A Curvature Criterion for Monotonicity of Domains

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Abstract

We extend the definition of monotonicity given in [?] to higher dimensional domains and establish a criterion which implies that a given domain is monotone with respect to at least one direction in terms of an inequality involving the total absolute Gauss-Kronecker curvature of its boundary. As a corollary of this result, we show that all polygons with five or fewer sides are monotone.

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1. Introduction

Let Ω be an open polygon in the plane, and let $\theta \in \mathbb{S}^1$ be a direction vector. We say that Ω is *monotone* with respect to θ if its intersection with every line ℓ orthogonal to θ is either empty or an interval in ℓ. Intuitively, this means one can completely "hatch" the region with a pen without having to lift the pen for any hatch line, as shown in Figure **??**. Monotonicity is a kind of generalization of convexity: if Ω is convex, then it is automatically monotone with respect to all directions. Typically, this notion is defined for polygons in terms of the number of crossings of a line with the boundary of the polygon, as done in [?] and [?]. The name is derived from the following fact: if Γ is polygon monotone with respect to a direction θ, then the edges of Γ may be split into two continguous chains of vertices $v_1, \dots, v_n$ and $w_1, \dots, w_m$ such that $\langle v_i, \theta \rangle$ and $\langle w_j, \theta \rangle$ are monotone in i and j respectively.

The definition of monotonicity may be extended to higher-dimensional domains (open sets) as follows. If $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and $\theta \in \mathbb{S}^{n-1}$, we say that Ω is *monotone* with respect to θ if its intersection with every hyperplane H orthogonal to θ is either empty or a homeomorphic to the $(n-1)$-ball $\mathbb{B}^n$. In this note, we establish criteria involving the boundary of a domain Ω which imply that it is monotone with respect to some direction.

1.1. Preliminaries and Conventions

We establish some usual notation, terminology, and conventions. Let $\mathbb{RP}^{n-1}$ be the quotient of $\mathbb{S}^{n-1}$ under the involution map $x \mapsto -x$ for all $x \in \mathbb{S}^{n-1}$. Since a domain $\Omega \subseteq \mathbb{R}^n$ is monotonic with respect to a direction $\theta \in \mathbb{S}^{n-1}$ if and only if it is monotonic with respect to $-\theta$, we may say that Ω is monotonic with respect to the direction $[\theta] \in \mathbb{RP}^{n-1}$. The involution map on the sphere is an isometry, so the quotient carries an induced Riemannian metric. Denote the quotient map by $\rho\colon \mathbb{S}^{n-1} \to \mathbb{RP}^{n-1}$, which is a double cover and a local isometry.

For an oriented smooth hypersurface $M \subseteq \mathbb{R}^n$, the Gauss map is a smooth map $n\colon M \to \mathbb{S}^{n-1}$ which maps a point $p \in M$ to the normal vector (defined by its orientation) at p. By composing with the double cover, we obtain a map $\nu\colon M \to \mathbb{RP}^{n-1}$ which we refer to as the *projectivized* Gauss map. The Gauss-Kronecker curvature of M is the unique real function K such that

$$n^*\omega_{\mathbb{S}^{n-1}} = K \cdot \omega_M, \tag{1.1}$$

where $\omega_{\mathbb{S}^{n-1}}$ and ω_M are the volume forms of $\mathbb{S}^{n-1}$ and M, respectively. The absolute Gauss-Kronecker curvature is given by

$$|K(p)| = \sqrt{\det(\mathrm{d}n_p^* \circ \mathrm{d}n_p)}, \tag{1.2}$$

where $[\cdot]^*$ is the adjoint of a linear map between inner product spaces.

Finally, if Γ is a nondegenerate polygon, we use the convention that all exterior angles are given in the range $(-\pi, \pi)$.

2. Monotone Domains

The main theorems of this paper rely on the following lemma which is analogous to [?], which concerns polygons in the plane. However, the result we give fails to be a complete characterization of monotonicity in a given direction, though it is enough for our purposes.

Proposition 2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary Γ, and let $\nu\colon \Gamma \to \mathbb{RP}^n$ be the projectivized Gauss map. If $\left|\nu^{-1}\{[\theta]\}\right| = 2$ for some $[\theta] \in \mathbb{RP}^{n-1}$, then Ω is monotone with respect to $[\theta]$.*

Proof. Consider the projection $\pi\colon \mathbb{R}^n \to \mathbb{R}$ defined by $\pi(x) = \langle x, \theta\rangle$ and let $h = \pi|_\Gamma$. At a critical point p of h, we have

$$\mathrm{d}h_p(v) = \langle v, \theta\rangle = 0$$

for all $v \in \mathrm{T}_p\Gamma$, which occurs if and only if $\nu(p) = [\theta]$. Since Γ is compact and not contained in a line, h attains a maximum and minimum at distinct points $p_+, p_- \in \Gamma$. By hypothesis, the preimage of $[\theta]$ under ν contains two elements, so there are no other critical values in $(\min h, \max h)$. By [?, Theorem A], $h^{-1}\{t\}$ is homeomorphic to $\mathbb{S}^{n-2}$

whenever t is not an extreme value of h. Therefore, by the generalized Schoenflies theorem (see [?]), each slice $\Omega \cap \pi^{-1}\{t\}$ is homeomorphic to $\mathbb{B}^{n-1}$ whenever it is nonempty. The conclusion follows. ∎

We now prove the main theorems of the paper.

