Gas Dynamics — Work 1

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1 Part 1

1.1 Initial Values

The freestream properties at 18.00 km and Mach 3, as taken from website AeroToolbox (https://aerotoolbox.com/atmcalc/), are:

Table 1 – Freestream flow properties.

Property	**Value**
P	7505 Pa
M	3.000
T	216.7 K
ρ	0.1207 kg/m^3

1.2 Equation system

The flow across the planar 3-shockwave diffuser is assumed to be constant in the control volumes ∞, 1, and 2. Inside control volume 3, the flow is subsonic and isentropic, but we are only interested in the values imediately after the shockwave, so we will not calculate the changes inside this volume.

Across the region boundaries, the flow goes through shockwaves. The flow after a shockwave can be calculated from the flow before the shockwave according to Eqs. (1.1) to (1.7) (ANDERSON JR., 2017), where the properties before and after the shock are represented by the subscripts 1 and 2, respectively. The right side of these equations depend only on the Mach number before the shockwave M_1 and the shockwave angle β. Alternatively, we can calculate β from θ, but that would lead to costlier calculations.

$$M_{n,1} = M_1 \sin\beta \tag{1.1}$$

$$M_{n,2}{}^2 = \frac{1 + \frac{\gamma-1}{2} M_{n,1}{}^2}{\gamma M_{n,1}{}^2 - \frac{\gamma-1}{2}} \tag{1.2}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1) M_{n,1}{}^2}{2 + (\gamma+1) M_{n,1}{}^2} \tag{1.3}$$

$$\frac{p_2}{p_1} = 1 + \frac{2}{\gamma+1}(M_{n,1}{}^2 - 1) \tag{1.4}$$

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} \tag{1.5}$$

$$M_2 = \frac{M_{n,2}}{\sin(\beta-\theta)} \tag{1.6}$$

$$\tan\theta = 2\cot\beta \frac{M_1{}^2 \sin^2\beta - 1}{M_1{}^2(\gamma + \cos(2\beta)) + 2} \tag{1.7}$$

We are also interested in the total properties (represented with subscript 0) and in the sonic speed a. In any point of the flow, they can be calculated from the static properties using Eqs. (1.8) to (1.11) (ANDERSON JR., 2017).

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2} M^2 \tag{1.8}$$

$$\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} \tag{1.9}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{1}{\gamma-1}} \tag{1.10}$$

$$a = \sqrt{\gamma R T} \tag{1.11}$$

Using these equations, we can calculate the values of the flow properties at each control volume for given freestream properties and shockwave angles. Then, the efficiency function for an 3-shockwave diffuser is simply $P_{0_3}/P_{0_\infty} = \mathrm{f}(M_\infty, \beta_2, \beta_2)$. This function can be calculated easily by repeated application of the shockwave equations, and a descent algorithm can be used to fing the local maximum.

1.3 Optimization

Finding the optimal diffuser is equivalent to finding the global maximum of the efficiency function $\eta(\beta_1, \beta_2)$. A script, shown in full in Appendix A, was used to find the optimal diffuser angles, and some important parts of the code will be discussed here.

The script uses a class, called `Flow`, to represent flow state. This class implements total/static properties conversions, and also a shockwave method. The shockwave method is shown in listing 1, though the actual implementation is a class method, not a plain function.

Listing 1: Shockwave function

```python
def _shockwave(self, beta) -> Flow:
    """
    Calculates flow properties after an oblique shockwave.
    Uses equations (9.13) through (9.18) @anderson2017.

    Parameters:
    beta: oblique angle of the shockwave [rad]

    Returns:
    The Flow object after the shockwave
    """

    # Checks shockwave is valid (supersonic and positive theta)
    # Maybe subsonic flow could be accepted, and spit a supersonic flow?
    if (self.mach < 1) or ((theta := self.velocity_deflection(beta)) <= 0):
        return Flow(np.nan, np.nan, np.nan, np.nan, self.fluid)

    gamma = self.fluid.gamma

    # Equation (9.13) @anderson2017
    M_n_1 = self.mach * np.sin(beta)
    M_n_1_sq = M_n_1 * M_n_1
    # Equation (9.14) @anderson2017 (modified)
    M_n_2 = np.sqrt((2 + (gamma - 1) * M_n_1_sq) / (2 * gamma * M_n_1_sq - (gamma - 1)))
    # Equation (9.15) @anderson2017
    rho_2 = self.density * (gamma + 1) * M_n_1_sq / (2 + (gamma - 1) * M_n_1_sq)
    # Equation (9.16) @anderson2017
    p_2 = self.pressure * (1 + 2 * gamma * (M_n_1_sq - 1) / (gamma + 1))
    # Equation (9.17) @anderson2017
    T_2 = self.temperature * p_2 * self.density / (self.pressure * rho_2)
    # Equation (9.18) @anderson2017
    M_2 = M_n_2 / np.sin(beta - theta)

    flow_2 = Flow(p_2, M_2, T_2, rho_2, self.fluid)
    return flow_2
```

Using the `Flow` class, we can create a function that calculates the total pressure efficiency $\eta(M_\infty, \beta_1, \beta_2)$. In fact, the function is easily generalized to n-shockwave diffusers, so the general version was implemented. The implementation used in the script is shown in listing 2.

Listing 2: Total Pressure efficiency in an n-shockwave diffuser

```python
def nShock_p0_eff(betas: tuple[float, ...], flow_in: Flow) -> float:
    """
    Calculates the total pressure efficiency in a diffuser with n-1 oblique and 1 normal
    shockwaves.

    Parameters:
    betas: angles of the n-1 oblique shockwaves [rad]
    flow_in: Flow object of the flown in the upstream

    Returns:
    Total pressure efficiency of the diffuser
    """

    flow_after = flow_in
    for beta in betas:
        flow_after = flow_after.shockwave(beta)
    flow_after = flow_after.shockwave(np.pi / 2)
    return flow_after.total_pressure / flow_in.total_pressure
```

Finally, the script uses the `SciPy` module to implement the optimization algorithm. The algorithm implements minimization on a scalar function of n-dimensional inputs, i.e. it finds a local minimum. Since we want the local maximum, the optimized function was the total pressure loss $1-\eta$, instead of the efficiency η. The optimization method chosen takes a initial guess, which was chosen to be just above the minimum β angles to ensure existence. The implementation is shown in listing 3.

Listing 3: Optimization algorithm

```python
# Finds minimum beta values, aka beta when theta = 0
min_beta_1 = root_scalar(flow_inf.velocity_deflection, x0=0.1, x1=0.11).root
assert 0 < min_beta_1 < np.pi / 2

# Initial guess based on minimum beta angle (scaled to avoid evaluating on the boundary)
initial_betas = (min_beta_1 * 1.01, min_beta_1 * 1.01)

print("Solving...")
# Minimizes the total pressure efficiency loss using a descent algorithm
sol_weak = minimize(
    lambda betas: 1 - nShock_p0_eff(betas, flow_inf),
    initial_betas,
    method="SLSQP",
    options={"disp": True},
)
print("Done!")
print(f"Solution: {sol_weak.x}")
```

```
Solving...
Optimization terminated successfully    (Exit mode 0)
            Current function value: 0.2563475732605983
            Iterations: 7
            Function evaluations: 24
            Gradient evaluations: 7
Done!
Solution: [0.5622804  0.78800818]
```

To investigate wether our optimization was correct and if the maximum is global, we can plot coarse contour graph of η, shown in Fig. 1. As can be seen, the maximum does appear to be correct and global.

We can then use our solution to extract the values we want, i.e. the optimal diffuser angles and the optimal flow properties in each control volume. This is implemented in listing 4, along with some assertions, using the `Flow` class methods.

