Homework G.7 - Solution

The longitudinal equations of motion are

$$(m_c + 2m_r)\ddot{h} = \tilde{F},$$

which implies that

$$\ddot{h} = \left(\frac{1}{m_c + 2m_r}\right)\tilde{F}.$$

Let $x = (h, \dot{h})^\top \stackrel{\triangle}{=} (x_1, x_2)^\top$ and $u = \tilde{F}$ and $y = h$, then

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{h} \\ \ddot{h} \end{pmatrix} = \begin{pmatrix} x_2 \\ \left(\frac{1}{m_c+2m_r}\right)\tilde{F} \end{pmatrix} = \begin{pmatrix} x_2 \\ \left(\frac{1}{m_c+2m_r}\right)u \end{pmatrix}.$$

Therefore, the state space equations are

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m_c+2m_r} \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + 0u. \end{aligned}$$

The lateral equations of motion are

$$\begin{aligned} (m_c + 2m_r)\ddot{z} &= -F_e\theta - \mu\dot{z} \\ (J_c + 2m_r d^2)\ddot{\theta} &= \tau. \end{aligned}$$

Solving for the highest order derivatives on the left hand side gives which implies that

$$\begin{aligned} \ddot{z} &= -\left(\frac{F_e}{m_c + 2m_r}\right)\theta - \left(\frac{\mu}{m_c + 2m_r}\right)\dot{z} \\ \ddot{\theta} &= \left(\frac{1}{J_c + 2m_r d^2}\right)\tau. \end{aligned}$$

Let $x = (z, \theta, \dot{z}, \dot{\theta})^\top \stackrel{\triangle}{=} (x_1, x_2, x_3, x_4)^\top$ and $u = \tau$ and $y = (z, \theta)^\top$, then

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} \dot{z} \\ \ddot{\theta} \\ -\left(\frac{F_e}{m_c+2m_r}\right)x_2 - \left(\frac{\mu}{m_c+2m_r}\right)x_3 \\ \left(\frac{1}{J_c+2m_r d^2}\right)u \end{pmatrix}.$$

Therefore, the state space equations are

$$\dot{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \left(-\frac{F_e}{m_c+2m_r}\right) & \left(-\frac{\mu}{m_c+2m_r}\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \left(\frac{1}{J_c+2m_r d^2}\right) \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u.$$