in particular integrating by parts the right hand side, one obtains:

     $$ye^{t^2/2} = te^{t^2/2} - \int e^{t^2/2}dt + C \Rightarrow y = t - e^{-t^2/2}\int e^{t^2/2}dt + Ce^{-t^2/2}$$

2. Find the general solutions to the following second order linear homogeneous ordinary differential equations:

   - $y'' + 5y' + 6y = 0$.
   - $y'' - 2y' + y = 0$.
   - $y'' - 2y' + 2y = 0$.
   - $t^2y'' - 2ty' + 2y = 0$ for $t > 0$.

3. Find the general solutions to the following second order linear non-homogeneous differential equations using both methods: (1) undetermined coefficients, and (2) variation of parameters.

   - $y'' - 5y' + 6y = 5e^{2t}$.
   - $y'' - 2y' + 2y = e^t\cos(t)$.

     Solution - Undetermined Coefficients: The auxiliary associated equation is $r^2 - 2r + 2 = 0$ which has solutions $r = 1 \pm i$, and thus the general solution to the homogeneous $y_{gh} = Ae^t\cos(t) + Be^t\sin(t)$. Now, the particular solution should be analogous to $g(t) = e^t\cos(t)$, since sin and cos are treated similarly, then $y_p = Ce^t\cos(t) + De^t\sin(t)$. However, this is the same as the general solution to the homogeneous, thus we assume the particular solution to be of the form:

     $$y_p = Cte^t\cos(t) + Dte^t\sin(t)$$

     Then

     $$y_p' = e^t\cos(t)\left[C + Ct + Dt\right] + e^t\sin(t)\left[D + Dt - Ct\right]$$

     And

     $$y_p'' = e^t\cos(t)\left[2Dt + 2C + 2D\right] + e^t\sin(t)\left[-2Ct - 2C + 2D\right]$$

     Therefore,

     $$e^t\cos(t) = y_p'' - 2y_p' + 2y_p = 2De^t\cos(t) - 2Ce^t\sin(t)$$

     Hence,

     $$y_p = \frac{1}{2}te^t\sin(t)$$

And
$$y_g = y_{gh} + y_p = Ae^t\cos(t) + Be^t\sin(t) + \frac{1}{2}te^t\sin(t)$$

Solution - Variation of Parameters: As well as in the previous case, $y_{gh} = Ae^t\cos(t) + Be^t\sin(t)$, i.e.: $y_1 = e^t\cos(t)$ and $y_2 = e^t\sin(t)$. Therefore,

$$W(y_1, y_2) = \det\begin{bmatrix} e^t\cos(t) & e^t\sin(t) \\ e^t\cos(t) - e^t\sin(t) & e^t\sin(t) + e^t\cos(t)\end{bmatrix} = e^{2t}$$

Therefore,

$$y_p = -e^t\cos(t)\int\frac{e^t\sin(t)e^t\cos(t)}{e^{2t}}dt + e^t\sin(t)\int\frac{e^t\cos(t)e^t\cos(t)}{e^{2t}}dt$$

Simplifying,

$$y_p = -\frac{1}{2}e^t\cos(t)\int\sin(2t)dt + \frac{1}{2}e^t\sin(t)\int 1 + \cos(2t)dt$$

i.e.
$$y_p = \frac{1}{4}e^t\left(\cos(t)\cos(2t) + \sin(t)\sin(2t)\right) + \frac{1}{2}te^t\sin(t)$$

Simplifying this is
$$y_p = \frac{1}{4}e^t\cos(t) + \frac{1}{2}te^t\sin(t)$$

And
$$y_g = y_{gh} + y_p = Ae^t\cos(t) + Be^t\sin(t) + \frac{1}{2}te^t\sin(t)$$

7 Worksheet 2: Review Linear Algebra

1. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are real numbers.

   - Find the determinant of A, i.e. $\det(A)$.
   - Find the characteristic polynomial of A, i.e. $\mathrm{char}(A)(x) = \det(A - xI)$ or in a different notation $\mathrm{char}(A)(\lambda) = \det(A - \lambda I)$.

   Challenge. *Find the conditions on a, b, c, d so A is not diagonalizable.*

   solution. *Since* $\dim(A) = 2$*, then the characteristic polynomial of A must have one solution $\lambda = -\frac{a+d}{2}$, i.e. $(a + d)^2 - 4(ad - bc) = (a - d)^2 + 4bc = 0$. And the $Nullity(A - \lambda I) = 1$.*

   - *Case I: $a = d$. Then either $b = 0$ or $c = 0$ or both. In the last case $b = c = 0$, $A - \lambda I = 0_{2\times 2}$ and Hence $Nullity(A - \lambda I) = 2$, so this is not the case. The other two cases can easily be verified. Hence $(a - d)^2 + 4bc = 0$ and $a = d$ and either $b = 0$ or $c = 0$ but not both.*
   - *Case II: $a \neq d$. Then $b \neq 0$ and $c \neq 0$ because otherwise $(a - d)^2 + 4bc \neq 0$. Again, it can be verified that this case always work. Hence $(a - d)^2 + 4bc = 0$ and $a \neq d$ and $b \neq 0$ and $c \neq 0$.*

2. Let $A = \begin{bmatrix} 1 + i & 1 \\ -1 & 1 - i \end{bmatrix}$.

   - Find the determinant of A, i.e. $\det(A)$.
   - Find the characteristic polynomial of A, i.e. $\mathrm{char}(A)(x) = \det(A - xI)$ or in a different notation $\mathrm{char}(A)(\lambda) = \det(A - \lambda I)$.
   - Find the eigenvalues and the associated eigenspaces of A. For each eigenvalue, find a basis and the dimension of the associated eigenspace.

3. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$.

   - Find the determinant of A, i.e. $\det(A)$ and find the characteristic polynomial of A.
   - The unique eigenvalues of A is 2 and a basis for the corresponding eigenspace is $E_2 = \left\langle \vec{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle$. Is the matrix diagonalizable?
   - Find a vector $\vec{b}$ such that $(A - 2I)\vec{b} = \vec{a}$. What is $(A - 2I)^2\vec{b}$? **Remark**: $\vec{b}$ is a generalized eigenvalue.
   - Show that $A = [\vec{a}, \vec{b}] \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} [\vec{a}, \vec{b}]^{-1}$. **Remark**: The middle matrix is not a diagonal matrix, but this is the closest decomposition to a diagonalization of A. Moreover, the middle matrix is called the Jordan canonical form of A.

8 Worksheet 3: Introduction to Linear Systems of Ordinary Differential Equations

1. **A single equation into a system of equations**: Consider a second order linear differential equation
$$au'' + bu' + cu = 0$$
where u depends on t and the derivatives are with respect to t. We are going to write this single equation as a system of linear differential equations on two functions x_1 and x_2.
   - Let $x_1 = u$ and $x_2 = u'$. What is x_1' in terms of both x_1 and x_2? In other words, let $x_1' = f(x_1, x_2)$, then what is f? This is your first equation.
   - Notice that $x_2' = u''$. Solve for u'' in the original equation and replace u and u' by x_1 and x_2 respectively. This is your second equation.
   - Write the system in the form $\vec{x}' = A\vec{x}$

2. **A system of two equations into a single equation**: Consider the system of linear equations
$$\begin{aligned} x_1' &= 3x_1 - 2x_2 \\ x_2' &= x_1 - x_2 \end{aligned}$$
where x_1 and x_2 depend on t and the derivatives are with respect to t. We are going to write this single equation as a system of linear differential equations on u.
   - Subtract 3 times equation 2 from equation 1.
   - Let $u = x_1 - 3x_2$ and $u' = x_2$. In other words, $x_1 = u + 3u'$ and $x_2 = u'$. Replace these in the equations obtained above.
   - Write the result as an equation of the form $au'' + bu' + cu = 0$. In other words, what are a, b and c?

   In views of the principle of superposition (linear combinations of solutions are solutions,) and the decomposition of the solution of a non-homogeneous system, first we are going to study homogeneous systems and fundamental sets of solutions (or basis for the general solution of the homogeneous system.) Remarkably if there is a point in the interval where a set of solutions resembles the standard basis for $\mathbb{R}^2$ (in general for $\mathbb{R}^n$,) then they are a fundamental set of solutions.

3. **Wronskian** Recall that the Wronskian of two vector functions x_1, x_2 is given by
$$W(x_1, x_2) = \det \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$
Now consider the vectors $x_1(t) = (t, 1)^T; x_2(t) = (t^2, 2t)^T$.
   - Compute the Wronskian of x_1 and x_2.
   - In what intervals are x_1 and x_2 linearly independent.
   - Find a system satisfied by x_1 and x_2.

   One possible Solution: The wronskian is t^2, so any solution that we find should work either on $(-\infty, 0)$ or in $(0, \infty)$. Sadly if one tries to fit these two solutions in terms of a system of the form $\vec{x}' = A\vec{x} + g(t)$ with A a constant matrix, it does not work. So relax the conditions and let A be a non-constant matrix, whose entries are $a(t), b(t), c(t), d(t)$. Since x_1 is a solution and $x_1' = (1,0)^T$, one has

$$\begin{aligned} 1 &= a_1(t) \cdot t + a_2(t) + g_1(t) \\ 0 &= a_3(t) \cdot t + a_4(t) + g_2(t) \end{aligned}$$

Since x_2 is a solution and $x_2' = (2t, 2)^T$, one has

$$\begin{aligned} 2t &= a_1(t) \cdot t^2 + a_2(t) \cdot (2t) + g_1(t) \\ 2 &= a_3(t) \cdot t^2 + a_4(t) \cdot (2t) + g_2(t) \end{aligned}$$

Both cases suggest to try $a_1(t) = a_3(t) = 0$, $a_2(t) = 1$ and $g_1(t) = 0$. Which guarantees the first equation of each system, and one only has the second equations to care for:

$$\begin{aligned} 0 &= a_4(t) + g_2(t) \\ 2 &= a_4(t) \cdot (2t) + g_2(t) \end{aligned}$$

which has a solution of the form: $a_4(t) = \frac{2}{2t-1}$ and $g_2(t) = \frac{2}{1-2t}$. Therefore, a possible system is letting $t \in (0, 1/2)$ and

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ 0 & \frac{2}{2t-1} \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ \frac{2}{1-2t} \end{bmatrix}$$

Note: The reason I picked the interval $(0, 1/2)$ is because the function entries of A are continuous in this interval, and all the function entries of A and g plus the Wronskian of the solutions are continuous in this interval. I could have also chose $(1/2, \infty)$, or any sub-interval of both choices.

