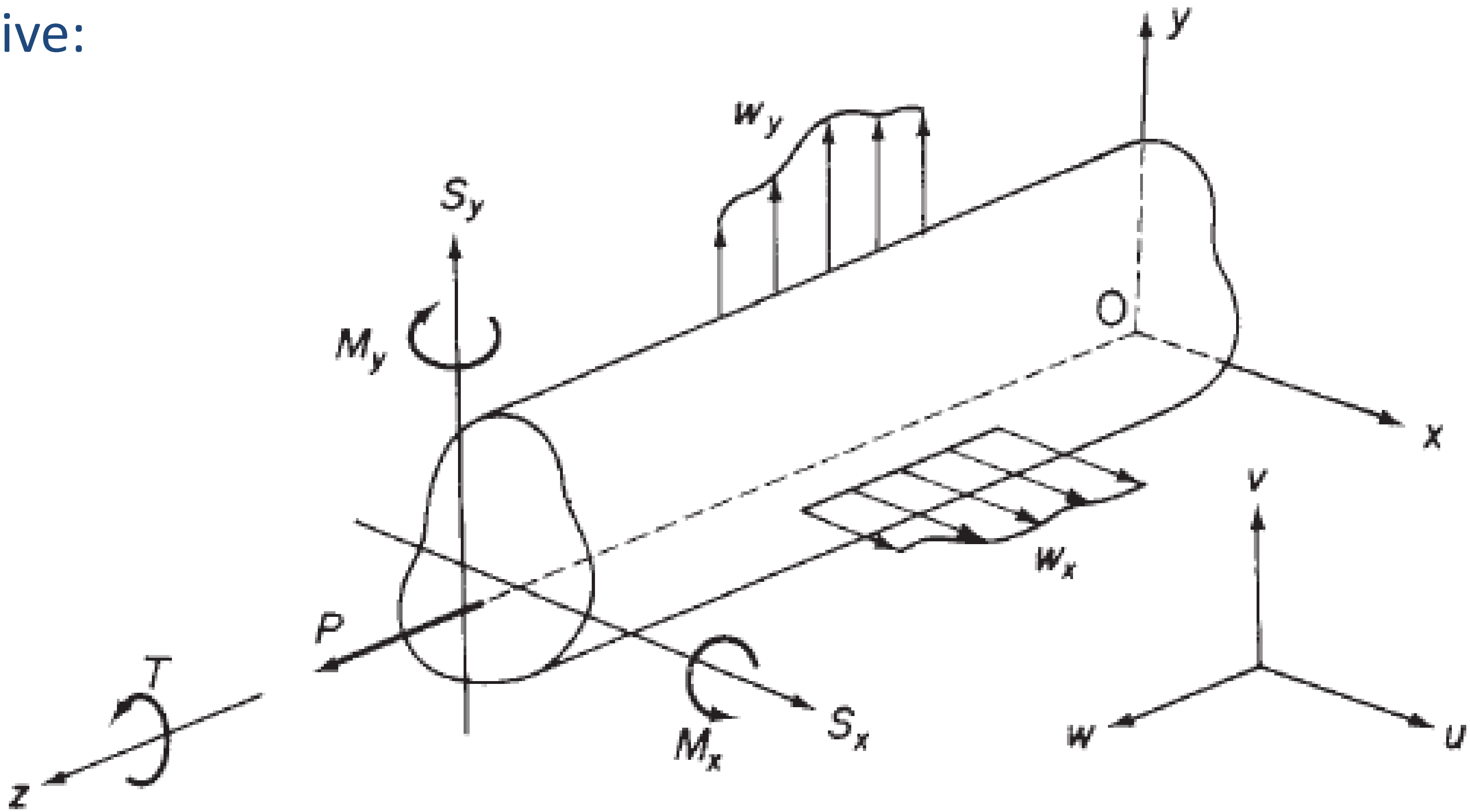
Shear of thin-walled beams

2013

Sign convention

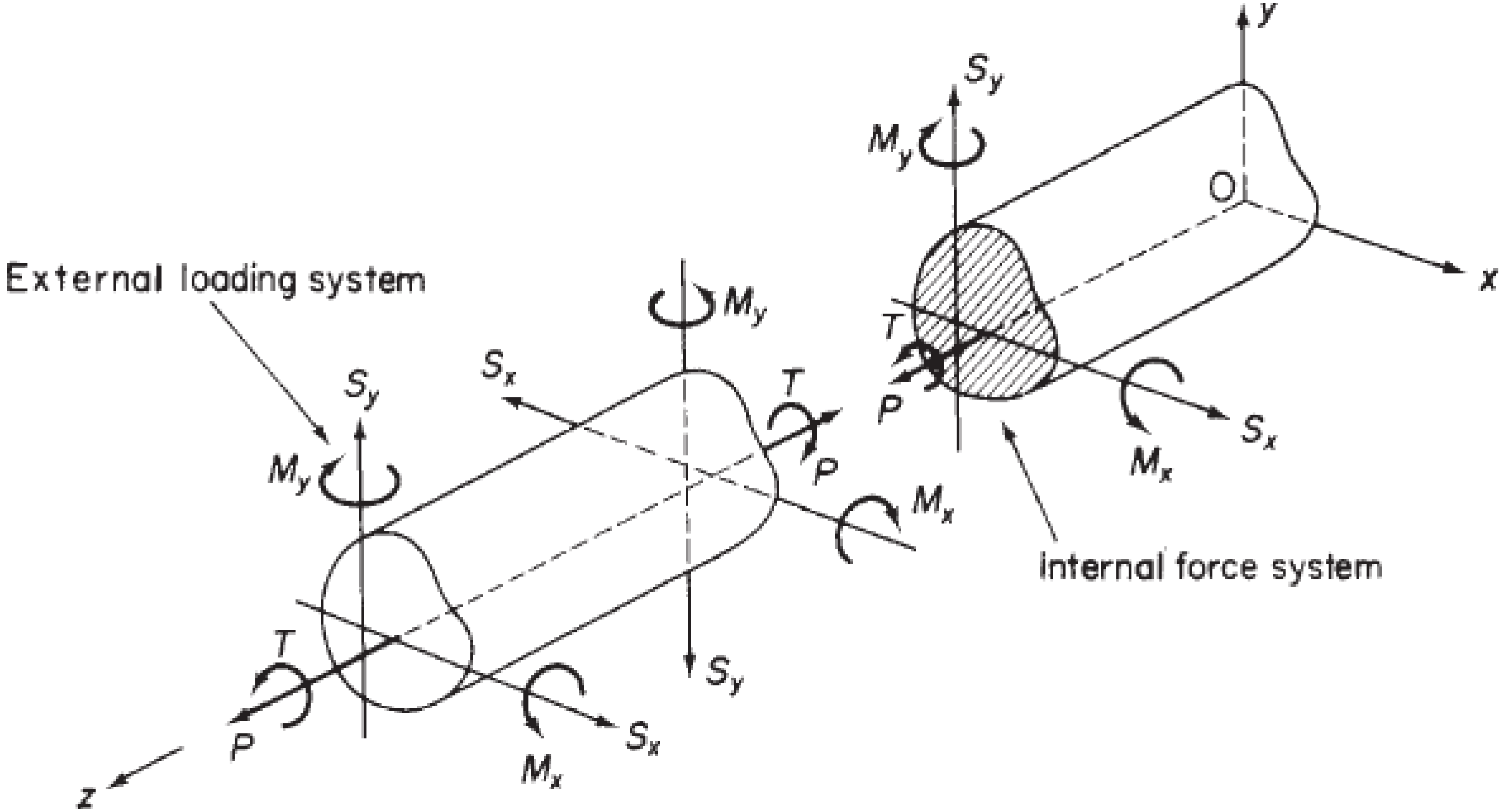
All directions for <u>external</u> loads and displacements given in figure are positive:



also bending moments $\boldsymbol{M_x}$ and $\boldsymbol{M_y}$ are positive when they induce tension in the positive ***xy*** quadrant of the beam cross-section.

Sign convention

All directions for <u>internal</u> loads given in figure are positive:

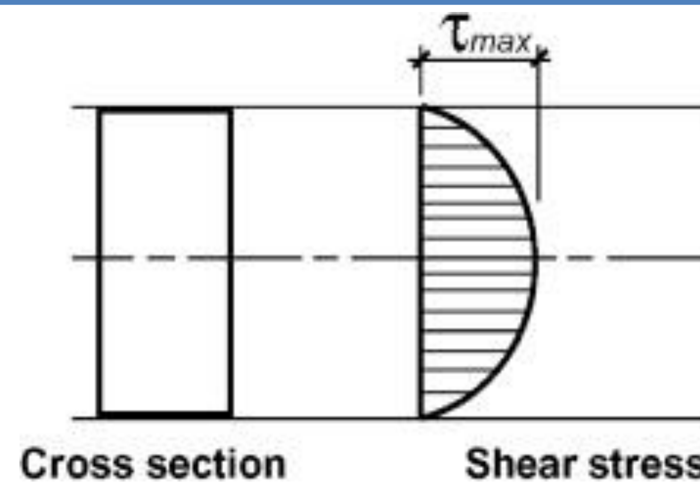


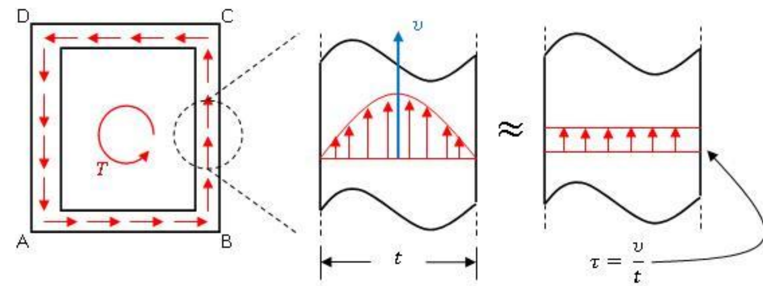
Positive internal forces and moments are in the same direction and sense as the externally applied loads (viewed in ***zO***-direction)

General stress, strain and displacement relationships

Assumptions:

- the shear stresses normal to the beam surface may be neglected since they are zero at each surface and the wall is thin;
- direct and shear stresses on planes normal to the beam surface are constant across the thickness;
- the beam is of uniform section so that the thickness may vary with distance around each section but is constant along the beam;
- in addition, we ignore squares and higher powers of the thickness ***t*** in the calculation of section properties.



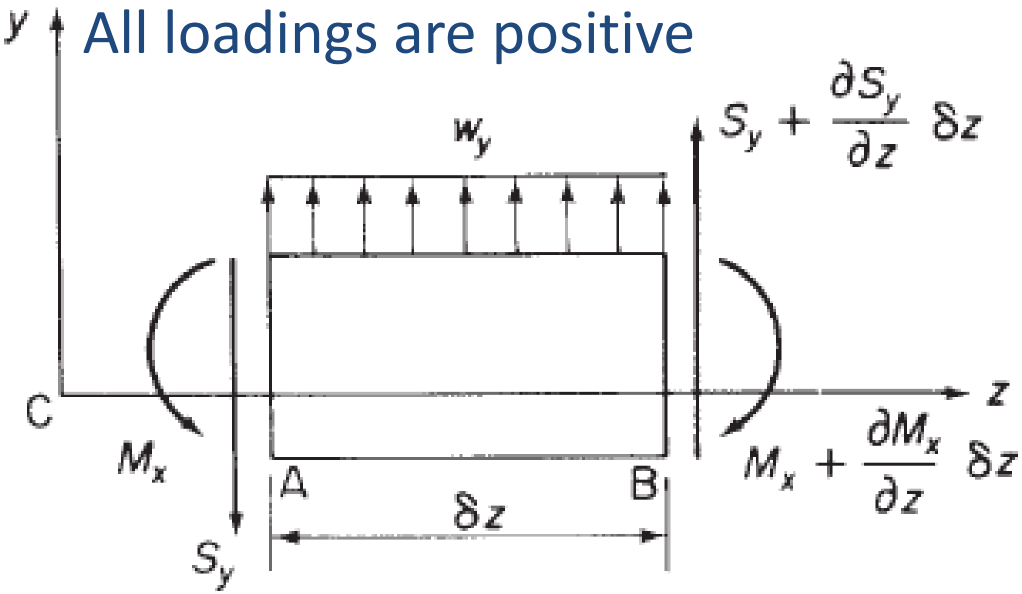


Unsymmetrical bending

Consider an element of length ***δz*** of a beam of unsymmetrical cross-section subjected in the ***yz*** plane to:

- shear forces,
- bending moments and
- a distributed load.

