PMATH 333: Introduction to Real Analysis

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Contents

These are my notes for my 2nd year course Introduction to Analysis (PMATH 333) at the University of Waterloo.
Reference text: Introduction to Analysis, William R. Wade

Chapter 1

Real Numbers

Definition. Let $A \subseteq \mathbf{R}$, we say that A is **bounded above** if $\exists M \in \mathbf{R}$ s.t. $a \leq M \, \forall a \in A$ we call M an **upper bound** for A
Similar definitions for a set to be **bounded below** and to have a **lower bound**

Definition. We say A is **bounded** if it is both bounded above and below.

Definition. Let $\emptyset \neq A \subseteq \mathbf{R}$. A number $M \in \mathbf{R}$ is called a **supremum (sup)** for A if

- M is an upper bound for A and
- If N is an upper bound for A then $M \leq N$

i.e. Supremum = least upper bound

Remark: Supremums are unique. $\emptyset \neq A \subseteq \mathbf{R}$
1) Suppose M, N are supremums of A, then

$$M \leq N, N \leq M \Rightarrow M = N$$

we write $M = \sup A$
2) If A is not bounded above, we write $\sup A = \infty$

Definition. Let $\emptyset \neq A \subseteq \mathbf{R}$. A number $M \in \mathbf{R}$ is called an **infimum (inf)** for A if

- M is a lower bound for A and
- If N is a lower bound for A then $N \leq M$

i.e. Infimum = greatest lower bound
Again, infimums are also unique, and are written as $\inf A = M$, if it is not bounded below

then $\inf A = -\infty$

Note: It is not always the case that $\sup A = \max A$, e.g. $A = [1, 2)$, here $\max A$ does not exist

Axiom. **[Least Upper Bound Property, LUB]**
If $\emptyset \neq A \subseteq \mathbf{R}$ is bounded above then $\sup A$ exists

Theorem. Let $\emptyset \neq A \subseteq \mathbf{R}$. If A is bounded below then $B = \{-a : a \in A\}$ is bounded above. Moreover,

$$\inf A = -\sup B$$

In particular, $\inf A$ exists

Theorem. **[Archimedian Principle]**
Let $a, b \in \mathbf{R}$ be positive. there exists $n \in \mathbf{N}$ s.t. $b < na$

Theorem. **[Density of the Rationals]**
Let $a < b$ be real numbers. There exists $q \in \mathbb{Q}$ s.t. $a < q < b$

Corollary. Let $a \in \mathbf{R}$. For every $\epsilon > 0$ there exists $q \in \mathbb{Q}$ s.t. $|a - q| < \epsilon$

Definition. A **sequence** of real numbers is an *infinite* list $(a_1, a_2, \dots)$ where each $a_i \in \mathbf{R}$
Notation: $(a_n)_{n=1}^{\infty}$ or (a_n). We write $(a_n) \subseteq \mathbf{R}$ (this does not mean sequence is a subset, it means terms in sequence are real)

Definition. $(a_n) \subseteq \mathbf{R}, a \in \mathbf{R}$ we say (a_n) **converges** to a, written $a_n \to a$,
if for all $\epsilon > 0$ there exists $N \in \mathbf{N}$ s.t. $|a_n - a| < \epsilon$ for all $n \geq N$. We call a the **limit** of the sequence

Definition. We say $(a_n) \subseteq \mathbf{R}$ is **bounded** if $\{a_1, a_2, \dots\}$ is bounded. i.e. If $\exists M \in \mathbf{R}$ s.t. $|a_n| \leq M$ for all $n \in \mathbf{N}$

Proposition. If $(a_n) \subseteq \mathbf{R}$ is convergent then (a_n) is bounded

Proposition. $(a_n), (b_n) \subseteq \mathbf{R}, a_n \to a, b_n \to b$

- $a_n + b_n \to a + b$
- If $\alpha \in \mathbf{R}$ then $\alpha a_n \to \alpha a$
- $a_n b_n \to ab$

- If $b_n \neq 0$ for all $n \in \mathbf{N}$ and $b \neq 0$ then $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$

Proposition. $(a_n), (b_n) \subseteq \mathbf{R}, a_n \to a, b_n \to b$. If there exists $N \in \mathbf{N}$ s.t. $a_n \leq b_n$ for all $n \geq N$, then $a \leq b$

Proposition. $(a_n) \subseteq [c, d]$ and $a_n \to a$, then $a \in [c, d]$

Definition. $(a_n) \subseteq \mathbf{R}$

- (a_n) is **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbf{N}$
- (a_n) is **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbf{N}$
- (a_n) is **monotone** if it is either **increasing** or **decreasing**

We call the sequences **strictly increasing** or **strictly decreasing** when the inequalities are strict

