The Universal Coefficient Theorems

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Contents

1. Introduction

2. Derived Functors

2.1. Tor

2.2. Ext

3. Universal Coefficient Theorems

3.1. Homology

Let R be a commutative ring and let $(C_\bullet, \partial_\bullet)$ be a chain complex of R-modules. One may add "coefficients" to the chain complex by tensoring each C_n with a fixed R-module

M. This yields a new chain complex

$$\cdots \to C_{n+1} \otimes_R M \xrightarrow{\partial_{n+1} \otimes 1_M} C_n \otimes_R M \xrightarrow{\partial_n \otimes 1_M} C_{n-1} \otimes_R M \to \cdots$$

of R-modules, denoted by $C_\bullet \otimes_R M$. Each chain $c \in C_n \otimes_R M$ may be written as an "M-linear" combination of chains

$$c = \sum_{i \in I} c_i \otimes m_i$$

where $c_i \in C_n$ and $m_i \in M$. Note that M is not required to have a ring structure, so after adding coefficients one should still regard $C_n \otimes_R M$ as an R-module. The purpose of the universal coefficient theorem for homology is to relate the homology of $C_\bullet \otimes_R M$ to the homology of $C_\bullet$. Note that there is a map

$$\delta' \colon \mathrm{H}_n(C_\bullet) \otimes_R M \to \mathrm{H}_n(C_\bullet \otimes_R M) \qquad ([c] \otimes m) \overset{\delta'}{\mapsto} [c \otimes m].$$

which should be thought of as an approximation of the structure of $\mathrm{H}_n(C_\bullet \otimes_R M)$; the universal coefficient theorem tells us how well this estimate works.

Theorem 3.1 (Universal Coefficient Theorem for Homology)**.** *Let R be a PID, and let $(C_\bullet, \partial_\bullet)$ be a chain complex of free R-modules. Then for any R-module M, there exists a short exact sequence of R-modules*

$$0 \to \mathrm{H}_n(C_\bullet) \otimes_R M \xrightarrow{\delta'} \mathrm{H}_n(C_\bullet \otimes_R M) \to \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(C_\bullet), M) \to 0 \tag{$\star$}$$

for each $n \in \mathbb{Z}$, which is natural in both $C_\bullet$ and M. Furthermore, this sequence splits, but not naturally.

Proof. Since R is a PID, and C_{n-1} is a free R-module, it follows that $\operatorname{im} \partial_n \subseteq C_{n-1}$ is also free. Thus, the short exact sequence of chain complexes

$$0 \to \ker \partial_\bullet \to C_\bullet \to \operatorname{im} \partial_\bullet \to 0$$

splits, where the subcomplexes $\ker \partial_\bullet$ and $\operatorname{im} \partial_\bullet$ are defined by taking kernels and images degree-wise. Since this sequence is split, we may apply $- \otimes_R M$ to obtain a new short exact sequence of chain complexes

$$0 \to \ker \partial_\bullet \otimes_R M \to C_\bullet \otimes_R M \to \operatorname{im} \partial_\bullet \otimes_R M \to 0.$$

For a chain complex with trivial boundary maps, the homology groups coincide with the chain groups in each degree, so the associated long exact sequence in homology may be written as

$$\begin{tikzcd}
& \cdots \arrow[r] & \operatorname{im}\partial_{n+1} \otimes_R M \arrow[dll, "\delta_{n+1}"'] \\
\ker\partial_n \otimes_R M \arrow[r] & \mathrm{H}_n(C_\bullet \otimes_R M) \arrow[r] & \operatorname{im}\partial_n \otimes_R M \arrow[dll, "\delta_n"'] \\
\ker\partial_{n-1} \otimes_R M \arrow[r] & \cdots &
\end{tikzcd}$$

where $\delta_n = \iota_n \otimes_R 1_M$ for the inclusion $\iota_n \colon \operatorname{im}\partial_n \hookrightarrow \ker\partial_{n-1}$. The weaving lemma yields short exact sequences