All loadings are positive



Equilibrium in the ***y***-direction:

$$\left(S_y + \frac{\partial S_y}{\partial z}\delta z\right) + w_y\delta z - S_y = 0$$ from which $$\boxed{w_y = -\frac{\partial S_y}{\partial z}}$$

Taking moments about ***A***:

$$\left(M_x + \frac{\partial M_x}{\partial z}\delta z\right) - \left(S_y + \frac{\partial S_y}{\partial z}\delta z\right)\delta z - w_y\frac{(\delta z)^2}{2} - M_x = 0$$ from which $$\boxed{S_y = \frac{\partial M_x}{\partial z}}$$

Combining results we have: $$\boxed{-w_y = \frac{\partial S_y}{\partial z} = \frac{\partial^2 M_x}{\partial z^2}}$$

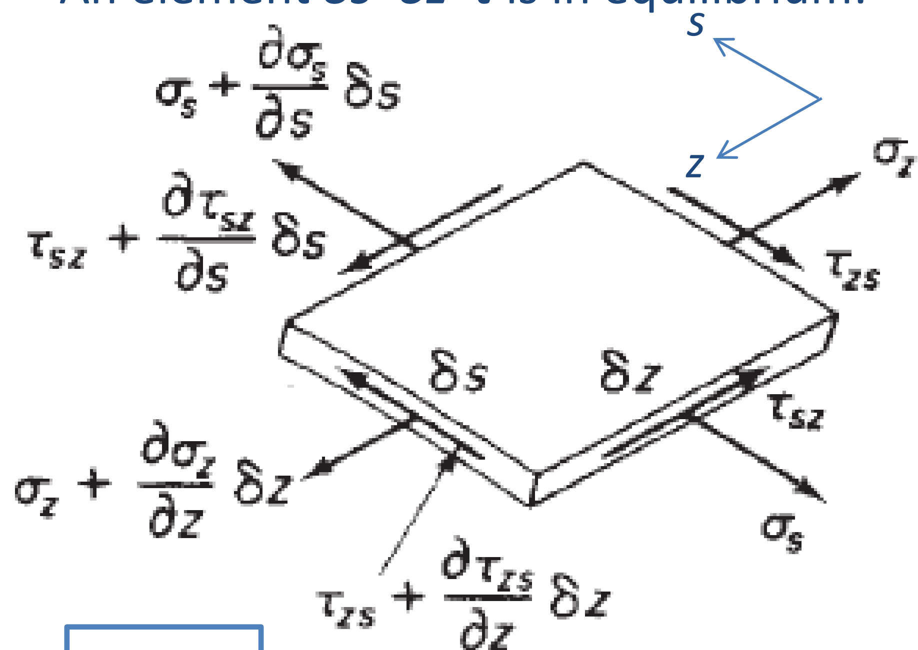
General stress, strain and displacement relationships

- ***s*** is distance around the c/s
- ***t*** is constant over the length ***δs.***

Stresses:

- $\boldsymbol{\sigma_z}$ due to bending moments or shear loads;
- $\boldsymbol{\tau}$ due to shear and/or torsion of a closed section beam or shear of an open section beam;
- the hoop stress $\boldsymbol{\sigma_s}$ is zero.

An element ***δs×δz×t*** is in equilibrium:



shear flow q (positive in ***s***-direction): $\boxed{q = \tau t}$

$\sum F_z = 0$:

$$\left(\sigma_z + \frac{\partial \sigma_z}{\partial z}\delta z\right) t\delta s - \sigma_z t\delta s + \left(q + \frac{\partial q}{\partial s}\delta s\right)\delta_z - q\delta z = 0 \quad \Longrightarrow \quad \boxed{\frac{\partial q}{\partial s} + t\frac{\partial \sigma_z}{\partial z} = 0}$$

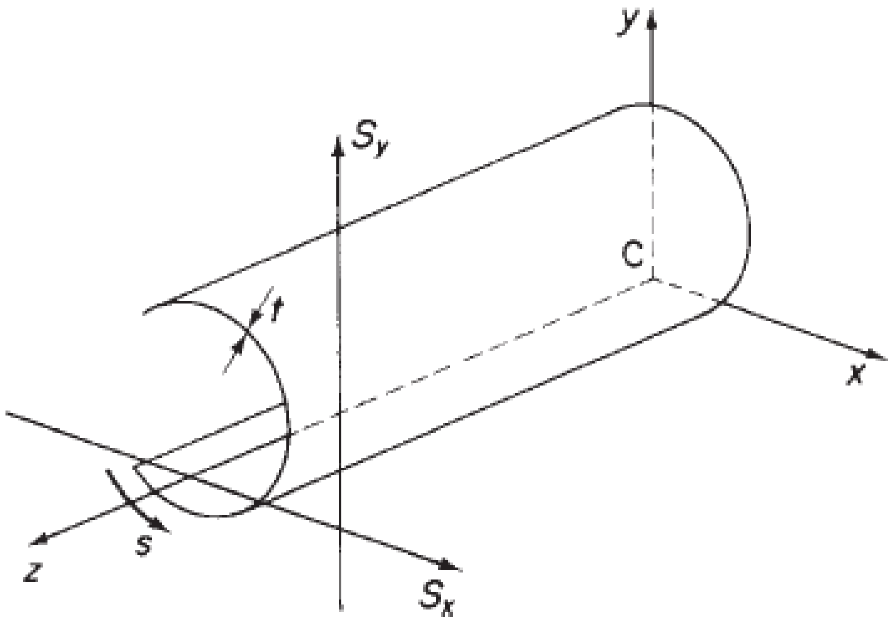
Similarly in the ***s*** direction (neglecting body forces): $\boxed{\frac{\partial q}{\partial z} + t\frac{\partial \sigma_s}{\partial s} = 0}$

Shear of open section beams

Open section beam is under shear loads S_x and S_y, which both pass through a ***shear centre*** (no twisting).