Theorem. **[Monotone Convergence Theorem]**
If $(a_n) \subseteq \mathbf{R}$ is increasing and $\{a_n : n \in \mathbf{N}\}$ is bounded above then $a_n \to \sup\{a_n : n \in \mathbf{N}\}$

Corollary. If $(a_n) \subseteq \mathbf{R}$ is decreasing and $\{a_n : n \in \mathbf{N}\}$ is bounded below, then $a_n \to \inf\{a_n : n \in \mathbf{N}\}$

Theorem. **[Nested Intervals Lemma]**
Let $I_1 \supseteq I_2 \supseteq \dots$ where each $I_i = [a_i, b_i]$, $a_i \leq b_i$. Then $\bigcap_{i-1}^{\infty} I_i \neq \emptyset$

Definition. $(a_n) \subseteq \mathbf{R}$. A **subsequence** of (a_n) is a sequence

$$(a_{n_k})_{k=1}^{\infty}$$

where $n_1 < n_2 < n_3 < \dots$

Theorem. **[Bolzano-Weierstrass Theorem]**
Every bounded sequence of real numbers has a convergent subsequence

Definition. $(a_n) \subseteq \mathbf{R}$. We say (a_n) is **Cauchy** if $\forall \epsilon > 0$, $\exists N \in \mathbf{N}$ s.t. $|a_n - a_m| < \epsilon$ for all $n, m \geq N$

Proposition. $(a_n) \subseteq \mathbf{R}$. If (a_n) is convergent then (a_n) is Cauchy

Proposition. $(a_n) \subseteq \mathbf{R}$. If (a_n) is Cauchy then (a_n) is bounded

Proposition. If $(a_n) \subseteq \mathbf{R}$ is Cauchy and has a subsequence $a_{n_k} \to a$ then $a_n \to a$

Theorem. **[Completeness of R]**

A sequence $(a_n) \subseteq \mathbf{R}$ is convergent $\iff$ it is Cauchy

Remark: Up till now the big theorems we have used have been

Least Upper Bound property

$\Rightarrow$ Monotone Convergence Theorem

$\Rightarrow$ Nested Intervals Theorem

$\Rightarrow$ Bolzano-Weierstrass theorem

$\Rightarrow$ Completeness (Cauchy $\iff$ Convergent)

Chapter 2

Normed Vector Spaces

Analysis, is the study of approximation of mathematical objects Idea: A Normed Vector Space is a vector space where we can measure the distance between vectors, it's useful for the purpose of approximations

Definition. Let $\mathbf{V}$ be a real vector space. A **norm** on $\mathbf{V}$ is a function

$$\|\cdot\| : \mathbf{V} \to \mathbf{R}$$

such that

1. $\|\mathbf{v}\| \geq 0$ for all $\mathbf{v} \in \mathbf{V}$
2. $\|\mathbf{v}\| = 0 \iff \mathbf{v} = 0$
3. For all $\alpha \in \mathbf{R}, \mathbf{v} \in \mathbf{V}$
$$\|\alpha \mathbf{v}\| = |\alpha| \, \|\mathbf{v}\|$$
4. [Triangle Inequality]: For all $\mathbf{u}, \mathbf{v} \in \mathbf{V}$,
$$\|u + v\| \leq \|u\| + \|v\|$$

Definition. Let $\|\cdot\|$ be a norm on $\mathbf{V}$. We call the pair $(\mathbf{V}, \|\cdot\|)$ a **normed vector space**

Convention: If $\|\cdot\|$ is understood, we write $\mathbf{V}$ instead of $(\mathbf{V}, \|\cdot\|)$

Aside: $\|\mathbf{v}\|$ can be seen as "length" of $\mathbf{v}$. And $\|\mathbf{v} - \mathbf{u}\|$ can be seen as the distance between two points

Examples of norms:

- The **absolute value** $(\mathbf{R}, |\cdot|)$
- The **Euclidean norm** $(\mathbf{R}^n, \|\cdot\|_2)$ which is

$$\|(x_1, \ldots, x_n)\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$$

- The **p-norm** $(\mathbf{R}^n, \|\cdot\|_p), p \geq 1 \in \mathbf{R}$ where

$$\|(x_1, \ldots, x_n)\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$$

- The **sup norm** $(\mathbf{R}^n, \|\cdot\|_\infty)$, where

$$\|(x_1, \ldots, x_n)\|_\infty = \sup\{|x_i| : i = 1, 2, \ldots, n\} = \max\{\|x_i| : i = 1, 2, \ldots, n\}$$

