PMATH 347: Groups & Rings

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These are my notes for my 2nd year course Groups & Rings (PMATH 347) at the University of Waterloo. They are pretty similar to the content you may see in the course notes provided by Professor Ross Willard.
You will find that these aren't very useful as notes, in the sense that they are not significantly shorter than the content in the course notes, these notes are really just a way for me to type down and absorb the content I am learning. Also, I won't be including the proofs, it's best to read the course notes for that.
Reference text: Abstract Algebra, David S. Dummit, Richard M. Foote.

Contents

Part I

Group Theory

Chapter 1

Dihedral Symmetries and Permutations

Let C_n denote a regular n-gon for $n \geq 3$ (in $\mathbb{R}^3$). **A dihedral symmetry** of C_n is any "rigid motion" that moves C_n back to itself (so that it looks unchanged).
For example, the dihedral symmetries of C_6 include; Rotations (by multiples of 60 deg), "flips" (**reflections**) along an axis, and the "identity" symmetry (which does nothing)

Definition. D_{2n} = the set of all dihedral symmetries of C_n Note. In geometry the set is called D_n

Definition. Let X be any non-empty set.

- A **permutation** of X is a bijection $\sigma : X \to X$
- S_X is the set of all permutations of X
- If $X = \{1, 2, 3, \ldots, n\}$ then we denote S_X by S_n

Special notation, terminology.

- id denotes the identity permutation in S_X (id$(x) = x$ for all $x \in X$)
- The cycle notation for id is () or just .
- Given $\sigma \in S_X$, the **support** of σ is the set

$$supp(\sigma) = \{x \in X : \sigma(x) \neq x\}$$

  That is, the supp(σ) is the set of elements in the cycle notation of σ
- σ, τ are **disjoint** if supp(σ) $\cap$ supp(τ)$= \emptyset$

Chapter 2

Definition of a Group

Definition. Let A be a non-empty set. A **binary operation on** A is a function $* : A \times A \to A$
Notice that a binary operation requires closure by definition

Definition. A **group** is an ordered pair $(G, *)$ where

- G is a non-empty set
- $*$ is a binary operation on G;

which jointly satisfy the following further conditions

1. $*$ is **associative**: $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$
2. There exists an **identity** element $e \in G : a * e = e * a = a$ for all $a \in G$
3. Every $a \in G$ has a 2-sided **inverse**, i.e., an element $a' \in G$ which satisfies $a * a' = a' * a = e$ (where e is the identity element from 2)

Note: A group, G, is called **abelian** (or **commutative**) if any $a, b \in G$ satisfy the equation $a * b = b * a$
Note that 2 ensures that a group is always non-empty

Notation: when discussing generic groups

- We often denote a group $(G, *)$ by just G. Unless we want to distinguish the group from its underlying set, e.g. then group is denoted by $\mathbb{G}$ and the set by just G
- We pften write ab or $a \cdot b$ for $a * b$
- Denote the identity element e of G by 1, often.
- Denote the inverse a' of an element a by a^{-1}, often
- The **order** of a group G, denoted $|G|$, is the number of its elements

Definition. In any group G, if $a \in G$ then define $a^0 = 1$ and $a^{n+1} = a \cdot a^n$ for $n \geq 0$. Also define $a^{-n} = (a^n)^{-1}$ for $n \geq 2$. This notation satisifes the usual rules of exponents.

Lemma 3.2. Let $(G, \cdot)$ be a group, $a \in G$, and $m, n \in \mathbb{Z}$

1. $a^1 = a$
2. $a^m \cdot a^n = a^{m+n}$
3. $(a^m)^n = a^{mn}$

Warning: in general $(ab)^n = a^n b^n$ is not true, since $(ab)^2 = abab$ and we need commutativity to get a^2b^2.
Also, *additive notation* is used for operations involving the symbol $+$. Since for groups like $(\mathbb{R}, +)$, writing $a^n = a + \cdots + a$ is awkward.
Additive Notation. When the group operation is denoted by $+$ (or whenever the operation is being thought of as something "like addition") we may

