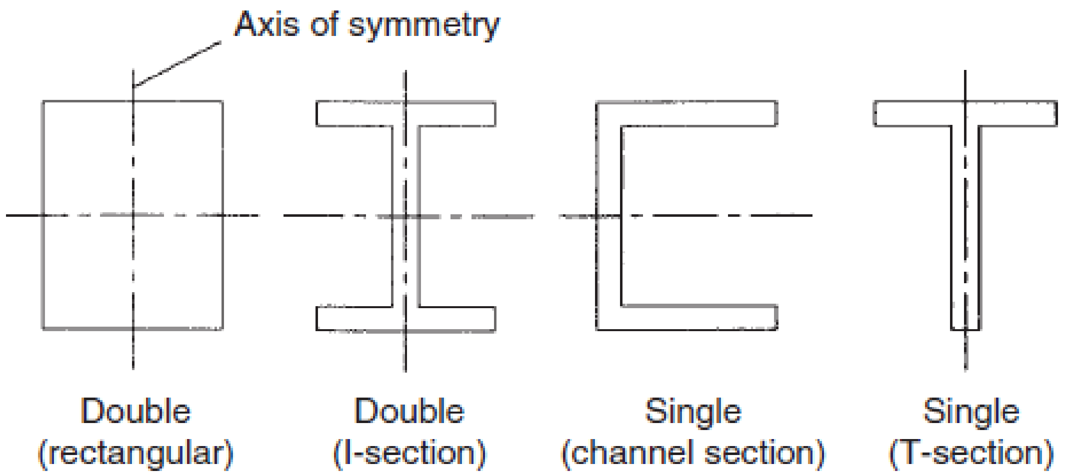
Bending of thin walled beams

2013

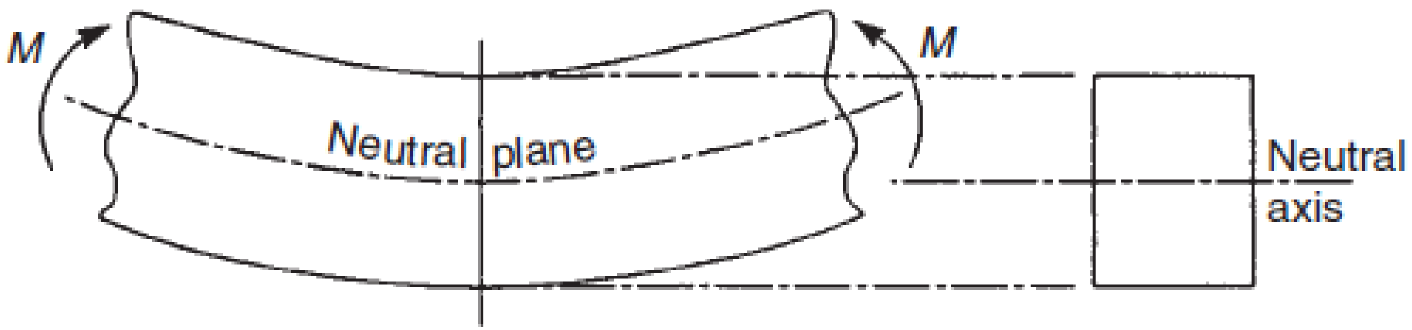
Symmetrical bending

Symmetrical bending arises in beams which have either singly or doubly symmetrical cross-sections.

The direct stress varies through the depth of the beam from compression in the upper fibres to tension in the lower.



The direct stress is zero for the fibres that do not change in length; i.e. in ***neutral plane***. The line of intersection of the neutral plane and any cross-section of the beam is termed the ***neutral axis***.

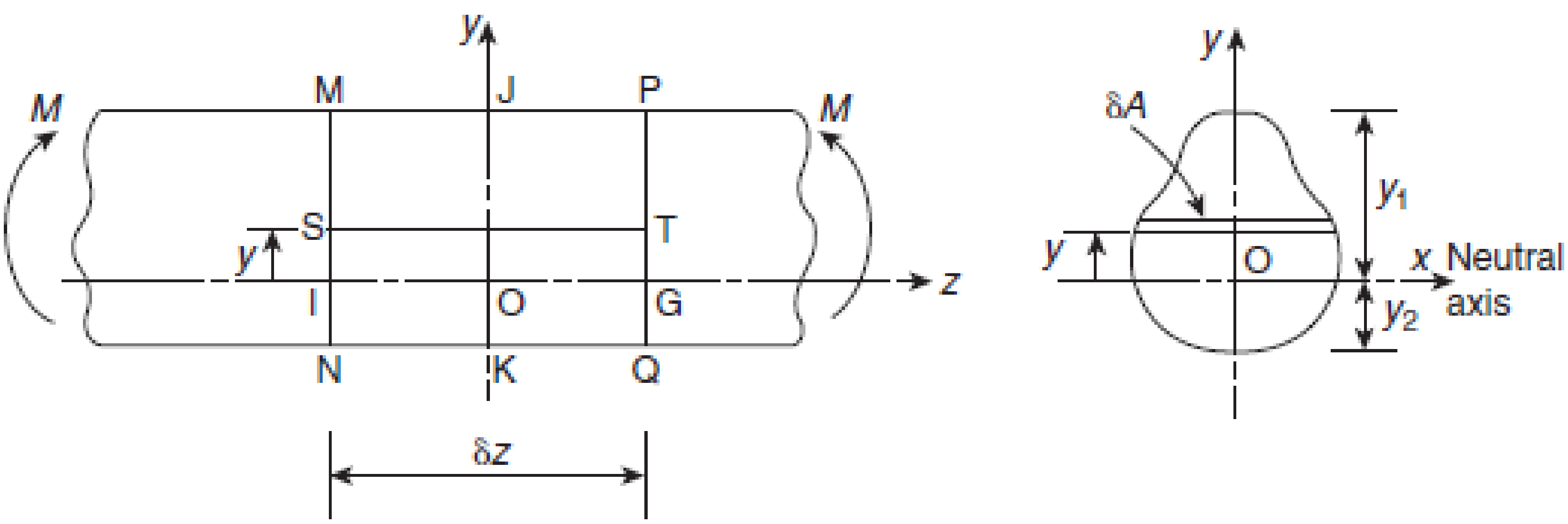


Symmetrical bending

Assumptions for pure bending:

- plane cross-sections of the beam remain plane and normal to the longitudinal fibres of the beam after bending;
- the material of the beam is linearly elastic (obeys Hooke's law),
- the material of the beam is homogeneous.

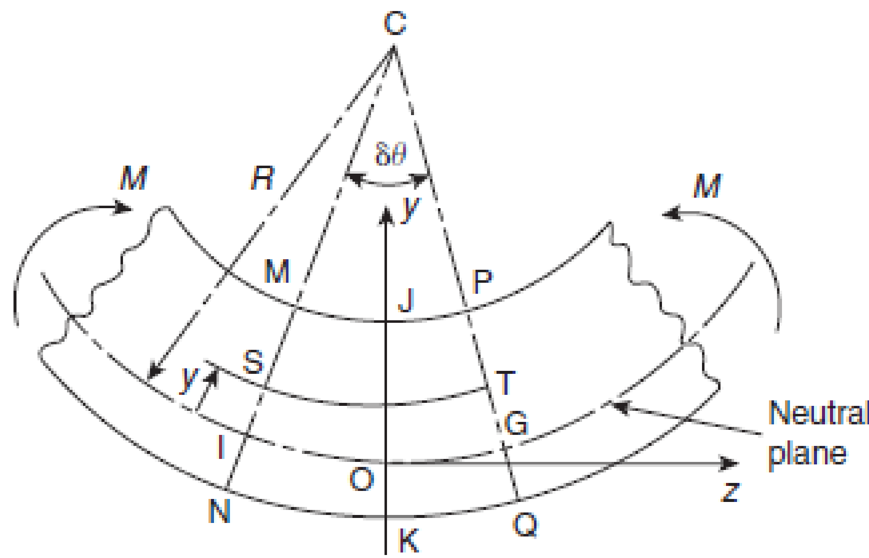
Consider a beam subjected to a pure, sagging bending moment, ***M***:



Before bending $MP = IG = ST = NQ = \delta z$

Symmetrical bending

The bending moment ***M*** causes the length of beam to bend about a centre of curvature ***C.***



The curved shape of the beam is circular with a radius of curvature ***R*** measured to the neutral plane.

