Bending of thin plates

2015

Bending of thin plates

Generally, we define **a thin plate** as a sheet of material

- whose thickness is small compared with its other dimensions
- but which is capable of resisting bending in addition to membrane forces.

Such a plate forms a basic part of an aircraft structure, being, for example, the area of stressed skin bounded

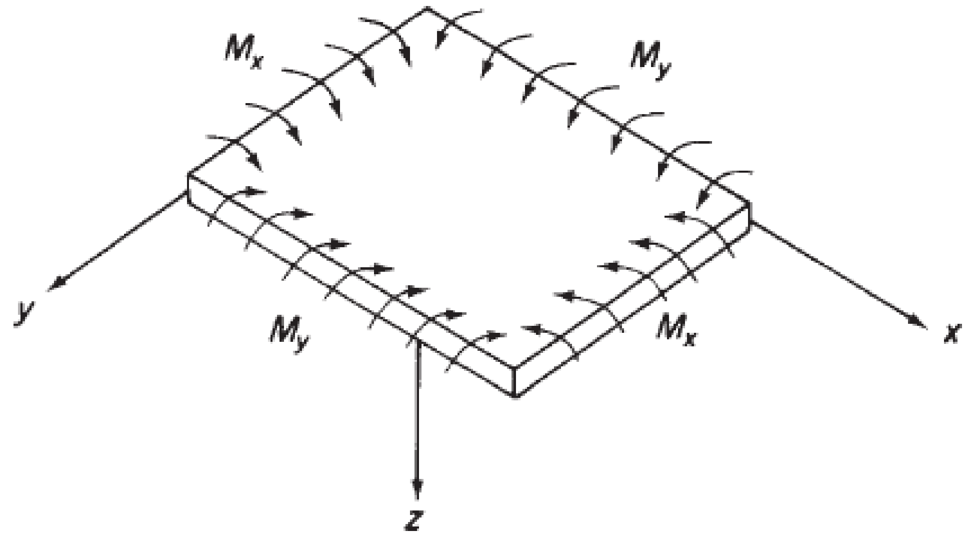
- by adjacent stringers and ribs in a wing structure or
- by adjacent stringers and frames in a fuselage.

Pure bending of thin plates

The thin rectangular plate is subjected to pure bending moments of intensity M_x and M_y per unit length uniformly distributed along its edges.

Sign convention

These bending moments are **positive** when they produce *compression* at the upper surface of the plate and *tension* at the lower.



We **assume**

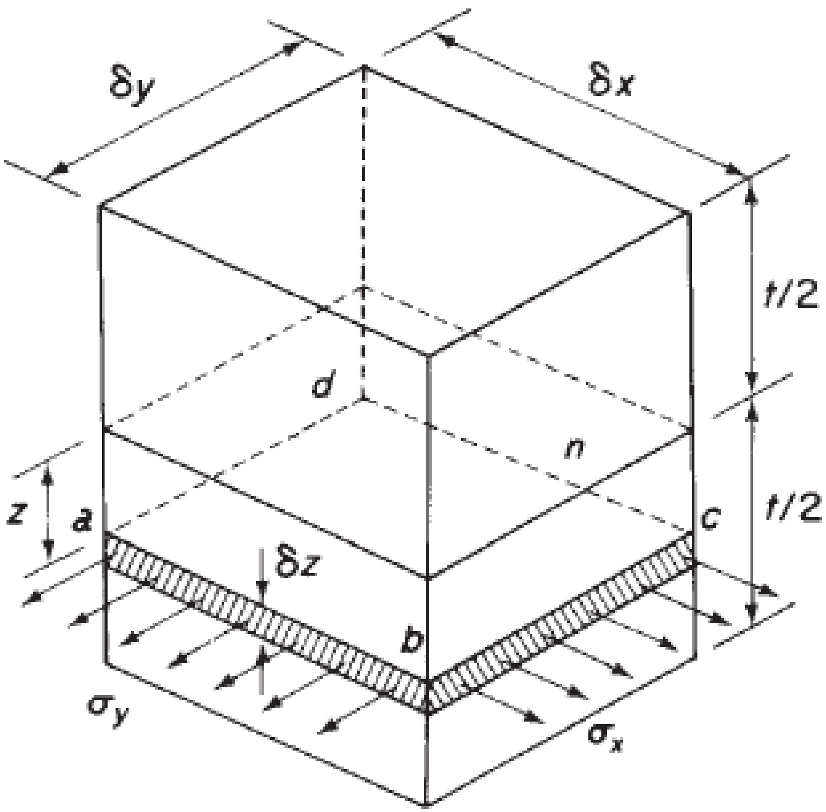
- the displacement of the plate in a direction *//* to the z axis is small compared with its thickness t and
- that plane sections remain plane after bending.

I.e. the middle plane is a neutral plane (**reference for system of axes).**

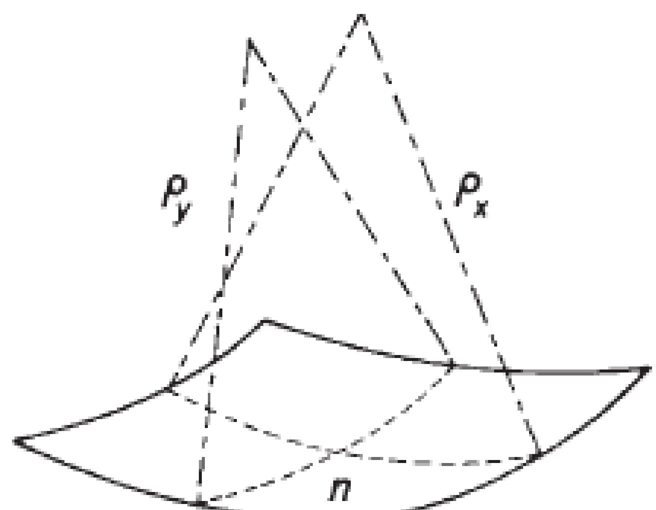
Pure bending of thin plates

Let us consider an element of the plate of side $\delta x \delta y$ and having a depth equal to the thickness t of the plate.

Suppose that the radii of curvature of the neutral plane n are ρ_x and ρ_y in the xz and yz planes respectively.



Positive curvature under positive bending moments corresponds to displacements in the positive ***z***-direction (downward).



Pure bending of thin plates

The $\boldsymbol{\varepsilon_x}$ and $\boldsymbol{\varepsilon_y}$ corresponding to $\boldsymbol{\sigma_x}$ and $\boldsymbol{\sigma_y}$ of an elemental lamina of thickness $\boldsymbol{\delta z}$ a distance $\boldsymbol{z}$ below the neutral plane are given by

$$\varepsilon_x = \frac{z}{\rho_x} \qquad \varepsilon_y = \frac{z}{\rho_y}$$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

plane stress

$$\left.\begin{aligned} \sigma_x &= \frac{Ez}{1-\nu^2}\left(\frac{1}{\rho_x} + \frac{\nu}{\rho_y}\right) \\ \sigma_y &= \frac{Ez}{1-\nu^2}\left(\frac{1}{\rho_y} + \frac{\nu}{\rho_x}\right) \end{aligned}\right\}$$

From assumption of plane sections remaining plane:

- the direct stresses vary linearly across the thickness of the plate,
- their magnitudes depending on the curvatures (i.e. bending moments) of the plate.

The internal direct stress distribution on each vertical surface of the element must be in equilibrium with the applied bending moments

$$M_x \delta y = \int_{-t/2}^{t/2} \sigma_x z \delta y \, \mathrm{d}z \qquad M_y \delta x = \int_{-t/2}^{t/2} \sigma_y z \delta x \, \mathrm{d}z$$

Pure bending of thin plates

$$M_x \delta y = \int_{-t/2}^{t/2} \sigma_x z \delta y \, dz \qquad \sigma_x = \frac{Ez}{1-\nu^2}\left(\frac{1}{\rho_x} + \frac{\nu}{\rho_y}\right) \qquad M_x = \int_{-t/2}^{t/2} \frac{Ez^2}{1-\nu^2}\left(\frac{1}{\rho_x} + \frac{\nu}{\rho_y}\right) dz$$

$$M_y \delta x = \int_{-t/2}^{t/2} \sigma_y z \delta x \, dz \qquad \sigma_y = \frac{Ez}{1-\nu^2}\left(\frac{1}{\rho_y} + \frac{\nu}{\rho_x}\right) \qquad M_y = \int_{-t/2}^{t/2} \frac{Ez^2}{1-\nu^2}\left(\frac{1}{\rho_y} + \frac{\nu}{\rho_x}\right) dz$$

