PMATH 333: Introduction to Real Analysis

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Contents

These are my notes for my 2nd year course Introduction to Analysis (PMATH 333) at the University of Waterloo.
Reference text: Introduction to Analysis, William R. Wade

Chapter 1

Real Numbers

Definition. Let $A \subseteq \mathbf{R}$, we say that A is **bounded above** if $\exists M \in \mathbf{R}$ s.t. $a \leq M \, \forall a \in A$ we call M an **upper bound** for A
Similar definitions for a set to be **bounded below** and to have a **lower bound**

Definition. We say A is **bounded** if it is both bounded above and below.

Definition. Let $\emptyset \neq A \subseteq \mathbf{R}$. A number $M \in \mathbf{R}$ is called a **supremum (sup)** for A if

- M is an upper bound for A and
- If N is an upper bound for A then $M \leq N$

i.e. Supremum = least upper bound

Remark: Supremums are unique. $\emptyset \neq A \subseteq \mathbf{R}$
1) Suppose M, N are supremums of A, then

$$M \leq N, N \leq M \Rightarrow M = N$$

we write $M = \sup A$
2) If A is not bounded above, we write $\sup A = \infty$

Definition. Let $\emptyset \neq A \subseteq \mathbf{R}$. A number $M \in \mathbf{R}$ is called an **infimum (inf)** for A if

- M is a lower bound for A and
- If N is a lower bound for A then $N \leq M$

i.e. Infimum = greatest lower bound
Again, infimums are also unique, and are written as $\inf A = M$, if it is not bounded below

then $\inf A = -\infty$

Note: It is not always the case that $\sup A = \max A$, e.g. $A = [1, 2)$, here $\max A$ does not exist

Axiom. [Least Upper Bound Property, LUB]
If $\emptyset \neq A \subseteq \mathbf{R}$ is bounded above then $\sup A$ exists

Theorem. Let $\emptyset \neq A \subseteq \mathbf{R}$. If A is bounded below then $B = \{-a : a \in A\}$ is bounded above. Moreover,

$$\inf A = -\sup B$$

In particular, $\inf A$ exists

Theorem. [Archimedian Principle]
Let $a, b \in \mathbf{R}$ be positive. there exists $n \in \mathbf{N}$ s.t. $b < na$

Theorem. [Density of the Rationals]
Let $a < b$ be real numbers. There exists $q \in \mathbb{Q}$ s.t. $a < q < b$

Corollary. Let $a \in \mathbf{R}$. For every $\epsilon > 0$ there exists $q \in \mathbb{Q}$ s.t. $|a - q| < \epsilon$

Definition. A **sequence** of real numbers is an *infinite* list $(a_1, a_2, \dots)$ where each $a_i \in \mathbf{R}$
Notation: $(a_n)_{n=1}^{\infty}$ or (a_n). We write $(a_n) \subseteq \mathbf{R}$ (this does not mean sequence is a subset, it means terms in sequence are real)

Definition. $(a_n) \subseteq \mathbf{R}, a \in \mathbf{R}$ we say (a_n) **converges** to a, written $a_n \to a$, if for all $\epsilon > 0$ there exists $N \in \mathbf{N}$ s.t. $|a_n - a| < \epsilon$ for all $n \geq N$. We call a the **limit** of the sequence

Definition. We say $(a_n) \subseteq \mathbf{R}$ is **bounded** if $\{a_1, a_2, \dots\}$ is bounded. i.e. If $\exists M \in \mathbf{R}$ s.t. $|a_n| \leq M$ for all $n \in \mathbf{N}$

Proposition. If $(a_n) \subseteq \mathbf{R}$ is convergent then (a_n) is bounded

Proposition. $(a_n), (b_n) \subseteq \mathbf{R}, a_n \to a, b_n \to b$

- $a_n + b_n \to a + b$
- If $\alpha \in \mathbf{R}$ then $\alpha a_n \to \alpha a$
- $a_n b_n \to ab$

- If $b_n \neq 0$ for all $n \in \mathbf{N}$ and $b \neq 0$ then $\frac{a_n}{b_n} \to \frac{a}{b}$

Proposition. $(a_n), (b_n) \subseteq \mathbf{R}, a_n \to a, b_n \to b$. If there exists $N \in \mathbf{N}$ s.t. $a_n \leq b_n$ for all $n \geq N$, then $a \leq b$

Proposition. $(a_n) \subseteq [c, d]$ and $a_n \to a$, then $a \in [c, d]$

Definition. $(a_n) \subseteq \mathbf{R}$

- (a_n) is **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$
- (a_n) is **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$
- (a_n) is **monotone** if it is either **increasing** or **decreasing**

We call the sequences **strictly increasing** or **strictly decreasing** when the inequalities are strict

Theorem. **[Monotone Convergence Theorem]**
If $(a_n) \subseteq \mathbf{R}$ is increasing and $\{a_n : n \in \mathbf{N}\}$ is bounded above then $a_n \to \sup\{a_n : n \in \mathbf{N}\}$

Corollary. If $(a_n) \subseteq \mathbf{R}$ is decreasing and $\{a_n : n \in \mathbf{N}\}$ is bounded below, then $a_n \to \inf\{a_n : n \in \mathbf{N}\}$

Theorem. **[Nested Intervals Lemma]**
Let $I_1 \supseteq I_2 \supseteq \dots$ where each $I_i = [a_i, b_i]$, $a_i \leq b_i$. Then $\bigcap_{i-1}^{\infty} I_i \neq \emptyset$

Definition. $(a_n) \subseteq \mathbf{R}$. A **subsequence** of (a_n) is a sequence

$$(a_{n_k})_{k=1}^{\infty}$$

where $n_1 < n_2 < n_3 < \dots$

Theorem. **[Bolzano-Weierstrass Theorem]**
Every bounded sequence of real numbers has a convergent subsequence

Definition. $(a_n) \subseteq \mathbf{R}$. We say (a_n) is **Cauchy** if $\forall \epsilon > 0$, $\exists N \in \mathbf{N}$ s.t. $|a_n - a_m| < \epsilon$ for all $n, m \geq N$

Proposition. $(a_n) \subseteq \mathbf{R}$. If (a_n) is convergent then (a_n) is Cauchy

Proposition. $(a_n) \subseteq \mathbf{R}$. If (a_n) is Cauchy then (a_n) is bounded

Proposition. If $(a_n) \subseteq \mathbf{R}$ is Cauchy and has a subsequence $a_{n_k} \to a$ then $a_n \to a$

Theorem. [Completeness of R]

A sequence $(a_n) \subseteq \mathbf{R}$ is convergent $\iff$ it is Cauchy

Remark: Up till now the big theorems we have used have been

Least Upper Bound property

$\Rightarrow$ Monotone Convergence Theorem

$\Rightarrow$ Nested Intervals Theorem

$\Rightarrow$ Bolzano-Weierstrass theorem

$\Rightarrow$ Completeness (Cauchy $\iff$ Convergent)

Chapter 2

Normed Vector Spaces

Analysis, is the study of approximation of mathematical objects Idea: A Normed Vector Space is a vector space where we can measure the distance between vectors, it's useful for the purpose of approximations

Definition. Let $\mathbf{V}$ be a real vector space. A **norm** on $\mathbf{V}$ is a function

$$\|\cdot\| : \mathbf{V} \to \mathbf{R}$$

such that

1. $\|\mathbf{v}\| \geq 0$ for all $\mathbf{v} \in \mathbf{V}$
2. $\|\mathbf{v}\| = 0 \iff \mathbf{v} = 0$
3. For all $\alpha \in \mathbf{R}, \mathbf{v} \in \mathbf{V}$
$$\|\alpha \mathbf{v}\| = |\alpha| \, \|\mathbf{v}\|$$
4. [Triangle Inequality]: For all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$,
$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

Definition. Let $\|\cdot\|$ be a norm on $\mathbf{V}$. We call the pair $(\mathbf{V}, \|\cdot\|)$ a **normed vector space**

Convention: If $\|\cdot\|$ is understood, we write $\mathbf{V}$ instead of $(\mathbf{V}, \|\cdot\|)$

Aside: $\|\mathbf{v}\|$ can be seen as "length" of $\mathbf{v}$. And $\|\mathbf{v} - \mathbf{u}\|$ can be seen as the distance between two points

Examples of norms:

- The **absolute value** $(\mathbf{R}, |\cdot|)$
- The **Euclidean norm** $(\mathbf{R}^n, \|\cdot\|_2)$ which is

$$\|(x_1, \ldots, x_n)\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$$

- The **p-norm** $(\mathbf{R}^n, \|\cdot\|_p), p \geq 1 \in \mathbf{R}$ where

$$\|(x_1, \ldots, x_n)\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$$

- The **sup norm** $(\mathbf{R}^n, \|\cdot\|_\infty)$, where

$$\|(x_1, \ldots, x_n)\|_\infty = \sup\{|x_i| : i = 1, 2, \ldots, n\} = \max\{\|x_i| : i = 1, 2, \ldots, n\}$$

- Let $\mathbf{R}^{\mathbf{N}} := \{(x_i)_{i=1}^\infty : x_i \in \mathbf{R}\}$. This is how we are defining an infinite dimensional real vector space, this is actually called a *real sequence space*
  $p \geq 1$ (real number)

$$\|(x_i)_{i=1}^\infty\|_p = \big(\sum_{i=1}^{\infty} |x_i|^p\big)^{1/p} \text{ but this could be } \stackrel{?}{=} \infty$$

  so to prevent this by removing sequences that give an infinite p-norm

$$l^p := \{(x_i) \in \mathbf{R}^{\mathbf{N}} : \|(x_i)\|_p < \infty\}$$

  Also, l^p is a subspace of $\mathbf{R}^{\mathbf{N}}$. And so $(l^p, \|\cdot\|_p)$ is a normed vector space and is called the **p-norm**. l^p is sometimes referred to as a *Lebesgue space*
- $(x_i) \in \mathbf{R}^{\mathbf{N}}$,

$$\|(x_i)\|_\infty = \sup\{|x_i| : i \in \mathbf{N}\} \text{ but again this could be } \stackrel{?}{=} \infty$$

  so we define the vector space

$$l^\infty = \{(x_i) \in \mathbf{R}^{\mathbf{N}} : \|(x_i)\|_\infty < \infty\}$$

  Again, l^∞ is the space of all bounded sequences, and is a subspace of $\mathbf{R}^{\mathbf{N}}$. And so $(l^\infty, \|\cdot\|_\infty)$ is a normed vector space with the norm called **sup norm** or **infinity norm**
- For real numbers $a < b$, the vector space of all continuous functions on $[a, b]$

$$C([a, b]) = \{f : [a, b] \to \mathbf{R} \text{ that are continuous}\}$$

$$\|f\|_\infty = \sup\{|f(x)| : x \in [a, b]\} \stackrel{EVT}{=} \max\{|f(x)| : x \in [a, b]\}$$

  $(C([a, b]), \|\cdot\|_\infty)$ is a normed vector space is called the **uniform norm**

2.1 Convergence

Definition. Let $\mathbf{V}$ be a normed vector space. A **sequence** in $\mathbf{V}$ is a right-infinite ordered list $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots)$ where each $\mathbf{v} \in \mathbf{V}$.
We denote this sequence by $(\mathbf{v}_i)_{i=1}^{\infty}$ or $(\mathbf{v}_i)$. Again, we write $(\mathbf{v}_i) \subseteq \mathbf{V}$ to mean each $\mathbf{v}_i \in \mathbf{V}$