Listing 4: Extracting the optimal diffuser angles and flow properties

```python
# Gets and prints all the solution values
betas = sol_weak.x
assert np.all(0 < betas) and np.all(betas < np.pi / 2)

p0_eff = 1 - sol_weak.fun
assert 0 < p0_eff < 1

print(f"Efficiency = {p0_eff:.4}")
print()

# Stores flow states into a list
flows = [flow_inf]
for n, beta in enumerate(betas):
    print(f"beta_{n+1}:  {np.rad2deg(beta)}")
    print(f"theta_{n+1}: {np.rad2deg(flows[-1].velocity_deflection(beta))}")
    flows.append(flows[-1].shockwave(beta))
    print(f"Flow_{n+1}:", flows[-1], "\n")
flows.append(flows[-1].shockwave(np.pi / 2))
print(f"Flow_out: {flows[-1]}")
# Garbage Collection
gc.collect()
```

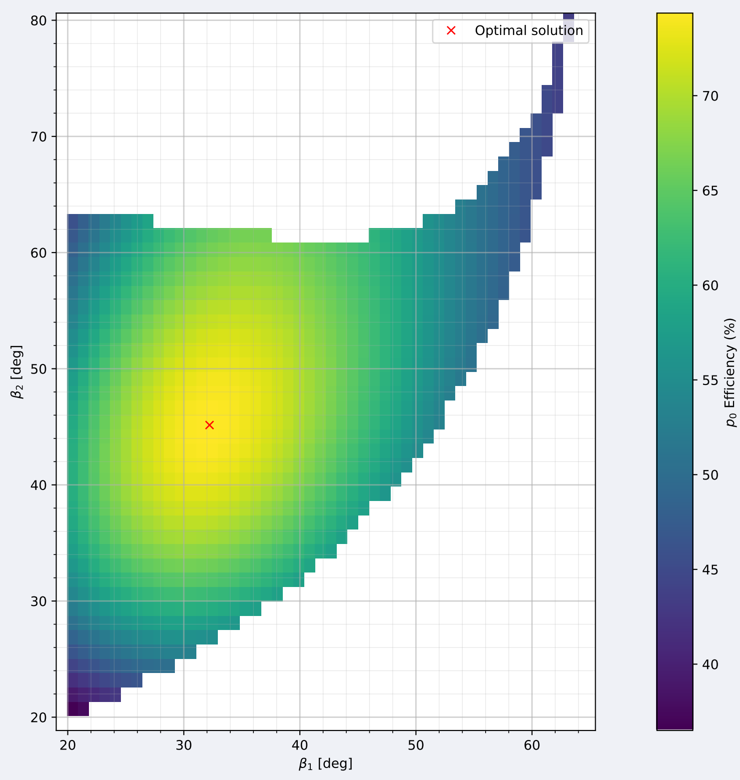
```
Efficiency = 0.7437

beta_1:  32.21629364825329
theta_1: 14.976701128986543
Flow_1: Flow(
        Pressure: 2.115e+04
        Mach: 2.256
        Temperature: 300.6
        Density: 0.2451
        fluid: Fluid(R=287.058, gamma=1.4)
)

beta_2:  45.14954292171187
theta_2: 18.814742838551123
Flow_2: Flow(
        Pressure: 5.959e+04
        Mach: 1.507
        Temperature: 417.1
        Density: 0.4977
        fluid: Fluid(R=287.058, gamma=1.4)
)

Flow_out: Flow(
        Pressure: 1.48e+05
        Mach: 0.6986
        Temperature: 552.7
        Density: 0.9328
        fluid: Fluid(R=287.058, gamma=1.4)
)
```

Figure 1: Contour plot of η function.

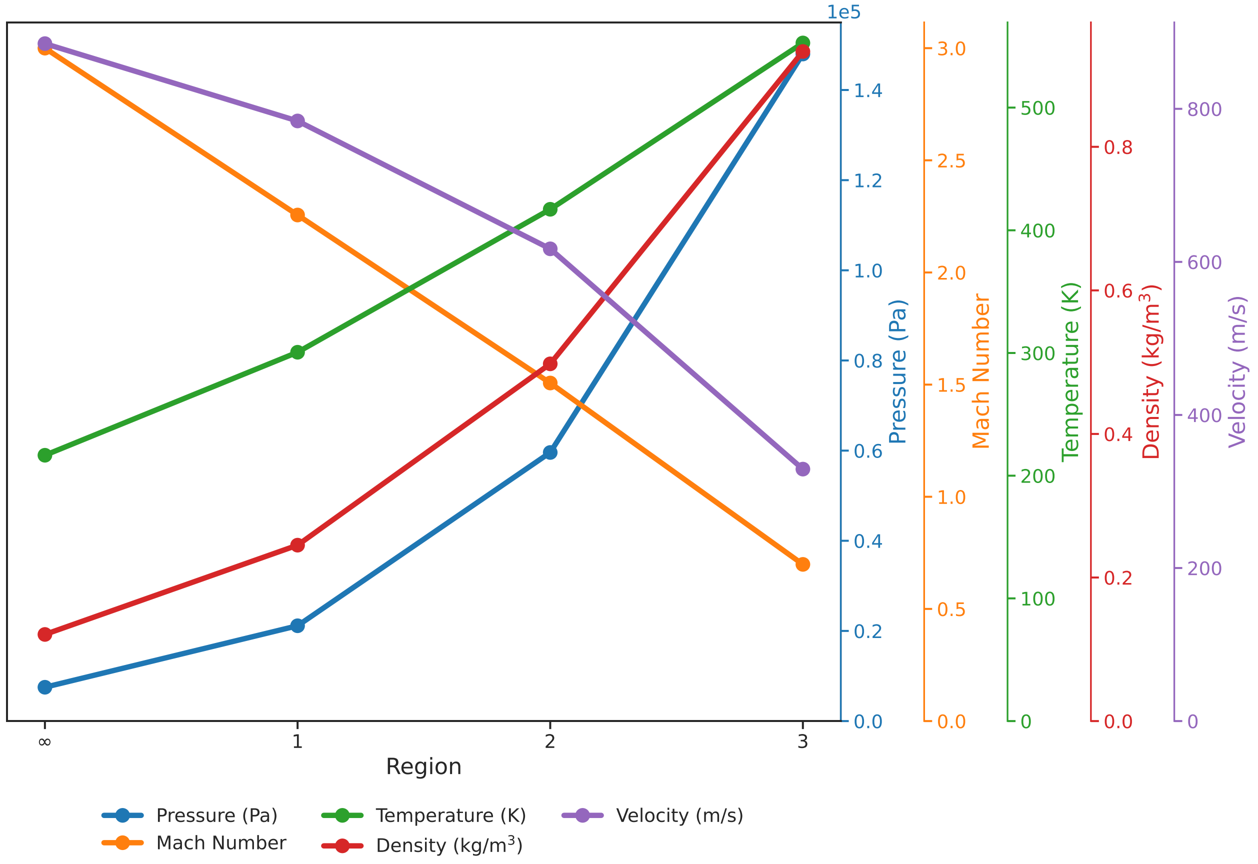


Then, we can plot the properties over each region. To better visualize relative changes in magnitude, we compared Pressure, Mach number, Temperature, Density, and Velocity in the Fig. 2. For easier lookup, the properties were also tabulated in Table 2.

Table 2 – Flow Properties in the 3-shockwave diffuser

Region	**Pressure** [Pa]	**Mach Number**	**Temperature** [K]	**Density** [kg/m^3]	**Velocity** [m/s]
∞	7505	3.000	216.7	0.1207	885.2
1	21 150	2.256	300.6	0.2451	784.2
2	59 590	1.507	417.1	0.4977	617.1
3	148 000	0.6986	552.7	0.9328	329.2

Figure 2: Properties in each control volume.