9 Worksheet 4: Homogeneous Linear System of Ordinary Differential Equations with constant real coefficients

For each of the following system, find the general solution of the system by finding the eigen-stuff of the constant matrix. Recall that if $(\lambda, \vec{v})$ is an eigen-pair, then

$$\vec{x} = \vec{v}e^{\lambda t} \Rightarrow \vec{x}' = \lambda\vec{v}e^{\lambda t} = A\vec{v}e^{\lambda t} = A\vec{x}$$

1. **Two different Eigenvalues**: Let $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$, and solve $\vec{x}' = A\vec{x}$.

    - Compute the eigenvalues and basis for each eigenspace of the matrix A, i.e. solve $\det(A - xI) = 0$ and for each λ find a basis for $E_\lambda = Null(A - \lambda I)$.
    - Write the general solution as

    $$\vec{x}_{gh} = C_1\vec{v}_1 e^{\lambda_1 t} + C_2\vec{v}_2 e^{\lambda_2 t}$$

2. **A single eigenvalue and two LI eigenvectors**: Let $A = I_2$, and solve $\vec{x}' = A\vec{x}$.

    - Compute the eigenvalue and a basis for the eigenspace of the matrix A, i.e. solve $\det(A - xI) = 0$ and find a basis for $E_\lambda = Null(A - \lambda I)$.
    - Write the general solution as

    $$\vec{x}_{gh} = C_1\vec{v}_1 e^{\lambda t} + C_2\vec{v}_2 e^{\lambda t}$$

3. **A single eigenvalue and one eigenvector**: Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$, and solve $\vec{x}' = A\vec{x}$. *Hint*: A fundamental set of solutions is of the form $\left\langle \vec{v}e^{\lambda t}, \vec{v}te^{\lambda t} + \vec{w}e^{\lambda t} \right\rangle$ where $\vec{v}$ and $\vec{w}$ are an eigenvector and a generalized eigenvector associated to λ.

    - Compute the eigenvalue and a basis for the eigenspace of the matrix A, i.e. solve $\det(A - xI) = 0$ and find a basis for $E_\lambda = Null(A - \lambda I) = \langle \vec{v} \rangle$.
    - Compute the generalized eigenvalue $\vec{w}$, i.e. Solve $A\vec{w} = \vec{v}$.
    - Write the general solution as

    $$\vec{x}_{gh} = C_1\vec{v}e^{\lambda t} + C_2\left(\vec{v}te^{\lambda t} + \vec{w}e^{\lambda t}\right)$$

4. **Two complex eigenvector**: Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, and solve $\vec{x}' = A\vec{x}$.

    - Compute the eigenvalues and basis for each eigenspace of the matrix A, i.e. solve $\det(A - xI) = 0$ and for each λ find a basis for $E_\lambda = Null(A - \lambda I)$. **Note**: Since A is real, one expects that the eigenvalues are conjugates of each other, and the basis eigenvectors are also conjugate of each other.
    - Take one eigenpair $(\lambda = a + bi, \vec{v} = \vec{\alpha} + i\vec{\beta}$, and find the real part and imaginary part of $\vec{v}e^{\lambda t}$.
    - Write the general solution as:

    $$\vec{x}_{gh} = C_1 Re\left(\vec{v}e^{\lambda t}\right) + C_2 Im\left(\vec{v}e^{\lambda t}\right)$$

10 Worksheet 4b: Alternative Methods

In the following two problems, we are going to solve the following system:

$$X' = A_{2\times 2}X$$

where $X_{2\times 2} = \Phi$, i.e. Φ is an invertible matrix whose columns are the fundamental solutions to the system $\vec{x}' = A\vec{x}$.

1. **Diagonalization or Jordan Canonical Form**: Let $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$.

   - Find the decomposition of A as $A = TJT^{-1}$ where J is either a diagonal matrix or a Jordan Canonical Form matrix, i.e. an upper triangular matrix with 1 in the top right corner. **Solution**: First, we need to find the characteristic polynomial of A:

     $$\text{char}(A)(x) = \det(A - xI) = x^2 - 2x - 3 = (x-3)(x+1)$$

     whose roots are the eigenvalues $\lambda = -1, 3$. Then we need to find basis for each eigenspace:

     $$E_{-1} = Null(A + I) = \left\langle \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\rangle$$

     $$E_3 = Null(A - 3I) = \left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle$$

     Then we know A is diagonalizable and thus the matrix T has columns equal to the basis eigenvectors, and J is a diagonal matrix with eigenvalues in the diagonal.

     $$A = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}^{-1}$$

   - Replace $\vec{x}$ by $\vec{x} = T\vec{y}$, so $\vec{x}' = T\vec{y'}$. Therefore

     $$\vec{y}' = T^{-1}AT\vec{y} = J\vec{y}$$

     Solve for $\vec{y}$. **Solution**: Then the system becomes $\vec{y}' = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \vec{y}$.

   - Solve for $\vec{y}$ either by solving two independent equations (if A is diagonalizable) or by applying backward substitution (if A is not diagonalizable). **Solution**: The first equation is $y_1' = -y_1 \Rightarrow y_1 = C_1 e^{-t}$. The second equation is $y_2' = 3y_1 \Rightarrow y_2 = C_2 e^{3t}$.

   - Plug back in $\vec{x} = T\vec{y}$ to find $\vec{x}$. **Solution**: Then $\vec{x} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} C_1 e^{-t} \\ C_2 e^{3t} \end{bmatrix}$. Therefore,

     $$x_1(t) = C_1 e^{-t} + C_2 e^{3t}$$

     $$x_2(t) = -2C_1 e^{-t} + 2C_2 e^{3t}$$

     .

   - Find the fundamental matrix $X = \Phi$ whose columns are the columns are the fundamental solutions of the system above.

     $$\Phi = \begin{bmatrix} e^{-t} & e^{3t} \\ -2e^{-t} & 2e^{3t} \end{bmatrix}$$

2. **Exponential Method**: Just like the initial value problem $x' = ax$ where $x(0) = x_0$ has solution $x = x_0 e^{at}$, then the problem $\vec{x}' = A\vec{x}$ where $\vec{x}(0) = \vec{x}_0$ has solution $\vec{x}_0 \exp(At)$.

   Define
$$\exp(At) = \sum_{t\geq 0} \frac{(At)^n}{n!} \Rightarrow \frac{d}{dt}\exp(At) = A\exp(At)$$
   And
$$\exp(At)|_{t=0} = I$$

   - Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Find $\exp(At)$. Note: for diagonal matrix, this is just the exponential of its entries times t. Moreover, the columns of the exponential matrix satisfy the IVP's whose common equation is $\vec{x}' = A\vec{x}$, and its IC's are $\vec{x}(0) = \vec{e}_1$ and $\vec{x}(0) = \vec{e}_2$ respectively.

   In general in order to find the exponential of a Jordan canonical form $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ write $J = \lambda I + N$ where $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. These upper triangular matrix whose only nonzero entries are 1 on some entries above the main diagonal are nilpotent, meaning $N^n = 0_{n\times n}$ where $n = \dim(A)$. Then by the binomial theorem:
$$J^m = \sum_{k=0}^{m} \binom{m}{k} \lambda^{m-k} I^{m-k} N^k = \sum_{k=0}^{\min(n,m)} \binom{m}{k} \lambda^{m-k} N^k$$
   which for the case of $n = \dim(A) = 2$, one has $J^m = \lambda^m I + m\lambda^{m-1} N$

11 Worksheet 5: Graphing Homogeneous Linear System of Ordinary Differential Equations with constant real coefficients

In each of the following problems, assume you are given a system of the form $\vec{x}' = A\vec{x}$. Assume that you have already found basis for each eigenspace of A. Graph the phase plane of each linear system: (1) Graph the eigenvector(s) and label it (them), (2) Graph a few trajectories describing the system.

1. **Two different eigenvalues:**
   - **Two Positive eigenvalues**: Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ whose eigenspaces are $E_3 = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$ and $E_2 = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$.
   - **Two Negative eigenvalues**: Let $A = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}$ whose eigenspaces are $E_{-3} = \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle$ and $E_{-2} = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$.
   - **One positive and one negative eigenvalue**: Let $A = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$ whose eigenspaces are $E_3 = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$ and $E_{-3} = \left\langle \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\rangle$.
2. **A single eigenvalue and a two dimensional eigenspace**:
   - **A positive eigenvalue**: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ whose only eigenspace is $E_1 = \mathbb{R}^2$.
   - **A negative eigenvalue**: Let $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2$ whose only eigenspace is $E_{-1} = \mathbb{R}^2$.
3. **A single eigenvalue and a one dimensional eigenspace**:
   - **A positive eigenvalue**: Let $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ whose only eigenspace is $E_2 = \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$.
   - **A negative eigenvalue**: Let $A = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix}$ whose only eigenspace is $E_{-2} = \left\langle \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\rangle$.
4. **Two Complex non-imaginary eigenvalues**:
   - **Positive real part**: Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ whose eigenspaces are $E_{1\pm i} = \left\langle \begin{bmatrix} 1 \\ \pm i \end{bmatrix} \right\rangle$.
   - **Negative real part**: Let $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$ whose eigenspaces are $E_{-1\pm i} = \left\langle \begin{bmatrix} 1 \\ \pm i \end{bmatrix} \right\rangle$.
5. **Two complex imaginary eigenvalues**: Let $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$ whose eigenspaces are $E_{\pm i} = \left\langle \begin{bmatrix} 1 \\ -1 \pm i \end{bmatrix} \right\rangle$.

