PMATH 333: Introduction to Real Analysis

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Chapter 1

Real Numbers

Definition. Let $A \subseteq \mathbf{R}$, we say that A is **bounded above** if $\exists M \in \mathbf{R}$ s.t. $a \leq M \, \forall a \in A$ we call M an **upper bound** for A
Similar definitions for a set to be **bounded below** and to have a **lower bound**

Definition. We say A is **bounded** if it is both bounded above and below.

Definition. Let $\emptyset \neq A \subseteq \mathbf{R}$. A number $M \in \mathbf{R}$ is called a **supremum (sup)** for A if

- M is an upper bound for A and
- If N is an upper bound for A then $M \leq N$

i.e. Supremum = least upper bound

Remark: Supremums are unique. $\emptyset \neq A \subseteq \mathbf{R}$
1) Suppose M, N are supremums of A, then

$$M \leq N, N \leq M \Rightarrow M = N$$

we write $M = \sup A$
2) If A is not bounded above, we write $\sup A = \infty$

Definition. Let $\emptyset \neq A \subseteq \mathbf{R}$. A number $M \in \mathbf{R}$ is called an **infimum (inf)** for A if

- M is a lower bound for A and
- If N is a lower bound for A then $N \leq M$

i.e. Infimum = greatest lower bound
Again, infimums are also unique, and are written as $\inf A = M$, if it is not bounded below

then $\inf A = -\infty$

Note: It is not always the case that $\sup A = \max A$, e.g. $A = [1, 2)$, here $\max A$ does not exist

Axiom. **[Least Upper Bound Property, LUB]**
If $\emptyset \neq A \subseteq \mathbf{R}$ is bounded above then $\sup A$ exists

Theorem. Let $\emptyset \neq A \subseteq \mathbf{R}$. If A is bounded below then $B = \{-a : a \in A\}$ is bounded above. Moreover,

$$\inf A = -\sup B$$

In particular, $\inf A$ exists

Theorem. **[Archimedian Principle]**
Let $a, b \in \mathbf{R}$ be positive. there exists $n \in \mathbf{N}$ s.t. $b < na$

Theorem. **[Density of the Rationals]**
Let $a < b$ be real numbers. There exists $q \in \mathbb{Q}$ s.t. $a < q < b$

Corollary. Let $a \in \mathbf{R}$. For every $\epsilon > 0$ there exists $q \in \mathbb{Q}$ s.t. $|a - q| < \epsilon$

Definition. A **sequence** of real numbers is an *infinite* list $(a_1, a_2, \dots)$ where each $a_i \in \mathbf{R}$
Notation: $(a_n)_{n=1}^{\infty}$ or (a_n). We write $(a_n) \subseteq \mathbf{R}$ (this does not mean sequence is a subset, it means terms in sequence are real)

Definition. $(a_n) \subseteq \mathbf{R}, a \in \mathbf{R}$ we say (a_n) **converges** to a, written $a_n \to a$,
if for all $\epsilon > 0$ there exists $N \in \mathbf{N}$ s.t. $|a_n - a| < \epsilon$ for all $n \geq N$. We call a the **limit** of the sequence

Definition. We say $(a_n) \subseteq \mathbf{R}$ is **bounded** if $\{a_1, a_2, \dots\}$ is bounded. i.e. If $\exists M \in \mathbf{R}$ s.t. $|a_n| \leq M$ for all $n \in \mathbf{N}$

Proposition. If $(a_n) \subseteq \mathbf{R}$ is convergent then (a_n) is bounded

Proposition. $(a_n), (b_n) \subseteq \mathbf{R}, a_n \to a, b_n \to b$

- $a_n + b_n \to a + b$
- If $\alpha \in \mathbf{R}$ then $\alpha a_n \to \alpha a$
- $a_n b_n \to ab$

- If $b_n \neq 0$ for all $n \in \mathbf{N}$ and $b \neq 0$ then $\frac{a_n}{b_n} \to \frac{a}{b}$

Proposition. $(a_n), (b_n) \subseteq \mathbf{R}, a_n \to a, b_n \to b$. If there exists $N \in \mathbf{N}$ s.t. $a_n \leq b_n$ for all $n \geq N$, then $a \leq b$

Proposition. $(a_n) \subseteq [c, d]$ and $a_n \to a$, then $a \in [c, d]$

Definition. $(a_n) \subseteq \mathbf{R}$

- (a_n) is **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$
- (a_n) is **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$
- (a_n) is **monotone** if it is either **increasing** or **decreasing**

We call the sequences **strictly increasing** or **strictly decreasing** when the inequalities are strict

Theorem. **[Monotone Convergence Theorem]**
If $(a_n) \subseteq \mathbf{R}$ is increasing and $\{a_n : n \in \mathbf{N}\}$ is bounded above then $a_n \to \sup\{a_n : n \in \mathbf{N}\}$