$$0 \to \operatorname{coker}(\iota_{n+1} \otimes_R 1_M) \xrightarrow{\delta'_{n+1}} \mathrm{H}_n(C_\bullet \otimes_R M) \xrightarrow{\partial'_n} \ker(\iota_n \otimes_R 1_M) \to 0.$$

where δ'_{n+1} and ∂'_n are the maps induced by δ_{n+1} and ∂_n, respectively. Since the tensor product preserves cokernels, the cokernel may be expressed as

$$\operatorname{coker}(\iota_{n+1} \otimes_R 1_M) \cong \operatorname{coker}\iota_{n+1} \otimes_R M \cong \mathrm{H}_n(C_\bullet) \otimes_R M.$$

Under this identification, δ'_{n+1} takes $([c] \otimes m) \mapsto [c \otimes m]$. Since the failure of the tensor product to preserve kernels is measured by the first torsion group, the kernel may be expressed as

$$\ker(\iota_n \otimes_R 1_M) \cong \operatorname{Tor}_1^R(\operatorname{coker}\iota_n, M) \cong \operatorname{Tor}_1^R(\mathrm{H}_{n-1}(C_\bullet), M).$$

Substituting these isomorphisms into the short exact sequence above yields the desired result. Since the connecting homomorphism and isomorphisms used were all natural, the entire sequence is natural in both $C_\bullet$ and M. ■

Remark 3.1. In the above proof, we implicitly used the fact that Tor was defined by taking a projective resolution of the first argument when computing the kernel of $\iota_n \otimes 1_M$. Since Tor is symmetric, one may also prove the theorem by taking a free resolution of the second argument M instead: given a free resolution

$$0 \to F_1 \to F_0 \to M \to 0$$

tensoring with $C_\bullet$ and applying the weaving lemma to the associated long exact sequence also yields the desired result.

3.2. Cohomology

Let R be a commutative ring and let $(C_\bullet, \partial_\bullet)$ be a chain complex of R-modules. The dual cochain complex $(C^\bullet, \delta^\bullet)$ is formed by taking the dual of each C_n, yielding the sequence of R-modules

$$\cdots \to \operatorname{Hom}_R(C_{n-1}, R) \xrightarrow{d^{n-1}} \operatorname{Hom}_R(C_n, R) \xrightarrow{d^n} \operatorname{Hom}_R(C_{n+1}, R) \to \cdots$$

where $d^n = (\partial_{n+1})^*$ is the pullback along ∂_{n+1}. More generally, one may bake coefficients into the cochain complex by instead taking the R-module of homomorphisms $\operatorname{Hom}_R(C_n, M)$ for a fixed R-module M. This yields a new cochain complex

$$\cdots \to \operatorname{Hom}_R(C_{n-1}, M) \xrightarrow{d^{n-1}} \operatorname{Hom}_R(C_n, M) \xrightarrow{d^n} \operatorname{Hom}_R(C_{n+1}, M) \to \cdots$$

of R-modules, denoted by $\operatorname{Hom}_R(C_\bullet, M)$. Note that this is a fundamentally different process than *adding* coefficients by tensoring the dual chain complex with M. The purpose of the universal coefficient theorem for cohomology is to relate the cohomology of $\operatorname{Hom}_R(C_\bullet, M)$ to the cohomology of the dual complex $C^\bullet$.

Theorem 3.2 (Universal Coefficient Theorem for Cohomology). *Let R be a PID, and let $(C_\bullet, \partial_\bullet)$ be a chain complex of free R-modules. Then for any R-module M, there exists a short exact sequence of R-modules*

$$0 \to \operatorname{Ext}^1_R(\mathrm{H}_{n-1}(C_\bullet), M) \to \mathrm{H}^n(\operatorname{Hom}_R(C_\bullet, M)) \to \operatorname{Hom}_R(\mathrm{H}_n(C_\bullet), M) \to 0$$

for each $n \in \mathbb{Z}$, which is natural in both $C_\bullet$ and M. Furthermore, this sequence splits, but not naturally.

Proof. ■

4. Künneth Theorems

4.1. Homology

4.2. Cohomology

5. Applications