Construction of $\mathbb{Z}$

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Assumptions

- There exists a set $\mathbb{N} = \{0, 1, 2, ...\}$.
- There exists $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $(\mathbb{N}, +)$ forms a commutative monoid with identity $0 \in \mathbb{N}$. [1]
- The function $succ : \mathbb{N} \to \mathbb{N}^+, n \mapsto n + 1$ is injective.

[1] We will use infix notation for $+$

Goals

- Constructing the set $\mathbb{Z}$.
- Defining $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ and $\cdot : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$[2]

[2] We will use infix notation for $+$ and $\cdot$

Defining the set $\mathbb{Z}$

Idea

We want $\overbrace{z}^{\in \mathbb{Z}} \equiv \overbrace{(a,b)}^{\in \mathbb{N}\times\mathbb{N}} \Leftrightarrow z = a - b$.
Issue: This representation is not unique. E.g: $0 = 1 - 1 = 2 - 2 = \dots$.

Definition: $\sim$

$(a,b) \sim (c,d) :\Leftrightarrow a + d = b + c$

Lemma

$\sim$ *is an equivalence relation*

Defining the set $\mathbb{Z}$

Proof.

Reflexivity:
$\forall (a,b) \in \mathbb{N} \times \mathbb{N} : a + b = b + a.$
Symmetry:

$$(a,b) \sim (c,d) \Rightarrow c + b = b + c \underset{(a,b)\sim(c,d)}{=} a + d = d + a \Rightarrow (c,d) \sim (a,b).$$

Transitivity:
Let $(a,b) \sim (c,d), (c,d) \sim (e,f)$. Then

$$succ^{c+d}(a+f) = \underbrace{a+d}_{=b+c} + \underbrace{c+f}_{=d+e} = b + c + d + e = succ^{c+d}(b+e)$$

$$\underset{\text{succ injective}}{\Rightarrow} a + f = b + e \Rightarrow (a,b) \sim (e,f)$$

$\square$

Defining the set $\mathbb{Z}$

Definition: $\mathbb{Z}$

$\mathbb{Z} := \{[(a, b)] \mid a, b \in \mathbb{N}\}$

Defining $+$ and $\cdot$

Definition : $+$

$[a, b] + [c, d] := [a + c, b + d]$

Remark

This gives us the usual Addition on $\mathbb{Z}$: $y = a - b, z = c - d \Rightarrow y + z = a + c - (b + d)$