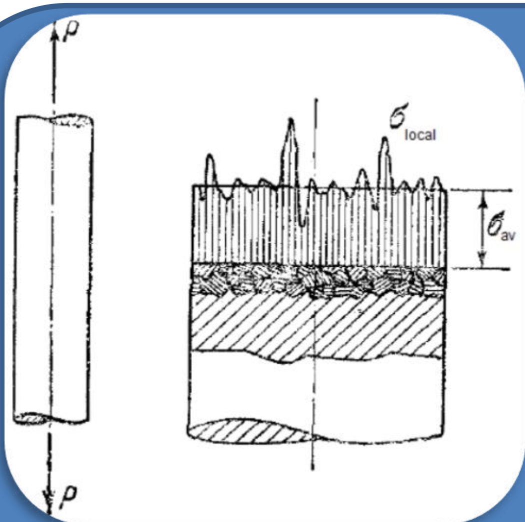
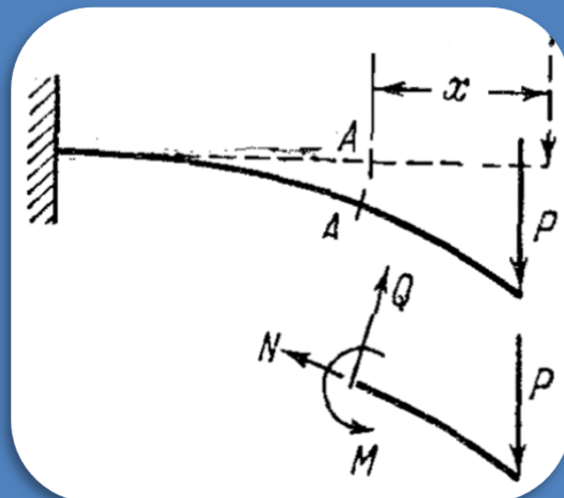
# Basic Equations of Solid Mechanics

2016

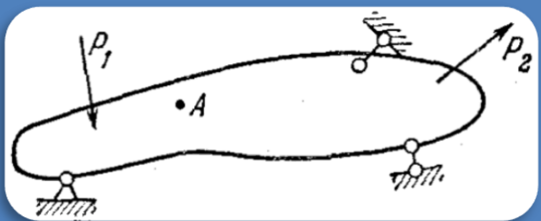
# Important principles in analysis




- **continuity** and **homogeneity** of the body;



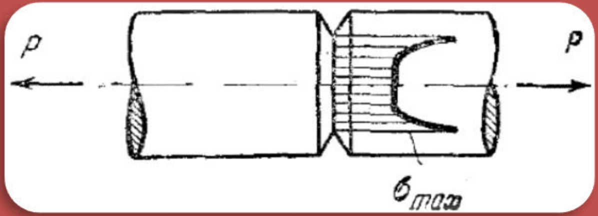

- **relative stiffness** of the body (system);

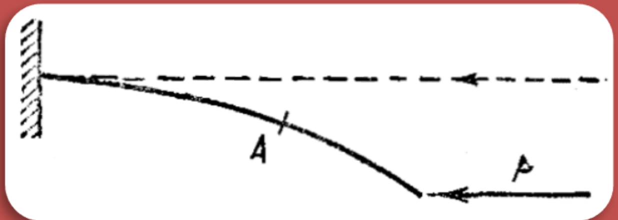


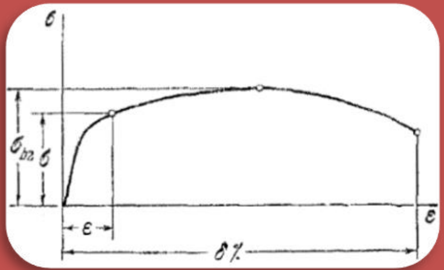

$$u_A = \delta_1 P_1 + \delta_2 P_2.$$

- principle of **superposition**.

exceptions



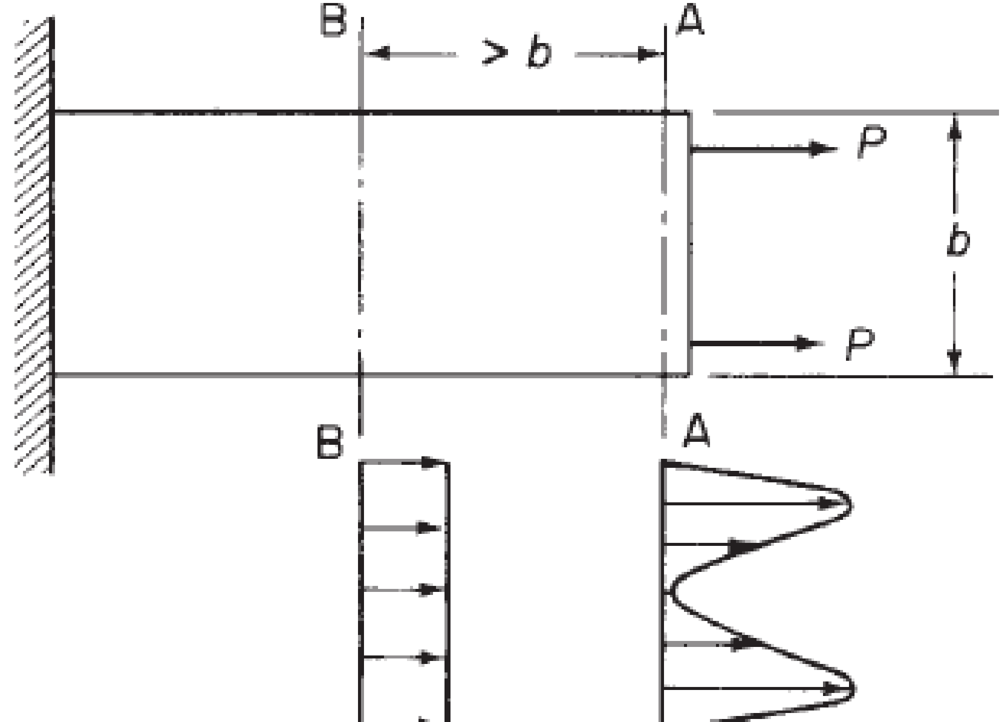
# St. Vernant's Principle

If we apply two loads to the end section of the cantilever we obtain two types of stress distribution in the cross-section of beam (along its length):

- **uniform** at some distance from the place of application of the loads,
- **non-uniform** near end section (in accordance with the distribution of applied loads).

**principle of St. Vernant:**

*while statically equivalent systems of forces acting on a body produce substantially different local effects the stresses at sections distant from the surface of loading are essentially the same.*



# St. Vernant’s Principle

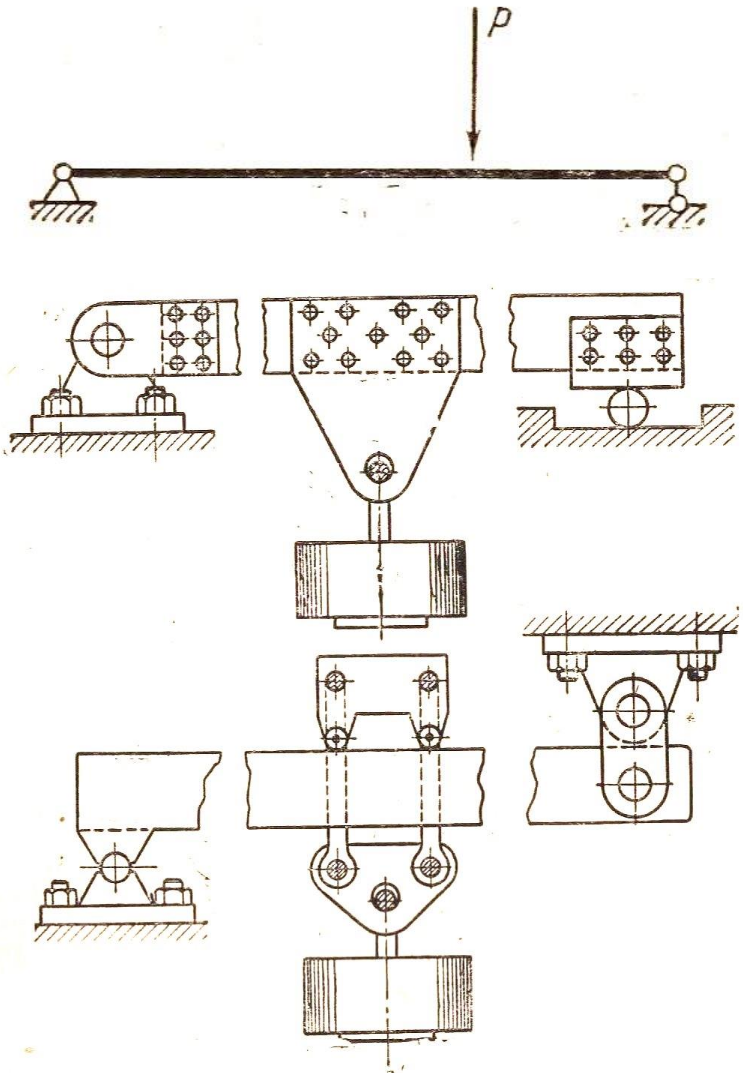
By other words the stress-distribution is defined by the resultant of the applied loads at the cross-section distant from the place of application of loads.


The distance at which the distribution of loads may be neglected is usually greater then the dimension of the surface to which the load is applied:

**L >> d**

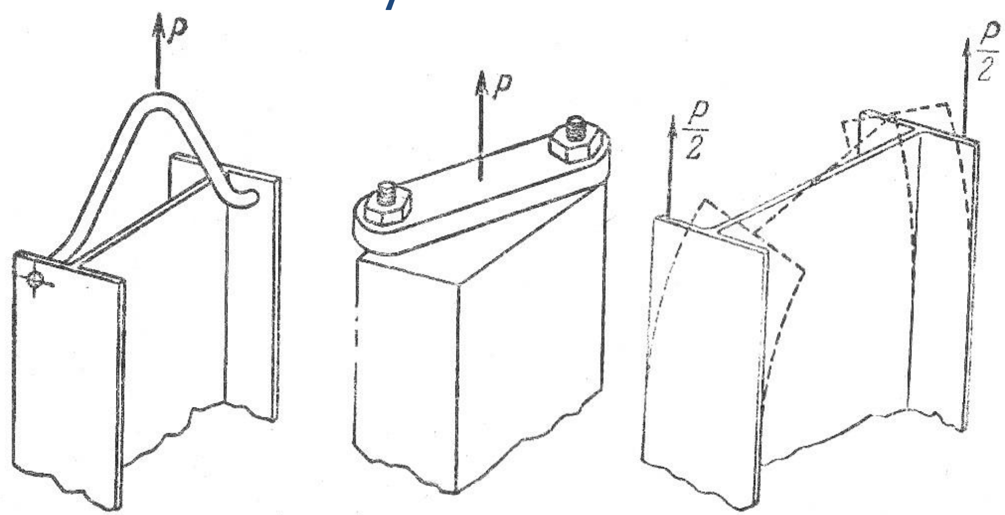
Stress analysis in regions close to the place of load application requires special treatment.

# St. Vernant’s Principle




**Consequence:** given diagram may be implemented by any of the presented variants of design (for stress analysis of the sections quite distant from the place of load application).

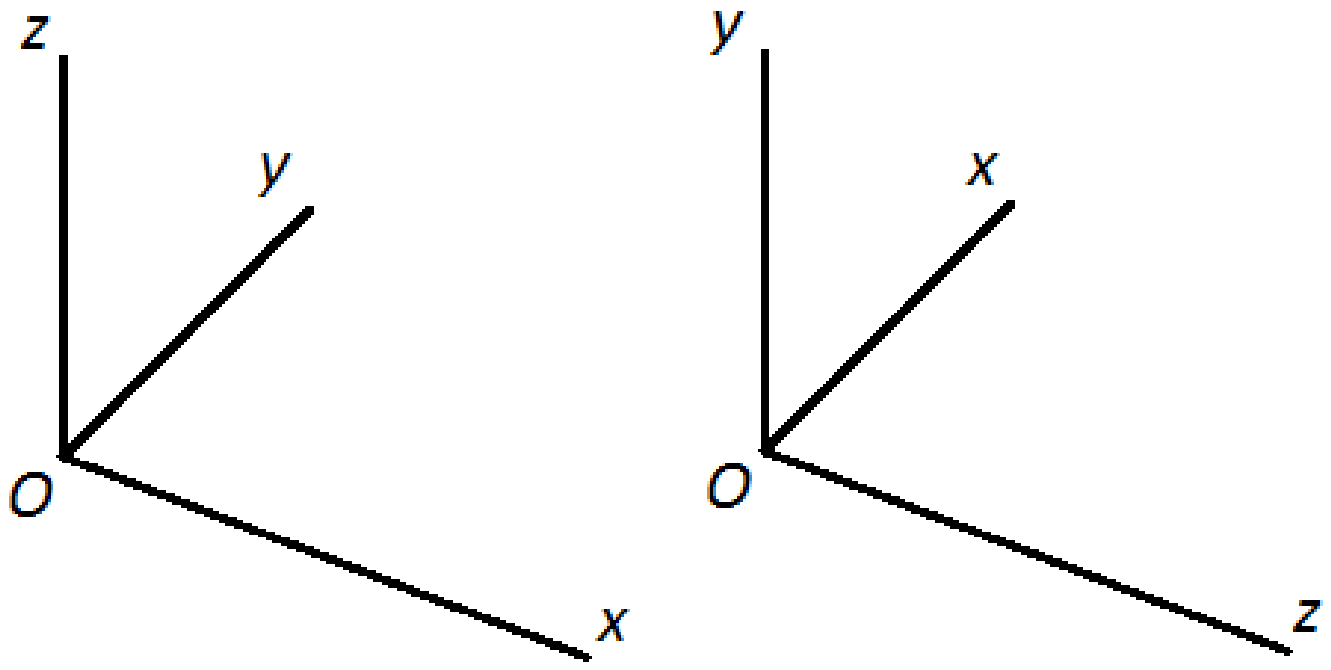
**Warning:** the region of non-uniform stress-distribution is greater for thin cross-sections relatively to solid cross-sections.



# Notation Conventions

**Orthogonal set of axes**

It is usually convenient to refer the state of stress at a point in a body to an orthogonal set of axes *Oxyz*.