12 Worksheet 6: Euler Homogeneous Linear System of Ordinary Differential Equations with constant real coefficients

For each of the following system, find the general solution of the system by finding the eigen-pairs $(\lambda, \vec{v})$ of the constant matrix. The solutions are of the form: $\vec{v}t^{\lambda}$. Indeed,

$$\vec{x} = \vec{v}t^{\lambda} \Rightarrow t\vec{x}' = \lambda\vec{v}t^{\lambda} = A\vec{v}t^{\lambda} = \vec{x}$$

1. **Two different Eigenvalues**: Let $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$, and solve $t\vec{x}' = A\vec{x}$ where $t > 0$. The general solution will be of the form:

$$\vec{x}_{gh} = C_1\vec{v}_1 t^{\lambda_1} + C_2\vec{v}_2 t^{\lambda_2}$$

2. **A single eigenvalue and two LI eigenvectors**: Let $A = I$, and solve $t\vec{x}' = A\vec{x}$ where $t > 0$. The general solution will be of the form:

$$\vec{x}_{gh} = C_1\vec{v}_1 t^{\lambda} + C_2\vec{v}_2 t^{\lambda}$$

3. **A single eigenvalue and one eigenvector**: Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$, and solve $t\vec{x}' = A\vec{x}$ where $t > 0$.

$$\vec{x}_{gh} = C_1\vec{v}t^{\lambda} + C_2(\vec{v}t^{\lambda}\ln(t) + \vec{w}t^{\lambda})$$

where $\vec{v}$ and $\vec{w}$ are a basis eigenvector and a generalized eigenvalue for A and λ.

4. **Two complex eigenvector**: Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, and solve $t\vec{x}' = A\vec{x}$ where $t > 0$. If the eigenvectors are decomposed as $\vec{\alpha} \pm i\vec{\beta}$ and the eigenvalues is decomposed as $\lambda = a \pm bi$, then the general solution will be of the form:

$$\vec{x}_{gh} = t^{a}\left[C_1\left(\vec{\alpha}\cos(b\ln(t)) - \vec{\beta}\sin(b\ln(t))\right) + C_2\left(\vec{\alpha}\sin(b\ln(t)) + \vec{\beta}\cos(b\ln(t))\right)\right]$$

13 Worksheet 7: Nonhomogeneous Linear System of Ordinary Differential Equations with constant real coefficients

$$\vec{x}' = A\vec{x} + g(t)$$

1. **Method 1**: Find the general solution to the following system using the *diagonalization method*

$$\vec{x}' = A\vec{x} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} e^t \\ t \end{bmatrix}$$

Also find the particular solution satisfying $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

- Find the general solution x_{gh} to the homogeneous system $\vec{x}' = A\vec{x}$. Assume it is of the form $\vec{x} = C_1\vec{x}^{(1)} + C_2\vec{x}^{(2)}$.
- Decompose the matrix $A = TJT^{-1}$ where J is a diagonal Matrix, $J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, or the Jordan canonical form of A, $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.
- Replace $\vec{x} = T\vec{y}$ to obtain $\vec{y} = J\vec{y} + T^{-1}\vec{g}$. Solve this Y-system using backward substitution: Write the second equation of this system. It should be a linear non-homogeneous first order ordinary differential equation on y_2. Solve it for y_2, and plug it in the first equation of this system. It should be another linear non-homogeneous first order ordinary differential equation now on y_1. Solve for y_1. Therefore, now you know $\vec{y}$. Plug it in $\vec{x} = T\vec{y}$ in order to find $\vec{x}$.
- Finally plug the initial condition in the solution for $\vec{x}$ and find the constants that satisfy the initial condition.

2. **Method 2**: Find the general solution the following system using the *undetermined coefficient method*

$$\vec{x}' = A\vec{x} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} e^t \\ t \end{bmatrix}$$

Also find the particular solution.

- Find the general solution $\vec{x}_{gh}$ to the homogeneous system $\vec{x}' = A\vec{x}$. Assume it is of the form $\vec{x} = C_1\vec{x}^{(1)} + C_2\vec{x}^{(2)}$.
- Assume the particular solution to the non-homogeneous system has a form similar to that of g taking into consideration $\vec{x}_{gh}$. The initial attempt should be $\vec{x}_p = A_1e^t + A_2t + A$, the first term from $g_1(t) = e^t$ and the other two terms from $g_2(t) = t$. However, if any of these terms is already part of $\vec{x}_{gh}$, then multiply the term by the smallest power of t that makes it a new term not part of $\vec{x}_{gh}$. Plug this and determine the coefficients A_1, A_2, A_3 coefficients.
- Form the general solution to the non-homogenous system $\vec{x}_g = \vec{x}_{gh} + \vec{x}_p$.
- Finally plug the initial condition in the solution for $\vec{x}_g$ and find the constants that satisfy the initial condition.

14 Sample Quiz

14.1 Problem 1

Consider the matrix $A = \begin{pmatrix} 5 & 2 \\ -4 & 1 \end{pmatrix}$ whose eigenspaces are: $E_{3\pm 2i} = \left\langle \begin{pmatrix} 1 \\ -1 \pm i \end{pmatrix} \right\rangle$. Write the general solution $\vec{x}_{gh}$ to the following 1st order linear homogeneous system: $t\frac{d\vec{x}}{dt} = A \cdot \vec{x}(t)$ for $t > 0$. Note: you must write it as the span of 2 real-valued solutions.

14.2 Solution

By the form of the system, we know that the each of the fundamental solutions come from a function of the form $\vec{v}t^{\lambda}$. Since the eigenvalues and basis-eigenvectors are complex, we can ignore one set and focus in only one of them:

$$\vec{x} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} t^{3+2i}$$

The vector factor is $\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} i$. The function factor splits as

$$t^{3+2i} = t^3 e^{2\ln(t)i} = t^3[\cos(2\ln(t)) + i\sin(2\ln(t))]$$

Therefore,

$$\vec{x} = \left[t^3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2\ln(t)) - t^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\ln(t))\right] + \left[t^3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(2\ln(t)) + t^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2\ln(t))\right] i$$

By the theorems of section 6.4, we know that both the real and imaginary parts are solutions to the system too. Moreover, they are linearly independent, and since any set of fundamental solutions has cardinality 2, then

$$\vec{x}_{gh} = C_1 \left[t^3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2\ln(t)) - t^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\ln(t))\right] + C_2 \left[t^3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(2\ln(t)) + t^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2\ln(t))\right]$$

14.3 Problem 2

Consider the matrix $A = \begin{pmatrix} 5 & 2 \\ -4 & 1 \end{pmatrix}$ whose eigenspaces are: $E_{3\pm 2i} = \left\langle \begin{pmatrix} 1 \\ -1 \pm i \end{pmatrix} \right\rangle$. Write the general solution $\vec{x}_{gh}$ to the following 1st order linear homogeneous system: $\frac{d\vec{x}}{dt} = A \cdot \vec{x}(t)$. Note: you must write it as the span of 2 real-valued solutions.

14.4 Solution

By the form of the system, we know that the each of the fundamental solutions come from a function of the form $\vec{v}e^{\lambda t}$. Since the eigenvalues and basis-eigenvectors are complex, we can ignore one set and focus in only one of them:

$$\vec{x} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} e^{(3+2i)t}$$

The vector factor is $\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} i$. The function factor splits as

$$e^{(3+2i)t} = e^{3t+2ti} = e^{3t}e^{2ti} = e^{3t}[\cos(2t) + i\sin(2t)]$$

Therefore,

$$\vec{x} = \left[e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(2t) - e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin(2t)\right] + \left[e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin(2t) + e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(2t)\right] i$$

By the theorems of section 6.4, we know that both the real and imaginary parts are solutions to the system too. Moreover, they are linearly independent, and since any set of fundamental solutions has cardinality 2, then

$$\vec{x}_{gh} = C_1\left[e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(2t) - e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin(2t)\right] + C_2\left[e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin(2t) + e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(2t)\right]$$

15 Midterm Review

15.1 Problem * similar to extra credit

Let $P = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$. Let $A = PDP^{-1} = \begin{bmatrix} -26 & 15 \\ -50 & 29 \end{bmatrix}$. Find the general solution to $A\vec{x}'' = \vec{x}$ where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

15.2 Solution

Notice that A is diagonalizable where the eigenspaces are $E_4 = \langle (1,2)^T \rangle$ and $E_{-1} = \langle (3,5)^T \rangle$.

One possible solution is to find what would be the square root of the matrix A, i.e. another matrix B such that $B^2 = A$, which is $PD^{1/2}P^{-1}$, i.e.:

$$B = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -10+6i & 6-3i \\ -20+10i & 12-5i \end{bmatrix}$$

Then we propose as a solution:

$$\vec{x} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{it}$$

First we prove that it is a solution. Let $\vec{x}^{(1)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$, then

$$\vec{x}^{(1)''} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} = A \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} = A\vec{x}^{(1)}$$

Similarly, let $\vec{x}^{(2)} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{it}$, then

$$\vec{x}^{(2)''} = -\begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{it} = A \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{it} = A\vec{x}^{(2)}$$

Then by the principle of superposition, any linear combination of these two is a solution. However, we are looking for a real solution, and the second one is a complex solution. So, right now we know that the following is a solution:

$$\vec{x} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} \cos(t) + C_3 \begin{bmatrix} 3 \\ 5 \end{bmatrix} \sin(t)$$

But does it make sense to have a family of three functions as a solution to a linear system of second order ordinary differential equations? Or should it be four rather than three?

