Columns

2013

Columns

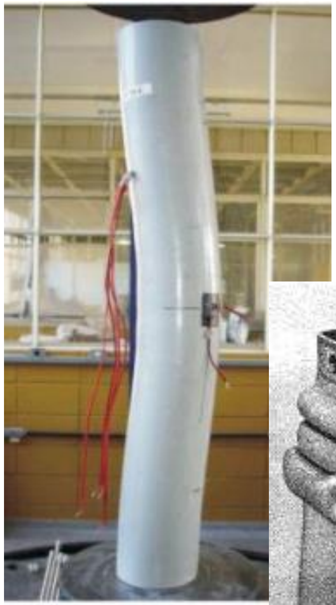
The most critical mode of failure for such aircraft structures as thin webs stiffened by slender longerons or stringers is **buckling**.

Extremely important to predict the buckling loads of the following structures: 1) columns, 2) thin plates, 3) stiffened panels.

Two types of structural instability arise:

Primary:

- no change in cross-sectional area;
- the wavelength of the buckle is of the same order as the length of the element;
- solid and thick-walled columns.



Secondary:

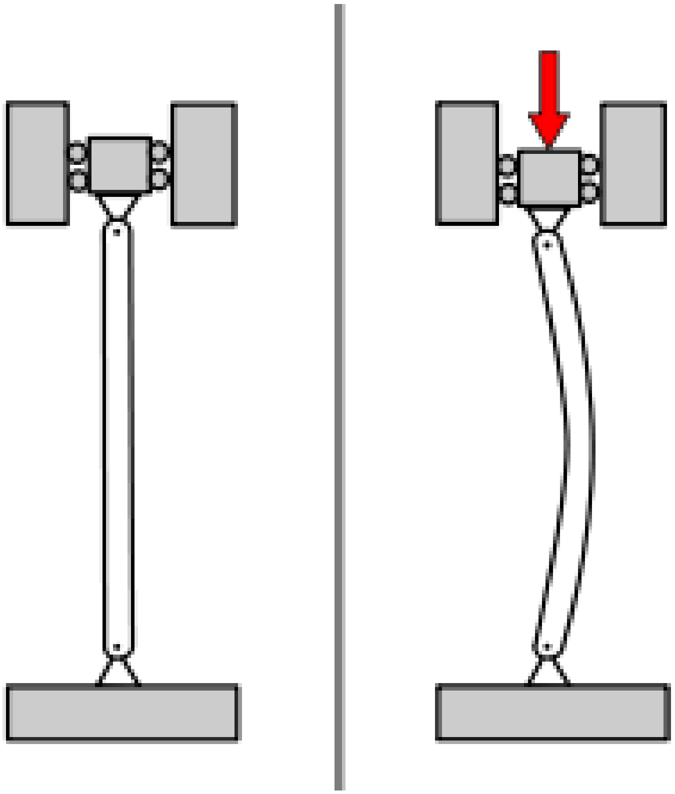
- changes in cross-sectional area occur;
- the wavelength of the buckle is of the order of the cross-sectional dimensions of the element;
- thin-walled columns and stiffened plates.

Euler buckling of columns

If an increasing axial compressive load is applied to a slender column there is a value of the load called **buckling load** at which the column will suddenly bow or buckle in some unpredetermined direction.

Direction of the buckling may depend:

- degree of asymmetry;
- geometrical imperfection;
- asymmetrical load application;
- material imperfection.



Consider a perfect column in which

- the load is applied precisely along the perfectly straight centroidal axis,
- there is perfect symmetry so that, theoretically, there can be no sudden bowing or buckling.

