Affine Connections, Curvature, and Torsion

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1. Torsion

Theorem 1.1. *Let ∇ be an affine connection on $\mathbb{R}^n$, and*

Proposition 1.2. *Let M be a smooth manifold with affine connection ∇. If γ is a piecewise-smooth path, then*

$$\tau_\nabla(\overline{\gamma}) = -\Gamma(\gamma)\tau_\nabla(\gamma). \tag{1.1}$$

Proof. The accumulated torsion of the reverse path γ is given by

$$\tau_\nabla(\overline{\gamma}) = \int_0^{t(\overline{\gamma})} \Gamma(\overline{\gamma})_t^0 \Big[\overline{\gamma}'(t)\Big]\,\mathrm{d}t = -\int_0^{t(\gamma)} \Gamma(\overline{\gamma})_{t(\overline{\gamma})}^0 \Gamma(\overline{\gamma})_t^{t(\overline{\gamma})} \Big[\gamma'\big(t(\gamma) - t\big)\Big]\,\mathrm{d}t. \tag{1.2}$$

Performing the substitution $t' = t(\gamma) - t$ and applying Proposition **??** yields

$$\tau_\nabla(\overline{\gamma}) = -\Gamma(\gamma)\int_0^{t(\gamma)} \Gamma(\gamma)_{t'}^0 \Big[\gamma'(t')\Big]\,\mathrm{d}t'. \tag{1.3}$$

The conclusion follows. ∎

Proposition 1.3. *Let M be a smooth manifold with affine connection ∇. If $\alpha\colon y \rightsquigarrow z$ and $\beta\colon x \rightsquigarrow y$ are smooth paths, then*

$$\tau_\nabla(\alpha\beta) = \Gamma(\overline{\beta})\big[\tau_\nabla(\alpha)\big] + \tau_\nabla(\beta). \tag{1.4}$$

Proof. The endpoint of the development of $\alpha\beta$ is given by

$$\tau_\nabla(\alpha\beta) = \int_{t(\beta)}^{t(\alpha)+t(\beta)} \Gamma(\alpha\beta)_t^0 \Big[(\alpha\beta)'(t)\Big]\,\mathrm{d}t + \int_0^{t(\beta)} \Gamma(\alpha\beta)_t^0 \Big[(\alpha\beta)'(t)\Big]\,\mathrm{d}t. \tag{1.5}$$

Applying Proposition **??** yields

$$\tau_\nabla(\alpha\beta) = \int_0^{t(\alpha)} \Gamma(\overline{\beta})\Gamma(\alpha)_t^0\Big[\alpha'(t)\Big]\,\mathrm{d}t + \int_0^{t(\beta)} \Gamma(\beta)_t^0\Big[\beta'(t)\Big]\,\mathrm{d}t \tag{1.6}$$

$$= \Gamma(\overline{\beta})\int_0^{t(\alpha)} \Gamma(\alpha)_t^0\Big[\alpha'(t)\Big]\,\mathrm{d}t + \int_0^{t(\beta)} \Gamma(\beta)_t^0\Big[\beta'(t)\Big]\,\mathrm{d}t. \tag{1.7}$$

The conclusion follows. ■

Note that this implies a kind of "right cancellation" law for τ_∇: $\tau_\nabla(\alpha\beta) = \tau_\nabla(\alpha'\beta)$ if and only if $\tau_\nabla(\alpha) = \tau_\nabla(\alpha')$. Left cancellation does not immediately hold, due to the prescence of the parallel transport term. However, if β is a loop such that $\Gamma(\beta)$ is the identity, then the torsion satisfies $\tau_\nabla(\alpha\beta) = \tau_\nabla(\alpha) + \tau_\nabla(\beta)$. In this sense, the accumulated torsion around loops is "additive" when a connection has trivial holonomy.

Proposition 1.4. *Let M be a smooth manifold with affine connection ∇. If $\alpha_1, \alpha_2 \colon x \rightsquigarrow y$ are piecewise smooth paths and $\gamma \colon x \rightsquigarrow y$ is another such path, then*

$$\tau_\nabla(\overline{\alpha}_2\alpha_1) = \tau_\nabla(\overline{\alpha}_2\gamma \cdot \overline{\gamma}\alpha_1). \tag{1.8}$$

Proof. By Proposition **??**, it suffies to show that $\tau_\nabla(\overline{\alpha}_2) = \tau_\nabla(\overline{\alpha}_2 \cdot \gamma\overline{\gamma})$. But $\gamma\overline{\gamma}$ has trivial parallel transport, so by Proposition **??** again we have

$$\tau_\nabla(\overline{\alpha}_2 \cdot \gamma\overline{\gamma}) = \tau_\nabla(\overline{\alpha}_2) + \tau_\nabla(\gamma\overline{\gamma}). \tag{1.9}$$

Finally, by Proposition **??** and **??**, we have

$$\tau_\nabla(\gamma\overline{\gamma}) = \Gamma(\gamma)\big[\tau_\nabla(\gamma)\big] + \tau_\nabla(\overline{\gamma}) = \Gamma(\gamma)\big[\tau_\nabla(\gamma)\big] - \Gamma(\gamma)\big[\tau_\nabla(\gamma)\big] = 0.$$

The conclusion follows. ■

By using the laws we have just established, we may express the accumulated torsion over a contractible loop in terms of an integral of the infinitesimal torsion within the region bounded by the loop. We adopt the following notation to describe paths in an interval $I \subseteq \mathbb{R}$: given $a, b \in I$, define $a \to b$ to be the straight-line path

$$[a \to b] \colon a \rightsquigarrow b; \qquad [a \to b](t) = (1-t) \cdot a + t \cdot b \tag{1.10}$$

from a to b.