Definition. Let $\mathbf{V}$ be a normed vector space, $(\mathbf{a}_n) \subseteq \mathbf{V}$, and $\mathbf{v} \in \mathbf{V}$
We say $(\mathbf{a}_n)$ **converges** to $\mathbf{v}$, written $\mathbf{a}_n \to \mathbf{v}$ if for all $\epsilon > 0$, if there exists $N \in \mathbf{N}$, such that if for all $n \geq N$ we have that $\|\mathbf{a}_n - \mathbf{v}\| < \epsilon$
We call $\mathbf{v}$ the limit of $(\mathbf{a}_n)$. If $(\mathbf{a}_n)$ does not converge to any $\mathbf{v} \in \mathbf{V}$, we say $(\mathbf{a}_n)$ **diverges** (in $\mathbf{V}$)

Example: Let $\mathbf{V} = l^{\infty} = \{(x_i) \in \mathbf{R}^{\mathbf{N}} : \sup\limits_{i \in \mathbf{N}}\{|x_i|\} < \infty\}$
Let $(\mathbf{a}_n) \subseteq \mathbf{V}$: $\mathbf{a}_n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, 0, \dots)$
Claim $\mathbf{a}_n \to (1, \frac{1}{2}, \frac{1}{3}, \dots)$
Hint: $\|\mathbf{a}_n - (1, 1/2, 1/3, \dots)\|_{\infty} = \left\|(0, \dots, 0, -\frac{1}{n+1}, -\frac{1}{n+1}, \dots)\right\| = \sup\{0, \frac{1}{n+1}, \frac{1}{n+2}, \dots\} = \frac{1}{n+1}$

Example: Let $\mathbf{V} = l^{\infty}, (\mathbf{a}_n) \subseteq l^{\infty}, \mathbf{a}_n = (1, 2, \dots, n, 0, 0, \dots)$
Claim: $(\mathbf{a}_n)$ diverges in l^{∞}
Hint: Often for contradiction style proofs involving epsilons, use a fixed explicit value for epsilon and go about showing a contradiction.

Example: We have the normed vector space $(C([0,1]), \|\cdot\|_{\infty})$ (note this is the uniform norm, not the infinity norm)
Let $(f_n) \subseteq C([0,1])$, $f_n(x) = (x - \frac{1}{n})^2$.
Claim: $f_n \to f$, where $f(x) = x^2$.
Hint: Using the absolute value as the norm for some x in the interval show that the difference is within epsilon. And then reconcile the result for the function f being the limit using the uniform norm

Proposition. Let $\mathbf{V}$ be a normed vector space, $(\mathbf{a}_n), (\mathbf{b}_n) \subseteq \mathbf{V}$.
Suppose $\mathbf{a}_n \to \mathbf{v} \in \mathbf{V}$ and $\mathbf{b}_n \to \mathbf{w} \in \mathbf{V}$. Then,

1. $\mathbf{a}_n + \mathbf{b}_n \to \mathbf{v} + \mathbf{w}$
2. $\alpha \mathbf{a}_n \to \alpha \mathbf{v}$ $(\alpha \in \mathbf{R})$

2.2 Cauchy Sequences

Problem: The definition of convergence requires us to know or guess the limit of the sequence.

Proposition. Let $\mathbf{V}$ be a normed vector space, $(\mathbf{a}_n) \subseteq \mathbf{V}$, and $\mathbf{a}_n \to \mathbf{a} \in \mathbf{V}$.
For all $\epsilon > 0$ there exists $N \in \mathbf{N}$ s.t. for all $n, m \geq N$

$$\|\mathbf{a}_n - \mathbf{a}_m\| < \epsilon$$

Definition. For a normed vector space $\mathbf{V}$, and $(\mathbf{a}_n) \subseteq \mathbf{V}$.
We say $(\mathbf{a}_n)$ is **Cauchy** if $\forall \epsilon > 0, \exists N \in \mathbf{N}$ s.t. whenever $n, m \geq N$

$$\|\mathbf{a}_n - \mathbf{a}_m\| < \epsilon$$

Remark: here we have only shown that for a sequence Convergent $\Rightarrow$ Cauchy

Example: Let $\mathbf{V} = C_{00} := \{(a_n) \in l^\infty : \exists N \in \mathbf{N}, \forall n \geq N, a_n = 0\}$
We equip $\mathbf{V}$ with $\|\cdot\|_\infty$ (the infinity norm)
The sequence $(\mathbf{a}_n) \subseteq \mathbf{V}$ given by $\mathbf{a}_n = (1, 1/2, \dots, 1/n, 0, 0, \dots)$. We showed that $\mathbf{a}_n \to \mathbf{a} \in l^\infty$, $\mathbf{a} = (1, 1/2, 1/3, \dots) \notin C_{00}$. Hence, $(\mathbf{a}_n) \subseteq C_{00}$ diverges.
Remark: We see that the convergence and divergence of sequences sometimes depends on the normed vector space that we are working in.
Claim: $(\mathbf{a}_n)$ is Cauchy
Rough proof: This is since the sequence is convergent in l^∞ so it is Cauchy in l^∞, and since it is Cauchy in l^∞ it is Cauchy in its subspace C_{00}. So it is Cauchy in C_{00} but not convergent

2.3 Completeness

Definition. Let $\mathbf{V}$ be a normed vector space, $(\mathbf{a}_n) \subseteq \mathbf{V}$. We say $(\mathbf{a}_n)$ is **bounded** (bd) if $\exists N \in \mathbf{N}$ s.t

$$\|\mathbf{a}_n\| < N$$

for all $n \in \mathbf{N}$

Proposition. Let $\mathbf{V}$ be a normed vector space. If $(\mathbf{a}_n) \subseteq \mathbf{V}$ is Cauchy then $(\mathbf{a}_n)$ is bounded
Idea: Here one may pick a fixed epsilon $(= 1)$, and find the max of the set $\{\|\mathbf{a}_1\|, \ldots, \|\mathbf{a}_{N-1}\|, 1+ \|\mathbf{a}_N\|\}$, where N satisfies the epsilon

So know that sequences being convergent is not the same as being Cauchy as there exist Cauchy sequences that don't converge. So we give the spaces where these notions are the same a name

Definition. Let $\mathbf{V}$ be a normed vector space
We say that $A \subseteq \mathbf{V}$ is **complete** if every Cauchy sequence $(\mathbf{a}_n) \subseteq A$ converges in A.
If $\mathbf{V}$ is complete itself (i.e. $A = \mathbf{V}$), we call $\mathbf{V}$ a **Banach space**

Remark: In a Banach space, a sequence is convergent $\iff$ the sequence is Cauchy
Also, only vector spaces that are complete can be called Banach spaces, a set that is not a vector space can be complete but not a Banach space.
Examples:

- $\mathbf{R}, \mathbf{R}^n, l^\infty$ are complete and so, are Banach spaces
- C_{00} is *not* a Banach space, though it is a vector space
- Let $(\frac{1}{n+1}) \subseteq (0,1) \subseteq \mathbf{R}$. $\frac{1}{n+1} \to 0 \notin (0,1)$. $(\frac{1}{n+1})$ is convergent in $\mathbf{R}$ so it is Cauchy. But since $0 \notin (0,1)$, $(0,1)$ is not complete

Chapter 3

Topology

Topology is the study of subsets of a set $\mathbf{X}$ which afford $\mathbf{X}$ meaningful analytic/ geometric properties.
What we'll do is; given a normed vector space $\mathbf{V}$, we want to investigate the way convergence/limits of sequences behave in subsets of $\mathbf{V}$

3.1 Closed and Open sets

Definition. Let $\mathbf{V}$ be a normed vector space. We say $C \subseteq \mathbf{V}$ is closed (in $\mathbf{V}$) if whenever $(\mathbf{x}_n) \subseteq C$ s.t. $\mathbf{x}_n \to \mathbf{x} \in \mathbf{V}$ then $\mathbf{x} \in C$
This is sort of saying that C is closed $\iff$ C is "closed" under taking limits.

Examples:

1. $\emptyset, \mathbf{V} \in \mathbf{V}$ are closed [1]
2. Let $\mathbf{V}$ be a NVS, for an $\mathbf{x} \in \mathbf{V}$ the set $\{\mathbf{x}\}$ is closed [2]
3. $[0,1) \subseteq \mathbf{R}$ is not closed [3]
4. $C_{00} \subseteq l^\infty$ is not closed in l^∞ **(*)**, the sequence $\mathbf{a}_n = (1, 1/2, ..., 1/n, 0, 0) \in C_{00}$ but $\mathbf{a} = (1, 1/2, 1/3, ...) \notin C_{00}$
5. $C_{00} \subseteq C_{00}$, C_{00} is closed in C_{00}. This is because C_{00} is a normed vector space itself and so here we see that by (1) C_{00} is closed when it is considered in the NVS which is itself. In (4) we saw that it was not closed since it was being considered in the NVS l^∞

[1]For the empty set, according to the definition, the hypothesis is false therefore the conclusion must be true
[2]there is only one sequence in this set, the sequence $\mathbf{x}_n = \mathbf{x}$ and this converges in the set
[3]Can be disproven by the sequence $(1 - 1/n)$

6. $A = \{f \in C([0,1]) : f(1/2) = 0\} \subseteq C([0,1])$
   Claim: A is closed in $C([0,1])$
   Proof: Let $(f_n) \subseteq A$ s.t. $f_n \to f \in C([0,1])$. We must show that $f \in A$
   Let $\epsilon > 0$ be given. Since $f_n \to f, \exists N \in \mathbf{N} : \|f_n - f\|_\infty < \epsilon$ for all $n \geq N$
   Then for any $n \geq N$, we have

$$\begin{aligned} |f(1/2)| &= |\underbrace{f_n(1/2)}_{=0} - f(1/2)| \\ &\leq \|f_n - f\|_\infty \text{ uniform norm} \\ &< \epsilon \end{aligned}$$

   Hence, $f(1/2) = 0 \implies f \in A$ and so A is closed
7. $\mathbf{V}$ NVS, $\mathbf{a} \in \mathbf{V}, r > 0$

   Definition. $\overline{B_r(\mathbf{a})} := \{\mathbf{x} \in \mathbf{V} : \|\mathbf{x} - \mathbf{a}\| \leq r\}$. This is called the **closed ball** at $\mathbf{a}$ of radius r
   Claim: $\overline{B_r(\mathbf{a})}$ is closed
   $(\mathbf{x}_n) \subseteq \overline{B_r(\mathbf{a})}, \mathbf{x}_n \to \mathbf{x} \in \mathbf{V}$
   We show $\mathbf{x} \in \overline{B_r(\mathbf{a})}$
   $\mathbf{x}_n - \mathbf{a} \to \mathbf{x} - \mathbf{a} \overset{hw}{\implies} \underbrace{\|\mathbf{x}_n - \mathbf{a}\|}_{\leq r} \to \underbrace{\|\mathbf{x} - \mathbf{a}\|}_{\leq r}$ Therefore, $\mathbf{x} \in \overline{B_r(\mathbf{a})}$
   Hw: We have that $\forall \epsilon > 0, \exists N \in \mathbf{N} : \forall n \geq N, \|\mathbf{x}_n - \mathbf{a} + (-\mathbf{x} + \mathbf{a})\| < \epsilon$
   By the reverse triangle inequality
   $\big| \|\mathbf{x}_n - \mathbf{a}\| - \|\mathbf{x} - \mathbf{a}\| \big| \leq \|\mathbf{x}_n - \mathbf{a} - \mathbf{x} + \mathbf{a}\| < \epsilon \implies \|\mathbf{x}_n - \mathbf{a}\| \to \|\mathbf{x} - \mathbf{a}\|$

Examples of Closed balls:
In $\mathbf{R} : \overline{B_r(a)} = [a - r, a + r]$
In $\mathbf{R}^2 : \overline{B_r(a)} =$ the solid disc centered at a of radius r
In $\mathbf{R}^3 : \overline{B_r(a)} =$ the solid/closed sphere centered at a of radius r

Proposition. $\mathbf{V}$ NVS, If $C \subseteq \mathbf{V}$ is complete then C is closed
Proof: Suppose C is complete. Take $(\mathbf{x}_n) \subseteq C : \mathbf{x}_n \to \mathbf{x} \in \mathbf{V}$. Since $(\mathbf{x}_n)$ is convergent, $(\mathbf{x}_n)$ is Cauchy. Thus, $\mathbf{x} \in C$ by completeness, therefore C is closed.
Converse is not true, if we took $V = C_{00}$ then it is closed as $C_{00} \subseteq C_{00}$ but it is not complete.