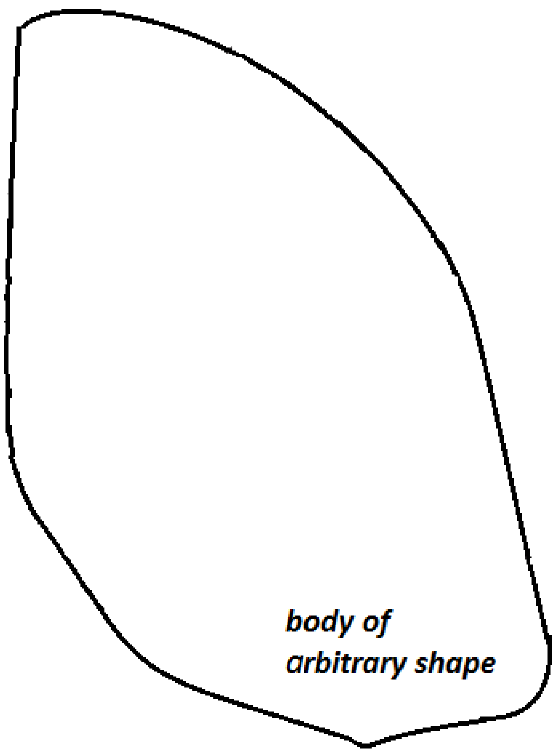



right-handed coordinate system

# Notation Conventions

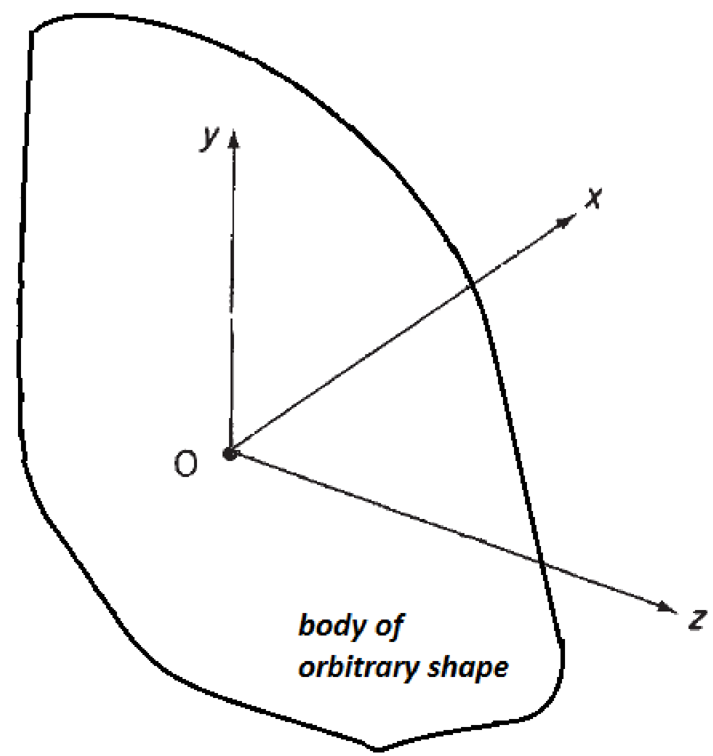
**For example**

Consider a body of an arbitrary shape



# Notation Conventions

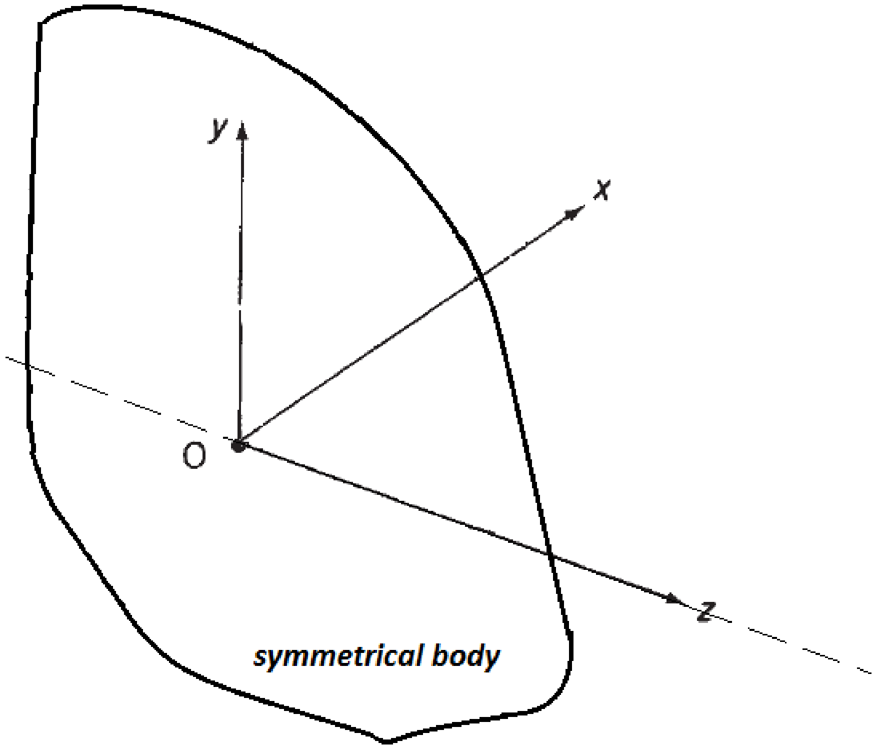
Assume a point O, which coincides with the origin of coordinates



# Notation Conventions

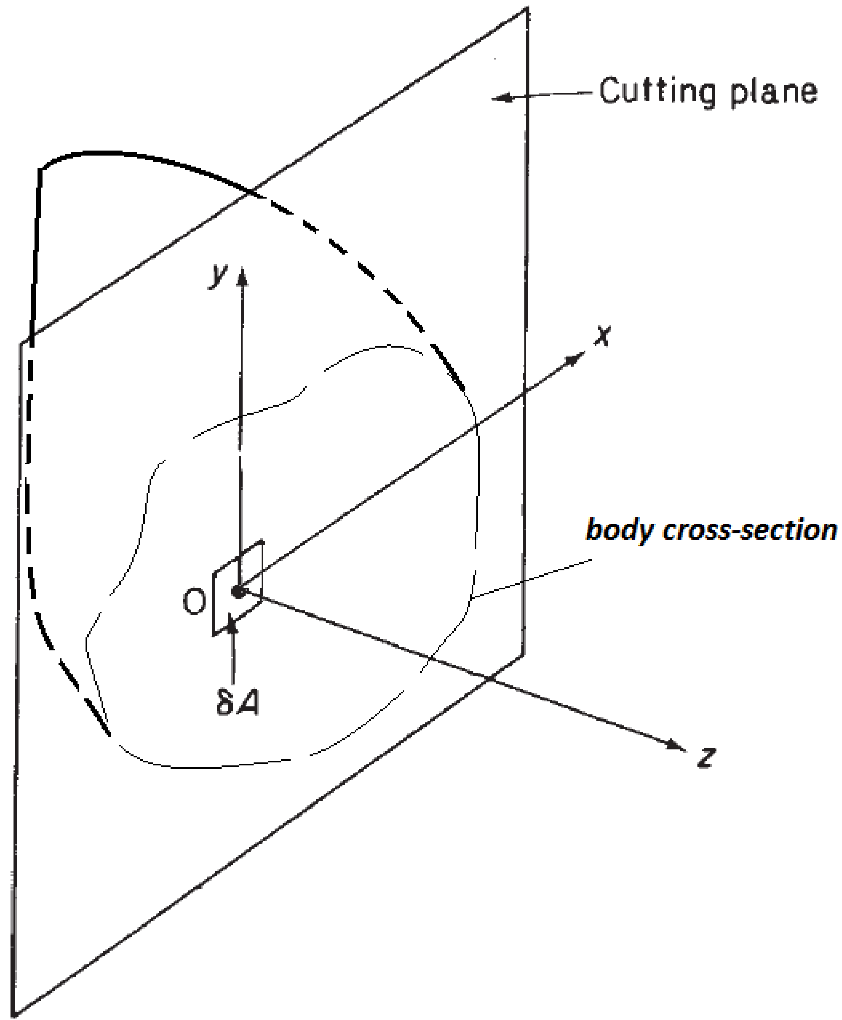
**In particular case**

it is very convenient to
unite one (or couple) of axes with
the axis(es) of symmetry
of the body.



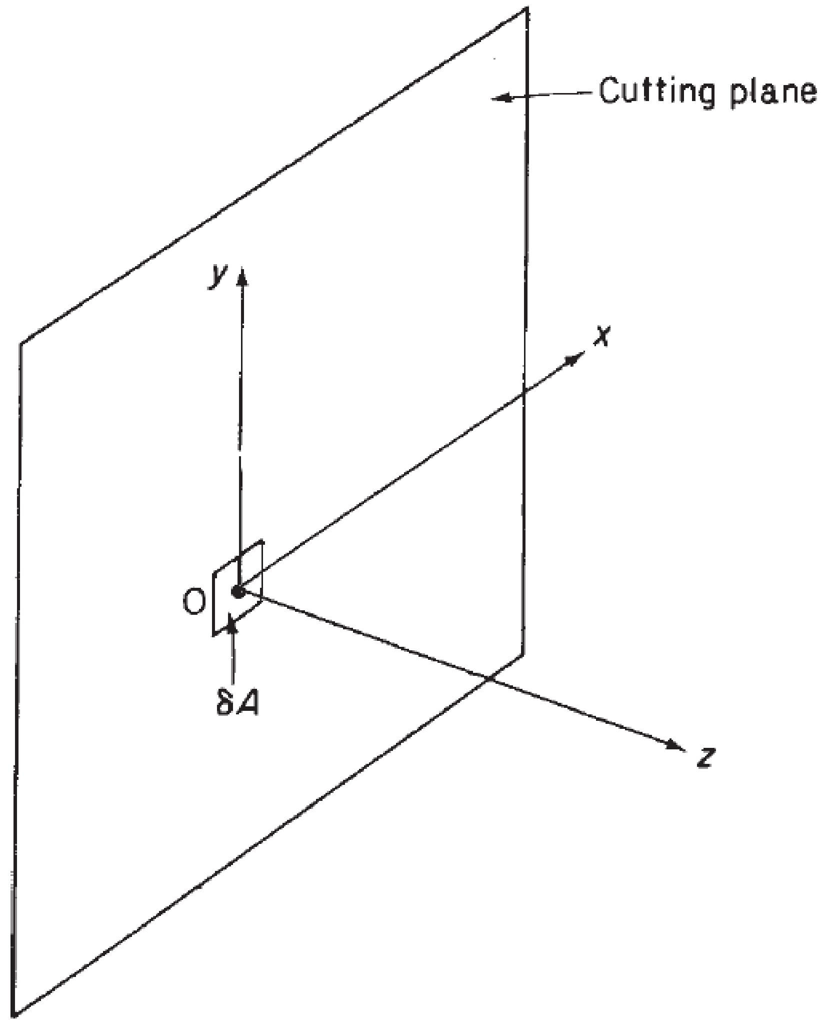
# Notation Conventions

For stress analysis
we may cut the body by planes parallel to the direction of the axes.



# Notation Conventions

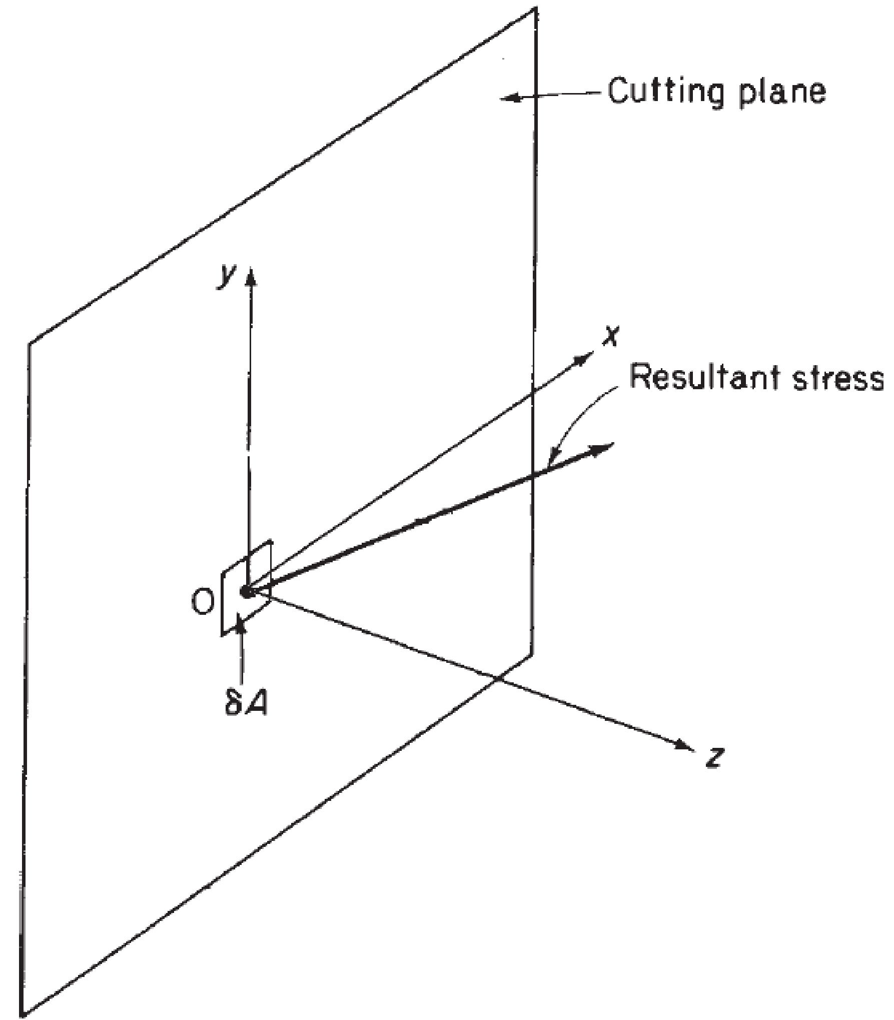
Afterwards we may analyze a vanishing section $\delta A$ of the body



# Notation Conventions

**Stress resolution**

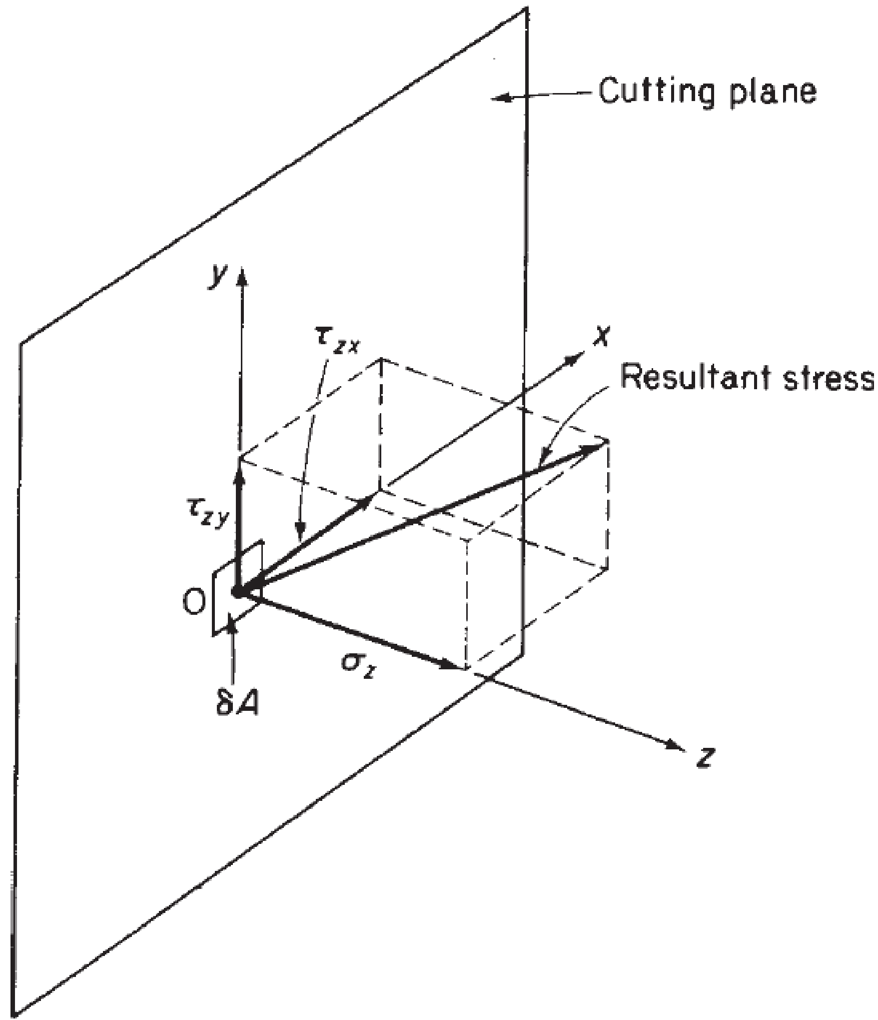
Assume a resultant stress acting at the point *O* on one of these cutting planes.