Let's try $\vec{x}^{(4)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} te^{2t}$ as the fourth alternative and see what happens:

$$\vec{x}^{(4)'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} (1+2t)e^{2t} \Rightarrow \vec{x}^{(4)''} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} (1+t)e^{2t}$$

which is clearly not a solution.

Now, let's look back at A and its eigenvalues 4 and -1. From them we obtained the eigenvalues of $B = A^{1/2}$, which are 2 and i. But what if we take the negative eigenvalues -2 and $-i$, we would obtain a solution of the form $\vec{x}^{(4)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}$. Therefore, the general solution is:

$$\vec{x}_{gh} = C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t} + C_3 \begin{bmatrix} 3 \\ 5 \end{bmatrix} \cos(t) + C_4 \begin{bmatrix} 3 \\ 5 \end{bmatrix} \sin(t)$$

15.3 Problem

Consider the matrix $A = \begin{pmatrix} 5 & 2 \\ -4 & 1 \end{pmatrix}$ whose eigenspaces are: $E_{3\pm 2i} = \left\langle \begin{pmatrix} 1 \\ -1 \pm i \end{pmatrix} \right\rangle$. Write the general solution $\vec{x}_g$ to the following 1st order linear nonhomogeneous system: $t\frac{d\vec{x}}{dt} = A \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ t \end{bmatrix}$ for $t > 0$. Note: you must write the general solution to the homogeneous system as the span of 2 real-valued solutions.

15.4 Solution

First, we solve the homogeneous system. We know that each of the fundamental solutions to the homogeneous system comes from a function of the form $\vec{v}t^{\lambda}$. Since the eigenvalues and basis-eigenvectors are complex, we can ignore one set and focus in only one of them:

$$\vec{x} = \begin{pmatrix} 1 \\ -1+i \end{pmatrix} t^{3+2i}$$

The vector factor is $\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} i$. The function factor splits as

$$t^{3+2i} = t^3 e^{2\ln(t)i} = t^3[\cos(2\ln(t)) + i\sin(2\ln(t))]$$

Therefore,

$$\vec{x} = \left[t^3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2\ln(t)) - t^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\ln(t))\right] + \left[t^3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(2\ln(t)) + t^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2\ln(t))\right] i$$

By the theorems of section 6.4, we know that both the real and imaginary parts are solutions to the system too. Moreover, they are linearly independent, and since any set of fundamental solutions has cardinality 2, then

$$\vec{x}_{gh} = C_1 \left[t^3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2\ln(t)) - t^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2\ln(t))\right] + C_2 \left[t^3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(2\ln(t)) + t^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2\ln(t))\right]$$

Notice that A is diagonalizable (even if the resulting diagonal matrix has complex entries.) Decompose $A = T^{-1}DT$ and replace $\vec{x} = T\vec{y}$. Then $tT\vec{y}' = AT\vec{y} + g(t)$ and thus

$$t\vec{y}' = \begin{bmatrix} 3+2i & 0 \\ 0 & 3-2i \end{bmatrix} \vec{y} + \begin{bmatrix} 1 & 1 \\ -1+i & -1-i \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 3+2i & 0 \\ 0 & 3-2i \end{bmatrix} \vec{y} + \begin{bmatrix} t/2i \\ -t/2i \end{bmatrix}$$

In other words:

$$ty_1' = (3+2i)y_1 + \frac{t}{2i}; ty_2' = (3-2i)y_2 - \frac{t}{2i}$$

Actually, one only needs to solve the equation on y_1 because they are conjugate of each other. Rewrite the equation as $y_1' - \frac{3+2i}{t}y_1 = \frac{1}{2i}$, and multiply by the integrating factor $\mu = t^{-(3+2i)}$ to obtain:

$$\frac{d}{dt}\left(y_1 t^{-(3+2i)}\right) = \frac{t^{-(3+2i)}}{2i} \Rightarrow y_1 t^{-(3+2i)} = t^{-(2+2i)}-2i(2+2i) + C$$

Therefore, a particular solution is obtained by letting the constant of integration to be 0:

$$(y_1)_p = \frac{t}{8}(1+i)$$

Therefore, $(y_2)_p = \frac{t}{8}(1-i)$, i.e. $\vec{y}_p = \frac{t}{8}\begin{bmatrix}1+i\\1-i\end{bmatrix}$. Yet we are looking for a particular solution $\vec{x}_p$, which is:

$$\vec{x}_p = \frac{t}{8}\begin{bmatrix}1 & 1\\-1+i & -1-i\end{bmatrix}\begin{bmatrix}1+i\\1-i\end{bmatrix} = \frac{t}{4}\begin{bmatrix}1\\-2\end{bmatrix}$$

Finally, the general solution to the non-homogeneous system is:

$$\begin{aligned}\vec{x}_g &= C_1\left[t^3\begin{pmatrix}1\\-1\end{pmatrix}\cos(2\ln(t)) - t^3\begin{pmatrix}0\\1\end{pmatrix}\sin(2\ln(t))\right]\\ &\quad +C_2\left[t^3\begin{pmatrix}1\\-1\end{pmatrix}\sin(2\ln(t)) + t^3\begin{pmatrix}0\\1\end{pmatrix}\cos(2\ln(t))\right]\\ &\quad +\tfrac{t}{4}\begin{bmatrix}1\\-2\end{bmatrix}\end{aligned}$$

15.5 Problem

Consider the matrix $A = \begin{pmatrix}5 & 2\\-4 & 1\end{pmatrix}$ whose eigenspaces are: $E_{3\pm 2i} = \left\langle\begin{pmatrix}1\\-1\pm i\end{pmatrix}\right\rangle$. Find the solution to the IVP with the following 1st order linear homogeneous system: $\frac{d\vec{x}}{dt} = A\cdot\vec{x}(t)$ and initial condition $\vec{x}(0) = \begin{bmatrix}\pi\\e\end{bmatrix}$. Also classify the critical point and determine its stability.

15.6 Solution

Since the eigenvalues are complex non-imaginary numbers $3\pm 2i$ with positive real part 3, then the critical point is a Spiral Unstable point. By the form of the system, we know that the each of the fundamental solutions come from a function of the form $\vec{v}e^{\lambda t}$. Since the eigenvalues and basis-eigenvectors are complex, we can ignore one set and focus in only one of them:

$$\vec{x} = \begin{pmatrix}1\\-1+i\end{pmatrix}e^{(3+2i)t}$$

The vector factor is $\begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}i$. The function factor splits as

$$e^{(3+2i)t} = e^{3t+2ti} = e^{3t}e^{2ti} = e^{3t}[\cos(2t) + i\sin(2t)]$$

Therefore,

$$\vec{x} = \left[e^{3t}\begin{pmatrix}1\\-1\end{pmatrix}\cos(2t) - e^{3t}\begin{pmatrix}0\\1\end{pmatrix}\sin(2t)\right] + \left[e^{3t}\begin{pmatrix}1\\-1\end{pmatrix}\sin(2t) + e^{3t}\begin{pmatrix}0\\1\end{pmatrix}\cos(2t)\right]i$$

By the theorems of section 6.4, we know that both the real and imaginary parts are solutions to the system too. Moreover, they are linearly independent, and since any set of fundamental solutions has cardinality 2, then

$$\vec{x}_{gh} = C_1 \left[e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2t) - e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2t) \right] + C_2 \left[e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(2t) + e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2t) \right]$$

By plugging the values $t = 0$ and $\vec{x} = \begin{bmatrix} \pi \\ e \end{bmatrix}$, we obtain that $C_1 = \pi$ and $C_2 = (e + \pi)$, so

$$\vec{x} = \pi e^{3t} \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2t) \right] + (e+\pi) e^{3t} \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(2t) \right]$$

15.7 Problem

Given the Matrix $A = \begin{bmatrix} 1 & 2 \\ 9 & 4 \end{bmatrix}$, find $\Phi = \exp(At)$.

15.8 Solution

Since the matrix is not diagonal, it is not true that $\exp(At) = \begin{bmatrix} e^{t} & e^{2t} \\ e^{9t} & e^{4t} \end{bmatrix}$. Recall that the matrix satisfies $\Phi' = A\Phi$ where $\Phi(0) = I_2$.

