Torsion of solid sections

2016

Stress function

As usual the loads applied to thin walled structures are so high that we can **neglect the body forces** of the structure (inertial force, weight).

Then the compatibility equation for plane stress

$$\boxed{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)=-(1+\nu)\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)} \Longrightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(\sigma_x+\sigma_y)=0$$

And equations of equilibrium for plane stress

$$\left.\begin{aligned}\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+X=0\\ \frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yx}}{\partial x}+Y=0\end{aligned}\right\} \Longrightarrow \left.\begin{aligned}\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0\\ \frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yx}}{\partial x}=0\end{aligned}\right\}$$

It seems like it possible to find such functions of $\boldsymbol{\sigma_x}$, $\boldsymbol{\sigma_y}$ and $\boldsymbol{\tau_{xy}}$ (which are really functions of ***x*** and ***y***), that satisfy

- the modified equilibrium equations and
- equation of compatibility.

Stress function

The English mathematician Airy proposed a stress function φ defined by the equations

$$\boxed{\sigma_x = \frac{\partial^2 \phi}{\partial y^2} \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x\, \partial y}}$$

These equations satisfy the equilibrium equations (Check it!).

And substituting these equations into modified equation of compatibility gives so called ***biharmonic equation***

$$\boxed{\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0}$$

Every stress function ***φ*** of ***x*** and ***y*** which satisfy

- *the **biharmonic equation** and*
- *given **boundary conditions***

may be used as a solution for two dimensional problem at all points in the body and at all points on the boundary of the body.

Inverse method

The main idea of the ***inverse method*** is following:

- find a stress function which satisfies the biharmonic equation;
- assume an arbitrary boundary conditions;
- determine the loading conditions which fit the assumed stress function and chosen boundary.

The most common solutions of stress function is expressed as a ***polynomial***.

Timoshenko and Goodier consider ***a variety of polynomials*** for $\boldsymbol{\varphi}$ and determine the associated loading conditions for a variety of rectangular sheets.

The obvious disadvantage of the inverse method is that we are *determining problems to fit assumed solutions*, whereas in structural analysis the reverse is the case.

Inverse Method

Direct Method
(usual problem solving)

Initial data analysis

General (equilibrium) equation

General solution

Boundary conditions

Partial solution

Inverse Method

Suggest stress function

Check biharmonic equation

Assume boundary conditions

Determine loading conditions

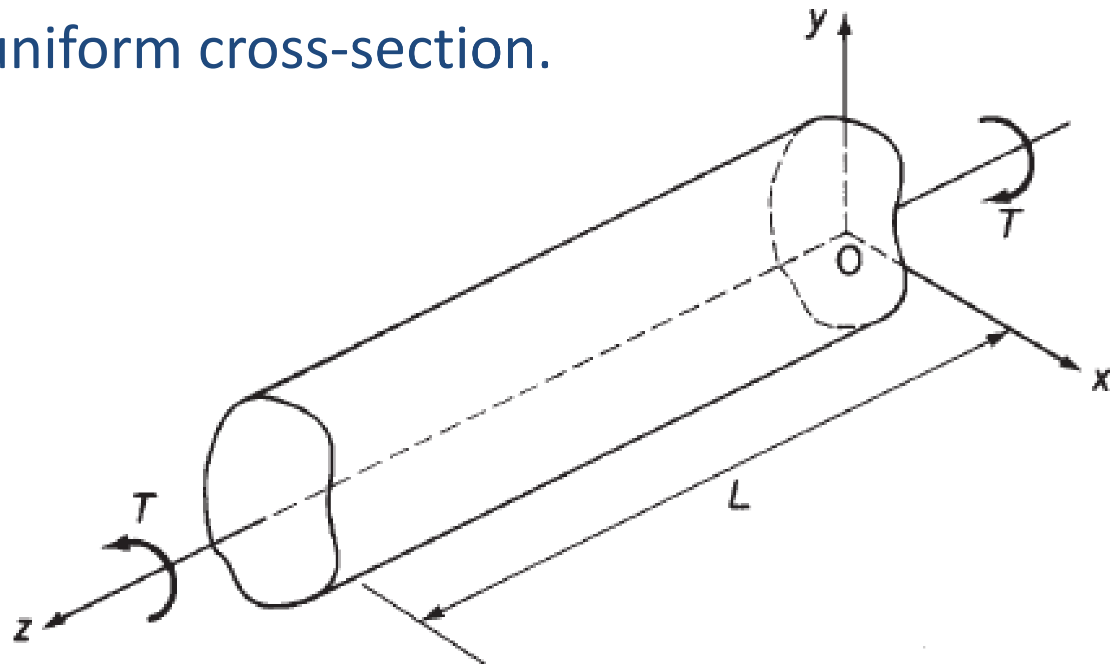
Find practical problem for the given loading conditions

Inverse method

Look
example on whiteboard
(Example 3-1 and 3-2 in Moodle)

Prandtl stress function solution

Consider the straight bar of uniform cross-section.