Euler buckling of columns

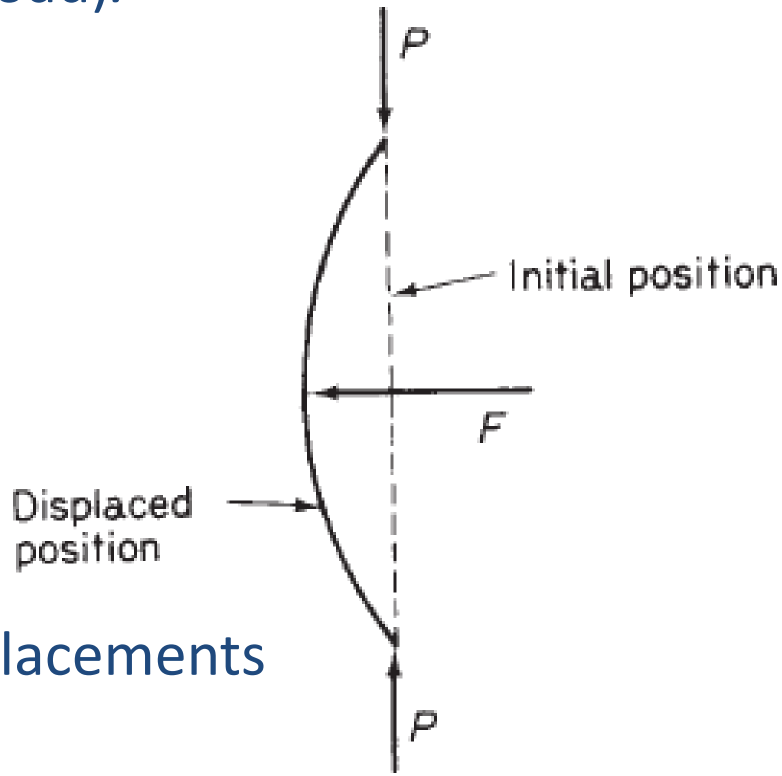
Properties of the “perfect” column

1. Only shortening of the perfect column occurs no matter what the value of compressive load ***P.***

2. The column is displaced a small amount by a lateral load ***F*** at values of $P < P_{CR}$ (P_{CR} – critical or buckling load).

3. Removal of ***F*** results in a return of the column to its undisturbed position (state of *stable equilibrium*).

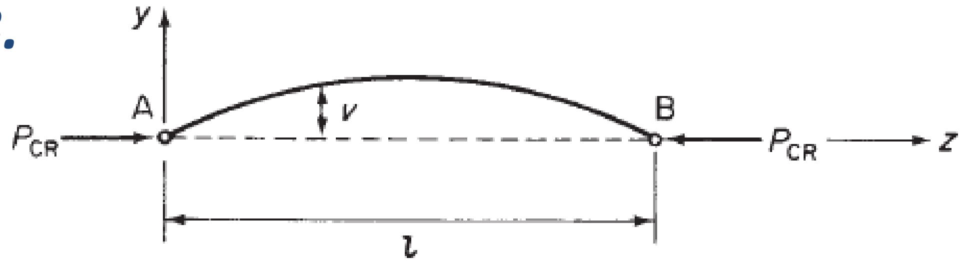
4. At the P_{CR} the displacement does not disappear and the column will remain in any displaced position as long as the displacement is small (state of a *neutral equilibrium*).



5. For $P > P_{CR}$ enforced lateral displacements increase and the column is *unstable.*

Euler buckling of columns

Consider the pin-ended column ***AB.*** Assume that it is in the displaced state of neutral equilibrium (P_{CR}). Simple bending theory gives:



$$\boxed{EI\frac{d^2v}{dz^2} = -M} \quad \Rightarrow \quad EI\frac{d^2v}{dz^2} = -P_{CR}\,v \quad \Rightarrow \quad \frac{d^2v}{dz^2} + \frac{P_{CR}}{EI}v = 0$$

The well-known solution $\boxed{v = A\cos\mu z + B\sin\mu z}$

where $\mu^2 = P_{CR}/EI$, ***A*** and ***B*** – unknown coefficients

The boundary conditions for this particular case are

$v = 0$ at $z = 0$ and l, thus $A = 0$ and $B\sin\mu l = 0$

For a non-trivial solution (i.e. $v \neq 0$) then

$\sin\mu l = 0 \quad \text{or} \quad \mu l = n\pi \quad \text{where } n = 1, 2, 3, \ldots$

giving $\dfrac{P_{CR}l^2}{EI} = n^2\pi^2$ or $\boxed{P_{CR} = \dfrac{n^2\pi^2 EI}{l^2}}$

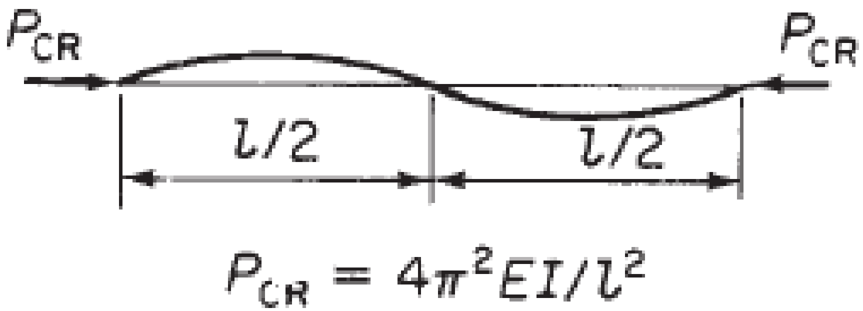
Euler buckling of columns

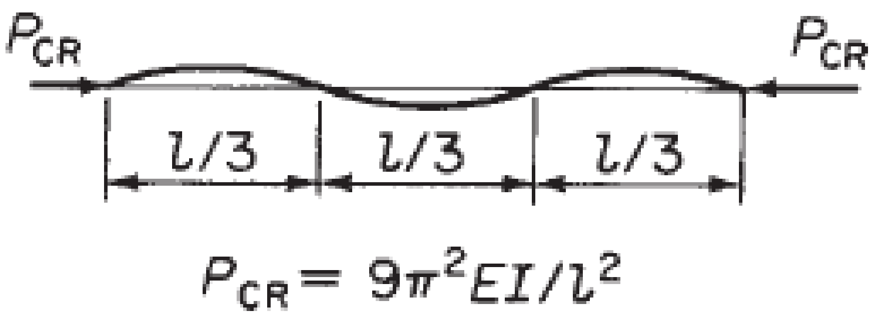
Note that solution of differential equation $\boxed{v = A\cos\mu z + B\sin\mu z}$ cannot be solved for ***v*** no matter how many of the available boundary conditions are inserted, since neutral state of equilibrium means that ***v*** is indeterminate.