# Notation Conventions

The resultant stress then may be resolved into the following components:

- one component of direct stress;
- two components of shear stress.

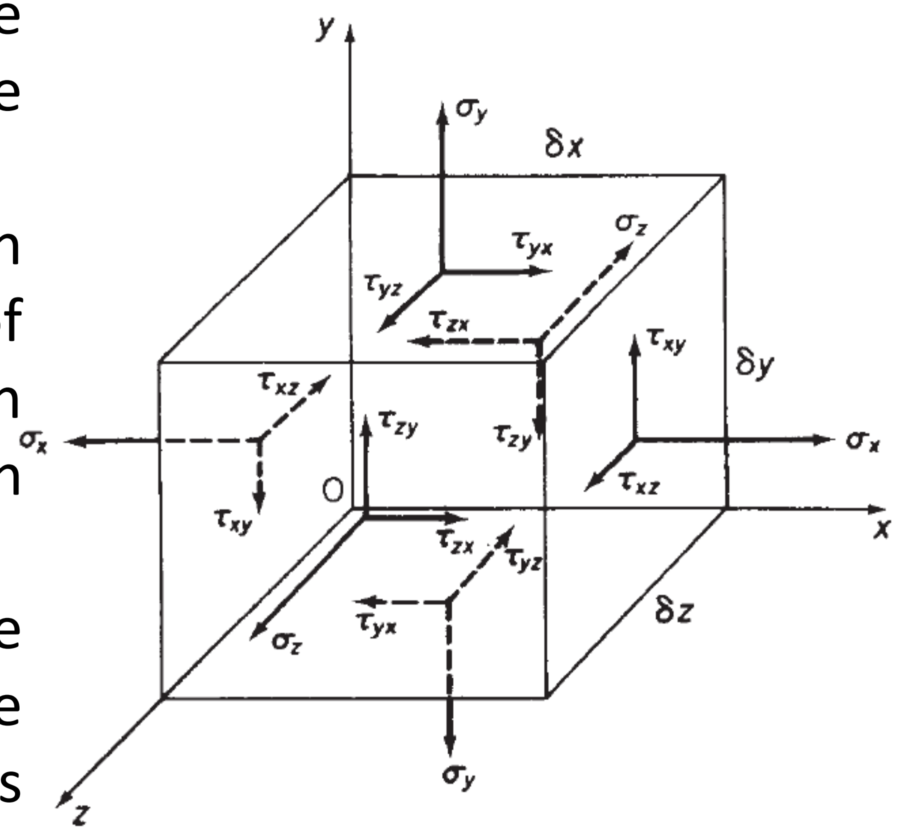



We allocate:

- a single subscript to direct stress to denote the plane on which it acts
- two subscripts to shear stress, the first specifying the plane, the second direction.

**Sign convention**

- normal stresses directed away from their related surfaces are tensile and positive, opposite compressive stresses are negative;
- shear stresses are positive when they act in the positive direction of the relevant axis in a plane on which the direct tensile stress is in the positive direction of the axis;
- if the tensile stress is in the opposite direction then positive shear stresses are in directions opposite to the positive directions of the appropriate axes.



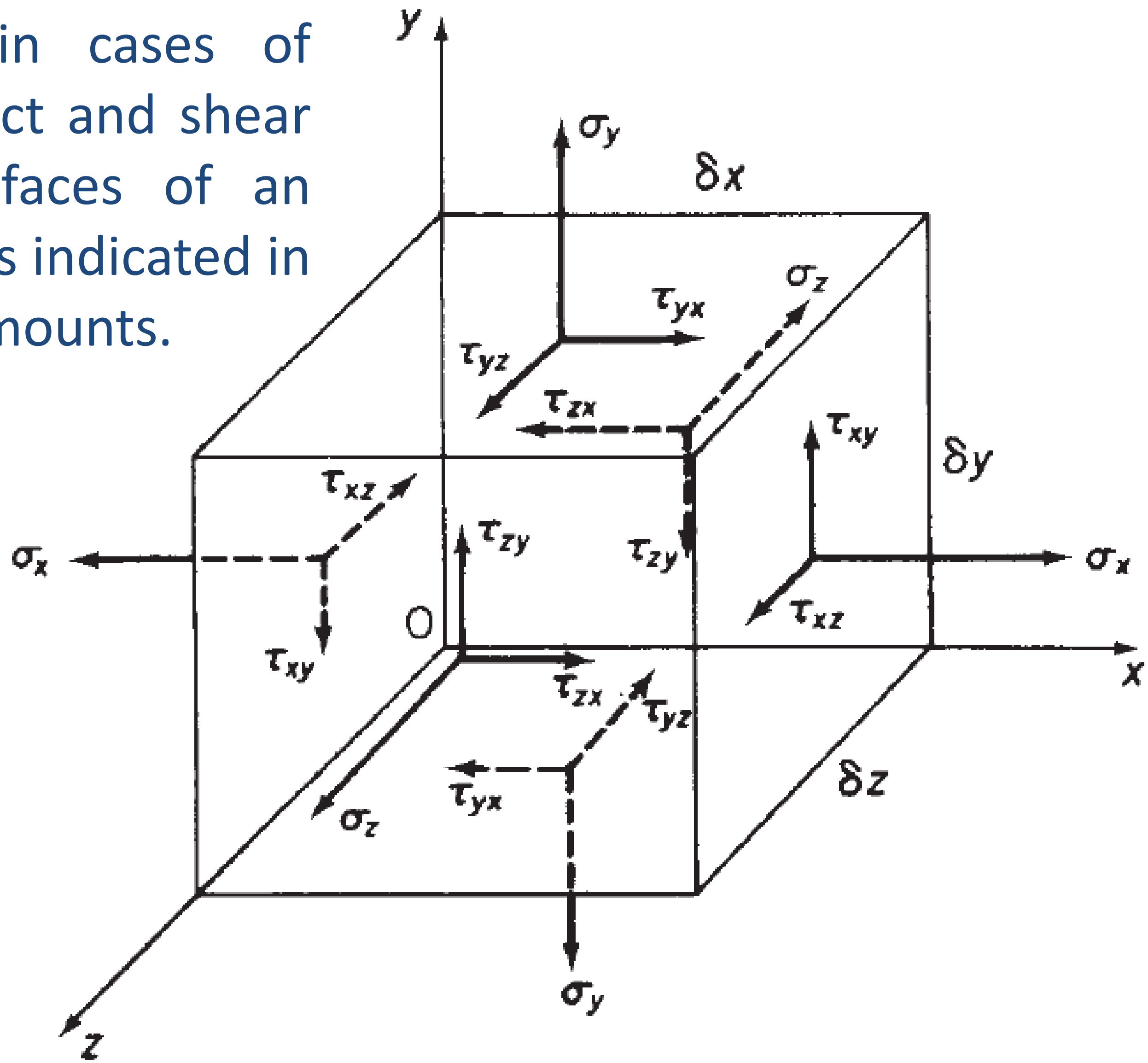
# Forces

- ***Surface forces*** such as $P_1$, $P_2$, . . . . , or *hydrostatic pressure*, are distributed over the surface area of the body. The surface force per unit area may be resolved into components parallel to our orthogonal system of axes and these are generally given the symbols ***X'***, ***Y'*** and ***Z'***.

- The second force system derives from gravitational and inertia effects and the forces are known as ***body forces***. These are distributed over the volume of the body and the components of body force per unit volume are designated ***X, Y*** and ***Z***.

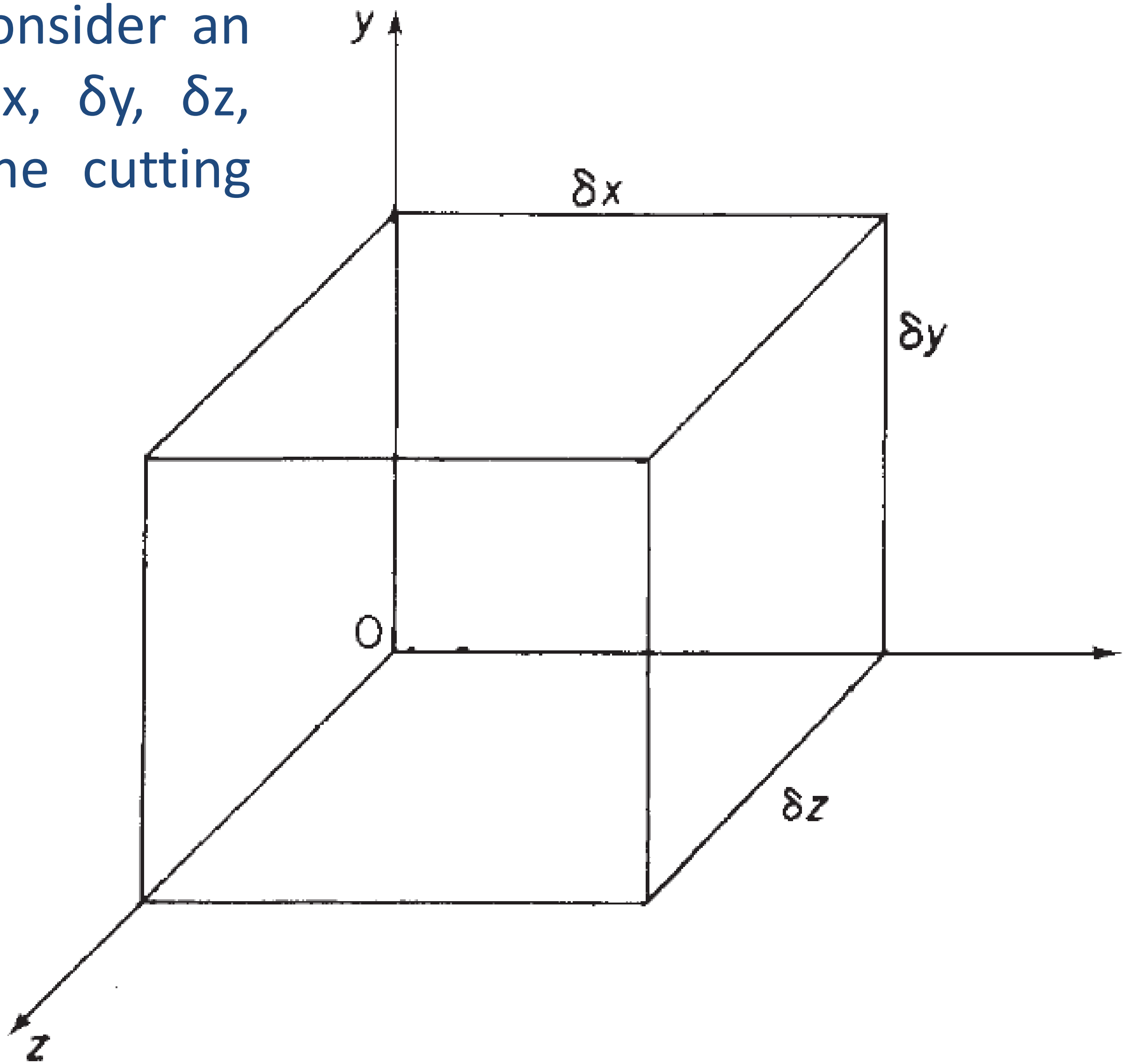
# Equilibrium Equations

Generally, except in cases of *uniform stress*, the direct and shear stresses on opposite faces of an element are not equal as indicated in Fig but differ by small amounts.