Assumptions:

- displacements of cross-sections normal to and out of their original planes, are unrestrained;
- no direct loads applied, i.e

$$\sigma_x = \sigma_y = \sigma_z = 0$$

- torque is resisted solely by shear stresses in the plane of the cross-section, i.e.

$\boldsymbol{\tau_{xy} = 0}$

To prove assumptions:

- check equilibrium and compatibility equations;
- fulfil the equilibrium boundary conditions.

Prandtl stress function solution

The equations of equilibrium

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+X=0\\ \frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\tau_{yz}}{\partial z}+Y=0\\ \frac{\partial\sigma_z}{\partial z}+\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{zy}}{\partial y}+Z=0\end{aligned}\right\}\xrightarrow{\tau_{xy}=0,\ \sigma_x=0,\ \sigma_y=0,\ \sigma_z=0}\quad \frac{\partial\tau_{xz}}{\partial z}=0\quad \frac{\partial\tau_{yz}}{\partial z}=0\quad \frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}=0$$

Shear stresses are constant at all points along the length of the bar which have the same ***x*** and ***y*** coordinates.

Prandtl introduced a stress function ***φ*** defined by $\boxed{\frac{\partial\phi}{\partial x}=-\tau_{zy}\qquad \frac{\partial\phi}{\partial y}=\tau_{zx}}$

which identically satisfies the third of the equilibrium equations whatever form ***φ*** may take

φ - ?

Prandtl stress function solution

The strain-stress equation for three-dimensional case are

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \quad \text{and} \quad \gamma_{xy} = \frac{\tau_{xy}}{G} \xrightarrow[\sigma_z = 0]{\tau_{xy} = 0,\ \sigma_x = 0,\ \sigma_y = 0} \varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)]$$

Six equations of compatibility

$$\frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x\, \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z} = \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y\, \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \quad \text{reduce to}$$

$$\frac{\partial^2 \gamma_{xz}}{\partial x\, \partial z} = \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial x\, \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

Prandtl stress function solution

$$\frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x}+\frac{\partial \gamma_{xz}}{\partial y}\right)=0$$
$$\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x}-\frac{\partial \gamma_{xz}}{\partial y}\right)=0$$

$$\gamma_{xz}=\frac{\tau_{xz}}{G} \quad \gamma_{yz}=\frac{\tau_{yz}}{G}$$

$$\frac{\partial \phi}{\partial x}=-\tau_{zy} \quad \frac{\partial \phi}{\partial y}=\tau_{zx}$$

Prandtl equations

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right)=0$$
$$-\frac{\partial}{\partial y}\left(\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}\right)=0$$

Or

$$\frac{\partial}{\partial x}\nabla^2\phi=0$$
$$-\frac{\partial}{\partial y}\nabla^2\phi=0$$

∇^2 is the two-dimensional Laplacian operator

The parameter $\nabla^2\varphi$ is therefore constant at any section of the bar

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=\text{constant}=F$$

Prandtl stress function solution

Boundary conditions *on cylindrical surface*

$$\left.\begin{aligned}\bar{X} &= \sigma_x l + \tau_{yx} m + \tau_{zx} n \\ \bar{Y} &= \sigma_y m + \tau_{xy} l + \tau_{zy} n \\ \bar{Z} &= \sigma_z n + \tau_{yz} m + \tau_{xz} l\end{aligned}\right\}$$

No surface forces

$X^{-} = Y^{-} = Z^{-} = 0$

n = 0

$$\tau_{yz} m + \tau_{xz} l = 0$$

$$l = \frac{\mathrm{d}y}{\mathrm{d}s} \qquad m = -\frac{\mathrm{d}x}{\mathrm{d}s}$$

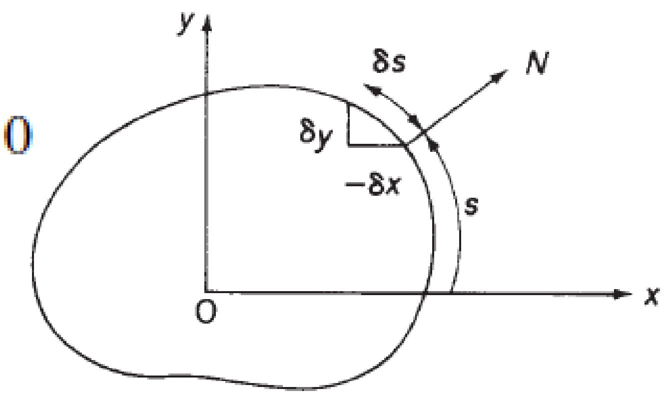
where ***l*** and ***m*** direction cosines of the normal ***N*** to any point on the surface (fig.).

$$\frac{\partial \phi}{\partial x} = -\tau_{zy} \qquad \frac{\partial \phi}{\partial y} = \tau_{zx}$$

$$\tau_{yz} m + \tau_{xz} l = 0$$

$$l = \frac{\mathrm{d}y}{\mathrm{d}s} \qquad m = -\frac{\mathrm{d}x}{\mathrm{d}s}$$

$$\frac{\partial \phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial \phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} = 0$$



or $$\frac{\partial \phi}{\mathrm{d}s} = 0$$

Thus *φ* is constant on the surface of the bar and since the actual value of this constant does not affect the stresses of Prandtl equations we may conveniently take **the constant to be zero**: $\phi = 0$