The smallest value of critical buckling load at ***n = 1*** $P_{CR} = \dfrac{\pi^2 EI}{l^2}$

Other values of $\boldsymbol{P_{CR}}$ (*at* ***n = 2, 3, …***) $P_{CR} = \dfrac{4\pi^2 EI}{l^2}, \dfrac{9\pi^2 EI}{l^2}, \ldots$

These values of buckling load cause more **complex modes** of buckling:





Can be produced by applying external restraints to a very slender column at the points of contraflexure to prevent lateral movement.

Euler buckling of columns

$$P_{CR} = \frac{\pi^2 EI}{l^2} \Rightarrow \sigma_{CR} = \frac{\pi^2 E}{(l/r)^2}$$

here ***r* = √*I*/*A*** is the ***radius of gyration*** of the cross-sectional area of the column.

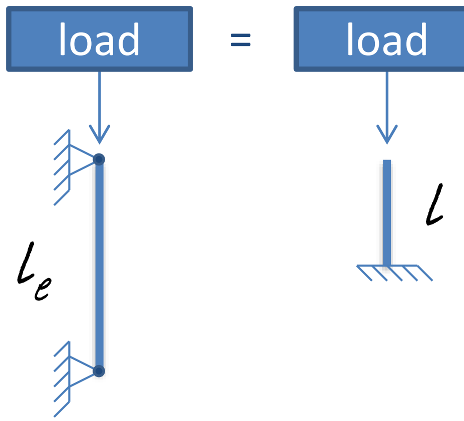
The term ***l/r*** is known as the ***slenderness ratio*** of the column.

- for unsymmetrical column ***r*** is the least (since the column will bend about an axis about which the flexural rigidity ***EI*** is least);
- for column with restraints ***EI*** is the flexural rigidity in unrestrained plane.

Can rewrite foregoing equations:

$$P_{CR} = \frac{\pi^2 EI}{l_e^2} \qquad \sigma_{CR} = \frac{\pi^2 E}{(l_e/r)^2}$$

where l_e is the ***effective length of the column***, i.e. the length of a pin-ended column that would have the same critical load as that of a column of length *l*, but with different end conditions.

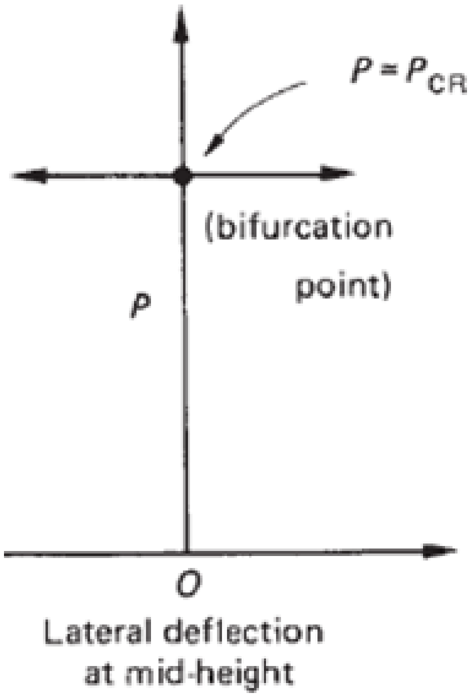


Euler buckling of columns

Ends	l_e/l	Boundary conditions
Both pinned	1.0	$v=0$ at $z=0$ and l
Both fixed	0.5	$v=0$ at $z=0$ and $z=l$, $dv/dz=0$ at $z=l$
One fixed, the other free	2.0	$v=0$ and $dv/dz=0$ at $z=0$
One fixed, the other pinned	0.6998	$dv/dz=0$ at $z=0$, $v=0$ at $z=l$ and $z=0$

If the lateral load ***F*** is removed the column (which is perfectly straight, homogeneous and loaded exactly along its axis) will suffer only axial compression as ***P*** is increased. This situation, theoretically, would continue until yielding of the material of the column occurred. However