1.4 Optimal Geometry

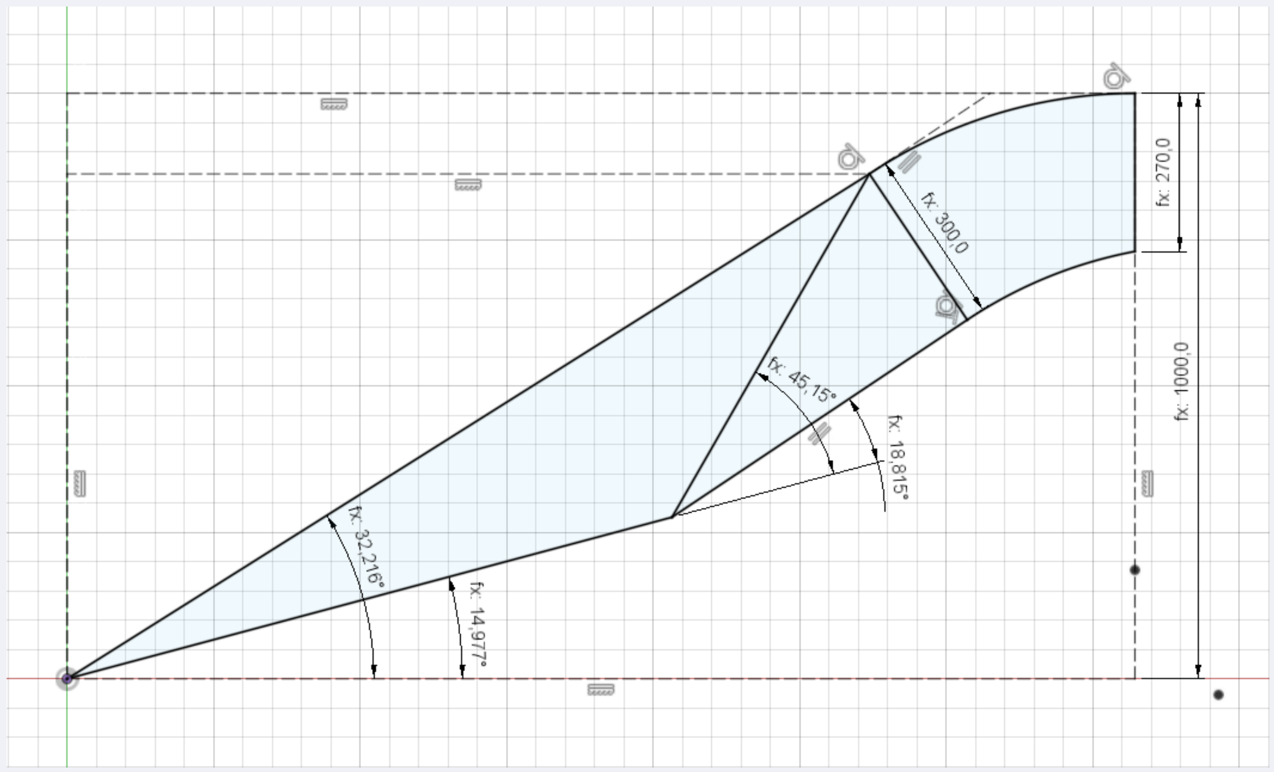
With the optimal angles, the geometry is still under-determined. The external wall size, D can be scaled arbitrarily, and will be set as unitary for a geometry generalization. Scaling this value will scale the geometry uniformly. For the walls to normal shockwave to form adequately, both the inner and outer wall must be (nearly) tangent to the flow in the normal shockwave vicinity. Additionaly, a constraint is put in place such that the outer wall ends in a horizontal direction, finishing the rotation of the flow.

The wedge walls can still be scaled (up to a limit) inside these constraints, changing the geometry of entry to the diffuser. To constrain this geometry, the cross-sectional length of the normal shockwave was set to $D_e = 0.3D$.

Lastly, the internal length D_{int} is also arbitrary. Oswatitsch (1944) and (HERMANN, 1956) show some restrictions to the throat formed here, but not a conclusive answer to its value. The critical value (where the normal shockwave happens right at the entrance) depends on the geometry further down the flow. Thus, as an initial approximation, the D_{int} was set such that the diffuser exit length is equal to $D_o = 0.9D_e$.

The final geometry, then, was modeled on the CAD Software Fusion 360, and shown in Fig. 3.

Figure 3: Geometry of the 3-shockwave diffuser.



2 Part 2

The flow inside the ramjet motor can be divided into 3 parts: Expansion, Heat Addition, and Nozzle.

2.1 Expansion

Quasi–one-dimensional, isentropic expansion follows the Area–Mach Number relationship (Eq. (2.1)). Even though a sonic throat is not present, the relationship still stands, and the Mach Number anywhere in the flow can be calculated from the area ratio with a point with known properties. In our case, the point with known properties will be the entrance right after the normal shockwave from Chapter 1. The area in the entrance is D_e, and the area after expansion is D.

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}} \tag{2.1}$$

To calculate the Mach Number, we can find A^* and numerically solve Eq. (2.1) for $A = D$. Thus, the implementation in the script can be reduced to the Area–Mach Number relationship implemented in listing 5 and the root search implemented in listing 6. Details of the implementation can be found in Appendix A.

Listing 5: Area–Mach Number relationship function

```
def area_mach(M: float, gamma: float) -> float:
    """
    Calculates the ratio A / A* in isentropic quasi-one-dimensional flow.
    Uses Equation (10.32) @anderson2017.

    Parameters:
    M: Mach number
    gamma: Specific heat ratio
    """
    g_ratio = (gamma + 1) / (gamma - 1)
    # Equation (10.32) @anderson2017 (modified)
    return (2 / (gamma + 1) + M * M / g_ratio) ** (g_ratio / 2) / M
```

Listing 6: Finding Mach number after expansion

```
# Calculates Mach number using the area-Mach relation
# Calculates virtual throat
D_virtual = D_entrance / area_mach(flows2[-1].mach, flows2[-1].fluid.gamma)
# Solves area-Mach relation for the new Mach number
expanded_mach = root_scalar(
    lambda M: area_mach(M, flows2[-1].fluid.gamma) - D / D_virtual,
    bracket=[1, 1e-2],
    method="brentq",
).root

# Appends new Flow to the list
flows2.append(Flow.from_total_values(p0, expanded_mach, T0, rho0, flows2[-1].fluid))
print("Flow after expansion:", flows2[-1])
```

```
Flow after expansion: Flow(
        Pressure: 2.013e+05
        Mach: 0.161
        Temperature: 603.5
        Density: 1.162
        fluid: Fluid(R=287.058, gamma=1.4)
)
```

2.2 Heat Addition

The temperature variation ΔT will be considered as a change in static temperature. Equation (2.2) gives the relationship between Mach Number and static temperature (ANDERSON, 2021). Since we know the temperature before and after heat addition, the new Mach Number can be found by solving this equation.

When the new Mach Number is known, the new pressure and density can be calculated using Eqs. (2.3) and (2.4) (ANDERSON, 2021). Thus, the flow after heat addition is completely defined.

$$\frac{T_2}{T_1} = \left(\frac{1+\gamma {M_1}^2}{\gamma {M_2}^2}\right)^2 \left(\frac{{M_2}^2}{{M_1}^2}\right) \tag{2.2}$$

$$\frac{p_2}{p_1} = \frac{1+\gamma {M_1}^2}{1+\gamma {M_2}^2} \tag{2.3}$$

$$\frac{\rho_2}{\rho_1} = \frac{p_2}{p_1}\frac{T_1}{T_2} \tag{2.4}$$

Using Eq. (2.2) the script can find the new Mach number as shown in listing 7, while the pressure and density were calculated as shown in listing 8, yielding a complete Flow object.

Listing 7: Finding Mach number after heat addition

```
# Calculates temperature after heat addition
expanded_temp = flows2[-1].temperature
heated_temp = expanded_temp + delta_temp

# Calculates Mach number after heat addition
# Part of Equation (3.81) @anderson2021 that doesn't depend on the new Mach number
const_part = (
    np.sqrt(heated_temp / expanded_temp)
    * expanded_mach
    / (1 + flows2[-1].fluid.gamma * expanded_mach * expanded_mach)
)
# Solves Equation (3.81) @anderson2021 (modified) for the new Mach number
heated_mach = root_scalar(
    lambda M: const_part - M / (1 + flows2[-1].fluid.gamma * M * M),
    bracket=[1, 0],
    method="brentq",
).root
print(f"New Mach number: {heated_mach:.4}")
```

```
New Mach number: 0.3358
```

Listing 8: Finding flow properties after heat addition

```python
# Calculates remaining Flow properties after heat addition
# Equation (3.78) @anderson2021
heated_pressure = (
    flows2[-1].pressure
    * (1 + flows2[-1].fluid.gamma * expanded_mach * expanded_mach)
    / (1 + flows2[-1].fluid.gamma * heated_mach * heated_mach)
)
# Equation (3.79) @anderson2021
heated_rho = (
    flows2[-1].density
    * heated_pressure
    / flows2[-1].pressure
    * expanded_temp
    / heated_temp
)

# Appends new Flow to the list
flows.append(
    Flow(heated_pressure, heated_mach, heated_temp, heated_rho, flows2[-1].fluid)
)
print("Flow after heat addition:", flows[-1])
```

```
Flow after heat addition: Flow(
        Pressure: 1.802e+05
        Mach: 0.3358
        Temperature: 2.103e+03
        Density: 0.2984
        fluid: Fluid(R=287.058, gamma=1.4)
)
```

2.3 Nozzle

Just as in Section 2.1, the nozzle is a quasi-one-dimensional flow. The nozzle is defined by the area ratio, and the Mach Number. Using Eq. (2.1), one can find the critical throat area, implemented in the script as shown in listing 9.