Corollary. If $(a_n) \subseteq \mathbf{R}$ is decreasing and $\{a_n : n \in \mathbf{N}\}$ is bounded below, then $a_n \to \inf\{a_n : n \in \mathbf{N}\}$

Theorem. **[Nested Intervals Lemma]**
Let $I_1 \supseteq I_2 \supseteq \dots$ where each $I_i = [a_i, b_i]$, $a_i \leq b_i$. Then $\bigcap_{i-1}^{\infty} I_i \neq \emptyset$

Definition. $(a_n) \subseteq \mathbf{R}$. A **subsequence** of (a_n) is a sequence

$$(a_{n_k})_{k=1}^{\infty}$$

where $n_1 < n_2 < n_3 < \dots$

Theorem. **[Bolzano-Weierstrass Theorem]**
Every bounded sequence of real numbers has a convergent subsequence

Definition. $(a_n) \subseteq \mathbf{R}$. We say (a_n) is **Cauchy** if $\forall \epsilon > 0$, $\exists N \in \mathbf{N}$ s.t. $|a_n - a_m| < \epsilon$ for all $n, m \geq N$

Proposition. $(a_n) \subseteq \mathbf{R}$. If (a_n) is convergent then (a_n) is Cauchy

Proposition. $(a_n) \subseteq \mathbf{R}$. If (a_n) is Cauchy then (a_n) is bounded

Proposition. If $(a_n) \subseteq \mathbf{R}$ is Cauchy and has a subsequence $a_{n_k} \to a$ then $a_n \to a$

Theorem. [Completeness of R]

A sequence $(a_n) \subseteq \mathbf{R}$ is convergent $\iff$ it is Cauchy

Remark: Up till now the big theorems we have used have been

Least Upper Bound property

$\Rightarrow$ Monotone Convergence Theorem

$\Rightarrow$ Nested Intervals Theorem

$\Rightarrow$ Bolzano-Weierstrass theorem

$\Rightarrow$ Completeness (Cauchy $\iff$ Convergent)

Chapter 2

Normed Vector Spaces

Analysis, is the study of approximation of mathematical objects Idea: A Normed Vector Space is a vector space where we can measure the distance between vectors, it's useful for the purpose of approximations

Definition. Let $\mathbf{V}$ be a real vector space. A **norm** on $\mathbf{V}$ is a function

$$\|\cdot\| : \mathbf{V} \to \mathbf{R}$$

such that

1. $\|\mathbf{v}\| \geq 0$ for all $\mathbf{v} \in \mathbf{V}$
2. $\|\mathbf{v}\| = 0 \iff \mathbf{v} = 0$
3. For all $\alpha \in \mathbf{R}, \mathbf{v} \in \mathbf{V}$
$$\|\alpha \mathbf{v}\| = |\alpha| \, \|\mathbf{v}\|$$
4. [Triangle Inequality]: For all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$,
$$\|u + v\| \leq \|u\| + \|v\|$$

Definition. Let $\|\cdot\|$ be a norm on $\mathbf{V}$. We call the pair $(\mathbf{V}, \|\cdot\|)$ a **normed vector space**

Convention: If $\|\cdot\|$ is understood, we write $\mathbf{V}$ instead of $(\mathbf{V}, \|\cdot\|)$

Aside: $\|\mathbf{v}\|$ can be seen as "length" of $\mathbf{v}$. And $\|\mathbf{v} - \mathbf{u}\|$ can be seen as the distance between two points

Examples of norms:

- The **absolute value** $(\mathbf{R}, |\cdot|)$
- The **Euclidean norm** $(\mathbf{R}^n, \|\cdot\|_2)$ which is

$$\|(x_1, \dots, x_n)\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$$

- The **p-norm** $(\mathbf{R}^n, \|\cdot\|_p), p \geq 1 \in \mathbf{R}$ where

$$\|(x_1, \dots, x_n)\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$$

- The **sup norm** $(\mathbf{R}^n, \|\cdot\|_\infty)$, where

$$\|(x_1, \dots, x_n)\|_\infty = \sup\{|x_i| : i = 1, 2, \dots, n\} = \max\{\|x_i| : i = 1, 2, \dots, n\}$$