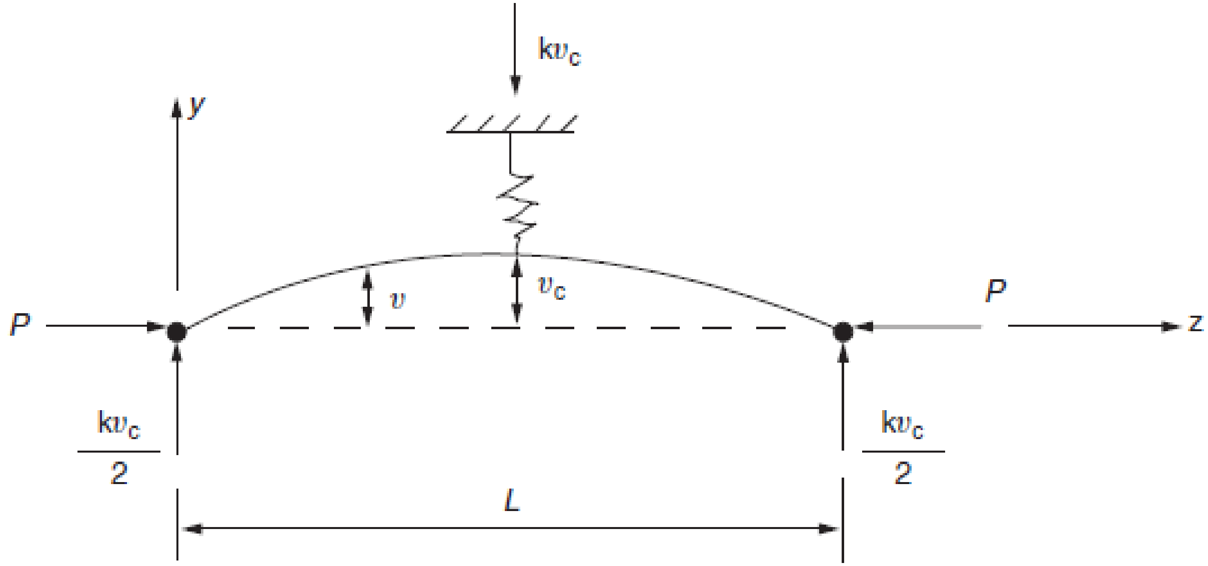
- for values of $\boldsymbol{P < P_{CR}}$ the column is in stable equilibrium;
- whereas for $\boldsymbol{P > P_{CR}}$ the column is unstable.



Euler buckling of columns

A uniform column of

- length ***L***
- flexural stiffness ***EI***
- is simply supported
- has an elastic support at midspan.



This support is such that if a lateral displacement v_c occurs at this point a restoring force kv_c is generated at the point.

Derive

- an equation giving the buckling load of the column.
- If the buckling load is $4\pi^2 EI/L^2$ find the value of k.
- Also if the elastic support is infinitely stiff show that the buckling load is given by the equation $\tan \lambda L/2 = \lambda L/2$ where $\lambda = \sqrt{(P/EI)}$.

Euler buckling of columns

The bending moment at any section of the column $M = Pv - \frac{kv_c}{2}z$

$$\boxed{EI\frac{d^2v}{dz^2} = -M} \xrightarrow{M = Pv - \frac{kv_c}{2}z} EI\frac{d^2v}{dz^2} = -Pv + \frac{kv_c}{2}z \longrightarrow \frac{d^2v}{dz^2} + \lambda^2 v = \frac{kv_c}{2EI}z$$

The solution $v = A\cos\lambda z + B\sin\lambda z + \frac{kv_c}{2P}z$

The constants ***A*** and ***B*** are found using the boundary conditions of the column which are:

- ***v = 0*** when ***z = 0***,
- ***v = v$_c$***, when ***z = L/2*** and
- ***(dv/dz) = 0*** when ***z = L/2***.

A = 0 $\quad B = \frac{v_c}{\sin(\lambda L/2)}\left(1 - \frac{k\lambda}{4P}\right)$

$$\left(1 - \frac{kL}{4P}\right)\cos\frac{\lambda L}{2} + \frac{k}{2P\lambda}\sin\frac{\lambda L}{2} = 0 \longrightarrow P = \frac{kL}{4}\left(1 - \frac{\tan(\lambda L/2)}{\lambda L/2}\right)$$

If ***P = 4π²EI/L²*** then ***λL/2 = π*** so that ***k = 4P/L*** (*as* ***tan π = 0***).

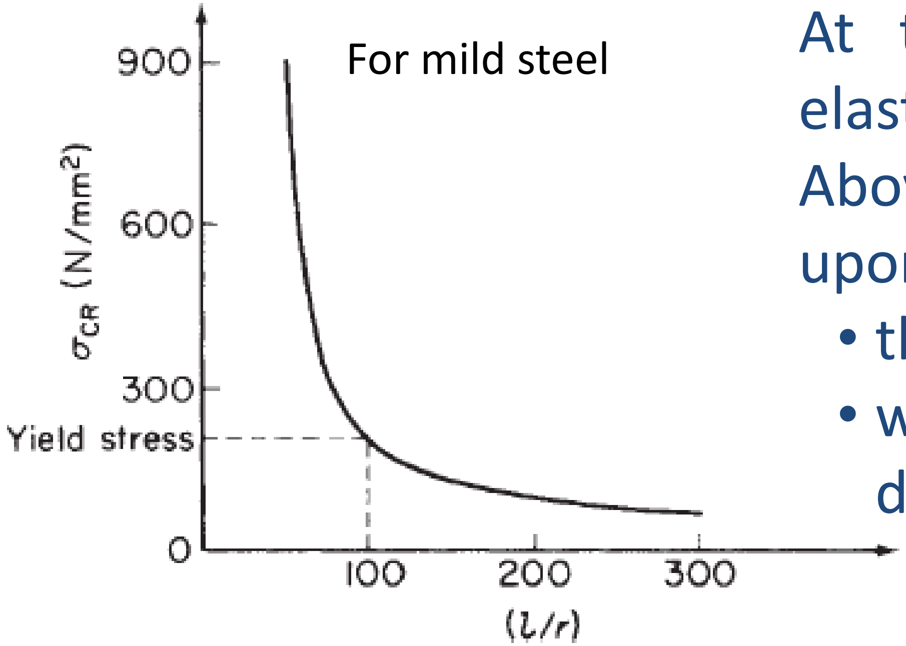
Finally, if ***k→∞*** then ***tan (λL/2) = (λL/2)***