Definition. $\mathbf{V}$ NVS, We say $\mathcal{U} \subseteq \mathbf{V}$ is **open** (in $\mathbf{V}$) if $\mathbf{V} \setminus \mathcal{U}$ is closed.
Note: $\mathbf{V} \setminus \mathcal{U} = \{\mathbf{x} \in \mathbf{V} : \mathbf{x} \notin \mathcal{U}\}$

Example: $\mathbf{V}$ NVS, $\mathbf{a} \in \mathbf{V}, r > 0, A = \{\mathbf{x} \in \mathbf{V} : \|\mathbf{x} - \mathbf{a}\| \geq r\}$ is closed $\implies \mathbf{V} \setminus A = \{\mathbf{x} \in \mathbf{V} : \|\mathbf{x} - \mathbf{a}\| < r\}$ is open

Definition. We call $\mathbf{V} \setminus A$ the **open ball**, $B_r(\mathbf{a})$, centered at $\mathbf{a}$ of radius r
Now we'll prove a proposition that would allow us to say that a set is open without having to say that some other set is closed.

Proposition. $\mathbf{V}$ NVS, $\mathcal{U} \subseteq \mathbf{V}$. The following are equivalent

1. $\mathcal{U}$ is open in $\mathbf{V}$
2. $\forall \mathbf{a} \in \mathcal{U}, \exists r > 0 : B_r(\mathbf{a}) \subseteq \mathcal{U}$

The idea here is that for any $\mathbf{a}$ we pick inside $\mathcal{U}$, (imagine is close the periphery of $\mathcal{U}$), there is always enough "wiggle room" for us to form an open ball around $\mathbf{a}$
Proof: It's an iff statement so
($\implies$) Suppose $\mathcal{U}$ is open in $\mathbf{V}$. Thus $\mathbf{V} \setminus \mathcal{U}$ is closed
Let $\mathbf{a} \in \mathcal{U}$. Assuming for a contradiction, suppose for all $r > 0$,
$B_r(\mathbf{a}) \not\subseteq \mathcal{U} \iff B_r(\mathbf{a}) \cap (\mathbf{V} \setminus \mathcal{U}) \neq \emptyset$
For every $n \in \mathbf{N}$, let $\mathbf{x}_n \in B_{\frac{1}{n}}(\mathbf{a}) \cap (\mathbf{V} \setminus \mathcal{U})$. Then $\|\mathbf{x}_n - \mathbf{a}\| < 1/n \to 0$ and so $\underset{\in \mathbf{V} \setminus \mathcal{U}}{\mathbf{x}_n} \to \underset{\in \mathcal{U}}{\mathbf{a}}$
This contradicts that $\mathbf{V} \setminus \mathcal{U}$ is closed, so it must be that $B_r(\mathbf{a}) \subseteq \mathcal{U}$
($\impliedby$) Suppose (2). We show $\mathbf{V} \setminus \mathcal{U}$ is closed. Take $(\mathbf{x}_n) \subseteq \mathbf{V} \setminus \mathcal{U} : \mathbf{x}_n \to \mathbf{x} \in \mathbf{V}$
For a contradiction, suppose $\mathbf{x} \notin \mathbf{V} \setminus \mathcal{U} \iff \mathbf{x} \in \mathcal{U}$. So by (2), we know that $\exists r > 0 : B_r(\mathbf{x}) \subseteq \mathcal{U}$
For large enough n, $\mathbf{x}_n \in B_r(\mathbf{x}) \subseteq \mathcal{U}$ since $\mathbf{x} \in \mathcal{U}$. This is a contradiction, so it must be that $\mathbf{x} \notin \mathcal{U}$ (i.e. $\mathbf{x} \in \mathbf{V} \setminus \mathcal{U}$). Therefore, $\mathbf{V} \setminus \mathcal{U}$ is closed and hence $\mathcal{U}$ is open in $\mathbf{V}$

Definition. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$.

1. We say $\mathbf{x} \in \mathbf{V}$ is a **limit point** of A If $\exists (\mathbf{a}_n) \in A : \mathbf{a}_n \to \mathbf{x}$
2. We say $\mathbf{x} \in A$ is an **interior point** of A if $\exists r > 0 : B_r(\mathbf{x}) \subseteq A$

Summary: $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$

1. A is closed in $\mathbf{V} \iff A$ contains all its limit points
2. A is open in $\mathbf{V} \iff$ every point in A is an interior point of A
3. Clearly all interior points are limit points

3.2 Unions and Intersections

Examples:

1. $\bigcap_{i=1}^{\infty} \underbrace{\left(-\frac{1}{i}, \frac{1}{i}\right)}_{\text{open}} = \underbrace{\{0\}}_{\text{closed}}$
2. $\bigcup_{i=1}^{\infty} \underbrace{[0, 1-1/n]}_{\text{closed}} = \underbrace{[0,1)}_{\text{not closed}}$

Proposition. $\mathbf{V}$ NVS

1. If $\{A_\alpha\}_{\alpha \in I}$ are open in $\mathbf{V}$ then $\bigcup_{\alpha \in I} A_\alpha$ is open
2. If $A_1, A_2, \dots, A_n$ are open in $\mathbf{V}$ then $\bigcap_{i=1}^n A_i$ is open (since n is finite)

Proof:
(1) $\mathbf{a} \in \bigcup_{\alpha \in I} A_\alpha \implies \exists \alpha \in I : \mathbf{a} \in A_\alpha$. Since A_α is open, $\exists \epsilon > 0 : B_\epsilon(\mathbf{a}) \subseteq A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha$
This does not require I to be a finite collection, it can be infinite too
(2) $\mathbf{a} \in A_1 \cap \cdots \cap A_n, \forall i, \mathbf{a} \in A_i \implies \forall i, \exists r_i > 0, B_{r_i}(\mathbf{a}) \subseteq A_i$
We take $r = \min\{r_i : \forall i\} \implies B_r(\mathbf{a}) \subseteq A_1 \cap \cdots \cap A_n$
For (2) it is the minimum argument that causes it to work for finite but not infinite size, as the min will have to be replaced with an inf which could be 0, this would cause the $B_r(\mathbf{a})$ to be invalid

Corollary. $\mathbf{V}$ NVS

1. If $\{A_\alpha\}_{\alpha \in I}$ are closed in $\mathbf{V}$ then $\bigcap_{\alpha \in I} A_\alpha$ is closed
2. If $A_1, \dots, A_n$ are closed in $\mathbf{V}$ then $\bigcup_{i=1}^n A_i$ is closed

This is can be seen by using De Morgan's laws
$$\underbrace{\mathbf{V} \setminus (\bigcap_{\alpha \in I} A_\alpha)}_{\text{open}} = \underbrace{\bigcup_{\alpha \in I} (\mathbf{V} \setminus A_\alpha)}_{\text{open}}$$
Similarly
$$\underbrace{\mathbf{V} \setminus (\bigcup_{i=1}^{n} A_i)}_{\text{open}} = \underbrace{\bigcap_{i=1}^{n} (\mathbf{V} \setminus A_i)}_{\text{open}}$$

3.3 Closures and Interiors

Definition. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$

1. The **closure** of A is defined as $\overline{A} := \bigcap_{A\subseteq C} C$ where C is closed. That is the intersection of all the closed sets that contain A
2. The **interior** of A is defined as $\text{int}(A) := \bigcup_{\mathcal{U}\subseteq A} \mathcal{U}$ where $\mathcal{U}$ is open. The union of all the open sets contained in A

Remark:

1. $\overline{A}$ is the smallest closed set in $\mathbf{V}$ containing A
2. $\text{Int}(A)$ is the largest open set in $\mathbf{V}$ contained in A
3. A is closed $\iff \overline{A} = A$
4. A is open $\iff \text{int}(A) = A$

Proposition. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$. $\overline{A} = \{\mathbf{x} \in \mathbf{V} : \mathbf{x} \text{ is a limit point of } A\}$

Proof: Let $X = \{\mathbf{x} \in \mathbf{V} : \mathbf{x} \text{ is a limit point of } A\}$

$(X \subseteq \overline{A})$. HW: By definition $\overline{A}$ is the smallest closed set containing A, and since it is closed this set contains all the limit points of $\overline{A}$ and since $A \subseteq \overline{A}$ thus $\overline{A}$ contains all limit points of A i.e. $X \subseteq \overline{A}$

$(\overline{A} \subseteq X)$. We must show X is closed and A is a subset of X, so that we can show $(\overline{A} \subseteq X)$

Indeed, $A \subseteq X$. **Claim:** X is closed.

Let $(\mathbf{x}_n) \subseteq X : \mathbf{x}_n \to \mathbf{x} \in \mathbf{V}$

For every $n \in \mathbf{N}, \mathbf{x}_n$ is a limit point of A (by simply belonging to set X) and so we may find $\mathbf{y}_n \in A : \|\mathbf{y}_n - \mathbf{x}_n\| < 1/n$

Then, $\mathbf{y}_n = \underbrace{\mathbf{y}_n - \mathbf{x}_n}_{\to 0} + \underbrace{\mathbf{x}_n}_{\to \mathbf{x}} \to \mathbf{x}$ and so $\mathbf{x} \in X$. Therefore, X is closed.