- Denote the identity element by 0 (instead of 1)
- Denote the inverses by $-a$ (instead of a^{-1})
- Denote $a + \cdots + a$ (n times) by na (instead of a^n), for any $n \geq 1$

This notation is seldom used for non-abelian groups

Definition. For a group G and element $a \in G$, the **order** of a (denoted $|a|$ or $\circ(a)$) is the least integer $n > 0$ such that $a^n = 1$, if it exists. If no such n exists (this requires G to be infinite), then the order of a is defined to be ∞ **Remark.** The word has been used in two different ways

- of a *group* (the number of elements of the group) or
- of an *element* of a group (the least positive exponent giving the identity element)

Proposition 3.3. Suppose G is a group, $a \in G$, and $\circ(a) = n < \infty$. Then for all $k \in \mathbb{Z}$, $a^k = 1 \iff n|k$

Chapter 3

Elementary Properties of Groups

Proposition 4.1. Let G be a group and $a, b, u, v \in G$

1. Left and right cancellation:

   (a) if $au = av$, then $u = v$

   (b) If $ub = vb$, then $u = v$

2. the equations $ax = b$ and $ya = b$ have unique solutions for $x, y \in G$

Corollary 4.2. In any group G, the identity element is unique

Proposition 4.3. Suppose G is a group

1. Each $a \in G$ has a unique inverse a^{-1}
2. $(a^{-1})^{-1} = a$ for all $a \in G$
3. $(ab)^{-1} = (b^{-1})(a^{-1})$ for all $a, b \in G$

Some terminology:

1. G is **abelian** if $ab = ba$ for all $a, b \in G$
2. If $a \in G$ then $\langle a \rangle$ denotes the set $\{a^n : n \in \mathbb{Z}\}$. Thus $\langle a \rangle \subseteq G$
3. G is **cyclic** if there exists $a \in G$ such that $G = \langle a \rangle$

   In this case we call a a **generator** of G

   Note: A cyclic group can have more than one generator

Chapter 4

Isomorphisms

The most fundamental relation between groups is that of *isomorphism*

Definition. Let $\mathbb{G} = (G, \star)$ and $(\mathbb{H}, \diamond)$ be groups. A function $\varphi : G \to H$ is an **isomorphism from $\mathbb{G}$ to $\mathbb{H}$** if φ is a bijection and

$$\varphi(x \star y) = \varphi(x) \diamond \varphi(y) \qquad \textit{for all } x, y \in G$$

Theology:

1. If φ is an isomorphism from $\mathbb{G}$ to $\mathbb{H}$, then the operation tables for $\mathbb{G}$ and $\mathbb{H}$ are "the same" (modulo the translation given by φ)
2. If the operation tables for $\mathbb{G}$ and $\mathbb{H}$ are "the same" in this sense, then $\mathbb{G}$ and $\mathbb{H}$ are "essentially the same group"

Definition. We say that groups $\mathbb{G}$ and $\mathbb{H}$ are **isomorphic** and write $\mathbb{G} \cong \mathbb{H}$ if ther exists an isomorphism $\varphi : G \to H$

Chapter 5

Subgroups

Definition. Let $\mathbb{G} = (G, \cdot)$ be a group. A **subgroup** of $\mathbb{G}$ is a subset $H \subseteq G$ satisfying

1. $H \neq \emptyset$
2. H is closed under products; i.e. $a, b \in H$ implies $ab \in H$
3. H is closed under inverses; i.e. $a \in H$ implies $a^{-1} \in H$

Proposition 6.2. If $\mathbb{G} = (G, \cdot)$ is a group and H is a subgroup of $\mathbb{G}$, then $\mathbb{H} = (H, \cdot \restriction_H)$ is a group in its own right. ($\cdot \restriction_H$ is the restriction of the operation $\cdot$ to pairs from H)

Conventions.