- Let $\mathbf{R}^{\mathbf{N}} := \{(x_i)_{i=1}^\infty : x_i \in \mathbf{R}\}$. This is how we are defining an infinite dimensional real vector space, this is actually called a *real sequence space*
  $p \geq 1$ (real number)

$$\|(x_i)_{i=1}^\infty\|_p = \Big(\sum_{i=1}^\infty |x_i|^p\Big)^{1/p} \text{ but this could be } \stackrel{?}{=} \infty$$

  so to prevent this by removing sequences that give an infinite p-norm

$$l^p := \{(x_i) \in \mathbf{R}^{\mathbf{N}} : \|(x_i)\|_p < \infty\}$$

  Also, l^p is a subspace of $\mathbf{R}^{\mathbf{N}}$. And so $(l^p, \|\cdot\|_p)$ is a normed vector space and is called the **p-norm**. l^p is sometimes referred to as a *Lebesgue space*
- $(x_i) \in \mathbf{R}^{\mathbf{N}}$,

$$\|(x_i)\|_\infty = \sup\{|x_i| : i \in \mathbf{N}\} \text{ but again this could be } \stackrel{?}{=} \infty$$

  so we define the vector space

$$l^\infty = \{(x_i) \in \mathbf{R}^{\mathbf{N}} : \|(x_i)\|_\infty < \infty\}$$

  Again, l^∞ is the space of all bounded sequences, and is a subspace of $\mathbf{R}^{\mathbf{N}}$. And so $(l^\infty, \|\cdot\|_\infty)$ is a normed vector space with the norm called **sup norm** or **infinity norm**
- For real numbers $a < b$, the vector space of all continuous functions on $[a, b]$

$$C([a, b]) = \{f : [a, b] \to \mathbf{R} \text{ that are continuous}\}$$

$$\|f\|_\infty = \sup\{|f(x)| : x \in [a, b]\} \stackrel{EVT}{=} \max\{|f(x)| : x \in [a, b]\}$$

  $(C([a, b]), \|\cdot\|_\infty)$ is a normed vector space is called the **uniform norm**

2.1 Convergence

Definition. Let $\mathbf{V}$ be a normed vector space. A **sequence** in $\mathbf{V}$ is a right-infinite ordered list $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots)$ where each $\mathbf{v} \in \mathbf{V}$.
We denote this sequence by $(\mathbf{v}_i)_{i=1}^{\infty}$ or $(\mathbf{v}_i)$. Again, we write $(\mathbf{v}_i) \subseteq \mathbf{V}$ to mean each $\mathbf{v}_i \in \mathbf{V}$

Definition. Let $\mathbf{V}$ be a normed vector space, $(\mathbf{a}_n) \subseteq \mathbf{V}$, and $\mathbf{v} \in \mathbf{V}$
We say $(\mathbf{a}_n)$ **converges** to $\mathbf{v}$, written $\mathbf{a}_n \to \mathbf{v}$ if for all $\epsilon > 0$, if there exists $N \in \mathbf{N}$, such that if for all $n \geq N$ we have that $\|\mathbf{a}_n - \mathbf{v}\| < \epsilon$
We call $\mathbf{v}$ the limit of $(\mathbf{a}_n)$. If $(\mathbf{a}_n)$ does not converge to any $\mathbf{v} \in \mathbf{V}$, we say $(\mathbf{a}_n)$ **diverges** (in $\mathbf{V}$)

Example: Let $\mathbf{V} = l^\infty = \{(x_i) \in \mathbf{R}^{\mathbf{N}} : \sup_{i \in \mathbf{N}}\{|x_i|\} < \infty\}$
Let $(\mathbf{a}_n) \subseteq \mathbf{V}\ : \mathbf{a}_n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, 0, \dots)$
Claim $\mathbf{a}_n \to (1, \frac{1}{2}, \frac{1}{3}, \dots)$
Hint: $\|\mathbf{a}_n - (1, 1/2, 1/3, \dots)\|_\infty = \left\|(0, \dots, 0, -\frac{1}{n+1}, -\frac{1}{n+1}, \dots)\right\| = \sup\{0, \frac{1}{n+1}, \frac{1}{n+2}, \dots\} = \frac{1}{n+1}$

Example: Let $\mathbf{V} = l^\infty, (\mathbf{a}_n) \subseteq l^\infty, \mathbf{a}_n = (1, 2, \dots, n, 0, 0, \dots)$
Claim: $(\mathbf{a}_n)$ diverges in l^∞
Hint: Often for contradiction style proofs involving epsilons, use a fixed explicit value for epsilon and go about showing a contradiction.

