Course: CSC707, Automata, Computability and Computational Theory
Homework 1: Functions, Sets, Proofs, Induction, Countability
Submission: Use Wolfware
File Format: Both LaTeX and PDF

Due Date: 2:00 AM, Tuesday, January 19, 2010

1. Specify an estimate time spent (minutes) on each sub-problem (**optional**).
2. No penatly for skipping up to three (3) sub-problems at your choosing.
3. What sub-problems you would like us to discuss in class (**optional**).
4. Provide any feedback/questions you may have on this homework (**optional**).
5. Using LaTeX is required.

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1. Prove of disprove the countability of each of the following sets:

   (a) ________ $S = \{f : N \to N \mid f \text{ is total and, } \forall i \in N,\ f(i) \leq 2i\}$
   (b) ________ The set of all finite sequences of natural numbers
   (c) ________ The set of all subsets of a countable set
   (d) ________ $S = \{1, 3, 5\}$
   (e) ________ The set of all ordered pairs of integers

   Answers:

   (a) **We hope to discuss this sub-problem in the class**.
   For $S = \{f : N \to N \mid f \text{ is total and, } \forall i \in N,\ f(i) \leq 2i\}$, every $x \in N$ maps to a finite subset of N:

   $$1 \to \{1, 2\}, 2 \to \{1, 2, 3, 4\}, \ldots, n \to \{1, 2, 3, \ldots, 2 * n\}, \ldots$$

   Since the countable union of countable sets: $\bigcup_{j=1}^{\infty} S_j$, S_j is countable for $\forall j$, $S = \{f : N \to N \mid f \text{ is total and, } \forall i \in N,\ f(i) \leq 2i\}$ is countable.

   (b) The set of all finite sequences of natural numbers consists of 1-length sequences of N, 2-length sequences of N, ..., n-length sequences of N and so on. All these n-length sequences of N, $n \in N$, are countable. Since the countable union of countable sets is countable, the set of all finite sequences of natural numbers is countable.

(c) The set of all subsets of a countable set is countable.
Assume a countable set $S = \{s_1, s_2, \ldots, s_n, \ldots\}$, we can create a function to map a sequence $A =< a_1, a_2, \ldots, a_n, \ldots >$ to a subset of S. For example, given a subset S' of S, if $s_1 \notin S'$, then $a_1 = 0$ in the corresponding sequence A' for S'. This function maps the countable set "sequence of S" onto the set of all subsets of a countable set S. Thus, the set of subsets of a countable set is countable.

(d) $S = \{1, 3, 5\}$ is countable since $|S|$ is finite, which is 3.

(e) I provide two solutions to it:

i. The set of all ordered pairs of integers is the cross-product of the set of integers, $Z \times Z$. Since the set of integers, Z, is coutable and $\prod_{j=1}^{\infty} S_j$ is countable for $\forall j$, $Z \times Z$ is countable. Thus, the set of all ordered pairs of integers is countable.

ii. I use a function that is similar to the one that counts the ordered pairs of natural numbers. For an ordered pair of integers, (i, j), I can compute their absolute values, $(|i|, |j|)$, and use the function $\frac{(|i|+|j|-2)(|i|+|j|-1)}{2} + |i|$ to compute the corresponding mapping value k. In this way, we can map the k value of the integer pairs, $(i, j), i > 0, j > 0$, to $4k + 1$, the k value of the integer pairs, $(i, j), i < 0, j > 0$, to $4k + 2$, the k value of the integer pairs, $(i, j), i < 0, j < 0$, to $4k + 3$, and the k value of the integer pairs, $(i, j), i > 0, j < 0$, to $4k + 3$. First I construct a function $f(i, j) : Z \to N, i \in Z, j \in Z, i \neq 0, j \neq 0$:

$$f(i,j) = \begin{cases} f(i,j) = 4 * \left(\frac{(|i|+|j|-2)(|i|+|j|-1)}{2} + |i|\right) + 1 \text{ if } i > 0, j > 0 \\ f(i,j) = 4 * \left(\frac{(|i|+|j|-2)(|i|+|j|-1)}{2} + |i|\right) + 2 \text{ if } i < 0, j > 0 \\ f(i,j) = 4 * \left(\frac{(|i|+|j|-2)(|i|+|j|-1)}{2} + |i|\right) + 3 \text{ if } i < 0, j < 0 \\ f(i,j) = 4 * \left(\frac{(|i|+|j|-2)(|i|+|j|-1)}{2} + |i|\right) + 4 \text{ if } i > 0, j < 0 \end{cases}$$

Our new function use the same function to compute the corresponding mapping value k by replacing i, j with $|i|, |j|$ and map (i, j) to $4k + x$ based on which quadrant (i, j) belong to. Since the function $f(i, j) = \frac{(i+j-2)(i+j-1)}{2} + i, i \in N, j \in N$ is bijection, our function is also bijection. Thus, the ordered pairs of $(i, j), i \in Z, j \in Z, i \neq 0, j \neq 0$ is countable. I can constuct the whole ordered pairs of integers by unioning the finite set of ordered pairs $(i, j), i \in Z, j \in Z, i = 0$ or $j = 0$. Since both sets are countable, their union is countable, too.

2. Define the contrapositive (NOT) for each of the two statements:

(a) ________ A AND (NOT B) $\to$ C OR (NOT D)

(b) ________ $\exists p : \forall x \in L\ \exists\ u, v, w$ such that:

   i. $x = uvw$
   ii. $|uv| \leq p$
   iii. $|v| > 0$
   iv. $\forall k \geq 0$, $uv^k w \in L$

Answer:

(a) (NOT C) AND D $\rightarrow$ (NOT A) OR B

(b) $\forall p : \exists x \in L\ \forall\ u, v, w$ such that:

   i. $x \neq uvw$ OR
   ii. $|uv| > p$ OR
   iii. $|v| \leq 0$ OR
   iv. $\exists k \geq 0$, $uv^k w \notin L$

3. Correct (if necessary) each of the following claims and prove the claims:

(a) ________ For $\forall x$, $2^x \geq x^2$

(b) ________ There is no pair of integers a and b : $a \bmod b = b \bmod a$

(c) ________ $\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$

Answer:

(a) The claim, $\forall x$, $2^x \geq x^2$, is not correct: When x = -2, $2^x = \frac{1}{4} < x^2 = 4$.
The corrected claim is: $\forall x \geq 4, 2^x \geq x^2$.
Prove: First I use induction to prove $2^x \geq 2x + 1, \forall x \geq 4$.
When $x = 4, 2^x = 16, 2x + 1 = 9, 2^x \geq 2x + 1$
Assume $x > 4, 2^x \geq 2x + 1$, then we have:
$2^{x+1} = 2 * 2^x \geq 2x + 1 + 2 = 2(x + 1) + 1$ Thus, $2^x \geq 2x + 1, \forall x \geq 4$.
Then I use induction to prove $\forall x \geq 4, 2^x \geq x^2$: When $x = 4, 2^x = 16, x^2 = 16, 2^x \geq 2x + 1$
Assume $x > 4, 2^x \geq x^2$, then we have:
$2^{x+1} = 2 * 2^x \geq x^2 + 2x + 1 = (x + 1)^2 \Rightarrow 2^{x+1} \geq (x + 1)^2$
Thus, $2^x \geq 2x + 1, \forall x \geq 4$.