1. In light of Proposition 6.2, we will return to beign lazy and not distinguish between $\mathbb{H}$ and H.
2. We will no longer write $(H, \cdot \restriction_H)$ and instead just write $(H, \cdot)$
3. We write $H \leq G$ or $\mathbb{H} \leq \mathbb{G}$ to mean H is a subgroup of $\mathbb{G}$

Claim. For $a \in G$, $\langle a \rangle \leq G$

Definition. If G is a group and $a \in G$, then $\langle a \rangle$ is called the **cyclic subgroup** of G **generated by** a

Cyclic subgroups are important and are the easiest subgroups to find. Note that if G is a group and $a \in G$, then $\langle a \rangle$ is the *smallest* subgroup of G containing a. Furthermore, any subgroup that contains a must also contain $\langle a \rangle$.

Proposition 6.6. Let G be a group and $a \in G$

1. If $\circ(a) = \infty$ then $a^i \neq a^j$ for all $i \neq j$ and $\langle a \rangle \cong (\mathbb{Z}, +)$

2. If $\circ(a) = n$, then $\langle a \rangle = \{a^0, a^1, \ldots, a^{n-1}\}$ and $\langle a \rangle \cong (\mathbb{Z}_n, +)$

Corollary 6.7. If G is a group and $a \in G$, then $\circ(a) = |\,\langle a \rangle\,|$. That is, the orders of an element and the cyclic subgroup generated by that element are the same

Chapter 6

Cosets and Lagrange's Theorem

Definition. Suppose G is a group, $H \leq G$, and $a \in G$. The **left coset of** H **determined by a** is the set

$$aH := \{ah : h \in H\}$$

E.g. $1H = H$.

Warning: aH is generally *not* a subgroup of G.

Note: When using additive notation, we write $a + H$ instead of aH

Lemma 7.2. For all $a \in G, |aH| = |H|$. Hence all left cosets of H have the same size as H

Caution: It can happen that $aH = bH$ even if $a \neq b$

Proposition 7.3. Suppose $H \leq G$. The set of left cosets of H partition G; that is

1. $\cup\{aH : a \in G\} = G$
2. If $aH \neq bH$ then $aH \cap bH = \emptyset$

Theorem 7.4. **(Lagrange's Theorem)**. Suppose G is a finite group and $H \leq G$. Then $|H|$ divides $|G|$

Corollary 7.5. Suppose G is a finite group and $a \in G$. Then $\circ(a)$ divides $|G|$

Corollary 7.6. If G is a finite group and $|G| = n$, then $x^n = 1$ for all $x \in G$

Corollary 7.7. If G is a finite group and $|G| = p$ is prime, then G is cyclic

Note: the proof of the previous corollary shows that if $|G|$ is prime then *every* non-identity element of G is a generator

Chapter 7

Cosets (continued), Normal Subgroups

The number of left and right cosets of a subgroup are the same, however, they aren't always the same. This is because there is a bijection between the collection of left cosets and right cosets of H

Definition. If G is a group and $H \leq G$, the **index** of H in G, denoted $[G : H]$, is the number of distinct left (or right) cosets of H.
If G is *finite*, then

$$[G : H] = \frac{|G|}{|H|}$$

Definition. Suppsose $H \leq G$. We say that H is **normal**, or is a **normal subgroup** and write $H \triangleleft G$, if $aH = Ha$ for all $a \in G$

Note that this definition does not talk about commutivity, it talks about the left and right cosets being equal
Of course if G is abelian then every subgroup is normal. It is generally tedious to check whether a subgroup is normal or not.
Notation: Generalizing the notation used for cosets: If A, B are nonempty subsets of a group G and $g \in G$ then

$$\begin{aligned} gA &:= \{ga : a \in A\} \\ Ag &:= \{ag : a \in A\} \\ AB &:= \{ab : a \in A \text{ and } b \in B\} \\ A^{-1} &:= \{a^{-1} : a \in A\} \end{aligned}$$

With this notation we can "multiply" and "invert" nonempty sets as well as elements of G. The notation obeys the associative property of multiplication $(Ag)B = A(gB)$, so we

can just write AgB, law of inverses $(AB)^{-1} = B^{-1}A^{-1}$ and $(gA)^{-1} = A^{-1}g^{-1}$. However, *cancellation laws do not work in this context, set cancellation is not a thing*, e.g. for any $a, b \in G$ it is true that $aG = bG$, but this does not imply that $a = b$ as the only thing the equation conveys is that the two sets generated are equal. Similarly, it is not true that $AA^{-1} = 1$ (or $\{1\}$). Inverses of sets are not true inverses
The notation shortens the definition of a subgroup;
Fact: If G is a group and $\varnothing \neq H \subseteq G$, then the following are equivalent