The shear flows and direct stresses are related by $\dfrac{\partial q}{\partial s} + t\dfrac{\partial \sigma_z}{\partial z} = 0$

The direct stresses are obtained from basic bending theory



$$\frac{\partial \sigma_z}{\partial z} = \frac{[(\partial M_y/\partial z)I_{xx} - (\partial M_x/\partial z)I_{xy}]}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{[(\partial M_x/\partial z)I_{yy} - (\partial M_y/\partial z)I_{xy}]}{I_{xx}I_{yy} - I_{xy}^2}y$$

$\partial M_y/\partial z = S_x$

$$\frac{\partial \sigma_z}{\partial z} = \frac{(S_x I_{xx} - S_y I_{xy})}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{(S_y I_{yy} - S_x I_{xy})}{I_{xx}I_{yy} - I_{xy}^2}y$$

$$\frac{\partial q}{\partial s} + t\frac{\partial \sigma_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial q}{\partial s} = -\frac{(S_x I_{xx} - S_y I_{xy})}{I_{xx}I_{yy} - I_{xy}^2}tx - \frac{(S_y I_{yy} - S_x I_{xy})}{I_{xx}I_{yy} - I_{xy}^2}ty$$

Shear of open section beams

Integrating with respect to ***s***

$$\int_0^s \frac{\partial q}{\partial s} \mathrm{d}s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s tx \, \mathrm{d}s - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s ty \, \mathrm{d}s$$

If the origin for ***s*** is taken at the open edge of the cross-section, then ***q = 0*** when ***s = 0***

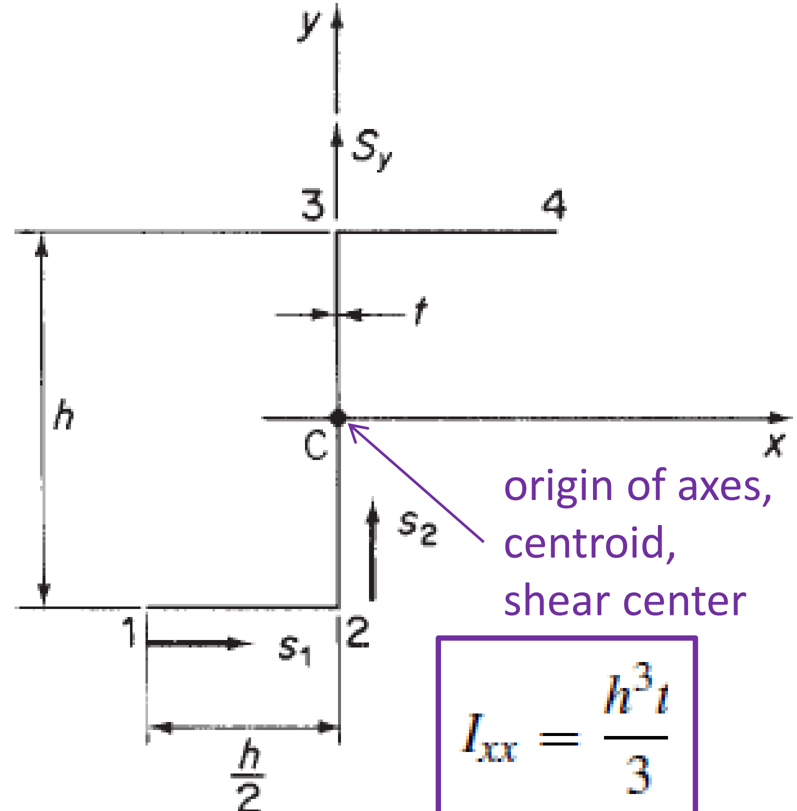
$$\boxed{q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s tx \, \mathrm{d}s - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s ty \, \mathrm{d}s}$$

For ***Cx*** or ***Cy*** as an axis of symmetry $I_{xy} = 0$, then

$$q_s = -\frac{S_x}{I_{yy}} \int_0^s tx \, \mathrm{d}s - \frac{S_y}{I_{xx}} \int_0^s ty \, \mathrm{d}s$$

Shear of open section beams

Determine the ***q*** distribution in the thin-walled ***Z***-section due to a shear load $\boldsymbol{S_y}$ applied through the shear centre of the section.



Shear flow distribution is given by

$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\int_0^s tx\,\mathrm{d}s - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\int_0^s ty\,\mathrm{d}s \xrightarrow{S_x = 0}$$

$$q_s = \frac{S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s tx\,\mathrm{d}s - \frac{S_y I_{yy}}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s ty\,\mathrm{d}s \Rightarrow$$

$$\Rightarrow \quad q_s = \frac{S_y}{I_{xx}I_{yy} - I_{xy}^2}\left(I_{xy}\int_0^s tx\,\mathrm{d}s - I_{yy}\int_0^s ty\,\mathrm{d}s\right)$$

$$\boxed{I_{xx} = \frac{h^3 t}{3} \qquad I_{yy} = \frac{h^3 t}{12} \qquad I_{xy} = \frac{h^3 t}{8}}$$

Shear of open section beams

$$q_s = \frac{S_y}{I_{xx}I_{yy} - I_{xy}^2}\left(I_{xy}\int_0^s tx\,\mathrm{d}s - I_{yy}\int_0^s ty\,\mathrm{d}s\right) \longrightarrow I_{xx} = \frac{h^3 t}{3},\quad I_{yy} = \frac{h^3 t}{12},\quad I_{xy} = \frac{h^3 t}{8}$$

$$\boxed{q_s = \frac{S_y}{h^3}\int_0^s (10.32x - 6.84y)\mathrm{d}s}$$

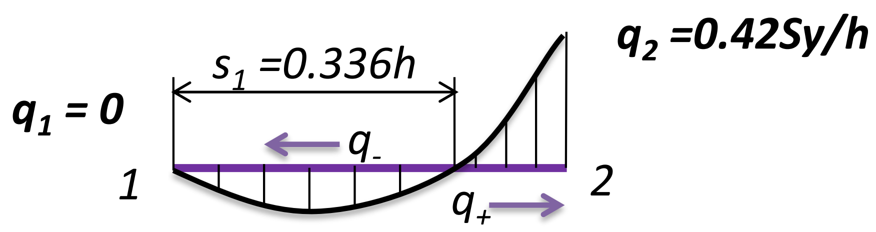
On the bottom flange 12, ***y = −h/2*** *and* ***x = −h/2+s₁***, *where* ***0 ≤ s₁ ≤ h/2***.