Notice that the characteristic polynomial of A is $x^2 - 5x - 14 = (x-7)(x+2)$. The eigenspaces are $E_{-2} = \langle (2, -3)^T \rangle$ and $E_7 = \langle (1, 3)^T \rangle$. Therefore, the general solution to the system $\vec{x}' = A\vec{x}$ is:

$$\vec{x}_g = A(2, -3)^T e^{-2t} + B(1, 3)^T e^{7t}$$

Solving the IVP where $\vec{x}(0) = \vec{e}_1$, we obtain the system:

$$\begin{aligned} 2A + B &= 1 \\ -3A + 3B &= 0 \end{aligned}$$

whose solution is $A = B = 1/3$. Solving the IVP where $\vec{x}(0) = \vec{e}_2$, we obtain the system:

$$\begin{aligned} 2A + B &= 0 \\ -3A + 3B &= 1 \end{aligned}$$

whose solution is $A = -1/9$ and $B = 2/9$. Therefore,

$$\Phi = \exp(At) = \begin{bmatrix} \frac{2}{3}e^{-2t} + \frac{1}{3}e^{7t} & -\frac{2}{9}e^{-2t} + \frac{2}{9}e^{7t} \\ -e^{-2t} + e^{7t} & \frac{1}{3}e^{-2t} + \frac{2}{3}e^{7t} \end{bmatrix}$$

Alternative, using the diagonal decomposition of A:

$$A = \frac{1}{9} \begin{bmatrix} 2 & 1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 3 & 2 \end{bmatrix}$$

Therefore,

$$\Phi = \exp(At) = \frac{1}{9} \begin{bmatrix} 2 & 1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{7t} \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}e^{-2t} + \frac{1}{3}e^{7t} & -\frac{2}{9}e^{-2t} + \frac{2}{9}e^{7t} \\ -e^{-2t} + e^{7t} & \frac{1}{3}e^{-2t} + \frac{2}{3}e^{7t} \end{bmatrix}$$

16 Complete Problem

Assume one wants to solve the following two problems: $\vec{x}' = A\vec{x} + \vec{g}(t)$ and $t\vec{X}' = A\vec{X} + \vec{g}(\log(t))$ where one knows:

$$A = \begin{bmatrix} 11 & 18 \\ -6 & -10 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}^{-1}; \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}; \vec{g} = \begin{bmatrix} e^{-t} \\ 1 \end{bmatrix}$$

1. Standard Solution:

$$\boxed{\vec{x}_{gh} = C_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-t}}$$

2. Exponential:

$$\begin{aligned} \exp(At) &= \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 4\,e^{(2\,t)} - 3\,e^{(-t)} & 6\,e^{(2\,t)} - 6\,e^{(-t)} \\ -2\,e^{(2\,t)} + 2\,e^{(-t)} & -3\,e^{(2\,t)} + 4\,e^{(-t)} \end{bmatrix} \end{aligned}$$

Therefore,

$$\boxed{\vec{x}_{gh} = C_1 \begin{bmatrix} 4\,e^{(2\,t)} - 3\,e^{(-t)} \\ -2\,e^{(2\,t)} + 2\,e^{(-t)} \end{bmatrix} + C_2 \begin{bmatrix} 6\,e^{(2\,t)} - 6\,e^{(-t)} \\ -3\,e^{(2\,t)} + 4\,e^{(-t)} \end{bmatrix}}$$

3. Partial Diagonalization: $\vec{x} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \vec{y} \Rightarrow \vec{y}' = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \vec{y} \Rightarrow \vec{y} = \begin{bmatrix} C_1 e^{2t} \\ C_2 e^{-t} \end{bmatrix} \Rightarrow \vec{x}_{gh} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} C_1 e^{2t} \\ C_2 e^{-t} \end{bmatrix} \Rightarrow$

$$\boxed{\vec{x}_{gh} = \begin{bmatrix} 2C_1 e^{2t} + 3C_2 e^{-t} \\ -C_1 e^{2t} - 2C_2 e^{-t} \end{bmatrix}}$$

4. Variation of Parameters: From Standard Solution $\Phi = \begin{bmatrix} 2e^{2t} + 3e^{-t} \\ -e^{2t} - 2e^{-t} \end{bmatrix}$

$$\Rightarrow \Phi^{-1} = \begin{bmatrix} 2\,e^{(-2\,t)} & 3\,e^{(-2\,t)} \\ -e^{t} & -2\,e^{t} \end{bmatrix} \Rightarrow \Phi^{-1}\vec{g} = \begin{bmatrix} 3\,e^{(-2\,t)} + 2\,e^{(-3\,t)} \\ -2\,e^{t} - 1 \end{bmatrix} \Rightarrow \int \Phi^{-1}\vec{g} = \begin{bmatrix} -\frac{3}{2}\,e^{(-2\,t)} + \frac{2}{3}\,e^{(-3\,t)} \\ -2\,e^{t} - t \end{bmatrix}$$

$$\Rightarrow \vec{x}_p = \Phi \int \Phi^{-1}\vec{g} = \begin{bmatrix} -\frac{1}{3}\left(9\,e^{(-2\,t)} - 4\,e^{(-3\,t)}\right)e^{(2\,t)} - 3\left(t + 2\,e^{t}\right)e^{(-t)} \\ \frac{1}{6}\left(9\,e^{(-2\,t)} - 4\,e^{(-3\,t)}\right)e^{(2\,t)} + 2\left(t + 2\,e^{t}\right)e^{(-t)} \end{bmatrix}$$

$$\Rightarrow \boxed{\vec{x}_p = \begin{bmatrix} -3te^{-t} - \frac{4}{3}e^{-t} - 9 \\ 2te^{-t} + \frac{2}{3}e^{-t} + \frac{11}{2} \end{bmatrix}}$$

5. Full Diagonalization (Performed at once): $\vec{x} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \vec{y} \Rightarrow \vec{y}' = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \vec{y} + \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \vec{g}$

$$\begin{aligned} y_1' &= 2y_1 + 2e^{-t} + 3 \\ y_2' &= -y_2 - e^{-t} - 2 \end{aligned}$$

$$y_1e^{-2t} = \int 2e^{-3t} + 3e^{-2t}dt \Rightarrow y_1 = -\frac{2}{3}e^{-t} - \frac{3}{2} + C_1e^{2t}$$

$$y_2e^{t} = \int -1 - 2e^{t}dt \Rightarrow y_2 = -te^{-t} - 2 + C_2e^{-t}$$

$$\Rightarrow \vec{x}_g = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -\frac{2}{3}e^{-t} - \frac{3}{2} + C_1e^{2t} \\ -te^{-t} - 2 + C_2e^{-t} \end{bmatrix}$$

$$\boxed{\vec{x}_g = \begin{bmatrix} -3te^{-t} - \frac{4}{3}e^{-t} - 9 + 2C_1e^{2t} + 3C_2e^{-t} \\ 2te^{-t} + \frac{2}{3}e^{-t} + \frac{11}{2} + C_1e^{2t} - 2C_2e^{-t} \end{bmatrix}}$$

6. Undetermined Coefficients: $\vec{x}_p = \vec{a}te^{-t} + \vec{b}e^{-t} + \vec{c}$. Rewrite $\vec{g} = \vec{e}_1e^{-t} + \vec{e}_2$

$$\Rightarrow \vec{a}e^{-t} - \vec{a}te^{-t} - \vec{b}e^{-t} = A\vec{a}te^{-t} + A\vec{b}e^{-t} + A\vec{c} + \vec{e}_1e^{-t} + \vec{e}_2$$

Coefficients of te^{-t} : $-\vec{a} = A\vec{a}$.
Coefficients of e^{-t} : $\vec{a} - \vec{b} = A\vec{b} + \vec{e}_1$.
Constant vectors: $\vec{0} = A\vec{c} + \vec{e}_2$.
Solve for $\vec{c}$ in the last equation to obtain: $\vec{c} = -A^{-1}\vec{e}_2 = -\begin{bmatrix} 9 \\ -\frac{11}{2} \end{bmatrix}$.
Solve for $\vec{a}$ in the first equation to obtain: $\vec{a} \in E_{-1} \Rightarrow \vec{a} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.
Solve for $\vec{b}$: $(A+I)\vec{b} = \vec{a} - \vec{e}_1$. The resulting system is:

$$\begin{aligned} 12b_1 + 18b_2 &= -4 \\ -6b_1 - 12b_2 &= 2 \end{aligned}$$

Add 3/2 of the second equation from the first: $3b_1 = 1 \Rightarrow b_1 = -\frac{1}{3} \Rightarrow b_2 = 0$. Therefore,

$$\boxed{\vec{x}_g = \begin{bmatrix} -3te^{-t} - \frac{1}{3}e^{-t} - 9 \\ 2te^{-t} + \frac{11}{2} \end{bmatrix}}$$

Notice the difference with the previous solutions is $-\frac{1}{3}\begin{bmatrix} 3 \\ -2 \end{bmatrix}e^{-t}$, which is part of the homogeneous solution.

7. Therefore,

$$\vec{x}_g = \begin{bmatrix} -3te^{-t} - \frac{4}{3}e^{-t} - 9 + 2C_1e^{2t} + 3C_2e^{-t} \\ 2te^{-t} + \frac{2}{3}e^{-t} + \frac{11}{2} + C_1e^{2t} - 2C_2e^{-t} \end{bmatrix}$$

The claim is that $\vec{X} = \vec{x}(\log(t))$ solves the system $t\vec{X}' = A\vec{X} + \vec{g}(\log(t))$ rather than the system where g does not get modified. Indeed,

$$t\vec{X}' = \vec{x}'(\log(t)) = A\vec{x}(\log(t)) + \vec{g}(e^{\log(t)}) = A\vec{X} + \vec{g}(t)$$

$$\boxed{\vec{X}_g = \begin{bmatrix} -3t^{-1}\log(t) - \frac{4}{3}t^{-1} - 9 + 2C_1t^2 + 3C_2t^{-1} \\ 2t^{-1}\log(t) + \frac{2}{3}t^{-1} + \frac{11}{2} + C_1t^2 - 2C_2t^{-1} \end{bmatrix}}$$

17 Chapter 10 Glossary and Map

- Boundary Value Problem: Solving $ax'' + bx' + cx = g(t)$ for t in $[\alpha, \beta]$ with the boundary conditions $x(\alpha) = x_0$ and $x(\beta) = x_1$.

- Eigenvalue problems: $y'' + \lambda y = 0$ with either (1) boundary conditions $x(0)$ and $x(L)$. Or initial conditions $x(0)$ and $x'(L)$; $x'(0)$ and $x(L)$; or $x'(0)$ and $x'(L)$.

- Fourier Series: A basis (up to some punctual differences) for periodic functions f where f and f' are piecewise continuous.

- Parity Reviewed: Odd and Even functions.

- The HEAT equation and separation of variables.