Let $D = \int_{-t/2}^{t/2} \frac{Ez^2}{1-\nu^2} dz = \frac{Et^3}{12(1-\nu^2)}$ then

$$\boxed{M_x = D\left(\frac{1}{\rho_x} + \frac{\nu}{\rho_y}\right)}$$

$$\boxed{M_y = D\left(\frac{1}{\rho_y} + \frac{\nu}{\rho_x}\right)}$$

where ***D*** is known as the *flexural rigidity* of the plate

We may relate ***w*** to the curvature of the plate in the same manner as the well-known expression for beam curvature:

$$\boxed{\frac{1}{\rho_y} = -\frac{\partial^2 w}{\partial y^2}} \qquad \boxed{\frac{1}{\rho_x} = -\frac{\partial^2 w}{\partial x^2}}$$

$$M_x = D\left(\frac{1}{\rho_x} + \frac{\nu}{\rho_y}\right) \qquad M_y = D\left(\frac{1}{\rho_y} + \frac{\nu}{\rho_x}\right)$$

$$\boxed{M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)}$$

$$\boxed{M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)}$$

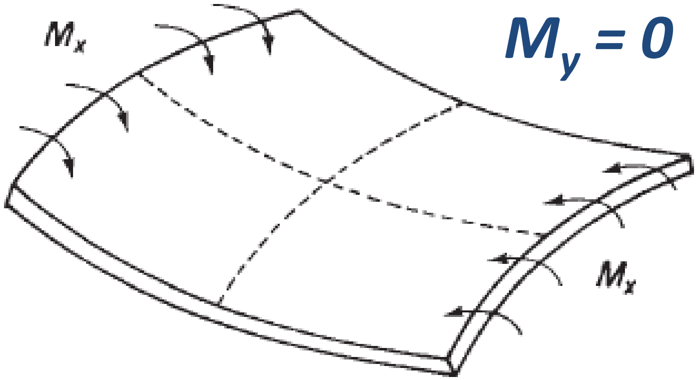
Pure bending of thin plates

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$
$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

M_x = 0 or ***M_y = 0***

$$\frac{\partial^2 w}{\partial x^2} = -\nu\frac{\partial^2 w}{\partial y^2}$$
$$\frac{\partial^2 w}{\partial y^2} = -\nu\frac{\partial^2 w}{\partial x^2}$$

and the plate has curvatures of opposite signs.



M_y = 0

A surface with two curvatures of opposite sign is ***anticlastic surface***, as opposed to a *synclastic* (the same sign).

$$M_x = D\left(\frac{1}{\rho_x} + \frac{\nu}{\rho_y}\right)$$
$$M_y = D\left(\frac{1}{\rho_y} + \frac{\nu}{\rho_x}\right)$$

M_x = M_y = M

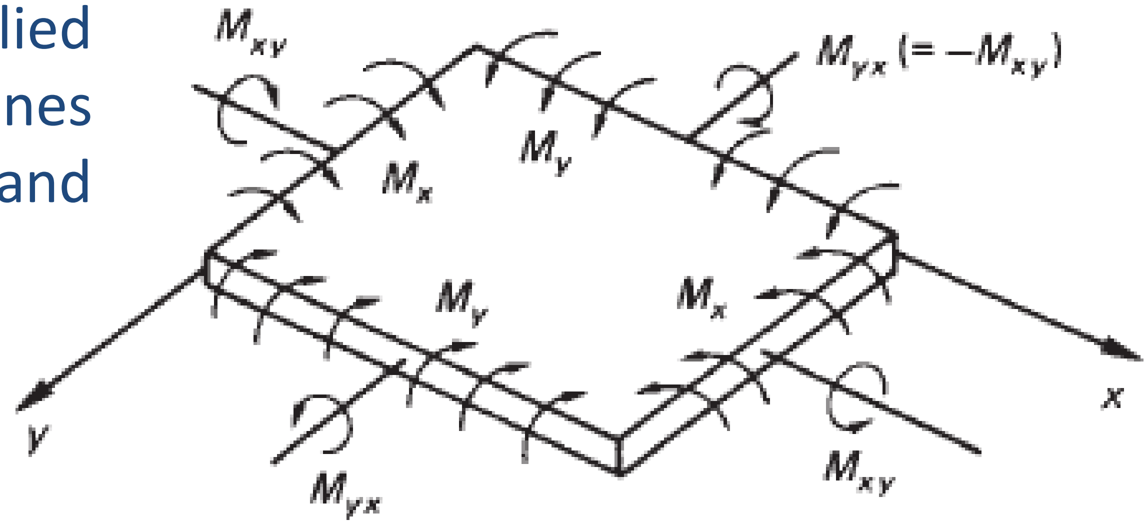
$$\frac{1}{\rho_x} = \frac{1}{\rho_y} = \frac{1}{\rho}$$
$$\frac{1}{\rho} = \frac{M}{D(1+\nu)}$$

Therefore, the deformed shape of the plate is ***spherical***.

Plates subjected to bending and twisting

The bending moments applied to the plate will not be in planes perpendicular to its edges and may be resolved into

- perpendicular and
- tangential components.



The perpendicular components – $\boldsymbol{M_x}$ and $\boldsymbol{M_y}$ as before, while the tangential $\boldsymbol{M_{xy}}$ and $\boldsymbol{M_{yx}}$ (per unit length) produce twisting of the plate about axes parallel to the $\boldsymbol{x}$ and $\boldsymbol{y}$ axes.

$\boldsymbol{M_{xy}}$ is a twisting moment in a vertical $\boldsymbol{x}$ plane parallel to the $\boldsymbol{y}$ axis, while $\boldsymbol{M_{yx}}$ – in a vertical $\boldsymbol{y}$ plane parallel to the $\boldsymbol{x}$ axis.

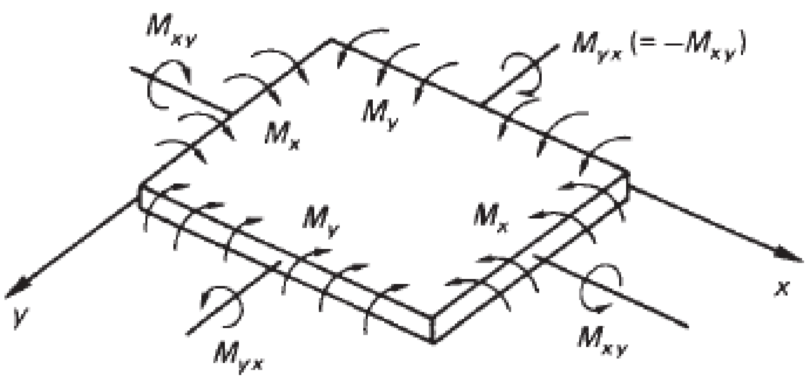
Positive twisting moments are being clockwise when viewed along their axes in directions parallel to the positive directions of the corresponding $\boldsymbol{x}$ or $\boldsymbol{y}$ axis.

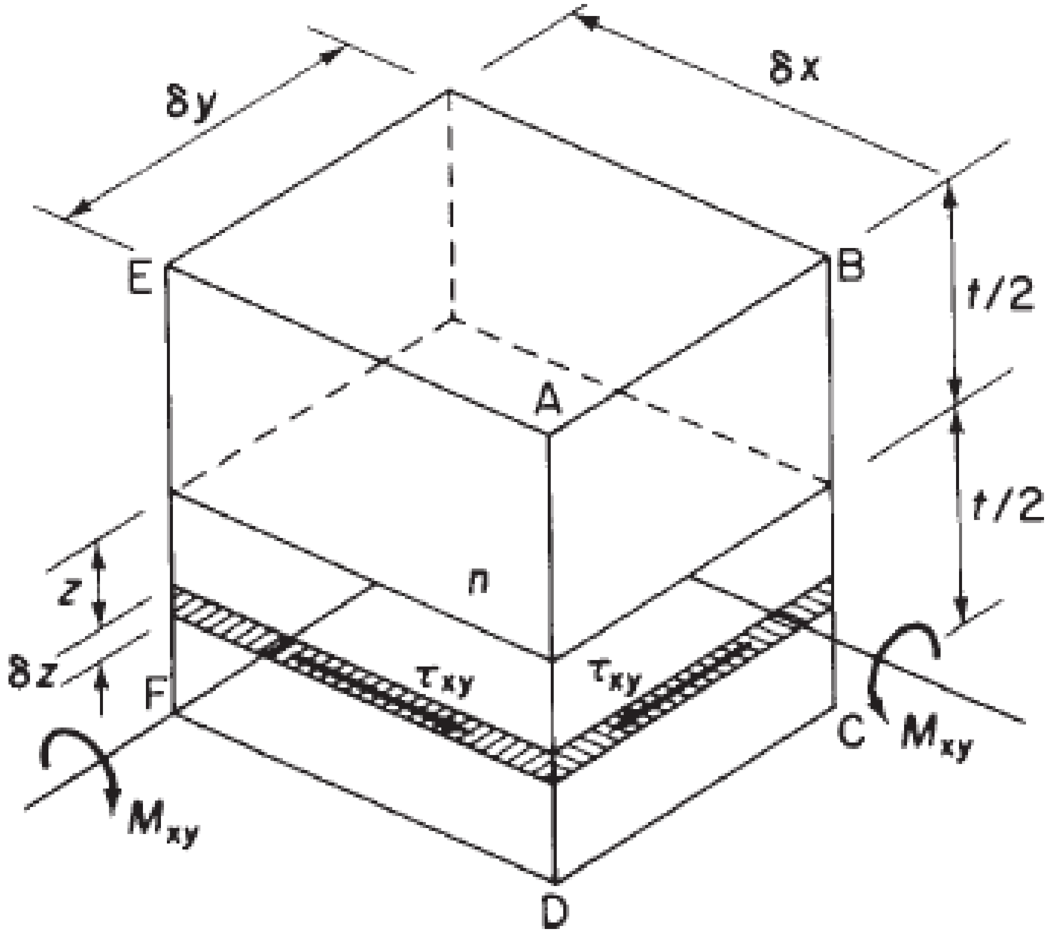
Plates subjected to bending and twisting

Since the twisting moments are tangential moments or torques they are resisted by a system of horizontal shear stresses τ_{xy}