Example: We have the normed vector space $(C([0,1]), \|\cdot\|_\infty)$ (note this is the uniform norm, not the infinity norm)
Let $(f_n) \subseteq C([0,1]),\ f_n(x) = (x - \frac{1}{n})^2$.
Claim: $f_n \to f$, where $f(x) = x^2$.
Hint: Using the absolute value as the norm for some x in the interval show that the difference is within epsilon. And then reconcile the result for the function f being the limit using the uniform norm

Proposition. Let $\mathbf{V}$ be a normed vector space, $(\mathbf{a}_n), (\mathbf{b}_n) \subseteq \mathbf{V}$.
Suppose $\mathbf{a}_n \to \mathbf{v} \in \mathbf{V}$ and $\mathbf{b}_n \to \mathbf{w} \in \mathbf{V}$. Then,

1. $\mathbf{a}_n + \mathbf{b}_n \to \mathbf{v} + \mathbf{w}$
2. $\alpha \mathbf{a}_n \to \alpha \mathbf{v}$ $(\alpha \in \mathbf{R})$

2.2 Cauchy Sequences

Problem: The definition of convergence requires us to know or guess the limit of the sequence.

Proposition. Let $\mathbf{V}$ be a normed vector space, $(\mathbf{a}_n) \subseteq \mathbf{V}$, and $\mathbf{a}_n \to \mathbf{a} \in \mathbf{V}$.
For all $\epsilon > 0$ there exists $N \in \mathbf{N}$ s.t. for all $n, m \geq N$

$$\|\mathbf{a}_n - \mathbf{a}_m\| < \epsilon$$

Definition. For a normed vector space $\mathbf{V}$, and $(\mathbf{a}_n) \subseteq \mathbf{V}$.
We say $(\mathbf{a}_n)$ is **Cauchy** if $\forall \epsilon > 0, \exists N \in \mathbf{N}$ s.t. whenever $n, m \geq N$

$$\|\mathbf{a}_n - \mathbf{a}_m\| < \epsilon$$

Remark: here we have only shown that for a sequence Convergent $\Rightarrow$ Cauchy

Example: Let $\mathbf{V} = C_{00} := \{(a_n) \in l^\infty : \exists N \in \mathbf{N}, \forall n \geq N, a_n = 0\}$
We equip $\mathbf{V}$ with $\|\cdot\|_\infty$ (the infinity norm)
The sequence $(\mathbf{a}_n) \subseteq \mathbf{V}$ given by $\mathbf{a}_n = (1, 1/2, \dots, 1/n, 0, 0, \dots)$. We showed that $\mathbf{a}_n \to \mathbf{a} \in l^\infty$, $\mathbf{a} = (1, 1/2, 1/3, \dots) \notin C_{00}$. Hence, $(\mathbf{a}_n) \subseteq C_{00}$ diverges.
Remark: We see that the convergence and divergence of sequences sometimes depends on the normed vector space that we are working in.
Claim: $(\mathbf{a}_n)$ is Cauchy
Rough proof: This is since the sequence is convergent in l^∞ so it is Cauchy in l^∞, and since it is Cauchy in l^∞ it is Cauchy in its subspace C_{00}. So it is Cauchy in C_{00} but not convergent

2.3 Completeness

Definition. Let $\mathbf{V}$ be a normed vector space, $(\mathbf{a}_n) \subseteq \mathbf{V}$. We say $(\mathbf{a}_n)$ is **bounded** (bd) if $\exists N \in \mathbf{N}$ s.t

$$\|\mathbf{a}_n\| < N$$

for all $n \in \mathbf{N}$

Proposition. Let $\mathbf{V}$ be a normed vector space. If $(\mathbf{a}_n) \subseteq \mathbf{V}$ is Cauchy then $(\mathbf{a}_n)$ is bounded
Idea: Here one may pick a fixed epsilon $(= 1)$, and find the max of the set $\{\|\mathbf{a}_1\|, \ldots, \|\mathbf{a}_{N-1}\|, 1+ \|\mathbf{a}_N\|\}$, where N satisfies the epsilon

So know that sequences being convergent is not the same as being Cauchy as there exist Cauchy sequences that don't converge. So we give the spaces where these notions are the same a name

Definition. Let $\mathbf{V}$ be a normed vector space
We say that $A \subseteq \mathbf{V}$ is **complete** if every Cauchy sequence $(\mathbf{a}_n) \subseteq A$ converges in A.
If $\mathbf{V}$ is complete itself (i.e. $A = \mathbf{V}$), we call $\mathbf{V}$ a **Banach space**

Remark: In a Banach space, a sequence is convergent $\iff$ the sequence is Cauchy
Also, only vector spaces that are complete can be called Banach spaces, a set that is not a vector space can be complete but not a Banach space.
Examples:

- $\mathbf{R}, \mathbf{R}^n, l^\infty$ are complete and so,are Banach spaces
- C_{00} is *not* a Banach space, though it is a vector space
- Let $(\frac{1}{n+1}) \subseteq (0,1) \subseteq \mathbf{R}$. $\frac{1}{n+1} \to 0 \notin (0,1)$. $(\frac{1}{n+1})$ is convergent in $\mathbf{R}$ so it is Cauchy. But since $0 \notin (0,1)$, $(0,1)$ is not complete

Chapter 3

Topology

Topology is the study of subsets of a set $\mathbf{X}$ which afford $\mathbf{X}$ meaningful analytic/ geometric properties.
What we'll do is; given a normed vector space $\mathbf{V}$, we want to investigate the way convergence/limits of sequences behave in subsets of $\mathbf{V}$

3.1 Closed and Open sets

Definition. Let $\mathbf{V}$ be a normed vector space. We say $C \subseteq \mathbf{V}$ is closed (in $\mathbf{V}$) if whenever $(x_n \subseteq C)$ s.t. $x_n \to x \in \mathbf{V}$ then $x \in C$
This is sort of saying that C is closed $\iff$ C is "closed" under taking limits.