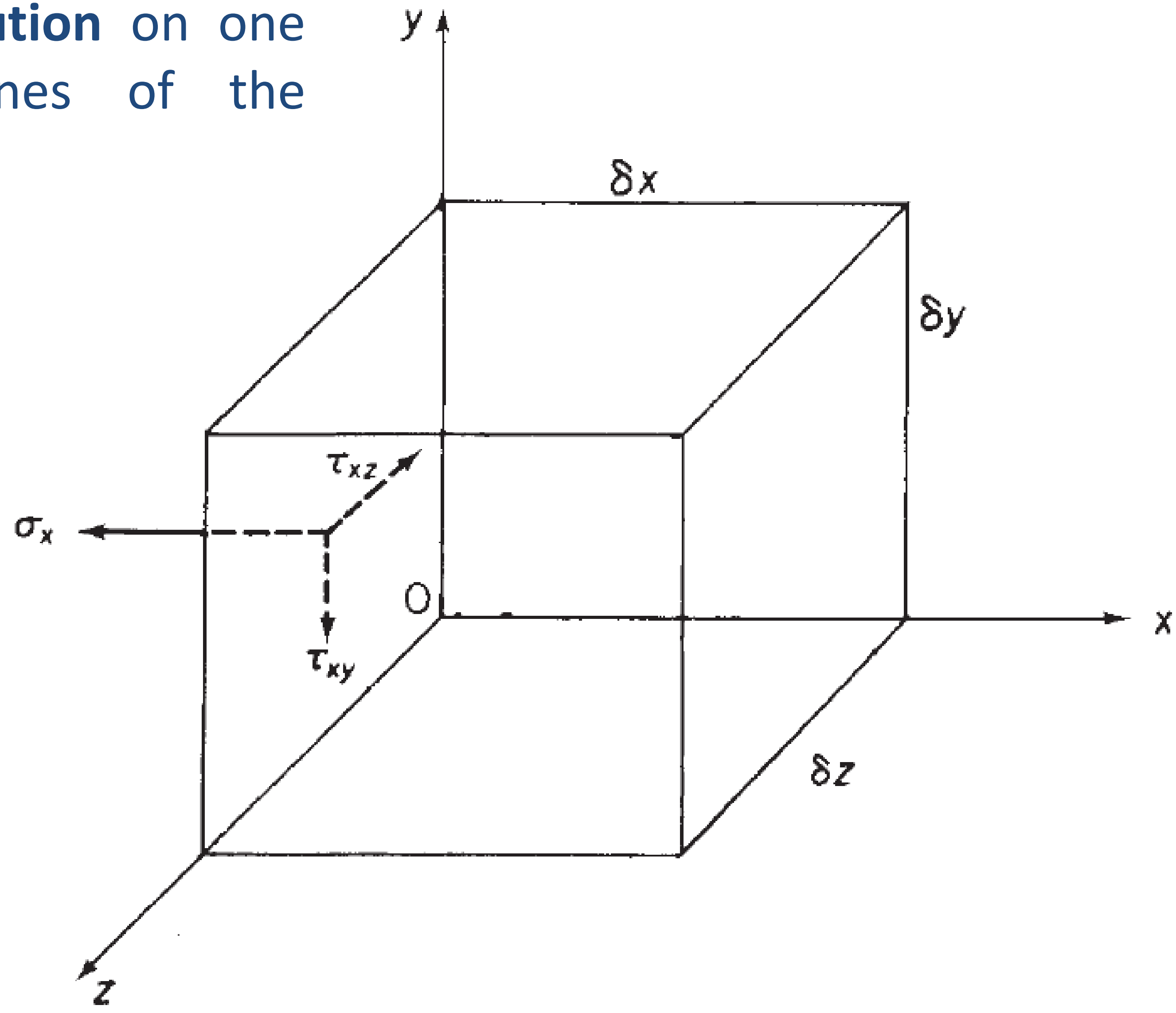
# Equilibrium Equations

**For example**, consider an element of side δx, δy, δz, formed at O by the cutting planes.



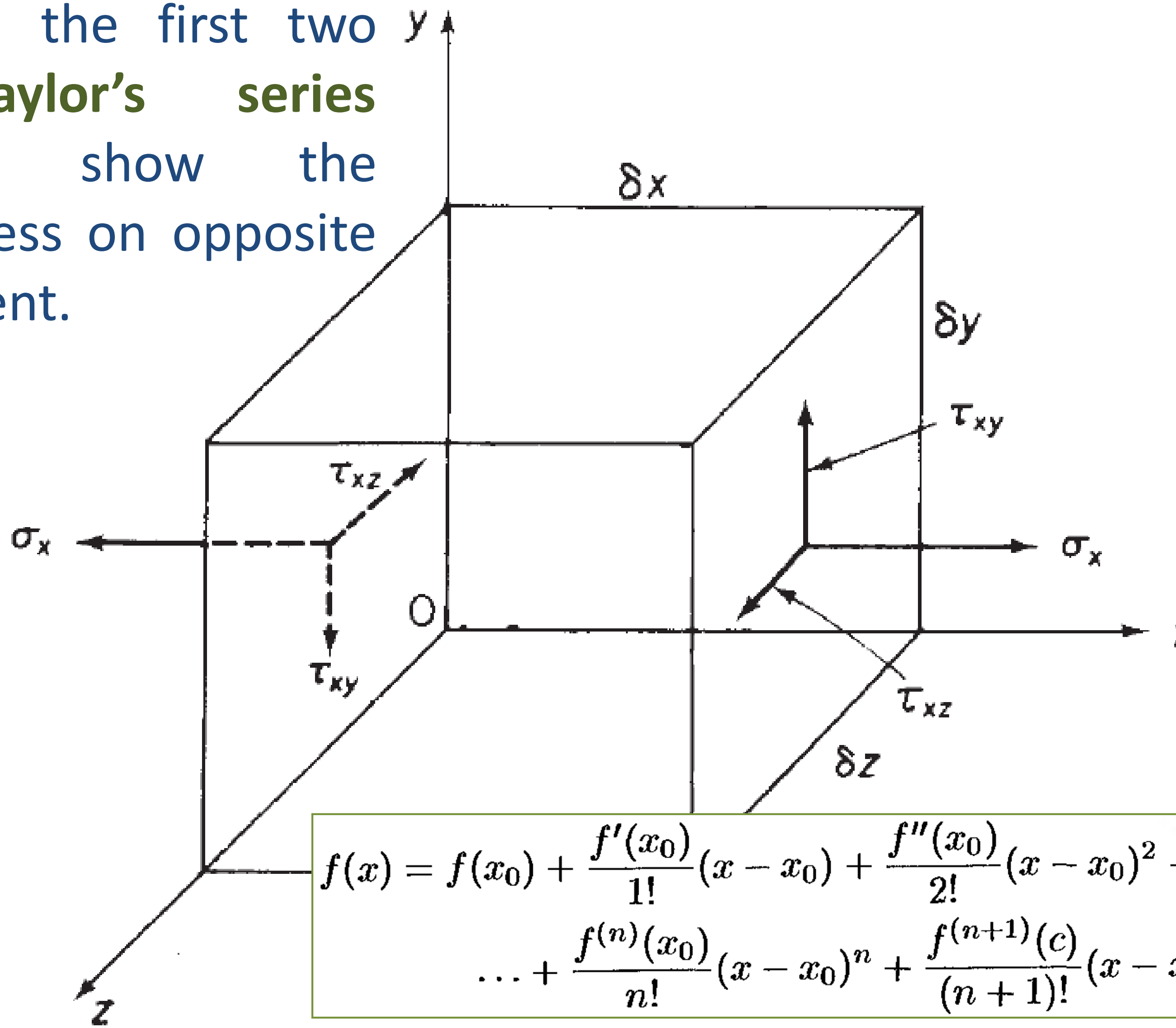
# Equilibrium Equations

**Stress resolution** on one of the x planes of the element.



# Equilibrium Equations

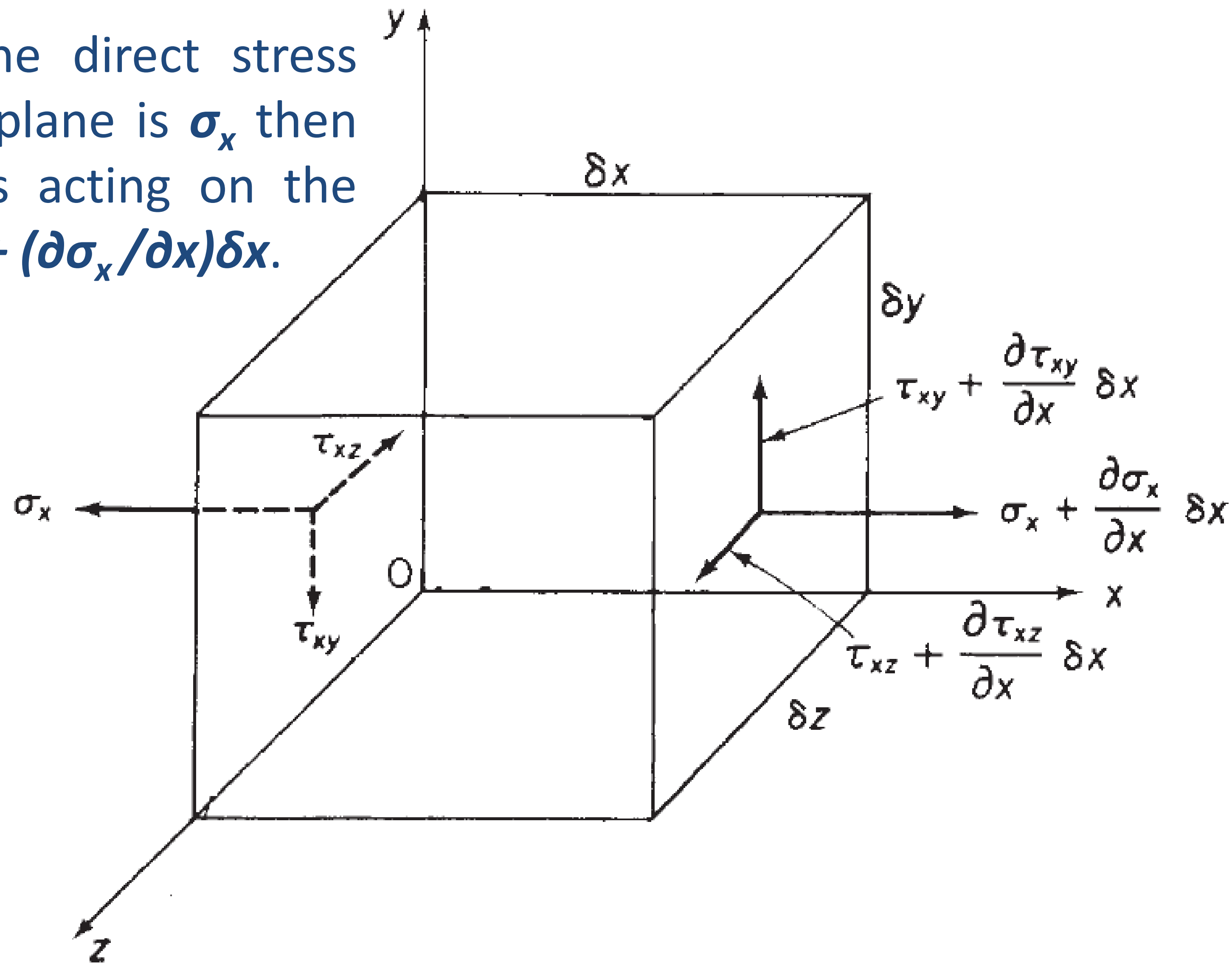
We will use the first two terms of **Taylor's series expansion** to show the difference in stress on opposite sides of an element.




$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots$$
$$\ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

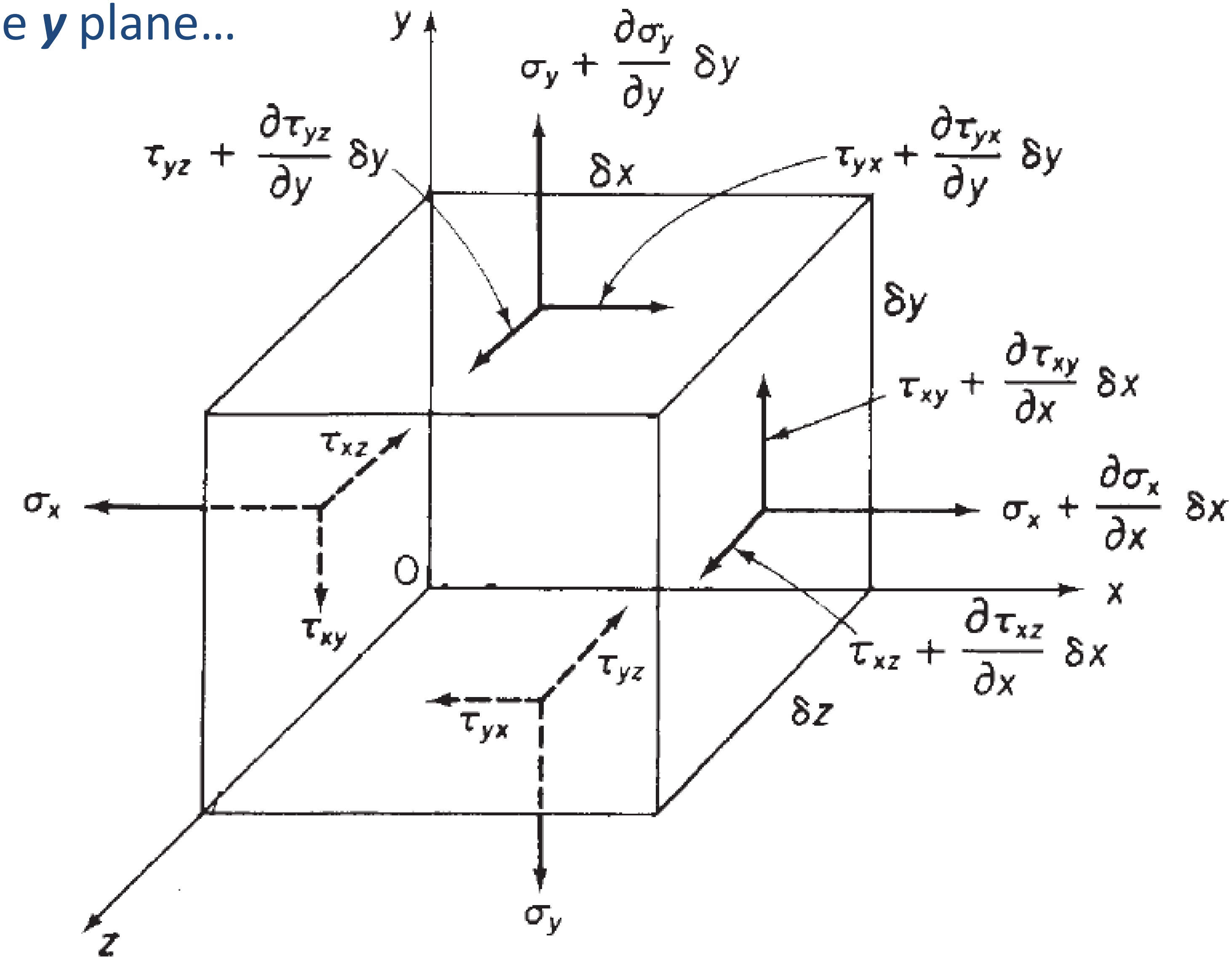
# Equilibrium Equations

Therefore, the direct stress acting on the *x* plane is $\boldsymbol{\sigma_x}$ then the direct stress acting on the $x+\delta$x plane is $\boldsymbol{\sigma_x + (\partial\sigma_x/\partial x)\delta x}$.