Plane sections ***MIN*** and ***PGQ*** remain plane as we have demonstrated but are now inclined at an angle ***δϑ*** to each other.

The fibre ***ST*** has a strain $\boldsymbol{\varepsilon_z}$ $\quad \varepsilon_z = \dfrac{\text{change in length}}{\text{original length}}$ or $\quad \varepsilon_z = \dfrac{(R-y)\delta\theta - \delta z}{\delta z}$

i.e. $\quad \varepsilon_z = \dfrac{(R-y)\delta\theta - R\delta\theta}{R\delta\theta}$ or $\quad \varepsilon_z = -\dfrac{y}{R}$

The negative sign indicates that fibres in the region where ***y*** is positive will shorten when the ***M*** is negative.

Optional content

Symmetrical bending

The direct stress $\boldsymbol{\sigma_z}$ in the fibre ***ST*** is given by $\sigma_z = -E\frac{y}{R}$

There are no axial load (the stress is due to bending), then $\int_A \sigma_z \, \mathrm{d}A = 0$

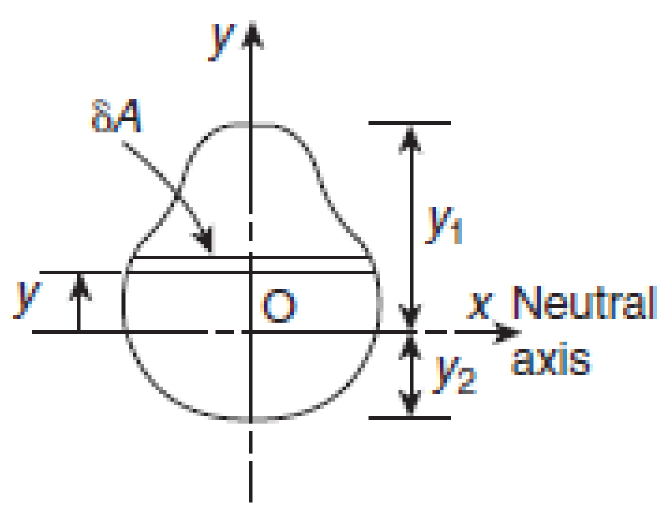
where ***A*** is the area of the beam cross-section.

$$\int_A \sigma_z \, \mathrm{d}A = 0 \xrightarrow{\sigma_z = -E\frac{y}{R}} -\frac{E}{R}\int_A y \, \mathrm{d}A = 0 \longrightarrow \int_A y \, \mathrm{d}A = 0$$ i.e.

the neutral axis passes through the centroid of area of the cross-section.

Consider the elemental strip ***δA*** (the cross-section of the fibre ***ST***), on which acts a compressive load ***(Ey/R)δA***.

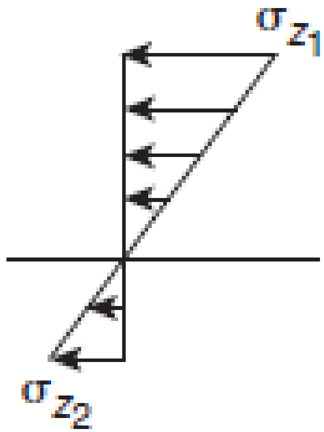
The moment resultant about the neutral axis of the stresses on all fibres must be equivalent to the applied negative moment ***M***.



Optional content

Symmetrical bending

$$M = -\int_A E\frac{y^2}{R}\mathrm{d}A \Rightarrow M = -\frac{E}{R}\int_A y^2\mathrm{d}A$$



$\int_A y^2\,\mathrm{d}A$ is the second moment of area of the cross-section of the beam about the neutral axis

$$M = -\frac{EI}{R} \Rightarrow \frac{M}{I} = -\frac{E}{R} = \frac{\sigma_z}{y} \Rightarrow \boxed{\sigma_z = \frac{My}{I}}$$

Sing convention for $\boldsymbol{\sigma_z}$ at positive bending moment:

- positive, i.e. tensile, when ***y*** is positive;
- negative, i.e. compressive, when ***y*** is negative.

The curvature, ***1/R***, of the beam is given by

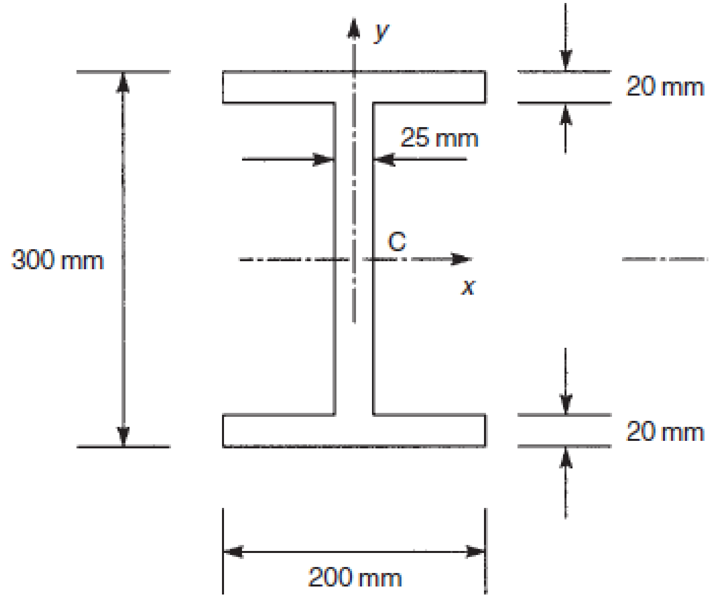
$$\frac{1}{R} = \frac{M}{EI}$$

Optional content

Symmetrical bending

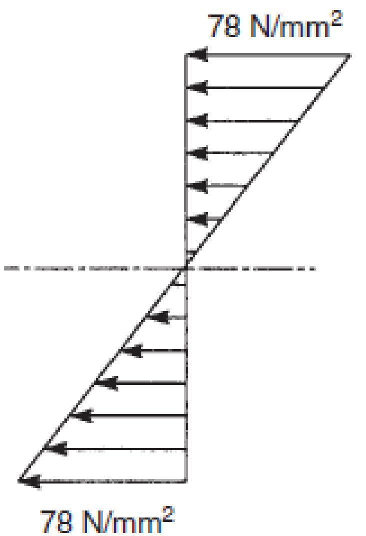
The beam with a given cross-section is subjected to a negative bending moment of 100 kNm applied in a vertical plane. Find stress distribution.