Examples:

1. $\emptyset, \mathbf{V} \in \mathbf{V}$ are closed [1]
2. Let $\mathbf{V}$ be a NVS, for an $x \in \mathbf{V}$ the set $\{x\}$ is closed [2]
3. $[0,1) \subseteq \mathbf{R}$ is not closed [3]
4. $C_{00} \subseteq l^\infty$ is not closed in l^∞ **(*)**, the sequence $a_n = (1, 1/2, ..., 1/n, 0, 0) \in C_{00}$ but $(a_n) = (1, 1/2, 1/3, ...) \notin C_{00}$
5. $C_{00} \subseteq C_{00}$, C_{00} is closed in C_{00}. This is because C_{00} is a normed vector space itself and so here we see that by (1) C_{00} is closed when it is considered in the NVS which is itself. In (4) we saw that it was not closed since it was being considered in the NVS l^∞

[1] For the empty set, according to the definition, the hypothesis is false therefore the conclusion must be true

[2] there is only one sequence in this set, the sequence $x_n = x$ and this converges in the set

[3] Can be disproven by the sequence $(1 - 1/n)$

6. $A = \{f \in C([0,1]) : f(1/2) = 0\} \subseteq C([0,1])$
   Claim: A is closed in $C([0,1])$
   Proof: Let $(f_n) \subseteq A$ s.t. $f_n \to f \in C([0,1])$. We must show that $f \in A$
   Let $\epsilon > 0$ be given. Since $f_n \to f, \exists N \in \mathbf{N} : \|f_n - f\|_\infty < \epsilon$ for all $n \geq N$
   Then for any $n \geq N$, we have
   $$\begin{aligned} |f(1/2)| &= |\underbrace{f_n(1/2)}_{=0} - f(1/2)| \\ &\leq \|f_n - f\|_\infty \text{ uniform norm} \\ &< \epsilon \end{aligned}$$
   Hence, $f(1/2) = 0 \implies f \in A$ and so A is closed
7. $\mathbf{V}$ NVS, $a \in \mathbf{V}, r > 0$

   Definition. $\overline{B_r(a)} := \{x \in \mathbf{V} : \|x - a\| \leq r\}$. This is called the **closed ball** at a of radius r
   Claim: $\overline{B_r(a)}$ is closed
   $(x_n) \subseteq \overline{B_r(a)}, x_n \to x \in \mathbf{V}$
   We show $x \in \overline{B_r(a)}$
   $x_n - a \to x - a \overset{hw}{\implies} \underbrace{\|x_n - a\|}_{\leq r} \to \underbrace{\|x - a\|}_{\leq r}$ Therefore, $x \in \overline{B_r(a)}$

Examples of Closed balls:
In $\mathbf{R} : \overline{B_r(a)} = [a - r, a + r]$
In $\mathbf{R}^2 : \overline{B_r(a)} =$ the solid disc centered at a of radius r
In $\mathbf{R}^3 : \overline{B_r(a)} =$ the solid/closed sphere centered at a of radius r

Proposition. $\mathbf{V}$ NVS, If $C \subseteq \mathbf{V}$ is complete then C is closed
Proof: Suppose C is complete. Take $(x_n) \subseteq C : x_n \to x \in \mathbf{V}$. Since (x_n) is convergent, (x_n) is Cauchy. Thus, $x \in C$ by completeness, therefore C is closed.
Converse is not true, if we took $V = C_{00}$ then it is closed as $C_{00} \subseteq C_{00}$ but it is not complete.

Definition. $\mathbf{V}$ NVS, We say $\mathcal{U} \subseteq \mathbf{V}$ is **open** (in $\mathbf{V}$) if $\mathbf{V} \setminus \mathcal{U}$ is closed.
Note: $\mathbf{V} \setminus \mathcal{U} = \{x \in \mathbf{V} : x \notin \mathcal{U}\}$

Example: $\mathbf{V}$ NVS, $a \in \mathbf{V}, r > 0, A = \{x \in \mathbf{V} : \|x - a\| \geq r\}$ is closed $\implies \mathbf{V} \setminus A = \{x \in \mathbf{V} : \|x - a\| < r\}$ is open

Definition. We call $\mathbf{V} \setminus A$ the **open ball**, $B_r(a)$, centered at a of radius r

Now we'll prove a proposition that would allow us to say that a set is open without having to say that something is closed.

