

THE UNIVERSAL COEFFICIENT THEOREMS

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1. Introduction

2. Derived Functors

2.1. Tor

2.2. Ext

3. Universal Coefficient Theorems

3.1. Homology

Let R be a commutative ring and let $(C_\bullet, \partial_\bullet)$ be a chain complex of R -modules. One may add “coefficients” to the chain complex by tensoring each C_n with a fixed R -module

M . This yields a new chain complex

$$\cdots \rightarrow C_{n+1} \otimes_R M \xrightarrow{\partial_{n+1} \otimes 1_M} C_n \otimes_R M \xrightarrow{\partial_n \otimes 1_M} C_{n-1} \otimes_R M \rightarrow \cdots$$

of R -modules, denoted by $C_\bullet \otimes_R M$. Each chain $c \in C_n \otimes_R M$ may be written as an “ M -linear” combination of chains

$$c = \sum_{i \in I} c_i \otimes m_i$$

where $c_i \in C_n$ and $m_i \in M$. Note that M is not required to have a ring structure, so after adding coefficients one should still regard $C_n \otimes_R M$ as an R -module. The purpose of the universal coefficient theorem for homology is to relate the homology of $C_\bullet \otimes_R M$ to the homology of C_\bullet .

Before stating the theorem, we make a few important notational remarks. Let $\phi: A \rightarrow B$ be a map of R -modules and let B_0 be a subspace containing the image of ϕ . Typically, one denotes the corestriction $\phi|^{B_0}: A \rightarrow B_0$ by the symbol “ ϕ ” again. However, this abuse of notation can lead to confusion when dealing with tensor products: the map

$$\phi|^{B_0} \otimes 1_M: A \otimes_R M \rightarrow B_0 \otimes_R M$$

is *not* a corestriction of $\phi \otimes 1_M$, unless one can identify $B_0 \otimes_R M$ with a submodule of $B \otimes_R M$ via the inclusion map $\iota: B_0 \hookrightarrow B$. In general, this is not possible, since $- \otimes_R M$ fails to preserve monomorphisms. Thus, to avoid confusion, we explicitly denote corestrictions in the proceeding argument as follows: given a chain complex $(C_\bullet, \partial_\bullet)$ and a map $\phi: A \rightarrow C_{n+1}$ such that $\text{im } \phi \subset \ker \partial_n$, let

$$\phi^Z = \phi|_{\ker \partial_n}^{\ker \partial_n}: A \rightarrow \ker \partial_n$$

be the corestriction of ϕ to the n -cycles in C_\bullet .

Theorem 3.1 (Universal Coefficient Theorem for Homology). *Let R be a PID, and let $(C_\bullet, \partial_\bullet)$ be a chain complex of free R -modules. Then for any R -module M , there exists a short exact sequence of R -modules*

$$0 \rightarrow H_n(C_\bullet) \otimes_R M \rightarrow H_n(C_\bullet \otimes_R M) \rightarrow \text{Tor}_1^R(H_{n-1}(C_\bullet), M) \rightarrow 0$$

for each $n \in \mathbb{Z}$, which is natural in both C_\bullet and M . Furthermore, this sequence splits, but not naturally.

Proof. Let $\iota: \ker \partial_n \hookrightarrow C_n$ be the inclusion map. Then $\partial_{n+1} = \iota \circ \partial_{n+1}^Z$, which implies that

$$(\partial_{n+1} \otimes 1_M)^Z = \tilde{\iota} \circ (\partial_{n+1}^Z \otimes 1_M)$$

where $\tilde{\iota}$ is the corestriction of $\iota \otimes 1_M$ to $\ker(\partial_n \otimes 1_M)$.

Since R is a PID, and C_{n-1} is a free R -module, it follows that $\text{im } \partial_n \subseteq C_{n-1}$ is also free. Thus, the short exact sequence

$$0 \rightarrow \ker \partial_n \xrightarrow{\iota} C_n \xrightarrow{\partial_n} \text{im } \partial_n \rightarrow 0$$

splits, which implies $\iota \otimes 1_M$ is a split monomorphism. The bottom right object in the commutative triangle

$$\begin{array}{ccc} C_{n+1} \otimes_R M & \xrightarrow{\partial_{n+1}^Z \otimes 1_M} & \ker(\partial_n) \otimes_R M \\ \parallel & & \downarrow \iota \otimes 1_M \\ C_{n+1} \otimes_R M & \xrightarrow{\partial_{n+1} \otimes 1_M} & C_n \otimes 1_M \end{array}$$

may be replaced with $\ker(\partial_n \otimes 1_M)$, since $\text{im}(\partial_{n+1} \otimes 1_M) \subseteq \ker(\partial_n \otimes 1_M)$. Takin ■

3.2. Cohomology

Let R be a commutative ring and let $(C_\bullet, \partial_\bullet)$ be a chain complex of R -modules. The dual cochain complex $(C^\bullet, \delta^\bullet)$ is formed by taking the dual of each C_n , yielding the sequence of R -modules

$$\cdots \rightarrow \text{Hom}_R(C_{n-1}, R) \xrightarrow{\delta^{n-1}} \text{Hom}_R(C_n, R) \xrightarrow{\delta^n} \text{Hom}_R(C_{n+1}, R) \rightarrow \cdots$$

where $\delta^n = \partial_{n+1}^*$ is the pullback along ∂_{n+1} . More generally, one may bake coefficients into the cochain complex by instead taking the R -module of homomorphisms $\text{Hom}_R(C_n, M)$ for a fixed R -module M . This yields a new cochain complex

$$\cdots \rightarrow \text{Hom}_R(C_{n-1}, M) \xrightarrow{\delta^{n-1}} \text{Hom}_R(C_n, M) \xrightarrow{\delta^n} \text{Hom}_R(C_{n+1}, M) \rightarrow \cdots$$

of R -modules, denoted by $\text{Hom}_R(C_\bullet, M)$. Note that this is a fundamentally different process than *adding* coefficients by tensoring the dual chain complex with M , since $\text{Hom}_R(C_n, M)$ is not generally isomorphic to $\text{Hom}_R(C_n, R) \otimes_R M$. The purpose of the universal coefficient theorem for cohomology is to relate the cohomology of $\text{Hom}_R(C_\bullet, M)$ to the cohomology of the dual complex C^\bullet .

Theorem 3.2 (Universal Coefficient Theorem for Cohomology). *Let R be a PID, and let $(C_\bullet, \partial_\bullet)$ be a chain complex of free R -modules. Then for any R -module M , there exists a*

short exact sequence of R -modules

$$0 \rightarrow \text{Ext}_R^1(H_{n-1}(C_\bullet), M) \rightarrow H^n(\text{Hom}_R(C_\bullet, M)) \rightarrow \text{Hom}_R(H_n(C_\bullet), M) \rightarrow 0$$

for each $n \in \mathbb{Z}$, which is natural in both C_\bullet and M . Furthermore, this sequence splits, but not naturally.

4. Künneth Theorems

4.1. Homology

4.2. Cohomology

5. Applications