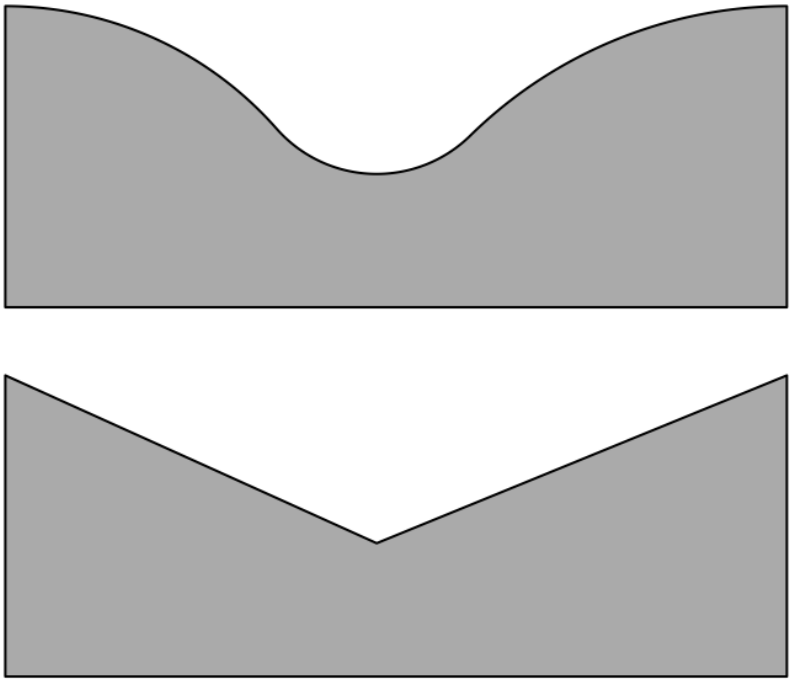
Listing 9: Finding the throat area

```python
# Calculates throat semi-area
D_throat = D / area_mach(heated_mach, flows[-1].fluid.gamma)
print(f"Throat area: {D_throat}")
```

```
Throat area: 0.542665144954579
```

With the nozzle throat area D_t known, the geometry of the nozzle is fully determined. The two nozzle shapes under study are shown in Fig. 4.

Figure 4: Nozzle shapes.



The walls for each shape were implemented as functions, and flow properties were calculated using Eq. (2.1) and Eqs. (1.8) and (1.11). The code implementing this calculation is shown in listing 10.

Listing 10: Calculating flow properties in the nozzle

```
nozzles_flows = np.empty([len(nozzle_curves), len(x)], Flow)
p0_nozzle, T0_nozzle, rho0_nozzle = flows2[-1].total_values()
for n, curve in enumerate(nozzle_curves):
    # Calculates the Mach number across the nozzle.
    crit_index = np.argmin(curve[1])
    D_crit = curve[1, crit_index]
    crit_x = curve[0, crit_index]
    machs = np.array(
        [
            root_scalar(
                lambda M: area_mach(M, flows2[-1].fluid.gamma) - y_ / D_crit,
                # The sign function selects solution branch according to starting flow
                # and position relative to critical length
                bracket=[1, 100 ** np.sign((1 - flows2[-1].mach) * (x_ - crit_x))],
                method="brentq",
            ).root
            for x_, y_ in curve.T
        ]
    )
    nozzles_flows[n, :] = np.array(
        [
            Flow.from_total_values(
                p0_nozzle, mach, T0_nozzle, rho0_nozzle, flows2[-1].fluid
            )
            for mach in machs
        ]
    )
```

Some flow properties were calculated and plotted in Fig. 5, where one can compare their relative changes as well as the differences between the two nozzle shapes.

Finally, the full ramjet design sketch can be seen in Fig. 6. Some dimensions remained unconstrained (namely the length of the expansion region and of the combustion chamber), and were set to D.

Figure 5: Flow properties in the nozzle. Dotted lines represent the Bell shaped nozzle, while solid lines represent the wedge shaped nozzle.

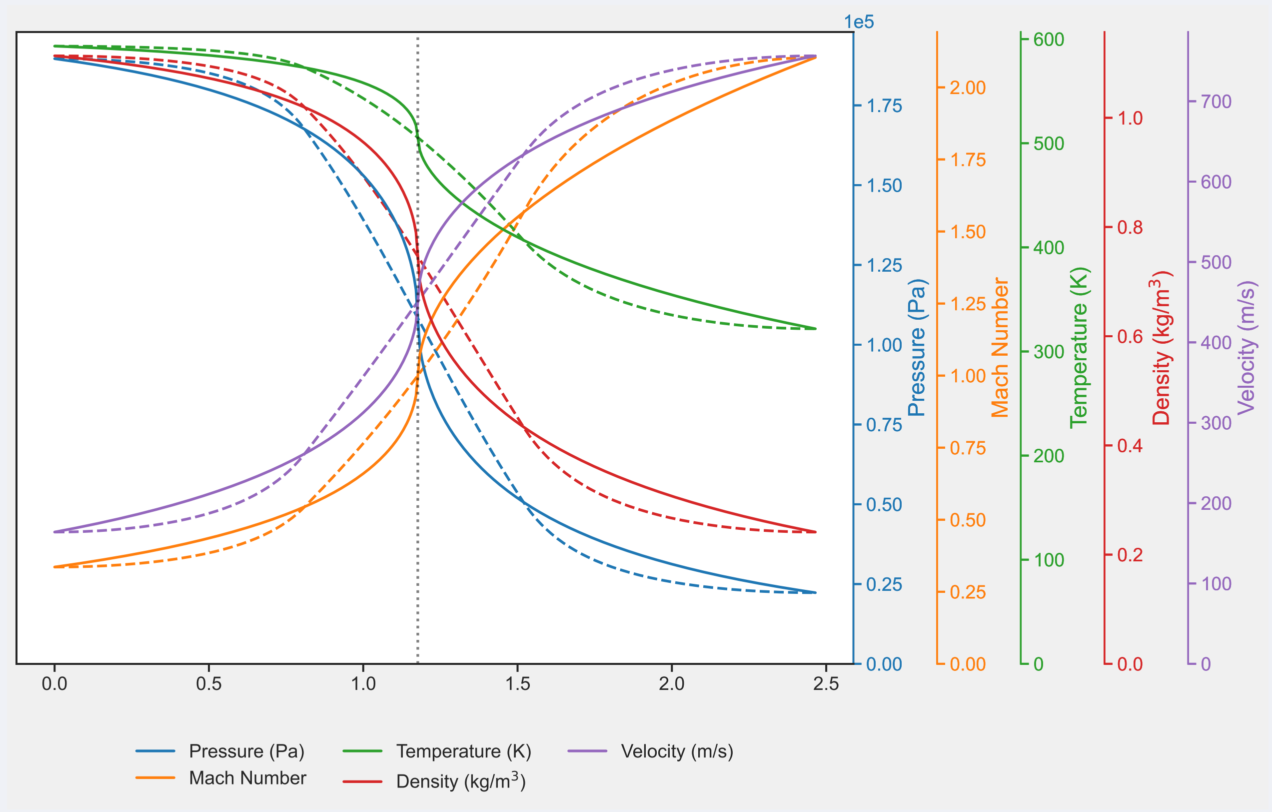
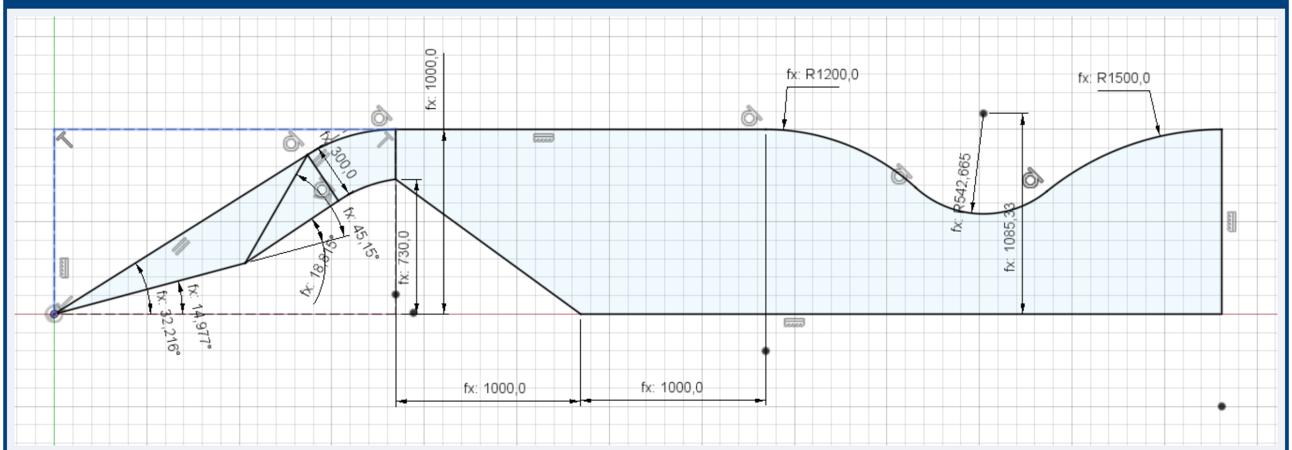