$$q_{12} = \frac{S_y}{h^3}\int_0^{s_1}(10.32s_1 - 1.74h)\mathrm{d}s_1 \longrightarrow q_{12} = \frac{S_y}{h^3}(5.16s_1^2 - 1.74hs_1)$$

Hence:

at 1 (***s_1 = 0***), ***q_1 = 0***

at 2 *(**s_1 = h/2**), **q_2 = 0.42Sy/h**.*



The shear flow distribution on the bottom flange is parabolic with a change of sign (i.e. direction) at ***s_1 =0.336h***.

Shear of open section beams

In the web 23, $y = -h/2 + s_2$, *where* $0 \le s_2 \le h$ and $x = 0$.

$$\boxed{q_s = \frac{S_y}{h^3}\int_0^s (10.32x - 6.84y)\mathrm{d}s} \Longrightarrow q_{23} = \frac{S_y}{h^3}\int_0^{s_2} (3.42h - 6.84s_2)\mathrm{d}s_2 + q_2$$

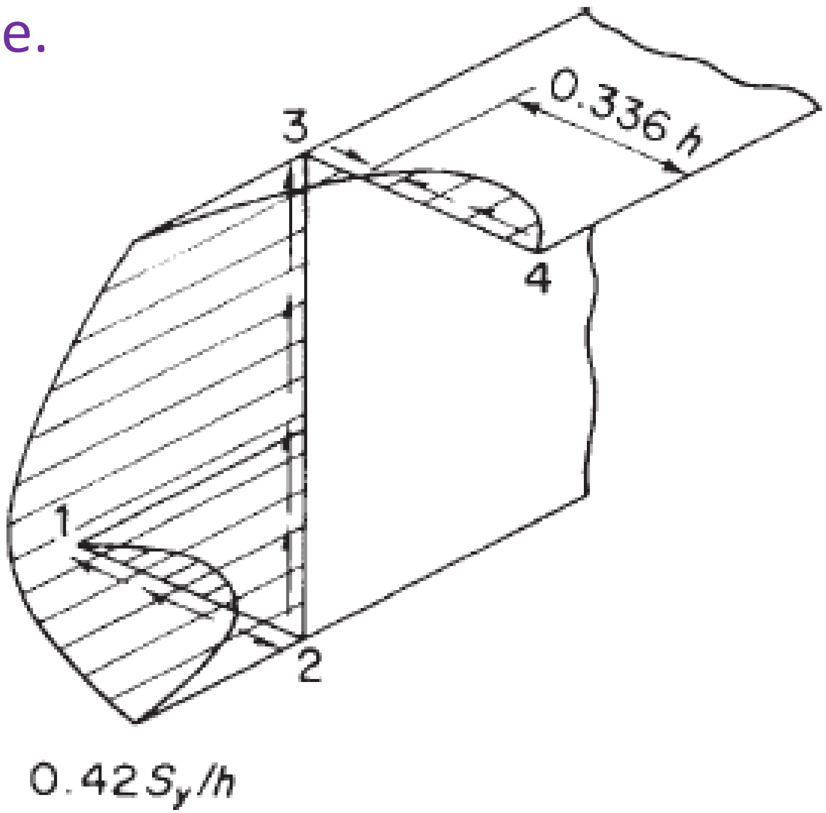
The shear flow is not zero when $s_2 = 0$ but equal to the value obtained by inserting $s_1 = h/2$ in equation for q_{12}, i.e.

$q_2 = 0.42Sy/h$

After integration

$$q_{23} = \frac{S_y}{h^3}(0.42h^2 + 3.42hs_2 - 3.42s_2^2)$$

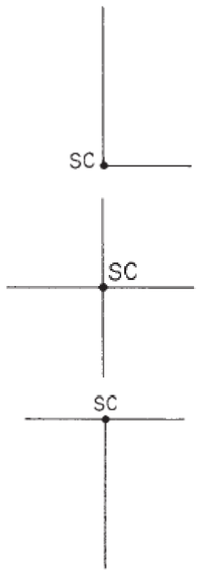
- q distribution is symmetrical about Cx with a maximum value at $s_2 = h/2$;
- and the shear flow is positive at all points in the web.



Shear centre

The shear centre ***S*** is the point in the cross-section through which shear loads produce ***no twisting***.