Prandtl stress function solution

Boundary conditions *on the ends* of the bar

The direction cosines of the normal to the surface ***l =0**, **m=0*** and ***n=1***.

$$\left.\begin{aligned}\bar{X} &= \sigma_x l + \tau_{yx} m + \tau_{zx} n \\ \bar{Y} &= \sigma_y m + \tau_{xy} l + \tau_{zy} n \\ \bar{Z} &= \sigma_z n + \tau_{yz} m + \tau_{xz} l\end{aligned}\right\} \xrightarrow[l=0,\ m=0,\ n=1]{\tau_{xy}=0,\ \sigma_x=0,\ \sigma_y=0,\ \sigma_z=0} \begin{aligned}\bar{X} &= \tau_{zx} \\ \bar{Y} &= \tau_{zy} \\ \bar{Z} &= 0\end{aligned}$$

Shear forces on each end of the bar are distributed over the ends of the bar in the same manner as the shear stresses over the cross-section.

The resultant shear force in the positive direction of the ***x*** axis, which we shall call S_x, is then

$$S_x = \iint \bar{X}\,\mathrm{d}x\,\mathrm{d}y = \iint \tau_{zx}\,\mathrm{d}x\,\mathrm{d}y$$ and with Prandtl equation $$\frac{\partial \phi}{\partial y} = \tau_{zx}$$

$$S_x = \iint \frac{\partial \phi}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = \int \mathrm{d}x \int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = 0$$ as ***φ=0*** at the boundary.

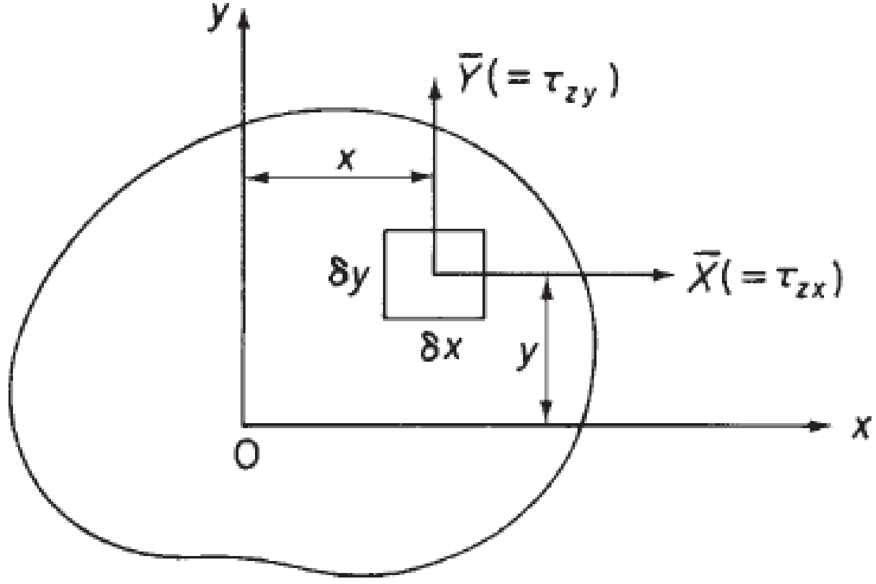
Prandtl stress function solution

As $$S_x = \iint \frac{\partial \phi}{\partial y}\, dx\, dy = \int dx \int \frac{\partial \phi}{\partial y}\, dy = 0$$ the same for ***S_y***

$$S_y = -\int dy \int \frac{\partial \phi}{\partial x}\, dx = 0$$

It follows that there is **no resultant shear force** on the ends of the bar and **the forces represent a torque of magnitude**

$$T = \iint (\tau_{zy} x - \tau_{zx} y)\, dx\, dy$$ positive in anticlockwise sense.



Putting the stress function ***φ***

$$T = -\iint \frac{\partial \phi}{\partial x} x\, dx\, dy - \iint \frac{\partial \phi}{\partial y} y\, dx\, dy$$

or after integration by parts and noting that ***φ = 0***

$$\boxed{T = 2\iint \phi\, dx\, dy}$$

Prandtl stress function solution

Solution is obtained by finding the stress function, which satisfies

$$\nabla^2 \varphi = const$$

at all points within the bar and vanishes on the surface of the bar, and providing that the external torques are distributed over the ends of the bar in an identical manner to the distribution of internal stress over the cross-section (but only in the end regions according to St. Venant's principle).

Therefore, the solution is applicable to sections at distances from the ends usually taken to be greater than the largest cross-sectional dimension.

