Course: CSC707, Automata, Computability and Computational Theory
Bonus Homework: Bonus points assignment
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Submission: Use Wolfware
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1. **Dominating Set is NP-complete**
   Decision Problem:
   $DS = \{< G, k > | G \text{ contains a dominating set with } k \text{ nodes}\}$
   A subset of nodes of a graph G is a **dominating set** if every other node of G is adjacent to some node in the subset.

   (a) **Step 1 (Verification):**
   Claim: $DS \in NP$ i.e. can be verified in polynomial time
   Given a graph $G(V, E)$ has vertexes $v \in V$ with size of n, edge $e = \{u, v\} \in E$ and Dominating Set DS with size k.
   Algorithm:

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for all v ∈ V ............................. O(|V|)
    if v is an isolate vertex
        if v ∈ DS then continue ......... O(|DS|)
        else return false
    else
        if v ∈ DS
            then continue
        else for all e ∈ E ⋀ v ∈ e ......... O(|E|)
            if u ∈ e − {v} ............... O(|DS|)
                then v is covered and break
            else continue
            if all edges are checked and v is not covered
                then return false
return true
```

   Algorithm's time complexity: $O(|V| \cdot |E| \cdot |DS|) = O(n^2 k)$
   So the Verification Algorithm will finished in Polynomial time, and hence the Dominating Set is a NP problem.

   (b) **Step 2 (Reduction): DS can be derived with Reduction from the $NP - complete$ problem "Vertex Cover"**
   P_{known} = Vertex Cover, $VC = \{< G, k > | G \text{ has a vertex cover of size } k\}$
   Claim: $VC \to DS$ reduction in polynomial time

1. G' construction : Given graph $G(V, E)$, construct G' such that:
for each $e = \{u, v\} \in E$
a. Add vertex w
b. Add edges $\{u, w\}$ and $\{v, w\}$
Complexity is $O(|V|) = O(n)$
2. Verify that G' has a dominating set of size k using Step 1
Complexity is $O(n^2k)$
Hence, $T(n, k) = O(n^2k)$, reduction is possible in polynomial time

(c) **Step 3 (Correctness):**
Claim: Graph G has a vertex cover of size k if and only if graph G' has a dominating set of size k.

$\Rightarrow$ Suppose that G has a vertex cover S of size k, we need to prove that either $v \in S$ or v has an adjacent node in S
$\forall v \in G'$, we have the following two options:
a. Original vertex : $v \in G$
Since $\{u, v\} \in G$ and S is the vertex cover, if $v \notin S$, then by definition of a vertex cover $u \in S$. Thus, an adjecent node of v is in S
b. Additional vertex w has two adjacent vertices $u, v \in G$ and by definition of a vertex cover, either $u \in S$ or $v \in S$. Hence, if G has a vertex cover of size k, then G' has a dominating set of at most size k

$\Leftarrow$ Suppose that G' has a dominating set D of size k.Consider the additional vertices $w \in D$. We note that w is connected to exactly 2 vertices $u, v \in G$.Since $u, v, w \in G'$ form a triangle, choosing any vertex in the triangle would dominate the other two. Hence we can construct D_{new} by replacing every $w \in D$ by u or v and still cover all the vertices in G'. Also every $w \in G'$ corresponds to an edge $\{u, v\} \in G$ and hence all edges in G are covered by D_{new}.
Thus, if G' has a dominating set of size k, then G has a vertex cover of at most size k