- Let $\mathbf{R}^{\mathbf{N}} := \{(x_i)_{i=1}^\infty : x_i \in \mathbf{R}\}$. This is how we are defining an infinite dimensional real vector space, this is actually called a *real sequence space*
  $p \geq 1$ (real number)

$$\|(x_i)_{i=1}^\infty\|_p = \Big(\sum_{i=1}^\infty |x_i|^p\Big)^{1/p} \text{ but this could be } \stackrel{?}{=} \infty$$

  so to prevent this by removing sequences that give an infinite p-norm

$$l^p := \{(x_i) \in \mathbf{R}^{\mathbf{N}} : \|(x_i)\|_p < \infty\}$$

  Also, l^p is a subspace of $\mathbf{R}^{\mathbf{N}}$. And so $(l^p, \|\cdot\|_p)$ is a normed vector space and is called the **p-norm**. l^p is sometimes referred to as a *Lebesgue space*
- $(x_i) \in \mathbf{R}^{\mathbf{N}}$,

$$\|(x_i)\|_\infty = \sup\{|x_i| : i \in \mathbf{N}\} \text{ but again this could be } \stackrel{?}{=} \infty$$

  so we define the vector space

$$l^\infty = \{(x_i) \in \mathbf{R}^{\mathbf{N}} : \|(x_i)\|_\infty < \infty\}$$

  Again, l^∞ is the space of all bounded sequences, and is a subspace of $\mathbf{R}^{\mathbf{N}}$. And so $(l^\infty, \|\cdot\|_\infty)$ is a normed vector space with the norm called **sup norm** or **infinity norm**
- For real numbers $a < b$, the vector space of all continuous functions on $[a, b]$

$$C([a, b]) = \{f : [a, b] \to \mathbf{R} \text{ that are continuous}\}$$

$$\|f\|_\infty = \sup\{|f(x)| : x \in [a, b]\} \stackrel{EVT}{=} \max\{|f(x)| : x \in [a, b]\}$$

  $(C([a, b]), \|\cdot\|_\infty)$ is a normed vector space is called the **uniform norm**

2.1 Convergence

Definition. Let $\mathbf{V}$ be a normed vector space. A **sequence** in $\mathbf{V}$ is a right-infinite ordered list $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots)$ where each $\mathbf{v} \in \mathbf{V}$.
We denote this sequence by $(\mathbf{v}_i)_{i=1}^{\infty}$ or $(\mathbf{v}_i)$. Again, we write $(\mathbf{v}_i) \subseteq \mathbf{V}$ to mean each $\mathbf{v}_i \in \mathbf{V}$

Definition. Let $\mathbf{V}$ be a normed vector space, $(\mathbf{a}_n) \subseteq \mathbf{V}$, and $\mathbf{v} \in \mathbf{V}$
We say $(\mathbf{a}_n)$ **converges** to $\mathbf{v}$, written $\mathbf{a}_n \to \mathbf{v}$ if for all $\epsilon > 0$, if there exists $N \in \mathbf{N}$, such that if for all $n \geq N$ we have that $\|\mathbf{a}_n - \mathbf{v}\| < \epsilon$
We call $\mathbf{v}$ the limit of $(\mathbf{a}_n)$. If $(\mathbf{a}_n)$ does not converge to any $\mathbf{v} \in \mathbf{V}$, we say $(\mathbf{a}_n)$ **diverges** (in $\mathbf{V}$)

Example: Let $\mathbf{V} = l^{\infty} = \{(x_i) \in \mathbf{R}^{\mathbf{N}} : \sup_{i \in \mathbf{N}}\{|x_i|\} < \infty\}$
Let $(\mathbf{a}_n) \subseteq \mathbf{V}$: $\mathbf{a}_n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, 0, \dots)$
Claim $\mathbf{a}_n \to (1, \frac{1}{2}, \frac{1}{3}, \dots)$
Hint: $\|\mathbf{a}_n - (1, 1/2, 1/3, \dots)\|_{\infty} = \left\|(0, \dots, 0, -\frac{1}{n+1}, -\frac{1}{n+1}, \dots)\right\| = \sup\{0, \frac{1}{n+1}, \frac{1}{n+2}, \dots\} = \frac{1}{n+1}$

Example: Let $\mathbf{V} = l^{\infty}, (\mathbf{a}_n) \subseteq l^{\infty}, \mathbf{a}_n = (1, 2, \dots, n, 0, 0, \dots)$
Claim: $(\mathbf{a}_n)$ diverges in l^{∞}
Hint: Often for contradiction style proofs involving epsilons, use a fixed explicit value for epsilon and go about showing a contradiction.

