Construction of $\mathbb{Z}$

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Assumptions

- There exists a set $\mathbb{N} = \{0, 1, 2, ...\}$.
- There exists $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $(\mathbb{N}, +)$ forms a commutative monoid with identity 0. [1]
- The function $succ : \mathbb{N} \to \mathbb{N}^+, n \mapsto n + 1$ is injective.
- There exists $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $(\mathbb{N}, \cdot)$ is a commutative monoid with identity 1
- The functions $\varphi_k : \mathbb{N} \to \{mathbbN, x \mapsto kx$ are injective for $k \in \mathbb{N}^+$
- There exists the usual order on $\mathbb{N}$

[1] We will use infix notation for $+$

Goals

- Constructing the set $\mathbb{Z}$.
- Defining $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ and $\cdot : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ [2]
- Showing $(\mathbb{Z}, +, \cdot)$ is a commutative ring with multiplicative identity 1.

[2] We will use infix notation for $+$ and $\cdot$

Defining the set $\mathbb{Z}$

Idea

We want $\overbrace{z}^{\in \mathbb{Z}} \equiv \overbrace{(a, b)}^{\in \mathbb{N}\times\mathbb{N}} \Leftrightarrow z = a - b$.
Issue: This representation is not unique. E.g: $0 = 1 - 1 = 2 - 2 = \ldots$.

Definition: $\sim$

$(a, b) \sim (c, d) :\Leftrightarrow a + d = b + c$

Lemma

$\sim$ *is an equivalence relation*

Defining the set $\mathbb{Z}$

Proof.

Reflexivity:
$\forall (a,b) \in \mathbb{N} \times \mathbb{N} : a + b = b + a.$
Symmetry:
$$(a,b) \sim (c,d) \Rightarrow c + b = b + c \underset{(a,b)\sim(c,d)}{=} a + d = d + a \Rightarrow (c,d) \sim (a,b).$$
Transitivity:
Let $(a,b) \sim (c,d), (c,d) \sim (e,f)$. Then
$$succ^{c+d}(a+f) = \underbrace{a+d}_{=b+c} + \underbrace{c+f}_{=d+e} = b + c + d + e = succ^{c+d}(b+e)$$
$$\underset{\text{succ injective}}{\Rightarrow} a + f = b + e \Rightarrow (a,b) \sim (e,f)$$
$\square$

Defining the set $\mathbb{Z}$

Definition: $\mathbb{Z}$

$$\mathbb{Z} := \mathbb{N} \times \mathbb{N} / \sim \; = \{[(a, b)] \mid a, b \in \mathbb{N}\}$$

Defining $+$

Remark

$(\mathbb{N} \times \mathbb{N}, +_2)$ as direct product of $(\mathbb{N}, +)$ with itself is a semigroup.

Lemma

$\sim$ is comptabile with $+_2$.

Proof.

Let $(a, b) \sim (a', b'), (c, d) \sim (c', d')$. Then

$$(a + c) + (b' + d') = \underbrace{(a + b')}_{=b+a'} + \underbrace{(c + d')}_{=d+c'} = (b + d) + (a' + c')$$

$$\Rightarrow (a, b) +_2 (c, d) = (a + c, b + d) \sim (a' + c', b' + d') = (a', b') +_2 (c', d') \qquad \square$$

Defining $+$

Corollary: Definition of $+$

$[(a, b)] +_3 [(c, d)] := [(a, b) +_2 (c, d)] = [(a + c, b + d)]$ is well-defined and makes $(\mathbb{Z}, +_3)$ a semigroup.

Remark

This gives us the usual Addition on $\mathbb{Z}$:
$y = a - b, z = c - d \Rightarrow y + z = a + c - (b + d)$ [a]

[a] From now on we will not distinguish between $+, +_2$ and $+_3$

Lemma

$(\mathbb{Z}, +)$ is an abelian group.

Defining $+$

Proof.

Commutativity: $\forall [(a,b)], [(c,d)] \in \mathbb{Z}:$
$[(a,b)] + [(c,d)] = [(a+c, b+d)] = [(c+a, d+b)] = [(c,d)] + [(a,b)]$

Neutral Element: $\forall [(a,b)] \in \mathbb{Z} : [(a,b)] + [(0,0)] = [(a,b)]$

Inverses: $\forall [(a,b)] \in \mathbb{Z} : [(a,b)] + [(b,a)] = [\underbrace{(a+b, b+a)}_{\sim(0,0)}] = [(0,0)]$ $\square$

Difference representation

Definition: -

For $\alpha, \beta \in \mathbb{Z}$ we define: $\alpha - \beta := \alpha + (-\beta)$

Identification of $\mathbb{N}$

The Map $\iota : \mathbb{N} \to \mathbb{Z}, n \mapsto [(n, 0)]$ is injective and compatible with $+$.

Proof.