  - Equation: $\alpha^2 u_{xx} = u_t$ for $0 < x < L$ and $t > 0$. Note: α^2 is called the thermal diffusivity. Homogeneous Boundary Conditions: $u(0,t) = 0$ and $u(L,t) = 0$ (or the alternatives.) Initial Condition: $u(x,0) = f(x)$.
  - Separation of Variables $u = XT$, and separation of equation: An eigenvalue problem $X'' + \lambda X = 0$, and a linear homogeneous equation $T' + \alpha^2 \lambda T = 0$.
  - Alternative Non-homogeneous Initial Conditions: $u(0,t) = T_1$ and $u(L,t) = T_2$. Then $u = (T_2 - T_1)\frac{x}{L} + T_1$ and use the Fourier expansion of $f(x) - \left((T_2 - T_1)\frac{x}{L} + T_1\right)$ in the computations.
  - Review the section of More general problems.

- The WAVE equation and separation of variables.

  - Equation: $\alpha^2 u_{xx} = u_{tt}$ for $0 < x < L$ and $t > 0$. Homogeneous Boundary Conditions: $u(0,t) = 0$ and $u(L,t) = 0$ (or the alternatives.) Boundary Conditions: $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.
  - Separation of Variables $u = XT$, and separation of equation: Two eigenvalue problems $X'' + \lambda X = 0$ and $T'' + \alpha^2 \lambda T = 0$.

- Two dimensional LAPLACE's equation.

  - Equation $u_{xx} + u_{yy} = 0$. **Dirichlet Problem** when one has prescribed boundary values or **Neumann problem** when one has prescribed normal derivative boundary values.
  - The initial conditions are usually 0 on three edges and nonzero in one edge. Nonzero conditions on more than one edge can be separated into multiple subproblems and the sum of their solutions is the solution of the problem.
  - **Dirichlet Problem on a Rectangle**: Separation of Variables $u = XY$ changes the problem into two eigenvalue problems $X'' - \lambda X = 0$ and $Y'' + \lambda Y = 0$.
  - **Dirichlet Problem on a Circle** (Only if time allows): $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$ with $0 \leq r \leq R$ and $0 \leq \theta < 2\pi$. The conditions on $u(R,\theta)$ being 2π-periodic.

18 Calendar of HW assignments for Chapter 10

6-th Friday	May 10th, 2019	Worksheet 1	Read 10.1
7-th Friday	May 17th, 2019	Worksheet 2,2b	Read 10.2-3
8-th Friday	May 24th, 2019	Worksheet 2c,3	Read 10.4
9-th Friday	May 31th, 2019	Worksheet 5,7	Read 10.5-7

Notice that I will not assignt worksheets 4 or 6 but it is your responsability to know how to solve this problems for the Final exam.

19 Worksheet 1: Eigenvalue Problems

Read section 10.1 of the textbook, in particular example 10.1.4, which is labeled as the eigen problem or the eigen equation. We always consider $\lambda \in \mathbb{R}$ and $L > 0$.

$$y'' + \lambda y = 0$$

with boundary conditions $y(0) = y(L) = 0$. If $\lambda \leq 0$, then there is only one solution, the trivial solution $y = 0$, so λ is not an eigenvalue.If $\lambda = \mu^2 > 0$, then $\lambda_n = n^2\pi^2/L^2$ and $y_n(x) = \sin(n\pi x/L)$ for $n = 1, 2, 3, \ldots$.

The method to solve this is by recalling how to solve a linear ordinary differential equation of order 2 with constant coefficients. If a set of fundamental solutions is formed by the pair of functions $e^{\pm kx}$, then it is better to replace them by $\sinh(kx)$ and $\cosh(kx)$.

1. Solve the eigen-problem given by $y'' + \lambda y = 0$ with initial conditions $y'(0) = 0$ and $y(L) = 0$. What is the solution to $y'' - \lambda y = 0$ with the same initial conditions?

2. Solve the eigen-problem given by $y'' + \lambda y = 0$ with initial conditions $y(0) = 0$ and $y'(L) = 0$. What is the solution to $y'' - \lambda y = 0$ with the same initial conditions?

3. Solve the eigen-problem given by $y'' + \lambda y = 0$ with initial conditions $y'(0) = 0$ and $y'(L) = 0$. What is the solution to $y'' - \lambda y = 0$ with the same initial conditions?

20 Worksheet 2: A Basis for periodic functions

Read the section 10.2 in the textbook. Recall that the inner product of two P-periodic functions u, v (where P is the fundamental period, i.e. the minimal $P > 0$ such that $u(x+P) = u(x)$) is denoted by $u \cdot v = (u, v)$ and it is given by:

$$(u, v) := \int_a^{a+P} u(x)v(x)dx$$

where a is any number of your choice.

1. Show that $u(x) = \cos\left(\frac{m\pi x}{L}\right)$ is orthogonal to any other function $v(x) = \cos\left(\frac{n\pi x}{L}\right)$ where $n \neq m$, i.e.: Show that $(u, v) = 0$. Also find the value of (u, v) whenever $n = m$.

2. Show that $u(x) = \sin\left(\frac{m\pi x}{L}\right)$ is orthogonal to any other function $v(x) = \sin\left(\frac{n\pi x}{L}\right)$ where $n \neq m$, i.e.: Show that $(u, v) = 0$. Also find the value of (u, v) whenever $n = m$.

3. Show that $u(x) = \cos\left(\frac{m\pi x}{L}\right)$ is orthogonal to any function $v(x) = \sin\left(\frac{n\pi x}{L}\right)$ regardless of whether $n \neq m$ or $n = m$.

Therefore, an (almost) orthogonal basis for L-periodic functions f where f and f' are piecewise continuous is

$$\left\langle 1, \sin\left(\frac{m\pi x}{L}\right), \cos\left(\frac{m\pi x}{L}\right)\right\rangle_{m\in\mathbb{Z}^+}$$

21 Worksheet 2b: Fourier Series Expansion

Read section 10.2 in the Textbook. An orthonormal basis for $\mathbb{R}^n$ is $\langle \vec{e}_i \rangle$. Additionally, any vector in $\mathbb{R}^n$ can always be expressed as $\vec{v} = \sum_i a_i \vec{e}_i$ where $a_i = (\vec{v}, \vec{e}_i)$. Similarly, if f is an $2L$-periodic smooth function, then

$$f(x) \approx \frac{a_0}{2} + \sum_{n \geq 1} a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)$$

where

$$\begin{array}{rcl} a_m & = & \frac{1}{L} \int_\alpha^{\alpha+2L} f(x) \cos\left(\frac{m\pi x}{L}\right) dx \\ b_m & = & \frac{1}{L} \int_\alpha^{\alpha+2L} f(x) \sin\left(\frac{m\pi x}{L}\right) dx \end{array}$$

The above decomposition is called the Fourier Series or Fourier expansion of f. The formulas are called Euler-Fourier formulas.

1. Find the Fourier series of a 6-periodic function f given by

$$f(x) = \begin{cases} 0 & \Leftarrow & -3 < x \leq -1 \\ 1 & \Leftarrow & -1 < x \leq 1 \\ 0 & \Leftarrow & 1 < x \leq 3 \end{cases}$$

2. Find the Fourier series of a $2L$-periodic function g given by

$$g(x) = \begin{cases} 0 & \Leftarrow & -L \leq x \leq 0 \\ x & \Leftarrow & 0 \leq x < L \end{cases}$$

3. Find the fundamental period and the Fourier expansion of the following functions:

   - $$h_1(x) = \cos(\pi x/2) + \cos(\pi x/3) + \cos(\pi x/5)$$
   - $$h_2(x) = \sin(2\pi x) + \sin(4\pi x) + \cos(6\pi x)$$

22 Worksheet 2c: Fourier Convergence

Read section 10.3 in the textbook. If f and f' are piecewise continuous and $2L$-periodic, then f has a Fourier series

$$\mathcal{F}(f)(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n\left(\frac{n\pi x}{L}\right)$$

Then the Fourier series $\mathcal{F}(f)$ converges to $f(x)$ are all points where f is continuous, and it converges to $[f(x+) + f(x-)]/2$ at all points where f is discontinuous.

1. For each of the following pairs (f, L) consisting on a function f whose minimal period is L, identify the set of points x where its Fourier transform $\mathcal{F}(f)$ differs from f. For each of these points x, compute $f(x)$ and $\mathcal{F}(f)(x)$.
   - $f(x) = \begin{cases} -1 & \Leftarrow -1 \leq x < 0 \\ 1 & \Leftarrow 0 \leq x < 1 \end{cases}$, $L = 2$.
   - $f(x) = \begin{cases} -x & \Leftarrow -1 \leq x < 0 \\ x & \Leftarrow 0 \leq x < 1 \end{cases}$, $L = 2$.
   - For $x \in [0, 3) : f(x) = \lfloor x \rfloor$, $L = 3$.
2. For each of the following functions f, identify if f is periodic. Identify the fundamental period of f, that means the minimal period $T > 0$ of f such that $f(x + T) = f(x)$, and there is not a smaller positive number than T satisfying this relation.
   - $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \sin(e\pi x)$.
   - $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \cos\left(\frac{15}{4}\pi x\right)$.
   - $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \sin(ax)+\cos(bx)$ where a and b are positive integers, in particular they are non-zero.
3. Let $f : \mathbb{R} \to \mathbb{R}$ be a 1-periodic function, i.e. $f(1+x) = f(x)$ where $f(x) = x$ for any $x \in [0, 1)$. Find the Fourier expansion $\mathcal{F}(f)(x)$ of $f(x)$, and determine the points where $\mathcal{F}(f)(x)$ differs from $f(x)$.

23 Worksheet 3: Using parity to extend the definition of a function

Read section 10.4 in the textbook.

1. Consider the function defined by $f_1(x) = x$ for $0 \leq x < 1$. Extend this definition of f_1 so that $f_1 : \mathbb{R} \to \mathbb{R}$ is even and 2-periodic. And compute its Fourier transform $\mathcal{F}(f)$.

2. Consider the function defined by $f_2(x) = x$ for $0 \leq x < 1$. Extend this definition of f_2 so that $f_2 : \mathbb{R} \to \mathbb{R}$ is odd and 2-periodic. And compute its Fourier transform $\mathcal{F}(f)$.