The neutral axis of the beam section pass through centroid of the beam.



$$I_{xx} = \frac{200 \times 300^3}{12} - \frac{175 \times 260^3}{12} = 193.7 \times 10^6 \text{ mm}^4$$

The direct stress $\sigma_z = \dfrac{My}{I}$ or $\sigma_z = -\dfrac{100 \times 10^6}{193.7 \times 10^6} y = -0.52y$



Maximum stresses are

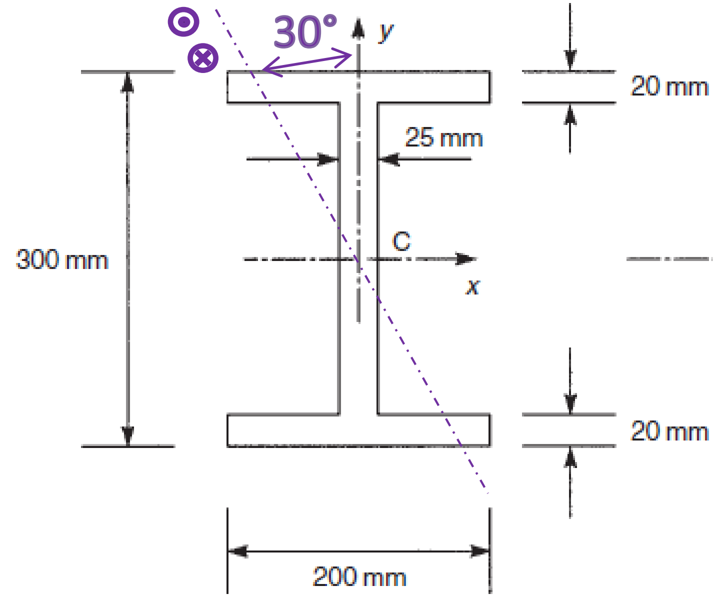
$$-0.52 \times (+150) = -78 \text{ N/mm}^2 \text{ (compression)}$$

$$-0.52 \times (-150) = +78 \text{ N/mm}^2 \text{ (tension)}$$

Optional content

Symmetrical bending

The beam with a given cross-section is subjected to a negative bending moment of 100 kNm applied plane inclined at 30° with respect to vertical.



Resolve moments in a vertical and a horizontal planes correspondingly:

$$\left.\begin{aligned} M_x &= 100\cos 30^\circ = 86.6\ \text{kN m} \\ M_y &= 100\sin 30^\circ = 50.0\ \text{kN m} \end{aligned}\right\}$$

Respective stresses: $\sigma_z = \dfrac{M_x}{I_{xx}}y \quad \sigma_z = \dfrac{M_y}{I_{yy}}x$ the total stress: $\sigma_z = \dfrac{M_x}{I_{xx}}y + \dfrac{M_y}{I_{yy}}x$

M_x is a positive bending moment producing tension in the upper half of the beam where ***y*** is positive.

M_y is a negative bending moment producing tension in the left-hand half of the beam where ***x*** is negative.

Optional content

Symmetrical bending

According to sign convention the total stress:

$$\sigma_z = \frac{86.6 \times 10^6}{193.7 \times 10^6} y - \frac{50.0 \times 10^6}{27.0 \times 10^6} x \quad \Rightarrow \quad \sigma_z = 0.45y - 1.85x$$

The value of stress in different points of the cross-section:

- at the top left corner of the top flange

  y = +150 mm, x = −100 mm

  $\boldsymbol{\sigma_z}$ **= +252.5** N/mm^2 (tension)

- at the top right corner of the top flange

  y = +150 mm, x = +100 mm

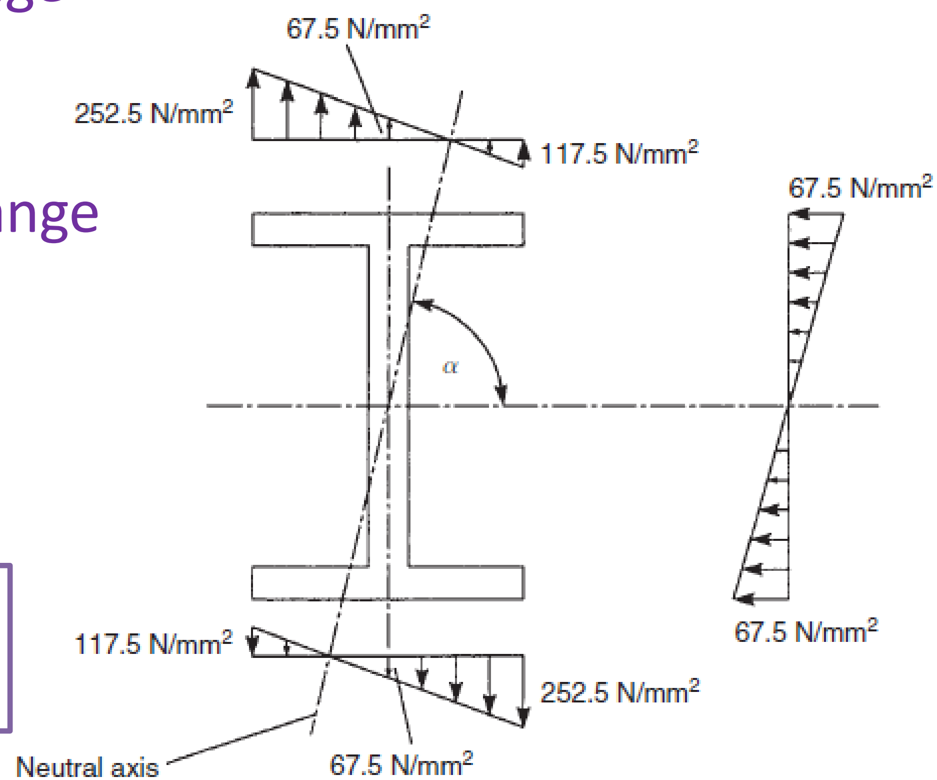
  $\boldsymbol{\sigma_z}$ **= -117.5** N/mm^2 (compression)

Position of the neutral axis at $\boldsymbol{\sigma_z = 0}$

$$0 = 0.45y - 1.85x$$

$$\frac{y}{x} = \frac{1.85}{0.45} = 4.11 = \tan\alpha \qquad \boxed{\tan\alpha = \frac{M_y I_{xx}}{M_x I_{yy}}}$$

