Course: CSC707, Automata, Computability and Computational Theory
Bonus Homework: Bonus points assignment
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Submission: Use Wolfware
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Grading: 9.5 out of 10. Besides few glitches (Highlighted), all steps are correct.
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1. **Dominating Set is NP-complete**
   Decision Problem:
   $DS = \{< G, k > | G$ contains a dominating set with k nodes$\}$
   A subset of nodes of a graph G is a **dominating set** if every other node of G is adjacent to some node in the subset.

   (a) **Step 1 (Verification):**
   Claim: $DS \in NP$ i.e. can be verified in polynomial time
   Given a graph $G(V, E)$ has vertexes $v \in V$ with size of n, edge $e = \{u, v\} \in E$ and Dominating Set DS with size k.
   Algorithm:

   for all $v \in V$ .... $O(|V|)$
       if v is an isolate vertex
           if $v \in DS$ then continue .... $O(|DS|)$
           else return false
       else
           if $v \in DS$
               then continue
           else for all $e \in E \bigwedge v \in e$ .... $O(|E|)$
               if $u \in DS$ and $u \in e - \{v\}$ .... $O(|DS|)$
                   then v is covered and break
               else continue
               if all edges are checked and v is not covered
                   then return false
   return true

   Algorithm's time complexity: $O(|V| \cdot |E| \cdot |DS|) = O(n^2 k)$
   So the Verification Algorithm will finished in Polynomial time, and hence the Dominating Set is a NP problem.

   (b) **Step 2 (Reduction):** DS **can be derived with Reduction from the** $NP - complete$ **problem "Vertex Cover"**

P_{known} = Vertex Cover, $VC = \{< G, k > |G$ has a vertex cover of size $k\}$
Claim: $VC \rightarrow DS$ reduction in polynomial time
1. G' construction : Given graph $G(V, E)$, construct G' such that: for each $e = \{u, v\} \in E$
a. Add vertex w
b. Add edges $\{u, w\}$ and $\{v, w\}$
Complexity is $O(|V|) = O(n)$
~~2. Verify that G' has a dominating set of size k using Step 1~~
~~Complexity is $O(n^2k)$~~
Hence, $T(n, k) = O(n)$, reduction is possible in polynomial time

(c) **Step 3 (Correctness):**
Claim: Graph G has a vertex cover of size k if and only if graph G' has a dominating set of size k.

$\Rightarrow$ Suppose that G has a vertex cover S of size k, we need to prove that either $v \in S$ or v has an adjacent node in S
$\forall v \in G'$, we have the following two options:
a. Original vertex : $v \in G$
Since $\{u, v\} \in G$ and S is the vertex cover, if $v \notin S$, then by definition of a vertex cover $u \in S$. Thus, an adjecent node of v is in S
b. Additional vertex w has two adjacent vertices $u, v \in G$; $(u, v) \in E$ and by definition of a vertex cover, either $u \in S$ or $v \in S$. Hence, if G has a vertex cover of size k, then G' has a dominating set of at most size k

$\Leftarrow$ Suppose that G' has a dominating set D of size k.Consider the additional vertices $w \in D$. We note that w is connected to exactly 2 vertices $u, v \in G$.Since $u, v, w \in G'$ form a triangle, choosing any vertex in the triangle would dominate the other two. Hence we can construct D_{new} by replacing every $w \in D$ by u or v and still cover all the vertices in G'. Also every $w \in G'$ corresponds to an edge $\{u, v\} \in G$ and hence all edges in G are covered by D_{new}.
Thus, if G' has a dominating set of size k, then G has a vertex cover of at most size k