Homework E.7 - Solution

The linearized equations are

$$\begin{aligned} m_1\ddot{\tilde{z}} + m_1 g\tilde{\theta} &= 0 \\ \left(\frac{m_2 l^2}{3} + m_1 z_e^2\right)\ddot{\tilde{\theta}} + m_1 g\left(\tilde{z}\right) &= l\tilde{F}. \end{aligned}$$

Solving for $\ddot{\tilde{z}}$ and $\ddot{\tilde{\theta}}$ gives

$$\begin{aligned} \ddot{\tilde{z}} &= -g\tilde{\theta} \\ \ddot{\tilde{\theta}} &= \frac{l}{\frac{m_2 l^2}{3} + m_1 z_e^2}\tilde{F} - \frac{m_1 g}{\frac{m_2 l^2}{3} + m_1 z_e^2}\tilde{z}. \end{aligned}$$

Let

$$\begin{aligned} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &\triangleq \begin{bmatrix} \tilde{\theta} \\ \tilde{z} \\ \dot{\tilde{\theta}} \\ \dot{\tilde{z}} \end{bmatrix} \\ u &\triangleq F \\ \mathbf{y} &\triangleq \begin{bmatrix} \tilde{z} \\ \tilde{\theta} \end{bmatrix} \end{aligned}$$

then

$$\begin{aligned} \dot{x}_1 &= \dot{\tilde{\theta}} = x_3 \\ \dot{x}_2 &= \dot{\tilde{z}} = x_4 \\ \dot{x}_3 &= \ddot{\tilde{\theta}} = \frac{l}{\frac{m_2 l^2}{3} + m_1 z_e^2}\tilde{F} - \frac{m_1 g}{\frac{m_2 l^2}{3} + m_1 z_e^2}\tilde{z} \\ &= \frac{l}{\frac{m_2 l^2}{3} + m_1 z_e^2}u - \frac{m_1 g}{\frac{m_2 l^2}{3} + m_1 z_e^2}x_2 \\ \dot{x}_4 &= \ddot{\tilde{z}} = -g\tilde{\theta} = -g x_1. \end{aligned}$$

In matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{m_1 g}{\frac{m_2 l^2}{3} + m_1 z_e^2} & 0 & 0 \\ -g & 0 & 0 & 0 \end{bmatrix}}_{\triangleq \mathbf{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{l}{\frac{m_2 l^2}{3} + m_1 z_e^2} \\ 0 \end{bmatrix}}_{\triangleq \mathbf{B}} u$$

$$\mathbf{y} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{\triangleq \mathbf{C}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\triangleq \mathbf{D}} u.$$