From a consideration of complementary shear stresses $M_{xy} = -M_{yx}$

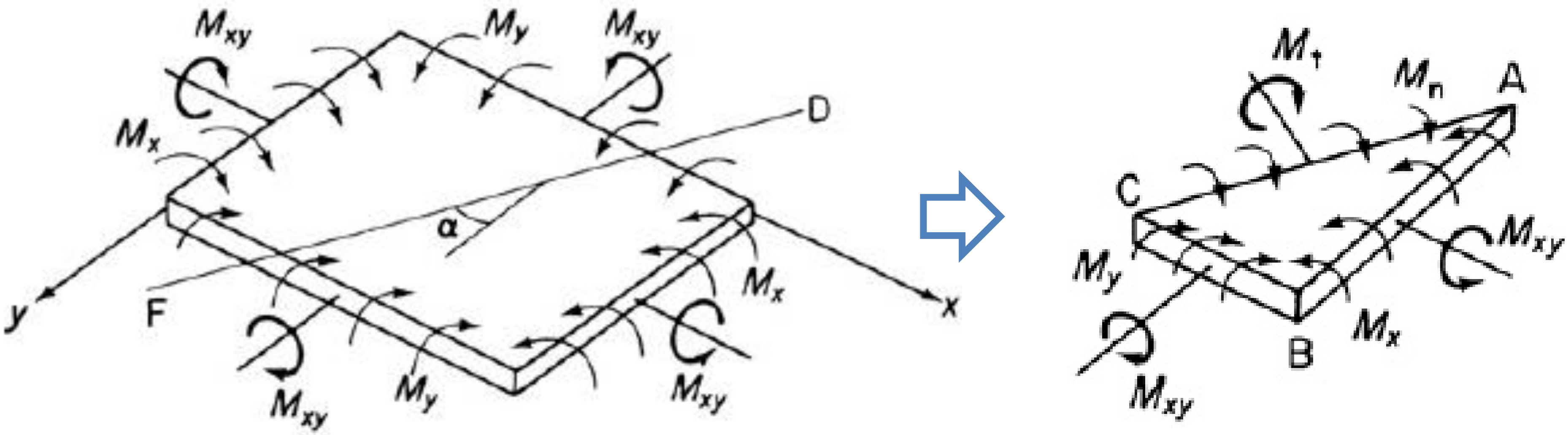
We may represent a general moment application to the plate in terms of M_x, M_y and M_{xy}





Plates subjected to bending and twisting

Moments $\boldsymbol{M_x}$, $\boldsymbol{M_y}$, $\boldsymbol{M_{xy}}$ produce tangential $\boldsymbol{M_t}$ and normal moments $\boldsymbol{M_n}$, on an arbitrarily chosen diagonal plane *FD*.



for equilibrium of the triangular element ***ABC*** in a plane perpendicular to ***AC***

$$M_\mathrm{n}\mathrm{AC} = M_x\mathrm{AB}\cos\alpha + M_y\mathrm{BC}\sin\alpha - M_{xy}\mathrm{AB}\sin\alpha - M_{xy}\mathrm{BC}\cos\alpha$$

$$\boxed{M_\mathrm{n} = M_x\cos^2\alpha + M_y\sin^2\alpha - M_{xy}\sin 2\alpha}$$

Plates subjected to bending and twisting

for equilibrium in a plane parallel to ***CA***

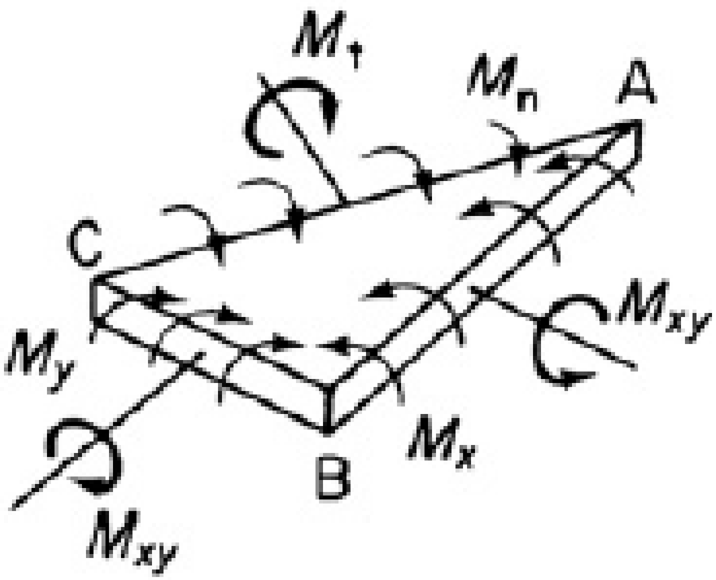
$$M_t AC = M_x AB \sin\alpha - M_y BC \cos\alpha + M_{xy} AB \cos\alpha - M_{xy} BC \sin\alpha$$

$$\boxed{M_t = \frac{(M_x - M_y)}{2}\sin 2\alpha + M_{xy}\cos 2\alpha}$$

There are two values of ***α***, differing by 90◦ and given by

$$\boxed{\tan 2\alpha = -\frac{2M_{xy}}{M_x - M_y}}$$

for which ***M***$\boldsymbol{M_t = 0}$, leaving normal moments of intensity $\boldsymbol{M_n}$ on two mutually perpendicular planes.



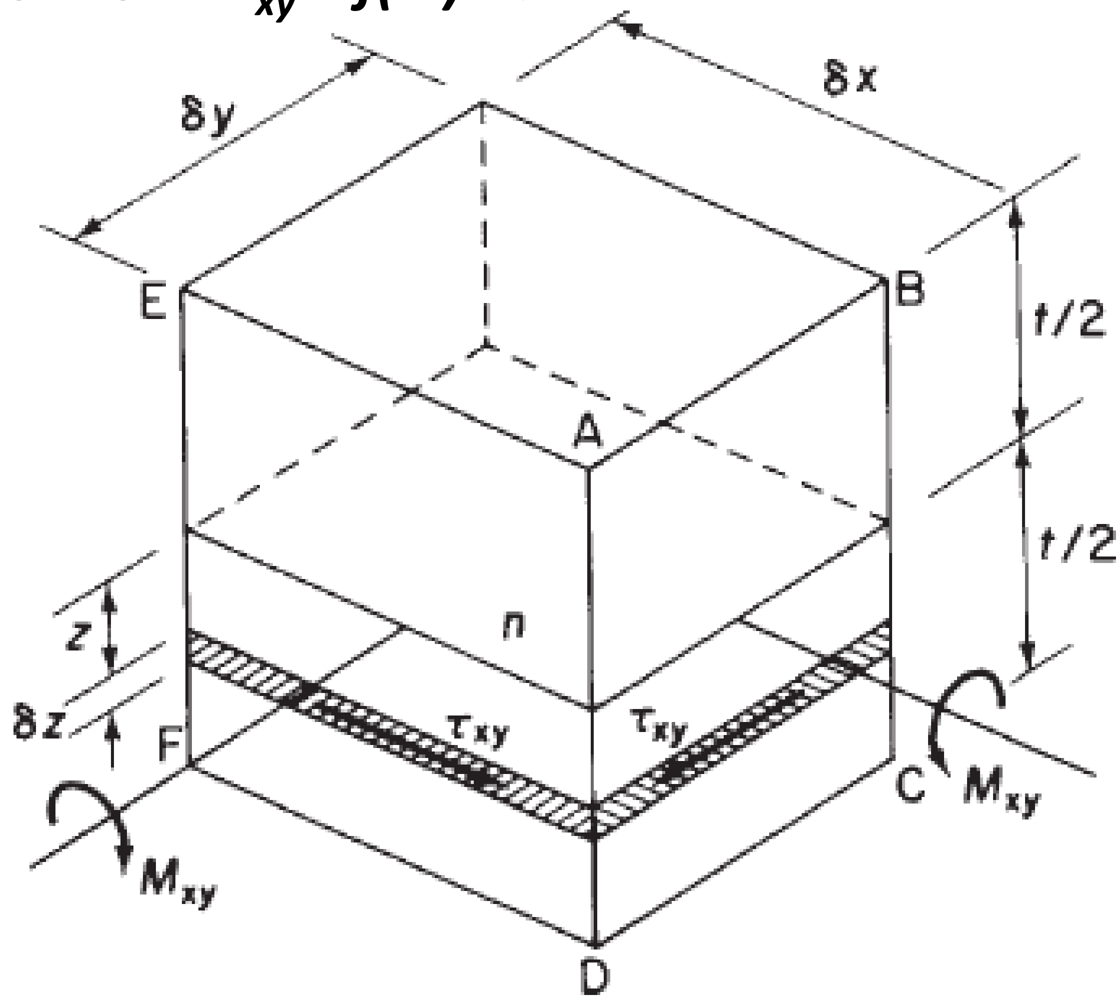
These moments are termed ***principal moments,*** which are the algebraically greatest and least moments in the plate.

There are no shear stresses on these planes and the corresponding direct stresses, for a given value of ***z*** and moment intensity, are the algebraically greatest and least values of direct stress in the plate.

Plates subjected to bending and twisting

From the principle of superposition we may consider M_{xy} acting separately from M_x and M_y. For now $M_{xy} = f(w)$ - ?