3. Consider the function defined by $f_3(x) = x$ for $0 \leq x < 1$. Extend this definition of f_3 so that $f_3 : \mathbb{R} \to \mathbb{R}$ is 1-periodic. And compute its Fourier transform $\mathcal{F}(f)$.

24 Worksheet 4: Some useful formulas

Read sections 10.3 and 10.4 in the textbook.

1. Problem 10.3.17 in version 10 of the textbook: Assume that $f(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$. Show that

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n\geq 1}(a_n^2 + b_n^2)$$

2. Problem 10.4.37 in version 10 of the textbook: Assume that f has a Fourier sine series $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)$ for $0 \leq x \leq L$. Show that:

$$\frac{2}{L}\int_{0}^{L} [f(x)]^2 dx = \sum_{n\geq 1} b_n^2$$

3. Assume $f(x)$ is a 2-periodic function whose Fourier series expansion is given by:

$$f(x) \approx \sum_{n\geq 1} \frac{1}{3^n} \sin(n\pi x)$$

Compute the exact value of $\int_0^1 [f(x)]^2 dx$.

25 Worksheet 5: Heat equation

Read sections 10.5 and 10.6 in the textbook.

1. Heat equation problem with homogeneous boundary conditions: Find the temperature $u(x,t)$ at any time (in seconds) in a metal road $L = 100$ cm long, which initially has a uniform temperature 20° C throughout and whose ends are maintained at 0° C for all $t > 0$. Assume $\alpha = 1$ cm/second which fits nicely into the equation.

   - Find the temperature of the hottest point after half of a minute. Give an exact value.
   - How long do you need to wait (in seconds) so the metal road has temperature less than 5° C at all points?

2. Heat equation problem with non-homogeneneous boundary conditions - Assume $u(0,t) = T_1$ and $u(L,t) = T_2$ for $t > 0$: Consider the heat conduction problem given by (1) The equation is $u_{xx} = u_t$ for $0 < x < 30 = L$ and $t > 0$. (2) The boundary conditions are $u(0,t) = T_1 = 10$ and $u(L = 30, t) = T_2 = 50$ for $t > 0$. (3) The initial condition is $u(x,0) = 60 - 2x$ for $0 < x < 30$.

26 Worksheet 6: The General Heat equation

Read section 10.6 in the textbook.

1. This is problem 10.6.18 in the 10 edition of the textbook. Consider the problem $X'' + \lambda X = 0$ where X is a function of x, and $X'(0) = 0$ and $X'(L) = 0$. Let $\lambda = \mu^2$, where $\mu = \nu + i\sigma$ with ν and σ real. Show that if $\sigma \neq 0$, then the only solution to these problem is the trivial solution $X(x) = 0$.

2. This is the last example in section 10.6 in the 10 edition of the textbook. Find all possible $\lambda \geq 0$ for which the following boundary value problem has a solution:

$$X'' + \lambda X = 0$$

$$X'(0) - h_1 X(0) = 0$$
$$X'(L) + h_2 X(L) = 0$$

where h_1, h_2 are positive real constants. Try to solve this problem for $h_1 = h_2 = 1/2$.

27 Worksheet 7: The Wave Equation

Read section 10.7 in the textbook.

1. Consider a vibrating string of Length $L = 20$ satisfying the wave equation $4u_{xx} = u_{tt}$ where $u(x,0) = f(x) = \begin{cases} x/10 & \Leftarrow \quad x \in [0,10] \\ 1 - x/10 & \Leftarrow \quad x \in (10,20] \end{cases}$

2. This is problem 10.7.21 in the textbook. The motion of a circular elastic membrane, such as a drumhead, is governed by the two dimensional wave equation in polar coordinates:

$$u_{rr} + (1/r)u_r + (1/r^2)u_{\theta\theta} = a^{-2}u_{tt}$$

Assuming that $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$, find ordinary differential equations satisfied by $R(r)$, $\Theta(\theta)$, and $T(t)$.

28 Sample Final

28.1 Problem 1

Find the Fourier series expansions of the following functions:

- $f(x) = x^2$ for $x \in [-1, 1)$ and f is periodic with fundamental period 2. **Solution**: x^2 is even, so it does not have sine terms, i.e.: $b_n = 0$.

$$a_0 = \int_{-1}^{1} x^2 dx = \frac{2}{3}; a_n =_{even} 2\int_0^1 x^2 \cos(n\pi x)dx = \frac{4}{n^2\pi^2} x\cos(n\pi x)|_0^1 = (-1)^n \frac{4}{n^2\pi^2}$$

Therefore,

$$f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2\pi^2} \cos(n\pi x)$$

- $g(x) = x^2$ for $x \in (0, 2)$, $g(0) < \infty$, and g is periodic with fundamental period 2. What should the value $g(0)$ be if we want g to agree at all points with its Fourier Expansion. **Solution**: Periodicity implies $g(x) = g(x + 2) = (x + 2)^2$ on $(-2, 0)$. So using the period $[-1, 1)$, g is neither odd nor even. Therefore $L = 1$

$$a_0 = \frac{1}{1}\int_0^2 x^2 dx = \frac{8}{3}$$

$$a_n = \int_0^2 x^2 \cos(n\pi x)dx = \frac{1}{n\pi} x^2 \sin(n\pi x) + \frac{2}{n^2\pi^2} x\cos(n\pi x) - \frac{2}{n^3\pi^3} \sin(n\pi x)\Big|_0^2 = (-1)^n \frac{4}{n^2\pi^2}$$

$$b_n = \int_0^2 x^2 \sin(n\pi x)dx = -\frac{1}{n\pi} x^2 \cos(n\pi x) + \frac{2}{n^2\pi^2} x\sin(n\pi x) + \frac{2}{n^3\pi^3} \cos(n\pi x)\Big|_0^2 = -\frac{4}{n\pi}$$

Therefore,

$$g(x) = \frac{4}{3} + \sum_{n\geq 1} (-1)^n \frac{4}{n^2\pi^2} \cos(n\pi x) - \frac{4}{n\pi}\sin(n\pi x)$$

- $h(x) = x^2$ for $x \in [0, 1]$, h is periodic of fundamental period 4, symmetric in $(0, 2)$ about the line $x = 1$, i.e.: for $x \in (1, 2] : f(x) = f(2 - x)$, and h is odd. Solution: because h is odd, then $a_0 = a_n = 0$. Because the period is 4, then $L = 2$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin(n\pi x/2)dx =_{odd} \int_0^2 f(x)\sin(n\pi x/2)dx =_{\substack{sym \\ n:odd}} 2\int_0^1 x^2 \sin(n\pi x/2)dx$$

$$b_n = 2\left(-\frac{2}{n\pi} x^2 \cos(\frac{n\pi x}{2}) + \frac{8}{n^2\pi^2} x\sin(\frac{n\pi x}{2}) + \frac{16}{n^3\pi^3}\cos(\frac{n\pi x}{2})\right)_0^1 = (-1)^{\frac{n-1}{2}} \frac{16}{n^2\pi^2} - \frac{32}{n^3\pi^3}$$

Therefore,

$$h(x) = \sum_{odd\, n\geq 1} \left((-1)^{\frac{n-1}{2}} \frac{16}{n^2\pi^2} - \frac{32}{n^3\pi^3}\right) \sin\left(\frac{n\pi x}{2}\right)$$

28.2 Problem 2

Find the temperature $u(x,t)$ at any time in a metal road of length 100 units, whose endpoints are held at constant temperatures of 50 degrees, which has an initial temperature given by:

$$u(x,0) = \begin{cases} x^2+50 & \text{for } x \in [0,50] \\ x^2-200x+10050 & \text{for } x \in [50,100] \end{cases}$$

Assume that the thermal diffusivity of the metal is 1 units of length square over units of time.
Solution: Notice that the steady solution will be $u^{\infty} = 50$ and $f - u^{\infty} = \begin{cases} x^2 \Leftarrow x \in [0,50] \\ (100-x)^2 \in [50,100] \end{cases}$ which is symmetric around $x = 50$ on the interval $[0,100]$. Now,

$$u(x,t) = 50 + \sum_{n\geq 1} c_n e^{-\frac{n^2\pi^2}{100^2}} \sin\left(\frac{n\pi x}{100}\right)$$

which forces us to extend $f - u^{\infty}$ to be odd and of period 200. Therefore,

$$c_n =_{\substack{sym \\ n:odd}} \frac{1}{25}\int_0^{50} x^2 \sin(\frac{n\pi x}{100})dx$$

$$c_n = \frac{1}{25}\left(-\frac{100}{n\pi}x^2\cos(\frac{n\pi x}{100}) + \frac{2*100^2}{n^2\pi^2}x\sin(\frac{n\pi x}{100}) + \frac{2*100^3}{n^3\pi^3}\cos(\frac{n\pi x}{100})\right)_0^{50} = \frac{100^3}{25n^2\pi^2}(-1)^{\frac{n-1}{2}} - \frac{2*100^3}{25n^3\pi^3}$$

Therefore,

$$u(x,t) = 50 + \frac{1}{25}\sum_{odd\, n\geq 1}\left(\frac{100^3}{n^2\pi^2}(-1)^{\frac{n-1}{2}} - \frac{2*100^3}{n^3\pi^3}\right)e^{-\frac{n^2\pi^2}{100^2}}\sin\left(\frac{n\pi x}{100}\right)$$

- Write a list of the first three nonzero coefficients of the series solution.

$$c_1 = \left(\frac{100^3}{25\pi^2} - \frac{2*100^3}{25\pi^3}\right); c_3 = \left(-\frac{100^3}{3^2*25\pi^2} - \frac{2*100^3}{3^3*25\pi^3}\right); c_5 = \left(\frac{100^3}{5^2*25\pi^2} - \frac{2*100^3}{5^3*25\pi^3}\right)$$

1. What is α?
2. What kind of function did we assume U is in order to find such solution?
3. What are the X problem and the T problem obtained from separation of variables?
4. How do we extend the function $u(x,0)$?
5. What is the 2019-th non-zero term in the sum part of the solution?

