# Homework E.5 - Solution

The equations of motion are

$$
\begin{aligned}
m_1\ddot{z} - m_1\dot{\theta}^2 + m_1 g\sin(\theta) &= 0 \\
\left(\frac{m_2 l^2}{3} + m_1 z^2\right)\ddot{\theta} + 2m_1 z\dot{z}\dot{\theta} + m_1 g z\cos\theta + \frac{m_2 g l}{2}\cos\theta &= lF\cos\theta.
\end{aligned}
$$

The equilibria are when $\dot{z} = \ddot{z} = \dot{\theta} = \ddot{\theta} = 0$ or

$$
\begin{bmatrix} m_1 g\sin(\theta_e) \\ m_1 g z_e\cos(\theta_e) + \frac{m_2 g l}{2}\cos(\theta_e) \end{bmatrix} = \begin{bmatrix} 0 \\ lF_e\cos(\theta_e) \end{bmatrix}
$$

Looking at the first equation $m_1 g\sin(\theta_e) = 0$ implies that $\theta_e = 0$ since the other configurations don't make sense. Looking at the second equation, and setting $\theta_e = 0$,

$$
\begin{aligned}
lF_e &= m_1 g z_e + \frac{m_2 g l}{2} \\
F_e &= \frac{m_1 g}{l} z_e + \frac{m_2 g}{2} \qquad (1)
\end{aligned}
$$

The equilibria points are all $(z_e, \theta_e, F_e)$ such that $\theta_e = 0$ and $(z_e, F_e)$ satisfy Eq (1). Let

$$
\begin{aligned}
z &= z_e + \tilde{z} \\
\theta &= \theta_e + \tilde{\theta} \\
F &= F_e + \tilde{F}
\end{aligned}
$$

then

$$
\begin{aligned}
z\dot{\theta}^2 &\approx z_e\dot{\theta}_e^2 + \frac{\partial}{\partial z}\left(z\dot{\theta}^2\right)\bigg|_e \tilde{z} + \frac{\partial}{\partial \dot{\theta}}\left(z\dot{\theta}^2\right)\bigg|_e \dot{\tilde{\theta}} \\
&= z_e\dot{\theta}_e^2 + \dot{\theta}_e^2\tilde{z} + 2z_e\dot{\theta}_e\dot{\tilde{\theta}} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\sin(\theta) &\approx \sin(\theta_e) + \frac{\partial}{\partial \theta}\left(\sin(\theta)\right)\bigg|_e \tilde{\theta} \\
&= \sin(\theta_e) + \cos(\theta_e)\tilde{\theta} \\
&= \tilde{\theta}
\end{aligned}
$$

$$
\begin{aligned}
z\dot{z}\dot{\theta} &\approx z_e\dot{z}_e\dot{\theta}_e + \frac{\partial}{\partial z}\left(z\dot{z}\dot{\theta}\right)\bigg|_e \tilde{z} + \frac{\partial}{\partial z}\left(z\dot{z}\dot{\theta}\right)\bigg|_e \dot{\tilde{z}} + \frac{\partial}{\partial z}\left(z\dot{z}\dot{\theta}\right)\bigg|_e \dot{\tilde{\theta}} \\
&= z_e\dot{z}_e\dot{\theta}_e + \dot{z}_e\dot{\theta}_e\tilde{z} + z_e\dot{\theta}_e\dot{\tilde{z}} + z_e\dot{z}_e\dot{\tilde{\theta}} \\
&= z_e\dot{z}_e\dot{\tilde{\theta}} \\
z\cos(\theta) &\approx z_e\cos(\theta_e) + \frac{\partial}{\partial z}\left(z\cos(\theta)\right)\bigg|_e \tilde{z} + \frac{\partial}{\partial \theta}\left(z\cos(\theta)\right)\bigg|_e \tilde{\theta} \\
&= z_e\cos(\theta_e) + \cos(\theta_e)\tilde{z} - z_e\sin(\theta_e)\tilde{\theta} \\
&= z_e + \tilde{z}
\end{aligned}
$$

$$
\begin{aligned}
\cos(\theta) &\approx \cos(\theta_e) + \frac{\partial}{\partial \theta}\left(\cos(\theta)\right)\bigg|_e \tilde{\theta} \\
&= \cos(\theta_e) - \sin(\theta_e)\tilde{\theta} \\
&= 1 \\
z^2\ddot{\theta} &\approx z_e^2\ddot{\theta}_e + \frac{\partial}{\partial z}\left(z^2\ddot{\theta}\right)\bigg|_e \tilde{z} + \frac{\partial}{\partial \ddot{\theta}}\left(z^2\ddot{\theta}\right)\bigg|_e \ddot{\tilde{\theta}} \\
&= z_e^2\ddot{\theta}_e + 2z_e\ddot{\theta}_e\tilde{z} + z_e^2\ddot{\tilde{\theta}} \\
&= z_e^2\ddot{\tilde{\theta}}
\end{aligned}
$$

The linearized equations are

$$
\begin{aligned}
m_1 \ddot{\tilde{z}} + m_1 g \tilde{\theta} &= 0 \\
\left( \frac{m_2 l^2}{3} + m_1 z_e^2 \right) \ddot{\tilde{\theta}} + m_1 g \left( z_e + \tilde{z} \right) + \frac{m_2 g l}{2} &= l \left( F_e + \tilde{F} \right).
\end{aligned}
$$

This can be simplified by using the equilibrium force given by (1) to obtain

$$
\begin{aligned}
m_1 \ddot{\tilde{z}} + m_1 g \tilde{\theta} &= 0 \\
\left( \frac{m_2 l^2}{3} + m_1 z_e^2 \right) \ddot{\tilde{\theta}} + m_1 g \tilde{z} &= l \tilde{F}.
\end{aligned}
$$