The complementary shear stresses on a lamina of the element a distance z below the neutral plane are τ_{xy}. Therefore, on the face **ABCD** and on the face **ADFE**



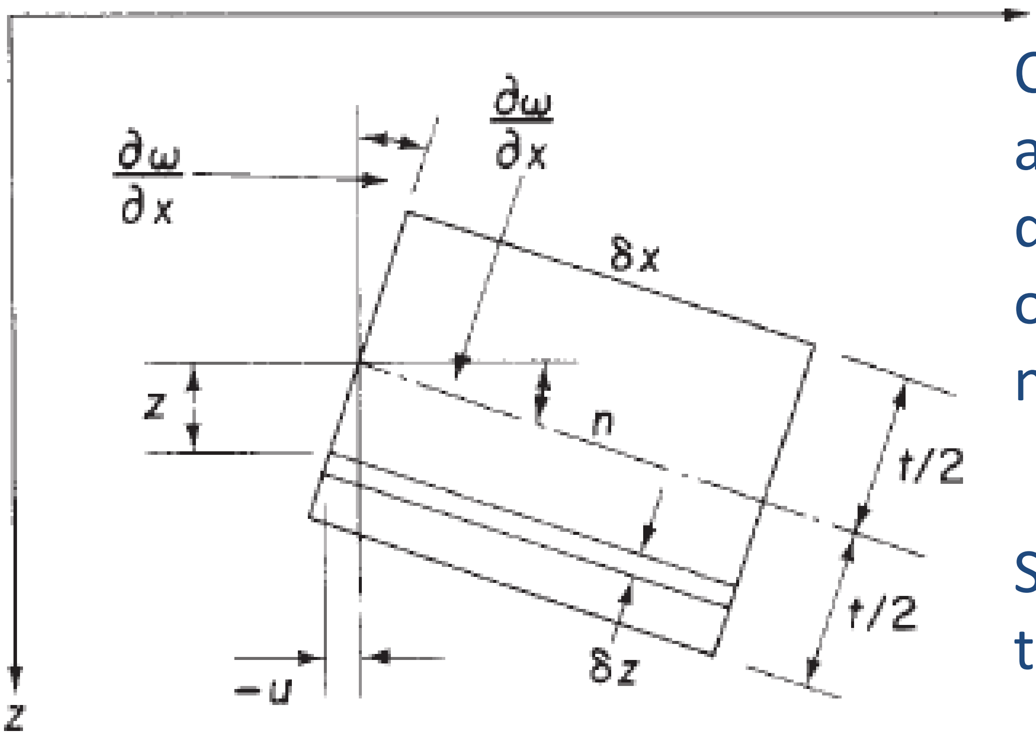
$$M_{xy}\delta x = -\int_{-t/2}^{t/2} \tau_{xy}\delta x z \, dz$$

$$M_{xy} = -\int_{-t/2}^{t/2} \tau_{xy} z \, dz$$

$$M_{xy} = -G\int_{-t/2}^{t/2} \gamma_{xy} z \, dz$$

Plates subjected to bending and twisting

An element taken through the thickness of the plate will suffer rotations equal to $\partial w/\partial x$ and $\partial w/\partial y$ in the ***xz*** and ***yz*** planes respectively.



Considering the rotation of such an element in the ***xz*** plane, the displacement ***u*** in the ***x*** direction of a point a distance ***z*** below the neutral plane is

$$u = -\frac{\partial w}{\partial x} z$$

Similarly, the displacement ***v*** in the ***y*** direction is

$$v = -\frac{\partial w}{\partial y} z$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \xrightarrow{u = -\frac{\partial w}{\partial x} z \quad v = -\frac{\partial w}{\partial y} z} \gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

Plates subjected to bending and twisting

$$M_{xy} = -G\int_{-t/2}^{t/2} \gamma_{xy} z \,\mathrm{d}z \xrightarrow{\gamma_{xy} = -2z\frac{\partial^2 w}{\partial x \partial y}} M_{xy} = G\int_{-t/2}^{t/2} 2z^2 \frac{\partial^2 w}{\partial x \partial y}\mathrm{d}z$$

$$M_{xy} = \frac{Gt^3}{6}\frac{\partial^2 w}{\partial x \partial y} \xrightarrow{G = E/2(1+v)} M_{xy} = \frac{Et^3}{12(1+\nu)}\frac{\partial^2 w}{\partial x \partial y}$$

$$M_{xy} = \frac{Et^3}{12(1+\nu)}\frac{\partial^2 w}{\partial x \partial y} \xrightarrow{\times (1-v) \quad D = \frac{Et^3}{12(1-\nu^2)}} \boxed{M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}}$$

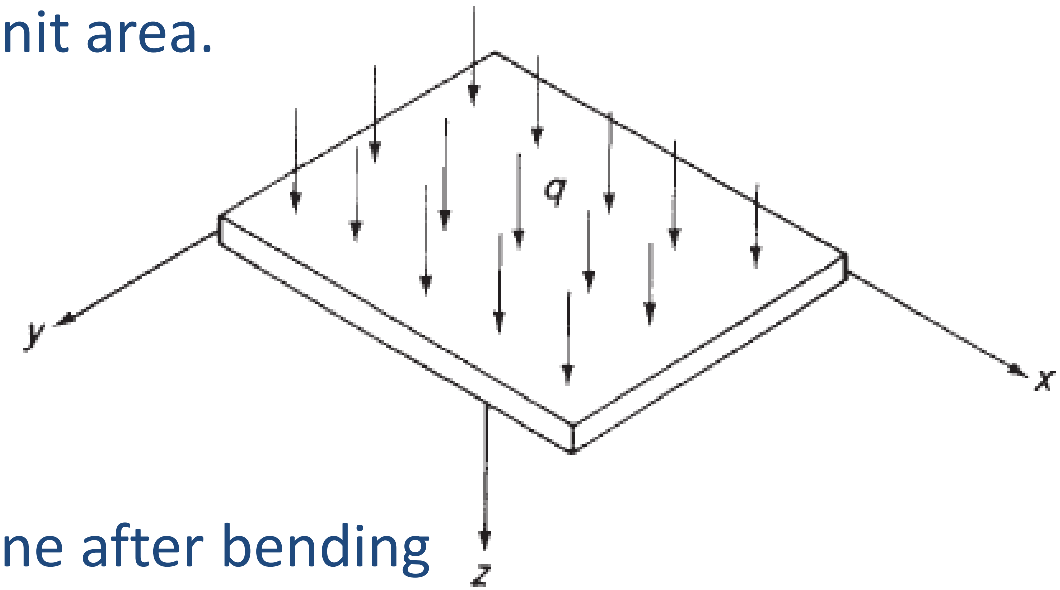
Equations

$$\boxed{M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)} \quad \boxed{M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)} \quad \boxed{M_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}}$$

relate the bending and twisting moments to the plate deflection and are analogous to the bending moment-curvature relationship for a beam.

Plates subjected to a distributed transverse load

Assume a thin rectangular plate, supporting a distributed transverse load of intensity $q = f\,(x,y)$ per unit area.



Also:

- middle plane – neutral;
- plane sections remain plane after bending.

Inconsistency:

- for plane sections remain plane after bending

$$\gamma_{xz} = \gamma_{yz} = 0$$

but

transverse load produces transverse shear forces (stresses).

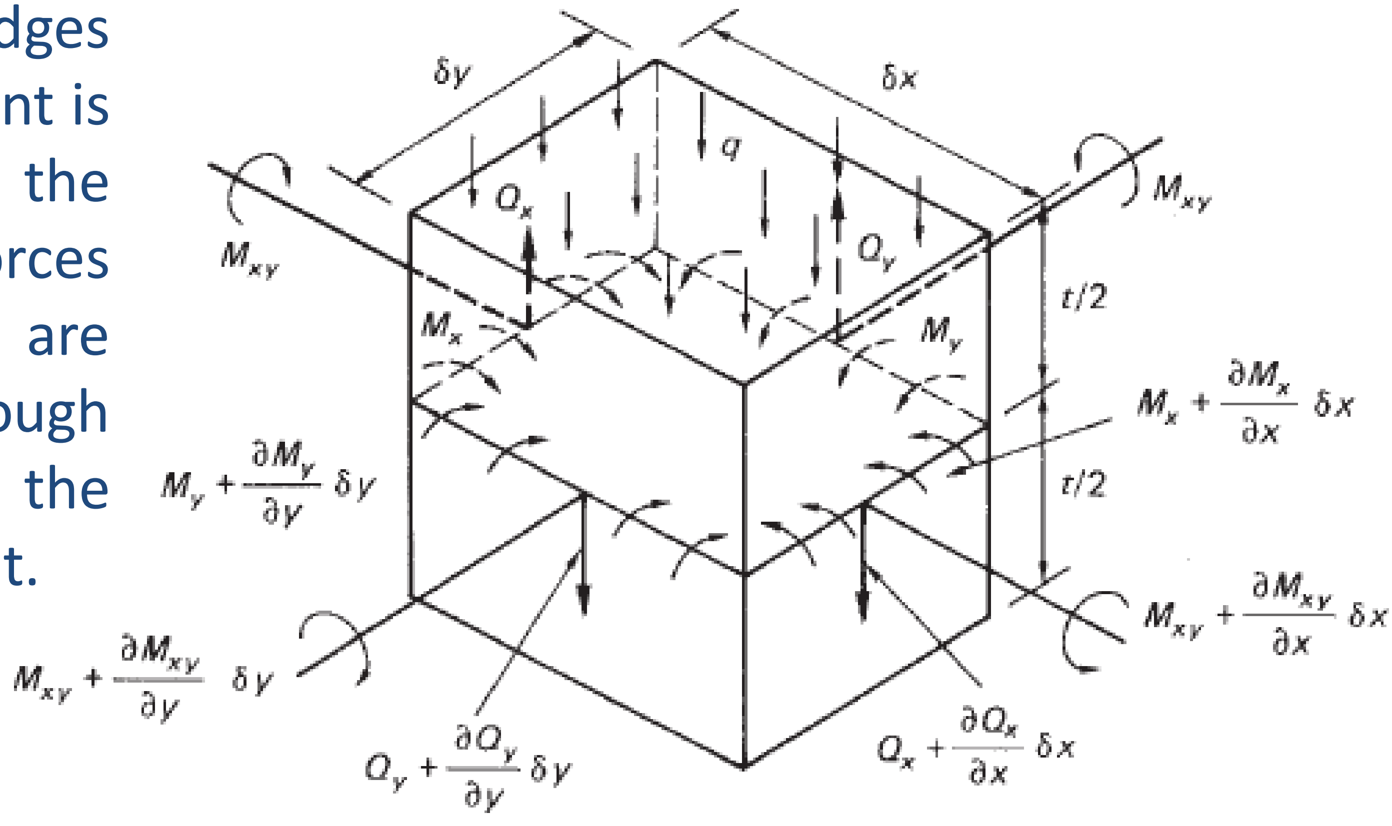
Assume (like in a slender beam theory):