Prandtl stress function solution

Determination of **displacements:**

- the angle of twist and
- the warping displacement of the cross-section

We have shown that $\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0$ and from

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} \\ \varepsilon_y &= \frac{\partial v}{\partial y} \\ \varepsilon_z &= \frac{\partial w}{\partial z} \end{aligned}\right\}$$

and $\gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}$ $\Rightarrow$ $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = \dfrac{\partial w}{\partial z} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} = 0$

- **Each cross-section rotates as a rigid body** in its own plane about a centre of rotation or twist, and
- that although cross-sections suffer warping displacements normal to their planes **the values of this displacement at points having the same coordinates along the length of the bar are equal**.
- **Each longitudinal fibre of the bar therefore remains unstrained**.

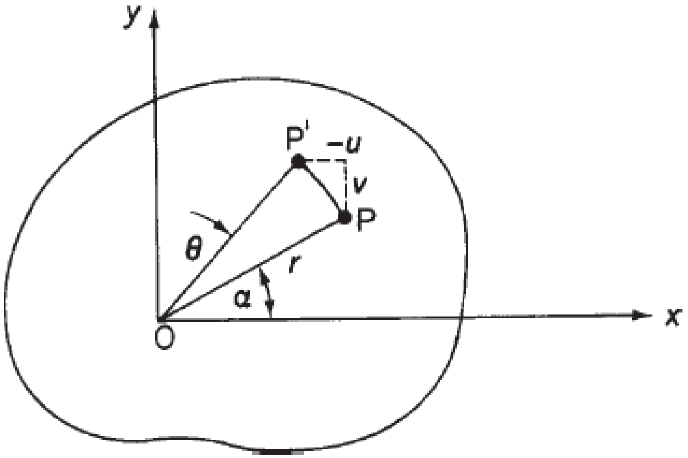
Prandtl stress function solution

Consider that a cross-section of the bar rotates through a small angle ϑ about its centre of twist assumed coincident with the origin of the axes.

P(r, α) will be displaced to ***P'(r, α+ϑ)***

$$u = -r\theta \sin\alpha \qquad v = r\theta\cos\alpha$$

$$u = -\theta y \qquad v = \theta x$$



Referring to

$$\left.\begin{aligned} \gamma_{xz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\ \gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \gamma_{yz} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \end{aligned}\right\}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \quad \gamma_{xz} = \frac{\tau_{xz}}{G} \quad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\left.\begin{aligned} \gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\tau_{zx}}{G} \\ \gamma_{zy} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{\tau_{zy}}{G} \end{aligned}\right]$$

Substituting for ***u*** and ***v*** we have $\displaystyle \frac{\partial w}{\partial x} = \frac{\tau_{zx}}{G} + \frac{d\theta}{dz}y \qquad \frac{\partial w}{\partial y} = \frac{\tau_{zy}}{G} - \frac{d\theta}{dz}x$

Prandtl stress function solution

These equations enable the warping displacement, and since each cross-section rotates as a rigid body ϑ is a function of z only.

Differentiating 1st and 2nd equations correspondingly with respect to y and x and subtracting we have

$$0 = \frac{1}{G}\left(\frac{\partial \tau_{zx}}{\partial y} - \frac{\partial \tau_{zy}}{\partial x}\right) + 2\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

Expressing τ_{zx} and τ_{zy} in terms of φ gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\frac{\mathrm{d}\theta}{\mathrm{d}z} \xrightarrow{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \text{constant} = F} -2G\frac{\mathrm{d}\theta}{\mathrm{d}z} = \nabla^2\phi = F \text{ (constant)}$$

Applying torsion constant J defined by the general torsion equation as

$$\boxed{T = GJ\frac{\mathrm{d}\theta}{\mathrm{d}z}}$$

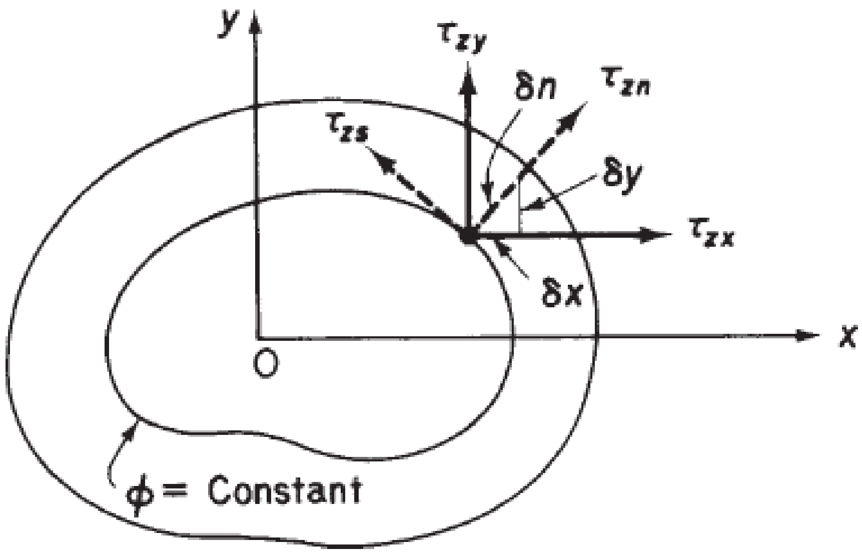
we determine *torsional rigidity of the bar* ***GJ***

$$GJ \longrightarrow \boxed{GJ = -\frac{4G}{\nabla^2\phi}\iint \phi \,\mathrm{d}x\,\mathrm{d}y}$$

$$-2G\frac{\mathrm{d}\theta}{\mathrm{d}z} = \nabla^2\phi = F \qquad T = 2\iint \phi\,\mathrm{d}x\,\mathrm{d}y$$