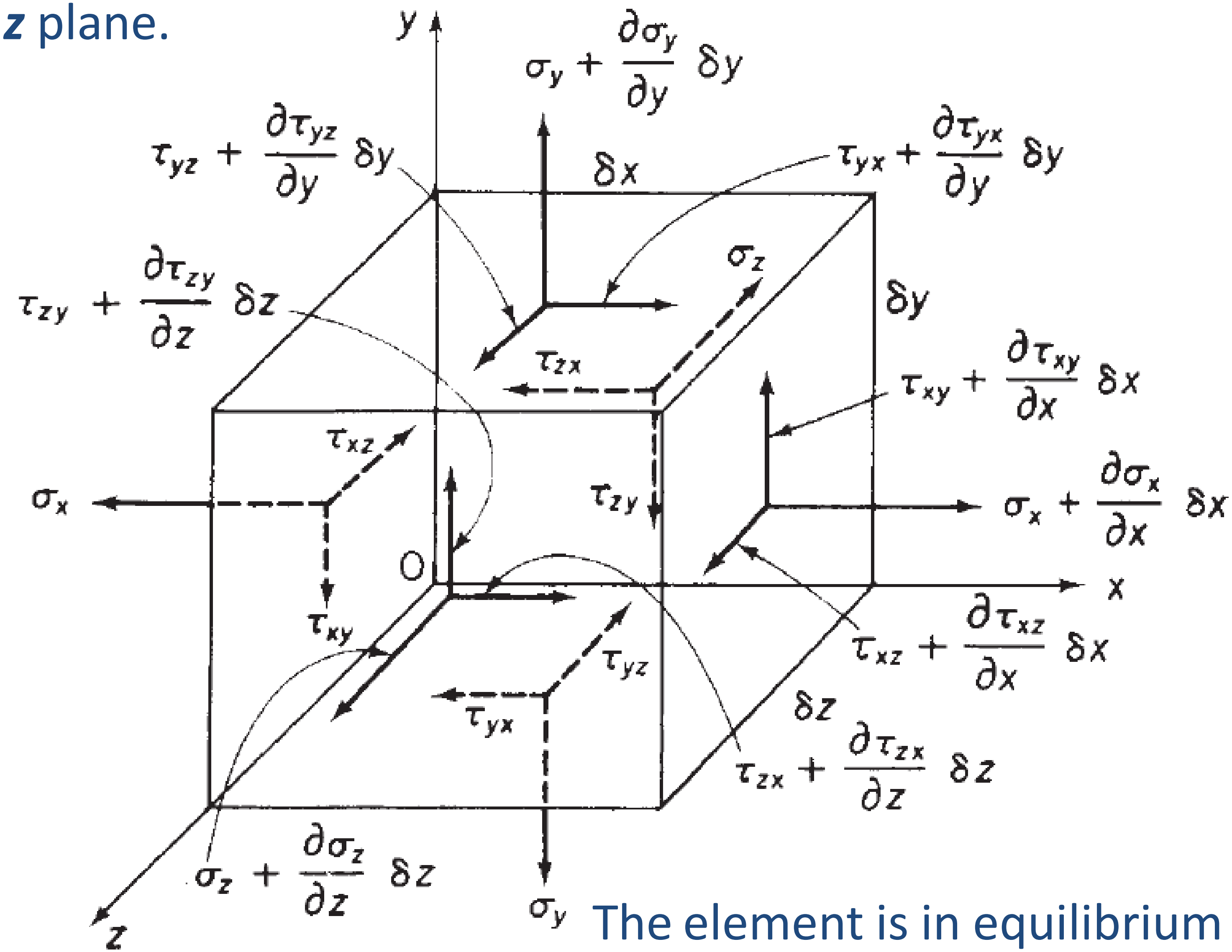
# Equilibrium Equations

Similar for the ***y*** plane…



# Equilibrium Equations

…and finally ***z*** plane.




The element is in equilibrium

# Equilibrium Equations

**Taking moments** about an axis through the center of the element parallel to the ***z* axis**

$$\tau_{xy} = \tau_{yx}$$

taking the limit as $\delta x$ and $\delta y$ approach zero.

Taking moments about other axis

$$\left.\begin{aligned} \tau_{xy} &= \tau_{yx} \\ \tau_{xz} &= \tau_{zx} \\ \tau_{yz} &= \tau_{zy} \end{aligned}\right\}$$

# Equilibrium Equations

Considering the **equilibrium** of the element in the ***x* direction**

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0$$

And **equilibrium** of the element in other directions

$$\boxed{\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + Z = 0 \end{aligned}\right\}}$$

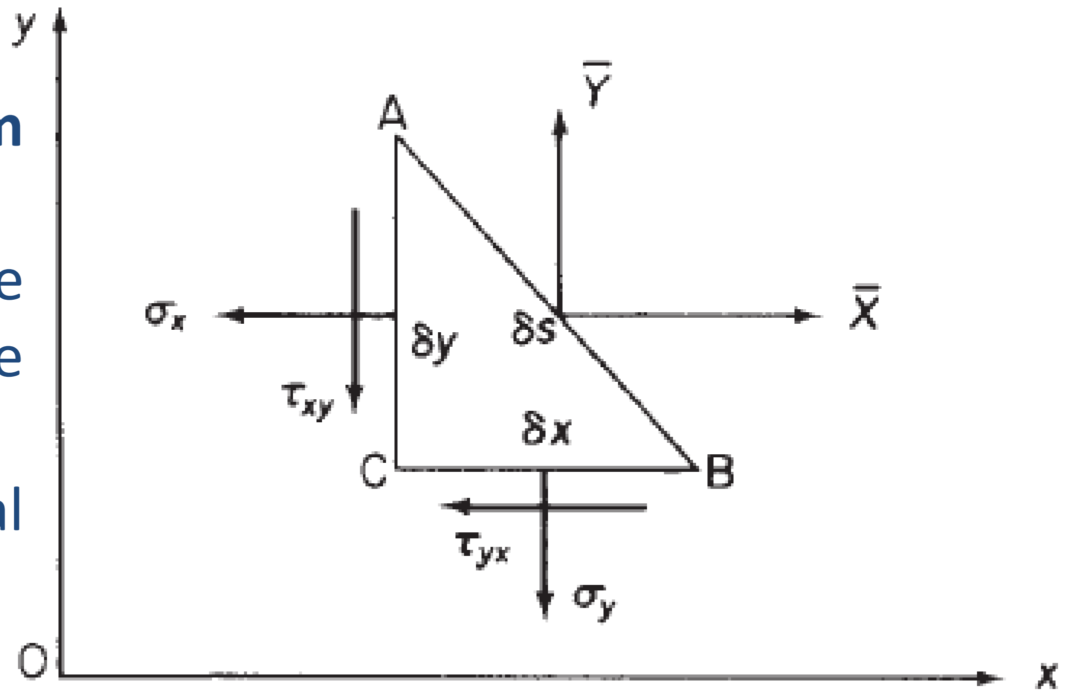
**The equations of equilibrium** must be satisfied at all interior points in a deformable body under a three-dimensional force system.

# Boundary conditions

**Equilibrium** must also be satisfied at all positions **on the boundary of the body** where the components of the surface force per unit area are $X'$, $Y'$ and $Z'$.

The triangle is in **equilibrium** under the action:

- of ***surface forces*** on the elemental length *AB* of the boundary and
- ***internal forces*** on internal faces *AC* and *CB*



# Boundary conditions

**Summation** of forces in the **x direction** (and taking the limit as ***δx → 0***) gives

$$\bar{X} = \sigma_x \frac{dy}{ds} + \tau_{yx} \frac{dx}{ds}$$

The derivatives *dy/ds* and *dx/ds* are the direction cosines *l* and *m* of the angles that a normal to *AB* makes with the *x* and *y* axes, respectively.

The boundary conditions for a three-dimensional body are

$$\boxed{\left.\begin{aligned} \bar{X} &= \sigma_x l + \tau_{yx} m + \tau_{zx} n \\ \bar{Y} &= \sigma_y m + \tau_{xy} l + \tau_{zy} n \\ \bar{Z} &= \sigma_z n + \tau_{yz} m + \tau_{xz} l \end{aligned}\right\}}$$

where *l*, *m* and *n* become the direction cosines of the angles that a normal to the surface of the body makes with the *x*, *y* and *z* axes, respectively.

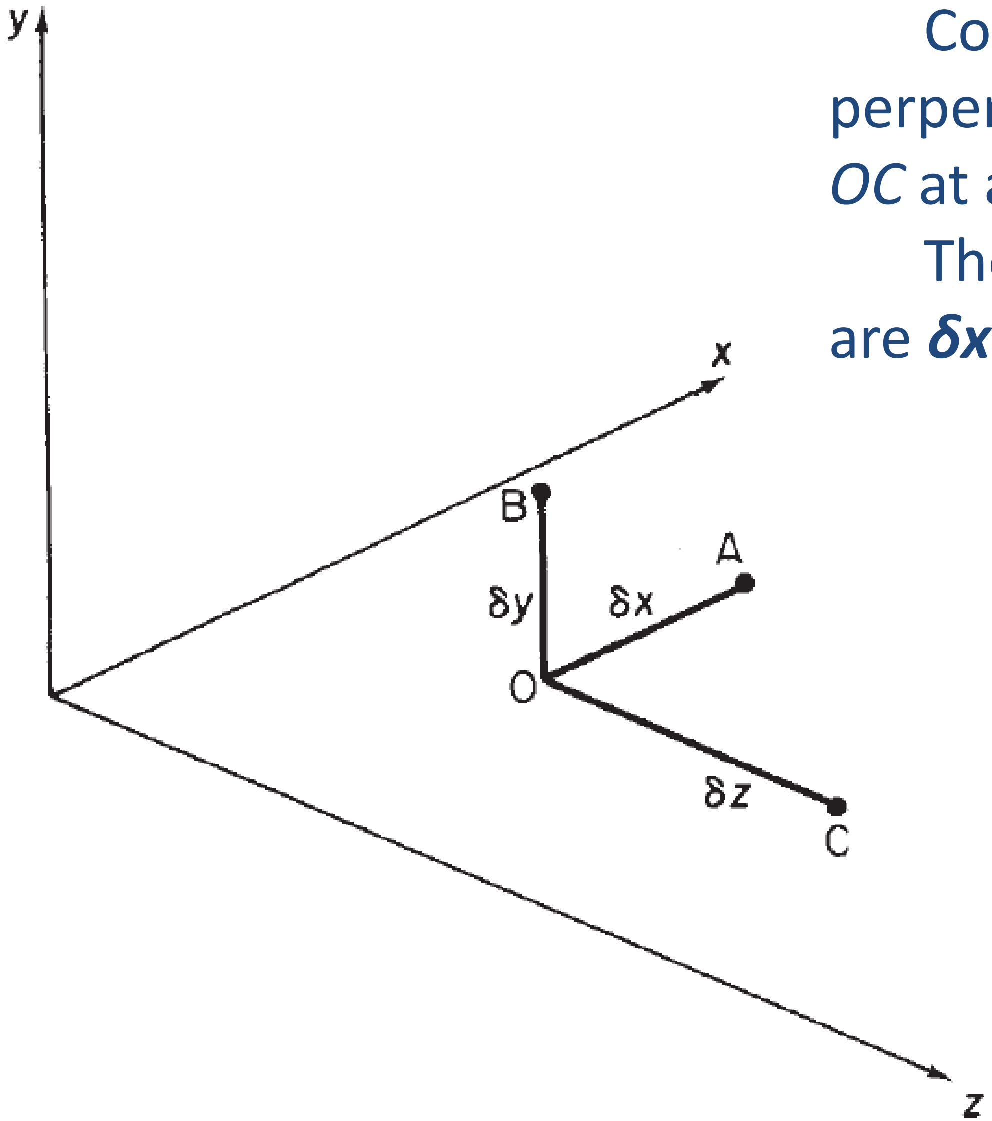
# Boundary conditions

Example
(whiteboard)

# Strain

- *longitudinal or **direct strains ε*** are associated with direct stresses *σ* and relate to changes in length
- while ***shear strains γ*** define changes in angle produced by shear stresses.

# Strain



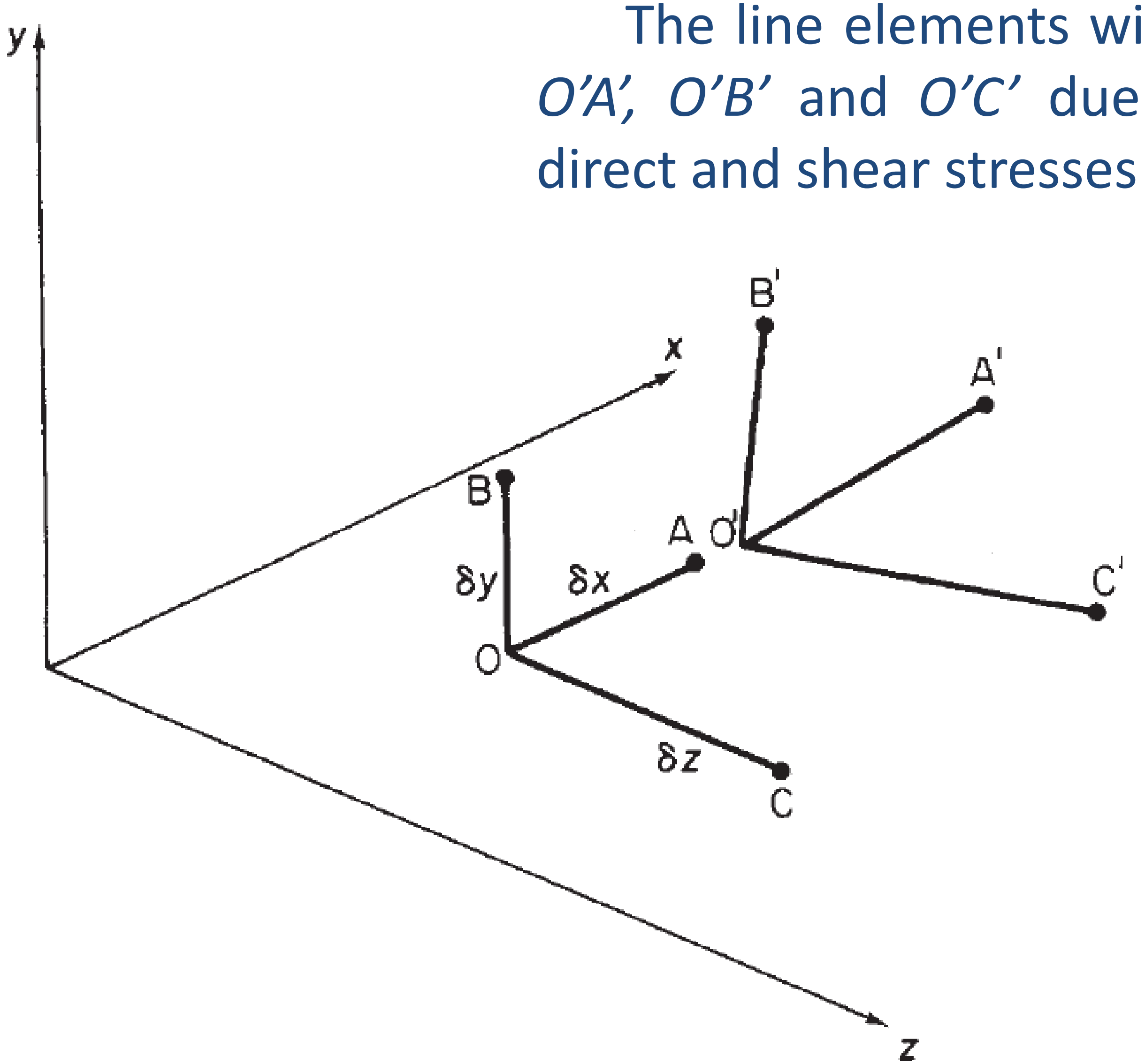

Consider three mutually perpendicular **line elements** *OA*, *OB* and *OC* at a point *O* in a deformable body.

Their original or unstrained lengths are $\boldsymbol{\delta x}$, $\boldsymbol{\delta y}$ and $\boldsymbol{\delta z}$, respectively.