Proposition. $\mathbf{V}$ NVS, $\mathcal{U} \subseteq \mathbf{V}$. The following are equivalent

1. $\mathcal{U}$ is open in $\mathbf{V}$
2. $\forall a \in \mathcal{U}, \exists r > 0 : B_r(a) \subseteq \mathcal{U}$

The idea here is that for any a we pick inside $\mathcal{U}$, (imagine is close the periphery of $\mathcal{U}$), there is always enough "wiggle room" for us to form an open ball around a

Proof: It's an iff statement so

$(\implies)$ Suppose $\mathcal{U}$ is open in $\mathbf{V}$. Thus $\mathbf{V} \setminus \mathcal{U}$ is closed

Let $a \in \mathcal{U}$. Assuming for a contradiction, suppose for all $r > 0$,

$B_r(a) \nsubseteq \mathcal{U} \iff B_r(a) \cap (\mathbf{V} \setminus \mathcal{U}) \neq \emptyset$

For every $n \in \mathbf{N}$, let $x_n \in B_{\frac{1}{n}}(a) \cap (\mathbf{V} \setminus \mathcal{U})$. Then $\|x_n - a\| < 1/n \to 0$ and so $\underset{\in \mathbf{V}\setminus\mathcal{U}}{x_n} \to \underset{\in \mathcal{U}}{a}$

This contradicts that $\mathbf{V} \setminus \mathcal{U}$ is closed, so it must be that $B_r(a) \subseteq \mathcal{U}$

$(\impliedby)$ Suppose (2). We show $\mathbf{V} \setminus \mathcal{U}$ is closed. Take $(x_n) \subseteq \mathbf{V} \setminus \mathcal{U} : x_n \to x \in \mathbf{V}$

For a contradiction, suppose $x \notin \mathbf{V} \setminus \mathcal{U} \iff x \in \mathcal{U}$. So by (2), we know that $\exists r > 0 :$ $B_r(a) \subseteq \mathcal{U}$

For large enough n, $x_n \in B_r(a) \subseteq \mathcal{U}$ since $x \in \mathcal{U}$. This is a contradiction, so it must be that $x \notin \mathcal{U}$ (i.e. $x \in \mathbf{V} \setminus \mathcal{U}$). Therefore, $\mathbf{V} \setminus \mathcal{U}$ is closed and hence $\mathcal{U}$ is open in $\mathbf{V}$

Definition. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$.

1. We say $x \in \mathbf{V}$ is a **limit point** of A If $\exists (a_n) \in A : a_n \to x$
2. We say $x \in A$ is an **interior point** of A if $\exists r > 0 : B_r(x) \subseteq A$

Summary: $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$

1. A is closed in $\mathbf{V} \iff A$ contains all its limit points
2. A is open in $\mathbf{V} \iff$ every point in A is an interior point of A

3.2 Unions and Intersections

Examples:

1. $\displaystyle\bigcap_{i=1}^{\infty} \underbrace{\left(-\frac{1}{i}, \frac{1}{i}\right)}_{\text{open}} = \underbrace{\{0\}}_{\text{closed}}$
2. $\displaystyle\bigcup_{i=1}^{\infty} \underbrace{[0, 1 - 1/n]}_{\text{closed}} = \underbrace{[0, 1)}_{\text{not closed}}$

Proposition. $\mathbf{V}$ NVS

1. If $\{A_\alpha\}_{\alpha \in I}$ are open in $\mathbf{V}$ then $\bigcup_{\alpha \in I} A_\alpha$ is open
2. If $A_1, A_2, \ldots, A_n$ are open in $\mathbf{V}$ then $\bigcap_{i=1}^{\infty} A_i$ is open (since n is finite)

Proof:
(1) $a \in \bigcup_{\alpha \in I} A_\alpha \implies \exists \alpha \in I : a \in A_\alpha$. Since A_α is open, $\exists \epsilon > 0 : B_\epsilon(a) \subseteq A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha$
This does not require I to be a finite collection, it can be infinite too
(2) $a \in A_1 \cap \cdots \cap A_n, \forall i, a \in A_i \implies \forall i, \exists r_i > 0, B_{r_i}(a) \subseteq A_i$
We take $r = \min\{r_i : \forall i\} \implies B_r(a) \subseteq A_1 \cap \cdots \cap A_n$
For (2) it is the minimum argument that causes it to work for finite but not infinite size, as the min will have to be replaced with an inf which could be 0, this would cause the $B_r(a)$ to be invalid