$$\alpha = 76.3^\circ$$

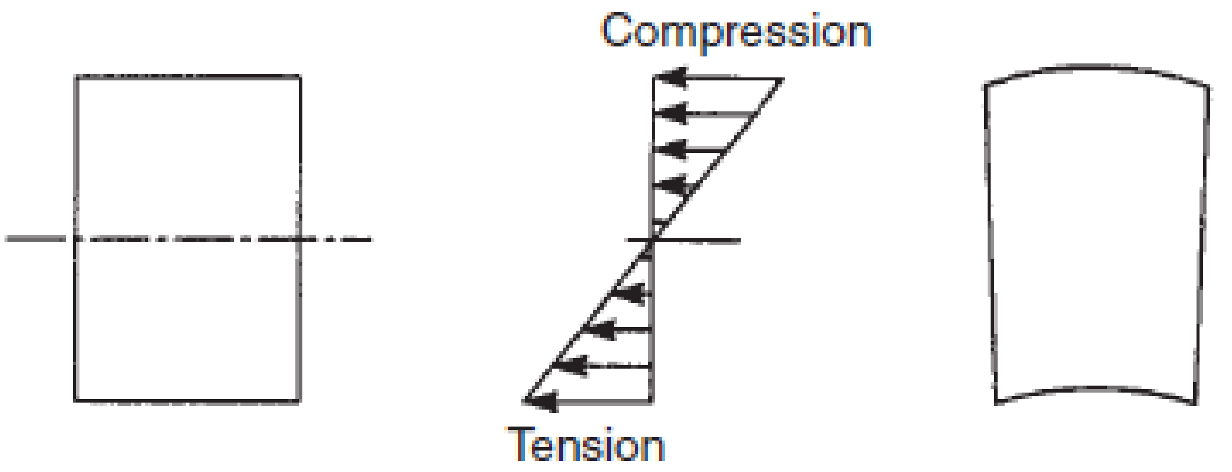


Anticlastic bending

Due to the Poisson effect

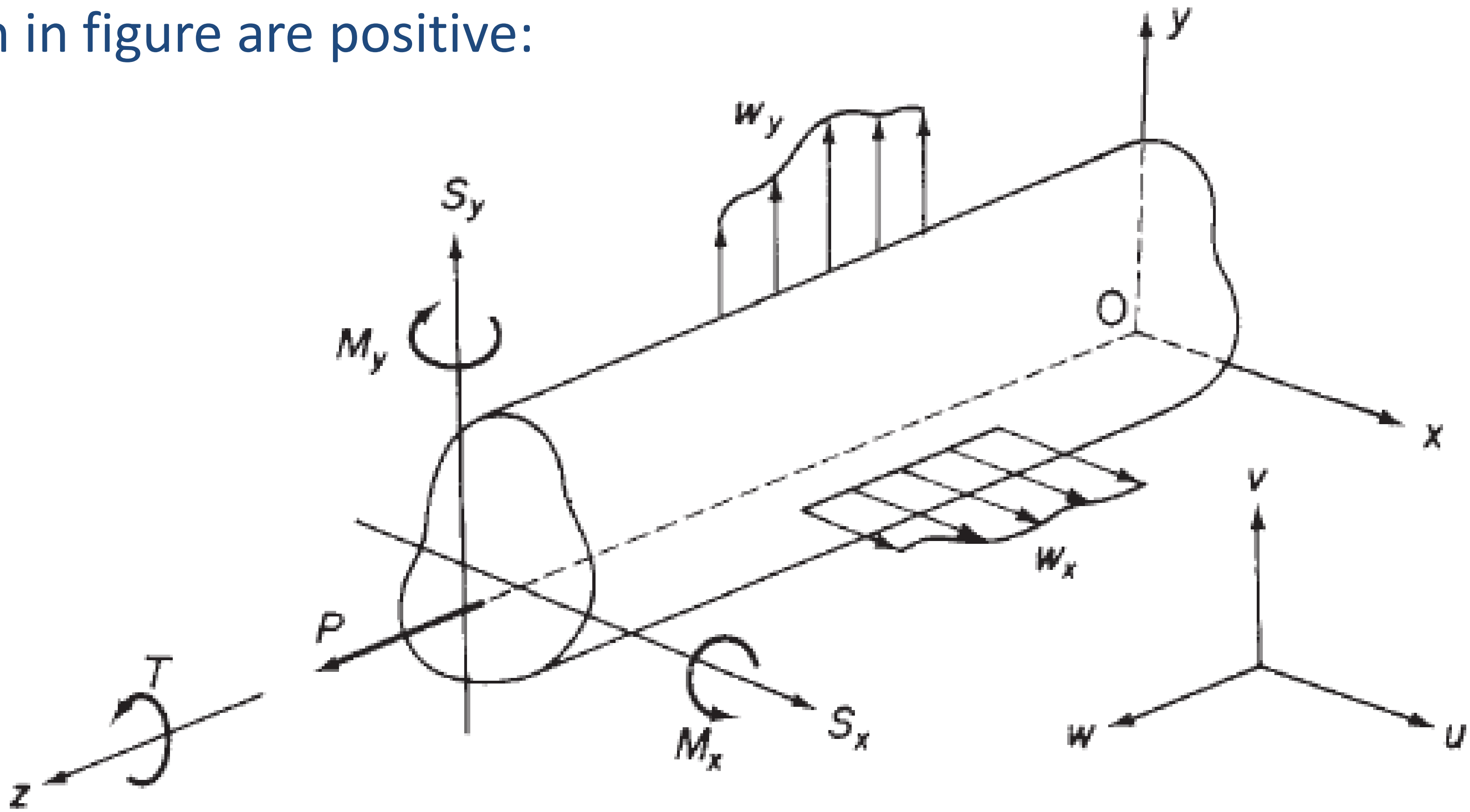
- the compressive stress produces a lateral elongation of the upper fibres of the beam section
- the tensile stress produces a lateral contraction of the lower.

The section does not therefore remain rectangular but distorts as shown in Figure:



Unsymmetrical bending

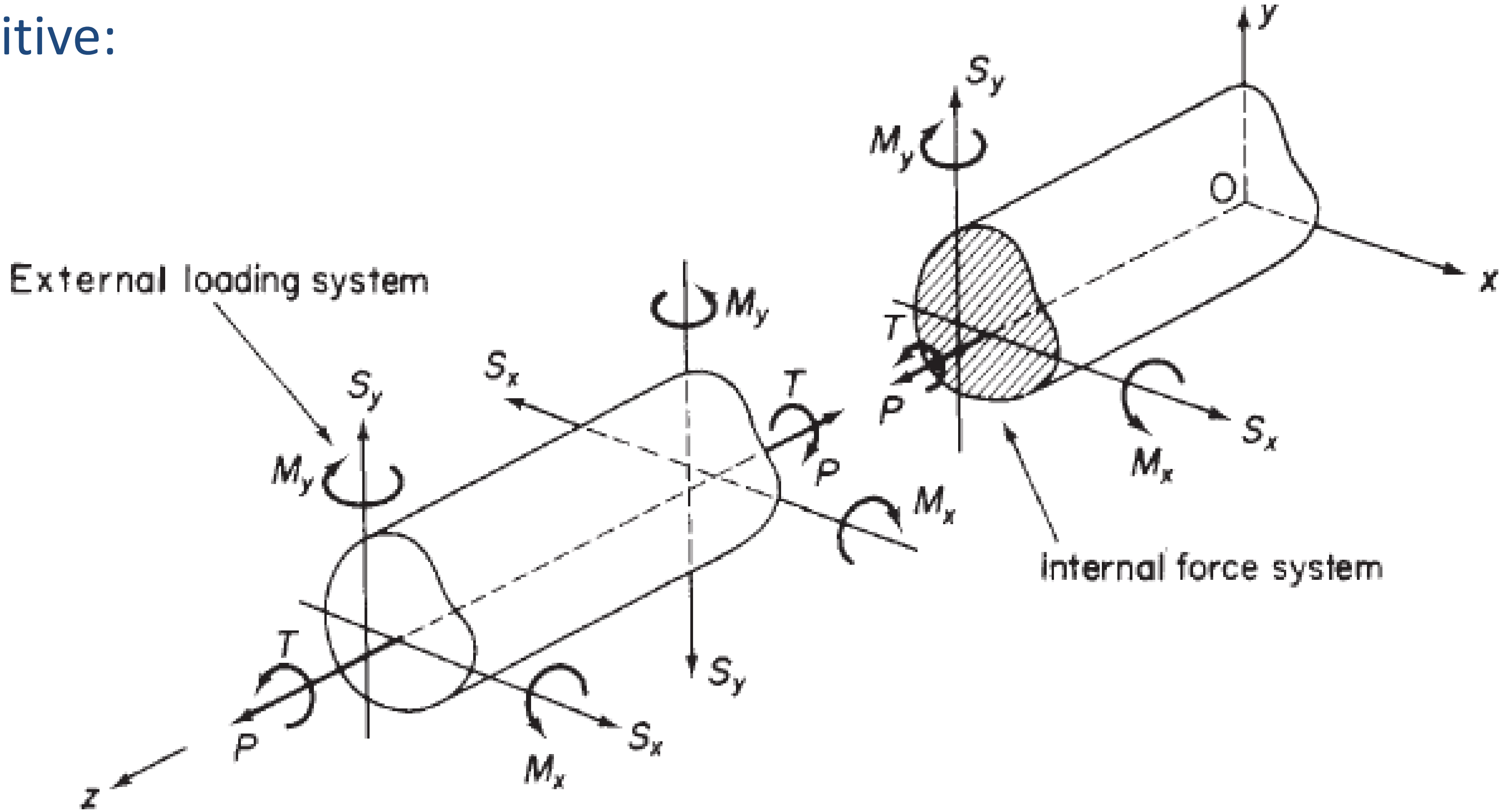
Sign convention. All directions for <u>external</u> loads and displacements given in figure are positive:



also bending moments $\boldsymbol{M_x}$ and $\boldsymbol{M_y}$ are positive when they induce tension in the positive ***xy*** quadrant of the beam cross-section.