*O* $(x, y, z)$
*A* $(x+\delta x, y, z)$
*B* $(x, y+\delta y, z)$
*C* $(x, y, z+\delta z)$

# Strain

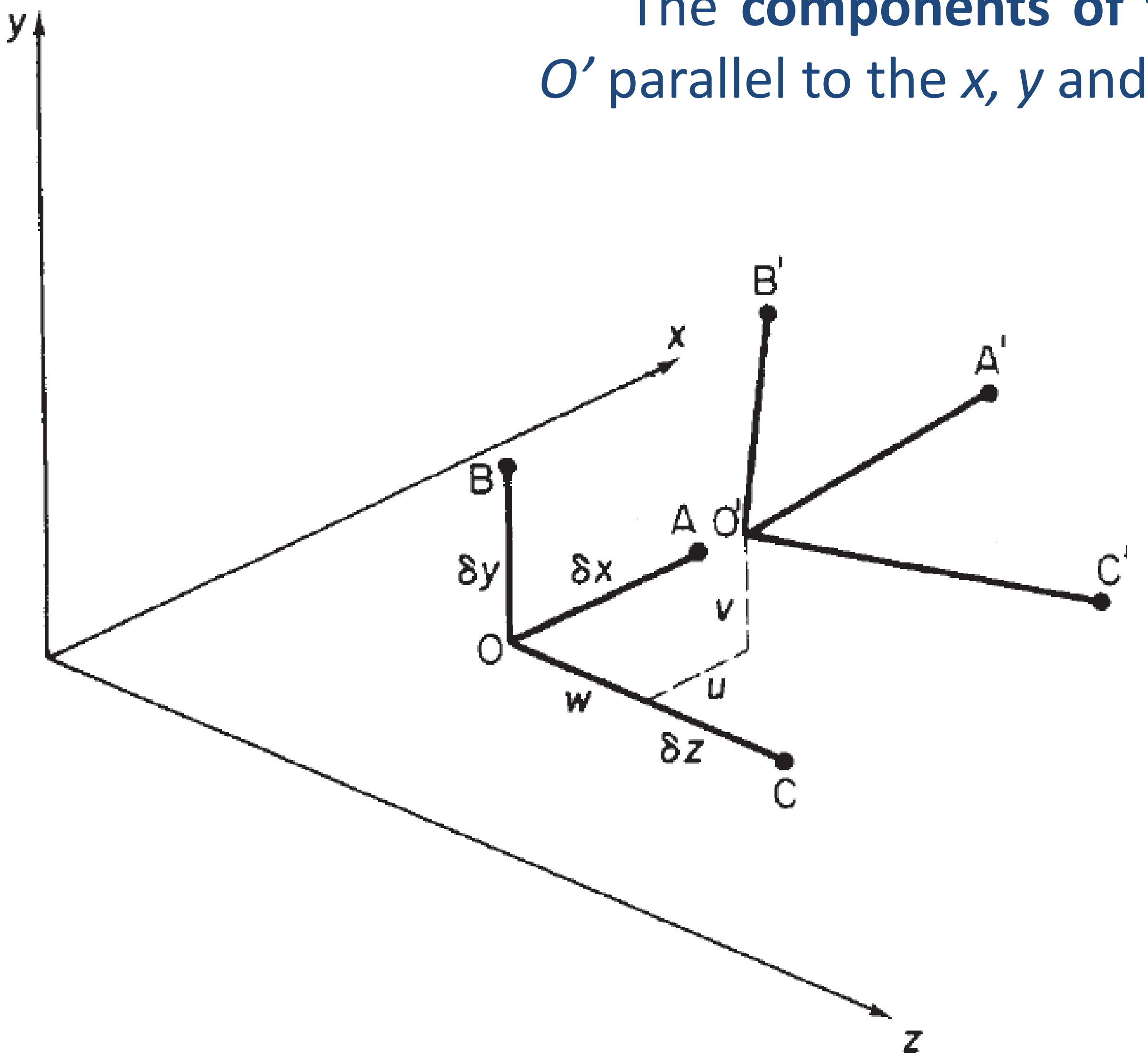
The line elements will **deform** to the positions *O'A', O'B'* and *O'C'* due to a complex system of direct and shear stresses at *O*.




*O (x, y, z)*
*A (x+δx, y, z)*
*B (x, y+δy, z)*
*C (x, y, z+δz)*

# Strain

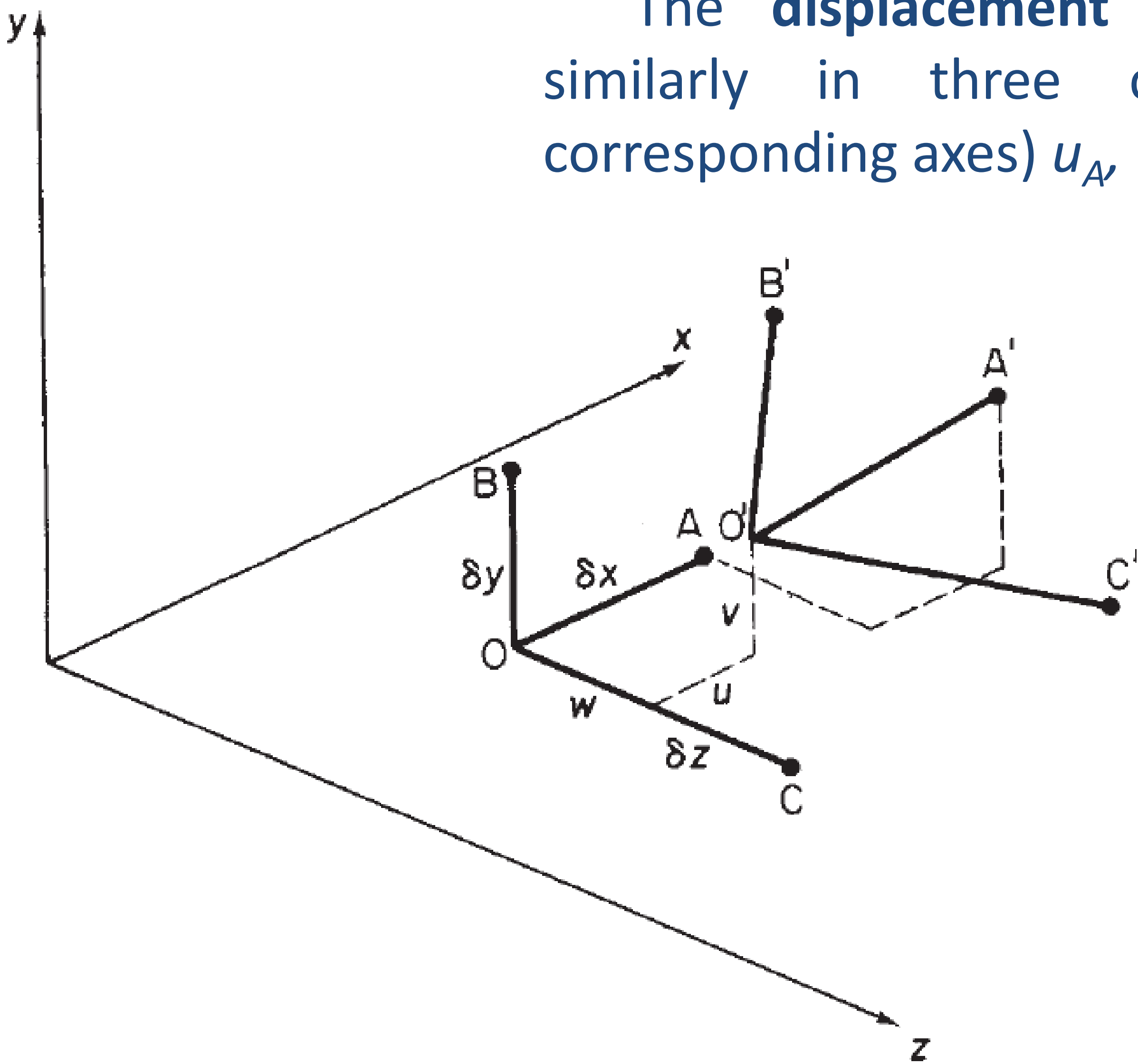
The **components of the displacement** of *O* to *O'* parallel to the *x, y* and *z* axes are *u, v* and *w*.



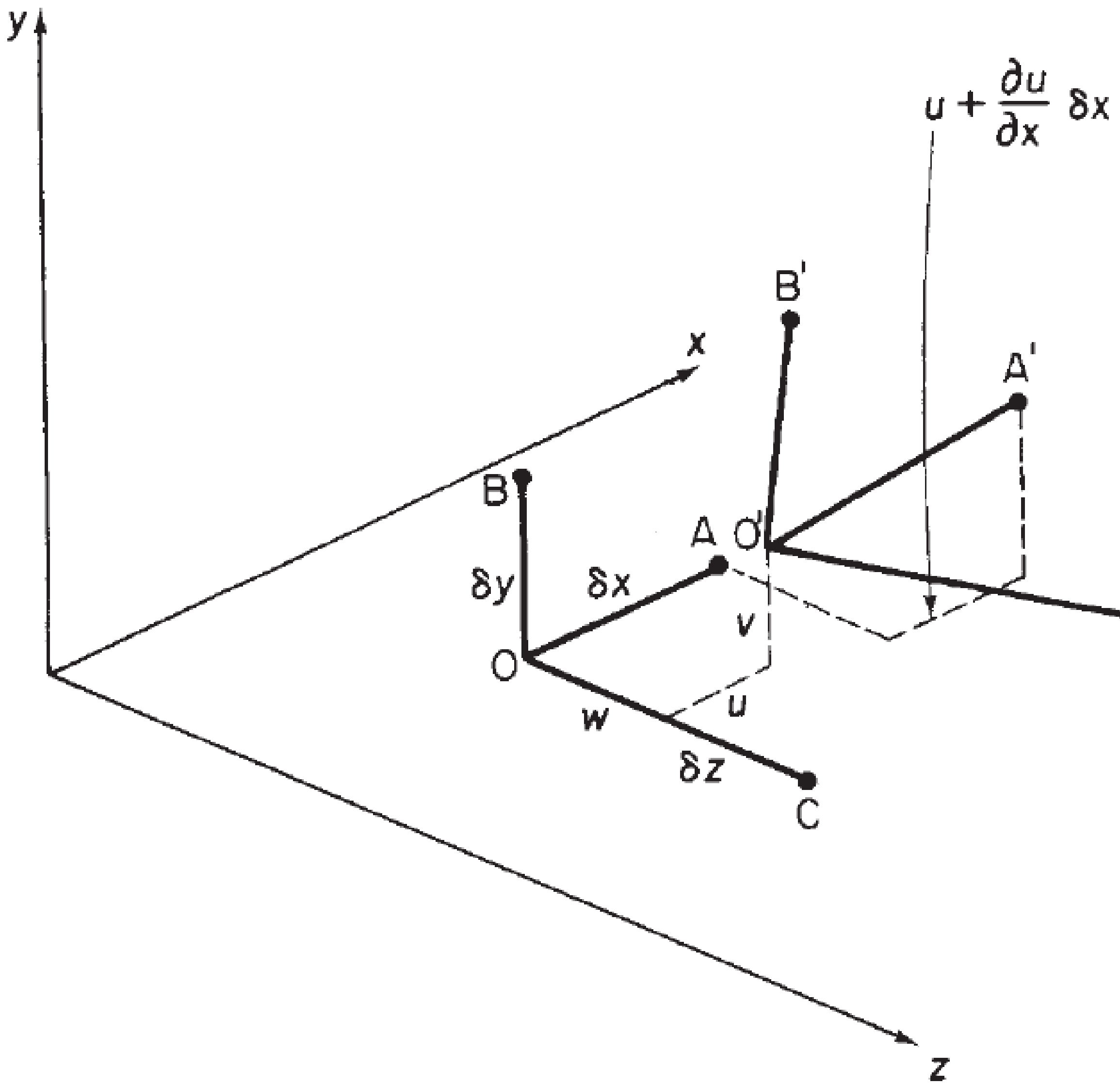

Displacements are defined as ***positive*** in the *positive directions of the axes*

# Strain

The **displacement of A** may be resolved similarly in three components (along the corresponding axes) $u_A$, $v_A$ and $w_A$.