Prandtl stress function solution

Consider now the line of constant *φ*. If ***s*** is the distance measured along this line from some arbitrary point then $\frac{\partial \phi}{\partial s} = 0 = \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial x}\frac{dx}{ds}$



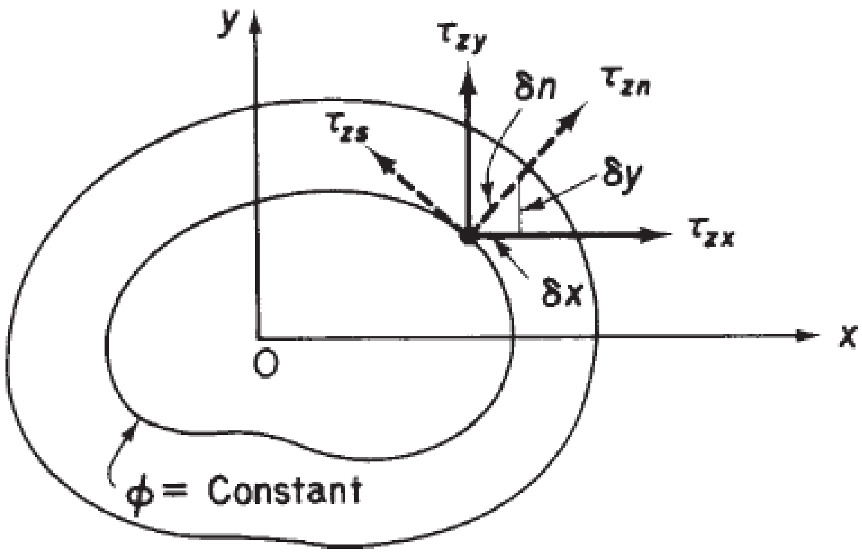
Using Prandtl equations and direction cosines we have

$$\frac{\partial \phi}{\partial s} = \tau_{zx} l + \tau_{zy} m = 0$$

According to figure the normal and tangential components of shear stress are $\tau_{zn} = \tau_{zx} l + \tau_{zy} m$ $\tau_{zs} = \tau_{zy} l - \tau_{zx} m$, i.e. the normal shear stress is zero so that the resultant shear stress at any point is tangential to a line of constant ***φ***.

Prandtl stress function solution

Lines with constant $\boldsymbol{\varphi}$ are known as ***lines of shear stress*** or ***shear lines***.



Substituting $\boldsymbol{\varphi}$ in equation for $\boldsymbol{\tau_{zs}}$

$$\tau_{zs} = -\frac{\partial \phi}{\partial x} l - \frac{\partial \phi}{\partial y} m$$

which may be written (from figure) as

$$\tau_{zx} = -\frac{\partial \phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}n} - \frac{\partial \phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}n} = -\frac{\partial \phi}{\partial n}$$

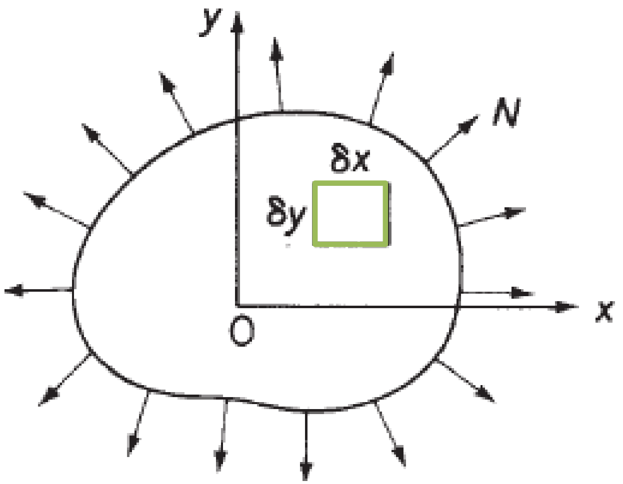
where, in this case, the direction cosines ***l*** and ***m*** are defined in terms of an elemental normal of length $\boldsymbol{\delta n}$.

Conclusions: - the resultant shear stress at any point is tangential to the line of shear stress through the point and

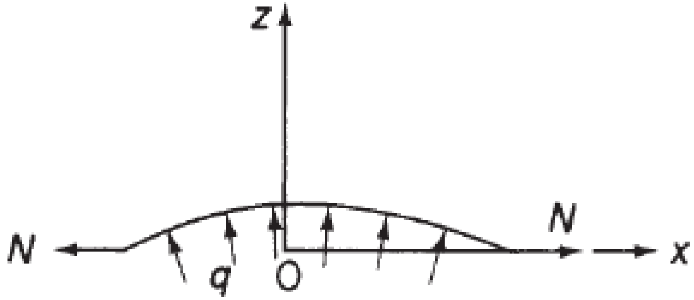
- has a value equal to minus the derivative of φ in a direction normal to the line.