Inelastic buckling

The critical stress $\sigma_{CR} = \frac{\pi^2 E}{(l_e/r)^2}$ (applicable for $\boldsymbol{\sigma < \sigma_y}$) depends only on

- the elastic modulus ***E*** of the material of the column and
- the slenderness ratio ***l*/r**.

The critical stress increases as the slenderness ratio decreases; i.e. as the column becomes shorter and thicker.



At the elastic region the modulus of elasticity ***E* (=*dσ/dε*)** is constant.

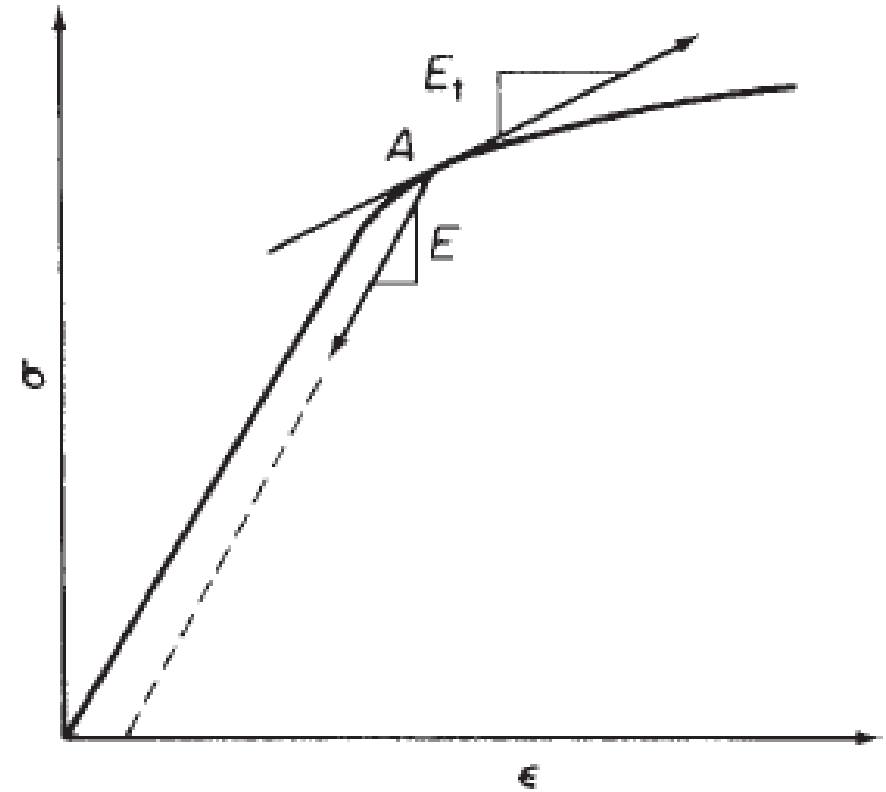
Above the elastic limit ***dσ/dε*** depends upon

- the value of stress and
- whether the stress is increasing or decreasing.

Inelastic buckling

The elastic modulus at the point ***A*** is

- the tangent modulus $\boldsymbol{E_t}$ if the stress is increasing but
- ***E*** if the stress is decreasing.



Consider

- a column having a plane of symmetry
- subjected to a compressive load ***P*** (so stress is above the elastic limit).

