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Prove LPATH is NP-Complete

Definition of Longest Path (LPATH):
LPATH $= \{<G, a, b, k> \mid G$ contains a simple path of length at least k from a to $b\}$

Step 1 (Verify): Given a LPATH of length k between points a and b on a graph G, verify that a simple path of length at least k exists from a to b in polynomial time.

-Check that the LPATH is simple

for $i = 1 \rightarrow n$
{
    for $j = (i+1) \rightarrow n$
    {
    make sure $vertex_i \neq vertex_j$
    }
}
This takes $O(n^2)$ time.

-Check that all edges in path actually exist

for $i = 1 \rightarrow |$edges in LPATH$|$
{
    for $j = (i+1) \rightarrow |$edges in $G|$
    {
    make sure $edge_i = edge_j$ for some $edge_j \in G$
    }
}
This takes $O(n^2)$ time.

-Count number of edges in LPATH
This takes $O(n)$ time.

Thus, the entire verify step takes $O(n^2)$ time which is polynomial.

Step 2 (Reduction): Show that LPATH is NP-Hard. We do this by showing reduction from known NP-Complete problem to the LPATH problem.
P_{known} = UHAMPATH
P_{new} = LPATH

$UHAMPATH = \{< G, s, t > | G$ is undirected graph that contains a Hamiltonian path from s to $t\}$

Reduction:
UHAMPATH($G = (V, E), s, t$)
k = |V| - 1
Return the answer to LPATH(G, s, t, k)

The reduction step occurs in polynomial time.

Step 3 (Correctness): Graph G has a Longest Path of size |V| - 1 between a and b iff G has a Hamiltonian Path between a and b.

-Case 1 Assume G has Longest Path of size |V| - 1 between a and b. Show G has a Hamiltonian Path between a and b.

There exists a simple path from a to b with length |V| - 1(from given Hamiltonian Path). Therefore, there exists a Longest Path of at least length |V| - 1 from a to b. This Longest Path is the Hamiltonian Path.

-Case 2 Assume G does not have a Hamiltonian Path between a and b. Show G does not have a Longest Path between a and b of size at least |V| - 1.

A longest path with length at least |V| - 1 from a to b would be a simple path which visits every vertex v, such that $v \in V$. A simple path that starts at a and ends at b, of length |V| - 1, is a Hamiltonian Path. However, no hamiltonian path from a to b exists (given). Therefore, no longest path of length |V| - 1 exists.