

AFFINE CONNECTIONS, CURVATURE, AND TORSION

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October 27, 2025

1. Torsion

Theorem 1.1. *Let ∇ be an affine connection on \mathbb{R}^n , and*

Proposition 1.2. *Let M be a smooth manifold with affine connection ∇ . If γ is a piecewise-smooth path, then*

$$\tau_{\nabla}(\bar{\gamma}) = -\Gamma(\gamma)\tau_{\nabla}(\gamma). \quad (1.1)$$

Proof. The accumulated torsion of the reverse path γ is given by

$$\tau_{\nabla}(\bar{\gamma}) = \int_0^{t(\bar{\gamma})} \Gamma(\bar{\gamma})_t^0 [\bar{\gamma}'(t)] dt = - \int_0^{t(\gamma)} \Gamma(\bar{\gamma})_{t(\bar{\gamma})}^0 \Gamma(\bar{\gamma})_t^{t(\bar{\gamma})} [\gamma'(t(\gamma) - t)] dt. \quad (1.2)$$

Performing the substitution $t' = t(\gamma) - t$ and applying Proposition ?? yields

$$\tau_{\nabla}(\bar{\gamma}) = -\Gamma(\gamma) \int_0^{t(\gamma)} \Gamma(\gamma)_{t'}^0 [\gamma'(t')] dt'. \quad (1.3)$$

The conclusion follows. ■

Proposition 1.3. *Let M be a smooth manifold with affine connection ∇ . If $\alpha: y \rightsquigarrow z$ and $\beta: x \rightsquigarrow y$ are smooth paths, then*

$$\tau_{\nabla}(\alpha\beta) = \Gamma(\bar{\beta})[\tau_{\nabla}(\alpha)] + \tau_{\nabla}(\beta). \quad (1.4)$$

Proof. The endpoint of the development of $\alpha\beta$ is given by

$$\tau_{\nabla}(\alpha\beta) = \int_{t(\beta)}^{t(\alpha)+t(\beta)} \Gamma(\alpha\beta)_t^0 [(\alpha\beta)'(t)] dt + \int_0^{t(\beta)} \Gamma(\alpha\beta)_t^0 [(\alpha\beta)'(t)] dt. \quad (1.5)$$

Applying Proposition ?? yields

$$\tau_{\nabla}(\alpha\beta) = \int_0^{t(\alpha)} \Gamma(\bar{\beta})\Gamma(\alpha)_t^0[\alpha'(t)] dt + \int_0^{t(\beta)} \Gamma(\beta)_t^0[\beta'(t)] dt \quad (1.6)$$

$$= \Gamma(\bar{\beta}) \int_0^{t(\alpha)} \Gamma(\alpha)_t^0[\alpha'(t)] dt + \int_0^{t(\beta)} \Gamma(\beta)_t^0[\beta'(t)] dt. \quad (1.7)$$

The conclusion follows. ■

Note that this implies a kind of “right cancellation” law for τ_{∇} : $\tau_{\nabla}(\alpha\beta) = \tau_{\nabla}(\alpha'\beta)$ if and only if $\tau_{\nabla}(\alpha) = \tau_{\nabla}(\alpha')$. Left cancellation does not immediately hold, due to the prescence of the parallel transport term. However, if β is a loop such that $\Gamma(\beta)$ is the identity, then the torsion satisfies $\tau_{\nabla}(\alpha\beta) = \tau_{\nabla}(\alpha) + \tau_{\nabla}(\beta)$. In this sense, the accumulated torsion around loops is “additive” when a connection has trivial holonomy.

Proposition 1.4. *Let M be a smooth manifold with affine connection ∇ . If $\alpha_1, \alpha_2: x \rightsquigarrow y$ are piecewise smooth paths and $\gamma: x \rightsquigarrow y$ is another such path, then*

$$\tau_{\nabla}(\bar{\alpha}_2\alpha_1) = \tau_{\nabla}(\bar{\alpha}_2\gamma \cdot \bar{\gamma}\alpha_1). \quad (1.8)$$

Proof. By Proposition ??, it suffies to show that $\tau_{\nabla}(\bar{\alpha}_2) = \tau_{\nabla}(\bar{\alpha}_2 \cdot \gamma\bar{\gamma})$. But $\gamma\bar{\gamma}$ has trivial parallel transport, so by Proposition ?? again we have

$$\tau_{\nabla}(\bar{\alpha}_2 \cdot \gamma\bar{\gamma}) = \tau_{\nabla}(\bar{\alpha}_2) + \tau_{\nabla}(\gamma\bar{\gamma}). \quad (1.9)$$

Finally, by Proposition ?? and ??, we have

$$\tau_{\nabla}(\gamma\bar{\gamma}) = \Gamma(\gamma)[\tau_{\nabla}(\gamma)] + \tau_{\nabla}(\bar{\gamma}) = \Gamma(\gamma)[\tau_{\nabla}(\gamma)] - \Gamma(\gamma)[\tau_{\nabla}(\gamma)] = 0.$$

The conclusion follows. ■

By using the laws we have just established, we may express the accumulated torsion over a contractible loop in terms of an integral of the infinitesimal torsion within the region bounded by the loop.

Theorem 1.5. *Let M be a smooth manifold with affine connection ∇ . If $\gamma: x \rightsquigarrow x$ is contractible via the homotopy¹ $H: [0, 1] \times [0, t(\gamma)] \rightarrow M$ with $H_1 = \gamma$, then*

$$\tau_{\nabla}(\gamma) = \iint_{[0,1] \times [0,t(\gamma)]} ds dt \quad (1.10)$$

¹ooga booga

Proof. We adopt the following notation to describe paths in an interval $I \subseteq \mathbb{R}$: given $a, b \in I$, define $a \rightarrow b$ to be the straight-line path

$$[a \rightarrow b]: a \rightsquigarrow b; \quad [a \rightarrow b](t) = (1-t) \cdot a + t \cdot b \quad (1.11)$$

from a to b . Given $s \in [0, 1]$ or $t \in [0, t(\gamma)]$, we also view the maps $H(s, -)$ and $H(-, t)$ as functors between path categories; when applied to straight-line paths in their respective domains, these yield “vertical” or “horizontal” paths in M .

For an integer $n > 0$, we divide the rectangle $[0, 1] \times [0, t(\gamma)]$ into $n \times n$ subrectangles. By Proposition ??, the fact that the homotopy fixes endpoints, and Proposition ??, the torsion accumulated over $\gamma = H_1$ is equal to

$$\tau_{\nabla}(H(1, 0 \rightarrow 1)) = \tau_{\nabla} \left[\prod_{k=0}^{n-1} H\left(1, \frac{k}{n} \rightarrow \frac{k+1}{n}\right) \right] \quad (1.12)$$

$$= \tau_{\nabla} \left[\prod_{k=0}^{n-1} H\left(1 \rightarrow 0, \frac{k+1}{n}\right) \cdot H\left(1, \frac{k}{n} \rightarrow \frac{k+1}{n}\right) \cdot H\left(0 \rightarrow 1, \frac{k}{n}\right) \right] \quad (1.13)$$

$$= \sum_{k=0}^{n-1} \tau_{\nabla} \left[H\left(1 \rightarrow 0, \frac{k+1}{n}\right) \cdot H\left(1, \frac{k}{n} \rightarrow \frac{k+1}{n}\right) \cdot H\left(0 \rightarrow 1, \frac{k}{n}\right) \right]. \quad (1.14)$$

For each “wedge” in the sum ??, we ■