MECÂNICA DE ESTRUTURAS AEROESPACIAIS

Problems of practical classes of the chapter
TORSION OF SOLID SECTIONS

Example 3.3

Determine the

- rate of twist and
- the stress distribution

in a circular section bar of radius *R*
which is subjected to equal and opposite torques *T* at each of its free ends.

If we assume an origin of axes at the centre of the bar the equation of its surface is given by

$$x^2 + y^2 = R^2$$

If we now choose a stress function of the form

$$\phi = C(x^2 + y^2 - R^2)$$

the boundary condition $\varphi = 0$ is satisfied at every point on the boundary of the bar and the constant *C* may be chosen to fulfill the remaining requirement of compatibility.

Therefore from Eqs

$$-2G\frac{\mathrm{d}\theta}{\mathrm{d}z} = \nabla^2\phi = F \text{ (constant)}$$

and the stress function Eq we have

$$4C = -2G\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

so that

$$C = -\frac{G}{2}\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

and

$$\phi = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}(x^2 + y^2 - R^2)|2$$

Substituting for φ in Eq.

$$T = 2\iint \phi \,\mathrm{d}x\,\mathrm{d}y$$

we have

$$T = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(\iint x^2\,\mathrm{d}x\,\mathrm{d}y + \iint y^2\,\mathrm{d}x\,\mathrm{d}y - R^2\iint \mathrm{d}x\,\mathrm{d}y\right)$$

The first and second integrals in this equation both have the value $\pi R^4/4$ while the third integral is equal to πR^2, the area of cross-section of the bar.

Then

$$T = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(\frac{\pi R^4}{4} + \frac{\pi R^4}{4} - \pi R^4\right)$$

which gives

$$T = \frac{\pi R^4}{2}G\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

i.e.

$$T = GJ\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

in which

$J = \pi R^4/2 = \pi D^4/32$

(D is the diameter), the *polar second moment of area* of the bar's cross-section.

Substituting for G(dθ/dz) in Eq.

$$\phi = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}(x^2 + y^2 - R^2)|2$$

from

$$T = GJ\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

we have

$$\phi = -\frac{T}{2J}(x^2 + y^2 - R^2)$$

and from Prandtl Eqs

$$\tau_{zy} = -\frac{\partial\phi}{\partial x} = \frac{Tx}{J} \qquad \tau_{zx} = \frac{\partial\phi}{\partial y} = -\frac{T}{J}y.$$

The resultant shear stress at any point on the surface of the bar is then given by

$$\tau = \sqrt{\tau_{zy}^2 + \tau_{zx}^2}$$

i.e.

$$\tau = \frac{T}{J}\sqrt{x^2 + y^2}$$

i.e.

$$\tau = \frac{TR}{J}$$

The above argument may be applied to any annulus of radius *r* within the cross-section of the bar so that the stress distribution is given by

$$\tau = \frac{TR}{J}$$

and therefore increases linearly from zero at the centre of the bar to a maximum *TR/J* at the surface.

Problem 3.4.

Determine

- the maximum shear stress and
- the rate of twist in terms of the applied torque T

for the section comprising narrow rectangular strips shown in Fig. P.3.5.

Solution

The torsion constant, J, for the complete cross-section is found by summing the torsion constants of the narrow rectangular strips which form the section.

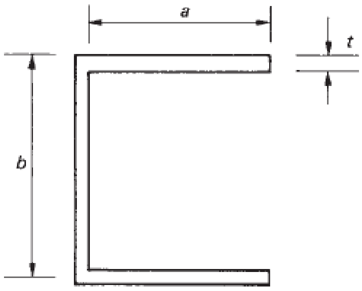

Fig. P.3.5

Ans. $\tau_{max} = 3T/(2a+b)t^2$, $d\theta/dz = 3T/G(2a+b)t^3$.

Then, from Eq.

$$J = \frac{st^3}{3}$$

we have

$$J = 2\frac{at^3}{3} + \frac{bt^3}{3} = \frac{(2a+b)t^3}{3}$$

Therefore, from the general torsion equation

$$T = GJ\frac{d\theta}{dz}$$

we have

$$\frac{d\theta}{dz} = \frac{3T}{G(2a+b)t^3}$$

The maximum shear stress follows from the previous Eq and Eq

$$\tau_{zy,max} = \pm Gt\frac{d\theta}{dz}$$

Hence $$\tau_{max} = \pm Gt\frac{d\theta}{dz} = \pm\frac{3T}{(2a+b)t^2}.$$