Course: CSC707, Automata, Computability and Computational Theory
Homework 4: Finite automata (FA), DFA, NFA, regular expressions, Pumping lemma, and closure properties
Submission: Use Wolfware
File Format: LaTeX and PDF
NOTE: If you create images, make sure you submit them as well.

Due Date: 11:00 AM, Saturday, March 13, 2010

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1. Assuming $L_1, L_2, ...$ are regular, which of the following languages are regular. Prove your answers.

   (a) $\bigcup_{i=1}^{n} L_i$

   (b) $\bigcup_{i=1}^{\infty} L_i$

   (c) $\bigcap_{i=1}^{n} L_i$

   (d) $\bigcap_{i=1}^{\infty} L_i$

   (a) **Theorem 1** $\bigcup_{i=1}^{n} L_i$ *is regular language.*

   Proof. Prove by induction.

   Basis: For $n = 1$, $\bigcup_{i=1}^{n} L_i = L_1$. Since L_1 is regular, $\bigcup_{i=1}^{n} L_i$ is regular.

   Inductive Hypothesis: Assume that $\bigcup_{i=1}^{n} L_i$ is regular for $\forall n \geq 1, n \in N$

   Inductive Step: Prove that $\bigcup_{i=1}^{n+1} L_i$ is regular.

   Note: $\bigcup_{i=1}^{n+1} L_i = \bigcup_{i=1}^{n} L_i \bigcup L_{i+1}$. By assumption, L_{i+1} is regular. By inductive hypothesis, $\bigcup_{i=1}^{n} L_i$ is regular. Since regular languages are closed under union, $\bigcup_{i=1}^{n} L_i \bigcup L_{i+1}$ is regular. Therefore, $\bigcup_{i=1}^{n+1} L_i$ is regular.

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   (b) **Theorem 2** $\bigcup_{i=1}^{\infty} L_i$ *may be regular language or non-regular language.*

   Proof. Prove by example.

i. **Regular:** Let $L_1 = 0^1, L_2 = 0^2, \ldots, L_n = 0^n, \ldots$. Each L is a finite sequence of 0s, so each L is a regular language. The union $\bigcup_{i=1}^{\infty} L_i = 0^*$, which is a regular language. Thus, $\bigcup_{i=1}^{\infty} L_i$ is regular.

ii. **Non-regular:** Let $L_1 = 0^4, L_2 = 0^6, L_3 = 0^8, \ldots, L_n = 0^{composite}, \ldots$. Since each L is a finite sequence of 0s, they are regular languages. The union of all these regular languages that have composite number of 0s is a non-regular language, $L = \{0^{composite}\}$, which is proved in the class.

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(c) **Theorem 3** $\bigcap_{i=1}^{n} L_i$ *is a regular language.*

Proof. Prove by induction.

Basis: For $n = 1$, $\bigcap_{i=1}^{n} L_i = L_1$. Since L_1 is regular, $\bigcap_{i=1}^{n} L_i$ is regular.

Inductive Hypothesis: Assume that $\bigcap_{i=1}^{n} L_i$ is regular for $\forall n \geq 1, n \in N$

Inductive Step: Prove that $\bigcap_{i=1}^{n+1} L_i$ is regular.

Note: $\bigcap_{i=1}^{n+1} L_i = \bigcap_{i=1}^{n} L_i \bigcap L_{i+1}$. By assumption, L_{i+1} is regular. By inductive hypothesis, $\bigcap_{i=1}^{n} L_i$ is regular. Since the intersection of two regular languages $L_1 \bigcap L_2 = \overline{\overline{L_1} \bigcup \overline{L_2}}$, and union and complement are the two closed operators for regular languages, the intersection of two regular languages is a regular language. Thus, $\bigcap_{i=1}^{n} L_i \bigcap L_{i+1}$ is regular, which means $\bigcap_{i=1}^{n+1} L_i$ is regular.

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(d) **Theorem 4** $\bigcap_{i=1}^{\infty} L_i$ *may be regular language or non-regular language.*

Proof. Prove by example.

i. **Regular:** Let $L_1 = 0^1, L_2 = 0^2, \ldots, L_n = 0^n, \ldots$. Each L is a finite sequence of 0s, thus a regular language. In this case, $\bigcap_{i=1}^{\infty} L_i = \emptyset$. Thus, $\bigcap_{i=1}^{\infty} L_i$ is regular.

ii. **Non-regular:** Let $L = \{0^{composite}\} = 0^4 \bigcup 0^6 \bigcup 0^8 \bigcup \ldots \bigcup 0^{composite} \bigcup \ldots$. The composite of L is $\overline{L} = \{\overline{0^{composite}}\} = \overline{0^4} \bigcap \overline{0^6} \bigcap \overline{0^8} \bigcap \ldots \bigcap \overline{0^{composite}} \bigcap \ldots$. Since $0^4, 0^6, \ldots, 0^{composite}, \ldots$ are all finite, they are all regular languages. Thus, the complement of them are also regular, since regular languages are closed on the operator of union. In this way, $\overline{L} = \{\overline{0^{composite}}\}$ is the infinite intersections of each $\overline{0^{composite}}$. Assuming $\overline{L} = \{\overline{0^{composite}}\}$ is regular, then its complement, $L = \{0^{composite}\}$, is also regular, since regular languages are closed on the operator intersection, which is proved in (c). However, we know that $L = \{0^{composite}\}$ is not regular, which is a contradiction. Therefore, $\overline{0^{composite}}$ is not regular. Thus, the infinite intersections of regular languages may produce a non-regular language.

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2. Prove that the following languages are regular:

   (a) $MIN(L) = \{x \in L|$ no prefix of x is in $L\}$

   (b) $L^R = \{x|$ reverse of x is in $L\}$

   (a) **Theorem 5** $MIN(L) = \{x \in L|$ *no prefix of* x *is in* $L\}$ *is a regular language.*

   Proof.

   Given a regular language L, construct a DFA for it. For each final state, we cut off all the outgoing edges from it. In this way, we build a DFA that recognizes no prefix of x is in L. Thus, $MIN(L) = \{x \in L|$ no prefix of x is in $L\}$ is a regular language.

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   (b) **Theorem 6** $L^R = \{x|$ *reverse of* x *is in* $L\}$ *is a regular language.*

   Proof. Given a regular language L that recognizes x, we can construct a DFA for it. Using this DFA, we can build a DFA for $L^R = \{x|$ reverse of x is in $L\}$ as follows:

   - Reverse each transition.
   - Turn the start state into a final state.
   - Add a new start state, and add a λ-transition from the start state to each final state.
   - Turn the original final states into normal states.

   In this way, we can recognize the reverse of x. Thus, $L^R = \{x|$ reverse of x is in $L\}$ is a regular language.

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