Proposition. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$. $\text{Int}(A) = \{\mathbf{x} \in A : \mathbf{x} \text{ is an interior point of } A\}$

Proof: Let $X = \{\mathbf{x} \in A : \mathbf{x} \text{ is an interior point of } A\}$

$(\text{Int}(A) \subseteq X)$ HW: Let $\mathbf{x} \in \text{Int}(A) \implies \exists r > 0 : B_r(\mathbf{x}) \subseteq A \implies \mathbf{x} \in X$

$(X \subseteq \text{Int}(A))$. We show X is open and $X \subseteq A$. Obviously, $X \subseteq A$

Claim: X is open

Let $\mathbf{x} \in X$. Thus $\exists r > 0 : B_r(\mathbf{x}) \subseteq A$. Now, since that $B_r(\mathbf{x})$ is open, $\forall \mathbf{y} \in B_r(\mathbf{x}), \exists r' > 0 : B_{r'}(\mathbf{y}) \subseteq B_r(\mathbf{x})$. Thus, $B_{r'}(\mathbf{y}) \subseteq A$, and so $\mathbf{y} \in X$. Hence, $B_r(\mathbf{x}) \subseteq X$ and so X is open

Examples:

1. $A = [0, 1), \bar{A} = [0, 1]$, $\text{int}(A) = (0, 1)$
2. Closure of $B_r(a)$ is $\overline{B_r(a)}$
3. Interior of $\overline{B_r(a)}$ is $B_r(a)$

4. $A = \mathbb{Q} \in \mathbf{R}$, $\text{int}(A) = \emptyset$, $\bar{A} = \mathbf{R}$
5. $A = \{(e^{-x}\cos(x), e^{-x}\sin(x)) : x \geq 0\} \subseteq \mathbf{R}^2$ this traces an inward spiral. $\text{Int}A = \emptyset$, $\bar{A} = A \cup \{(0,0)\}$
6. $\mathbf{V} = l^\infty$

   $C_{00} = \{(\mathbf{x}_n) \in \mathbf{V} :$ entries eventually all 0's$\}$

   $C_0 = \{(\mathbf{x}_n) \in \mathbf{V} : \mathbf{x}_n \to \mathbf{0}\}$

   (a) Show C_0 is closed in l^∞

   Let $(\mathbf{x}_n) \subseteq C_0 : \mathbf{x}_n \to \mathbf{x} \in l^\infty$

   Claim: $\mathbf{x} \in C_0$

   Say for $n \in \mathbf{N}, \mathbf{x}_n = (x_n^{(1)}, x_n^{(2)}, \dots)$ and $\mathbf{x} = (a_1, a_2, \dots)$

   We know for every $n \in \mathbf{N}, \mathbf{x}_n \in C_0$ and so $x_n^{(k)} \to 0$, as $k \to \infty$

   Let $\epsilon > 0$ be given. We can find $N \in \mathbf{N} : \|\mathbf{x}_n - \mathbf{x}\|_\infty < \epsilon/2$ for $n \geq N$

   Also, we can find $K \in \mathbf{N} : |x_N^{(k)}| < \epsilon/2$ for $k \geq K$.

   Now, for $k \geq K$, we want the $a_k \to 0$ so

   $|a_k| = |a_k - x_N^{(k)} + x_N^{(k)}| \leq |a_k - x_N^{(k)}| + |x_N^{(k)}| \leq \|\mathbf{x} - \mathbf{x}_N\|_\infty + |x_N^{(k)}| < \epsilon/2 + \epsilon/2 = \epsilon$

   Hence, $\mathbf{x} \in C_0$ and so C_0 is closed.

   (b) Show $\overline{C_{00}} = C_0$.

   We have that $C_{00} \subseteq C_0$ and C_0 is closed. Hence, $\overline{C_{00}} \subseteq C_0$

   Claim: $C_0 \subseteq \overline{C_{00}}$

   Let $\mathbf{x} \in C_0$ say $\mathbf{x} = (a_1, a_2, \dots)$ hence by definition of $C_0, a_k \to 0$, as $k \to \infty$.

   For every $n \in \mathbf{N}$ let $\mathbf{x}_n = (a_1, a_2, \dots, a_n, 0, 0, \dots) \in C_{00}$

   Let $\epsilon > 0$ be given. We may find $N \in \mathbf{N} : |a_n| < \epsilon/2$ for $n \geq N$.

   For $n \geq N$, $\|\mathbf{x}_n - \mathbf{x}\|_\infty = \|\mathbf{x} - \mathbf{x}_n\|_\infty = \|(0, \dots, 0, a_{n+1}, a_{n+2}, \dots)\| \leq \epsilon/2 < \epsilon$

   Therefore, $\underbrace{\mathbf{x}_n}_{\in C_{00}} \to \mathbf{x}$ that is, every $\mathbf{x} \in C_0$ is a limit point of C_{00} and since that is how we've defined closure $\mathbf{x} \in \overline{C_{00}}$ and so $C_0 \subseteq \overline{C_{00}}$

3.4 Properties of Closures and Interiors

Proposition. $\mathbf{V}$ NVS, $A, B \subseteq \mathbf{V}$

1. $\text{Int}(A \cup B) \supseteq \text{Int}(A) \cup \text{Int}(B)$
2. $\text{Int}(A \cap B) = \text{Int}(A) \cap \text{Int}(B)$
3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
4. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

Example:

1. $A = [0, 1), B = (1, 2]$. $\overline{A \cap B} = \overline{\emptyset} = \emptyset$. $\bar{A} \cap \bar{B} = \{1\}$
2. $A = [0, 1], B = [1, 2]$. $\text{Int}(A \cup B) = (0{,}2)$. $\text{Int}(A) \cup \text{Int}(B) = (0, 1) \cup (1, 2)$

Proposition. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$

1. $\text{Int}(\mathbf{V} \setminus A) = \mathbf{V} \setminus \overline{A}$
2. $\overline{\mathbf{V} \setminus A} = \mathbf{V} \setminus \text{Int}(A)$

Proof:

1. Observe that $A \subseteq \overline{A}$ and so $\mathbf{V} \setminus \overline{A} \subseteq \mathbf{V} \setminus A$. Since $\mathbf{V} \setminus \overline{A}$ is open and $\mathbf{V} \setminus \overline{A} \subseteq \mathbf{V} \setminus A$, $\mathbf{V} \setminus \overline{A} \subseteq \text{Int}(\mathbf{V} \setminus A)$.
   Now, $\text{Int}(\mathbf{V} \setminus A) \subseteq \mathbf{V} \setminus A$ and so inverting this previous statement (by subtracting from $\mathbf{V}$ gives) $\mathbf{V} \setminus (\mathbf{V} \setminus A) = A \subseteq \mathbf{V} \setminus \text{Int}(\mathbf{V} \setminus A)$. And since $\mathbf{V} \setminus \text{Int}(\mathbf{V} \setminus A)$ is closed and contains A, we have by the smallness of the closure
   $\overline{A} \subseteq \mathbf{V} \setminus \text{Int}(\mathbf{V} \setminus A)$ then taking complements again $\text{Int}(\mathbf{V} \setminus A) \subseteq \mathbf{V} \setminus \overline{A}$
2. Let $B = \mathbf{V} \setminus A$, we have from (1)
   $\text{Int}(\mathbf{V} \setminus B) = \mathbf{V} \setminus \overline{B} \iff \text{Int}(A) = \mathbf{V} \setminus \overline{(\mathbf{V} \setminus A)}$ Taking complements again we get $\iff \mathbf{V} \setminus \text{Int}(A) = \overline{\mathbf{V} \setminus A}$

Definition. Let $\mathbf{V}$ NVS and $A \subseteq \mathbf{V}$. We define the **boundary** of A to be $\partial A = \overline{A} \setminus \text{Int}(A)$

3.5 Compactness and Heine-Borel Theorem

Definition. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$. We say A is **compact** if every sequence in A has a subsequence which converges in A.

Definition. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$. We say A is **bounded** if $\exists M > 0 : \|\mathbf{a}\| \leq M, \forall \mathbf{a} \in A$

Proposition. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$. If A is compact, then A is closed and bounded.
Proof sketch:
Assume A to be compact.
For every sequence $(\mathbf{a}_n)$ in A there will be a subsequence $(\mathbf{a}_{n_k})$ that converges in A. If $(\mathbf{a}_n)$ is convergent, then by Assignment 1, the sequence itself will converge in A hence A is closed.
Assume for contradiction that A is unbounded, so we can find a sequence whose norm

keeps increasing, any subsequence of that sequence is divergent and that contradicts compactness, so it must be bounded.

Theorem. **[Heine-Borel Theorem]**
A set $A \subseteq \mathbf{R}^n$ is compact $\iff$ A is closed and bounded
The proof for this theorem requires a few lemmas
Recall [Bolzano Weierstrass theorem]
Every bounded sequence of real numbers has a convergent subsequence.
So this proves Heine-Borel theorem for $\mathbf{R}^1$

Lemma. $\mathbf{V}$ NVS, $A \subseteq B \subseteq \mathbf{V}$. If A is closed and B is compact, then A is compact
Proof: Let $(\mathbf{a}_n) \subseteq A \subseteq B$. By compactness of B, $\mathbf{a}_{n_k} \to \mathbf{b} \in B$ but A is closed so $\mathbf{b} \in A$. So every $(\mathbf{a}_n) \subseteq A$ has a subsequence that converges in A, so A is compact

Lemma. $A, B \subseteq \mathbf{R}$. If A, B are compact then $A \times B \subseteq \mathbf{R}^2$ is compact
Proof: Suppose $A, B \in \mathbf{R}$ are compact. Let $(a_n, b_n) \subseteq A \times B$ be a sequence. Since $(a_n) \subseteq A$ and A is compact, we may find a subsequence, $a_{n_k} \to a \in A$. And similarly for $(b_{n_k}) \subseteq B$ we may find a subsequence $b_{n_{k_l}} \to b \in B$. By Assignment 1, $a_{n_{k_l}} \to a$. Therefore, $(a_{n_{k_l}}, b_{n_{k_l}}) \to (a, b) \in A \times B$. Hence $A \times B$ is compact.

Now the Corollary follows from induction on this method

Corollary. If $A_1, \ldots, A_n \subseteq \mathbf{R}$ are compact, then $A_1 \times \cdots \times A_n \subseteq \mathbf{R}^n$ is compact

Theorem. **[Heine-Borel Theorem]**
A set $A \subseteq \mathbf{R}^n$ is compact $\iff$ A is closed and bounded
Proof:
($\implies$) by the Proposition at the start of the section.
($\impliedby$) Suppose $A \subseteq \mathbf{R}^n$ is closed and bounded. Since A is bounded, $A \subseteq [-M, M]^n$, for some $M > 0$ (this is a "cube" that bounds A). By the previous corollary, $[-M, M]^n$ is compact since each $[-M, M] \subseteq \mathbf{R}$ is compact. And since A is closed and a subset of the compact set $[-M, M]^n$, by the Lemma above A is compact.

3.6 Open Covers

Definition. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$.