Theorem 2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary Γ. If the total absolute Gauss curvature satisfies $\int_\Gamma |K| \, \mathrm{d}\Gamma < 2\,\mathrm{vol}(\mathbb{S}^{n-1})$, then Ω is monotone with respect to some direction.*

Proof. Let $n \colon \Gamma \to \mathbb{S}^{n-1}$ be the Gauss map and $\nu = \rho \circ n$, where $\rho \colon \mathbb{S}^{n-1} \to \mathbb{RP}^{n-1}$ is the projection map. Since ρ is a local isometry, the absolute Gauss-Kronecker curvature at a point $p \in \Gamma$ is given by the normal Jacobian

$$|K(p)| = \sqrt{\det(\mathrm{d}n_p^* \circ \mathrm{d}n_p)} = \sqrt{\det(\mathrm{d}\nu_p^* \circ \mathrm{d}\nu_p)} = |\mathrm{J}_p \nu|. \tag{2.1}$$

Define $\mu \colon \mathbb{RP}^{n-1} \to \mathbb{N}_{>0}$ by $\mu([\theta]) = |\nu^{-1}\{[\theta]\}|$ and let $P \subseteq \mathbb{RP}^{n-1}$ be the set of regular values of ν. Since ρ is a double cover and P has full measure by Sard's theorem, its volume is given by $\mathrm{vol}(P) = \frac{1}{2}\,\mathrm{vol}(\mathbb{S}^{n-1})$. Then by (**??**) and the smooth coarea formula [?], we have

$$\frac{1}{\mathrm{vol}(P)} \int_P \mu \, \mathrm{d}P = \frac{1}{\frac{1}{2}\,\mathrm{vol}(\mathbb{S}^{n-1})} \int_\Gamma |K| \, \mathrm{d}\Gamma < 4. \tag{2.2}$$

Thus, the average multiplicity of a direction $[\theta] \in \mathbb{RP}^{n-1}$ is strictly bounded above by 4. Furthermore, $\deg_2(\nu) = 0$ since it factors through a double cover. It follows that μ only takes on positive even values, so such an average can attained only if $\mu([\theta]) = 2$ for some $\theta \in \mathbb{S}^{n-1}$. The conclusion follows from Proposition **??**. ∎

By applying a standard smoothing argument, one may obtain an analogous result for polygons in the plane.

Theorem 2.3. *Let $\Omega \subseteq \mathbb{R}^2$ be a domain with polygonal boundary Γ. If the sum of the absolute values of the exterior angles of Γ is less than 4π, then Ω is monotone with respect to some direction.*

Proof. Let $\theta_1, \dots, \theta_n$ be the exterior angles of Γ. By "rounding" each of the corners of Ω, one may obtain a sequence $\Omega_i \to \Omega$ of domains with smooth boundaries Γ_i such that the multiplicities of the projective Gauss maps $\mu_i \colon \mathbb{RP}^1 \to \mathbb{N}_{>0}$ remain constant, and

$$\int_{\Gamma_i} |K| \, \mathrm{d}\Gamma_i = \sum_{k=1}^{n} \theta_k. \tag{2.3}$$

It follows from the proof of Theorem **??** that there exists a consistent direction $[\theta]$ for which each Ω_i is monotone. Thus, for a line ℓ normal to $[\theta]$, the slices $\Omega_i \cap \ell$ are intervals which converge to $\Omega \cap \ell$. Since the limit of a sequence of intervals is also an interval, and $\Omega \cap \ell$ is open in ℓ, it must either be empty or homeomorphic to $\mathbb{B}^1$. The conclusion follows. ∎

As a corollary, we obtain a resolution of the question posed in [?] concerning monotone polygons.

Corollary 2.4. *Let $\Omega \subseteq \mathbb{R}^2$ be a domain with n-sided polygonal boundary. If $n \leq 5$, then Ω is monotone with respect to some direction.*

Proof. Let $\theta_1, \dots, \theta_n \in (-\pi, \pi)$ be the exterior angles of the boundary polygon. Suppose that j angles are nonnegative and k angles are negative. Since the sum of exterior angles is 2π, we have

$$\sum_{i=1}^{n} |\theta_i| = \sum_{\theta_i \geq 0} \theta_i - \sum_{\theta_i < 0} \theta_i < 2\pi \min(j-1, k+1) \leq 4\pi, \tag{2.4}$$

so Ω is monotone with respect to some direction. ∎