Corollary. $\mathbf{V}$ NVS

1. If $\{A_\alpha\}_{\alpha \in I}$ are closed in $\mathbf{V}$ then $\bigcap_{\alpha \in I} A_\alpha$ is closed
2. If $A_1, \ldots, A_n$ are closed in $\mathbf{V}$ then $\bigcup_{i=1}^{n} A_i$ is closed

This is can be seen by using De Morgan's laws
$$\underbrace{\mathbf{V} \setminus (\bigcap_{\alpha \in I} A_\alpha)}_{\text{open}} = \underbrace{\bigcup_{\alpha \in I} (\mathbf{V} \setminus A_\alpha)}_{\text{open}}$$
Similarly
$$\underbrace{\mathbf{V} \setminus (\bigcup_{i=1}^{n} A_i)}_{\text{open}} = \underbrace{\bigcap_{i=1}^{n} (\mathbf{V} \setminus A_i)}_{\text{open}}$$

3.3 Closures and Interiors

Definition. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$

1. The **closure** of A is defined as $\overline{A} := \bigcap_{A \subseteq C} C$ where C is closed. That is the intersection of all the closed sets that contain A
2. The **interior** of A is defined as $\text{int}(A) := \bigcup_{\mathcal{U} \subseteq A} \mathcal{U}$ where $\mathcal{U}$ is open. The union of all the open sets contained in A

Remark:

1. $\overline{A}$ is the smallest closed set in $\mathbf{V}$ containing A
2. $\text{Int}(A)$ is the largest open set in $\mathbf{V}$ contained in A

3. A is closed $\iff$ $\overline{A} = A$
4. A is open $\iff$ $\text{int}(A) = A$

Proposition. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$. $\overline{A} = \{x \in \mathbf{V} : x \text{ limit point of } A\}$

Proof: Let $X = \{x \in \mathbf{V} : x \text{ is a limit point of } A\}$

$(X \subseteq \overline{A})$ HW.

We must show X is closed and X is a subset of X, so that we can show $(\overline{A} \subseteq X)$

Indeed, $A \subseteq X$. **Claim:** X is closed.

Let $x_n \subseteq X : x_n \to x \in \mathbf{V}$

For every $n \in \mathbf{N}, x_n$ is a limit point of A and so we may find $y_n \in A : \|y_n - x_n\| < 1/n$

Then, $y_n = \underbrace{y_n - x_n}_{\to 0} + \underbrace{x_n}_{\to x} \to x$ and so $x \in X$. Therefore, X is closed.

Proposition. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$. $\text{Int}(A) = \{x \in A : x \text{ is an interior point of A}\}$

Proof: Let $X = \{x \in A : x \text{ is an interior point of A}\}$

$(\text{Int}(A) \subseteq X)$ HW.

$(X \subseteq \text{Int}(A))$. We show X is open and $X \subseteq A$. Obviously, $X \subseteq A$

Claim: X is open

Let $x \in X$. Thus $\exists r > 0 : B_r(x) \subseteq A$. Now, since that $B_r(x)$ is open $\forall y \in B_r(x), \exists r' > 0 :$ $B_{r'}(y) \subseteq B_r(x)$. Thus, $B_{r'}(y) \subseteq A$, and so $y \in X$. Hence, $B_r(x) \subseteq X$ and so X is open

Examples:

1. $A = [0,1), \bar{A} = [0,1]$, $\text{int}(A) = (0,1)$
2. Closure of $B_r(a)$ is $\overline{B_r(a)}$
3. interior of $\overline{B_r(a)}$ is $B_r(a)$
4. $A = \mathbb{Q} \in \mathbf{R}$, $\text{int}(A) = \emptyset$, $\bar{A} = \mathbf{R}$
5. $A = \{(e^{-x}\cos(x), e^{-x}\sin(x)) : x \geq 0\} \subseteq \mathbf{R}^2$ this traces an inward spiral. $\text{Int}A = \emptyset$, $\bar{A} = A \cup \{(0,0)\}$
6. $\mathbf{V} = l^\infty$

   $C_{00} = \{(x_n) \in \mathbf{V} : \text{ eventually all 0's}\}$

   $C_0 = \{(x_n \in \mathbf{V} : x_n \to 0)\}$

   (a) Show C_0 is closed in l^∞

   Let $(x_n) \subseteq C_0 : x_n \to x \in l^\infty$

   Claim: $x \in C_0$

   Say for $n \in \mathbf{N}, x_n = (x_n^{(1)}, x_n^{(2)}, \dots)$ and $x = (a_1, a_2, \dots)$

   We know for every $n \in \mathbf{N}, x_n \in C_0$ and so $x_n^{(k)} \to 0$, as $k \to \infty$

Let $\epsilon > 0$ be given. We can find $N \in \mathbf{N} : \|x_n - x\|_\infty < \epsilon/2$ for $n \geq N$

Also, we can find $K \in \mathbf{N} : |x_N^{(k)}| < \epsilon/2$ for $k \geq K$.