Theorem 1.5. *Let M be a smooth manifold with flat affine connection ∇. If $\gamma \colon x \rightsquigarrow x$ is contractible via the homotopy*[1] *$H \colon [0,1] \times [0, t(\gamma)] \to M$ with $H_1 = \gamma$, then*

$$\tau_\nabla(\gamma) = \iint_{[0,1]\times[0,t(\gamma)]} \Gamma H(s \to 0, t) \left[T_\nabla \left(\frac{\partial H}{\partial s}, \frac{\partial H}{\partial t} \right) \right] \mathrm{d}s \, \mathrm{d}t \tag{1.11}$$

Proof. Given $s \in [0,1]$ or $t \in [0, t(\gamma)]$, we view the maps $H(s, -)$ and $H(-, t)$ as functors between path categories; when applied to straight-line paths in their respective domains, these yield "vertical" or "horizontal" paths in M. Let $n > 0$ be an integer. By Proposition **??**, the fact that the homotopy fixes endpoints, and Proposition **??**, the torsion accumulated over $\gamma = H_1$ is equal to

$$\tau_\nabla(H(1, 0 \to 1)) = \tau_\nabla \left[\prod_{k=0}^{n-1} H(1, \tfrac{k}{n} \to \tfrac{k+1}{n}) \right] \tag{1.12}$$

$$= \tau_\nabla \left[\prod_{k=0}^{n-1} H\left(1 \to 0, \tfrac{k+1}{n}\right) \cdot H\left(1, \tfrac{k}{n} \to \tfrac{k+1}{n}\right) \cdot H\left(0 \to 1, \tfrac{k}{n}\right) \right] \tag{1.13}$$

$$= \sum_{k=0}^{n-1} \tau_\nabla \left[H\left(1 \to 0, \tfrac{k+1}{n}\right) \cdot H\left(1, \tfrac{k}{n} \to \tfrac{k+1}{n}\right) \cdot H\left(0 \to 1, \tfrac{k}{n}\right) \right]. \tag{1.14}$$

Each "wedge" in the sum **??** may be broken into individual rectangular units as follows. For integers $j \in [\![1, n]\!]$ and $k \in [\![0, n-1]\!]$, define "subwedges" $\omega_{j,k} \colon x \rightsquigarrow x$ by

$$\omega_{j,k} = \underbrace{H\left(\tfrac{j}{n} \to 0, \tfrac{k+1}{n}\right) \cdot H\left(\tfrac{j}{n}, \tfrac{k}{n} \to \tfrac{k+1}{n}\right)}_{\alpha_{j,k}} \cdot \underbrace{H\left(0 \to \tfrac{j}{n}, \tfrac{k}{n}\right)}_{\beta_{j,k}}, \tag{1.15}$$

and "cells" $\rho_{j,k}$ by

$$\rho_{j,k} = H\left(\tfrac{j-1}{n}, \tfrac{k+1}{n} \to \tfrac{k}{n}\right) \cdot H\left(\tfrac{j}{n} \to \tfrac{j-1}{n}, \tfrac{k+1}{n}\right) \cdot H\left(\tfrac{j}{n}, \tfrac{k}{n} \to \tfrac{k+1}{n}\right) \cdot H\left(\tfrac{j-1}{n} \to \tfrac{j}{n}, \tfrac{k}{n}\right). \tag{1.16}$$

[1] ooga booga

Then by Proposition **??**, the accumulated torsion is

$$\begin{aligned}
\tau_\nabla(\omega_{j,k}) &= \Gamma(\overline{\beta}_{j-1,k})[\tau_\nabla(\alpha_{j-1,k}\cdot\rho_{j,k})] + \tau_\nabla(\beta_{j-1,k}) && (1.17)\\
&= \Gamma(\overline{\beta}_{j-1,k})[\tau_\nabla(\alpha_{j-1,k}) + \tau_\nabla(\rho_{j,k})] + \tau_\nabla(\beta_{j-1,k}) && (1.18)\\
&= \Big[\underbrace{\Gamma(\overline{\beta}_{j-1,k})[\tau_\nabla(\alpha_{j-1,k})] + \tau_\nabla(\beta_{j-1,k})}_{\tau_\nabla(\omega_{j-1,k})}\Big] + \Gamma(\overline{\beta}_{j-1,k})[\tau_\nabla(\rho_{j,k})] && (1.19)
\end{aligned}$$

It follows that

$$\tau_\nabla(\gamma) = \sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\left[\Gamma H\left(\tfrac{j}{n}\to 0, \tfrac{k}{n}\right)\right]\tau_\nabla(\rho_{j,k}). \qquad (1.20)$$

By Theorem **??**, one recovers the desired expression as $n\to\infty$. ■