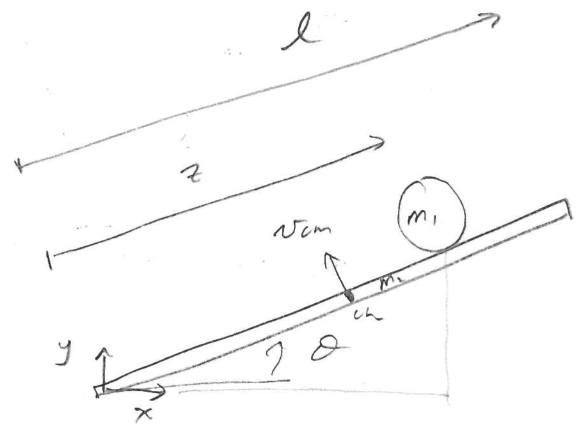
V. 1



$$K = \tfrac{1}{2} m_2 v_{cm,beam}^T v_{cm,beam} + \tfrac{1}{2} \omega_{beam}^T J_{beam} \omega_{beam}$$

$$+ \tfrac{1}{2} m_2 v_{ball}^T v_{ball} + \tfrac{1}{2} \omega_{ball}^T J_{ball} \omega_{ball}$$

→ Assume zero
– move without rolling

position of ball

$$p_{ball} = \begin{pmatrix} z\cos\theta \\ z\sin\theta \\ 0 \end{pmatrix}$$

position of cm of beam

$$p_{cm,beam} = \begin{pmatrix} \ell/2 \cos\theta \\ \ell/2 \sin\theta \\ 0 \end{pmatrix}$$

$$\therefore \quad v_{ball} = \begin{pmatrix} \dot z\cos\theta - z\dot\theta\sin\theta \\ \dot z\sin\theta + z\dot\theta\cos\theta \\ 0 \end{pmatrix}, \quad v_{beam} = \begin{pmatrix} -\dot\theta\,\ell/2\,\sin\theta \\ \dot\theta\,\ell/2\,\cos\theta \\ 0 \end{pmatrix}$$

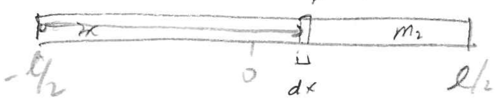
The angular velocity of the beam is

$$\omega_{beam} = \begin{pmatrix} 0 \\ 0 \\ \dot\theta \end{pmatrix}$$

Therefore we only need to know the (3,3) element of J_{beam}

Let J_z be the (3,3) element

elemental mass is $m_2/_\ell \, dx$



$$J = \int (r^T r I - r r^T)\, dm$$

where $r = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$ and $dm = \frac{m_2}{\ell} dx$

Note that $r^T r I - r r^T = \begin{pmatrix} x^2 & 0 & 0 \\ 0 & x^2 & 0 \\ 0 & 0 & x^2 \end{pmatrix} - \begin{pmatrix} x^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x^2 & 0 \\ 0 & 0 & x^2 \end{pmatrix}$

$\therefore$ The $(3,3)$ element is

$$J_z = \int_{x=-\ell/2}^{\ell/2} x^2 \left(\frac{m_2}{\ell}\right) dx$$

$$= \frac{m_2}{\ell} \int_{-\ell/2}^{\ell/2} x^2\, dx = \frac{m_2}{\ell} \left. \frac{x^3}{3} \right|_{-\ell/2}^{\ell/2} = \frac{m_2}{3\ell}\left[\frac{\ell^3}{8} + \frac{\ell^3}{8}\right]$$

$$= \frac{m_2 \ell^2}{12}$$

$$\therefore \quad K = \tfrac{1}{2} m_2 \left[\left(-\dot{\theta} \tfrac{\ell}{2} \sin\theta\right)^2 + \left(\dot{\theta} \tfrac{\ell}{2} \cos\theta\right)^2 \right] + \tfrac{1}{2}\left(\frac{m_2 \ell^2}{12}\right)\dot{\theta}^2$$

$$+ \tfrac{1}{2} m_1 \left[\left(\dot{z}\cos\theta - z\dot{\theta}\sin\theta\right)^2 + \left(\dot{z}\sin\theta + z\dot{\theta}\cos\theta\right)^2 \right]$$

$$= \tfrac{1}{2} m_2 \frac{\ell^2}{4} \dot{\theta}^2 + \tfrac{1}{2} m_2 \frac{\ell^2}{12} \dot{\theta}^2$$

$$+ \tfrac{1}{2} m_1 \Big[\dot{z}^2 \cos^2\theta - 2 z \dot{z} \dot{\theta} \sin\theta \cos\theta + z^2 \dot{\theta}^2 \sin^2\theta$$
$$+ \dot{z}^2 \sin^2\theta + 2 z \dot{z} \dot{\theta} \sin\theta\cos\theta + z^2 \dot{\theta}^2 \cos^2\theta \Big]$$

$$= \frac{1}{2} \frac{m_2}{3} l^2 \dot{\theta}^2 + \frac{1}{2} m_1 \left(\dot{z}^2 + z^2 \dot{\theta}^2 \right)$$

$$= \frac{1}{2} m_1 \dot{z}^2 + \frac{1}{2} \left(\frac{m_2}{3} l^2 + m_1 z^2 \right) \dot{\theta}^2$$