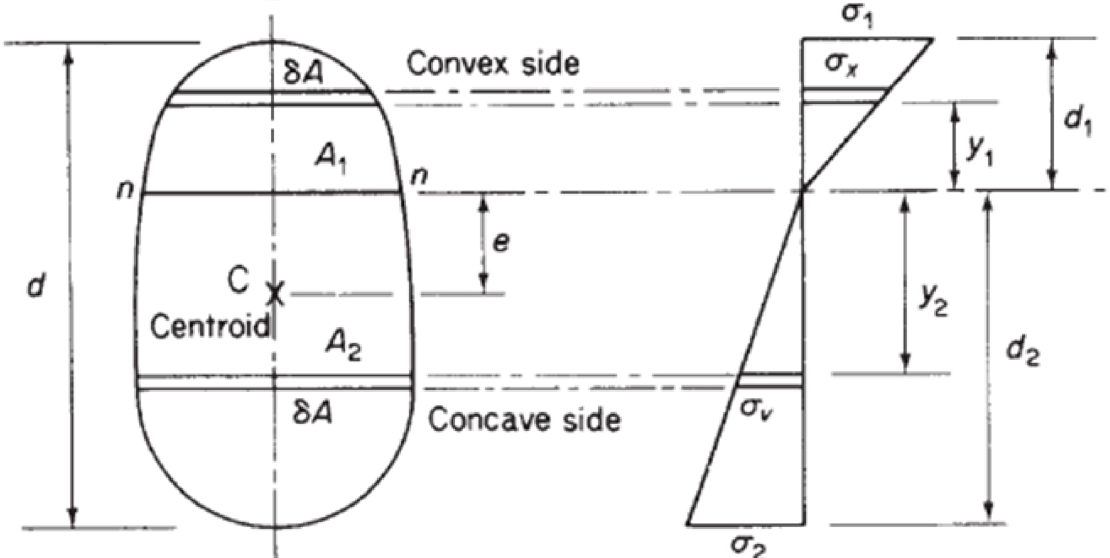
If the column is given a small deflection, ***v***, in its plane of symmetry, then the stress on the concave side increases while the stress on the convex side decreases.

Inelastic buckling

In the cross-section of the column the compressive stress decreases in the area $\boldsymbol{A_1}$ and increases in the area $\boldsymbol{A_2}$, while the stress on the line ***nn*** is unchanged.

The modulus of elasticity of the material in the area $\boldsymbol{A_1}$ is $\boldsymbol{E}$ while that in $\boldsymbol{A_2}$ is $\boldsymbol{E_t}$.

The homogeneous column now behaves as if it were non-homogeneous.



The linearity of the distribution follows from an assumption that plane sections remain plane.

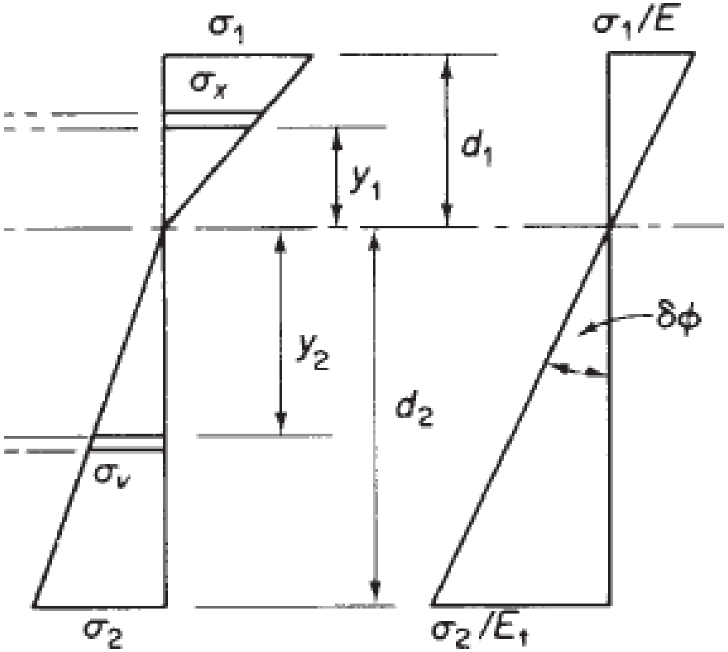
$$\int_0^{d_1} \sigma_x \, \mathrm{d}A = \int_0^{d_2} \sigma_v \, \mathrm{d}A$$ as load *P* is unchanged by the disturbance.

$$\int_0^{d_1} \sigma_x (y_1 + e) \, \mathrm{d}A + \int_0^{d_2} \sigma_v (y_2 - e) \, \mathrm{d}A = -Pv$$ as *P* is applied at C.

Inelastic buckling

According to given figure (left)

$$\sigma_x = \frac{\sigma_1}{d_1} y_1 \quad \sigma_v = \frac{\sigma_2}{d_2} y_2$$



The change of slope equals to the angle between two close sections of the column, and equal to angle ***δφ*** in the strain diagram.

$$\frac{d^2 v}{dz^2} = \frac{\sigma_1}{E d_1} = \frac{\sigma_2}{E_t d_2}$$

$$\int_0^{d_1} \sigma_x \, dA = \int_0^{d_2} \sigma_v \, dA$$

$$\frac{d^2 v}{dz^2} = \frac{\sigma_1}{E d_1} = \frac{\sigma_2}{E_t d_2}$$

$$\sigma_x = \frac{\sigma_1}{d_1} y_1 \quad \sigma_v = \frac{\sigma_2}{d_2} y_2$$

$$E \frac{d^2 v}{dz^2} \int_0^{d_1} y_1 \, dA - E_t \frac{d^2 v}{dz^2} \int_0^{d_2} y_2 \, dA = 0$$

Inelastic buckling

In similar manner

$$\int_0^{d_1} \sigma_x (y_1 + e)\,\mathrm{d}A + \int_0^{d_2} \sigma_v (y_2 - e)\,\mathrm{d}A = -Pv$$