28.3 Problem 3

Find the temperature $u(x,t)$ at any time in a metal road of length 20 units that is insulated on the sides, whose end at $x=0$ is held at constant temperature $u(0,t)=20$ degrees, whose end at $x=20$ is insulated or $u_x(20,t)=0$, and whose initial temperature is given by:

$$u(x,0)=f(x)=x^2 \text{ for } x\in[0,20)$$

Solution: First $L=20$. Notice that the steady solution in this case is $u^\infty=20$. Then we already know that the will only have terms with factors $\sin(n\pi x/2L)$ for n odd. Indeed

$$u(x,t)=20+\sum_{odd\, n\geq 1} c_n \sin\left(\frac{n\pi x}{40}\right)e^{-\frac{n^2\pi^2\alpha^2}{40^2}t}$$

which forces to extend $f-u^\infty$ to be odd, periodic of period $4L=80$, and symmetric on $x\in(0,40)$ around $x=20$. Hence,

$$c_n=\frac{1}{10}\int_0^{20}(x^2-20)\sin(n\pi x/40)dx=\frac{-80}{n\pi}+(-1)^{\frac{n-1}{2}}\frac{6400}{n^2\pi^2}-\frac{12800}{n^3\pi^3}$$

Therefore,

$$u(x,t)=20+\sum_{odd\, n\geq 1}\left(\frac{-80}{n\pi}+(-1)^{\frac{n-1}{2}}\frac{6400}{n^2\pi^2}-\frac{12800}{n^3\pi^3}\right)\sin\left(\frac{n\pi x}{40}\right)e^{-\frac{n^2\pi^2\alpha^2}{40^2}t}$$

- Write a list of the first three nonzero coefficients of the series solution.

$$c_1=\frac{-80}{\pi}+\frac{6400}{\pi^2}-\frac{12800}{\pi^3}$$
$$c_3=\frac{-80}{3\pi}-\frac{6400}{9\pi^2}-\frac{12800}{27\pi^3}$$
$$c_5=\frac{-80}{5\pi}-\frac{6400}{25\pi^2}-\frac{12800}{125\pi^3}$$

28.4 Problem 4

Consider a vibrating string of length $L=40$ units whose endpoints are fixed and that satisfies the wave equation $16u_{xx}=u_{tt}$ for $x\in(0,40)$ and $t>0$. Assume that the string starts at the initial vertical displacement

$$u(x,0)=f(x)=\begin{cases} x & \text{for } x\in[0,20] \\ 40-x & \text{for } x\in[20,40]\end{cases}$$

and that the initial vertical velocity (**Note for the practice**: I changed this because it requires to be 0 on the endpoints, because they are fixed)

$$u_t(x,0)=g(x)=x(x-40) \text{ for } x\in(0,40)$$

find the vertical displacement $u(x,t)$ at any time $t>0$ and $x\in(0,40)$. **Solution**: we know $L=40$.

$$u(x,t)=\sum_{n\geq 1}\sin(n\pi x/40)\left[c_n\cos(4n\pi t/40)+d_n\sin(4n\pi t/40)\right]$$

After distribution, the terms with coefficients c_n correspond to assume that $g(x) = 0$, which forces f to be extended odd and of period $2L = 80$. Also the terms with coefficients d_n correspond to assume $f(x) = 0$, which forces g to be extended odd and of period $2L = 80$.

$$c_n = \frac{1}{20}\int_0^{40} f(x)\sin(n\pi x/40)dx =_{\substack{sym\\ n:odd}} \frac{1}{10}\int_0^{20} x\sin(n\pi x/40)dx$$

$$c_n = \frac{1}{10}\left[-\frac{40}{n\pi}x\cos(\frac{n\pi x}{40}) + \frac{40^2}{n^2\pi^2}\sin(\frac{n\pi x}{40})\right]_0^{20} = \frac{160}{n^2\pi^2}(-1)^{\frac{n-1}{2}}$$

Now, let's work on the d_n's:

$$d_n = \frac{1}{2n\pi}\int_0^{40} g(x)\sin(n\pi x/40)dx$$

Do we enjoy the same kind of symmetry for g on $(0, 40)$ around $x = 20$? It wouldn't save us that much time because g is not piecewise defined, it is a single rule, so we just compute:

$$d_n = \frac{1}{2n\pi}\int_0^{40}(x^2 - 40x)\sin(n\pi x/40)dx$$

$$d_n = \frac{1}{2n\pi}\left(\frac{(-1)^{n+1}40^3}{n\pi} + \frac{2*40^3}{n^3\pi^3}((-1)^n - 1) + (-1)^n\frac{40^3}{n\pi}\right) =_{n:odd} \frac{-4*40^3}{n^3\pi^3}$$

Therefore,

$$u(x,t) = \sum_{odd\, n\geq 1}\sin(n\pi x/40)\left[(-1)^{\frac{n-1}{2}}\frac{160}{n^2\pi^2}\cos(4n\pi t/40) - \frac{4*40^3}{n^3\pi^3}\sin(4n\pi t/40)\right]$$

- For each list of coefficients in the series solution, write the first five non-zero coefficients. **Solution**:

$$c_1 = \frac{160}{\pi^2}, c_3 = -\frac{160}{3^2\pi^2}, c_5 = \frac{160}{5^2\pi^2}, c_7 = -\frac{160}{7^2\pi^2}, c_9 = \frac{160}{9^2\pi^2}$$

$$d_1 = \frac{-4*40^3}{\pi^3}, d_3 = \frac{-4*40^3}{3^3\pi^3}, d_5 = \frac{-4*40^3}{5^3\pi^3}, d_7 = \frac{-4*40^3}{7^3\pi^3}, d_9 = \frac{-4*40^3}{9^3\pi^3}$$

28.5 Problem 5

Dirichlet Rectangle Problem: Find a function $u(x, y)$ satisfying $u_{xx} + u_{yy} = 0$ on the rectangle $R = \{(x, y) : x \in (0, 10), y \in (0, 10)\}$ with boundary conditions $u(0, y) = 0, u(10, y) = 0, u(x, 0) = \sin(\pi x/10)$, and

$$u(x,b) = f(x) = \begin{cases} x & \text{for } x \in [0,5] \\ 10 - x & \text{for } x \in [5,10] \end{cases}$$

Solution: This is the second type in the summary where $i(x) = \sin(\pi x/10)$ and $j(x) = \begin{cases} x & \text{for } x \in [0,5] \\ 10 - x & \text{for } x \in [5,10] \end{cases}$. Hence,

$$u(x,y) = \sum_{n\geq 1}(e_n\sinh(n\pi y/10) + f_n\sinh(n\pi(y - 10)/10))\sin(n\pi x/10)$$

where the e_n's terms correspond to assume $i(x) = 0$ and thus we need to extend $f(x)$ to be odd and of period $2L$. The f_n's terms correspond to assume $j(x) = 0$ and thus we need to extend $u(x, 0)$ to be odd and of period $2L$.

$$e_n \sinh(n\pi) = \frac{1}{5}\int_0^{10} j(x)\sin(n\pi x/10)dx \underset{n:odd}{=_{sym}} \frac{2}{5}\int_0^5 x\sin(n\pi x/10)dx$$

$$e_n \sinh(n\pi) = \frac{2}{5}\left(-\frac{10}{n\pi}\,x\,\cos(n\pi x/10) + \frac{100}{n^2\pi^2}\,\sin(n\pi x/10)\right)_0^5 = (-1)^{\frac{n-1}{2}}\frac{40}{n^2\pi^2}$$

$$\Rightarrow e_n = (-1)^{\frac{n-1}{2}}\frac{40}{n^2\pi^2\sinh(n\pi)}$$

Additionally,

$$-f_n \sinh(n\pi) = \frac{1}{5}\int_0^{10} \sin(\pi x/10)\sin(n\pi x/10)dx$$

$$-f_n \sinh(n\pi) = \frac{1}{10}\int_0^{10} \cos((n-1)\pi x/10) - \cos((n+1)\pi x/10)dx$$

Here we have to differentiate between the case $n = 1$:

$$-f_1 \sinh(\pi) = \frac{1}{10}\int_0^{10} 1 - \cos(\pi x/5)dx = 1 - \frac{1}{2\pi}\sin(\pi x/5)|_0^{10} = 1 \Rightarrow f_1 = -\frac{1}{\sinh(\pi)}$$

And the case $n > 1$:

$$-f_n \sinh(n\pi) = \frac{1}{10}\left(\frac{10}{(n-1)\pi}\sin((n-1)\pi x/10) - \frac{10}{(n+1)\pi}\sin((n+1)\pi x/10)\right)_0^{10} = 0$$

$$\Rightarrow f_n = 0$$

Therefore,

$$u(x,y) = -\frac{1}{\sinh(\pi)}\sinh(\frac{\pi(y-10)}{10})\sin(\frac{\pi x}{10}) + \sum_{odd\, n\geq 1}(-1)^{\frac{n-1}{2}}\frac{40}{n^2\pi^2\sinh(n\pi)}\sinh(\frac{n\pi y}{10})$$

- Write the first five non-zero coefficients for the series solution.

$$e_1 = \frac{40}{\pi^2\sinh(\pi)}, e_3 = -\frac{40}{9\pi^2\sinh(3\pi)}, e_5 = \frac{40}{25\pi^2\sinh(5\pi)}, e_7 = -\frac{40}{49^2\pi^2\sinh(7\pi)}, e_9 = \frac{40}{81\pi^2\sinh(9\pi)}$$