Figure 6: Complete geometry of the ramjet.



APPENDIX A – Source Code

```python
# %% [markdown]
# # SETUP
#

# %% [markdown]
# ## Imports

# %%
from __future__ import annotations
from scipy.optimize import minimize, root_scalar
import numpy as np
import matplotlib.pyplot as plt
from functools import lru_cache
import gc


# %% [markdown]
# ## Classes

# %%
class Fluid:
    """
    Stores the constant properties R and gamma of a fluid
    """

    def __init__(self, R: float, gamma: float) -> None:
        self.R = R
        self.gamma = gamma

    def __str__(self) -> str:
        return f"Fluid(R={self.R}, gamma={self.gamma})"


# %% [markdown]
# Flow class uses a few equations from @anderson2017, repeated here as they appear in the book:

# %% [markdown]
#
# From chapter 8:
#
# $$\tag{8.25}
#   a = \sqrt{\gamma R T}
# $$
```

```
#
# $$\tag{8.40}
#   \frac{T_0}{T} = 1 + \frac{\gamma - 1}{2}M^2
# $$
#
# $$\tag{8.42}
#   \frac{p_0}{p} = \left( 1 + \frac{\gamma - 1}{2}M^2 \right) ^ {\frac{\gamma}{\gamma - 1}}
# $$
#
# $$\tag{8.43}
#   \frac{\rho_0}{\rho} = \left( 1 + \frac{\gamma - 1}{2}M^2 \right) ^ {\frac{1}{\gamma - 1}}
# $$
#

# %% [markdown]
# From chapter 9
# $$\tag{9.13}
#   M_{n,1} = M_1 \sin\beta
# $$
#
# $$\tag{9.14}
#   {M_{n,2}}^2 = \frac{
#     1 + \frac{\gamma - 1}{2} {M_{n,1}}^2
#   }{
#     \gamma {M_{n,1}}^2 - \frac{\gamma - 1}{2}
#   }
# $$
#
# $$\tag{9.15}
#   \frac{\rho_2}{\rho_1} = \frac{ (\gamma + 1) {M_{n,1}}^2 }{ 2 + (\gamma + 1) {M_{n,1}}^2 }
# $$
#
# $$\tag{9.16}
#   \frac{p_2}{p_1} = 1 + \frac{2}{\gamma + 1} ( {M_{n,1}}^2 - 1 )
# $$
#
# $$\tag{9.17}
#   \frac{T_2}{T_1} = \frac{p_2}{p_1} \frac{\rho_1}{\rho_2}
# $$
#
# $$\tag{9.18}
#   M_2 = \frac{M_{n,2}}{\sin(\beta - \theta)}
# $$
#
# $$\tag{9.23}
#   \tan\theta = 2 \cot\beta \frac{ {M_1}^2\sin^2\beta - 1 }{ {M_1}^2 (\gamma + \cos(2\beta)) + 2 }
```

```python
# $$

# %%
class Flow:
    """
    Stores the defining properties of a flow:

    Pressure (p)
    Mach Number (M)
    Temperature (T)
    Density (rho)
    Fluid (fluid)
    Speed of sound (a)
    Velocity (V)

    Care should be taken to use consistent units.
    """

    def __init__(self, p: float, M: float, T: float, rho: float, fluid: Fluid) -> None:
        """
        Constructs a Flow object from the defining properties and calculates some
        auxiliary properties.
        Uses Equations (8.25), (8.40), (8.42), (8.43) @anderson2017.

        Parameters:
        p: pressure [Pa]
        M: Mach number
        T: temperature [K]
        rho: density [kg/m^3]
        fluid: Fluid object
        """

        self.pressure = p
        self.mach = M
        self.temperature = T
        self.density = rho
        self.fluid = fluid

        # Equation (8.25) @anderson2017
        self.snd_spd = np.sqrt(fluid.gamma * fluid.R * self.temperature)
        # From the definition of Mach Number
        self.vel = M * self.snd_spd

        aux_const = 1 + (fluid.gamma - 1) * M * M / 2
        # Equation (8.40) @anderson2017
        self.total_temperature = T * aux_const
        # Equation (8.42) @anderson2017
        self.total_pressure = p * aux_const ** (fluid.gamma / (fluid.gamma - 1))
        # Equation (8.43) @anderson2017
```

```python
        self.total_density = rho * aux_const ** (1 / (fluid.gamma - 1))

        self.shockwave = lru_cache(maxsize=8)(self._shockwave)

    @classmethod
    def from_total_values(
        cls, p0: float, M: float, T0: float, rho0: float, fluid: Fluid
    ):
        f = 1 / Flow(1 / p0, M, 1 / T0, 1 / rho0, fluid).total_values()
        return Flow(f[0], M, f[1], f[2], fluid)

    def _p_ratio(M, gamma):
        """
        Calculates the pressure ratio of a shockwave.
        Uses equation (9.19) @anderson2017.

        Parameters:
        M: Mach number
        gamma: ratio of specific heats

        Returns:
        Pressure ratio
        """

        return (1 + (gamma - 1) * M * M / 2) ** (gamma / (gamma - 1))

    def __str__(self) -> str:
        return (
            "\n\t".join(
                [
                    "Flow(",
                    f"Pressure: {self.pressure:.4}",
                    f"Mach: {self.mach:.4}",
                    f"Temperature: {self.temperature:.4}",
                    f"Density: {self.density:.4}",
                    f"fluid: {self.fluid}",
                ]
            )
            + "\n)"
        )

    def static_values(self) -> np.ndarray:
        """
        Outputs the defining properties of a flow as an ndarray.
        Uses same order as __init__, but excludes the fluid property.
        [p, M, T, rho]
        """
        return np.array([self.pressure, self.mach, self.temperature, self.density])
```

```python
    def total_values(self) -> np.ndarray:
        """
        Outputs the stagnation properties of a flow as an ndarray.
        [p0, T0, rho0]
        """
        return np.array(
            [self.total_pressure, self.total_temperature, self.total_density]
        )

    def aux_values(self) -> np.ndarray:
        """
        Outputs the auxiliary properties of a flow as an ndarray.
        [a, V]
        """
        return np.array([self.snd_spd, self.vel])

    def _shockwave(self, beta) -> Flow:
        """
        Calculates flow properties after an oblique shockwave.
        Uses equations (9.13) through (9.18) @anderson2017.

        Parameters:
        beta: oblique angle of the shockwave [rad]

        Returns:
        The Flow object after the shockwave
        """

        # Checks shockwave is valid (supersonic and positive theta)
        # Maybe subsonic flow could be accepted, and spit a supersonic flow?
        if (self.mach < 1) or ((theta := self.velocity_deflection(beta)) <= 0):
            return Flow(np.nan, np.nan, np.nan, np.nan, self.fluid)

        gamma = self.fluid.gamma

        # Equation (9.13) @anderson2017
        M_n_1 = self.mach * np.sin(beta)
        M_n_1_sq = M_n_1 * M_n_1
        # Equation (9.14) @anderson2017 (modified)
        M_n_2 = np.sqrt(
            (2 + (gamma - 1) * M_n_1_sq) / (2 * gamma * M_n_1_sq - (gamma - 1))
        )
        # Equation (9.15) @anderson2017
        rho_2 = self.density * (gamma + 1) * M_n_1_sq / (2 + (gamma - 1) * M_n_1_sq)
        # Equation (9.16) @anderson2017
        p_2 = self.pressure * (1 + 2 * gamma * (M_n_1_sq - 1) / (gamma + 1))
        # Equation (9.17) @anderson2017
        T_2 = self.temperature * p_2 * self.density / (self.pressure * rho_2)
        # Equation (9.18) @anderson2017
```

```python
        M_2 = M_n_2 / np.sin(beta - theta)

        flow_2 = Flow(p_2, M_2, T_2, rho_2, self.fluid)
        return flow_2

    def velocity_deflection(self, beta: float) -> float:
        """
        Calculates the velocity deflection angle theta for an oblique shockwave
        Uses equations (9.13), (9.23) @anderson2017

        Parameters:
        beta: oblique angle of the shockwave [rad]

        Returns:
        angle of deflection of velocity [rad]
        """

        M = self.mach
        gamma = self.fluid.gamma

        # Equation (9.13) @anderson2017
        M_n_1 = M * np.sin(beta)

        # Equation (9.23) @anderson2017
        theta = np.arctan(
            2
            / np.tan(beta)
            * (M_n_1 * M_n_1 - 1)
            / (M * M * (gamma + np.cos(2 * beta)) + 2)
        )
        return theta


# %% [markdown]
# ## Functions

# %% [markdown]
# Area-Mach number relation @anderson2017:
# $$\tag{10.32}
# \left( \frac{A}{A^*} \right)^2 = \frac{1}{M^2} \left[ \frac{2}{\gamma + 1} \left( 1 +
↪  \frac{\gamma - 1}{2} M^2 \right) \right]^{\frac{\gamma+1}{\gamma-1}}
# $$

# %%
def area_mach(M: float, gamma: float) -> float:
    """
    Calculates the ratio A / A* in isentropic quasi-one-dimensional flow.
    Uses Equation (10.32) @anderson2017.

