Course: CSC707, Automata, Computability and Computational Theory
Homework 2: Complexity Theory, polynomial time reduction, P vs NP, NP-hard and NP-complete problems.
Submission: Use Wolfware
File Format: LaTeX and PDF

Due Date: 2:00 A.M. (EST), Thursday, February 11, 2010

1. Provide any feedback/questions you may have on this homework (**optional**).
2. Using LaTeX is required.

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1. Let $\sum$ denote $\{0,1\}$. Let $\propto$ denote polynomial-time reducibility. A set $L \subseteq \sum^*$ is P-complete if $L \in P$ and $M \propto L$ for all M, where $M \subseteq \sum^*$ and $M \in P$.
   (a) Show that $\emptyset$ and $\sum^*$ are not P-complete.
   (b) Show that if $L \in P$ and $\emptyset \neq L \neq \sum^*$, then L is P-complete.
2. Show that the Vertex Cover remains NP-complete even when all the vertices in the graph are restricted to have even degree.
   Vertex Cover is defined as follows:
   INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
   QUESTION: Is there a subset $V' \subseteq V$ such that $|V'| \leq k$, and for each edge $\{u, v\} \in E$ at least one of u and v belongs to V'?
   Proof:
   Decisition Problem: $VC = \{< G, k >$: graph G has a vertex cover of size k where $\forall e \in E, degree(e) = 2\}$
   Step 1:
   Given a vertex cover of size k of a graph $G = (V, E), \forall v \in V degree(v)$is even, we can verify that it is indeed the vertex cover of graph $G(n, k)$ in polynomial time:
   $$T(n, k) = O(|E|k) = O(n^2k)$$
   Step 2:
   Given a $G = (V, E)$, we can construct a graph $G' = (V', E')$ such that:
   $V' = V \bigcup \{v_1, v_2, v_3\}$, and
   $E' = E \bigcup \{< v_1, v_2 >, < v_1, v_3 >, < v_2, v_3 >\} \bigcup \{< v, v_1 > | v \in V, degree(v)$is odd $>$ $\}$

   We claim that all the vertices in G' have even degree since the number of vertices that has odd degree is even.

Proof:
The sum of the degree of all vertices in a graph G with n edges is $2n$ since each edge has two end points and add 1 to the degree of two end points(vertices). Assume the set of odd-degree vertices is V_o and the set of even-degree vertices is V_e, then, we have

$$\sum_{v \in V} degree(v) = \sum_{v' \in V_o} degree(v') + \sum_{v'' \in V_e} degree(v'')$$

$\sum_{v \in V} degree(v) = 2n$ is even and $\sum_{v'' \in V_e} degree(v'')$ is even (sum of even numbers is even) $\Rightarrow \sum_{v' \in V_o} degree(v')$ is even $\Rightarrow$ the number of odd number must be even

Thus, all the vertices of G' has even degrees.

The construction of G' can be completed in polynomial time since we only need to visit every vertex in G for one time:

$$T(n, k) = O(|E|) = O(n)$$

Step 3:
Claim: The graph G has a vertex cover of size k if and only if G' has a vertex cover of size $k + 2$
Suppose graph $G = (V, E)$ has vertex cover of size k, $V' \subseteq V : |V'| = k$, and we choose a set of vertices $V'' = V' \bigcup \{v_1, v_2\}$:

(a) $v_1 \in V'' \Rightarrow$ edges $e' \in \{< v, v_1 > | v \in V, degree(v) isodd >\}$ is covered by v_1.
(b) $v_1 \in V'', v_2 \in V'' \Rightarrow < v_1, v_2 >, < v_1, v_3 >, < v_2, v_3 >$ is covered by v_1, v_2.
(c) $\forall e \in E$ is covered by $V', V' \subseteq V''$. $\Rightarrow V''$ cover all the edges in E

Combining (a),(b), and (c) $\Rightarrow V''$ is the vertex cover of G'.

Suppose set G' has a set cover V'' of size $k + 2$
To cover $< v_1, v_2 >, < v_1, v_3 >, < v_2, v_3 >$, at least two vertices of $\{v_1, v_2, v_3\}$ are included in V''
Since all egdes in the E' are covered by V'', the edges except $< v_1, v_2 >$ $, < v_1, v_3 >, < v_2, v_3 >$ must be covered by k vertices. These edges are exactly the same edges in E.
Thus, G has a vertex cover of k.

3. Show that the Set Cover problem is NP-complete using the reduction from Vertex Cover.
Set Cover problem is defined as follows:
INSTANCE: A set X of n elements, a family F of subsets of X, and a

positive integer k.
QUESTION: Is there a set k or fewer subsets from F whose union is X?
For example, if $X = \{1, 2, 3, 4\}$ and $F = \{\{1, 2\}, \{2, 3\}, \{4\}, \{2, 4\}\}$, a solution does NOT exist for $k = 2$ but does exist for $k = 3$ (e.g., $\{\{1, 2\}, \{2, 3\}, \{4\}\}$.

Proof:
Decision Problem: $SC = \{< S, S_1, S_2, ..., S_m, k >$: Given a set S of n elements, and $S_1, S_2, \ldots, S_m$ are the subsets of S, S has a set cover of size k such that $S_{i1} \bigcup S_{i2} \bigcup \ldots \bigcup S_{ik} = S$

Step 1:
Given k subsets, we can verify that it is the set cover of S by unioning k subsets in polynomial time, since unioning two subsets can be completed in $O(n^2)$.

$$T(S, n, k) = O(kn^2)$$

Step 2:
Given a $G = (V, E)$, let each edge $e \in E$ represent an element in the set S and each vertex $v_i \in V$ represent a set S_i contains edges incident to v_i, i.e. $\forall e \in E, e =< u, v >, e \in S_u, e \in S_v$. Thus, $S_i \subseteq S$. The time of the transform procedure is $T(V, E) = |V| + |E|$, which is polynomial in time.

Step 3:
Claim: The graph G has a vertex cover of size k if and only if the set S has a set cover of size k.
Suppose graph $G = (V, E)$ has vertex cover of size k, $V' \subseteq V : |V'| = k$:
$\forall (u, v) \in E \Rightarrow$ either $u \in V'$ or $v \in V'$ or both $\Rightarrow$
$\forall v_i \in V' \Rightarrow \{S_{i1} \bigcup S_{i2} \bigcup \ldots \bigcup S_{ik} = S\}$

Suppose set S has a set cover of size k
$S_{i1} \bigcup S_{i2} \bigcup \ldots \bigcup S_{ik} = S$
$\Rightarrow \forall < u, v >\in E$, either u or v or both are in $\{S_{i1}, S_{i2}, \ldots, S_{ik}\}$
$\Rightarrow$ every subset S_i represent a vertex $v \in V$
$\Rightarrow$ there is a vertex cover of size k in G

4. The Independent Set problem is defined as follows.
Set Cover problem is defined as follows:
INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
QUESTION: Does G contain an indepenent set of size k or more, i.e., a subset $V' \subseteq V$ and $|V'| \geq k$ such that no two vertices in V' are joined by an edge in E?
Suppose you are given a graph, $G = (V, E)$, and an integer k as input with $|V| = n$. And suppose you are given an algorithm, D, that solves the decision version of the Independent Set problem in time $T(n, k)$.

    (a) Use D to find the size of the maximum independent set, and state the time complexity involved.

(b) Use D in a self-reduction to solve the search version of the independent set problem, and state the time complexity involved.