The membrane analogy

The membrane is a thin sheet of material which relies for its resistance to transverse loads on internal in-plane or membrane forces.



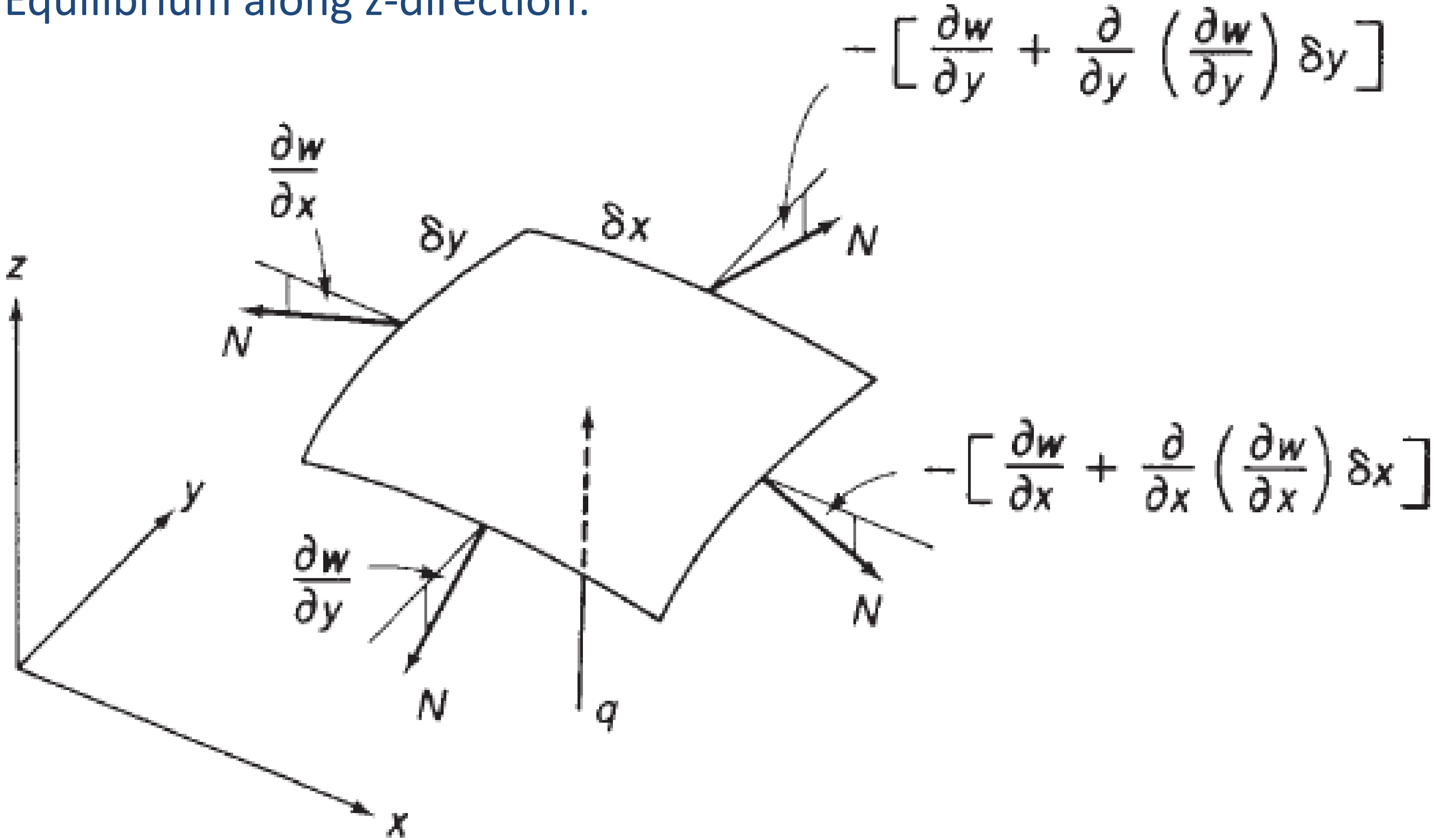
- membrane has the same external shape as the cross-section of a torsion bar;
- it supports a transverse uniform pressure ***q;***
- it is restrained along its edges by a uniform tensile force ***N/unit length;***
- ***N*** remains unchanged as the membrane deflects (small transverse displacements).



Consider the equilibrium of an element ***δxδy*** of the membrane.