```

```
    Parameters:
    M: Mach number
    gamma: Specific heat ratio
    """
    g_ratio = (gamma + 1) / (gamma - 1)
    # Equation (10.32) @anderson2017 (modified)
    return (2 / (gamma + 1) + M * M / g_ratio) ** (g_ratio / 2) / M


# %%
def nShock_p0_eff(betas: tuple[float, ...], flow_in: Flow) -> float:
    """
    Calculates the total pressure efficiency in a diffuser with n-1 oblique and 1 normal
    shockwaves.

    Parameters:
    betas: angles of the n-1 oblique shockwaves [rad]
    flow_in: Flow object of the flown in the upstream

    Returns:
    Total pressure efficiency of the diffuser
    """

    flow_after = flow_in
    for beta in betas:
        flow_after = flow_after.shockwave(beta)
    flow_after = flow_after.shockwave(np.pi / 2)
    return flow_after.total_pressure / flow_in.total_pressure


# %%
def bellNozzleGenerator(R_in, R_throat, R_out, L_throat, L_in=1, L_out=None):
    """
    Generates a function describing the geometry of a planar nozzle defined by 3 arcs.

    Parameters:
    R_in: Radius of the inlet arc
    R_throat: Radius of the throat arc
    R_out: Radius of the outlet arc
    L_throat: semi-length of the throat
    L_in: semi-length of the inlet arc
    L_out: semi-length of the outlet arc

    Returns:
    bellCurve: The function describing the geometry of the nozzle
    crit_len: The length where minimum area occurs
    max_len: The length of the nozzle
    """
    if L_out is None:
```

```python
        L_out = L_in
    L1 = (
        np.sqrt((R_in + R_throat) ** 2 - (R_in - L_in + R_throat + L_throat) ** 2)
        * R_in
        / (R_in + R_throat)
    )
    l1 = L1 * R_throat / R_in
    L2 = (
        np.sqrt((R_throat + R_out) ** 2 - (R_out - L_out + R_throat + L_throat) ** 2)
        * R_out
        / (R_throat + R_out)
    )
    l2 = L2 * R_throat / R_out
    max_len = L1 + l1 + l2 + L2
    crit_len = L1 + l1

    def bellCurve(x):
        if not isinstance(x, np.ndarray):
            x = np.array(x)
        y = np.zeros_like(x)
        # First arc
        i1 = (0 <= x) & (x <= L1)
        y[i1] = L_in - R_in + np.sqrt(R_in**2 - (x[i1] - 0) ** 2)
        # Second arc
        i2 = (L1 < x) & (x < max_len - L2)
        y[i2] = L_throat + R_throat - np.sqrt(R_throat**2 - (x[i2] - (L1 + l1)) ** 2)
        # Third arc
        i3 = (max_len - L2 <= x) & (x <= max_len)
        y[i3] = L_out - R_out + np.sqrt(R_out**2 - (x[i3] - max_len) ** 2)
        return y

    return bellCurve, crit_len, max_len


# %%
def coneNozzleGenerator(crit_len, max_len, L_throat, L_in=1, L_out=None):
    """
    Generates a function describing the geometry of a planar nozzle defined by 2 slopes.

    Parameters:
    max_len: Length of the nozzle
    crit_len: Length where minimum area occurs
    L_throat: Semi-length of the throat
    L_in: Semi-length of the inlet
    L_out: Semi-length of the outlet

    Returns:
    coneCurve: The function describing the geometry of the nozzle
    crit_len: The length where minimum area occurs
```

```python
    max_len: The length of the nozzle
    """
    if L_out is None:
        L_out = L_in

    def coneCurve(x):
        if not isinstance(x, np.ndarray):
            x = np.array(x)
        y = np.zeros_like(x)
        # First slope
        i1 = (0 <= x) & (x <= crit_len)
        y[i1] = L_in + x[i1] * (L_throat - L_in) / crit_len
        # Second slope
        i2 = (crit_len < x) & (x <= max_len)
        y[i2] = L_throat + (x[i2] - crit_len) * (L_out - L_throat) / (
            max_len - crit_len
        )
        return y

    return coneCurve, crit_len, max_len


# %%
styles = ["default", "fivethirtyeight", "seaborn-white"]
colors = plt.rcParams["axes.prop_cycle"].by_key()["color"]


def plot_flow_evolution(flow_data: np.ndarray, pos: np.ndarray, fig_ax=None, cols=None):
    # Plots the relevant properties of the flow
    if fig_ax is None:
        fig = plt.figure(figsize=(10, 10))
        ax = plt.axes()
    else:
        fig, ax = fig_ax

    if cols is None:
        cols = list(range(flow_data.shape[-1]))

    # Different vertical axes
    v_axes = [ax.twinx() for _ in cols]

    # Line labels
    labels = [
        "Pressure (Pa)",
        "Mach Number",
        "Temperature (K)",
        "Density (kg/m$^3$)",
        "Speed of Sound (m/s)",
        "Velocity (m/s)",
```

```
        "Total Pressure (Pa)",
        "Total Temperature (K)",
        "Total Density (kg/m$^3$)",
    ]

    labels = [labels[c] for c in cols]

    flow_data = flow_data[:, cols]

    # Generates the actual plot lines
    lines = [
        v_ax.plot(pos, property, color=color, label=label)[0]
        for v_ax, property, label, color in zip(v_axes, flow_data.T, labels, colors)
    ]

    # Tick appearance
    tick_params = dict(size=6, width=1.5)

    # Configures the scale axes
    for v_ax, label, line, n in zip(v_axes, labels, lines, range(len(lines))):
        v_ax.set_ylabel(label)
        v_ax.yaxis.label.set_color(line.get_color())
        v_ax.spines.right.set_color(line.get_color())
        v_ax.tick_params(axis="y", colors=line.get_color(), **tick_params)
        v_ax.ticklabel_format(axis="y", scilimits=(-3, 3))
        v_ax.yaxis.offsetText.set_position((1.025 + 0.10 * n, 0))
        v_ax.spines.right.set_position(("axes", 1 + 0.10 * n))
        v_ax.set_ylim(bottom=0)

    # Ticks on main ax
    ax.tick_params(axis="y", which="both", left=False, right=False)
    ax.tick_params(axis="x", **tick_params)

    # fig.legend(loc="upper left", handles=lines, ncol=3)
    box = ax.get_position()
    ax.set_position([box.x0, box.y0 + box.height * 0.1, box.width, box.height * 0.9])

    # Put a legend below current axis
    ax.legend(
        handles=lines,
        loc="upper center",
        bbox_to_anchor=(0.5, -0.1),
        fancybox=True,
        ncol=3,
    )
    ax.set_yticks([])
    return fig, [ax, *v_axes]
```

```python
# %% [markdown]
# # PART 1

# %% [markdown]
# ## Initial data

# %%
# Gets data from problem conditions
M_inf = 3.0
h = 18e3  # [m]

# Atmospheric conditions @aerotoolbox
p_inf = 7505.00  # [Pa]
T_inf = 216.65  # [K]
rho_inf = 0.12068  # [kg/m^3]

# Air constants
gamma_air = 1.4
R_air = 287.058


# %%
# Creates initial Objects
air = Fluid(R_air, gamma_air)
flow_inf = Flow(p_inf, M_inf, T_inf, rho_inf, air)

print(f"Freestream flow: {flow_inf}")


# %% [markdown]
# ## Minimization

# %% [markdown]
# Minimizes over beta angles instead of over theta angles, as `theta(beta)` is a much less
↪  expensive function to calculate than `beta(theta)`

# %%
# Finds minimum beta values, aka beta when theta = 0
min_beta_1 = root_scalar(flow_inf.velocity_deflection, x0=0.1, x1=0.11).root
assert 0 < min_beta_1 < np.pi / 2


# %%
# Solves the minimization problem

# Initial guess based on minimum beta angle (scaled to avoid evaluating on the boundary)
initial_betas = (min_beta_1 * 1.01, min_beta_1 * 1.01)

print("Solving...")
```

```python
# Minimizes the total pressure efficiency loss using a descent algorithm
sol_weak = minimize(
    lambda betas: 1 - nShock_p0_eff(betas, flow_inf),
    initial_betas,
    method="SLSQP",
    options={"disp": True},
)
print("Done!")
print(f"Solution: {sol_weak.x}")


# %% [markdown]
# ## Results

# %% [markdown]
# ### Optimal geometry and flow

# %%
# Gets and prints all the solution values
betas = sol_weak.x
assert np.all(0 < betas) and np.all(betas < np.pi / 2)

p0_eff = 1 - sol_weak.fun
assert 0 < p0_eff < 1

print(f"Efficiency = {p0_eff:.4}")
print()

# Stores flow states into a list
flows = [flow_inf]
for n, beta in enumerate(betas):
    print(f"beta_{n+1}:  {np.rad2deg(beta)}")
    print(f"theta_{n+1}: {np.rad2deg(flows[-1].velocity_deflection(beta))}")
    flows.append(flows[-1].shockwave(beta))
    print(f"Flow_{n+1}:", flows[-1], "\n")
flows.append(flows[-1].shockwave(np.pi / 2))
print(f"Flow_out: {flows[-1]}")
# Garbage Collection
gc.collect()


# %%
print(flows[-1].temperature)
print(flows[-1].pressure)

# %% [markdown]
# ### Plot minimization locus

# %%
```

```python