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = \frac{\sigma_1}{Ed_1} = \frac{\sigma_2}{E_t d_2}$$

$$\sigma_x = \frac{\sigma_1}{\mathrm{d}_1} y_1 \quad \sigma_v = \frac{\sigma_2}{\mathrm{d}_2} y_2$$

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2}\left(E\int_0^{d_1} y_1^2\,\mathrm{d}A + E_t \int_0^{d_2} y_2^2\,\mathrm{d}A\right) + e\frac{\mathrm{d}^2 v}{\mathrm{d}z^2}\underbrace{\left(E\int_0^{d_1} y_1\,\mathrm{d}A - E_t \int_0^{d_2} y_2\,\mathrm{d}A\right)}_{=0} = -Pv$$

Therefore we have $$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2}(EI_1 + E_t I_2) = -Pv$$

where $$I_1 = \int_0^{d_1} y_1^2\,\mathrm{d}A \quad \text{and} \quad I_2 = \int_0^{d_2} y_2^2\,\mathrm{d}A$$

the second moments of area about ***nn*** of the convex and concave sides of the column respectively.

Inelastic buckling

Putting $E_r I = E I_1 + E_t I_2$ or $E_r = E\frac{I_1}{I} + E_t\frac{I_2}{I}$ ***E_r - reduced modulus***

$$\frac{d^2 v}{dz^2}(E I_1 + E_t I_2) = -Pv \xrightarrow{E_r I = E I_1 + E_t I_2} E_r I \frac{d^2 v}{dz^2} + Pv = 0$$

Comparing with equation of bending of the column $\frac{d^2 v}{dz^2} + \frac{P_{CR}}{EI} v = 0$

we see that if ***P*** is the critical load ***P_{CR}*** then $P_{CR} = \frac{\pi^2 E_r I}{l_e^2}$ $\sigma_{CR} = \frac{\pi^2 E_r}{(l_e/r)^2}$

The above method for predicting critical loads and stresses outside the elastic range is known as the *reduced modulus theory*

$$E\frac{d^2 v}{dz^2}\int_0^{d_1} y_1\,dA - E_t\frac{d^2 v}{dz^2}\int_0^{d_2} y_2\,dA = 0 \Longrightarrow E\int_0^{d_1} y_1\,dA - E_t\int_0^{d_2} y_2\,dA = 0$$

The last equation together with the relationship ***$d = d_1 + d_2$***
enables the position of ***nn*** to be found

Inelastic buckling

Also there are two possible cases:

the axial load ***P*** is increased at the time of the lateral disturbance of the column:

- no strain reversal on convex side
- the compressive stress therefore increases over the complete section
- the tangent modulus applies over the whole cross-section

$$P_{\mathrm{CR}} = \frac{\pi^2 E_{\mathrm{t}} I}{l_{\mathrm{e}}^2}$$

$$\sigma_{\mathrm{CR}} = \frac{\pi^2 E_{\mathrm{t}}}{(l_{\mathrm{e}}/r^2)}$$

reduction in ***P*** could result in a decrease in stress over the whole cross-section:

- strain reversal on both sides
- the compressive stress therefore decreases over the complete section
- usual modulus applies over the whole cross-section

$$P_{\mathrm{CR}} = \frac{\pi^2 E I}{l_{\mathrm{e}}^2}$$

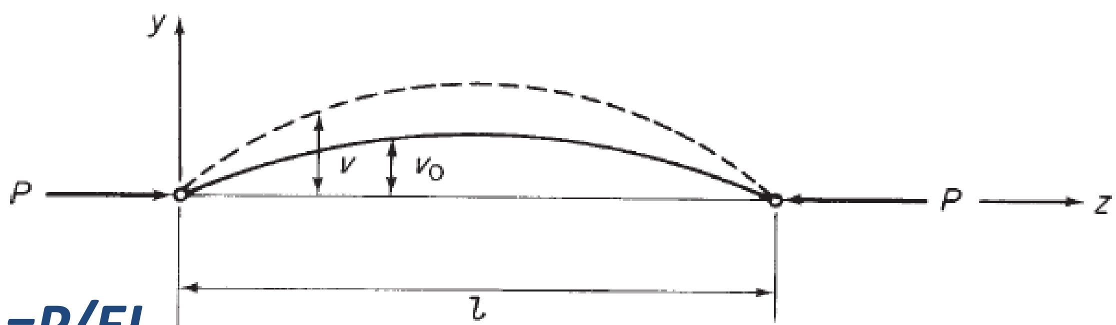
$$\sigma_{\mathrm{CR}} = \frac{\pi^2 E}{(l_{\mathrm{e}}/r)^2}$$

The buckling load of columns is given most accurately for practical purposes by the **tangent modulus theory**

Effect of initial imperfections

Possible imperfections influence to a large degree the behaviour of the column which, unlike the perfect column, begins to bend immediately the axial load is applied.

