The Universal Coefficient Theorems

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3.1. Homology

Let R be a commutative ring and let $(C_\bullet, \partial_\bullet)$ be a chain complex of R-modules. One may add "coefficients" to the chain complex by tensoring each C_n with a fixed R-module

M. This yields a new chain complex

$$\cdots \to C_{n+1} \otimes_R M \xrightarrow{\partial_{n+1} \otimes 1_M} C_n \otimes_R M \xrightarrow{\partial_n \otimes 1_M} C_{n-1} \otimes_R M \to \cdots$$

of R-modules, denoted by $C_\bullet \otimes_R M$. Each chain $c \in C_n \otimes_R M$ may be written as an "M-linear" combination of chains

$$c = \sum_{i \in I} c_i \otimes m_i$$

where $c_i \in C_n$ and $m_i \in M$. Note that M is not required to have a ring structure, so after adding coefficients one should still regard $C_n \otimes_R M$ as an R-module. The purpose of the universal coefficient theorem for homology is to relate the homology of $C_\bullet \otimes_R M$ to the homology of $C_\bullet$.

Before stating the theorem, we make a few important notational remarks. Let $\phi\colon A \to B$ be a map of R-modules and let B_0 be a subspace containing the image of ϕ. Typically, one denotes the corestriction $\phi|^{B_0}\colon A \to B_0$ by the symbol "ϕ" again. However, this abuse of notation can lead to confusion when dealing with tensor products: the map

$$\phi|^{B_0} \otimes 1_M\colon A \otimes_R M \to B_0 \otimes_R M$$

is *not* a corestriction of $\phi \otimes 1_M$, unless one can identify $B_0 \otimes_R M$ with a submodule of $B \otimes_R M$ via the inclusion map $\iota\colon B_0 \hookrightarrow B$. In general, this is not possible, since $- \otimes_R M$ fails to preserve monomorphisms. Thus, to avoid confusion, we explicitly denote corestrictions in the proceeding argument as follows: given a chain complex $(C_\bullet, \partial_\bullet)$ and a map $\phi\colon A \to C_{n+1}$ such that $\operatorname{im}\phi \subset \ker \partial_n$, let

$$\phi^{\mathrm{Z}} = \phi|^{\ker \partial_n}\colon A \to \ker \partial_n$$

be the corestriction of ϕ to the n-cycles in $C_\bullet$.

Theorem 3.1 (Universal Coefficient Theorem for Homology)**.** *Let R be a PID, and let $(C_\bullet, \partial_\bullet)$ be a chain complex of free R-modules. Then for any R-module M, there exists a short exact sequence of R-modules*

$$0 \to \mathrm{H}_n(C_\bullet) \otimes_R M \to \mathrm{H}_n(C_\bullet \otimes_R M) \to \mathrm{Tor}_1^R(\mathrm{H}_{n-1}(C_\bullet), M) \to 0 \qquad (\star)$$

for each $n \in \mathbb{Z}$, which is natural in both $C_\bullet$ and M. Furthermore, this sequence splits, but not naturally.

Proof. Let $\iota\colon \ker \partial_n \hookrightarrow C_n$ be the inclusion map. Then $\partial_{n+1} = \iota \circ \partial_{n+1}^{\mathrm{Z}}$, which implies that

$$(\partial_{n+1} \otimes_R 1_M)^{\mathrm{Z}} = (\iota \otimes 1_M)^{\mathrm{Z}} \circ (\partial_{n+1}^{\mathrm{Z}} \otimes_R 1_M).$$

In other words, the diagram

$$\begin{tikzcd}
C_{n+1} \otimes_R M \arrow[r, "\partial^{\mathrm{Z}}_{n+1} \otimes 1_M"] \arrow[d, equal] & \ker(\partial_n) \otimes_R M \arrow[d, "(\iota \otimes 1_M)^{\mathrm{Z}}"] \\
C_{n+1} \otimes_R M \arrow[r, "(\partial_{n+1} \otimes 1_M)^{\mathrm{Z}}"'] & \ker(\partial_n \otimes 1_M)
\end{tikzcd}$$

commutes, so forming cokernels and using the fact that $- \otimes_R M$ preserves cokernels yields a map

$$j \colon \mathrm{H}_n(C_\bullet) \otimes_R M \to \mathrm{H}_n(C_\bullet \otimes_R M); \qquad ([c] \otimes m) \overset{j}{\longmapsto} [c \otimes m]$$

Since R is a PID, and C_{n-1} is a free R-module, it follows that $\operatorname{im} \partial_n \subseteq C_{n-1}$ is also free. Thus, the short exact sequence

$$0 \to \ker \partial_n \overset{\iota}{\to} C_n \to \operatorname{im} \partial_n \to 0$$

splits. By functoriality, $\iota \otimes 1_M$ is a split monomorphism, and so is its corestriction $(\iota \otimes 1_M)^{\mathrm{Z}}$. Taking cokernels is once again functorial, so j is also a split monomorphism. This yields the first half of the sequence ($\star$); it remains to determine the cokernel of j.

