CSC-707: NP-Complete Proof

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Prove that the TSP (Traveling Salesman) problem is NP-Complete by reduction to the Hamiltonian Cycle problem.

Hamiltonian Cycle Problem:
Input: An un-weighted graph G
Output: Does there exist a **simple** tour that visits every vertex of G without repetition?

Prove that TSP is NP-Complete via reduction

Step 1: (Verification) Show that $TSP \in NP$

The TSP problem: $TSP = \{< G, c, k >: G = (V, E)$ is a complete graph , c is a function $V \times V \to Z$, $k \in Z$ and G has a traveling salesman tour with cost at most $k\}$
Given a tour $h = (v_1, v_2, v_3, \cdots, v_n)$ and a cost k, we need to check for two conditions

1. The tour visits every city (vertex) in the graph and returns to the starting point
2. Cost k is not exceeded by the tour

Pseudo-code:

```
Part 1:  ---   O(n^2)
// identify that every city is visited
for j = 1 ... |V|:
  vertex_present = false
  for i = 1 ... n
     // test if present
     if ( h[i] contains vertex(j) ):
         vertex_present = true
  if vertex_present == false:
     fails to be a tour

Part 2:  ---   O(1)
// test that we end where we began
if ( h[0] = h[n] ):
  valid_tour = true

Part 3:  ---   O(n)
// determine total cost for tour h
cost = 0
for i = 0 ... (h-1):
  cost = cost + d( v_i, v_(i+1) )
if ( cost > k ):
  fails to be a tour of cost <= k
```

Therefore given a graph, we can decide if a tour has cost at most k in $T(n) = O(n^2)$. Therefore, TSP is verifiable in polynomial time.

Step 2: Reduction
We will use the HAM-CYC problem which is known to be NP-Complete and show there exists a polynomial reduction to the TSP problem.

Let $G = (V, E)$ be an input instance of the HAM-CYC problem with $|V| = n$. Then we will construct a complete weighted graph $G' = (V, E')$ such that if an edge $e_i \in E$ then the weight of $e_i = 1$ in E' and if the edge was not in E then the weight of $e_i = 2$.

The reduction to TSP can be implemented in polynomial time $O(|V| \times |V|) = O(n^2)$

Pseudocode:

```
for i = 1 to |V| do
   for j = 1 to |V| do
      if ( i,j ) in E then w(i,j) = 1 else w(i,j) = 2
```

Therefore, $G = (V, E) \rightarrow T(n) = O(n^2) \rightarrow G' = (V, E')$
Step 3: Correctness
Claim: The graph G has a Hamiltonian cycle if and only if there is a TSP tour of G' of weight exactly n.

Suppose that the graph G' has a Hamiltonian cycle h. Then each edge in h belongs to E and has a cost associated with it of 1 in G'. Also, by definition, every vertex in V is visited by the cycle h. Thus, h is a tour in G' of cost $|V| = n$.

Now suppose that the graph G' has a tour h' of cost exactly n. Then, since the edges in h' are found in E' and the total cost is exactly n, then each edge in h' has cost 1. This means that every edge in h' is also in E. And since the tour implies that every city is visited, we conclude that h' must also be a Hamiltonian cycle in graph G.

□