Unsymmetrical bending

Sign convention. All directions for <u>internal</u> loads given in figure are positive:



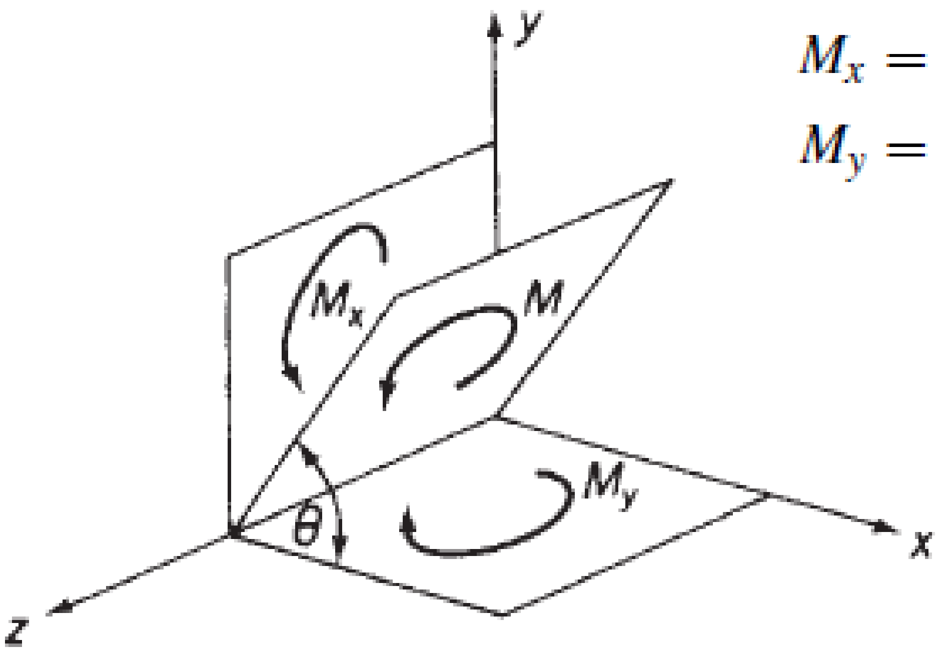
Positive internal forces and moments are in the same direction and sense as the externally applied loads (viewed in ***zO***-direction)

Unsymmetrical bending

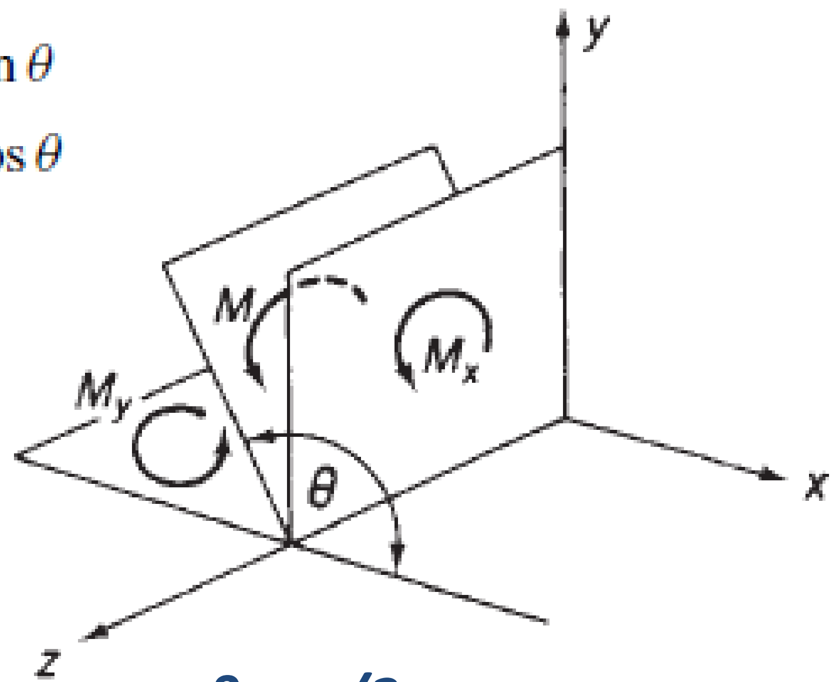
A bending moment ***M*** applied in any longitudinal plane parallel to the ***z*** axis may be resolved into components $\boldsymbol{M_x}$ and $\boldsymbol{M_y}$ by the normal rules of vectors.

$$M_x = M \sin\theta$$

$$M_y = M \cos\theta$$



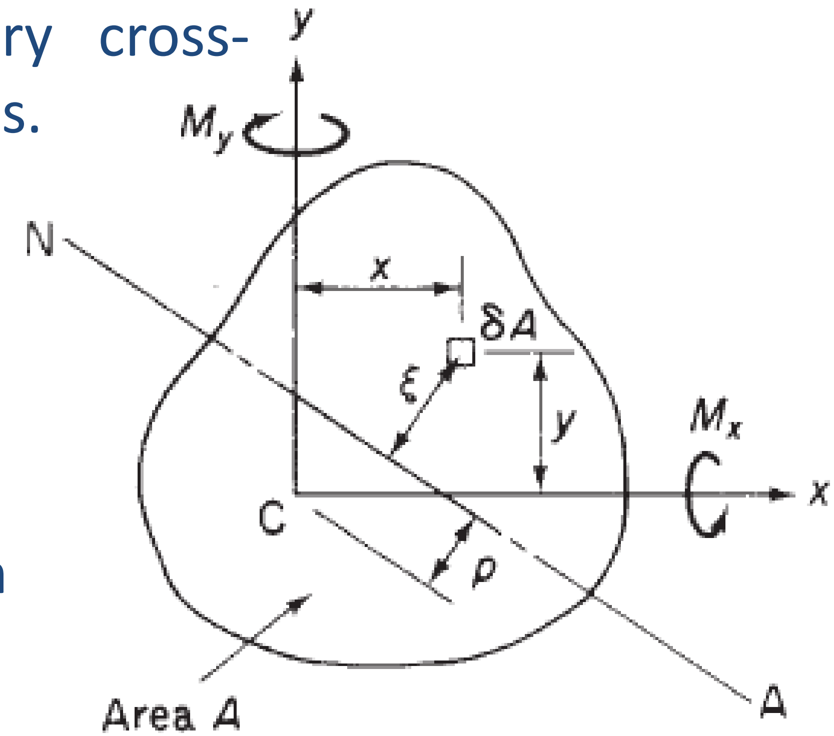
ϑ < π/2
$\boldsymbol{M_x}$ *positive*
$\boldsymbol{M_y}$ *positive*



ϑ > π/2
$\boldsymbol{M_x}$ *positive*
$\boldsymbol{M_y}$ *negative*

Unsymmetrical bending

Consider a beam having the arbitrary cross-section under action of bending moments.