To this end, observe that the short exact sequence

$$0 \to \operatorname{im} \partial_{n+1} \to \ker \partial_n \to \mathrm{H}_n(C_\bullet) \to 0$$

is a free resolution of $\mathrm{H}_n(C_\bullet)$, since $\operatorname{im} \partial_{n+1}$ and $\ker \partial_n$ are submodules of the free R-module C_n, and R is a PID. Tensoring this sequence with M yields the long exact sequence

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Since R is a PID, and C_{n-1} is a free R-module, it follows that $\operatorname{im} \partial_n \subseteq C_{n-1}$ is also free. Thus, the short exact sequence of chain complexes

$$0 \to \ker \partial_\bullet \to C_\bullet \to \operatorname{im} \partial_\bullet \to 0$$

splits, where the subcomplexes $\ker \partial_\bullet$ and $\operatorname{im} \partial_\bullet$ are defined by taking kernels and images degree-wise. The boundary maps on these subcomplexes are all zero. Since this sequence is split, we may apply $- \otimes_R M$ to obtain a new short exact sequence of chain complexes

$$0 \to \ker \partial_\bullet \otimes_R M \to C_\bullet \otimes_R M \to \operatorname{im} \partial_\bullet \otimes_R M \to 0.$$

The associated long exact sequence in homology is

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3.2. Cohomology

Let R be a commutative ring and let $(C_\bullet, \partial_\bullet)$ be a chain complex of R-modules. The dual cochain complex $(C^\bullet, \delta^\bullet)$ is formed by taking the dual of each C_n, yielding the sequence of R-modules

$$\cdots \to \mathrm{Hom}_R(C_{n-1}, R) \xrightarrow{\delta^{n-1}} \mathrm{Hom}_R(C_n, R) \xrightarrow{\delta^n} \mathrm{Hom}_R(C_{n+1}, R) \to \cdots$$

where $\delta^n = \partial^*_{n+1}$ is the pullback along ∂_{n+1}. More generally, one may bake coefficients into the cochain complex by instead taking the R-module of homomorphisms $\mathrm{Hom}_R(C_n, M)$ for a fixed R-module M. This yields a new cochain complex

$$\cdots \to \mathrm{Hom}_R(C_{n-1}, M) \xrightarrow{\delta^{n-1}} \mathrm{Hom}_R(C_n, M) \xrightarrow{\delta^n} \mathrm{Hom}_R(C_{n+1}, M) \to \cdots$$

of R-modules, denoted by $\mathrm{Hom}_R(C_\bullet, M)$. Note that this is a fundamentally different process than *adding* coefficients by tensoring the dual chain complex with M, since $\mathrm{Hom}_R(C_n, M)$ is not generally isomorphic to $\mathrm{Hom}_R(C_n, R) \otimes_R M$. The purpose of the universal coefficient theorem for cohomology is to relate the cohomology of $\mathrm{Hom}_R(C_\bullet, M)$ to the cohomology of the dual complex $C^\bullet$.

Theorem 3.2 (Universal Coefficient Theorem for Cohomology)**.** *Let R be a PID, and let $(C_\bullet, \partial_\bullet)$ be a chain complex of free R-modules. Then for any R-module M, there exists a short exact sequence of R-modules*

$$0 \to \mathrm{Ext}^1_R(\mathrm{H}_{n-1}(C_\bullet), M) \to \mathrm{H}^n(\mathrm{Hom}_R(C_\bullet, M)) \to \mathrm{Hom}_R(\mathrm{H}_n(C_\bullet), M) \to 0$$

for each $n \in \mathbb{Z}$, which is natural in both $C_\bullet$ and M. Furthermore, this sequence splits, but not naturally.

4. Künneth Theorems

4.1. Homology

4.2. Cohomology

5. Applications