- γ_{xz}, γ_{yz} – negligible, but the shear forces are of the same order of magnitude as the applied load q and the moments M_x, M_y, M_{xy}

Plates subjected to a distributed transverse load

The element of plate supports bending and twisting moments and, in addition, vertical shear forces $\boldsymbol{Q_x}$ and $\boldsymbol{Q_y}$ per unit length on faces perpendicular to the $\boldsymbol{x}$ and $\boldsymbol{y}$ axes, respectively.

The variation $\boldsymbol{\tau_{xz}}$ and $\boldsymbol{\tau_{yz}}$ along the small edges $\boldsymbol{\delta x}$, $\boldsymbol{\delta y}$ of the element is neglected and the resultant shear forces $\boldsymbol{Q_x \delta y}$ and $\boldsymbol{Q_y \delta x}$ are assumed to act through the centroid of the faces of the element.



Plates subjected to a distributed transverse load

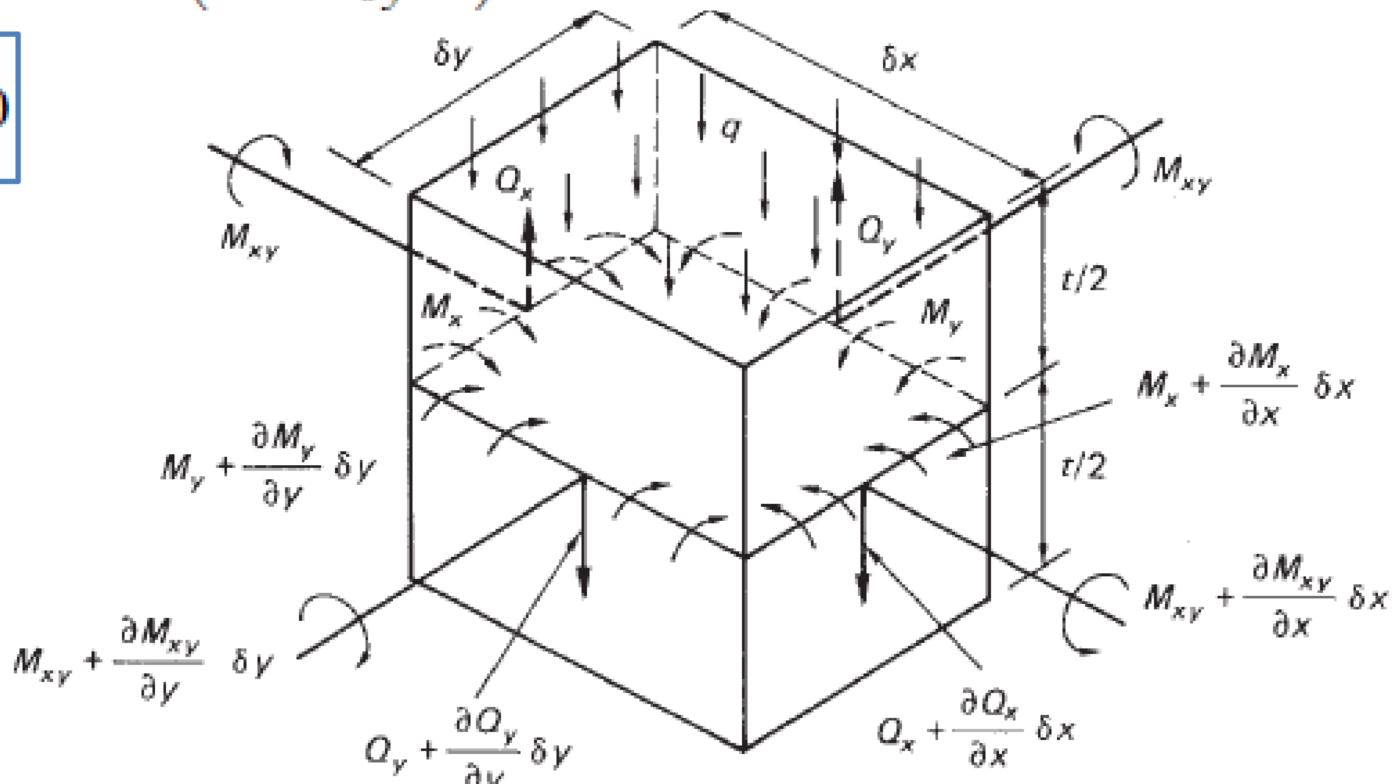
According to the given figure:

$$Q_x = \int_{-t/2}^{t/2} \tau_{xz}\,\mathrm{d}z \quad Q_y = \int_{-t/2}^{t/2} \tau_{yz}\,\mathrm{d}z$$

For equilibrium of the element parallel to ***Oz*** (weight included in ***q***):

$$\left(Q_x + \frac{\partial Q_x}{\partial x}\delta x\right)\delta y - Q_x\delta y + \left(Q_y + \frac{\partial Q_y}{\partial y}\delta y\right)\delta x - Q_y\delta x + q\delta x\delta y = 0$$

$$\boxed{\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0}$$



Plates subjected to a distributed transverse load

Taking moments about the ***x*** axis

$$M_{xy}\delta y - \left(M_{xy} + \frac{\partial M_{xy}}{\partial x}\delta x\right)\delta y - M_y\delta x + \left(M_y + \frac{\partial M_y}{\partial y}\delta y\right)\delta x$$

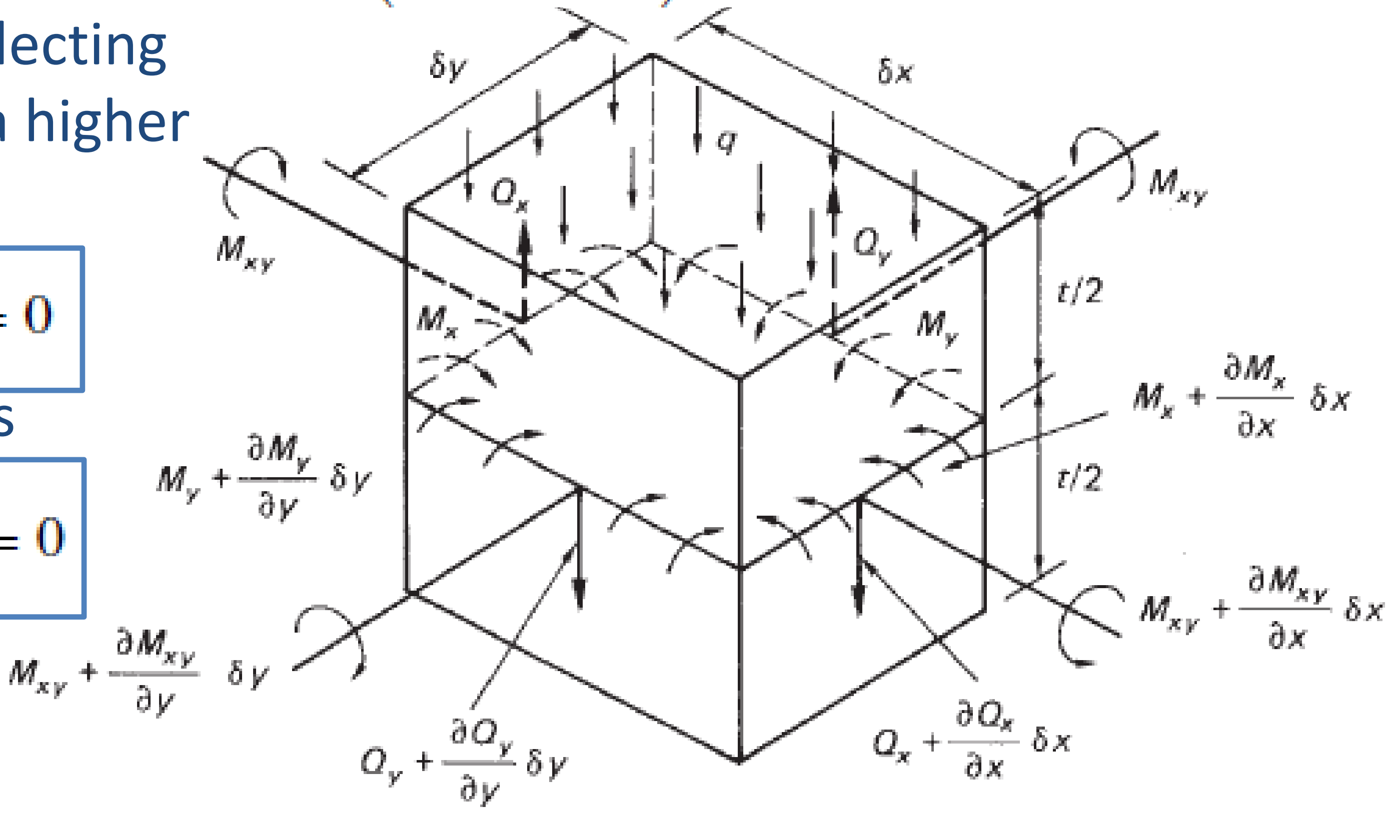
$$-\left(Q_y + \frac{\partial Q_y}{\partial y}\delta y\right)\delta x\delta y + Q_x\frac{\delta y^2}{2} - \left(Q_x + \frac{\partial Q_x}{\partial x}\delta x\right)\frac{\delta y^2}{2} - q\delta x\frac{\delta y^2}{2} = 0$$