The direct stress $\boldsymbol{\sigma_z}$ on an element of area $\boldsymbol{\delta A}$

$$\varepsilon_z = \frac{\xi}{\rho}$$

$$\sigma_z = E\varepsilon_z \Rightarrow \sigma_z = \frac{E\xi}{\rho}$$

The resultant normal load on any section must be zero:

$$\int_A \sigma_z \, \mathrm{d}A = 0 \Rightarrow \int_A \xi \, \mathrm{d}A = 0$$ i.e.

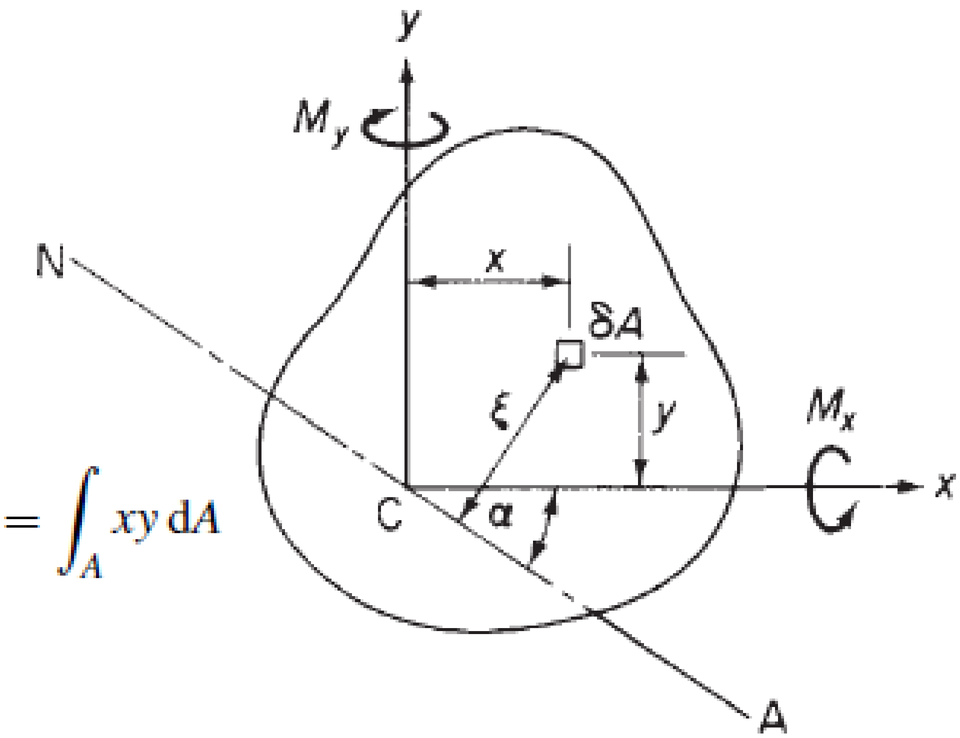
the neutral axis passes through the centroid of the cross-section.

Unsymmetrical bending

If the inclination equal to $\boldsymbol{\alpha}$, then $\xi = x\sin\alpha + y\cos\alpha$ and direct stress:

$$\sigma_z = \frac{E}{\rho}(x\sin\alpha + y\cos\alpha)$$

The moment resultants of the internal direct stress:



$$M_x = \int_A \sigma_z y\,\mathrm{d}A, \quad M_y = \int_A \sigma_z x\,\mathrm{d}A$$

$$I_{xx} = \int_A y^2\,\mathrm{d}A, \quad I_{yy} = \int_A x^2\,\mathrm{d}A, \quad I_{xy} = \int_A xy\,\mathrm{d}A$$

$$\sigma_z = \frac{E}{\rho}(x\sin\alpha + y\cos\alpha)$$

$$M_x = \frac{E\sin\alpha}{\rho}I_{xy} + \frac{E\cos\alpha}{\rho}I_{xx}, \quad M_y = \frac{E\sin\alpha}{\rho}I_{yy} + \frac{E\cos\alpha}{\rho}I_{xy}$$

or in matrix form: $\begin{Bmatrix} M_x \\ M_y \end{Bmatrix} = \frac{E}{\rho}\begin{bmatrix} I_{xy} & I_{xx} \\ I_{yy} & I_{xy} \end{bmatrix}\begin{Bmatrix} \sin\alpha \\ \cos\alpha \end{Bmatrix}$

Unsymmetrical bending

$$\begin{Bmatrix} M_x \\ M_y \end{Bmatrix} = \frac{E}{\rho} \begin{bmatrix} I_{xy} & I_{xx} \\ I_{yy} & I_{xy} \end{bmatrix} \begin{Bmatrix} \sin\alpha \\ \cos\alpha \end{Bmatrix} \Rightarrow \frac{E}{\rho} \begin{Bmatrix} \sin\alpha \\ \cos\alpha \end{Bmatrix} = \begin{bmatrix} I_{xy} & I_{xx} \\ I_{yy} & I_{xy} \end{bmatrix}^{-1} \begin{Bmatrix} M_x \\ M_y \end{Bmatrix} \Rightarrow$$

$$\frac{E}{\rho} \begin{Bmatrix} \sin\alpha \\ \cos\alpha \end{Bmatrix} = \frac{1}{I_{xx}I_{yy} - I_{xy}^2} \begin{bmatrix} -I_{xy} & I_{xx} \\ I_{yy} & -I_{xy} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \end{Bmatrix}$$

The direct stress:

$$\sigma_z = \frac{E}{\rho}(x\sin\alpha + y\cos\alpha) \qquad \sigma_z = \left(\frac{M_y I_{xx} - M_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)x + \left(\frac{M_x I_{yy} - M_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)y$$

$$\boxed{\sigma_z = \frac{M_x(I_{yy}y - I_{xy}x)}{I_{xx}I_{yy} - I_{xy}^2} + \frac{M_y(I_{xx}x - I_{xy}y)}{I_{xx}I_{yy} - I_{xy}^2}}$$

If ***M_y = 0,*** the ***M_x*** produces a stress which varies with both ***x*** and ***y***.

If ***Cx*** or ***Cy*** are axes of symmetry and ***I_{xy} = 0*** *and* ***Cxy*** are principal axes.

Then $\sigma_z = \dfrac{M_x}{I_{xx}}y + \dfrac{M_y}{I_{yy}}x$

M_x = 0 $\Rightarrow$ $\sigma_z = \dfrac{M_y}{I_{yy}}x$

M_y = 0 $\Rightarrow$ $\sigma_z = \dfrac{M_x}{I_{xx}}y$

symmetrical case

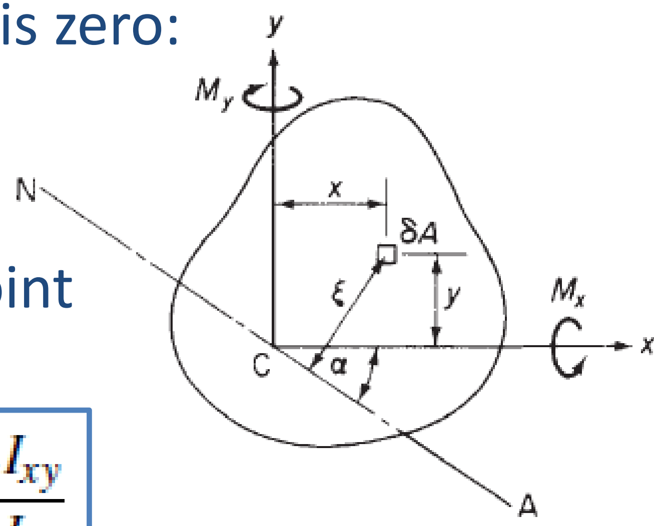
Unsymmetrical bending

At pure bending:

- the neutral axis always passes through the centroid;
- its inclination $\boldsymbol{\alpha}$ depends on
  - the form of the applied loading and
  - the geometrical properties of the beam's cross-section.