# Calculates the efficiency over a grid of beta_1 and beta_2
b1v = np.linspace(min_beta_1, np.deg2rad(65), 50)
b2v = np.linspace(min_beta_1, np.deg2rad(80), 50)
d = np.empty((len(b2v), len(b1v)))
for j, b1 in enumerate(b1v):
    for i, b2 in enumerate(b2v):
        d[i, j] = nShock_p0_eff((b1, b2), flow_inf)
assert np.nanmax(d) < p0_eff
assert np.nanmin(d) > 0
# Garbage collection for all the dereferenced values
gc.collect()


# %%
# Plots the efficiency distribution
plt.figure(figsize=(10, 10))
plt.pcolormesh(
    np.rad2deg(b1v),
    np.rad2deg(b2v),
    100 * d,
    cmap="viridis",
)
cb = plt.colorbar(label="$p_0$ Efficiency (%)")

# Plots optimal solution in red X
plt.plot(*np.rad2deg(betas[:2]), "rx", label="Optimal solution")

# plt.title(r"Total Pressure Efficiency as a function of $\beta_1$ and $\beta_2$")
plt.xlabel(r"$\beta_1$ [deg]")
plt.ylabel(r"$\beta_2$ [deg]")
plt.legend()

# Grid and ticks
plt.axis("scaled")
plt.minorticks_on()
plt.grid(visible=True, axis="both", which="major", linewidth=1, alpha=0.6)
plt.grid(visible=True, axis="both", which="minor", linewidth=0.5, alpha=0.3)

# Color bar
plt.savefig("gradient.pdf", bbox_inches="tight")
plt.show()


# %% [markdown]
# ### Plot optimal flow properties

# %%
# Prepares the data as a matrix
data = np.stack(
```

```python
    [(*f.static_values(), *f.aux_values(), *f.total_values()) for f in flows]
)
# Saves data to CSV file
np.savetxt("data.csv", data, fmt="%8.5g", delimiter=",")


# %%
# Plots the flow variation over the diffuser
xt = np.arange(len(data))
with plt.style.context(styles), plt.rc_context(
    rc={"lines.marker": "o", "lines.markersize": 10}
):
    fig, axes = plot_flow_evolution(data, xt, cols=[0, 1, 2, 3, 5])
    # Sets the lower axis
    axes[0].set_xlabel("Region")
    axes[0].set_xticks(xt)
    axes[0].set_xticklabels([r"$\infty$", *np.arange(1, len(data))])
plt.savefig("diffuser_properties.pdf", bbox_inches="tight")
plt.show()


# %% [markdown]
# # PART 2

# %% [markdown]
# ## Initial data

# %%
# Sets problem parameters
# The cross-sectional area after the normal shockwave can be arbitrarily small. For this
# geometry, it can also be as large as about 0.3 * D.

# D was chosen to be unitary;
D = 1
D_inlet = 1 * D
D_outlet = 1 * D
# The cross sectional area was arbitrarily chosen to be 0.25 * D
D_entrance = 0.3 * D
# Temperature change
delta_temp = 1500
# Total Values from Part 1
p0, T0, rho0 = flows[-1].total_values()
# Creates new Flow list for Part 2
flows2 = [flows[-1]]
print("Flow after diffuser:", flows2[-1])


# %% [markdown]
# ## Expansion
```

```
# %%
# Calculates Mach number using the area-Mach relation
# Calculates virtual throat
D_virtual = D_entrance / area_mach(flows2[-1].mach, flows2[-1].fluid.gamma)
# Solves area-Mach relation for the new Mach number
expanded_mach = root_scalar(
    lambda M: area_mach(M, flows2[-1].fluid.gamma) - D / D_virtual,
    bracket=[1, 1e-2],
    method="brentq",
).root

# Appends new Flow to the list
flows2.append(Flow.from_total_values(p0, expanded_mach, T0, rho0, flows2[-1].fluid))
print("Flow after expansion:", flows2[-1])


# %%
print(D_virtual / D)

# %% [markdown]
# ## Heat addition

# %% [markdown]
# To calculate the properties after heat addition, equations (3.78), (3.79), (3.81) from
↪ @anderson2021. These equations are written down here as needed before they are
↪ implemented.

# %% [markdown]
# $$\tag{3.81}
# \frac{T_2}{T_1} = \left( \frac{1 + \gamma {M_1}^2}{\gamma {M_2}^2} \right)^2
#                   \left( \frac{{M_2}^2}{{M_1}^2} \right)
# $$

# %%
# Calculates temperature after heat addition
expanded_temp = flows2[-1].temperature
heated_temp = expanded_temp + delta_temp

# Calculates Mach number after heat addition
# Part of Equation (3.81) @anderson2021 that doesn't depend on the new Mach number
const_part = (
    np.sqrt(heated_temp / expanded_temp)
    * expanded_mach
    / (1 + flows2[-1].fluid.gamma * expanded_mach * expanded_mach)
)
# Solves Equation (3.81) @anderson2021 (modified) for the new Mach number
heated_mach = root_scalar(
    lambda M: const_part - M / (1 + flows2[-1].fluid.gamma * M * M),
```

```python
    bracket=[1, 0],
    method="brentq",
).root
print(f"New Mach number: {heated_mach:.4}")


# %% [markdown]
# $$\tag{3.78}
#  \frac{p_2}{p_1} = \frac{1 + \gamma {M_1}^2}{1 + \gamma {M_2}^2}
# $$
#
# $$\tag{3.79}
#  \frac{\rho_2}{\rho_1} = \frac{p_2}{p_1} \frac{T_1}{T_2}
# $$

# %%
# Calculates remaining Flow properties after heat addition
# Equation (3.78) @anderson2021
heated_pressure = (
    flows2[-1].pressure
    * (1 + flows2[-1].fluid.gamma * expanded_mach * expanded_mach)
    / (1 + flows2[-1].fluid.gamma * heated_mach * heated_mach)
)
# Equation (3.79) @anderson2021
heated_rho = (
    flows2[-1].density
    * heated_pressure
    / flows2[-1].pressure
    * expanded_temp
    / heated_temp
)

# Appends new Flow to the list
flows.append(
    Flow(heated_pressure, heated_mach, heated_temp, heated_rho, flows2[-1].fluid)
)
print("Flow after heat addition:", flows[-1])


# %% [markdown]
# ## Nozzle

# %% [markdown]
# ### Geometry

# %%
# Calculates throat semi-area
D_throat = D / area_mach(heated_mach, flows[-1].fluid.gamma)
print(f"Throat area: {D_throat}")
```

```python


# %%
# Creates the 2 nozzle geometries
bellFunc, crit_len, max_len = bellNozzleGenerator(
    1.2 * D_inlet, D_throat, 1.5 * D_outlet, D_throat, D_inlet, D_outlet
)
coneFunc, _, _ = coneNozzleGenerator(crit_len, max_len, D_throat, D_inlet, D_outlet)

x = np.hstack(
    [
        np.linspace(0, crit_len, 200, endpoint=False),
        np.linspace(crit_len, max_len, 200),
    ],
)
bell = np.array([x, bellFunc(x)])
cone = np.array([x, coneFunc(x)])
nozzle_curves = [bell, cone]


# %%
for x, y in nozzle_curves:
    plt.figure(figsize=(10, 10))
    plt.fill_between(x, y, 0, facecolor="#aaa", edgecolor="black", linewidth=2)
    plt.axis("scaled")
    plt.axis("off")
    plt.tight_layout()
plt.show()


# %% [markdown]
# ### Flow properties

# %%
nozzles_flows = np.empty([len(nozzle_curves), len(x)], Flow)
p0_nozzle, T0_nozzle, rho0_nozzle = flows2[-1].total_values()
for n, curve in enumerate(nozzle_curves):
    # Calculates the Mach number across the nozzle.
    crit_index = np.argmin(curve[1])
    D_crit = curve[1, crit_index]
    crit_x = curve[0, crit_index]
    machs = np.array(
        [
            root_scalar(
                lambda M: area_mach(M, flows2[-1].fluid.gamma) - y_ / D_crit,
                # The sign function selects solution branch according to starting flow
                # and position relative to critical length
                bracket=[1, 100 ** np.sign((1 - flows2[-1].mach) * (x_ - crit_x))],
                method="brentq",
```

```
            ).root
            for x_, y_ in curve.T
        ]
    )
    nozzles_flows[n, :] = np.array(
        [
            Flow.from_total_values(
                p0_nozzle, mach, T0_nozzle, rho0_nozzle, flows2[-1].fluid
            )
            for mach in machs
        ]
    )


# %% [markdown]
# ### Contour plots

# %%
def plot_nozzle_prop(prop_name, flow_list, nozzle_curve, ax=None):
    from mpl_toolkits.axes_grid1.inset_locator import inset_axes

    if ax is None:
        ax = plt.gca()
    prop = np.array([flow.__getattribute__(prop_name) for flow in flow_list])
    x, y = nozzle_curve
    plot = plt.contourf([x, x], [np.zeros_like(y), y], [prop, prop], levels=10)
    plt.axis("scaled")
    # cb = plt.colorbar(label=prop_name, orientation="vertical")
    axins = inset_axes(
        plt.gca(),
        width="5%",
        height="100%",
        loc="lower left",
        bbox_to_anchor=(1.02, 0.0, 1, 1),
        bbox_transform=ax.transAxes,
        borderpad=0,
    )
    cb = plt.gcf().colorbar(plot, cax=axins, label=prop_name)
    cb.formatter.set_powerlimits((-3, 3))
    cb.formatter.set_scientific(True)


def plot_nozzles_props(prop_list, nozzle_flow_list, curve_list):
    for property in prop_list:
        for nozzle_flow, curve in zip([*nozzle_flow_list], curve_list):
            plt.figure()
            plot_nozzle_prop(property, nozzle_flow, curve)
    plt.show()