1. An **open cover** of A is a collection of open sets $\{\mathcal{U}_\alpha : \alpha \in I\} : A \subseteq \bigcup_{\alpha \in I} \mathcal{U}_\alpha$
2. An open cover $A \subseteq \bigcup_{\alpha \in I} \mathcal{U}_\alpha$ is called a **finite open cover** if $|I| < \infty$
3. A subset of an open cover of A, $\{\mathcal{U}_\alpha : \alpha \in I\}$, which is also an open cover of A is called a **subcover** of $\{\mathcal{U}_\alpha : \alpha \in I\}$

Theorem. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$. $A \subseteq \mathbf{V}$ is compact $\iff$ every open cover of A has a finite subcover.
We prove this after a few lemmas

Lemma. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$ compact. Let $A \subseteq \bigcup_{\alpha \in I} \mathcal{U}_\alpha$ be an open cover of A. There exists $R > 0 : \forall \mathbf{a} \in A$, $B_R(\mathbf{a}) \subseteq \mathcal{U}_\alpha$ for some $\alpha \in I$
Proof: Suppose no such $R > 0$ exists. In particular, for all $n \in \mathbf{N}, \exists \mathbf{a}_n \in A : B_{\frac{1}{n}}(\mathbf{a}_n) \nsubseteq \mathcal{U}_\alpha$ for all $\alpha \in I$
Since $(\mathbf{a}_n) \subseteq A$ and A is compact, there exists a subsequence $\mathbf{a}_{n_k} \to \mathbf{a} \in A$
Say $\mathbf{a} \in \mathcal{U}_{\alpha_0}$ for some $\alpha_0 \in I$. Pick $M \in \mathbf{N} : B_{\frac{2}{M}}(\mathbf{a}) \subseteq \mathcal{U}_{\alpha_0}$
Moreover, since $\mathbf{a}_{n_k} \to \mathbf{a}$, we may find $N \in \mathbf{N} : \mathbf{a}_{n_k} \in B_{1/M}(\mathbf{a})$ for $k \geq N$. Then for $k \geq N : n_k > M$: Take $\mathbf{x} \in B_{1/M}(\mathbf{a}_{n_k})$
$\implies \|\mathbf{x} - \mathbf{a}\| = \|\mathbf{x} - \mathbf{a}_{n_k} + \mathbf{a}_{n_k} - \mathbf{a}\| \leq \|\mathbf{x} - \mathbf{a}_{n_k}\| + \|\mathbf{a}_{n_k} - \mathbf{a}\| < \frac{1}{M} + \frac{1}{M} = \frac{2}{M}$
Therefore, $B_{1/M}(\mathbf{a}_{n_k}) \subseteq B_{2/M}(\mathbf{a}) \subseteq \mathcal{U}_{\alpha_0}$
Since $n_k > M$, $B_{1/n_k}(\mathbf{a}_{n_k}) \subseteq B_{1/M}(\mathbf{a}_{n_k}) \subseteq \mathcal{U}_{\alpha_0}$ contradiction. Since we said that none of the $B_{1/n}(\mathbf{a}_n)$ are in $\mathcal{U}_\alpha$ for all $\alpha \in I$
Note: The R that satisifes this statement is called the **Lebesgue** number of the compact set

Proposition. [Part 1]
$\mathbf{V}$ NVS. If $A \subseteq \mathbf{V}$ is compact, then every open cover of A has a finite subcover
Proof: Suppose $A \subseteq \mathbf{V}$ is compact. Let $A \subseteq \bigcup_{\alpha \in I} \mathcal{U}_\alpha$ be an open cover of A. We may find $R > 0$ as in the lemma above. If $\exists \mathbf{a}_1, \ldots, \mathbf{a}_n \in A : A \subseteq B_R(\mathbf{a}_1) \cup \cdots \cup B_R(\mathbf{a}_n)$ we are done. Suppose no such covering exists. Find:
$\mathbf{a}_1 \in A$
$\mathbf{a}_2 \in A, \mathbf{a}_2 \notin B_R(\mathbf{a}_1)$
$\mathbf{a}_3 \in A, \mathbf{a}_3 \notin B_R(\mathbf{a}_1) \cup B_R(\mathbf{a}_2)$
$\vdots$
Since $(\mathbf{a}_n) \subseteq A$ and A is compact, $(\mathbf{a}_n)$ has a convergent subsequence. However for $n < m$,

$\mathbf{a}_m \notin B_R(\mathbf{a}_n) \implies \|\mathbf{a}_n - \mathbf{a}_m\| \geq R$
Therefore, $(\mathbf{a}_n)$ has no Cauchy subsequences, and hence has not convergent subsequences.
Contradiction

Lemma. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$. Suppose every open cover of A has a finite subcover. Then if $A \subseteq \bigcup_{\alpha \in I} \mathcal{U}_\alpha$, where each $\mathcal{U}_\alpha$ is **relatively open** in A then $\exists \alpha_1, \dots, \alpha_n \in I$ s.t. $A \subseteq \mathcal{U}_{\alpha_1} \cup \dots \cup \mathcal{U}_{\alpha_n}$
Proof: $A \subseteq \bigcup_{\alpha \in I} \mathcal{U}_\alpha$, where each $\mathcal{U}_\alpha$ are relatively open. That is $\mathcal{U}_\alpha = A \cap O_\alpha$ where $O_\alpha \subseteq \mathbf{V}$ is some open set. $\implies A \subseteq \bigcup_{\alpha \in I}(A \cap O_\alpha) = A \cap (\bigcup_{\alpha \in I} O_\alpha) \subseteq \bigcup_{\alpha \in I} O_\alpha$
But by the hypothesis $\implies A \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n} \implies A \subseteq \mathcal{U}_{\alpha_1} \cup \dots \cup \mathcal{U}_{\alpha_n}$

Proposition. [Part 2]
$\mathbf{V}$ NVS. If every open cover of $A \subseteq \mathbf{V}$ has a finite subcover then A is compact
Proof: Suppose $A \subseteq \mathbf{V}$ s.t. every open cover of A has a finite subcover. Consider $(\mathbf{a}_n) \subseteq A$. For $K \in \mathbf{N}$, consider $C_K = \overline{\{\mathbf{a}_n : n \geq K\}} \cap A$. We want to show $\bigcap_{K=1}^{\infty} C_K \neq \emptyset$. We see that each C_K is **relatively closed** in A. Hence, every $\mathcal{U}_K = A \setminus C_K$ is relatively open in A
For a contradiction, assume the $\bigcap C_K = \emptyset$. Then, $A = A \setminus \emptyset = A \setminus (\bigcap C_K) \underset{\text{by De Morgan's laws}}{=} \bigcup (A \setminus C_K) = \bigcup_{K=1}^{\infty} \mathcal{U}_K$ so our assumptions gives us a relatively open cover of A. By the lemma, we can get a finite relatively open subcover of A, i.e. $\exists i_1 < \dots < i_l$ s.t. $A \subseteq \mathcal{U}_{i_1} \cup \dots \cup \mathcal{U}_{i_l}$
Since $C_1 \supseteq C_2 \supseteq \dots$, then $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \dots$ then that just means $A \subseteq \mathcal{U}_{i_l} \subseteq A$, last relation since U_K are relatively open in A. $\implies A = \mathcal{U}_{i_l} \implies C_{i_l} = A \setminus \mathcal{U}_{i_l} = A \setminus A = \emptyset$
Hence, $\mathbf{a}_{i_l} \in C_{i_l} = \emptyset$ *Contradiction.*
Thus, we may find $\mathbf{a} \in \bigcap_{K=1}^{\infty} C_K$
Piazza discussion
Therefore, we may find $n_1 < n_2 < \dots$ s.t. $\|\mathbf{a}_{n_k} - \mathbf{a}\| < \frac{1}{k}$ for every $k \in \mathbf{N}$
Hence, $(\mathbf{a}_{n_k}) \subseteq A$ with $\mathbf{a}_{n_k} \to \mathbf{a} \in A$

Chapter 4

Limits and Continuity

Definition. $\mathbf{V}, \mathbf{W}$ NVS, $A \subseteq \mathbf{V}$. Let $f : A \to \mathbf{W}$ be a function and let $\mathbf{a} \in \overline{A \setminus \{\mathbf{a}\}}$ (explained in later remark)

We say the **limit** of f as $\mathbf{x}$ approaches $\mathbf{a}$ is $\mathbf{v} \in \mathbf{W}$, written

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = \mathbf{v}$$

If for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\text{For } \mathbf{x} \in A \text{ with } 0 < \|\mathbf{x} - \mathbf{a}\| < \delta \implies \|f(\mathbf{x}) - \mathbf{v}\| < \epsilon$$

Remark: Why $\mathbf{a} \in \overline{A \setminus \{\mathbf{a}\}}$? If $\mathbf{a} \notin \overline{A \setminus \{\mathbf{a}\}}$ then $\nexists \mathbf{x} \in A$ s.t. $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$ for small enough δ

Piazza discussion: $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}, a \in A$

The following are equivalent:

1. $a \notin \overline{A \setminus \{a\}}$
2. $\exists R > 0 : B_r(a) \cap A = \{a\}$ (such points are called **Isolated points**)

Example: $\lim\limits_{(x,y) \to (0,0)} \frac{xy^2}{x^2+2y^2} = 0$

Remark: If the domain is not mentioned, we assume that the domain is wherever the expression or function is defined

Let $\epsilon > 0$ be given. Choose $\boxed{\delta = 2\epsilon}$ and suppose $(x, y) \neq (0, 0)$ s.t. $0 < \|(x, y) - (0, 0)\| < \delta$

Then, $\left|\dfrac{xy^2}{x^2 + 2y^2} - 0\right| = \dfrac{|x|y^2}{x^2 + 2y^2} \leq \dfrac{|x|y^2}{2y^2} = \dfrac{|x|}{2} = \dfrac{\sqrt{x^2}}{2} \leq \dfrac{\sqrt{x^2 + y^2}}{2} = \dfrac{\|(x, y)\|}{2} < \frac{\delta}{2} = \epsilon$

Definition. $\mathbf{V}, \mathbf{W}$ NVS, $A \subseteq \mathbf{V}$. We say $f : A \to \mathbf{W}$ is **continuous at** $\mathbf{a} \in A$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\|\mathbf{x} - \mathbf{a}\| < \delta$ then $\|f(\mathbf{x}) - f(\mathbf{a})\| < \epsilon$
We simply say f is **continuous** if f is continuous at every $\mathbf{a} \in A$

Remark: $f : A \to \mathbf{W}, A \subseteq \mathbf{V}$
Suppose $\mathbf{a} \in A$ with $\mathbf{a} \notin \overline{A \setminus \{a\}}$. Then $\exists r > 0$ s.t. $\overline{B_r(a)} \cap A = \{a\}$
(i.e. $\mathbf{a}$ is an isolated point of A).
Let $\epsilon > 0$ be given. Then…
Choose $\delta = r$. Then if $\mathbf{x} \in A$ with $\|\mathbf{x} - \mathbf{a}\| < \delta$, then $\mathbf{x} = \mathbf{a}$. Thus, $\|f(\mathbf{x}) - f(\mathbf{a})\| = \|f(\mathbf{a}) - f(\mathbf{a})\| = 0 < \epsilon$
Therefore, f is continuous at $\mathbf{a}$
Note The takeaway from this is that we always trivially have that a function is continuous at isolated points
Remark: $f : A \to \mathbf{W}, A \subseteq \mathbf{V}$.
Now assume $\mathbf{a} \in \overline{A \setminus \{\mathbf{a}\}}$. Then, by definition, f is continuous at $\mathbf{a} \in A$ iff $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$

Summary: $f : A \to \mathbf{W}$ is continuous iff $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$ for all $\mathbf{a} \in A$ with $\mathbf{a} \in \overline{A \setminus \{\mathbf{a}\}}$

Proposition. $f : A \to \mathbf{W}, A \subseteq \mathbf{V}, \mathbf{a} \in A$
The following are equivalent:

1. f is continuous at $\mathbf{a} \in A$
2. If $(\mathbf{a}_n) \subseteq A$ with $\mathbf{a}_n \to \mathbf{a}$ then $f(\mathbf{a}_n) \to f(\mathbf{a})$