At all points on the neutral axis the direct stress is zero:

$$0 = \left(\frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) x_{\mathrm{NA}} + \left(\frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) y_{\mathrm{NA}}$$



where $\boldsymbol{x_{NA}}$ and $\boldsymbol{y_{NA}}$ are the coordinates of any point on the neutral axis.

$$\frac{y_{\mathrm{NA}}}{x_{\mathrm{NA}}} = -\frac{M_y I_{xx} - M_x I_{xy}}{M_x I_{yy} - M_y I_{xy}} \quad \Rightarrow \quad \boxed{\tan\alpha = \frac{M_y I_{xx} - M_x I_{xy}}{M_x I_{yy} - M_y I_{xy}}}$$

$\boldsymbol{\alpha}$ is positive, then $\boldsymbol{x}$ and $\boldsymbol{y}$ of the opposite sign;

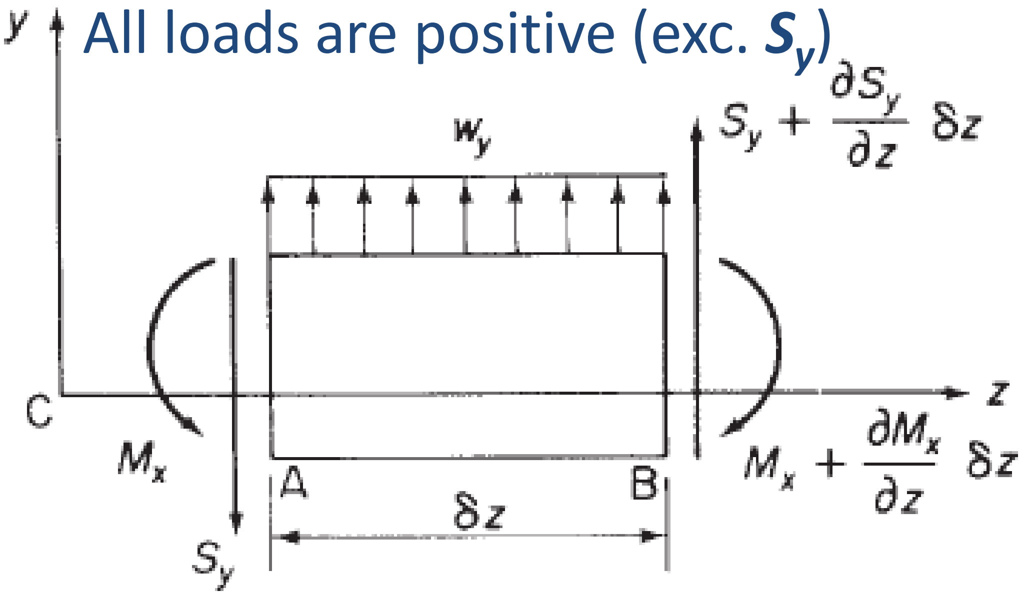
$\boldsymbol{\alpha}$ is negative, then $\boldsymbol{x}$ and $\boldsymbol{y}$ of the same sign.

Unsymmetrical bending

Consider an element of length ***δz*** of a beam of unsymmetrical cross-section subjected in the ***yz*** plane to:

- shear forces,
- bending moments and
- a distributed load.

All loads are positive (exc. S_y)



Equilibrium in the ***y***-direction:

$$\left(S_y + \frac{\partial S_y}{\partial z}\delta z\right) + w_y\delta z - S_y = 0$$ from which $$\boxed{w_y = -\frac{\partial S_y}{\partial z}}$$

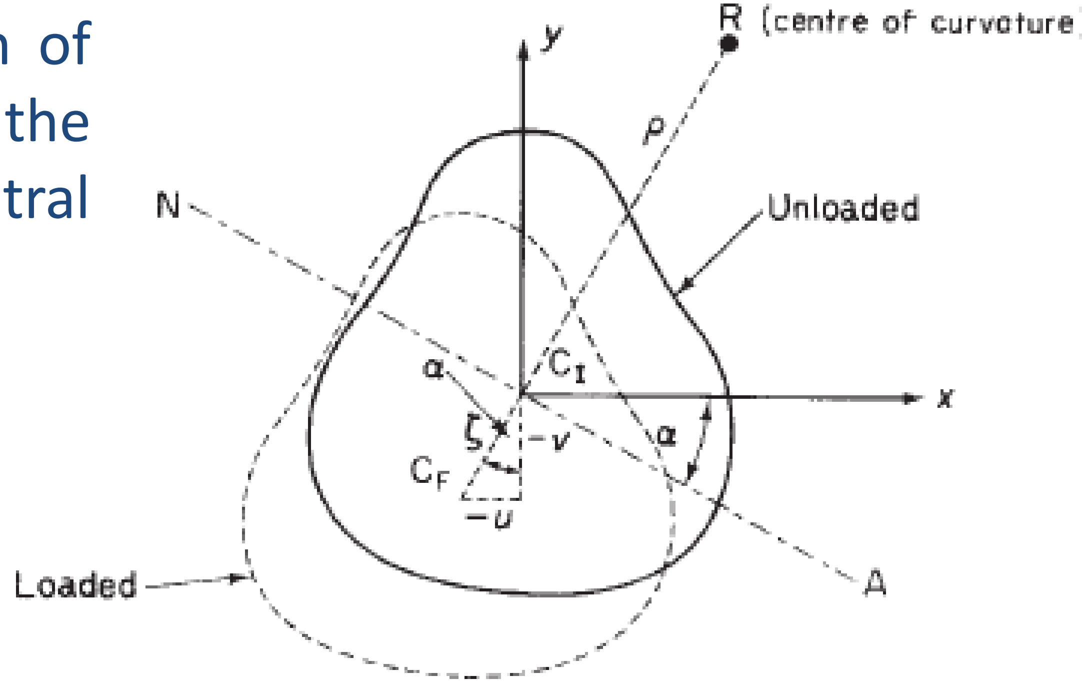
Taking moments about ***A***:

$$\left(M_x + \frac{\partial M_x}{\partial z}\delta z\right) - \left(S_y + \frac{\partial S_y}{\partial z}\delta z\right)\delta z - w_y\frac{(\delta z)^2}{2} - M_x = 0$$ from which $$\boxed{S_y = \frac{\partial M_x}{\partial z}}$$

Combining results we have: $$\boxed{-w_y = \frac{\partial S_y}{\partial z} = \frac{\partial^2 M_x}{\partial z^2}}$$

Deflections due to bending

Suppose that at some section of an unsymmetrical beam the deflection normal to the neutral axis is $\boldsymbol{\zeta}$: $\quad \dfrac{1}{\rho} = \dfrac{\mathrm{d}^2\zeta}{\mathrm{d}z^2}$



