Homework E.3 - Solution

The generalized coordinates are

$$\begin{aligned} y_1 &= z \\ y_2 &= \theta \end{aligned}$$

and the generalized forces are

$$\begin{aligned} \tau_1 &= 0 \\ \tau_2 &= \underbrace{lF\cos(\theta)}_{\text{torque}}. \end{aligned}$$

The kinetic energy of the system is

$$K = \frac{1}{2}m_1\dot{z}^2 + \frac{1}{2}\left(\frac{m_2 l^2}{3} + m_1 z^2\right)\dot{\theta}^2$$

and the potential energy of the system is

$$P = P_0 + m_1 g z\sin(\theta) + \frac{m_2 g l}{2}\sin\theta.$$

Note that $\frac{m_2 l^2}{3}$ is the moment of inertia for a rod. The Lagrangian is

$$L = K - P = \frac{1}{2}m_1\dot{z}^2 + \frac{1}{2}\left(\frac{m_2 l^2}{3} + m_1 z^2\right)\dot{\theta}^2 - P_0 - m_1 g z\sin(\theta) - \frac{m_2 g l}{2}\sin(\theta).$$

The Euler Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} &= \tau_1 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= \tau_2, \end{aligned}$$

where

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{z}} &= m_1 \dot{z} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) &= m_1 \ddot{z} \\
\frac{\partial L}{\partial z} &= m_1 z \dot{\theta}^2 - m_1 g \sin\theta \\
\frac{\partial L}{\partial \dot{\theta}} &= \left(\frac{m_2 l^2}{3} + m_1 z^2\right)\dot{\theta} \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \left(\frac{m_2 l^2}{3} + m_1 z^2\right)\ddot{\theta} + 2 m_1 z \dot{z} \dot{\theta} \\
\frac{\partial L}{\partial \theta} &= -m_1 g z \cos(\theta) - \frac{m_2 g l}{2}\cos(\theta).
\end{aligned}
$$

This means that the equations of motion are

$$
\begin{aligned}
m_1 \ddot{z} - m_1 \dot{\theta}^2 + m_1 g \sin(\theta) &= 0 \\
\left(\frac{m_2 l^2}{3} + m_1 z^2\right)\ddot{\theta} + 2 m_1 z \dot{z} \dot{\theta} + m_1 g z \cos\theta + \frac{m_2 g l}{2}\cos\theta &= l F \cos\theta.
\end{aligned}
$$