Proof:
(1) $\implies$ (2).
Suppose f is continuous at $\mathbf{a}$. Let $(\mathbf{a}) \subseteq A$ s.t. $\mathbf{a}_n \to \mathbf{a}$
To show $f(\mathbf{a}_n) \to f(\mathbf{a})$. Let $\epsilon > 0$ be given. We know $\exists \delta > 0$ s.t. if $\mathbf{x} \in A$ with $\|\mathbf{x} - \mathbf{a}\| < \delta$ then $\|f(\mathbf{x}) - f(\mathbf{a})\| < \epsilon$
Choose $N \in \mathbf{N}$ s.t. $\|\mathbf{a}_n - \mathbf{a}\| < \delta$ for all $n \geq N$. For $n \geq N$, $\|\mathbf{a}_n - \mathbf{a}\| < \delta \implies \|f(\mathbf{a}_n) - f(\mathbf{a})\| < \epsilon$
(2) $\implies$ (1).
Suppose (2). Assume f is not continuous at $\mathbf{a}$. Therefore, $\exists \epsilon > 0$, and $\mathbf{a}_n \in A$ $(n \in \mathbf{N})$ s.t. $\|\mathbf{a}_n - \mathbf{a}\| < 1/n$ but $\|f(\mathbf{a}_n) - f(\mathbf{a})\| \geq \epsilon$. But $\mathbf{a}_n \to \mathbf{a}$ and so $f(\mathbf{a}_n) \to f(\mathbf{a})$ Contradiction

Proposition. $f : A \to \mathbf{W}, A \subseteq \mathbf{V}$

The following are equivalent:

1. f is continuous
2. If $(\mathbf{a}_n) \subseteq A$ with $\mathbf{a}_n \to \mathbf{a} \in A$ then $f(\mathbf{a}_n) \to f(\mathbf{a})$
3. If $\mathcal{U} \subseteq \mathbf{W}$ is open then $f^{-1}(\mathcal{U})$ is relatively open in A

Proof: By Assignment 2, (2) $\iff$ (3)

Remark: (2) If $(\mathbf{a}_n) \subseteq A$ with $\mathbf{a}_n \to \mathbf{a} \in A$ then $f(\mathbf{a}_n) \to f(\mathbf{a})$
Is the most useful characterisation of continuity in this course.

Proposition. Let $f, g : A \to \mathbf{W}$ be continuous, $A \subseteq \mathbf{V}$

1. $f + g$ is continuous
2. $\forall \alpha \in \mathbf{R}, \alpha f$ is continuous
   Moreover, if $\mathbf{W} = \mathbf{R}$ then
3. fg is continuous
4. f/g is continuous, provided $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in A$

1,2 are proven by simply using the sequential characterisation

Proposition. If f, g are continuous with $f \circ g$ defined, then $f \circ g$ is continuous
Why? $\mathbf{a}_n \to \mathbf{a}$ so $g(\mathbf{a}_n) \to g(\mathbf{a})$ by continuity of g and $f(g(\mathbf{a}_n)) \to f(g(\mathbf{a}))$ by continuity of f
Note: That the previous two propositions show that for functions that are well defined ($g(\mathbf{x}) \neq 0$) that polynomials and rational functions are continuous etc.

4.1 Compactness and Continuity

Proposition. $\mathbf{V}, \mathbf{W}$ NVS, $C \subseteq \mathbf{V}$ is compact. If $f : C \to W$ is continuous then $f(C)$ is compact
Proof: Assignment 2

Theorem. [Extreme Value Theorem]
$\mathbf{V}$ NVS, $C \subseteq \mathbf{V}$ compact. If $f : C \to \mathbf{R}$ is continuous then f attains its max and min values on C
Proof: From before, $f(C)$ is compact. So $f(C)$ is closed and bounded. In particular, we can consider $y_1 = \inf f(C) < \infty$ and $y_2 = \sup f(C) < \infty$. Moreover, $\inf f(C), \sup f(C) \in \overline{f(C)} = f(C)$

Therefore, $\exists \mathbf{a}, \mathbf{b} \in C$ s.t. $y_1 = f(\mathbf{a}), y_2 = f(\mathbf{b})$
$\implies f(\mathbf{a}) = \min f(C), f(\mathbf{b}) = \max f(C)$

Remark

$\mathbf{V}, \mathbf{W}$ NVS, $K \subseteq \mathbf{V}$ compact. $C(K, \mathbf{W}) = \{f : K \to \mathbf{W} \,|\, f \text{ is continuous}\}$ is a NVS when equipped with the **uniform norm**
$\|f\|_\infty = \sup\{\|f(\mathbf{x})\| : \mathbf{x} \in K\} = \max\{\|f(\mathbf{x})\|_\infty : \mathbf{x} \in K\}$
Why? $f : K \to \mathbf{W}$ is continuous and $\|\cdot\| : \mathbf{W} \to \mathbf{R}$ is continuous (proved in previous lectures via sequential characterisation, if $\mathbf{a}_n \to \mathbf{a}$ then $\|\mathbf{a}_n\| \to \|\mathbf{a}\|$)
$\implies \|\cdot\| \circ f$ is continuous. So $(\|\cdot\| \circ f)(\mathbf{x}) = \|f(\mathbf{x})\|$.

Notation: If $\mathbf{W} = \mathbf{R}$ we write $C(K)$ instead of writing $C(K, \mathbf{W})$

4.1.1 Uniform Continuity

Let $\mathbf{V}, \mathbf{W}$ NVS, $A \subseteq \mathbf{V}$ and $f : A \to \mathbf{W}$ unless specified otherwise
We recall that f is continuous iff $\forall \mathbf{a} \in A, \forall \epsilon > 0, \exists \delta > 0 : \forall \mathbf{x} \in A$

$$\|\mathbf{x} - \mathbf{a}\| < \delta \implies \|f(\mathbf{x}) - f(\mathbf{a})\| < \epsilon$$

We have known of cases where the δ depends on epsilon but there could possibly be cases in which the δ depended on the $\mathbf{a}$ already chosen because of the order of the quantifiers. For all $\mathbf{a}$ and for all ϵ there exists a δ.
So now we see a stronger form of continuity

Definition. f is **uniformly continuous** $\iff \forall \epsilon > 0, \exists \delta > 0, \forall \mathbf{a}, \mathbf{b} \in A$

$$\|\mathbf{a} - \mathbf{b}\| < \delta \implies \|f(\mathbf{a}) - f(\mathbf{b})\| < \epsilon$$

This means that for each ϵ, there is a δ such that for *all* the $\mathbf{a}$'s and $\mathbf{b}$'s within that δ of each other, their function values are within ϵ of each other.
The idea here is that δ works *uniformly* for continuity at $\mathbf{a}$, for all $\mathbf{a} \in A$
And so, *uniform continuity* $\implies$ *continuity.*

Example: $f : \mathbf{R} \to \mathbf{R}, f(x) = x^2$ is continuous but not uniformly continuous
Suppose f is uniformly cts. For $\epsilon = 1, \exists \delta > 0$ such that if $a, b \in \mathbf{R}$ with $|a - b| < \delta$ then $|a^2 - b^2| < 1$
We take $N \in \mathbf{N} : 1/N < \delta$. So $|N + \frac{1}{N} - N| < \delta$ and so
$|(N + \frac{1}{N})^2 - N^2| < 1 \implies (N + \frac{1}{N})^2 - N^2 < 1 \implies 2 + \frac{1}{N^2} < 1$ *Contradiction*
Example:$f L(0, 2) \to \mathbf{R}, f(x) = \ln(x)$, is continuous but not uniformly continuous

For a contradiction, assume f is uniformly continuous. Let $\epsilon = 1$ so that $\exists \delta > 0$ s.t. for $a, b \in (0,2)$ with $|a - b| < \delta$ then $|\ln(a) - \ln(b)| < 1$
So we want to pick values very close together that are close to 0 so that the ϵ doesn't work
Take $N \in \mathbf{N} : 1/N < \delta$ we have $\frac{1}{N} - \frac{1}{N^2} < \frac{1}{N} < \delta$ so
$\implies \ln(1/N) - \ln(1/N^2) < 1 \implies \ln(N) < 1$, this is false for large enough N

Theorem. $C \subseteq \mathbf{V}$ is compact. If $f : C \to \mathbf{W}$ is continuous then f is uniformly continuous

Proof: Suppose f is continuous but not uniformly continuous. Therefore, $\exists \epsilon > 0$ and $(\mathbf{a}_k), (\mathbf{b}_k) \subseteq C$ s.t. $\|\mathbf{a}_k - \mathbf{b}_k\| < 1/k$ and $\|f(\mathbf{a}_k) - f(\mathbf{b}_k)\| \geq \epsilon$
Since C is compact, $\exists \mathbf{a}_{k_l} \to \mathbf{a} \in C$
Note: $\mathbf{b}_{k_l} = \underbrace{\mathbf{a}_{k_l}}_{\to \mathbf{a}} + \underbrace{\mathbf{b}_{k_l} - \mathbf{a}_{k_l}}_{\to 0} \to a$
And so, since f is continuous, we have that $f(\mathbf{a}_{k_l}) \to f(\mathbf{a})$ and $f(\mathbf{b}_{k_l}) \to f(\mathbf{a})$
$\implies \|f(\mathbf{a}_{k_l}) - f(\mathbf{b}_{k_l})\| \to 0$ *contradiction*

4.2 Uniform and Pointwise Convergence

Here we talk about Spaces of functions and Uniform convergence. So we focus on NVS's which consist of functions. E.g $C(K, \mathbf{W})$
Let $\mathbf{V}, \mathbf{W}$ be NVS, $A \subseteq \mathbf{V}$ unless specified otherwise

Motivation: Let (f_n) be a sequence of functions from $A \to \mathbf{W}$. What should it mean for (f_n) to "converge" to some $f : A \to \mathbf{W}$?
Two ideas:

1. Pointwise convergence
2. Uniform convergence

Remark: Here, we are not claiming that the f_n's and f belong to a particular NVS

Notation for the norm like thing we will use to denote a sense of distance between two functions: $f, g : A \to \mathbf{W}$. $\|f - g\|_\infty = \sup\{\|f(\mathbf{x}) - g(\mathbf{x})\| : \mathbf{x} \in A\}$
Remark: We may have $\|f\|_\infty = \infty$. E.g. $f : \mathbf{R} \to \mathbf{R}, f(x) = x$ so this actually a norm as it can include things like infinite distance

Recall: If $f \in C(K, \mathbf{W})$, where K is compact, then $\|f\|_\infty < \infty$. In fact, we know $(C(K, \mathbf{W}), \|\cdot\|_\infty)$ is a NVS

Definition. $(f_n) : A \to \mathbf{W}, f : A \to \mathbf{W}$

1. We say (f_n) converges to f **pointwise** (written $f_n \to f$ pointwise)
   If $\forall x \in A, f_n(x) \to f(x)$
2. We say (f_n) converges to f **uniformly** (written $f_n \to f$ uniformly)
   If $\forall \epsilon > 0, \exists N \in \mathbf{N}, \forall x \in A$
   $$(n \geq N) \implies \|f_n(x) - f(x)\| < \epsilon$$

The reason the word uniform appears in the definition of convergence of functions, this is because of the order of the quantifiers, just like in the case of uniform continuity. The definition reads that for each ϵ *there exists an* N, such that for all n satisfying the inequality the norm is within ϵ. Here you may see the parallels between δ and N for each given ϵ

So the idea is that some N works uniformly for all $x \in A$ so that $\|f_n(x) - f(x)\| < \epsilon$ for $n \geq N$