Example: We have the normed vector space $(C([0,1]), \|\cdot\|_{\infty})$ (note this is the uniform norm, not the infinity norm)
Let $(f_n) \subseteq C([0,1])$, $f_n(x) = (x - \frac{1}{n})^2$.
Claim: $f_n \to f$, where $f(x) = x^2$.
Hint: Using the absolute value as the norm for some x in the interval show that the difference is within epsilon. And then reconcile the result for the function f being the limit using the uniform norm

Proposition. Let $\mathbf{V}$ be a normed vector space, $(\mathbf{a}_n), (\mathbf{b}_n) \subseteq \mathbf{V}$.
Suppose $\mathbf{a}_n \to \mathbf{v} \in \mathbf{V}$ and $\mathbf{b}_n \to \mathbf{w} \in \mathbf{V}$. Then,

1. $\mathbf{a}_n + \mathbf{b}_n \to \mathbf{v} + \mathbf{w}$
2. $\alpha\mathbf{a}_n \to \alpha\mathbf{v}$ $(\alpha \in \mathbf{R})$

2.2 Cauchy Sequences

Problem: The definition of convergence requires us to know or guess the limit of the sequence.

Proposition. Let $\mathbf{V}$ be a normed vector space, $(\mathbf{a}_n) \subseteq \mathbf{V}$, and $\mathbf{a}_n \to \mathbf{a} \in \mathbf{V}$.
For all $\epsilon > 0$ there exists $N \in \mathbf{N}$ s.t. for all $n, m \geq N$

$$\|\mathbf{a}_n - \mathbf{a}_m\| < \epsilon$$

Definition. For a normed vector space $\mathbf{V}$, and $(\mathbf{a}_n) \subseteq \mathbf{V}$.
We say $(\mathbf{a}_n)$ is **Cauchy** if $\forall \epsilon > 0, \exists N \in \mathbf{N}$ s.t. whenever $n, m \geq N$

$$\|\mathbf{a}_n - \mathbf{a}_m\| < \epsilon$$

Remark: here we have only shown that for a sequence Convergent $\Rightarrow$ Cauchy

Example: Let $\mathbf{V} = C_{00} := \{(a_n) \in l^\infty : \exists N \in \mathbf{N}, \forall n \geq N, a_n = 0\}$
We equip $\mathbf{V}$ with $\|\cdot\|_\infty$ (the infinity norm)
The sequence $(\mathbf{a}_n) \subseteq \mathbf{V}$ given by $\mathbf{a}_n = (1, 1/2, \ldots, 1/n, 0, 0, \ldots)$. We showed that $\mathbf{a}_n \to \mathbf{a} \in l^\infty$, $\mathbf{a} = (1, 1/2, 1/3, \ldots) \notin C_{00}$. Hence, $(\mathbf{a}_n) \subseteq C_{00}$ diverges.
Remark: We see that the convergence and divergence of sequences sometimes depends on the normed vector space that we are working in.
Claim: $(\mathbf{a}_n)$ is Cauchy
Rough proof: This is since the sequence is convergent in l^∞ so it is Cauchy in l^∞, and since it is Cauchy in l^∞ it is Cauchy in its subspace C_{00}. So it is Cauchy in C_{00} but not convergent

2.3 Completeness

Definition. Let $\mathbf{V}$ be a normed vector space, $(\mathbf{a}_n) \subseteq \mathbf{V}$. We say $(\mathbf{a}_n)$ is **bounded** (bd) if $\exists N \in \mathbf{N}$ s.t

$$\|\mathbf{a}_n\| < N$$

for all $n \in \mathbf{N}$

Proposition. Let $\mathbf{V}$ be a normed vector space. If $(\mathbf{a}_n) \subseteq \mathbf{V}$ is Cauchy then $(\mathbf{a}_n)$ is bounded

Idea: Here one may pick a fixed epsilon $(= 1)$, and find the max of the set $\{\|\mathbf{a}_1\|, \ldots, \|\mathbf{a}_{N-1}\|, 1+ \|\mathbf{a}_N\|\}$, where N satisfies the epsilon

So know that sequences being convergent is not the same as being Cauchy as there exist Cauchy sequences that don't converge. So we give the spaces where these notions are the same a name

Definition. Let $\mathbf{V}$ be a normed vector space

We say that $A \subseteq \mathbf{V}$ is **complete** if every Cauchy sequence $(\mathbf{a}_n) \subseteq A$ converges in A.

If $\mathbf{V}$ is complete itself (i.e. $A = \mathbf{V}$), we call $\mathbf{V}$ a **Banach space**

Remark: In a Banach space, a sequence is convergent $\iff$ the sequence is Cauchy

Also, only vector spaces that are complete can be called Banach spaces, a set that is not a vector space can be complete but not a Banach space.

Examples:

- $\mathbf{R}, \mathbf{R}^n, l^\infty$ are complete and so,are Banach spaces
- C_{00} is *not* a Banach space, though it is a vector space
- Let $(\frac{1}{n+1}) \subseteq (0,1) \subseteq \mathbf{R}$. $\frac{1}{n+1} \to 0 \notin (0,1)$. $(\frac{1}{n+1})$ is convergent in $\mathbf{R}$ so it is Cauchy. But since $0 \notin (0,1)$, $(0,1)$ is not complete

Appendix