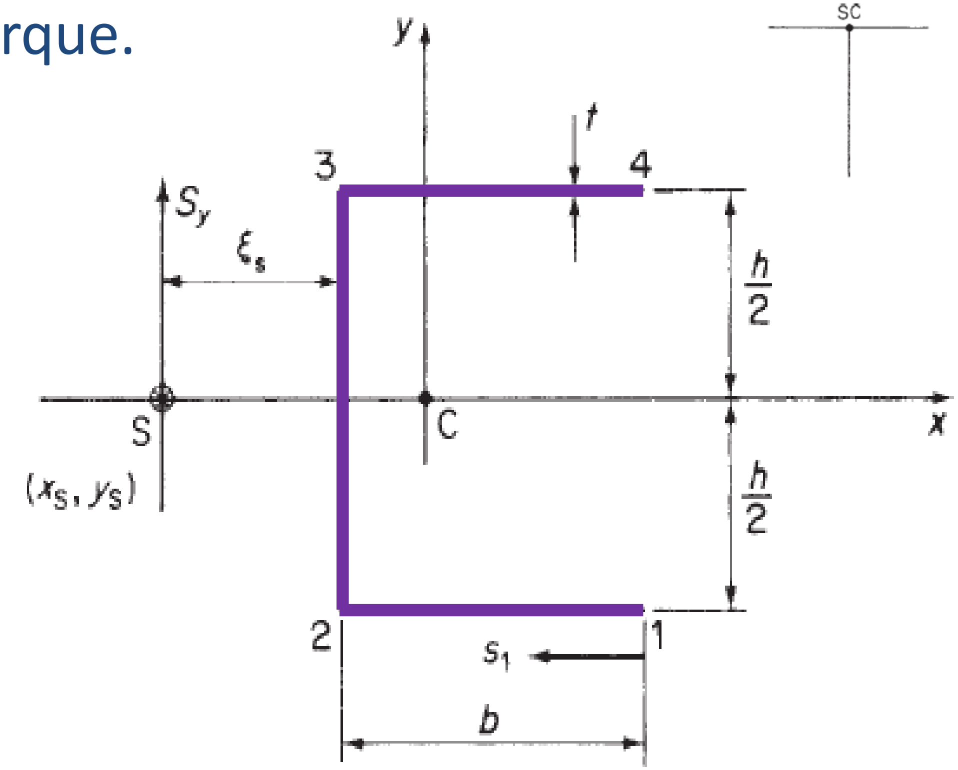
- ***S*** is also the centre of twist of sections subjected to torsion.
- Shear load, which does not pass through ***S***, may be represented by the combination of the shear load applied through the shear centre and a torque.
- ***S*** lies on the axis of symmetry.



Example. Find the position of the ***S***:

If we apply an arbitrary shear load S_y through the shear centre:

- the moment about any point by shear flows is equivalent to the moment of the applied shear load.



Shear centre

Shear flow $q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s tx\,\mathrm{d}s - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s ty\,\mathrm{d}s$

$$\xrightarrow[S_x = 0]{I_{xy} = 0} q_s = -\frac{S_y}{I_{xx}} \int_0^s ty\,\mathrm{d}s \xrightarrow{I_{xx} = 2bt\left(\frac{h}{2}\right)^2 + \frac{th^3}{12} = \frac{h^3 t}{12}\left(1 + \frac{6b}{h}\right)} q_s = \frac{-12S_y}{h^3(1 + 6b/h)} \int_0^s y\,\mathrm{d}s$$

The moment by shear flow in the web equal to zero if it is taken about:

- the center of the web;
- web/flange junction.

On the bottom flange ***y* = *–h*/2**, then $q_{12} = \frac{6S_y}{h^2(1 + 6b/h)} s_1$

Equating the clockwise moments:

$$S_y \xi_s = 2 \int_0^b q_{12} \frac{h}{2} \mathrm{d}s_1 \xrightarrow{q_{12}} S_y \xi_s = 2 \int_0^b \frac{6S_y}{h^2(1 + 6b/h)} \frac{h}{2} s_1 \mathrm{d}s_1$$

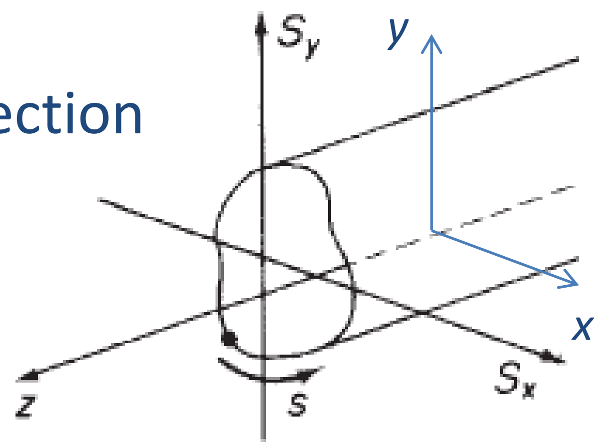
Then $\xi_s = \frac{3b^2}{h(1 + 6b/h)}$

Shear of closed section beams

Differences from open section beams:

- the shear loads may be applied through points in the cross-section other than the shear centre so that torsional effects as well as shear effects are included;
- it is generally not possible to choose an origin for ***s*** at which the value of shear flow is known.

Consider the closed section beam of arbitrary section



The shear loads $\boldsymbol{S_x}$ and $\boldsymbol{S_y}$ are applied through any point in the cross-section and cause

- direct bending stresses and
- shear flows;
- hoop stresses and body forces are absent.

$$\frac{\partial q}{\partial s} + t\frac{\partial \sigma_z}{\partial z} = 0 \quad \Rightarrow \quad \int_0^s \frac{\partial q}{\partial s}\,\mathrm{d}s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\int_0^s tx\,\mathrm{d}s - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\int_0^s ty\,\mathrm{d}s$$

Shear of closed section beams

At the origin of ***s*** the shear flow has the unknown value $\boldsymbol{q_{s,0}}$.

$$q_s - q_{s,0} = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s tx \, \mathrm{d}s - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s ty \, \mathrm{d}s$$

$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s tx \, \mathrm{d}s - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s ty \, \mathrm{d}s + q_{s,0}$$

The first two terms represent the shear flow distribution in an open section beam $\boldsymbol{q_b}$ loaded through its shear centre.