1. $H \leq G$
2. $HH \subseteq H$ and $H^{-1} \subseteq H$
3. $HH = H$ and $H^{-1} = H$

This is since $HH \subseteq H \iff H$ is closed under products, $H^{-1} \subseteq H \iff H$ is closed under inverses

Chapter 8

Applications of Normality

Proposition 9.1. Suppose $H \leq G$. The following are equivalent (TFAE):

1. $H \lhd G$
2. $aHa^{-1} = H$ for all $a \in G$
3. $aHa^{-1} \subseteq H$ for all $a \in G$
4. If $h \in H$, then $aha^{-1} \in H$ for all $a \in G$

Lemma 9.2. Suppose $H, K \lhd G$ and $H \cap K = \{1\}$. Then $hk = kh$ for all $h \in H, k \in K$

Useful trick: For $a, b \in G$, their **commutator** is $[a.b] = a^{-1}b^{-1}ab$. This makes it easy to show $ab = ba \iff [a, b] = 1$ by using prop 9.1. and showing the expression is in the intersection

If we have that H, K are two subgroups of G, then $H \subseteq HK$ (since $1 \in K$) and $K \subseteq HK$ (because $1 \in H$), but HK does not need to be a subgroup.

Proposition 9.3. Suppose G is a group and $H, K \leq G$. If either $H \lhd G$ or $K \lhd G$, then $HK \leq G$

Proof sketch:

Sub-claim: $HK = KH$. Proof: assume H is normal, since $HK = \cup_{k \in K} Hk = \cup_{k \in K} kH = KH$

Show that $(HK)(HK) \subseteq HK$ and $(HK)^{-1} \subseteq HK$, by making use of $HH = H = H^{-1}$

Definition. Suppose G is a group and $H \leq G$. The **normalizer** of H, denoted $N_G(H)$, is the set

$$N_G(H) = \{a \in G : aH = Ha\}$$

Normalizers are useful in many contexts, a couple of them are

1. $N_G(H) \leq G$
2. $H \lhd G \iff N_G(H) = G$

Corollary 9.4. Suppose G is a group and $H, K \leq G$. If $K \subseteq N_G(H)$ (or $H \subseteq N_G(K)$) then $HK \leq G$

Chapter 9

Direct Products

Definition. Let $(G_1, \star)$ and $(G_2, \diamond)$ be groups. Their **direct product** is $(G_1 \times G_2, *)$ where

$$(a_1, a_2) * (b_1, b_2) = (a_1 \star b_1, a_2 \diamond b_2)$$

More clarification: We are defining $(G_1 \times G_2, *)$, which means there is some new binary operation $*$ on the set of all the ordered pairs (tuple), i.e. of all cartesian products, of elements from G_1 and G_2. Whereby, applying the binary operation on elements from $(G_1 \times G_2, *)$ means that we evaluate individual entries/components with respect to the binary operation associated with the group they come from.
E.g. letting $G_1 = G_2 = \mathbb{R}$ would give us the euclidean plane $(\mathbb{R}^2)$ with the appropriate restrictions.

Fact: If G_1, G_2 are groups, then $G_1 \times G_2$ is also a group

Notation:

- If both $\star$ and $\diamond$ are written as $+$ then we may also write $*$ as $+$
- Products of more factors are defined analogously. $G^n = \underbrace{G \times \cdots \times G}_{n}$

Theorem 10.1. Let G be a group. Suppose there exists $H, K \lhd G$ satisfying

1. $H \cap K = \{1\}$
2. $HK = G$

Then $G \cong H \times K$

Corollary 10.3. $(\mathbb{Z}_{mn}, +) \cong (\mathbb{Z}_m, +) \times (\mathbb{Z}_n, +)$ provided $\gcd(m, n) = 1$

Appendix