Injective:
$[(a, 0)] = [(b, 0)] \Rightarrow a + 0 = b + 0 \Rightarrow a = b$

Compatible: $\forall a, b \in \mathbb{N}$:
$\iota(a + b) = [(a + b, 0)] = [(a, 0)] + [(b, 0)] = \iota(a) + \iota(b)$ $\square$

Difference representation

Identification of $\mathbb{N}$

We identify $\mathbb{N}$ with the isomorphic set $\iota(\mathbb{N}) \subseteq \mathbb{Z}$.

Difference representation

We can now represent integers as
$[(a, b)] = [(a, 0)] + [(0, b)] = [(a, 0)] - [(b, 0)] = a - b$

Definition of $\cdot$

Idea

We want $(a-b)(c-d) = ac - ad - bc + bd = ac + bd - (ad + bc)$

Definition: $\cdot$

$[(a,b)] \cdot [(c,d)] := [(ac+bd, ad+bc)] = [(ca+db, da+cb)] = [(c,d)] \cdot [(a,b)]$

$\cdot$ is well-defined

Let $[(a,b)] = [(a',b')], [(c,d)]$. We have

$[(a',b')] \cdot [(c,d)] = [(a'c + b'd, a'd + b'c)] = \underbrace{[(a'c, b'c)]}_{\sim(ac,bc)} + \underbrace{[(b'd, a'd)]}_{\sim(bd,ad)} =$

$[(ac+bd, ad+bc)]$

By symmetry $\cdot$ is also invariant under changes of representative in the 2nd argument.

Definition of $\cdot$

Lemma

$(\mathbb{Z}, \cdot)$ is a commutative monoid.

Proof.

Associativity: $\forall [(a,b)], [(c,d)], [(e,f)] \in \mathbb{Z}$:
$[(a,b)] \cdot ([(c,d)] \cdot [(e,f)]) = [(a,b)] \cdot [(ce+df, cf+de)]$
$= [(e(ac+bd)+f(ad+bc), f(ac+bd)+e(ad+bc))]$
$= [(ac+bd, ad+bc)] \cdot [(e,f)]$
$= ([(a,b)] \cdot [(c,d)]) \cdot [(e,f)]$

Neutral Element: $\forall [(a,b)] \in \mathbb{Z}$:
$[(a,b)] \cdot [(1,0)] = [(a \cdot 1 + b \cdot 0, a \cdot 0 + b \cdot 1)] = [(a,b)]$ $\square$

$\mathbb{Z}$ is an integral domain

Corollary

$\mathbb{Z}$ is a commutative ring with identity.

Proof.

$(\mathbb{Z}, +)$ is an abelian group and $(\mathbb{Z}, \cdot)$ is a commutative monoid. We also have:

Distributivity: $\forall [(a,b)], [(c,d)], [(e,f)]$:
$[(a,b)] \cdot ([(c,d)] + [(e,f)]) = [(a(c+e) + b(d+f), a(d+f) + b(c+e))] =$
$[(a,b)] \cdot [(c,d)] + [(a,b)] \cdot [(e,f)]$ $\square$

$\mathbb{Z}$ is an integral doman

Corollay

$\mathbb{Z}$ is an integral domain

Proof.

Let $[(a,b)][(c,d)] = [(0,0)], [(a,b)] \neq [(0,0)]$

$\Rightarrow ac + bd = ad + bc$

$\Rightarrow (a-b)c = (a-b)d$

Assume $a > b$, so $a - b = k$ for some $k \geq 1$

$\Rightarrow kc = kd \underset{\substack{\varphi_k : \mathbb{N} \to \mathbb{N}, x \mapsto kx \\ \text{injective for } k \in \mathbb{N}^+}}{\Rightarrow} c = d \Rightarrow [(c,d)] = [(0,0)]$

If $a < b$, then for $k := a - b$ the map $\varphi_{-k} : \mathbb{N} \to \mathbb{N}, x \mapsto (-k)x$ is injective. □

Ordering on $\mathbb{Z}$

Definition

For $a, b \in \mathbb{Z}$ we define $a \leq b :\Leftrightarrow b - a \in \mathbb{N}$

Lemma

$\leq$ *is an order relation*

Ordering on $\mathbb{Z}$

Proof.

Reflexivity: $\forall[(a,b)] \in \mathbb{Z} : [(a,b)] - [(a,b)] = [(0,0)] \in \mathbb{N}$

Antisymmetry: Let $[(a,b)] \leq [(c,d)], [(c,d)] \leq [(a,b)]$.
$\Rightarrow \exists k_1, k_2 \in \mathbb{N} : a + d = b + c + k_1,\ b + c = a + d + k_2$.
$\Rightarrow k_1 = a + d - (b + c) = -k_2$
$\Rightarrow k_1 = k_2 = 0 \Rightarrow [(a,b)] = [(c,d)]$

Transitivity: Let $a \leq b, b \leq c$.
$\Rightarrow \exists k_1, k_2 \in \mathbb{N} : a + k_1 = b,\ b + k_2 = c$
$\Rightarrow \exists k_3 \in \mathbb{N} : a + k_3 = c$
$\Rightarrow c - a \in \mathbb{N} \Rightarrow a \leq c$ $\square$

Ordering on $\mathbb{Z}$

Lemma

$\leq$ *is a total order.*