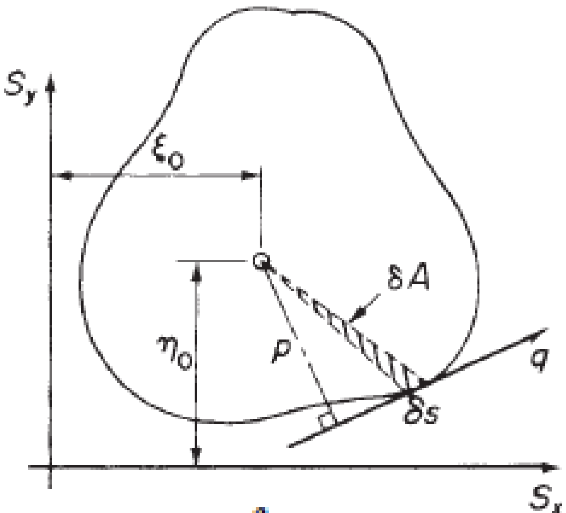
$$q_s = q_{\mathrm{b}} + q_{s,0}$$

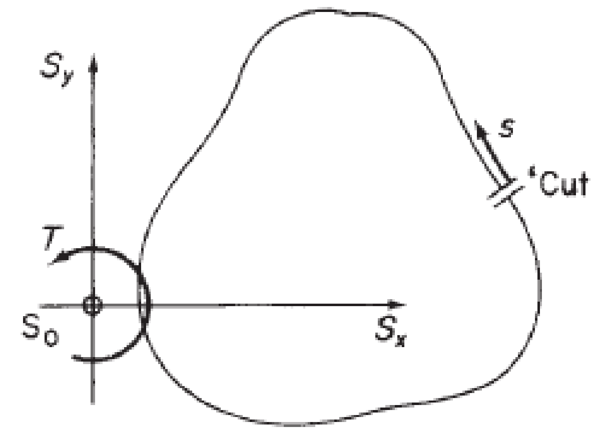
We obtain $\boldsymbol{q_b}$ by supposing that the closed beam section is ‘cut’ at some convenient point thereby producing an ‘open’ section.

$$q_{\mathrm{b}} = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s tx \, \mathrm{d}s - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{xy} - I_{xy}^2}\right) \int_0^s ty \, \mathrm{d}s$$

Shear of closed section beams

The value of $q_{s,0}$ at the cut ($s = 0$) is then found by equating applied and internal moments taken about some convenient moment centre:





$$S_x\eta_0 - S_y\xi_0 = \oint pq\,\mathrm{d}s = \oint pq_\mathrm{b}\,\mathrm{d}s + q_{s,0}\oint p\,\mathrm{d}s$$

As $\delta A = \frac{1}{2}\delta s p$ or $\oint \mathrm{d}A = \frac{1}{2}\oint p\,\mathrm{d}s$, then $\oint p\mathrm{d}s = 2A$

A is the area enclosed by the mid-line of the beam section wall…

Hence $S_x\eta_0 - S_y\xi_0 = \oint pq_\mathrm{b}\mathrm{d}s + 2Aq_{s,0}$

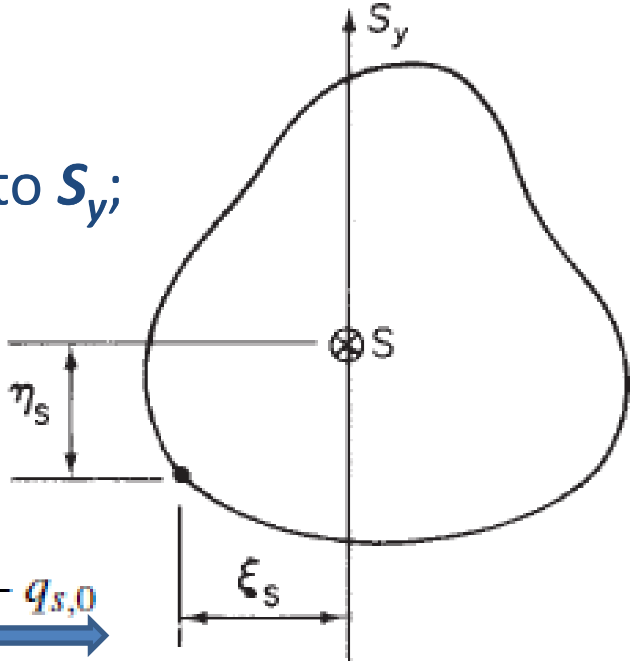
If the moment centre is the point of intersection of S_x and S_y, then

$$0 = \oint pq_\mathrm{b}\,\mathrm{d}s + 2Aq_{s,0}$$

Shear center

Procedure to find ξ_S:

- apply an arbitrary shear load S_y through S;
- calculate the distribution of shear flow q_s due to S_y;
- equate internal and external moments.



To obtain $q_{s,0}$ use the condition that a shear load acting through the shear centre of a section produces zero twist.

$$\boxed{\frac{d\theta}{dz} = \frac{1}{2A}\oint \frac{q_s}{Gt}ds} \xrightarrow{d\vartheta/dz = 0} 0 = \oint \frac{q_s}{Gt}ds \xrightarrow{q_s = q_b + q_{s,0}}$$

$$0 = \oint \frac{1}{Gt}(q_b + q_{s,0})ds$$

$$q_{s,0} = -\frac{\oint (q_b/Gt)ds}{\oint ds/Gt} \xrightarrow{Gt = const} q_{s,0} = -\frac{\oint q_b\, ds}{\oint ds}$$

The coordinate η_S is found in a similar manner by applying S_x through S.

Optional content

Shear center

In a given cross-section each wall of the section is flat and has the same thickness ***t*** and shear modulus ***G***.

For the chosen position of ***x***-axis: $I_{xy} = 0$

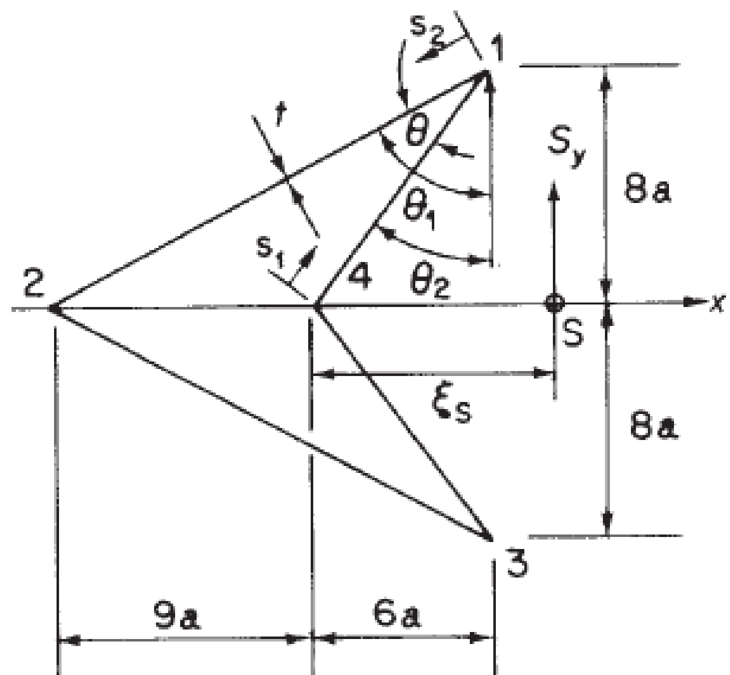
$$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\int_0^s tx\,\mathrm{d}s - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\int_0^s ty\,\mathrm{d}s + q_{s,0}$$