```

```python

# %%
with plt.rc_context({"figure.figsize": (10, 10)}):
    plot_nozzles_props(
        ["pressure", "mach", "temperature", "density", "vel"],
        nozzles_flows,
        nozzle_curves,
    )


# %% [markdown]
# ### Line plot

# %%
with plt.style.context(styles):
    fig, ax = plt.subplots(figsize=(10, 10))
    for nozzle_flows, line_style in zip(nozzles_flows, ["dashed", "solid", "-."]):
        with plt.rc_context(
            {"lines.linestyle": line_style, "lines.marker": "", "lines.linewidth": 2}
        ):
            data2 = np.stack(
                [
                    (*f.static_values(), *f.aux_values(), *f.total_values())
                    for f in nozzle_flows
                ]
            )
            _, axes = plot_flow_evolution(data2, x, (fig, ax), cols=[0, 1, 2, 3, 5])
    plt.axvline(crit_len, color="black", linestyle="dotted", linewidth=2, alpha=0.5)
    plt.savefig("nozzle_evolution.pdf", bbox_inches="tight")
    plt.show()


# %%
with plt.style.context("default"):
    plt.figure(figsize=(10, 10))
    for prop, color in zip(
        ["pressure", "mach", "temperature", "density", "vel"], colors
    ):
        plt.title(prop)
        for nozzle_flows, nozzle_names, line_style in zip(
            nozzles_flows, ["Bell", "Cone"], ["-", "--"]
        ):
            y = [f.__getattribute__(prop) for f in nozzle_flows]
            y = y / y[crit_index]
            plt.plot(
                x,
                y,
                linestyle=line_style,
                color=color,
```

```
                label=f"{prop} {nozzle_names}",
            )
        plt.ticklabel_format(scilimits=(-3, 3))
        plt.autoscale()
        plt.ylim(bottom=0)
        plt.legend()
```