The membrane analogy

Equilibrium along z-direction:



The membrane analogy

$$-N\delta y\frac{\partial w}{\partial x} - N\delta y\left(-\frac{\partial w}{\partial x} - \frac{\partial^2 w}{\partial x^2}\delta x\right) - N\delta x\frac{\partial w}{\partial y} - N\delta x\left(-\frac{\partial w}{\partial y} - \frac{\partial^2 w}{\partial y^2}\delta x\right) + q\delta x\delta y = 0$$

or $\boxed{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \nabla^2 w = -\frac{q}{N}}$ must be satisfied at all points within the boundary of the membrane

Furthermore, at all points on the boundary ***w = 0***

Comparing

$$\boxed{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \nabla^2 w = -\frac{q}{N}} \qquad \boxed{-2G\frac{d\theta}{dz} = \nabla^2\phi = F \text{ (constant)}}$$

w is analogous to ***φ*** when ***q*** is constant.

If the membrane has the same external shape as the cross-section of the bar, then $w(x, y) = \phi(x, y)$ $\boxed{\frac{q}{N} = -F = 2G\frac{d\theta}{dz}}$

The membrane analogy

Relating the deflected form of the membrane to the state of stress in the bar, we have:

- contour lines (***w = const***) correspond to lines of **constant *φ*** in the bar;
- the resultant shear stress at any point is tangential to the membrane contour line;
- the resultant shear stress equal in value to the negative of the membrane slope, ***∂w/∂n***, at that point, the direction ***n*** being normal to the contour line;
- the volume between the membrane and the ***xy*** plane is

$$\mathrm{Vol} = \iint w \, \mathrm{d}x \, \mathrm{d}y$$

and knowing that $T = 2 \iint \phi \, \mathrm{d}x \, \mathrm{d}y$

then ***T = 2 Volume***

method is useful of analysing torsion bars possessing irregular cross-sections for which ***φ*** forms are not known.

Torsion of a narrow rectangular strip

Membrane surface has the same cross-sectional shape at all points along its length except for small regions near its ends.

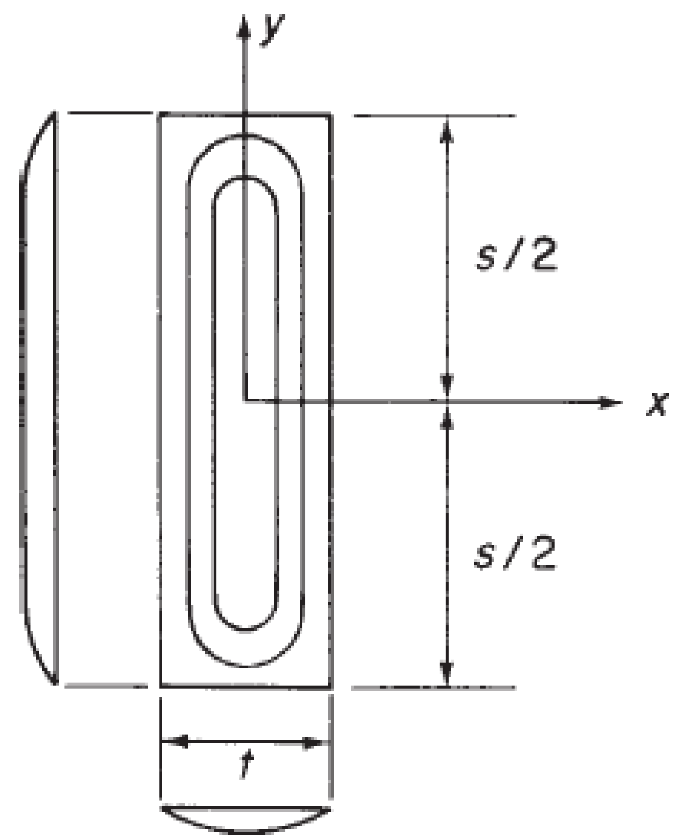
If we ignore these regions and assume that the shape of the membrane is independent of ***y***, than equation

$$-2G\frac{\mathrm{d}\theta}{\mathrm{d}z} = \nabla^2\phi = F \text{ (constant)}$$

simplifies to

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = -2G\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

Integration gives



Torsion of a narrow rectangular strip

$$\phi = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}x^2 + Bx + C$$

Substituting the boundary conditions ***φ=0*** at ***x = ± t/2***

$$\phi = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left[x^2 - \left(\frac{t}{2}\right)^2\right]$$

Although $\varphi \neq 0$ along the short edges of the strip, the actual volume of the membrane differs only slightly from the assumed volume, i.e. corresponding torque and shear stresses are reasonably accurate.

Stress distribution (putting φ into Prandtl equation):

$$\tau_{zy} = 2Gx\frac{\mathrm{d}\theta}{\mathrm{d}z} \quad \tau_{zx} = 0$$

Maximal value is attained at x=±t/2: $\tau_{zy,\max} = \pm Gt\frac{\mathrm{d}\theta}{\mathrm{d}z}$

Torsion of a narrow rectangular strip

The torsion constant ***J*** follows from

$$\phi = -G\frac{d\theta}{dz}\left[x^2 - \left(\frac{t}{2}\right)^2\right]$$

$$GJ = -\frac{4G}{\nabla^2\phi}\iint \phi\, dx\, dy \longrightarrow J = \frac{st^3}{3}$$

$$\tau_{zy,\max} = \pm Gt\frac{d\theta}{dz} \xrightarrow{J = \frac{st^3}{3}} \boxed{\tau_{zy,\max} = \frac{3T}{st^3}}$$

These equations represent exact solutions when ***s/t* →*ꝏ***.
At ***s/t = 10*** the error is of the order of only ***6%***.

