Course: CSC707, Automata, Computability and Computational Theory
Group Problem: Clique $\propto$ Independent Set.
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Minus Points: -19

Input: A graph G =(V,E), an integer k $\leq |V|$
Output: If there is a subset S of k vertices in G such that no pair of vertices in S is connected by an edge in G?
Decision Problem:
IS = $\{< G, k >$: Graph G has a Independent set of size k $\}$
Step 1:
We need to show that Independent Set (IS) $\in$ NP. Suppose we have been given a set of vertices V'. We can verify that V' is IS of Graph G as follows:

1. Verify $|V'|$ = k. This is O(1).
2. For each edge (u,v) $\in$ E(G), we check whether u $\in V'$ and v$\in V'$. If none of the edge satisfies this relation, then V' is IS of G, otherwise it is not. This is $O(k^2 \, |E|)$.

So the verification step is polynomial. Hence IS is in NP.
Step2:
Given the Clique problem is NP-Complete, we show that Clique $\propto$ IS. Given an instance of Clique problem $< G, k >$, following steps reduces it to IS problem.

1. We compute the complement $\overline{G}$ of given Graph. This take $\cancel{O(n^2)}O(|V|^2)$. -2
2. Calculate $k' = k$.

The output of reduction algorithm is an instance $< \overline{G}, k' >$ of IS problem.
Step3:
~~Graph G has Clique $V' \in$ V~~G has a clique of size k iff ~~$\overline{G}$ has IS V'~~$\overline{G}$ has IS of size k. -1
$\rightarrow$
~~Suppose G has Clique $V' \subset$ V. If (u,v) be an edge in G and u $\in V'$ and v $\in V'$, then by definition of Clique and Complement, (u,v) $\notin \overline{E}$. Since u,v were chosen arbitrarily, the vertices that forms a clique in G forms IS in $\overline{G}$.~~ -8
Consider any two vertices $u, v \in Clique(G)$, suppose that edge $(u, v) \in \overline{G}$. By the definition of complement graph, $(u, v) \notin G$, which goes contradict with that $u, v \in Clique(G)$. Therefore for $\forall u, v \in Clique(G)$, $(u, v) \notin \overline{E}$. $\overline{G}$ has a IS of size k.
$\leftarrow$

~~Suppose G has IS $V' \subset$ V. Take any two verices u,v in G such that u $\in V'$ and v $\in V'$, then (u,v) $\notin E$ by defition of Independent Set. This implies that (u,v) $\in \overline{E}$. Since u,v were chosen arbitrarily, the vertices that forms a IS in G forms a Clique of same size in $\overline{G}$.~~ -8

Suppose $\overline{G}$ has an IS $V' \subset V, |V'| = k$, $u \in V', v \in V'$, $\Rightarrow (u, v) \notin \overline{E}$. By the definition of complement graph, $(u, v) \in E$. Therefore, V' is an Clique of $G \Rightarrow$ G has a clique of size k.

This proves that IS is NP-Complete problem.