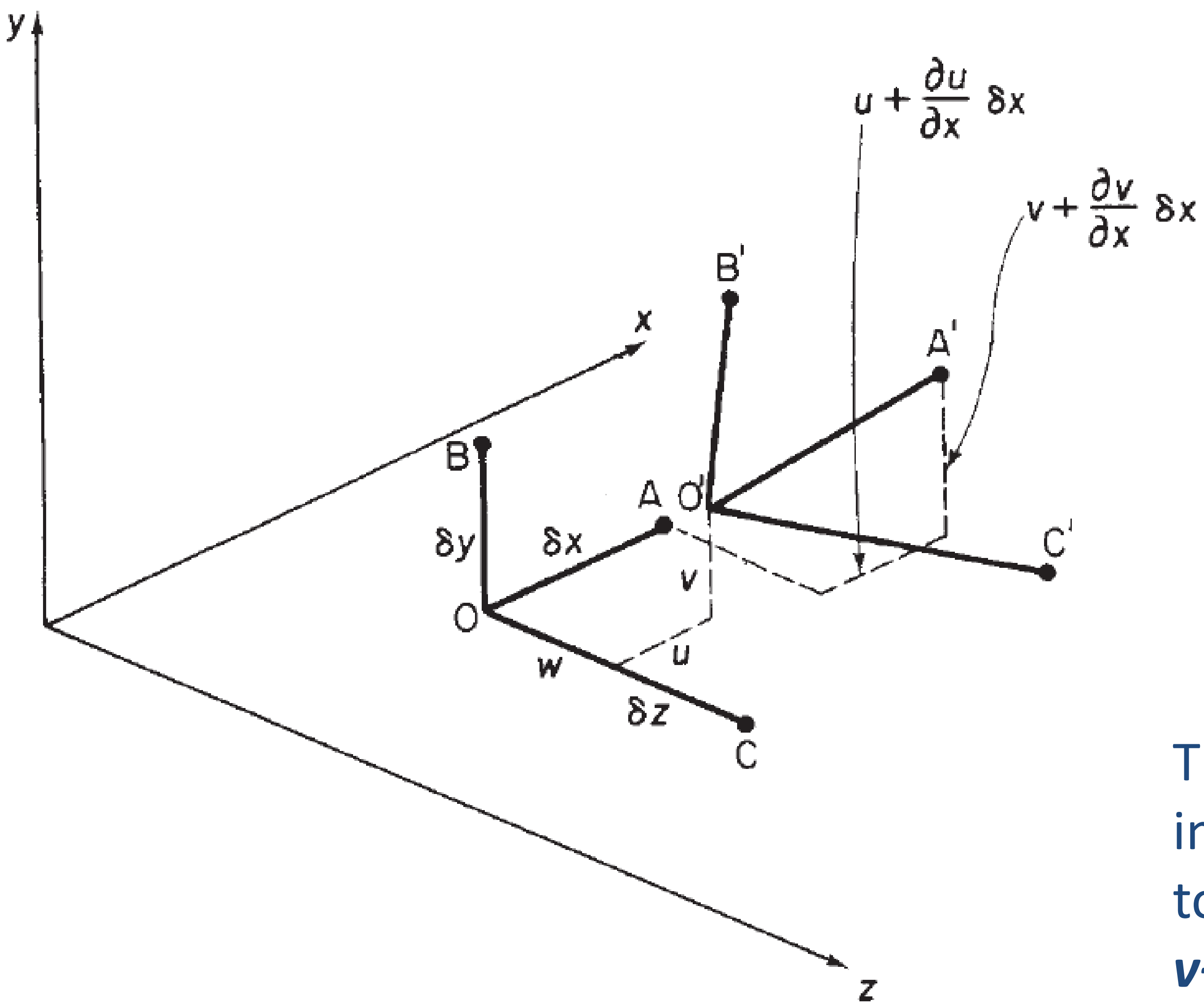


# Strain



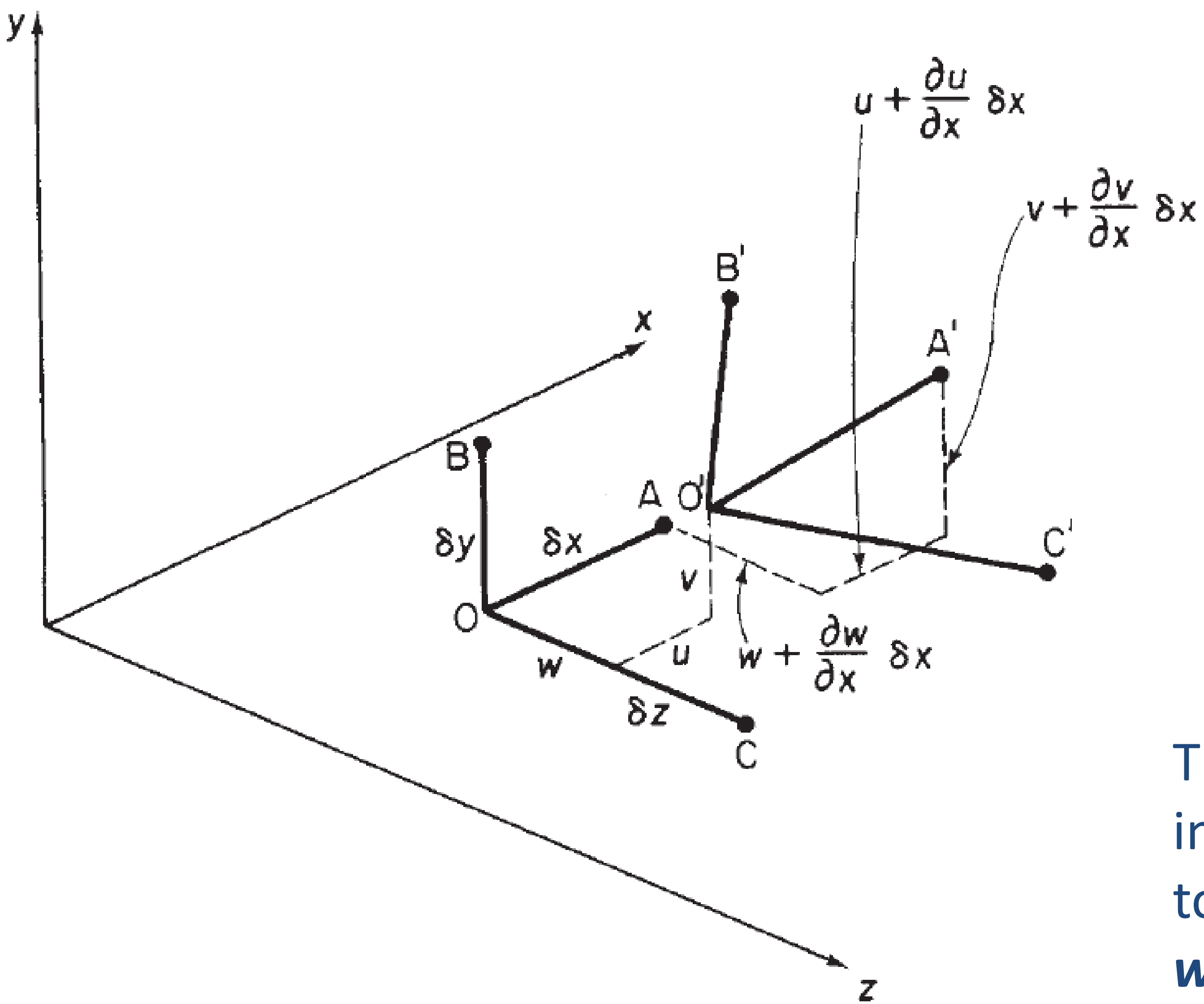

The **displacement of *A*** in a direction parallel to the ***x*** **axis** is ***u+(∂u/∂x)δx***
(according to the first two terms of Taylor 's series expansion)

# Strain



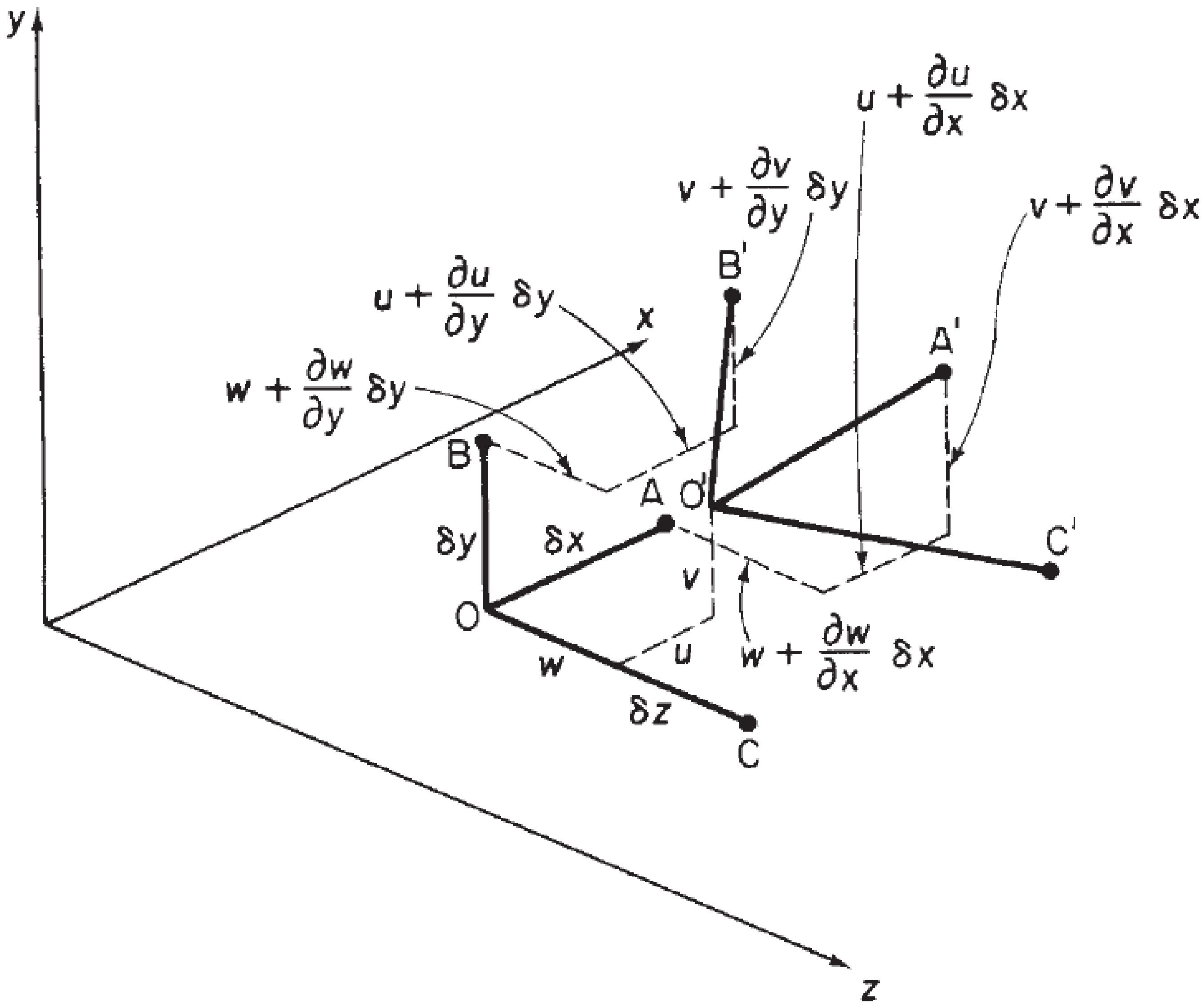

The **displacement of *A*** in a direction parallel to the ***y* axis** is ***v+(∂v/∂x)δx***

# Strain




The **displacement of *A*** in a direction parallel to the ***z* axis** is ***w+(∂w/∂x)δx***

# Strain



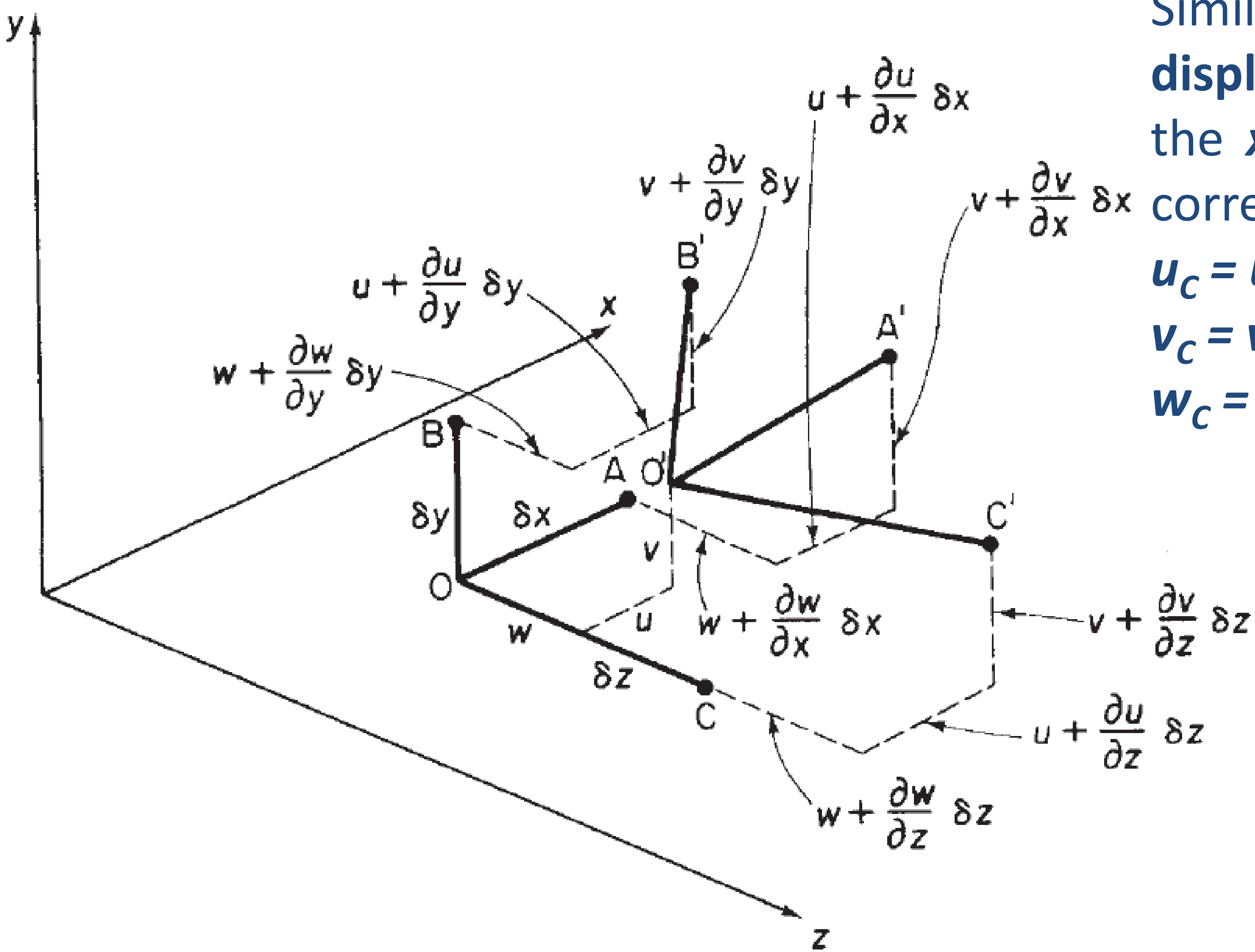

Similarly for the **displacement of *B*** in the ***x, y, z*** **direction** correspondingly

***$u_B = u + (\partial u/\partial y)\delta y$***

***$v_B = v + (\partial v/\partial y)\delta y$***

***$w_B = w + (\partial w/\partial y)\delta y$***

# Strain




Similarly for the **displacement of *C*** in the ***x, y, z* direction** correspondingly

$u_C = u+(\partial u/\partial z)\delta z$

$v_C = v+(\partial v/\partial z)\delta z$

$w_C = w+(\partial w/\partial z)\delta z$

# Strain

The longitudinal strain in the direction of the line element is

$$\varepsilon = \lim_{L \to 0} \frac{\Delta L}{L}$$

Then the direct strain at *O* in the *x* direction is obtained from

$$\varepsilon_x = \frac{\mathrm{O'A'} - \mathrm{OA}}{\mathrm{OA}} = \frac{\mathrm{O'A'} - \delta x}{\delta x}$$

Knowing coordinates of the endpoints of A'O' we may find its length

$$r^2 = (x-x_0)^2+(y-y_0)^2+(z-z_0)^2$$

where $x, y, z$ – coordinates of the *A'*
$x_0, y_0, z_0$ – coordinates of the *O' (for example* $x_0 = u$*)*

# Strain

Then we have

$$(\mathrm{O'A'})^2 = \left(\delta x + u + \frac{\partial u}{\partial x}\delta x - u\right)^2 + \left(v + \frac{\partial v}{\partial x}\delta x - v\right)^2 + \left(w + \frac{\partial w}{\partial x}\delta x - w\right)^2$$

or

$$\mathrm{O'A'} = \delta x\sqrt{\left(1 + \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2}$$

after neglecting of second-order terms

$$\mathrm{O'A'} = \delta x\left(1 + 2\frac{\partial u}{\partial x}\right)^{\frac{1}{2}}$$

$$(1+x)^n \equiv 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

applying ***binominal expansion*** we have

$$\mathrm{O'A'} = \delta x\left(1 + \frac{\partial u}{\partial x}\right)$$

(squares and higher powers of *∂u/∂x* are ignored)

# Strain

Substituting for *O′A′* in equation for $\varepsilon_x$

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

Similarly for other elements we have

$$\left.\boxed{\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \varepsilon_z &= \frac{\partial w}{\partial z} \end{aligned}}\right\}$$