(b) The claim, there is no pair of integers a and b : $a \bmod b = b \bmod a$, is not correct.
Prove: If $a = b$, then $a \bmod b = 0, b \bmod a = 0$ and $a \bmod b = b \bmod a$, which means the claim is not correct. Therefore, the claim should be changed to: there is no pair of distinct non-zero integers a and b : $a \bmod b = b \bmod a$. Prove: When $ab = 0$, $a \bmod b$, $b \bmod a$, or both $a \bmod b$ and $b \bmod a$ is invalid since the modulo can not be 0. When $ab < 0$, then $(a \bmod b) * (b \bmod a) < 0$ and $(a \bmod b) \neq (b \bmod a)$. When $ab > 0$ and assume $|a| > b$, $b \bmod a = b$ and

$a \bmod b \neq b$. Therefore, the claim, there is no pair of distinct non-zero integers a and b : $a \bmod b = b \bmod a$, is correct.

(c) Prove: The sum of cubes can be represented as: $\sum_{i=0}^{n} i^3 = \frac{n^2 * (n+1)^2}{4}$.

Since $\sum_{i=0}^{n} i = \frac{n*(n+1)}{2}$, $\left(\sum_{i=0}^{n} i\right)^2 = (\frac{n*(n+1)}{2})^2 = \frac{n^2*(n+1)^2}{4}$, $\sum_{i=0}^{n} i^3 = \left(\sum_{i=0}^{n} i\right)^2$. The claim is correct.

4. Prove or disprove each of the following claims:

   (a) ________ $\sqrt{2}$ is a rational number

   (b) ________ Any even number is composite (i.e. not prime)

   Answer:

   (a) Disprove: Assume $\sqrt{2}$ is a rational number. Thus, we can have: $\sqrt{2} = \frac{a}{b}$ such that a and b do not have common divisor. From $\sqrt{2} = \frac{a}{b}$, we can have: $2 = \frac{a^2}{b^2}$, from which we can further get $a^2 = 2*b^2$. Then we know a^2 is an even number and a must be an even number, since the square of an odd number can not be an even number. Knowing that a is an even number, we can replace a with $2 * k$ and we have: $4 * k^2 = 2 * b^2 \Rightarrow b^2 = 2 * k^2$, which means that b is also an even number. If a and b are both even numbers, then they have a common divisor, 2, which means that we have a conflict with our assumption that a and b do not have common divisor. Therefore, $\sqrt{2}$ is not a rational number.

   (b) Disprove: 2 is an even number since it can be divided by itself. 2 is also a prime number since it has exactly two distinct natural number divisors, 1 and 2. Thus, not every even number is composite.

5. State (without proof) if a definition is *one-to-one*, *onto*, or a *bijection*.

   (a) ________ $f : Z \to Z \mid f(i) = |i|$

   (b) ________ $f : N \to \{0, 1, 2, 3\} \mid f(i) \equiv i \bmod 4$

   (c) ________ $f : \{1, 2, 3\} \to \{1, 2, 3\} \mid f(1) = 3,\ f(2) = 1,\ f(3) = 2,$ $f(2) = 3$

   Answer:

   (a) $f : Z \to Z \mid f(i) = |i|$ is not *one-to-one*, *onto*, or a *bijection*.

   (b) $f : N \to \{0, 1, 2, 3\} \mid f(i) \equiv i \bmod 4$ is *onto*.

   (c) $f : \{1, 2, 3\} \to \{1, 2, 3\} \mid f(1) = 3,\ f(2) = 1,\ f(3) = 2,\ f(2) = 3$ is *onto*.

6. **The ugly proof**.

(a) Choose any of the above sub-problems.

(b) Provide an **ugly** solution to this sub-problem.

The '**ugliest solution**' is the one that contains either *common* or *hard-to-catch* mistakes. The more mistakes the better, but the *elegancy* of the mistakes will be valued higher.

Answers:

(a) I choose problem 1(c): prove or disprove the set of all subsets of a countable set.

(b) The set of all subsets of a countable set is countable. A subset of a countable set is countable and the set of all subsets of a countable set can be viewed as the countable union of its subsets. Since the coutable union of countable sets: $\bigcup_{j=1}^{\infty} S_j$ is countable for $\forall j$, the set of all subsets of a countable set is countable.

The problem of this ugly proof is: the union of all its subsets equals itself.