Let us suppose that a column, initially bent, is subjected to an increasing axial load ***P***.



$$EI\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} - EI\frac{\mathrm{d}^2 v_0}{\mathrm{d}z^2} = -Pv$$

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \lambda^2 v = \frac{\mathrm{d}^2 v_0}{\mathrm{d}z^2}$$ where $\lambda^2 = P/EI$.

The final deflected shape, v, of the column depends upon the form of its unloaded shape, v_0.

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \lambda^2 v = \frac{\mathrm{d}^2 v_0}{\mathrm{d}z^2} \xrightarrow{v_0 = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi z}{l}} \frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \lambda^2 v = -\frac{\pi^2}{l^2}\sum_{n=1}^{\infty} n^2 A_n \sin\frac{n\pi z}{l}$$

Effect of initial imperfections

Solution
$$v = B\cos\lambda z + D\sin\lambda z + \sum_{n=1}^{\infty}\frac{n^2 A_n}{n^2-\alpha}\sin\frac{n\pi z}{l}$$
where ***B*** and ***D*** are constants of integration and $\boldsymbol{\alpha = \lambda^2 l^2/\pi^2}$

The boundary conditions

v = 0 at ***z = 0*** then ***B = 0***

v = 0 at ***z = l*** then ***D = 0***

$$v = \sum_{n=1}^{\infty}\frac{n^2 A_n}{n^2-\alpha}\sin\frac{n\pi z}{l} \qquad \alpha = \frac{Pl^2}{\pi^2 EI} = \frac{P}{P_{\mathrm{CR}}}$$

$\alpha < 1 \rightarrow$ first term in series dominates

A good approximation when the axial load is close to critical load

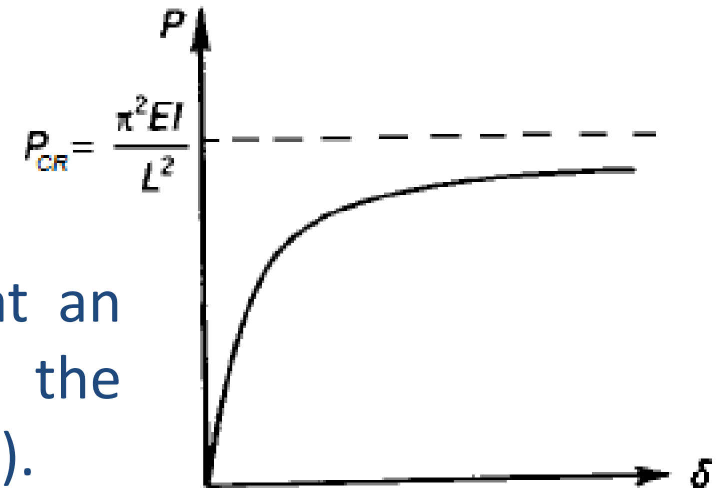
$$v = \frac{A_1}{1-\alpha}\sin\frac{\pi z}{l}$$
or at the centre of the column ***z = l/2***
$$v = \frac{A_1}{1 - P/P_{\mathrm{CR}}}$$

$\boldsymbol{A_1}$ – initial central deflection

Effect of initial imperfections

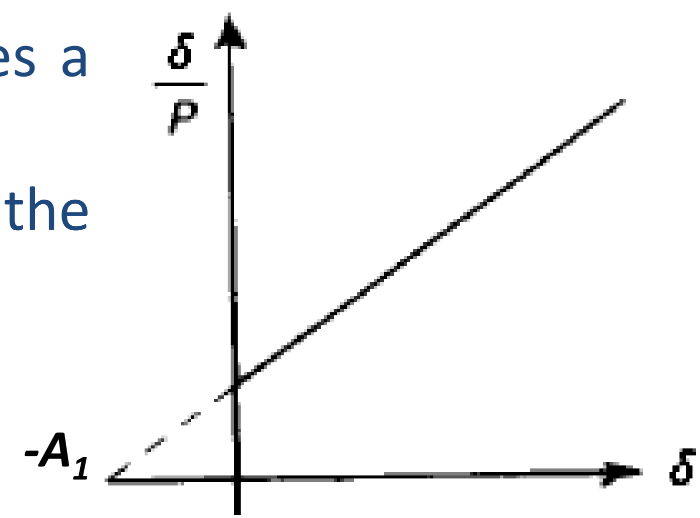
Imagine that we measure central deflections from the initially bowed position of the column, i.e. $\boldsymbol{\delta = v - A_1}$

$$\frac{A_1}{1 - P/P_{CR}} - A_1 = \delta \quad \Longrightarrow \quad \delta = P_{CR}\frac{\delta}{P} - A_1$$



Theoretically load $\boldsymbol{P_{CR}}$ can be attained at an infinitely large deflection (in practice the column breaks before $\boldsymbol{P_{CR}}$ could be attained).

Plotting $\boldsymbol{\delta}$ against $\boldsymbol{\delta/P}$ in a column test gives a value of initial central deflection $\boldsymbol{A_1}$.
The slope of this straight line defines $\boldsymbol{P_{CR}}$ (the buckling load for a perfect-straight column).



Obrigado!