Remark: Uniform convergence is equivalent to convergence with the uniform norm

$\forall x \in A, \|f(x) - g(x)\| \leq \epsilon \iff \|f - g\|_\infty \leq \epsilon$

Therefore, $f_n \to f$ uniformly $\iff$

1. $\|f_n - f\|_\infty < \infty$ eventually ($n \geq M$ for some M) and
2. $\|f_n - f\|_\infty \to 0$ as $n \to \infty$

Note: Uniform convergence implies pointwise convergence

Examples:

1. $f_n : [0,1] \to \mathbf{R}, f_n(x) = x^n$
   For $x \in [0,1], f_n(x) \to f(x)$, where $f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$ So $f_n \to f$ pointwise.
   Now, $\|f_n - f\|_\infty = 1 \nrightarrow 0$ so the convergence is not uniform

2. $f_n : (0,\infty) \to \mathbf{R}, f_n(x) = \dfrac{nx}{e^{nx}}$
   We see that $\displaystyle\lim_{n\to\infty} \frac{nx}{e^{nx}} \overset{LHR}{=} \lim_{n\to\infty} \frac{x}{xe^{nx}} = \lim_{n\to\infty} \frac{1}{e^{nx}} = 0$ So $f_n \to 0$ pointwise
   Now, $\|f_n - 0\|_\infty = \sup\{|nx/e^{nx}| : x > 0\} \geq \dfrac{n(1/n)}{e^{n(1/n)}} = \dfrac{1}{e}$
   So $\|f_n - 0\|_\infty \nrightarrow 0$ convergence is not uniform
3. $f : A = [0,1] \times [0,1] \to \mathbf{R}^2$ wtih $f_n(x,y) = \left(\dfrac{x+y}{n}, \dfrac{\sin(xy)}{x^2+n^2}\right)$
   For $(x,y) \in A$, $\displaystyle\lim_{n\to\infty} f_n(x,y) = 0$ so $f_n \to f$ pointwise

Now, $\|f_n - 0\|_\infty = \sup\left\{ \left\| \left(\frac{x+y}{n}, \frac{\sin(xy)}{x^2+n^2} \right) \right\| : (x,y) \in A \right\}$

$= \sup\left\{ \sqrt{\left(\frac{x+y}{n}\right)^2 + \left(\frac{\sin(xy)}{x^2+n^2}\right)^2} : (x,y) \in A \right\}$

$\leq \sqrt{\left(\frac{2}{n}\right)^2 + \left(\frac{1}{n^2}\right)^2} \to 0$ so $f_n \to 0$ uniformly

Theorem. $(f_n) : A \to \mathbf{W}$ are continuous. If $f_n \to f$ uniformly, then f is continuous (This is not true for pointwise, as seen in example 1 above.)

Proof: Suppose f_n is continuous and $f_n \to f$ uniformly

Let $(a_n) \subseteq A$ s.t $a_n \to a \in A$

Claim: $f(a_n) \to f(a)$

Let $\epsilon > 0$ be given. Since $f_k \to f$ uniformly, $\exists N_1 \in \mathbf{N}$ s.t. $\|f_k - f\|_\infty < \boxed{\epsilon/3}$ for $k \geq N_1$

Moreover, since f_{N_1} is continuous, $\exists N_2 \in \mathbf{N}$ s.t. $\|f_{N_1}(a_n) - f_{N_1}(a)\| < \boxed{\epsilon/3}$ for all $n \geq N_2$

Then for $n \geq N_2$,

$\|f(a_n) - f(a)\| \leq \|f(a_n) - f_{N_1}(a_n)\| + \|f_{N_1}(a_n) - f_{N_1}(a)\| + \|f_{N_1}(a) - f(a)\|$

$\leq \|f - f_{N_1}\|_\infty + \|f_{N_1}(a_n) - f_{N_1}(a)\|_\infty + \|f_{N_1} - f\|_\infty < \epsilon$

Therefore $f(a_n) \to f(a) \implies f$ is continuous, as required

Theorem. $K \subseteq \mathbf{V}$ is compact, $\mathbf{W}$ is a Banach space, then $(C(K, \mathbf{W}), \|\cdot\|_\infty)$ is a Banach space

Proof: Let $(f_n) \subseteq C(K, \mathbf{W})$ be Cauchy.

So take $x \in K$ and consider $(f_n(x)) \subseteq \mathbf{W}$ (which is a Banach space)

Claim: $(f_n(x))$ is Cauchy

Let $\epsilon > 0$ be given. $\exists N \in \mathbf{N}$ s.t. $\|f_n - f_m\| < \epsilon$ for $n, m \geq N$.

Then for $n, m \geq N$, $\|f_n(x) - f_m(x)\| \leq \|f_n - f_m\| < \epsilon$. This proves the claim.

(Now we find a candidate for the limit using pointwise convergence)

Since $\mathbf{W}$ is a Banach space, $f_n(x) \to f(x) \in \mathbf{W}$. By doing this for all $x \in K$, we have created a function $f : K \to \mathbf{W}$ s.t. $f_n \to f$ pointwise

Claim: $f_n \to f$ uniformly

Let $\epsilon > 0$ be given. $\exists M \in \mathbf{N}$ s.t. if $n, m \geq M$ then $\|f_n - f_m\|_\infty < \epsilon/2$

Let $n \geq M$ and let $x \in K$. Then, $\|f_n(x) - f(x)\| = \lim_{m\to\infty} \underbrace{\|f_n(x) - f_m(x)\|}_{\text{each } <\epsilon/2} \leq \epsilon/2 < \epsilon$

Since the above is true for all $x \in K$ for $n \geq M$ so $\|f_n - f\| < \epsilon$, this proves the claim

By previous theorem, $f \in C(K, \mathbf{W})$. Hence, $f_n \to f \in C(K, \mathbf{W})$ this proves the theorem.

Chapter 5

Multivariable Calculus

5.1 Partial Derivatives

Idea: Given a function $f : A \to \mathbf{R}^m$, $A \subseteq \mathbf{R}^n$, we will try to develop a theory of multivariable calculus using as much single variable calculus as possible

Definition. A **scalar function** is a function of the form $f : A \to \mathbf{R}, A \subseteq \mathbf{R}^n$
Remark: If $f : A \to \mathbf{R}^m$ is a function, then $\exists$ scalar functions $f_1, f_2, \ldots, f_m$ on A such that $f = (f_1, f_2, \ldots, f_m)$

Definition. $A \subseteq \mathbf{R}^n, f : A \to \mathbf{R}$.
Let $\{\mathbf{e}_1, \cdots \mathbf{e}_n\}$ be the standard basis for $\mathbf{R}^n$. For $1 \leq i \leq n$, we define the i^{th} **partial derivative** of f at $\mathbf{a} = (a_1, \ldots, a_n) \in A$ by

$$f_{x_i}(\mathbf{a}) = \frac{\partial f}{\partial x_i}(\mathbf{a}) := \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{e}_i) - f(\mathbf{a})}{h}$$

provided the limit exists

For $a \in A \subseteq \mathbf{R}^n, f : A \to \mathbf{R}$. We write $f(x_1, x_2, \cdots x_n)$

1. $f_{x_i}(\mathbf{a})$ is the derivative of f at $\mathbf{a}$ w.r.t. the variable x_i (treating the other x_i's as constants)
2. $f_{x_i}(\mathbf{a})$ is the slope of the tangent line to the surface $y = f(x_1, \ldots, x_n)$ which is parallel to $\mathbf{e}_i$

Partial derivatives for vector-valued functions

Definition. $A \subseteq \mathbf{R}^n, f : A \to \mathbf{R}^m, f = (f_1, \dots f_m)$. For $\mathbf{a} \in A$,

$$\frac{\partial f}{\partial x_i}(\mathbf{a}) = f_{x_i}(\mathbf{a}) := \left(\frac{\partial f_1}{\partial x_i}(\mathbf{a}), \frac{\partial f_2}{\partial x_i}(\mathbf{a}), \dots, \frac{\partial f_m}{\partial x_i}(\mathbf{a})\right)$$

provided it exists

5.2 Differentiability

What should it mean for a function $f : A \to \mathbf{R}^m, A \subseteq \mathbf{R}^n$ to be differentiable

Recall: $A \subseteq \mathbf{R}, f : A \to \mathbf{R}. f$ is differentiable at $a \in A \iff$

1. $a \in \text{Int}(A)$ and
2. $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$ exists $\iff \lim\limits_{h \to 0} \frac{f(a+h)-f(a)-mh}{h} = 0$ for some $m \in \mathbf{R}$

Remark: Since $a \in \text{Int}(A)$, $a + h \in A$ for small enough $h \in \mathbf{R}$

Remark: $T : \mathbf{R} \to \mathbf{R}$ is a linear transformation $\iff T(x) = mx$ for some $m \in \mathbf{R}$. Reason is that we may set $m = T(1)$ and see that $T(x) = T(x \cdot 1) = xT(1) = mx$

Idea: $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)-mh}{h} = \lim\limits_{h \to 0} \left(\frac{f(a+h)-f(a)}{h} - \frac{mh}{h}\right) = 0 \iff f$ over $[a, a+h]$ can be approximated arbitrarily well by the line $T(x) = mx$

Notation: $\mathcal{L}(\mathbf{R}^n, \mathbf{R}^m) = \{T : \mathbf{R}^n \to \mathbf{R}^m : T \text{ is linear}\}$

Definition. $\mathbf{a} \in A \subseteq \mathbf{R}^n, \mathbf{f} : A \to \mathbf{R}^m$. We say $\mathbf{f}$ is **differentiable** at $\mathbf{a} \in A$ if

1. $\mathbf{a} \in \text{Int}(A)$ and
2. There exists $T \in \mathcal{L}(\mathbf{R}^n, \mathbf{R}^m)$ s.t.