$S_x = 0$

$$q_s = -\frac{S_y}{I_{xx}}\int_0^s ty\,\mathrm{d}s + q_{s,0}$$

$$I_{xx} = 2\left[\int_0^{10a} t\left(\frac{8}{10}s_1\right)^2 \mathrm{d}s_1 + \int_0^{17a} t\left(\frac{8}{17}s_2\right)^2 \mathrm{d}s_2\right]$$

$$I_{xx} = 1152a^3 t.$$



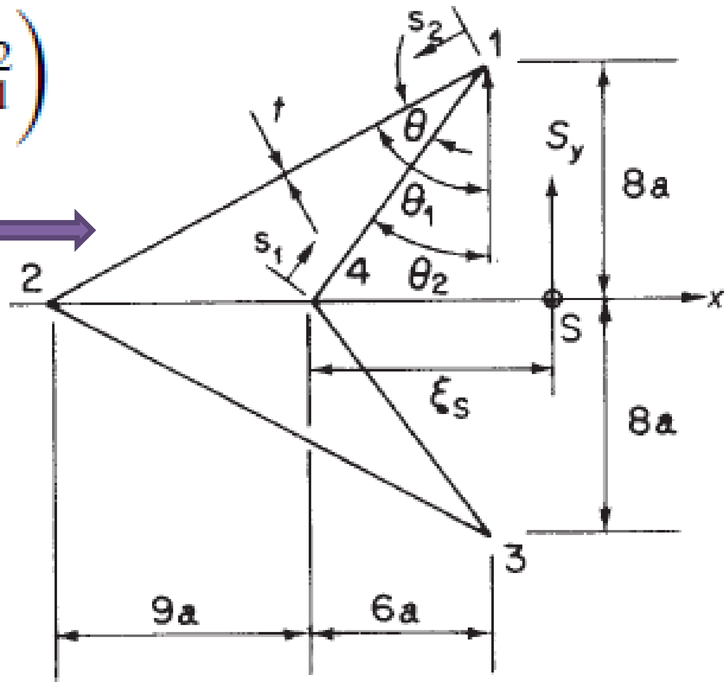
Optional content

Shear center

$$q_{b,41} = \frac{-S_y}{1152a^3 t}\int_0^{s_1} t\left(\frac{8}{10}s_1\right) ds_1 = \frac{-S_y}{1152a^3}\left(\frac{2}{5}s_1^2\right)$$

$$q_{b,12} = \frac{-S_y}{1152a^3}\left[\int_0^{s_2}(17a - s_2)\frac{8}{17}ds_2 + 40a^2\right]$$

$$q_{b,12} = \frac{-S_y}{1152a^3}\left(-\frac{4}{17}s_2^2 + 8as_2 + 40a^2\right)$$



$$q_{s,0} = -\frac{\oint q_b\, ds}{\oint ds}$$

$$q_{s,0} = \frac{2S_y}{54a \times 1152a^3}\left[\int_0^{10a}\frac{2}{5}s_1^2\, ds_1 + \int_0^{17a}\left(-\frac{4}{17}s_2^2 + 8as_2 + 40a^2\right) ds_2\right]$$

$$q_{s,0} = \frac{S_y}{1152a^3}(58.7a^2)$$

Optional content

Shear center

Taking moments about the point ***2*** we have

$$S_y(\xi_S + 9a) = 2\int_0^{10a} q_{41}\, 17a \sin\theta \,\mathrm{d}s_1$$

$$S_y(\xi_S + 9a) = \frac{S_y 34a \sin\theta}{1152a^3}\int_0^{10a}\left(-\frac{2}{5}s_1^2 + 58.7a^2\right)\mathrm{d}s_1$$

$\sin\theta = \sin(\theta_1 - \theta_2) = \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2$

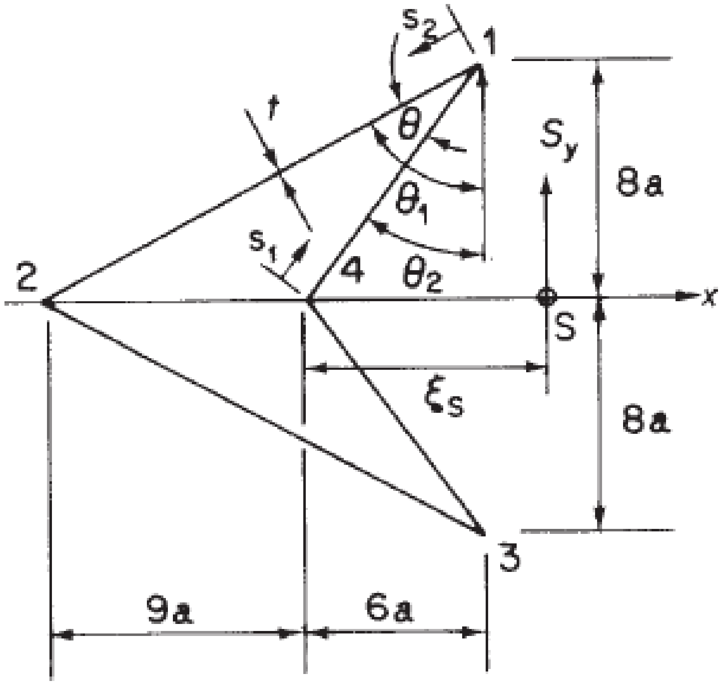
$\sin\theta_1 = 15/17$,

$\cos\theta_2 = 8/10$,

$\cos\theta_1 = 8/17$,

$\sin\theta_2 = 6/10$.

$$\boxed{\xi_S = -3.35a}$$



Obrigado!