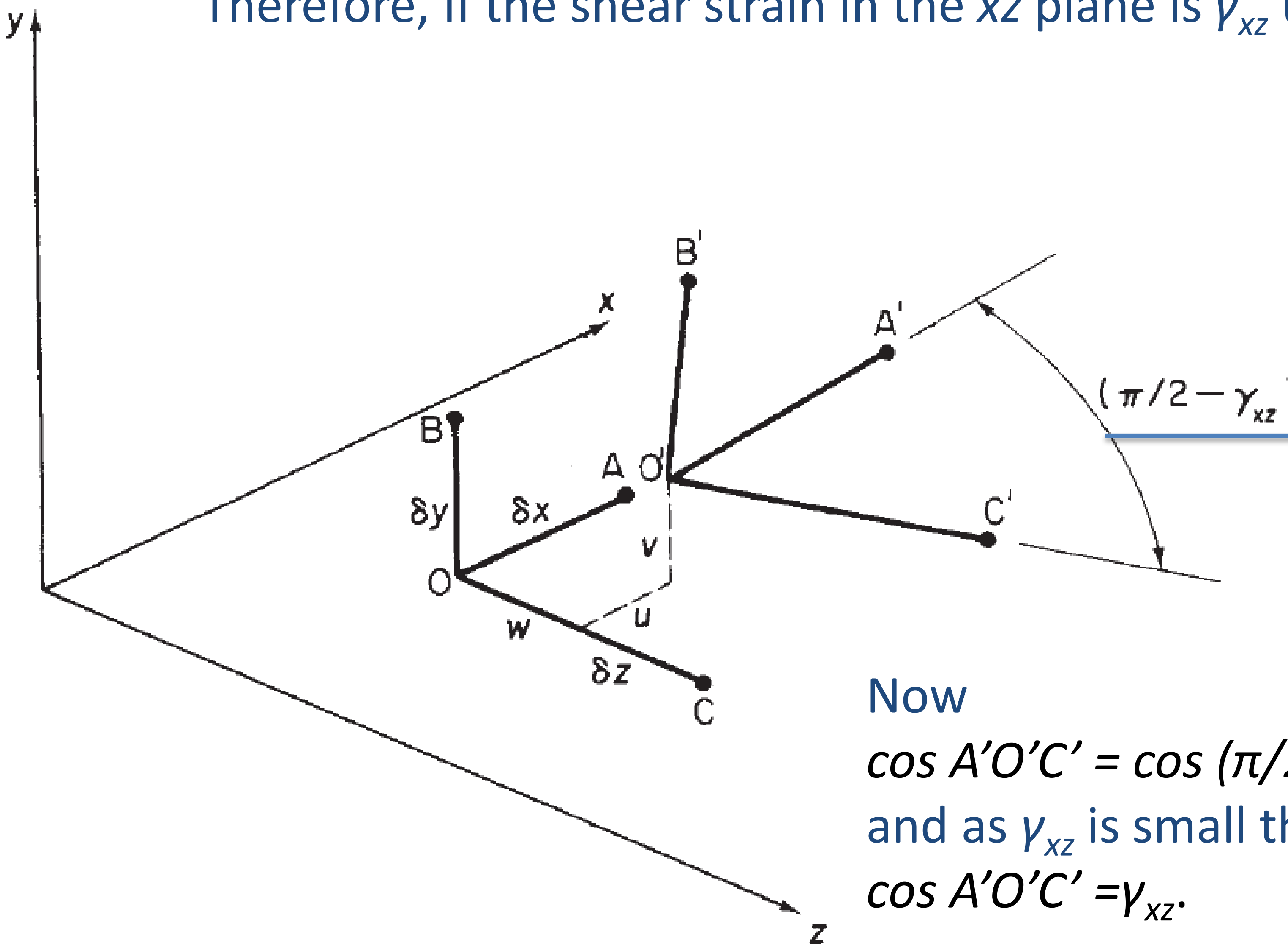
**strain** equations (or strain-displacement equations)

The **shear strain** at a point in a body is defined as

- the change in the angle between two mutually perpendicular lines at the point.

# Strain

Therefore, if the shear strain in the *xz* plane is $\gamma_{xz}$ then the angle between the displaced line elements O'A' and O'C' is




Now

$\cos A'O'C' = \cos(\pi/2-\gamma_{xz}) = \sin \gamma_{xz}$

and as $\gamma_{xz}$ is small then

$\cos A'O'C' = \gamma_{xz}$.

# Strain

From the trigonometrical relationships for a triangle

$$\cos \mathrm{A'O'C'} = \frac{(\mathrm{O'A'})^2 + (\mathrm{O'C'})^2 - (\mathrm{A'C'})^2}{2(\mathrm{O'A'})(\mathrm{O'C'})}$$

Previously it was shown

$$\mathrm{O'A'} = \delta x\left(1 + \frac{\partial u}{\partial x}\right) \qquad \mathrm{O'C'} = \delta z\left(1 + \frac{\partial w}{\partial z}\right)$$

But for small displacements the derivatives of *u, v* and *w* are small compared with 1, so that

$O'A' \approx \delta x$
$O'C' \approx \delta z$

To a first approximation $$(\mathrm{A'C'})^2 = \left(\delta z - \frac{\partial w}{\partial x}\delta x\right)^2 + \left(\delta x - \frac{\partial u}{\partial z}\delta z\right)^2$$

# Strain

Substituting for *O'A', O'C'* and *A'C'* in equation for *cos A'O'C'* we have

$$\cos \mathrm{A'O'C'} = \frac{(\delta x^2) + (\delta z)^2 - [\delta z - (\partial w/\partial x)\delta x]^2 - [\delta x - (\partial u/\partial z)\delta z]^2}{2\delta x \delta z}$$

Expanding and neglecting second-order powers gives

$$\cos \mathrm{A'O'C'} = \frac{2(\partial w/\partial x)\delta x \delta z + 2(\partial u/\partial z)\delta x \delta z}{2\delta x \delta z}$$

or

$$\boxed{\left.\begin{aligned} \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \end{aligned}\right\}}$$

**Strain equations** were derived on the assumption that the **displacements** involved **are small**

# Compatibility equations

The body remains *continuous* during the deformation so that no voids are formed. It follows that each component, ***u***, ***v*** and ***w***, must be a ***continuous, single-valued function***:

$$u = f_1(x, y, z)$$
$$v = f_2(x, y, z)$$
$$w = f_3(x, y, z)$$

If voids were formed then displacements in regions of the body separated by the voids would be expressed as *different functions* of $x$, $y$ and $z$.

The existence, therefore, of just three single-valued functions for displacement is an expression of the continuity or *compatibility* of displacement.

Since the **six strains** are defined in terms of three displacement functions then they **must bear some relationship to each other** and cannot have arbitrary values.

# Compatibility equations

Differentiating $\gamma_{xy}$ from equation

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

with respect to *x* and *y* gives

$$\frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = \frac{\partial^2}{\partial x\, \partial y}\frac{\partial v}{\partial x} + \frac{\partial^2}{\partial x\, \partial y}\frac{\partial u}{\partial y}$$

or, since the functions of *u* and *v* are continuous

$$\frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = \frac{\partial^2}{\partial x^2}\frac{\partial v}{\partial y} + \frac{\partial^2}{\partial y^2}\frac{\partial u}{\partial x}$$

which may be written, using equations $\varepsilon_x = \frac{\partial u}{\partial x}$ $\quad \varepsilon_y = \frac{\partial v}{\partial y}$

$$\boxed{\frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2}}$$

***Equations of compatibility***

In a similar manner

$$\boxed{\frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z} = \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2}} \qquad \boxed{\frac{\partial^2 \gamma_{xz}}{\partial x\, \partial z} = \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2}}$$

# Compatibility equations

If we now differentiate $\gamma_{xy}$ with respect to *x* and *z* and add the result to $\gamma_{zx}$, differentiated with respect to *y* and *x*, we obtain

$$\frac{\partial^2 \gamma_{xy}}{\partial x\,\partial z}+\frac{\partial^2 \gamma_{xz}}{\partial y\,\partial x}=\frac{\partial^2}{\partial x\,\partial z}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{\partial^2}{\partial y\,\partial x}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \gamma_{xy}}{\partial z}+\frac{\partial \gamma_{xz}}{\partial y}\right)=\frac{\partial^2}{\partial z\,\partial y}\frac{\partial u}{\partial x}+\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)+\frac{\partial^2}{\partial y\,\partial z}\frac{\partial u}{\partial x}$$

Substituting from equations $\varepsilon_x=\dfrac{\partial u}{\partial x}$ , $\gamma_{yz}=\dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}$ and rearranging

$$\boxed{2\frac{\partial^2 \varepsilon_x}{\partial y\,\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x}+\frac{\partial \gamma_{xz}}{\partial y}+\frac{\partial \gamma_{xy}}{\partial z}\right)}$$

Similarly

$$\boxed{2\frac{\partial^2 \varepsilon_z}{\partial x\,\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x}+\frac{\partial \gamma_{xz}}{\partial y}-\frac{\partial \gamma_{xy}}{\partial z}\right)}$$

$$\boxed{2\frac{\partial^2 \varepsilon_y}{\partial x\,\partial z}=\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x}-\frac{\partial \gamma_{xz}}{\partial y}+\frac{\partial \gamma_{xy}}{\partial z}\right)}$$

***Equations of compatibility***

# Current status

6 stresses,

6 strains and

3 displacements.

3 equations of equilibrium

6 strain–displacement relationships

3 compatibility equations

more 3 equations

stress-strain relationships

# Stress-strain relationship

Problem scope:

- **linearly elastic isotropic materials** for which stress is directly proportional to strain;
- elastic properties of the materials are the same in all directions (**homogeneous**).

The application of a uniform direct stress, say $\sigma_x$, does not produce any shear distortion of the material , then according to Hooke's Law

$$\varepsilon_x = \frac{\sigma_x}{E}$$

Further, $\varepsilon_x$ is accompanied by lateral strains

$$\varepsilon_y = -\nu \frac{\sigma_x}{E} \quad \varepsilon_z = -\nu \frac{\sigma_x}{E}$$

$\nu$ is a constant termed *Poisson's ratio*

# Stress-strain relationship

For a body subjected to direct stresses $\sigma_x$, $\sigma_y$ and $\sigma_z$ the direct strains are, from the *principle of superposition*

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)]$$
$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

rearrangement

where

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_x$$
$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_y$$
$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}e + \frac{E}{(1+\nu)}\varepsilon_z$$

Also, we know

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{xz} = \frac{\tau_{xz}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

*where G is the modulus of rigidity*

***G = E/2(1+v)***

# Basic equations of Solid Mechanics

Foundation

Fail

Gain

# Basic equations of Solid Mechanics

## Equilibrium equations:

$$\left.\begin{aligned}\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+X=0\\ \frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yz}}{\partial z}+Y=0\\ \frac{\partial \sigma_z}{\partial z}+\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zy}}{\partial y}+Z=0\end{aligned}\right\}$$

## Strain-deformation equations:

$$\left.\begin{aligned}\varepsilon_x=\frac{\partial u}{\partial x}\\ \varepsilon_y=\frac{\partial v}{\partial y}\\ \varepsilon_z=\frac{\partial w}{\partial z}\end{aligned}\right\}$$

$$\left.\begin{aligned}\gamma_{xz}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\\ \gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\\ \gamma_{yz}=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\end{aligned}\right\}$$

## Boundary conditions:

$$\left.\begin{aligned}\bar{X}=\sigma_x l+\tau_{yx} m+\tau_{zx} n\\ \bar{Y}=\sigma_y m+\tau_{xy} l+\tau_{zy} n\\ \bar{Z}=\sigma_z n+\tau_{yz} m+\tau_{xz} l\end{aligned}\right\}$$

## Stress-strain relationship:

$$\begin{aligned}\varepsilon_x&=\frac{1}{E}[\sigma_x-\nu(\sigma_y+\sigma_z)]\\ \varepsilon_y&=\frac{1}{E}[\sigma_y-\nu(\sigma_x+\sigma_z)]\\ \varepsilon_z&=\frac{1}{E}[\sigma_z-\nu(\sigma_x+\sigma_y)]\end{aligned}$$

$$\gamma_{xy}=\frac{\tau_{xy}}{G}\quad \gamma_{xz}=\frac{\tau_{xz}}{G}\quad \gamma_{yz}=\frac{\tau_{yz}}{G}$$

## Compatibility equations:

$$\frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y}=\frac{\partial^2 \varepsilon_y}{\partial x^2}+\frac{\partial^2 \varepsilon_x}{\partial y^2}$$

$$\frac{\partial^2 \gamma_{yz}}{\partial y\,\partial z}=\frac{\partial^2 \varepsilon_y}{\partial z^2}+\frac{\partial^2 \varepsilon_z}{\partial y^2}$$

$$\frac{\partial^2 \gamma_{xz}}{\partial x\,\partial z}=\frac{\partial^2 \varepsilon_z}{\partial x^2}+\frac{\partial^2 \varepsilon_x}{\partial z^2}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y\,\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x}+\frac{\partial \gamma_{xz}}{\partial y}+\frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x\,\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x}+\frac{\partial \gamma_{xz}}{\partial y}-\frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial x\,\partial z}=\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x}-\frac{\partial \gamma_{xz}}{\partial y}+\frac{\partial \gamma_{xy}}{\partial z}\right)$$

# Plane stress

Most **aircraft structural components** are fabricated from thin metal sheet so that stresses across the thickness of the sheet are usually negligible.

$$\left.\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \\
\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + Z = 0
\end{aligned}\right\}
\xrightarrow[\tau_{yz} = 0]{\sigma_z = 0,\ \tau_{xz} = 0}
\left.\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \\
\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + Y = 0
\end{aligned}\right\}$$

The condition (case) of **plane stress**

$$\begin{aligned}
\varepsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\
\varepsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \\
\varepsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]
\end{aligned}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \quad \gamma_{xz} = \frac{\tau_{xz}}{G} \quad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\xrightarrow[\tau_{yz} = 0]{\sigma_z = 0,\ \tau_{xz} = 0}
\left.\begin{aligned}
\varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) \\
\varepsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) \\
\varepsilon_z &= \frac{-\nu}{E}(\sigma_x - \sigma_y) \\
\gamma_{xy} &= \frac{\tau_{xy}}{G}
\end{aligned}\right\}
\xrightarrow{\text{rearrangement}}
\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \\
\sigma_y &= \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)
\end{aligned}$$

# Plane stress

Look
example on whiteboard
(Example 2-1 in Moodle)

# Plane strain

The state of strain, in which it is assumed that particles of the body suffer displacements in one plane only, is known as plane strain (let it be plane *xy*), then

$\boldsymbol{\varepsilon_z = 0}$

$\boldsymbol{\gamma_{xz} = 0}$

$\boldsymbol{\gamma_{yz} = 0}$

and equations for longitudinal and shear strain reduce to

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y}} \qquad \boxed{\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}}$$

By substituting $\boldsymbol{\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0}$ in the six equations of compatibility and noting that $\varepsilon_x$, $\varepsilon_y$ and $\gamma_{xy}$ are now purely functions of *x* and *y*

$$\boxed{\frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2}}$$

# Obrigado!