Displacement of the centroid:

$$u = -\zeta \sin\alpha \qquad v = -\zeta\cos\alpha$$

Differentiating twice

Substituting for $\boldsymbol{\zeta}$

$$\frac{\sin\alpha}{\rho} = -\frac{\mathrm{d}^2 u}{\mathrm{d}z^2}, \qquad \frac{\cos\alpha}{\rho} = -\frac{\mathrm{d}^2 v}{\mathrm{d}z^2}$$

$$\frac{1}{\rho}\begin{Bmatrix}\sin\alpha\\ \cos\alpha\end{Bmatrix} = \frac{1}{E(I_{xx}I_{yy}-I_{xy}^2)}\begin{bmatrix}-I_{xy} & I_{xx}\\ I_{yy} & -I_{xy}\end{bmatrix}\begin{Bmatrix}M_x\\ M_y\end{Bmatrix}$$

$$\frac{\sin\alpha}{\rho} = -\frac{\mathrm{d}^2 u}{\mathrm{d}z^2}, \qquad \frac{\cos\alpha}{\rho} = -\frac{\mathrm{d}^2 v}{\mathrm{d}z^2}$$

$$\begin{Bmatrix}u''\\ v''\end{Bmatrix} = \frac{-1}{E(I_{xx}I_{yy}-I_{xy}^2)}\begin{bmatrix}-I_{xy} & I_{xx}\\ I_{yy} & -I_{xy}\end{bmatrix}\begin{Bmatrix}M_x\\ M_y\end{Bmatrix}$$

Deflections due to bending

$$\begin{Bmatrix} u'' \\ v'' \end{Bmatrix} = \frac{-1}{E(I_{xx}I_{yy} - I_{xy}^2)} \begin{bmatrix} -I_{xy} & I_{xx} \\ I_{yy} & -I_{xy} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \end{Bmatrix} \xrightarrow{\text{Rearranging}}$$

$$\boxed{\begin{Bmatrix} M_x \\ M_y \end{Bmatrix} = -E \begin{bmatrix} I_{xy} & I_{xx} \\ I_{yy} & I_{xy} \end{bmatrix} \begin{Bmatrix} u'' \\ v'' \end{Bmatrix}} \quad \text{i.e.} \quad \boxed{\begin{matrix} M_x = -EI_{xy}u'' - EI_{xx}v'' \\ M_y = -EI_{yy}u'' - EI_{xy}v'' \end{matrix}}$$

M_x produces curvatures, i.e. deflections, in both the ***xz*** and ***yz*** planes even though ***$M_y = 0$***.

Thus, an unsymmetrical beam will deflect both vertically and horizontally even though the loading is entirely in a vertical plane.

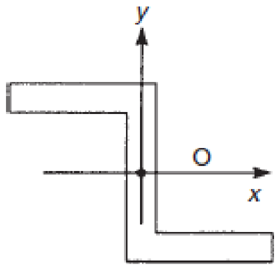
For particular case:

$$\begin{Bmatrix} u'' \\ v'' \end{Bmatrix} = \frac{-1}{E(I_{xx}I_{yy} - I_{xy}^2)} \begin{bmatrix} -I_{xy} & I_{xx} \\ I_{yy} & -I_{xy} \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \end{Bmatrix} \xrightarrow{I_{xy} = 0} \boxed{u'' = -\frac{M_y}{EI_{yy}}, \quad v'' = -\frac{M_x}{EI_{xx}}}$$

Calculation of thin-wall section properties

Product second moment of area

$$\boxed{I_{xy} = \int_A xy \, \mathrm{d}A}$$



Properties:

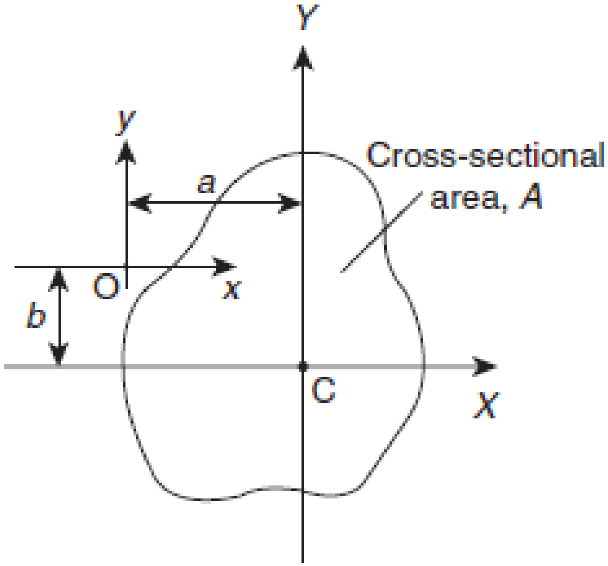
•can be negative (if area lies predominantly in the ***II*** and ***IV*** quadrants);

•equal to zero for cross-section having at least one axis of symmetry;

•parallel axes theorem: $I_{xy} = I_{XY} + abA$

where $\boldsymbol{I_{XY}}$ – product second moment of area about axes ***CXY*** passed through the centroid;

$\boldsymbol{I_{xy}}$ - product second moment of area about some axes ***Oxy***.



If CX or CY is axis of symmetry, then $I_{XY} = 0$ and $\boxed{I_{xy} = abA}$

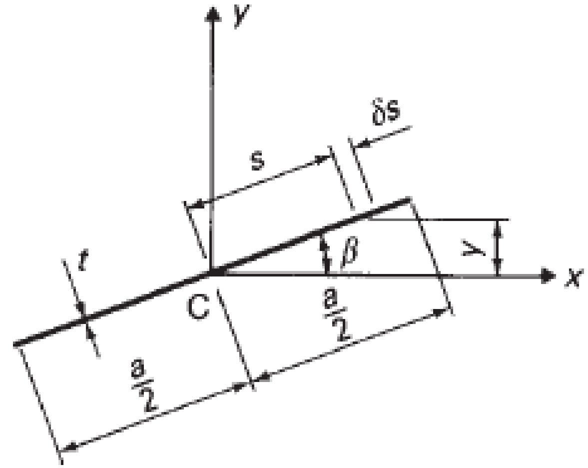
Calculation of thin-wall section properties

Optional content

Section properties for ***the inclined thin section***

$$I_{xx} = 2\int_0^{a/2} ty^2\,ds = 2\int_0^{a/2} t(s\sin\beta)^2\,ds \quad \text{or}$$

$$\boxed{I_{xx} = \frac{a^3 t\sin^2\beta}{12}} \quad \boxed{I_{yy} = \frac{a^3 t\cos^2\beta}{12}}$$



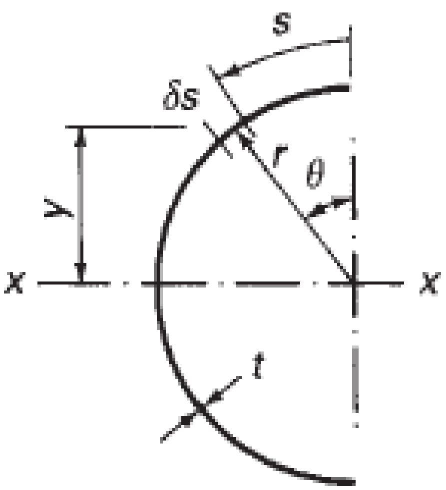
Product second moment of area

$$I_{xy} = 2\int_0^{a/2} txy\,ds = 2\int_0^{a/2} t(s\cos\beta)(s\sin\beta)\,ds \quad \text{or} \quad \boxed{I_{xy} = \frac{a^3 t\sin 2\beta}{24}}$$

Section properties for ***the thin-walled curved section***

$$I_{xx} = \int_0^{\pi r} ty^2\,ds \xrightarrow[ds\,=\,r\,d\vartheta]{y\,=\,r\cos\vartheta} I_{xx} = \int_0^{\pi} t(r\cos\theta)^2 r\,d\theta$$

$$\boxed{I_{xx} = \frac{\pi r^3 t}{2}}$$



Obrigado!