Homework G.3 - Solution

The generalized coordinates are

$$\begin{aligned} y_1 &= z \\ y_2 &= h \\ y_3 &= \theta \end{aligned}$$

and the generalized forces are

$$\begin{aligned} \tau_1 &= -(f_r + f_l)\sin(\theta) - \mu\dot{z} \\ \tau_2 &= (f_r + f_l)\cos(\theta) \\ \tau_3 &= d(f_r - f_l). \end{aligned}$$

The kinetic energy of the system is

$$K = \frac{1}{2}(m_c + 2m_r)\dot{z}^2 + \frac{1}{2}(m_c + 2m_r)\dot{h}^2 + \frac{1}{2}\left(J_c + 2m_r d^2\right)\dot{\theta}^2$$

and the potential energy of the system is

$$\begin{aligned} P &= P_0 + m_c g h + m_r g(h + d\sin(\theta) + m_l g(h - d\sin(\theta)) \\ &= (m_c + 2m_r)gh + P_0. \end{aligned}$$

The Lagrangian is

$$\begin{aligned} L &= K - P \\ &= \frac{1}{2}(m_c + 2m_r)\dot{z}^2 + \frac{1}{2}(m_c + 2m_r)\dot{h}^2 + \frac{1}{2}(J_c + 2m_r d^2)\dot{\theta}^2 - (m_c + 2m_r)gh - P_0. \end{aligned}$$

The Euler Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} &= \tau_1 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{h}}\right) - \frac{\partial L}{\partial h} &= \tau_2 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= \tau_3, \end{aligned}$$

where

$$\begin{aligned}
\frac{\partial L}{\partial \dot{z}} &= (m_c + 2m_r)\dot{z} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) &= (m_c + 2m_r)\ddot{z} \\
\frac{\partial L}{\partial z} &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{h}} &= (m_c + 2m_r)\dot{h} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{h}}\right) &= (m_c + 2m_r)\ddot{h} \\
\frac{\partial L}{\partial h} &= -(m_c + 2m_r)g
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} &= (J_c + 2m_r d^2)\dot{\theta} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= (J_c + 2m_r d^2)\ddot{\theta} \\
\frac{\partial L}{\partial \theta} &= 0.
\end{aligned}$$

This means that the equations of motion are

$$\begin{aligned}
(m_c + 2m_r)\ddot{z} &= -(f_r + f_l)\sin(\theta) - \mu\dot{z} \\
(m_c + 2m_r)\ddot{h} + (m_c + 2m_r)g &= (f_r + f_l)\cos(\theta) \\
\left(J_c + 2m_r d^2\right)\ddot{\theta} &= d\left(f_r - f_l\right)
\end{aligned}$$

or in matrix form

$$\begin{bmatrix} m_c + 2m_r & 0 & 0 \\ 0 & m_c + 2m_r & 0 \\ 0 & 0 & J_x + 2m_r d^2 \end{bmatrix} \begin{bmatrix} \ddot{z} \\ \ddot{h} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -(f_r + f_l)\sin(\theta) - \mu\dot{z} \\ -(m_c + 2m_r)g + (f_r + f_l)\cos(\theta) \\ d\left(f_r - f_l\right) \end{bmatrix}.$$