Now, for $k \geq K$, we want the $a_k \to 0$ so

$|a_k| = |a_k - x_N^{(k)} + x_N^{(k)}| \leq |a_k - x_N^{(k)}| + |x_N^{(k)}| \leq \|x - x_N\|_\infty + |x_N^{(k)}| < \epsilon/2 + \epsilon/2 = \epsilon$

Hence, $x \in C_0$ and so C_0 is closed.

(b) Show $\overline{C_{00}} = C_0$.

We have that $C_{00} \subseteq C_0$ and C_0 is closed. Hence, $\overline{C_{00}} \subseteq C_0$

Claim: $C_0 \subseteq \overline{C_{00}}$

Let $x \in C_0$ say $x = (a_1, a_2, \dots)$ hence $a_k \to 0, k \to \infty$.

For every $n \in \mathbf{N}$ let $x_n = (a_1, a_2, \dots, a_n, 0, 0, \dots) \in C_{00}$

Let $\epsilon > 0$ be given. We may find $N \in \mathbf{N} : |a_n| < \epsilon/2$ for $n \geq N$.

For $n \geq N$, $\|x_n - x\|_\infty = \|x - x_n\|_\infty = \|(0, \dots, 0, a_{n+1}, a_{n+2}, \dots)\| \leq \epsilon/2 < \epsilon$

Therefore, $\underbrace{x_n}_{\in C_{00}} \to x$ that is, every $x \in C_0$ is a limit point of C_{00} and since that is how we've defined closure $x \in \overline{C_{00}}$ and so $C_0 \subseteq \overline{C_{00}}$

3.4 Properties of Closures and Interiors

Proposition. $\mathbf{V}$ NVS, $A, B \subseteq \mathbf{V}$

1. $\text{Int}(A \cup B) \supseteq \text{Int}(A) \cup \text{Int}(B)$
2. $\text{Int}(A \cap B) = \text{Int}(A) \cap \text{Int}(B)$
3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
4. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

Example:

1. $A = [0,1), B = (1,2]$. $\overline{A \cap B} = \overline{\emptyset} = \emptyset$. $\bar{A} \cap \bar{B} = \{1\}$
2. $A = [0,1], B = [1,2]$. $\text{Int}(A \cup B) = (0,2)$. $\text{Int}(A) \cup \text{Int}(B) = (0,1) \cup (1,2)$

Proposition. $\mathbf{V}$ NVS, $A \subseteq \mathbf{V}$

1. $\text{Int}(\mathbf{V} \setminus A) = \mathbf{V} \setminus \overline{A}$
2. $\overline{\mathbf{V} \setminus A} = \mathbf{V} \setminus \text{Int}(A)$

Proof:

1. Observe that $A \subseteq \overline{A}$ and so $\mathbf{V} \setminus \overline{A} \subseteq \mathbf{V} \setminus A$. Since $\mathbf{V} \setminus \overline{A}$ is open and $\mathbf{V} \setminus \overline{A} \subseteq \mathbf{V} \setminus A$, $\mathbf{V} \setminus \overline{A} \subseteq \operatorname{Int}(\mathbf{V} \setminus A)$
   Then, $\operatorname{Int}(\mathbf{V} \setminus A) \subseteq \mathbf{V} \setminus A$ and so inverting this previous statement (by subtracting from $\mathbf{V}$ gives) $\mathbf{V} \setminus (\mathbf{V} \setminus A) = A \subseteq \mathbf{V} \setminus \operatorname{Int}(\mathbf{V} \setminus A)$. And since $\mathbf{V} \setminus \operatorname{Int}(\mathbf{V} \setminus A)$ is closed and contains A, we have by the smallness of the closure
   $\overline{A} \subseteq \mathbf{V} \setminus \operatorname{Int}(\mathbf{V} \setminus A$ then taking complements again $\operatorname{Int}(\mathbf{V} \setminus A) \subseteq \mathbf{V} \setminus \overline{A}$
2. Let $B = \mathbf{V} \setminus A$, we have from (1)
   $\operatorname{Int}(\mathbf{V} \setminus B) = \mathbf{V} \setminus \overline{B} \iff \operatorname{Int}(A) = \mathbf{V} \setminus \overline{(\mathbf{V} \setminus A)}$ Taking complements again we get $\iff \mathbf{V} \setminus \operatorname{Int}(A) = \overline{\mathbf{V} \setminus A}$

Definition. Let $\mathbf{V}$ NVS and $A \subseteq \mathbf{V}$. We define the **boundary** of A to be
$\partial A = \overline{A} \setminus \operatorname{Int}(A)$

Appendix