

Construction of \mathbb{Z}

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Assumptions

- There exists a set $\mathbb{N} = \{0, 1, 2, \dots\}$.
- There exists $+$: $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that $(\mathbb{N}, +)$ forms a commutative monoid with identity $0 \in \mathbb{N}$.¹
- The function $\text{succ} : \mathbb{N} \rightarrow \mathbb{N}^+, n \mapsto n + 1$ is injective.

¹We will use infix notation for $+$

Goals

- Constructing the set \mathbb{Z} .
- Defining $+: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ and $\cdot: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ ²
- Showing $(\mathbb{Z}, +, \cdot)$ is a commutative ring with multiplicative identity 1.

²We will use infix notation for $+$ and \cdot .

Defining the set \mathbb{Z}

Idea

We want $\underbrace{}_{\in \mathbb{Z}} \equiv \underbrace{(a, b)}_{\in \mathbb{N} \times \mathbb{N}} \Leftrightarrow z = a - b.$

Issue: This representation is not unique. E.g: $0 = 1 - 1 = 2 - 2 = \dots$

Definition: \sim

$$(a, b) \sim (c, d) :\Leftrightarrow a + d = b + c$$

Lemma

\sim is an equivalence relation

Defining the set \mathbb{Z}

Proof.

Reflexivity:

$$\forall (a, b) \in \mathbb{N} \times \mathbb{N} : a + b = b + a.$$

Symmetry:

$$(a, b) \sim (c, d) \Rightarrow c + b = b + c \stackrel{(a,b) \sim (c,d)}{=} a + d = d + a \Rightarrow (c, d) \sim (a, b).$$

Transitivity:

Let $(a, b) \sim (c, d), (c, d) \sim (e, f)$. Then

$$\text{succ}^{c+d}(a + f) = \underbrace{a + d}_{=b+c} + \underbrace{c + f}_{=d+e} = b + c + d + e = \text{succ}^{c+d}(b + e)$$

$$\stackrel{\text{succ injective}}{\Rightarrow} a + f = b + e \Rightarrow (a, b) \sim (e, f)$$



Defining the set \mathbb{Z}

Definition: \mathbb{Z}

$$\mathbb{Z} := \mathbb{N} \times \mathbb{N} / \sim = \{[(a, b)] \mid a, b \in \mathbb{N}\}$$

Defining $+$

Remark

$(\mathbb{N} \times \mathbb{N}, +_2)$ as direct product of $(\mathbb{N}, +)$ with itself is a semigroup.

Lemma

\sim is compatible with $+_2$.

Proof.

Let $(a, b) \sim (a', b'), (c, d) \sim (c', d')$. Then

$$(a + c) + (b' + d') = \underbrace{(a + b')}_{=b+a'} + \underbrace{(c + d')}_{=d+c'} = (b + d) + (a' + c')$$

$$\Rightarrow (a, b) +_2 (c, d) = (a + c, b + d) \sim (a' + c', b' + d') = (a', b') +_2 (c', d')$$



Defining $+$

Corollary: Definition of $+$

$[(a, b)] +_3 [(c, d)] := [(a, b) +_2 (c, d)] = [(a + c, b + d)]$ is well-defined and makes $(\mathbb{Z}, +_3)$ a semigroup.

Remark

This gives us the usual Addition on \mathbb{Z} :

$$y = a - b, z = c - d \Rightarrow y + z = a + c - (b + d)^a$$

^aFrom now on we will not distinguish between $+$, $+_2$ and $+_3$

Lemma

$(\mathbb{Z}, +)$ is an abelian group.

Defining +

Proof.

Commutativity: $\forall [(a, b)], [(c, d)] \in \mathbb{Z} :$

$$[(a, b)] + [(c, d)] = [(a + c, b + d)] = [(c + a, d + b)] = [(c, d)] + [(a, b)]$$

Neutral Element: $\forall [(a, b)] \in \mathbb{Z} : [(a, b)] + [(0, 0)] = [(a, b)]$

Inverses: $\forall [(a, b)] \in \mathbb{Z} : [(a, b)] + [(b, a)] = [(a + b, b + a)] = [(0, 0)]$
 $\sim_{(0,0)}$