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a}) - T(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0}$$

Remark: This means that for sufficiently small $\|\mathbf{h}\|$ the function acts like a linear transformation

Remark:

1. Let $T \in \mathcal{L}(\mathbf{R}^n \to \mathbf{R}^m)$. Letting B be the matrix of T relative to the standard basis, $T(x) = B\mathbf{x}$ for all $\mathbf{x} \in \mathbf{R}^n$
2. *Class discussion*

   Let $A \in M_{m \times n}(\mathbf{R})$, $\|A\|_{op} = \sup\{\|A\mathbf{x}\|_2 : \mathbf{x} \in \mathbf{R}^n, \|\mathbf{x}\|_2 = 1\}$ is a norm on $M_{m \times n}(\mathbf{R})$. We call it the **operator norm**

3. If $A \in M_{m\times n}(\mathbf{R})$ and $\mathbf{x} \in \mathbf{R}^n$ then $\|A\mathbf{x}\| \leq \|A\|_{op} \|\mathbf{x}\|$
   Why? Clear if $\mathbf{x} = \mathbf{0}$. Otherwise, $\left\|\frac{\mathbf{x}}{\|\mathbf{x}\|}\right\| = 1$, so $\|A\|_{op} \geq \left\|A\frac{\mathbf{x}}{\|\mathbf{x}\|}\right\| = \|A\mathbf{x}\| \frac{1}{\|\mathbf{x}\|}$
   $\implies \|A\mathbf{x}\| \leq \|A\|_{op} \|\mathbf{x}\|$

Theorem. $\mathbf{a} \in A \subseteq \mathbf{R}^n, f : A \to \mathbf{R}^m$.
If $\mathbf{f}$ is differentiable at $\mathbf{a}$ then $\mathbf{f}$ is continuous at $\mathbf{a}$
Proof sketch: Since $\mathbf{f}$ is differentiable at $\mathbf{a}$, $\exists T \in \mathcal{L}(\mathbf{R}^n \to \mathbf{R}^m)$:

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a}) - T(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0} \implies \lim_{\mathbf{h}\to\mathbf{0}} \frac{\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a}) - B\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$$

$\implies$ we may find $\delta > 0$: if $0 \leq \|\mathbf{h} \cdot 0\| < \|\mathbf{h}\| < \delta$ then

$$\left\|\frac{\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a}) - B\mathbf{h}}{\|\mathbf{h}\|}\right\| < \epsilon$$
$$\implies \|\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a}) - B\mathbf{h}\| < \|\mathbf{h}\| \epsilon$$
$$\implies \|\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a})\| - \|B\mathbf{h}\| < \|\mathbf{h}\| \epsilon$$
$$\implies \|\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a})\| < \|B\mathbf{h}\| + \|\mathbf{h}\| \epsilon \leq \|B\|_{op} \|\mathbf{h}\| + \|\mathbf{h}\| \epsilon$$

As $\mathbf{h} \to \mathbf{0}, \|B\|_{op} \|\mathbf{h}\| + \|\mathbf{h}\| \epsilon \to 0$
Therefore, by the squeeze theorem, $\lim_{\mathbf{h}\to\mathbf{0}} \mathbf{f}(\mathbf{a}+\mathbf{h}) = \mathbf{f}(\mathbf{a}) \iff \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$,
where $\mathbf{x} = \mathbf{a} + \mathbf{h}$
`(I will now relax the boldface notation for vector valued functions)`

Definition. $\mathcal{U} \subseteq \mathbf{R}^n$ be open, $f : \mathcal{U} \to \mathbf{R}^m$. We say f is **differentiable** (on $\mathcal{U}$) if f is differentiable at every point in $\mathcal{U}$

5.3 Total Derivative

For any given function $f : A \to \mathbf{R}^m, A \subseteq \mathbf{R}^n$ which is differentiable at $\mathbf{a}$, how do we find B such that
$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - B\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$$
Since f is differentiable at $\mathbf{a}$ so $\exists B \in M_{m\times n}(\mathbf{R})$ such that the above limit holds.
Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis for $\mathbf{R}^n$. As $t \to 0, t\mathbf{e}_i \to \mathbf{0}$
$\implies \lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{e}_i) - f(\mathbf{a}) - Bt\mathbf{e}_i}{|t|} = \mathbf{0}$
We see that
$$\lim_{t\to 0^+} \frac{f(\mathbf{a}+t\mathbf{e}_i) - f(\mathbf{a}) - Bt\mathbf{e}_i}{t} = \mathbf{0} \text{ and } \lim_{t\to 0^+} \frac{tB\mathbf{e}_i}{t} = B\mathbf{e}_i \implies \lim_{t\to 0^+} \frac{f(\mathbf{a}+t\mathbf{e}_i) - f(\mathbf{a})}{t} = B\mathbf{e}_i$$

$$\text{similarly } \lim_{t\to 0^-} \frac{f(\mathbf{a}+t\mathbf{e}_i)-f(\mathbf{a})}{t} = B\mathbf{e}_i$$

$$\implies B\mathbf{e}_i = \lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{e}_i)-f(\mathbf{a})}{t} = \frac{\partial f}{\partial x_i}$$

$$\implies B\mathbf{e}_i = \left(\frac{\partial f_1}{\partial x_i}(a), \dots, \frac{\partial f_m}{\partial x_i}\right) = (b_{1,i}, \dots, b_{m,i}) = i^{th} \text{ column of } B$$

$$\implies B\mathbf{e}_j = \left(\frac{\partial f_1}{\partial x_j}(a), \dots, \frac{\partial f_m}{\partial x_j}\right) = (b_{1,j}, \dots, b_{m,j}) = j^{th} \text{ column of } B$$

$$b_{ij} = \frac{\partial f_i}{\partial x_j}(a)$$

Definition. $\mathbf{a} \in A \subseteq \mathbf{R}^n, f : A \to \mathbf{R}^m$. We call the matrix

$$Df(a) = \left[\frac{\partial f_i}{\partial x_j}(a)\right]_{m\times n}$$

the **total derivative** of f at $\mathbf{a}$, provided it exists

Theorem. $\mathbf{a} \in A \subseteq \mathbf{R}^n, f : A \to \mathbf{R}^m$. If f is differentiable at $\mathbf{a}$, then

1. For all $1 \leq j \leq n$, $\frac{\partial f}{\partial x_j}(\mathbf{a})$ exists $(= B\mathbf{e}_j)$
2. $\displaystyle\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h})-f(\mathbf{a})-Df(\mathbf{a})\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$

Definition. $\mathbf{a} \in A \subseteq \mathbf{R}^n, \mathbf{f} : A \to \mathbf{R}^m$. We call $D\mathbf{f}(\mathbf{a})$ the **gradient** of $\mathbf{f}$ at $\mathbf{a}$ and label it by $\nabla\mathbf{f}(\mathbf{a})$, i.e. $D\mathbf{f}(\mathbf{a}) := \nabla\mathbf{f}(\mathbf{a}) = \left(\frac{\partial \mathbf{f}}{\partial x_1}(\mathbf{a}), \dots, \frac{\partial \mathbf{f}}{\partial x_n}(\mathbf{a})\right)$

5.4 Continuous Partials

The converse of the theorem above is not always true. That is, the existence of partial derivatives does not imply that the function is differentiable. This can be easily verfied with some examples of piecewise functions that are not continuous.
However, we do have that the converse works in some cases.

Theorem. $\mathcal{U} \subseteq \mathbf{R}^n$ is open, $f : \mathcal{U} \to \mathbf{R}$. If $\mathbf{a} \in \mathcal{U}$ and $\forall 1 \leq j \leq n$, $\frac{\partial f}{\partial x_j}$ exist on $\mathcal{U}$ and *are continuous* at $\mathbf{a}$, then f is differentiable at $\mathbf{a}$
Proof: provided as handout
Remark: Do not use the false converse of *this* theorem. A function may be differentiable on an open set and at $\mathbf{a}$, but the partial derivatives may not always be continuous at $\mathbf{a}$

Chapter 6

Definitions and Results from Assignments and Tests

6.1 Assignment 1

Proposition. If A, B are non-empty subsets of $\mathbf{R}$ which are bounded below, then the set $A + B := \{a + b : a \in A, b \in B\}$, is bounded below and $\inf(A + B) = \inf(A) + \inf(B)$

Proposition. If $(a_n) \subseteq \mathbf{R}$ is such that $a_n \to a \in \mathbf{R}$, then $|a_n| \to |a|$

Proposition. If (a_n) be a sequence of non-negative real number with $a_n \to a \in \mathbf{R}$, then $\sqrt{a_n} \to \sqrt{a}$

Proposition. If (a_n) be a sequence of non-zero real number with $a_n \to a \in \mathbf{R}$ and $a \neq 0$, then $\frac{1}{a_n} \to \frac{1}{a}$

Proposition. For $(a_n) \subseteq \mathbf{R}$ such that $a_n \to a \in \mathbf{R}$ then

1. Every subsequence of (a_n) converges to a
2. If $b \in \mathbf{R}$ such that $b \neq a$ then there exists $M > 0$ and $N \in \mathbf{N}$ such that $|a_n| > M$ for all $n \geq N$

Proposition. If $(a_n) \subseteq \mathbf{R}$ is a Cauchy sequence and a subsequence of $a_{n_k} \to a \in \mathbf{R}$, then $a_n \to a$

Definition. A sequence $(a_n) \subseteq \mathbf{R}$ is said to be **strongly-Cauchy** if there exists a convergent series $\sum_{n=1}^{\infty} \epsilon_n$ of positive real numbers such that $|a_{n+1} - a_n| < \epsilon_n$ for all $n \in \mathbf{N}$

Proposition. Every strongly Cauchy sequence is Cauchy

6.2 Assignment 2

Definition. Let $\mathbf{V}$ be a vector space. Two norms, $\|\cdot\|_a$ and $\|\cdot\|_b$, are said to be **equivalent** if there exists $C, D > 0$ such that for all $\mathbf{v} \in \mathbf{V}$,

$$C \|\mathbf{v}\|_a \leq \|\mathbf{v}\|_b \leq D \|\mathbf{v}\|_a$$

Proposition. Let $\mathbf{V}_1 = (\mathbf{V}, \|\cdot\|_a), \mathbf{V}_2 = (\mathbf{V}, \|\cdot\|_b)$ and assume the two norms are equivalent

1. $(\mathbf{a}_n) \to \mathbf{v} \in \mathbf{V}_1 \iff (\mathbf{a}_n) \to \mathbf{v} \in \mathbf{V}_2$
2. $(\mathbf{a}_n)$ is Cauchy in $\mathbf{V}_1 \iff (\mathbf{a}_n)$ is Cauchy in $\mathbf{V}_2$
3. A is a complete subset of $\mathbf{V}_1 \iff A$ is a complete subset of $\mathbf{V}_2$

Definition. Let A be a subset of a normed vector space $\mathbf{V}$. We say $B \subseteq A$ is **relatively open** in A if there exists an open set U in $\mathbf{V}$ such that $B = A \cap U$. Similarly we say $B \subseteq A$ is **relatively closed** in A if there exists a closed set C in $\mathbf{V}$ such that $B = A \cap C$

Proposition. Let $\mathbf{V}$ NVS and $A \subseteq \mathbf{V}$

1. $B \subseteq A$ is relatively open in $A \iff A \setminus B$ is relatively closed in A
2. $B \subseteq A$ is relatively closed in $A \iff$ whenever $(\mathbf{b}_n)$ is a sequence in B which converges to some $\mathbf{a} \in A$, then $\mathbf{a} \in B$

Proposition. Let $\mathbf{V}$ and $\mathbf{W}$ be two normed vector spaces. Let $A \subseteq \mathbf{V}$ and $f : A \to \mathbf{W}$. Whenever U is open in $\mathbf{W}$, f is continuous $\iff f^{-1}(U)$ is relatively open in A

Proposition. Let $\mathbf{V}, \mathbf{W}$ NVS and $f : \mathbf{V} \to \mathbf{W}$ be continuous. If $C \subseteq \mathbf{V}$ is compact then $f(C) \subseteq \mathbf{W}$ is compact

Definition. Let A be a subset of $\mathbf{V}$. We say two nonempty relatively open sets $U, W \subseteq A$ **separate** A if $U \cap W = \emptyset$ and $A = U \cup W$. We say A is **connected** if A cannot be separated by any pair of nonempty relatively open sets

Definition. We say $A \subseteq \mathbf{V}$ is **path connected** if for all $\mathbf{a}, \mathbf{b} \in A$ there exists a continuous function $f : [0, 1] \to A$ such that $f(0) = \mathbf{a}$ and $f(1) = \mathbf{b}$. We say f is a **path** from $\mathbf{a}$ to $\mathbf{b}$

Proposition. If $f : A \to \mathbf{W}$ is continuous and $A \subseteq \mathbf{V}$ is connected, then $f(A)$ is connected

Appendix