Simplifying and neglecting small quantities of a higher order gives

$$\boxed{\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + Q_y = 0}$$

And about the ***y*** axis

$$\boxed{\frac{\partial M_{xy}}{\partial y} - \frac{\partial M_x}{\partial x} + Q_x = 0}$$



Plates subjected to a distributed transverse load

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q=0 \xrightarrow[\frac{\partial M_{xy}}{\partial y}-\frac{\partial M_x}{\partial x}+Q_x=0]{\frac{\partial M_{xy}}{\partial x}-\frac{\partial M_y}{\partial y}+Q_y=0} \frac{\partial^2 M_x}{\partial x^2}-\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}-\frac{\partial^2 M_{xy}}{\partial x\partial y}=-q$$

$$\frac{\partial^2 M_x}{\partial x^2}-2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}=-q \xrightarrow[\text{as functions of } \boldsymbol{w}]{M_x,\ M_y \text{ and } M_{xy}} \boxed{\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q}{D}}$$

$$\boxed{M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)} \qquad \boxed{M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)} \qquad \boxed{M_{xy}=D(1-\nu)\frac{\partial^2 w}{\partial x\partial y}}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=\frac{q}{D} \Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 w=\frac{q}{D} \Rightarrow \boxed{(\nabla^2)^2 w=\frac{q}{D}}$$

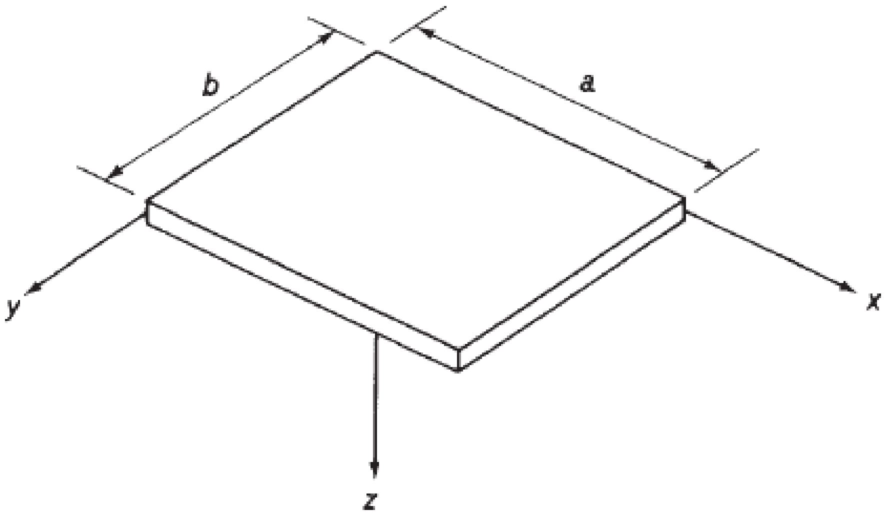
$$\left.\begin{aligned}\frac{\partial M_{xy}}{\partial x}-\frac{\partial M_y}{\partial y}+Q_y=0\\ \frac{\partial M_{xy}}{\partial y}-\frac{\partial M_x}{\partial x}+Q_x=0\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}&\boxed{Q_y=\frac{\partial M_y}{\partial y}-\frac{\partial M_{xy}}{\partial x}=-D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)}\\ &\boxed{Q_x=\frac{\partial M_x}{\partial x}-\frac{\partial M_{xy}}{\partial y}=-D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)}\end{aligned}\right.$$

It is necessary to establish also **boundary conditions …**

Plates subjected to a distributed transverse load

THE SIMPLY SUPPURTED EDGE (Boundary conditions**)**

Let us suppose that the edge ***x = 0*** of the thin plate is free to rotate but not to deflect:



- the bending moment along this edge must be zero;

$$(M_x)_{x=0} = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)_{x=0} = 0$$

- the deflection ***w = 0.***

$$(w)_{x=0} = 0$$

The last conditions gives $\dfrac{\partial w}{\partial y} = \dfrac{\partial^2 w}{\partial y^2} = 0$ along edge ***x = 0***.

The above boundary conditions therefore reduce to

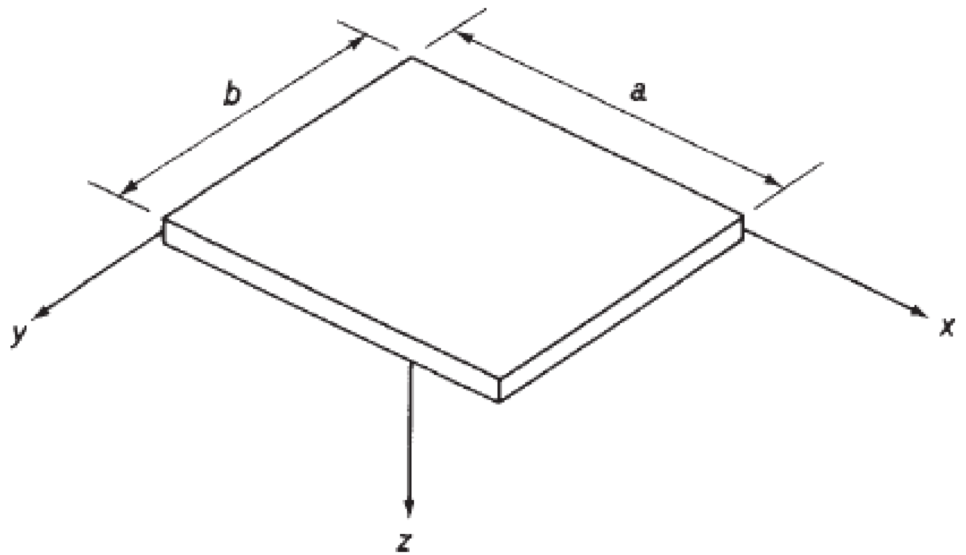
$$(w)_{x=0} = 0 \quad \left(\frac{\partial^2 w}{\partial x^2}\right)_{x=0} = 0$$

Plates subjected to a distributed transverse load

THE BUILT-IN EDGE (Boundary conditions**)**

If the edge ***x = 0*** is firmly clamped

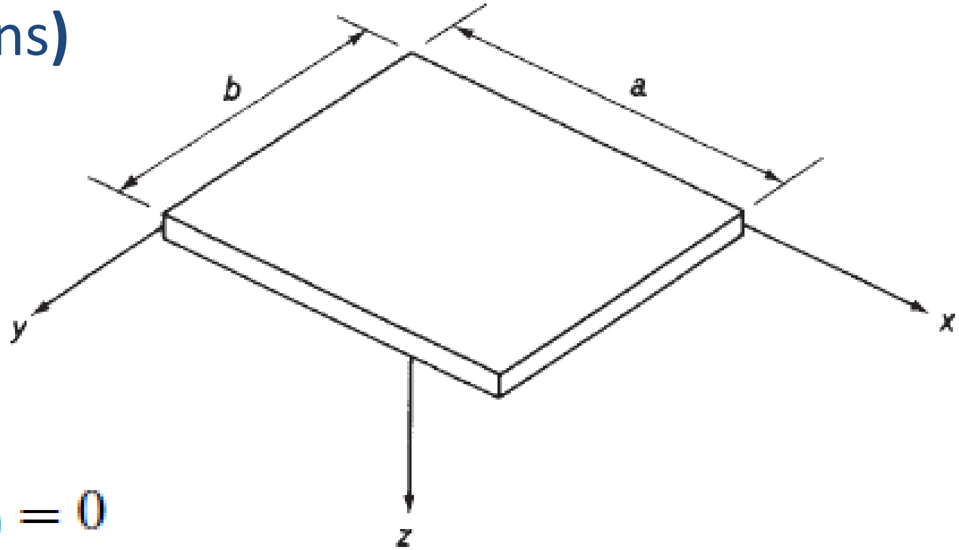
- it can neither rotate nor deflect,
- ***w = 0***,
- also the slope of the middle plane of the plate normal to this edge must be zero.