In order to retain the usefulness of the analysis, a factor *μ* is included

$$J = \frac{\mu st^3}{3}$$

Values of *μ* for different types of section are found experimentally and quoted in various references. As *s/t* →*ꝏ* then *μ* → 1.

Torsion of a narrow rectangular strip

The cross-section suffers warping displacements normal to its plane:

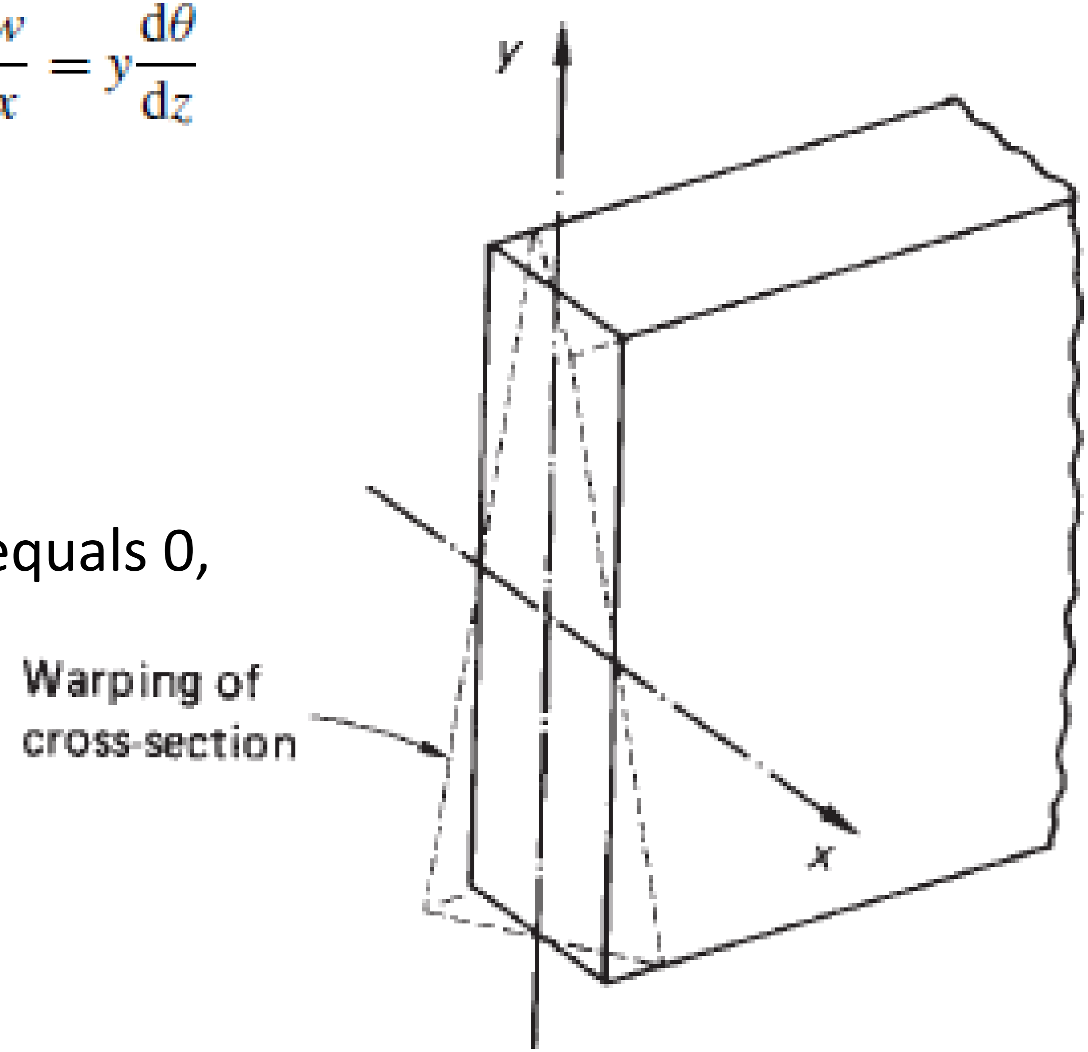
$$\frac{\partial w}{\partial x} = \frac{\tau_{zx}}{G} + \frac{\mathrm{d}\theta}{\mathrm{d}z}y \xrightarrow{\tau_{zx} = 0} \frac{\partial w}{\partial x} = y\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

After integration, we have

$$w = xy\frac{\mathrm{d}\theta}{\mathrm{d}z} + \text{constant}$$

As ***w = 0*** at ***x = y = 0*** constant equals 0, therefore

$$\boxed{w = xy\frac{\mathrm{d}\theta}{\mathrm{d}z}}$$



Torsion of a narrow rectangular strip

Obrigado!