The boundary conditions are $(w)_{x=0} = 0 \quad \left(\frac{\partial w}{\partial x}\right)_{x=0} = 0$

Plates subjected to a distributed transverse load

THE FREE EDGE (Boundary conditions**)**



Along a free edge there are no

- bending moments,
- twisting moments or
- vertical shearing forces,

so that if ***x = 0*** is the free edge then

$$(M_x)_{x=0} = 0 \quad (M_{xy})_{x=0} = 0 \quad (Q_x)_{x=0} = 0$$

But only two boundary conditions are necessary to obtain a solution

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

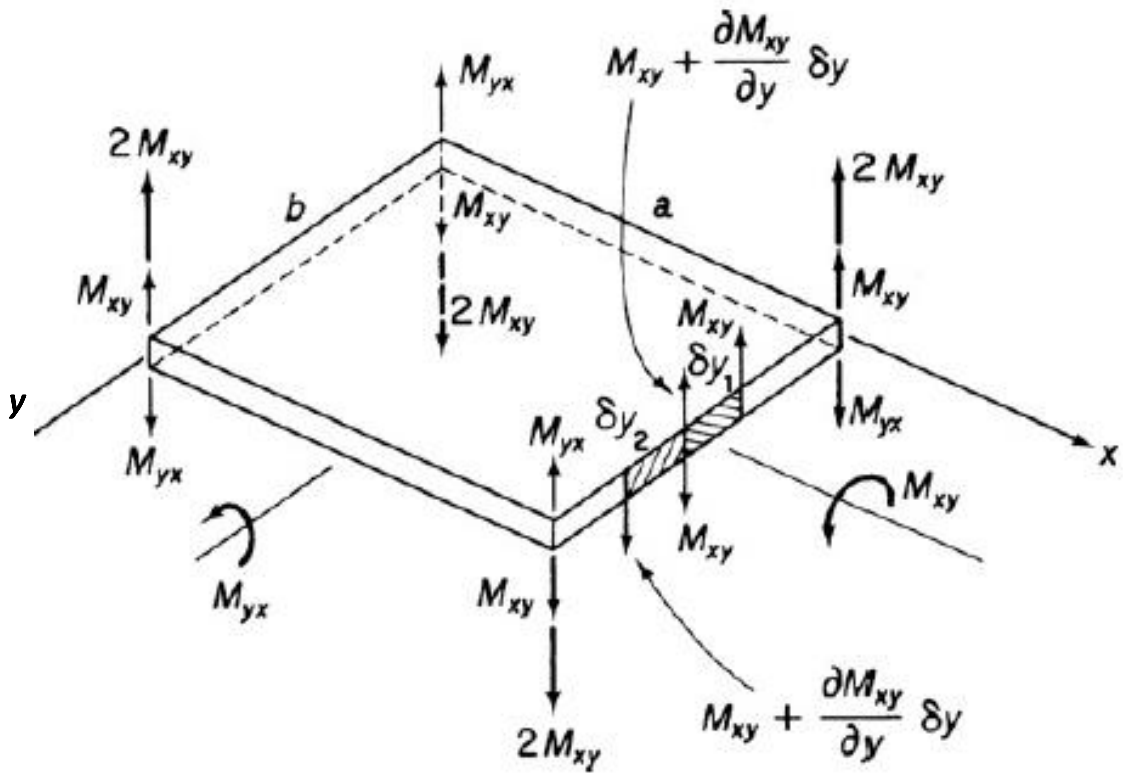
Two last conditions may be replaced by a single equivalent condition: the horizontal force system equilibrating the twisting moment $\boldsymbol{M_{xy}}$ may be replaced along the edge of the plate by a *vertical force system*.

Plates subjected to a distributed transverse load

Consider two adjacent elements δy_1 and δy_2 along the edge of the plate

The twisting moment $M_{xy}\delta y_1$ on the element δy_1 may be replaced by forces M_{xy} a distance δy_1 apart.

The moment on δy_2 $[M_{xy}+(\partial M_{xy}/\partial y)\delta y]\delta y_2$ may be replaced by forces $M_{xy}+(\partial M_{xy}/\partial y)\delta y$.



At the common surface of the two adjacent elements there is now

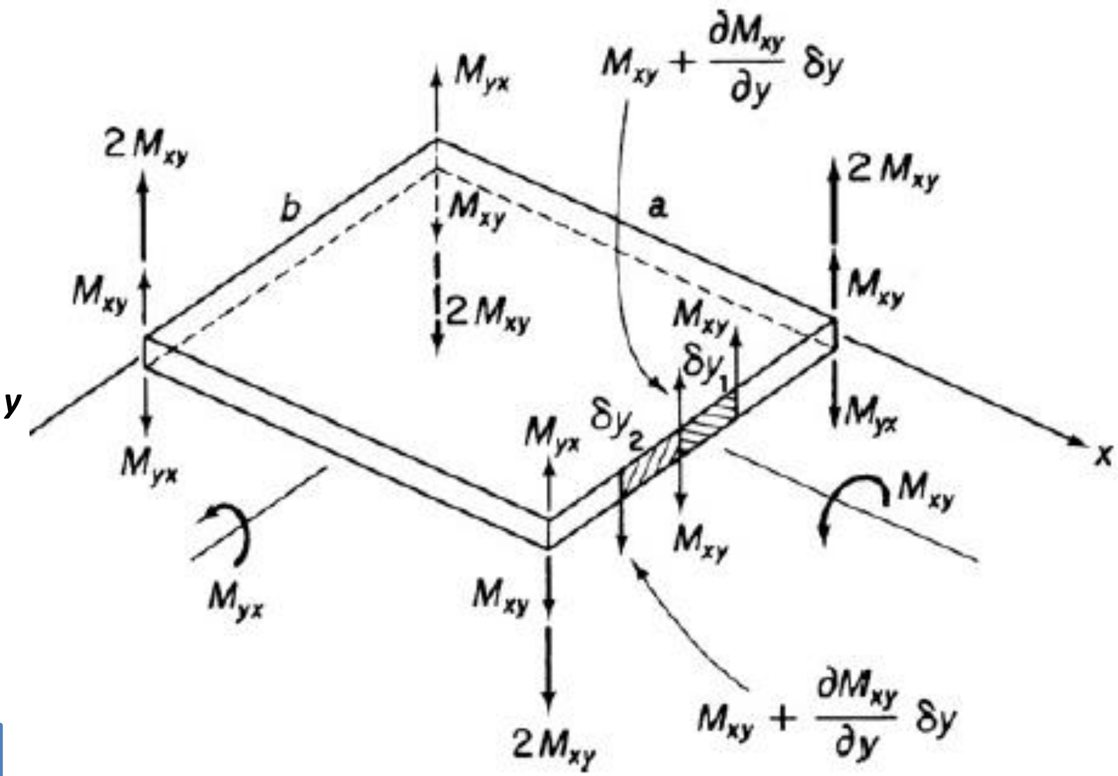
- a resultant force $(\partial M_{xy}/\partial y)\delta y$ or
- a vertical force per unit length of $\partial M_{xy}/\partial y$.

Plates subjected to a distributed transverse load

For the edge ***x = 0*** we have a statically equivalent vertical force per unit length $(Q_x - \partial M_{xy}/\partial y)$

According to boundary conditions

- $(M_{xy})_{x=0} = 0$
- $(Q_x)_{x=0} = 0$



we have $\boxed{\left(Q_x - \frac{\partial M_{xy}}{\partial y}\right)_{x=0} = 0}$

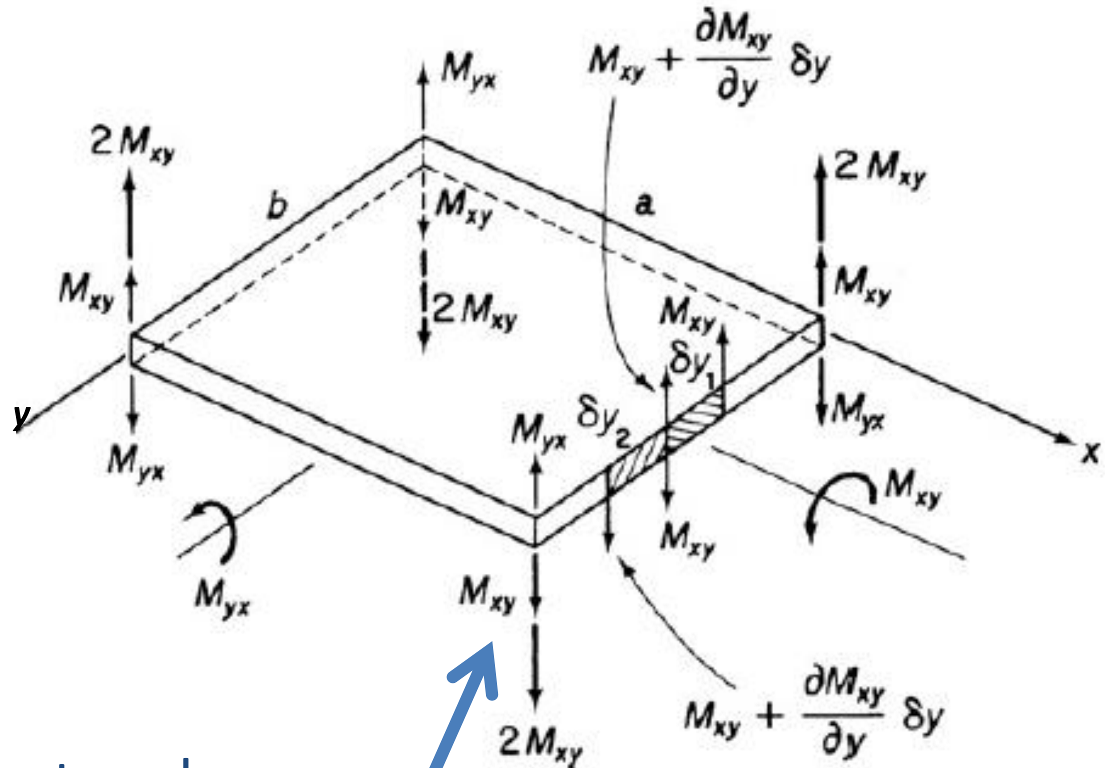
or in terms of deflection ***w***

$$\boxed{\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]_{x=0} = 0}$$

bending moment along the free edge $(M_x)_{x=0} = \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)_{x=0} = 0$

Plates subjected to a distributed transverse load

The replacement of the M_{xy} along the edges $x = 0$ and $x = a$ of a plate by a vertical force distribution results in leftover concentrated forces at the corners of M_{xy}



The same will happen with a twisting moment M_{yx}

Since

$M_{xy} = -M_{yx}$,

then resultant forces $2M_{xy}$ act at each corner

The directions of these forces are easily obtained if the deflected shape of the plate is known.

Plates subjected to a distributed transverse load

Consider a thin rectangular plate of dimensions ***a×b***, simply supported along each of its four edges and carrying a distributed load ***q(x, y)***:

- the deflected form of the plate must satisfy the differential equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D}$$

- with boundary conditions

$$(w)_{x=0,a} = 0 \quad \left(\frac{\partial^2 w}{\partial x^2}\right)_{x=0,a} = 0 \qquad (w)_{y=0,b} = 0 \quad \left(\frac{\partial^2 w}{\partial y^2}\right)_{x=0,b} = 0$$

These conditions are satisfied by representing the deflection ***w*** as an infinite trigonometrical or *Fourier series*

$$\boxed{w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}$$

$\boldsymbol{A_{mn}}$ are unknown coefficients

m represents the number of half waves in the ***x*** direction and

n the corresponding number in the ***y*** direction

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We may also represent the load ***q(x, y)*** by a Fourier series, thus

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

A particular coefficient $\boldsymbol{a_{m'n'}}$ is calculated by multiplying both sides by ***sin(m'πx/a) sin(n'πy/b)***
and integrating with:

- respect to ***x*** from ***0*** to ***a*** and
- with respect to ***y*** from ***0*** to ***b***.

Thus

$$\int_0^a\int_0^b q(x,y)\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\,\mathrm{d}x\,\mathrm{d}y$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^a\int_0^b a_{mn}\sin\frac{m\pi x}{a}\sin\frac{m'\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{n'\pi y}{b}\,\mathrm{d}x\,\mathrm{d}y$$

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Integrating the right part of the equation

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^a\int_0^b a_{mn}\sin\frac{m\pi x}{a}\sin\frac{m'\pi x}{a}\sin\frac{n\pi y}{b}\sin\frac{n'\pi y}{b}\,dx\,dy$$

we have $\displaystyle\int_0^a \sin\frac{m\pi x}{a}\sin\frac{m'\pi x}{a}\,dx = 0 \quad \text{when} \quad m \neq m'$

$$= \frac{a}{2} \quad \text{when} \quad m = m'$$

and $\displaystyle\int_0^b \sin\frac{n\pi y}{b}\sin\frac{n'\pi y}{b}\,dy = 0 \quad \text{when} \quad n \neq n'$

$$= \frac{b}{2} \quad \text{when} \quad n = n'$$

Then $\displaystyle\int_0^a\int_0^b q(x,y)\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\,dx\,dy = \frac{ab}{4}a_{m'n'}$

And finally $$\boxed{a_{m'n'} = \frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\,dx\,dy}$$

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$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D}$$

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{A_{mn}\left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right] - \frac{a_{mn}}{D}\right\} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = 0$$

For different values of ***x*** and ***y***

$$A_{mn}\left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right] - \frac{a_{mn}}{D} = 0$$

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Transforming last equation we have

$$A_{mn}\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2-\frac{a_{mn}}{D}=0$$

from which

$$A_{mn}=\frac{1}{\pi^4 D}\frac{a_{mn}}{[(m^2/a^2)+(n^2/b^2)]^2}$$

Hence the general solution for a thin rectangular plate under a transverse load ***q(x, y)***.

$$\boxed{w=\frac{1}{\pi^4 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{mn}}{[(m^2/a^2)+(n^2/b^2)]^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}$$

in which a_{mn} is obtained from

$$\boxed{a_{m'n'}=\frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\,\mathrm{d}x\,\mathrm{d}y}$$

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Example

A thin rectangular plate:

- has dimensions ***a×b***,
- simply supported along its edges ,
- carries a uniformly distributed load of intensity q_0.

Determine

- the deflected form of the plate and
- the distribution of bending moment.

Solution

Since $\boldsymbol{q_0}$ does not depend on ***x*** and ***y***

$$a_{mn} = \frac{4q_0}{ab}\int_0^a\int_0^b \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,\mathrm{d}x\,\mathrm{d}y = \frac{16q_0}{\pi^2 mn}$$

Where ***m*** and ***n*** odd integers, since for even integers $\boldsymbol{a_{mn} = 0}$

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$$w = \frac{1}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{mn}}{[(m^2/a^2) + (n^2/b^2)]^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

$$a_{mn} = \frac{16 q_0}{\pi^2 mn}$$

$$w = \frac{16 q_0}{\pi^6 D} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\sin(m\pi x/a) \sin(n\pi y/b)}{mn[(m^2/a^2) + (n^2/b^2)]^2}$$

The maximum deflection occurs at the centre (***x = a/2, y = b/2***)

$$w_{\max} = \frac{16 q_0}{\pi^6 D} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\sin(m\pi/2) \sin(n\pi/2)}{mn[(m^2/a^2) + (n^2/b^2)]^2}$$

The first few terms giving a satisfactory answer.

For a square plate (***a = b***), taking ***v = 0.3*** the first four terms gives

$$w_{\max} = 0.0443 q_0 \frac{a^4}{E t^3}$$

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$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$w = \frac{16q_0}{\pi^6 D}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{\sin(m\pi x/a)\sin(n\pi y/b)}{mn[(m^2/a^2)+(n^2/b^2)]^2}$$

$$M_x = \frac{16q_0}{\pi^4}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{[(m^2/a^2)+\nu(n^2/b^2)]}{mn[(m^2/a^2)+(n^2/b^2)]^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$M_y = \frac{16q_0}{\pi^4}\sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{[\nu(m^2/a^2)+(n^2/b^2)]}{mn[(m^2/a^2)+(n^2/b^2)]^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

Maximum values occur at the centre of the plate (for a square plate and the first five terms)

$$M_{x,\max} = M_{y,\max} = 0.0479q_0a^2$$

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$$\left.\begin{aligned}\sigma_x &= \frac{Ez}{1-\nu^2}\left(\frac{1}{\rho_x}+\frac{\nu}{\rho_y}\right)\\ \sigma_y &= \frac{Ez}{1-\nu^2}\left(\frac{1}{\rho_y}+\frac{\nu}{\rho_x}\right)\end{aligned}\right\} \quad \begin{aligned}M_x &= D\left(\frac{1}{\rho_x}+\frac{\nu}{\rho_y}\right)\\ M_y &= D\left(\frac{1}{\rho_y}+\frac{\nu}{\rho_x}\right)\end{aligned} \quad \Longrightarrow \quad \begin{aligned}\sigma_x &= \frac{12M_x z}{t^3}\\ \sigma_y &= \frac{12M_y z}{t^3}\end{aligned}$$

The maximum values occur at the centre of the plate at ***z= ±t/2***

$$\sigma_{x,\max} = \frac{6M_x}{t^2}$$

$$\sigma_{y,\max} = \frac{6M_y}{t^2}$$

for a square plate: $\sigma_{x,\max} = \sigma_{y,\max} = 0.287 q_0 \frac{a^2}{t^2}$

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The infinite series

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

assumed for the deflected shape of a plate **gives an exact solution** for displacements and stresses.

However, a more rapid, but approximate, solution may be obtained by assuming a displacement function in the form of a ***polynomial***.

The polynomial must, of course, satisfy

- the governing differential equation $$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$
- and the boundary conditions of the specific problem.

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Example 2

Show that the deflection function $w = A(x^2y^2 - bx^2y - axy^2 + abxy)$

is valid for

- rectangular plate of sides ***a*** and ***b***,
- built in on all four edges and
- subjected to a uniformly distributed load of intensity ***q***.

If the material of the plate

- has a Young's modulus ***E*** and
- is of thickness ***t***

determine the distributions of bending moment along the edges of the plate.

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Solution

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

$$\frac{\partial^4 w}{\partial x^4} = 0 \quad \frac{\partial^4 w}{\partial y^4} = 0 \quad \frac{\partial^4 w}{\partial x^2 \partial y^2} = 4A$$

$$0 + 2 \times 4A + 0 = \text{constant} = \frac{q}{D}$$

The deflection function is therefore valid and $A = \dfrac{q}{8D}$

The bending moment distributions given by

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \quad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

are

$$M_x = -\frac{q}{4}[y^2 - by + \nu(x^2 - ax)]$$

$$M_y = -\frac{q}{4}[x^2 - ax + \nu(y^2 - by)]$$

for the edges ***x = 0*** and ***x = a*** $\quad M_x = -\dfrac{q}{4}(y^2 - by) \quad M_y = -\dfrac{\nu q}{4}(y^2 - by)$

for the edges ***y = 0*** and ***y = b*** $\quad M_x = -\dfrac{\nu q}{4}(x^2 - ax) \quad M_y = -\dfrac{q}{4}(x^2 - ax)$

Obrigado!